# Homotopy Theory of Enriched Mackey Functors

Closed Multicategories, Permutative Enrichments, and Algebraic Foundations for Spectral Mackey Functors

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ABSTRACT. Mackey functors provide the coefficient systems for equivariant cohomology theories. More generally, enriched presheaf categories provide a classification and organization for many stable model categories of interest. Changing enrichments along *K*-theory multifunctors provides an important tool for constructing spectral Mackey functors from Mackey functors enriched in algebraic structures such as permutative categories.

This work gives a detailed development of diagrams, presheaves, and Mackey functors enriched over closed multicategories. Change of enrichment, including the relevant compositionality, is treated with care. This framework is applied to the homotopy theory of enriched diagram and Mackey functor categories, including equivalences of homotopy theories induced by *K*-theory multifunctors. Particular applications of interest include diagrams and Mackey functors enriched in pointed multicategories, permutative categories, and symmetric spectra.

The first author dedicates this book to Nemili, Linus, and Kavya.
The second author dedicates this book to Jacqueline.

# Contents

# Preface

*"Just as the social insects build marvellously designed intricate structures by apparently carrying materials around at random so have the mathematicians built a marvellously articulated body of abstract concepts by following their individual instincts with an eye to what their colleagues are doing."*

– George W. Mackey, *What Do Mathematicians Do? Paris, 1982*

This work develops techniques and basic results concerning the homotopy theory of enriched diagrams and enriched Mackey functors. Presentation of a category of interest as a diagram category has become a standard and powerful technique in a range of applications. Diagrams that carry enriched structures provide deeper and more robust applications. With an eye to such applications, we provide further development of both the categorical algebra of enriched diagrams, and the homotopy theoretic applications in *K*-theory spectra.

The title of this work refers to certain enriched presheaves, known as Mackey functors, whose homotopy theory classifies that of equivariant spectra. More generally, certain stable model categories are classified as modules—in the form of enriched presheaves—over categories of generating objects. We provide further review of this motivating context in Chapter 0 below.

The main body of this work provides a detailed study of enriched diagrams, including enriched presheaves and enriched Mackey functors, and their homotopy theory. Part 1 provides background on the homotopy-theoretic context, including *K*-theory functors and the homotopy theory of multicategories. Part 2 extends this material to the homotopy theory of pointed multicategories, providing for the later applications a setting that is both conceptually and technically more natural.

The categorical algebra of enriched diagrams is the subject of Part 3 and may be of independent interest. It extends the theory of enrichment over a symmetric monoidal category in two ways. First, it gives a careful exposition of the theory of enrichment over a multicategory, including fundamental definitions of closed multicategory and self-enrichment. Second, it carefully explains change of enrichment along a multifunctor and the resulting diagram change of enrichment. Compositionality of the latter is more subtle, and treated in detail.

Applications to the homotopy theory of enriched diagrams, Mackey functors, and change of enrichment are the focus of Part 4. These arise from enrichments over permutative categories and pointed multicategories, with change of enrichment along *K*-theory multifunctors.

## Audience

This work is aimed at graduate students and researchers with an interest in category theory, algebraic *K*-theory, and homotopy theory. Our highly detailed exposition is designed to make this work accessible to a wide audience.

## Part and Chapter Summaries

This work consists of the following.

- Chapter 0: Motivations from Equivariant Topology
- Part 1: Background on Multicategories and *K*-Theory Functors
- Part 2: Homotopy Theory of Pointed Multicategories, $\mathcal{M}\underline{1}$-Modules, and Permutative Categories
- Part 3: Enrichment of Diagrams and Mackey Functors in Closed Multi-categories
- Part 4: Homotopy Theory of Enriched Diagrams and Mackey Functors
- Appendices A through C on Categories, Enriched Category Theory, and Multicategories
- Appendix D: Open Questions

Below is a brief summary of each part and chapter. Following these summaries, we outline the interdependence of their content. In the main text, each chapter starts with an introduction that describes more thoroughly its content and connections with other chapters.

### Chapter 0: Motivations from Equivariant Topology

This chapter reviews the use of enriched diagrams and enriched Mackey functors in equivariant homotopy theory and the theory of stable model categories. The results outlined here are not used directly in the main body of the work below, but they provide an important motivating context. The goal of this chapter is, therefore, to outline some of the main ideas and provide numerous references to the literature for further treatment.

**Part 1**. Background on Multicategories and *K*-Theory Functors
This part provides background that is essential for this text, and is more special-ized than that of the appendices. The main inputs for modern *K*-theory spectra are multicategories and permutative categories reviewed in Chapter 1. The con-struction of *K*-theory functors—also known as infinite loop space machinery—is reviewed in Chapter 2. More modern applications to the homotopy theory of mul-ticategories are summarized in Chapter 3.

### Chapter 1: Categorically Enriched Multicategories

This chapter reviews the 2-category of small multicategories, including three important special cases. These are pointed multicategories, left $\mathcal{M}\underline{1}$-modules, and permutative categories with multilinear functors. These variants are related by various free, forgetful, and endomorphism functors that will be used throughout the rest of this work.

### Chapter 2: Infinite Loop Space Machines

This chapter reviews the *K*-theory functors $K^{Se}$ and $K^{EM}$, due to Segal and Elmendorf-Mandell, respectively. These are also called infinite loop space machines because they produce connective spectra from permutative categories and multicategories. Each is constructed as a composite of other functors, via certain diagram categories, that we describe.

### Chapter 3: Homotopy Theory of Multicategories

This chapter reviews equivalences of homotopy theories between Multicat, the category of small multicategories and multifunctors, PermCat$^{st}$, the category of small permutative categories and strict monoidal functors, and PermCat$^{su}$, the category of small permutative categories and strictly unital symmetric monoidal functors. These equivalences are given by a free left adjoint to the endomorphism functor. This material provides important foundation for that of Part 2.

**Part 2**. Homotopy Theory of Pointed Multicategories, $\mathcal{M}\underline{1}$-Modules, and Permutative Categories

The purpose of these two chapters is to extend the equivalences of homotopy theories from Chapter 3 to the context of pointed multicategories. Chapter 4 develops the essential extensions to the pointed case. Chapter 5 develops the multifunctoriality results of the pointed free functor, along with multinaturality of the adjunction unit and counit.

### Chapter 4: Pointed Multicategories and $\mathcal{M}\underline{1}$-Modules Model All Connective Spectra

This chapter extends the material of Chapter 3 to a pointed free construction, $F_\bullet$, from pointed multicategories to permutative categories. This is not a restriction, along the inclusion of pointed multicategories among all multicategories, but an extension, along the functor that adjoins a disjoint basepoint. Essential results, such as the adjunction with the endomorphism construction and compatibility with stable equivalences, are likewise extended from Chapter 3.

### Chapter 5: Multiplicative Homotopy Theory

This chapter shows that the pointed free construction from Chapter 4, $F_\bullet$, is a non-symmetric multifunctor. Furthermore, $F_\bullet$ provides equivalences of homotopy theories between categories of non-symmetric algebras in pointed multicategories and permutative categories. This is the basis for applications to enriched diagrams in Chapter 12.

**Part 3**. Enrichment of Diagrams and Mackey Functors in Closed Multicategories

This part covers the categorical algebra of enrichment over multicategories (Chapter 6), change of enrichment along a multifunctor (Chapter 7), closed multicategories (Chapter 8), and self-enrichment thereof (Chapter 9). These are combined in Chapter 10 to develop the diagram change of enrichment for non-symmetric multifunctors (Theorem 10.3.1) and the presheaf change of enrichment for multifunctors (Theorem 10.3.4). These results are the foundation for applications to homotopy theory in Part 4.

### Chapter 6: Multicategorically Enriched Categories

This chapter gives the basic definitions and results for enrichment in a non-symmetric multicategory M. Proposition 6.2.1 shows that this material agrees, in the case that M is the endomorphism multicategory of a monoidal category, with classical enriched category theory over V. The main application takes M to be the multicategory of permutative categories and strictly unital symmetric monoidal functors, which is not an endomorphism multicategory. Sections 6.3 through 6.5 treat this case in detail.

### Chapter 7: Change of Multicategorical Enrichment

This chapter develops the first collection of results around change of enrichment along a (possibly non-symmetric) multifunctor. As in Chapter 6, it is shown that this theory extends the classical theory for enrichment over (possibly symmetric) monoidal categories. Compositionality and 2-functoriality for the change-of-enrichment constructions are treated in Sections 7.4 and 7.5, respectively.

### Chapter 8: The Closed Multicategory of Permutative Categories

This chapter extends the general multicategorical enrichment theory from Chapter 6 to enrichment over *closed* multicategories. Because enrichment over permutative categories is both illustrative of the general theory and essential for the further applications, this chapter focuses on that case in detail.

### Chapter 9: Self-Enrichment and Standard Enrichment

This chapter describes the theory of self-enrichment for closed multicategories, and of standard enrichment for multifunctors between closed multicategories. The self-enrichment of the multicategory of permutative categories, from Chapter 8, is a special case. Compositionality of standard enrichment is discussed in Section 9.3, and applied to the factorization of Elmendorf-Mandell $K$-theory in Section 9.4.

### Chapter 10: Enriched Mackey Functors of Closed Multicategories

This chapter provides the main results of Part 3. These make use of the preceding material on enrichment over (closed) multicategories, and apply it to categories of enriched diagrams and enriched Mackey functors. A key detail, both here and in the homotopical applications of Part 4 is that non-symmetric multifunctors provide a diagram change of enrichment, but not necessarily a change of enrichment for enriched Mackey functors (presheaves). The essential reason is that symmetry of a multifunctor is required for commuting the $(-)^{\mathsf{op}}$ in the domain of enriched presheaves with change of enrichment. Sections 10.5 and 10.6 give applications to Elmendorf-Mandell $K$-theory, with attention to the relevant symmetry conditions among other details.

**Part 4**. Homotopy Theory of Enriched Diagrams and Mackey Functors
The two final chapters of this work apply the preceding categorical algebra and homotopical constructions. Chapter 11 develops applications to the homotopy theory of enriched diagrams and Mackey functors in general. Chapter 12 gives further detailed applications to enriched diagrams and Mackey functors of pointed multicategories and permutative categories.

### Chapter 11: Homotopy Equivalences between Enriched Diagram Categories

This chapter establishes the general theory for a pair of non-symmetric multifunctors $(E, F)$ to provide inverse equivalences of homotopy theories between enriched diagram categories. The main result is Theorem 11.4.14 and does not require $E$ or $F$ to satisfy the symmetry condition of a multifunctor. A similar result for enriched Mackey functor categories, in Theorem 11.4.24, requires that $E$, but not necessarily $F$, is a multifunctor. This is important for the applications, Theorems 12.1.6 and 12.4.6 below. There, $E$ is an endomorphism multifunctor and $F$ is a corresponding free non-symmetric multifunctor.

## Chapter 12: Applications to Multicategories and Permutative Categories

This chapter applies the general theory from Chapter 11 to change of enrichment along the inverse equivalences of homotopy theories developed in Part 2. The main results, Theorems 12.1.6, 12.4.6, and 12.6.6, establish equivalences of homotopy theories for enriched diagram categories and Mackey functor categories over pointed multicategories, permutative categories, and $\mathcal{M}\underline{1}$-modules.

## Appendices

This work includes the following four appendices of supplemental background material and further open questions.

**Appendix A. Categories:** This appendix reviews basic concepts related to monoidal categories and 2-categories.

**Appendix B. Enriched Category Theory:** Here we review the classical theory of categories enriched over monoidal categories.

**Appendix C. Multicategories:** This appendix gives background on multicategories, including enriched multicategories, endomorphism multicategories, and pointed multicategories.

**Appendix D. Open Questions:** In this appendix we discuss a number of open questions related to the topics of this work. They provide further motivation for the main text.

## Chapter Interdependence

The material in Chapter 0 is not prerequisite for the main text, but is part of the broader context in which this work is situated. Appendices A, B, and C contain background material that will be used throughout.

As noted in the summaries above, part of this work involves abstract categorical algebra of multicategorical enrichments, which may be of independent interest. In the following table, we separate chapters and sections into columns according to whether they involve only (multi-)categorical algebra, or additional homotopy-theoretic concepts.

|          | (Multi-) Categorical Algebra | Homotopy Theory |
|----------|------------------------------|-----------------|
|          | Appendices A, B, and C       |                 |
| Part 1   | Chapter 1                    | Chapters 2 and 3 |
| Part 2   |                              | Chapters 4 and 5 |
| Part 3   | Chapters 6, 7, and 8<br>Sections 9.1 through 9.3<br>Sections 10.1 through 10.4 | Section 9.4<br>Sections 10.5 and 10.6 |
| Part 4   | Sections 11.1 through 11.3   | Section 11.4<br>Chapter 12 |

Each entry in the above table depends on those to its left and above. Thus, the material in the left column may be read independently of that in the right. The introduction of each chapter contains subsections titled *Connection with Other Chapters* and *Background* that give more detailed discussions of the respective dependencies.

## Related Literature

Here we list a selection of references for background or further reading.

2-Dimensional Categories: **[JY21]**

Monoidal Categories and Enriched Multicategories: **[JY∞, Yau16]**

Stable Homotopy Theory: **[BR20, May99]**

Equivariant Homotopy Theory: **[tD79, LMS86, May96, HHR21]**

Algebraic *K*-Theory Spectra: **[JY∞, JY22c, BF78, May78, Man10, Qui73, Seg74, Tho95, Wal85]**

Multifunctorial *K*-Theory: **[EM06, EM09, JY∞, JY22a, JY22b, JY22c, JY23]**

Spectral Mackey Functors: **[SS03, BO15, Bar17, MM19, BGS20, MM22, GM22, GMMO23]**

CHAPTER 0

# Motivations from Equivariant Topology

In this chapter we describe context from equivariant topology and the theory of stable model categories that motivates our further study of multicategorically enriched categories, enriched diagrams, enriched Mackey functors, and change of enrichment.

**Convention 0.0.1.** Assume throughout this chapter that $G$ is a finite group. See Remarks 0.1.1 and 0.2.8 for further comments on this convention.                    ⋄

**Connection with Main Content.** The purpose of this chapter is to indicate the role that categorical diagrams—particularly Mackey functors—play in equivariant homotopy theory. None of the mathematics in this present work depends on the content of this chapter, but the attendant applications are a key motivation.

For example, the Burnside 2-category $G\mathcal{E}$ (Definition 0.3.5) is enriched in the multicategory of permutative categories, PermCat$^{\mathsf{su}}$ (Section 1.4). We give a treatment of

- categories enriched in closed multicategories, in Chapter 6,
- change of enrichment, in Chapter 7,
- the closed multicategory structure of PermCat$^{\mathsf{su}}$ in Chapter 8, and
- self-enrichment for closed multicategories in Chapter 9.

In the Guillou-May Theorem 0.3.9, the domain of spectral Mackey functors, $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$, is given by a change of enrichment $(-)_{\mathbb{K}}$ and requires a distinction between enriched diagrams, with domain $G\mathcal{E}_{\mathbb{K}}$, and enriched Mackey functors, with domain $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$. We describe the relevant subtleties further in Remarks 0.3.7 and 10.5.5.

Similarly, but in a more abstract context, the spectral presheaves in the Schwede-Shipley Characterization Theorem 0.4.3 have domain $\mathcal{E}(P)^{\mathsf{op}}$. The input $\mathcal{E}(P)$ is the spectral endomorphism category of a set of compact generators $P$ for a simplicial, cofibrantly generated, proper, and stable model category M.

We give a general treatment of enriched diagrams and enriched Mackey functors, including interactions with change of enrichment, in Chapter 10. We develop techniques and applications for the corresponding homotopy theory in Chapters 11 and 12.

**Chapter Summary.** A substantive treatment of equivariant homotopy theory is well beyond our current scope. At the end of this introduction we give a list of key references. The remaining content in this chapter is restricted to those definitions and results that provide motivating context for our work below.

Section 0.1 concerns equivariant spaces.

- The orbit category of $G$ is denoted $\mathcal{O}_G$; see Definition 0.1.3.

- Elmendorf's Theorem 0.1.9 shows that the homotopy theory of $G$-spaces is equivalent to that of topological presheaves on $\mathcal{O}_G$.

Section 0.2 concerns Abelian Mackey functors.

- The Burnside ring of $G$ is denoted $G\mathcal{A}$. Its elements are isomorphism classes of finite $G$-sets with disjoint union and Cartesian product; see Definition 0.2.5.
- The Burnside category of $G$ is denoted $G\mathcal{B}$. Its morphisms are isomorphism classes of spans between finite $G$-sets. Disjoint union provides an enrichment over Abelian groups; see Definition 0.2.5.
- Abelian Mackey functors are enriched presheaves on the Burnside category; see Definition 0.2.9.

Sections 0.3 and 0.4 concern spectral Mackey functors.

- The Burnside 2-category of $G$ is denoted $G\mathcal{E}$. Its 1- and 2-cells are categories of spans between finite $G$-sets. Disjoint union, together with a choice of pullbacks and whiskering by a strict unit, provides an enrichment over permutative categories; see Definition 0.3.5.
- Spectral Mackey functors are enriched presheaves on a spectral enrichment of the Burnside 2-category; see Definition 0.3.8.
- The Guillou-May Theorem 0.3.9 shows that the homotopy theory of $G$-spectra is equivalent to that of spectral Mackey functors.
- The Schwede-Shipley Characterization Theorem 0.4.3 shows that the homotopy theory of a simplicial, cofibrantly generated, proper, and stable model category is equivalent to that of spectral presheaves on an endomorphism category of generating objects.

**References.** Main references for equivariant homotopy theory include, at least, the following. We include further specialized references at relevant points in the discussion below.

- The text by tom Dieck [**tD79**] lays the foundations for equivariant homotopy theory of spaces, including equivariant (co)homology theories known as Bredon cohomology.
- The monograph [**LMS86**], by Lewis, May, and Steinberger, gives the foundational treatment of equivariant stable homotopy theory, particularly equivariant spectra.
- The CBMS Alaska conference proceedings [**May96**] refines and significantly extends the preceding theory, including more development of the closed monoidal structure for equivariant spectra.
- The recent textbook account by Hill-Hopkins-Ravenel [**HHR21**] provides a more modern perspective, with thorough treatment of norm operations and the slice filtration that are essential in their solution of the Kervaire invariant problem [**HHR16**].

## 0.1. Equivariant Spaces and Presheaves on the Orbit Category

Recall Convention 0.0.1 that $G$ is assumed to be a finite group.

**Remark 0.1.1.** Many, but not all, of the concepts below extend to more general cases of interest, such as $G$ being a compact Lie group or a general topological group. The most important exception is our definition of the Burnside category in

Definitions 0.2.5 and 0.3.5, which depends on finiteness of $G$. See Remark 0.2.8 for further comments and references regarding that point. ⋄

**Definition 0.1.2.** Suppose C and M are categories, with C small. A *diagram of shape* C, or C-*diagram* in M is a functor

$$C \longrightarrow M.$$

A *presheaf on* C or C-*presheaf* in M is a diagram of shape $C^{op}$ in M, where $C^{op}$ is the opposite category of C. That is, a presheaf on C is a functor

$$C^{op} \longrightarrow M.$$

The phrase "in M" is often omitted when M is clear from context. Morphisms between diagrams and presheaves are natural transformations, and so

$$Cat(C, M) \quad and \quad Cat(C^{op}, M)$$

are the respective categories of diagrams and presheaves on C. If M is a symmetric monoidal closed category (Definition A.1.19) or, more generally, a closed multicategory (Definition 8.1.1), then there are corresponding enriched variants described in Definition 10.1.1. ⋄

**Definition 0.1.3.** The *orbit category* of a group $G$, denoted $\mathcal{O}_G$, consists of the following. Its objects are the $G$-orbits $G/H$, where $H$ is a subgroup of $G$, and its morphisms are the $G$-equivariant morphisms. ⋄

**Remark 0.1.4.** Note that each $G$-equivariant map

$$f : G/H \longrightarrow G/K \quad in \quad \mathcal{O}_G$$

determines and is determined by an element $g \in G$, where $f(eH) = gK$, such that $g^{-1}Hg \subset K$. Thus, the morphisms in $\mathcal{O}_G$ are given by subconjugacy relations. ⋄

**Definition 0.1.5** (*G*-Spaces)**.** A *G-space* is a topological space on which $G$ acts continuously. Morphisms of $G$-spaces are continuous functions that commute with the $G$-action. The category of $G$-spaces and their morphisms is denoted $\mathsf{Top}^G$. ⋄

**Definition 0.1.6** (Fixed Points)**.** For each $G$-space $X$, and for each subgroup $H$ in $G$, the *H-fixed point space*, denoted $X^H$, consists of the subspace of points on which $H$ acts trivially. As a $G$-space, $X^H$ can be defined equivalently as the space of $G$-equivariant morphisms

$$\mathsf{Top}^G(G/H, X),$$

where $G/H$ has the discrete topology. The assignment

$$G/H \longmapsto X^H$$

determines a presheaf of spaces on the orbit category $\mathcal{O}_G$,

(0.1.7) $$\Phi X : \mathcal{O}_G^{op} \longrightarrow \mathsf{Top}^G$$

called the *fixed point functor*. ⋄

Discussion of equivariant homotopy and (co)homology is beyond our current scope, but the following gives an indication of the role that the orbit category plays in equivariant topology.

**Explanation 0.1.8.** The coefficient systems for Bredon cohomology of $G$-spaces are given by presheaves

$$A : \mathcal{O}_G^{\mathsf{op}} \longrightarrow \mathsf{Ab},$$

where Ab is the category of Abelian groups and group homomorphisms. In particular, for a $G$-space $X$, the composite with $\pi_n$ for $n \geq 2$ yields a coefficient system

$$\mathcal{O}_G^{\mathsf{op}} \xrightarrow{\ \Phi X\ } \mathsf{Top} \xrightarrow{\ \pi_n\ } \mathsf{Ab}. \qquad\qquad \diamond$$

The following result due to Elmendorf [**Elm83**] gives a different indication of the importance of presheaves on the orbit category.

**Theorem 0.1.9** ([**Elm83, GMR19**]). *The fixed points functor, $\Phi$, induces a Quillen equivalence*

$$\Phi : \mathsf{Top}^G \xrightarrow{\ \simeq_Q\ } (\mathsf{Top}^G\text{-}\mathsf{Cat})(\mathcal{O}_G^{\mathsf{op}}, \mathsf{Top})$$

*between the category of $G$-equivariant topological spaces and the category of topological presheaves on $\mathcal{O}_G$.*

As we outline below, presheaves on the Burnside (2-)category, which are known as Mackey functors, fill an analogous role in the generalization to stable equivariant homotopy.

## 0.2. The Burnside Category and Abelian $G$-Mackey Functors

The Burnside category (Definition 0.2.5 below) extends the orbit category of $G$ using spans of finite $G$-sets. The key motivation for this expansion of $\mathcal{O}_G$ is to account for the restriction, induction, and transfer morphisms on finite $G$-sets. Further explanation and examples of this perspective can be found in [**Web00**] and [**HHR21**, Sections 8.1 and 8.2].

**Definition 0.2.1** (Finite $G$-Sets). Let $\mathcal{N}_G$ denote the following skeleton of the category of finite $G$-sets. The objects of $\mathcal{N}_G$ are pairs $(\overline{n}, \alpha)$, where $n$ is a natural number, $\overline{n} = \{1, \ldots, n\}$, and

$$\alpha : G \longrightarrow \Sigma_n$$

is a group homomorphism. We regard $X = (\overline{n}, \alpha)$ as a $G$-set with the action

$$g \cdot i = \alpha(g)(i)$$

for $g \in G$ and $i \in \overline{n}$. The morphisms $f : (\overline{n}, \alpha) \longrightarrow (\overline{p}, \beta)$ in $\mathcal{N}_G$ are $G$-equivariant morphisms. That is, $f$ is a map of sets $\overline{n} \longrightarrow \overline{p}$ such that

$$\beta(g)(f(i)) = f\big(\alpha(g)(i)\big) \quad \text{for} \quad g \in G \quad \text{and} \quad i \in \overline{n}.$$

We call $n$ the *cardinality* of $X = (\overline{n}, \alpha)$ and write $|X| = n$. Additionally, we define the following.

(1) The disjoint union of finite $G$-sets, $\amalg$, defines a permutative structure with unit given by the empty $G$-set. We write $(\overline{0}, !)$ for the empty finite set and the unique action homomorphism $G \longrightarrow \Sigma_0$.

(2) The Cartesian product, together with the lexicographic ordering

$$(0.2.2) \qquad\qquad \overline{n} \times \overline{p} \cong \overline{np} \quad \text{via} \quad (i, j) \longmapsto p(i - 1) + j,$$

defines a second permutative structure on $\mathcal{N}_G$. Its unit is the terminal $G$-set $(\overline{1}, !)$, consisting of the terminal set and the unique action homomorphism $G \longrightarrow \Sigma_1$. $\qquad \diamond$

**Definition 0.2.3** (Bicategory of Spans)**.** Suppose C is a small category with pull-backs, equipped with a choice of pullbacks for each pair of morphisms having a common codomain. The *bicategory of spans in* C is denoted $\mathsf{Span}(\mathsf{C})$ and consists of the following.

**0-Cells:** The 0-cells are objects $X \in \mathsf{C}$.

**1-Cells:** The 1-cells with domain $X$ and codomain $Y$ are triples $(A, f, g)$ that are spans

(0.2.4)
$$X \xleftarrow{f} A \xrightarrow{g} Y \quad \text{in} \quad \mathsf{C}.$$

Since the object $A$ is determined by the two morphisms, a span is some-times denoted by its pair of morphisms, $(f, g)$.

**2-Cells:** The 2-cells $(A, f, g) \longrightarrow (A', f', g')$ are morphisms $w : A \longrightarrow A'$ in C that make the following diagram commute in C.

$$
\begin{array}{ccc}
 & A & \\
f \swarrow & \downarrow w & \searrow g \\
X \leftarrow & & \rightarrow Y \\
f' \nwarrow & & \nearrow g' \\
 & A' & \\
\end{array}
$$

**Identities:** The identity 1-cell on a 0-cell $X$ is the triple $\Delta_X = (X, 1_X, 1_X)$ given by the identity morphisms in C. The identity 2-cell on a 1-cell $(A, f, g)$ is the identity morphism $1_A$ in C.

**Composition:** For objects $X, Y$, and $Z$ in C, the composition functor

$$\mathsf{Span}(\mathsf{C})(Y, Z) \times \mathsf{Span}(\mathsf{C})(X, Y) \longrightarrow \mathsf{Span}(\mathsf{C})(X, Z)$$

sends a composable pair to the span given by their chosen pullback, as shown below.

$$
\begin{array}{ccccc}
 & & A \times_Y B & & \\
 & \swarrow & & \searrow & \\
 & A & & B & \\
f \swarrow & & g \searrow \quad h \swarrow & & \searrow k \\
X & & Y & & Z \\
\end{array}
$$

Having a chosen pullback for each pair of morphisms with a common codomain makes the composition of 1-cells well defined. Universality of pullbacks makes it associative and unital up to isomorphisms that satisfy the axioms of bicategorical composition. See [**JY21**, Example 2.1.22] for further details of this construction. ⬦

Now we use Definitions 0.2.1 and 0.2.3 to define the Burnside category and its specialization, the Burnside ring. In Definition 0.3.5 below we generalize further to a Burnside 2-category.

**Definition 0.2.5** (Burnside Category and Burnside Ring)**.** The *Burnside category* of a finite group $G$, denoted $G\mathcal{B}$, is an Ab-enriched category defined as follows. The objects of $G\mathcal{B}$ are the finite $G$-sets $X \in \mathcal{N}_G$. The Abelian group $G\mathcal{B}(X, Y)$ for $X, Y \in \mathcal{N}_G$ is the Grothendieck group of isomorphism classes of spans

$$X \longleftarrow A \longrightarrow Y \quad \text{in} \quad \mathcal{N}_G.$$

Thus, $G\mathcal{B}$ is the category obtained from $\mathsf{Span}(\mathcal{N}_G)$ by taking isomorphism classes of 1-cells and then group-completing each set of morphisms with respect to the Abelian monoid structure given by disjoint union.

The *Burnside ring* of $G$, denoted $G\mathcal{A}$, is obtained by taking isomorphism classes of objects in $G\mathcal{B}$. Equivalently, the additive group of elements is given by the Grothendieck group of isomorphism classes of finite $G$-sets, with addition given by disjoint union. Its multiplication is induced by Cartesian product.                $\diamond$

**Lemma 0.2.6** (Self-Duality of $G\mathcal{B}$)**.** *There is an isomorphism of* Ab*-enriched categories*

$$(0.2.7) \qquad\qquad\qquad G\mathcal{B} \xrightarrow{\ \cong\ } G\mathcal{B}^{\mathsf{op}}$$

*that is the identity on objects and is induced on hom Abelian groups by the isomorphism*

$$\mathsf{Span}(\mathcal{N}_G)(X, Y) \xrightarrow{\ \cong\ } \mathsf{Span}(\mathcal{N}_G)(Y, X)$$

*that sends a span $(f, g)$ to its reverse, $(g, f)$.*

*Proof.* Functoriality of the indicated isomorphism follows from universality of the pullbacks defining composition.                $\square$

We warn the reader that the 2-categorical analog of the self-duality (0.2.7) does *not* hold for the Burnside 2-category $G\mathcal{E}$ in Definition 0.3.5 below. Sending $(f, g)$ to its reverse $(g, f)$ does not define a 2-functor in that context; see Remark 0.3.7.

**Remark 0.2.8** (Self-Duality and Stable Orbit Spectra)**.** Self-duality of the Burnside category $G\mathcal{B}$ (0.2.7) is nearly transparent in its simplicity, but it is an algebraic artifact of a much deeper topological phenomenon. Each orbit $G/H$ has an equivariant suspension spectrum, $\Sigma^\infty G/H_+$, and there is an equivalent definition of $G\mathcal{B}$ with morphisms given by stable equivariant morphisms $\Sigma^\infty G/H_+ \longrightarrow \Sigma^\infty G/K_+$; see [**May96**, Section XIX.3]. The stable orbit spectra $\Sigma^\infty G/H_+$ satisfy an equivariant self-duality ([**May96**, Section XVI.7] or [**HHR21**, Section 8.0C]) that implies that of Lemma 0.2.6.

The definition of the Burnside category in terms of stable orbit spectra is the more general one, with origins in work of tom Dieck [**tD79**]; see [**May96**, Section XVII.2]. The proofs that this definition can be given equivalently by spans of finite $G$-sets, as in Definition 0.2.5, depend on the assumption that $G$ is finite. In more general cases, the definition of the Burnside category in terms of stable orbit spectra is necessary.                $\diamond$

**Definition 0.2.9.** An *Abelian $G$-Mackey functor* is an Ab-enriched presheaf

$$G\mathcal{B}^{\mathsf{op}} \longrightarrow \mathsf{Ab}.$$

Because $G\mathcal{B}$ is isomorphic to $G\mathcal{B}^{\mathsf{op}}$ (Lemma 0.2.6), an Abelian $G$-Mackey functor is equivalently defined as a functor $G\mathcal{B} \longrightarrow \mathsf{Ab}$.                $\diamond$

**Remark 0.2.10.** Each Abelian $G$-Mackey functor $M$ has an associated Eilenberg-Mac Lane $G$-spectrum, $HM$. See [**May96**, Section V.4] or [**HHR21**, Theorem 8.8.4] for constructions via Elmendorf's Theorem 0.1.9. Such Mackey functors $M$, and their associated $G$-spectra $HM$, are the coefficient systems for Bredon cohomology of $G$-spectra.                $\diamond$

**Explanation 0.2.11.** An Abelian $G$-Mackey functor $M$ can be defined equivalently as a pair of functors

$$M_* : \mathcal{N}_G \longrightarrow \mathsf{Ab} \quad \text{and} \quad M^* : \mathcal{N}_G^{\mathsf{op}} \longrightarrow \mathsf{Ab}$$

that agree on objects and are subject to the following two axioms, where

$$MX = M_* X = M^* X$$

denotes the common value on objects.

(1) For each pair of objects $X$ and $Y$ in $\mathcal{N}_G$, applying $M_*$ to the structure morphisms of the coproduct

$$X \longrightarrow X \amalg Y \longleftarrow Y$$

induces a universal morphism with domain $MX \oplus MY$ that is an isomorphism

$$MX \oplus MY \xrightarrow{\ \cong\ } M(X \amalg Y).$$

(2) For each pullback diagram in $\mathcal{N}_G$,

$$
\begin{array}{ccc}
W & \xrightarrow{\ p\ } & X \\
{\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\
Y & \xrightarrow{\ g\ } & Z
\end{array}
$$

the following equality of composite morphisms holds in Ab:

$$(M^* f)(M_* g) = (M_* p)(M^* q).$$

See [**Web00**, Section 2] and [**HHR21**, Definition 8.2.3] for further discussion of this perspective, explanation of the equivalence with Definition 0.2.5, and several compelling examples.                                                                ◇

## 0.3. Equivariant Spectra and Presheaves on the Burnside 2-Category

For the category $\mathsf{C} = \mathcal{N}_G$, there is a choice of pullbacks that makes $\mathsf{Span}(\mathcal{N}_G)$ nearly a 2-category. Following Guillou-May [**GM22**, Remark 1.8 and Definition 6.2], the following will be used in the definition of the Burnside 2-category (Definition 0.3.5) below. A more general approach to such strictifications can be found in [**Gui10**].

**Explanation 0.3.1** (Choices of Pullbacks in $\mathcal{N}_G$). Recall the lexicographic ordering of products from (0.2.2). We use this to determine choices of pullbacks in $\mathcal{N}_G$, as follows. Suppose given the following composable pair of spans in $\mathcal{N}_G$,

$$
\begin{array}{ccccc}
 & A & & B & \\
{\scriptstyle f}\swarrow & & {\scriptstyle g}\searrow\ \ {\scriptstyle h}\swarrow & & \searrow{\scriptstyle k} \\
X & & Y & & Z,
\end{array}
$$

where

$$X = (\overline{n}_X, \alpha_X), \quad A = (\overline{n}_A, \alpha_A), \quad Y = (\overline{n}_Y, \alpha_Y), \quad B = (\overline{n}_B, \alpha_B), \text{ and } Z = (\overline{n}_Z, \alpha_Z).$$

Let

$$A \times_Y B = \{(a,b) \in \overline{n}_A \times \overline{n}_B \mid g(a) = h(b)\}$$

denote the pullback of $G$-sets, with its ordering induced by the lexicographic ordering on $\overline{n}_A \times \overline{n}_B$. This determines a unique order-preserving isomorphism of finite $G$-sets

$$(0.3.2) \qquad\qquad\qquad (\overline{p}, \rho) \xrightarrow{\ \cong\ } A \times_Y B$$

with $(\overline{p}, \rho) \in \mathcal{N}_G$.

We write $A \circ B = (\overline{p}, \rho)$ to denote this choice of pullback in $\mathcal{N}_G$ and let $\pi_A$ and $\pi_B$ denote the indicated composites below, where the unlabeled isomorphism is that of (0.3.2).

(0.3.3)



We note three consequences of these choices via lexicographic ordering.

(1) These choices for pullbacks make composition in $\mathsf{Span}(\mathcal{N}_G)$ strictly associative.
(2) The morphism $\pi_A$ is always order-preserving.
(3) The morphism $\pi_B$ is generally not order-preserving.

For each $Y = (\overline{n}_Y, \alpha_Y)$ in $\mathcal{N}_G$, let $\Delta_Y$ denote the unit 1-cell for $Y$ in $\mathsf{Span}(\mathcal{N}_G)$:

$$\Delta_Y = \left( Y \xleftarrow{\ 1_Y\ } Y \xrightarrow{\ 1_Y\ } Y \right).$$

In (0.3.3) above, if the span $(h, k)$ is the unit $\Delta_Y$, then $B = Y$ and we have

$$A \circ B = A, \qquad \pi_A = 1_A, \quad \text{and} \quad \pi_B = g.$$

Thus, $\Delta_Y$ is a strict right unit.

Now suppose, instead, that the span $(f, g)$ in (0.3.3) is the unit $\Delta_Y$. Then $A = Y$, but $\pi_B = g$ if and only if $h$ is an order-preserving $G$-map. In general, $\pi_B$ is an isomorphism of finite $G$-sets determined by the re-ordering of $\overline{n}_B$ that is induced by the fibers of $h$. ⋄

To construct a 2-category from $\mathsf{Span}(\mathcal{N}_G)$, the identity 1-cells $\Delta_X$ are augmented by new strict identities via the following construction.

**Definition 0.3.4** (Whiskering a Category). Suppose given a small category $\mathsf{D}$ with a distinguished object $\Delta \in \mathsf{D}$. Define the *whiskering at* $\Delta$, denoted $\mathsf{D}^\dagger$, as a category whose objects consist of those of $\mathsf{D}$, together with a new object $I$ and an isomorphism

$$I \xrightarrow[\cong]{\ \zeta_\Delta\ } \Delta.$$

The morphisms in $\mathsf{D}^\dagger$ are generated by those of $\mathsf{D}$ and composition with $\zeta_\Delta$ and its inverse. Thus, $\mathsf{D}^\dagger$ is the pushout in $\mathsf{Cat}$ of the two inclusions

$$\mathsf{D} \longleftarrow \{\Delta\} \longrightarrow \{\zeta_\Delta^{\pm 1}\}$$

where $\{\Delta\}$ denotes the discrete category on $\Delta$ and the right hand side denotes the category generated by the isomorphism $\zeta_\Delta$ and its inverse. A further elaboration of the whiskering construction is given in [**GM22**, Definition 6.1]. ⋄

**Definition 0.3.5** (The Burnside 2-Category). The *Burnside 2-category* of a finite group $G$ is a $\mathsf{PermCat}^{\mathsf{su}}$-enriched category (Explanation 6.3.2) denoted $G\mathcal{E}$ and defined as follows. Its objects are the finite $G$-sets $X = (\overline{n}, \alpha)$ of $\mathcal{N}_G$ (Definition 0.2.1). For each pair of objects

$$X = (\overline{n}, \alpha) \quad \text{and} \quad Y = (\overline{p}, \beta) \quad \text{in} \quad \mathcal{N}_G,$$

the category of 1- and 2-cells is given by

$$
(0.3.6) \qquad G\mathcal{E}(X,Y) = \begin{cases} \mathsf{Span}(\mathcal{N}_G)(X,Y) & \text{if} \quad X \neq Y \quad \text{or} \quad |X| \leq 1, \\ \mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger & \text{if} \quad X = Y \quad \text{and} \quad |X| \geq 2 \end{cases}
$$

where $\mathsf{Span}(\mathcal{N}_G)$ is the bicategory of spans (Definition 0.2.3) with the lexicographic choice of pullbacks from Explanation 0.3.1 and $\mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger$ is the whiskering of the category $\mathsf{Span}(\mathcal{N}_G)(X,X)$ as in Definition 0.3.4 at the unit 1-cell $\Delta_X$.

The horizontal composition of $\mathsf{Span}(\mathcal{N}_G)$ extends uniquely to $G\mathcal{E}$ such that the 1-cells $I_{\Delta_X} \in \mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger$ are strictly unital. The permutative structure of each $\mathsf{Span}(\mathcal{N}_G)(X,Y)$ given by disjoint union (Definition 0.2.1 (1)) also extends uniquely such that $(\overline{0}, !)$ remains its unit and for $Y \neq (\overline{0}, !)$ we have

$$
I_{\Delta_X} \coprod Y = X \coprod Y \quad \text{and} \quad Y \coprod I_{\Delta_X} = Y \coprod X.
$$

For further explanation of this structure, see [**GM22**, Definition 6.2], where our $G\mathcal{E}$ is denoted $G\mathcal{E}'$. ◇

**Remark 0.3.7** (Non-Self-Duality of $G\mathcal{E}$). Recall that the Burnside 1-category, $G\mathcal{B}$ in Definition 0.2.5 is self-dual (Lemma 0.2.6). However, the assignment that sends a span $(f,g)$ as in (0.2.4) to its reverse $(g,f)$ does not define a 2-functor

$$
G\mathcal{E} \longrightarrow G\mathcal{E}^{\mathsf{op}}
$$

because it does not preserve composition strictly. It is natural to consider the generalization from 2-functors to pseudofunctors, but the latter structure does not provide a $\mathsf{PermCat}^{\mathsf{su}}$-enriched functor in the sense of Explanation 6.3.12. This subtlety has further implications to be noted in Remark 10.5.5 below. ◇

The following is a special case of more general enriched Mackey functors introduced in Definition 10.1.1.

**Definition 0.3.8.** Suppose given a (possibly non-symmetric) $K$-theory multifunctor

$$
K : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}
$$

from permutative categories to spectra, and let $(-)_K$ denote the corresponding change of enrichment (Definition 7.1.1). The category of *spectral $G$-Mackey functors* for $K$ is the enriched presheaf category

$$
\mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}_K)^{\mathsf{op}}, \mathsf{Sp}\big),
$$

consisting of $\mathsf{Sp}$-enriched functors and transformations, as in (10.1.3). ◇

Note that if $K$ is a multifunctor in the symmetric sense—for example, if $K$ is the Elmendorf-Mandell $K$-theory, $\mathsf{K}^{\mathsf{EM}}$, in (2.5.8)—then $(G\mathcal{E}_K)^{\mathsf{op}}$ and $(G\mathcal{E}^{\mathsf{op}})_K$ are equal as $\mathsf{Sp}$-categories by Proposition 7.2.1. In such a case, the category of spectral $G$-Mackey functors is equal to $\mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}^{\mathsf{op}})_K, \mathsf{Sp}\big)$. However, if $K$ is not symmetric, then there is no such identification. See, e.g., Theorem 10.5.1 and Remark 10.5.5 for particular uses of these details.

For further development of both theory and applications of spectral Mackey functors in equivariant algebraic $K$-theory, the reader is referred to [**BO15, Bar17, MM19, BGS20, MM22, GM22, GMMO23**]. The key result for our purposes is the following from Guillou-May [**GM22**], which is a stable analog of Elmendorf's Theorem 0.1.9. Here, $\mathbb{K}$ denotes the non-symmetric $K$-theory multifunctor in [**GM22, GMMO23**].

**Theorem 0.3.9** ([**GM22**, Theorem 0.1])**.** *There is a zigzag of Quillen equivalences*

$$G\text{-Sp} \simeq_Q \mathsf{Sp}\text{-Cat}\big((G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}, \mathsf{Sp}\big)$$

*where* $G$-$\mathsf{Sp}$ *is the category of* $G$-*spectra.*

Thus, the Guillou-May theorem shows that the homotopy theory of $G$-spectra is equivalent to that of spectral $G$-Mackey functors for $\mathbb{K}$.

## 0.4. Stable Model Categories as Spectral Presheaf Categories

**Definition 0.4.1.** Suppose given a model category M. We recall the following terms briefly and refer the reader to [**Hov99, Hir03**] for more detailed descriptions.

(1) We say M is *simplicial* if it is enriched, tensored, and cotensored over simplicial sets, such that the following *pullback powering* condition holds. For each cofibration $i : A \longrightarrow B$ and fibration $p : X \longrightarrow Y$ in M, the universal morphism induced by $\mathsf{M}(i, X)$ and $\mathsf{M}(B, p)$,

$$\mathsf{M}(B, X) \longrightarrow \mathsf{M}(A, X) \times_{\mathsf{M}(A,Y)} \mathsf{M}(B, Y),$$

is a Kan fibration that is acyclic whenever either $i$ or $p$ is acyclic.

(2) We say M is *cofibrantly generated* if it is equipped with two sets of morphisms, $\mathcal{I}$ and $\mathcal{J}$, such that the following three statements hold.
   - Both $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument.
   - A morphism of M is a fibration if and only if it has the right lifting property with respect to every element of $\mathcal{J}$
   - A morphism of M is an acyclic fibration if and only if it has the right lifting property with respect to every element of $\mathcal{I}$.

(3) We say that M is *proper* if the following two conditions hold.
   - Every pushout of a weak equivalence along a cofibration is a weak equivalence.
   - Every pullback of a weak equivalence along a fibration is a weak equivalence.

(4) We say that M is *stable* if the suspension and loop functors on its homotopy category are inverse equivalences.                                   ◇

For the remainder of this section we suppose that M is a simplicial, cofibrantly generated, proper, and stable model category. The category of symmetric spectra over M [**SS03**, Definition 3.6.1] is denoted $\mathsf{Sp}^{\mathsf{M}}$. The following, from [**SS03**, Definition 3.7.5], describes an $\mathsf{Sp}$-enriched category generalizing the endomorphism spectrum associated to an object of M.

**Definition 0.4.2.** Suppose $P$ is a set of cofibrant objects in M. The *spectral endomorphism category* $\mathcal{E}(P)$ is the full $\mathsf{Sp}$-subcategory of $\mathsf{Sp}^{\mathsf{M}}$ with objects given by the fibrant replacements, relative to the level model structure on $\mathsf{Sp}^{\mathsf{M}}$, of the symmetric suspension spectra of the objects in $P$.                                   ◇

The following result of Schwede-Shipley gives a characterization of M via $\mathsf{Sp}$-enriched presheaves. In this result, $\mathsf{Sp}\text{-Cat}\big(\mathcal{E}(P)^{\mathsf{op}}, \mathsf{Sp}\big)$ denotes the $\mathcal{E}(P)$-presheaf category of $\mathsf{Sp}$ as in (10.1.3).

**Theorem 0.4.3** ([**SS03**, Theorem 3.3.3])**.** *Suppose $P$ is a set of compact generators of a simplicial, cofibrantly generated, proper, and stable model category* M. *Then there is a chain of simplicial Quillen equivalences*

$$\mathsf{M} \simeq_Q \mathsf{Sp}\text{-Cat}\big(\mathcal{E}(P)^{\mathsf{op}}, \mathsf{Sp}\big).$$

The work of Schwede-Shipley goes on to give a number of applications in (derived) Morita theory and equivariant stable homotopy. In each case, their work characterizes the relevant stable model category as a category of spectral presheaves, also called enriched Mackey functors (see Definition 10.1.1).

**Part 1**

# Background on Multicategories and *K*-Theory Functors

CHAPTER 1

# Categorically Enriched Multicategories

In this chapter we discuss the following four categorically-enriched multicategories that are central to this work:

- Multicat of small multicategories (Section 1.1),
- Multicat$_*$ of small pointed multicategories (Section 1.2),
- Mod$^{\mathcal{M}\underline{1}}$ of left $\mathcal{M}\underline{1}$-modules (Section 1.3), and
- PermCat$^{\mathsf{su}}$ of small permutative categories (Section 1.4).

Each of the Cat-multicategories,

$$\text{Multicat}, \quad \text{Multicat}_*, \quad \text{and} \quad \text{Mod}^{\mathcal{M}\underline{1}},$$

is induced by a corresponding symmetric monoidal Cat-category structure. On the other hand, the Cat-multicategory structure on PermCat$^{\mathsf{su}}$ is not induced by the monoidal structure on Mod$^{\mathcal{M}\underline{1}}$ or Multicat$_*$. The next table summarizes the various structures of these categories.

| | Multicat | Multicat$_*$ | Mod$^{\mathcal{M}\underline{1}}$ | PermCat$^{\mathsf{su}}$ |
|---|---|---|---|---|
| 2-category | C.1.33 | C.4.9 | 1.3.13 | A.2.3 |
| symmetric monoidal Cat-category | 1.1.19 | 1.2.8 | 1.3.23 | — |
| Cat-multicategory | 1.1.20 | 1.2.9 | 1.3.24 | 1.4.29 |

These four Cat-multicategories are related by several Cat-multifunctors

(1.0.1)



that we will explain in Theorem 1.4.38. Here is a summary table.

| | End | End$_{\bullet}$ | End$_{\mathcal{M}\underline{1}}$ | U$_{\bullet}$ | U$_{\mathcal{M}\underline{1}}$ |
|---|---|---|---|---|---|
| 2-functor | C.3.6 | C.4.10 | 1.3.16 | C.4.11 | 1.3.17 (3) and (4) |
| symmetric monoidal Cat-functor | — | — | — | 1.2.10 | 1.3.27 |
| Cat-multifunctor | 1.4.40 | 1.4.32 | 1.4.41 | 1.2.14 | 1.3.29 |

**Connection with Other Chapters.**

*Infinite Loop Space Machines.* As summarized in (2.5.1), the Cat-multicategory PermCat$^{\mathsf{su}}$ is connected to several categories in Segal $K$-theory and Elmendorf-Mandell $K$-theory via enriched multifunctors, including End$_{\mathcal{M}\underline{1}}$.

*Equivalences of Homotopy Theories.* In Chapter 3 we discuss the fact that the endomorphism multicategory construction End in (1.0.1) is an equivalence of homotopy theories. Moreover, in Chapters 4 and 5 we extend this observation to the pointed setting by showing that each of End., End$_{\mathcal{M}1}$, and U$_{\mathcal{M}1}$ is an equivalence of homotopy theories. In Part 4 we further extend these equivalences of homotopy theories to the respective categories of enriched diagrams and enriched Mackey functors. See Theorems 12.1.6, 12.4.6, and 12.6.6.

**Background.** Definitions about permutative categories, enriched multicategories, and pointed multicategories are in Appendices A.1 and C. Definitions for 2-categories and enriched categories are in Appendices A.2 and B.1. Symmetric monoidal enriched categories are discussed in Appendices B.2 through B.4.

**Chapter Summary.** The following table lists the main content in this chapter.

| Section 1.1. Multicategories | |
|---|---|
| multicategories as monadic algebras | 1.1.4 |
| Boardman-Vogt tensor product | 1.1.12 and 1.1.19 |
| Multicat as a Cat-multicategory | 1.1.20 |
| internal hom multicategories | 1.1.23 and 1.1.26 |
| Section 1.2. Pointed Multicategories | |
| smash product and pointed internal hom | 1.2.1 |
| pointed transformations | 1.2.6 |
| Multicat$_*$ as a Cat-multicategory | 1.2.9 |
| Cat-multifunctor U. : Multicat$_*$ $\longrightarrow$ Multicat | 1.2.14 |
| Section 1.3. $\mathcal{M}1$-Modules | |
| partition multicategories, $\mathcal{M}1$, and partition product | 1.3.1, 1.3.3, and 1.3.4 |
| 2-category Mod$^{\mathcal{M}1}$ of $\mathcal{M}1$-modules | 1.3.13 |
| endomorphism $\mathcal{M}1$-modules End$_{\mathcal{M}1}$ | 1.3.15 |
| free-forgetful 2-adjunction $\mathcal{M}1 \wedge -$ : Multicat$_*$ $\overrightarrow{\longleftarrow}$ Mod$^{\mathcal{M}1}$ : U$_{\mathcal{M}1}$ | 1.3.19 |
| Mod$^{\mathcal{M}1}$ as a Cat-multicategory | 1.3.24 |
| Cat-multifunctor U$_{\mathcal{M}1}$ : Mod$^{\mathcal{M}1}$ $\longrightarrow$ Multicat$_*$ | 1.3.29 |
| Section 1.4. Permutative Categories | |
| multilinear functors and transformations | 1.4.2 and 1.4.10 |
| PermCat$^{\mathsf{su}}$ as a Cat-multicategory | 1.4.15, 1.4.16, and 1.4.21 |
| End = U. $\circ$ End. = U. $\circ$ U$_{\mathcal{M}1}$ $\circ$ End$_{\mathcal{M}1}$ | 1.4.32, 1.4.39, 1.4.40, and 1.4.41 |

The material in this chapter is adapted from [**JY**∞, Chapters 5, 6, 8, and 10], which has all the detailed proofs. We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

## 1.1. Multicategories

There is a 2-category Multicat of small multicategories, multifunctors, and multinatural transformations (Theorem C.1.33). In this section we review

(1) the symmetric monoidal closed structure (Theorems 1.1.19 and 1.1.26) and
(2) the Cat-multicategory structure (Explanation 1.1.20)

on Multicat.

- The monoidal product is in Definition 1.1.12 after some preliminary constructions. This monoidal product is often called the *Boardman-Vogt tensor product* in the literature because of its origin in [**BV73**].
- The closed structure is given by the internal hom multicategory in Definition 1.1.23.

The material in this section is adapted from [**JY∞**, Chapters 5 and 6].

**Multicategories as Monadic Algebras.** The tensor product on small multicategories requires some preliminary constructions, which we recall first. The first fact we need is that small multicategories are algebras over a monad. We use the following notation for input profiles and output. Recall the class of profiles Prof (Definition C.1.1).

**Definition 1.1.1.** Suppose $C$ and $D$ are two classes. Given profiles
$$\langle c \rangle = \langle c_i \rangle_{i=1}^m \in \mathsf{Prof}(C) \quad \text{and} \quad \langle d \rangle = \langle d_j \rangle_{j=1}^n \in \mathsf{Prof}(D),$$
we define the following in $\mathsf{Prof}(C \times D)$:
$$\langle c \rangle \times d_j = \langle (c_i, d_j) \rangle_{i=1}^m$$
$$c_i \times \langle d \rangle = \langle (c_i, d_j) \rangle_{j=1}^n$$
$$\langle c \rangle \otimes \langle d \rangle = \langle \langle (c_i, d_j) \rangle_{i=1}^m \rangle_{j=1}^n$$
$$\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle = \langle \langle (c_i, d_j) \rangle_{j=1}^n \rangle_{i=1}^m$$

Denote by

(1.1.2) $$\zeta^\otimes = \zeta^\otimes_{m,n} : \langle c \rangle \otimes \langle d \rangle \xrightarrow{\ \cong\ } \langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle$$

the *transpose permutation* induced by changing order of indexing.  ◇

**Definition 1.1.3.** A *multigraph* $X$ consists of
- a class $\mathsf{Vt}X$ of *vertices* and
- a set $X(\langle x \rangle; x')$ for each tuple of vertices $\langle x \rangle$ and $x'$.

We refer to the elements of $X(\langle x \rangle; x')$ as *multiedges*, with *source* $\langle x \rangle$ and *target* $x'$. We let $\mathsf{Prof}(X)$ denote $\mathsf{Prof}(\mathsf{Vt}X)$.

A *morphism* of multigraphs
$$f : X \longrightarrow Y$$
consists of
- a function
$$f : \mathsf{Vt}X \longrightarrow \mathsf{Vt}Y$$
on vertices and
- a function
$$f : X(\langle x \rangle; x') \longrightarrow Y(f\langle x \rangle; f(x'))$$
on multiedges for each $(\langle x \rangle; x') \in \mathsf{Prof}(X) \times \mathsf{Vt}X$, with $f\langle x \rangle = \langle fx_j \rangle_{j=1}^n$ if $\langle x \rangle = \langle x_j \rangle_{j=1}^n$.

Moreover, we define the following.

- A multigraph is *small* if its class of vertices is a set.
- The collection of small multigraphs and their morphisms form a category, denoted MGraph.

This finishes the definition.                                                    ◇

The following result combines [**JY∞**, 5.5.9 and 5.5.11].

**Theorem 1.1.4.** *There is a free-forgetful adjunction*

$$
\text{MGraph} \underset{U}{\overset{L}{\underset{\perp}{\rightleftarrows}}} \text{Multicat}
$$

*that is strictly monadic.*

**Two Auxiliary Products.**

**Definition 1.1.5.** For multigraphs $X$ and $Y$ with vertex classes $C$ and $D$, respectively, we define a multigraph $X \& Y$ with vertex class $C \times D$ as follows. Given

$$
\langle c, d \rangle = \big\langle (c_j, d_j) \big\rangle_{j=1}^{n} \in \mathsf{Prof}(C \times D) \quad \text{and} \quad (c', d') \in C \times D,
$$

the set of multiedges with source $\langle c, d \rangle$ and target $(c', d')$ is given by the coproduct

$$
(1.1.6) \qquad (X \& Y)\big(\langle c, d \rangle; (c', d')\big) = \coprod_{\langle c'' \rangle \otimes \langle d'' \rangle = \langle c, d \rangle} X\big(\langle c'' \rangle; c'\big) \times Y\big(\langle d'' \rangle; d'\big).
$$

The coproduct is indexed by pairs

$$
\big(\langle c'' \rangle, \langle d'' \rangle\big) \in \mathsf{Prof}(C) \times \mathsf{Prof}(D) \quad \text{such that} \quad \langle c'' \rangle \otimes \langle d'' \rangle = \langle c, d \rangle
$$

with the tensor product of profiles in Definition 1.1.1.                          ◇

**Definition 1.1.7.** For small multicategories M and N, we define the *sharp product* M # N as the pushout in Multicat

$$
(1.1.8)
$$

$$
\begin{array}{ccc}
\mathrm{Ob}\,\mathsf{M} \times \mathrm{Ob}\,\mathsf{N} & \longrightarrow & \coprod_{d \in \mathrm{Ob}\,\mathsf{N}} \mathsf{M} \\
\downarrow & & \downarrow \\
\coprod_{c \in \mathrm{Ob}\,\mathsf{M}} \mathsf{N} & \longrightarrow & \mathsf{M} \# \mathsf{N}
\end{array}
$$

along morphisms induced by the inclusions

$$
\mathrm{Ob}\,\mathsf{M} \hookrightarrow \mathsf{M} \quad \text{and} \quad \mathrm{Ob}\,\mathsf{N} \hookrightarrow \mathsf{N}.
$$

                                                                                ◇

**Explanation 1.1.9** (Sharp Product)**.** Restricting Definition 1.1.7 to objects, there is a canonical bijection

$$
\mathrm{Ob}(\mathsf{M} \# \mathsf{N}) \cong \mathrm{Ob}\,\mathsf{M} \times \mathrm{Ob}\,\mathsf{N}.
$$

The operations of M # N are generated by operations of the form

$$
\phi \times d \in \mathsf{M} \times \{d\} \quad \text{and} \quad c \times \psi \in \{c\} \times \mathsf{N}
$$

subject to the axioms (i) through (v) below, which are determined by the pushout (1.1.8).

  (i) For $(c, d) \in \mathsf{M} \# \mathsf{N}$, there are equalities

$$
1_c \times d = 1_{(c,d)} = c \times 1_d.
$$

(ii) For operations $\phi, \phi_1, \ldots, \phi_n$ in $\mathsf{M}$ such that the composite below is defined, there is an equality
$$\gamma\big(\phi \times d \,;\, \langle \phi_j \times d \rangle_{j=1}^n\big) = \gamma\big(\phi \,;\, \langle \phi_j \rangle_{j=1}^n\big) \times d.$$

(iii) For $\sigma \in \Sigma_n$, there is an equality
$$(\phi \times d) \cdot \sigma = (\phi \cdot \sigma) \times d.$$

(iv) For operations $\psi, \psi_1, \ldots, \psi_m$ in $\mathsf{N}$ such that the composite below is defined, there is an equality
$$\gamma\big(c \times \psi \,;\, \langle c \times \psi_i \rangle_{i=1}^m\big) = c \times \gamma\big(\psi \,;\, \langle \psi_i \rangle_{i=1}^m\big).$$

(v) For $\sigma \in \Sigma_m$, there is an equality
$$(c \times \psi) \cdot \sigma = c \times (\psi \cdot \sigma).$$

These conditions are equivalent to the requirement that a multifunctor
$$F : \mathsf{M} \# \mathsf{N} \longrightarrow \mathsf{P}$$
consists of an assignment on objects,
$$F(c, d) \in \mathsf{Ob}\,\mathsf{P} \quad \text{for} \quad (c, d) \in \mathsf{Ob}\,\mathsf{M} \times \mathsf{Ob}\,\mathsf{N},$$
such that each of
$$F(c, -) : \mathsf{N} \longrightarrow \mathsf{P} \quad \text{and} \quad F(-, d) : \mathsf{M} \longrightarrow \mathsf{P}$$
is a multifunctor. $\diamond$

**The Boardman-Vogt Tensor Product of Multicategories.**

**Definition 1.1.10.** Suppose given small multicategories $\mathsf{M}$ and $\mathsf{N}$ along with operations
$$\phi \in \mathsf{M}\big(\langle c \rangle \,;\, c'\big) \quad \text{and} \quad \psi \in \mathsf{N}\big(\langle d \rangle \,;\, d'\big).$$
We define the following:
$$\phi \times \langle d \rangle = \langle \phi \times d_j \rangle_j \in \prod_j \mathsf{M}\big(\langle c \rangle \,;\, c'\big) \times \{d_j\}$$
$$\langle c \rangle \times \psi = \langle c_i \times \psi \rangle_i \in \prod_i \{c_i\} \times \mathsf{N}\big(\langle d \rangle \,;\, d'\big)$$
$$\phi \otimes \psi = \gamma\big(c' \times \psi \,;\, \phi \times \langle d \rangle\big) \in (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle \,;\, (c', d')\big)$$
$$\phi \otimes^{\mathsf{t}} \psi = \gamma\big(\phi \times d' \,;\, \langle c \rangle \times \psi\big) \in (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle \,;\, (c', d')\big)$$
Denote by $\xi^\otimes$ the bijection

(1.1.11) $\qquad (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes^{\mathsf{t}} \langle d \rangle \,;\, (c', d')\big) \xrightarrow{\;\cong\;} (\mathsf{M} \# \mathsf{N})\big(\langle c \rangle \otimes \langle d \rangle \,;\, (c', d')\big)$

induced by the transpose permutation $\xi^\otimes$ in (1.1.2) that interchanges order of indexing. $\diamond$

**Definition 1.1.12** (Tensor Product of Multicategories). For small multicategories $\mathsf{M}$ and $\mathsf{N}$, the tensor products of Definition 1.1.10 give two canonical morphisms of multigraphs
$$(U\mathsf{M}) \,\&\, (U\mathsf{N}) \xrightarrow[\;\xi^\otimes \circ \otimes^{\mathsf{t}}\;]{\;\otimes\;} U(\mathsf{M} \# \mathsf{N}).$$
Taking adjoints, we obtain the two morphisms in Multicat below. We define $\mathsf{M} \otimes \mathsf{N}$ to be their coequalizer in Multicat:

(1.1.13) $\qquad L\big((U\mathsf{M}) \,\&\, (U\mathsf{N})\big) \rightrightarrows \mathsf{M} \# \mathsf{N} \dashrightarrow \mathsf{M} \otimes \mathsf{N}$

For an object $(c, d) \in M \# N$, we denote by $c \otimes d$ its image in $M \otimes N$. Moreover, the tensor product $\otimes$ extends naturally to multifunctors.                                                ◇

**Explanation 1.1.14** (Unpacking the Tensor Product). Restricting Definition 1.1.12 to objects, there are canonical bijections

$$(1.1.15) \qquad \mathsf{Ob}(M \otimes N) \cong \mathsf{Ob}(M \# N) \cong \mathsf{Ob}\,M \times \mathsf{Ob}\,N.$$

The operations of $M \otimes N$ are generated by

$$\phi \otimes d \in M\big(\langle c \rangle ; c'\big) \times \{d\} \quad \text{and} \quad c \otimes \psi \in \{c\} \times N\big(\langle d \rangle ; d'\big)$$

subject to the relations of $M \# N$ in Explanation 1.1.9 along with one additional *interchange relation*

$$(1.1.16) \qquad \qquad \phi \otimes \psi = (\phi \otimes^{\mathsf{t}} \psi) \cdot \zeta^{\otimes}.$$

A multifunctor

$$F : M \otimes N \longrightarrow P$$

consists of an assignment on objects,

$$F(c, d) \in \mathsf{Ob}\,P \quad \text{for} \quad (c, d) \in \mathsf{Ob}\,M \times \mathsf{Ob}\,N,$$

such that the following two conditions hold.

- Each of

$$F(c, -) : N \longrightarrow P \quad \text{and} \quad F(-, d) : M \longrightarrow P$$

  is a multifunctor.
- There is an equality

$$(1.1.17) \qquad \qquad F(\phi \otimes \psi) = F(\phi \otimes^{\mathsf{t}} \psi) \cdot \zeta^{\otimes}$$

  for each $\phi \in M\big(\langle c \rangle ; c'\big)$ and $\psi \in N\big(\langle d \rangle ; d'\big)$.                                     ◇

**Definition 1.1.18** (Braiding on Multicategories). For small multicategories $M$ and $N$, suppose

$$\beta : M \# N \longrightarrow N \# M$$

is the multifunctor given

- on objects by $\beta(c, d) = (d, c)$ and
- on generating operations by

$$\beta(\phi \times d) = d \times \phi \quad \text{and} \quad \beta(c \times \psi) = \psi \times c.$$

Define the *braiding*

$$\beta : M \otimes N \longrightarrow N \otimes M$$

as the induced multifunctor on tensor products.                                              ◇

Recall the following.

- A symmetric monoidal $V$-category (Definition B.2.16) is a symmetric monoidal category in the $V$-enriched sense. Theorem 1.1.19 below involves the case $V = (\mathsf{Cat}, \times, \mathbf{1})$.
- There is a 2-category Multicat, which is Theorem C.1.33 with $V = \mathsf{Set}$.
- The initial operad $I$ in Example C.1.35 (i) has a single object $*$ and a single unit operation $1_* \in I(* ; *)$.

The following result combines [**JY**∞, 5.6.18 and 6.4.3].

**Theorem 1.1.19.** *The following quadruple is a symmetric monoidal category, with the associativity and unit isomorphisms induced by those of the sharp product,* #.

$$\big(\mathsf{Multicat}, \otimes, \mathsf{I}, \beta\big)$$

*Moreover, the tensor product* $\otimes$ *extends componentwise to multinatural transformations such that the quadruple above becomes a symmetric monoidal* Cat-*category.*

**Explanation 1.1.20** (Multicat as a Cat-Multicategory)**.** Since Multicat is a symmetric monoidal Cat-category, it has the structure of a Cat-multicategory by Proposition C.3.9, with the following data.

- Its objects are small multicategories.
- For small multicategories $\langle \mathsf{M}_j \rangle_{j=1}^n$ and $\mathsf{N}$, the $n$-ary multimorphism category is

(1.1.21) $$\mathsf{Multicat}\big(\langle \mathsf{M}_j \rangle_{j=1}^n \,;\, \mathsf{N}\big) = \begin{cases} \mathsf{Multicat}\big(\otimes_{j=1}^n \mathsf{M}_j \,,\, \mathsf{N}\big) & \text{if } n > 0 \text{ and} \\ \mathsf{Multicat}(\mathsf{I}, \mathsf{N}) & \text{if } n = 0. \end{cases}$$

  If $n > 0$, then this category has
    - multifunctors
    $$\otimes_{j=1}^n \mathsf{M}_j \longrightarrow \mathsf{N}$$
    as objects and
    - multinatural transformations between such multifunctors as morphisms.

  If $n = 0$, then the objects in
  $$\mathsf{Multicat}\big(\langle\rangle \,;\, \mathsf{N}\big) = \mathsf{Multicat}\,(\mathsf{I}, \mathsf{N})$$
  are multifunctors
  $$\mathsf{I} \longrightarrow \mathsf{N}.$$
  Each such multifunctor is determined by a choice of an object in $\mathsf{N}$. Thus, $\mathsf{Multicat}\big(\langle\rangle \,;\, \mathsf{N}\big)$ is canonically isomorphic to the underlying category of $\mathsf{N}$ as in Example C.1.16.
- The symmetric group action is induced by the braiding $\beta$ of the tensor product (Definition 1.1.18).
- The multicategorical composition is given by tensor product and composition of multifunctors, and likewise for multinatural transformations.

This finishes the description of the Cat-multicategory Multicat. ◇

**Internal Hom for Multicategories.** We use the following notation for a tuple of multifunctors $\langle F \rangle$.

**Definition 1.1.22.** Suppose given multicategories $\mathsf{M}$ and $\mathsf{N}$ together with a tuple of multifunctors $\langle F \rangle = \langle F_i \rangle_{i=1}^m$ with each $F_i : \mathsf{M} \longrightarrow \mathsf{N}$. Then we use the following notation.

- For $c \in \mathsf{Ob}\,\mathsf{M}$, denote by
$$\langle F \rangle c = \langle F_i c \rangle_{i=1}^m.$$

- For $\langle c \rangle = \langle c_j \rangle_{j=1}^n \in \mathsf{Prof}(\mathsf{Ob}\,\mathsf{M})$, denote by
$$\langle Fc \rangle = \langle\langle F_i c_j \rangle_{i=1}^m \rangle_{j=1}^n \quad \text{and} \quad \langle Fc \rangle^{\mathsf{t}} = \langle\langle F_i c_j \rangle_{j=1}^n \rangle_{i=1}^m.$$

- For an $n$-ary operation $\phi \in \mathsf{M}(\langle c \rangle; c')$, denote by

$$\langle F \rangle \phi = \langle F_i \phi \rangle_{i=1}^{m} \in \prod_{i=1}^{m} \mathsf{N}(F_i \langle c \rangle; F_i c').$$

This finishes the definition.                                                    ◇

**Definition 1.1.23.** For small multicategories $\mathsf{M}$ and $\mathsf{N}$, the *internal hom multicategory*

$$\mathsf{Hom}(\mathsf{M}, \mathsf{N})$$

is defined as follows.

- The objects of $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$ are multifunctors $\mathsf{M} \longrightarrow \mathsf{N}$.
- The $m$-ary operations

(1.1.24)                     $\theta : \langle F \rangle = \langle F_i \rangle_{i=1}^{m} \longrightarrow G$

    in $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$ are called *transformations* and are given by component $m$-ary operations

$$\theta_c \in \mathsf{N}(\langle F \rangle c; Gc) \quad \text{for} \quad c \in \mathsf{Ob}\,\mathsf{M}.$$

    For each operation $\phi \in \mathsf{M}(\langle c \rangle; c')$ with $\langle c \rangle = \langle c_j \rangle_{j=1}^{n}$ and $\theta_{\langle c \rangle} = \langle \theta_{c_j} \rangle_{j=1}^{n}$, the following *naturality condition* is required to hold:

(1.1.25)                  $\gamma(G\phi; \theta_{\langle c \rangle}) = \gamma(\theta_{c'}; \langle F \rangle \phi)) \cdot \xi_{m,n}^{\otimes}.$

- The unit operation

$$1_G : G \longrightarrow G$$

    is given by the identity multinatural transformation whose component at $c$ is the $(Gc)$-colored unit $1_{Gc}$ in $\mathsf{N}$.
- The composition and symmetric group action of $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$ are given componentwise by those of $\mathsf{N}$.

This finishes the definition of the internal hom multicategory $\mathsf{Hom}(\mathsf{M}, \mathsf{N})$. Moreover, $\mathsf{Hom}(-, -)$ extends naturally to multifunctors, contravariantly in the first argument and covariantly in the second argument.                            ◇

Recall from Definition A.1.19 that a symmetric monoidal category is *closed* if, for each object $x$, the functor $- \otimes x$ admits a right adjoint. The following is [**JY**∞, 5.7.14].

**Theorem 1.1.26.** *Equipped with the internal hom of Definition 1.1.23, the symmetric monoidal category* Multicat *in Theorem 1.1.19 is closed.*

## 1.2. Pointed Multicategories

There is a 2-category Multicat$_*$ of small pointed multicategories, pointed multifunctors, and pointed multinatural transformations (Appendix C.4). In this section we review

(1) the symmetric monoidal closed structure (Theorem 1.2.8) and
(2) the Cat-multicategory structure (Explanation 1.2.9)

on Multicat$_*$. The material in this section is adapted from [**JY**∞, Chapters 5 and 6].

**Definition 1.2.1.** Suppose $(\mathsf{M}, i^{\mathsf{M}})$ and $(\mathsf{N}, i^{\mathsf{N}})$ are small pointed multicategories. We define the following small pointed multicategories.

(1) The *wedge product*, also called the *wedge sum*, is the pointed multicategory defined by the following coequalizer in Multicat.

(1.2.2)
$$\mathsf{T} \xrightarrow[\ i^{\mathsf{N}}\ ]{\ i^{\mathsf{M}}\ } \mathsf{M} \amalg \mathsf{N} \dashrightarrow \mathsf{M} \vee \mathsf{N}$$

(2) The *smash product* is the pointed multicategory defined by the following pushout in Multicat.

(1.2.3)
$$\begin{array}{ccc}
(\mathsf{M} \otimes \mathsf{T}) \amalg (\mathsf{T} \otimes \mathsf{N}) & \longrightarrow & \mathsf{M} \otimes \mathsf{N} \\
\downarrow & & \downarrow {\scriptstyle \omega_{\mathsf{M},\mathsf{N}}} \\
\mathsf{T} & \xrightarrow{\ i^{\mathsf{M} \wedge \mathsf{N}}\ } & \mathsf{M} \wedge \mathsf{N}
\end{array}$$

In (1.2.3), $\otimes$ is the tensor product in (1.1.13), and $\mathsf{T}$ is the terminal multicategory in Example C.1.17. Moreover, the smash product extends naturally to pointed multifunctors.

(3) The *smash unit* $\mathsf{I}_+$ is the pointed multicategory

(1.2.4)
$$\mathsf{I}_+ = \mathsf{I} \amalg \mathsf{T}$$

with $\mathsf{I}$ the initial operad in Example C.1.35 (i). The pointed structure $\mathsf{T} \longrightarrow \mathsf{I}_+$ is given by the $\mathsf{T}$ summand in $\mathsf{I}_+$.

(4) The *pointed internal hom* is the pointed multicategory defined by the following pullback in Multicat, where $\mathrm{Hom}(-,-)$ is the internal hom multicategory in Definition 1.1.23.

(1.2.5)
$$\begin{array}{ccc}
\mathrm{Hom}_*(\mathsf{M},\mathsf{N}) & \longrightarrow & \mathsf{T} \\
\downarrow & & \downarrow \\
\mathrm{Hom}(\mathsf{M},\mathsf{N}) & \longrightarrow & \mathrm{Hom}(\mathsf{T},\mathsf{N})
\end{array}$$

The pointed structure $\mathsf{T} \longrightarrow \mathrm{Hom}_*(\mathsf{M},\mathsf{N})$ is induced by the composite

$$\mathsf{T} \cong \mathrm{Hom}(\mathsf{M},\mathsf{T}) \xrightarrow{(i^{\mathsf{N}})_*} \mathrm{Hom}(\mathsf{M},\mathsf{N}) \xrightarrow{(i^{\mathsf{M}})^*} \mathrm{Hom}(\mathsf{T},\mathsf{N}),$$

which is equal to the right vertical morphism in (1.2.5). Moreover, $\mathrm{Hom}_*(-,-)$ extends naturally to pointed multifunctors, contravariantly in the first argument and covariantly in the second argument.

This finishes the definition. ⋄

**Explanation 1.2.6** (Pointed Internal Hom Multicategory). Unpacking the pullback (1.2.5), we describe the pointed multicategory $\mathrm{Hom}_*(\mathsf{M},\mathsf{N})$ explicitly as follows.

- Its objects are pointed multifunctors $(\mathsf{M}, i^{\mathsf{M}}) \longrightarrow (\mathsf{N}, i^{\mathsf{N}})$.
- For pointed multifunctors

$$\langle F \rangle = \langle F_i \rangle_{i=1}^m, \ G : \mathsf{M} \longrightarrow \mathsf{N},$$

an *m*-ary operation in $\mathrm{Hom}_*(\mathsf{M},\mathsf{N})\big(\langle F \rangle; G\big)$ is a transformation

$$\theta = \{\theta_c\}_{c \in \mathrm{Ob}\,\mathsf{M}} : \langle F \rangle \longrightarrow G$$

as in (1.1.24) such that the component at the basepoint object $* \in \mathrm{Ob}\,\mathsf{M}$,

(1.2.7)
$$\theta_* = \iota_m \in \mathsf{N}\big(\langle F \rangle *; G(*)\big) = \mathsf{N}\big(\langle * \rangle_{i=1}^m; *\big),$$

is equal to the $m$-ary basepoint operation in $\mathsf{N}$.

A transformation that satisfies the basepoint condition (1.2.7) is called a *pointed transformation*.                                                                                                    ◇

The following result combines [**JY∞**, 5.7.22 and 6.4.4] and is the pointed variant of Theorems 1.1.19 and 1.1.26.

**Theorem 1.2.8.** *In the context of Definition 1.2.1, the quadruple*

$$\left(\mathsf{Multicat}_*, \wedge, \mathsf{I}_+, \mathsf{Hom}_*\right)$$

*is a complete and cocomplete symmetric monoidal closed category, with the associativity and unit isomorphisms induced by those of* $\mathsf{Multicat}$ *in Theorem 1.1.19.*

*Moreover, the smash product* $\wedge$ *extends componentwise to pointed multinatural transformations such that* $\mathsf{Multicat}_*$ *becomes a symmetric monoidal* $\mathsf{Cat}$-*category.*

Next is the pointed variant of Explanation 1.1.20.

**Explanation 1.2.9** ($\mathsf{Multicat}_*$ as a $\mathsf{Cat}$-Multicategory). Since $\mathsf{Multicat}_*$ is a symmetric monoidal $\mathsf{Cat}$-category, it has the structure of a $\mathsf{Cat}$-multicategory by Proposition C.3.9, with the following data.

- Its objects are small pointed multicategories.
- For small pointed multicategories $\left\langle (\mathsf{M}_j, i^{\mathsf{M}_j}) \right\rangle_{j=1}^n$ and $(\mathsf{N}, i^{\mathsf{N}})$, the $n$-ary multimorphism category

$$\mathsf{Multicat}_* \left( \left\langle (\mathsf{M}_j, i^{\mathsf{M}_j}) \right\rangle_{j=1}^n ; (\mathsf{N}, i^{\mathsf{N}}) \right) = \mathsf{Multicat}_* \left( \wedge_{j=1}^n \mathsf{M}_j , \mathsf{N} \right)$$

  has
  - pointed multifunctors

$$\wedge_{j=1}^n \mathsf{M}_j \longrightarrow \mathsf{N}$$

    as objects and
  - pointed multinatural transformations between such pointed multifunctors as morphisms.
  If $n = 0$, then the objects in

$$\mathsf{Multicat}_* \left( \langle \rangle ; (\mathsf{N}, i^{\mathsf{N}}) \right) = \mathsf{Multicat}_* (\mathsf{I}_+, \mathsf{N})$$

  are pointed multifunctors

$$\mathsf{I}_+ = \mathsf{I} \amalg \mathsf{T} \longrightarrow (\mathsf{N}, i^{\mathsf{N}}).$$

  Each such pointed multifunctor is determined by a choice of an object in $\mathsf{N}$. Thus, $\mathsf{Multicat}_* \left( \langle \rangle ; (\mathsf{N}, i^{\mathsf{N}}) \right)$ is canonically isomorphic to the underlying category of $\mathsf{N}$ as in Example C.1.16.
- The symmetric group action is induced by the braiding of the smash product, which, in turn, is induced by the braiding of the tensor product (Definition 1.1.18).
- The multicategorical composition is given by smash product and composition of pointed multifunctors, and likewise for pointed multinatural transformations.

This finishes the description of the $\mathsf{Cat}$-multicategory $\mathsf{Multicat}_*$.                                                              ◇

**Explanation 1.2.10** (Forgetting Basepoints). The forgetful 2-functor

(1.2.11) $$\mathsf{U}_{\bullet} : (\mathsf{Multicat}_*, \wedge, \mathsf{I}_+) \longrightarrow (\mathsf{Multicat}, \otimes, \mathsf{I})$$

in Proposition C.4.11 is a symmetric monoidal Cat-functor (Definition B.2.24) with the following structure morphisms:

**Unit Constraint:** It is the multifunctor

(1.2.12) $$i : \mathsf{I} \longrightarrow \mathsf{I}_+ = \mathsf{I} \coprod \mathsf{T}$$

given by the inclusion of the $\mathsf{I}$ summand in $\mathsf{I}_+$.

**Monoidal Constraint:** Its component for small pointed multicategories M and N is the multifunctor

(1.2.13) $$\varpi_{\mathsf{M},\mathsf{N}} : \mathsf{M} \otimes \mathsf{N} \longrightarrow \mathsf{M} \wedge \mathsf{N}$$

given by the right vertical arrow in the pushout (1.2.3) that defines the smash product. ◇

**Explanation 1.2.14** (U$_{\bullet}$ as a Cat-Multifunctor). Regarding Multicat and Multicat$_*$ as Cat-multicategories as in Explanations 1.1.20 and 1.2.9, respectively, the symmetric monoidal Cat-functor U$_{\bullet}$ in (1.2.11) induces a Cat-multifunctor

(1.2.15) $$\mathsf{U}_{\bullet} : \mathsf{Multicat}_* \longrightarrow \mathsf{Multicat}$$

in the sense of Definition C.1.19 with the following structure:

**Object Assignment:** U$_{\bullet}$ sends a small pointed multicategory $(\mathsf{M}, i)$ to the multicategory M.

**Multimorphism Functors:** Suppose given small pointed multicategories $\langle \mathsf{M} \rangle = \langle \mathsf{M}_j \rangle_{j=1}^n$ and N. The $n$-ary multimorphism functor

$$\mathsf{U}_{\bullet} : \mathsf{Multicat}_*\big( \textstyle\bigwedge_{j=1}^n \mathsf{M}_j , \mathsf{N} \big) \longrightarrow \mathsf{Multicat}\big( \textstyle\bigotimes_{j=1}^n \mathsf{M}_j , \mathsf{N} \big)$$

sends a pointed multifunctor

$$P : \textstyle\bigwedge_{j=1}^n \mathsf{M}_j \longrightarrow \mathsf{N}$$

to the composite multifunctor

$$\textstyle\bigotimes_{j=1}^n \mathsf{M}_j \xrightarrow{\ \varpi\ } \textstyle\bigwedge_{j=1}^n \mathsf{M}_j \xrightarrow{\ P\ } \mathsf{N}$$

if $n > 0$, where $\varpi$ is an iterate of the monoidal constraint in (1.2.13). If $n = 0$, then U$_{\bullet}$ sends $P$ to the composite multifunctor

$$\mathsf{I} \xrightarrow{\ i\ } \mathsf{I}_+ \xrightarrow{\ P\ } \mathsf{N}$$

where $i$ is the unit constraint in (1.2.12). For a pointed multinatural transformation, U$_{\bullet}$ is defined similarly by whiskering with $\varpi$ if $n > 0$ and with $i$ if $n = 0$. ◇

## 1.3. $\mathcal{M}\underline{1}$-Modules

In this section we review a full sub-2-category of Multicat$_*$ given by the left modules over a small pointed multicategory $\mathcal{M}\underline{1}$ (Example 1.3.3).

- In Definitions 1.3.1 and 1.3.4 we discuss the partition multicategory $\mathcal{M}a$ of a pointed finite set $a$ and a pairing called the partition product.
- The partition multicategory $\mathcal{M}\underline{1}$ is equipped with the structure of a commutative monoid in $(\mathsf{Multicat}_*, \wedge, \mathsf{I}_+)$ in Definition 1.3.12.

- The 2-category of left $\mathcal{M}\underline{1}$-modules is in Definition 1.3.13. Its main properties are summarized in Proposition 1.3.17.
- The symmetric monoidal Cat-category of left $\mathcal{M}\underline{1}$-modules is in Definition 1.3.23. Its induced Cat-multicategory is discussed in Explanation 1.3.24.

The material in this section is adapted from [**JY∞**, Chapters 8 and 10].

**Partition Multicategories.** Recall from Definition A.1.17 the permutative category

$$\left(\mathcal{F}, \wedge, \underline{1}, \xi\right)$$

of pointed finite sets and pointed functions with the smash product as the monoidal product.

**Definition 1.3.1.** For a pointed finite set $a$ with basepoint $*$, the unpointed finite set $a^\flat$ is obtained from $a$ by removing its basepoint:

(1.3.2)                    $$a^\flat = a \smallsetminus \{*\}.$$

The *partition multicategory*, denoted $\mathcal{M}a$, is the pointed multicategory defined as follows.

**Objects:** $\mathsf{Ob}(\mathcal{M}a) = \mathcal{P}(a^\flat)$, the set of basepoint-free subsets of $a$.

**Multimorphisms:** For an $n$-tuple $\langle s \rangle = \langle s_j \rangle_{j=1}^n$ with each $s_j \in \mathcal{P}(a^\flat)$ and $t \in \mathcal{P}(a^\flat)$, the set of $n$-ary operations is

$$(\mathcal{M}a)\big(\langle s \rangle ; t\big) = \begin{cases} \big\{\iota_{\langle s \rangle}\big\} & \text{if } \langle s \rangle \text{ is a partition of } t \text{ and} \\ \varnothing & \text{otherwise.} \end{cases}$$

In the first case, $\big\{\iota_{\langle s \rangle}\big\}$ is a one-element set.

**Pointed Structure:** It is the multifunctor $i : \mathsf{T} \longrightarrow \mathcal{M}a$ given by
- the empty set $\varnothing \in \mathcal{P}(a^\flat)$ and
- the unique operations

$$\iota^n = \iota_{\langle \varnothing \rangle_{j=1}^n} \in (\mathcal{M}a)\big(\langle \varnothing \rangle_{j=1}^n ; \varnothing\big) \quad \text{for} \quad n \geq 0.$$

**Other Structure:** The colored units, symmetric group action, and composition are uniquely defined by the terminal property of a one-element set.

This finishes the definition of the partition multicategory $\mathcal{M}a$.                    ◇

An important pointed multicategory for this work is the partition multicategory $\mathcal{M}\underline{1}$ of the pointed finite set $\underline{1} = \{0, 1\}$.

**Example 1.3.3.** We explain in detail the partition multicategory $\mathcal{M}\underline{1}$.

- Its object set is

$$\mathsf{Ob}(\mathcal{M}\underline{1}) = \mathcal{P}(\underline{1}^\flat) = \big\{\varnothing, \{1\}\big\}.$$

- Its nonempty sets of operations are

$$\mathcal{M}\underline{1}\big(\langle \varnothing \rangle_{j=1}^n ; \varnothing\big) = \{\iota^n\}$$

$$\mathcal{M}\underline{1}\big((\varnothing, \ldots, \{1\}, \ldots, \varnothing) ; \{1\}\big) = \{\pi_j^n\}$$

for $n \geq 0$ and $j \in \{1, \ldots, n\}$. In the definition of $\pi_j^n$ above,

$$(\varnothing, \ldots, \{1\}, \ldots, \varnothing)$$

has length $n$ with $\{1\}$ in the $j$th entry and $\varnothing$ in other entries. The operations $\iota^n$ for $n \geq 0$ are closed under the symmetric group action and composition.

- The $\{1\}$-colored unit is

$$\pi_1^1 \in \mathcal{M}\underline{1}(\{1\}\,;\{1\}).$$

- The right $\Sigma_n$-action on $\pi_j^n$ is given by

$$\pi_j^n \cdot \sigma = \pi_{\sigma^{-1}(j)}^n \quad \text{for} \quad \sigma \in \Sigma_n.$$

- The composition involving $\pi_j^n$ is given by

$$\gamma\left(\pi_j^n\,;\left(\langle \iota^{k_i}\rangle_{i=1}^{j-1}, \pi_p^{k_j}, \langle \iota^{k_i}\rangle_{i=j+1}^n\right)\right) = \pi_{k_1+\cdots+k_{j-1}+p}^{k_1+\cdots+k_n}$$

for $k_1, \ldots, k_n \geq 0$ and $p \in \{1, \ldots, k_j\}$.

This finishes the description of the partition multicategory $\mathcal{M}\underline{1}$.                    ◇

**The Commutative Monoid $\mathcal{M}\underline{1}$.** The partition multicategory $\mathcal{M}\underline{1}$ becomes a commutative monoid (Definition A.1.18) via the following multiplication. Recall the smash product defined in (1.2.3).

**Definition 1.3.4.** For a pair of pointed finite sets $a$ and $b$, the *partition product* is the pointed multifunctor

(1.3.5)                    $$\textstyle\prod_{a,b} : \mathcal{M}a \wedge \mathcal{M}b \longrightarrow \mathcal{M}(a \wedge b)$$

defined as follows.

**Object Assignment:** This is given by the Cartesian product of subsets, noting that

$$s \times t \subset (a^\flat \times b^\flat) \cong (a \wedge b)^\flat \quad \text{for} \quad s \subset a^\flat \quad \text{and} \quad t \subset b^\flat.$$

**Multimorphism Assignment:** By Explanation 1.1.14, the generating operations of the tensor product

$$\mathcal{M}a \otimes \mathcal{M}b$$

are of the form

$$\iota_{\langle s\rangle} \otimes t \in \mathcal{M}a\big(\langle s\rangle\,;s'\big) \times \{t\} \quad \text{and} \quad s \otimes \iota_{\langle t\rangle} \in \{s\} \times \mathcal{M}b\big(\langle t\rangle\,;t'\big)$$

with
- $s \subset a^\flat$, $t \subset b^\flat$,
- $\langle s\rangle$ a partition of $s'$ in $\mathcal{M}a$, and
- $\langle t\rangle$ a partition of $t'$ in $\mathcal{M}b$.

We define partitions of $s' \times t$ and $s \times t'$ by, respectively,

$$\langle s\rangle \times t = \langle s_i \times t\rangle_i \quad \text{and} \quad s \times \langle t\rangle = \langle s \times t_j\rangle_j.$$

Then we define $\prod_{a,b}$ on generating operations by

$$\iota_{\langle s\rangle} \otimes t \longmapsto \iota_{\langle s\rangle \times t} \quad \text{and} \quad s \otimes \iota_{\langle t\rangle} \longmapsto \iota_{s \times \langle t\rangle}.$$

The definition of $\prod_{a,b}$ descends to the smash product $\mathcal{M}a \wedge \mathcal{M}b$ because the Cartesian product of any set with the empty set is empty. Therefore, each generating operation of the form

$$\iota_{\langle s\rangle} \times \varnothing \quad \text{or} \quad \varnothing \times \iota_{\langle t\rangle}$$

is sent to a partition of the empty set, which is a basepoint operation of $\mathcal{M}(a \wedge b)$. This finishes the definition of the partition product $\prod_{a,b}$.                    ◇

**Lemma 1.3.6.** *For each pointed finite set $b$, the partition products for $\underline{1}$ and $b$ are isomorphisms*

$$(1.3.7) \qquad \textstyle\prod_{\underline{1},b} : \mathcal{M}\underline{1} \wedge \mathcal{M}b \xrightarrow{\;\cong\;} \mathcal{M}(\underline{1} \wedge b) \cong \mathcal{M}b$$

$$\textstyle\prod_{b,\underline{1}} : \mathcal{M}b \wedge \mathcal{M}\underline{1} \xrightarrow{\;\cong\;} \mathcal{M}(b \wedge \underline{1}) \cong \mathcal{M}b.$$

Recall from Theorem 1.2.8 the symmetric monoidal category $\mathsf{Multicat}_*$.

**Proposition 1.3.8.** *The partition multicategory $\mathcal{M}$ defines a symmetric monoidal functor*

$$\mathcal{M} : (\mathcal{F}^{\mathsf{op}}, \wedge, \underline{1}) \longrightarrow (\mathsf{Multicat}_*, \wedge, \mathsf{I}_+)$$

*with the following structure morphisms.*

**Unit Constraint:** *It is the pointed multifunctor*

$$(1.3.9) \qquad \mathcal{M}^0 : \mathsf{I}_+ = \mathsf{I} \coprod \mathsf{T} \longrightarrow \mathcal{M}\underline{1}$$

*determined by sending the unique object of $\mathsf{I}$ to $\{1\} \in \mathcal{P}(\underline{1}^\flat)$.*

**Monoidal Constraint:** *It is the composite of the partition product $\prod$ with the lexicographic isomorphisms*

$$(1.3.10) \qquad \mathcal{M}^2_{\underline{m},\underline{n}} : \mathcal{M}\underline{m} \wedge \mathcal{M}\underline{n} \xrightarrow{\;\prod_{\underline{m},\underline{n}}\;} \mathcal{M}(\underline{m} \wedge \underline{n}) \cong \mathcal{M}(\underline{mn}).$$

**Explanation 1.3.11.** The symmetric monoidal functor $\mathcal{M}$ is neither strong nor strictly unital because the unit and monoidal constraints, $\mathcal{M}^0$ and $\mathcal{M}^2$ in (1.3.9) and (1.3.10), are not isomorphisms.                                                  ⋄

Recall from Definition A.1.18 the notion of a *commutative monoid* in a symmetric monoidal category. The following definition uses the symmetric monoidal category $(\mathsf{Multicat}_*, \wedge, \mathsf{I}_+)$ in Theorem 1.2.8.

**Definition 1.3.12.** We define the commutative monoid

$$\left( \mathcal{M}\underline{1}, \textstyle\prod_{\underline{1},\underline{1}}, \mathcal{M}^0 \right)$$

in the symmetric monoidal category $(\mathsf{Multicat}_*, \wedge, \mathsf{I}_+)$ as follows.

**Object:** It is the partition multicategory $\mathcal{M}\underline{1}$ (Example 1.3.3).

**Multiplication:** It is the partition product for $\underline{1}$ and $\underline{1}$ in (1.3.7):

$$\textstyle\prod_{\underline{1},\underline{1}} : \mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1} \xrightarrow{\;\cong\;} \mathcal{M}\underline{1}.$$

**Unit:** It is the unit constraint $\mathcal{M}^0 : \mathsf{I}_+ \longrightarrow \mathcal{M}\underline{1}$ in (1.3.9).                          ⋄

**The 2-Category of $\mathcal{M}\underline{1}$-Modules.** Recall from Definition A.1.9 that each monoid in a monoidal category has an associated category of left modules. The following definition uses the commutative monoid $\mathcal{M}\underline{1}$ in $\mathsf{Multicat}_*$ in Definition 1.3.12.

**Definition 1.3.13.** We define the 2-category of left $\mathcal{M}\underline{1}$-modules, denoted $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, as follows.

- It has objects and 1-cells given by, respectively, left $\mathcal{M}\underline{1}$-modules and their morphisms as in Definition A.1.9.
- For left $\mathcal{M}\underline{1}$-module morphisms

$$F, F' : \mathsf{N} \longrightarrow \mathsf{N}',$$

the set of *left $\mathcal{M}\underline{1}$-module 2-cells* consists of pointed multinatural transformations (Definition C.4.1)

$$\theta : F \longrightarrow F'$$

such that the two whiskerings in (1.3.14) below are equal, where $\mu$ and $\mu'$ denote the left $\mathcal{M}\underline{1}$-module structures for N and N', respectively.

(1.3.14)



- Identities and compositions in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are given by those of $\mathsf{Multicat}_*$ in Theorem C.4.9.

We use the same notation for the underlying 1-category of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. $\diamond$

**Example 1.3.15** (Endomorphism $\mathcal{M}\underline{1}$-Modules). Each small permutative category $(\mathsf{C}, \oplus, e, \xi)$ has an associated left $\mathcal{M}\underline{1}$-module

$$\mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{C}) = \big(\mathsf{End}_\bullet(\mathsf{C}), \mu\big)$$

defined as follows.

- $\mathsf{End}_\bullet(\mathsf{C})$ is the pointed endomorphism multicategory in Example C.4.8.
- The left $\mathcal{M}\underline{1}$-module structure

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{End}_\bullet(\mathsf{C}) \longrightarrow \mathsf{End}_\bullet(\mathsf{C})$$

is given by the following assignments for $a \in \mathsf{Ob}\,\mathsf{C}$ and multimorphisms $f$ in $\mathsf{End}_\bullet(\mathsf{C})$.

$$(\varnothing, a) \longmapsto e \qquad \iota^n \wedge a \longmapsto \iota^n = 1_e \qquad \varnothing \wedge f \longmapsto 1_e$$
$$(\{1\}, a) \longmapsto a \qquad \pi_j^n \wedge a \longmapsto 1_a \qquad \{1\} \wedge f \longmapsto f$$

As in Examples C.3.1 and C.4.8, it is enough to assume that $(\mathsf{C}, \otimes, 1)$ is a symmetric monoidal category. In this more general case,

- the image of $\iota^n \wedge a$ is an iterate of the right unit isomorphism $\rho$ in C, and
- the image of $\pi_j^n \wedge a$ is an iterate of the left unit isomorphism $\lambda$ and the right unit isomorphism $\rho$ in C.

Moreover, the following statements hold:

(1) Each *strictly unital* symmetric monoidal functor between symmetric monoidal categories

$$(P, P^2, P^0 = 1) : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a left $\mathcal{M}\underline{1}$-module morphism

$$\mathsf{End}_{\mathcal{M}\underline{1}}(P) : \mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{C}) \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{D})$$

given by the pointed multifunctor $\mathsf{End}_\bullet(P)$ in Example C.4.8. The latter is given by $\mathsf{End}(P)$ in (C.3.3).

(2) Each monoidal natural transformation between strictly unital symmetric monoidal functors between symmetric monoidal categories

$$\theta : (P, P^2, P^0 = 1) \longrightarrow (Q, Q^2, Q^0 = 1) : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a left $\mathcal{M}\underline{1}$-module 2-cell

$$\mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{C}) \overset{\mathsf{End}_{\mathcal{M}\underline{1}}(P)}{\underset{\mathsf{End}_{\mathcal{M}\underline{1}}(Q)}{\Downarrow \mathsf{End}_{\mathcal{M}\underline{1}}(\theta)}} \mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{D})$$

given by the pointed multinatural transformation $\mathsf{End}_{\bullet}(\theta)$ in Example C.4.8.

If there is no danger of confusion, we denote $\mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{C})$ by $\mathsf{C}$.                    ◇

**Proposition 1.3.16.** *The endomorphism left $\mathcal{M}\underline{1}$-module in Example 1.3.15 defines a 2-functor*

$$\mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

We also denote by $\mathsf{End}_{\mathcal{M}\underline{1}}$ the restriction of the 2-functor in Proposition 1.3.16 to the locally-full sub-2-category $\mathsf{PermCat}^{\mathsf{st}}$ in Definition A.2.3.

The following result from [**JY∞**, 10.1.14, 10.1.28, 10.2.5, and 10.2.22] summarizes some of the main properties of $\mathcal{M}\underline{1}$-modules.

**Proposition 1.3.17.**

(1) *Suppose* $\mathsf{N}$ *is a left $\mathcal{M}\underline{1}$-module in* $\mathsf{Multicat}_*$. *Then the structure morphism*

$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{N} \longrightarrow \mathsf{N}$$

*is an isomorphism. Its inverse is given by the unit*

(1.3.18)       $$\mathsf{N} \xrightarrow[\cong]{\lambda^{-1}} \mathsf{I}_+ \wedge \mathsf{N} \xrightarrow{\mathcal{M}^0 \wedge 1} \mathcal{M}\underline{1} \wedge \mathsf{N}$$

*with*
- $\lambda$ *the left unit isomorphism for* $\wedge$ *and*
- $\mathcal{M}^0 : \mathsf{I}_+ \longrightarrow \mathcal{M}\underline{1}$ *the unit constraint in (1.3.9).*

(2) *The structure morphism for a right $\mathcal{M}\underline{1}$-module is an isomorphism with inverse given by* $(1 \wedge \eta) \circ \rho^{-1}$, *where* $\rho$ *is the right unit isomorphism for* $\wedge$.

(3) *Each small pointed multicategory* $\mathsf{N}$ *admits at most one left $\mathcal{M}\underline{1}$-module structure and at most one right $\mathcal{M}\underline{1}$-module structure.*

(4) *The 2-category of left, respectively right, $\mathcal{M}\underline{1}$-modules is a full sub-2-category of* $\mathsf{Multicat}_*$ *in Theorem C.4.9.*

(5) *For a left $\mathcal{M}\underline{1}$-module* $(\mathsf{N}, \mu^{\mathsf{N}})$ *and a right $\mathcal{M}\underline{1}$-module* $(\mathsf{P}, \mu^{\mathsf{P}})$, *the two morphisms in* $\mathsf{Multicat}_*$

$$(\mathsf{P} \wedge \mathcal{M}\underline{1}) \wedge \mathsf{N} \xrightarrow[\ (1 \wedge \mu^{\mathsf{N}}) \circ \alpha\ ]{\ \mu^{\mathsf{P}} \wedge 1\ } \mathsf{P} \wedge \mathsf{N}$$

*are equal. Therefore, the canonical morphism to the coequalizer*

$$\mathsf{P} \wedge \mathsf{N} \xrightarrow{\cong} \mathsf{P} \wedge_{\mathcal{M}\underline{1}} \mathsf{N}$$

*is an isomorphism in* $\mathsf{Multicat}_*$.

(6) *For each small permutative category* C, *there are natural isomorphisms of pointed categories*

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}\big(\mathcal{M}\underline{1},\, \mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{C})\big) \cong \mathsf{Multicat}_*\big(\mathsf{I}_+,\, \mathsf{End}_{\bullet}(\mathsf{C})\big) \cong (\mathsf{C}, e).$$

(7) *There is a complete, cocomplete, symmetric monoidal, and closed category*

$$\big(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \wedge, \mathcal{M}\underline{1}, \mathsf{Hom}_*\big).$$

- *The monoidal product and internal hom are those of* $(\mathsf{Multicat}_*, \wedge, \mathsf{Hom}_*)$ *in Theorem 1.2.8.*
- *The monoidal unit is* $\mathcal{M}\underline{1}$.
- *The unit isomorphisms are given by the left* $\mathcal{M}\underline{1}$*-module structure and braiding for* $\wedge$.

Recall from Definition A.2.11 the notion of a 2-adjunction.

**Proposition 1.3.19.** *There is a free-forgetful 2-adjunction*

$$\mathsf{Multicat}_* \underset{\mathsf{U}_{\mathcal{M}\underline{1}}}{\overset{\mathcal{M}\underline{1} \wedge -}{\rightleftarrows}} \perp \; \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

**Explanation 1.3.20** (Unit and Counit)**.** For the 2-adjunction $(\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}$, the unit

(1.3.21)                     $\hat{\eta} : 1_{\mathsf{Multicat}_*} \longrightarrow \mathsf{U}_{\mathcal{M}\underline{1}} \circ (\mathcal{M}\underline{1} \wedge -)$

has component pointed multifunctor given by the composite

$$\mathsf{M} \xrightarrow[\cong]{\lambda^{-1}} \mathsf{I}_+ \wedge \mathsf{M} \xrightarrow{\mathcal{M}^0 \wedge 1} \mathcal{M}\underline{1} \wedge \mathsf{M}$$

with the overarching arrow labeled $\hat{\eta}_\mathsf{M}$

in (1.3.18) for each small pointed multicategory M. This component is, in general, *not* an isomorphism because M need not be a left $\mathcal{M}\underline{1}$-module.

The counit

(1.3.22)                     $\hat{\varepsilon} : (\mathcal{M}\underline{1} \wedge -) \circ \mathsf{U}_{\mathcal{M}\underline{1}} \longrightarrow 1_{\mathsf{Mod}^{\mathcal{M}\underline{1}}}$

of the 2-adjunction $(\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}$ has component

$$\hat{\varepsilon}_\mathsf{N} = \mu : \mathcal{M}\underline{1} \wedge \mathsf{N} \xrightarrow{\cong} \mathsf{N}$$

given by the left $\mathcal{M}\underline{1}$-module structure morphism for each left $\mathcal{M}\underline{1}$-module $(\mathsf{N}, \mu)$. This component is an isomorphism by Proposition 1.3.17 (1).                     ⋄

**The Symmetric Monoidal** Cat**-Category of** $\mathcal{M}\underline{1}$**-Modules.** Recall the notion of a symmetric monoidal V-category (Definition B.2.16). The following definition involves the case $\mathsf{V} = (\mathsf{Cat}, \times, \mathbf{1})$.

**Definition 1.3.23.** Define the symmetric monoidal Cat-category

$$\big(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \wedge, \mathcal{M}\underline{1}\big)$$

with the following data.

- The base Cat-category is the 2-category of left $\mathcal{M}\underline{1}$-modules in Definition 1.3.13. By Proposition 1.3.17 (4), it has hom categories

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}(\mathsf{N}, \mathsf{N}') = \mathsf{Multicat}_*(\mathsf{N}, \mathsf{N}')$$

  for left $\mathcal{M}\underline{1}$-modules $\mathsf{N}$ and $\mathsf{N}'$.
- The monoidal composition is given by the smash product, $\wedge$, in $\mathsf{Multicat}_*$. This is well defined by Proposition 1.3.17 (5).
- The identity object is $\mathcal{M}\underline{1}$ (Example 1.3.3).
- The monoidal unitors and monoidal associator are given by those of $(\mathsf{Multicat}_*, \wedge)$ in Theorem 1.2.8.

This finishes the definition.                                                    ◇

**Explanation 1.3.24** ($\mathsf{Mod}^{\mathcal{M}\underline{1}}$ as a Cat-Multicategory). Since $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a symmetric monoidal Cat-category, it has the structure of a Cat-multicategory by Proposition C.3.9, with the following data.

- Its objects are left $\mathcal{M}\underline{1}$-modules.
- For left $\mathcal{M}\underline{1}$-modules $\langle \mathsf{N}_j \rangle_{j=1}^n$ and $\mathsf{N}'$, the $n$-ary multimorphism category is

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}\left( \langle \mathsf{N}_j \rangle_{j=1}^n ; \mathsf{N}' \right) = \mathsf{Mod}^{\mathcal{M}\underline{1}}\left( \wedge_{j=1}^n \mathsf{N}_j , \mathsf{N}' \right)$$

$$= \begin{cases} \mathsf{Multicat}_* \left( \wedge_{j=1}^n \mathsf{N}_j , \mathsf{N}' \right) & \text{if } n > 0 \text{ and} \\ \mathsf{Multicat}_* \left( \mathcal{M}\underline{1}, \mathsf{N}' \right) & \text{if } n = 0. \end{cases}$$

  If $n > 0$, then this category has
  – pointed multifunctors

(1.3.25)                              $$\wedge_{j=1}^n \mathsf{N}_j \longrightarrow \mathsf{N}'$$

  as objects and
  – pointed multinatural transformations between such pointed multifunctors as morphisms.
  If $n = 0$, then an empty $\wedge$ in (1.3.25) means $\mathcal{M}\underline{1}$, the monoidal unit in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.
- The symmetric group action is induced by the braiding of the smash product in $\mathsf{Multicat}_*$.
- The multicategorical composition is given by smash product and composition of pointed multifunctors, and likewise for pointed multinatural transformations.

This finishes the description of the Cat-multicategory $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. We note the subtle difference between the Cat-multicategories $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and $\mathsf{Multicat}_*$ in Explanation 1.2.9, especially in arity 0.                                                    ◇

**Proposition 1.3.26.** *The 2-functor*

$$\mathcal{M}\underline{1} \wedge - : \mathsf{Multicat}_* \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

*is a strong symmetric* Cat-*monoidal functor, hence also a* Cat-*multifunctor.*

*Proof.* The unit constraint for $\mathcal{M}\underline{1} \wedge -$ is the isomorphism

$$\mathcal{M}\underline{1} \xrightarrow[\cong]{\rho^{-1}} \mathcal{M}\underline{1} \wedge \mathsf{I}_+$$

where $\rho$ is the right unit isomorphism for $\wedge$. The monoidal constraint is the composite isomorphism for $M, N \in \mathsf{Multicat}_*$

$$(\mathcal{M}\underline{1} \wedge M) \wedge (\mathcal{M}\underline{1} \wedge N) \xrightarrow{\;\cong\;} (\mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1}) \wedge (M \wedge N) \xrightarrow{\;\Pi_{1,1}\;} \mathcal{M}\underline{1} \wedge (M \wedge N),$$

where the first isomorphism permutes the factors and the second isomorphism is the partition product from Lemma 1.3.6 with $b = 1$. The symmetric $\mathsf{Cat}$-monoidal functor axioms of Definitions B.2.20 and B.2.24 then follow because $\mathcal{M}\underline{1}$ is a commutative monoid in $\mathsf{Multicat}_*$. $\qquad\square$

**Explanation 1.3.27** (Forgetting $\mathcal{M}\underline{1}$-Module Structure)**.** The forgetful 2-functor

$$(1.3.28) \qquad\qquad U_{\mathcal{M}\underline{1}} : \left( \mathsf{Mod}^{\mathcal{M}\underline{1}}, \wedge, \mathcal{M}\underline{1} \right) \longrightarrow \left( \mathsf{Multicat}_*, \wedge, I_+ \right)$$

is a symmetric monoidal $\mathsf{Cat}$-functor (Definition B.2.24) with the following structure morphisms:

**Monoidal Constraint:** It is the identity.
**Unit Constraint:** It is the pointed multifunctor in (1.3.9),

$$\mathcal{M}^0 : I_+ \longrightarrow \mathcal{M}\underline{1}.$$

By Proposition 1.3.17 (4) and (5), the underlying forgetful 2-functor

$$U_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \hookrightarrow \mathsf{Multicat}_*$$

is an inclusion between the underlying 2-categories. However, as a symmetric monoidal $\mathsf{Cat}$-functor, it is neither unital nor strong, since $\mathcal{M}^0$ is not an isomorphism. Therefore, $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is *not* a symmetric monoidal $\mathsf{Cat}$-subcategory of $\mathsf{Multicat}_*$. $\qquad\diamond$

**Explanation 1.3.29** ($U_{\mathcal{M}\underline{1}}$ as a $\mathsf{Cat}$-Multifunctor)**.** Regarding $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and $\mathsf{Multicat}_*$ as $\mathsf{Cat}$-multicategories using Explanations 1.2.9 and 1.3.24, there is an induced $\mathsf{Cat}$-multifunctor (Definition C.1.19)

$$(1.3.30) \qquad\qquad U_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathsf{Multicat}_*$$

with the following structure:

**Object Assignment:** $U_{\mathcal{M}\underline{1}}$ sends a left $\mathcal{M}\underline{1}$-module $(M, \mu)$ to the pointed multicategory M.
**Multimorphism Functors:** Suppose given left $\mathcal{M}\underline{1}$-modules $\langle M \rangle = \langle M_j \rangle_{j=1}^n$ and N. The $n$-ary multimorphism functor

$$U_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \left( \textstyle\bigwedge_{j=1}^n M_j , N \right) \longrightarrow \mathsf{Multicat}_* \left( \textstyle\bigwedge_{j=1}^n M_j , N \right)$$

is an isomorphism if $n > 0$ by Proposition 1.3.17 (4) and (5). If $n = 0$, then $U_{\mathcal{M}\underline{1}}$ sends a left $\mathcal{M}\underline{1}$-module morphism

$$P : \mathcal{M}\underline{1} \longrightarrow N$$

to the composite pointed multifunctor

$$I_+ \xrightarrow{\;\mathcal{M}^0\;} \mathcal{M}\underline{1} \xrightarrow{\;P\;} N$$

where $\mathcal{M}^0$ is the unit constraint in (1.3.9). For a left $\mathcal{M}\underline{1}$-module 2-cell (Definition 1.3.13), $U_{\mathcal{M}\underline{1}}$ is defined similarly by whiskering with $\mathcal{M}^0$. $\qquad\diamond$

**Proposition 1.3.31.** *The unit $\hat{\eta}$ in (1.3.21) and the counit $\hat{\varepsilon}$ in (1.3.22) are monoidal* $\mathsf{Cat}$-*natural transformations, hence also* $\mathsf{Cat}$-*multinatural transformations.*

*Proof.* As in Proposition 1.3.26, the assertions about $\hat{\eta}$ and $\hat{\varepsilon}$ follow from the commutative monoid structure of $\mathcal{M}\underline{1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## 1.4. Permutative Categories

There is a 2-category $\mathsf{PermCat}^{\mathsf{su}}$ of small permutative categories, strictly unital symmetric monoidal functors, and monoidal natural transformations (Definition A.2.3). In this section we extend the 2-category $\mathsf{PermCat}^{\mathsf{su}}$ to a Cat-multicategory.

- In Definitions 1.4.2 and 1.4.10 we define $n$-linear functors and $n$-linear transformations. They generalize strictly unital symmetric monoidal functors and monoidal natural transformations, respectively.
- The Cat-multicategory $\mathsf{PermCat}^{\mathsf{su}}$ is in Definitions 1.4.15, 1.4.16, and 1.4.21.
- Theorem 1.4.38 shows that the Cat-multicategories $\mathsf{PermCat}^{\mathsf{su}}$, $\mathsf{Multicat}$, $\mathsf{Multicat}_*$, and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are related by the various endomorphism multicategory constructions and forgetful functors.

The material in this section is adapted from [**JY∞**, Sections 6.5 and 6.6].

Throughout this section, suppose $\langle \mathsf{C} \rangle = \langle \mathsf{C}_j \rangle_{j=1}^n$ and $\mathsf{D}$ are permutative categories. In each of these permutative categories, the monoidal product, monoidal unit, and braiding are denoted by $\oplus$, $e$, and $\xi$, respectively.

**Multilinear Functors and Transformations.**

**Notation 1.4.1.** Suppose $\langle x \rangle = \langle x_j \rangle_{j=1}^n$ is an $n$-tuple of symbols, and $y$ is a symbol with $k \in \{1, \dots, n\}$. We denote by

$$\langle x \circ_k y \rangle = \langle x \rangle \circ_k y = \Big( \underbrace{x_1, \dots, x_{k-1}}_{\text{empty if } k=1}, y, \underbrace{x_{k+1}, \dots, x_n}_{\text{empty if } k=n} \Big)$$

the $n$-tuple obtained from $\langle x \rangle$ by replacing its $k$-th entry by $y$. Similarly, for $k \neq \ell \in \{1, \dots, n\}$ and a symbol $z$, we denote by

$$\langle x \circ_k y \circ_\ell z \rangle = \langle x \rangle \circ_k y \circ_\ell z$$

the $n$-tuple obtained from $\langle x \circ_k y \rangle$ by replacing its $\ell$-th entry by $z$. $\qquad\qquad$ $\diamond$

**Definition 1.4.2.** An *$n$-linear functor*

$$\textstyle\prod_{j=1}^n \mathsf{C}_j \xrightarrow{\big( P, \{P_j^2\}_{j=1}^n \big)} \mathsf{D}$$

consists of the following data.

- $P : \prod_{j=1}^n \mathsf{C}_j \longrightarrow \mathsf{D}$ is a functor.
- For each $j \in \{1, \dots, n\}$, $P_j^2$ is a natural transformation, called the *$j$-th linearity constraint*, with component morphisms

$$(1.4.3) \qquad P\langle x \circ_j x_j \rangle \oplus P\langle x \circ_j x_j' \rangle \xrightarrow{\ P_j^2\ } P\langle x \circ_j (x_j \oplus x_j') \rangle \in \mathsf{D}$$

for objects $\langle x \rangle \in \prod_{j=1}^n \mathsf{C}_j$ and $x_j' \in \mathsf{C}_j$.

These data are required to satisfy the axioms (1.4.4) through (1.4.8) below.

**Unity:** For objects $\langle x \rangle$ and morphisms $\langle f \rangle$ in $\prod_{j=1}^{n} \mathsf{C}_j$, the following object and morphism equalities hold for each $j \in \{1, \ldots, n\}$.

$$(1.4.4) \qquad \begin{cases} P\langle x \circ_j \mathsf{e} \rangle = \mathsf{e} \\ P\langle f \circ_j 1_{\mathsf{e}} \rangle = 1_{\mathsf{e}} \end{cases}$$

**Constraint Unity:**

$$(1.4.5) \qquad P_j^2 = 1 \qquad \text{if any } x_i = \mathsf{e} \text{ or if } x'_j = \mathsf{e}.$$

**Constraint Associativity:** The following diagram commutes for each $i \in \{1, \ldots, n\}$ and objects $\langle x \rangle \in \prod_{j=1}^{n} \mathsf{C}_j$, with $x'_i, x''_i \in \mathsf{C}_i$.

$(1.4.6)$

$$\begin{array}{ccc}
P\langle x \circ_i x_i \rangle \oplus P\langle x \circ_i x'_i \rangle \oplus P\langle x \circ_i x''_i \rangle & \xrightarrow{\; 1 \oplus P_i^2 \;} & P\langle x \circ_i x_i \rangle \oplus P\langle x \circ_i (x'_i \oplus x''_i) \rangle \\
{\scriptstyle P_i^2 \oplus 1} \downarrow & & \downarrow {\scriptstyle P_i^2} \\
P\langle x \circ_i (x_i \oplus x'_i) \rangle \oplus P\langle x \circ_i x''_i \rangle & \xrightarrow{\; P_i^2 \;} & P\langle x \circ_i (x_i \oplus x'_i \oplus x''_i) \rangle
\end{array}$$

**Constraint Symmetry:** The following diagram commutes for each $i \in \{1, \ldots, n\}$ and objects $\langle x \rangle \in \prod_{j=1}^{n} \mathsf{C}_j$, with $x'_i \in \mathsf{C}_i$.

$(1.4.7)$

$$\begin{array}{ccc}
P\langle x \circ_i x_i \rangle \oplus P\langle x \circ_i x'_i \rangle & \xrightarrow{\; P_i^2 \;} & P\langle x \circ_i (x_i \oplus x'_i) \rangle \\
{\scriptstyle \xi} \downarrow & & \downarrow {\scriptstyle P\langle 1 \circ_i \xi \rangle} \\
P\langle x \circ_i x'_i \rangle \oplus P\langle x \circ_i x_i \rangle & \xrightarrow{\; P_i^2 \;} & P\langle x \circ_i (x'_i \oplus x_i) \rangle
\end{array}$$

**Constraint 2-By-2:** The following diagram commutes for each

$$i \neq k \in \{1, \ldots, n\}, \quad \langle x \rangle \in \prod_{j=1}^{n} \mathsf{C}_j, \quad x'_i \in \mathsf{C}_i, \quad \text{and} \quad x'_k \in \mathsf{C}_k.$$

$(1.4.8)$



This finishes the definition of an $n$-linear functor.

Moreover, we define the following.

- If $n = 0$, then a 0-linear functor is a functor $\mathbf{1} \longrightarrow \mathsf{D}$, which is also regarded as a choice of an object in $\mathsf{D}$.
- An $n$-linear functor $(P, \{P_j^2\})$ is
  - *strong* if each $P_j^2$ is a natural isomorphism and
  - *strict* if each $P_j^2$ is an identity natural transformation.
- A *multilinear functor* is an $n$-linear functor for some $n \geq 0$.                      $\diamond$

**Example 1.4.9.** A 1-linear functor $\mathsf{C} \longrightarrow \mathsf{D}$ is precisely a strictly unital symmetric monoidal functor (Definition A.1.22). ◇

**Definition 1.4.10.** Suppose $P, Q$ are $n$-linear functors as displayed below.

$$(1.4.11) \qquad \prod_{j=1}^{n} \mathsf{C}_j \overset{(P, \{P_j^2\})}{\underset{(Q, \{Q_j^2\})}{\Longrightarrow \theta}} \mathsf{D}$$

An *$n$-linear transformation* $\theta : P \longrightarrow Q$ is a natural transformation of underlying functors that satisfies the following two *multilinearity conditions*.

**Unity:**

$$(1.4.12) \qquad\qquad \theta_{\langle x \rangle} = 1_{\mathsf{e}} \qquad \text{if any } x_i = \mathsf{e} \in \mathsf{C}_i.$$

**Constraint Compatibility:** The following diagram commutes for each $\langle x \rangle \in \prod_{j=1}^{n} \mathsf{C}_j$ and $x_i' \in \mathsf{C}_i$ with $i \in \{1, \ldots, n\}$.

$$(1.4.13)$$

$$\begin{array}{ccc}
P\langle x \circ_i x_i \rangle \oplus P\langle x \circ_i x_i' \rangle & \xrightarrow{\;P_i^2\;} & P\langle x \circ_i (x_i \oplus x_i') \rangle \\
{\scriptstyle \theta \oplus \theta} \downarrow & & \downarrow {\scriptstyle \theta} \\
Q\langle x \circ_i x_i \rangle \oplus Q\langle x \circ_i x_i' \rangle & \xrightarrow{\;Q_i^2\;} & Q\langle x \circ_i (x_i \oplus x_i') \rangle
\end{array}$$

This finishes the definition of an *$n$-linear transformation*. Moreover, we define the following.

- A *multilinear transformation* is an $n$-linear transformation for some $n \geq 0$.
- Identities and compositions of multilinear transformations are defined componentwise. ◇

**Example 1.4.14.** A 1-linear transformation between 1-linear functors is precisely a monoidal natural transformation (Definition A.1.27) between corresponding strictly unital symmetric monoidal functors. ◇

Cat**-Multicategory Structure.** Next we define the Cat-multicategory (Definition C.1.3) PermCat$^{\mathsf{su}}$ whose objects are small permutative categories. For the rest of this section, $\langle \mathsf{C} \rangle = \langle \mathsf{C}_j \rangle_{j=1}^{n}$ and $\mathsf{D}$ are small permutative categories. The notation in the following definition is chosen to match with the notation in Definition A.2.3 in the 1-linear case; see Examples 1.4.9 and 1.4.14.

**Definition 1.4.15** (Multimorphism Categories)**.** We define the following categories of $n$-linear functors and transformations.

- PermCat$^{\mathsf{su}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ is the category with
  - $n$-linear functors $\langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$ as objects and
  - $n$-linear transformations between them as morphisms.
- PermCat$^{\mathsf{sus}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ is the full subcategory of strong $n$-linear functors.
- PermCat$^{\mathsf{st}}\big(\langle \mathsf{C} \rangle ; \mathsf{D}\big)$ is the full subcategory of strict $n$-linear functors. ◇

**Definition 1.4.16** (Symmetric Group Action). Suppose given $n$-linear functors $P$ and $Q$ together with an $n$-linear transformation $\theta$ as displayed below.

(1.4.17)
$$\prod_{j=1}^{n} C_j \underset{(Q,\{Q_j^2\})}{\overset{(P,\{P_j^2\})}{\theta\Downarrow}} D$$

For a permutation $\sigma \in \Sigma_n$, the symmetric group action

(1.4.18)
$$\mathsf{PermCat}^{\mathsf{su}}\left(\langle C\rangle ; D\right) \xrightarrow[\cong]{\sigma} \mathsf{PermCat}^{\mathsf{su}}\left(\langle C\rangle\sigma ; D\right)$$

sends the data (1.4.17) to the following composites and whiskerings, where $\sigma$ permutes the coordinates according to $\sigma$.

(1.4.19)
$$\prod_{j=1}^{n} C_{\sigma(j)} \xrightarrow{\sigma} \prod_{j=1}^{n} C_j \underset{(Q,\{Q_j^2\})}{\overset{(P,\{P_j^2\})}{\theta\Downarrow}} D$$

For objects

$$\langle a\rangle = \langle a_j\rangle_{j=1}^{n} \in \prod_{j=1}^{n} C_{\sigma(j)} \quad \text{and} \quad a_j' \in C_{\sigma(j)},$$

the $j$-th linearity constraint of $P^\sigma = P \circ \sigma$ has component given by the following composite in D.

(1.4.20)
$$
\begin{array}{ccc}
P^\sigma\langle a\rangle \oplus P^\sigma\langle a \circ_j a_j'\rangle & \xrightarrow{(P^\sigma)_j^2} & P^\sigma\langle a \circ_j (a_j \oplus a_j')\rangle \\
\| & & \| \\
P\big(\sigma\langle a\rangle\big) \oplus P\big(\sigma\langle a\rangle \circ_{\sigma(j)} a_j'\big) & \xrightarrow{P_{\sigma(j)}^2} & P\big(\sigma\langle a\rangle \circ_{\sigma(j)} (a_j \oplus a_j')\big)
\end{array}
$$

If $P$ is strong, respectively strict, with each $P_j^2$ a natural isomorphism, respectively identity, then $P^\sigma$ is also strong, respectively strict. ◇

**Definition 1.4.21** (Multicategorical Composition). Suppose given, for each $j \in \{1,\ldots,n\}$,

- permutative categories $\langle B_j\rangle = \langle B_{j,i}\rangle_{i=1}^{k_j}$,
- $k_j$-linear functors $P_j', Q_j' : \langle B_j\rangle \longrightarrow C_j$, and
- a $k_j$-linear transformation $\theta_j : P_j' \longrightarrow Q_j'$ as follows.

(1.4.22)
$$\prod_{i=1}^{k_j} B_{j,i} \underset{Q_j'}{\overset{P_j'}{\theta_j\Downarrow}} C_j$$

With $\langle B\rangle = \big\langle\langle B_j\rangle\big\rangle_{j=1}^{n}$, the multicategorical composition functor

(1.4.23)
$$\mathsf{PermCat}^{\mathsf{su}}\left(\langle C\rangle ; D\right) \times \prod_{j=1}^{n} \mathsf{PermCat}^{\mathsf{su}}\left(\langle B_j\rangle ; C_j\right) \xrightarrow{\gamma} \mathsf{PermCat}^{\mathsf{su}}\left(\langle B\rangle ; D\right)$$

sends the data (1.4.17) and (1.4.22) to the composites

(1.4.24)
$$\prod_{j=1}^{n}\prod_{i=1}^{k_j} \mathsf{B}_{j,i} \; \begin{array}{c} P \circ \prod_j P'_j \\ \Downarrow \theta \otimes (\prod_j \theta_j) \\ Q \circ \prod_j Q'_j \end{array} \; \mathsf{D}$$

defined as follows.

**Composite Multilinear Functor:** Suppose given tuples of objects

(1.4.25)
$$\langle w_j \rangle = \langle w_{j,i} \rangle_{i=1}^{k_j} \in \prod_{i=1}^{k_j} \mathsf{B}_{j,i} \qquad \text{for } j \in \{1, \ldots, n\} \text{ and}$$
$$\langle w \rangle = \langle \langle w_j \rangle \rangle_{j=1}^{n} \in \prod_{j=1}^{n} \prod_{i=1}^{k_j} \mathsf{B}_{j,i}.$$

Then we have the object

(1.4.26)
$$\left( P \circ \prod_j P'_j \right) \langle w \rangle = P \left\langle P'_j \langle w_j \rangle \right\rangle_{j=1}^{n}$$
$$= P \left( P'_1 \langle w_1 \rangle, \ldots, P'_n \langle w_n \rangle \right) \quad \text{in} \quad \mathsf{D}.$$

For the linearity constraints of the composite $P \circ \prod_j P'_j$ in (1.4.24), in addition to the objects in (1.4.25), consider
  • an object $w'_{j,i} \in \mathsf{B}_{j,i}$ for some choice of $(j,i)$ with

$$\ell = k_1 + \cdots + k_{j-1} + i$$

  and
  • $\langle P'w \rangle = \left\langle P'_j \langle w_j \rangle \right\rangle_{j=1}^{n} \in \prod_{j=1}^{n} \mathsf{C}_j.$
The following objects appear in (1.4.27) below.

$$\langle w_j \circ_i w'_{j,i} \rangle = \big( \overbrace{w_{j,1}, \ldots, w_{j,i-1}}^{\text{empty if } i=1}, w'_{j,i}, \overbrace{w_{j,i+1}, \ldots, w_{j,k_j}}^{\text{empty if } i=k_j} \big)$$
$$\langle w_j \circ_i (w_{j,i} \oplus w'_{j,i}) \rangle = \big( \underbrace{w_{j,1}, \ldots, w_{j,i-1}}_{\text{empty if } i=1}, w_{j,i} \oplus w'_{j,i}, \underbrace{w_{j,i+1}, \ldots, w_{j,k_j}}_{\text{empty if } i=k_j} \big).$$

The $\ell$-th linearity constraint $\left( P \circ \prod_j P'_j \right)_{\ell}^{2}$ is defined as the following composite in $\mathsf{D}$.

(1.4.27)

If $P$ and each $P'_j$ are strong, respectively strict, then each linearity constraint $\left(P \circ \prod_j P'_j\right)^2_\ell$ is componentwise invertible, respective identity, and, therefore, the composite $P \circ \prod_j P'_j$ is also strong, respective strict.

**Composite Multinatural Transformation:** The $n$-linear transformation $\theta \otimes \left(\prod_j \theta_j\right)$ in (1.4.24) is the horizontal composite of the natural transformations $\prod_j \theta_j$ and $\theta$. More explicitly, the component morphism $\left(\theta \otimes (\prod_j \theta_j)\right)_{\langle w \rangle}$ is the following composite in D.

$$(1.4.28) \qquad P\langle P'_j \langle w_j \rangle\rangle^n_{j=1} \xrightarrow{P\langle(\theta_j)_{\langle w_j\rangle}\rangle^n_{j=1}} P\langle Q'_j\langle w_j\rangle\rangle^n_{j=1} \xrightarrow{\theta_{\langle Q'_j\langle w_j\rangle\rangle^n_{j=1}}} Q\langle Q'_j\langle w_j\rangle\rangle^n_{j=1}$$

The finishes the definition of the multicategorical composition in PermCat$^{\mathsf{su}}$. $\quad \diamond$

**Theorem 1.4.29.** *There is a* Cat*-multicategory*

$$\mathsf{PermCat^{su}}$$

*defined by the following data.*

- *The objects are small permutative categories.*
- *The multimorphism categories are in Definition 1.4.15.*
- *The colored units are identity symmetric monoidal functors.*
- *The symmetric group action is in Definition 1.4.16.*
- *The multicategorical composition is in Definition 1.4.21.*

*Moreover, there are sub-*Cat*-multicategories*

$$\mathsf{PermCat^{st}} \lhook\joinrel\longrightarrow \mathsf{PermCat^{sus}} \lhook\joinrel\longrightarrow \mathsf{PermCat^{su}}$$

*with the multimorphism categories in Definition 1.4.15.*

**Explanation 1.4.30.** The underlying 2-categories, in the sense of Example C.1.16 with $\mathsf{V} = (\mathsf{Cat}, \times, \mathbf{1})$, of the Cat-multicategories

$$\mathsf{PermCat^{su}}, \quad \mathsf{PermCat^{sus}}, \quad \text{and} \quad \mathsf{PermCat^{st}}$$

are the corresponding 2-categories in Definition A.2.3. $\quad \diamond$

The following result combines [**JY**$\infty$, 6.5.10 and 6.5.13].

**Proposition 1.4.31.** *For small permutative categories $\langle \mathsf{C}_j \rangle^n_{j=1}$ and* D, *the 2-functor*

$$\mathsf{End_{\bullet}} : \mathsf{PermCat^{su}} \longrightarrow \mathsf{Multicat_*}$$

*in Proposition C.4.10 induces an isomorphism of multimorphism categories*

$$\mathsf{PermCat^{su}}\left(\langle\mathsf{C}\rangle ; \mathsf{D}\right) \xrightarrow[\cong]{\mathsf{End_{\bullet}}} \mathsf{Multicat_*}\left(\langle\mathsf{End_{\bullet}(C)}\rangle ; \mathsf{End_{\bullet}(D)}\right)$$

$$= \mathsf{Multicat_*}\left(\wedge^n_{j=1} \mathsf{End_{\bullet}(C_j)}, \mathsf{End_{\bullet}(D)}\right)$$

*between*

- *the category of n-linear functors and transformations $\langle\mathsf{C}\rangle \longrightarrow \mathsf{D}$ and*
- *the category of pointed multifunctors*

$$\wedge^n_{j=1} \mathsf{End_{\bullet}(C_j)} \longrightarrow \mathsf{End_{\bullet}(D)}$$

*and pointed multinatural transformations.*

*Therefore,* End$_{\bullet}$ *is a* Cat*-multifunctor.*

**Explanation 1.4.32** (End$_\bullet$ as a Cat-Multifunctor). The Cat-multifunctor End$_\bullet$ in Proposition 1.4.31 is given explicitly as follows.

*Object Assignment.* It sends a small permutative category C to the pointed endomorphism multicategory

$$\mathsf{End}_\bullet(\mathsf{C}) = \big(\mathsf{End}(\mathsf{C}), i\big)$$

in Example C.4.8.

*Multimorphism Functor on Objects.* An object in the $n$-ary multimorphism category $\mathsf{PermCat}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle; \mathsf{D}\big)$ is an $n$-linear functor (Definition 1.4.2)

$$\prod_{j=1}^n \mathsf{C}_j \xrightarrow{\big(P, \{P_j^2\}_{j=1}^n\big)} \mathsf{D}.$$

The image pointed multifunctor

$$(1.4.33) \qquad\qquad \mathsf{End}_\bullet(P) : \bigwedge_{j=1}^n \mathsf{End}_\bullet(\mathsf{C}_j) \longrightarrow \mathsf{End}_\bullet(\mathsf{D})$$

has object assignment induced by the object assignment of $P$. This makes sense because, by the pushout (1.2.3) that defines the smash product $\bigwedge_{j=1}^n \mathsf{End}_\bullet(\mathsf{C}_j)$, its objects are represented by elements in

$$\mathsf{Ob}\Big(\bigotimes_{j=1}^n \mathsf{End}(\mathsf{C}_j)\Big) = \prod_{j=1}^n \mathsf{Ob}\big(\mathsf{End}(\mathsf{C}_j)\big) = \prod_{j=1}^n \mathsf{Ob}(\mathsf{C}_j).$$

The object unity axiom (1.4.4) ensures that it descends to the objects of the smash product.

By the pushout (1.2.3) again, each multimorphism in $\bigwedge_{j=1}^n \mathsf{End}_\bullet(\mathsf{C}_j)$ is represented by a multimorphism in $\bigotimes_{j=1}^n \mathsf{End}(\mathsf{C}_j)$. By

- Explanation 1.1.14 of the tensor product and
- the definition of End (Example C.3.1),

the multimorphisms in $\bigotimes_{j=1}^n \mathsf{End}(\mathsf{C}_j)$ are generated by

$$(1.4.34) \qquad\qquad c_1 \otimes \cdots \otimes c_{j-1} \otimes \psi \otimes c_{j+1} \otimes \cdots \otimes c_n$$

for some $j \in \{1, \ldots, n\}$, objects $c_i \in \mathsf{C}_i$ for $i \neq j$, and multimorphism

$$\psi \in \mathsf{End}(\mathsf{C}_j)\big(\langle x_k\rangle_{k=1}^r ; y\big) = \mathsf{C}_j\big(\oplus_{k=1}^r x_k, y\big).$$

We use the following notation.

$$(1.4.35) \qquad\qquad \begin{aligned} \langle c\rangle \circ_j ? &= \big(c_1, \ldots, c_{j-1}, ?, c_{j+1}, \ldots, c_n\big) \\ 1_{\langle c\rangle} \circ_j ? &= \big(1_{c_1}, \ldots, 1_{c_{j-1}}, ?, 1_{c_{j+1}}, \ldots, 1_{c_n}\big) \end{aligned}$$

Then End$_\bullet(P)$ sends the multimorphism in (1.4.34) to the following composite morphism in D.

$$(1.4.36)$$

$$\begin{array}{ccc}
\oplus_{k=1}^r P\big(\langle c\rangle \circ_j x_k\big) & & P\big(\langle c\rangle \circ_j y\big) \\
& P_j^2 \searrow \quad \nearrow P\big(1_{\langle c\rangle} \circ_j \psi\big) & \\
& P\big(\langle c\rangle \circ_j \big(\oplus_{k=1}^r x_k\big)\big) &
\end{array}$$

This is an $r$-ary multimorphism in

$$\mathsf{End}(\mathsf{D})\Big(\big\langle P(\langle c\rangle \circ_j x_k)\big\rangle_{k=1}^r ; P\big(\langle c\rangle \circ_j y\big)\Big).$$

This assignment descends to the smash product by the unity axioms (1.4.4) and (1.4.5). These object and multimorphism assignments yield a pointed multifunctor $\mathsf{End}_\bullet(P)$ as in (1.4.33) by the other axioms of an $n$-linear functor (Definition 1.4.2).

*Multimorphism Functor on Morphisms.* A morphism

$$\theta : P \longrightarrow Q \quad \text{in} \quad \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{C}\rangle; \mathsf{D}\big)$$

is an $n$-linear transformation (Definition 1.4.10). A morphism in

$$\mathsf{Multicat}_*\big(\textstyle\bigwedge_{j=1}^n \mathsf{End}_\bullet(\mathsf{C}_j), \mathsf{End}_\bullet(\mathsf{D})\big)$$

is a pointed multinatural transformation (Definition C.4.1). For each object $\langle c\rangle \in \prod_{j=1}^n \mathsf{C}_j$, the component of

$$\mathsf{End}_\bullet(\theta) : \mathsf{End}_\bullet(P) \longrightarrow \mathsf{End}_\bullet(Q)$$

at the object of $\bigwedge_{j=1}^n \mathsf{End}_\bullet(\mathsf{C}_j)$ represented by $\langle c\rangle$ is the component morphism

(1.4.37) $$\theta_{\langle c\rangle} : P\langle c\rangle \longrightarrow Q\langle c\rangle \quad \text{in} \quad \mathsf{D}.$$

This defines a pointed multinatural transformation $\mathsf{End}_\bullet(\theta)$ by the multilinearity conditions (1.4.12) and (1.4.13).                                                    ◇

The following result combines [**JY∞**, 5.3.6, 5.3.9, 6.5.1, and 10.2.14].

**Theorem 1.4.38.** *There is a commutative diagram of* $\mathsf{Cat}$*-multifunctors*

(1.4.39)



*defined as follows.*

- $\mathsf{U}_\bullet$, $\mathsf{U}_{\mathcal{M}\underline{1}}$*, and* $\mathsf{End}_\bullet$ *are the* $\mathsf{Cat}$*-multifunctors in Explanations 1.2.14, 1.3.29, and 1.4.32, respectively.*
- $\mathsf{End}$ *is the composite* $\mathsf{Cat}$*-multifunctor* $\mathsf{U}_\bullet \circ \mathsf{End}_\bullet$*, which restricts to the 2-functor in Proposition C.3.6.*
- $\mathsf{End}_{\mathcal{M}\underline{1}}$ *is defined on objects in Example 1.3.15. It extends to a* $\mathsf{Cat}$*-multifunctor satisfying*

$$\mathsf{U}_{\mathcal{M}\underline{1}} \circ \mathsf{End}_{\mathcal{M}\underline{1}} = \mathsf{End}_\bullet$$

*by Proposition 1.3.17 (4) through (6) and Proposition 1.4.31.*

**Explanation 1.4.40** (End as a $\mathsf{Cat}$-Multifunctor)**.** The $\mathsf{Cat}$-multifunctor

$$\mathsf{End} = \mathsf{U}_\bullet\mathsf{End}_\bullet : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}$$

in (1.4.39) sends a small permutative category $\mathsf{C}$ to the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ in Example C.3.1. For small permutative categories $\langle \mathsf{C}\rangle$ and $\mathsf{D}$, the composite multimorphism functor



is as described in Explanation 1.4.32 before descending to the smash product.     ◇

**Explanation 1.4.41** ($\text{End}_{\mathcal{M}\underline{1}}$ as a Cat-Multifunctor). The Cat-multifunctor

$$\text{End}_{\mathcal{M}\underline{1}} : \text{PermCat}^{\text{su}} \longrightarrow \text{Mod}^{\mathcal{M}\underline{1}}$$

in (1.4.39) sends a small permutative category $\text{C}$ to the endomorphism left $\mathcal{M}\underline{1}$-module

$$\text{End}_{\mathcal{M}\underline{1}}(\text{C}) = \big(\text{End}_{\bullet}(\text{C}), \mu\big)$$

in Example 1.3.15. For small permutative categories $\langle\text{C}\rangle$ and $\text{D}$, the multimorphism functor $\text{End}_{\mathcal{M}\underline{1}}$ is the following composite isomorphism.

$$\text{PermCat}^{\text{su}}\big(\langle\text{C}\rangle; \text{D}\big) \xrightarrow[\cong]{\text{End}_{\bullet}} \text{Multicat}_{*}\big(\wedge_{j=1}^{n} \text{End}_{\bullet}(\text{C}_j), \text{End}_{\bullet}(\text{D})\big)$$

$$\text{End}_{\mathcal{M}\underline{1}} \searrow \qquad \stackrel{\cong}{\nearrow} (\text{U}_{\mathcal{M}\underline{1}})^{-1}$$

$$\text{Mod}^{\mathcal{M}\underline{1}}\big(\wedge_{j=1}^{n} \text{End}_{\mathcal{M}\underline{1}}(\text{C}_j), \text{End}_{\mathcal{M}\underline{1}}(\text{D})\big)$$

- $\text{End}_{\bullet}$ is an isomorphism by Proposition 1.4.31.
- $\text{U}_{\mathcal{M}\underline{1}}$ is an isomorphism by
  - Proposition 1.3.17 (4) and (5) if $n > 0$ and
  - Proposition 1.3.17 (6) if $n = 0$.                                   ◇

# Infinite Loop Space Machines

In this chapter we review two *K*-theory functors,

- $\mathsf{K}^{\mathsf{Se}}$ due to Segal [**Seg74**] and
- $\mathsf{K}^{\mathsf{EM}}$ due to Elmendorf-Mandell [**EM06, EM09**],

from small permutative categories, $\mathsf{PermCat}^{\mathsf{su}}$, to connective symmetric spectra, $\mathsf{Sp}_{\geq 0}$:

$$\mathsf{PermCat}^{\mathsf{su}} \xrightarrow[\mathsf{K}^{\mathsf{EM}}]{\mathsf{K}^{\mathsf{Se}}} \mathsf{Sp}_{\geq 0}\,.$$

Each of $\mathsf{K}^{\mathsf{Se}}$ and $\mathsf{K}^{\mathsf{EM}}$ is an equivalence of homotopy theories. This implies that each of them induces an equivalence between the respective stable homotopy categories, which are obtained by inverting the respective classes of stable equivalences.

Moreover, there are decompositions

$$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathsf{Se}} \quad \text{and}$$

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}$$

as in the following diagram, which we explain in more detail in Section 2.5.

(2.0.1)


Each category in (2.0.1) is an enriched symmetric monoidal category, except for $\mathsf{PermCat}^{\mathsf{su}}$, which is a Cat-multicategory. However, Segal *K*-theory, $\mathsf{K}^{\mathsf{Se}}$, is *not* a multifunctor because $\mathsf{J}^{\mathsf{Se}}$ is not a multifunctor. On the other hand, each constituent functor that comprises Elmendorf-Mandell *K*-theory, $\mathsf{K}^{\mathsf{EM}}$, is an enriched multifunctor, so $\mathsf{K}^{\mathsf{EM}}$ itself is an enriched multifunctor.

Each functor in (2.0.1), *except* $\mathsf{J}^{\mathcal{T}}$ and $\mathsf{K}^{\mathcal{G}}$, is an equivalence of homotopy theories. Each of the three pairs along the top row of (2.0.1),

$$(\mathcal{P}, \mathsf{J}^{\mathsf{Se}}), \quad (S_*, \mathsf{Ner}_*), \quad \text{and} \quad (\mathbb{A}, \mathsf{K}^{\mathcal{F}}),$$

induces mutually inverse equivalences between the respective stable homotopy categories. Among the three homotopy inverses $\mathcal{P}$, $S_*$, and $\mathbb{A}$, only $\mathcal{P}$ is a non-symmetric Cat-multifunctor. Each of $S_*$ and $\mathbb{A}$ is incompatible with the multiplicative structures of its domain and codomain.

**Connection with Other Chapters.**

*Equivalences of Homotopy Theories.* We use equivalences of homotopy theories (Section 2.1) in several subsequent chapters.

- In Chapter 3 we observe that there are equivalences of homotopy theories between small multicategories, Multicat, and small permutative categories, PermCat$^{\mathsf{su}}$. Together with Segal $K$-theory, $\mathsf{K}^{\mathsf{Se}}$ (Section 2.5), we obtain an equivalence of homotopy theories from Multicat to $\mathsf{Sp}_{\geq 0}$.
- In Chapters 4 and 5 we extend the equivalences of homotopy theories between Multicat and PermCat$^{\mathsf{su}}$ first to *pointed* multicategories, Multicat$_*$, and then further to left $\mathcal{M}\underline{1}$-modules, $\mathsf{Mod}^{\mathcal{M}\underline{1}}$. Together with Segal $K$-theory, this implies that there is an equivalence of homotopy theories from each of Multicat$_*$ and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ to $\mathsf{Sp}_{\geq 0}$.

*Enriched Mackey Functors.*

- In Sections 9.4, 10.5, and 10.6 we apply our general results about multicategorical standard enrichment, enriched diagrams, and enriched presheaves to the Cat-multifunctors that constitute Elmendorf-Mandell $K$-theory, $\mathsf{K}^{\mathsf{EM}}$. In particular, Theorems 10.5.1 and 10.6.2 prove that $\mathsf{K}^{\mathsf{EM}}$ induces a change-of-enrichment functor $\mathsf{K}^{\mathsf{EM}}_\star$ that produces spectral Mackey functors from permutative Mackey functors based on any small category C enriched in PermCat$^{\mathsf{su}}$. Moreover, this functor factors through categories of enriched Mackey functors based on left $\mathcal{M}\underline{1}$-modules, $\mathcal{G}_*$-categories, and $\mathcal{G}_*$-simplicial sets.
- In Part 4 we further extend the equivalences of homotopy theories between Multicat$_*$, $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, and PermCat$^{\mathsf{su}}$ to their respective categories of enriched diagrams and enriched Mackey functors. See Theorems 12.1.6, 12.4.6, and 12.6.6.

**Background.** We use the notions of enriched (monoidal) categories and multicategories in Appendices B.1, B.2, and C.1. We also use the symmetric monoidal Cat-category $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and the Cat-multicategory PermCat$^{\mathsf{su}}$ in Sections 1.3 and 1.4, respectively.

**Chapter Summary.** In Section 2.1 we review homotopy theories and their equivalences in the context of complete Segal spaces. In Section 2.2 we discuss pointed diagram categories and their symmetric monoidal closed structure. In Section 2.3 we review $\Gamma$-objects, which are pointed diagrams on the category $\mathcal{F}$ of pointed finite sets. In Section 2.4 we review $\mathcal{G}_*$-objects, which are pointed diagrams on the category $\mathcal{G}$ with finite tuples of pointed finite sets as objects. In Section 2.5 we review the categories and functors that constitute Segal $K$-theory, its homotopy inverse, and Elmendorf-Mandell $K$-theory. Here is a summary table.

| Section 2.1. Homotopy Theories via Complete Segal Spaces | |
|---|---|
| complete Segal space model structure | 2.1.5 |
| relative categories, functors, and natural transformations | 2.1.6 |
| (criteria for) equivalences of homotopy theories | 2.1.7 (2.1.9) |
| Section 2.2. Category of Pointed Diagrams | |
| symmetric monoidal closed category of pointed objects | 2.2.7 |
| pointed unitary enrichment | 2.2.11 |
| pointed Day convolution and monoidal unit | 2.2.14 |
| symmetric monoidal closed category of pointed diagrams | 2.2.19 |
| Section 2.3. $\Gamma$-Objects | |
| category $\mathcal{F}$ and $\Gamma$-objects | A.1.17 and 2.3.1 |
| symmetric monoidal closed category of $\Gamma$-objects | 2.3.2 |
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| category $\mathcal{G}$ and $\mathcal{G}_*$-objects | 2.4.5, 2.4.7, and 2.4.10 |
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| Section 2.5. Segal and Elmendorf-Mandell $K$-theory | |
| Segal $K$-theory $\mathsf{K}^{\mathsf{Se}}$ | 2.5.3 |
| homotopy inverses $\mathcal{P}$, $S_*$, and $\mathbb{A}$ | 2.5.17, 2.5.18, and 2.5.19 |
| Elmendorf-Mandell $K$-theory $\mathsf{K}^{\mathsf{EM}}$ | 2.5.8 |

Most of the material in this chapter is adapted from [**JY∞**, Chapters 4 and 8–10] and [**JY22b, JY22c**]. We provide more references below. We remind the reader of Conventions A.1.2 and A.1.30.

## 2.1. Homotopy Theories via Complete Segal Spaces

In this section we review equivalences of homotopy theories in terms of complete Segal spaces in the sense of [**Rez01**]. See [**BK12, DK80, Toë05**] for further development. Practical criteria for checking equivalences of homotopy theories are discussed in [**GJO17a, GJO17b, JY22c**]. General references for model category theory are [**Hir03, Hov99**].

**Complete Segal Spaces.** The category sSet of simplicial sets is equipped with the standard *Kan model structure*. The *nerve* functor is denoted

(2.1.1) $$\mathsf{Ner} : \mathsf{Cat} \longrightarrow \mathsf{sSet}.$$

See [**JY∞**, Section 7.2] for an elementary discussion of the nerve. For $n \geq 0$, we denote by

(2.1.2) $$\underline{n} = \{0, 1, \ldots, n\}$$

the pointed finite set with $n + 1$ elements and basepoint 0.

**Definition 2.1.3.**

(1) Denote by **2** the nerve of the category consisting of two isomorphic objects.

(2) For $n \geq 2$ and $j \in \{1, \ldots, n\}$, the *j-th characteristic map* is the pointed function

$$\chi_j : \underline{1} \longrightarrow \underline{n} \quad \text{such that} \quad \chi_j(1) = j.$$

(3) A *bisimplicial set* is a simplicial object in the category of simplicial sets.

(4) A bisimplicial set is *Reedy fibrant* if it is a fibrant object in the Reedy model structure.

(5) For a bisimplicial set $A$, the *n-th Segal morphism* is the simplicial map

$$A_n \longrightarrow \overbrace{A_1 \times_{A_0} \cdots \times_{A_0} A_1}^{n \text{ copies of } A_1}$$

whose composite with the *j*-th coordinate projection is $A(\chi_j)$ for each $j \in \{1, \ldots, n\}$.

(6) We say that a bisimplicial set *satisfies the Segal condition* if the *n*-th Segal morphism is a weak equivalence of simplicial sets for each $n \geq 2$.          ⋄

**Definition 2.1.4.** A *complete Segal space* is a bisimplicial set $A$ that satisfies the following three conditions.

**Fibrancy:** $A$ is Reedy fibrant.
**Segal Condition:** $A$ satisfies the Segal condition.
**Path Condition:** The morphism

$$A_0 \cong \mathsf{Map}(\Delta[0], A) \longrightarrow \mathsf{Map}(\mathbf{2}, A)$$

induced by the unique morphism $\mathbf{2} \longrightarrow \Delta[0]$ is a weak equivalence of simplicial sets.          ⋄

The following is [**Rez01**, Theorem 7.2].

**Theorem 2.1.5.** *There is a simplicial model structure on the category of bisimplicial sets, called the* complete Segal space model structure, *that is given as a left Bousfield localization of the Reedy model structure and in which the fibrant objects are precisely the complete Segal spaces.*

A weak equivalence in the complete Segal space model structure of bisimplicial sets is called a *Rezk weak equivalence*.

**Homotopy Theories of Relative Categories.** A *wide subcategory* of a category $\mathsf{C}$ is a subcategory that contains all of the objects of $\mathsf{C}$.

**Definition 2.1.6.**

(1) A *relative category* is a pair $(\mathsf{C}, \mathcal{W})$ consisting of
   • a category $\mathsf{C}$ and
   • a wide subcategory $\mathcal{W}$ of $\mathsf{C}$.
   We refer to morphisms in $\mathcal{W}$ as *stable equivalences*. If there is no danger of confusion, we denote a relative category $(\mathsf{C}, \mathcal{W})$ by $\mathsf{C}$.

(2) A *relative functor* between relative categories

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{D}, \mathcal{X})$$

is a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ that restricts to a functor $\mathcal{W} \longrightarrow \mathcal{X}$.

(3) Suppose $(\mathsf{C}, \mathcal{W})$ and $(\mathsf{D}, \mathcal{X})$ are relative categories. A functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ *creates morphisms in* $\mathcal{W}$ if $\mathcal{W} = F^{-1}(\mathcal{X})$.

(4) For relative functors

$$F, G : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{D}, \mathcal{X}),$$

a *relative natural transformation*

$$\theta : F \longrightarrow G$$

is a natural transformation such that each component of $\theta$ is a morphism in $\mathcal{X}$.

(5) A subclass $\mathcal{W} \subset \mathsf{C}$ of morphisms in a category $\mathsf{C}$ has the *2-out-of-3 property* if, for each pair of morphisms $f, g \in \mathsf{C}$ with $gf$ defined, whenever any two of

$$f, \quad g, \quad \text{and} \quad gf$$

belong to $\mathcal{W}$, then so does the third morphism.

(6) A *category with weak equivalences* is a relative category $(\mathsf{C}, \mathcal{W})$ such that $\mathcal{W}$
   - contains all the isomorphisms and
   - has the 2-out-of-3 property. ◇

For example, the class of weak equivalences in each model category contains all the isomorphisms and has the 2-out-of-3 property.

**Definition 2.1.7.** Suppose $(\mathsf{C}, \mathcal{W})$ is a relative category.

(1) For a small category $\mathcal{D}$, the *relative diagram category*

$$(\mathsf{C}, \mathcal{W})^{\mathcal{D}}$$

is the wide subcategory of the diagram category $\mathsf{C}^{\mathcal{D}} = [\mathcal{D}, \mathsf{C}]$ in which a morphism is a natural transformation with each component in $\mathcal{W}$.

(2) The *classification diagram* of $(\mathsf{C}, \mathcal{W})$ is the bisimplicial set

$$\mathsf{Ner}^{\Delta}(\mathsf{C}, \mathcal{W}) = \mathsf{Ner}\left((\mathsf{C}, \mathcal{W})^{\Delta[?]}\right)$$

with $\Delta[n]$ denoting the category with $n$ composable arrows for $n \geq 0$.

(3) A *homotopy theory of* $(\mathsf{C}, \mathcal{W})$ is a fibrant replacement of $\mathsf{Ner}^{\Delta}(\mathsf{C}, \mathcal{W})$ in the complete Segal space model structure.

(4) A relative functor

$$F : (\mathsf{C}, \mathcal{W}) \longrightarrow (\mathsf{D}, \mathcal{X})$$

is an *equivalence of homotopy theories* if the bisimplicial map $\mathsf{R}\left(\mathsf{Ner}^{\Delta}(F)\right)$ is a Rezk weak equivalence, where $\mathsf{R}$ denotes fibrant replacement in the complete Segal space model structure. We sometimes denote an equivalence of homotopy theories by $\xrightarrow{\sim}$ . ◇

**Definition 2.1.8.** Suppose given
   - relative categories $(\mathsf{C}, \mathcal{W})$ and $(\mathsf{D}, \mathcal{X})$ and
   - functors

$$L : \mathsf{C} \rightleftarrows \mathsf{D} : R.$$

Then we say that $L$ and $R$ are *inverse equivalences of homotopy theories* if the following three conditions hold:

(i) $L$ and $R$ are relative functors.

(ii) $RL$ and $1_{\mathsf{C}}$ are connected by a zigzag of relative natural transformations.

(iii) $LR$ and $1_{\mathsf{D}}$ are connected by a zigzag of relative natural transformations.

This finishes the definition. ◇

We emphasize that the functors $L$ and $R$ in Definition 2.1.8 do *not* have to form an adjunction. The following result from [**GJO17b**, Corollary 2.9] is our main tool for proving equivalences of homotopy theories.

**Proposition 2.1.9.** *Suppose given inverse equivalences of homotopy theories (Definition 2.1.8)*

$$L : (\mathsf{C}, \mathcal{W}) \rightleftarrows (\mathsf{D}, \mathcal{X}) : R.$$

*Then $L$ and $R$ are equivalences of homotopy theories in the sense of Definition 2.1.7 (4).*

The following definition from [**GJO17a**, 1.8] strengthens Definition 2.1.8.

**Definition 2.1.10.** Suppose $(\mathsf{C}, \mathcal{W})$ and $(\mathsf{D}, \mathcal{X})$ are relative categories, and suppose

$$L : \mathsf{C} \rightleftarrows \mathsf{D} : R$$

is an adjunction of categories with left adjoint $L$. We call $L \dashv R$ an *adjoint equivalence of homotopy theories* if the following three conditions hold:

(i)  $L$ and $R$ are relative functors.
(ii)  The unit of $L \dashv R$ is a relative natural transformation $1_\mathsf{C} \longrightarrow RL$.
(iii)  The counit of $L \dashv R$ is a relative natural transformation $LR \longrightarrow 1_\mathsf{D}$.

This finishes the definition.                                                                   ⋄

If $L \dashv R$ is an adjoint equivalence of homotopy theories, then $L$ and $R$ are inverse equivalences of homotopy theories in the sense of Definition 2.1.8. Proposition 2.1.9 implies that $L$ and $R$ are equivalences of homotopy theories.

## 2.2. Category of Pointed Diagrams

To prepare for the discussion of $\Gamma$-objects and $\mathcal{G}_*$-objects in Sections 2.3 and 2.4, in this section we review pointed diagrams. The main results, Theorems 2.2.19 and 2.2.21, state that the category of pointed functors and natural transformations is complete, cocomplete, symmetric monoidal closed, enriched, tensored, and cotensored. The material in this section is adapted from [**JY∞**, Chapter 4].

### Symmetric Monoidal Closed Category of Pointed Objects.

**Definition 2.2.1** (Pointed Objects)**.** Suppose $\mathsf{C}$ is a category with a chosen terminal object $\mathsf{t}$. We denote by $\mathsf{C}_*$ the category under $\mathsf{t}$, which is defined as follows.

- An object in $\mathsf{C}_*$ is a pair $(a, i^a)$ consisting of
  - an object $a$ in $\mathsf{C}$ and
  - a morphism
  $$i^a : \mathsf{t} \longrightarrow a \in \mathsf{C}.$$
  We call $(a, i^a)$ a *pointed object* with *pointed structure* $i^a$.
- For pointed objects $(a, i^a)$ and $(b, i^b)$, a morphism
  $$f : (a, i^a) \longrightarrow (b, i^b) \quad \text{in} \quad \mathsf{C}_*$$
  is a morphism $f : a \longrightarrow b$ in $\mathsf{C}$ such that the following diagram in $\mathsf{C}$ commutes.

(2.2.2)



We call a morphism in $\mathsf{C}_*$ a *pointed morphism*.

This finishes the definition of $C_*$. ◇

Note that if $C$ is complete and cocomplete, then so is $C_*$.

**Definition 2.2.3** (Smash Product and Pointed Hom). Suppose $(C, \otimes, 1, \mathsf{Hom})$ is a complete and cocomplete symmetric monoidal closed category (Definition A.1.19) with a chosen terminal object $t$. Suppose $(a, i^a)$ and $(b, i^b)$ are pointed objects. We define the following pointed objects.

(1) The *wedge* $a \vee b$ is the pushout in $C$ of the span

$$a \xleftarrow{\ i^a\ } t \xrightarrow{\ i^b\ } b$$

with pointed structure given by the composite

$$t \xrightarrow{\ i^a\ } a \longrightarrow a \vee b.$$

(2) The *smash product* $a \wedge b$ is the following pushout in $C$.

(2.2.4)
$$
\begin{array}{ccc}
(a \otimes t) \amalg (t \otimes b) & \xrightarrow{(1_a \otimes i^b) \amalg (i^a \otimes 1_b)} & a \otimes b \\
\downarrow & & \downarrow{\scriptstyle \pi_{a,b}} \\
t & \xrightarrow{\quad i^{a \wedge b}\quad} & a \wedge b
\end{array}
$$

(3) The *smash unit* $1_+$ is the coproduct in $C$

(2.2.5)
$$1_+ = 1 \amalg t$$

with pointed structure given by the inclusion of the $t$ summand.

(4) The *pointed hom* $\mathsf{Hom}_*(a, b)$ is the following pullback in $C$.

(2.2.6)
$$
\begin{array}{ccc}
\mathsf{Hom}_*(a, b) & \longrightarrow & t \\
\downarrow & & \downarrow \\
\mathsf{Hom}(a, b) & \xrightarrow{(i^a)^*} & \mathsf{Hom}(t, b)
\end{array}
$$

The right vertical morphism in (2.2.6) is equal to the composite

$$t \cong \mathsf{Hom}(a, t) \xrightarrow{(i^b)_*} \mathsf{Hom}(a, b) \xrightarrow{(i^a)^*} \mathsf{Hom}(t, b).$$

This induces the pointed structure of $\mathsf{Hom}_*(a, b)$.

This finishes the definition. ◇

The following is [**EM09**, 4.20], as presented in [**JY∞**, 4.2.3].

**Theorem 2.2.7.** *In the context of Definition 2.2.3, the quadruple*

$$(C_*, \wedge, 1_+, \mathsf{Hom}_*)$$

*is a complete and cocomplete symmetric monoidal closed category.*

We mainly use Theorem 2.2.7 when $(C, \otimes, 1, t)$ is

- $(\mathsf{Cat}, \times, \mathbf{1}, \mathbf{1})$, the category of small categories (Example A.1.21), and
- $(\mathsf{sSet}, \times, *, *)$, the category of simplicial sets.

**Pointed Unitary Enrichment.**

**Definition 2.2.8** (Zero Objects and Zero Morphisms). A *zero object* in a category $C$ is an object $0$ that is both initial and terminal. Moreover, for a given zero object $0$ in $C$, we define the following.

- A *zero morphism* in $C$ is a morphism that factors through $0$. In other words, for objects $x, y \in C$, the zero morphism is the composite
$$x \longrightarrow 0 \longrightarrow y.$$

- A *nonzero morphism* is a morphism that does not factor through $0$.
- For objects $x, y \in C$, we denote by

(2.2.9)                          $C^\flat(x, y) = C(x, y) \smallsetminus \{0\}$

the set of nonzero morphisms $x \longrightarrow y$.                                    ◇

**Definition 2.2.10.** Suppose $(\mathcal{D}, \boxdot, e)$ is a symmetric monoidal category. A zero object $t^{\mathcal{D}}$ in $\mathcal{D}$ is called a *null object* if there are natural isomorphisms
$$a \boxdot t^{\mathcal{D}} \cong t^{\mathcal{D}} \cong t^{\mathcal{D}} \boxdot a$$

for objects $a$ in $\mathcal{D}$.                                    ◇

The following definition uses Theorem 2.2.7 on $V$ and categories enriched in $V_*$ as in Definition B.1.1.

**Definition 2.2.11** (Pointed Unitary Enrichment). Suppose

- $(\mathcal{D}, \boxdot, e, t^{\mathcal{D}})$ is a small symmetric monoidal category with a chosen null object $t^{\mathcal{D}}$ and
- $(V, \otimes, 1, [,], t^V)$ is a complete and cocomplete symmetric monoidal closed category with a chosen terminal object $t^V$.

The *pointed unitary enrichment* of $\mathcal{D}$ over $V_*$, denoted $\widehat{\mathcal{D}}$, is the $V_*$-category with

- the same class of objects as $\mathcal{D}$ and
- for any pair of objects $a, b \in \mathcal{D}$, the morphism object

(2.2.12)                          $\displaystyle \widehat{\mathcal{D}}(a, b) = \bigvee_{f \in \mathcal{D}^\flat(a, b)} 1_+ \quad \text{in} \quad V_*$

with the notation as follows.

    – The wedge and the smash unit, $1_+ = 1 \coprod t^V$, are as in Definition 2.2.3.
    – An empty wedge means $t^V \in V_*$.
    – $\mathcal{D}^\flat(a, b)$ is the set of nonzero morphisms as in (2.2.9).

Moreover, we denote by
$$\widehat{\mathcal{D}} \wedge \widehat{\mathcal{D}}$$

the tensor product $V_*$-category as in Definition B.2.1.                                    ◇

By [**JY∞**, 2.4.10], the symmetric monoidal structure on $\mathcal{D}$ induces a symmetric monoidal $V_*$-category structure on $\widehat{\mathcal{D}}$ (Definition B.2.16).

**Pointed Diagrams.**

**Definition 2.2.13** (Pointed Diagram Categories).

    (1) A *pointed category* is a pair $(C, *)$ consisting of
- a category $C$ and
- a chosen object $* \in C$, which is called the *basepoint*.

(2) For pointed categories $(\mathsf{C}, *)$ and $(\mathsf{D}, *)$, a *pointed functor*
$$F : (\mathsf{C}, *) \longrightarrow (\mathsf{D}, *)$$
is a functor $F : \mathsf{C} \longrightarrow \mathsf{D}$ such that $F(*) = *$.

(3) For pointed functors $F, G : (\mathsf{C}, *) \longrightarrow (\mathsf{D}, *)$, a *pointed natural transformation*
$$\theta : F \longrightarrow G$$
is a natural transformation such that
$$\theta_* = 1_* : F(*) = * \longrightarrow * = G(*) \quad \text{in} \quad \mathsf{D}.$$

Moreover, suppose $\mathsf{C}$ is small. We define the *pointed diagram category* $\mathsf{C}_*\text{-}\mathsf{D}$ with
- pointed functors $(\mathsf{C}, *) \longrightarrow (\mathsf{D}, *)$ as objects,
- pointed natural transformations between them as morphisms,
- identity functors as identity morphisms, and
- vertical composition of natural transformations as composition.

This finishes the definition. ◇

In what follows, as in Definition 2.2.11, an empty wedge means the chosen terminal object $\mathsf{t}^\vee \in \mathsf{V}_*$.

**Definition 2.2.14** (Pointed Day Convolution). In the context of Definition 2.2.11, suppose given pointed functors
$$A, B : (\mathcal{D}, \mathsf{t}^{\mathcal{D}}) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\vee).$$
We define the following pointed functors $(\mathcal{D}, \mathsf{t}^{\mathcal{D}}) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\vee)$.

(1) The *monoidal unit diagram* is the pointed functor
$$(2.2.15) \qquad \mathbf{j} = \bigvee_{\mathcal{D}^\flat(\mathsf{e}, -)} 1_+ : (\mathcal{D}, \mathsf{t}^{\mathcal{D}}) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\vee)$$
with $\mathcal{D}^\flat(-, -)$ the set of nonzero morphisms as in (2.2.9).

(2) The *pointed Day convolution* is the $\mathsf{V}_*$-coend
$$(2.2.16) \qquad A \wedge B = \int^{(a,b) \in \widehat{\mathcal{D}} \wedge \widehat{\mathcal{D}}} \bigvee_{\mathcal{D}^\flat(a \square b, -)} (Aa \wedge Bb)$$
with $\widehat{\mathcal{D}}$ the pointed unitary enrichment in Definition 2.2.11. If the input object is $\mathsf{t}^{\mathcal{D}}$ in (2.2.16), we choose $\mathsf{t}^\vee$ for the coend.

(3) The *pointed hom diagram* is the $\mathsf{V}_*$-end
$$(2.2.17) \qquad \mathsf{Hom}_{\mathcal{D}_*}(A, B) = \int_{b \in \widehat{\mathcal{D}}} \left[ Ab, B(- \square b) \right]_*$$
where $[,]_*$ is the pointed hom (2.2.6) for $\mathsf{V}_*$. If the input object is $\mathsf{t}^{\mathcal{D}}$ in (2.2.17), we choose $\mathsf{t}^\vee$ for the end.

(4) The *pointed mapping object* is the $\mathsf{V}_*$-end
$$(2.2.18) \qquad \mathsf{Map}_{\mathcal{D}_*}(A, B) = \int_{b \in \widehat{\mathcal{D}}} [Ab, Bb]_* \cong \left( \mathsf{Hom}_{\mathcal{D}_*}(A, B) \right)(\mathsf{e}).$$

Each of (2.2.16) through (2.2.18) extends componentwise to pointed natural transformations. ◇

**Theorem 2.2.19.** *In the context of Definitions 2.2.11, 2.2.13, and 2.2.14, the quadruple*
$$\left( \mathcal{D}_*\text{-}\mathsf{V}, \wedge, \mathbf{j}, \mathsf{Hom}_{\mathcal{D}_*} \right)$$
*is a complete and cocomplete symmetric monoidal closed category.*

It follows from Theorems 2.2.19 and B.3.7 that $\mathcal{D}_*$-V is a symmetric monoidal $(\mathcal{D}_*$-V$)$-category.

**Definition 2.2.20.** In the context of Theorem 2.2.19, evaluation at the monoidal unit of $\mathcal{D}$ defines a symmetric monoidal functor

$$\mathsf{ev}_{\mathsf{e}} : \mathcal{D}_*\text{-V} \longrightarrow \text{V}_*.$$

It admits a strong symmetric monoidal left adjoint, denoted $L_{\mathsf{e}}$.                    ◇

**Theorem 2.2.21.** *In the context of Theorem 2.2.19 and Definition 2.2.20, the adjunction*

$$(2.2.22) \qquad\qquad \text{V}_* \underset{\mathsf{ev}_{\mathsf{e}}}{\overset{L_{\mathsf{e}}}{\rightleftarrows}} \mathcal{D}_*\text{-V}$$

*makes the pointed diagram category $\mathcal{D}_*$-V enriched, tensored, and cotensored over $\text{V}_*$, with mapping objects given by $\mathsf{Map}_{\mathcal{D}_*}$ in (2.2.18). In particular, $\mathcal{D}_*$-V is*

- *a symmetric monoidal V-category (Definition B.2.16) and*
- *a V-multicategory.*

**Explanation 2.2.23.** In Theorem 2.2.21, the assertion about symmetric monoidal V-category follows from

- the fact that $\mathcal{D}_*$-V is a symmetric monoidal $(\mathcal{D}_*$-V$)$-category (Theorems 2.2.19 and B.3.7) and
- Corollary B.4.11 applied to the symmetric monoidal functor $\mathsf{ev}_{\mathsf{e}}$ (Definition 2.2.20).

The assertion about V-multicategory follows from Proposition C.3.9.            ◇

## 2.3. Γ-Objects

In this section we review the symmetric monoidal closed category of Γ-objects. The material in this section is adapted from [**JY∞**, Section 8.1].

Recall from Definition A.1.14 that a *permutative category* is a strict symmetric monoidal category. Recall from Definition A.1.17 the permutative category

$$\big(\mathcal{F}, \wedge, \underline{1}, \xi\big)$$

of pointed finite sets and pointed functions with the smash product as the monoidal product. Note that $\underline{0} \in \mathcal{F}$ is a null object (Definition 2.2.10). For V = sSet, the following definition is due to Segal [**Seg74**].

**Definition 2.3.1** (Γ-Objects). Suppose $(\text{V}, \otimes, 1, [,], \mathsf{t}^{\vee})$ is a complete and cocomplete symmetric monoidal closed category with a chosen terminal object $\mathsf{t}^{\vee}$. With the small pointed category $(\mathcal{F}, \underline{0})$, we define the *category of Γ-objects* in V as the pointed diagram category

$$\Gamma\text{-V} = \mathcal{F}_*\text{-V}$$

as in Definition 2.2.13. Moreover, we define the following.

- With
$$\big(\text{V}, \otimes, 1, \mathsf{t}^{\vee}\big) = \big(\mathsf{Cat}, \times, \mathbf{1}, \mathbf{1}\big),$$
we call an object in Γ-Cat a Γ-*category*.
- With
$$\big(\text{V}, \otimes, 1, \mathsf{t}^{\vee}\big) = \big(\mathsf{sSet}, \times, *, *\big),$$
we call an object in Γ-sSet a Γ-*simplicial set*.            ◇

**Explanation 2.3.2** (Symmetric Monoidal Closed Structure)**.**  A $\Gamma$-object in $\mathsf{V}$ is a pointed functor

$$(\mathcal{F}, \underline{0}) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\mathsf{V}).$$

Morphisms of $\Gamma$-objects are pointed natural transformations between such pointed functors. With the indexing permutative category

$$\left(\mathcal{D}, \boxdot, \mathsf{e}, \mathsf{t}^\mathcal{D}\right) = \left(\mathcal{F}, \wedge, \underline{1}, \underline{0}\right),$$

by Theorem 2.2.19 there is a complete and cocomplete symmetric monoidal closed category

$$(2.3.3) \qquad\qquad \left(\Gamma\text{-}\mathsf{V}, \wedge, \mathbf{j}, \mathsf{Hom}_{\mathcal{F}_*}\right).$$

Moreover, by Theorem 2.2.21, $\Gamma$-$\mathsf{V}$ is enriched, tensored, and cotensored over $\mathsf{V}_*$. In particular, $\Gamma$-$\mathsf{V}$ is

- a symmetric monoidal $\mathsf{V}$-category (Definition B.2.16) and
- a $\mathsf{V}$-multicategory (Proposition C.3.9).

Therefore,

- $\Gamma$-Cat is a Cat-multicategory, and
- $\Gamma$-sSet is an sSet-multicategory.

Specifying Definition 2.2.14 to the case $\mathcal{D} = \mathcal{F}$, the symmetric monoidal closed structure and $\mathsf{V}_*$-enrichment of $\Gamma$-$\mathsf{V}$ is given as follows.  An empty wedge means the chosen terminal object $\mathsf{t}^\mathsf{V} \in \mathsf{V}_*$.

(1) The *monoidal unit diagram* is the pointed functor

$$(2.3.4) \qquad\qquad \mathbf{j} = \bigvee_{\mathcal{F}^\flat(\underline{1}, -)} 1_+ : (\mathcal{F}, \underline{0}) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\mathsf{V}).$$

(2) The *pointed Day convolution* is the $\mathsf{V}_*$-coend

$$(2.3.5) \qquad\qquad A \wedge B = \int^{(\underline{m},\underline{n}) \in \widehat{\mathcal{F}} \wedge \widehat{\mathcal{F}}} \bigvee_{\mathcal{F}^\flat(\underline{m} \wedge \underline{n}, -)} (A\underline{m} \wedge B\underline{n}).$$

If the input object is $\underline{0}$, we choose $\mathsf{t}^\mathsf{V}$ for the coend.

(3) The *pointed hom diagram* is the $\mathsf{V}_*$-end

$$(2.3.6) \qquad\qquad \mathsf{Hom}_{\mathcal{F}_*}(A, B) = \int_{\underline{n} \in \widehat{\mathcal{F}}} \left[ A\underline{n}, B(- \wedge \underline{n}) \right]_*.$$

If the input object is $\underline{0}$, we choose $\mathsf{t}^\mathsf{V}$ for the end.

(4) The *pointed mapping object* is the $\mathsf{V}_*$-end

$$(2.3.7) \qquad\qquad \mathsf{Map}_{\mathcal{F}_*}(A, B) = \int_{\underline{n} \in \widehat{\mathcal{F}}} [A\underline{n}, B\underline{n}]_* \cong \left( \mathsf{Hom}_{\mathcal{F}_*}(A, B) \right)(\underline{1}).$$

To understand (2.3.4), we observe that there is a canonical bijection

$$\mathcal{F}^\flat(\underline{1}, \underline{n}) \cong \underline{n}^\flat = \{1, \ldots, n\}$$

for each $n \geq 0$, where $\mathcal{F}^\flat(\underline{1}, \underline{0}) = \varnothing$.                                         $\diamond$

## 2.4. $\mathcal{G}_*$-**Objects**

In this section we review the symmetric monoidal closed category of $\mathcal{G}_*$-objects. The material in this section is adapted from [**JY**$\infty$, Sections 9.1 and 9.2].

**Smash Powers of** $\mathcal{F}$**.** The indexing category $\mathcal{G}$ involves the category $\mathcal{F}$ in Definition A.1.17. To define $\mathcal{G}$, first we need some preliminary definitions.

**Definition 2.4.1** (Injections)**.** We define the category $\mathsf{Inj}$ as follows.

- Its objects are *unpointed finite sets*

$$(2.4.2) \qquad \overline{n} = \begin{cases} \{1, \ldots, n\} & \text{if} \quad n > 0, \\ \varnothing & \text{if} \quad n = 0, \end{cases}$$

  for $n \geq 0$.
- Its morphisms are injections.

Suppose $f : \overline{q} \hookrightarrow \overline{p}$ is an injection. We define a functor

$$f_* : \mathcal{F}^q \longrightarrow \mathcal{F}^p$$

called the *reindexing injection* as follows. Suppose given $q$-tuples of pointed finite sets or pointed functions

$$(2.4.3) \qquad \langle \underline{n} \rangle = \langle \underline{n}_k \rangle_{k=1}^q \quad \text{or} \quad \langle \psi \rangle = \langle \psi_k \rangle_{k=1}^q \in \mathcal{F}^q,$$

respectively. We define

$$f_* \langle \underline{n} \rangle = \left\langle \underline{n}_{f^{-1}(j)} \right\rangle_{j=1}^p \quad \text{and} \quad f_* \langle \psi \rangle = \left\langle \psi_{f^{-1}(j)} \right\rangle_{j=1}^p \in \mathcal{F}^p,$$

where

$$\underline{n}_\varnothing = \underline{1} \quad \text{and} \quad \psi_\varnothing = 1_{\underline{1}}.$$

This finishes the definition. $\qquad \qquad \diamond$

**Definition 2.4.4** (Smash Powers of $\mathcal{F}$)**.** For $q \geq 0$, we define the pointed category $\mathcal{F}^{(q)}$, called the *$q$-th smash power of $\mathcal{F}$*, as follows.

**The Case** $q = 0$**:** We define

$$\mathsf{Ob}\, \mathcal{F}^{(0)} = \big\{ *, \langle \rangle \big\},$$

which consists of the basepoint object $*$ and the empty tuple $\langle \rangle$. We define the morphisms of $\mathcal{F}^{(0)}$ such that
- $*$ is both initial and terminal and
- the only nonzero morphism is the identity morphism of $\langle \rangle$.

**The Case** $q > 0$**:** With $\mathcal{F}$ having basepoint $\underline{0}$, we define the $q$-fold smash power of pointed categories

$$\mathcal{F}^{(q)} = \mathcal{F}^{\wedge q}$$

as in (2.2.4) applied to

$$(\mathsf{C}, \otimes, 1) = (\mathsf{Cat}, \times, \mathbf{1}).$$

We denote the objects and morphisms of $\mathcal{F}^{(q)}$ as $q$-tuples as in (2.4.3).
- A $q$-tuple $\langle \underline{n} \rangle$ is identified with the basepoint of $\mathcal{F}^{(q)}$ if any $\underline{n}_k = \underline{0}$. We call $q$ the *length* of a $q$-tuple $\langle \underline{n} \rangle$.
- A $q$-tuple $\langle \psi \rangle$ is a *zero morphism* if any $\psi_k$ factors through $\underline{0} \in \mathcal{F}$.

This finishes the definition. $\qquad \qquad \diamond$

**The Permutative Category $\mathcal{G}$.** Next we define the indexing category $\mathcal{G}$.

**Definition 2.4.5** (Tuples of Pointed Finite Sets). We define a small pointed category

$$(\mathcal{G}, \star)$$

as follows.

**Objects:** The set of objects is the wedge of pointed sets

$$\mathrm{Ob}\,\mathcal{G} = \bigvee_{q \geq 0} \mathrm{Ob}\left(\mathcal{F}^{(q)}\right)$$

as in Definition 2.2.3 applied to $(\mathsf{C}, \mathsf{t}) = (\mathsf{Set}, *)$. The basepoint object $\star$ is both initial and terminal in $\mathcal{G}$.

**Morphisms:** For a pair of objects

$$\langle \underline{n} \rangle = \langle \underline{n}_k \rangle_{k=1}^q \quad \text{and} \quad \langle \underline{m} \rangle = \langle \underline{m}_j \rangle_{j=1}^p,$$

the set of morphisms in $\mathcal{G}$ is

$$(2.4.6) \qquad \mathcal{G}\big(\langle \underline{n} \rangle, \langle \underline{m} \rangle\big) = \bigvee_{f \in \mathsf{Inj}(\overline{q}, \overline{p})} \left( \mathcal{F}^{(p)}\big(f_*\langle \underline{n} \rangle, \langle \underline{m} \rangle\big) \right)$$

$$= \bigvee_{f \in \mathsf{Inj}(\overline{q}, \overline{p})} \left( \bigwedge_{j=1}^p \mathcal{F}\big(\underline{n}_{f^{-1}(j)}, \underline{m}_j\big) \right).$$

In (2.4.6) for $p > 0$, we denote a morphism by a pair $(f, \langle \psi \rangle)$ with

$$\overline{q} \xrightarrow{\;f\;} \overline{p} \in \mathsf{Inj} \quad \text{and} \quad f_*\langle \underline{n} \rangle \xrightarrow{\;\langle \psi \rangle\;} \langle \underline{m} \rangle \in \mathcal{F}^{(p)}.$$

A morphism $(f, \langle \psi \rangle)$ is identified with the *zero morphism* in $\mathcal{G}(\langle \underline{n} \rangle, \langle \underline{m} \rangle)$ if there exists a component morphism

$$\psi_j : \underline{n}_{f^{-1}(j)} \longrightarrow \underline{m}_j \quad \text{for some} \quad j \in \{1, \ldots, p\}$$

that factors through $\underline{0} \in \mathcal{F}$.

**Identities:** The identity morphism of a $q$-tuple $\langle \underline{n} \rangle$ is the pair $(1_{\overline{q}}, 1_{\langle \underline{n} \rangle})$.

**Composition:** The composite of morphisms

$$\langle \underline{n} \rangle \xrightarrow{\;(f, \langle \psi \rangle)\;} \langle \underline{m} \rangle \xrightarrow{\;(g, \langle \phi \rangle)\;} \langle \underline{\ell} \rangle$$

in $\mathcal{G}$ is the pair

$$\big(gf, \langle \phi \rangle \circ g_*\langle \psi \rangle\big).$$

This finishes the definition of $(\mathcal{G}, \star)$. $\diamond$

**Definition 2.4.7** (Permutative Structure on $\mathcal{G}$). We define a permutative category

$$\big(\mathcal{G}, \oplus, \langle\rangle, \xi\big)$$

as follows.

**Monoidal Unit:** It is the empty tuple $\langle\rangle$.

**Monoidal Product:** The *concatenation product*

$$\mathcal{G} \times \mathcal{G} \xrightarrow{\;\oplus\;} \mathcal{G}$$

is the concatenation of tuples

$$\langle \underline{n} \rangle \oplus \langle \underline{n}' \rangle \quad \text{and} \quad \langle \psi \rangle \oplus \langle \psi' \rangle$$

for tuples of pointed finite sets and pointed functions, respectively.

- The basepoint $\star \in \mathcal{G}$ is defined as a null object for $\oplus$ (Definition 2.2.10).
- The concatenation product of any morphism with a morphism from, respectively to, $\star$ is uniquely determined because $\star$ is a null object.

Given morphisms

$$(f, \langle \psi \rangle) \in \mathcal{G}(\langle \underline{n} \rangle, \langle \underline{m} \rangle) \quad \text{and} \quad (f', \langle \psi' \rangle) \in \mathcal{G}(\langle \underline{n'} \rangle, \langle \underline{m'} \rangle),$$

the concatenation of injections

$$f : \overline{q} \hookrightarrow \overline{p} \quad \text{and} \quad f' : \overline{q'} \hookrightarrow \overline{p'}$$

is the injection

$$(f \oplus f')(i) = \begin{cases} f(i) & \text{for} \quad i \in \{1, \ldots, q\} \\ p + f'(i - q) & \text{for} \quad i \in \{q + 1, \ldots, q + q'\}. \end{cases}$$

Then we define the morphism

$$(f, \langle \psi \rangle) \oplus (f', \langle \psi' \rangle) = (f \oplus f', \langle \psi \rangle \oplus \langle \psi' \rangle).$$

**Symmetry:** For non-basepoint objects $\langle \underline{n} \rangle$ and $\langle \underline{n'} \rangle$ in $\mathcal{G}$, the symmetry component

$$(2.4.8) \qquad\qquad \xi_{\langle \underline{n} \rangle, \langle \underline{n'} \rangle} : \langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle \overset{\cong}{\longrightarrow} \langle \underline{n'} \rangle \oplus \langle \underline{n} \rangle$$

is the pair $(\tau_{q,q'}, 1)$ defined as follows.
- The first entry

$$(2.4.9) \qquad\qquad \tau_{q,q'} : \overline{q + q'} \overset{\cong}{\longrightarrow} \overline{q' + q}$$

is the block permutation that swaps the first $q$ elements with the last $q'$ elements.
- The second entry is the identity morphism on

$$(\tau_{q,q'})_* (\langle \underline{n} \rangle \oplus \langle \underline{n'} \rangle) = \langle \underline{n'} \rangle \oplus \langle \underline{n} \rangle.$$

Each component of $\xi$ involving the null basepoint $\star$ is $1_\star$.

This finishes the definition of the permutative structure on $\mathcal{G}$.                              $\diamond$

$\mathcal{G}_*$**-Objects.** For $\mathsf{V} = \mathsf{Cat}$ and $\mathsf{sSet}$, the following definition is due to Elmendorf and Mandell [**EM06**].

**Definition 2.4.10** ($\mathcal{G}_*$-Objects)**.** Suppose $(\mathsf{V}, \otimes, 1, [,], \mathsf{t}^\mathsf{V})$ is a complete and cocomplete symmetric monoidal closed category with a chosen terminal object $\mathsf{t}^\mathsf{V}$. With the small pointed category $(\mathcal{G}, \star)$, we define the *category of $\mathcal{G}_*$-objects* in $\mathsf{V}$ as the pointed diagram category

$$\mathcal{G}_* \text{-} \mathsf{V}$$

as in Definition 2.2.13. Moreover, we define the following.
- With

$$(\mathsf{V}, \otimes, 1, \mathsf{t}^\mathsf{V}) = (\mathsf{Cat}, \times, \mathbf{1}, \mathbf{1}),$$

we call an object in $\mathcal{G}_*$-$\mathsf{Cat}$ a $\mathcal{G}_*$-*category*.
- With

$$(\mathsf{V}, \otimes, 1, \mathsf{t}^\mathsf{V}) = (\mathsf{sSet}, \times, *, *),$$

we call an object in $\mathcal{G}_*$-$\mathsf{sSet}$ a $\mathcal{G}_*$-*simplicial set*.                              $\diamond$

**Explanation 2.4.11** (Symmetric Monoidal Closed Structure). A $\mathcal{G}_*$-object in $\mathsf{V}$ is a pointed functor

$$(\mathcal{G}, \star) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\mathsf{V}).$$

Morphisms of $\mathcal{G}_*$-objects are pointed natural transformations between such pointed functors. With the indexing permutative category

$$\left(\mathcal{D}, \Box, \mathsf{e}, \mathsf{t}^\mathcal{D}\right) = \left(\mathcal{G}, \oplus, \langle\rangle, \star\right),$$

by Theorem 2.2.19 there is a complete and cocomplete symmetric monoidal closed category

(2.4.12) $$\left(\mathcal{G}_*\text{-}\mathsf{V}, \wedge, \mathbf{j}, \mathrm{Hom}_{\mathcal{G}_*}\right).$$

Moreover, by Theorem 2.2.21, $\mathcal{G}_*$-$\mathsf{V}$ is enriched, tensored, and cotensored over $\mathsf{V}_*$. In particular, $\mathcal{G}_*$-$\mathsf{V}$ is

- a symmetric monoidal $\mathsf{V}$-category (Definition B.2.16) and
- a $\mathsf{V}$-multicategory (Proposition C.3.9).

Therefore,

- $\mathcal{G}_*$-Cat is a Cat-multicategory, and
- $\mathcal{G}_*$-sSet is an sSet-multicategory.

Specifying Definition 2.2.14 to the case $\mathcal{D} = \mathcal{G}$, the symmetric monoidal closed structure and $\mathsf{V}_*$-enrichment of $\mathcal{G}_*$-$\mathsf{V}$ is given as follows. An empty wedge means the chosen terminal object $\mathsf{t}^\mathsf{V} \in \mathsf{V}_*$.

(1) The *monoidal unit diagram* is the pointed functor

(2.4.13) $$\mathbf{j} = \bigvee_{\mathcal{G}^\flat(\langle\rangle, -)} 1_+ : (\mathcal{G}, \star) \longrightarrow (\mathsf{V}_*, \mathsf{t}^\mathsf{V}).$$

(2) The *pointed Day convolution* is the $\mathsf{V}_*$-coend

(2.4.14) $$A \wedge B = \int^{((\underline{m}), (\underline{n})) \in \widehat{\mathcal{G}} \wedge \widehat{\mathcal{G}}} \bigvee_{\mathcal{G}^\flat((\underline{m}) \oplus (\underline{n}), -)} \left(A\langle\underline{m}\rangle \wedge B\langle\underline{n}\rangle\right).$$

If the input object is $\star$, we choose $\mathsf{t}^\mathsf{V}$ for the coend.

(3) The *pointed hom diagram* is the $\mathsf{V}_*$-end

(2.4.15) $$\mathrm{Hom}_{\mathcal{G}_*}(A, B) = \int_{\langle\underline{n}\rangle \in \widehat{\mathcal{G}}} \left[A\langle\underline{n}\rangle, B(- \oplus \langle\underline{n}\rangle)\right]_*.$$

If the input object is $\star$, we choose $\mathsf{t}^\mathsf{V}$ for the end.

(4) The *pointed mapping object* is the $\mathsf{V}_*$-end

(2.4.16) $$\mathrm{Map}_{\mathcal{G}_*}(A, B) = \int_{\langle\underline{n}\rangle \in \widehat{\mathcal{G}}} \left[A\langle\underline{n}\rangle, B\langle\underline{n}\rangle\right]_* \cong \left(\mathrm{Hom}_{\mathcal{G}_*}(A, B)\right)\langle\rangle.$$

To understand (2.4.13), we observe that there are canonical bijections

$$\mathcal{G}^\flat\left(\langle\rangle, \langle\underline{n}\rangle\right) \cong \prod_{k=1}^q \mathcal{F}^\flat(\underline{1}, \underline{n}_k) \cong \prod_{k=1}^q \underline{n}_k^\flat = \prod_{k=1}^q \overline{n}_k$$

if $\langle\underline{n}\rangle = \langle\underline{n}_k\rangle_{k=1}^q$ with each $\underline{n}_k \in \mathcal{F}$. $\diamond$

**Functors between $\mathcal{F}$ and $\mathcal{G}$.** The indexing categories $\mathcal{F}$ and $\mathcal{G}$ in, respectively, Definitions 2.4.5 and A.1.17 are related by the following functors.

**Definition 2.4.17.** We define the following pointed functors.

**Length-One Inclusion:** The pointed functor

$$(2.4.18) \qquad\qquad i : (\mathcal{F}, \underline{0}) \longrightarrow (\mathcal{G}, \star)$$

sends
- each pointed finite set $\underline{n} \in \mathcal{F}$ to the length-one tuple $(\underline{n}) \in \mathcal{G}$ and
- each morphism $\psi$ in $\mathcal{F}$ to the pair $(1_{\overline{1}}, (\psi))$.

**Smash Product:** The strict symmetric monoidal pointed functor

$$(2.4.19) \qquad\qquad \wedge : \big(\mathcal{G}, \oplus, \langle\rangle, \star\big) \longrightarrow \big(\mathcal{F}, \wedge, \underline{1}, \underline{0}\big)$$

is defined on objects as follows:

$$\begin{cases} \wedge \star = \underline{0}, \\ \wedge \langle\rangle = \underline{1}, \quad \text{and} \\ \wedge \langle \underline{n}_k \rangle_{k=1}^q = \wedge_{k=1}^q \underline{n}_k = \underline{n_1 \cdots n_q} \quad \text{for} \quad q > 0. \end{cases}$$

The image of a morphism

$$(f, \langle \psi \rangle) : \langle \underline{n}_k \rangle_{k=1}^q \longrightarrow \langle \underline{m}_j \rangle_{j=1}^p \quad \text{in} \quad \mathcal{G}$$

under $\wedge$ is the following composite in $\mathcal{F}$.

$$\wedge_{k=1}^q \underline{n}_k \xrightarrow[\cong]{f_*} \wedge_{k=1}^q \underline{n}_{f^{-1}(k)} \xrightarrow{\cong} \wedge_{j=1}^p \underline{n}_{f^{-1}(j)} \xrightarrow{\wedge_{j=1}^p \psi_j} \wedge_{j=1}^p \underline{m}_j$$

**Induced Functors on Diagrams:** In the context of Definitions 2.3.1 and 2.4.10, $\wedge$ in (2.4.19) and $i$ in (2.4.18) induce the functors

$$(2.4.20) \qquad\qquad \Gamma\text{-}\mathsf{V} \xrightarrow{\wedge^*} \mathcal{G}_*\text{-}\mathsf{V} \xrightarrow{i^*} \Gamma\text{-}\mathsf{V}$$

given by precomposition and whiskering with $\wedge$ and $i$, respectively.        $\diamond$

**Explanation 2.4.21.** Consider Definition 2.4.17.

(1) The functor $i$ is fully faithful, and

$$\wedge \circ i = 1_{\mathcal{F}} : \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}.$$

This implies that the composite in (2.4.20), $i^* \circ \wedge^*$, is the identity functor on $\Gamma\text{-}\mathsf{V}$.

(2) While $\wedge$ is strict symmetric monoidal, $i$ is neither monoidal nor oplax monoidal.        $\diamond$

## 2.5. Segal and Elmendorf-Mandell $K$-theory

In this section we briefly review
- Segal $K$-theory

$$\mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0} \,;$$

- the homotopy inverse functors $\mathcal{P}$, $S_*$, and $\mathbb{A}$; and
- Elmendorf-Mandell $K$-theory

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0} \,.$$

They are summarized in the diagram (2.5.1) below.

(2.5.1)



Each arrow in (2.5.1), *except* $J^{\mathcal{T}}$ and $K^{\mathcal{G}}$, is an equivalence of homotopy theories. The following table summarizes their multicategorical and symmetric monoidal properties.

| PermCat$^{\mathsf{su}}$ | Cat-multicategory | 1.4.29 |
|---|---|---|
| Mod$^{\mathcal{M}\underline{1}}$, $\Gamma$-Cat, $\mathcal{G}_*$-Cat | symmetric monoidal Cat-categories | 1.3.23, 2.3.2, 2.4.11 |
| $\Gamma$-sSet, $\mathcal{G}_*$-sSet, Sp$_{\geq 0}$ | symmetric monoidal sSet-categories | 2.3.2, 2.4.11, 2.5.2 |
| $\mathcal{P}$ | *non-symmetric* Cat-multifunctor | 2.5.17 |
| $K^{\mathsf{EM}}$ | sSet-multifunctor | 2.5.8 |
| $J^{\mathsf{EM}}$, End$_{\mathcal{M}\underline{1}}$ | Cat-multifunctors | 2.5.10, 1.4.41 |
| $\wedge^*$ | enriched symmetric monoidal functors | 2.4.20 |
| $J^{\mathcal{T}}$ | symmetric monoidal Cat-functor | 2.5.9 |
| Ner$_*$, $K^{\mathcal{F}}$, $K^{\mathcal{G}}$ | symmetric monoidal sSet-functors | 2.5.5, 2.5.11, 2.5.6, 2.5.12 |
| $K^{\mathsf{Se}}$, $J^{\mathsf{Se}}$, $i^*$, $L$, $S_*$, $\mathbb{A}$ | *not* multifunctors | 2.5.3, 2.5.4, 2.4.20, 2.5.13, 2.5.18, 2.5.19 |

Later in this work, when we apply our general results to some of these functors, we only need to use some of their categorical properties, which we recall below. Detailed discussion of these functors are in [**JY∞**, Chapters 8–10], [**JY22b, JY22c**], and the references cited below.

**Categories.** The categories in (2.5.1) are defined as follows. Recall symmetric monoidal V-category and V-multicategory from Definitions B.2.16 and C.1.3, respectively.

- PermCat$^{\mathsf{su}}$ is the category of small permutative categories and strictly unital symmetric monoidal functors in Definition A.2.3. By Theorem 1.4.29 it is a Cat-multicategory.
- Mod$^{\mathcal{M}\underline{1}}$ is the symmetric monoidal Cat-category of left $\mathcal{M}\underline{1}$-modules in Definition 1.3.23.
- $\Gamma$-Cat and $\Gamma$-sSet are the categories of $\Gamma$-categories and $\Gamma$-simplicial sets, respectively, in Definition 2.3.1. They are symmetric monoidal categories enriched in Cat and sSet, respectively, by Explanation 2.3.2.
- $\mathcal{G}_*$-Cat and $\mathcal{G}_*$-sSet are the categories of $\mathcal{G}_*$-categories and $\mathcal{G}_*$-simplicial sets, respectively, in Definition 2.4.10. They are symmetric monoidal categories enriched in Cat and sSet, respectively, by Explanation 2.4.11.
- Sp$_{\geq 0}$ is the category of connective symmetric spectra. It is the full subcategory of the category

(2.5.2)                                    Sp

of all symmetric spectra [**HSS00**]. Connective means that the negative degree homotopy groups are trivial. The category Sp is complete, cocomplete, and symmetric monoidal closed. Moreover, it is enriched, tensored, and cotensored over sSet$_*$. In particular, Sp$_{\geq 0}$ is a symmetric monoidal sSet-category. See [**JY∞**, Chapter 7] for an elementary discussion of symmetric spectra.

**Segal $K$-Theory.** The left-to-right composite functor along the top row of (2.5.1),

$$(2.5.3) \qquad \mathsf{K}^{\mathsf{Se}} = \mathsf{K}^{\mathcal{F}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0},$$

is called *Segal $K$-theory*. We also denote by $\mathsf{K}^{\mathsf{Se}}$ its composite with the subcategory inclusion Sp$_{\geq 0}$ $\longhookrightarrow$ Sp. The constituent functors are as follows.

- The first functor is *Segal J-theory* [**Seg74, May78**]

$$(2.5.4) \qquad \mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-}\mathsf{Cat}$$

that sends each small permutative category C to the $\Gamma$-category

$$(\mathsf{J}^{\mathsf{Se}}\mathsf{C})(-) = \mathsf{Multicat}_*\big(\mathcal{M}(-), \mathsf{End}_\bullet\mathsf{C}\big) : \mathcal{F} \longrightarrow \mathsf{Cat}_*.$$

In this definition,
  - $\mathcal{M}(-)$ is the partition multicategory in Definition 1.3.1, and
  - $\mathsf{End}_\bullet\mathsf{C}$ is the pointed endomorphism multicategory in Example C.4.8.
  While both its domain and codomain are Cat-multicategories, $\mathsf{J}^{\mathsf{Se}}$ is *not* a multifunctor because it is incompatible with the multiplicative structures. See [**JY∞**, Section 8.5] for a thorough discussion.
- The second functor,

$$(2.5.5) \qquad \mathsf{Ner}_* : \Gamma\text{-}\mathsf{Cat} \longrightarrow \Gamma\text{-}\mathsf{sSet},$$

is induced by precomposing and whiskering with the nerve functor, Ner, in (2.1.1). Since the nerve is a right adjoint, it preserves all small limits, in particular, terminal objects and finite products. Therefore, $\mathsf{Ner}_*$ is a symmetric monoidal sSet-functor by [**JY∞**, 3.7.28].
- The third functor [**BF78, Seg74**],

$$(2.5.6) \qquad \mathsf{K}^{\mathcal{F}} : \Gamma\text{-}\mathsf{sSet} \longrightarrow \mathsf{Sp}_{\geq 0},$$

sends each $\Gamma$-simplicial set $X$ to the connective symmetric spectrum

$$(2.5.7) \qquad \mathsf{K}^{\mathcal{F}} X = \Big\{ (\mathsf{K}^{\mathcal{F}} X)k = \big| X(S^k) \big| \Big\}_{k \geq 0}$$

with $|-|$ denoting the diagonal and $S^k = (S^1)^{\wedge k}$ denoting the standard simplicial $k$-sphere. Thus $(\mathsf{K}^{\mathcal{F}} X)k$ is the pointed simplicial set whose set of $n$-simplices is given by

$$\big((\mathsf{K}^{\mathcal{F}} X)k\big)_n = \big( X(S^k_n) \big)_n = \big( X\underline{n}^k \big)_n.$$

For $k \geq 1$ the structure morphism

$$\Sigma \big| X(S^{k-1}) \big| \longrightarrow \big| X(S^k) \big|$$

is induced by the inclusions

$$S^{k-1}_n \cong \{0, i\} \wedge S^{k-1}_n \longhookrightarrow S^1_n \wedge S^{k-1}_n \cong S^k_n$$

for $i \in \{1, \ldots, n\}$. The $\Sigma_k$-action on $|X(S^k)|$ is induced by the $\Sigma_k$-action on $S^k = (S^1)^{\wedge k}$ that permutes the $k$ smash factors. See [**JY∞**, Section 8.2] for a thorough discussion. Moreover, with Sp as the codomain, $\mathsf{K}^{\mathcal{F}}$ is a symmetric monoidal sSet-functor by [**JY∞**, 9.4.18].

We discuss the functors $\mathcal{P}$, $S_*$, and $\mathbb{A}$ in (2.5.17) through (2.5.19) below when we discuss equivalences of homotopy theories.

**Elmendorf-Mandell $K$-Theory.** The composite along the bottom of (2.5.1),

$$(2.5.8) \quad \begin{aligned} \mathsf{K}^{\mathsf{EM}} &= \mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathsf{EM}} \\ &= \mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0}, \end{aligned}$$

is called *Elmendorf-Mandell $K$-theory* [**EM06, EM09**]. We also denote by $\mathsf{K}^{\mathsf{EM}}$ its composite with the subcategory inclusion $\mathsf{Sp}_{\geq 0} \hookrightarrow \mathsf{Sp}$. The constituent enriched multifunctors are as follows.

- The endomorphism left $\mathcal{M}\underline{1}$-module Cat-multifunctor

$$\mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

  is as in Explanation 1.4.41.
- The symmetric monoidal Cat-functor

$$(2.5.9) \quad \mathsf{J}^{\mathcal{T}} = \mathsf{Multicat}_*\big(\mathcal{T}, -\big) : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-Cat}$$

  sends each left $\mathcal{M}\underline{1}$-module M to the $\mathcal{G}_*$-category

$$(\mathsf{J}^{\mathcal{T}}\mathsf{M})(-) = \mathsf{Multicat}_*\big(\mathcal{T}(-), \mathsf{M}\big) : \mathcal{G} \longrightarrow \mathsf{Cat}_*.$$

  Here $\mathcal{T}$ is defined by

$$\mathcal{T}\langle \underline{m}_j \rangle_{j=1}^p = \wedge_{j=1}^p \mathcal{M}\underline{m}_j \quad \text{for} \quad \langle \underline{m}_j \rangle_{j=1}^p \in \mathcal{G}$$

  with $\mathcal{M}(-)$ the partition multicategory in Definition 1.3.1. See [**JY∞**, 10.3.17] for a detailed discussion.
- *Elmendorf-Mandell $J$-theory* is the composite Cat-multifunctor

$$(2.5.10) \quad \mathsf{J}^{\mathsf{EM}} = \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathcal{G}_*\text{-Cat},$$

  which associates to each small permutative category a $\mathcal{G}_*$-category. See [**JY∞**, Section 10.3] for a thorough discussion.
- The symmetric monoidal sSet-functor [**JY∞**, 9.2.19]

$$(2.5.11) \quad \mathsf{Ner}_* : \mathcal{G}_*\text{-Cat} \longrightarrow \mathcal{G}_*\text{-sSet}$$

  is induced by precomposing and whiskering with the nerve functor, Ner.
- The symmetric monoidal sSet-functor [**EM06**, Definition 4.5]

$$(2.5.12) \quad \mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-sSet} \longrightarrow \mathsf{Sp}$$

  sends each $\mathcal{G}_*$-simplicial set $X$ to the connective symmetric spectrum

$$\mathsf{K}^{\mathcal{G}}X = \Big\{ (\mathsf{K}^{\mathcal{G}}X)k = \big| X\big( \underbrace{S^1, \ldots, S^1}_{k} \big) \big| \Big\}_{k \geq 0}.$$

  The structure morphisms and symmetric group action are analogous to those for $\mathsf{K}^{\mathcal{F}}(-)$ in (2.5.7). See [**JY∞**, Sections 9.3 and 9.4] for a thorough discussion.

It follows that $\mathsf{K}^{\mathsf{EM}}$ is an sSet-multifunctor.

**Relating Segal and Elmendorf-Mandell $K$-Theory.** The other functors in (2.5.1) are defined as follows.

- The functors

$$\Gamma\text{-Cat} \xrightarrow{\;\wedge^*\;} \mathcal{G}_*\text{-Cat} \xrightarrow{\;i^*\;} \Gamma\text{-Cat}$$
$$\Gamma\text{-sSet} \xrightarrow{\;\wedge^*\;} \mathcal{G}_*\text{-sSet} \xrightarrow{\;i^*\;} \Gamma\text{-sSet}$$

  are induced by
  - the smash product, $\wedge : \mathcal{G} \longrightarrow \mathcal{F}$, and
  - the length-one inclusion functor, $i : \mathcal{F} \longrightarrow \mathcal{G}$, as in (2.4.20).

  Since $\wedge$ is a strict symmetric monoidal functor, each $\wedge^*$ is a symmetric monoidal functor in the enriched sense by [**JY∞**, 9.4.18]. However, neither $i^*$ is a multifunctor because $i$ is not compatible with the permutative structures of its domain and codomain.

- In the middle square in (2.5.1), associativity of composition of functors and whiskering implies the following equalities.

$$\mathsf{Ner}_* \circ \wedge^* = \wedge^* \circ \mathsf{Ner}_* : \Gamma\text{-Cat} \longrightarrow \mathcal{G}_*\text{-sSet}$$
$$\mathsf{Ner}_* \circ i^* = i^* \circ \mathsf{Ner}_* : \mathcal{G}_*\text{-Cat} \longrightarrow \Gamma\text{-sSet}.$$

- The right region in (2.5.1) commutes,

$$\mathsf{K}^{\mathcal{G}} \circ \wedge^* = \mathsf{K}^{\mathcal{F}} : \Gamma\text{-sSet} \longrightarrow \mathsf{Sp}_{\geq 0}.$$

  See [**JY∞**, 9.3.16] for a proof.

- In the left region in (2.5.1), there is an equality

$$i^* \circ \mathsf{J}^{\mathsf{EM}} = \mathsf{J}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \Gamma\text{-Cat}$$

  by the definitions of the functors involved; see [**JY∞**, 8.5.1, 10.3.1, and 10.3.27]. Moreover, there is a natural transformation [**EM06**, Theorem 4.6]

$$\Pi^* : \wedge^* \circ \mathsf{J}^{\mathsf{Se}} \longrightarrow \mathsf{J}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathcal{G}_*\text{-Cat}.$$

  See [**JY∞**, Section 10.6] for a detailed discussion.

- Each of the functors

(2.5.13)           $$\Gamma\text{-Cat} \xrightarrow{\;L\;} \mathcal{G}_*\text{-Cat} \quad \text{and} \quad \Gamma\text{-sSet} \xrightarrow{\;L\;} \mathcal{G}_*\text{-sSet}$$

  is the left adjoint of the respective functor $i^*$. See [**JY22c**, Section 3] for a detailed construction of $L$. By the explicit formulas of the pointed Day convolution, (2.3.5) and (2.4.14), neither $L$ is compatible with the multiplicative structures of its domain and codomain.

**Stable Equivalences.** Each category in (2.5.1) is equipped with the structure of a relative category (Definition 2.1.6) as follows.

- The pair

$$\left( \mathsf{Sp}_{\geq 0} , \mathcal{S} \right)$$

  is a relative category, where $\mathcal{S}$ is the wide subcategory of *stable equivalences* of connective symmetric spectra.

- For each of PermCat$^{su}$, Γ-Cat, and Γ-sSet, we denote by $\mathcal{S}$ the wide sub-category of morphisms created by the indicated functor:

(2.5.14) $\left( \text{PermCat}^{su}, \mathcal{S} \right) \xrightarrow{J^{Se}} \left( \text{Γ-Cat}, \mathcal{S} \right) \xrightarrow{\text{Ner}_*} \left( \text{Γ-sSet}, \mathcal{S} \right) \xrightarrow{K^{\mathcal{F}}} \left( \text{Sp}_{\geq 0}, \mathcal{S} \right).$

In each case, we call morphisms in $\mathcal{S}$ *stable equivalences*. In particular, stable equivalences in PermCat$^{su}$ are created by Segal *K*-theory $K^{Se}$ (2.5.3).

- We denote by

$$\mathcal{S}^i \subset \mathcal{G}_*\text{-Cat} \quad \text{and} \quad \mathcal{S}^i \subset \mathcal{G}_*\text{-sSet}$$

the wide subcategories created by, respectively, the functors

$$\mathcal{G}_*\text{-Cat} \xrightarrow{i^*} \text{Γ-Cat} \quad \text{and} \quad \mathcal{G}_*\text{-sSet} \xrightarrow{i^*} \text{Γ-sSet}.$$

We refer to morphisms in $\mathcal{S}^i$ as *$i^*$-stable equivalences*.

- The natural transformation $\Pi^*$ is componentwise an $i^*$-stable equivalence in $\mathcal{G}_*$-Cat; see [**JY22c**, 4.10] for an explanation.

In Definition 4.7.1 we equip $\text{Mod}^{\mathcal{M}1}$ with the structure of a relative category.

**Remark 2.5.15** (Stable Equivalences). The following two key properties of stable equivalences will be used repeatedly below.

(1) Each class of stable equivalences includes isomorphisms, is closed under composition, and has the 2-out-of-3 property. These follow from the stronger statement that there is a Quillen model structure on Sp whose weak equivalences are the stable equivalences [**HSS00**, 3.4.4 and 5.3.8].

(2) Suppose given

$$P : C \longrightarrow D \quad \text{in} \quad \text{PermCat}^{su}.$$

If the underlying functor of $P$ is a left or right adjoint, then $P$ is a stable equivalence. This follows from the observations that (i) an adjunction of categories induces a homotopy equivalence on nerves [**JY∞**, 7.2.5] and (ii) the stable equivalences of symmetric spectra contain the level equivalences [**JY∞**, 7.8.8]. ◇

**Equivalences of Homotopy Theories.** Equipped with the relative category structures above, each arrow in (2.5.1), *except* $J^{\mathcal{T}}$ and $K^{\mathcal{G}}$, is an equivalence of homotopy theories in the sense of Definition 2.1.7.

*Segal K-Theory.*

- Each of the two relative functors

$$\text{PermCat}^{su} \xrightarrow{J^{Se}} \text{Γ-Cat} \xrightarrow{\text{Ner}_*} \text{Γ-sSet}$$

is an equivalence of homotopy theories by the work of Mandell [**Man10**], which sharpens earlier work of Thomason [**Tho95**].

- The relative functor

$$K^{\mathcal{F}} : \text{Γ-sSet} \longrightarrow \text{Sp}_{\geq 0}$$

is an equivalence of homotopy theories by the work of Segal [**Seg74**] and Bousfield-Friedlander [**BF78**]. Therefore, Segal *K*-theory

(2.5.16) $\qquad K^{Se} = K^{\mathcal{F}} \circ \text{Ner}_* \circ J^{Se} : \text{PermCat}^{su} \longrightarrow \text{Sp}_{\geq 0}$

is an equivalence of homotopy theories.

*Homotopy Inverses.*  Each constituent functor in Segal $K$-theory admits a homotopy inverse functor given by the right-to-left functors along the top row of (2.5.1).

- The work of Mandell [**Man10**] constructs the relative functor $\mathcal{P}$ in

$$(2.5.17) \qquad \mathsf{PermCat^{su}} \; \underset{\mathsf{J^{Se}}}{\overset{\mathcal{P}}{\longleftrightarrow}} \; \mathsf{\Gamma\text{-}Cat}$$

  and shows that the pair $(\mathcal{P}, \mathsf{J^{Se}})$ forms inverse equivalences of homotopy theories as in Definition 2.1.8. Thus $\mathcal{P}$ is an equivalence of homotopy theories by Proposition 2.1.9.

  While $\mathsf{J^{Se}}$ is not a multifunctor even in the non-symmetric sense, $\mathcal{P}$ *is* a non-symmetric Cat-multifunctor by [**JY22b**, 1.3]. Moreover, $\mathcal{P}$ is a *pseudo symmetric* Cat-multifunctor by [**Yau24**, 10.12]. This means that $\mathcal{P}$ preserves the symmetric group action up to natural isomorphisms that satisfy further coherence axioms. Since we will not use the pseudo symmetry of $\mathcal{P}$ in this work, we refer the reader to [**Yau24**] for detailed definitions and discussion.

- The work of Mandell [**Man10**] also constructs the relative functor $S_*$ in

$$(2.5.18) \qquad \mathsf{\Gamma\text{-}Cat} \; \underset{\mathsf{Ner_*}}{\overset{S_*}{\longleftrightarrow}} \; \mathsf{\Gamma\text{-}sSet}$$

  and shows that the pair $(S_*, \mathsf{Ner_*})$ forms inverse equivalences of homotopy theories as in Definition 2.1.8. Thus $S_*$ is an equivalence of homotopy theories by Proposition 2.1.9. In contrast to the symmetric monoidal sSet-functor $\mathsf{Ner_*}$, the functor $S_*$ is *not* a multifunctor even in the non-symmetric sense. See the introduction of [**JY22c**] for an explanation.

- The relative functor $\mathbb{A}$ in

$$(2.5.19) \qquad \mathsf{\Gamma\text{-}sSet} \; \underset{\mathsf{K^{\mathcal{F}}}}{\overset{\mathbb{A}}{\longleftrightarrow}} \; \mathsf{Sp}_{\geq 0}$$

  is constructed in [**Seg74**, Def. 3.1]. The work of Bousfield-Friedlander [**BF78**, Theorem 5.8] shows that the adjoint pair $\mathsf{K^{\mathcal{F}}} \dashv \mathbb{A}$ is a Quillen equivalence. In contrast to the symmetric monoidal sSet-functor $\mathsf{K^{\mathcal{F}}}$, the functor $\mathbb{A}$ is *not* compatible with the multiplicative structures of its domain and codomain. Thus $\mathbb{A}$ is not a multifunctor even in the non-symmetric sense.

*Elmendorf-Mandell K-Theory.*

- The authors show in [**JY22c**] that each of the three relative functors

$$(2.5.20) \qquad \left\{ \begin{array}{l} \mathsf{J^{EM}} : \mathsf{PermCat^{su}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{Cat}, \\[4pt] \mathsf{Ner_*} : \mathcal{G}_*\text{-}\mathsf{Cat} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}, \quad \text{and} \\[4pt] \mathsf{K^{EM}} = \mathsf{K^{\mathcal{G}}} \circ \mathsf{Ner_*} \circ \mathsf{J^{EM}} : \mathsf{PermCat^{su}} \longrightarrow \mathsf{Sp}_{\geq 0} \end{array} \right.$$

  is an equivalence of homotopy theories. Therefore, the composite

$$\mathsf{Ner_*} \circ \mathsf{J^{EM}} : \mathsf{PermCat^{su}} \longrightarrow \mathcal{G}_*\text{-}\mathsf{sSet}$$

  is also an equivalence of homotopy theories.

- The work of [**JY22c**] also shows that each of the six relative functors

$$\Gamma\text{-Cat} \qquad\qquad \Gamma\text{-sSet}$$

$$\wedge^* \left\downarrow\uparrow \left(i^* \vdash\right) L \qquad L\left(\dashv\ i^* \right\uparrow\downarrow \wedge^*\right.$$

$$\mathcal{G}_*\text{-Cat} \qquad\qquad \mathcal{G}_*\text{-sSet}$$

is an equivalence of homotopy theories.

In Theorems 4.8.3 and 5.5.12 we show that

$$\mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

is an equivalence of homotopy theories. We emphasize that the functors

$$\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-Cat} \quad \text{and}$$

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-sSet} \longrightarrow \mathsf{Sp}_{\geq 0}$$

are probably not equivalences of homotopy theories because they are not known to be relative functors.

CHAPTER 3

# Homotopy Theory of Multicategories

Recall the following 2-categories from Definition A.2.3 and Theorem C.1.33.

- PermCat$^{\mathsf{st}}$ is the 2-category of small permutative categories, *strict* symmetric monoidal functors, and monoidal natural transformations.
- PermCat$^{\mathsf{su}}$ is the larger 2-category with *strictly unital* symmetric monoidal functors as 1-cells.
- Multicat is the 2-category of small multicategories, multifunctors, and multinatural transformations.

To prepare for Chapters 4 and 5, in this chapter we review equivalences of homotopy theories between these three 2-categories. Here is a summary diagram.

$$(3.0.1) \qquad \left( \mathsf{Multicat}, \mathcal{S}^{\mathsf{F}} \right) \underset{\mathsf{End}}{\overset{\mathsf{F}}{\underset{\perp}{\rightleftarrows}}} \left( \mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I \right) \overset{I}{\hookrightarrow} \left( \mathsf{PermCat}^{\mathsf{su}}, \mathcal{S} \right) \overset{\mathsf{K}^{\mathsf{Se}}}{\longrightarrow} \left( \mathsf{Sp}_{\geq 0}, \mathcal{S} \right)$$

Each arrow in the diagram (3.0.1) is an equivalence of homotopy theories, where

$$\mathsf{K}^{\mathsf{Se}} : \left( \mathsf{PermCat}^{\mathsf{su}}, \mathcal{S} \right) \overset{\sim}{\longrightarrow} \left( \mathsf{Sp}_{\geq 0}, \mathcal{S} \right)$$

is Segal $K$-theory. Thus, there is a composite equivalence of homotopy theories

$$\left( \mathsf{Multicat}, \mathcal{S}^{\mathsf{F}} \right) \overset{\mathsf{F}}{\underset{\sim}{\longrightarrow}} \left( \mathsf{PermCat}^{\mathsf{su}}, \mathcal{S} \right) \overset{\mathsf{K}^{\mathsf{Se}}}{\underset{\sim}{\longrightarrow}} \left( \mathsf{Sp}_{\geq 0}, \mathcal{S} \right).$$

Moreover, for each small non-symmetric Cat-multicategory Q, there are equivalences of homotopy theories

$$(3.0.2) \qquad \left( \mathsf{Multicat}^{\mathsf{Q}}, (\mathcal{S}^{\mathsf{F}})^{\mathsf{Q}} \right) \underset{\mathsf{End}^{\mathsf{Q}}}{\overset{\mathsf{F}^{\mathsf{Q}}}{\underset{\sim}{\rightleftarrows}}} \left( (\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}, \mathcal{S}^{\mathsf{Q}} \right)$$

between categories of non-symmetric Q-algebras. See Theorem 3.5.5.

**Connection with Other Chapters.**

*Pointed Extension.* In Chapter 4 we extend the 2-adjunction and adjoint equivalence of homotopy theories

$$(3.0.3) \qquad \left( \mathsf{Multicat}, \mathcal{S}^{\mathsf{F}} \right) \underset{\mathsf{End}}{\overset{\mathsf{F}}{\underset{\perp}{\rightleftarrows}}} \left( \mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I \right)$$

to small *pointed* multicategories, Multicat$_*$, and left $\mathcal{M}\underline{1}$-modules, Mod$^{\mathcal{M}\underline{1}}$. One subtlety of this pointed extension is that it is *not* achieved through the free-forgetful adjunction between Multicat and Multicat$_*$ in Proposition C.4.16. Instead, the key ingredient is a detailed analysis of the pointed variant of F, denoted F$_\bullet$ in Theorem 4.1.17. Note that F in (3.0.3) is *not* a multifunctor because a multifunctor

structure on F requires using strictly unital, but generally non-strict, symmetric monoidal functors. See Theorem 3.4.31.

*Pointed Multifunctorial Extension.* With PermCat$^{su}$ in place of PermCat$^{st}$, in Chapter 5 we extend

- the Cat-multifunctors F and End (non-symmetric in the case of F) and
- the equivalences of homotopy theories in (3.0.2)

to small pointed multicategories and left $\mathcal{M}\underline{1}$-modules. This requires a nontrivial extension of the machinery in Section 3.4 and key results in Chapter 4.

*Enriched Mackey Functors.* In Part 4 we further extend the main results in Chapter 5 to equivalences of homotopy theories between

- enriched Mackey functors based on permutative categories,
- enriched Mackey functors based on small pointed multicategories, and
- enriched Mackey functors based on left $\mathcal{M}\underline{1}$-modules.

See Theorems 12.1.6, 12.4.6, and 12.6.6.

**Background.** We use 2-adjunctions in Definition A.2.11, enriched multifunctors and multinatural transformations in Appendix C.1, multilinear functors in Definition 1.4.2, and equivalences of homotopy theories in Section 2.1. Discussion of Segal $K$-theory is in Section 2.5. The material in this chapter is adapted from [**JY22a, JY23**], where we refer the reader for detailed proofs.

**Chapter Summary.** In Section 3.1 we discuss the 2-functor F in (3.0.3), which we call the *free permutative category* construction. In Section 3.2 we discuss the 2-adjunction F ⊣ End in (3.0.3). In Section 3.3 we discuss a componentwise right adjoint $\varrho$ of the counit $\varepsilon$ of F ⊣ End. This componentwise right adjoint is, furthermore, a symmetric monoidal functor. In Section 3.4 we extend the 2-functor F to a non-symmetric Cat-multifunctor with codomain PermCat$^{su}$. In Section 3.5 we define the subcategories $\mathcal{S}^I$ and $\mathcal{S}^F$ and review the equivalences of homotopy theories in (3.0.1) and (3.0.2). Here is a summary table.

| | |
|---|---|
| F on multicategories, multifunctors, and multinatural transformations | 3.1.5, 3.1.16, and 3.1.19 |
| unit $\eta : 1 \longrightarrow$ End F and counit $\varepsilon :$ F End $\longrightarrow 1$ for F ⊣ End | 3.2.1 and 3.2.4 |
| componentwise right adjoint $\varrho_C$ of $\varepsilon_C$ | 3.3.1 |
| symmetric monoidal functor $\left(\varrho_C, \varrho_C^2, \varrho_C^0\right)$ | 3.3.9 |
| multilinear functor $F^n : \langle FM_i \rangle \longrightarrow F(\otimes_i M_i)$ | 3.4.14 |
| non-symmetric Cat-multifunctor F | 3.4.26 |
| non-symmetric Cat-multinatural unit $\eta : 1 \longrightarrow$ End F | 3.4.34 |
| stable equivalences $\mathcal{S}^I$ and $\mathcal{S}^F$ | 3.5.1 |
| equivalences of homotopy theories in (3.0.1) and (3.0.2) | 3.5.3, 3.5.5, 3.5.7, and 3.5.9 |

We remind the reader of Conventions A.1.2 and A.1.30.

## 3.1. Free Permutative Categories

In this section we describe a 2-functor

$$F : \mathsf{Multicat} \longrightarrow \mathsf{PermCat^{st}} .$$

In Section 3.2 we discuss unit and counit that make the pair $(\mathsf{F}, \mathsf{End})$ into a 2-adjunction.

$$\mathsf{Multicat} \underset{\mathsf{End}}{\overset{\mathsf{F}}{\rightleftarrows}} \perp \mathsf{PermCat}^{\mathsf{st}}$$

**Free Permutative Category of a Multicategory.** Recall from (2.4.2) that

$$\overline{n} = \{1, \ldots, n\}$$

denotes the unpointed finite set with $n$ elements, where $\overline{0} = \varnothing$. Concatenation of tuples is denoted by $\oplus$. The definition of $\mathsf{F}$ uses the following notation for sub-tuples of objects and morphisms.

**Definition 3.1.1** (Sub-tuples). Suppose $\mathsf{M}$ is a multicategory (Definition C.1.3), and $\langle x \rangle = \langle x_i \rangle_{i=1}^r$ is an $r$-tuple of objects in $\mathsf{M}$. Suppose

$$\overline{r} \xrightarrow{f} \overline{s} \xrightarrow{g} \overline{t}$$

are functions of unpointed finite sets with $r, s, t \geq 0$. We define the following.

(1) For $j \in \overline{s}$, we define the sub-tuple of $\langle x \rangle$,

(3.1.2)
$$\langle x \rangle_{f^{-1}(j)} = \begin{cases} \langle x_i \rangle_{i \in f^{-1}(j)} & \text{if } f^{-1}(j) \neq \varnothing \text{ and} \\ \langle \rangle & \text{if } f^{-1}(j) = \varnothing, \end{cases}$$

consisting of those objects $x_i$ such that $f(i) = j$.

(2) For an $s$-tuple $\langle \phi \rangle = \langle \phi_j \rangle_{j=1}^s$ of multimorphisms in $\mathsf{M}$ and $k \in \overline{t}$, we define the sub-tuple of $\langle \phi \rangle$,

(3.1.3)
$$\langle \phi \rangle_{g^{-1}(k)} = \begin{cases} \langle \phi_j \rangle_{j \in g^{-1}(k)} & \text{if } g^{-1}(k) \neq \varnothing \text{ and} \\ \langle \rangle & \text{if } g^{-1}(k) = \varnothing. \end{cases}$$

(3) For $k \in \overline{t}$, we define $\sigma_{g,f}^k \in \Sigma_t$ to be the unique permutation determined by the equality

(3.1.4)
$$\left[ \bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} \right] \cdot \sigma_{g,f}^k = \langle x \rangle_{(gf)^{-1}(k)}.$$

- The tuple $\bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)}$ in (3.1.4) is the concatenation of the tuples $\langle x \rangle_{f^{-1}(j)}$ in the order specified by $j \in g^{-1}(k)$.
- The right-hand side of (3.1.4) is a sub-tuple of $\langle x \rangle$, defined as in (3.1.2).

This finishes the definition. ◇

Recall from Definition C.1.1 that $\mathsf{Prof}(S)$ means the class of finite tuples in $S$. The free permutative category in the next definition is sketched in [**EM09**, Theorem 4.2].

**Definition 3.1.5** (Free Permutative Category). Given a multicategory $(\mathsf{M}, \gamma, 1)$, we define a permutative category

$$(\mathsf{FM}, \oplus, \langle \rangle, \xi),$$

which is called the *free permutative category on* $\mathsf{M}$, as follows.

**Objects:** $\mathsf{Ob}(\mathsf{FM}) = \mathsf{Prof}(\mathsf{M})$, the class of finite tuples $\langle x \rangle = \langle x_i \rangle_{i=1}^r$ with each $x_i \in \mathsf{Ob}\,\mathsf{M}$ and $r \geq 0$.

**Morphisms:** Given finite tuples $\langle x \rangle = \langle x_i \rangle_{i=1}^r$ and $\langle y \rangle = \langle y_j \rangle_{j=1}^s$, a morphism

$$(3.1.6) \qquad\qquad (f, \langle \phi \rangle) : \langle x \rangle \longrightarrow \langle y \rangle \quad \text{in} \quad \mathsf{FM}$$

is a pair consisting of
- a function
$$f : \overline{r} \longrightarrow \overline{s},$$
called the *index map*, and
- an $s$-tuple of multimorphisms
$$\langle \phi \rangle = \langle \phi_j \rangle_{j=1}^s \quad \text{with} \quad \phi_j \in \mathsf{M}\left( \langle x_i \rangle_{i \in f^{-1}(j)} ; y_j \right).$$

**Identities:** The identity morphism for an object $\langle x \rangle = \langle x_i \rangle_{i=1}^r$ in $\mathsf{FM}$ is the pair
$$1_{\langle x \rangle} = \left( 1_{\overline{r}}, \langle 1_{x_i} \rangle_{i=1}^r \right).$$

**Composition:** Given a pair of morphisms

$$\langle x \rangle = \langle x_i \rangle_{i=1}^r \xrightarrow{(f, \langle \phi \rangle)} \langle y \rangle = \langle y_j \rangle_{j=1}^s \xrightarrow{(g, \langle \psi \rangle)} \langle z \rangle = \langle z_k \rangle_{k=1}^t$$

their composite is the morphism

$$(3.1.7) \qquad\qquad \left( gf, \left\langle \theta_k \cdot \sigma_{g,f}^k \right\rangle_{k=1}^t \right) : \langle x \rangle \longrightarrow \langle z \rangle$$

with

$$(3.1.8) \qquad\qquad \theta_k = \gamma\left( \psi_k ; \langle \phi_j \rangle_{j \in g^{-1}(k)} \right) \in \mathsf{M}\left( \bigoplus_{j \in g^{-1}(k)} \langle x \rangle_{f^{-1}(j)} ; z_k \right)$$

for each $k \in \overline{t}$ and $\sigma_{g,f}^k$ as in (3.1.4).

**Monoidal Product on Objects:** The monoidal product

$$(3.1.9) \qquad\qquad \oplus : \mathsf{FM} \times \mathsf{FM} \longrightarrow \mathsf{FM}$$

is given by concatenation of finite tuples on objects:

$$(3.1.10) \qquad\qquad \langle x_i \rangle_{i=1}^r \oplus \langle y_j \rangle_{j=1}^s = \left( \langle x_i \rangle_{i=1}^r, \langle y_j \rangle_{j=1}^s \right).$$

**Monoidal Product on Morphisms:** Given a pair of morphisms

$$\left( f, \langle \phi_j \rangle_{j=1}^s \right) : \langle x_i \rangle_{i=1}^r \longrightarrow \langle y_j \rangle_{j=1}^s \quad \text{and} \quad \left( f', \langle \phi'_j \rangle_{j=1}^{s'} \right) : \langle x'_i \rangle_{i=1}^{r'} \longrightarrow \langle y'_j \rangle_{j=1}^{s'}$$

in $\mathsf{FM}$, their monoidal product is the morphism

$$(3.1.11) \qquad\qquad \left( f \oplus f', \langle \phi \rangle \oplus \langle \phi' \rangle \right) : \langle x \rangle \oplus \langle x' \rangle \longrightarrow \langle y \rangle \oplus \langle y' \rangle.$$

In (3.1.11) the index map is the composite

$$\overline{r + r'} \xrightarrow[\cong]{} \overline{r} \amalg \overline{r'} \xrightarrow{f \amalg f'} \overline{s} \amalg \overline{s'} \xrightarrow[\cong]{} \overline{s + s'}$$

with $f \oplus f'$ over the top, given by
- the canonical order-preserving isomorphisms and
- the disjoint union of $f$ with $f'$.

**Monoidal Unit:** The strict monoidal unit is the empty sequence $\langle\,\rangle$. The associativity and unit isomorphisms for $\oplus$ are identity natural transformations.

**Braiding:** The braiding for objects $\langle x_i \rangle_{i=1}^r$ and $\langle y_j \rangle_{j=1}^s$ is

(3.1.12) $$\xi_{\langle x \rangle, \langle y \rangle} = \big(\tau_{r,s}, \langle 1 \rangle\big) : \langle x \rangle \oplus \langle y \rangle \xrightarrow{\ \cong\ } \langle y \rangle \oplus \langle x \rangle.$$

In (3.1.12) the index map is the composite

$$\overline{r+s} \xrightarrow{\ \cong\ } \overline{r} \amalg \overline{s} \xrightarrow[\text{swap}]{\ \cong\ } \overline{s} \amalg \overline{r} \xrightarrow{\ \cong\ } \overline{s+r}$$

with $\tau_{r,s}$ the overall map.

given by
- the canonical order-preserving isomorphisms and
- the block permutation that swaps $\overline{r}$ and $\overline{s}$, keeping the relative order within each block unchanged.

Each entry in the $(r+s)$-tuple $\langle 1 \rangle$ in (3.1.12) is a colored unit of some $x_i$ or $y_j$.

This finishes the definition of $\big(\mathsf{FM}, \oplus, \langle \rangle, \xi\big)$. ◇

A detailed proof of the following is in [**JY23**, 5.7].

**Proposition 3.1.13.** *For each multicategory* $\mathsf{M}$, *the quadruple in Definition 3.1.5*

$$\big(\mathsf{FM}, \oplus, \langle \rangle, \xi\big)$$

*is a permutative category.*

**Example 3.1.14** (Free Permutative Category of the Initial Operad). The initial operad $\mathsf{I}$ in Example C.1.35 (i) has a single object $*$ and a single operation $1_* \in \mathsf{I}_1$. The free permutative category $\mathsf{F}(\mathsf{I})$ is isomorphic to the permutation category defined as follows.

- Its objects are natural numbers, $n \geq 0$, corresponding to length-$n$ sequences of the object $* \in \mathsf{I}$.
- Its morphisms are permutations

$$\mathsf{F}(\mathsf{I})(p,q) = \begin{cases} \Sigma_p & \text{if} \quad p = q, \\ \varnothing & \text{if} \quad p \neq q \end{cases}$$

for $p, q \geq 0$.
- The permutative structure $\oplus$ is given by addition on objects and block sums on morphisms.
- The monoidal unit is the object $0$.
- The braiding

$$\xi_{p,q} : p + q \xrightarrow{\ \cong\ } q + p$$

is the block permutation in $\Sigma_{p+q}$ that swaps the first $p$ elements with the last $q$ elements. This is denoted $\tau_{p,q}$ in (2.4.9). ◇

**Example 3.1.15** (Free Permutative Category of the Terminal Multicategory). The terminal multicategory $\mathsf{T}$ in Example C.1.17 has a single object and a unique $n$-ary operation for each $n$. The free permutative category $\mathsf{FT}$ is isomorphic to the natural number category $\mathbf{N}$, whose objects are natural numbers and whose morphisms are given by morphisms of finite sets

$$\mathbf{N}(r,s) = \mathsf{Set}(\overline{r}, \overline{s}).$$

The natural number $r \in \mathbf{N}$ corresponds to the sequence of length $r$ where each term is the unique object of T. A morphism $f : \overline{r} \longrightarrow \overline{s}$ in $\mathbf{N}$ corresponds to the morphism

$$(f, \langle \phi \rangle) \in \mathsf{FT}$$

where $\phi_j$ is the unique operation in T of arity $|f^{-1}(j)|$.                                          ⋄

**Free Permutative Category as a 2-Functor.**  Next we define F on multifunctors (Definition C.1.19) and multinatural transformations (Definition C.1.25).

**Definition 3.1.16** (F on Multifunctors). Given a multifunctor $H : \mathsf{M} \longrightarrow \mathsf{N}$, we define a *strict* symmetric monoidal functor

$$\mathsf{F}H : \mathsf{FM} \longrightarrow \mathsf{FN}$$

as follows.

**Object Assignment:**  For an object $\langle x_i \rangle_{i=1}^r$ in FM, we define the object

$$(3.1.17) \qquad\qquad (\mathsf{F}H)\langle x_i \rangle_{i=1}^r = \langle H x_i \rangle_{i=1}^r \quad \text{in} \quad \mathsf{FN}.$$

**Morphism Assignment:**  For a morphism

$$\left( f, \langle \phi_j \rangle_{j=1}^s \right) : \langle x_i \rangle_{i=1}^r \longrightarrow \langle y_j \rangle_{j=1}^s \quad \text{in} \quad \mathsf{FM}$$

as in (3.1.6), we define the morphism

$$(3.1.18) \qquad (\mathsf{F}H)(f, \langle \phi \rangle) = \left( f, \langle H\phi_j \rangle_{j=1}^s \right) : \langle H x_i \rangle_{i=1}^r \longrightarrow \langle H y_j \rangle_{j=1}^s$$

in FN.

**Constraints:**  The unit and monoidal constraints for $\mathsf{F}H$ are identities.

This finishes the definition of $\mathsf{F}H$.                                          ⋄

**Definition 3.1.19** (F on Multinatural Transformations). Suppose $H, K : \mathsf{M} \longrightarrow \mathsf{N}$ are multifunctors. Given a multinatural transformation

$$\omega : H \longrightarrow K,$$

we define a monoidal natural transformation

$$\mathsf{F}\omega : \mathsf{F}H \longrightarrow \mathsf{F}K$$

with component morphism

$$(3.1.20) \qquad (\mathsf{F}\omega)_{\langle x \rangle} = \left( 1_{\overline{r}}, \langle \omega_{x_i} \rangle_{i=1}^r \right) : \langle H x_i \rangle_{i=1}^r \longrightarrow \langle K x_i \rangle_{i=1}^r \quad \text{in} \quad \mathsf{FN}$$

for each object $\langle x_i \rangle_{i=1}^r$ in FM.                                          ⋄

The following result is [**JY23**, 5.13].

**Proposition 3.1.21.**  *The constructions in Definitions 3.1.5, 3.1.16, and 3.1.19 provide a 2-functor*

$$\mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}.$$

We also use F to denote the composite of the 2-functor in Proposition 3.1.21 with any one of the inclusion 2-functors in (A.2.6).

## 3.2. Free Permutative Category as a Left 2-Adjoint

In this section we recall the fact that the 2-functor F in Proposition 3.1.21 is a left 2-adjoint of the endomorphism multicategory 2-functor in Proposition C.3.6

$$\mathsf{End} : \mathsf{PermCat^{st}} \longrightarrow \mathsf{Multicat}.$$

Recall from Example C.3.1 that, for each permutative category C, the endomorphism multicategory $\mathsf{End}(\mathsf{C})$ has the same objects as C. Next we define the unit and counit for the 2-adjunction $(\mathsf{F}, \mathsf{End})$ in the sense of Definition A.2.11.

**Definition 3.2.1** (Unit). Given a multicategory M, we define a multifunctor

$$\eta_\mathsf{M} : \mathsf{M} \longrightarrow \mathsf{End\,FM}$$

as follows.

**Object Assignment:** For an object $y$ in M, we define the object

(3.2.2) $$\eta_\mathsf{M}(y) = (y) \quad \text{in} \quad \mathsf{End\,FM},$$

where $(y)$ on the right-hand side is the length-1 tuple consisting of the object $y$.

**Multimorphism Assignment:** For an $r$-ary multimorphism

$$\phi : \langle x \rangle = \langle x_i \rangle_{i=1}^r \longrightarrow y \quad \text{in} \quad \mathsf{M},$$

we define the $r$-ary multimorphism

(3.2.3) $$\eta_\mathsf{M}(\phi) = \big(\iota_r, (\phi)\big) : \langle x \rangle \longrightarrow (y)$$

in

$$\big(\mathsf{End\,FM}\big)\Big(\big\langle (x_i) \big\rangle_{i=1}^r ; (y)\Big) = (\mathsf{FM})\big(\langle x \rangle, (y)\big).$$

On the right-hand side of (3.2.3),
- $\iota_r : \overline{r} \longrightarrow \overline{1}$ is the unique function, and
- $(\phi)$ is the length-1 tuple consisting of $\phi$.

This finishes the definition of $\eta_\mathsf{M}$. Multifunctoriality of $\eta_\mathsf{M}$ follows from the definitions of monoidal sum and composition in FM. The 2-naturality of $\eta$ follows from the termwise definitions of $\mathsf{F}H$ and $\mathsf{F}\omega$ in Definitions 3.1.16 and 3.1.19, respectively. ◇

**Definition 3.2.4** (Counit). Given a permutative category $(\mathsf{C}, \oplus, \mathsf{e}, \xi)$, we define a *strict* symmetric monoidal functor

$$\varepsilon_\mathsf{C} : \mathsf{F\,End}(\mathsf{C}) \longrightarrow \mathsf{C}$$

as follows.

**Object Assignment:** For an $r$-tuple $\langle x_i \rangle_{i=1}^r$ of objects in C, we define the object

(3.2.5) $$\varepsilon_\mathsf{C} \langle x \rangle = \bigoplus_{i=1}^r x_i \quad \text{in} \quad \mathsf{C},$$

where an empty $\oplus$ means the monoidal unit $\mathsf{e}$.

**Morphism Assignment:** Suppose given a morphism

(3.2.6) $$\big(f, \langle \phi_j \rangle_{j=1}^s\big) : \langle x_i \rangle_{i=1}^r \longrightarrow \langle y_j \rangle_{j=1}^s \quad \text{in} \quad \mathsf{F\,End}(\mathsf{C})$$

with each

$$\phi_j \in \mathsf{End}(\mathsf{C})\Big(\langle x \rangle_{f^{-1}(j)} ; y_j\Big) = \mathsf{C}\Big(\bigoplus_{i \in f^{-1}(j)} x_i, y_j\Big).$$

We define the morphism

$$\varepsilon_{\mathsf{C}}\big(f, \langle \phi \rangle\big) : \varepsilon_{\mathsf{C}}\langle x \rangle \longrightarrow \varepsilon_{\mathsf{C}}\langle y \rangle \quad \text{in} \quad \mathsf{C}$$

as the following composite.

(3.2.7)
$$\overset{\varepsilon_{\mathsf{C}}(f, \langle \phi \rangle)}{\overbrace{\bigoplus_{i=1}^{r} x_i \xrightarrow[\cong]{\xi_f} \bigoplus_{j=1}^{s} \bigoplus_{i \in f^{-1}(j)} x_i \xrightarrow{\oplus_{j=1}^{s} \phi_j} \bigoplus_{j=1}^{s} y_j}}$$

In (3.2.7) $\xi_f$ is the unique coherence isomorphism in $\mathsf{C}$ that permutes the terms of the sum. Existence and uniqueness of this coherence isomorphism follows from the symmetric monoidal Coherence Theorem [**ML98**, XI.1, Th. 1].

**Constraints:** The unit and monoidal constraints of $\varepsilon_{\mathsf{C}}$ are defined as the identities.

This finishes the definition of $\varepsilon_{\mathsf{C}}$. Verification that $\varepsilon_{\mathsf{C}}$ is strict symmetric monoidal follows from strictness of concatenation and uniqueness of the coherence isomorphisms $\xi_f$. The 2-naturality of $\varepsilon$ follows because strict symmetric monoidal functors preserve the monoidal sums $\oplus_i x_i$ and coherence isomorphisms $\xi_f$ in (3.2.5) and (3.2.7). ⬦

The following result combines [**JY23**, 6.2, 6.8, and 6.11]. Recall the notion of a 2-adjunction from Definition A.2.11.

**Theorem 3.2.8.** *There is a 2-adjunction*

$$\mathsf{Multicat} \underset{\mathsf{End}}{\overset{\mathsf{F}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{PermCat}^{\mathsf{st}}$$

*consisting of the following data.*

- $\mathsf{PermCat}^{\mathsf{st}}$ *is the 2-category in Definition A.2.3.*
- $\mathsf{Multicat}$ *is the 2-category in Theorem C.1.33.*
- *The left adjoint is* $\mathsf{F}$ *in Proposition 3.1.21.*
- *The right adjoint is* $\mathsf{End}$ *in Proposition C.3.6 restricted to* $\mathsf{PermCat}^{\mathsf{st}}$.
- *The unit*

$$\eta : 1_{\mathsf{Multicat}} \longrightarrow \mathsf{End}\, \mathsf{F}$$

  *has components in Definition 3.2.1.*
- *The counit*

$$\varepsilon : \mathsf{F}\, \mathsf{End} \longrightarrow 1_{\mathsf{PermCat}^{\mathsf{st}}}$$

  *has components in Definition 3.2.4.*

**Remark 3.2.9** (Naturality of the Counit). The counit $\varepsilon$ in Theorem 3.2.8 is only natural with respect to *strict* symmetric monoidal functors. This implies that the 2-adjunction $\mathsf{F} \dashv \mathsf{End}$ does *not* extend to a 2-adjunction, or even an adjunction of underlying categories, to any of the larger 2-categories in Definition A.2.3 in which the 1-cells are, in general, not strict symmetric monoidal. ⬦

## 3.3. Componentwise Right Adjoint of the Counit

In this section we discuss a componentwise right adjoint of the counit of the 2-adjunction $\mathsf{F} \dashv \mathsf{End}$ in Theorem 3.2.8. We also discuss a symmetric monoidal structure on this componentwise right adjoint; see Lemma 3.3.12.

**Definition 3.3.1.** For each permutative category $(\mathsf{C}, \oplus)$, we define an adjunction

$$\mathsf{F}\,\mathsf{End}(\mathsf{C}) \underset{\varrho_{\mathsf{C}}}{\overset{\varepsilon_{\mathsf{C}}}{\rightleftarrows}} \mathsf{C}$$

as follows, where $\varepsilon_{\mathsf{C}}$ is the strict symmetric monoidal functor in Definition 3.2.4.
**Right Adjoint:** The functor $\varrho_{\mathsf{C}}$ is defined by the object and morphism assignments

$$(3.3.2) \qquad \begin{cases} \varrho_{\mathsf{C}}(x) = (x) & \text{for} \quad x \in \mathsf{Ob}\,\mathsf{C} \quad \text{and} \\ \varrho_{\mathsf{C}}(\phi) = \left(1_{\overline{1}}, (\phi)\right) : (x) \longrightarrow (y) & \text{for} \quad \phi \in \mathsf{C}(x, y). \end{cases}$$

On the right-hand side of (3.3.2), $(x)$, $(y)$, and $(\phi)$ are tuples of length 1.
**Counit:** The following composite is the identity functor.

$$(3.3.3) \qquad\qquad \mathsf{C} \xrightarrow{\varrho_{\mathsf{C}}} \mathsf{F}\,\mathsf{End}(\mathsf{C}) \xrightarrow{\varepsilon_{\mathsf{C}}} \mathsf{C}$$

We define the counit for the adjunction $(\varepsilon_{\mathsf{C}}, \varrho_{\mathsf{C}})$ as the identity functor,

$$\epsilon = 1_{\mathsf{C}}.$$

**Unit:** The composite

$$(3.3.4) \qquad\qquad \mathsf{F}\,\mathsf{End}(\mathsf{C}) \xrightarrow{\varepsilon_{\mathsf{C}}} \mathsf{C} \xrightarrow{\varrho_{\mathsf{C}}} \mathsf{F}\,\mathsf{End}(\mathsf{C})$$

is given by the following assignments for each object $\langle x_i \rangle_{i=1}^{r}$ and morphism $\left(f, \langle \phi_j \rangle_{j=1}^{s}\right)$ in $\mathsf{F}\,\mathsf{End}(\mathsf{C})$ as in (3.2.6).

$$(3.3.5) \qquad \begin{cases} \langle x \rangle \longmapsto \left(\oplus_{i=1}^{r} x_i\right) \\ (f, \langle \phi \rangle) \longmapsto \left(1_{\overline{1}}, \left(\oplus_{j=1}^{s} \phi_j\right) \circ \xi_f\right) : \left(\oplus_{i=1}^{r} x_i\right) \longrightarrow \left(\oplus_{j=1}^{s} y_j\right) \end{cases}$$

We define the unit for the adjunction $(\varepsilon_{\mathsf{C}}, \varrho_{\mathsf{C}})$ as the natural transformation

$$v : 1_{\mathsf{F}\,\mathsf{End}(\mathsf{C})} \longrightarrow \varrho_{\mathsf{C}} \varepsilon_{\mathsf{C}}$$

with component morphism

$$(3.3.6) \qquad v_{\langle x \rangle} = \left(\iota_r, 1_{\oplus_{i=1}^{r} x_i}\right) : \langle x_i \rangle_{i=1}^{r} \longrightarrow \left(\oplus_{i=1}^{r} x_i\right) \quad \text{in} \quad \mathsf{F}\,\mathsf{End}(\mathsf{C})$$

for each length-$r$ tuple $\langle x_i \rangle_{i=1}^{r}$ of objects in $\mathsf{C}$. In (3.3.6),
- $\iota_r : \overline{r} \longrightarrow \overline{1}$ is the unique function, and
- the identity morphism

$$1_{\oplus_{i=1}^{r} x_i} \in \left(\mathsf{End}(\mathsf{C})\right)\left(\langle x_i \rangle_{i=1}^{r}; \oplus_{i=1}^{r} x_i\right) = \mathsf{C}\left(\oplus_{i=1}^{r} x_i, \oplus_{i=1}^{r} x_i\right)$$

is an $r$-ary multimorphism in $\mathsf{End}(\mathsf{C})$.
This finishes the definition. $\diamond$

The following result is [**JY23**, 6.13].
**Proposition 3.3.7.** *In the context of Definition 3.3.1, there is an adjunction of categories*

$$(3.3.8) \qquad\qquad \mathsf{F}\,\mathsf{End}(\mathsf{C}) \underset{\varrho_{\mathsf{C}}}{\overset{\varepsilon_{\mathsf{C}}}{\rightleftarrows}} \mathsf{C}$$

*with the following data.*
- *The left adjoint is $\varepsilon_{\mathsf{C}}$ in Definition 3.2.4.*
- *The right adjoint is $\varrho_{\mathsf{C}}$ in (3.3.2).*

- *The counit $\epsilon$ is the identity functor on $\mathsf{C}$.*
- *The unit $\upsilon$ has components in (3.3.6).*

**Symmetric Monoidal Structure.** The right adjoint $\varrho_\mathsf{C}$ in (3.3.8) is a symmetric monoidal functor with the following structure morphisms. The next definition uses the permutative category structure on $\mathsf{F}\,\mathrm{End}(\mathsf{C})$ in Definition 3.1.5, with $\mathsf{M} = \mathrm{End}(\mathsf{C})$ in Example C.3.1.

**Definition 3.3.9.** For each permutative category $(\mathsf{C}, \oplus, e, \xi)$, we define unit constraint $\varrho_\mathsf{C}^0$ and monoidal constraint $\varrho_\mathsf{C}^2$ for the functor in (3.3.8)

$$\varrho_\mathsf{C} : \mathsf{C} \longrightarrow \mathsf{F}\,\mathrm{End}(\mathsf{C})$$

as follows.

**Unit Constraint:** It is the morphism

$$(3.3.10) \qquad \varrho_\mathsf{C}^0 = (\iota_0\,,\, 1_e) : \langle\rangle \longrightarrow (e) = \varrho_\mathsf{C}(e) \quad \text{in} \quad \mathsf{F}\,\mathrm{End}(\mathsf{C})$$

defined as follows.
- $\iota_0 : \overline{0} = \varnothing \longrightarrow \overline{1}$ is the unique function.
- The identity morphism

$$1_e \in \big(\mathrm{End}(\mathsf{C})\big)\big(\langle\rangle\,;\, e\big) = \mathsf{C}(e, e)$$

is a nullary multimorphism in $\mathrm{End}(\mathsf{C})$.

**Monoidal Constraint:** For each pair of objects $x, y \in \mathsf{C}$, the monoidal constraint has a component morphism

$$(3.3.11) \qquad (\varrho_\mathsf{C}^2)_{x,y} = \big(\iota_2\,,\, 1_{x \oplus y}\big) : (x) \oplus (y) = (x, y) \longrightarrow (x \oplus y) \quad \text{in} \quad \mathsf{F}\,\mathrm{End}(\mathsf{C})$$

defined as follows.
- $\iota_2 : \overline{2} \longrightarrow \overline{1}$ is the unique function.
- The identity morphism

$$1_{x \oplus y} \in \big(\mathrm{End}(\mathsf{C})\big)\big(x, y\,;\, x \oplus y\big) = \mathsf{C}\big(x \oplus y, x \oplus y\big)$$

is a binary multimorphism in $\mathrm{End}(\mathsf{C})$.

This finishes the definition of $\varrho_\mathsf{C}^0$ and $\varrho_\mathsf{C}^2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad \diamond$

The terminal property of $\overline{1} = \{1\}$ and the permutative category axioms of $\mathsf{C}$ imply the following.

**Lemma 3.3.12.** *In the context of Definition 3.3.9, the triple*

$$\big(\varrho_\mathsf{C}, \varrho_\mathsf{C}^2, \varrho_\mathsf{C}^0\big) : \mathsf{C} \longrightarrow \mathsf{F}\,\mathrm{End}(\mathsf{C})$$

*is a symmetric monoidal functor.*

**Remark 3.3.13** (Not Strictly Unital)**.** The symmetric monoidal functor $\big(\varrho_\mathsf{C}, \varrho_\mathsf{C}^2, \varrho_\mathsf{C}^0\big)$ in Lemma 3.3.12 is *neither* strictly unital nor strong because the unit constraint, which is the morphism

$$\varrho_\mathsf{C}^0 = (\iota_0\,,\, 1_e) : \langle\rangle \longrightarrow (e)$$

in (3.3.10), is not an isomorphism in $\mathsf{F}\,\mathrm{End}(\mathsf{C})$. This stands in stark contrast with its left adjoint, $\varepsilon_\mathsf{C}$ in Definition 3.2.4, which is a *strict* symmetric monoidal functor. $\quad \diamond$

### 3.4. Free Permutative Category as a Non-Symmetric Cat-Multifunctor

In this section we extend the 2-functor in Proposition 3.1.21

$$\mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}$$

to a *non-symmetric* Cat-multifunctor

$$\mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$

in the sense of Definition C.1.19 with $(\mathsf{V}, \otimes) = (\mathsf{Cat}, \times)$. This is one of the main results in [**JY22a**].

- The domain Multicat is the Cat-multicategory in Explanation 1.1.20. It is induced by a symmetric monoidal Cat-category structure (Theorem 1.1.19), with the tensor product $\otimes$ in Definition 1.1.12.
- The codomain PermCat$^{\mathsf{su}}$ is the Cat-multicategory in Theorem 1.4.29. Its multimorphism categories have multilinear functors as objects and multilinear transformations as morphisms (Definition 1.4.15).

Here is an outline of this section.

- The non-symmetric Cat-multifunctor $\mathsf{F}$ involves strong $n$-linear functors $\mathsf{F}^n$ in Definition 3.4.14. The definition of each $\mathsf{F}^n$, in turn, requires some auxiliary constructions involving tuples in Definitions 3.4.1 and 3.4.7.
- The multimorphism functors of $\mathsf{F}$ are in Definition 3.4.26.

**Tensor Product of Tuples.** Suppose $\langle \mathsf{M}_i \rangle_{i=1}^n$ and $\mathsf{N}$ are small multicategories for some $n \geq 0$. Recall from Definition 3.1.5 that objects in the free permutative category $\mathsf{FN}$ are finite tuples of objects in $\mathsf{N}$.

**Definition 3.4.1** (Tensor Product of Tuples of Objects). Suppose given objects

$$(3.4.2) \qquad \langle x^i \rangle = \langle x^i_j \rangle_{j=1}^{r_i} \in \mathsf{FM}_i \quad \text{for} \quad i \in \{1, \ldots, n\}$$

with each $x^i_j$ an object in $\mathsf{M}_i$.

- For each $n$-tuple of indices $\langle j_i \rangle_{i=1}^n$ with each $j_i \in \{1, \ldots, r_i\}$, we define the object

$$(3.4.3) \qquad x^{1 \cdots n}_{j_1, \ldots, j_n} = \langle x^i_{j_i} \rangle_{i=1}^n \quad \text{in} \quad \otimes_{i=1}^n \mathsf{M}_i$$

using the canonical bijection (1.1.15) for objects in the tensor product.

- We define an object

$$(3.4.4) \qquad \langle x^{1 \cdots n} \rangle = \otimes_{i=1}^n \langle x^i \rangle \quad \text{in} \quad \mathsf{F}\big( \otimes_{i=1}^n \mathsf{M}_i \big)$$

using the tensor product of tuples in Definition 1.1.1.

In other words, with

$$(3.4.5) \qquad r_{1 \cdots n} = \prod_{i=1}^n r_i,$$

the object $\langle x^{1 \cdots n} \rangle$ in (3.4.4) is the $r_{1 \cdots n}$-tuple

$$(3.4.6) \qquad \langle x^{1 \cdots n} \rangle = \left\langle \cdots \left\langle x^{1 \cdots n}_{j_1, \ldots, j_n} \right\rangle_{j_1=1}^{r_1} \cdots \right\rangle_{j_n=1}^{r_n}$$

with each entry as in (3.4.3). ◇

The following definition extends the construction $\langle x^{1 \cdots n} \rangle$ to morphisms.

**Definition 3.4.7** (Tensor Product of Morphisms). Suppose given morphisms

$$(3.4.8) \qquad \left(f^i, \langle\phi^i\rangle\right) : \langle x^i\rangle = \langle x^i_j\rangle^{r_i}_{j=1} \longrightarrow \langle y^i\rangle = \langle y^i_k\rangle^{s_i}_{k=1} \quad \text{in} \quad \mathsf{FM}_i$$

for $i \in \{1,\ldots,n\}$ with

- each $f^i : \overline{r_i} \longrightarrow \overline{s_i}$ an index map,
- each $\langle\phi^i\rangle = \langle\phi^i_k\rangle^{s_i}_{k=1}$ an $s_i$-tuple, and
- each

$$\phi^i_k \in \mathsf{M}_i\left(\langle x^i_j\rangle_{j\in(f^i)^{-1}(k)} ; y^i_k\right)$$

an $|(f^i)^{-1}(k)|$-ary multimorphism in $\mathsf{M}_i$.

We define the following objects and (multi)morphisms.

- First we define an index map $f^{1\cdots n}$ as the composite function

$$(3.4.9) \qquad \overbrace{\overline{r_{1\cdots n}} \xrightarrow{\ \cong\ } \prod^n_{i=1}\overline{r_i} \xrightarrow{\ \prod^n_{i=1}f^i\ } \prod^n_{i=1}\overline{s_i} \xrightarrow{\ \cong\ } \overline{s_{1\cdots n}}}^{f^{1\cdots n}}.$$

  In (3.4.9),
  - the integers

$$r_{1\cdots n} = \prod^n_{i=1}r_i \quad \text{and} \quad s_{1\cdots n} = \prod^n_{i=1}s_i$$

  are as in (3.4.5), and
  - the two unlabeled isomorphisms are given by the reverse lexicographic ordering of the products.
- For each $n$-tuple of indices $\langle k_i\rangle^n_{i=1}$ with each $k_i \in \{1,\ldots,s_i\}$, we define the object in $\mathsf{F}\left(\bigotimes^n_{i=1}\mathsf{M}_i\right)$

$$(3.4.10) \qquad \begin{aligned} \langle x^{1\cdots n}\rangle_{f;k_1,\ldots,k_n} &= \bigotimes^n_{i=1}\langle x^i_j\rangle_{j\in(f^i)^{-1}(k_i)} \\ &= \left\langle \cdots \left\langle x^{1\cdots n}_{j_1,\ldots,j_n}\right\rangle_{j_1\in(f^1)^{-1}(k_1)} \cdots \right\rangle_{j_n\in(f^n)^{-1}(k_n)} \end{aligned}$$

  as a sub-tuple of $\langle x^{1\cdots n}\rangle$ in (3.4.6). Then we define the multimorphism

$$(3.4.11) \qquad \phi^{1\cdots n}_{k_1,\ldots,k_n} = \bigotimes^n_{i=1}\phi^i_{k_i} : \langle x^{1\cdots n}\rangle_{f;k_1,\ldots,k_n} \longrightarrow y^{1\cdots n}_{k_1,\ldots,k_n} \quad \text{in} \quad \bigotimes^n_{i=1}\mathsf{M}_i.$$

- We define the $s_{1\cdots n}$-tuple of multimorphisms

$$(3.4.12) \qquad \begin{aligned} \langle\phi^{1\cdots n}\rangle &= \bigotimes^n_{i=1}\langle\phi^i\rangle \\ &= \left\langle \cdots \left\langle \phi^{1\cdots n}_{k_1,\ldots,k_n}\right\rangle^{s_1}_{k_1=1} \cdots \right\rangle^{s_n}_{k_n=1} \end{aligned}$$

  with each entry as in (3.4.11).
- Finally, we define the morphism

$$(3.4.13) \qquad \left(f^{1\cdots n}, \langle\phi^{1\cdots n}\rangle\right) : \langle x^{1\cdots n}\rangle \longrightarrow \langle y^{1\cdots n}\rangle \quad \text{in} \quad \mathsf{F}\left(\bigotimes^n_{i=1}\mathsf{M}_i\right)$$

  with
  - the objects $\langle x^{1\cdots n}\rangle$ and $\langle y^{1\cdots n}\rangle$ as in (3.4.4),
  - the index map $f^{1\cdots n}$ in (3.4.9), and
  - $\langle\phi^{1\cdots n}\rangle$ the $s_{1\cdots n}$-tuple in (3.4.12).

This finishes the definition of $\left(f^{1\cdots n}, \langle\phi^{1\cdots n}\rangle\right)$. ⋄

**The Strong Multilinear Functor** $\mathsf{F}^n$. Recall

- from Definition 1.4.2 the notion of an *n-linear functor* and
- from Example 3.1.14 the free permutative category $\mathsf{F}(\mathsf{I})$ of the initial operad $\mathsf{I}$.

**Definition 3.4.14** (Multilinear Functor $\mathsf{F}^n$)**.** Suppose $\langle \mathsf{M}_i \rangle_{i=1}^n$ are small multicategories. We define the data of an *n*-linear functor

$$\left( \mathsf{F}^n, \left\langle (\mathsf{F}^n)_p^2 \right\rangle_{p=1}^n \right) : \prod_{i=1}^n \mathsf{FM}_i \longrightarrow \mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right)$$

as follows. For $n = 0$, we define the 0-linear functor

$$\mathsf{F}^0 : \mathbf{1} \longrightarrow \mathsf{F}(\mathsf{I})$$

by the choice of the length-one tuple $(*) \in \mathsf{F}(\mathsf{I})$.

Suppose $n > 0$ for the rest of this definition. Suppose given morphisms

$$\left( f^i, \langle \phi^i \rangle \right) : \langle x^i \rangle = \langle x^i_j \rangle_{j=1}^{r_i} \longrightarrow \langle y^i \rangle = \langle y^i_k \rangle_{k=1}^{s_i} \quad \text{in} \quad \mathsf{FM}_i$$

for $i \in \{1, \dots, n\}$ as in (3.4.8).

**Object Assignment:** We define the object

$$(3.4.15) \qquad \mathsf{F}^n \left\langle \langle x^i \rangle \right\rangle_{i=1}^n = \langle x^{1 \cdots n} \rangle \quad \text{in} \quad \mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right)$$

using (3.4.4).

**Morphism Assignment:** We define the morphism

$$(3.4.16) \qquad \mathsf{F}^n \left( \left( f^i, \langle \phi^i \rangle \right) \right)_{i=1}^n = \left( f^{1 \cdots n}, \langle \phi^{1 \cdots n} \rangle \right) \quad \text{in} \quad \mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right)$$

using (3.4.13).

**Linearity Constraints:** For $p \in \{1, \dots, n\}$, suppose given an object

$$\langle \hat{x}^p \rangle = \left\langle \hat{x}^p_j \right\rangle_{j=1}^{\hat{r}_p} \quad \text{in} \quad \mathsf{FM}_p$$

with each $\hat{x}^p_j$ an object in $\mathsf{M}_p$. We first define the object

$$\langle \tilde{x}^p \rangle = \langle x^p \rangle \oplus \langle \hat{x}^p \rangle \quad \text{in} \quad \mathsf{FM}_p$$

with length $r_p + \hat{r}_p$. Then we define the objects

$$(3.4.17) \qquad \langle \hat{x}^{1 \cdots n} \rangle \quad \text{and} \quad \langle \tilde{x}^{1 \cdots n} \rangle \quad \text{in} \quad \mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right)$$

as in (3.4.4), using $\langle \hat{x}^p \rangle$ and $\langle \tilde{x}^p \rangle$, respectively, in place of $\langle x^p \rangle$.

The $p$th linearity constraint, $(\mathsf{F}^n)_p^2$, is defined by component isomorphisms in $\mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right)$

$$(3.4.18) \qquad (\mathsf{F}^n)_p^2 = \left( \rho_{r_p, \hat{r}_p}, \langle 1 \rangle \right) : \langle x^{1 \cdots n} \rangle \oplus \langle \hat{x}^{1 \cdots n} \rangle \xrightarrow{\ \cong\ } \langle \tilde{x}^{1 \cdots n} \rangle.$$

In (3.4.18) the two components of $(\mathsf{F}^n)_p^2$ are as follows.

- The first component

$$(3.4.19) \qquad \rho_{r_p, \hat{r}_p} \in \Sigma_{r_1 \cdots (r_p + \hat{r}_p) \cdots r_n}$$

is the unique permutation of entries determined by the domain and codomain of $(\mathsf{F}^n)_p^2$.

- Each entry in $\langle 1 \rangle$ is a colored unit of an entry in either $\langle x^{1 \cdots n} \rangle$ or $\langle \hat{x}^{1 \cdots n} \rangle$.

This finishes the definition of $\left( \mathsf{F}^n, \left\langle (\mathsf{F}^n)_p^2 \right\rangle_{p=1}^n \right)$. ◇

**Remark 3.4.20.**

(1) For $n = 1$, the 1-linear functor $\mathsf{F}^1$ is the identity symmetric monoidal functor on $\mathsf{FM}_1$.

(2) For $n \geq 2$, the permutation $\rho_{r_p, \hat{r}_p}$ in (3.4.19) is the identity if $p = n$, but it is not the identity in general.

(3) In [**JY22a**, Section 7], the multilinear functors $(\mathsf{F}^n, (\mathsf{F}^n)_p^2)$ are denoted $(\mathsf{S}, \mathsf{S}_p^2)$. ◇

The following result combines [**JY22a**, 7.12, 7.14, and 7.16].

**Proposition 3.4.21.** *Suppose* $\langle \mathsf{M}_i \rangle_{i=1}^n$ *are small multicategories. Then the following statements hold.*

(1) *The data in Definition* 3.4.14,

$$\left( \mathsf{F}^n, \left\langle (\mathsf{F}^n)_p^2 \right\rangle_{p=1}^n \right) : \prod_{i=1}^n \mathsf{FM}_i \longrightarrow \mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right),$$

*form a strong n-linear functor.*

(2) *Each* $\mathsf{F}^n$ *is 2-natural with respect to multifunctors and multinatural transformations.*

**Explanation 3.4.22** (2-Naturality of $\mathsf{F}^n$)**.** In Proposition 3.4.21 (2), the 2-naturality of $\mathsf{F}^n$ with respect to multifunctors between small multicategories

$$H_i : \mathsf{M}_i \longrightarrow \mathsf{N}_i \quad \text{for} \quad i \in \{1, \dots, n\}$$

means that the following two composite $n$-linear functors are equal.

(3.4.23)

$$\begin{array}{ccc}
\prod_{i=1}^n \mathsf{FM}_i & \xrightarrow{\prod_{i=1}^n \mathsf{F}H_i} & \prod_{i=1}^n \mathsf{FN}_i \\
\mathsf{F}^n \downarrow & & \downarrow \mathsf{F}^n \\
\mathsf{F}\left( \otimes_{i=1}^n \mathsf{M}_i \right) & \xrightarrow{\mathsf{F}(\otimes_{i=1}^n H_i)} & \mathsf{F}\left( \otimes_{i=1}^n \mathsf{N}_i \right)
\end{array}$$

The 2-naturality of $\mathsf{F}^n$ with respect to multinatural transformations

$$\theta_i : H_i \longrightarrow K_i \quad \text{for} \quad i \in \{1, \dots, n\}$$

means the following equality of $n$-linear transformations.

(3.4.24) $$1_{(\mathsf{F}^n)} * \left( \prod_{i=1}^n \mathsf{F}\theta_i \right) = \mathsf{F}\left( \otimes_{i=1}^n \theta_i \right) * 1_{(\mathsf{F}^n)}$$

This equality is obtained from the diagram (3.4.23) by replacing each $H_i$ with $\theta_i$. ◇

**The Non-Symmetric** $\mathsf{Cat}$**-Multifunctor** $\mathsf{F}$**.** Next we extend the 2-functor $\mathsf{F}$ in Proposition 3.1.21 to multimorphism categories.

**Convention 3.4.25.** To avoid confusion in Definition 3.4.26 below, for small multicategories $\mathsf{M}$ and $\mathsf{N}$, we denote by

$$\overline{\mathsf{F}} : \mathsf{Multicat}(\mathsf{M}, \mathsf{N}) \longrightarrow \mathsf{PermCat}^{\mathsf{st}}(\mathsf{FM}, \mathsf{FN})$$

the assignment of $\mathsf{F}$ on multifunctors and multinatural transformations as in Definitions 3.1.16 and 3.1.19, respectively. ◇

In (3.4.28) below, we use the multilinear functor $\mathsf{F}^n$ (Proposition 3.4.21).

**Definition 3.4.26.** Suppose $\langle \mathsf{M} \rangle = \langle \mathsf{M}_i \rangle_{i=1}^{n}$ and $\mathsf{N}$ are small multicategories. We define a functor between multimorphism categories

(3.4.27) $\qquad \mathsf{F} : \mathsf{Multicat}\big(\langle \mathsf{M} \rangle ; \mathsf{N}\big) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{FM} \rangle ; \mathsf{FN}\big)$

as follows. Suppose given multifunctors $H$ and $K$ and a multinatural transformation $\theta$ as in the diagram below.

$$\langle \mathsf{M} \rangle \overset{H}{\underset{K}{\Downarrow \theta}} \mathsf{N}$$

Then $\mathsf{F}$ sends these data to the following composite $n$-linear functors and whiskering.

(3.4.28) $\qquad \langle \mathsf{FM} \rangle \xrightarrow{\;\mathsf{F}^n\;} \mathsf{F}\big( \otimes_{i=1}^{n} \mathsf{M}_i \big) \overset{\overline{\mathsf{F}}H}{\underset{\overline{\mathsf{F}}K}{\Downarrow \overline{\mathsf{F}}\theta}} \mathsf{FN}$

This finishes the definition of the multimorphism functor $\mathsf{F}$. $\qquad \diamond$

**Explanation 3.4.29.** In the case $n = 1$, the 1-linear functor $\mathsf{F}^1$ is the identity symmetric monoidal functor (Remark 3.4.20 (1)). The multimorphism functor $\mathsf{F}$ in (3.4.27) reduces to the hom functor of the 2-functor $\mathsf{F}$ in Proposition 3.1.21. Therefore, there is no ambiguity in reusing the notation $\mathsf{F}$ in Definition 3.4.26. $\qquad \diamond$

**Explanation 3.4.30.** In the case $n = 0$, recall from Explanation 1.1.20 that a nullary operation $H \in \mathsf{Multicat}\big(\langle \rangle ; \mathsf{N}\big)$ consists of a choice of object $H* \in \mathsf{N}$, where $*$ is the unique object of $\mathsf{I}$. The 0-linear functor $\mathsf{F}^0 : \mathbf{1} \longrightarrow \mathsf{F}(\mathsf{I})$ chooses the length-one tuple $(*) \in \mathsf{F}(\mathsf{I})$, and hence $\mathsf{F}H = (\overline{\mathsf{F}}H) \circ \mathsf{F}^0$ is determined by the length-one tuple $(H*) \in \mathsf{FN}$. $\qquad \diamond$

Recall from Definition C.1.19 that a *non-symmetric* Cat-*multifunctor* between Cat-multicategories preserve colored units and composition, but it is *not* required to preserve the symmetric group action as in (C.1.20). The following result is [**JY22a**, 8.1].

**Theorem 3.4.31.** *There is a non-symmetric* Cat-*multifunctor*

(3.4.32) $\qquad \mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$

*defined by the following data.*
- Multicat *is the* Cat-*multicategory in Explanation 1.1.20.*
- $\mathsf{PermCat}^{\mathsf{su}}$ *is the* Cat-*multicategory in Theorem 1.4.29.*
- *The object assignment of* $\mathsf{F}$ *is the free permutative category in Proposition 3.1.13.*
- *The multimorphism functors of* $\mathsf{F}$ *are in (3.4.27).*

**Explanation 3.4.33.** Consider Theorem 3.4.31.

(1) As explained in [**JY22a**, 8.2], the non-symmetry of $\mathsf{F}$ in (3.4.28) is due to the incompatibility of $\mathsf{F}^n$ with permutations.

(2) As in Definition 3.1.16, $\overline{\mathsf{F}}$ sends each multifunctor to a *strict* symmetric monoidal functor. Moreover, by Proposition 3.4.21 (1), each $\mathsf{F}^n$ is a *strong* multilinear functor. Therefore, the composite in (3.4.28)

$$\mathsf{F}H = \overline{\mathsf{F}}H \circ \mathsf{F}^n$$

is a *strong $n$-linear functor* by definition (1.4.27). $\qquad \diamond$

Recall from Explanation 1.4.40 the Cat-multifunctor

$$\mathsf{End} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}.$$

Also recall from Explanation C.2.2 an explicit description of a (non-symmetric) Cat-multinatural transformation. The following result is [**JY22a**, 9.2], where F is the non-symmetric Cat-multifunctor in Theorem 3.4.31.

**Lemma 3.4.34.** *The unit in Theorem 3.2.8*

$$\mathsf{Multicat} \underset{\mathsf{End\ F}}{\overset{1_{\mathsf{Multicat}}}{\rightleftarrows}} \Downarrow\eta \quad \mathsf{Multicat}$$

*is a non-symmetric* Cat-*multinatural transformation.*

**Remark 3.4.35** (Non-Existence of Counit Analog)**.** The counit in Theorem 3.2.8 does *not* yield a non-symmetric Cat-multinatural transformation

$$\varepsilon : \mathsf{F\ End} \longrightarrow 1_{\mathsf{PermCat}^{\mathsf{su}}}.$$

As we mentioned in Remark 3.2.9, that counit $\varepsilon$ is only natural with respect to *strict* symmetric monoidal functors but not strictly unital symmetric monoidal functors in general. Thus, the analog of Lemma 3.4.34 does not hold for the counit.          ◇

## 3.5. Homotopy Equivalences between Multicategories and Permutative Categories

In this section we review equivalences of homotopy theories between the categories Multicat, PermCat$^{\mathsf{st}}$, and PermCat$^{\mathsf{su}}$.

- Theorem 3.5.3, which relates Multicat and PermCat$^{\mathsf{st}}$, is the main result of [**JY23**]. The morphisms in the category PermCat$^{\mathsf{st}}$ are *strict* symmetric monoidal functors. The proof of this theorem relies on the componentwise right adjoint of the counit $\varepsilon$ in Proposition 3.3.7. In Chapter 4 we use a pointed version of this componentwise right adjoint to extend the equivalence of homotopy theories to *pointed* multicategories and left $\mathcal{M}\underline{1}$-modules.
- Theorem 3.5.5 relates Q-algebras in Multicat and Q-algebras in PermCat$^{\mathsf{su}}$ for a small non-symmetric Cat-multicategory Q. It is the main result of [**JY22a**]. In the larger category PermCat$^{\mathsf{su}}$, the morphisms are *strictly unital* symmetric monoidal functors. In Chapter 5 we extend this equivalence of homotopy theories between categories of non-symmetric algebras, as well as the Cat-multifunctoriality of F (non-symmetric in the case of F) and End, to pointed multicategories and left $\mathcal{M}\underline{1}$-modules.
- Theorem 3.5.7, which relates Multicat and PermCat$^{\mathsf{su}}$, is the special case of Theorem 3.5.5 when Q is the initial operad.
- A consequence of Theorems 3.5.3 and 3.5.7 is that the inclusion functor from PermCat$^{\mathsf{st}}$ to PermCat$^{\mathsf{su}}$ is an equivalence of homotopy theories. See Corollary 3.5.9.

**Stable Equivalences.** Recall from Section 2.5 that Segal $K$-theory

$$\mathsf{K}^{\mathsf{Se}} : \left( \mathsf{PermCat}^{\mathsf{su}} , \mathcal{S} \right) \overset{\sim}{\longrightarrow} \left( \mathsf{Sp}_{\geq 0} , \mathcal{S} \right)$$

is an equivalence of homotopy theories with the following relative category structures. Recall that a subcategory is *wide* if it contains all the objects of the larger category.

- In the codomain, the wide subcategory

$$\mathcal{S} \subset \mathsf{Sp}_{\geq 0}$$

  consists of stable equivalences of connective symmetric spectra.
- In the domain, the wide subcategory of stable equivalences

$$\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$$

  is created by $\mathsf{K}^{\mathsf{Se}}$. In other words, a *stable equivalence* in $\mathsf{PermCat}^{\mathsf{su}}$ is a strictly unital symmetric monoidal functor $P$ between small permutative categories such that $\mathsf{K}^{\mathsf{Se}} P \in \mathsf{Sp}_{\geq 0}$ is a stable equivalence of connective symmetric spectra.

To relate the homotopy theories of Multicat, $\mathsf{PermCat}^{\mathsf{st}}$, and $\mathsf{PermCat}^{\mathsf{su}}$, first we specify classes of stable equivalences in Multicat and $\mathsf{PermCat}^{\mathsf{st}}$.

**Definition 3.5.1** (Stable Equivalences). We define the wide subcategories

$$\mathcal{S}^I = I^{-1}(\mathcal{S}) \subset \mathsf{PermCat}^{\mathsf{st}} \quad \text{and}$$
$$\mathcal{S}^{\mathsf{F}} = \mathsf{F}^{-1}(\mathcal{S}^I) \subset \mathsf{Multicat}$$

as the subcategories created by the indicated functors below.

$$(3.5.2) \qquad \left(\mathsf{Multicat}, \mathcal{S}^{\mathsf{F}}\right) \xrightarrow{\ \mathsf{F}\ } \left(\mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I\right) \xhookrightarrow{\ I\ } \left(\mathsf{PermCat}^{\mathsf{su}}, \mathcal{S}\right)$$

- F is the underlying functor of the 2-functor in Proposition 3.1.21.
- $I$ is the underlying functor of the inclusion 2-functor in (A.2.6).

We call morphisms in $\mathcal{S}^I$ and $\mathcal{S}^{\mathsf{F}}$ *stable equivalences* and F-*stable equivalences*, respectively. ◇

**Equivalences of Homotopy Theories.** Recall from Definition 2.1.10 the notion of an *adjoint equivalence of homotopy theories*. The following result is [**JY23**, 7.3]. Its proof makes crucial use of Proposition 3.3.7 on the componentwise right adjoint of the counit.

**Theorem 3.5.3.** *The adjunction in Theorem 3.2.8*

$$\left(\mathsf{Multicat}, \mathcal{S}^{\mathsf{F}}\right) \underset{\mathsf{End}}{\overset{\mathsf{F}}{\rightleftarrows}} {\scriptstyle \perp} \left(\mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I\right)$$

*is an adjoint equivalence of homotopy theories.*

In particular, each of the unit and the counit of $\mathsf{F} \dashv \mathsf{End}$ in Theorem 3.5.3 is a relative natural transformation with respect to $\mathcal{S}^{\mathsf{F}}$ and $\mathcal{S}^I$, respectively.

The equivalences of homotopy theories—but not the adjunction—in Theorem 3.5.3 can be extended to the larger category $\mathsf{PermCat}^{\mathsf{su}}$, together with algebraic structures in the following sense.

**Definition 3.5.4** (Symmetric and Non-Symmetric Algebras). Suppose P is a small Cat-multicategory, and N is a Cat-multicategory.

- A P-*algebra* in N is a Cat-multifunctor

$$\mathsf{P} \longrightarrow \mathsf{N}.$$

- A *morphism* of P-algebras in N is a Cat-multinatural transformation (Explanation C.2.2).
- The category of P-algebras and their morphisms in N is denoted $N^P$.

There is also a non-symmetric analog, for which we use the same notation and similar terminology. Suppose Q is a small non-symmetric Cat-multicategory, and N as above is a Cat-multicategory.

- A *non-symmetric Q-algebra* in N is a non-symmetric Cat-multifunctor

$$Q \longrightarrow N.$$

- A *morphism* of non-symmetric Q-algebras in N is a non-symmetric Cat-multinatural transformation.
- The category of non-symmetric Q-algebras and their morphisms in N is denoted $N^Q$.

If the underlying 1-category of N is a relative category $(N, \mathcal{W})$, then the categories of P-algebras, respectively non-symmetric Q-algebras, have an induced relative structure.

- We define the wide subcategories

$$\mathcal{W}^P \subset N^P \quad \text{and} \quad \mathcal{W}^Q \subset N^Q$$

to be those that contain all the morphisms with each component in $\mathcal{W}$.
- We consider the pairs $(N^P, \mathcal{W}^P)$ and $(N^Q, \mathcal{W}^Q)$ as relative categories.

This finishes the definition.                                                                    ◇

Recall the Cat-multifunctors (non-symmetric for F)

$$F : \text{Multicat} \xrightarrow{\quad\longrightarrow\quad} \text{PermCat}^{su} : \text{End}$$

in Theorem 3.4.31 and Explanation 1.4.40. Theorem 3.5.5 below extends the equivalences of homotopy theories in Theorem 3.5.3 to non-symmetric algebras. It is the main result in [**JY22a**, 1.1].

**Theorem 3.5.5.** *Suppose* Q *is a small non-symmetric* Cat-*multicategory. Then the functors*

$$F^Q : \left(\text{Multicat}^Q, (\mathcal{S}^F)^Q\right) \xrightarrow{\ \sim\ } \left((\text{PermCat}^{su})^Q, \mathcal{S}^Q\right) : \text{End}^Q,$$

*induced by post-composition and whiskering with, respectively,* F *and* End, *are inverse equivalences of homotopy theories in the sense of Definition 2.1.8.*

Thus, by Proposition 2.1.9, $F^Q$ and $\text{End}^Q$ are equivalences of homotopy theories between categories of non-symmetric Q-algebras.

**Remark 3.5.6** (Subtleties)**.** There are two main subtleties of Theorem 3.5.5.

(1) Unlike Theorem 3.5.3, the two functors in Theorem 3.5.5 do *not* form an adjunction in general, as discussed in Remark 3.2.9. Nevertheless, the unit

$$\eta : 1_{\text{Multicat}} \longrightarrow \text{End} \, F$$

in Theorem 3.2.8 is
- componentwise an F-stable equivalence by Theorem 3.5.3 and
- a non-symmetric Cat-multinatural transformation by Lemma 3.4.34.

(2) For a permutative category $C$, the functor in (3.3.8)

$$\varrho_C : C \longrightarrow F \, End(C)$$

is *not* strictly unital, as discussed in Remark 3.3.13. Therefore, we cannot use $\varrho$ directly to compare the identity functor on $(PermCat^{su})^Q$ and the composite $F^Q \, End^Q$. To use Proposition 2.1.9, the proof in [**JY22a**] compares these two functors via a zigzag of relative natural transformations.

As we discuss in Chapter 4, the pointed variant of $\varrho_C$ *is* strictly unital. In this sense, the pointed variant of Theorem 3.5.5 is more natural than the unpointed version here. ◇

Taking $Q$ as the initial operad in Example C.1.35 (i), Theorem 3.5.5 yields the following special case.

**Theorem 3.5.7.** *The relative functors*

$$F : \big(Multicat, \mathcal{S}^F\big) \xrightarrow{\quad \sim \quad} \big(PermCat^{su}, \mathcal{S}\big) : End$$

*are equivalences of homotopy theories.*

Corollary 3.5.9 below follows from

- the commutative diagram

(3.5.8)
$$
\begin{array}{ccc}
& \xrightarrow{\quad F \quad} & \\
Multicat \xrightarrow{\;\; F \;\;} PermCat^{st} & \xhookrightarrow{\;\; I \;\;} & PermCat^{su},
\end{array}
$$

- Theorems 3.5.3 and 3.5.7, and
- the 2-out-of-3 property of Rezk weak equivalences (Theorem 2.1.5).

**Corollary 3.5.9.** *The inclusion relative functor in (3.5.2)*

$$\big(PermCat^{st}, \mathcal{S}^I\big) \xrightarrow[\sim]{\;\; I \;\;} \big(PermCat^{su}, \mathcal{S}\big)$$

*is an equivalence of homotopy theories.*

Combining (2.5.16), Theorems 3.5.3 and 3.5.7, and Corollary 3.5.9, we see that each arrow in the diagram

(3.5.10)
$$
\big(Multicat, \mathcal{S}^F\big) \underset{End}{\overset{F}{\rightleftarrows}} \big(PermCat^{st}, \mathcal{S}^I\big) \xhookrightarrow{\;\; I \;\;} \big(PermCat^{su}, \mathcal{S}\big) \xrightarrow{\;\; K^{Se} \;\;} \big(Sp_{\geq 0}, \mathcal{S}\big)
$$

*is an equivalence of homotopy theories.*

**Part 2**

# Homotopy Theory of Pointed Multicategories, $\mathcal{M}\underline{1}$-Modules, and Permutative Categories

# Pointed Multicategories and $\mathcal{M}\underline{1}$-Modules Model All Connective Spectra

The title of this chapter refers to equivalences of homotopy theories

$$\mathsf{K}^{\mathsf{Se}} \circ I \circ \mathsf{F}_{\bullet} : \left(\mathsf{Multicat}_* , \mathcal{S}_{\bullet}\right) \xrightarrow{\;\sim\;} \left(\mathsf{Sp}_{\geq 0} , \mathcal{S}\right) \quad \text{and}$$

$$\mathsf{K}^{\mathsf{Se}} \circ I \circ \mathsf{F}_{\mathcal{M}\underline{1}} : \left(\mathsf{Mod}^{\mathcal{M}\underline{1}} , \mathcal{S}^{\mathcal{M}\underline{1}}\right) \xrightarrow{\;\sim\;} \left(\mathsf{Sp}_{\geq 0} , \mathcal{S}\right)$$

that are established below. The main results and their context are summarized in the following diagram of 2-adjunctions and functors.



(4.0.1)

Each of the arrows in (4.0.1), *except* $\mathsf{U}_{\bullet}$, is a relative functor.

- $\mathsf{K}^{\mathsf{Se}}$ is Segal *K*-theory in (2.5.1) and (2.5.3).
- *I* is the inclusion functor in (A.2.6), which is the identity on objects.
- The 2-adjunctions

$$(-)_+ \dashv \mathsf{U}_{\bullet}, \quad \mathsf{F} \dashv \mathsf{End}, \quad \text{and} \quad (\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}$$

  are in Proposition C.4.16, Theorem 3.2.8, and Proposition 1.3.19, respectively.
- We establish the 2-adjunctions

$$\mathsf{F}_{\bullet} \dashv \mathsf{End}_{\bullet} \quad \text{and} \quad \mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$$

  in Theorems 4.3.11 and 4.4.1, respectively, below.
- The diagram involving the right adjoints

$$\mathsf{End} = \mathsf{U}_{\bullet} \circ \mathsf{End}_{\bullet} = \mathsf{U}_{\bullet} \circ \mathsf{U}_{\mathcal{M}\underline{1}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}$$

  is the restriction of the commutative diagram (1.4.39) to $\mathsf{PermCat}^{\mathsf{st}}$ and underlying 2-functors.
- The diagram involving the left adjoints commutes up to 2-natural isomorphisms:

$$\mathsf{F} \cong \mathsf{F}_{\bullet} \circ (-)_+ \cong \mathsf{F}_{\mathcal{M}\underline{1}} \circ (\mathcal{M}\underline{1} \wedge -) \circ (-)_+.$$

Each arrow in (4.0.1), *except* $\mathsf{U}_\bullet$, is an equivalence of homotopy theories in the sense of Definition 2.1.7 (4).

- The inclusion functor $I$ and Segal $K$-theory $\mathsf{K}^{\mathsf{Se}}$ are equivalences of homotopy theories by Corollary 3.5.9 and (2.5.16), respectively.
- $\mathsf{F} \dashv \mathsf{End}$ is an adjoint equivalence of homotopy theories by Theorem 3.5.3.
- We show that $(-)_+$ is an equivalence of homotopy theories in Corollary 4.7.4 below.
- We establish the adjoint equivalences of homotopy theories

$$\mathsf{F}_\bullet \dashv \mathsf{End}_\bullet, \quad (\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}, \quad \text{and} \quad \mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$$

  in Theorems 4.7.3, 4.8.1, and 4.8.3, respectively, below.
- We do not know whether $\mathsf{U}_\bullet$ is a relative functor with respect to the subcategories $\mathcal{S}_\bullet$ and $\mathcal{S}^\mathsf{F}$, and thus cannot conclude that it is an equivalence of homotopy theories. See Question D.2 for further discussion of this point.

The definition of $\mathsf{F}_\bullet$ and some of its properties follow formally from the adjunction $\mathsf{F} \dashv \mathsf{End}$. In particular, Proposition 4.2.5 shows that $\mathsf{F}_\bullet$ can be constructed as a 2-categorical pushout that collapses basepoint operations. However, the adjoint equivalences of homotopy theories are slightly more subtle because our arguments depend on particular details of $\mathsf{F}_\bullet$ described in Section 4.5. Explanations 4.6.3 and 4.6.11 contain further comments on this point.

**Connection with Other Chapters.** Chapter 5 extends $\mathsf{F}_\bullet$ and $\mathsf{F}_{\mathcal{M}\underline{1}}$ to Cat-enriched multifunctors and extends the equivalences of homotopy theories in this chapter to categories of algebras. This is further extended in Chapter 12 to enriched diagrams and Mackey functors. See Remark 4.6.12 for more detailed technical comments about these extensions.

**Background.** The material in Chapter 3 describes the equivalences of homotopy theories given by the unpointed free construction $\mathsf{F}$ and its right 2-adjoint $\mathsf{End}$. Pointed multicategories are reviewed in Section 1.2. The symmetric monoidal Cat-category of $\mathcal{M}\underline{1}$-modules, and its relation to pointed multicategories, is described in Section 1.3. Equivalences of homotopy theories are described in Section 2.1.

**Chapter Summary.** In Section 4.1 we define the *pointed free permutative category* construction, $\mathsf{F}_\bullet$, via certain equivalence relations on $\mathsf{F}$. Section 4.2 further develops the relationships between $\mathsf{F}$ and $\mathsf{F}_\bullet$. Section 4.3 shows that $\mathsf{F}_\bullet$ has a right 2-adjoint $\mathsf{End}_\bullet$. Section 4.4 explains how $\mathsf{F}_\bullet$ and related constructions restrict to the sub 2-category of $\mathcal{M}\underline{1}$-modules. In Section 4.5 we compute several examples $\mathsf{F}_\bullet\mathsf{M}$ for certain pointed multicategories M. Section 4.6 gives a componentwise right adjoint, providing a pointed analog of Section 3.3. Sections 4.7 and 4.8 then establish the equivalences of homotopy theories described above. Here is a summary table.

| | |
|---|---|
| Underlying category and permutative structure on $\mathsf{F}_\bullet\mathsf{M}$ | 4.1.4 and 4.1.11 |
| $\mathsf{F}_\bullet$ on pointed multifunctors and multinatural transformations | 4.1.12 and 4.1.15 |
| $\mathsf{p}_\mathsf{M} : \mathsf{FM} \longrightarrow \mathsf{F}_\bullet\mathsf{M}$ | 4.2.1 |
| $\mathsf{F}_\bullet$ as a 2-pushout | 4.2.5 |
| unit $\eta^\bullet : 1 \longrightarrow \mathsf{End}_\bullet\mathsf{F}_\bullet$ and counit $\varepsilon^\bullet : \mathsf{F}_\bullet\mathsf{End}_\bullet \longrightarrow 1$ | 4.3.1 and 4.3.6 |
| isomorphism $\mathsf{FM} \cong \mathsf{F}_\bullet(\mathsf{M}_+)$ | 4.3.16 |
| 2-adjunction $\mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$ | 4.4.1 |
| symmetric monoidal functor $\varrho_\mathsf{C}^\bullet : \mathsf{C} \longrightarrow \mathsf{F}_\bullet\mathsf{End}_\bullet\mathsf{C}$ | 4.6.1 |
| stable equivalences $\mathcal{S}^{\mathcal{M}\underline{1}}$ and $\mathcal{S}_\bullet$ | 4.7.1 |
| equivalences of homotopy theories | 4.7.3, 4.7.4, 4.8.1, 4.8.3 |

We remind the reader of Convention A.1.2 about universes.

## 4.1. Pointed Free Permutative Categories

This section defines the pointed free construction $\mathsf{F}_\bullet$ in details that will be useful below. Section 4.2 shows that $\mathsf{F}_\bullet$ is a certain pushout constructed from $\mathsf{F}$.

**Underlying Category $\mathsf{F}_\bullet\mathsf{M}$.** The objects and morphisms of $\mathsf{F}_\bullet\mathsf{M}$ are determined by the following equivalence relations on objects and morphisms of $\mathsf{F}$.

**Definition 4.1.1** (Removing Basepoints). Suppose $(\mathsf{M}, i^\mathsf{M})$ is a pointed multicategory with basepoint $* = i^\mathsf{M}(*)$ and $n$-ary basepoint operations $\iota^n \in \mathsf{M}(\langle * \rangle ; *)$.

(1) For each tuple of objects $\langle x_i \rangle_{i=1}^r \in \mathsf{FM}$, with $x_i \in \mathsf{M}$, let $\langle x \rangle^\wedge$ be the sub-tuple consisting of non-basepoint objects, $x_i \neq * = i^\mathsf{M}(*)$.

(2) For each morphism

$$\langle x_i \rangle_{i=1}^r \xrightarrow{(f, \langle \phi \rangle)} \langle y_j \rangle_{j=1}^s, \quad \text{in} \quad \mathsf{FM}$$

define

$$\langle x \rangle' \xrightarrow{(f', \langle \phi \rangle')} \langle y \rangle' \quad \text{in} \quad \mathsf{FM}$$

as follows. An index $j \in s$ is called *removable* if $\phi_j$ is a basepoint operation in $\mathsf{M}$. That is, if

- $y_j = *$,
- $x_i = *$ for each $i \in f^{-1}(j)$, and
- $\phi_j = \iota^{|f^{-1}(j)|} \in \mathsf{M}(\langle * \rangle ; *)$.

An index $j$ is *irremovable* if it is not removable. Let $\overline{s}' \subset \overline{s}$ be the subset consisting of irremovable $j$. Let $\overline{r}' = f^{-1}(\overline{s}')$ and then define

$$(4.1.2) \qquad \langle y \rangle' = \langle y_j \rangle_{j \in \overline{s}'}, \quad \langle x \rangle' = \langle x_i \rangle_{i \in \overline{r}'}, \quad \langle \phi \rangle' = \langle \phi_j \rangle_{j \in \overline{s}'}, \quad \text{and} \quad f' = f|_{\overline{r}'}. \qquad \diamond$$

**Definition 4.1.3** (Up-To-Basepoint Equivalence). Suppose $(\mathsf{M}, i^\mathsf{M})$ is a pointed multicategory. Define the following equivalence relations, called *up-to-basepoint equivalence* on objects and morphisms of $\mathsf{FM}$.

**Objects:** Up-to-basepoint equivalence is denoted $\overset{\text{ob}}{\sim}$ on $\mathsf{Ob}(\mathsf{FM})$ and defined by

$$\langle x \rangle \overset{\text{ob}}{\sim} \langle y \rangle \quad \text{if and only if} \quad \langle x \rangle^\wedge = \langle y \rangle^\wedge.$$

Thus, $\langle x \rangle \overset{\mathrm{ob}}{\sim} \langle y \rangle$ if the tuples $\langle x \rangle$ and $\langle y \rangle$ agree up to insertion or deletion of basepoints. The equivalence class of $\langle x \rangle$ is denoted $[\langle x \rangle]$.

**Morphisms:** An $m$-tuple of morphisms $\big((f_1, \langle \phi_1 \rangle), \ldots, (f_m, \langle \phi_m \rangle)\big)$ is $\overset{\mathrm{ob}}{\sim}$-*composable* if

$$\mathrm{cod}(f_i, \langle \phi_i \rangle) \overset{\mathrm{ob}}{\sim} \mathrm{dom}(f_{i+1}, \langle \phi_{i+1} \rangle) \quad \text{for each} \quad 1 \leq i \leq m-1.$$

Let $\widetilde{\mathrm{Mor}}(\mathsf{FM}) \subset \coprod_{m \geq 1} \mathrm{Mor}(\mathsf{FM})^{\times m}$ denote the collection of $\overset{\mathrm{ob}}{\sim}$-composable tuples of morphisms.

For $\overset{\mathrm{ob}}{\sim}$-composable tuples of morphisms

$$\underline{f} = \big((f_1, \langle \phi_1 \rangle), \ldots, (f_m, \langle \phi_m \rangle)\big) \quad \text{and} \quad \underline{g} = \big((g_1, \langle \psi_1 \rangle), \ldots, (g_n, \langle \psi_n \rangle)\big),$$

up-to-basepoint equivalence is denoted $\simeq$ and is generated by $\overset{1}{\sim}$ and $\overset{2}{\sim}$ as follows.

(1) For relation one, $\underline{f} \overset{1}{\sim} \underline{g}$ if $m = n+1$ and there is some $1 \leq i \leq m-1$ such that
   - for $j \notin \{i, i+1\}$:
   
   $$(f_j, \langle \phi_j \rangle) = (g_j, \langle \psi_j \rangle) \quad \text{if} \quad 1 \leq j < i$$
   $$(f_j, \langle \phi_j \rangle) = (g_{j-1}, \langle \psi_{j-1} \rangle) \quad \text{if} \quad i+1 < j \leq m$$
   
   - $\mathrm{cod}(f_i, \langle \phi_i \rangle) = \mathrm{dom}(f_{i+1}, \langle \phi_{i+1} \rangle)$ in $\mathsf{FM}$, and
   
   $$(g_i, \langle \psi_i \rangle) = (f_{i+1}, \langle \phi_{i+1} \rangle) \circ (f_i, \langle \phi_i \rangle),$$
   
   the composite in $\mathsf{FM}$ (3.1.7).

(2) For relation two, $\underline{f} \overset{2}{\sim} \underline{g}$ if $m = n$ and there is some $1 \leq i \leq m$ such that
   - $(f_j, \langle \phi_j \rangle) = (g_j, \langle \psi_j \rangle)$ for $j \neq i$, and
   - $(f_i, \langle \phi_i \rangle)' = (g_i, \langle \psi_i \rangle)'$ as in (4.1.2) above.

Thus, two $\overset{\mathrm{ob}}{\sim}$-composable tuples of morphisms are equivalent if they differ by either
   - composition in $\mathsf{FM}$ of adjacent entries, or
   - insertion or deletion of basepoint operations.

The equivalence class of $\underline{f}$ is denoted $[\underline{f}]$.     ◇

**Definition 4.1.4** (Pointed Free Permutative Category)**.** Suppose $(\mathsf{M}, i^{\mathsf{M}})$ is a pointed multicategory. Define the data of the *pointed free permutative category* $\mathsf{F}_\bullet\mathsf{M}$ as follows.

**Objects:** The objects are given by $\overset{\mathrm{ob}}{\sim}$-equivalence classes:

$$\mathrm{Ob}(\mathsf{F}_\bullet\mathsf{M}) = \mathrm{Ob}(\mathsf{FM})/\overset{\mathrm{ob}}{\sim}.$$

**Morphisms:** The morphisms are given by $\simeq$-equivalence classes of $\overset{\mathrm{ob}}{\sim}$-composable tuples:

$$\mathrm{Mor}(\mathsf{F}_\bullet\mathsf{M}) = \widetilde{\mathrm{Mor}}(\mathsf{FM})/\simeq.$$

For $\underline{f} = \big((f_1, \langle \phi_1 \rangle), \ldots, (f_m, \langle \phi_m \rangle)\big)$ in $\widetilde{\mathrm{Mor}}(\mathsf{FM})$, define

$$\mathrm{dom}([\underline{f}]) = [\mathrm{dom}(f_1, \langle \phi_1 \rangle)] \quad \text{and} \quad \mathrm{cod}([\underline{f}]) = [\mathrm{cod}(f_m, \langle \phi_m \rangle)].$$

Note that these are well defined since both relations $\overset{1}{\sim}$ and $\overset{2}{\sim}$ preserve $\overset{\mathrm{ob}}{\sim}$-equivalence classes of (co)domain.

**Identities:** For an $\overset{\text{ob}}{\sim}$-equivalence class $[\langle x \rangle]$ in $\mathsf{F}_{\bullet}\mathsf{M}$, define

$$1_{[\langle x \rangle]} = [1_{\langle x \rangle}].$$

This is well defined by relation $\overset{2}{\sim}$ for $1_{\langle x \rangle} = (1, \langle 1_{x_i} \rangle_i)$.

**Composition:** Composition of equivalence classes

$$[\langle x \rangle] \xrightarrow{\;[f]\;} [\langle y \rangle] \xrightarrow{\;[g]\;} [\langle z \rangle] \in \mathsf{F}_{\bullet}\mathsf{M}$$

is given by concatenation of representative tuples,

$$[\underline{g}] \circ [\underline{f}] = [\underline{f}, \underline{g}] = \big[ \big( (f_1, \langle \phi_1 \rangle), \dots, (f_m, \langle \phi_m \rangle), (g_1, \langle \psi_1 \rangle), \dots, (g_n, \langle \psi_n \rangle) \big) \big],$$

where

$$\underline{f} = \big( (f_1, \langle \phi_1 \rangle), \dots, (f_m, \langle \phi_m \rangle) \big) \quad \text{and} \quad \underline{g} = \big( (g_1, \langle \psi_1 \rangle), \dots, (g_n, \langle \psi_n \rangle) \big).$$

This concatenation is $\overset{\text{ob}}{\sim}$-composable because

$$\mathsf{cod}([\underline{f}]) = [\mathsf{cod}(f_m, \langle \phi_m \rangle)] \quad \text{and} \quad \mathsf{dom}([\underline{g}]) = [\mathsf{dom}(g_1, \langle \phi_1 \rangle)].$$

The composite $[\underline{g}] \circ [\underline{f}]$ is well defined by the definitions of $\overset{1}{\sim}$ and $\overset{2}{\sim}$.

This finishes the definition of $\mathsf{F}_{\bullet}\mathsf{M}$.                                          ◇

**Remark 4.1.5.** The description of $\mathsf{F}_{\bullet}$ in Definitions 4.1.3 and 4.1.4 is an extension of the explicit description for coequalizers in Cat from [**Yau20b**, Section 1.4]. Proposition 4.2.5 shows that $\mathsf{F}_{\bullet}\mathsf{M}$ can equivalently be constructed as a pushout in $\mathsf{PermCat}^{\mathsf{su}}$.                                          ◇

**Lemma 4.1.6.** *In the context of Definition 4.1.4, $\mathsf{F}_{\bullet}\mathsf{M}$ is a category. If $\mathsf{M}$ is a small multicategory, then $\mathsf{F}_{\bullet}\mathsf{M}$ is a small category.*

*Proof.* Associativity of composition holds because concatenation of sequences is associative. The composition with identities is unital by relation $\overset{1}{\sim}$. If $\mathsf{M}$ is small, then $\mathsf{FM}$, and hence also $\mathsf{F}_{\bullet}\mathsf{M}$, is small because its objects are finite tuples of objects of $\mathsf{M}$ (Definition 3.1.5).                                          □

**Permutative Structure for $\mathsf{F}_{\bullet}\mathsf{M}$.** Recall from Definition 3.1.5 that the unpointed free construction $\mathsf{FM}$ has monoidal sum $\oplus$ given by concatenation and monoidal unit the empty tuple, $\langle \rangle$. There is an induced permutative structure on $\mathsf{F}_{\bullet}\mathsf{M}$, described as follows.

**Definition 4.1.7.** In the context of Definition 4.1.4, suppose given morphisms

$$[\underline{f}] : [\langle x \rangle] \longrightarrow [\langle y \rangle] \quad \text{and} \quad [\underline{f'}] : [\langle x' \rangle] \longrightarrow [\langle y' \rangle]$$

in $\mathsf{F}_{\bullet}\mathsf{M}$, with

$$\underline{f} = \big( (f_1, \langle \phi_1 \rangle), \dots, (f_m, \langle \phi_m \rangle) \big) \quad \text{and}$$
$$\underline{f'} = \big( (f'_1, \langle \phi'_1 \rangle), \dots, (f'_n, \langle \phi'_n \rangle) \big).$$

Define

(4.1.8)     $\underline{f} \oplus 1_{\langle x' \rangle} = \big( (f_1, \langle \phi_1 \rangle) \oplus 1_{\langle x' \rangle}, \dots, (f_m, \langle \phi_m \rangle) \oplus 1_{\langle x' \rangle} \big) \quad \text{and}$

(4.1.9)     $1_{\langle y \rangle} \oplus \underline{f'} = \big( 1_{\langle y \rangle} \oplus (f'_1, \langle \phi'_1 \rangle), \dots, 1_{\langle y \rangle} \oplus (f'_n, \langle \phi'_n \rangle) \big),$

where the sums at right are those of $\mathsf{FM}$.                                          ◇

**Explanation 4.1.10.** Observe that the $\simeq$-equivalence classes of (4.1.8) and (4.1.9) are well defined in the following senses.

- If $\langle x' \rangle \overset{\mathrm{ob}}{\sim} \langle w' \rangle$, then $\underline{f} \oplus 1_{\langle x' \rangle} \overset{2}{\sim} \underline{f} \oplus 1_{\langle w' \rangle}$.
- If $\underline{f} \overset{1}{\sim} \underline{g}$, then $\underline{f} \oplus 1_{\langle x' \rangle} \overset{1}{\sim} \underline{g} \oplus 1_{\langle x' \rangle}$ by functoriality of $\oplus$.
- If $\underline{f} \overset{2}{\sim} \underline{g}$, then $\underline{f} \oplus 1_{\langle x' \rangle} \overset{2}{\sim} \underline{g} \oplus 1_{\langle x' \rangle}$.

The corresponding statements for (4.1.9) hold likewise.                                          $\diamond$

**Definition 4.1.11** (Permutative Structure on $\mathsf{F}_{\bullet}\mathsf{M}$). Suppose $(\mathsf{M}, i^{\mathsf{M}})$ is a pointed multicategory. Define the data of a permutative category

$$\big( \mathsf{F}_{\bullet}\mathsf{M}, \oplus, [\langle \rangle], \xi \big)$$

as follows.

**Monoidal Sum:** The monoidal sum functor

$$\mathsf{F}_{\bullet}\mathsf{M} \times \mathsf{F}_{\bullet}\mathsf{M} \overset{\oplus}{\longrightarrow} \mathsf{F}_{\bullet}\mathsf{M}$$

is defined by that of $\mathsf{FM}$ on representative objects and morphisms, as follows.

- For objects $[\langle x \rangle]$ and $[\langle x' \rangle]$ in $\mathsf{F}_{\bullet}\mathsf{M}$, let

$$[\langle x \rangle] \oplus [\langle x' \rangle] = [\langle x \rangle \oplus \langle x' \rangle],$$

where the monoidal sum at right is that of $\mathsf{FM}$. This is well defined because concatenation of representatives in $\mathsf{FM}$ preserves $\overset{\mathrm{ob}}{\sim}$-equivalence classes.

- For morphisms $[\underline{f}] : [\langle x \rangle] \longrightarrow [\langle y \rangle]$ and $[\underline{f'}] : [\langle x' \rangle] \longrightarrow [\langle y' \rangle]$ in $\mathsf{F}_{\bullet}\mathsf{M}$, use Definition 4.1.7 and let

$$[\underline{f}] \oplus [\underline{f'}] = [1_{\langle y \rangle} \oplus \underline{f'}] \circ [\underline{f} \oplus 1_{\langle x' \rangle}]$$
$$= [\underline{f} \oplus 1_{\langle y' \rangle}] \circ [1_{\langle x \rangle} \oplus \underline{f'}].$$

The $\simeq$-equivalence classes at right are well defined by Explanation 4.1.10. The two indicated composites are equal as morphisms in $\mathsf{F}_{\bullet}\mathsf{M}$ by $\overset{1}{\sim}$-equivalence and functoriality of $\oplus$ in $\mathsf{FM}$.

**Monoidal Unit:** The monoidal unit is $[\langle \rangle]$, the equivalence class of the empty tuple of objects.

**Symmetry:** The symmetry isomorphism for $\mathsf{F}_{\bullet}\mathsf{M}$ is given by the equivalence class of the symmetry for $\mathsf{FM}$, as below for objects $[\langle x \rangle]$ and $[\langle x' \rangle]$.

$$
\begin{array}{ccc}
[\langle x \rangle] \oplus [\langle x' \rangle] & \xrightarrow{\ \xi_{[\langle x \rangle],[\langle x' \rangle]}\ } & [\langle x' \rangle] \oplus [\langle x \rangle] \\
\| & & \| \\
[\langle x \rangle \oplus \langle x' \rangle] & \xrightarrow{\ [\xi_{\langle x \rangle, \langle x' \rangle}]\ } & [\langle x' \rangle \oplus \langle x \rangle]
\end{array}
$$

This is well defined because $\xi_{\langle w \rangle, \langle w' \rangle} \overset{2}{\sim} \xi_{\langle x \rangle, \langle x' \rangle}$ for $\langle x \rangle \overset{\mathrm{ob}}{\sim} \langle w \rangle$ and $\langle x' \rangle \overset{\mathrm{ob}}{\sim} \langle w' \rangle$.

The monoidal sum is strictly associative and unital because concatenation of tuples is so. The symmetry and hexagon axioms (A.1.15) follow from those of $\mathsf{FM}$.     $\diamond$

**The 2-Functor F.** Recall the descriptions of F on multifunctors and multi-natural transformations from Definitions 3.1.16 and 3.1.19. These induce pointed variants for F$_\bullet$, which we now describe.

**Definition 4.1.12** (F$_\bullet$ on 1-Cells). Suppose given a pointed multifunctor between pointed multicategories

$$H : (\mathsf{M}, i^{\mathsf{M}}) \longrightarrow (\mathsf{N}, i^{\mathsf{N}}).$$

Define a strict symmetric monoidal functor

$$\mathsf{F}_\bullet H : \mathsf{F}_\bullet \mathsf{M} \longrightarrow \mathsf{F}_\bullet \mathsf{N}$$

induced by F$H$ as follows.

**Object Assignment:** For an object $[\langle x_i \rangle_{i=1}^r]$ in F$_\bullet$M, define the object

(4.1.13) $$(\mathsf{F}_\bullet H)[\langle x_i \rangle_{i=1}^r] = [\langle Hx_i \rangle_{i=1}^r] \quad \text{in} \quad \mathsf{F}_\bullet \mathsf{N}.$$

The assumption that $H$ is a pointed multifunctor ensures that (4.1.13) is well defined with respect to $\overset{\mathrm{ob}}{\sim}$-equivalence.

**Morphism Assignment:** For a morphism

$$[\underline{f}] : [\langle x \rangle] \longrightarrow [\langle y \rangle] \quad \text{in} \quad \mathsf{F}_\bullet \mathsf{M}$$

with

$$\underline{f} = \big( (f_1, \langle \phi_1 \rangle), \ldots, (f_m, \langle \phi_m \rangle) \big),$$

define

(4.1.14) $$(\mathsf{F}_\bullet H)[\underline{f}] = \big[ (\mathsf{F}H)(f_1, \langle \phi_1 \rangle), \ldots, (\mathsf{F}H)(f_m, \langle \phi_m \rangle) \big],$$
$$= \big[ (f_1, \langle H\phi_{1,j_1} \rangle_{j_1=1}^{s_1}), \ldots, (f_m, \langle H\phi_{m,j_m} \rangle_{j_m=1}^{s_m}) \big],$$

where each $\langle \phi_i \rangle$ has length $s_i$. Multifunctoriality of $H$ ensures that (4.1.14) is well defined with respect to $\overset{1}{\sim}$-equivalence. The assumption that $H$ is pointed, and therefore preserves basepoint operations, ensures that (4.1.14) is well defined with respect to $\overset{2}{\sim}$-equivalence.

**Constraints:** The unit and monoidal constraints for F$_\bullet H$ are identities.

Functoriality and the strict symmetric monoidal conditions for F$_\bullet H$ follow from those of F$H$ on representatives. This finishes the definition of F$_\bullet H$. ⋄

**Definition 4.1.15** (F$_\bullet$ on 2-Cells). Suppose given a pointed multinatural transformation $\theta$ between pointed multifunctors between pointed multicategories

$$(\mathsf{M}, i^{\mathsf{M}}) \underset{K}{\overset{H}{\rightrightarrows}} {\Downarrow \theta} (\mathsf{N}, i^{\mathsf{N}}).$$

Define a monoidal natural transformation

$$\mathsf{F}_\bullet \mathsf{M} \underset{\mathsf{F}_\bullet K}{\overset{\mathsf{F}_\bullet H}{\rightrightarrows}} {\Downarrow \mathsf{F}_\bullet \theta} \mathsf{F}_\bullet \mathsf{N}$$

induced by F$\theta$ with component morphism in F$_\bullet$N

(4.1.16) $$(\mathsf{F}_\bullet \theta)_{[\langle x \rangle]} = [(\mathsf{F}\theta)_{\langle x \rangle}] = \big[ (1_{\overline{r}}, \langle \theta_{x_i} \rangle_{i=1}^r) \big] : [\langle Hx_i \rangle_{i=1}^r] \longrightarrow [\langle Kx_i \rangle_{i=1}^r]$$

for each object $[\langle x_i \rangle_{i=1}^r]$ in F$_\bullet$M. The assumption that $\theta$ is a pointed multifunctor, and hence its basepoint component is an identity operation, ensures that (4.1.16)

is well defined with respect to $\overset{\text{ob}}{\sim}$-equivalence. The monoidal naturality conditions for $\mathsf{F}_\bullet\theta$ follow from those of $\mathsf{F}\theta$ on representatives. This finishes the definition of $\mathsf{F}_\bullet\theta$.                                                                           ◇

The 2-functoriality of $\mathsf{F}$ (Proposition 3.1.21) implies that the assignments above determine a 2-functor.

**Theorem 4.1.17.** *The assignments on objects, 1-cells, and 2-cells given in Definitions 4.1.11, 4.1.12, and 4.1.15, respectively, determine a 2-functor*

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{st}}.$$

### 4.2. Relating Unpointed and Pointed Free Permutative Categories

In this section we define a 2-natural transformation

$$\mathsf{Multicat}_* \overset{\mathsf{F}\circ\mathsf{U}_\bullet}{\underset{\mathsf{F}_\bullet}{\Longrightarrow\Downarrow\mathsf{p}}} \mathsf{PermCat}^{\mathsf{st}}.$$

We usually suppress $\mathsf{U}_\bullet$ and abbreviate $\mathsf{F}\circ\mathsf{U}_\bullet$ to $\mathsf{F}$. Using $\mathsf{p}$, we show in Proposition 4.2.5 that $\mathsf{F}_\bullet\mathsf{M}$ is the pushout in $\mathsf{PermCat}^{\mathsf{st}}$ of the span

$$\mathbf{1} \longleftarrow \mathsf{FT} \overset{\mathsf{F}i^{\mathsf{M}}}{\longrightarrow} \mathsf{FM},$$

where $\mathbf{1}$ is the terminal permutative category, $\mathsf{T}$ is the terminal multicategory (Example C.1.17), and $(\mathsf{M}, i^{\mathsf{M}})$ is a small pointed multicategory.

**Definition 4.2.1.** For a small pointed multicategory $(\mathsf{M}, i^{\mathsf{M}})$, define a strict symmetric monoidal functor

$$\mathsf{p}_{\mathsf{M}} : \mathsf{FM} \longrightarrow \mathsf{F}_\bullet\mathsf{M}$$

on objects $\langle x \rangle$ and morphisms $(f, \langle \phi \rangle)$ by the assignments

(4.2.2)
$$\begin{cases} \langle x \rangle \longmapsto [\langle x \rangle] & \text{and} \\ (f, \langle \phi \rangle) \longmapsto [(f, \langle \phi \rangle)], \end{cases}$$

where a morphism of $\mathsf{FM}$ is regarded as an $\overset{\text{ob}}{\sim}$-composable tuple of length one. These assignments are functorial by definition of $\overset{1}{\sim}$. Recall that the monoidal sum in $\mathsf{F}_\bullet\mathsf{M}$ is given by concatenation, and the monoidal sum in $\mathsf{FM}$ is functorial. These imply that $\mathsf{p}_{\mathsf{M}}$ is a strict symmetric monoidal functor.                              ◇

**Proposition 4.2.3.** *The strict symmetric monoidal functors* $\mathsf{p}_{\mathsf{M}}$ *of Definition 4.2.1 are components of a 2-natural transformation*

(4.2.4)
$$\mathsf{p} : \mathsf{F} \longrightarrow \mathsf{F}_\bullet.$$

*Proof.* Suppose given a pointed multinatural transformation $\theta$ between pointed multifunctors between small pointed multicategories

$$(\mathsf{M}, i^{\mathsf{M}}) \overset{H}{\underset{K}{\Longrightarrow\Downarrow\theta}} (\mathsf{N}, i^{\mathsf{N}}).$$

In the following diagram we have

$$\mathsf{p}_{\mathsf{N}} \circ (\mathsf{F}H) = (\mathsf{F}_\bullet H) \circ \mathsf{p}_{\mathsf{M}} \quad \text{and} \quad \mathsf{p}_{\mathsf{N}} \circ (\mathsf{F}K) = (\mathsf{F}_\bullet K) \circ \mathsf{p}_{\mathsf{M}}$$

by definition of $\mathsf{p}$ and Definition 4.1.12 for $\mathsf{F}_\bullet$ on 1-cells.

$$
\begin{array}{ccc}
\mathsf{FM} & \overset{\mathsf{F}H}{\underset{\mathsf{F}K}{\Longrightarrow}\mathsf{F}\theta} & \mathsf{FN} \\
\mathsf{p_M} \downarrow & & \downarrow \mathsf{p_N} \\
\mathsf{F_\bullet M} & \overset{\mathsf{F_\bullet}H}{\underset{\mathsf{F_\bullet}K}{\Longrightarrow}\mathsf{F_\bullet}\theta} & \mathsf{F_\bullet N}
\end{array}
$$

By Definition 4.1.15, we also have the equality of whiskerings

$$(\mathsf{F_\bullet}\theta) * \mathsf{p_M} = \mathsf{p_N} * (\mathsf{F}\theta).$$

This completes the proof that $\mathsf{p}$ is 2-natural.                $\square$

Recall from Example 3.1.15 that the free permutative category $\mathsf{FT}$ is the natural number category $\mathbf{N}$ whose objects are natural numbers and morphisms are given by morphisms of finite sets.

**Proposition 4.2.5.** *For each small pointed multicategory* $(\mathsf{M}, i^\mathsf{M})$, *the diagram*

(4.2.6)
$$
\begin{array}{ccc}
\mathsf{FT} & \overset{\mathsf{F}i^\mathsf{M}}{\longrightarrow} & \mathsf{FM} \\
\downarrow & & \downarrow \mathsf{p_M} \\
\mathbf{1} & \longrightarrow & \mathsf{F_\bullet M}
\end{array}
$$

*is a 2-pushout in* $\mathsf{PermCat^{st}}$.

*Proof.* Suppose given a permutative category $\mathsf{C}$ and a strict symmetric monoidal functor $Q$ such that the outer diagram below commutes.

(4.2.7)
$$
\begin{array}{ccc}
\mathsf{FT} & \overset{\mathsf{F}i^\mathsf{M}}{\longrightarrow} & \mathsf{FM} \\
\downarrow & & \downarrow \mathsf{p_M} \quad\searrow^{Q} \\
\mathbf{1} & \longrightarrow & \mathsf{F_\bullet M} \overset{\overline{Q}}{\dashrightarrow} \mathsf{C}
\end{array}
$$

Commutativity of the outer diagram implies that $Q$ sends the objects and morphisms of $\mathsf{FT}$ to, respectively, the monoidal unit $\mathsf{e} \in \mathsf{C}$ and identities on that unit.

If $\langle x \rangle \overset{\mathrm{ob}}{\sim} \langle y \rangle$ in $\mathsf{FM}$ then, because $Q$ is strict symmetric monoidal, we have

$$Q\langle x \rangle = \oplus_i Q x_i = \oplus_j Q y_j = Q\langle y \rangle,$$

where the middle equality holds because $\langle x \rangle \overset{\mathrm{ob}}{\sim} \langle y \rangle$ and $\mathsf{e} \in \mathsf{C}$ is a strict unit. Therefore, $Q$ uniquely determines a well-defined assignment on objects $\overline{Q}$ that commutes with $\mathsf{p_M}$.

Now consider

$$[\underline{f}] = \big[(f_1, \langle \phi_1 \rangle), \ldots, (f_m, \langle \phi_m \rangle)\big] \quad \text{in} \quad \mathrm{Mor}(\mathsf{F_\bullet M}) = \widetilde{\mathrm{Mor}}(\mathsf{FM})/\simeq.$$

For $\overline{Q}$ to be functorial and commute with $\mathsf{p_M}$, we must have

$$\overline{Q}[\underline{f}] = Q(f_m, \langle \phi_m \rangle) \circ \cdots \circ Q(f_1, \langle \phi_1 \rangle).$$

This is well defined with respect to $\overset{1}{\sim}$-equivalence classes by functoriality of $Q$. The definition of $\overline{Q}$ on $\overset{2}{\sim}$-equivalence classes is well defined because the morphisms in FM that are induced by basepoint operations will be sent by $Q$ to identity morphisms $1_e$ in C.

Thus, there is a unique strict symmetric monoidal functor $\overline{Q}$ in (4.2.7) that commutes with $p_M$. This proves that $F_{\bullet}M$ satisfies the desired 1-dimensional universal property.

For the 2-dimensional aspect of the universal property, suppose given a monoidal natural transformation $\kappa$ as in the diagram below, such that the whiskering $\kappa * (Fi^M)$ is equal to the identity 2-cell of the constant functor given by the left-bottom composite.

(4.2.8)



Hence, the component of $\kappa$ at the basepoint object is an identity morphism. Because $\kappa$ is monoidal natural, this implies that

$$\kappa_{\langle x \rangle} = \kappa_{\langle y \rangle}$$

whenever $\langle x \rangle \overset{ob}{\sim} \langle y \rangle$. Therefore, $\overline{\kappa}$ is uniquely determined and well defined with components

$$\overline{\kappa}_{[\langle x \rangle]} = \kappa_{\langle x \rangle}.$$

This completes the proof.                                                    □

**Definition 4.2.9.** In the context of Proposition 4.2.5, say that a strict symmetric monoidal functor

$$Q : FM \longrightarrow C$$

is *basepoint-collapsing* if the outer diagram (4.2.7) commutes. Say that a monoidal natural transformation

$$\kappa : Q \longrightarrow R$$

is *basepoint-collapsing* if $Q$ and $R$ are basepoint-collapsing and

$$\kappa * Fi^M = 1_{1_e}$$

as in (4.2.8). Thus, $Q$ is basepoint-collapsing if and only if $Q\iota_n = 1_e$ for each basepoint operation $\iota_n$. Similarly, $\kappa$ is basepoint-collapsing if and only if $\kappa_* = 1_e$. With these terms, Proposition 4.2.5 asserts that basepoint-collapsing 1- and 2-cells of PermCat$^{st}$ extend uniquely along $p_M$.                                    ◇

## 4.3. Pointed Free Permutative Category as a Left 2-Adjoint

In this section we show that the adjunction $F \dashv End$ from Theorem 3.2.8 induces an adjunction $F_{\bullet} \dashv End_{\bullet}$. As above, we usually suppress the forgetful $U_{\bullet}$.

**Definition 4.3.1** (Unit). Suppose M is a small pointed multicategory. We define the composite multifunctor

(4.3.2)
$$M \xrightarrow{\ \eta_M\ } \mathsf{End}\,FM = \mathsf{End}_\bullet\,FM \xrightarrow{\ \mathsf{End}_\bullet(p_M)\ } \mathsf{End}_\bullet\,F_\bullet M$$

$$\eta_M^\bullet$$

as follows.

- $\eta_M$ is the unit of the 2-adjunction $F \dashv \mathsf{End}$ in Definition 3.2.1.
- $\mathsf{End}_\bullet$ is the Cat-multifunctor in Explanation 1.4.32.
- $p_M : FM \longrightarrow F_\bullet M$ is the strict symmetric monoidal functor in Definition 4.2.1.

This finishes the definition of $\eta_M^\bullet$.                                        ◇

**Explanation 4.3.3.** The multifunctor $\eta_M^\bullet$ in (4.3.2) is given explicitly by the assignments

$$y \longmapsto [(y)], \quad \text{for} \quad y \in M, \quad \text{and}$$
$$\phi \longmapsto [(\iota_r, (\phi))] \quad \text{for} \quad \phi \in M(\langle x_i \rangle_{i=1}^r ; y),$$

where $\iota_r : \overline{r} \longrightarrow \overline{1}$ is the unique index map. Note, in particular, that $\eta_M^\bullet$ sends each basepoint operation of M to the equivalence class of the identity $1_{[\langle\rangle]}$ by relation $\overset{2}{\sim}$.                                        ◇

**Explanation 4.3.4.** In (4.3.2), even though M and $\mathsf{End}_\bullet\,FM$ are pointed multicategories, we emphasize that $\eta_M$ is *not* pointed because $\eta_M$ sends the basepoint object $* \in M$ to the length-one tuple $(*) \neq \langle\rangle$. The composite, $\eta_M^\bullet$ in (4.3.2) *is* pointed because $[(*)] = [\langle\rangle]$ in $F_\bullet M$.                                        ◇

**Lemma 4.3.5.** *In the context of Definition 4.3.1,*

$$\eta_M^\bullet : M \longrightarrow \mathsf{End}_\bullet\,F_\bullet M$$

*is a pointed multifunctor that is, moreover, 2-natural in* M.

*Proof.* Multifunctoriality of $\eta_M^\bullet$ follows from that of $\eta_M$. As noted in Explanations 4.3.3 and 4.3.4, $\eta_M^\bullet$ is pointed by relations $\overset{\mathrm{ob}}{\sim}$ on objects and $\overset{2}{\sim}$ on morphisms in $F_\bullet M$. The 2-naturality of $\eta^\bullet$ follows from that of $\eta_M$ and $p_M$, together with Cat-multifunctoriality of $\mathsf{End}_\bullet$.                                        □

For a small permutative category C, recall the counit

$$\varepsilon_C : F\,\mathsf{End}\,C \longrightarrow C$$

from Definition 3.2.4. Note that $\varepsilon_C$ is basepoint-collapsing (Definition 4.2.9) by its definition on morphisms (3.2.7).

**Definition 4.3.6** (Counit). Suppose $(C, \oplus, e)$ is a small permutative category. We define a strict symmetric monoidal functor $\varepsilon_C^\bullet$ as the unique dashed arrow

(4.3.7)
$$F\,\mathsf{End}\,C = F\,\mathsf{End}_\bullet\,C \xrightarrow{\ p_{\mathsf{End}_\bullet C}\ } F_\bullet\,\mathsf{End}_\bullet\,C \dashrightarrow^{\ \varepsilon_C^\bullet\ } C$$

$$\varepsilon_C$$

induced by the pushout (4.2.7), where

- $M = \mathsf{End}_\bullet C$,
- $p_{\mathsf{End}_\bullet C}$ is the strict symmetric monoidal functor in Definition 4.2.1, and

- $Q = \varepsilon_{\mathsf{C}}$ is the counit of the 2-adjunction $\mathsf{F} \dashv \mathsf{End}$ in Definition 3.2.4.

This finishes the definition of $\varepsilon_{\mathsf{C}}^{\bullet}$. ◇

**Explanation 4.3.8.** The proof of Proposition 4.2.5 explains that the multifunctor $\varepsilon_{\mathsf{C}}^{\bullet}$ in (4.3.7) is given by $\varepsilon_{\mathsf{C}}$ on representative objects and operations of $\mathsf{FEnd}_{\bullet}\mathsf{C}$. ◇

**Lemma 4.3.9.** *In the context of Definition 4.3.6, $\varepsilon_{\mathsf{C}}^{\bullet}$ is 2-natural in $\mathsf{C}$.*

*Proof.* The 2-naturality of $\varepsilon^{\bullet}$ follows from the universality of the pushouts in Proposition 4.2.5. Indeed, given a strict symmetric monoidal functor $R : \mathsf{C} \longrightarrow \mathsf{D}$, consider the following.

(4.3.10)

$$
\begin{array}{ccccc}
& & \varepsilon_{\mathsf{C}} & & \\
\mathsf{F}\,\mathsf{End}_{\bullet}\mathsf{C} & \xrightarrow{\ \mathsf{p}_{\mathsf{End}_{\bullet}\mathsf{C}}\ } & \mathsf{F}_{\bullet}\,\mathsf{End}_{\bullet}\mathsf{C} & \xrightarrow{\ \varepsilon_{\mathsf{C}}^{\bullet}\ } & \mathsf{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{F}\,\mathsf{End}_{\bullet}\mathsf{D} & \xrightarrow{\ \mathsf{p}_{\mathsf{End}_{\bullet}\mathsf{D}}\ } & \mathsf{F}_{\bullet}\,\mathsf{End}_{\bullet}\mathsf{D} & \xrightarrow{\ \varepsilon_{\mathsf{D}}^{\bullet}\ } & \mathsf{D} \\
& & \varepsilon_{\mathsf{D}} & &
\end{array}
$$

In the above diagram, the rectangle at left and the outer rectangle commute by naturality of $\mathsf{p}$ and $\varepsilon$, respectively. This implies that both composites around the rectangle at right are basepoint-collapsing. Therefore, the two composites are equal by uniqueness of the universal dashed functor in (4.3.10). Similar reasoning, using the 2-dimensional aspect of the pushout, verifies 2-naturality of $\varepsilon^{\bullet}$ with respect to monoidal natural transformations. □

The diagram (4.3.12) below uses the 2-adjunctions

$$(-)_{+} \dashv \mathsf{U}_{\bullet} \quad \text{and} \quad \mathsf{F} \dashv \mathsf{End}$$

in Proposition C.4.16 and Theorem 3.2.8, respectively.

**Theorem 4.3.11.** *There is a 2-adjunction*

$$
\mathsf{Multicat}_{*} \underset{\mathsf{End}_{\bullet}}{\overset{\mathsf{F}_{\bullet}}{\rightleftarrows}} \perp \mathsf{PermCat}^{\mathsf{st}}
$$

*with unit $\eta^{\bullet}$ and counit $\varepsilon^{\bullet}$ in Definitions 4.3.1 and 4.3.6, respectively. Moreover, (i) and (ii) below hold regarding the following diagram.*

(4.3.12)

$$
\begin{array}{ccc}
& \mathsf{Multicat} & \\
(-)_{+} \nearrow \ \mathsf{U}_{\bullet} & & \mathsf{End} \ \nwarrow \ \mathsf{F} \\
\mathsf{Multicat}_{*} & \underset{\mathsf{End}_{\bullet}}{\overset{\mathsf{F}_{\bullet}}{\rightleftarrows}} & \mathsf{PermCat}^{\mathsf{st}}
\end{array}
$$

  (i) *There is an equality of 2-functors*

(4.3.13)    $$\mathsf{End} = \mathsf{U}_{\bullet} \circ \mathsf{End}_{\bullet} : \mathsf{PermCat}^{\mathsf{st}} \longrightarrow \mathsf{Multicat}$$

     *given by restricting the diagram (1.4.39) to $\mathsf{PermCat}^{\mathsf{st}}$.*

  (ii) *There is a 2-natural isomorphism*

(4.3.14)    $$\mathsf{F} \cong \mathsf{F}_{\bullet} \circ (-)_{+} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}.$$

*Proof.* Throughout this proof we write $\mathsf{E} = \mathsf{End}$ and $\mathsf{E}_\bullet = \mathsf{End}_{\bullet\bullet}$. The unit $\eta^\bullet$ and counit $\varepsilon^\bullet$ are shown to be 2-natural in Lemmas 4.3.5 and 4.3.9, respectively. We now verify that the two triangle identities for $\eta^\bullet$ and $\varepsilon^\bullet$ follow from those of $\eta$ and $\varepsilon$.

Consider the following diagram, for a small pointed multicategory $\mathsf{M}$. As above, we suppress the forgetful $\mathsf{U}_{\bullet\bullet}$.

(4.3.15)



In the above diagram, the two upper regions commute by definition of $\eta^\bullet$ and $\varepsilon^\bullet$. The lower left quadrilateral commutes by 2-naturality of $\mathsf{p}$ with respect to $\eta^\bullet_{\mathsf{M}}$. The middle quadrilateral commutes by 2-naturality of $\mathsf{p}$ with respect to $\mathsf{E}_\bullet\mathsf{p}_\mathsf{M}$. The lower right quadrilateral commutes by 2-naturality of $\varepsilon^\bullet$ with respect to $\mathsf{p}_\mathsf{M}$.

The composite along the top of (4.3.15) is equal to $1_{\mathsf{FM}}$ by the triangle identity for $\eta$ and $\varepsilon$. Thus, the composite along the top and right, which is equal to $\mathsf{p}_\mathsf{M}$, is basepoint-collapsing in the sense of Definition 4.2.9. Therefore, by the universal property of the pushout (Proposition 4.2.5), the bottom composite of (4.3.15) is equal to $1_{\mathsf{F}_\bullet\mathsf{M}}$.

The other triangle identity is simpler. For each small permutative category $\mathsf{C}$, the following diagram commutes by definition of $\eta^\bullet$ and $\varepsilon^\bullet$.



Commutativity of the above diagram, together with the respective triangle identity for $\eta$ and $\varepsilon$, implies that the bottom composite is equal to $1_{\mathsf{E}_\bullet\mathsf{C}}$.

The equality (i) holds by definition. Uniqueness of left adjoints implies the existence of an isomorphism as in (ii). $\qquad\square$

**Explanation 4.3.16.** An explicit description of the isomorphism

$$\mathsf{F} \cong \mathsf{F}_\bullet \circ (-)_+ : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}$$

in Theorem 4.3.11 (ii) can be given as follows. For a small multicategory $\mathsf{M}$, the canonical inclusion into $\mathsf{M}_+$ induces a strict symmetric monoidal functor

$$\mathsf{F}(\mathsf{M}) \longrightarrow \mathsf{F}(\mathsf{M}_+) \xrightarrow{\ \mathsf{p}_\mathsf{M}\ } \mathsf{F}_\bullet(\mathsf{M}_+).$$

For the reverse direction, there is a strict symmetric monoidal functor

(4.3.17) $$\mathsf{F}(\mathsf{M}_+) \longrightarrow \mathsf{FM}$$

that is given on length-one tuples by

$$(x) \longmapsto \begin{cases} (x) & \text{if} \quad x \neq * \\ \langle\rangle & \text{if} \quad x = *. \end{cases}$$

On generating morphisms $(\iota_r, (\phi))$, where $\phi$ is an $r$-ary operation and $\iota_r : \overline{r} \longrightarrow \overline{1}$ is the unique index map, (4.3.17) is given by

$$(\iota_r, (\phi)) \longmapsto \begin{cases} (\iota_r, (\phi)) & \text{if} \quad \phi \in \mathsf{M}\big(\langle x \rangle; y\big) \\ 1_{\langle\rangle} & \text{otherwise,} \end{cases}$$

where the second case applies if $\phi$ is a basepoint operation of $\mathsf{M}_+$. These assignments uniquely determine a strict symmetric monoidal functor that is basepoint-collapsing, and hence induce a unique

(4.3.18)                           $\mathsf{F}_\bullet(\mathsf{M}_+) \longrightarrow \mathsf{FM}.$

The composite

$$\mathsf{FM} \longrightarrow \mathsf{F}(\mathsf{M}_+) \longrightarrow \mathsf{F}_\bullet(\mathsf{M}_+) \longrightarrow \mathsf{FM}$$

is the identity by definition. The other composite is the identity by uniqueness of (4.3.18). The 2-naturality of this isomorphism follows from 2-naturality of the canonical inclusion $\mathsf{M} \longrightarrow \mathsf{M}_+$ and the 2-dimensional universality of the pushout $\mathsf{F}_\bullet(\mathsf{M}_+)$. ◇

## 4.4. Free Permutative Categories of $\mathcal{M}\underline{1}$-Modules

In this section we use the 2-adjunctions

$$(\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}} \quad \text{and} \quad \mathsf{F}_\bullet \dashv \mathsf{End}_\bullet.$$

from Proposition 1.3.19 and Theorem 4.3.11, respectively, to describe an induced adjunction for $\mathcal{M}\underline{1}$-modules.

**Theorem 4.4.1.** *There is a 2-adjunction*

$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \underset{\mathsf{End}_{\mathcal{M}\underline{1}}}{\overset{\mathsf{F}_{\mathcal{M}\underline{1}}}{\rightleftarrows}} \perp \mathsf{PermCat}^{\mathsf{st}}$$

*such that (i) through (iii) below hold regarding the following diagram.*

(4.4.2)

$$\mathsf{Multicat}_* \underset{\mathsf{End}_\bullet}{\overset{\mathsf{F}_\bullet}{\rightleftarrows}} \mathsf{PermCat}^{\mathsf{st}}$$

$$\mathcal{M}\underline{1} \wedge - \quad \mathsf{U}_{\mathcal{M}\underline{1}} \quad \mathsf{End}_{\mathcal{M}\underline{1}} \quad \mathsf{F}_{\mathcal{M}\underline{1}}$$

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}$$

(i) $\mathsf{F}_{\mathcal{M}\underline{1}}$ *is defined as the following composite 2-functor.*

(4.4.3)

$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \overset{\mathsf{U}_{\mathcal{M}\underline{1}}}{\longrightarrow} \mathsf{Multicat}_* \overset{\mathsf{F}_\bullet}{\longrightarrow} \mathsf{PermCat}^{\mathsf{st}}$$

with top label $\mathsf{F}_{\mathcal{M}\underline{1}}$

(ii) *There is an equality of 2-functors*

$$(4.4.4) \qquad\qquad \mathsf{End}_{\bullet} = \mathsf{U}_{\mathcal{M}\underline{1}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}$$

*given by restricting (1.4.39) to* $\mathsf{PermCat}^{\mathsf{st}}$.

(iii) *There is a 2-natural isomorphism*

$$(4.4.5) \qquad\qquad \mathsf{F}_{\bullet} \cong \mathsf{F}_{\mathcal{M}\underline{1}} \circ (\mathcal{M}\underline{1} \wedge -).$$

*Proof.* The 2-adjunction

$$\mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$$

is given by the following 2-natural isomorphisms, explained below, for each $\mathsf{N} \in \mathsf{Mod}^{\mathcal{M}\underline{1}}$ and $\mathsf{Q} \in \mathsf{PermCat}^{\mathsf{st}}$:

$$
\begin{aligned}
\mathsf{PermCat}^{\mathsf{st}}\left(\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{Q}\right) &= \mathsf{PermCat}^{\mathsf{st}}\left(\mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{Q}\right) \\
&\cong \mathsf{Multicat}_{*}\left(\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{End}_{\bullet}\mathsf{Q}\right) \\
&= \mathsf{Multicat}_{*}\left(\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{N}, \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{Q}\right) \\
&\cong \mathsf{Mod}^{\mathcal{M}\underline{1}}\left(\mathsf{N}, \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{Q}\right).
\end{aligned}
$$

In the above computation, the two equalities follow from (4.4.3) and (4.4.4). The first isomorphism is given by the 2-adjunction $\mathsf{F}_{\bullet} \dashv \mathsf{End}_{\bullet}$ of Theorem 4.3.11. The last isomorphism holds because $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a full sub-2-category of $\mathsf{Multicat}_{*}$ (Proposition 1.3.17 (4)).

The 2-natural isomorphism (4.4.5) follows from uniqueness of left adjoints. This completes the proof. $\qquad\qquad\square$

**Explanation 4.4.6.** The unit and counit of the 2-adjunction $\mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$ in Theorem 4.4.1 are given as follows.

**Unit:** The unit is the 2-natural transformation

$$(4.4.7) \qquad\qquad \eta^{\mathcal{M}\underline{1}} : 1_{\mathsf{Mod}^{\mathcal{M}\underline{1}}} \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}$$

with component at each left $\mathcal{M}\underline{1}$-module M given by the morphism

$$
\begin{aligned}
\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}} : \mathsf{M} &\longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{M} \\
&= \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M} \quad \text{in} \quad \mathsf{Mod}^{\mathcal{M}\underline{1}}.
\end{aligned}
$$

This morphism is uniquely determined by the image

$$
\begin{aligned}
\mathsf{U}_{\mathcal{M}\underline{1}}\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}} = \eta_{\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}}^{\bullet} : \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M} &\longrightarrow \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M} \\
&= \mathsf{End}_{\bullet}\mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}
\end{aligned}
$$

in $\mathsf{Multicat}_{*}$ using Proposition 1.3.17 (4), with $\eta^{\bullet}$ the unit of $\mathsf{F}_{\bullet} \dashv \mathsf{End}_{\bullet}$ in (4.3.2).

**Counit:** The counit is the 2-natural transformation

$$(4.4.8) \qquad\qquad \varepsilon^{\mathcal{M}\underline{1}} : \mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}} \longrightarrow 1_{\mathsf{PermCat}^{\mathsf{st}}}$$

with components defined by $\varepsilon^{\bullet}$ as follows:

$$(4.4.9)$$

$$
\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C} = \mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{E}_{\mathcal{M}\underline{1}}\mathsf{C} = \mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C} \xrightarrow[\varepsilon_{\mathsf{C}}^{\bullet}]{} \mathsf{C}
$$

with $\varepsilon_{\mathsf{C}}^{\mathcal{M}\underline{1}}$ the upper composite arrow to $\mathsf{C}$.

for each $\mathsf{C} \in \mathsf{PermCat}^{\mathsf{st}}$. $\qquad\qquad\diamond$

## 4.5. Examples of Pointed Free Permutative Categories

In this section we describe some examples of $\mathsf{F}_\bullet\mathsf{M}$ for pointed multicategories $\mathsf{M}$. Our first two examples make use of the isomorphism (4.3.14):

$$\mathsf{F}_\bullet(\mathsf{M}_+) \cong \mathsf{F}(\mathsf{M})$$

for unpointed multicategories $\mathsf{M}$. Recall, from Example 3.1.14, that $\mathsf{F}(\mathsf{I})$ is isomorphic to the permutation category with objects given by natural numbers and morphisms given by symmetric groups.

**Example 4.5.1** (The Smash Unit). Recall from Theorem 1.2.8 that $\mathsf{I}_+$ is the unit for the smash product of pointed multicategories. Using (4.3.14) we have

(4.5.2) $$\mathsf{F}_\bullet(\mathsf{I}_+) \cong \mathsf{F}(\mathsf{I}).$$

Each object of $\mathsf{F}_\bullet(\mathsf{I}_+)$ is an $\overset{\text{ob}}{\sim}$-equivalence class of tuples of objects of $\mathsf{I}_+$ and, therefore, is identified with a natural number for its non-basepoint entries. Each morphism of $\mathsf{F}(\mathsf{I}_+)$ is $\overset{2}{\sim}$-equivalent to one with only non-basepoint operations. Because the only such operations are the unit operation in $\mathsf{I}$, each morphism of $\mathsf{F}(\mathsf{I}_+)$ is $\overset{2}{\sim}$-equivalent to one given by a permutation of non-basepoint entries. $\diamond$

**Example 4.5.3** (The Terminal Multicategory). Let $\varnothing$ denote the empty multicategory, with no objects and no operations. Then $\mathsf{T} = \varnothing_+$ and we have

$$\mathsf{F}_\bullet(\mathsf{T}) \cong \mathsf{F}(\varnothing) \cong \mathbf{1},$$

where the last isomorphism identifies the unique object of $\mathbf{1}$ with the empty tuple $\langle\rangle \in \mathsf{F}(\varnothing)$. $\diamond$

For further examples, we will make use of the following observation for general pointed multicategories.

**Explanation 4.5.4** (Morphisms Arising from Nullary Operations). For each pointed multicategory $\mathsf{M}$, with basepoint object $*$, there is a nullary operation

$$\iota_0 \in \mathsf{M}\big(\langle\rangle\,;\,*\big).$$

For each object $x \in \mathsf{M}$, this gives a morphism in $\mathsf{F}\mathsf{M}$

$$\big(f^{(x)},\,(1_x,\iota_0)\big) : (x) \longrightarrow (x,*),$$

where the index map $f^{(x)}$ is the standard inclusion $\overline{1} \lhook\joinrel\longrightarrow \overline{2}$.

More generally, suppose given an object $\langle x \rangle \in \mathsf{F}\mathsf{M}$. Recall that $\langle x \rangle^\wedge$ denotes the sub-tuple consisting of non-basepoint objects, $x_i \neq *$. The nullary operations give rise to a morphism in $\mathsf{F}\mathsf{M}$

(4.5.5) $$w^{\langle x \rangle} = \big(f^{\langle x \rangle},\,\langle \delta^{x_i} \rangle_i\big) : \langle x \rangle^\wedge \longrightarrow \langle x \rangle$$

defined as follows.

- The index map $f^{\langle x \rangle}$ is the inclusion to the subset that indexes non-basepoint objects of $\langle x \rangle$, preserving their order.
- For each entry $x_i$ in $\langle x \rangle$,

$$\delta^{x_i} = \begin{cases} 1_{x_i} & \text{if} \quad x_i \neq * \\ \iota_0 & \text{if} \quad x_i = *. \end{cases}$$

Note that we have
$$w^{\langle x \rangle} \overset{2}{\sim} 1_{\langle x \rangle}.$$
Therefore, for any morphism
$$(f, \langle \phi \rangle) : \langle x \rangle \longrightarrow \langle x' \rangle \quad \text{in} \quad \mathsf{FM},$$
we have

(4.5.6) $\qquad (f, \langle \phi \rangle) \overset{1}{\sim} \left( 1_{\langle x \rangle}, (f, \langle \phi \rangle) \right) \overset{2}{\sim} \left( w^{\langle x \rangle}, (f, \langle \phi \rangle) \right) \quad \text{in} \quad \widetilde{\mathsf{Mor}}(\mathsf{FM}).$

We will make use of these equivalences now in the following examples, and later in the proof of Proposition 4.6.6. $\qquad \diamond$

We next consider $\mathsf{F} \bullet \mathsf{M} = \mathsf{F}_{\mathcal{M}\underline{1}} \mathsf{M}$ for $\mathcal{M}\underline{1}$-modules M. We use the following characterization, from [**JY∞**], to construct a partial inverse for $w^{\langle x \rangle}$. Recall the interchange permutation $\xi^{\otimes}$ from (1.1.16).

**Proposition 4.5.7** ([**JY∞**, 10.2.7]). *Suppose* M *is a pointed multicategory with basepoint* $*$. *Then a left* $\mathcal{M}\underline{1}$-*module structure on* M
$$\mu : \mathcal{M}\underline{1} \wedge \mathsf{M} \longrightarrow \mathsf{M}.$$
*determines and is uniquely determined by operations*
$$\pi_1^2(x) \in \mathsf{M}\big(x, *; x\big) \quad \text{for} \quad x \in \mathsf{M}$$
*such that the following basepoint, unit, and interchange conditions hold for objects* $x$ *in* M *and operations* $\phi$ *in* $\mathsf{M}(\langle y \rangle; x)$ *with* $\langle y \rangle = (y_1, \dots, y_m)$.

(4.5.8) $\qquad\qquad \pi_1^2(*) = \iota^2 \qquad\qquad\qquad\qquad\qquad in \quad \mathsf{M}\big(*, *; *\big)$

(4.5.9) $\qquad \gamma\big(\pi_1^2(x); 1_x, \iota^0\big) = 1_x \qquad\qquad\qquad in \quad \mathsf{M}\big(x; x\big)$

(4.5.10) $\qquad \gamma\big(\phi; \langle \pi_1^2(y_j) \rangle_j\big) = \gamma\big(\pi_1^2(x); \phi, \iota^m\big) \cdot \xi^{\otimes} \qquad in \quad \mathsf{M}\big(y_1, *, \dots, y_m, *; x\big)$

**Remark 4.5.11.** Using equivariance on operations in M and induction, the operations $\pi_1^2$ determine a larger family with similar properties
$$\pi_j^n(x) \in \mathsf{M}\big(*, \dots, x, \dots, *; x\big),$$
where the input profile has $x$ in position $j$ and the $n - 1$ other entries are basepoints. See [**JY∞**, 8.4.5,10.2.4] for further discussion of these operations. $\qquad \diamond$

**Example 4.5.12** (Operations Arising from Strict Units). If $\mathsf{M} = \mathcal{M}a$ for a pointed finite set $a$, then the operations $\pi_1^2(x)$ arise because $* = \varnothing$ is a strict unit for the disjoint union in $\mathcal{P}(a^\flat)$. If $\mathsf{M} = \mathsf{End} \bullet \mathsf{C}$ for a permutative category C, then the operations $\pi_1^2(x)$ arise because $* = \mathsf{e}$ is strict unit for the monoidal sum in C. $\qquad \diamond$

We will show that the operations $\pi_1^2(x_i)$, for entries $x_i$ in a tuple $\langle x \rangle$, provide a partial inverse to $w^{\langle x \rangle}$. Note, however, that the only morphism in FM with codomain $\langle \rangle$ is the identity $1_{\langle \rangle}$. Therefore, cases where $\langle x \rangle$ consists entirely of basepoint objects must be treated separately, as in the following.

**Definition 4.5.13.** Suppose M is a pointed multicategory with basepoint object $*$. Let $(*^r)$ denote the $r$-tuple consisting entirely of basepoints. Define the following for $\langle x \rangle \in \mathsf{FM}$:

(4.5.14) $\qquad \langle x \rangle^{\sim} = \begin{cases} (*) & \text{if } \langle x \rangle = (*^r) \text{ for } r > 0 \\ \langle \rangle & \text{if } \langle x \rangle = \langle \rangle \\ \langle x \rangle^{\wedge} & \text{if } \langle x \rangle \text{ has any non-basepoint entries.} \end{cases}$

Thus, $\langle x \rangle^{\sim}$ is equal to $\langle x \rangle^{\wedge}$ unless $\langle x \rangle = (*^r)$ for $r > 0$.                         ◇

**Lemma 4.5.15.** *Suppose* M *is a left* $\mathcal{M}\underline{1}$*-module. Then, for each* $\langle x \rangle \in$ FM *there is a morphism*

$$c^{\langle x \rangle} : \langle x \rangle \longrightarrow \langle x \rangle^{\sim} \quad in \quad \mathsf{FM}$$

*such that the composite*

$$\langle x \rangle^{\wedge} \xrightarrow{\ w^{\langle x \rangle}\ } \langle x \rangle \xrightarrow{\ c^{\langle x \rangle}\ } \langle x \rangle^{\sim}$$

*is* $\simeq$*-equivalent to the identity* $1_{\langle x \rangle}$ *in* $\widetilde{\mathrm{Mor}}(\mathsf{FM})$.

*Proof.* Suppose $\langle x \rangle$ has length $r$. We use the operations $\pi_1^2(x)$ described in Proposition 4.5.7 to show that there is a morphism $c^{\langle x \rangle}$ such that

$$(4.5.16) \qquad c^{\langle x \rangle} \circ w^{\langle x \rangle} = \begin{cases} (\varnothing \xrightarrow{\ !\ } \overline{1}, \iota_0) & \text{if } \langle x \rangle = (*^r) \text{ for } r > 0, \\ 1_{\langle x \rangle^{\wedge}} & \text{otherwise,} \end{cases}$$

where ! denotes the unique index map. In either of the two cases (4.5.16), we have the conclusion

$$[c^{\langle x \rangle} \circ w^{\langle x \rangle}] = 1_{[\langle x \rangle]} \quad \text{in} \quad \mathsf{F_\bullet M} = \mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{M}.$$

First we discuss three trivial cases, where $\langle x \rangle$ consists of either $r > 0$ basepoint entries, or is empty, or consists of $r > 0$ non-basepoint entries.

- If $\langle x \rangle = (*^r)$ for $r > 0$, then the $r$-ary basepoint operation $\iota_r$ defines a morphism

$$c^{(*^r)} = (\overline{r} \xrightarrow{\ !\ } \overline{1}, \iota_r) : (*^r) \longrightarrow (*)$$

  as desired.
- If $r = 0$, note $w^{\langle\rangle} = 1_{\langle\rangle}$ and choose

$$c^{\langle\rangle} = 1_{\langle\rangle}.$$

- If $r > 0$ and $\langle x \rangle$ has no basepoint entries, then $\langle x \rangle = \langle x \rangle^{\wedge} = \langle x \rangle^{\sim}$ and we choose

$$c^{\langle x \rangle} = 1_{\langle x \rangle} = 1_{\langle x \rangle^{\wedge}}.$$

Each of these choices satisfies (4.5.16).

For the remainder of the proof we suppose $r \geq 2$ and $\langle x \rangle$ has at least one basepoint entry that is adjacent to one non-basepoint entry $x_i \neq *$. Note, therefore, that $\langle x \rangle^{\sim} = \langle x \rangle^{\wedge}$. If $r = 2$ and $\langle x \rangle = (x_1, *)$, then there is an operation

$$\pi = \pi_1^2(x_1) \in \mathsf{M}\big((x_1, *); x_1\big)$$

determined by Proposition 4.5.7. In this case, choose

$$c^{(x_1, *)} = (\overline{2} \xrightarrow{\ !\ } \overline{1}, \pi).$$

If $r = 2$ and $\langle x \rangle = (*, x_2)$ then choose $c^{(*, x_2)}$ similarly, using the operation

$$\pi = \pi_1^2(x_2)\tau \in \mathsf{M}\big((*, x_2); x_2\big)$$

given by the right action of the transposition $\tau \in \Sigma_2$. Note that

$$(4.5.17) \qquad c^{(x_1, *)} \circ w^{(x_1, *)} = 1_{(x_1)} \quad \text{and} \quad c^{(*, x_2)} \circ w^{(*, x_2)} = 1_{(x_2)} \quad \text{in} \quad \widetilde{\mathrm{Mor}}(\mathsf{FM})$$

by property (4.5.9) of the operations $\pi$ and hence these choices satisfy (4.5.16).

If $r > 2$, then $\langle x \rangle$ decomposes as a sum

$$\langle x \rangle = \langle x' \rangle \oplus \langle x'' \rangle \oplus \langle x''' \rangle,$$

where the first and last summands are possibly empty and the middle summand $\langle x'' \rangle$ is either $(x_i, *)$ or $(*, x_i)$. Then choose $c^{\langle x \rangle}$ inductively as the composite along the top and right of the following diagram, where $\langle y \rangle = \langle x' \rangle \oplus (x_i) \oplus \langle x''' \rangle$, so $\langle y \rangle$ has length $r - 1$, and $c^{\langle y \rangle}$ is any choice of morphism satisfying (4.5.16).



In the above diagram, the region at left commutes by definition of $w^{\langle x \rangle}$ and composition in FM and the lower right region commutes by inductive hypothesis on $c^{\langle y \rangle}$. The upper triangle commutes by (4.5.17) for $c^{\langle x'' \rangle}$, and hence the composite $c^{\langle x \rangle}$ also satisfies (4.5.16). This completes the proof. $\square$

Combining Lemma 4.5.15 with (4.5.6) gives an application that we explain in Lemma 4.5.19 below.

**Explanation 4.5.18** (Non-Uniqueness of $c^{\langle x \rangle}$). Note that the morphisms $c^{\langle x \rangle}$ constructed in the proof of Lemma 4.5.15 generally depend on a choice of non-basepoint entry $x_i$ in $\langle x \rangle$ and are not necessarily unique. However, the condition

$$c^{\langle x \rangle} \circ w^{\langle x \rangle} \simeq 1_{\langle x \rangle} \quad \text{in} \quad \widetilde{\text{Mor}}(\text{FM}),$$

together with the observation $w^{\langle x \rangle} \overset{2}{\sim} 1_{\langle x \rangle}$ from Explanation 4.5.4, implies that $c^{\langle x \rangle} \simeq 1_{\langle x \rangle}$ for any choice of $c^{\langle x \rangle}$. Therefore, all such choices result in identity morphisms in F.M. This is the key property of the morphisms $c^{\langle x \rangle}$. $\diamond$

The following result uses the notation of Explanation 4.5.4, Definition 4.5.13, and Lemma 4.5.15 above.

**Lemma 4.5.19.** *Suppose* M *is a pointed multicategory and suppose that, for each object* $\langle x \rangle \in$ FM*, there is a morphism*

$$c^{\langle x \rangle} : \langle x \rangle \longrightarrow \langle x \rangle^{\sim} \quad \text{in} \quad \text{FM}$$

*such that the composite*

$$\langle x \rangle^{\wedge} \xrightarrow{\ w^{\langle x \rangle}\ } \langle x \rangle \xrightarrow{\ c^{\langle x \rangle}\ } \langle x \rangle^{\sim}$$

*is* $\simeq$-*equivalent to the identity* $1_{\langle x \rangle}$ *in* $\widetilde{\text{Mor}}(\text{FM})$. *Then each morphism* $[\langle x \rangle] \longrightarrow [\langle y \rangle]$ *in* F.M *is represented by a length-one sequence consisting of a morphism*

$$(f', \langle \phi' \rangle) : \langle x \rangle^{\wedge} \longrightarrow \langle y \rangle^{\sim} \quad \text{in} \quad \text{FM}.$$

*Proof.* Applying the observation (4.5.6) to $c^{\langle x \rangle}$, we have the following equivalences in $\widetilde{\mathrm{Mor}}(\mathrm{FM})$:

$$\left( c^{\langle x \rangle} \right) \simeq \left( w^{\langle x \rangle}, c^{\langle x \rangle} \right) \overset{1}{\sim} \left( c^{\langle x \rangle} \circ w^{\langle x \rangle} \right) \simeq \left( 1_{\langle x \rangle} \right),$$

where the final relation holds by hypothesis on $c^{\langle x \rangle}$. Now suppose given a morphism

$$\left( f, \langle \phi \rangle \right) : \langle x \rangle \longrightarrow \langle y \rangle \quad \text{in} \quad \mathrm{FM}.$$

Let $(f', \langle \phi' \rangle)$ and $(f'', \langle \phi'' \rangle)$ denote the composites indicated below.

(4.5.20)



In the above diagram, either

- $\langle x \rangle$ has some non-basepoint operation, in which case the unlabeled morphism $\langle x \rangle^{\wedge} \longrightarrow \langle x \rangle^{\sim}$ is the identity and $\widetilde{w}^{\langle x \rangle}$ is equal to $w^{\langle x \rangle}$, or
- $\langle x \rangle = (*^r)$ consists of all basepoint entries, in which case the unlabeled morphism, $\widetilde{w}^{\langle x \rangle}$, and $w^{\langle x \rangle}$, are all determined by basepoint operations and a choice of index map $\overline{1} \longrightarrow \overline{r}$ for $\widetilde{w}^{\langle x \rangle}$.

In either case, note that $\widetilde{w}^{\langle x \rangle}$ is also $\simeq$-equivalent to an identity. This conclusion holds independently of which index map $\overline{1} \longrightarrow \overline{r}$ we choose for $\widetilde{w}^{\langle x \rangle}$ in the case $\langle x \rangle = (*^r)$.

Since each of $w^{\langle x \rangle}$, $\widetilde{w}^{\langle x \rangle}$, and $c^{\langle y \rangle}$ is $\simeq$-equivalent to an identity, we have

$$\left( f', \langle \phi' \rangle \right) \simeq \left( f, \langle \phi \rangle \right) \simeq \left( f'', \langle \phi'' \rangle \right) \quad \text{in} \quad \widetilde{\mathrm{Mor}}(\mathrm{FM}).$$

This proves the assertion for morphisms of the form $\left[ \left( f, \langle \phi \rangle \right) \right]$ in $\mathrm{F}_{\bullet}\mathrm{M}$.

For a pair of $\overset{\mathrm{ob}}{\sim}$-composable morphisms,

$$\langle x \rangle \xrightarrow{(f, \langle \phi \rangle)} \langle y \rangle \overset{\mathrm{ob}}{\sim} \langle u \rangle \xrightarrow{(g, \langle \psi \rangle)} \langle v \rangle,$$

note that we have $\langle y \rangle^{\wedge} = \langle u \rangle^{\wedge}$ and $\langle y \rangle^{\sim} = \langle u \rangle^{\sim}$ by definition of $\overset{\mathrm{ob}}{\sim}$ and (4.5.14). Considering the following diagram,

we conclude that

$$\big[(f, \langle \phi \rangle), (g, \langle \psi \rangle)\big] \simeq \big[(f'', \langle \phi'' \rangle), (g', \langle \psi' \rangle)\big] \overset{1}{\sim} \big[(g', \langle \psi' \rangle) \circ (f'', \langle \phi'' \rangle)\big].$$

Therefore, by induction, each tuple of $\overset{ob}{\sim}$-composable morphisms in $\widetilde{\text{Mor}}(\text{FM})$ is equivalent to a single morphism in FM, with domain of the form $\langle x \rangle^\wedge$ and codomain of the form $\langle y \rangle^\sim$. This completes the proof. $\qquad\square$

Combining Lemmas 4.5.15 and 4.5.19 yields the following result.

**Corollary 4.5.21.** *Suppose* M *is a left* $\mathcal{M}\underline{1}$-module. Then each morphism $[\langle x \rangle] \longrightarrow [\langle y \rangle]$ *in* $\text{F}_{\mathcal{M}\underline{1}}\text{M} = \text{F} \bullet \text{U}_{\mathcal{M}\underline{1}}\text{M}$ *is represented by a length-one sequence consisting of a morphism*

$$(f', \langle \phi' \rangle) : \langle x \rangle^\wedge \longrightarrow \langle y \rangle^\sim \quad in \quad \text{FM}.$$

**Example 4.5.22** (Partition Multicategories). Applying Corollary 4.5.21 for a partition multicategory $\text{M} = \mathcal{M}a$, note that the only partitions of the empty subset $\varnothing \subset a^\flat$ are those consisting of empty subsets. Therefore, the only morphisms with codomain $(\varnothing) \in \text{F}(\mathcal{M}a)$ are those that are $\overset{2}{\sim}$-equivalent to $1_{()}$. Thus Corollary 4.5.21 simplifies slightly to give the following description of the permutative category $\text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}a) = \text{F} \bullet \text{U}_{\mathcal{M}\underline{1}}(\mathcal{M}a)$.

**Objects:** Each object of $\text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}a)$ is uniquely represented by a tuple $\langle s \rangle$ such that each entry $s_i$ is a nonempty subset of $a^\flat$.

**Morphisms:** Each morphism of $\text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}a)$ is uniquely represented by a morphism

$$(f, \langle \phi \rangle) : \langle s \rangle \longrightarrow \langle t \rangle$$

where each $s_i$ and $t_j$ is nonempty and each $\langle s \rangle_{f^{-1}(j)}$ is a partition of $t_j$, with corresponding operation $\phi_j = \iota_{\langle s \rangle_{f^{-1}(j)}}$ in $\mathcal{M}a$.

As a special case, with $a = \underline{1}$, the discussion above shows that $\text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1})$ is isomorphic to the permutation category with

- objects given by natural numbers $n \geq 0$, corresponding to length-$n$ tuples of the subset $s = \{1\}$, and
- morphisms given by symmetric groups, whose elements permute the entries of a tuple.

Thus, we conclude

$$(4.5.23) \qquad\qquad\qquad \text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1}) \cong \text{F}(\text{I}).$$

In fact, this isomorphism is a special case $\text{M} = \text{I}$ combining the general isomorphisms

$$\text{FM} \cong \text{F} \bullet (\text{M}_+) \cong \text{F}_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1} \wedge \text{M}_+)$$

from (4.3.14) and (4.4.5) above. $\qquad\qquad\diamond$

Corollary 4.5.21 can be used for an application similar to that of Example 4.5.22 in the case $\text{M} = \text{End} \bullet \text{C}$, although there may be nontrivial morphisms $x \longrightarrow e$ in C and hence this case does not admit the same simplification as in Example 4.5.22. An alternative application is given in Proposition 4.6.6 below, where we use Explanation 4.5.4 to show that there is an adjunction in $\text{PermCat}^{\text{su}}$ between C and $\text{F} \bullet \text{End} \bullet \text{C}$.

## 4.6. Componentwise Right Adjoint of the Pointed Adjunction

In this section we extend the componentwise right adjoint $\varrho_C$ from Proposition 3.3.7 to the pointed context.

**Definition 4.6.1.** For each small permutative category C, we define the symmetric monoidal functor $\varrho_C^\bullet$ as the composite

$$
(4.6.2) \qquad \overbrace{C \xrightarrow{\;\varrho_C\;} F\,\mathsf{End}(C) = F\,\mathsf{End}_\bullet C \xrightarrow{\;p_{\mathsf{End}_\bullet C}\;} F_\bullet\,\mathsf{End}_\bullet C}^{\varrho_C^\bullet}
$$

of

- the symmetric monoidal functor $\varrho_C$ in Lemma 3.3.12 and
- the strict symmetric monoidal functor $p_{\mathsf{End}_\bullet C}$ in Definition 4.2.1.     ◇

**Explanation 4.6.3.** The symmetric monoidal functor $\varrho_C^\bullet$ in (4.6.2) is given explicitly by combining the definition of $\varrho_C$ from (3.3.2) with that of $p$ from (4.2.2):

$$
(4.6.4) \qquad \begin{cases} \varrho_C^\bullet(x) = [(x)] & \text{for} \quad x \in \mathsf{Ob}\,C \quad \text{and} \\ \varrho_C^\bullet(\phi) = \left[\left(1_{\overline{1}}, (\phi)\right)\right] : [(x)] \longrightarrow [(y)] & \text{for} \quad \phi \in C(x,y). \end{cases}
$$

We emphasize that $\varrho_C$ is *not* strictly unital, as discussed in Remark 3.3.13. Despite this, Lemma 4.6.5 below shows that $\varrho_C^\bullet$ *is* strictly unital.     ◇

**Lemma 4.6.5.** *For each small permutative category* C, *the symmetric monoidal functor*

$$
\varrho_C^\bullet : C \longrightarrow F_\bullet\,\mathsf{End}_\bullet C
$$

*in (4.6.2) is strictly unital.*

*Proof.* The monoidal and unit constraints for $\varrho_C$ are given in Definition 3.3.9. Since $p_{\mathsf{End}_\bullet C}$ is strict symmetric monoidal, the monoidal and unit constraints of $\varrho_C^\bullet$ are given by $p_{\mathsf{End}_\bullet C}(\varrho_C^2)$ and $p_{\mathsf{End}_\bullet C}(\varrho_C^0)$, respectively. By (3.3.10),

$$
\varrho_C^0 = (\iota_0, 1_e) : \langle\rangle \longrightarrow (e).
$$

Therefore $p_{\mathsf{End}C}(\varrho_C^0) = \left[(\iota_0, 1_e)\right] = 1_{[\langle\rangle]}$, by relation $\overset{2}{\sim}$.     □

**Proposition 4.6.6.** *For each small permutative category* C, *the adjunction* $\varepsilon_C \dashv \varrho_C$ *in (3.3.8) extends along* $p_{\mathsf{End}_\bullet(C)}$ *to an adjunction in* $\mathsf{PermCat}^{\mathsf{su}}$.

$$
(4.6.7) \qquad F_\bullet\,\mathsf{End}_\bullet C \underset{\varrho_C^\bullet}{\overset{\varepsilon_C^\bullet}{\underset{\bot}{\rightleftarrows}}} C
$$

*with*

- $\varepsilon_C^\bullet$ *the counit in (4.3.7) and*
- $\varrho_C^\bullet$ *the strictly unital symmetric monoidal functor in Lemma 4.6.5.*

*Proof.* Throughout this proof we write

$$
E = \mathsf{End}, \quad E_\bullet = \mathsf{End}_\bullet, \quad \text{and} \quad p = p_{E_\bullet C}.
$$

Recalling Proposition 3.3.7, the adjunction $\varepsilon_C \dashv \varrho_C$ has unit and counit

$$
1_{\mathsf{FEC}} \xrightarrow{\;v\;} \varrho_C \varepsilon_C \quad \text{and} \quad \varepsilon_C \varrho_C \xrightarrow{\;1\;} 1_C,
$$

respectively, where $v$ is defined in (3.3.6). Lemma 3.3.12 and Remark 3.3.13 explain that $\rho_C$ is a symmetric monoidal functor that is generally not strong monoidal.

To show that $v$ gives a well defined a unit

$$v^\bullet : 1_{\mathsf{F\bullet E\bullet C}} \longrightarrow \varrho_C^\bullet \varepsilon_C^\bullet,$$

first recall from (4.5.6) with $\mathsf{M} = \mathsf{E\bullet C}$ that we have

(4.6.8) $$(f, \langle\phi\rangle) \simeq \left(w^{\langle x\rangle}, (f, \langle\phi\rangle)\right) \quad \text{in} \quad \widetilde{\mathsf{Mor}}(\mathsf{FE\bullet C}),$$

where $(f, \langle\phi\rangle) : \langle x\rangle \longrightarrow \langle z\rangle$ is any morphism in $\mathsf{FE\bullet C}$ and

$$w^{\langle x\rangle} : \langle x\rangle^\wedge \longrightarrow \langle x\rangle$$

is determined by the nullary basepoint operations (Explanation 4.5.4).

Now suppose $\langle x\rangle \overset{\mathrm{ob}}{\sim} \langle y\rangle$ in $\mathsf{FE\bullet C}$. Then

- $\langle x\rangle^\wedge = \langle y\rangle^\wedge$ by definition of $\overset{\mathrm{ob}}{\sim}$,
- $\oplus_i x_i = \oplus_j y_j$ by strictness of the monoidal unit in $\mathsf{C}$, and
- the following equalities hold in $\mathsf{FE\bullet C}$:

$$v_{\langle x\rangle} \circ w^{\langle x\rangle} = v_{\langle x\rangle^\wedge} = v_{\langle y\rangle^\wedge} = v_{\langle y\rangle} \circ w^{\langle y\rangle}.$$

Therefore, by (4.6.8) with $\mathsf{M} = \mathsf{E\bullet C}$ we have

$$\left(v_{\langle x\rangle}\right) \simeq \left(w^{\langle x\rangle}, v_{\langle x\rangle}\right) \simeq \left(w^{\langle y\rangle}, v_{\langle y\rangle}\right) \simeq \left(v_{\langle y\rangle}\right).$$

Hence, the components

(4.6.9) $$v^\bullet_{[\langle x\rangle]} = [v_{\langle x\rangle}] : [\langle x\rangle] \longrightarrow [(\oplus_i x_i)] \quad \text{in} \quad \mathsf{F\bullet E\bullet C}$$

are well defined with respect to $\overset{\mathrm{ob}}{\sim}$-equivalence classes of objects. With this definition, naturality of $v$ implies that of $v^\bullet$ and we have

$$\mathsf{p} * v = v^\bullet * \mathsf{p}$$

in the following diagram in $\mathsf{PermCat}^{\mathsf{su}}$.

(4.6.10)



Since $\varepsilon_C^\bullet \mathsf{p} = \varepsilon_C$, we also have

$$\varepsilon_C^\bullet \varrho_C^\bullet = 1_C.$$

One of the triangle identities for $(\varepsilon_C^\bullet, \varrho_C^\bullet, v^\bullet, 1_C)$ is, therefore, immediate. The other follows from the corresponding identity for $(\varepsilon_C, \rho_C, v, 1_C)$ on representative objects and morphisms. $\square$

**Explanation 4.6.11.** Note that our proof of Proposition 4.6.6 depends on specific details from Explanation 4.5.4 instead of the 2-categorical pushout description of $\mathsf{F\bullet M}$ in Proposition 4.2.5. This is because Proposition 4.2.5 describes a pushout in $\mathsf{PermCat}^{\mathsf{st}}$, while (4.6.10) is a diagram in $\mathsf{PermCat}^{\mathsf{su}}$. Thus, a more general proof of Proposition 4.6.6 using Proposition 4.2.5 would require a comparison of

2-dimensional pushouts. The strictification theory of [**BKP89**] provides some comparisons of this sort, and may be one way of approaching more general versions of the results here and below. ⋄

**Remark 4.6.12.** Our first main application of Proposition 4.6.6 is Theorem 4.7.3, showing that the components of $\varepsilon^{\bullet}$ are stable equivalences. The further applications of Proposition 4.6.6, in Theorems 5.4.1, 5.5.12, 12.1.6, and 12.4.6 use the same result for $\varrho^{\bullet}$ instead. The reason for this change is that $\varrho^{\bullet}$ is shown to be Cat-multinatural in Lemma 5.3.3, while the corresponding result for $\varepsilon^{\bullet}$ does not hold (see [**JY22a**, Remark 10.4]). ⋄

Recall from Example C.4.8 (i) that each *strictly unital* symmetric monoidal functor $P$ induces a *pointed* multifunctor $\mathsf{End}_{\bullet}(P)$. Since the components of $\varrho^{\bullet}$ are strictly unital, by Lemma 4.6.5, we may consider $\mathsf{End}_{\bullet}\varrho^{\bullet}$. For comparison, recall the description of $\eta^{\bullet}$ from Explanation 4.3.3. The following result is used in the proof of Theorem 12.1.6, step (v).

**Lemma 4.6.13.** *Suppose* $\mathsf{C}$ *is a small permutative category. Then the two pointed multifunctors below are equal.*

$$\mathsf{End}_{\bullet}\mathsf{C} \xrightarrow[\ \mathsf{End}_{\bullet}\varrho^{\bullet}_{\mathsf{C}}\ ]{\ \eta^{\bullet}_{\mathsf{End}_{\bullet}\mathsf{C}}\ } \mathsf{End}_{\bullet}\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C}$$

*Proof.* We need to show that $\eta^{\bullet}_{\mathsf{End}_{\bullet}\mathsf{C}}$ and $\mathsf{End}_{\bullet}\varrho^{\bullet}_{\mathsf{C}}$ have (i) the same object assignment and (ii) the same multimorphism assignment. For each object $x$ in $\mathsf{C}$, there are object equalities

$$\eta^{\bullet}_{\mathsf{End}_{\bullet}\mathsf{C}}(x) = [(x)] = (\mathsf{End}_{\bullet}\varrho^{\bullet}_{\mathsf{C}})(x) \quad \text{in} \quad \mathsf{End}_{\bullet}\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C}.$$

This proves (i).

To prove (ii), consider an $r$-ary multimorphism

$$\psi \in (\mathsf{End}_{\bullet}\mathsf{C})\big(\langle x_i\rangle^r_{i=1}\,;\, y\big) = \mathsf{C}\big(\textstyle\bigoplus^r_{i=1} x_i\,,\, y\big).$$

Then there are equalities

(4.6.14) $$\eta^{\bullet}_{\mathsf{End}_{\bullet}\mathsf{C}}(\psi) = \big[\big(\iota_r\,,\,(\psi)\big)\big] = (\mathsf{End}_{\bullet}\varrho^{\bullet}_{\mathsf{C}})(\psi)$$

in

$$(\mathsf{End}_{\bullet}\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C})\big(\langle[(x_i)]\rangle^r_{i=1}\,;\, [(y)]\big) = (\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C})\big([\langle x_i\rangle^r_{i=1}]\,,\, [(y)]\big)$$

for the following reasons.

- The first equality in (4.6.14) follows from definition (3.2.3) of

$$\eta_{\mathsf{End}\mathsf{C}}(\psi) = \big(\iota_r\,,\,(\psi)\big) : \langle x_i\rangle^r_{i=1} \longrightarrow (y)$$

  and the fact that each component of $\mathsf{p}$ is a strict symmetric monoidal functor (Definition 4.2.1).

- By definition (3.3.2), we have the equality

$$\varrho_{\mathsf{C}}(\psi) = \big(1_{\overline{1}}\,,\,(\psi)\big) : \big(\textstyle\bigoplus^r_{i=1} x_i\big) \longrightarrow (y).$$

  The symmetric monoidal structure of $\varrho_{\mathsf{C}}$ in Lemma 3.3.12 implies that $(\mathsf{End}_{\bullet}\varrho^{\bullet}_{\mathsf{C}})(\psi)$ is the following composite.

$$[\langle x_i\rangle^r_{i=1}] \xrightarrow{\ [(\iota_r\,,\,1_{\oplus_i x_i})]\ } [(\textstyle\bigoplus^r_{i=1} x_i)] \xrightarrow{\ [(1_{\overline{1}}\,,\,(\psi))]\ } [(y)]$$

This composite is equal to the middle entry in (4.6.14) because

$$1_{\overline{1}} \circ \iota_r = \iota_r : \overline{r} \longrightarrow \overline{1} \quad \text{and}$$

$$\psi \circ 1_{\oplus_i x_i} = \psi : \bigoplus_{i=1}^r x_i \longrightarrow y.$$

This proves (4.6.14). □

## 4.7. Homotopy Theory of Pointed Multicategories

Recall the notions about relative categories in Definition 2.1.6. We extend the relative category structure on PermCat$^{st}$ to Multicat$_*$ and Mod$^{\mathcal{M}\underline{1}}$ as follows.

**Definition 4.7.1** (Stable Equivalences). We define the wide subcategories

$$\mathcal{S}^{\mathcal{M}\underline{1}} = (F_{\mathcal{M}\underline{1}})^{-1}(\mathcal{S}^I) \subset \text{Mod}^{\mathcal{M}\underline{1}} \quad \text{and}$$

$$\mathcal{S}_{\bullet} = F_{\bullet}^{-1}(\mathcal{S}^I) \subset \text{Multicat}_*$$

as the subcategories created by the indicated functors below.

(4.7.2)
$$
\begin{array}{c}
\left( \text{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}} \right) \xrightarrow{\ F_{\mathcal{M}\underline{1}}\ } \\[2mm]
\left( \text{Multicat}_*, \mathcal{S}_{\bullet} \right) \xrightarrow{\ F_{\bullet}\ }
\end{array}
\left( \text{PermCat}^{st}, \mathcal{S}^I \right)
$$

- $\mathcal{S}^I = I^{-1}(\mathcal{S}) \subset \text{PermCat}^{st}$ is the wide subcategory in Definition 3.5.1.
- $F_{\mathcal{M}\underline{1}}$ is the underlying functor of the 2-functor in (4.4.3).
- $F_{\bullet}$ is the underlying functor of the 2-functor in Theorem 4.1.17.

We refer to morphisms in $\mathcal{S}^{\mathcal{M}\underline{1}}$ and $\mathcal{S}_{\bullet}$ as $F_{\mathcal{M}\underline{1}}$-*stable equivalences* and $F_{\bullet}$-*stable equivalences*, respectively. ◇

Note that $F_{\bullet}$-stable equivalences are the preimages of the stable equivalences in $\mathcal{S} \subset \text{PermCat}^{su}$ (2.5.14) under the following composite.

$$\text{Multicat}_* \xrightarrow{\ F_{\bullet}\ } \text{PermCat}^{st} \xhookrightarrow{\ I\ } \text{PermCat}^{su}$$

Recall the notion of an *adjoint equivalence of homotopy theories* in Definition 2.1.10.

**Theorem 4.7.3.** *The adjunction in Theorem 4.3.11*

$$
\left( \text{Multicat}_*, \mathcal{S}_{\bullet} \right) \underset{\text{End}_{\bullet}}{\overset{F_{\bullet}}{\underset{\bot}{\rightleftarrows}}} \left( \text{PermCat}^{st}, \mathcal{S}^I \right)
$$

*is an adjoint equivalence of homotopy theories.*

*Proof.* The left adjoint $F_{\bullet}$ is a relative functor by definition of $\mathcal{S}_{\bullet}$. To see that End$_{\bullet}$ is a relative functor, first recall that each $\varepsilon_C^{\bullet}$ is a left adjoint in PermCat$^{su}$ by Proposition 4.6.6. Thus, the components of $\varepsilon^{\bullet}$ are stable equivalences by Remark 2.5.15 (2). Naturality of $\varepsilon^{\bullet}$ and the 2-out-of-3 property for stable equivalences (Remark 2.5.15 (1)) then imply that

$$F_{\bullet}\text{End}_{\bullet}P : F_{\bullet}\text{End}_{\bullet}C \longrightarrow F_{\bullet}\text{End}_{\bullet}D$$

is a stable equivalence whenever $P$ is a stable equivalence. This, in turn, implies that End$_{\bullet}P$ is an $F_{\bullet}$-stable equivalence. Hence End$_{\bullet}$ is a relative functor.

The triangle identities for $\eta^{\bullet}$ and $\varepsilon^{\bullet}$, together with the 2-out-of-3 property, imply that the components of $\eta^{\bullet}$ are also stable equivalences. This completes the proof. □

Recall from Definition 3.5.1 the wide subcategory

$$\mathcal{S}^{\mathsf{F}} = \mathsf{F}^{-1}(\mathcal{S}^I) \subset \mathsf{Multicat}$$

of F-stable equivalences. Also recall the 2-functor in (C.4.14)

$$(-)_+ : \mathsf{Multicat} \longrightarrow \mathsf{Multicat}_*$$

given by adjoining a basepoint. Corollary 4.7.4 below uses the underlying functor of this 2-functor.

**Corollary 4.7.4.** *The functor*

$$(-)_+ : \left(\mathsf{Multicat}, \mathcal{S}^{\mathsf{F}}\right) \xrightarrow{\;\sim\;} \left(\mathsf{Multicat}_*, \mathcal{S}_\bullet\right)$$

*is an equivalence of homotopy theories, with $\mathcal{S}_\bullet$ as in Definition 4.7.1.*

*Proof.* The result follows by the 2-out-of-3 property for equivalences of homotopy theories, using Theorems 3.5.3 and 4.7.3 together with Theorem 4.3.11 (ii).    □

## 4.8. Homotopy Theory of $\mathcal{M}\underline{1}$-Modules

**Theorem 4.8.1.** *The adjunction in Proposition 1.3.19*

$$(4.8.2) \qquad \left(\mathsf{Multicat}_*, \mathcal{S}_\bullet\right) \underset{\mathsf{U}_{\mathcal{M}\underline{1}}}{\overset{\mathcal{M}\underline{1} \wedge -}{\rightleftarrows}} \bot \left(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}}\right)$$

*is an adjoint equivalence of homotopy theories, with $\mathcal{S}_\bullet$ and $\mathcal{S}^{\mathcal{M}\underline{1}}$ as in Definition 4.7.1.*

*Proof.* The left adjoint, $\mathcal{M}\underline{1} \wedge -$, creates $\mathsf{F}_\bullet$-stable equivalences because, by Theorem 4.4.1 (iii), there is a 2-natural isomorphism

$$\mathsf{F}_\bullet \cong \mathsf{F}_{\mathcal{M}\underline{1}} \circ (\mathcal{M}\underline{1} \wedge -).$$

The right adjoint, $\mathsf{U}_{\mathcal{M}\underline{1}}$, creates $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalences by definition of $\mathsf{F}_{\mathcal{M}\underline{1}}$ (4.4.3).

Recalling Explanation 1.3.20, the counit of the adjunction (4.8.2) is a componentwise isomorphism and, therefore, a componentwise $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalence. Using the 2-out-of-3 property and the fact that $\mathcal{M}\underline{1} \wedge -$ creates stable equivalences, the left triangle identity then shows that the unit of (4.8.2) is a componentwise $\mathsf{F}_\bullet$-stable equivalence.    □

**Theorem 4.8.3.** *The adjunction in Theorem 4.4.1*

$$\left(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}}\right) \underset{\mathsf{End}_{\mathcal{M}\underline{1}}}{\overset{\mathsf{F}_{\mathcal{M}\underline{1}}}{\rightleftarrows}} \bot \left(\mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I\right)$$

*is an adjoint equivalence of homotopy theories, with $\mathcal{S}^{\mathcal{M}\underline{1}}$ as in Definition 4.7.1.*

*Proof.* Recall from Theorem 4.7.3 that the components of $\eta^\bullet$ and $\varepsilon^\bullet$ are stable equivalences. By Explanation 4.4.6 we have

$$\mathsf{U}_{\mathcal{M}\underline{1}}\eta_\mathsf{M}^{\mathcal{M}\underline{1}} = \eta^\bullet_{\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}}.$$

This shows that $\eta^{\mathcal{M}\underline{1}}$ is a componentwise $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalence because $\mathsf{U}_{\mathcal{M}\underline{1}}$ creates $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalences (4.4.3). Likewise, $\varepsilon^{\mathcal{M}\underline{1}}$ is a componentwise stable equivalence by (4.4.9).    □

CHAPTER 5

# Multiplicative Homotopy Theory of Pointed Multicategories and $\mathcal{M}\underline{1}$-Modules

The main results of this chapter extend the equivalences of homotopy theories between categories of non-symmetric Q-algebras, for Q a small non-symmetric Cat-multicategory, in Theorem 3.5.5,

$$\mathsf{F}^{\mathsf{Q}} : \big(\mathsf{Multicat}^{\mathsf{Q}}, (\mathcal{S}^{\mathsf{F}})^{\mathsf{Q}}\big) \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \big((\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}, \mathcal{S}^{\mathsf{Q}}\big) : \mathsf{End}^{\mathsf{Q}},$$

from $\mathsf{Multicat}^{\mathsf{Q}}$ to $\mathsf{Multicat}^{\mathsf{Q}}_{*}$ and $(\mathsf{Mod}^{\mathcal{M}\underline{1}})^{\mathsf{Q}}$. The three pairs of functors in the diagram below are shown to be inverse equivalences of homotopy theories in Theorems 5.4.1, 5.5.12, and 5.5.14.

(5.0.1)



**Connection with Other Chapters.** In Chapter 12 we show, after developing the relevant basic theory in the intervening chapters, that the multifunctors $\mathsf{F}_{\bullet}$ and $\mathsf{F}_{\mathcal{M}\underline{1}}$ also induce equivalences of homotopy theories between enriched diagram categories.

**Background.** Recall Definition 2.1.8 for inverse equivalences of homotopy theories. Chapter 4 describes the underlying adjoint pairs $\mathsf{F}_{\bullet} \dashv \mathsf{End}_{\bullet}$ and $\mathsf{F}_{\mathcal{M}\underline{1}} \dashv \mathsf{End}_{\mathcal{M}\underline{1}}$ in Theorems 4.3.11 and 4.4.1, respectively. Note that these adjunctions are restricted to $\mathsf{PermCat}^{\mathsf{st}}$ in Chapter 4, but the natural domain of $\mathsf{End}_{\bullet}$, and hence also $\mathsf{End}_{\mathcal{M}\underline{1}}$, is $\mathsf{PermCat}^{\mathsf{su}}$ (see Explanation 1.4.32). The necessity of expanding to $\mathsf{PermCat}^{\mathsf{su}}$ for the codomain of $\mathsf{F}_{\bullet}$ as a non-symmetric Cat-multifunctor is described in Explanation 5.2.5. Also recall Remark 4.6.12 for further technical remarks about how Proposition 4.6.6 is used below. The pair $(\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}$ is described in Proposition 1.3.19.

**Chapter Summary.** Section 5.1 defines the strong multilinear functors $\mathsf{F}^{n}_{\bullet}$ with their linearity constraints $(\mathsf{F}^{n}_{\bullet})^{2}_{p}$. Section 5.2 uses $\mathsf{F}^{n}_{\bullet}$ to define $\mathsf{F}_{\bullet}$ on multimorphism categories, showing that $\mathsf{F}_{\bullet}$ is a non-symmetric Cat-multifunctor. Section 5.3 develops the (non-symmetric) Cat-multinaturality of $\eta^{\bullet}$ and $\varrho^{\bullet}$. Sections 5.4 and 5.5 develop the three inverse equivalences of homotopy theories shown in the diagram (5.0.1) above. Here is a summary table.

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| $n$-linear functors $\big(\mathsf{F}^n_\bullet,\,(\mathsf{F}^n_\bullet)^2_p\big)$ | 5.1.1, 5.1.7 |
|---|---|
| non-symmetric Cat multifunctor $\mathsf{F}_\bullet$ | 5.2.2, 5.2.6 |
| non-symmetric Cat-multinatural transformations | |
| $\mathsf{p} : \mathsf{F}\mathsf{U}_\bullet \longrightarrow \mathsf{F}_\bullet$ | 5.2.8 |
| $\eta^\bullet : 1_{\mathsf{Multicat}_*} \longrightarrow \mathsf{End}_\bullet\mathsf{F}_\bullet$ | 5.3.2 |
| $\varrho^\bullet : 1_{\mathsf{PermCat}^{\mathsf{su}}} \longrightarrow \mathsf{F}_\bullet\mathsf{End}_\bullet$ | 5.3.3 |
| inverse equivalences of homotopy theories | |
| $\big(\mathsf{F}^\mathsf{Q}_\bullet,\,\mathsf{End}^\mathsf{Q}_\bullet\big)$ | 5.4.1 |
| $\big(\mathsf{F}^\mathsf{Q}_{\mathcal{M}\underline{1}},\,\mathsf{End}^\mathsf{Q}_{\mathcal{M}\underline{1}}\big)$ | 5.5.12 |
| $\big((\mathcal{M}\underline{1}\wedge-)^\mathsf{Q},\,\mathsf{U}^\mathsf{Q}_{\mathcal{M}\underline{1}}\big)$ | 5.5.14 |

We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

## 5.1. The Strong Multilinear Functor $\mathsf{F}^n_\bullet$

This section defines pointed variants of the strong multilinear functors $\mathsf{F}^n$ from Proposition 3.4.21. These are determined by commutativity with the strict symmetric monoidal functors

$$\mathsf{p}_\mathsf{M} : \mathsf{F}\mathsf{M} \longrightarrow \mathsf{F}_\bullet\mathsf{M}$$

for small pointed multicategories M, from Definition 4.2.1.

To begin, the definition of underlying functors $\mathsf{F}^n_\bullet$ depends on the tensor products of objects (3.4.4) and morphisms (3.4.13) defining $\mathsf{F}^n$. Also recall the universal morphism $\varpi$ (1.2.3) from a tensor product of small pointed multicategories to their corresponding smash product. For the case $n = 0$, recall from Example 4.5.1 that $\mathsf{F}_\bullet(\mathsf{I}_+) \cong \mathsf{F}(\mathsf{I})$ is isomorphic to the permutation category.

**Definition 5.1.1** (The Functors $\mathsf{F}^n_\bullet$)**.** Suppose $\langle\mathsf{M}\rangle = \langle\mathsf{M}_i\rangle^n_{i=1}$ are small pointed multicategories. We define the data of a functor $\mathsf{F}^n_\bullet$ such that the following diagram commutes.

(5.1.2)

$$
\begin{array}{ccc}
\prod_{i=1}^n \mathsf{F}\mathsf{M}_i & \xrightarrow{\quad \mathsf{F}^n \quad} & \mathsf{F}\big(\otimes_{i=1}^n \mathsf{M}_i\big) \\
\Big\downarrow{\scriptstyle \prod_i \mathsf{p}_{\mathsf{M}_i}} & & \Big\downarrow{\scriptstyle \mathsf{F}\varpi} \\
 & & \mathsf{F}\big(\wedge_{i=1}^n \mathsf{M}_i\big) \\
 & & \Big\downarrow{\scriptstyle \mathsf{p}_{(\wedge_i \mathsf{M}_i)}} \\
\prod_{i=1}^n \mathsf{F}_\bullet\mathsf{M}_i & \dashrightarrow[\mathsf{F}^n_\bullet] & \mathsf{F}_\bullet\big(\wedge_{i=1}^n \mathsf{M}_i\big)
\end{array}
$$

If $n = 0$, then recall $\mathsf{T}$ is the empty product in Multicat and $\mathsf{I}_+$ is the empty smash product in $\mathsf{Multicat}_*$. The functor

$$\mathsf{F}^0_\bullet : \mathbf{1} \longrightarrow \mathsf{F}_\bullet(\mathsf{I}_+)$$

is defined by the choice of object $[(1)] \in \mathsf{F}_\bullet(\mathsf{I}_+)$, where $(1)$ is the length-one tuple consisting of the unique object of the initial operad, I.

Now suppose $n > 0$.

**Assignment on Objects:** Define $\mathsf{F}^n_\bullet$ on objects by

(5.1.3) $$\mathsf{F}^n_\bullet\big([\langle x^1\rangle],\ldots,[\langle x^n\rangle]\big) = [\langle x^{1\cdots n}\rangle] \in \mathsf{F}_\bullet\big(\wedge_{i=1}^n \mathsf{M}_i\big),$$

using the tensor product of tuples

$$(5.1.4) \qquad \langle x^{1\cdots n}\rangle = \left\langle \cdots \left\langle x^{1\cdots n}_{j_1,\ldots,j_n}\right\rangle^{r_1}_{j_1=1} \cdots \right\rangle^{r_n}_{j_n=1}$$

from (3.4.4). This assignment is well defined on $\overset{\text{ob}}{\sim}$-equivalence classes because the term

$$x^{1\cdots n}_{j_1,\ldots,j_n} = \langle x^i_{j_i}\rangle^n_{i=1} \quad \text{in} \quad \otimes^n_{i=1} \mathsf{M}_i$$

is sent to the basepoint object of $\wedge_i \mathsf{M}_i$ whenever any $x^i_{j_i} = * \in \mathsf{M}_i$. Thus, any basepoint terms of $\langle x^i\rangle$ produce corresponding basepoint terms of $\langle x^{1\cdots n}\rangle$.

**Assignment on Morphisms:** For morphisms, suppose given length-one sequences

$$\left[\left(f^i, \langle \phi^i\rangle\right)\right] : [\langle x^i\rangle] \longrightarrow [\langle y^i\rangle] \quad \text{in} \quad \mathsf{F}_\bullet\mathsf{M}_i.$$

Define

$$(5.1.5) \qquad \mathsf{F}^n_\bullet\left(\left[(f^1, \langle\phi^1\rangle)\right],\ldots,\left[(f^n, \langle\phi^n\rangle)\right]\right) = \left[(f^{1\cdots n}, \langle\phi^{1\cdots n}\rangle)\right],$$

using the index map $f^{1\cdots n}$ and the tensor product of morphisms

$$(5.1.6) \qquad \langle\phi^{1\cdots n}\rangle = \otimes^n_{i=1}\langle\phi^i\rangle$$

from (3.4.13). This assignment is well defined on $\overset{2}{\sim}$-equivalence classes because the operation

$$\otimes^n_{i=1} \phi^i_{k_i} \quad \text{in} \quad \otimes^n_{i=1} \mathsf{M}_i$$

is sent to a basepoint operation of $\wedge_i \mathsf{M}_i$ whenever any $\phi^i_{k_i}$ is a basepoint operation in $\mathsf{M}_i$.

Composition in $\mathsf{F}_\bullet\mathsf{M}_i$ is given by concatenation of representative sequences, so the definition of $\mathsf{F}^n_\bullet$ on morphisms with representative sequences of length greater than one is determined by the desired functoriality of $\mathsf{F}^n_\bullet$. These assignments are well defined on $\overset{1}{\sim}$-equivalence classes by functoriality of $\mathsf{F}^n$, $\mathsf{p}$, and $\mathsf{F}_\bullet(\varpi)$.

The assignments above show that (5.1.2) commutes on objects and morphisms. The definition of $\mathsf{F}^n_\bullet$ on morphisms implies that it is functorial. $\diamond$

**Definition 5.1.7** (Linearity Constraints of $\mathsf{F}^n_\bullet$). Suppose $\langle\mathsf{M}\rangle = \langle\mathsf{M}_i\rangle^n_{i=1}$ is a tuple of small pointed multicategories. For each $p \in \{1,\ldots,n\}$, we define the data of a natural transformation $(\mathsf{F}^n_\bullet)^2_p$ with components determined on representatives by $(\mathsf{F}^n)^2_p$. That is, we define

$$(5.1.8) \qquad (\mathsf{F}^n_\bullet)^2_p = \left[(\mathsf{F}^n)^2_p\right] = \left[\left(\rho_{r_p,\hat{r}_p}, \langle 1\rangle\right)\right] : [\langle x^{1\cdots n}\rangle] \oplus [\langle\hat{x}^{1\cdots n}\rangle] \overset{\cong}{\longrightarrow} [\langle\tilde{x}^{1\cdots n}\rangle]$$

for

$$\langle\hat{x}^p\rangle \in \mathsf{FM}_p, \quad \text{and} \quad \langle x^{1\cdots n}\rangle, \langle\hat{x}^{1\cdots n}\rangle, \langle\tilde{x}^{1\cdots n}\rangle \in \mathsf{F}\left(\otimes^n_{i=1}\mathsf{M}_i\right)$$

as in (3.4.17). The components (5.1.8) are well defined because the operations determining $(\mathsf{F}^n)^2_p$ are colored units and, therefore, $\overset{\text{ob}}{\sim}$-equivalent representatives in the domain or codomain of (5.1.8) will result in $\overset{2}{\sim}$-equivalent components $(\mathsf{F}^n)^2_p$.

Naturality of (5.1.8) follows, for length-one morphism sequences, from that of $(\mathsf{F}^n)_p^2$. This implies naturality with respect to general $\overset{\mathrm{ob}}{\sim}$-composable morphism sequences because composition in $\mathsf{F}_\bullet\big(\wedge_{i=1}^n \mathsf{M}_i\big)$ is given by concatenation.            ◇

Now we show that the constructions above assemble to form multilinear functors (Definition 1.4.2).

**Proposition 5.1.9.** *Suppose* $\langle \mathsf{M} \rangle = \langle \mathsf{M}_i \rangle_{i=1}^n$ *is a tuple of small pointed multicategories for* $n \geq 0$. *The data in Definitions 5.1.1 and 5.1.7*

$$(5.1.10) \qquad \Big( \mathsf{F}_\bullet^n, \big\langle (\mathsf{F}_\bullet^n)_p^2 \big\rangle_{p=1}^n \Big) : \textstyle\prod_{i=1}^n \mathsf{F}_\bullet \mathsf{M}_i \longrightarrow \mathsf{F}_\bullet \big( \wedge_{i=1}^n \mathsf{M}_i \big)$$

*form a strong $n$-linear functor.*

*Proof.* Each of the multilinearity axioms in Definition 1.4.2 follows from the corresponding axiom for $\Big( \mathsf{F}^n, \big\langle (\mathsf{F}^n)_p^2 \big\rangle_{p=1}^n \Big)$ on representatives. The components of each $(\mathsf{F}_\bullet^n)_p^2$ are isomorphisms because their representatives $(\mathsf{F}^n)_p^2$ are so.            □

**Proposition 5.1.11.** *For $n \geq 0$, the strong $n$-linear functor in (5.1.10),*

$$\Big( \mathsf{F}_\bullet^n, \big\langle (\mathsf{F}_\bullet^n)_p^2 \big\rangle_{p=1}^n \Big) : \textstyle\prod_{i=1}^n \mathsf{F}_\bullet \mathsf{M}_i \longrightarrow \mathsf{F}_\bullet \big( \wedge_{i=1}^n \mathsf{M}_i \big),$$

*is 2-natural with respect to pointed multifunctors and pointed multinatural transformations.*

*Proof.* Suppose given a tuple $\langle H \rangle = \langle H_i \rangle_{i=1}^n$ of pointed multifunctors

$$H_i : \mathsf{M}_i \longrightarrow \mathsf{N}_i \quad \text{for} \quad 1 \leq i \leq n.$$

Naturality of $\mathsf{F}_\bullet^n$ with respect to $\langle H_i \rangle$ is verified by commutativity of the inner rectangle in the following diagram.



The functors $\mathsf{p}$ are surjective on objects and on length-one tuples of morphisms. Since morphisms in each $\mathsf{F}_\bullet \mathsf{M}_i$ are generated under composition (concatenation) by those of length one, it suffices to verify

$$\mathsf{F}_\bullet^n \circ \big( \textstyle\prod_i \mathsf{F}_\bullet H_i \big) \circ \big( \textstyle\prod_i \mathsf{p}_{\mathsf{M}_i} \big) = \mathsf{F}_\bullet \big( \wedge_i H_i \big) \circ \mathsf{F}_\bullet^n \circ \big( \textstyle\prod_i \mathsf{p}_{\mathsf{M}_i} \big).$$

The equality above holds by commutativity of the following in (5.1.12).

- The trapezoids at left and right commute by (5.1.2).

- The bottom trapezoid commutes by naturality of $\varpi$ (1.2.3).
- The remaining two trapezoid regions commute by naturality of $\mathsf{p}$ (4.2.4).
- The outer rectangle commutes by naturality of $\mathsf{F}^n$.

This shows that $\mathsf{F}^n_\bullet$ is natural with respect to $\langle H \rangle$. A similar analysis for pointed multinatural transformations shows that $\mathsf{F}^n_\bullet$ is 2-natural. $\qquad\square$

## 5.2. Pointed Free Permutative Category as a Non-Symmetric Cat-Multifunctor

In this section we extend the 2-functor

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{st}}$$

in Theorem 4.1.17 to multimorphism categories.

**Convention 5.2.1.** To avoid confusion in Definition 5.2.2 below, for small pointed multicategories M and N, we denote by

$$\overline{\mathsf{F}}_\bullet : \mathsf{Multicat}_*(\mathsf{M},\mathsf{N}) \longrightarrow \mathsf{PermCat}^{\mathsf{st}}(\mathsf{F}_\bullet\mathsf{M}, \mathsf{F}_\bullet\mathsf{N})$$

the assignment of $\mathsf{F}_\bullet$ on pointed multifunctors and pointed multinatural transformations as in Definitions 4.1.12 and 4.1.15, respectively. This is the pointed analog of Convention 3.4.25 for F. $\qquad\diamond$

In (5.2.4) below, we use the multilinear functor $\mathsf{F}^n_\bullet$ (Proposition 5.1.9).

**Definition 5.2.2.** Suppose $\langle \mathsf{M} \rangle = \langle \mathsf{M}_i \rangle_{i=1}^n$ and N are small pointed multicategories. We define a functor between multimorphism categories

(5.2.3) $\qquad \mathsf{F}_\bullet : \mathsf{Multicat}_*\big(\langle \mathsf{M} \rangle \,;\, \mathsf{N}\big) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\big(\langle \mathsf{F}_\bullet\mathsf{M} \rangle \,;\, \mathsf{F}_\bullet\mathsf{N}\big)$

as follows. Suppose given pointed multifunctors $H$ and $K$ and a pointed multinatural transformation $\theta$ as in the diagram below.

$$\langle \mathsf{M} \rangle \underset{K}{\overset{H}{\rightrightarrows}} \Downarrow\theta \; \mathsf{N}$$

Then $\mathsf{F}_\bullet$ sends these data to the following composite $n$-linear functors and whiskering.

(5.2.4) $\qquad \langle \mathsf{F}_\bullet\mathsf{M} \rangle \xrightarrow{\;\mathsf{F}^n_\bullet\;} \mathsf{F}_\bullet\big( \wedge_{i=1}^n \mathsf{M}_i \big) \underset{\overline{\mathsf{F}}_\bullet K}{\overset{\overline{\mathsf{F}}_\bullet H}{\rightrightarrows}} \Downarrow\overline{\mathsf{F}}_\bullet\theta \; \mathsf{F}_\bullet\mathsf{N}$

This finishes the definition of the multimorphism functor $\mathsf{F}_\bullet$. $\qquad\diamond$

**Explanation 5.2.5** (Codomain Not Strict). In (5.2.3) above and Theorem 5.2.6 below, we stress that the codomain uses $\mathsf{PermCat}^{\mathsf{su}}$ and *not* $\mathsf{PermCat}^{\mathsf{st}}$, which is the codomain of $\overline{\mathsf{F}}_\bullet$ in Convention 5.2.1. The reason is that the definition of $\mathsf{F}_\bullet$ in (5.2.4) involves the strong $n$-linear functor $\mathsf{F}^n_\bullet$ in (5.1.10). The latter is *not* strict because the components of its linearity constraints $(\mathsf{F}^n_\bullet)^2_p$ in Definition 5.1.7 are not identities in general. $\qquad\diamond$

**Theorem 5.2.6.** *There is a non-symmetric* Cat-*multifunctor*

(5.2.7) $\qquad\qquad\qquad\qquad \mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$

*with*

- *object assignment in Definition 4.1.11 and*
- *multimorphism functors in Definition 5.2.2.*

*Moreover,* $F_\bullet$ *extends the 2-functor in Theorem 4.1.17.*

*Proof.* To see that $F_\bullet$ preserves units, note that $F_\bullet^1$ is the identity monoidal functor. Since $\overline{F}_\bullet$ is functorial, we have

$$F_\bullet(1_M) = 1_{F_\bullet M}$$

for each small pointed multicategory M.

To see that $F_\bullet$ preserves composition, suppose given

$$H_i \in \mathsf{Multicat}_*\big(\langle M_i \rangle ; M_i'\big) \quad \text{for} \quad 1 \le i \le n, \quad \text{and}$$

$$H' \in \mathsf{Multicat}_*\big(\langle M' \rangle ; M''\big).$$

Let $k_i$ denote the length of $\langle M_i \rangle$. The two multilinear functors

$$F_\bullet\big(\gamma(H'; \langle H \rangle)\big) \quad \text{and} \quad \gamma\big(F_\bullet H' ; \langle F_\bullet H \rangle\big)$$

are given by the two composites around the boundary in the following diagram, where the unlabeled isomorphisms are given by reordering terms.

$$
\begin{array}{ccc}
\prod_{i,j} F_\bullet M_{i,j} & \xrightarrow{\ \cong\ } & \prod_i \prod_j F_\bullet M_{i,j} \\
\end{array}
$$

In the above diagram, the two composites around the middle rectangle are equal as multilinear functors by naturality of $F_\bullet^n$ (Proposition 5.1.11) with respect to the multifunctors $H_i$. The rectangle at left commutes on objects and length-one sequences of morphisms because the corresponding diagram for F and $(F^n, (F^n)_p^2)$ commutes (Theorem 3.4.31 c.f., proof of [**JY22a**, 8.1]). The commutativity for general morphisms then follows from functoriality of $F_\bullet^n$. A similar diagram for multinatural transformations $H_i \longrightarrow K_i$ and $H_i' \longrightarrow K_i'$ commutes by the 2-naturality of $F_\bullet^n$.  $\square$

Recall (non-symmetric) Cat-multinatural transformation from Explanation C.2.2.

**Lemma 5.2.8.** *The 2-natural transformation* p *of Proposition 4.2.3 extends to a non-symmetric* Cat-*multinatural transformation*

$$\mathsf{Multicat}_* \underset{F_\bullet}{\overset{F \circ U_\bullet}{\rightrightarrows}} \Downarrow p \ \mathsf{PermCat^{su}}.$$

*Proof.* Recall from Explanation 1.2.14 that the forgetful

$$U_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{Multicat}$$

is a Cat-multifunctor with $n$-ary multimorphism functors

$$\mathsf{Multicat}_*\big(\langle M \rangle ; N\big) \longrightarrow \mathsf{Multicat}\big(U_\bullet \langle M \rangle ; U_\bullet N\big)$$

given by composition and whiskering with $\omega : \otimes_i \mathsf{M}_i \longrightarrow \wedge_i \mathsf{M}_i$ (1.2.13). The Cat-multifunctoriality of $\mathsf{F}$ is described in Theorem 3.4.31 and its multimorphism functors are given by precomposition and whiskering with the $n$-linear functors $\mathsf{F}^n$ from Definition 3.4.14.

For objects of $\mathsf{Multicat}_*\big(\langle \mathsf{M} \rangle ; \mathsf{N}\big)$, i.e., multifunctors

$$H : \wedge_i \mathsf{M}_i \longrightarrow \mathsf{N},$$

the object Cat-naturality diagram (C.2.8) for $\mathsf{p}$ is the following. In this diagram, we suppress $\mathsf{U}_\bullet$ except in the lower left arrow.

$$
\begin{array}{ccc}
\prod_i \mathsf{FM}_i & \xrightarrow{\ \prod_i \mathsf{p}_{\mathsf{M}_i}\ } & \prod_i \mathsf{F}_\bullet \mathsf{M}_i \\
{\scriptstyle \mathsf{F}^n}\big\downarrow & & \big\downarrow{\scriptstyle \mathsf{F}^n_\bullet} \\
\mathsf{F}(\otimes_i \mathsf{M}_i) \ \xrightarrow{\ \overline{\mathsf{F}}\omega\ } \ \mathsf{F}(\wedge_i \mathsf{M}_i) \ \xrightarrow{\ \mathsf{p}_{\wedge_i \mathsf{M}_i}\ } \ \mathsf{F}_\bullet(\wedge_i \mathsf{M}_i) \\
{\scriptstyle \overline{\mathsf{F}}(\mathsf{U}_\bullet H)}\big\downarrow \quad {\scriptstyle \overline{\mathsf{F}}H} & & \big\downarrow{\scriptstyle \mathsf{F}_\bullet H} \\
\mathsf{FN} \ \xrightarrow{\qquad \mathsf{p}_\mathsf{N} \qquad} \ \mathsf{F}_\bullet \mathsf{N}
\end{array}
$$

In the above diagram, the top rectangle commutes by construction (5.1.2), the lower left triangle commutes by definition of $\mathsf{U}_\bullet H$, and the lower right trapezoid commutes by naturality of $\mathsf{p}$. Therefore, the object Cat-naturality condition (C.2.8) holds for each $H \in \mathsf{Multicat}_*\big(\langle \mathsf{M} \rangle ; \mathsf{N}\big)$. For morphisms, i.e., multinatural transformations

$$\theta : H \longrightarrow H' : \wedge_i \mathsf{M}_i \longrightarrow \mathsf{N},$$

a similar argument applies to verify the morphism Cat-naturality condition (C.2.9), using 2-naturality of $\mathsf{p}$. $\qquad\square$

**Example 5.2.9** (Partition Products). Recall from Definition 1.3.4 the partition product multifunctor

$$\Pi_{a,b} : \mathcal{M}a \wedge \mathcal{M}b \longrightarrow \mathcal{M}(a \wedge b)$$

for finite pointed sets $a$ and $b$. It is given on objects by the Cartesian product, and preserves operations because a pairwise Cartesian product of partitions $\langle s \rangle$ and $\langle t \rangle$ provides a partition of $(\cup_i s_i) \times (\cup_j t_j)$. Using the description from Example 4.5.22, and omitting the forgetful $\mathsf{U}_{\mathcal{M}\underline{1}}$, the composite

$$\mathsf{F}_\bullet(\mathcal{M}a) \times \mathsf{F}_\bullet(\mathcal{M}b) \ \xrightarrow{\ \mathsf{F}^2_\bullet\ } \ \mathsf{F}_\bullet(\mathcal{M}a \wedge \mathcal{M}b) \ \xrightarrow{\ \mathsf{F}_\bullet(\Pi_{a,b})\ } \ \mathsf{F}_\bullet(\mathcal{M}(a \wedge b))$$

is given as follows for two tuples of nonempty subsets $s^1_i \subset a^\flat$ and $s^2_j \subset b^\flat$:

$$[\langle s^1 \rangle], [\langle s^2 \rangle] \longmapsto [\langle s^{12} \rangle] \longmapsto [\langle \langle s^1_i \times s^2_j \rangle_i \rangle_j]. \qquad \diamond$$

## 5.3. Comparison Transformations

Consider the diagram

(5.3.1) $$\mathsf{F}_\bullet : \mathsf{Multicat}_* \ \underset{\longleftarrow}{\overset{\longrightarrow}{\rule{0pt}{0pt}}} \ \mathsf{PermCat}^{\mathsf{su}} : \mathsf{End}_\bullet$$

consisting of the following data.

- $\mathsf{Multicat}_*$ is the Cat-multicategory in Explanation 1.2.9.
- $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-multicategory in Theorem 1.4.29.
- $\mathsf{End}_\bullet$ is the Cat-multifunctor in Explanation 1.4.32.
- $\mathsf{F}_\bullet$ is the non-symmetric Cat-multifunctor in Theorem 5.2.6.

Recall from Theorem 4.3.11 that the 2-adjunction

$$\mathsf{Multicat}_* \underset{\mathsf{End}_\bullet}{\overset{\mathsf{F}_\bullet}{\rightleftarrows}} \mathsf{PermCat}^{\mathsf{st}}$$

has unit

$$\eta^\bullet : 1_{\mathsf{Multicat}_*} \longrightarrow \mathsf{End}_\bullet \mathsf{F}_\bullet$$

in Definition 4.3.1. Recall *(non-symmetric)* $\mathsf{Cat}$-*multinatural transformation* from Explanation C.2.2.

**Lemma 5.3.2.** *In the context of (5.3.1),*

$$\eta^\bullet : 1_{\mathsf{Multicat}_*} \longrightarrow \mathsf{End}_\bullet \mathsf{F}_\bullet$$

*is a non-symmetric* $\mathsf{Cat}$-*multinatural transformation.*

*Proof.* In this proof we will omit the adjective non-symmetric for the Cat-multinatural transformations under discussion. Recall from (4.3.2) that $\eta^\bullet$ is the composite of $\mathsf{End}_\bullet(\mathsf{p}) = \mathsf{End}_\bullet * \mathsf{p}$ with $\eta$. Thus, Cat-multinaturality of $\eta^\bullet$ follows from that of $\eta$ (Lemma 3.4.34) and that of $\mathsf{p} * \mathsf{End}_\bullet$ (Lemma 5.2.8 and Explanation 1.4.32). Indeed, as a 2-natural transformation, $\eta^\bullet$ satisfies the following equality of pasting diagrams.



The right hand side is also a diagram of Cat-multinatural transformations, and thus $\mathsf{U}_\bullet * \eta^\bullet$ is Cat-multinatural. Now since

 (i) the two conditions for Cat-multinaturality in Explanation C.2.2 consist of certain equalities (C.2.8) and (C.2.9) involving composition of operations, and

 (ii) and such equalities are detected on underlying multicategories, by applying $\mathsf{U}_\bullet$,

we conclude that $\eta^\bullet$ also satisfies the conditions for Cat-multinaturality.  $\square$

Recall from Lemma 4.6.5 the strictly unital symmetric monoidal functor

$$\varrho^\bullet_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{F}_\bullet \mathsf{End}_\bullet \mathsf{C}$$

for each small permutative category $\mathsf{C}$.

**Lemma 5.3.3.** *In the context of (5.3.1),*

$$\varrho^\bullet : 1_{\mathsf{PermCat}^{\mathsf{su}}} \longrightarrow \mathsf{F}_\bullet \mathsf{End}_\bullet$$

*is a non-symmetric* $\mathsf{Cat}$-*multinatural transformation.*

*Proof.* For objects of $\mathsf{PermCat}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle ; \mathsf{D}\big)$, i.e., multilinear functors

$$Q : \textstyle\prod_{i=1}^n \mathsf{C}_i \longrightarrow \mathsf{D},$$

the object Cat-naturality diagram (C.2.8) for $\varrho^{\bullet}$ is the following, where we use the abbreviation $\mathsf{E}_{\bullet} = \mathrm{End}_{\bullet \bullet}$.

(5.3.4)

$$
\begin{array}{ccccc}
\prod_i \mathsf{C}_i & \xrightarrow{\prod_i \varrho_{\mathsf{C}_i}} & \prod_i \mathsf{FE}_{\bullet}\mathsf{C}_i & \xrightarrow{\prod_i \mathsf{PE}_{\bullet}\mathsf{C}_i} & \prod_i \mathsf{F}_{\bullet}\mathsf{E}_{\bullet}\mathsf{C}_i \\
\Big\downarrow Q & & \Big\downarrow \mathsf{F}^n & & \Big\downarrow \mathsf{F}^n_{\bullet} \\
& & \mathsf{F}(\otimes_i \mathsf{E}_{\bullet}\mathsf{C}_i) & & \mathsf{F}_{\bullet}(\wedge_i \mathsf{E}_{\bullet}\mathsf{C}_i) \\
& & \Big\downarrow \mathsf{F}(\mathsf{E}_{\bullet}Q) & & \Big\downarrow \mathsf{F}_{\bullet}(\mathsf{E}_{\bullet}Q) \\
\mathsf{D} & \xrightarrow{\varrho_{\mathsf{D}}} & \mathsf{FE}_{\bullet}\mathsf{D} & \xrightarrow{\mathsf{PE}_{\bullet}\mathsf{D}} & \mathsf{F}_{\bullet}\mathsf{E}_{\bullet}\mathsf{D}
\end{array}
$$

In the above diagram, the rectangle at right commutes by multifunctoriality of $\mathsf{p}$ (Lemma 5.2.8). To verify that the rectangle at left commutes, we first consider the underlying functors and then check linearity constraints.

For morphisms

$$\phi^i : b^i \longrightarrow c^i \quad \text{in} \quad \mathsf{C}_i,$$

the composite to $\mathsf{FE}_{\bullet}\mathsf{D}$ along the top and vertical middle functors of (5.3.4) is given as follows:

$$
\begin{aligned}
(\phi^1, \cdots, \phi^n) &\longmapsto \Big( \big(1_{\underline{1}}, (\phi^1)\big), \ldots \big(1_{\underline{1}}, (\phi^n)\big) \Big) && \text{by (3.3.2) for } \varrho, \\
&\longmapsto \Big( 1_{\underline{1}}, \big(\phi^{1\cdots n}\big) \Big) && \text{by (3.4.16) for } \mathsf{F}^n, \\
&\longmapsto \Big( 1_{\underline{1}}, \big(H\phi^{1\cdots n}\big) \Big) && \text{by (3.1.18) for } H = \mathsf{E}_{\bullet}Q,
\end{aligned}
$$

where $(\phi^{1\cdots n})$ is the length-one tuple consisting of the morphism $\phi^{1\cdots n} = \otimes_i \phi^i$ in $\otimes_i \mathsf{C}_i$. By (1.4.36) we have

$$H\phi^{1\cdots n} = (\mathsf{E}_{\bullet}Q)(\phi^1, \ldots, \phi^n) = Q(\phi^1, \ldots, \phi^n);$$

the linear constraints $Q^2$ do not appear because each $\phi^i$ is a unary operation in $\mathsf{E}_{\bullet}\mathsf{C}$. This verifies that the left side of (5.3.4) commutes as a diagram of underlying functors.

Next we consider the $j$th linearity constraints on the left side of (5.3.4), for $1 \le j \le n$. Suppose given

$$c^i \in \mathsf{C}_i \quad \text{for each} \quad 1 \le i \le n, \quad \text{and} \quad \hat{c}^j \in \mathsf{C}_j.$$

Recall the notation $\circ_j$ from Notation 1.4.1. We will use the following:

$$
\begin{aligned}
\tilde{c}^j &= c^j \oplus \hat{c}^j \quad \text{in} \quad \mathsf{C}_j, \\
\{c\} &= (c^1, \ldots, c^n) \quad \text{in} \quad \prod_i \mathsf{C}_i, \\
\{\hat{c}\} &= \{c \circ_j \hat{c}^j\} = (c^1, \ldots, \hat{c}^j, \ldots, c^n) \quad \text{in} \quad \prod_i \mathsf{C}_i, \quad \text{and} \\
\{\tilde{c}\} &= \{c \circ_j \tilde{c}^j\} = (c^1, \ldots, \tilde{c}^j, \ldots, c^n) \quad \text{in} \quad \prod_i \mathsf{C}_i.
\end{aligned}
$$

So each of $\{\hat{c}\}$ and $\{\tilde{c}\}$ has the same entries as $\{c\}$ in all but the $j$th position. Next we introduce further notation:

$$\Pi\varrho = \Pi_i \varrho_{\mathsf{C}_i},$$
$$(c^{1\cdots n}) = \otimes_i(c^i) = \mathsf{F}^n\big(\Pi\varrho\{c\}\big) \quad \text{in} \quad \mathsf{F}\big(\otimes_i \mathsf{E}.\mathsf{C}_i\big),$$
$$(\hat{c}^{1\cdots n}) = \mathsf{F}^n\big(\Pi\varrho\{\hat{c}\}\big) \quad \text{in} \quad \mathsf{F}\big(\otimes_i \mathsf{E}.\mathsf{C}_i\big),$$
$$(\tilde{c}^{1\cdots n}) = \mathsf{F}^n\big(\Pi\varrho\{\tilde{c}\}\big) \quad \text{in} \quad \mathsf{F}\big(\otimes_i \mathsf{E}.\mathsf{C}_i\big), \quad \text{and}$$
$$\overline{c} = \Pi\varrho\{c\} \circ_j (c^j, \hat{c}^j) \quad \text{in} \quad \Pi_i \mathsf{FE}.\mathsf{C}_i.$$

So each of $(c^{1\cdots n})$, $(\hat{c}^{1\cdots n})$, and $(\tilde{c}^{1\cdots n})$ is a length-one tuple and

$$\overline{c} = \big((c^1), \ldots, (c^j, \hat{c}^j), \ldots, (c^n)\big)$$

has the same entries as $\Pi\varrho\{c\}$ except entry $j$, which is $\Pi\varrho(c) \oplus \Pi\varrho(\hat{c}^j) = (c^j, \hat{c}^j)$. With this notation, note that $\mathsf{F}^n\overline{c}$ is the length-two tuple $(c^{1\cdots n}, \hat{c}^{1\cdots n})$.

Applying (1.4.27) and (3.4.18), the $j$th linearity constraint of $\mathsf{F}^n \circ \Pi\varrho$ at the objects $\{c\}$ and $\{\hat{c}\}$ is the composite in $\mathsf{F}^n(\otimes_i \mathsf{E}.\mathsf{C}_i)$

(5.3.5) $\qquad (c^{1\cdots n}, \hat{c}^{1\cdots n}) \xrightarrow{\ (\mathsf{F}^n)^2_j = 1\ } \mathsf{F}^n\overline{c} \xrightarrow{\ \mathsf{F}^n(\langle 1 \circ_j \varrho^2_{\mathsf{C}_j}\rangle)\ } (\tilde{c}^{1\cdots n}),$

where we note

$$\mathsf{F}^n\big(\Pi\varrho\{c\}\big) \oplus \mathsf{F}^n\big(\Pi\varrho\{\hat{c}\}\big) = (c^{1\cdots n}, \hat{c}^{1\cdots n}) \quad \text{and}$$
$$\mathsf{F}^n\big(\Pi\varrho\{\tilde{c}\}\big) = (\tilde{c}^{1\cdots n}).$$

Recall (3.3.11) for the monoidal constraint $\rho^2_{\mathsf{C}_j}$. Since $\mathsf{F}(\mathsf{E}.Q)$ is strict symmetric monoidal, the $j$th linearity constraint of the top and vertical middle composite of (5.3.4) is given by applying $\mathsf{F}(\mathsf{E}.Q)$ to (5.3.5), resulting in the following:

$$Q(c^{1\cdots n}) \oplus Q(\hat{c}^{1\cdots n}) \xrightarrow{\ 1\ } Q(c^{1\cdots n}) \oplus Q(\hat{c}^{1\cdots n}) \xrightarrow{\ (\iota_2, (Q^2_j))\ } Q(\tilde{c}^{1\cdots n}).$$

Now since

$$\big(\iota_2, (Q^2_j)\big) = \varrho_{\mathsf{D}}\big(Q^2_j\big),$$

we conclude that the $j$th linearity constraints of the two composites in the left half of (5.3.4) are equal. This completes verification of the object Cat-naturality condition (C.2.5) for $\varrho$.

Verification of the morphism Cat-naturality condition (C.2.6) is similar. Given a multilinear transformation

$$\theta : Q \longrightarrow Q',$$

one verifies

$$\mathsf{F}.(\mathsf{E}.\theta) * (\mathsf{F}^n_\bullet \circ \Pi_i \mathsf{p}_{\mathsf{E}.\mathsf{C}_i} \circ \Pi_i \varrho_{\mathsf{C}_i}) = \mathsf{p}_{\mathsf{E}.\mathsf{D}} * \mathsf{F}(\mathsf{E}.\theta) * (\mathsf{F}^n \circ \Pi_i \varrho_{\mathsf{C}_i}) = (\mathsf{p}_{\mathsf{E}.\mathsf{D}} \circ \varrho_{\mathsf{D}}) * \theta,$$

using Lemma 5.2.8 for the first equality and (1.4.37) with (3.1.20) for the second. This completes the proof. $\qquad\square$

## 5.4. Multiplicative Homotopy Theory of Pointed Multicategories

Recall from Definition 4.7.1 the wide subcategory

$$\mathcal{S}_\bullet \subset \mathsf{Multicat}_*$$

of $\mathsf{F}_\bullet$-stable equivalences created by

$$\mathsf{F}_\bullet : \big(\mathsf{Multicat}_* , \mathcal{S}_\bullet\big) \longrightarrow \big(\mathsf{PermCat}^{\mathsf{st}} , \mathcal{S}^I\big).$$

Recall from Definition 3.5.4 the notion of *non-symmetric algebras*.

Theorem 5.4.1 below considers the Cat-multifunctors (non-symmetric for $\mathsf{F}_\bullet$)

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \xrightleftharpoons{\quad\longrightarrow\quad} \mathsf{PermCat}^{\mathsf{su}} : \mathsf{End}_\bullet$$

in (5.3.1). This result simultaneously extends

- Theorem 3.5.5 from Multicat to $\mathsf{Multicat}_*$ and
- Theorem 4.7.3 to non-symmetric algebras.

As in Theorem 3.5.5, each of the two induced functors $\mathsf{F}_\bullet^{\mathsf{Q}}$ and $\mathsf{End}_\bullet^{\mathsf{Q}}$ is given by post-composition and whiskering with the respective functor.

**Theorem 5.4.1.** *Suppose* Q *is a small non-symmetric* Cat-*multicategory. In the context of (5.3.1), the induced functors*

$$\mathsf{F}_\bullet^{\mathsf{Q}} : \big(\mathsf{Multicat}_*^{\mathsf{Q}} , \mathcal{S}_\bullet^{\mathsf{Q}}\big) \xrightleftharpoons{\quad\sim\quad} \big((\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}} , \mathcal{S}^{\mathsf{Q}}\big) : \mathsf{End}_\bullet^{\mathsf{Q}}$$

*are inverse equivalences of homotopy theories in the sense of Definition 2.1.8.*

*Proof.* We first verify that $\mathsf{F}_\bullet^{\mathsf{Q}}$ and $\mathsf{End}_\bullet^{\mathsf{Q}}$ are relative functors. The functor $\mathsf{F}_\bullet^{\mathsf{Q}}$ is a relative functor because the stable equivalences $\mathcal{S}_\bullet^{\mathsf{Q}}$ are determined componentwise and $\mathsf{F}_\bullet$ creates the stable equivalences $\mathcal{S}_\bullet$.

In Theorem 4.7.3, $\mathsf{End}_\bullet$ is shown to be a relative functor with respect to *strict* symmetric monoidal functors. For more general strictly unital symmetric monoidal functors, recall from Lemma 4.6.5 the strictly unital

$$\varrho_\mathsf{C}^\bullet : \mathsf{C} \longrightarrow \mathsf{F}_\bullet\mathsf{End}_\bullet\mathsf{C} \quad \text{for} \quad \mathsf{C} \in \mathsf{PermCat}^{\mathsf{st}}.$$

Proposition 4.6.6 shows that each $\varrho_\mathsf{C}^\bullet$ is a right adjoint and, hence, a stable equivalence of permutative categories. Lemma 5.3.3 shows that $\varrho^\bullet$ is Cat-multinatural and, in particular, 2-natural with respect to strictly unital symmetric monoidal functors. Therefore, if $P : \mathsf{C} \longrightarrow \mathsf{D}$ is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$, then the naturality diagram

(5.4.2)

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\ \varrho_\mathsf{C}^\bullet\ }_{\sim} & \mathsf{F}_\bullet\mathsf{End}_\bullet\mathsf{C} \\
{\scriptstyle P}\downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \mathsf{F}_\bullet\mathsf{End}_\bullet P} \\
\mathsf{D} & \xrightarrow[\sim]{\ \varrho_\mathsf{D}^\bullet\ } & \mathsf{F}_\bullet\mathsf{End}_\bullet\mathsf{D}
\end{array}
$$

shows that $\mathsf{F}_\bullet\mathsf{End}_\bullet P$ is a stable equivalence by the 2-out-of-3 property. Since $\mathsf{F}_\bullet$ creates stable equivalences, this implies that $\mathsf{End}_\bullet P$ is a stable equivalence. Therefore, $\mathsf{End}_\bullet$ is a relative functor with respect to the strictly unital stable equivalences. Thus it follows that $\mathsf{End}_\bullet^{\mathsf{Q}}$ is also a relative functor with respect to $\mathcal{S}^{\mathsf{Q}}$.

Since $\varrho^\bullet$ is 2-natural and componentwise a stable equivalence, the induced $(\varrho^\bullet)^{\mathsf{Q}}$ is a natural stable equivalence

$$1_{(\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}} \xrightarrow[\sim]{\ (\varrho^\bullet)^{\mathsf{Q}}\ } \mathsf{F}_\bullet^{\mathsf{Q}}\mathsf{End}_\bullet^{\mathsf{Q}}.$$

For the reverse composite, recall from Lemma 5.3.2 that $\eta^\bullet$ is Cat-multinatural and, in particular, 2-natural with respect to pointed multifunctors. The components of $\eta^\bullet$ are shown to be stable equivalences in the proof of Theorem 4.7.3, and hence the induced $(\eta^\bullet)^\mathsf{Q}$ is a natural stable equivalence

$$1_{\mathsf{Multicat}_*^\mathsf{Q}} \xrightarrow[\sim]{(\eta^\bullet)^\mathsf{Q}} \mathsf{End}_\bullet^\mathsf{Q}\mathsf{F}_\bullet^\mathsf{Q}.$$

Thus, by Proposition 2.1.9, $\mathsf{F}_\bullet^\mathsf{Q}$ and $\mathsf{End}_\bullet^\mathsf{Q}$ are equivalences of homotopy theories between categories of non-symmetric Q-algebras. This completes the proof.  $\square$

## 5.5. Multiplicative Homotopy Theory of $\mathcal{M}\underline{1}$-Modules

Consider the diagram

(5.5.1)                          $\mathsf{F}_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \rightleftarrows \mathsf{PermCat}^{\mathsf{su}} : \mathsf{End}_{\mathcal{M}\underline{1}}$

consisting of the following.

- $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is the Cat-multicategory in Explanation 1.3.24.
- $\mathsf{PermCat}^{\mathsf{su}}$ is the Cat-multicategory in Theorem 1.4.29.
- $\mathsf{End}_{\mathcal{M}\underline{1}}$ is the Cat-multifunctor in Explanation 1.4.41.
- $\mathsf{F}_{\mathcal{M}\underline{1}}$ is the non-symmetric Cat-multifunctor given by the following composite.

(5.5.2)
$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\mathsf{U}_{\mathcal{M}\underline{1}}} \mathsf{Multicat}_* \xrightarrow{\mathsf{F}_\bullet} \mathsf{PermCat}^{\mathsf{su}}$$
with top arc $\mathsf{F}_{\mathcal{M}\underline{1}}$.

In (5.5.2),
  – $\mathsf{U}_{\mathcal{M}\underline{1}}$ is the Cat-multifunctor in Explanation 1.3.29, and
  – $\mathsf{F}_\bullet$ is the non-symmetric Cat-multifunctor in Theorem 5.2.6.

**Explanation 5.5.3** (2-Functors). Considering the underlying 2-functors, the diagram (5.5.2) factors as follows.

(5.5.4)
$$\mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\mathsf{U}_{\mathcal{M}\underline{1}}} \mathsf{Multicat}_* \xrightarrow{\mathsf{F}_\bullet} \mathsf{PermCat}^{\mathsf{st}} \xhookrightarrow{I} \mathsf{PermCat}^{\mathsf{su}}$$
with interior arrows $\mathsf{F}_{\mathcal{M}\underline{1}}$ and $\mathsf{F}_\bullet$ and top arc $\mathsf{F}_{\mathcal{M}\underline{1}}$.

- The interior $\mathsf{F}_{\mathcal{M}\underline{1}}$ is the 2-functor in (4.4.3).
- The interior $\mathsf{F}_\bullet$ is the 2-functor in Theorem 4.1.17.
- $I$ is the inclusion 2-functor in (A.2.6).

Thus the non-symmetric Cat-multifunctor $\mathsf{F}_{\mathcal{M}\underline{1}}$ in (5.5.2) extends the 2-functor $\mathsf{F}_{\mathcal{M}\underline{1}}$ in (4.4.3).  $\diamond$

### Comparison Transformations.

**Definition 5.5.5** (Comparing $\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}$ and the Identity). In the context of (5.5.1), we define the non-symmetric Cat-multinatural transformation

(5.5.6)                          $\eta^{\mathcal{M}\underline{1}} : 1_{\mathsf{Mod}^{\mathcal{M}\underline{1}}} \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}$

with, for each left $\mathcal{M}\underline{1}$-module M, the component morphism

$$\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}} : \mathsf{M} \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{M}$$
$$= \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\bullet}\, \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M} \quad \text{in} \quad \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

given by the unit in (4.4.7). ◇

**Explanation 5.5.7.** As explained in (4.4.7), the left $\mathcal{M}\underline{1}$-module morphism $\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}}$ is uniquely determined by its image in $\mathsf{Multicat}_*$:

$$\mathsf{U}_{\mathcal{M}\underline{1}}\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}} = \eta_{\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}}^{\bullet} : \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M} \longrightarrow \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\bullet}\, \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}$$
$$= \mathsf{End}_{\bullet}\, \mathsf{F}_{\bullet}\, \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{M}.$$

The components $\eta_{\mathsf{M}}^{\mathcal{M}\underline{1}}$ define a non-symmetric Cat-multinatural transformation $\eta^{\mathcal{M}\underline{1}}$ as in (5.5.6) by the following facts.

- $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a full sub-2-category of $\mathsf{Multicat}_*$ (Proposition 1.3.17 (4)).
- $\mathsf{U}_{\mathcal{M}\underline{1}}$ is a Cat-multifunctor (Explanation 1.3.29).
- $\eta^{\bullet}$ is a non-symmetric Cat-multinatural transformation (Lemma 5.3.2). ◇

**Definition 5.5.8** (Comparing $\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}$ and the Identity). In the context of (5.5.1), we define the non-symmetric Cat-multinatural transformation $\varrho^{\mathcal{M}\underline{1}}$,

(5.5.9)
$$\varrho^{\mathcal{M}\underline{1}} = \varrho^{\bullet} : 1_{\mathsf{PermCat}^{\mathsf{su}}} \longrightarrow \mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}$$
$$= \mathsf{F}_{\bullet}\, \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}} = \mathsf{F}_{\bullet}\, \mathsf{End}_{\bullet},$$

as $\varrho^{\bullet}$ in Lemma 5.3.3. ◇

**Explanation 5.5.10.** For each small permutative category C, the component strictly unital symmetric monoidal functor

$$\varrho_{\mathsf{C}}^{\mathcal{M}\underline{1}} = \varrho_{\mathsf{C}}^{\bullet} : \mathsf{C} \longrightarrow \mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C} = \mathsf{F}_{\bullet}\, \mathsf{End}_{\bullet}\mathsf{C}$$

is in Definition 4.6.1 and Explanation 4.6.3. ◇

The following result is the $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ analog of Lemma 4.6.13. It is used in Theorem 12.4.6, which is the $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ analog of Theorem 12.1.6.

**Lemma 5.5.11.** *Suppose C is a small permutative category. Then the two left $\mathcal{M}\underline{1}$-module morphisms below are equal.*

$$\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C} \xrightarrow[\mathsf{End}_{\mathcal{M}\underline{1}}\varrho_{\mathsf{C}}^{\mathcal{M}\underline{1}}]{\eta_{\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}}^{\mathcal{M}\underline{1}}} \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}$$

*Proof.* Left $\mathcal{M}\underline{1}$-module morphisms are determined by their underlying multifunctors. Thus it suffices to show that the two arrows in question are equal as multifunctors. In this case, the equality between them follows from Lemma 4.6.13, which gives the equality

$$\eta_{\mathsf{End}_{\bullet}\mathsf{C}}^{\bullet} = \mathsf{End}_{\bullet}\varrho_{\mathsf{C}}^{\bullet},$$

along with Example 1.3.15 about $\mathsf{End}_{\mathcal{M}\underline{1}}$, (4.4.3) about $\mathsf{F}_{\mathcal{M}\underline{1}}$, (4.4.7) about $\eta^{\mathcal{M}\underline{1}}$, and (5.5.9) about $\varrho^{\mathcal{M}\underline{1}}$. □

**Equivalences of Homotopy Theories.** Recall from Definition 4.7.1 the wide subcategory

$$\mathcal{S}^{\mathcal{M}\underline{1}} \subset \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

of $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalences created by

$$\mathsf{F}_{\mathcal{M}\underline{1}} : \left( \mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}} \right) \longrightarrow \left( \mathsf{PermCat}^{\mathsf{st}}, \mathcal{S}^I \right).$$

Recall from Definition 3.5.4 the notion of *non-symmetric algebras*. Theorem 5.5.12 below simultaneously extends

- Theorem 5.4.1 from $\mathsf{Multicat}_*$ to $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and
- Theorem 4.8.3 to non-symmetric algebras.

As in Theorems 3.5.5 and 5.4.1, each of the two induced functors $\mathsf{F}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$ and $\mathsf{End}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$ is given by post-composition and whiskering with the respective functor.

**Theorem 5.5.12.** *Suppose $\mathsf{Q}$ is a small non-symmetric $\mathsf{Cat}$-multicategory. In the context of (5.5.1), the induced functors*

$$\mathsf{F}^{\mathsf{Q}}_{\mathcal{M}\underline{1}} : \left( (\mathsf{Mod}^{\mathcal{M}\underline{1}})^{\mathsf{Q}}, (\mathcal{S}^{\mathcal{M}\underline{1}})^{\mathsf{Q}} \right) \xrightleftharpoons{\sim} \left( (\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}, \mathcal{S}^{\mathsf{Q}} \right) : \mathsf{End}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$$

*are inverse equivalences of homotopy theories in the sense of Definition 2.1.8.*

*Proof.* This proof is similar to that of Theorem 5.4.1. The functor $\mathsf{F}_{\mathcal{M}\underline{1}}$ creates stable equivalences by definition of $\mathcal{S}^{\mathcal{M}\underline{1}}$, and thus the forgetful $\mathsf{U}_{\mathcal{M}\underline{1}}$ also creates stable equivalences, because $\mathsf{F}_{\mathcal{M}\underline{1}} = \mathsf{F} \cdot \mathsf{U}_{\mathcal{M}\underline{1}}$. Also recall from Proposition 1.3.17 (4) that $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a full sub-2-category of $\mathsf{Multicat}_*$. Together these imply that

- the components of $\varrho^{\mathcal{M}\underline{1}}$ in (5.5.9) are natural stable equivalences and
- the components of $\eta^{\mathcal{M}\underline{1}}$ in (5.5.6) are natural stable equivalences.

The naturality diagram (5.4.2) then shows that $\mathsf{End}_{\mathcal{M}\underline{1}}$ is a relative functor.

Therefore, we conclude that $(\mathsf{F}_{\mathcal{M}\underline{1}})^{\mathsf{Q}}$ and $(\mathsf{End}_{\mathcal{M}\underline{1}})^{\mathsf{Q}}$ are both relative functors and the induced $(\varrho^{\mathcal{M}\underline{1}})^{\mathsf{Q}}$ and $(\eta^{\mathcal{M}\underline{1}})^{\mathsf{Q}}$ are natural stable equivalences

$$1_{(\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}} \xrightarrow[\sim]{(\varrho^{\mathcal{M}\underline{1}})^{\mathsf{Q}}} \mathsf{F}^{\mathsf{Q}}_{\mathcal{M}\underline{1}} \mathsf{End}^{\mathsf{Q}}_{\mathcal{M}\underline{1}} \quad \text{and} \quad 1_{(\mathsf{Mod}^{\mathcal{M}\underline{1}})^{\mathsf{Q}}} \xrightarrow[\sim]{(\eta^{\mathcal{M}\underline{1}})^{\mathsf{Q}}} \mathsf{End}^{\mathsf{Q}}_{\mathcal{M}\underline{1}} \mathsf{F}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}.$$

Thus, by Proposition 2.1.9, $\mathsf{F}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$ and $\mathsf{End}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$ are inverse equivalences of homotopy theories between categories of non-symmetric $\mathsf{Q}$-algebras. This completes the proof. $\square$

Recall that there is an adjoint equivalence of homotopy theories

(5.5.13)     $$\mathcal{M}\underline{1} \wedge - : \mathsf{Multicat}_* \xrightleftharpoons{\sim} \mathsf{Mod}^{\mathcal{M}\underline{1}} : \mathsf{U}_{\mathcal{M}\underline{1}}$$

by Theorem 4.8.1 Recall, furthermore, that

- $\mathcal{M}\underline{1} \wedge -$ is $\mathsf{Cat}$-multifunctorial by Proposition 1.3.26 and
- $\mathsf{U}_{\mathcal{M}\underline{1}}$ is $\mathsf{Cat}$-multifunctorial by Explanation 1.3.29.

Since $\mathcal{M}\underline{1} \wedge -$ and $\mathsf{U}_{\mathcal{M}\underline{1}}$ are $\mathsf{Cat}$-multifunctors in the symmetric sense, induced by symmetric $\mathsf{Cat}$-monoidal functors, the following result holds for both symmetric and non-symmetric $\mathsf{Q}$. Recall from Definition 3.5.4 that, for a $\mathsf{Cat}$-multicategory $\mathsf{N}$, the notation $\mathsf{N}^{\mathsf{Q}}$ denotes the category of $\mathsf{Q}$-algebras (in the symmetric sense) if $\mathsf{Q}$ is a small $\mathsf{Cat}$-multicategory in the symmetric sense.

**Theorem 5.5.14.** *Suppose* $\mathsf{Q}$ *is either a symmetric or non-symmetric small multicategory. The induced functors between categories of algebras (non-symmetric if* $\mathsf{Q}$ *is non-symmetric)*

$$(5.5.15) \qquad (\mathcal{M}\underline{1} \wedge -)^{\mathsf{Q}} : \left( \mathsf{Multicat}^{\mathsf{Q}}_* , \mathcal{S}^{\mathsf{Q}}_\bullet \right) \underset{\longleftarrow}{\overset{\sim}{\longrightarrow}} \left( (\mathsf{Mod}^{\mathcal{M}\underline{1}})^{\mathsf{Q}} , (\mathcal{S}^{\mathcal{M}\underline{1}})^{\mathsf{Q}} \right) : \mathsf{U}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$$

*are inverse equivalences of homotopy theories in the sense of Definition 2.1.8.*

*Proof.* Recall from Explanation 1.3.20 and Proposition 1.3.31 the unit $\hat{\eta}$ and counit $\hat{\varepsilon}$ for the adjunction $(\mathcal{M}\underline{1} \wedge -) \dashv \mathsf{U}_{\mathcal{M}\underline{1}}$ are monoidal Cat-natural transformations, and hence Cat-multinatural. It follows, therefore, that $(\mathcal{M}\underline{1} \wedge -)^{\mathsf{Q}}$ and $\mathsf{U}^{\mathsf{Q}}_{\mathcal{M}\underline{1}}$ are inverse equivalences of homotopy theories because $\hat{\eta}$ and $\hat{\varepsilon}$ are shown to be component-wise stable equivalences in Theorem 4.8.1. $\qquad\square$

# Part 3

# Enrichment of Diagrams and Mackey Functors in Closed Multicategories

CHAPTER 6

# Multicategorically Enriched Categories

This chapter defines and develops the basic properties of enrichment in a non-symmetric multicategory M. This is similar to, but more general than, the concept of enrichment in a monoidal category V from Appendix B. For the special case M = End V, Proposition 6.2.1 shows that the two notions of enrichment agree. Example 6.2.3 lists a number of symmetric monoidal closed categories for which enrichment over End V applies.

The main application for our further work is M = PermCat$^{\mathsf{su}}$. Theorem 6.4.20 shows that PermCat$^{\mathsf{su}}$ is a PermCat$^{\mathsf{su}}$-category. Propositions 6.5.7 and 6.5.8 provide further details that will be used in Chapters 8 and 9 to discuss a corresponding closed structure for PermCat$^{\mathsf{su}}$.

In the case that M has a symmetric group action making it a multicategory, there is a notion of opposite M-category described in Section 6.6. For the special case M = End V with V a symmetric monoidal category, Proposition 6.6.8 shows that the notions of opposite M-category and opposite V-category agree. This will be used in Chapter 10 for the discussion of enriched diagrams and enriched Mackey functors.

The abstract theory of multicategorical enrichment developed here and in the remaining chapters of Part 3 will find homotopy-theoretic applications in Part 4. Chapter 11 develops conditions for change of enrichment functors, between enriched diagram and Mackey functor categories, to induce equivalences of homotopy theories. Chapter 12 applies this to enrichment over, and diagrams in, PermCat$^{\mathsf{su}}$, Multicat$_*$, and Mod$^{\mathcal{M}1}$.

**Connection with Other Chapters.** The remaining chapters in this work depend on the multicategorical enrichment developed here. Change of enrichment is discussed in Chapter 7. Chapter 8 develops the basic theory of closed multicategories and extends the results about PermCat$^{\mathsf{su}}$ in Sections 6.4 and 6.5 to a closed multicategory structure. In Chapter 9 the theory of self-enriched multicategories is developed further, with additional applications to PermCat$^{\mathsf{su}}$. Opposite enriched categories (Section 6.6) are important in Chapters 10 through 12, where they are the domains of enriched Mackey functor categories (10.1.3).

**Background.** The content of this chapter depends only on that of Chapter 1 and Appendices A through C.

**Chapter Summary.** Section 6.1 defines enrichment in a non-symmetric multicategory. Section 6.2 shows that enrichment in an endomorphism multicategory agrees with enrichment in the underlying monoidal category. Section 6.3 describes enrichment in PermCat$^{\mathsf{su}}$, the Cat-multicategory of permutative categories. As an

important special case, Section 6.4 describes the self-enrichment of PermCat$^{su}$. Section 6.5 explains the bilinear evaluation for PermCat$^{su}$. Section 6.6 develops the basic theory for opposites of multicategorically-enriched categories. Here is a summary table.

| 2-category of M-categories | 6.1.27 |
|:---:|:---:|
| V-categories and End(V) categories | 6.2.1 and 6.2.3 |
| 2-category of PermCat$^{su}$-categories | 6.3.2, 6.3.12, and 6.3.16 |
| self-enrichment of PermCat$^{su}$ | 6.4.19 and 6.4.20 |
| bilinear evaluation for PermCat$^{su}$ | 6.5.1, 6.5.7, and 6.5.8 |
| opposite M-categories | 6.6.1 and 6.6.8 |

We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

## 6.1. Enrichment in a Multicategory

Throughout this section, we assume that

$$(M, \gamma, 1)$$

is a non-symmetric multicategory (Definition C.1.3). This means that M is a non-symmetric Set-multicategory, with $(Set, \times, *)$ the symmetric monoidal category of sets and functions with the Cartesian product as the monoidal product. In this section we define categories, functors, and natural transformations enriched in M. This section is organized as follows.

- M-categories, M-functors, and M-natural transformations are in Definitions 6.1.1, 6.1.7, and 6.1.14.
- Vertical composition of M-natural transformations are discussed in Definition 6.1.18 and Lemma 6.1.21.
- Horizontal composition of M-natural transformations are discussed in Definition 6.1.22 and Lemma 6.1.24.
- Theorem 6.1.27 proves the existence of a 2-category with small M-categories as objects.

We discuss enrichment in the endomorphism multicategory of a monoidal category in Section 6.2. In particular, Proposition 6.2.1 proves that, for a monoidal category V, the 2-category of small V-categories and the 2-category of small (End V)-categories are the same. Thus the two notions of enrichment over a monoidal category coincide.

### M-Categories.

**Definition 6.1.1.** An M-*category* $(C, m, i)$, which is also called a *category enriched in* M, consists of the following data.

**Objects:** C is equipped with a class Ob C of *objects*. We usually write $x \in C$ instead of $x \in Ob C$.

**Hom Objects:** For each pair of objects $x, y \in C$, C is equipped with an object

$$C(x, y) \in Ob M,$$

which is called the *hom object* with *domain* $x$ and *codomain* $y$. We sometimes abbreviate $C(x,y)$ to $C_{x,y}$ to save space.

**Composition:** For objects $x,y,z \in C$, $C$ is equipped with a binary multimorphism

$$(6.1.2) \qquad m_{x,y,z} : \big(C(y,z),\, C(x,y)\big) \longrightarrow C(x,z) \quad \text{in} \quad M,$$

which is called the *composition*.

**Identities:** Each object $x \in C$ is equipped with a nullary multimorphism

$$(6.1.3) \qquad i_x : \langle\rangle \longrightarrow C(x,x) \quad \text{in} \quad M,$$

which is called the *identity* of $x$. Here $\langle\rangle$ denotes the empty sequence.

These data are required to make the following associativity and unity diagrams in $M$ commute for objects $w,x,y,z \in C$.

$$(6.1.4) \qquad \begin{array}{ccc}
\big(C(y,z),\, C(x,y),\, C(w,x)\big) & \xrightarrow{\ (m_{x,y,z},\, 1)\ } & \big(C(x,z),\, C(w,x)\big) \\[2pt]
{\scriptstyle (1,\, m_{w,x,y})}\Big\downarrow & & \Big\downarrow{\scriptstyle m_{w,x,z}} \\[2pt]
\big(C(y,z),\, C(w,y)\big) & \xrightarrow{\quad m_{w,y,z}\quad } & C(w,z)
\end{array}$$

$$(6.1.5) \qquad \begin{array}{ccccc}
\big(C(x,y),\, \langle\rangle\big) & =\!=\!=\!= & C(x,y) & =\!=\!=\!= & \big(\langle\rangle,\, C(x,y)\big) \\[2pt]
{\scriptstyle (1,\, i_x)}\Big\downarrow & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle (i_y,\, 1)} \\[2pt]
\big(C(x,y),\, C(x,x)\big) & \xrightarrow{\ m_{x,x,y}\ } & C(x,y) & \xleftarrow{\ m_{x,y,y}\ } & \big(C(y,y),\, C(x,y)\big)
\end{array}$$

This finishes the definition of an $M$-category. An $M$-category is *small* if its class of objects is a set. $\diamond$

**Explanation 6.1.6** ($M$-Categories). The composition $m_{x,y,z}$ in (6.1.2) is an element in

$$M\big(C_{y,z},\, C_{x,y}\,;\, C_{x,z}\big).$$

The identity $i_x$ of $x$ in (6.1.3) is an element in

$$M\big(\langle\rangle\,;\, C_{x,x}\big).$$

The associativity diagram (6.1.4) means the equality of 3-ary multimorphisms

$$\gamma\big(m_{w,y,z}\,;\, 1_{C_{y,z}},\, m_{w,x,y}\big) = \gamma\big(m_{w,x,z}\,;\, m_{x,y,z},\, 1_{C_{w,x}}\big)$$

in $M\big(C_{y,z},\, C_{x,y},\, C_{w,x}\,;\, C_{w,z}\big)$. The unity diagram (6.1.5) means the equalities of unary multimorphisms

$$\gamma\big(m_{x,x,y}\,;\, 1_{C_{x,y}},\, i_x\big) = 1_{C_{x,y}} = \gamma\big(m_{x,y,y}\,;\, i_y,\, 1_{C_{x,y}}\big)$$

in $M\big(C_{x,y}\,;\, C_{x,y}\big)$. Other commutative diagrams in $M$ below are interpreted similarly. $\diamond$

### $M$-**Functors.**

**Definition 6.1.7.** Suppose $(C, m, i)$ and $(D, m, i)$ are $M$-categories. An $M$-*functor*

$$F : C \longrightarrow D,$$

which is also called a *functor enriched in* $M$, consists of the following data.

**Object Assignment:** $F$ is equipped with a function

$$F : \mathsf{Ob}\,\mathsf{C} \longrightarrow \mathsf{Ob}\,\mathsf{D},$$

which is called the *object assignment*.

**Component Morphisms:** For each pair of objects $x, y \in \mathsf{C}$, $F$ is equipped with a unary multimorphism

(6.1.8) $$F_{x,y} : \mathsf{C}(x, y) \longrightarrow \mathsf{D}(Fx, Fy) \quad \text{in} \quad \mathsf{M},$$

which is called the $(x, y)$-*component* of $F$. We sometimes abbreviate $F_{x,y}$ to $F$.

These data are required to make the following two diagrams in $\mathsf{M}$ commute for objects $x, y, z \in \mathsf{C}$.

(6.1.9)
$$
\begin{array}{ccc}
\big(\mathsf{C}(y,z)\,,\,\mathsf{C}(x,y)\big) & \xrightarrow{\ m_{x,y,z}\ } & \mathsf{C}(x,z) \\
{\scriptstyle (F_{y,z}\,,\,F_{x,y})}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{x,z}} \\
\big(\mathsf{D}(Fy,Fz)\,,\,\mathsf{D}(Fx,Fy)\big) & \xrightarrow{\ m_{Fx,Fy,Fz}\ } & \mathsf{D}(Fx,Fz)
\end{array}
\qquad
\begin{array}{ccc}
\langle\rangle & \xrightarrow{\ i_x\ } & \mathsf{C}(x,x) \\
{\scriptstyle i_{Fx}}\searrow & & \Big\downarrow{\scriptstyle F_{x,x}} \\
& & \mathsf{D}(Fx,Fx)
\end{array}
$$

This finishes the definition of an $\mathsf{M}$-functor.

Moreover, we define the following.

- The *identity* $\mathsf{M}$-*functor*

(6.1.10) $$1_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{C}$$

is defined by the identity object assignment and unit $(x, y)$-component

$$(1_{\mathsf{C}})_{x,y} = 1_{\mathsf{C}(x,y)} : \mathsf{C}(x, y) \longrightarrow \mathsf{C}(x, y) \quad \text{in} \quad \mathsf{M}$$

for each pair of objects $x, y \in \mathsf{C}$.

- For $\mathsf{M}$-functors $F : \mathsf{C} \longrightarrow \mathsf{D}$ and $G : \mathsf{D} \longrightarrow \mathsf{E}$, the *composite* $\mathsf{M}$-functor

(6.1.11) $$GF : \mathsf{C} \longrightarrow \mathsf{E}$$

is defined by composing the object assignments. The $(x, y)$-component of $GF$ is the composite unary multimorphism

(6.1.12) $$\mathsf{C}(x, y) \xrightarrow{\ F_{x,y}\ } \mathsf{D}(Fx, Fy) \xrightarrow{\ G_{Fx,Fy}\ } \mathsf{E}(GFx, GFy)$$

in $\mathsf{M}$. The commutativity of the diagrams (6.1.9) for $GF$ follows from the commutativity of the corresponding diagrams for $F$ and $G$. Composition of $\mathsf{M}$-functors is associative and unital with respect to identity $\mathsf{M}$-functors by the associativity and unity axioms of $\mathsf{M}$ in (C.1.8) through (C.1.10).

This finishes the definition.                                                                      ⬦

**Remark 6.1.13** (History). Enrichment in a non-symmetric multicategory goes back to the beginning of multicategory theory. It is mentioned in [**Lam69**, page 106], immediately after the definition of a non-symmetric multicategory. The explicit data and axioms of an $\mathsf{M}$-category and an $\mathsf{M}$-functor in Definitions 6.1.1 and 6.1.7 are from [**BO15**, Section 2].                                                                      ⬦

### M-Natural Transformations.

**Definition 6.1.14.** Suppose $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ are M-functors between M-categories. An M-*natural transformation*

$$\theta : F \longrightarrow G$$

consists of, for each object $x$ in $\mathsf{C}$, a nullary multimorphism

(6.1.15) $$\theta_x : \langle\rangle \longrightarrow \mathsf{D}(Fx, Gx) \quad \text{in} \quad \mathsf{M},$$

which is called the *x-component of $\theta$*. The following *naturality diagram* in M is required to commute for each pair of objects $x, y \in \mathsf{C}$, with m denoting the composition of D.

(6.1.16)

$$
\begin{array}{ccc}
\big(\langle\rangle, \mathsf{C}(x,y)\big) & \xrightarrow{(\theta_y, F_{x,y})} & \big(\mathsf{D}(Fy, Gy), \mathsf{D}(Fx, Fy)\big) \\
\big\| & & \downarrow{\scriptstyle m} \\
\big(\mathsf{C}(x,y), \langle\rangle\big) & & \\
{\scriptstyle (G_{x,y}, \theta_x)}\big\downarrow & & \\
\big(\mathsf{D}(Gx, Gy), \mathsf{D}(Fx, Gx)\big) & \xrightarrow{\quad m \quad} & \mathsf{D}(Fx, Gy)
\end{array}
$$

We also call $\theta$ a *natural transformation enriched in* M.

Moreover, the *identity* M-*natural transformation*

(6.1.17) $$1_F : F \longrightarrow F$$

is defined by the identity components:

$$(1_F)_x = i_{Fx} : \langle\rangle \longrightarrow \mathsf{D}(Fx, Fx) \quad \text{for} \quad x \in \mathsf{C}.$$

We use the 2-cell notation (A.1.29) for M-natural transformations. ◇

Next we define vertical and horizontal compositions for M-natural transformations.

**Definition 6.1.18.** Suppose $\theta$ and $\psi$ are M-natural transformations between M-functors between M-categories as in the left diagram below.

(6.1.19)



The *vertical composite* M-natural transformation $\psi\theta$, as in the right diagram above, has, for each object $x$ in $\mathsf{C}$, $x$-component given by the following composite in M, with m denoting the composition of D.

(6.1.20) $$\langle\rangle = \big(\langle\rangle, \langle\rangle\big) \xrightarrow{(\psi_x, \theta_x)} \big(\mathsf{D}(Gx, Hx), \mathsf{D}(Fx, Gx)\big) \xrightarrow{\ m\ } \mathsf{D}(Fx, Hx)$$

with overbrace $(\psi\theta)_x$.

This finishes the definition of $\psi\theta$. ◇

**Lemma 6.1.21.** *In the context of Definition 6.1.18, the following statements hold.*

(1) *The vertical composite $\psi\theta$ is a well-defined* M-*natural transformation.*

(2) *Vertical composition of* M-*natural transformations is associative.*

(3) *Identity* M-*natural transformations (6.1.17) are two-sided units for vertical composition.*

*Proof. Assertion ([1]).* Suppose $x$ and $y$ are objects in $\mathsf{C}$. By the definition ([6.1.20]) of $(\psi\theta)_x$, the naturality diagram ([6.1.16]) for $\psi\theta$ is the boundary of the following diagram in $\mathsf{M}$, where we abbreviate $\mathsf{C}(x,y)$ to $\mathsf{C}_{x,y}$ and likewise for $\mathsf{D}$.



The diagram above is commutative for the following reasons.

- The top left and right pentagons are commutative by the naturality ([6.1.16]) of $\psi$ and $\theta$, respectively.
- The other three sub-regions are commutative by the associativity ([6.1.4]) of $\mathsf{D}$.

This proves that $\psi\theta$ is an $\mathsf{M}$-natural transformation.

*Assertion ([2]).* Suppose $\varphi : H \longrightarrow I$ is an $\mathsf{M}$-natural transformation for an $\mathsf{M}$-functor $I : \mathsf{C} \longrightarrow \mathsf{D}$. We must show that, for each object $x$ in $\mathsf{C}$, the $x$-components of $(\varphi\psi)\theta$ and $\varphi(\psi\theta)$ are equal. Consider the following diagram in $\mathsf{M}$.



The following statements hold for the diagram above.

- The composite along the top is the $x$-component of $(\varphi\psi)\theta$.
- The composite along the bottom is the $x$-component of $\varphi(\psi\theta)$.
- The right quadrilateral commutes by the associativity ([6.1.4]) of $\mathsf{D}$.

This proves that the $\mathsf{M}$-natural transformations $(\varphi\psi)\theta$ and $\varphi(\psi\theta)$ are equal.

*Assertion ([3]).* We must show that $\theta 1_F$ and $1_G \theta$ have, for each object $x$ in $\mathsf{C}$, the same $x$-component as $\theta$. We consider the following diagram in $\mathsf{M}$.



The following statements hold for the diagram above.

- The left-bottom composite, $\gamma\big(\mathsf{m}\,;\theta_x,i_{Fx}\big)$, is the $x$-component of $\theta 1_F$. This composite is equal to the Z-shaped composite by the associativity ([C.1.8]) and left unity ([C.1.10]) of $\mathsf{M}$.

- The right-bottom composite, $\gamma\big(\mathsf{m}\,;\,i_{Gx},\theta_x\big)$, is the $x$-component of $1_G\theta$. This composite is equal to the S-shaped composite by the associativity (C.1.8) and left unity (C.1.10) of M.
- The composite $1\theta_x$ is equal to $\theta_x$ by the left unity (C.1.10) of M.
- The middle two triangles are commutative by the unity (6.1.5) of D.

This proves that $\theta 1_F$ and $1_G\theta$ are both equal to $\theta$. □

**Definition 6.1.22.** Suppose $\theta$ and $\theta'$ are M-natural transformations between M-functors between M-categories as in the left diagram below.

$$\mathsf{C} \xrightarrow[G]{\overset{F}{\Downarrow\theta}} \mathsf{D} \xrightarrow[G']{\overset{F'}{\Downarrow\theta'}} \mathsf{E} \qquad\qquad \mathsf{C} \xrightarrow[G'G]{\overset{F'F}{\Downarrow\theta'\ast\theta}} \mathsf{E}$$

The *horizontal composite* M-natural transformation $\theta'\ast\theta$, as in the right diagram above, has, for each object $x$ in C, $x$-component given by the following composite in M, with m denoting the composition of E.

(6.1.23)

$$
\begin{array}{ccc}
\langle\rangle & \xrightarrow{\ \ (\theta'\ast\theta)_x\ \ } & \mathsf{E}(F'Fx, G'Gx) \\[2mm]
\Big\downarrow{\scriptstyle\theta_x} & & \Big\uparrow{\scriptstyle\mathsf{m}} \\[2mm]
 & \big(\mathsf{E}(F'Gx, G'Gx)\,,\,\mathsf{E}(F'Fx, F'Gx)\big) & \\[2mm]
 & & \Big\uparrow{\scriptstyle(\theta'_{Gx}\,,\,F'_{Fx,Gx})} \\[2mm]
\mathsf{D}(Fx, Gx) & =\!=\!=\!=\!=\!=\!= & \big(\langle\rangle\,,\,\mathsf{D}(Fx, Gx)\big)
\end{array}
$$

This finishes the definition of $\theta'\ast\theta$. ◇

**Lemma 6.1.24.** *In the context of Definition 6.1.22, the following statements hold.*

(1) $(\theta'\ast\theta)_x$ *in (6.1.23) is equal to the following composite in* M.

(6.1.25)

$$
\begin{array}{ccc}
\langle\rangle & & \mathsf{E}(F'Fx, G'Gx) \\[2mm]
\Big\downarrow{\scriptstyle\theta_x} & & \Big\uparrow{\scriptstyle\mathsf{m}} \\[2mm]
 & \big(\mathsf{E}(G'Fx, G'Gx)\,,\,\mathsf{E}(F'Fx, G'Fx)\big) & \\[2mm]
 & & \Big\uparrow{\scriptstyle(G'_{Fx,Gx}\,,\,\theta'_{Fx})} \\[2mm]
\mathsf{D}(Fx, Gx) & =\!=\!=\!=\!=\!=\!= & \big(\mathsf{D}(Fx, Gx)\,,\,\langle\rangle\big)
\end{array}
$$

(2) $\theta'\ast\theta$ *is a well-defined* M-*natural transformation.*

(3) *Horizontal composition of* M-*natural transformations is associative.*

(4) *Identity* M-*natural transformations (6.1.17) of identity* M-*functors (6.1.10) are two-sided units for horizontal composition.*

(5) *Horizontal composition preserves identity* M-*natural transformations (6.1.17).*

(6) *Horizontal composition preserves vertical composition of* M-*natural transformations (6.1.19).*

*Proof. Assertion (1).* This follows from the naturality of $\theta'$ (6.1.16), which implies that the right composites in (6.1.23) and (6.1.25) are equal.

*Assertion (2)*. By (6.1.12) and (6.1.23), for objects $x, y \in \mathsf{C}$ the naturality diagram (6.1.16) for $\theta' * \theta$ is the boundary of the following diagram in $\mathsf{M}$, with $\mathsf{C}(x,y)$ abbreviated to $\mathsf{C}_{x,y}$ and likewise for $\mathsf{D}$ and $\mathsf{E}$.



The following statements hold for the diagram above.

- The sub-regions labeled $\star$ and $\diamondsuit$ are commutative by the naturality (6.1.16) of $\theta$ and $\theta'$, respectively.
- The sub-regions labeled $\spadesuit$ and $\clubsuit$ are commutative by the compatibility of $F'$ with composition (6.1.9).
- The three unlabeled sub-regions are commutative by the associativity of the composition of $\mathsf{E}$ (6.1.4).

This proves that $\theta' * \theta$ is an M-natural transformation.

*Assertion (3)*. To show that horizontal composition is associative, consider horizontally composable M-natural transformations as follows.

To show that, for each object $x$ in B, $\theta'' * (\theta' * \theta)$ and $(\theta'' * \theta') * \theta$ have the same $x$-components, we consider the following diagram in M.



The following statements hold for the diagram above.

- The left-bottom composite is $\left(\theta'' * (\theta' * \theta)\right)_x$.
- The other composite is $\left((\theta'' * \theta') * \theta\right)_x$.
- The left quadrilateral is commutative by the compatibility of $F''$ with composition (6.1.9).
- The right quadrilateral is commutative by the associativity of the composition of E (6.1.4).

This proves that $\theta'' * (\theta' * \theta)$ is equal to $(\theta'' * \theta') * \theta$.

*Assertion (4)*. Consider the following M-natural transformations.



To show that $1_{1_D} * \theta$ is equal to $\theta$, we consider, for each object $x \in$ C, the following diagram in M.



The following statements hold for the diagram above.

- The left-bottom-right composite is the $x$-component of $1_{1_D} * \theta$.
- The upper left triangle is commutative by the left unity of M (C.1.10).
- The lower right triangle is commutative by the unity of D (6.1.5).

This proves that $1_{1_D} * \theta$ is equal to $\theta$.

To show that $\theta * 1_{1_C}$ is equal to $\theta$, we consider, for each object $x \in C$, the following diagram in M.



The following statements hold for the diagram above.

- The left-bottom-right composite is the $x$-component of $\theta * 1_{1_C}$.
- The bottom sub-region and the middle quadrilateral are commutative by definition.
- The top sub-region is commutative by the left unity of M (C.1.10).
- The left triangle is commutative by the compatibility of $F$ with identities (6.1.9).
- The right triangle is commutative by the unity of D (6.1.5).

This proves that $\theta * 1_{1_C}$ is equal to $\theta$.

*Assertion (5)*. Consider two horizontally composable identity M-natural transformations as follows.



To show that $1_{F'} * 1_F$ is equal to $1_{F'F}$, we consider, for each object $x \in C$, the following diagram in M.



The following statements hold for the diagram above.

- The left-bottom-right composite is the $x$-component of $1_{F'} * 1_F$.
- The bottom sub-region is commutative by definition.
- The left triangle is commutative by the compatibility of $F'$ with identities (6.1.9).
- The middle triangle is commutative by the left unity of M (C.1.10).
- The right triangle is commutative by the unity of E (6.1.5).

This proves that horizontal composition preserves identity M-natural transformations.

*Assertion (6).* To show that horizontal composition preserves vertical composition, consider the following M-natural transformations.



We must show that, for each object $x$ in C, the following equality holds in M.

$$(6.1.26) \qquad \big((\psi' * \psi)(\theta' * \theta)\big)_x = \big((\psi'\theta') * (\psi\theta)\big)_x$$

To prove (6.1.26) we use the notation

$$X = \big(\mathsf{E}_{G'Hx,H'Hx}, \mathsf{E}_{G'Gx,G'Hx}, \mathsf{E}_{F'Gx,G'Gx}, \mathsf{E}_{F'Fx,F'Gx}\big),$$

$$Y = \big(\mathsf{E}_{G'Gx,H'Hx}, \mathsf{E}_{F'Gx,G'Gx}, \mathsf{E}_{F'Fx,F'Gx}\big), \quad \text{and}$$

$$Z = \big(\mathsf{E}_{G'Hx,H'Hx}, \mathsf{E}_{F'Hx,G'Hx}, \mathsf{E}_{F'Fx,F'Hx}\big)$$

and consider the following diagram in M.



The following statements hold for the diagram above.

- The left-bottom composite is the left-hand side of (6.1.26).
- The top-right composite is the right-hand side of (6.1.26).
- The upper left quadrilateral is commutative by the naturality of $\theta'$ (6.1.16).
- The upper right quadrilateral is commutative by the compatibility of $F'$ with composition (6.1.9).
- The lower left triangle is commutative by definition.
- The other five sub-regions are commutative by the associativity of the composition of E (6.1.4).

This proves the desired equality (6.1.26). $\qquad\square$

Recall the notion of a 2-category in Definition A.2.1.

**Theorem 6.1.27.** *For each non-symmetric multicategory* $(M, \gamma, 1)$*, there is a 2-category*

$$\mathsf{M}\text{-}\mathsf{Cat}$$

*defined by the following data.*

- *The objects are small* M*-categories (Definition 6.1.1).*
- *The 1-cells are* M*-functors (Definition 6.1.7).*
- *Identity 1-cells are identity* M*-functors (6.1.10).*
- *Horizontal composition of 1-cells is composition of* M*-functors (6.1.11).*
- *The 2-cells are* M*-natural transformations (Definition 6.1.14).*
- *Identity 2-cells are identity* M*-natural transformations (6.1.17).*
- *Vertical and horizontal compositions of 2-cells are those of* M*-natural transformations (Definitions 6.1.18 and 6.1.22).*

*Proof.* Axioms (i) through (iv) in Definition A.2.1 of a 2-category hold for M-Cat by, respectively,

(i) Lemma 6.1.21,
(ii) Lemma 6.1.24 (5) and (6),
(iii) Definition 6.1.7, and
(iv) Lemma 6.1.24 (3) and (4).

This finishes the proof.                                                    □

**Example 6.1.28.** By Theorem 1.4.29 PermCat$^{\mathsf{su}}$ is a Cat-multicategory, in particular a multicategory. By Theorem 6.1.27 there is a 2-category PermCat$^{\mathsf{su}}$-Cat of small categories, functors, and natural transformations enriched in the multicategory PermCat$^{\mathsf{su}}$. We describe this 2-category more explicitly in Section 6.3.          ◇

## 6.2. Enrichment in an Endomorphism Multicategory

For a monoidal category V (Definition A.1.3), there are two notions of enrichment over V.

(1) By Example B.1.12 there is a 2-category V-Cat of small V-categories, V-functors, and V-natural transformations.
(2) By Example C.3.1 there is a non-symmetric endomorphism multicategory End V, which is, furthermore, a multicategory if V is symmetric monoidal. By Theorem 6.1.27 there is a 2-category (End V)-Cat of small (End V)-categories, (End V)-functors, and (End V)-natural transformations.

Next we observe that these two notions of enrichment are the same. This result is stated in [**BO15**, Remark 2.10].

**Proposition 6.2.1.** *For each monoidal category* $(V, \otimes, 1)$*, there is an equality of 2-categories*

$$\mathsf{V}\text{-}\mathsf{Cat} = (\mathsf{End}\,\mathsf{V})\text{-}\mathsf{Cat}.$$

*Proof.* The objects in V-Cat and (End V)-Cat are small V-categories and small (End V)-categories, respectively. The identification of these objects follows by comparing

- Definition 6.1.1 for (End V)-categories and
- Definition B.1.1 for V-categories.

More explicitly, suppose C is a V-category.

- The identity of an object $x \in C$ is a morphism

$$i_x : 1 \longrightarrow C(x, x) \in V.$$

  By the definition (C.3.2) of $\mathsf{End}\, V$, such a morphism $i_x$ is the same as a nullary multimorphism in

$$(\mathsf{End}\, V)\big(\langle\rangle\, ;\, C(x, x)\big) = V\big(1, C(x, x)\big).$$

- For objects $x, y, z \in C$, the multiplication of $C$ is a morphism

$$m_{x,y,z} : C(y, z) \otimes C(x, y) \longrightarrow C(x, z) \quad \text{in} \quad V.$$

  This is the same as a binary multimorphism in

$$(\mathsf{End}\, V)\big(C(y, z), C(x, y)\, ;\, C(x, z)\big) = V\big(C(y, z) \otimes C(x, y)\, ,\, C(x, z)\big).$$

- Under the above identifications, the associativity axiom (B.1.4) and the unity axiom (B.1.5) of a $V$-category are equivalent to those of an $(\mathsf{End}\, V)$-category in (6.1.4) and (6.1.5).

Thus $C$ is equivalently an $(\mathsf{End}\, V)$-category.

A similar comparison of

- Definitions 6.1.7, 6.1.14, 6.1.18, and 6.1.22 for $(\mathsf{End}\, V)$-Cat and
- Definitions B.1.8 and B.1.10 for $V$-Cat

proves that the rest of the 2-category structures in $V$-Cat and $(\mathsf{End}\, V)$-Cat—namely, (identity) 1-cells, (identity) 2-cells, vertical composition of 2-cells, and horizontal composition of 1-cells and 2-cells—are the same. For example, the two diagrams in (B.1.9) for a $V$-functor are equivalent to the diagrams in (6.1.9) for an $(\mathsf{End}\, V)$-functor. The naturality diagram (B.1.11) for a $V$-natural transformation is equivalent to the naturality diagram (6.1.16) for an $(\mathsf{End}\, V)$-natural transformation. $\square$

For a monoidal category $V$, Proposition 6.2.1 identifies $V$-enrichment and $(\mathsf{End}\, V)$-enrichment, with $\mathsf{End}\, V$ the non-symmetric endomorphism multicategory. In what follows, we use them interchangeably.

**Example 6.2.2** (Self-Enrichment)**.** Suppose $V$ is a symmetric monoidal closed category (Definition A.1.19). Then $V$ is also a symmetric monoidal $V$-category with the canonical self-enrichment (Theorem B.3.7). By Proposition 6.2.1 the canonical self-enrichment of $V$ is equal to an $(\mathsf{End}\, V)$-enrichment, making $V$ into an $(\mathsf{End}\, V)$-category. $\diamond$

**Example 6.2.3.** Proposition 6.2.1 and Example 6.2.2 apply to the following symmetric monoidal closed categories:

- Multicat of small multicategories (Theorem 1.1.26);
- Multicat$_*$ of small pointed multicategories (Theorem 1.2.8);
- Mod$^{\mathcal{M}\underline{1}}$ of left $\mathcal{M}\underline{1}$-modules (Proposition 1.3.17 (7));
- $C_*$ of pointed objects in a complete and cocomplete symmetric monoidal closed category $C$ (Theorem 2.2.7);
- $\mathcal{D}_*$-$V$ of pointed diagrams in a complete and cocomplete symmetric monoidal closed category $V$ with a chosen terminal object (Theorem 2.2.19);
- $\Gamma$-$V$ of $\Gamma$-objects in $V$ (2.3.3);
- $\mathcal{G}_*$-$V$ of $\mathcal{G}_*$-objects in $V$ (2.4.12); and
- Sp of symmetric spectra (2.5.2).

However, Proposition 6.2.1 and Example 6.2.2 do not apply to $\mathsf{PermCat^{su}}$ (Theorem 1.4.29) because its multicategory structure is not induced by a monoidal structure. See [**JY∞**, 5.7.23 and 10.2.17]. We discuss its self-enrichment in Theorem 6.4.20.                                                                                    ◇

## 6.3. Enrichment in the Multicategory of Permutative Categories

Recall from Theorem 1.4.29 that $\mathsf{PermCat^{su}}$ is a Cat-multicategory, hence also a multicategory.

- Its objects are small permutative categories (Definition A.1.14).
- Its $n$-ary multimorphisms are $n$-linear functors (Definition 1.4.2).
- Its colored units are identity symmetric monoidal functors.
- Its symmetric group action and composition are in Definitions 1.4.16 and 1.4.21, respectively.

In this section we explicitly describe categories, functors, and natural transformations enriched in $\mathsf{PermCat^{su}}$ in, respectively, Explanations 6.3.2, 6.3.12, and 6.3.16. We discuss the self-enrichment of $\mathsf{PermCat^{su}}$ in Section 6.4.

We use the shortened notation

$$(6.3.1) \qquad\qquad\qquad \mathsf{P^{su}} = \mathsf{PermCat^{su}}$$

to simplify the presentation.

**Explanation 6.3.2** ($\mathsf{P^{su}}$-Categories)**.** Unpacking Definition 6.1.1 with $\mathsf{M} = \mathsf{P^{su}}$, a $\mathsf{P^{su}}$-category $(\mathsf{C}, \mathsf{m}, i)$ consists of the following data.

**Objects:** $\mathsf{C}$ is equipped with a class $\mathsf{Ob}\,\mathsf{C}$ of objects.

**Hom Permutative Categories:** For each pair of objects $x, y \in \mathsf{C}$, $\mathsf{C}$ is equipped with a small permutative category (Definition A.1.14)

$$\big(\mathsf{C}(x,y), \oplus, \mathsf{e}, \tilde{\xi}\big),$$

which is also denoted $\mathsf{C}_{x,y}$.

**Composition:** For each triple of objects $x, y, z \in \mathsf{C}$, $\mathsf{C}$ is equipped with a bilinear functor (Definition 1.4.2)

$$\mathsf{m}_{x,y,z} : \mathsf{C}(y,z) \times \mathsf{C}(x,y) \longrightarrow \mathsf{C}(x,y).$$

Its first and second linearity constraints are denoted $\mathsf{m}^1_{x,y,z}$ and $\mathsf{m}^2_{x,y,z}$, respectively. If there is no danger of confusion, we sometimes omit the subscripts.

**Identities:** Recalling that a 0-linear functor is a choice of an object in the codomain category, each object $x \in \mathsf{C}$ is equipped with an object

$$i_x \in \mathsf{C}(x,x).$$

This is also regarded as a functor $\mathbf{1} \longrightarrow \mathsf{C}(x,x)$ from the terminal category. We emphasize that $i_x$ is *not* required to be the monoidal unit $\mathsf{e}$ of $\mathsf{C}(x,x)$.

Below, we describe some implications of this structure. We abbreviate each $\mathsf{m}(-,-)$ to concatenation, so

$$(6.3.3) \qquad\qquad\qquad \mathsf{m}_{x,y,z}(f,g) = fg.$$

The constraint 2-by-2 axiom (1.4.8) for $\mathsf{m}$ says that, for objects $x, y, z \in \mathsf{C}$, with

$$(6.3.4) \qquad\qquad f, f' \in \mathsf{C}(y,z), \quad \text{and} \quad g, g' \in \mathsf{C}(x,y),$$

the following diagram in $\mathsf{C}(x,z)$ commutes.

$$(6.3.5)$$



The associativity axiom (6.1.4) of a $\mathsf{P}^{\mathsf{su}}$-category says that, for objects $w, x, y, z \in \mathsf{C}$, the following two composite 3-linear functors are equal, with each 1 denoting the identity symmetric monoidal functor.

$$(6.3.6)$$

$$
\begin{array}{ccc}
\mathsf{C}(y,z) \times \mathsf{C}(x,y) \times \mathsf{C}(w,x) & \xrightarrow{\;\mathsf{m}_{x,y,z} \times 1\;} & \mathsf{C}(x,z) \times \mathsf{C}(w,x) \\
{\scriptstyle 1 \times \mathsf{m}_{w,x,y}} \downarrow & & \downarrow {\scriptstyle \mathsf{m}_{w,x,z}} \\
\mathsf{C}(y,z) \times \mathsf{C}(w,y) & \xrightarrow{\;\mathsf{m}_{w,y,z}\;} & \mathsf{C}(w,z)
\end{array}
$$

This means, first of all, that the two composites in (6.3.6) are equal as functors. Furthermore, their respective linearity constraints are equal. To make this explicit, consider objects

$$(6.3.7) \qquad f, f' \in \mathsf{C}(y,z), \quad g, g' \in \mathsf{C}(x,y), \quad \text{and} \quad h, h' \in \mathsf{C}(w,x).$$

Linearity constraints of composite multilinear functors are defined in (1.4.27). The equality of, respectively, the first, second, and third linearity constraints of the two composites in (6.3.6) are the following three commutative diagrams in $\mathsf{C}(w,z)$.

$$(6.3.8)$$



$$(6.3.9)$$

The unity axiom (6.1.5) of a $\mathsf{P}^{\mathsf{su}}$-category says that, for objects $x, y \in \mathsf{C}$, the following diagram of symmetric monoidal functors commutes.

(6.3.10)

$$
\begin{array}{ccccc}
& & \mathsf{C}(x,y) & & \\
& \overset{\cong}{\swarrow} & \Big\downarrow{\scriptstyle 1} & \overset{\cong}{\searrow} & \\
\mathsf{C}(x,y) \times \mathbf{1} & & & & \mathbf{1} \times \mathsf{C}(x,y) \\
{\scriptstyle 1 \times i_x}\Big\downarrow & & & & \Big\downarrow{\scriptstyle i_y \times 1} \\
\mathsf{C}(x,y) \times \mathsf{C}(x,x) & \overset{\mathsf{m}_{x,x,y}}{\longrightarrow} \mathsf{C}(x,y) \overset{\mathsf{m}_{x,y,y}}{\longleftarrow} & & & \mathsf{C}(y,y) \times \mathsf{C}(x,y)
\end{array}
$$

Both boundary composites in (6.3.10) are strictly unital by the unity axiom (1.4.4) of each bilinear functor $\mathsf{m}$. In terms of the monoidal constraints, the commutative diagram (6.3.10) means the following commutative diagram in $\mathsf{C}(x,y)$ for objects $g, g' \in \mathsf{C}(x,y)$, where we use concatenation to denote $\mathsf{m}$ as in (6.3.3).

(6.3.11)

$$
\begin{array}{ccccc}
g i_x \oplus g' i_x & =\!=\!= & g \oplus g' & =\!=\!= & i_y g \oplus i_y g' \\
{\scriptstyle \mathsf{m}^1_{x,x,y}}\Big\downarrow & & \Big\downarrow{\scriptstyle 1} & & \Big\downarrow{\scriptstyle \mathsf{m}^2_{x,y,y}} \\
(g \oplus g') i_x & =\!=\!= & g \oplus g' & =\!=\!= & i_y (g \oplus g')
\end{array}
$$

This finishes the description of a $\mathsf{P}^{\mathsf{su}}$-category. $\diamond$

**Explanation 6.3.12** ($\mathsf{P}^{\mathsf{su}}$-Functors). Unpacking Definition 6.1.7 with $\mathsf{M} = \mathsf{P}^{\mathsf{su}}$, a $\mathsf{P}^{\mathsf{su}}$-functor

$$
F : (\mathsf{C}, \mathsf{m}, i) \longrightarrow (\mathsf{D}, \mathsf{m}, i)
$$

between $\mathsf{P}^{\mathsf{su}}$-categories consists of

- an object assignment $F : \mathrm{Ob}\,\mathsf{C} \longrightarrow \mathrm{Ob}\,\mathsf{D}$ and
- for each pair of objects $x, y \in \mathsf{C}$, a strictly unital symmetric monoidal functor (Definition A.1.22)

$$
\big(F_{x,y},\, F^2_{x,y},\, F^0_{x,y} = 1\big) : \mathsf{C}(x,y) \longrightarrow \mathsf{D}(Fx, Fy).
$$

The compatibility of $F$ with composition (6.1.9) is the following commutative diagram of bilinear functors.

(6.3.13)

$$
\begin{array}{ccc}
\mathsf{C}(y,z) \times \mathsf{C}(x,y) & \overset{\mathsf{m}_{x,y,z}}{\longrightarrow} & \mathsf{C}(x,z) \\
{\scriptstyle F_{y,z} \times F_{x,y}}\Big\downarrow & & \Big\downarrow{\scriptstyle F_{x,z}} \\
\mathsf{D}(Fy, Fz) \times \mathsf{D}(Fx, Fy) & \overset{\mathsf{m}_{Fx,Fy,Fz}}{\longrightarrow} & \mathsf{D}(Fx, Fz)
\end{array}
$$

This means, first of all, that (6.3.13) is a commutative diagram of functors. Moreover, the equality of, respectively, the first and second linearity constraints of the two composites in (6.3.13) are the following two commutative diagrams in $\mathsf{D}(Fx, Fz)$ for objects $f, f' \in \mathsf{C}(y,z)$ and $g, g' \in \mathsf{C}(x,y)$, with $\mathsf{m}$ shortened to concatenation as in (6.3.3).

(6.3.14)

$$
\begin{array}{ccc}
F(fg) \oplus F(f'g) & \overset{F^2}{\longrightarrow} & F(fg \oplus f'g) \\
\| & & {\scriptstyle F(\mathsf{m}^2_1)}\Big\downarrow \\
(Ff)(Fg) \oplus (Ff')(Fg) & & F((f \oplus f')g) \\
{\scriptstyle \mathsf{m}^2_1}\Big\downarrow & & \| \\
(Ff \oplus Ff')(Fg) & \overset{F^2 1_{Fg}}{\longrightarrow} & (F(f \oplus f'))(Fg)
\end{array}
\qquad
\begin{array}{ccc}
F(fg) \oplus F(fg') & \overset{F^2}{\longrightarrow} & F(fg \oplus fg') \\
\| & & {\scriptstyle F(\mathsf{m}^2_2)}\Big\downarrow \\
(Ff)(Fg) \oplus (Ff)(Fg') & & F(f(g \oplus g')) \\
{\scriptstyle \mathsf{m}^2_2}\Big\downarrow & & \| \\
(Ff)(Fg \oplus Fg') & \overset{1_{Ff} F^2}{\longrightarrow} & (Ff)F(g \oplus g')
\end{array}
$$

The compatibility of $F$ with identities (6.1.9) means the object equality

$$(6.3.15) \qquad F_{x,x}(i_x) = i_{Fx} \quad \text{in} \quad \mathsf{D}(Fx, Fx)$$

for each object $x$ in C. This finishes the description of a $\mathsf{P}^{\mathsf{su}}$-functor. $\diamond$

**Explanation 6.3.16** ($\mathsf{P}^{\mathsf{su}}$-Natural Transformations). Unpacking Definition 6.1.14 with $\mathsf{M} = \mathsf{P}^{\mathsf{su}}$, a $\mathsf{P}^{\mathsf{su}}$-natural transformation

$$\mathsf{C} \underset{G}{\overset{F}{\rightrightarrows}} \Downarrow\theta \;\; \mathsf{D}$$

between $\mathsf{P}^{\mathsf{su}}$-functors between $\mathsf{P}^{\mathsf{su}}$-categories consists of, for each object $x$ in C, an object

$$\theta_x \in \mathsf{D}(Fx, Gx).$$

This is also regarded as a functor $\mathbf{1} \longrightarrow \mathsf{D}(Fx, Gx)$.

The naturality diagram (6.1.16) for a $\mathsf{P}^{\mathsf{su}}$-natural transformation is the following commutative diagram of symmetric monoidal functors for each pair of objects $x, y \in \mathsf{C}$.

$$(6.3.17)$$

$$
\begin{array}{ccc}
\mathsf{C}(x,y) \xrightarrow{\;\cong\;} \mathbf{1} \times \mathsf{C}(x,y) \xrightarrow{\theta_y \times F_{x,y}} \mathsf{D}(Fy, Gy) \times \mathsf{D}(Fx, Fy) \\
{\scriptstyle\cong}\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mathsf{m}_{Fx,Fy,Gy}} \\
\mathsf{C}(x,y) \times \mathbf{1} \\
{\scriptstyle G_{x,y} \times \theta_x}\downarrow \\
\mathsf{D}(Gx, Gy) \times \mathsf{D}(Fx, Gx) \xrightarrow{\;\mathsf{m}_{Fx,Gx,Gy}\;} \mathsf{D}(Fx, Gy)
\end{array}
$$

Both composites in (6.3.17) are strictly unital because

- $F_{x,y}$ and $G_{x,y}$ are strictly unital and
- both bilinear functors $\mathsf{m}$ satisfy the unity axiom (1.4.4).

In addition to being a commutative diagram of functors, the equality of the monoidal constraints of the two composites in (6.3.17) means the following commutative diagram in $\mathsf{D}(Fx, Gy)$ for objects $g, g' \in \mathsf{C}(x, y)$, with $\mathsf{m}$ denoted by concatenation as in (6.3.3).

$$(6.3.18)$$

$$
\begin{array}{ccc}
\theta_y(Fg) \oplus \theta_y(Fg') & \xrightarrow{\;\mathsf{m}_2^2\;} & \theta_y(Fg \oplus Fg') \\
{\scriptstyle /\!/} & & \downarrow{\scriptstyle 1_{\theta_y}F^2} \\
(Gg)\theta_x \oplus (Gg')\theta_x & & \theta_y F(g \oplus g') \\
{\scriptstyle \mathsf{m}_1^2}\downarrow & & \| \\
(Gg \oplus Gg')\theta_x & \xrightarrow{\;G^2 1_{\theta_x}\;} & (G(g \oplus g'))\theta_x
\end{array}
$$

This finishes the description of a $\mathsf{P}^{\mathsf{su}}$-natural transformation. $\diamond$

## 6.4. Self-Enrichment of the Multicategory of Permutative Categories

In this section we observe that the 2-category $\mathsf{PermCat}^{\mathsf{su}}$ (Definition A.2.3) of small permutative categories has the additional structure of a category enriched in the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 1.4.29) in the sense of Definition 6.1.1. See Theorem 6.4.20. There are two ways to think about this result.

(1) It is a permutative extension of the 2-category structure on PermCat$^{\mathsf{su}}$. So we have hom *permutative* categories and composition *bilinear* functors between permutative categories. From this viewpoint, this section is an expanded version of [**BO15**, Section 5]; see also [**Gui10**, Example 3.4].

(2) Theorem 6.4.20 is a precursor of Theorem 8.4.15, which extends the Cat-multicategory PermCat$^{\mathsf{su}}$ and its self-enrichment to a *closed* multicategory structure. One may consider this section as a warm-up exercise for Theorem 8.4.15. Each non-symmetric closed multicategory has a canonical self-enrichment (Theorem 9.1.7), so, in particular, PermCat$^{\mathsf{su}}$ has a canonical self-enrichment. As we discuss in more detail in Proposition 9.1.8, the self-enrichment of PermCat$^{\mathsf{su}}$ in Theorem 6.4.20 is the same as its canonical self-enrichment obtained from its closed multicategory structure. However, a non-symmetric closed multicategory has more structure than its self-enrichment.

This section is organized as follows.

- The hom permutative categories are constructed in Definition 6.4.1 and verified in Lemma 6.4.11.
- The composition bilinear functors are constructed in Definition 6.4.12 and verified in Lemma 6.4.17.
- The self-enrichment of PermCat$^{\mathsf{su}}$ is constructed in Definition 6.4.19 and verified in Theorem 6.4.20.

To simplify the presentation, we also use the shortened notation in (6.3.1):

$$\mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.$$

In a typical permutative category, the monoidal product, monoidal unit, and braiding are denoted $\oplus$, $\mathsf{e}$, and $\xi$, respectively.

**Definition 6.4.1** (Hom Permutative Categories). Given small permutative categories C and D, we define a small permutative category

$$(6.4.2) \qquad\qquad \left(\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}), \oplus, \mathsf{e}, \xi\right)$$

as follows. The small category $\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D})$ is a hom category of the 2-category PermCat$^{\mathsf{su}}$ in Definition A.2.3.

- Its objects are strictly unital symmetric monoidal functors $\mathsf{C} \longrightarrow \mathsf{D}$ (Definition A.1.22).
- Its morphisms are monoidal natural transformations (Definition A.1.27).
- Identity morphisms and composition are those of monoidal natural transformations.

The monoidal product

$$(6.4.3) \qquad\qquad \oplus : \mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \times \mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \longrightarrow \mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D})$$

is defined as follows.

*Objects*. It sends a pair of strictly unital symmetric monoidal functors

$$\left(F, F^2\right), \left(G, G^2\right) : \mathsf{C} \longrightarrow \mathsf{D}$$

to the following composite functor, with diag denoting the diagonal functor.

$$(6.4.4)$$

$$
\mathsf{C} \xrightarrow{\ \mathrm{diag}\ } \mathsf{C} \times \mathsf{C} \xrightarrow{\ F \times G\ } \mathsf{D} \times \mathsf{D} \xrightarrow{\ \oplus\ } \mathsf{D}
$$
$$\overset{F \oplus G}{\overbrace{\hspace{6cm}}}$$

In other words, the monoidal product is defined pointwise in D:

(6.4.5) $$(F \oplus G)(x) = Fx \oplus Gx \quad \text{for} \quad x \in C.$$

The unit constraint and monoidal constraint of $F \oplus G$ are defined as follows.

*Unit Constraint*: It is the identity morphism

$$1_e : e \longrightarrow e = e \oplus e = (F \oplus G)(e) \quad \text{in} \quad D.$$

*Monoidal Constraint*: It is the following composite in D for objects $x, y \in C$.

(6.4.6)

$$
\begin{array}{ccc}
(F \oplus G)(x) \oplus (F \oplus G)(y) & \xrightarrow{(F \oplus G)^2_{x,y}} & (F \oplus G)(x \oplus y) \\
\| & & \| \\
Fx \oplus Gx \oplus Fy \oplus Gy & & F(x \oplus y) \oplus G(x \oplus y)
\end{array}
$$

$$1 \oplus \zeta_{Gx,Fy} \oplus 1 \searrow \quad Fx \oplus Fy \oplus Gx \oplus Gy \quad \nearrow F^2_{x,y} \oplus G^2_{x,y}$$

The naturality of $(F \oplus G)^2$ follows from

- the naturality of the braiding $\zeta$ of D, $F^2$, and $G^2$, and
- the functoriality of $\oplus$ for D.

*Morphisms*: For a pair of monoidal natural transformations

$$C \overset{F}{\underset{G}{\Rightarrow \theta}} D \quad \text{and} \quad C \overset{F'}{\underset{G'}{\Rightarrow \theta'}} D$$

between strictly unital symmetric monoidal functors, their monoidal product

$$\theta \oplus \theta' : F \oplus F' \longrightarrow G \oplus G'$$

is the following whiskering.

$$C \xrightarrow{\text{diag}} C \times C \overset{F \times F'}{\underset{G \times G'}{\Rightarrow \theta \times \theta'}} D \times D \xrightarrow{\oplus} D$$

In other words, for each object $x \in C$, the $x$-component of $\theta \oplus \theta'$ is

(6.4.7) $$(\theta \oplus \theta')_x = \theta_x \oplus \theta'_x : Fx \oplus F'x \longrightarrow Gx \oplus G'x.$$

This defines a monoidal natural transformation for the following reasons.

- The naturality of $\theta \oplus \theta'$ follows from the naturality of $\theta$ and $\theta'$, together with the functoriality of $\oplus$ in D.
- $\theta \oplus \theta'$ satisfies the unity constraint axiom in (A.1.28) because, if $x = e$ in C, then $\theta_e = 1_e = \theta'_e$.
- $\theta \oplus \theta'$ satisfies the monoidal constraint axiom in (A.1.28) by
  - the naturality of the braiding $\zeta$ in D and
  - the monoidal constraint axiom (A.1.28) for $\theta$ and $\theta'$.

The functoriality of $\oplus$ for $\mathsf{P}^{\mathsf{su}}(C, D)$ follows from the functoriality of $\oplus$ for D.

*Monoidal Unit*. It is the constant functor

(6.4.8) $$e : C \longrightarrow D$$

at the monoidal unit of D. Its unit constraint and monoidal constraint are both given by the identity morphism $1_e$ in D.

*Braiding.* For strictly unital symmetric monoidal functors $F$ and $G$ as in (6.4.4), the $(F, G)$-component of the braiding $\xi$ is the natural isomorphism

(6.4.9)
$$C \underset{G \oplus F}{\overset{F \oplus G}{\Longrightarrow \xi_{F,G}}} D$$

with, for each object $x \in C$, $x$-component given by the braiding

(6.4.10)
$$(F \oplus G)(x) = Fx \oplus Gx \xrightarrow[\cong]{\xi_{Fx,Gx}} Gx \oplus Fx = (G \oplus F)(x)$$

in D. The naturality of the braiding in D implies the naturality of both

- $(\xi_{F,G})_x$ with respect to $x \in C$ and
- $\xi_{F,G}$ with respect to $F$ and $G$ in $\mathsf{P}^{\mathsf{su}}(C, D)$.

This finishes the definition.                                        ◇

**Lemma 6.4.11.** *For small permutative categories* C *and* D, *the quadruple in (6.4.2)*

$$\left(\mathsf{P}^{\mathsf{su}}(C, D), \oplus, e, \xi\right)$$

*is a small permutative category.*

*Proof.* We already explained some of the required conditions in Definition 6.4.1. It remains to prove statements (i) through (v) below.

(i) The pair defined in (6.4.4) and (6.4.6)

$$\left(F \oplus G, (F \oplus G)^2\right) : C \longrightarrow D$$

is a strictly unital symmetric monoidal functor.

(ii) The functor $\oplus$ in (6.4.3) is associative.

(iii) The constant functor e in (6.4.8) is a strict two-sided unit for $\oplus$.

(iv) The natural transformation $\xi_{F,G}$ in (6.4.9) is monoidal (Definition A.1.27).

(v) $\mathsf{P}^{\mathsf{su}}(C, D)$ satisfies the symmetry and hexagon axioms (A.1.15).

*Statement (i).* We need to check the unity axiom (A.1.23), the associativity axiom (A.1.24), and the braiding axiom (A.1.25) for $F \oplus G$.

Since its unit constraint is $1_e$, the unity axiom (A.1.23) for $F \oplus G$ means the equalities

$$(F \oplus G)^2_{e,?} = 1_{F? \oplus G?} = (F \oplus G)^2_{?,e} \quad \text{in} \quad D.$$

These equalities follow from the following equalities in D.

$$\xi_{e,?} = 1_? = \xi_{?,e} \qquad F^2_{e,?} = 1_{F?} = F^2_{?,e} \qquad G^2_{e,?} = 1_{G?} = G^2_{?,e}$$

For objects $x, y, z \in C$, the associativity diagram (A.1.24) for $F \oplus G$ is the boundary of the following diagram in D.

$$
\begin{array}{c}
Fx \oplus Gx \oplus Fy \oplus Gy \oplus Fz \oplus Gz \\
\end{array}
$$

with arrows labeled $1 \oplus \xi \oplus 1$, $1 \oplus \xi \oplus 1$, leading to

$Fx \oplus Fy \oplus Gx \oplus Gy \oplus Fz \oplus Gz$ and $Fx \oplus Gx \oplus Fy \oplus Fz \oplus Gy \oplus Gz$

with $1 \oplus \xi \oplus 1$, $1 \oplus \xi \oplus 1$ to $Fx \oplus Fy \oplus Fz \oplus Gx \oplus Gy \oplus Gz$

$F^2 \oplus G^2 \oplus 1$ to $F(x \oplus y) \oplus G(x \oplus y) \oplus Fz \oplus Gz$

$1 \oplus F^2 \oplus G^2$ to $Fx \oplus Gx \oplus F(y \oplus z) \oplus G(y \oplus z)$

$1 \oplus \xi \oplus 1$, $F^2 \oplus 1 \oplus G^2 \oplus 1$ and $1 \oplus F^2 \oplus 1 \oplus G^2$, $1 \oplus \xi \oplus 1$

$F(x \oplus y) \oplus Fz \oplus G(x \oplus y) \oplus Gz$ and $Fx \oplus F(y \oplus z) \oplus Gx \oplus G(y \oplus z)$

$F^2 \oplus G^2$ and $F^2 \oplus G^2$ to $F(x \oplus y \oplus z) \oplus G(x \oplus y \oplus z)$

The following statements hold for the diagram above.

- The top sub-region commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].
- The left and right sub-regions commute by the naturality of the braiding $\xi$ in D.
- The bottom sub-region commutes by the axiom (A.1.24) for $F$ and $G$.

This proves the associativity axiom (A.1.24) for $F \oplus G$.

For objects $x, y \in C$, the braiding diagram (A.1.25) for $F \oplus G$ is the boundary of the following diagram in D.

$$
\begin{array}{ccc}
(F \oplus G)(x) \oplus (F \oplus G)(y) & & (F \oplus G)(y) \oplus (F \oplus G)(x) \\
\| & & \| \\
Fx \oplus Gx \oplus Fy \oplus Gy & \xrightarrow{\ \xi\ } & Fy \oplus Gy \oplus Fx \oplus Gx \\
{\scriptstyle 1 \oplus \xi \oplus 1}\big\downarrow & & \big\downarrow{\scriptstyle 1 \oplus \xi \oplus 1} \\
Fx \oplus Fy \oplus Gx \oplus Gy & \xrightarrow{\ \xi \oplus \xi\ } & Fy \oplus Fx \oplus Gy \oplus Gx \\
{\scriptstyle F^2 \oplus G^2}\big\downarrow & & \big\downarrow{\scriptstyle F^2 \oplus G^2} \\
F(x \oplus y) \oplus G(x \oplus y) & \xrightarrow{\ F\xi \oplus G\xi\ } & F(y \oplus x) \oplus G(y \oplus x)
\end{array}
$$

The following statements hold for the diagram above.

- The top rectangle commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].
- The bottom rectangle commutes by the axiom (A.1.25) for $F$ and $G$.

This proves the braiding axiom (A.1.25) for $F \oplus G$.

*Statement (ii).* To verify that $\oplus$ in (6.4.3) is associative on objects, we consider strictly unital symmetric monoidal functors

$$
(F, F^2), (G, G^2), (H, H^2) : C \longrightarrow D.
$$

The definition (6.4.5) of $(F \oplus G)(x)$ and the associativity of $\oplus$ in D imply that $(F \oplus G) \oplus H$ is equal to $F \oplus (G \oplus H)$ as functors. Moreover, each of them has unit

constraint given by $1_e$. To show that their monoidal constraints are the same, we consider the following diagram in D for objects $x, y \in C$.

$$Fx \oplus (Gx \oplus Hx) \oplus Fy \oplus (Gy \oplus Hy) \xrightarrow{\ 1 \oplus \zeta \oplus 1\ } Fx \oplus Fy \oplus Gx \oplus (Hx \oplus Gy) \oplus Hy$$

$$\parallel$$

$$(Fx \oplus Gx) \oplus Hx \oplus (Fy \oplus Gy) \oplus Hy$$

$$1 \oplus \zeta \oplus 1 \downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \oplus \zeta \oplus 1 \downarrow$$

$$Fx \oplus (Gx \oplus Fy) \oplus Gy \oplus Hx \oplus Hy \xrightarrow{\ 1 \oplus \zeta \oplus 1\ } Fx \oplus Fy \oplus Gx \oplus Gy \oplus Hx \oplus Hy$$

$$F^2 \oplus G^2 \oplus H^2 \downarrow$$

$$F(x \oplus y) \oplus G(x \oplus y) \oplus H(x \oplus y)$$

The following statements hold for the diagram above.

- The left-bottom composite is the monoidal constraint $\big((F \oplus G) \oplus H\big)^2_{x,y}$.

- The top-right composite is the monoidal constraint $\big(F \oplus (G \oplus H)\big)^2_{x,y}$.

- The top sub-region commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].

This shows that $\oplus$ in (6.4.3) is associative on objects. Its associativity on morphisms follows from the definition (6.4.7) of $(\theta \oplus \theta')_x$ and the associativity of $\oplus$ in D. Thus the monoidal product of $\mathsf{P}^{\mathsf{su}}(C, D)$ is associative.

*Statement (iii).* The constant functor e in (6.4.8), with $1_e$ as the unit and monoidal constraints, is a strict two-sided unit for $\oplus$ in $\mathsf{P}^{\mathsf{su}}(C, D)$ for the following reasons.

- $F \oplus e$ and $e \oplus F$ are both equal to $F$ as functors because the monoidal unit e in D is strict.

- The equalities (A.1.13)

$$\zeta_{e,?} = 1_? = \zeta_{?,e} \quad \text{in} \quad D$$

imply that for $F \oplus e$ and $e \oplus F$, the morphism $1 \oplus \zeta \oplus 1$ in (6.4.6) is the identity. The second morphism in (6.4.6) is $F^2_{x,y} \oplus 1_e$ or $1_e \oplus F^2_{x,y}$, which are both equal to $F^2_{x,y}$ by the strict unity of e in D.

This shows that $\big(\mathsf{P}^{\mathsf{su}}(C, D), \oplus, e\big)$ is a strict monoidal category.

*Statement (iv).* The natural transformation $\xi_{F,G}$ in (6.4.9) satisfies the unity axiom in (A.1.28) because its e-component is

$$\xi_{Fe,Ge} = \xi_{e,e} = 1_e$$

by either unity properties in (A.1.13).

To show that $\xi_{F,G}$ is compatible with the monoidal constraints of its domain and codomain in the sense of (A.1.28), we consider the following diagram in D for

objects $x, y \in C$.

$$
\begin{array}{ccc}
(F \oplus G)(x) \oplus (F \oplus G)(y) & & (G \oplus F)(x) \oplus (G \oplus F)(y) \\
\| & & \| \\
Fx \oplus Gx \oplus Fy \oplus Gy \xrightarrow{\ \zeta \oplus \zeta\ } & & Gx \oplus Fx \oplus Gy \oplus Fy \\
\downarrow{\scriptstyle 1 \oplus \zeta \oplus 1} & & \downarrow{\scriptstyle 1 \oplus \zeta \oplus 1} \\
Fx \oplus Fy \oplus Gx \oplus Gy \xrightarrow{\ \zeta\ } & & Gx \oplus Gy \oplus Fx \oplus Fy \\
\downarrow{\scriptstyle F^2 \oplus G^2} & & \downarrow{\scriptstyle G^2 \oplus F^2} \\
F(x \oplus y) \oplus G(x \oplus y) \xrightarrow{\ \zeta\ } & & G(x \oplus y) \oplus F(x \oplus y)
\end{array}
$$

The following statements hold for the diagram above.

- The left vertical composite is the monoidal constraint $(F \oplus G)^2_{x,y}$.
- The right vertical composite is the monoidal constraint $(G \oplus F)^2_{x,y}$.
- The top rectangle commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].
- The bottom rectangle commutes by the naturality of the braiding in D.

This shows that $\zeta_{F,G}$ is a monoidal natural transformation (Definition A.1.27).

   *Statement (v)*. The symmetry and hexagon axioms (A.1.15) hold in $\mathsf{P}^{\mathsf{su}}(C, D)$ by

- the componentwise definitions (6.4.5), (6.4.7), and (6.4.10), and
- the corresponding axioms in D.

This finishes the proof.                                      $\square$

   From now on $\mathsf{P}^{\mathsf{su}}(C, D)$ is a permutative category as in Lemma 6.4.11.

**Definition 6.4.12** (Composition). For small permutative categories B, C, and D, we define the data of a bilinear functor (Definition 1.4.2)

$$(6.4.13) \qquad m_{B,C,D} : \mathsf{P}^{\mathsf{su}}(C, D) \times \mathsf{P}^{\mathsf{su}}(B, C) \longrightarrow \mathsf{P}^{\mathsf{su}}(B, D)$$

as follows, where we abbreviate $m_{B,C,D}$ to m.

**Objects and Morphisms:** The underlying functor of m is given by

- composition of strictly unital symmetric monoidal functors on objects and
- horizontal composition of monoidal natural transformations on morphisms,

as displayed below.

$$(6.4.14) \qquad B \underset{G}{\overset{F}{\Rightarrow \theta}} C \underset{I}{\overset{H}{\Rightarrow \psi}} D \quad \xmapsto{\ m\ } \quad B \underset{IG}{\overset{HF}{\Rightarrow \psi * \theta}} D$$

   This is part of the 2-category structure on $\mathsf{PermCat}^{\mathsf{su}}$ (Definition A.2.3).

**First Monoidal Constraint:** It is the identity natural transformation

$$(6.4.15) \qquad HF \oplus IF \xrightarrow{\ m_1^2 = 1\ } (H \oplus I)F.$$

   Thus for each object $x \in B$, its $x$-component is the identity morphism

$$HFx \oplus IFx \xrightarrow{\ 1\ } (H \oplus I)(Fx).$$

**Second Monoidal Constraint:** It is the natural transformation

$$(6.4.16) \qquad\qquad HF \oplus HG \xrightarrow{\ \mathsf{m}_2^2\ } H(F \oplus G)$$

with, for each object $x \in \mathsf{B}$, $x$-component given by the monoidal constraint

$$HFx \oplus HGx \xrightarrow{\ H^2_{Fx,Gx}\ } H(Fx \oplus Gx).$$

The naturality of $\mathsf{m}_2^2$ with respect to $x \in \mathsf{B}$ follows from the naturality of $H^2$.

This finishes the definition of $\mathsf{m}_{\mathsf{B},\mathsf{C},\mathsf{D}}$.                        ◇

**Lemma 6.4.17.** *In the context of Definition 6.4.12,*

$$\left(\mathsf{m}_{\mathsf{B},\mathsf{C},\mathsf{D}}\,,\, \mathsf{m}_1^2 = 1\,,\, \mathsf{m}_2^2\right) : \mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \times \mathsf{P}^{\mathsf{su}}(\mathsf{B},\mathsf{C}) \longrightarrow \mathsf{P}^{\mathsf{su}}(\mathsf{B},\mathsf{D})$$

*is a bilinear functor.*

*Proof.* We prove statements (i) through (v) below.

   (i) The natural transformation $\mathsf{m}_1^2$ in (6.4.15) is monoidal.
  (ii) The natural transformation $\mathsf{m}_2^2$ in (6.4.16) is monoidal.
 (iii) $\mathsf{m}_1^2$ is natural in $F$, $H$, and $I$.
 (iv) $\mathsf{m}_2^2$ is natural in $F$, $G$, and $H$.
  (v) $(\mathsf{m}, \mathsf{m}_1^2, \mathsf{m}_2^2)$ satisfies the axioms of a bilinear functor (Definition 1.4.2).

*Statement (i).* To check that the natural transformation $\mathsf{m}_1^2$ in (6.4.15) is monoidal (Definition A.1.27), first recall that each of its components is an identity morphism. Thus $\mathsf{m}_1^2$ satisfies the unit axiom (A.1.28) because its e-component is $1_e$. Moreover, for objects $x, y \in \mathsf{B}$, the $(x, y)$-component of the monoidal constraint of each of $HF \oplus IF$ and $(H \oplus I)F$ is the following composite in $\mathsf{D}$.

$$
\begin{array}{ccc}
HFx \oplus IFx \oplus HFy \oplus IFy & & HF(x \oplus y) \oplus IF(x \oplus y) \\[2pt]
{\scriptstyle 1 \oplus \zeta \oplus 1}\big\downarrow & & \big\uparrow{\scriptstyle H(F^2) \oplus I(F^2)} \\[2pt]
HFx \oplus HFy \oplus IFx \oplus IFy & \xrightarrow{\ H^2 \oplus I^2\ } & H(Fx \oplus Fy) \oplus I(Fx \oplus Fy)
\end{array}
$$

This shows that $\mathsf{m}_1^2$ in (6.4.15) is a monoidal natural transformation.

*Statement (ii).* To check that the natural transformation $\mathsf{m}_2^2$ in (6.4.16) is monoidal, first note that the unity axioms (A.1.23) of the strictly unital symmetric monoidal functor $(H, H^2, H^0 = 1_e)$ imply the equalities

$$(6.4.18) \qquad\qquad H^2_{e,?} = 1_{H?} = H^2_{?,e}.$$

Since the domain and codomain of $\mathsf{m}_2^2$ are strictly unital, the unit axiom (A.1.28) for $\mathsf{m}_2^2$ is the equality

$$H^2_{Fe,Ge} = 1_e,$$

which holds by (6.4.18) and the strict unity of $F$, $G$, and $H$.

Next we check the compatibility of $m_2^2$ with the monoidal constraints of its domain and codomain (A.1.28). For objects $x, y \in B$, we use the notation

$$X_1 = HFx \oplus H(Gx \oplus Fy) \oplus HGy,$$
$$X_2 = H(Fx \oplus Gx \oplus Fy) \oplus HGy,$$
$$Y_1 = HFx \oplus H(Fy \oplus Gx) \oplus HGy, \quad \text{and}$$
$$Y_2 = H(Fx \oplus Fy \oplus Gx) \oplus HGy,$$

and consider the following diagram in D.



The following statements hold for the diagram above.

- The left vertical composite is the $(x, y)$-component of the monoidal constraint of $HF \oplus HG$, which is the domain of $m_2^2$.
- The right vertical composite is the $(x, y)$-component of the monoidal constraint of $H(F \oplus G)$, which is the codomain of $m_2^2$.
- The sub-region labeled ♠ is commutative by the compatibility of $H$ with the braiding (A.1.25).
- The three sub-regions labeled ♣ are commutative by the naturality of $H^2$.
- The two unlabeled triangles are commutative by the associativity of $H^2$ (A.1.24).

This shows that $m_2^2$ in (6.4.16) is a monoidal natural transformation.

*Statement (iii).* To show that $m_1^2$ is natural in $F$, $H$, and $I$, we consider monoidal natural transformations

between strictly unital symmetric monoidal functors. The corresponding naturality diagram for $m_1^2$ commutes because, for each object $x \in B$, both $x$-components

$$\big((\psi * \theta) \oplus (\varphi * \theta)\big)_x \quad \text{and} \quad \big((\psi \oplus \varphi) * \theta\big)_x$$

are given by the composite

$$HFx \oplus IFx \xrightarrow{H(\theta_x) \oplus I(\theta_x)} HF'x \oplus IF'x \xrightarrow{\psi_{F'x} \oplus \varphi_{F'x}} H'F'x \oplus I'F'x$$

in D.

*Statement (iv).* To show that $m_2^2$ is natural in $F$, $G$, and $H$, we consider monoidal natural transformations



between strictly unital symmetric monoidal functors and the corresponding naturality diagram below.



For each object $x \in B$, the $x$-component of the diagram above is the boundary of the following diagram in D.



The top rectangle commutes by the naturality of $H^2$. The bottom rectangle commutes by the monoidality of $\psi$ (A.1.28). This shows that $m_2^2$ is natural in $F$, $G$, and $H$.

*Statement (v).* Next we check the axioms of a bilinear functor for $(m, m_1^2, m_2^2)$.

*Unity (1.4.4).* It holds for m in (6.4.13) for the following reasons.

- The monoidal unit in each hom permutative category $P^{su}(C, D)$ is the constant functor at the monoidal unit of D (6.4.8).
- Objects in $P^{su}(C, D)$ are *strictly unital* symmetric monoidal functors.
- Each morphism in $P^{su}(C, D)$ has e-component given by $1_e$ by the unit axiom of a monoidal natural transformation (A.1.28).

*Constraint Unity (1.4.5)*. It holds for $m_1^2$ in (6.4.15) because each of its compo-
nents is an identity morphism. For the same reason, $m_1^2$ also satisfies the next two
axioms, (1.4.6) and (1.4.7).

The constraint unity axiom holds for $m_2^2$ in (6.4.16) for the following reasons.

- If $H$ is the monoidal unit e (6.4.8), then its monoidal constraint $H^2$ is $1_e$
  by definition.
- If either $F$ or $G$ is the monoidal unit e, then (6.4.18) yields the desired
  equalities
  $$H_{e,Gx}^2 = 1_{HGx} \quad \text{and} \quad H_{Fx,e}^2 = 1_{HFx}.$$

*Constraint Associativity (1.4.6) and Symmetry (1.4.7)*. They hold for $m_2^2$ by the
associativity (A.1.24) and symmetry (A.1.25) of the monoidal constraint $H^2$.

*Constraint 2-By-2 (1.4.8)*. We consider strictly unital symmetric monoidal func-
tors as follows.

$$\text{B} \underset{G}{\overset{F}{\rightrightarrows}} \text{C} \underset{I}{\overset{H}{\rightrightarrows}} \text{D}$$

In the constraint 2-by-2 axiom, if $i = 1$ and $k = 2$, then the diagram (1.4.8) is the left
pentagon below, which commutes by the definition of $(H \oplus I)^2$ in (6.4.6).



If $i = 2$ and $k = 1$, then the constraint 2-by-2 diagram (1.4.8) is the right pentagon
above. For each object $x \in$ B, the $x$-component of this pentagon is the following
diagram in D.



This diagram commutes by the symmetry axiom (A.1.15) and the functoriality of
$\oplus$ in the permutative category D.

This finishes the proof that $(m, m_1^2, m_2^2)$ is a bilinear functor.        $\square$

Recall the description of a PermCat$^{su}$-category in Explanation 6.3.2.

**Definition 6.4.19.** The *self-enrichment* of P$^{su}$, which we also denote by P$^{su}$, is the
P$^{su}$-category defined as follows. Theorem 6.4.20 verifies that P$^{su}$ is a P$^{su}$-category.

**Objects:** The objects are small permutative categories (Definition A.1.14).

**Hom Permutative Categories:** For small permutative categories C and D, the hom
permutative category P$^{su}$(C, D) is the one in Lemma 6.4.11.

**Composition:** For small permutative categories B, C, and D, the composition bi-
linear functor

$$\left(m_{B,C,D}\,,\, m_1^2, m_2^2\right) : P^{su}(C,D) \times P^{su}(B,C) \longrightarrow P^{su}(B,D)$$

is the one in Lemma 6.4.17.

**Identities:** Each small permutative category B is equipped with the identity
symmetric monoidal functor $1_B$, which is also regarded as an object in
$P^{su}(B,B)$.

This finishes the definition of the self-enrichment of $P^{su}$.                    ◇

**Theorem 6.4.20.** *Equipped with the self-enrichment in Definition 6.4.19,* PermCat$^{su}$ *is
a* PermCat$^{su}$*-category.*

*Proof.* We need to check that $P^{su}$ satisfies the associativity axiom (6.1.4) and the
unity axiom (6.1.5) of a $P^{su}$-category.

*Associativity (6.1.4)*. For $P^{su}$ this axiom means the commutativity of the follow-
ing diagram of composite 3-linear functors for small permutative categories A, B,
C, and D.

(6.4.21)

$$
\begin{array}{ccc}
P^{su}(C,D) \times P^{su}(B,C) \times P^{su}(A,B) & \xrightarrow{\; m_{B,C,D} \times 1 \;} & P^{su}(B,D) \times P^{su}(A,B) \\
\Big\downarrow {\scriptstyle 1 \times m_{A,B,C}} & & \Big\downarrow {\scriptstyle m_{A,B,D}} \\
P^{su}(C,D) \times P^{su}(A,C) & \xrightarrow{\quad m_{A,C,D} \quad} & P^{su}(A,D)
\end{array}
$$

As a diagram of functors, (6.4.21) is commutative because

- composition of strictly unital symmetric monoidal functors and
- horizontal composition of monoidal natural transformations

are both associative.

To show that the two composites in (6.4.21) have the same linearity constraints,
as in (6.3.8) and (6.3.9), we consider strictly unital symmetric monoidal functors as
follows.

$$A \underset{H'}{\overset{H}{\rightrightarrows}} B \underset{G'}{\overset{G}{\rightrightarrows}} C \underset{F'}{\overset{F}{\rightrightarrows}} D$$

We consider the three diagrams in (6.3.8) and (6.3.9) in the current context of $P^{su}$.

- The diagram for the first linearity constraint (6.3.8) is commutative be-
  cause each arrow is the identity, since $m_1^2 = 1$ by definition (6.4.15).
- The diagram for the second linearity constraint (= left diagram in (6.3.9))
  is commutative because $m_1^2 = 1$ and each $m_2^2$ is given by the monoidal
  constraint $F^2$ of $F$.
- The diagram for the third linearity constraint (= right diagram in (6.3.9))
  is as follows.

$$
\begin{array}{ccc}
(FG)H \oplus (FG)H' & \xrightarrow{\;(FG)^2\;} & (FG)(H \oplus H') \\
\rotatebox{90}{$=$}\!\!\!\!\!\! & & \\
F(GH) \oplus F(GH') & & \Big\| \\
\Big\downarrow {\scriptstyle F^2} & & \\
F(GH \oplus GH') & \xrightarrow{\; F(G^2) \;} & F\big(G(H \oplus H')\big)
\end{array}
$$

This diagram commutes by the definition of the monoidal constraint $(FG)^2$ (Definition A.1.26).

This proves the associativity axiom (6.1.4) for $\mathsf{P}^{\mathsf{su}}$.

*Unity (6.1.5).* In the current context of $\mathsf{P}^{\mathsf{su}}$, the unity diagram (6.3.10) commutes as a diagram of functors because, for each small permutative category B, the identity $i_\mathsf{B}$ is the identity symmetric monoidal functor $1_\mathsf{B}$ (Definition 6.4.19). Moreover, the diagram (6.3.11) for the monoidal constraints is commutative for the following reasons.

- The left half of (6.3.11) is commutative because $\mathsf{m}_1^2 = 1$.
- The right half of (6.3.11) is commutative because the monoidal constraint of $1_\mathsf{B}$ is the identity.

This finishes the proof that $\mathsf{P}^{\mathsf{su}}$ is a $\mathsf{P}^{\mathsf{su}}$-category. $\qquad\square$

## 6.5. Bilinear Evaluation for Permutative Categories

In this section we discuss a bilinear evaluation for permutative categories. The bilinear evaluation in (6.5.2) below is an analog of the evaluation in a symmetric monoidal closed category (B.3.2). Along with the self-enrichment of $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 6.4.20), the bilinear evaluation in this section is also a part of the *closed* multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$, which we discuss in Chapter 8. We will use the bilinear evaluation in Explanations 9.4.5, 9.4.9, and 12.2.9 below.

This section is organized as follows.

- The evaluation bilinear functor for small permutative categories is constructed in Definition 6.5.1 and verified in Proposition 6.5.7.
- Proposition 6.5.8 shows that evaluation is compatible with the composition bilinear functor m in Lemma 6.4.17.

Recall *bilinear functors* between permutative categories (Definition 1.4.2) and the permutative category $\mathsf{PermCat}^{\mathsf{su}}(\mathsf{C}, \mathsf{D})$ in Lemma 6.4.11. To simplify the presentation, we use the shortened notation in (6.3.1):

$$\mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.$$

**Definition 6.5.1.** For small permutative categories C and D, we define the data of a bilinear functor

$$(6.5.2) \qquad \left(\mathsf{ev}_{\mathsf{C},\mathsf{D}}, (\mathsf{ev}_{\mathsf{C},\mathsf{D}})_1^2, (\mathsf{ev}_{\mathsf{C},\mathsf{D}})_2^2\right) : \mathsf{P}^{\mathsf{su}}(\mathsf{C}, \mathsf{D}) \times \mathsf{C} \longrightarrow \mathsf{D},$$

which is called the *evaluation*, as follows.

**Objects:** For a strictly unital symmetric monoidal functor $(F, F^2) : \mathsf{C} \longrightarrow \mathsf{D}$ and an object $x \in \mathsf{C}$, the object assignment is defined as

$$(6.5.3) \qquad \mathsf{ev}_{\mathsf{C},\mathsf{D}}(F, x) = Fx \quad \text{in} \quad \mathsf{D}.$$

**Morphisms:** Consider
- a monoidal natural transformation $\theta : F \longrightarrow G$ between strictly unital symmetric monoidal functors $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ and
- a morphism $f : x \longrightarrow y$ in C.

The morphism

$$\mathsf{ev}_{\mathsf{C},\mathsf{D}}(F, x) = Fx \xrightarrow{\;\mathsf{ev}_{\mathsf{C},\mathsf{D}}(\theta, f)\;} Gy = \mathsf{ev}_{\mathsf{C},\mathsf{D}}(G, y)$$

is defined as either one of the following two composites in D, which are equal by the naturality of $\theta$.

(6.5.4)

$$
\begin{array}{ccc}
Fx & \xrightarrow{\ \theta_x\ } & Gx \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
Fy & \xrightarrow{\ \theta_y\ } & Gy
\end{array}
$$

**First Linearity Constraint:** It is the identity natural transformation with components as follows.

(6.5.5)

$$
\begin{array}{ccc}
\mathrm{ev}_{\mathsf{C},\mathsf{D}}(F,x) \oplus \mathrm{ev}_{\mathsf{C},\mathsf{D}}(G,x) & \xrightarrow{\ (\mathrm{ev}_{\mathsf{C},\mathsf{D}})_1^2\ } & \mathrm{ev}_{\mathsf{C},\mathsf{D}}(F \oplus G, x) \\
\| & & \| \\
Fx \oplus Gx & \xrightarrow{\ \ \ 1\ \ \ } & (F \oplus G)x
\end{array}
$$

**Second Linearity Constraint:** It is given componentwise by the monoidal constraint of the first variable as follows.

(6.5.6)

$$
\begin{array}{ccc}
\mathrm{ev}_{\mathsf{C},\mathsf{D}}(F,x) \oplus \mathrm{ev}_{\mathsf{C},\mathsf{D}}(F,y) & \xrightarrow{\ (\mathrm{ev}_{\mathsf{C},\mathsf{D}})_2^2\ } & \mathrm{ev}_{\mathsf{C},\mathsf{D}}(F, x \oplus y) \\
\| & & \| \\
Fx \oplus Fy & \xrightarrow{\ F_{x,y}^2\ } & F(x \oplus y)
\end{array}
$$

This finishes the definition of the evaluation. If there is no danger of confusion, we omit the subscripts in $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$.  ◇

   Propositions 6.5.7 and 6.5.8 below show that evaluation is bilinear and has the expected property with respect to composition. Proposition 6.5.7 is analogous to the composition bilinear functor in Lemma 6.4.17. In Lemma 8.3.8 we extend it to a multilinear functor as part of the closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$.

**Proposition 6.5.7.** *For small permutative categories* C *and* D*, the triple*

$$
\left(\mathrm{ev}_{\mathsf{C},\mathsf{D}},\, (\mathrm{ev}_{\mathsf{C},\mathsf{D}})_1^2,\, (\mathrm{ev}_{\mathsf{C},\mathsf{D}})_2^2\right) : \mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \times \mathsf{C} \longrightarrow \mathsf{D}
$$

*in (6.5.2) is a bilinear functor.*

*Proof.* We prove statements (i) through (iii) below.

   (i) $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$ is a functor.
  (ii) $(\mathrm{ev}_{\mathsf{C},\mathsf{D}})_1^2$ in (6.5.5) and $(\mathrm{ev}_{\mathsf{C},\mathsf{D}})_2^2$ in (6.5.6) are natural transformations.
 (iii) $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$ satisfies the bilinear functor axioms.

   *Statement (i).* The assignment $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$ preserves identity morphisms because all four morphisms in (6.5.4) are identity morphisms if $\theta = 1_F$ and $f = 1_x$.

   To see that $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$ preserves composition, consider a monoidal natural transformation $\psi : G \longrightarrow H$ and a morphism $g : y \longrightarrow z$ in C. In the following diagram in D, the rectangle is commutative by the naturality of $\psi$.

$$
\begin{array}{ccccc}
Fx & \xrightarrow{\ \theta_x\ } & Gx & \xrightarrow{\ \psi_x\ } & Hx \\
 & & {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Hf} \\
 & & Gy & \xrightarrow{\ \psi_y\ } & Hy \\
 & & & & \downarrow{\scriptstyle Hg} \\
 & & & & Hz
\end{array}
$$

The top-right composite is

$$\big(H(gf)\big)(\psi\theta)_x = \mathsf{ev}_{\mathsf{C},\mathsf{D}}(\psi\theta, gf).$$

The other composite is

$$(Hg)\psi_y(Gf)\theta_x = \mathsf{ev}_{\mathsf{C},\mathsf{D}}(\psi, g)\mathsf{ev}_{\mathsf{C},\mathsf{D}}(\theta, f).$$

This shows that $\mathsf{ev}_{\mathsf{C},\mathsf{D}}$ is a functor.

*Statement (ii)*. The morphism $(\mathsf{ev}_{\mathsf{C},\mathsf{D}})_1^2$ in (6.5.5) is natural

- with respect to $F$ and $G$ by (6.4.7) and
- with respect to $x$ by (6.4.5).

The morphism $(\mathsf{ev}_{\mathsf{C},\mathsf{D}})_2^2$ in (6.5.6) is natural

- with respect to $F$ by the monoidality of $\theta$ (A.1.28) and
- with respect to $x$ and $y$ by the naturality of the monoidal constraint $F^2$.

*Statement (iii)*. The unity axiom (1.4.4) follows from the definitions (6.5.3) and (6.5.4).

The first linearity constraint $(\mathsf{ev}_{\mathsf{C},\mathsf{D}})_1^2$ satisfies the constraint unity, associativity, and symmetry axioms, (1.4.5) through (1.4.7), because its components are identity morphisms.

The second linearity constraint $(\mathsf{ev}_{\mathsf{C},\mathsf{D}})_2^2$ satisfies the constrain unity axiom (1.4.5) for the following reasons.

- The monoidal constraint of the monoidal unit (6.4.8) in $\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D})$ is the identity morphism $1_\mathsf{e}$ in $\mathsf{D}$.
- If either $x$ or $y$ is the monoidal unit $\mathsf{e}$ in $\mathsf{C}$, then

$$F_{\mathsf{e},y}^2 = 1_{Fy} \quad \text{and} \quad F_{x,\mathsf{e}}^2 = 1_{Fx}$$

by the unity of $(F, F^2, F^0 = 1_\mathsf{e})$ in (A.1.23).

The constraint associativity and symmetry axioms, (1.4.6) and (1.4.7), hold for $(\mathsf{ev}_{\mathsf{C},\mathsf{D}})_2^2$ by the associativity (A.1.24) and symmetry (A.1.25) of the monoidal constraint $F^2$.

The constraint 2-by-2 axiom (1.4.8) is

- the left pentagon below if $(i,k) = (1,2)$ and
- the right pentagon below if $(i,k) = (2,1)$.



The left pentagon is commutative by the definition of $(F \oplus G)^2$ in (6.4.6). The right pentagon is commutative by the symmetry axiom (A.1.15) and the functoriality of $\oplus$ in the permutative category $\mathsf{D}$. □

The following result is the permutative categorical analog of the left diagram in (B.3.9) for symmetric monoidal closed categories. We will use this result in Proposition 9.1.8 as part of the identification of the two self-enrichment of PermCat$^{\mathsf{su}}$; see (v) in that proof. Recall the composition of multilinear functors in Definition 1.4.21 and the composition bilinear functor $\mathsf{m}_{\mathsf{B},\mathsf{C},\mathsf{D}}$ in Lemma 6.4.17.

**Proposition 6.5.8.** *For small permutative categories* B, C, *and* D, *the following two composite 3-linear functors are equal.*

(6.5.9)

$$
\begin{array}{ccc}
\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \times \mathsf{P}^{\mathsf{su}}(\mathsf{B},\mathsf{C}) \times \mathsf{B} & \xrightarrow{\ \mathsf{m}_{\mathsf{B},\mathsf{C},\mathsf{D}} \times 1\ } & \mathsf{P}^{\mathsf{su}}(\mathsf{B},\mathsf{D}) \times \mathsf{B} \\
{\scriptstyle 1 \times \mathsf{ev}_{\mathsf{B},\mathsf{C}}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{ev}_{\mathsf{B},\mathsf{D}}} \\
\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D}) \times \mathsf{C} & \xrightarrow[\ \mathsf{ev}_{\mathsf{C},\mathsf{D}}\ ]{} & \mathsf{D}
\end{array}
$$

*Proof.* We prove statements (i) and (ii) below.

    (i) The two composites in (6.5.9) are equal as functors.

   (ii) Their three respective linearity constraints are equal.

*Statement (i).* Consider a morphism $f : x \longrightarrow y$ in B and monoidal natural transformations between strictly unital symmetric monoidal functors as follows.

$$
\mathsf{B} \underset{G}{\overset{F}{\rightrightarrows}} {\Downarrow\theta}\ \mathsf{C}\ \underset{I}{\overset{H}{\rightrightarrows}} {\Downarrow\psi}\ \mathsf{D}
$$

Each composite in (6.5.9) sends the triple $(H, F, x)$ to the object $HFx \in \mathsf{D}$. For morphisms we consider the following diagram in D.

$$
\begin{array}{ccccc}
HFx & \xrightarrow{\ H\theta_x\ } & HGx & \xrightarrow{\ \psi_{Gx}\ } & IGx \\
& {\scriptstyle H\mathsf{ev}_{\mathsf{B},\mathsf{C}}(\theta,f)}\searrow & \Big\downarrow{\scriptstyle HGf} & & \Big\downarrow{\scriptstyle IGf} \\
& & HGy & \xrightarrow[\ \psi_{Gy}\ ]{} & IGy
\end{array}
$$

The following statements hold for the diagram above.

- The top-right composite is

$$
(IGf)(\psi * \theta)_x = \mathsf{ev}_{\mathsf{B},\mathsf{D}}(\psi * \theta, f) = \mathsf{ev}_{\mathsf{B},\mathsf{D}}\big(\mathsf{m}_{\mathsf{B},\mathsf{C},\mathsf{D}}(\psi,\theta), f\big).
$$

- The bottom composite is $\mathsf{ev}_{\mathsf{C},\mathsf{D}}\big(\psi, \mathsf{ev}_{\mathsf{B},\mathsf{C}}(\theta, f)\big)$ by the left-bottom composite in (6.5.4) applied to $\mathsf{ev}_{\mathsf{C},\mathsf{D}}(\psi, -)$.
- The left triangle is commutative by the definition (6.5.4) of $\mathsf{ev}_{\mathsf{B},\mathsf{C}}(\theta, f)$.
- The right rectangle is commutative by the naturality of $\psi$.

This shows that (6.5.9) is commutative as a diagram of functors.

*Statement (ii).* Recall from (1.4.27) the definition of the linearity constraints of a composite multilinear functor. Now we consider the three linearity constraints of the two composites in (6.5.9).

- Their first linearity constraints are equal because both first linearity constraints $\mathsf{m}_1^2$ in (6.4.15) and $\mathsf{ev}_1^2$ in (6.5.5) are the identity.
- Their second linearity constraints are equal because both second linearity constraints $\mathsf{m}_2^2$ in (6.4.16) and $\mathsf{ev}_2^2$ in (6.5.6) are given by the monoidal constraint of the monoidal functor in question.

- Their third linearity constraints are equal because the monoidal constraint of the composite monoidal functor $HF$ is $H(F^2) \circ H^2$ by Definition A.1.26.

This proves that the two composite 3-linear functors in (6.5.9) are equal. □

## 6.6. Opposite Enriched Categories

In this section we discuss the opposite of an M-category when $(M, \gamma, 1)$ is a multicategory; see Proposition 6.6.7. By Definition C.1.3 M is a Set-multicategory, where $(\mathsf{Set}, \times, *)$ is the symmetric monoidal category of sets and functions with the monoidal product given by the Cartesian product. We emphasize that a multicategory has a symmetric group action (C.1.4), which is necessary to define opposite enriched categories.

When the enriching category is symmetric monoidal, we observe that the opposite enriched category in Definition B.1.13 is the same as the one in this section; see Proposition 6.6.8. Looking ahead we consider change of enrichment of opposite enriched categories in Proposition 7.2.1. Moreover, opposite enriched categories are important in Chapter 10 and Part 4, where they are the domains of enriched presheaf categories (10.1.3).

Recall the notion of an M-category in Definition 6.1.1.

**Definition 6.6.1.** Suppose $(M, \gamma, 1)$ is a multicategory, and $(C, m, i)$ is an M-category. The *opposite* M-*category*

$$(C^{op}, m^{op}, i)$$

is the M-category defined as follows.

**Objects:** It has the same class of objects as C.

**Hom Objects:** For objects $x, y \in C^{op}$, its hom object is the object

$$C^{op}(x, y) = C(y, x) \quad \text{in} \quad M.$$

**Composition:** For objects $x, y, z \in C^{op}$, its composition is the binary multimorphism

(6.6.2) $$m^{op}_{x,y,z} : \big(C^{op}(y, z), C^{op}(x, y)\big) \longrightarrow C^{op}(x, z) \quad \text{in} \quad M$$

given by the image of the composition binary multimorphism

$$m_{z,y,x} : \big(C(y, x), C(z, y)\big) \longrightarrow C(z, x)$$

under the symmetric group action of the nonidentity permutation $\tau \in \Sigma_2$:

(6.6.3) $$M\big(C(y, x), C(z, y); C(z, x)\big) \xrightarrow[\cong]{\tau} M\big(C(z, y), C(y, x); C(z, x)\big).$$

**Identities:** The identity of an object $x \in C^{op}$ is the nullary multimorphism

(6.6.4) $$i_x : \langle\rangle \longrightarrow C(x, x) = C^{op}(x, x) \quad \text{in} \quad M.$$

This is the same as the identity of $x$ as an object in C.

This finishes the definition of $(C^{op}, m^{op}, i)$. Proposition 6.6.7 proves that $C^{op}$ is actually an M-category. ◇

**Explanation 6.6.5.** We denote the composition $m_{x,y,z}^{\mathsf{op}}$ in (6.6.2) diagrammatically as follows.

(6.6.6)

$$
\begin{array}{ccc}
\left(\mathsf{C}^{\mathsf{op}}(y,z),\mathsf{C}^{\mathsf{op}}(x,y)\right) & \xrightarrow{\ m_{x,y,z}^{\mathsf{op}}\ } & \mathsf{C}^{\mathsf{op}}(x,z) \\
\| & & \| \\
\left(\mathsf{C}(z,y),\mathsf{C}(y,x)\right) & & \mathsf{C}(z,x)
\end{array}
$$

with $\tau$ diagonal to $\left(\mathsf{C}(y,x),\mathsf{C}(z,y)\right)$ and $m_{z,y,x}$.

We understand this composite as given by the symmetric group action (6.6.3) in M. This diagram is an analog of the composite (B.1.14) that defines the composition of an opposite V-category with V a braided monoidal category. ◇

We now check that $\mathsf{C}^{\mathsf{op}}$ satisfies the axioms of an M-category. The following observation is stated in [**BO15**, Remark 2.8].

**Proposition 6.6.7.** *In the context of Definition 6.6.1, $(\mathsf{C}^{\mathsf{op}},\mathsf{m}^{\mathsf{op}},i)$ is an M-category.*

*Proof.* We need to prove the associativity axiom (6.1.4) and the unity axiom (6.1.5) for $\mathsf{C}^{\mathsf{op}}$.

For objects $w,x,y,z \in \mathsf{C}^{\mathsf{op}}$, the associativity diagram (6.1.4) for $\mathsf{C}^{\mathsf{op}}$ is the boundary of the following diagram in M, with $\mathsf{C}_{x,y}$ denoting $\mathsf{C}(x,y)$ and $\tau \in \Sigma_2$ denoting the nonidentity permutation.



The following statements hold for the diagram above.

- The right and bottom strips are the definition of $\mathsf{m}^{\mathsf{op}}$ (6.6.6).
- The left and top strips are commutative by (6.6.6) and the right unity axiom (C.1.9) for M.
- Each $\tau\langle r,s\rangle \in \Sigma_3$ is the block permutation (C.1.12) that permutes a block of length $r$ with a block of length $s$. There are equalities of permutations

$$\tau\langle 1,2\rangle \circ (1 \times \tau) = (1,3) = \tau\langle 2,1\rangle \circ (\tau \times 1) \quad \text{in} \quad \Sigma_3,$$

  where $(1,3)$ denotes the transposition of 1 and 3. The upper left rectangle composed with either $\gamma(\mathsf{m};\mathsf{m},1)$ or $\gamma(\mathsf{m};1,\mathsf{m})$ is commutative by the symmetric group axiom (C.1.7) for M.
- The lower left rectangle composed with the lower right horizontal $\mathsf{m}$ is commutative by the top equivariance axiom (C.1.11) for M.
- The lower right rectangle is commutative by the associativity axiom (6.1.4) for C.

- The upper right rectangle composed with the lower right vertical $m$ is commutative by the top equivariance axiom (C.1.11) for $M$.

This proves the associativity axiom for $\mathsf{C}^{\mathsf{op}}$.

For objects $x, y \in \mathsf{C}^{\mathsf{op}}$, the unity diagram (6.1.5) for $\mathsf{C}^{\mathsf{op}}$ is the boundary of the following diagram in $M$.

$$
\begin{array}{ccccc}
(\mathsf{C}_{y,x}, \langle\rangle) & \xrightarrow{\ \tau\langle 1,0\rangle\, =\, 1\ } & (\langle\rangle, \mathsf{C}_{y,x}) = (\mathsf{C}_{y,x}, \langle\rangle) & \xleftarrow{\ \tau\langle 0,1\rangle\, =\, 1\ } & (\langle\rangle, \mathsf{C}_{y,x}) \\
\downarrow{\scriptstyle(1,i_x)} & {\scriptstyle(i_x,1)} & \downarrow{\scriptstyle 1} & {\scriptstyle(1,i_y)} & \downarrow{\scriptstyle(i_y,1)} \\
(\mathsf{C}_{y,x}, \mathsf{C}_{x,x}) & & & & (\mathsf{C}_{y,y}, \mathsf{C}_{y,x}) \\
{\scriptstyle\tau}\searrow & & & & \swarrow{\scriptstyle\tau} \\
& (\mathsf{C}_{x,x}, \mathsf{C}_{y,x}) \xrightarrow{\ m\ } & \mathsf{C}_{y,x} & \xleftarrow{\ m\ } (\mathsf{C}_{y,x}, \mathsf{C}_{y,y}) &
\end{array}
$$

The following statements hold for the diagram above.

- The middle two sub-regions are commutative by the unity axiom (6.1.5) for $\mathsf{C}$.
- Since the block permutations $\tau\langle 1,0\rangle$ and $\tau\langle 0,1\rangle \in \Sigma_1$ are the identity, they act on $M$ as the identity by the first part of the symmetric group axiom (C.1.7) for $M$.
- The left sub-region composed with the bottom left $m$ is commutative by the top equivariance axiom (C.1.11) for $M$.
- The right sub-region composed with the bottom right $m$ is commutative by the top equivariance axiom (C.1.11) for $M$.

This proves the unity axiom for $\mathsf{C}^{\mathsf{op}}$.                                          $\square$

For a monoidal category $\mathsf{V}$, Proposition 6.2.1 shows that a $\mathsf{V}$-category is the same thing as an $(\mathsf{End}\,\mathsf{V})$-category for the non-symmetric endomorphism multi-category $\mathsf{End}\,\mathsf{V}$. Next we observe that, if $\mathsf{V}$ is symmetric monoidal, then we can also identify opposite categories enriched in $\mathsf{V}$ and in $\mathsf{End}\,\mathsf{V}$.

**Proposition 6.6.8.** *For each symmetric monoidal category* $(\mathsf{V}, \otimes, 1)$ *and* $\mathsf{V}$*-category* $\mathsf{C}$,

- *the opposite* $\mathsf{V}$*-category* $\mathsf{C}^{\mathsf{op}}$ *in Definition B.1.13 and*
- *the opposite* $(\mathsf{End}\,\mathsf{V})$*-category* $\mathsf{C}^{\mathsf{op}}$ *in Definition 6.6.1*

*are the same.*

*Proof.* A comparison of Definitions 6.6.1 and B.1.13 shows that $\mathsf{C}^{\mathsf{op}}$ in these two definitions have

- the same objects, namely, the objects of $\mathsf{C}$;
- the same hom objects, namely,

$$\mathsf{C}^{\mathsf{op}}(x,y) = \mathsf{C}(y,x)$$

for objects $x, y \in \mathsf{C}^{\mathsf{op}}$; and
- the same identity for each object $x \in \mathsf{C}^{\mathsf{op}}$, namely,

$$i_x \in (\mathsf{End}\,\mathsf{V})\big(\langle\rangle;\, \mathsf{C}(x,x)\big) = \mathsf{V}\big(1, \mathsf{C}(x,x)\big).$$

Their compositions, (6.6.6) and (B.1.14), are also the same because the symmetric group action of the multicategory $\mathsf{End}\,\mathsf{V}$ is induced by the braiding of the symmetric monoidal category $\mathsf{V}$.                                          $\square$

**Example 6.6.9.** Proposition 6.6.8 applies to all the symmetric monoidal categories in Example 6.2.3. For example, consider the symmetric monoidal closed category $\mathsf{Mod}^{\mathcal{M}_1}$ in Proposition 1.3.17 (7), with associated Cat-multicategory in Explanation 1.3.24. For a $\mathsf{Mod}^{\mathcal{M}_1}$-category $\mathsf{C}$, the opposite $\mathsf{Mod}^{\mathcal{M}_1}$-category $\mathsf{C}^{\mathsf{op}}$ in the sense of Definitions 6.6.1 and B.1.13 are the same. $\diamond$

CHAPTER 7

# Change of Multicategorical Enrichment

This chapter defines and develops the basic properties of the change-of-enrichment 2-functor induced by a non-symmetric multifunctor

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

between non-symmetric multicategories M and N. If M and N are multicategories—equipped with the necessary symmetric group actions—then change of enrichment is shown to preserve opposites in Proposition 7.2.1. Composing change of enrichment functors is treated in Proposition 7.4.1, and Theorem 7.5.6 extends this to show that there is a 2-functor

$$\mathbf{E} : \mathsf{Multicat}^{\mathsf{ns}} \longrightarrow \mathsf{2Cat}$$

given by the assignments

- $\mathbf{E}\mathsf{M} = \mathsf{M}\text{-Cat}$ (Theorem 6.1.27) on objects,
- $\mathbf{E}F = (-)_F$ (Proposition 7.1.9) on 1-cells, and
- $\mathbf{E}\theta = (-)_\theta$ (Proposition 7.5.5) on 2-cells.

Our main motivation for these results is for application to $K$-theory multifunctors, discussed in Examples 7.2.3, 7.3.2, and 7.4.2.

**Connection with Other Chapters.** The material in this chapter is used in each of Chapters 9 through 12. Of these, Chapters 9 and 10 extend the theory here to that of self-enrichments and enriched diagrams, respectively. The factorization of Elmendorf-Mandell $K$-theory (7.4.3) is discussed further

- in Theorem 9.4.2 in the context of standard enrichment and
- in Theorem 10.6.2 in the context of presheaf change of enrichment.

Chapters 11 and 12 give homotopy-theoretic applications, with Chapter 12 focusing on diagrams and Mackey functors enriched in $\mathsf{Multicat}_*$ and $\mathsf{PermCat}^{\mathsf{su}}$.

**Background.** The content of this chapter depends on the multicategorical enrichment developed in Chapter 6.

**Chapter Summary.** Section 7.1 defines the change-of-enrichment 2-functor along a non-symmetric multifunctor. Section 7.2 specializes to the symmetric case and shows that change of enrichment along a multifunctor preserves opposite enriched categories. Section 7.3 shows that the two notions of change of enrichment along a monoidal functor—monoidal or multicategorical—agree. Section 7.4 describes compositionality for change of enrichment, and Section 7.5 extends this to show that change of enrichment is 2-functorial. Here is a summary table.

| change of enrichment along a non-symmetric multifunctor | 7.1.1 and 7.1.9 |
|---|---|
| examples of change of enrichment | 7.1.10 and 7.1.12 |
| preservation of opposite enriched categories | 7.2.1 and 7.2.3 |
| change of enrichment along a monoidal functor | 7.3.1 |
| composition of change-of-enrichment | 7.4.1 |
| 2-functoriality of change of enrichment | 7.5.6 |

We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

## 7.1. Change of Enrichment along a Multifunctor

Each monoidal functor between monoidal categories

$$U : \mathsf{V} \longrightarrow \mathsf{W}$$

induces a change-of-enrichment 2-functor (Proposition B.4.6)

$$(-)_U : \mathsf{V}\text{-Cat} \longrightarrow \mathsf{W}\text{-Cat}.$$

In this section we generalize this construction from a monoidal functor to a non-symmetric multifunctor. This section is organized as follows.

- The change of enrichment along a non-symmetric multifunctor $F$ is constructed in Definition 7.1.1 and is shown to be a 2-functor in Proposition 7.1.9.
- Example 7.1.10 illustrates Proposition 7.1.9 with $K$-theory multifunctors, some of which are non-symmetric.
- As a further illustration of change of enrichment, in Explanation 7.1.12 we explicitly describe the change-of-enrichment 2-functor induced by the non-symmetric multifunctor (Theorem 5.2.6)

$$\mathsf{F}_{\bullet} : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$

given by the pointed free permutative category construction.

**Defining Change of Enrichment.** Recall

- non-symmetric multicategories (Definition C.1.3),
- non-symmetric multifunctors (Definition C.1.19), and
- for a non-symmetric multicategory M, the 2-category M-Cat of small M-categories, M-functors, and M-natural transformations (Theorem 6.1.27).

We emphasize that a non-symmetric multifunctor preserves colored units (C.1.21) and composition (C.1.22), but not necessarily the symmetric group action even if its domain and codomain are multicategories.

First we define the object, 1-cell, and 2-cell assignments of change of enrichment. Recall the notion of a 2-functor (Definition A.2.4).

**Definition 7.1.1.** Suppose given a non-symmetric multifunctor between non-symmetric multicategories

$$F : (\mathsf{M}, \gamma, 1) \longrightarrow (\mathsf{N}, \gamma, 1).$$

We define the data of a 2-functor

$$(-)_F : \mathsf{M}\text{-Cat} \longrightarrow \mathsf{N}\text{-Cat},$$

which is called the *change of enrichment* or the *change-of-enrichment 2-functor* along $F$, as follows.

**Object Assignment:** The image of an M-category $(\mathsf{C}, m, i)$ (Definition 6.1.1) under $(-)_F$ is the N-category

$$(7.1.2) \qquad\qquad \left( \mathsf{C}_F, m_F, i_F \right)$$

consisting of the following data.
- The objects of $\mathsf{C}_F$ are those of $\mathsf{C}$.
- For each pair of objects $x, y \in \mathsf{C}_F$, the hom object is

$$(7.1.3) \qquad\qquad (\mathsf{C}_F)(x, y) = F\mathsf{C}(x, y) \quad \text{in} \quad \mathsf{N}.$$

- For objects $x, y, z \in \mathsf{C}_F$, the composition binary multimorphism

$$(7.1.4) \qquad \left( F\mathsf{C}(y, z), F\mathsf{C}(x, y) \right) \xrightarrow{(m_F)_{x,y,z} = F(m_{x,y,z})} F\mathsf{C}(x, z) \quad \text{in} \quad \mathsf{N}$$

is the image under $F$ of the composition $m_{x,y,z}$ in (6.1.2).
- For each object $x \in \mathsf{C}$, the identity nullary multimorphism

$$(7.1.5) \qquad\qquad \langle\rangle \xrightarrow{(i_F)_x = F(i_x)} F\mathsf{C}(x, x) \quad \text{in} \quad \mathsf{N}$$

is the image under $F$ of the identity $i_x$ in (6.1.3).
This finishes the definition of the N-category $(\mathsf{C}_F, m_F, i_F)$ in (7.1.2). Its associativity diagram (6.1.4) and unity diagram (6.1.5) are obtained from those for $\mathsf{C}$ by applying $F$, which preserves colored units and composition.

**1-Cell Assignment:** The image of an M-functor between M-categories (Definition 6.1.7)

$$H : (\mathsf{C}, m, i) \longrightarrow (\mathsf{D}, m, i)$$

under $(-)_F$ is the N-functor

$$(7.1.6) \qquad\qquad H_F : \left( \mathsf{C}_F, m_F, i_F \right) \longrightarrow \left( \mathsf{D}_F, m_F, i_F \right)$$

defined as follows.
- The object assignment of $H_F$ is the object assignment of $H$.
- For each pair of objects $x, y \in \mathsf{C}_F$, the component unary multimorphism

$$(7.1.7) \qquad F\mathsf{C}(x, y) \xrightarrow{(H_F)_{x,y} = F(H_{x,y})} F\mathsf{D}(Hx, Hy) \quad \text{in} \quad \mathsf{N}$$

is the image under $F$ of the component $H_{x,y}$ in (6.1.8).
This finishes the definition of the N-functor $H_F$ in (7.1.6). Its compatibility axioms (6.1.9) are obtained from those for $H$ by applying the nonsymmetric multifunctor $F$.

**2-Cell Assignment:** Suppose $\theta : H \longrightarrow G$ is an M-natural transformation (Definition 6.1.14) as in the left diagram below.

$$\mathsf{C} \underset{G}{\overset{H}{\Longrightarrow}} \mathsf{D} \qquad\qquad \mathsf{C}_F \underset{G_F}{\overset{H_F}{\Longrightarrow}} \mathsf{D}_F$$

We define $\theta_F$ as the N-natural transformation, as in the right diagram above, with, for each object $x \in C_F$, $x$-component nullary multimorphism

(7.1.8)
$$\langle\rangle \xrightarrow{\;(\theta_F)_x = F(\theta_x)\;} FD(Hx, Gx) \quad \text{in} \quad \text{N}.$$

This is the image under $F$ of the $x$-component

$$\theta_x : \langle\rangle \longrightarrow D(Hx, Gx)$$

of $\theta$ in (6.1.15), which is a nullary multimorphism in M. The naturality diagram (6.1.16) for $\theta_F$ is obtained from the naturality diagram for $\theta$ by applying $F$.

This finishes the definition of $(-)_F$.                                              $\diamond$

The following result is stated in [**BO15**, Proposition 2.11].

**Proposition 7.1.9.** *For each non-symmetric multifunctor $F : \text{M} \longrightarrow \text{N}$ between non-symmetric multicategories, the change of enrichment along $F$ in Definition 7.1.1 is a 2-functor*

$$(-)_F : \text{M-Cat} \longrightarrow \text{N-Cat}.$$

*Proof.* We need to check that the assignment $(-)_F$ preserves

- identity 1-cells (6.1.10),
- horizontal composition of 1-cells (6.1.11),
- identity 2-cells (6.1.17),
- vertical composition of 2-cells (6.1.20), and
- horizontal composition of 2-cells (6.1.23).

These preservation properties of $(-)_F$ follow from

(1) the componentwise definitions (7.1.3) through (7.1.5), (7.1.7), and (7.1.8), and

(2) the fact that $F$ preserves colored units and composition.

This finishes the proof.                                                            $\square$

### Examples of Change of Enrichment.

**Example 7.1.10** (*K*-Theory Multifunctors). The change-of-enrichment 2-functor exists for each of the multifunctors in the following diagram from (1.4.39), (2.5.1), and (3.4.32) along with (5.2.7) and (5.5.2). Among these arrows, F, F•, $F_{\mathcal{M}_1}$, and $\mathcal{P}$ are non-symmetric.

(7.1.11)

In other words, each of these multifunctors (non-symmetric for F, F•, $F_{\mathcal{M}_1}$, and $\mathcal{P}$) induces a change-of-enrichment 2-functor as in Proposition 7.1.9. We emphasize that *none* of the arrows in (7.1.11) is a monoidal functor because the multicategory structure on PermCat$^{\text{su}}$ is not induced by a monoidal structure. Thus Proposition B.4.6 does not apply to the arrows in (7.1.11). In Example 7.3.2 below we extend this example to include other *K*-theoretic symmetric monoidal functors.    $\diamond$

**Explanation 7.1.12** (Change of Enrichment along $\mathsf{F}_\bullet$). To illustrate Definition 7.1.1, we describe the change-of-enrichment 2-functor

(7.1.13) $$(-)_{\mathsf{F}_\bullet} : \mathsf{Multicat}_*\text{-}\mathsf{Cat} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}$$

induced by the non-symmetric multifunctor (Theorem 5.2.6)

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}.$$

To simplify the presentation, we use the shortened notation

$$\mathsf{M}_* = \mathsf{Multicat}_* \quad \text{and} \quad \mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.$$

Since $\mathsf{M}_*$ is a symmetric monoidal closed category (Theorem 1.2.8), by Proposition 6.2.1 the 2-category $\mathsf{M}_*$-Cat is the same regardless of whether we consider it in the sense of Example B.1.12 or Theorem 6.1.27.

$(-)_{\mathsf{F}_\bullet}$ *on Objects*. Suppose $(\mathsf{C}, \mathsf{m}, i)$ is an $\mathsf{M}_*$-category (Definition B.1.1). Besides its class of objects, $\mathsf{C}$ consists of the following data.

- For each pair of objects $x, y \in \mathsf{C}$, the hom object $\mathsf{C}(x, y)$ is a small pointed multicategory (Definition C.4.1).
- For objects $x, y, z \in \mathsf{C}$, the composition is a pointed multifunctor

(7.1.14) $$\mathsf{m}_{x,y,z} : \mathsf{C}(y, z) \wedge \mathsf{C}(x, y) \longrightarrow \mathsf{C}(x, z).$$

- The identity of each object $x \in \mathsf{C}$ is a pointed multifunctor

(7.1.15) $$i_x : \mathsf{I}_+ = \mathsf{I} \amalg \mathsf{T} \longrightarrow \mathsf{C}(x, x)$$

from the smash unit $\mathsf{I}_+$ in (1.2.4). Since $i_x$ preserves the basepoint, it sends the unique object $* \in \mathsf{T}$ to the basepoint of $\mathsf{C}(x, x)$. Thus $i_x$ is determined by the object

$$i_x(1) \in \mathsf{C}(x, x)$$

with 1 denoting the unique object in the initial operad $\mathsf{I}$ in Example C.1.35 (i). To simplify the notation, we also denote the object $i_x(1)$ by $i_x$.

The associativity diagram (B.1.4) and the unity diagram (B.1.5) are required to commute.

Applying (7.1.3) through (7.1.5) to $\mathsf{F}_\bullet$, the $\mathsf{P}^{\mathsf{su}}$-category (Explanation 6.3.2)

$$\left(\mathsf{C}_{\mathsf{F}_\bullet}, \mathsf{m}_{\mathsf{F}_\bullet}, i_{\mathsf{F}_\bullet}\right)$$

consists of the following data.

- $\mathrm{Ob}\left(\mathsf{C}_{\mathsf{F}_\bullet}\right) = \mathrm{Ob}\,\mathsf{C}$.
- For each pair of objects $x, y \in \mathsf{C}_{\mathsf{F}_\bullet}$, the hom object

$$\mathsf{C}_{\mathsf{F}_\bullet}(x, y) = \mathsf{F}_\bullet\mathsf{C}(x, y)$$

is the pointed free permutative category (Definitions 4.1.4 and 4.1.11) of the small pointed multicategory $\mathsf{C}(x, y)$.

- For objects $x, y, z \in \mathsf{C}_{\mathsf{F}_\bullet}$, applying the $n = 2$ case of (5.2.4) to $\mathsf{m}_{x,y,z}$, the composition bilinear functor

$$\mathsf{F}_\bullet\mathsf{C}(y,z) \times \mathsf{F}_\bullet\mathsf{C}(x,y) \xrightarrow{\;\mathsf{F}^2_\bullet\;} \mathsf{F}_\bullet\big(\mathsf{C}(y,z) \wedge \mathsf{C}(x,y)\big) \xrightarrow{\;\mathsf{F}_\bullet(\mathsf{m}_{x,y,z})\;} \mathsf{F}_\bullet\mathsf{C}(x,z)$$

$$\overset{(\mathsf{m}_{\mathsf{F}_\bullet})_{x,y,z}}{\overbrace{\hspace{11cm}}}$$

is the composite of

– the strong bilinear functor $\mathsf{F}^2_\bullet$ in Proposition 5.1.9 and

– the strict symmetric monoidal functor $F_\bullet(m_{x,y,z})$ in Definition 4.1.12.

- For each object $x \in C$, we also regard the pointed multifunctor $i_x$ in (7.1.15) as an object in the pointed multicategory $C(x,x)$, given by the image of the unique object $1 \in I$. The identity of an object $x \in C_{F_\bullet}$ is the $\overset{ob}{\sim}$-equivalence class

$$(i_{F_\bullet})_x = [(i_x)] \in F_\bullet C(x,x)$$

of the length-one sequence $(i_x) \in FC(x,x)$ (Definition 4.1.4). More explicitly, applying the $n = 0$ case of (5.2.4) to $i_x$, the identity $(i_{F_\bullet})_x$ is given by the 0-linear functor

(7.1.16) $$\langle\rangle \xrightarrow{\ F_\bullet^0\ } F_\bullet(I_+) \xrightarrow{\ F_\bullet(i_x)\ } F_\bullet C(x,x).$$

By Definition 5.1.1 $F_\bullet^0$ is given by the $\overset{ob}{\sim}$-equivalence class $[(1)] \in F_\bullet(I_+)$. Then $F_\bullet(i_x)$ sends $[(1)]$ to $[(i_x)]$ by definition (4.1.13) applied to the pointed multifunctor $i_x$.

$(-)_{F_\bullet}$ *on 1-Cells.* Consider an $M_*$-functor (Definition B.1.8) between $M_*$-categories

$$H : (C, m, i) \longrightarrow (D, m, i).$$

Besides its object assignment, $H$ has, for each pair of objects $x, y \in C$, a component pointed multifunctor

(7.1.17) $$H_{x,y} : C(x,y) \longrightarrow D(Hx, Hy).$$

These pointed multifunctors are compatible with composition and identities in the sense of (B.1.9). In particular, compatibility with identities means that, for each object $x \in C$, the identity $i_x \in C(x,x)$ is sent to the identity

$$H_{x,x}(i_x) = i_{Hx} \in D(Hx, Hx).$$

Applying the change of enrichment along $F_\bullet$, the $P^{su}$-functor (Explanation 6.3.12)

$$H_{F_\bullet} : \left(C_{F_\bullet}, m_{F_\bullet}, i_{F_\bullet}\right) \longrightarrow \left(D_{F_\bullet}, m_{F_\bullet}, i_{F_\bullet}\right)$$

has the same object assignment as $H$. For objects $x, y \in C_{F_\bullet}$, $H_{F_\bullet}$ has the component strict symmetric monoidal functor (7.1.7)

$$F_\bullet C(x,y) \xrightarrow{\ F_\bullet H_{x,y}\ } F_\bullet D(Hx, Hy).$$

This is obtained from the pointed multifunctor $H_{x,y}$ in (7.1.17) by applying the 1-cell assignment of $F_\bullet$ in Definition 4.1.12.

$(-)_{F_\bullet}$ *on 2-Cells.* Consider an $M_*$-natural transformation (Definition B.1.10)

$$(C, m, i) \underset{G}{\overset{H}{\underset{\Downarrow\theta}{\rightrightarrows}}} (D, m, i)$$

between $M_*$-functors between $M_*$-categories. For each object $x \in C$, the $x$-component of $\theta$ is a pointed multifunctor

$$\theta_x : I_+ = I \amalg T \longrightarrow D(Hx, Gx).$$

Similar to (7.1.15), the pointed multifunctor $\theta_x$ is determined by the object

$$\theta_x(1) \in D(Hx, Gx)$$

with $1 \in I$ denoting the unique object. As before we abbreviate $\theta_x(1)$ to $\theta_x$. For objects $x, y \in C$, the naturality diagram (B.1.11) for $\theta$ is the following commutative diagram of pointed multifunctors.

$$(7.1.18)$$

$$
\begin{array}{ccc}
I_+ \wedge C(x,y) & \xrightarrow{\;\theta_y \wedge H_{x,y}\;} & D(Hy,Gy) \wedge D(Hx,Hy) \\
{\scriptstyle \lambda^{-1}} \uparrow {\scriptstyle \cong} & & \downarrow {\scriptstyle m} \\
C(x,y) & & D(Hx,Gy) \\
{\scriptstyle \rho^{-1}} \downarrow {\scriptstyle \cong} & & \uparrow {\scriptstyle m} \\
C(x,y) \wedge I_+ & \xrightarrow{\;G_{x,y} \wedge \theta_x\;} & D(Gx,Gy) \wedge D(Hx,Gx)
\end{array}
$$

The commutative diagram (7.1.18) is equivalent to the following equality in $D(Hx,Gy)$ for objects and multimorphisms $f \in C(x,y)$.

$$m\big(\theta_y \wedge (H_{x,y}f)\big) = m\big((G_{x,y}f) \wedge \theta_x\big)$$

Applying the change of enrichment along $F_\bullet$ to $\theta$ yields the following $P^{su}$-natural transformation (Explanation 6.3.16).

$$
\big(C_{F_\bullet}, m_{F_\bullet}, i_{F_\bullet}\big) \;\overset{H_{F_\bullet}}{\underset{G_{F_\bullet}}{\rightrightarrows}}\; \Downarrow \theta_{F_\bullet} \;\big(D_{F_\bullet}, m_{F_\bullet}, i_{F_\bullet}\big)
$$

For each object $x \in C_{F_\bullet}$, its $x$-component is the $\overset{ob}{\sim}$-equivalence class (7.1.8)

$$(\theta_{F_\bullet})_x = \big[(\theta_x)\big] \in F_\bullet D(Hx, Gx)$$

of the length-one sequence $(\theta_x) \in FD(Hx, Gx)$. The reasoning is the same as in (7.1.16), with $i_x$ replaced by $\theta_x$.

This finishes our description of the change-of-enrichment 2-functor $(-)_{F_\bullet}$. We will use $(-)_{F_\bullet}$ in Example 9.2.14 and Explanations 12.3.1, 12.3.5, and 12.3.11. ⋄

## 7.2. Preservation of Opposite Enriched Categories

By Proposition 7.1.9 each non-symmetric multifunctor $F$ yields a change-of-enrichment 2-functor $(-)_F$. Recall that a multifunctor preserves colored units, composition, *and* symmetric group action (Definition C.1.19). In this section we observe that change of enrichment along a multifunctor preserves opposite enriched categories (Definition 6.6.1); see Proposition 7.2.1. We will use this opposite-preservation property in (10.2.4) to define the presheaf change of enrichment of $F$. In that context, we will use Proposition 7.2.1 in Theorems 10.3.4, 10.4.5, and 11.4.24 and Remark 10.5.4.

**Proposition 7.2.1.** *Suppose given a multifunctor between multicategories*

$$F : (M, \gamma, 1) \longrightarrow (N, \gamma, 1)$$

*and an* M*-category* C*. Then there is an equality of* N*-categories*

$$(C^{op})_F = (C_F)^{op}$$

*with* $(-)_F$ *the change of enrichment in (7.1.2).*

*Proof.* For an M-category $(C, m, i)$, we need to show that the following two N-categories are the same.

- $(\mathsf{C}^{\mathsf{op}})_F$ is the change of enrichment along $F$ (7.1.2) of the opposite M-category $\mathsf{C}^{\mathsf{op}}$ (Definition 6.6.1).
- $(\mathsf{C}_F)^{\mathsf{op}}$ is the opposite N-category of $\mathsf{C}_F$, which is the change of enrichment of $\mathsf{C}$ along $F$.

In both $(\mathsf{C}^{\mathsf{op}})_F$ and $(\mathsf{C}_F)^{\mathsf{op}}$, the objects are those of $\mathsf{C}$. For objects $x, y \in \mathsf{C}$, the hom objects are equal:

$$(\mathsf{C}^{\mathsf{op}})_F(x, y) = F\mathsf{C}^{\mathsf{op}}(x, y) = F\mathsf{C}(y, x) = \mathsf{C}_F(y, x) = (\mathsf{C}_F)^{\mathsf{op}}(x, y).$$

By (6.6.4) and (7.1.5), the identity of an object $x$ in $(\mathsf{C}^{\mathsf{op}})_F$ and $(\mathsf{C}_F)^{\mathsf{op}}$ is the nullary multimorphism

$$F(i_x) : \langle\rangle \longrightarrow F\mathsf{C}(x, x) \quad \text{in} \quad \mathsf{N}.$$

For objects $x, y, z \in (\mathsf{C}^{\mathsf{op}})_F$ and $(\mathsf{C}_F)^{\mathsf{op}}$, the compositions are equal:

(7.2.2)
$$\begin{aligned}
\left((\mathsf{m}^{\mathsf{op}})_F\right)_{x,y,z} &= F(\mathsf{m}_{z,y,x} \cdot \tau) \\
&= (F\mathsf{m}_{z,y,x}) \cdot \tau \\
&= \left((\mathsf{m}_F)^{\mathsf{op}}\right)_{x,y,z}.
\end{aligned}$$

The first and third equalities above use the definitions (6.6.2) and (7.1.4). The second equality uses the fact that $F$ is a multifunctor, which preserves the symmetric group action (C.1.20). □

We emphasize that Proposition 7.2.1 does *not* extend to non-symmetric multifunctors between multicategories because the second equality in (7.2.2) requires that $F$ preserves the symmetric group action in the strict sense.

**Example 7.2.3** (*K*-Theory Multifunctors). Proposition 7.2.1 applies to the multifunctors in the following sub-diagram of (7.1.11).

(7.2.4)

$$\begin{array}{ccccc}
\mathsf{Multicat} & \xleftarrow{\;\text{End}\;} & \mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\;K^{\mathsf{EM}}\;} & \mathsf{Sp} \\
& {}^{\text{End}_\bullet}\swarrow & \;\;\downarrow{\scriptstyle \text{End}_{\mathcal{M}_1}} & {}^{J^{\mathsf{EM}}}\searrow & \\
\mathsf{Multicat}_* & & \mathsf{Mod}^{\mathcal{M}_1} & & \mathcal{G}_*\text{-}\mathsf{Cat}
\end{array}$$

In other words, the change of enrichment induced by each multifunctor in (7.2.4) preserves opposite enriched categories. For example, consider a $\mathsf{PermCat}^{\mathsf{su}}$-category $\mathsf{C}$ (Explanation 6.3.2). Then Proposition 7.2.1 yields the equality of $\mathsf{Sp}$-categories

$$(\mathsf{C}^{\mathsf{op}})_{K^{\mathsf{EM}}} = (\mathsf{C}_{K^{\mathsf{EM}}})^{\mathsf{op}}.$$

The opposite on the left-hand side is taken in $\mathsf{PermCat}^{\mathsf{su}}$-$\mathsf{Cat}$. The opposite on the right-hand side is taken in $\mathsf{Sp}$-$\mathsf{Cat}$, after the change of enrichment along $K^{\mathsf{EM}}$. The above equality of $\mathsf{Sp}$-categories is important in the context of presheaf change of enrichment; see Theorem 10.5.1 and Remarks 10.5.4 and 10.5.5.

Note that $\mathsf{F}$, $\mathsf{F}_\bullet$, $\mathsf{F}_{\mathcal{M}_1}$, and $\mathcal{P}$ in (7.1.11) are excluded from the diagram (7.2.4) because they are *non-symmetric* multifunctors and do not preserve opposite enriched categories in general. Specifically, each of these non-symmetric multifunctors fails to satisfy the second equality in (7.2.2), which requires strict preservation of symmetric group action. ◇

## 7.3. Change of Enrichment along a Monoidal Functor

For a monoidal functor between monoidal categories (Definition A.1.22)

$$U : \mathsf{V} \longrightarrow \mathsf{W},$$

there are two change-of-enrichment 2-functors as follows.

(1) By Proposition B.4.6 there is a change-of-enrichment 2-functor

$$(-)_U : \mathsf{V}\text{-Cat} \longrightarrow \mathsf{W}\text{-Cat}.$$

(2) By Example C.3.1 there is a non-symmetric multifunctor

$$\mathsf{End}\, U : \mathsf{End}\, \mathsf{V} \longrightarrow \mathsf{End}\, \mathsf{W}.$$

By Propositions 6.2.1 and 7.1.9 there is a change-of-enrichment 2-functor

$$(-)_{\mathsf{End}\, U} : (\mathsf{End}\, \mathsf{V})\text{-Cat} = \mathsf{V}\text{-Cat} \longrightarrow (\mathsf{End}\, \mathsf{W})\text{-Cat} = \mathsf{W}\text{-Cat}.$$

In this section we observe that these two change-of-enrichment 2-functors are equal.

**Proposition 7.3.1.** *For each monoidal functor between monoidal categories*

$$(U, U^2, U^0) : (\mathsf{V}, \otimes, 1) \longrightarrow (\mathsf{W}, \otimes, 1),$$

*there is an equality of change-of-enrichment 2-functors*

$$(-)_U = (-)_{\mathsf{End}\, U} : \mathsf{V}\text{-Cat} \longrightarrow \mathsf{W}\text{-Cat}.$$

*Proof.* We need to show that $(-)_U$ and $(-)_{\mathsf{End}\, U}$ are equal on the objects, 1-cells, and 2-cells of the 2-category (Proposition 6.2.1)

$$\mathsf{V}\text{-Cat} = (\mathsf{End}\, \mathsf{V})\text{-Cat}.$$

First we consider a V-category $(\mathsf{C}, \mathsf{m}, i)$ (Definition B.1.1).

- By Definitions 7.1.1 and B.4.1, both $\mathsf{C}_U$ and $\mathsf{C}_{\mathsf{End}\, U}$ have
  - the same objects as $\mathsf{C}$ and
  - hom objects $U\mathsf{C}(x, y)$ for objects $x, y \in \mathsf{C}$.
- The identity of an object $x \in \mathsf{C}_U$ is given by the composite (B.4.3)

$$1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ Ui_x\ } U\mathsf{C}(x, x).$$

  Regarding $x$ as an object in $\mathsf{C}_{\mathsf{End}\, U}$, (7.1.5) implies that its identity is also given by the composite above because $\mathsf{End}\, U$ on a nullary multimorphism (C.3.4) is the composite of the unit constraint $U^0$ followed by $U(-)$.
- For objects $x, y, z \in \mathsf{C}_U$ the composition is given by the composite (B.4.2)

$$U\mathsf{C}(y, z) \otimes U\mathsf{C}(x, y) \xrightarrow{\ U^2\ } U\big(\mathsf{C}(y, z) \otimes \mathsf{C}(x, y)\big) \xrightarrow{\ U\mathsf{m}\ } U\mathsf{C}(x, z).$$

  Regarding $x, y$, and $z$ as objects in $\mathsf{C}_{\mathsf{End}\, U}$, (7.1.4) implies that the composition is also given by the composite above because $\mathsf{End}\, U$ on a binary multimorphism (C.3.4) is the composite of the monoidal constraint $U^2$ followed by $U(-)$.

This proves that $(-)_U$ and $(-)_{\mathsf{End}\, U}$ are equal on objects.

Next we consider a V-functor between V-categories (Definition B.1.8)

$$H : \mathsf{C} \longrightarrow \mathsf{D}.$$

By (7.1.7), (B.4.4), and (C.3.4), for objects $x, y \in C_U$ and $C_{\mathsf{End}\, U}$, there are equalities of component morphisms in W:

$$(H_U)_{x,y} = U(H_{x,y}) = (\mathsf{End}\, U)(H_{x,y}) = (H_{\mathsf{End}\, U})_{x,y}.$$

Thus $(-)_U$ and $(-)_{\mathsf{End}\, U}$ are equal on 1-cells.

By (7.1.8) and (B.4.5), the proof that $(-)_U$ and $(-)_{\mathsf{End}\, U}$ are equal on each V-natural transformation $\theta$ is the same as the argument above for the identity of an object, with $i_x$ replaced by the component $\theta_x$ for objects $x \in C$.  $\square$

**Example 7.3.2** (*K*-Theory Symmetric Monoidal Functors). Proposition 7.3.1 applies to the symmetric monoidal functors in the following commutative diagram from (1.2.11), (1.3.28), (2.4.20), and (2.5.1).

(7.3.3)



In other words, for each of these symmetric monoidal functors, the induced change-of-enrichment 2-functors in Propositions 7.1.9 and B.4.6 are the same. Moreover, each of these change-of-enrichment 2-functors preserves opposite enriched categories in the sense of Proposition 7.2.1.                         ⋄

## 7.4. Composition of Change-of-Enrichment 2-Functors

By Proposition 7.1.9 each non-symmetric multifunctor $F$ has an associated change-of-enrichment 2-functor $(-)_F$. In this section we observe that change of enrichment respects composition of non-symmetric multifunctors; see Proposition 7.4.1. This is an analog of Proposition B.4.7 for change of enrichment along monoidal functors. Proposition 7.4.1 is important in the context of standard enrichment, diagram change of enrichment, and Mackey functor change of enrichment; see Theorems 9.3.6, 10.4.1, and 10.4.5 and (9.4.13), (11.1.4), (11.2.7), and (12.5.9).

**Proposition 7.4.1.** *For non-symmetric multifunctors between non-symmetric multicategories*

$$M \xrightarrow{\;F\;} N \xrightarrow{\;G\;} P,$$

*the following diagram of change-of-enrichment 2-functors commutes.*

$$\mathsf{M}\text{-Cat} \xrightarrow{(-)_F} \mathsf{N}\text{-Cat} \xrightarrow{(-)_G} \mathsf{P}\text{-Cat}$$

with $(-)_{GF}$ the composite.

*Proof.* We need to show that $(-)_{GF}$ and $(-)_G \circ (-)_F$ are equal on the objects, 1-cells, and 2-cells of the 2-category M-Cat (Theorem 6.1.27).

First we consider an M-category $(C, m, i)$ (Definition 6.1.1).

- Both $(C_F)_G$ and $C_{GF}$ have the same objects as C and hom objects

$$GFC(x, y) \in P \quad \text{for} \quad x, y \in C.$$

- For each object $x \in (C_F)_G$ and $C_{GF}$, by (7.1.5) and (C.1.23) the identity is the nullary multimorphism

$$\langle\rangle \xrightarrow{GF(i_x)} GFC(x,x) \quad \text{in} \quad P.$$

- For objects $x, y, z \in (C_F)_G$ and $C_{GF}$, by (7.1.4) and (C.1.23) the composition is the binary multimorphism

$$\big(GFC(y,z), GFC(x,y)\big) \xrightarrow{GF(\mathsf{m}_{x,y,z})} GFC(x,z) \quad \text{in} \quad P.$$

This shows that $(-)_{GF}$ and $(-)_G \circ (-)_F$ are equal on M-categories.

Next we consider an M-functor $H : C \longrightarrow D$ between M-categories (Definition 6.1.7).

- Both $(H_F)_G$ and $H_{GF}$ have the same object assignment as $H$.
- For objects $x, y \in (C_F)_G$ and $C_{GF}$, by (7.1.7) there are equalities of component unary multimorphisms

$$\big((H_F)_G\big)_{x,y} = GF(H_{x,y}) = (H_{GF})_{x,y} \quad \text{in} \quad P.$$

This shows that $(-)_{GF}$ and $(-)_G \circ (-)_F$ are equal on M-functors.

For an M-natural transformation $\theta$ and an object $x \in (C_F)_G$ and $C_{GF}$, by (7.1.8) there are equalities of $x$-component nullary multimorphisms

$$\big((\theta_F)_G\big)_x = GF(\theta_x) = (\theta_{GF})_x \quad \text{in} \quad P.$$

Thus $(-)_{GF}$ and $(-)_G \circ (-)_F$ are equal on M-natural transformations. $\qquad\square$

We emphasize that Proposition 7.4.1 does *not* require $F$ and $G$ to preserve the symmetric group action even if M, N, and P are multicategories.

**Example 7.4.2** (*K*-Theory Multifunctors). Proposition 7.4.1 applies to all the (non-symmetric) multifunctors in (7.1.11) and (7.3.3). For example, consider the following commutative sub-diagram of (2.5.1) consisting of multifunctors.

(7.4.3)



By Proposition 7.4.1 the associated diagram of change-of-enrichment 2-functors is commutative. In particular, the change-of-enrichment 2-functor along $\mathsf{K}^{\mathsf{EM}}$ is equal to the composite of the change-of-enrichment 2-functors along $\mathrm{End}_{\mathcal{M}_1}$, $\mathsf{J}^{\mathcal{T}}$, $\mathrm{Ner}_*$, and $\mathsf{K}^{\mathcal{G}}$. We discuss the factorization (7.4.3) of $\mathsf{K}^{\mathsf{EM}}$ further

- in Theorem 9.4.2 in the context of standard enrichment and
- in Theorem 10.6.2 in the context of presheaf change of enrichment.   ⋄

## 7.5. 2-Functoriality of Change of Enrichment

In this section we show that the (change of) enrichment constructions

$$\mathsf{M} \longmapsto \mathsf{M}\text{-Cat} \quad \text{and} \quad F \longmapsto (-)_F$$

in Theorem 6.1.27 and Proposition 7.1.9, respectively, are part of a 2-functor; see Theorem 7.5.6. To explain this precisely, first we construct the change of

enrichment of multinatural transformations (Definition 7.5.1). We show in Proposition 7.5.5 that this yields a well-defined 2-natural transformation.

Beyond its immediate usage in Theorem 7.5.6, we will use Proposition 7.5.5 in (11.1.3) to construct a functor that goes in the backward direction as a diagram change of enrichment. Then we use that functor to construct an equivalence of homotopy theories in that context; see Theorem 11.4.14.

Recall 2-natural transformations and multinatural transformations in Definitions A.2.7 and C.1.25, respectively.

**Definition 7.5.1.** Suppose $\theta : F \longrightarrow G$ is a multinatural transformation between non-symmetric multifunctors between non-symmetric multicategories, as in the left diagram below.

$$(7.5.2) \qquad (\mathsf{M},\gamma,1) \underset{G}{\overset{F}{\Longrightarrow}} \Downarrow\theta \; (\mathsf{N},\gamma,1) \qquad\qquad \mathsf{M\text{-}Cat} \underset{(-)_G}{\overset{(-)_F}{\Longrightarrow}} \Downarrow(-)_\theta \; \mathsf{N\text{-}Cat}$$

We define the data of a 2-natural transformation $(-)_\theta$, as in the right diagram in (7.5.2), as follows. Here

- M-Cat and N-Cat are the 2-categories in Theorem 6.1.27, and
- $(-)_F$ and $(-)_G$ are the change-of-enrichment 2-functors in Proposition 7.1.9.

For each small M-category $(\mathsf{C},\mathsf{m},i)$ (Definition 6.1.1), we define the data of an N-functor (Definition 6.1.7)

$$(7.5.3) \qquad\qquad \mathsf{C}_\theta : (\mathsf{C}_F,\mathsf{m}_F,i_F) \longrightarrow (\mathsf{C}_G,\mathsf{m}_G,i_G)$$

as follows.

- $\mathsf{C}_\theta$ is the identity assignment on objects. This is well defined since $\mathsf{C}_F$ and $\mathsf{C}_G$ both have the same objects as $\mathsf{C}$.
- For each pair of objects $x,y \in \mathsf{C}$, the $(x,y)$-component of $\mathsf{C}_\theta$ is defined as the $\mathsf{C}(x,y)$-component of $\theta$:

$$(7.5.4) \qquad \mathsf{C}_F(x,y) = F\mathsf{C}(x,y) \xrightarrow{\;(\mathsf{C}_\theta)_{x,y}\,=\,\theta_{\mathsf{C}(x,y)}\;} \mathsf{C}_G(x,y) = G\mathsf{C}(x,y).$$

   This is a unary multimorphism in N.

This finishes the definition of $(-)_\theta$.                                                 ◇

Now we check that $(-)_\theta$ is well defined.

**Proposition 7.5.5.** *In the context of Definition 7.5.1, $(-)_\theta$ is a 2-natural transformation.*

*Proof.* We prove statements (i) through (iii) below.

   (i) $\mathsf{C}_\theta$ in (7.5.3) is an N-functor (Definition 6.1.7).
   (ii) $(-)_\theta$ is natural with respect to M-functors (A.2.8).
   (iii) $(-)_\theta$ is natural with respect to M-natural transformations (A.2.9).

*Statement (i)*. We abbreviate $\mathsf{C}(x,y)$ to $\mathsf{C}_{x,y}$. For objects $x,y,z \in \mathsf{C}_F$, the two compatibility diagrams in (6.1.9) for $\mathsf{C}_\theta$ are as follows.

$$
\begin{array}{ccc}
\left(F\mathsf{C}_{y,z}, F\mathsf{C}_{x,y}\right) & \xrightarrow{F\mathsf{m}_{x,y,z}} & F\mathsf{C}_{x,z} \\
{\scriptstyle(\theta_{\mathsf{C}_{y,z}}, \theta_{\mathsf{C}_{x,y}})}\downarrow & & \downarrow{\scriptstyle\theta_{\mathsf{C}_{x,z}}} \\
\left(G\mathsf{C}_{y,z}, G\mathsf{C}_{x,y}\right) & \xrightarrow{G\mathsf{m}_{x,y,z}} & G\mathsf{C}_{x,z}
\end{array}
\qquad
\begin{array}{ccc}
\langle\rangle & \xrightarrow{Fi_x} & F\mathsf{C}_{x,x} \\
& {\scriptstyle Gi_x}\searrow & \downarrow{\scriptstyle\theta_{\mathsf{C}_{x,x}}} \\
& & G\mathsf{C}_{x,x}
\end{array}
$$

These two diagrams commute by the naturality of $\theta$ (C.1.26) for, respectively, the binary multimorphism $\mathsf{m}_{x,y,z}$ and the nullary multimorphism $i_x$ in $\mathsf{M}$.

*Statement (ii)*. For an $\mathsf{M}$-functor $P : \mathsf{C} \longrightarrow \mathsf{D}$ (Definition 6.1.7), the 1-cell naturality diagram (A.2.8) for $(-)_\theta$ is the left diagram of $\mathsf{N}$-functors below, where $P_F$ and $P_G$ are defined in (7.1.6).

$$
\begin{array}{ccc}
\mathsf{C}_F & \xrightarrow{\mathsf{C}_\theta} & \mathsf{C}_G \\
{\scriptstyle P_F}\downarrow & & \downarrow{\scriptstyle P_G} \\
\mathsf{D}_F & \xrightarrow{\mathsf{D}_\theta} & \mathsf{D}_G
\end{array}
\qquad
\begin{array}{ccc}
F\mathsf{C}_{x,y} & \xrightarrow{\theta_{\mathsf{C}_{x,y}}} & G\mathsf{C}_{x,y} \\
{\scriptstyle FP_{x,y}}\downarrow & & \downarrow{\scriptstyle GP_{x,y}} \\
F\mathsf{D}_{Px,Py} & \xrightarrow{\theta_{\mathsf{D}_{Px,Py}}} & G\mathsf{D}_{Px,Py}
\end{array}
$$

Both $P_G\mathsf{C}_\theta$ and $\mathsf{D}_\theta P_F$ have the same object assignment as $P$, since $\mathsf{C}_\theta$ and $\mathsf{D}_\theta$ are the identity functions on objects. For objects $x,y \in \mathsf{C}_F$, the $(x,y)$-component of the left diagram above is the right diagram in $\mathsf{N}$. The latter commutes by the naturality of $\theta$ (C.1.26) for the unary multimorphism

$$
P_{x,y} : \mathsf{C}(x,y) \longrightarrow \mathsf{D}(Px,Py) \quad \text{in} \quad \mathsf{M}.
$$

*Statement (iii)*. Consider an $\mathsf{M}$-natural transformation $\psi : P \longrightarrow Q$ (Definition 6.1.14) for $\mathsf{M}$-functors $P, Q : \mathsf{C} \longrightarrow \mathsf{D}$. The 2-cell naturality diagram (A.2.9) for $(-)_\theta$ is the left diagram of $\mathsf{N}$-natural transformations below, where $\psi_F$ and $\psi_G$ are defined in (7.1.8).

$$
\begin{array}{ccc}
\mathsf{C}_F & \xrightarrow{\mathsf{C}_\theta} & \mathsf{C}_G \\
{\scriptstyle P_F}\left(\overset{\psi_F}{\Rightarrow}\right){\scriptstyle Q_F} & & {\scriptstyle P_G}\left(\overset{\psi_G}{\Rightarrow}\right){\scriptstyle Q_G} \\
\mathsf{D}_F & \xrightarrow{\mathsf{D}_\theta} & \mathsf{D}_G
\end{array}
\qquad
\begin{array}{ccc}
\langle\rangle & \xrightarrow{G\psi_x} & \\
{\scriptstyle F\psi_x}\downarrow & & \searrow \\
F\mathsf{D}_{Px,Qx} & \xrightarrow{\theta_{\mathsf{D}_{Px,Qx}}} & G\mathsf{D}_{Px,Qx}
\end{array}
$$

For each object $x \in \mathsf{C}_F$, since $\mathsf{C}_\theta$ is the identity on objects, the $x$-component of the left diagram above is the right diagram in $\mathsf{N}$. The latter commutes by the naturality of $\theta$ (C.1.26) for the nullary multimorphism

$$
\psi_x : \langle\rangle \longrightarrow \mathsf{D}(Px,Qx) \quad \text{in} \quad \mathsf{M}.
$$

This finishes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

To state the main result of this section, recall the following 2-categories.

- 2Cat is the 2-category of small 2-categories, 2-functors, and 2-natural transformations (Example A.2.10).
- Multicat$^{\mathrm{ns}}$ is the 2-category of non-symmetric small multicategories, multifunctors, and multinatural transformations (C.1.34).

We now observe that change of enrichment is a 2-functor between these 2-categories. The following result is a multicategorical analog of [**JY∞**, 2.2.7], which is about enrichment in monoidal categories. See Explanation 7.5.7 for a discussion related to universes.

**Theorem 7.5.6.** *There is a 2-functor*

$$\mathbf{E} : \mathsf{Multicat}^{\mathsf{ns}} \longrightarrow \mathsf{2Cat}$$

*given by the assignments*

- **E**M = M-Cat *(Theorem 6.1.27) on objects,*
- **E**F = $(-)_F$ *(Proposition 7.1.9) on 1-cells, and*
- **E**$\theta$ = $(-)_\theta$ *(Proposition 7.5.5) on 2-cells.*

*Proof.* We prove statements (i) through (iv) below.

- (i) **E** preserves identity 1-cells and horizontal composition of 1-cells.
- (ii) **E** preserves identity 2-cells.
- (iii) **E** preserves vertical composition of 2-cells.
- (iv) **E** preserves horizontal composition of 2-cells.

*Statement (i).* For a small non-symmetric multicategory M, the identity non-symmetric multifunctor $1_\mathsf{M}$ (Definition C.1.19) consists of the identity functions on objects and multimorphisms. The change of enrichment along $1_\mathsf{M}$,

$$(-)_{1_\mathsf{M}} : \mathsf{M\text{-}Cat} \longrightarrow \mathsf{M\text{-}Cat},$$

is the identity 2-functor on M-Cat by the componentwise definitions (7.1.3) through (7.1.5), (7.1.7), and (7.1.8). Moreover, **E** preserves horizontal composition of 1-cells by Proposition 7.4.1.

*Statement (ii).* For a non-symmetric multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$, the identity multinatural transformation $1_F$ (Definition C.1.25) has, for each object $x \in \mathsf{M}$, $x$-component given by the colored unit $1_{Fx}$. By definitions (6.1.10) and (7.5.4), each component of the 2-natural transformation

$$(-)_{1_F} : (-)_F \longrightarrow (-)_F$$

is an identity N-functor, so $(-)_{1_F} = 1_{(-)_F}$.

*Statement (iii).* We consider vertically composable multinatural transformations between non-symmetric multifunctors, as in the left diagram below.



The desired equality of 2-natural transformations is

$$(-)_\psi (-)_\theta = (-)_{\psi\theta},$$

where $\psi\theta : F \longrightarrow H$ is the vertical composition (C.1.29). We need to show that, for each small M-category C, the following diagram of N-functors commutes.

On objects of $\mathsf{C}_F$, each of the three arrows in the diagram above is the identity function. For objects $x, y \in \mathsf{C}_F$, the $(x, y)$-component is the following diagram of unary multimorphisms in $\mathsf{N}$.

$$FC(x,y) \xrightarrow{\;\theta_{C(x,y)}\;} GC(x,y) \xrightarrow{\;\psi_{C(x,y)}\;} HC(x,y)$$

with top arrow $(\psi\theta)_{C(x,y)}$.

This diagram commutes because $\psi\theta$ is defined componentwise (C.1.30).

*Statement (iv)*. We consider horizontally composable multinatural transformations $\theta$ and $\theta'$ between non-symmetric multifunctors (C.1.31), as in the left diagram below.

$$\mathsf{M} \underset{G}{\overset{F}{\Rightarrow\theta\Rightarrow}} \mathsf{N} \underset{G'}{\overset{F'}{\Rightarrow\theta'\Rightarrow}} \mathsf{P} \qquad \mathsf{M} \underset{G'G}{\overset{F'F}{\Rightarrow\theta'*\theta\Rightarrow}} \mathsf{P}$$

This yields the following 2-natural transformations.

$$\mathsf{M\text{-}Cat} \underset{(-)_G}{\overset{(-)_F}{\Rightarrow(-)_\theta\Rightarrow}} \mathsf{N\text{-}Cat} \underset{(-)_{G'}}{\overset{(-)_{F'}}{\Rightarrow(-)_{\theta'}\Rightarrow}} \mathsf{P\text{-}Cat} \qquad \mathsf{M\text{-}Cat} \underset{(-)_{G'G}}{\overset{(-)_{F'F}}{\Rightarrow(-)_{\theta'*\theta}\Rightarrow}} \mathsf{P\text{-}Cat}$$

The desired equality of 2-natural transformations is

$$(-)_{\theta'} * (-)_\theta = (-)_{\theta'*\theta}.$$

For a small $\mathsf{M}$-category $\mathsf{C}$, the $\mathsf{C}$-component of $(-)_{\theta'} * (-)_\theta$ is the bottom composite $\mathsf{P}$-functor in the following diagram, while the $\mathsf{C}$-component of $(-)_{\theta'*\theta}$ is the top $\mathsf{P}$-functor.

$$\begin{array}{ccc}
\mathsf{C}_{F'F} & \xrightarrow{\quad\quad\mathsf{C}_{\theta'*\theta}\quad\quad} & \mathsf{C}_{G'G} \\
\| & & \| \\
(\mathsf{C}_F)_{F'} & \xrightarrow[\;(\mathsf{C}_\theta)_{F'}\;]{} (\mathsf{C}_G)_{F'} \xrightarrow[\;(\mathsf{C}_G)_{\theta'}\;]{} & (\mathsf{C}_G)_{G'}
\end{array}$$

Each of these two $\mathsf{P}$-functors is the identity on objects. For objects $x, y \in \mathsf{C}_{F'F}$, the $(x, y)$-components yield the following diagram of unary multimorphisms in $\mathsf{P}$.

$$F'FC(x,y) \xrightarrow{\;F'\theta_{C(x,y)}\;} F'GC(x,y) \xrightarrow{\;\theta'_{GC(x,y)}\;} G'GC(x,y)$$

with top arrow $(\theta' * \theta)_{C(x,y)}$.

This diagram commutes by the definition (C.1.32) of each component of $\theta' * \theta$.   □

**Explanation 7.5.7** (Universes). In Theorem 7.5.6, for a small non-symmetric multi-category $\mathsf{M}$ with respect to a given universe $\mathcal{U}$, the 2-category $\mathsf{M\text{-}Cat}$ is, in general, not small with respect to $\mathcal{U}$. We implicitly use Grothendieck's Axiom of Universes (Convention A.1.2) to choose a larger universe $\mathcal{U}'$ such that, for each $\mathcal{U}$-small non-symmetric multicategory $\mathsf{M}$, $\mathrm{Ob}(\mathsf{M\text{-}Cat})$ is a member of $\mathcal{U}'$. Then we consider the 2-category $2\mathsf{Cat}'$, whose objects are $\mathcal{U}'$-small 2-categories. The precise codomain of $\mathbf{E}$ in Theorem 7.5.6 is $2\mathsf{Cat}'$. For related discussion in the context of enrichment in monoidal categories, the reader is referred to [**JY**∞, 2.2.6 and 2.2.8].   ◇

CHAPTER 8

# The Closed Multicategory of Permutative Categories

This chapter defines and develops the basic properties of closed multicategories M. This is similar to, but more general than, the concept of symmetric monoidal closed categories V from Definition A.1.19. For the special case M = End V, Proposition 8.1.16 shows that the two notions agree. Sections 8.2 through 8.4 develop the special case of PermCat$^{\mathsf{su}}$, the multicategory of permutative categories. This, along with the closed symmetric monoidal categories Multicat$_*$ and Mod$^{\mathcal{M}\underline{1}}$, will be the main case of interest for applications in Chapters 11 and 12.

More general closed structures for PermCat, PermCat$^{\mathsf{st}}$, and PermCat$^{\mathsf{sus}}$ are discussed in Section 8.5. These structures involve notions of lax, respectively strong, respectively strictly unital strong, multilinear functors.

**Connection with Other Chapters.** Chapter 9 develops the general theory of self-enrichment for closed multicategories, with PermCat$^{\mathsf{su}}$ being one of the key examples for further applications. Chapter 10 develops the theory of enriched diagrams and enriched Mackey functors in a closed multicategory M. Chapter 12 applies theory from Chapter 11 to PermCat$^{\mathsf{su}}$-enriched categories and diagrams.

**Background.** The Cat-multicategory structure for PermCat$^{\mathsf{su}}$ is discussed in Section 1.4. The self-enrichment of PermCat$^{\mathsf{su}}$ is discussed in Section 6.4.

**Chapter Summary.** Section 8.1 gives the basic definitions for closed multicategories. Section 8.2 describes the internal hom for PermCat$^{\mathsf{su}}$, and Section 8.3 describes the multicategorical evaluation. In Section 8.4 these are combined to give the closed multicategory structure for PermCat$^{\mathsf{su}}$. Section 8.5 discusses how the previous structures can be generalized to PermCat and other multicategories of permutative categories. Here is a summary table.

| definition of a closed multicategory | 8.1.1 |
|---|---|
| internal hom for PermCat$^{\mathsf{su}}$ | 8.2.1 and 8.2.13 |
| symmetric group action on internal hom | 8.2.14 and 8.2.16 |
| multicategorical ev and $\chi$ | 8.3.1, 8.3.8, 8.3.9, and 8.3.16 |
| multicategorical evaluation axioms | 8.4.1 and 8.4.9 |
| lax multilinear functors | 8.5.6, 8.5.14, 8.5.34, and 8.5.36 |
| closed structure in lax case | 8.5.41, 8.5.46, 8.5.48, and 8.5.50 |
| main results | 8.4.15 and 8.5.56 |

We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

### 8.1. Closed Multicategories

Recall from Definition C.1.3 that a *multicategory* means a Set-multicategory for the symmetric monoidal category $(\mathsf{Set}, \times)$ of sets and functions with the Cartesian product as the monoidal product. In this section we define *closed multicategories*, which are multicategories equipped with internal hom objects and compatible evaluations. This concept provides a common setting for

- symmetric monoidal closed categories (Proposition 8.1.16) and
- the closed structure on the multicategory $\mathsf{PermCat}^{\mathsf{su}}$ (Theorem 8.4.15).

We emphasize that, just like multicategories, closed multicategories have symmetric group action compatible with the closed structure.

In the absence of a symmetric monoidal closed structure, a closed multicategory structure is the closest substitute that is still sufficient for a robust theory of self-enrichment (Section 9.1), standard enrichment (Section 9.2), enriched diagrams, enriched Mackey functors (Section 10.1), and change of enrichment in those contexts (Sections 9.3 and 10.2). In short, closed multicategories are the focus of the rest of this work.

After defining closed multicategories (Definition 8.1.1), in Remarks 8.1.12 and 8.1.13 we discuss the relationship between our definition and those in the literature [**Lam69, Man12, Zak18**]. Proposition 8.1.16 shows that each symmetric monoidal closed category yields a closed multicategory via the endomorphism construction.

**Definition 8.1.1.** A *closed multicategory* is a triple

$$\bigl(\mathsf{M}, \underline{\mathsf{M}}, \mathsf{ev}\bigr)$$

consisting of the following data.

**Underlying Multicategory:** $\mathsf{M} = (\mathsf{M}, \gamma, 1)$ is a multicategory (Definition C.1.3).

**Internal Hom Objects:** For $n \geq 0$ and each $(n+1)$-tuple of objects $\langle x \rangle = \langle x_i \rangle_{i=1}^n, y$ in $\mathsf{M}$, it is equipped with an object

$$(8.1.2) \qquad\qquad \underline{\mathsf{M}}\bigl(\langle x \rangle ; y\bigr) \in \mathsf{M},$$

which is called an *$n$-ary internal hom object* and also denoted $\underline{\mathsf{M}}_{\langle x \rangle ; y}$.

**Symmetric Group Action on Internal Hom:** For objects $\langle x \rangle, y \in \mathsf{M}$ as above and each permutation $\sigma \in \Sigma_n$, it is equipped with an invertible unary multimorphism

$$(8.1.3) \qquad\qquad \underline{\mathsf{M}}\bigl(\langle x \rangle ; y\bigr) \xrightarrow[\cong]{\sigma} \underline{\mathsf{M}}\bigl(\langle x \rangle \sigma ; y\bigr) \quad \text{in} \quad \mathsf{M}$$

with $\langle x \rangle \sigma = \langle x_{\sigma(i)} \rangle_{i=1}^n$. It is called the *right symmetric group action* or the *right $\sigma$-action* on internal hom objects.

**Multicategorical Evaluation:** For objects $\langle x \rangle, y \in \mathsf{M}$ as above, it is equipped with an $(n+1)$-ary multimorphism

$$(8.1.4) \qquad\qquad \bigl(\underline{\mathsf{M}}\bigl(\langle x \rangle ; y\bigr), \langle x \rangle\bigr) \xrightarrow{\mathsf{ev}_{\langle x \rangle ; y}} y \quad \text{in} \quad \mathsf{M},$$

which is called the *multicategorical evaluation* or the *evaluation* at $(\langle x \rangle ; y)$.

The above data are required to satisfy the axioms (8.1.5) through (8.1.8) below.

**Equivariance of Internal Hom:** For the identity permutation $\mathrm{id}_n \in \Sigma_n$, the right $\mathrm{id}_n$-action

$$\underline{M}(\langle x \rangle; y) \xrightarrow{\ \mathrm{id}_n\, =\, 1\ } \underline{M}(\langle x \rangle; y) \tag{8.1.5}$$

is the colored unit of the internal hom object $\underline{M}(\langle x \rangle; y)$. Moreover, for $\sigma, \tau \in \Sigma_n$, the following diagram of unary multimorphisms in M commutes.

$$\tag{8.1.6}$$

$$
\begin{array}{ccc}
\underline{M}(\langle x \rangle; y) & \xrightarrow{\ \sigma\ } & \underline{M}(\langle x \rangle \sigma; y) \\
& {\scriptstyle \sigma\tau}\searrow & \ \downarrow{\scriptstyle \tau} \\
& & \underline{M}(\langle x \rangle \sigma\tau; y)
\end{array}
$$

**Evaluation Bijection:** For objects $\langle x \rangle = \langle x_i \rangle_{i=1}^n$, $\langle y \rangle = \langle y_j \rangle_{j=1}^p$, $z \in M$, the function

$$\tag{8.1.7}$$
$$
\begin{array}{ccc}
M\left(\langle x \rangle; \underline{M}(\langle y \rangle; z)\right) & \xrightarrow[\cong]{\ \chi_{\langle x \rangle; \langle y \rangle; z}\ } & M\left(\langle x \rangle, \langle y \rangle; z\right) \\
f & \longmapsto & \gamma\!\left(\mathrm{ev}_{\langle y \rangle; z}; f, \langle 1_{y_j} \rangle_{j=1}^p \right)
\end{array}
$$

is a bijection, which is called the *evaluation bijection*. Two multimorphisms in M that correspond under this bijection are called *partners*. We write $f^{\#}$ for the partner of a multimorphism $f$, so

$$\chi(f) = f^{\#} \quad \text{and} \quad \chi^{-1}(g) = g^{\#}$$

for $f \in M\left(\langle x \rangle; \underline{M}(\langle y \rangle; z)\right)$ and $g \in M\left(\langle x \rangle, \langle y \rangle; z\right)$.

**Equivariance of Evaluation Bijection:** For objects $\langle x \rangle, \langle y \rangle, z \in M$ as above and permutations $\sigma \in \Sigma_n$ and $\varsigma \in \Sigma_p$, the following diagram of bijections commutes.

$$\tag{8.1.8}$$

$$
\begin{array}{ccc}
M\left(\langle x \rangle; \underline{M}(\langle y \rangle; z)\right) & \xrightarrow{\ \chi_{\langle x \rangle; \langle y \rangle; z}\ } & M\left(\langle x \rangle, \langle y \rangle; z\right) \\
{\scriptstyle \sigma}\downarrow & & \\
M\left(\langle x \rangle \sigma; \underline{M}(\langle y \rangle; z)\right) & & \Big\downarrow{\scriptstyle \sigma \times \varsigma} \\
{\scriptstyle \gamma(\varsigma;\,-)}\downarrow & & \\
M\left(\langle x \rangle \sigma; \underline{M}(\langle y \rangle \varsigma; z)\right) & \xrightarrow{\ \chi_{\langle x \rangle \sigma; \langle y \rangle \varsigma; z}\ } & M\left(\langle x \rangle \sigma, \langle y \rangle \varsigma; z\right)
\end{array}
$$

In (8.1.8) the arrows are defined as follows.
- The top left arrow $\sigma$ and the right vertical arrow $\sigma \times \varsigma$ are right symmetric group action of M (C.1.4).
- In the lower left arrow, $\varsigma$ is the right $\varsigma$-action on the internal hom object $\underline{M}(\langle y \rangle; z)$ in (8.1.3).
- $\gamma(\varsigma; -)$ is composition with the unary multimorphism $\varsigma$ in M.
- The two horizontal arrows $\chi$ are the evaluation bijections in (8.1.7).

This finishes the definition of a closed multicategory.

Moreover, a *non-symmetric closed multicategory* is a triple $\left(M, \underline{M}, \mathrm{ev}\right)$ as above with the changes (i) through (iii) below.

(i) $(M, \gamma, 1)$ is a non-symmetric multicategory.

(ii) The internal hom objects $\underline{M}(\langle x \rangle; y)$ are not equipped with the right symmetric group action (8.1.3).
(iii) We do not require the equivariance of
  - internal hom objects, (8.1.5) and (8.1.6), and
  - evaluation bijection, (8.1.8).
  Thus the only axiom is the evaluation bijection axiom (8.1.7).

This finishes the definition of a non-symmetric closed multicategory.                    ◇

**Example 8.1.9** (Waldhausen Categories)**.** The 2-category of small Waldhausen categories, exact functors, and natural transformations extends to a closed multicategory Wald by [**Zak18**, 5.6]. Since we do not use that result in this work, we refer the reader to [**Zak18**] for further discussion of the closed multicategory Wald. All the results in this work about (non-symmetric) closed multicategories apply to Wald. See, for example, Theorem 9.1.7.                    ◇

We discuss more examples below after some explanation and remarks.

**Explanation 8.1.10** (Closed Multicategories)**.** Suppose $(M, \underline{M}, \mathsf{ev})$ is a non-symmetric closed multicategory.

(1) If M is a closed multicategory, then the right $\sigma$-action on internal hom object (8.1.3) is an element

$$\sigma \in M\Big(\underline{M}(\langle x \rangle; y); \underline{M}(\langle x \rangle \sigma; y)\Big).$$

It is a unary multimorphism in M regardless of the length of $\langle x \rangle = \langle x_i \rangle_{i=1}^{n}$.
(2) The evaluation at $(\langle x \rangle; y)$ in (8.1.4), which is an $(n+1)$-ary multimorphism, is an element

$$\mathsf{ev}_{\langle x \rangle; y} \in M\Big(\underline{M}(\langle x \rangle; y), \langle x \rangle; y\Big).$$

This is an analog of the evaluation (B.3.2) in a symmetric monoidal closed category.
(3) If $\langle x \rangle = \langle \rangle$, then evaluation at $(\langle \rangle; y)$ is a unary multimorphism

$$\underline{M}(\langle \rangle; y) \xrightarrow{\ \mathsf{ev}_{\langle \rangle; y}\ } y \quad \text{in} \quad M.$$

We do *not* require this to be the colored unit of $y$. We elaborate on this point in Remark 8.1.13 below.
(4) In the evaluation bijection $\chi_{\langle x \rangle; \langle y \rangle; z}$ in (8.1.7) with $\langle x \rangle = \langle x_i \rangle_{i=1}^{n}$ and $\langle y \rangle = \langle y_j \rangle_{j=1}^{p}$,
  - the domain $M\big(\langle x \rangle; \underline{M}(\langle y \rangle; z)\big)$ is an $n$-ary multimorphism set in M, and
  - the codomain $M\big(\langle x \rangle, \langle y \rangle; z\big)$ is an $(n+p)$-ary multimorphism set in M.

The evaluation bijection is an analog of the $\otimes$-Hom adjunction in a symmetric monoidal closed category (Definition A.1.19). Partners—which mean multimorphisms that correspond to each other under the evaluation bijection—are analogs of adjoints.

(5) If $\langle x \rangle = \langle \rangle$, then the evaluation bijection $\chi_{\langle \rangle; \langle y \rangle; z}$ is the following bijection.

(8.1.11)
$$\mathsf{M}\big(\langle \rangle; \underline{\mathsf{M}}(\langle y \rangle; z)\big) \xrightarrow[\cong]{\chi_{\langle \rangle; \langle y \rangle; z}} \mathsf{M}\big(\langle y \rangle; z\big)$$
$$f \longmapsto \gamma\big(\mathrm{ev}_{\langle y \rangle; z}; f, \langle 1_{y_j} \rangle_{j=1}^{p}\big)$$

Thus, via this evaluation bijection, each multimorphism set $\mathsf{M}(\langle y \rangle; z)$ is in bijection with the nullary multimorphism set with output given by the internal hom object $\underline{\mathsf{M}}(\langle y \rangle; z)$. ◇

**Remark 8.1.12** (Variants). There are other variants of closed multicategories in the literature. We briefly discuss some of them here.

(1) A closed multicategory as in Definition 8.1.1 is called a *closed symmetric multicategory* in [**Zak18**, 1.2]. There are no other differences between the two definitions. In particular, the main result in [**Zak18**]—that small Waldhausen categories form a closed multicategory Wald (Example 8.1.9)—still holds with our Definition 8.1.1. The permutative analog of this observation is Theorem 8.4.15.

(2) A *biclosed monoidal multicategory* in [**Lam69**, page 106] is analogous to a non-symmetric closed multicategory as in Definition 8.1.1. However, Lambek's definition has more structure, denoted i and m there, which roughly correspond to a monoidal unit and a monoidal product.

(3) A *closed multicategory* in [**Man12**, 3.6, 3.7] is a more restrictive version of a non-symmetric closed multicategory as in Definition 8.1.1. We discuss this nontrivial difference in more detail in Remark 8.1.13.

Our terminology is in line with Definition C.1.3, where a V-multicategory is equipped with a symmetric group action. We add the adjective *non-symmetric* to the variant without the symmetric group action. ◇

**Remark 8.1.13** (Important Differences with Manzyuk's Definition). A closed multicategory as in [**Man12**, 3.7] is more restrictive than a non-symmetric closed multicategory as in Definition 8.1.1. Specifically, the definition in [**Man12**, 3.7] requires (i) the object equality

(8.1.14)
$$\underline{\mathsf{M}}\big(\langle \rangle; y\big) = y \quad \text{for} \quad y \in \mathrm{Ob}\,\mathsf{M}$$

and (ii) the unary multimorphism equality

(8.1.15)
$$\big(\underline{\mathsf{M}}(\langle \rangle; y), \langle \rangle\big) \xrightarrow{\mathrm{ev}_{\langle \rangle; y} = 1_y} y \quad \text{in} \quad \mathsf{M}(y; y).$$

On the other hand, Definition 8.1.1 does *not* require these two equalities. The reason that we do not impose the equalities (8.1.14) and (8.1.15) is that they are *not* satisfied by our most basic examples of endomorphism multicategories in Proposition 8.1.16 below. As stated in (8.1.17), the nullary internal hom object is the internal hom object

$$\underline{\mathrm{End}\,\mathsf{V}}\big(\langle \rangle; y\big) = [1, y] \quad \text{for} \quad y \in \mathsf{V}$$

because an empty $\otimes$ is, by definition, the monoidal unit 1 in V. The object $[1, y]$ is, in general, not equal to $y$. The nullary multicategorical evaluation $\mathrm{ev}_{\langle \rangle; y}$ in (8.1.21) is also not an identity in general. Due to this nontrivial difference between Definition 8.1.1 and the one in [**Man12**], in this work we will not use any of the results in [**Man12**]. ◇

Proposition 8.1.16 below says that the endomorphism multicategory (Example C.3.1) of a symmetric monoidal closed category (Definition A.1.19) is a closed multicategory, as one would expect. It is briefly mentioned in [**Zak18**, Section 1]; we provide a proof here for completeness. We remind the reader that an iterated monoidal product $\otimes_{i=1}^n$ is left normalized (Convention A.1.30), and an empty $\otimes$ is the monoidal unit 1.

**Proposition 8.1.16.** *For each symmetric monoidal closed category* $(\mathsf{V}, \otimes, 1, \xi, [,])$, *the endomorphism multicategory* $\mathsf{End}\,\mathsf{V}$ *becomes a closed multicategory*

$$\big(\mathsf{End}\,\mathsf{V}\,, \underline{\mathsf{End}\,\mathsf{V}}\,, \mathsf{ev}\big)$$

*when it is equipped with the following data.*

- *For objects* $\langle x_i \rangle_{i=1}^n, y \in \mathsf{V}$, *the n-ary internal hom object (8.1.2) is defined as the object*

(8.1.17) $$\underline{\mathsf{End}\,\mathsf{V}}\big(\langle x_i \rangle_{i=1}^n\,; y\big) = \big[\, \otimes_{i=1}^n x_i\,, y \,\big] \quad in \quad \mathsf{V}.$$

- *For a permutation* $\sigma \in \Sigma_n$, *the right $\sigma$-action (8.1.3) on internal hom objects*

(8.1.18) $$\big[\, \otimes_{i=1}^n x_i\,, y \,\big] \xrightarrow[\cong]{\sigma} \big[\, \otimes_{i=1}^n x_{\sigma(i)}\,, y \,\big] \quad in \quad \mathsf{V}$$

*is the image under* $[-, y]$ *of the unique symmetry coherence isomorphism*

(8.1.19) $$\otimes_{i=1}^n x_{\sigma(i)} \xrightarrow[\cong]{\sigma} \otimes_{i=1}^n x_i$$

*that permutes the n factors according to $\sigma$.*

- *The multicategorical evaluation* $\mathsf{ev}_{\langle x \rangle; y}$ *(8.1.4) is defined as the following composite in* $\mathsf{V}$ *if $n > 0$.*

(8.1.20)

$$
\begin{array}{ccc}
\big[\, \otimes_{i=1}^n x_i\,, y \,\big] \otimes x_1 \otimes \cdots \otimes x_n & \xrightarrow{\;\mathsf{ev}_{\langle x \rangle; y}\;} & y \\[2mm]
\alpha \downarrow \cong & & \nearrow \mathsf{ev}_{\otimes_{i=1}^n x_i, y} \\[2mm]
\big[\, \otimes_{i=1}^n x_i\,, y \,\big] \otimes \big( \otimes_{i=1}^n x_i \big) & &
\end{array}
$$

*In (8.1.20) $\alpha$ is the unique coherence isomorphism in* $\mathsf{V}$ *that moves parentheses, which is the identity if $n = 1$, and* $\mathsf{ev}_{\otimes_{i=1}^n x_i, y}$ *is the evaluation in* $\mathsf{V}$ *(B.3.2). If $n = 0$, then the multicategorical evaluation* $\mathsf{ev}_{\langle \rangle; y}$ *is defined as the following composite, with $\rho$ the right unit isomorphism and* $\mathsf{ev}_{1,y}$ *the evaluation in* $\mathsf{V}$.

(8.1.21)

$$
[1, y] \xrightarrow[\cong]{\rho^{-1}} [1, y] \otimes 1 \xrightarrow{\;\mathsf{ev}_{1,y}\;} y \qquad \overset{\mathsf{ev}_{\langle \rangle; y}}{\frown}
$$

*Proof.* We check the axioms (8.1.5) through (8.1.8) for $\mathsf{End}\,\mathsf{V}$. For the rest of this proof, $\langle x \rangle = \langle x_i \rangle_{i=1}^n$, $\langle y \rangle = \langle y_j \rangle_{j=1}^p$, $y$, and $z$ are objects in $\mathsf{V}$.

*Equivariance of Internal Hom.* For the identity permutation $\mathsf{id}_n \in \Sigma_n$, the symmetry coherence isomorphism (8.1.19) is the identity morphism. Thus the right $\mathsf{id}_n$-action in (8.1.18) is the identity morphism, proving the axiom (8.1.5).

For permutations $\sigma, \tau \in \Sigma_n$, the axiom (8.1.6) requires the commutativity of the following diagram in $\mathsf{V}$.

(8.1.22) $$\big[\, \otimes_{i=1}^n x_i\,, y \,\big] \xrightarrow{\sigma} \big[\, \otimes_{i=1}^n x_{\sigma(i)}\,, y \,\big] \xrightarrow{\tau} \big[\, \otimes_{i=1}^n x_{\sigma\tau(i)}\,, y \,\big] \qquad \overset{\sigma\tau}{\frown}$$

The diagram (8.1.22) is the image under $[-, y]$ of the following diagram in $\mathsf{V}$.

$$\bigotimes_{i=1}^{n} x_{\sigma\tau(i)} \xrightarrow{\ \tau\ } \bigotimes_{i=1}^{n} x_{\sigma(i)} \xrightarrow{\ \sigma\ } \bigotimes_{i=1}^{n} x_i$$

with the top arc labeled $\sigma\tau$.

This diagram commutes by the uniqueness of the symmetry coherence isomorphism that permutes the $n$ factors according to $\sigma\tau$ [**ML98**, XI.1 Theorem 1]. Thus the diagram (8.1.22) is also commutative.

*Evaluation Bijection.*  Note that each of the two cases of $\mathrm{ev}_{\langle x \rangle; y}$ in (8.1.20) and (8.1.21) consists of a coherence isomorphism followed by an instance of the evaluation in $\mathsf{V}$.  For the rest of this proof, the symbol $\cong$ denotes a coherence isomorphism.

The function $\chi = \chi_{\langle x \rangle; \langle y \rangle; z}$ in (8.1.7)—which we want to show is a bijection—sends a morphism

$$(8.1.23) \qquad\qquad \bigotimes_{i=1}^{n} x_i \xrightarrow{\ f\ } \left[ \bigotimes_{j=1}^{p} y_j, z \right] \quad \text{in} \quad \mathsf{V}$$

to the following composite.

$$(8.1.24)$$

$$\begin{array}{ccc}
\left( \bigotimes_{i=1}^{n} x_i \right) \otimes y_1 \otimes \cdots \otimes y_p & \xrightarrow{\ \ \chi(f)\ \ } & z \\[2mm]
{\scriptstyle f \otimes 1_{y_1} \otimes \cdots \otimes 1_{y_p}} \Big\downarrow & & \Big\uparrow {\scriptstyle \mathrm{ev}_{\bigotimes_{j=1}^{p} y_j, z}} \\[2mm]
\left[ \bigotimes_{j=1}^{p} y_j, z \right] \otimes y_1 \otimes \cdots \otimes y_p & \xrightarrow{\ \cong\ } & \left[ \bigotimes_{j=1}^{p} y_j, z \right] \otimes \left( \bigotimes_{j=1}^{p} y_j \right)
\end{array}$$

The desired inverse of $\chi$ sends a morphism

$$\left( \bigotimes_{i=1}^{n} x_i \right) \otimes y_1 \otimes \cdots \otimes y_p \xrightarrow{\ g\ } z \quad \text{in} \quad \mathsf{V}$$

to the adjoint

$$\bigotimes_{i=1}^{n} x_i \xrightarrow{\ \overline{g}^{\#}\ } \left[ \bigotimes_{j=1}^{p} y_j, z \right]$$

of the following composite $\overline{g}$.

$$\left( \bigotimes_{i=1}^{n} x_i \right) \otimes \left( \bigotimes_{j=1}^{p} y_j \right) \xrightarrow{\ \cong\ } \left( \bigotimes_{i=1}^{n} x_i \right) \otimes y_1 \otimes \cdots \otimes y_p \xrightarrow{\ g\ } z$$

with the top arc labeled $\overline{g}$.

The assignments, $f \longmapsto \chi(f)$ and $g \longmapsto \overline{g}^{\#}$, are inverses of each other by the uniqueness of adjoints and the fact that the evaluation in $\mathsf{V}$ (B.3.2) is the counit of the $\otimes$-$[,]$ adjunction. Thus $\chi$ is a bijection.

*Equivariance of Evaluation Bijection.* For permutations $\sigma \in \Sigma_n$ and $\varsigma \in \Sigma_p$ and a morphism $f$ as in (8.1.23), the desired commutative diagram (8.1.8) is the boundary of the following diagram in V.

$$
\begin{array}{c}
\left(\otimes_{i=1}^n x_{\sigma(i)}\right) \otimes y_{\varsigma(1)} \otimes \cdots \otimes y_{\varsigma(p)} \xrightarrow{\ \sigma \otimes \varsigma\ } \left(\otimes_{i=1}^n x_i\right) \otimes y_1 \otimes \cdots \otimes y_p
\end{array}
$$

The following statements hold for the diagram above.

- The top triangle and the quadrilateral under it commute by the functoriality of $\otimes$.
- The middle quadrilateral commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1].
- The lower left quadrilateral commutes by the naturality of coherence isomorphisms.
- The lower right quadrilateral commutes because each of the two composites has adjoint

$$
\left[\otimes_{j=1}^p y_j, z\right] \longrightarrow \left[\otimes_{j=1}^p y_{\varsigma(j)}, z\right]
$$

given by the image under $[-, z]$ of the symmetry coherence isomorphism

$$
\otimes_{j=1}^p y_{\varsigma(j)} \xrightarrow[\cong]{\ \varsigma\ } \otimes_{j=1}^p y_j.
$$

This finishes the proof of the axioms (8.1.5) through (8.1.8) for $\mathsf{End}\,\mathsf{V}$. $\qquad\square$

**Example 8.1.25.** Proposition 8.1.16 applies to each of the symmetric monoidal closed categories listed in Example 6.2.3. In other words, via the endomorphism multicategory construction, each of those symmetric monoidal closed categories is a closed multicategory. However, Proposition 8.1.16 does not apply to $\mathsf{PermCat^{su}}$ because it is not a monoidal category. $\qquad\diamond$

## 8.2. Internal Hom Permutative Categories

Recall the following two results about permutative categories.

(1) By Theorem 1.4.29 $\mathsf{PermCat^{su}}$ is a Cat-multicategory. As a multicategory, it has
   - small permutative categories (Definition A.1.14) as objects and
   - multilinear functors (Definition 1.4.2) as multimorphisms.
(2) By Theorem 6.4.20 the category $\mathsf{PermCat^{su}}$ is a $\mathsf{PermCat^{su}}$-category.

In Sections 8.2 through 8.4, we generalize these two facts by showing that $\mathsf{PermCat}^{\mathsf{su}}$ is a closed multicategory (Definition 8.1.1); see Theorem 8.4.15. In this section we construct the internal hom objects and their symmetry group action in the closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$. Here is an outline of this section.

- The internal hom objects are constructed in Definition 8.2.1 and verified in Lemma 8.2.13.
- The symmetric group action on the internal hom objects are constructed in Definition 8.2.14 and verified in Lemma 8.2.16.

We discuss the multicategorical evaluation and the closed multicategory axioms of $\mathsf{PermCat}^{\mathsf{su}}$ in subsequent sections.

**Internal Hom Objects.** To construct the closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$, first we define the $n$-ary internal hom objects (8.1.2). For a generic permutative category, we denote the monoidal product, monoidal unit, and braiding by $\oplus$, $e$, and $\xi$, respectively.

**Definition 8.2.1.** For small permutative categories $\mathsf{D}$ and $\langle \mathsf{C} \rangle = \langle \mathsf{C}_i \rangle_{i=1}^n$ for $n \geq 0$, we define the data of a small permutative category

$$(8.2.2) \qquad \left( \underline{\mathsf{PermCat}^{\mathsf{su}}}(\langle \mathsf{C} \rangle; \mathsf{D}), \oplus, \underline{e}, \underline{\xi} \right),$$

which is called an *internal hom permutative category*, as follows. We also use the shortened notation

$$\mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}} \quad \text{and} \quad \underline{\mathsf{P}^{\mathsf{su}}} = \underline{\mathsf{PermCat}^{\mathsf{su}}}.$$

**Underlying Category:** It is the category $\mathsf{P}^{\mathsf{su}}(\langle \mathsf{C} \rangle; \mathsf{D})$ in Definition 1.4.15.
- Its objects are $n$-linear functors $\langle \mathsf{C} \rangle \longrightarrow \mathsf{D}$ (Definition 1.4.2).
- Its morphisms are $n$-linear transformations (Definition 1.4.10).

**Monoidal Product on Objects:** For two $n$-linear functors

$$(8.2.3) \qquad \left( P, \{P_i^2\}_{i=1}^n \right), \left( Q, \{Q_i^2\}_{i=1}^n \right) : \textstyle\prod_{i=1}^n \mathsf{C}_i \longrightarrow \mathsf{D},$$

their monoidal product has underlying functor $P \oplus Q$ given by the composite functor below.

$$(8.2.4) \qquad \textstyle\prod_{i=1}^n \mathsf{C}_i \xrightarrow{(P,Q)} \mathsf{D} \times \mathsf{D} \xrightarrow{\oplus} \mathsf{D}$$

with $P \oplus Q$ spanning the top.

In other words, the functor $P \oplus Q$ is given by the objectwise monoidal product

$$(P \oplus Q)\langle x \rangle = P\langle x \rangle \oplus Q\langle x \rangle$$

for objects and morphisms $\langle x \rangle$ in $\prod_{i=1}^n \mathsf{C}_i$.

For each $i \in \{1, \ldots, n\}$, the $i$-th linearity constraint of $P \oplus Q$, denoted $(P \oplus Q)_i^2$, is defined as follows. For objects $\langle x \rangle \in \prod_{i=1}^n \mathsf{C}_i$ and $x_i' \in \mathsf{C}_i$, using Notation 1.4.1 we denote by

$$(8.2.5) \qquad \langle x' \rangle = \langle x \rangle \circ_i x_i' \quad \text{and} \quad \langle x'' \rangle = \langle x \rangle \circ_i (x_i \oplus x_i').$$

The corresponding component of $(P \oplus Q)_i^2$ is the following composite in D, with $\zeta$ denoting the braiding in D.

(8.2.6)

$$
\begin{array}{ccc}
(P \oplus Q)\langle x\rangle \oplus (P \oplus Q)\langle x'\rangle & \xrightarrow{\ (P \oplus Q)_i^2\ } & (P \oplus Q)\langle x''\rangle \\
\| & & \| \\
P\langle x\rangle \oplus Q\langle x\rangle \oplus P\langle x'\rangle \oplus Q\langle x'\rangle & & P\langle x''\rangle \oplus Q\langle x''\rangle \\
\end{array}
$$

$1 \oplus \zeta \oplus 1 \searrow \qquad P\langle x\rangle \oplus P\langle x'\rangle \oplus Q\langle x\rangle \oplus Q\langle x'\rangle \nearrow P_i^2 \oplus Q_i^2$

The naturality of $(P \oplus Q)_i^2$ follows from the naturality of $P_i^2$, $Q_i^2$, and $\zeta$, together with the functoriality of $\oplus$ in D.

**Monoidal Product on Morphisms:** For $n$-linear functors $P$ and $Q$ as above, suppose given $n$-linear transformations $\theta$ and $\psi$ as follows.

(8.2.7)

$$
\prod_{i=1}^n C_i \overset{P}{\underset{P'}{\Longrightarrow\theta}} D \qquad\qquad \prod_{i=1}^n C_i \overset{Q}{\underset{Q'}{\Longrightarrow\psi}} D
$$

Their monoidal product

$$
\theta \oplus \psi : P \oplus Q \longrightarrow P' \oplus Q'
$$

is the natural transformation defined by the whiskering below.

(8.2.8)

$$
\prod_{i=1}^n C_i \overset{(P,Q)}{\underset{(P',Q')}{\Longrightarrow(\theta,\psi)}} D \times D \xrightarrow{\ \oplus\ } D
$$

In other words, for each object $\langle x\rangle \in \prod_{i=1}^n C_i$, $\theta \oplus \psi$ has $\langle x\rangle$-component given by the monoidal product

$$
(\theta \oplus \psi)_{\langle x\rangle} = \theta_{\langle x\rangle} \oplus \psi_{\langle x\rangle} : P\langle x\rangle \oplus Q\langle x\rangle \longrightarrow P'\langle x\rangle \oplus Q'\langle x\rangle.
$$

This defines an $n$-linear transformation $\theta \oplus \psi$ for the following reasons.
- The naturality of $\theta \oplus \psi$ follows from the naturality of $\theta$ and $\psi$, together with the functoriality of $\oplus$ in D.
- The unity axiom (1.4.12) holds for $\theta \oplus \psi$ because, if any $x_i = e$ in $C_i$, then $\theta_{\langle x\rangle} = 1_e = \psi_{\langle x\rangle}$.
- The constraint compatibility axiom (1.4.13) holds for $\theta \oplus \psi$ by
  - the naturality of the braiding $\zeta$ in D and
  - the axiom (1.4.13) for $\theta$ and $\psi$.

The construction $\oplus$ for $\underline{P}^{\mathsf{su}}(\langle C\rangle\,;\,D)$ preserves identities and composition of $n$-linear transformations by the functoriality of $\oplus$ in D. Moreover, $\oplus$ is associative on $n$-linear transformations because $\oplus$ in D is associative.

**Monoidal Unit:** The monoidal unit is the constant functor

(8.2.9)
$$
\underline{e} : \prod_{i=1}^n C_i \longrightarrow D
$$

at the monoidal unit $e$ in D. Each of its $n$ linearity constraints is given componentwise by the identity morphism

$$
1_e : e \oplus e = e \longrightarrow e \quad \text{in} \quad D.
$$

The axioms of an $n$-linear functor hold for $\underline{e}$ because
- e is a strict monoidal unit in D and

- there are morphism equalities

$$(8.2.10) \qquad \xi_{e,?} = 1_? = \xi_{?,e} \quad \text{in} \quad D.$$

These two facts also imply that $\underline{e}$ is a strict two-sided unit for $\oplus$.

**Braiding:** For $n$-linear functors $P$ and $Q$ as above, the $(P,Q)$-component of the braiding

$$\underline{\xi}_{P,Q} : P \oplus Q \longrightarrow Q \oplus P$$

is the natural isomorphism given by the following pasting diagram, with $\tau$ swapping the two factors and $\xi$ denoting the braiding in D.

$$(8.2.11)$$



In other words, for each object $\langle x \rangle \in \prod_{i=1}^n C_i$, $\underline{\xi}_{P,Q}$ has $\langle x \rangle$-component given by the braiding in D

$$(\underline{\xi}_{P,Q})_{\langle x \rangle} = \xi_{P\langle x \rangle, Q\langle x \rangle} : P\langle x \rangle \oplus Q\langle x \rangle \xrightarrow{\cong} Q\langle x \rangle \oplus P\langle x \rangle.$$

The naturality of the braiding $\xi$ in D implies the naturality of each of

- $(\underline{\xi}_{P,Q})_{\langle x \rangle}$ with respect to $\langle x \rangle$ and
- $\underline{\xi}_{P,Q}$ with respect to $P$ and $Q$.

This finishes the definition of $\underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle; D)$. Lemma 8.2.13 proves that it is a permutative category. $\diamond$

**Explanation 8.2.12.** Consider Definition 8.2.1.

- If $n = 0$, then $\langle C \rangle$ is the empty sequence and

$$\underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle \rangle; D) = D$$

as permutative categories.

- If $n = 1$, then

$$\underline{\mathsf{PermCat}}^{\mathsf{su}}(C; D) = \mathsf{PermCat}^{\mathsf{su}}(C, D),$$

the hom permutative category in Lemma 6.4.11. $\diamond$

**Lemma 8.2.13.** *For small permutative categories* $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ *and* D, *the quadruple*

$$\left( \underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle; D), \oplus, \underline{e}, \underline{\xi} \right)$$

*in Definition 8.2.1 is a small permutative category.*

*Proof.* By Explanation 8.2.12 and Lemma 6.4.11 for the case $n = 1$, we only need to check the cases for $n > 1$. We already explained some of the required conditions in Definition 8.2.1. It remains to check statements (i) through (iii) below.

(i) The data defined in (8.2.4) and (8.2.6)

$$\left( P \oplus Q, \left\{ (P \oplus Q)_i^2 \right\}_{i=1}^n \right) : \prod_{i=1}^n C_i \longrightarrow D$$

satisfy the axioms (1.4.4) through (1.4.8) of an $n$-linear functor.

(ii) The construction $\oplus$ is associative on $n$-linear functors $\prod_{i=1}^n C_i \longrightarrow D$.

(iii) For $n$-linear functors $P$ and $Q$, the natural isomorphism defined in (8.2.11)

$$\underline{\xi}_{P,Q} : P \oplus Q \longrightarrow Q \oplus P$$

satisfies the axioms (1.4.12) and (1.4.13) of an $n$-linear transformation.

Once we establish statements (i) through (iii) above, the symmetry and hexagon axioms (A.1.15) for $\underline{\mathsf{P}}^{\mathsf{su}}(\langle \mathsf{C} \rangle ; \mathsf{D})$ follow from those for $\mathsf{D}$.

*Statement (i).* The unity axiom (1.4.4) for $P \oplus Q$ follows from the unity axiom for $P$ and $Q$, together with the strict unity of $\mathsf{e}$ in $\mathsf{D}$. The constraint unity, associativity, and symmetry axioms, (1.4.5) through (1.4.7), are proved using the argument for statement (i) in the proof of Lemma 6.4.11, with an appropriate change of notation.

For the constraint 2-by-2 axiom (1.4.8), suppose $i \neq k \in \{1, \ldots, n\}$ and consider objects

$$\langle x \rangle = \langle x_j^0 \rangle_{j=1}^n \in \prod_{j=1}^n \mathsf{C}_j, \quad x_i^1 \in \mathsf{C}_i, \quad \text{and} \quad x_k^1 \in \mathsf{C}_k.$$

We define the following objects for $\ell \in \{i, k\}$, $a, b \in \{0, 1, 2\}$, and $R \in \{P, Q\}$.

$$x_\ell^2 = x_\ell^0 \oplus x_\ell^1 \in \mathsf{C}_\ell \qquad R_{a,b} = R\langle x \circ_i x_i^a \circ_k x_k^b \rangle \in \mathsf{D}$$

For example, we have the objects

$$P_{0,0} = P\langle x \rangle \quad \text{and} \quad Q_{1,2} = Q\langle x \circ_i x_i^1 \circ_k (x_k^0 \oplus x_k^1) \rangle.$$

In the following diagram, we omit all the $\oplus$ symbols to save space. With these conventions, the constraint 2-by-2 diagram (1.4.8) for $P \oplus Q$ is the boundary of the following diagram in $\mathsf{D}$.



The following statements hold for the diagram above.

- The top and bottom quadrilaterals commute by the naturality of the braiding $\xi$ in $\mathsf{D}$.
- The left sub-region commutes by the coherence theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].
- The right pentagon commutes by the constraint 2-by-2 axiom for $P$ and $Q$.

This proves that $P \oplus Q$ is an $n$-linear functor.

*Statement (ii)*. The associativity of $\oplus$ for $n$-linear functors

$$\left(P, \{P_i^2\}_{i=1}^n\right), \left(Q, \{Q_i^2\}_{i=1}^n\right), \left(R, \{R_i^2\}_{i=1}^n\right) : \textstyle\prod_{i=1}^n C_i \longrightarrow D$$

is proved using the argument for statement (ii) in the proof of Lemma 6.4.11, with an appropriate change of notation.

*Statement (iii)*. The axioms (1.4.12) and (1.4.13) of an $n$-linear transformation for

$$\underline{\varsigma}_{P,Q} : P \oplus Q \longrightarrow Q \oplus P$$

are proved using the argument for statement (iv) in the proof of Lemma 6.4.11, with an appropriate change of notation. $\qquad\square$

From now on, $\underline{\mathsf{P}}^{\mathsf{su}}(\langle C \rangle; D)$ is a permutative category as in Lemma 8.2.13.

**Symmetric Group Action on Internal Hom.** Next we define the right symmetric group action (8.1.3) on the internal hom permutative categories $\underline{\mathsf{P}}^{\mathsf{su}}(\langle C \rangle; D)$. Recall from Definition 8.2.1 that the underlying category of $\underline{\mathsf{P}}^{\mathsf{su}}(\langle C \rangle; D)$ is the category $\mathsf{P}^{\mathsf{su}}(\langle C \rangle; D)$ in Definition 1.4.15, with $n$-linear functors as objects and $n$-linear transformations as morphisms.

**Definition 8.2.14.** For small permutative categories $D$ and $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ for $n \geq 0$ and a permutation $\sigma \in \Sigma_n$, we define the functor

$$(8.2.15) \qquad \underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle; D) \xrightarrow[\cong]{\sigma} \underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle\sigma; D)$$

as the isomorphism

$$\mathsf{PermCat}^{\mathsf{su}}(\langle C \rangle; D) \xrightarrow[\cong]{\sigma} \mathsf{PermCat}^{\mathsf{su}}(\langle C \rangle\sigma; D)$$

in (1.4.18). $\hfill\diamond$

**Lemma 8.2.16.** *In the context of Definition 8.2.14, the following statements hold.*

(1) *Equipped with identity monoidal and unit constraints, $\sigma$ in (8.2.15) is a strict symmetric monoidal isomorphism.*
(2) *The equivariance axioms for internal hom objects, (8.1.5) and (8.1.6), hold.*

*Proof.* Once statement (1) is proved, statement (2) follows from (i) the definition (1.4.19) and (ii) the associativity of functor composition and horizontal composition of natural transformations.

For statement (1), we need to show that $\sigma$ preserves (i) the monoidal unit and (ii) the monoidal product of its domain and codomain. We denote $\sigma(-)$ by $(-)^\sigma$.

*Preservation of Monoidal Unit*. The isomorphism $\sigma$ preserves the monoidal unit $\underline{e}$ in (8.2.9) because the composite

$$\textstyle\prod_{i=1}^n C_{\sigma(i)} \xrightarrow{\sigma} \prod_{i=1}^n C_i \xrightarrow{e} D$$

is the constant functor at the monoidal unit $e$ in $D$, since $\underline{e}$ is constant at $e$. By definition (1.4.20), for each $i \in \{1, \ldots, n\}$ the $i$-th linearity constraint of $\underline{e}^\sigma$ is the $\sigma(i)$-th linearity constraint of $\underline{e}$, which is componentwise given by $1_e$ in $D$. Thus $\underline{e}^\sigma$ is the monoidal unit in $\underline{\mathsf{P}}^{\mathsf{su}}(\langle C \rangle\sigma; D)$.

*Preservation of Monoidal Product.* For $n$-linear functors $P, Q : \langle C \rangle \longrightarrow D$ as in (8.2.3), there is an equality of $n$-linear functors

$$P^\sigma \oplus Q^\sigma = (P \oplus Q)^\sigma : \langle C \rangle \sigma \longrightarrow D$$

for the following reasons. First, by definitions (1.4.19) and (8.2.4), each of the functors $P^\sigma \oplus Q^\sigma$ and $(P \oplus Q)^\sigma$ is given by the following composite.

$$(8.2.17) \qquad \prod_{i=1}^n C_{\sigma(i)} \xrightarrow{\ \sigma\ } \prod_{i=1}^n C_i \xrightarrow{\ (P,Q)\ } D \times D \xrightarrow{\ \oplus\ } D$$

Next, to show that $P^\sigma \oplus Q^\sigma$ and $(P \oplus Q)^\sigma$ have the same $i$-th linearity constraint for each $i \in \{1, \dots, n\}$, we consider objects

$$\langle x \rangle = \langle x_j \rangle_{j=1}^n \in \prod_{j=1}^n C_{\sigma(j)} \quad \text{and} \quad x_i' \in C_{\sigma(i)}$$

and use the notation $x_i'' = x_i \oplus x_i'$. For these objects, by definition (1.4.20) and (8.2.6), each of $(P^\sigma \oplus Q^\sigma)_i^2$ and $\left((P \oplus Q)^\sigma\right)_i^2$ is given by the following composite morphism in D.

$$
\begin{array}{ccc}
(P \oplus Q)^\sigma \langle x \rangle \oplus (P \oplus Q)^\sigma \langle x \circ_i x_i' \rangle & & (P \oplus Q)^\sigma \langle x \circ_i x_i'' \rangle \\
\| & & \| \\
P(\sigma\langle x \rangle) \oplus Q(\sigma\langle x \rangle) \oplus P\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i'\big) \oplus Q\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i'\big) & & P\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i''\big) \oplus Q\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i''\big) \\
\end{array}
$$

$$
1 \oplus \zeta \oplus 1 \searrow \quad
\begin{array}{c}
P(\sigma\langle x \rangle) \oplus P\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i'\big) \\
\oplus Q(\sigma\langle x \rangle) \oplus Q\big(\sigma\langle x \rangle \circ_{\sigma(i)} x_i'\big)
\end{array}
\quad \nearrow P_{\sigma(i)}^2 \oplus Q_{\sigma(i)}^2
$$

This proves that $P^\sigma \oplus Q^\sigma$ and $(P \oplus Q)^\sigma$ are equal as $n$-linear functors.

Finally, by (1.4.19), (8.2.8), and (8.2.17) with the $n$-linear functors $(P, Q)$ replaced by $n$-linear transformations $(\theta, \psi)$, the functor $(-)^\sigma$ preserves the monoidal product of morphisms. $\qquad \square$

## 8.3. Multicategorical Evaluation for Permutative Categories

In Section 8.2 we constructed internal hom permutative categories and their symmetric group action. To continue the construction of the closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$, in this section we construct its multicategorical evaluation.

- We define the multicategorical evaluation for small permutative categories in Definition 8.3.1. We verify their multilinearity in Lemma 8.3.8.
- Explanations 8.3.9 and 8.3.16 provide a thorough description of the function $\chi$ (8.1.7) in the definition of a closed multicategory, in the context of small permutative categories. We use this discussion in the next section to establish the evaluation bijection and its equivariance axiom; see Lemmas 8.4.1 and 8.4.9.

**Multicategorical Evaluation.** Now we define the evaluation (8.1.4) for the internal hom permutative categories $\underline{\mathsf{P}^{\mathsf{su}}}\big(\langle C \rangle ; D\big)$ in Lemma 8.2.13.

**Definition 8.3.1.** For small permutative categories D and $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ with $n \geq 0$, we define the data of an $(n+1)$-linear functor

$$(8.3.2) \qquad \underline{\mathsf{PermCat}^{\mathsf{su}}}\big(\langle C \rangle ; D\big) \times \prod_{i=1}^n C_i \xrightarrow{\ \mathsf{ev}_{\langle C \rangle ; D}\ } D$$

as follows.

**Underlying Functor:** For an $n$-linear functor

$$\left(P, \{P_i^2\}_{i=1}^n\right) : \prod_{i=1}^n C_i \longrightarrow D$$

and an object $\langle x \rangle \in \prod_{i=1}^n C_i$, we define the object

(8.3.3)
$$ev_{\langle C \rangle; D}\left(P, \langle x \rangle\right) = P\langle x \rangle \quad \text{in} \quad D.$$

For an $n$-linear transformation $\theta : P \longrightarrow Q$ between $n$-linear functors

$$\left(P, \{P_i^2\}_{i=1}^n\right), \left(Q, \{Q_i^2\}_{i=1}^n\right) : \prod_{i=1}^n C_i \longrightarrow D$$

and a morphism $\langle f \rangle : \langle x \rangle \longrightarrow \langle y \rangle$ in $\prod_{i=1}^n C_i$, we define the morphism

$$ev_{\langle C \rangle; D}\left(\theta, \langle f \rangle\right) : P\langle x \rangle \longrightarrow Q\langle y \rangle \quad \text{in} \quad D$$

as either one of the following two composites.

(8.3.4)
$$
\begin{array}{ccc}
P\langle x \rangle & \xrightarrow{\;\theta_{\langle x \rangle}\;} & Q\langle x \rangle \\
{\scriptstyle P\langle f \rangle}\big\downarrow & & \big\downarrow{\scriptstyle Q\langle f \rangle} \\
P\langle y \rangle & \xrightarrow{\;\theta_{\langle y \rangle}\;} & Q\langle y \rangle
\end{array}
$$

The diagram (8.3.4) commutes by the naturality of $\theta$.

**Linearity Constraints:** Suppose $P$ and $Q$ are $n$-linear functors as above, and suppose $\langle x \rangle \in \prod_{i=1}^n C_i$ and $x_i' \in C_i$ are objects.

- The first linearity constraint of $ev_{\langle C \rangle; D}$ is given by the following identity morphism in D.

(8.3.5)
$$
\begin{array}{ccc}
ev_{\langle C \rangle; D}\left(P, \langle x \rangle\right) \oplus ev_{\langle C \rangle; D}\left(Q, \langle x \rangle\right) & \xrightarrow{\;(ev_{\langle C \rangle; D})_1^2\;} & ev_{\langle C \rangle; D}\left(P \oplus Q, \langle x \rangle\right) \\
\| & & \| \\
P\langle x \rangle \oplus Q\langle x \rangle & \xrightarrow{\quad\quad 1 \quad\quad} & (P \oplus Q)\langle x \rangle
\end{array}
$$

- For each $i \in \{1, \ldots, n\}$, the $(i+1)$-st linearity constraint of $ev_{\langle C \rangle; D}$ is given by the following $i$-th linearity constraint of $P$.

(8.3.6)
$$
\begin{array}{ccc}
ev_{\langle C \rangle; D}\left(P, \langle x \rangle\right) \oplus ev_{\langle C \rangle; D}\left(P, \langle x \rangle \circ_i x_i'\right) & \xrightarrow{\;(ev_{\langle C \rangle; D})_{i+1}^2\;} & ev_{\langle C \rangle; D}\left(P, \langle x \rangle \circ_i (x_i \oplus x_i')\right) \\
\| & & \| \\
P\langle x \rangle \oplus P\left(\langle x \rangle \circ_i x_i'\right) & \xrightarrow{\quad\quad P_i^2 \quad\quad} & P\left(\langle x \rangle \circ_i (x_i \oplus x_i')\right)
\end{array}
$$

This finishes the definition of $ev_{\langle C \rangle; D}$.                                                  $\diamond$

**Explanation 8.3.7.** Consider Definition 8.3.1.

- If $n = 0$, then $\langle C \rangle$ is empty, and

$$ev_{\langle \rangle; D} : \underline{PermCat}^{su}\left(\langle \rangle; D\right) = D \longrightarrow D$$

is the identity symmetric monoidal functor on D.

- If $n = 1$, then

$$ev_{C; D} : \underline{P}^{su}(C; D) \times C \longrightarrow D$$

is equal to the bilinear functor $ev_{C, D}$ in Proposition 6.5.7.            $\diamond$

**Lemma 8.3.8.** *For small permutative categories* D *and* $\langle C \rangle = \langle C_i \rangle_{i=1}^n$, *the data*

$$\left(ev_{\langle C \rangle; D}, \left\{(ev_{\langle C \rangle; D})_i^2\right\}_{i=1}^{n+1}\right) : \underline{PermCat}^{su}\left(\langle C \rangle; D\right) \times \prod_{i=1}^n C_i \longrightarrow D$$

*in Definition 8.3.1 form an $(n+1)$-linear functor.*

*Proof.* By Explanation 8.3.7 and Proposition 6.5.7 for the case $n = 1$, we only need to check statements (i) through (iii) below for $n > 1$.

(i) $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}$ is a functor.
(ii) $(\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}})_i^2$ is a natural transformation for each $i \in \{1, \ldots, n+1\}$.
(iii) $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}$ satisfies the axioms (1.4.4) through (1.4.8) for an $(n+1)$-linear functor.

*Statements (i) and (ii).* These assertions are proved using the proofs for statements (i) and (ii), respectively, in the proof of Proposition 6.5.7, with an appropriate change of notation.

*Statement (iii).* The unity axiom (1.4.4) holds for $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}$ by the definitions (8.2.9) of $\underline{\mathsf{e}}$, (8.3.3) of $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}(P, \langle x \rangle)$, and (8.3.4) of $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}(\theta, \langle f \rangle)$.

The constraint unity, associativity, and symmetry axioms, (1.4.5) through (1.4.7), hold for $(\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}})_1^2$ because it is the identity natural transformation. For $(\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}})_{i+1}^2$ with $i \in \{1, \ldots, n\}$, these axioms hold by the definition (8.2.9) of $\underline{\mathsf{e}}$ and the corresponding axioms for $n$-linear functors.

For the constraint 2-by-2 axiom (1.4.8) for $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}$, we consider distinct indices $i \neq k \in \{1, \ldots, n+1\}$ as follows.

- If either $i = 1$ or $k = 1$, then the desired diagram (1.4.8) commutes by the definition (8.2.6) of $(P \oplus Q)_i^2$, the symmetry axiom (A.1.15), and the functoriality of $\oplus$ in $\mathsf{D}$, as in the proof of statement (iii) in the proof of Proposition 6.5.7.
- If $i, k > 1$, then the desired diagram (1.4.8) is the constraint 2-by-2 diagram for an $n$-linear functor $P$ and the indices $i - 1 \neq k - 1 \in \{1, \ldots, n\}$, which is commutative.

This proves that $\mathsf{ev}_{\langle\mathsf{C}\rangle;\mathsf{D}}$ is an $(n+1)$-linear functor.                    $\square$

**The Function $\chi$ for Permutative Categories.** To prepare for the proofs of the other two axioms of a closed multicategory, (8.1.7) and (8.1.8), for $\mathsf{PermCat}^{\mathsf{su}}$ in the next section, here we explain the function $\chi$ in (8.1.7) in the current context, starting with its domain.

**Explanation 8.3.9** (Domain of $\chi$). We consider the following context:

- $\mathsf{M} = \mathsf{P}^{\mathsf{su}}$, the multicategory of small permutative categories and multilinear functors in Theorem 1.4.29;
- $\underline{\mathsf{M}}(?;?) = \underline{\mathsf{P}}^{\mathsf{su}}(?;?)$, the internal hom permutative categories in Lemma 8.2.13;
- the right symmetric group action on internal hom in Lemma 8.2.16; and
- the multilinear evaluation $\mathsf{ev}_{?;?}$ in Lemma 8.3.8.

In this context, we consider small permutative categories

$$\mathsf{B}, \quad \langle\mathsf{C}\rangle = \langle\mathsf{C}_i\rangle_{i=1}^n, \quad \text{and} \quad \langle\mathsf{D}\rangle = \langle\mathsf{D}_j\rangle_{j=1}^p;$$

morphisms

(8.3.10) $\qquad \langle w \rangle \xrightarrow{\langle f \rangle} \langle x \rangle \in \prod_{i=1}^n \mathsf{C}_i \quad \text{and} \quad \langle y \rangle \xrightarrow{\langle g \rangle} \langle z \rangle \in \prod_{j=1}^p \mathsf{D}_j;$

and an $n$-linear functor (Definition 1.4.2)

(8.3.11) $\qquad (P, \{P_i^2\}_{i=1}^n) : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \underline{\mathsf{P}}^{\mathsf{su}}(\langle\mathsf{D}\rangle; \mathsf{B}).$

We explain the $n$-linear functor $(P, \{P_i^2\}_{i=1}^n)$ and establish some notation.

*Underlying Functor.* By Definition 8.2.1, the object $P\langle w \rangle$ in $\underline{\mathsf{P}}^{\mathsf{su}}(\langle \mathsf{D} \rangle; \mathsf{B})$ is a $p$-linear functor, and $P\langle f \rangle$ is a $p$-linear transformation (Definition 1.4.10), as in the left diagram below.

(8.3.12)
$$\prod_{j=1}^{p} \mathsf{D}_j \overset{(P\langle w \rangle, \{P\langle w \rangle_j^2\}_{j=1}^p)}{\underset{(P\langle x \rangle, \{P\langle x \rangle_j^2\}_{j=1}^p)}{\overbrace{\underset{\Downarrow P\langle f \rangle}{\hspace{2cm}}}}} \mathsf{B} \qquad \prod_{j=1}^{p} \mathsf{D}_j \overset{P\langle w \rangle \oplus P\langle w' \rangle}{\underset{P\langle w'' \rangle}{\overbrace{\underset{\Downarrow (P_i^2)_{\langle w \rangle; x_i}}{\hspace{2cm}}}}} \mathsf{B}$$

*Linearity Constraints.* For indices $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, p\}$, we use the following notation.

(8.3.13)
$$\begin{aligned} \langle w' \rangle &= \langle w \circ_i x_i \rangle & \langle y' \rangle &= \langle y \circ_j z_j \rangle \\ \langle w'' \rangle &= \langle w \circ_i (w_i \oplus x_i) \rangle & \langle y'' \rangle &= \langle y \circ_j (y_j \oplus z_j) \rangle \\ (P\langle w \rangle)\langle y \rangle &= P\langle w \rangle \langle y \rangle \end{aligned}$$

For each $i \in \{1, \ldots, n\}$, the $i$-th linearity constraint $P_i^2$ of $P$ is a natural transformation, with a typical component given by a $p$-linear transformation, as in the right diagram in (8.3.12) above. Next we unpack its multilinearity axioms.

*Unity.* The axiom (1.4.12) for the $p$-linear transformation $(P_i^2)_{\langle w \rangle; x_i}$ says that, if any $y_j = \mathsf{e}$ in $\mathsf{D}_j$, then the $\langle y \rangle$-component

(8.3.14)
$$P\langle w \rangle \langle y \rangle \oplus P\langle w' \rangle \langle y \rangle \xrightarrow{\;\;((P_i^2)_{\langle w \rangle; x_i})_{\langle y \rangle}\;\;} P\langle w'' \rangle \langle y \rangle$$

is equal to $1_{\mathsf{e}}$ in $\mathsf{B}$.

*Constraint Compatibility.* For the axiom (1.4.13) for the $p$-linear transformation $(P_i^2)_{\langle w \rangle; x_i}$, first recall the linearity constraints of $P\langle w \rangle \oplus P\langle w' \rangle$ defined in (8.2.6). The diagram (1.4.13) for $(P_i^2)_{\langle w \rangle; x_i}$ is the following commutative diagram in $\mathsf{B}$.

(8.3.15)


This finishes our description of the $n$-linear functor $(P, \{P_i^2\}_{i=1}^n)$.                  ◇

**Explanation 8.3.16** (The Function $\chi$). In the context of Explanation 8.3.9, we consider the function defined in (8.1.7)

(8.3.17)
$$\mathsf{P}^{\mathsf{su}}(\langle \mathsf{C} \rangle; \underline{\mathsf{P}}^{\mathsf{su}}(\langle \mathsf{D} \rangle; \mathsf{B})) \xrightarrow{\;\chi_{\langle \mathsf{C} \rangle; \langle \mathsf{D} \rangle; \mathsf{B}}\;} \mathsf{P}^{\mathsf{su}}(\langle \mathsf{C} \rangle, \langle \mathsf{D} \rangle; \mathsf{B}),$$

which we abbreviate to $\chi$. This function sends an $n$-linear functor

$$(P, \{P_i^2\}_{i=1}^n) : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \underline{\mathsf{P}}^{\mathsf{su}}(\langle \mathsf{D} \rangle; \mathsf{B})$$

to the $(n + p)$-linear functor $\chi P$ given by the following composite, where we suppress the associativity isomorphism for the Cartesian product.

(8.3.18)
$$\left(\chi P, \{(\chi P)^2_r\}_{r=1}^{n+p}\right)$$
$$\prod_{i=1}^n C_i \times \prod_{j=1}^p D_j \xrightarrow{P \times 1} \underline{P}^{\mathsf{su}}(\langle D \rangle ; B) \times \prod_{j=1}^p D_j \xrightarrow{\mathsf{ev}_{\langle D \rangle; B}} B$$

Next we unpack this $(n + p)$-linear functor, using the notation in (8.3.13).

*Underlying Functor.* By Definition 8.3.1, on objects $\chi P$ is given by

(8.3.19)
$$(\chi P)\big(\langle w \rangle, \langle y \rangle\big) = \mathsf{ev}_{\langle D \rangle; B}\big(P\langle w \rangle, \langle y \rangle\big)$$
$$= P\langle w \rangle\langle y \rangle \quad \text{in} \quad B.$$

The morphism
$$(\chi P)\big(\langle f \rangle, \langle g \rangle\big) = \mathsf{ev}_{\langle D \rangle; B}\big(P\langle f \rangle, \langle g \rangle\big)$$
is given by either one of the following two boundary composites in B, which are equal by the naturality of $P\langle f \rangle$.

(8.3.20)
$$
\begin{array}{ccc}
P\langle w \rangle\langle y \rangle & \xrightarrow{\;\;P\langle f \rangle_{\langle y \rangle}\;\;} & P\langle x \rangle\langle y \rangle \\[4pt]
{\scriptstyle P\langle w \rangle\langle g \rangle}\big\downarrow & \searrow^{\;(\chi P)_{(\langle f \rangle, \langle g \rangle)}} & \big\downarrow{\scriptstyle P\langle x \rangle\langle g \rangle} \\[4pt]
P\langle w \rangle\langle z \rangle & \xrightarrow{\;\;P\langle f \rangle_{\langle z \rangle}\;\;} & P\langle x \rangle\langle z \rangle
\end{array}
$$

*Linearity Constraints.* The linearity constraints for composite multilinear functors are defined in (1.4.27). By (8.3.5), for each $i \in \{1, \ldots, n\}$, the $i$-th linearity constraint $(\chi P)^2_i$ is a natural transformation, with a typical component given by a component morphism in B of $(P^2_i)_{\langle w \rangle; x_i}$—which is a component of the $i$-th linearity constraint $P^2_i$ in (8.3.12)—as follows.

(8.3.21)
$$
\begin{array}{ccc}
(\chi P)\big(\langle w \rangle, \langle y \rangle\big) \oplus (\chi P)\big(\langle w' \rangle, \langle y \rangle\big) & \xrightarrow{\;\;((\chi P)^2_i)_{\langle w \rangle, \langle y \rangle; x_i}\;\;} & (\chi P)\big(\langle w'' \rangle, \langle y \rangle\big) \\[4pt]
\| & & \| \\[4pt]
P\langle w \rangle\langle y \rangle \oplus P\langle w' \rangle\langle y \rangle & \xrightarrow{\;\;((P^2_i)_{\langle w \rangle; x_i})_{\langle y \rangle}\;\;} & P\langle w'' \rangle\langle y \rangle
\end{array}
$$

By (8.3.6), for each $j \in \{1, \ldots, p\}$, the $(n + j)$-th linearity constraint $(\chi P)^2_{n+j}$ is a natural transformation, with a typical component given by a component morphism in B of $P\langle w \rangle^2_j$—which is the $j$-th linearity constraint of $P\langle w \rangle$ in (8.3.12)—as follows.

(8.3.22)
$$
\begin{array}{ccc}
(\chi P)\big(\langle w \rangle, \langle y \rangle\big) \oplus (\chi P)\big(\langle w \rangle, \langle y' \rangle\big) & \xrightarrow{\;\;((\chi P)^2_{n+j})_{\langle w \rangle, \langle y \rangle; z_j}\;\;} & (\chi P)\big(\langle w \rangle, \langle y'' \rangle\big) \\[4pt]
\| & & \| \\[4pt]
P\langle w \rangle\langle y \rangle \oplus P\langle w \rangle\langle y' \rangle & \xrightarrow{\;\;(P\langle w \rangle^2_j)_{\langle y \rangle; z_j}\;\;} & P\langle w \rangle\langle y'' \rangle
\end{array}
$$

*Multilinearity Axioms.* The axioms (1.4.4) through (1.4.8) of an $(n + p)$-linear functor hold for $\chi P$ for the following reasons.

- The unity axiom (1.4.4) holds for $\chi P$ by the same axiom for the $n$-linear functor $P$ and each $p$-linear functor $P\langle w \rangle$.
- The constraint unity axiom (1.4.5) holds for $\chi P$ by
  - the axiom (1.4.5) for the $n$-linear functor $P$ and each $p$-linear functor $P\langle w \rangle$,

  - the unity axiom (8.3.14) for the $p$-linear transformation $(P_i^2)_{\langle w \rangle; x_i}$, and
  - the unity axiom (1.4.4) for $P$.
- The constraint associativity and symmetry axioms, (1.4.6) and (1.4.7), hold for $\chi P$ by the same axioms for the $n$-linear functor $P$ and each $p$-linear functor $P\langle w \rangle$.
- The constraint 2-by-2 axiom (1.4.8) for $\chi P$ and distinct indices

$$r \neq s \in \{1, \ldots, n+p\}$$

holds for the following reasons.
  - If both $r, s \in \{1, \ldots, n\}$, then, by (8.3.21), the axiom (1.4.8) for $\chi P$ follows from the same axiom for the $n$-linear functor $P$.
  - If both $r, s \in \{n+1, \ldots, n+p\}$, then, by (8.3.22), the axiom (1.4.8) for $\chi P$ follows from the same axiom for the $p$-linear functor $P\langle w \rangle$ and indices

$$r - n \neq s - n \in \{1, \ldots, p\}.$$

  - If $r \in \{1, \ldots, n\}$ and $s \in \{n+1, \ldots, n+p\}$, then the axiom (1.4.8) for $\chi P$ follows from the commutative diagram (8.3.15).
  - If $r \in \{n+1, \ldots, n+p\}$ and $s \in \{1, \ldots, n\}$, then we again use the commutative diagram (8.3.15) but with the arrow $1 \oplus \xi \oplus 1$ reversed, which is possible by the symmetry axiom (A.1.15) for B.

This finishes our description of the function $\chi$ in (8.3.17). $\diamond$

## 8.4. The Closed Multicategory Structure

In this section we complete the construction of the closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$.

- In Lemma 8.4.1 we prove the evaluation bijection axiom.
- In Lemma 8.4.9 we prove the equivariance axiom for evaluation bijection.
- The closed multicategory structure on $\mathsf{PermCat}^{\mathsf{su}}$ is stated in Theorem 8.4.15.

**Multicategorical Evaluation Axioms.**

**Lemma 8.4.1.** *The evaluation bijection axiom (8.1.7) holds for*

$$\left( \mathsf{PermCat}^{\mathsf{su}}, \underline{\mathsf{PermCat}^{\mathsf{su}}}, \mathsf{ev} \right)$$

*defined in Theorem 1.4.29 and (8.2.2) and (8.3.2)*

*Proof.* We need to show that, for small permutative categories B, $\langle C \rangle = \langle C_i \rangle_{i=1}^n$, and $\langle D \rangle = \langle D_j \rangle_{j=1}^p$, the function $\chi = \chi_{\langle C \rangle; \langle D \rangle; B}$ in (8.3.17), as displayed below, is a bijection.

$$(8.4.2) \qquad \mathsf{P}^{\mathsf{su}}\left( \langle C \rangle; \underline{\mathsf{P}^{\mathsf{su}}}(\langle D \rangle; B) \right) \xrightarrow[\Psi]{\chi} \mathsf{P}^{\mathsf{su}}\left( \langle C \rangle, \langle D \rangle; B \right)$$

We show that $\chi$ is a bijection by constructing an explicit inverse $\Psi$ in the following two steps.

(1) We construct a function $\Psi$ that goes in the opposite direction as $\chi$.
(2) We show that $\chi$ and $\Psi$ are inverses of each other.

**Step (1): The Function $\Psi$**

Given an $(n+p)$-linear functor

$$(8.4.3) \qquad \left(R, \{R_r^2\}_{r=1}^{n+p}\right) : \prod_{i=1}^n \mathsf{C}_i \times \prod_{j=1}^p \mathsf{D}_j \longrightarrow \mathsf{B},$$

we define an $n$-linear functor

$$(8.4.4) \qquad \left(\Psi R, \{(\Psi R)_i^2\}_{i=1}^n\right) : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \underline{\mathsf{P}^{\mathsf{su}}}(\langle \mathsf{D} \rangle\, ; \mathsf{B})$$

in steps (i) through (iv) below.

- (i) We define $\Psi R$ on objects in (8.4.5) and (8.4.6).
- (ii) We define $\Psi R$ on morphisms in (8.4.7).
- (iii) We define the linearity constraints $(\Psi R)_i^2$ in (8.4.8).
- (iv) We check the $n$-linear functor axioms for $\left(\Psi R, \{(\Psi R)_i^2\}_{i=1}^n\right)$.

In the rest of this proof, we use the notation in (8.3.10) and (8.3.13) for objects and morphisms, so we ask the reader to briefly review them.

*Step (1)(i): Objects.* For an object $\langle w \rangle$ in $\prod_{i=1}^n \mathsf{C}_i$, we define the functor

$$(8.4.5) \qquad \prod_{j=1}^p \mathsf{D}_j \xrightarrow{\ (\Psi R)\langle w \rangle \,=\, R(\langle w \rangle, -)\ } \mathsf{B}.$$

For each $j \in \{1, \ldots, p\}$, its $j$-th linearity constraint $(\Psi R)\langle w \rangle_j^2$ is defined componentwise by the $(n+j)$-th linearity constraint of $R$, as indicated below.

$$(8.4.6) \quad \begin{array}{ccc}
(\Psi R)\langle w\rangle\langle y\rangle \oplus (\Psi R)\langle w\rangle\langle y'\rangle & \xrightarrow{\ \left((\Psi R)\langle w\rangle_j^2\right)_{\langle y\rangle;\, z_j}\ } & (\Psi R)\langle w\rangle\langle y''\rangle \\
\shortparallel & & \shortparallel \\
R\big(\langle w\rangle, \langle y\rangle\big) \oplus R\big(\langle w\rangle, \langle y'\rangle\big) & \xrightarrow{\ (R_{n+j}^2)_{\langle w\rangle,\langle y\rangle;\, z_j}\ } & R\big(\langle w\rangle, \langle y''\rangle\big)
\end{array}$$

This is natural in $\langle y \rangle$ and $z_j$ because $R_{n+j}^2$ is a natural transformation. The $p$-linear functor axioms—(1.4.4) through (1.4.8)—for $(\Psi R)\langle w \rangle$ defined in (8.4.5) and (8.4.6) follow from the corresponding axioms for the given $(n+p)$-linear functor $R$ in (8.4.3).

*Step (1)(ii): Morphisms.* For a morphism $\langle f \rangle : \langle w \rangle \longrightarrow \langle x \rangle$ in $\prod_{i=1}^n \mathsf{C}_i$, we define the natural transformation

$$\prod_{j=1}^p \mathsf{D}_j \underset{(\Psi R)\langle x\rangle}{\overset{(\Psi R)\langle w\rangle}{\Longrightarrow_{\ \Downarrow (\Psi R)\langle f\rangle\ }}} \mathsf{B}$$

with, for each object $\langle y \rangle$ in $\prod_{j=1}^p \mathsf{D}_j$, $\langle y \rangle$-component given by the following morphism in $\mathsf{B}$.

$$(8.4.7) \quad \begin{array}{ccc}
(\Psi R)\langle w\rangle\langle y\rangle & \xrightarrow{\ (\Psi R)\langle f\rangle_{\langle y\rangle}\ } & (\Psi R)\langle x\rangle\langle y\rangle \\
\shortparallel & & \shortparallel \\
R\big(\langle w\rangle, \langle y\rangle\big) & \xrightarrow{\ R(\langle f\rangle, 1_{\langle y\rangle})\ } & R\big(\langle x\rangle, \langle y\rangle\big)
\end{array}$$

Its naturality in $\langle y \rangle$ follows from the functoriality of $R$.

Moreover, $(\Psi R)\langle f \rangle$ is a $p$-linear transformation (Definition 1.4.10) for the following reasons.

- By definition (8.4.7), the unity axiom (1.4.12) for $(\Psi R)\langle f \rangle$ says that, if any $y_j = e$ in $D_j$, then

$$R\big(\langle f \rangle, 1_{\langle y \rangle}\big) = 1_e \quad \text{in} \quad B.$$

This is true by the unity axiom (1.4.4) for the $(n+p)$-linear functor $R$.
- By definition (8.4.6), the constraint compatibility axiom (1.4.13) for $(\Psi R)\langle f \rangle$ is the following diagram in $B$, which commutes by the naturality of $R^2_{n+j}$.

$$
\begin{array}{ccc}
R\big(\langle w \rangle, \langle y \rangle\big) \oplus R\big(\langle w \rangle, \langle y' \rangle\big) & \xrightarrow{(R^2_{n+j})_{\langle w \rangle, \langle y \rangle; z_j}} & R\big(\langle w \rangle, \langle y'' \rangle\big) \\
{\scriptstyle R(\langle f \rangle, 1_{\langle y \rangle}) \oplus R(\langle f \rangle, 1_{\langle y' \rangle})}\downarrow & & \downarrow{\scriptstyle R(\langle f \rangle, 1_{\langle y'' \rangle})} \\
R\big(\langle x \rangle, \langle y \rangle\big) \oplus R\big(\langle x \rangle, \langle y' \rangle\big) & \xrightarrow{(R^2_{n+j})_{\langle x \rangle, \langle y \rangle; z_j}} & R\big(\langle x \rangle, \langle y'' \rangle\big)
\end{array}
$$

Thus $(\Psi R)\langle f \rangle$ is a $p$-linear transformation. The functoriality of $\Psi R$ follows from the definition (8.4.7) and the functoriality of $R$.

*Step (1)(iii): Linearity Constraints.* For each $i \in \{1, \ldots, n\}$ and objects $\langle w \rangle \in \prod_{i=1}^{n} C_i$ and $x_i \in C_i$, we define the $i$-th linearity constraint $(\Psi R)^2_i$ as having the following component $p$-linear transformation.

$$
\prod_{j=1}^{p} D_j \underset{(\Psi R)\langle w'' \rangle}{\overset{(\Psi R)\langle w \rangle \oplus (\Psi R)\langle w' \rangle}{\rightrightarrows}} B \qquad \Downarrow ((\Psi R)^2_i)_{\langle w \rangle; x_i}
$$

For each object $\langle y \rangle \in \prod_{j=1}^{p} D_j$, its $\langle y \rangle$-component is the corresponding component morphism of the $i$-th linearity constraint $R^2_i$, as indicated below.

$$
\text{(8.4.8)} \qquad
\begin{array}{ccc}
(\Psi R)\langle w \rangle\langle y \rangle \oplus (\Psi R)\langle w' \rangle\langle y \rangle & \xrightarrow{\left(((\Psi R)^2_i)_{\langle w \rangle; x_i}\right)_{\langle y \rangle}} & (\Psi R)\langle w'' \rangle\langle y \rangle \\
\| & & \| \\
R\big(\langle w \rangle, \langle y \rangle\big) \oplus R\big(\langle w' \rangle, \langle y \rangle\big) & \xrightarrow{(R^2_i)_{\langle w \rangle, \langle y \rangle; x_i}} & R\big(\langle w'' \rangle, \langle y \rangle\big)
\end{array}
$$

The naturality of

- $\left(((\Psi R)^2_i)_{\langle w \rangle; x_i}\right)_{\langle y \rangle}$ with respect to $\langle y \rangle$ and
- $((\Psi R)^2_i)_{\langle w \rangle; x_i}$ with respect to $\langle w \rangle$ and $x_i$

follows from the naturality of $R^2_i$.

The components (8.4.8) define a $p$-linear transformation $((\Psi R)^2_i)_{\langle w \rangle; x_i}$ for the following reasons.

- The unity axiom (1.4.12) for $((\Psi R)^2_i)_{\langle w \rangle; x_i}$ says that, if any $y_j = e$ in $D_j$, then

$$(R^2_i)_{\langle w \rangle, \langle y \rangle; x_i} = 1_e \quad \text{in} \quad B.$$

This holds by the constraint unity axiom (1.4.5) for $R^2_i$.

- Using (8.2.6) for $(\Psi R)\langle w\rangle \oplus (\Psi R)\langle w'\rangle$ and (8.4.6), the constraint compatibility axiom (1.4.13) for $\big((\Psi R)_i^2\big)_{\langle w\rangle; x_i}$ is the following diagram in B.



This diagram commutes by the constraint 2-by-2 axiom (1.4.8) for $R$.

Thus $\big((\Psi R)_i^2\big)_{\langle w\rangle; x_i}$ is a $p$-linear transformation.

*Step (1)(iv): Multilinearity Axioms.* The data $\big(\Psi R, \{(\Psi R)_i^2\}\big)$ satisfy the axioms (1.4.4) through (1.4.8) for an $n$-linear functor for the following reasons.

- The unity axiom (1.4.4) for $\Psi R$ follows from
  - the definitions (8.4.5) through (8.4.7),
  - the unity axiom (1.4.4), and the constraint unity axiom (1.4.5) for $R$.
- Each of the constraint unity, associativity, symmetry, and 2-by-2 axioms—(1.4.5) through (1.4.8)—for $\Psi R$ follows from the same axiom for $R$ and the definition (8.4.8).

This finishes the construction of the $n$-linear functor $\big(\Psi R, \{(\Psi R)_i^2\}\big)$ in (8.4.4) and completes step (1).

**Step (2): $\chi$ and $\Psi$ are Mutual Inverses**

This consists of the following two steps.

(i) We show the equality

$$\Psi\chi = 1 : \mathsf{P}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle; \underline{\mathsf{P}^{\mathsf{su}}}(\langle\mathsf{D}\rangle; \mathsf{B})\big) \longrightarrow \mathsf{P}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle; \underline{\mathsf{P}^{\mathsf{su}}}(\langle\mathsf{D}\rangle; \mathsf{B})\big).$$

(ii) We show the equality

$$\chi\Psi = 1 : \mathsf{P}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle, \langle\mathsf{D}\rangle; \mathsf{B}\big) \longrightarrow \mathsf{P}^{\mathsf{su}}\big(\langle\mathsf{C}\rangle, \langle\mathsf{D}\rangle; \mathsf{B}\big).$$

*Step (2)(i).* For an $n$-linear functor

$$\big(P, \{P_i^2\}_{i=1}^n\big) : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \underline{\mathsf{P}^{\mathsf{su}}}(\langle\mathsf{D}\rangle; \mathsf{B})$$

as in (8.3.11), we want to show that $\Psi\chi P$ is equal to $P$ as $n$-linear functors.

- $\Psi\chi P$ is equal to $P$ on the objects of $\prod_{i=1}^n \mathsf{C}_i$ by (8.3.19) and (8.4.5).
- $\Psi\chi P$ is equal to $P$ on each morphism $\langle f\rangle$ in $\prod_{i=1}^n \mathsf{C}_i$ because

$$(\Psi\chi P)\langle f\rangle_? = (\chi P)\big(\langle f\rangle, 1_?\big) = P\langle f\rangle_?$$

by (8.3.20) and (8.4.7).
- For each $i \in \{1, \ldots, n\}$, the $i$-th linearity constraints of $\Psi\chi P$ and $P$ are equal by (8.3.21) and (8.4.8).

This shows that $\Psi\chi$ is equal to the identity function.

*Step (2)(ii).* For an $(n + p)$-linear functor

$$\left(R, \{R_r^2\}_{r=1}^{n+p}\right) : \prod_{i=1}^n C_i \times \prod_{j=1}^p D_j \longrightarrow B$$

as in (8.4.3), we want to show that $\chi\Psi R$ is equal to $R$ as $(n + p)$-linear functors.

- $\chi\Psi R$ is equal to $R$ on the objects of $\prod_{i=1}^n C_i \times \prod_{j=1}^p D_j$ by (8.3.19) and (8.4.5).
- $\chi\Psi R$ is equal to $R$ on each morphism

$$\left(\langle f\rangle, \langle g\rangle\right) : \left(\langle w\rangle, \langle y\rangle\right) \longrightarrow \left(\langle x\rangle, \langle z\rangle\right) \quad \text{in} \quad \prod_{i=1}^n C_i \times \prod_{j=1}^p D_j$$

by the following computation.

$$
\begin{aligned}
&(\chi\Psi R)\left(\langle f\rangle, \langle g\rangle\right) \\
&= (\Psi R)\langle x\rangle\langle g\rangle \circ (\Psi R)\langle f\rangle_{\langle y\rangle} && \text{by (8.3.20)} \\
&= R\left(1_{\langle x\rangle}, \langle g\rangle\right) \circ R\left(\langle f\rangle, 1_{\langle y\rangle}\right) && \text{by (8.4.5) and (8.4.7)} \\
&= R\left(\langle f\rangle, \langle g\rangle\right) && \text{by functoriality of } R
\end{aligned}
$$

- To check that $\chi\Psi R$ and $R$ have the same $n + p$ linearity constraints, we consider the following two cases.
  - For each $i \in \{1, \dots, n\}$, their $i$-th linearity constraints are equal by (8.3.21) and (8.4.8).
  - For each $j \in \{1, \dots, p\}$, their $(n + j)$-th linearity constraints are equal by (8.3.22) and (8.4.6).

This shows that $\chi\Psi$ is equal to the identity function and completes step (2). The proof of the Lemma is now complete. $\qquad\square$

**Lemma 8.4.9.** *The equivariance axiom for evaluation bijection (8.1.8) holds for*

$$\left( \mathsf{PermCat}^{\mathsf{su}}, \underline{\mathsf{PermCat}}^{\mathsf{su}}, \mathrm{ev} \right)$$

*defined in Theorem 1.4.29 and (8.2.2), (8.2.15), and (8.3.2).*

*Proof.* For small permutative categories $B$, $\langle C\rangle = \langle C_i\rangle_{i=1}^n$, and $\langle D\rangle = \langle D_j\rangle_{j=1}^p$ and permutations $(\sigma, \varsigma) \in \Sigma_n \times \Sigma_p$, the desired commutative diagram is the following.

(8.4.10)

$$
\begin{array}{ccc}
\mathsf{P}^{\mathsf{su}}\left(\langle C\rangle; \underline{\mathsf{P}}^{\mathsf{su}}(\langle D\rangle; B)\right) & \xrightarrow{\;\;\chi_{\langle C\rangle; \langle D\rangle; B}\;\;} & \mathsf{P}^{\mathsf{su}}\left(\langle C\rangle, \langle D\rangle; B\right) \\
{\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma\times\varsigma} \\
\mathsf{P}^{\mathsf{su}}\left(\langle C\rangle\sigma; \underline{\mathsf{P}}^{\mathsf{su}}(\langle D\rangle; B)\right) & & \\
{\scriptstyle\gamma(\varsigma;-)}\downarrow & & \\
\mathsf{P}^{\mathsf{su}}\left(\langle C\rangle\sigma; \underline{\mathsf{P}}^{\mathsf{su}}(\langle D\rangle\varsigma; B)\right) & \xrightarrow{\;\;\chi_{\langle C\rangle\sigma; \langle D\rangle\varsigma; B}\;\;} & \mathsf{P}^{\mathsf{su}}\left(\langle C\rangle\sigma, \langle D\rangle\varsigma; B\right)
\end{array}
$$

An object in the upper left node in (8.4.10) is an $n$-linear functor

$$\left(P, \{P_i^2\}_{i=1}^n\right) : \prod_{i=1}^n C_i \longrightarrow \underline{\mathsf{P}}^{\mathsf{su}}(\langle D\rangle; B)$$

as in (8.3.11). We need to check the following two statements.

(i) The two composites in (8.4.10) applied to $P$ are equal as functors.

(ii) The two functors in (i) have the same $n + p$ linearity constraints.

*Statement (i).* To show that the two composites in (8.4.10) applied to $P$ are equal as functors, we consider the following diagram.



By definitions (1.4.19) and (8.3.18), the following statements hold for the diagram above.

- Denoting $\chi = \chi_{\langle C \rangle; \langle D \rangle; B}$, the top boundary composite is the functor

$$(8.4.11) \qquad\qquad\qquad (\chi P)(\sigma \times \varsigma).$$

  Denoting $\chi' = \chi_{\langle C \rangle \sigma; \langle D \rangle \varsigma; B}$, the other boundary composite is the functor

$$(8.4.12) \qquad\qquad\qquad \chi'\big(\gamma(\varsigma; P\sigma)\big).$$

- The left quadrilateral and the lower middle triangle commute by functoriality of the Cartesian product.
- The right quadrilateral commutes by the definition (1.4.19) of the right $\varsigma$-action on $\mathsf{P}^{\mathsf{su}}(\langle D \rangle; B)$, which is the same as the right $\varsigma$-action on $\underline{\mathsf{P}^{\mathsf{su}}}(\langle D \rangle; B)$ (8.2.15).

This proves statement (i).

*Statement (ii).* We consider indices $r \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, p\}$ and objects

$$\big(\langle a \rangle, \langle b \rangle\big) \in \prod_{i=1}^{n} C_{\sigma(i)} \times \prod_{j=1}^{p} D_{\varsigma(j)} \quad \text{and} \quad (a'_r, b'_t) \in C_{\sigma(r)} \times D_{\varsigma(t)}.$$

The following equalities follow from (1.4.20), (8.3.21), and (8.3.22).

$$(8.4.13) \qquad \Big(\big((\chi P)(\sigma \times \varsigma)\big)_r^2\Big)_{\langle a \rangle, \langle b \rangle; a'_r} = \Big(\big(P_{\sigma(r)}^2\big)_{\sigma\langle a \rangle; a'_r}\Big)_{\varsigma\langle b \rangle}$$
$$= \Big(\big(\chi'(\gamma(\varsigma; P\sigma))\big)_r^2\Big)_{\langle a \rangle, \langle b \rangle; a'_r}$$

$$(8.4.14) \qquad \Big(\big((\chi P)(\sigma \times \varsigma)\big)_{n+t}^2\Big)_{\langle a \rangle, \langle b \rangle; b'_t} = \Big(P(\sigma\langle a \rangle)_{\varsigma(t)}^2\Big)_{\varsigma\langle b \rangle; b'_t}$$
$$= \Big(\big(\chi'(\gamma(\varsigma; P\sigma))\big)_{n+t}^2\Big)_{\langle a \rangle, \langle b \rangle; b'_t}$$

Thus the functors in (8.4.11) and (8.4.12) have the same $n + p$ linearity constraints, proving statement (ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We are now ready to show that small permutative categories form a closed multicategory (Definition 8.1.1).

**Theorem 8.4.15.** *There is a closed multicategory*

$$\big(\, \mathsf{PermCat}^{\mathsf{su}}, \, \underline{\mathsf{PermCat}^{\mathsf{su}}}, \, \mathrm{ev}\big)$$

*consisting of the following data.*

- *The underlying multicategory is* $\mathsf{PermCat}^{\mathsf{su}}$ *in Theorem 1.4.29.*

- *The internal hom objects are the permutative categories* $\underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle ; D)$ *in Lemma 8.2.13.*
- *The symmetric group action on internal hom is given in Lemma 8.2.16.*
- *The multicategorical evaluations are the multilinear functors*

$$\underline{\mathsf{PermCat}}^{\mathsf{su}}(\langle C \rangle ; D) \times \prod_{i=1}^{n} C_i \xrightarrow{\; \mathsf{ev}_{\langle C \rangle ; D} \;} D$$

*in Lemma 8.3.8.*

*Proof.* The data are well defined by the indicated results, Theorem 1.4.29 and Lemmas 8.2.13, 8.2.16, and 8.3.8. The closed multicategory axioms, (8.1.5) through (8.1.8), hold by Lemmas 8.2.16, 8.4.1, and 8.4.9. □

## 8.5. Closed Multicategories of Lax and Strong Multilinear Functors

In this section we describe closed multicategory structures for PermCat, PermCat$^{\mathsf{sg}}$, and PermCat$^{\mathsf{sus}}$. This material extends that of Sections 1.4 and 8.4 to multilinear functors with generally non-trivial unit constraints. Our treatment below describes the relevant data and axioms generalizing those of the strictly unital case.

Recall from Proposition 1.4.31 that multilinear functors can be identified with multifunctors out of a smash product of endomorphism multicategories. In Lemma 8.5.14 and Proposition 8.5.34 we show that lax multilinear functors have a corresponding description using tensor products in place of smash products. The main results of this section are stated in Theorems 8.5.36 and 8.5.56.

Cat**-Multicategory Structure.** Recall from Notation 1.4.1 that $\langle x \circ_j y \rangle$ denotes the tuple obtained by replacing $x_j$ with $y$. For the unit constraint below, we require the following similar notation, where the existence of $x_j$ is not assumed.

**Notation 8.5.1.** Suppose given $j \in \{1, \ldots, n\}$ together with symbols

$$x_i \quad \text{for} \quad 1 \le i \le n \quad \text{with} \quad i \ne j,$$

and another symbol $y$. We denote by

$$(8.5.2) \qquad \langle x_i \rangle_{i \ne j} \bullet_j y = \big( \underbrace{x_1, \ldots, x_{j-1}}_{\text{empty if } j = 1}, y, \underbrace{x_{j+1}, \ldots, x_n}_{\text{empty if } j = n} \big)$$

the $n$-tuple that has $x_i$ in each entry $i \ne j$, and has $y$ as its $j$-th entry. Thus, in the tuple (8.5.2), no entry of the $(n-1)$-tuple $\langle x_i \rangle_{i \ne j}$ is *replaced* by $y$. Instead, $y$ is inserted as a new entry of $\langle x_i \rangle_{i \ne j}$, shifting the positions of the entries in positions $i > j$.

We also use the following further variants:

$(8.5.3) \quad \langle x_i \rangle_{i \ne j} \circ_k x'_k \bullet_j \mathsf{e}$         has $x'_k$ in position $k$ and $\mathsf{e}$ in position $j$,

$(8.5.4) \quad \langle x_i \rangle_{i \ne j} \circ_k (x_k \oplus x'_k) \bullet_j \mathsf{e}$    has $x_k \oplus x'_k$ in position $k$ and $\mathsf{e}$ in position $j$, and

$(8.5.5) \quad \langle x_i \rangle_{i \ne j,k} \bullet_k \mathsf{e} \bullet_j \mathsf{e}$         has $\mathsf{e}$ in positions $k$ and $j$.

◇

The following generalizes Definition 1.4.2.

**Definition 8.5.6.** A *lax n-linear functor*

$$\prod_{j=1}^{n} C_j \xrightarrow{\left( P, \{P_j^2\}_{j=1}^n, \{P_j^0\}_{j=1}^n \right)} D$$

consists of the following data.

- $P : \prod_{j=1}^{n} C_j \longrightarrow D$ is a functor.
- For each $j \in \{1,\ldots,n\}$, $P_j^2$ is a natural transformation, called the *j-th linearity constraint*, with component morphisms

(8.5.7) $$P\langle x \circ_j x_j \rangle \oplus P\langle x \circ_j x'_j \rangle \xrightarrow{P_j^2} P\langle x \circ_j (x_j \oplus x'_j) \rangle \in D$$

for objects $\langle x \rangle \in \prod_{j=1}^{n} C_j$ and $x'_j \in C_j$.

- For each $j \in \{1,\ldots,n\}$, $P_j^0$ is a natural transformation, called the *j-th unit constraint*, with component morphisms

(8.5.8) $$e \xrightarrow{P_j^0} P\big( \langle x_i \rangle_{i \neq j} \bullet_j e \big) \in D$$

for objects $\langle x_i \rangle_{i \neq j} \in \prod_{i \neq j} C_i$. In (8.5.8), $e$ denotes the monoidal unit of $D$, respectively $C_j$, in the domain, respectively codomain of $P_j^0$.

These data are required to satisfy six axioms. Of these, three are axioms for $n$-linear functors: constraint associativity (1.4.6), constraint symmetry (1.4.7), and constraint 2-by-2 (1.4.8). The remaining three axioms are as follows, using Notation 8.5.1.

**Lax Unity:** For each $j \in \{1,\ldots,n\}$, and for objects $\langle x_i \rangle_{i \neq j} \in \prod_{i \neq j} C_i$, the following diagrams commute in $D$.

(8.5.9)



**Constraint 0-by-2:** For each pair $j, k \in \{1,\ldots,n\}$ with $j \neq k$, and for objects $\langle x_i \rangle_{i \neq j} \in \prod_{i \neq j} C_i$ and $x'_k \in C_k$, the following diagram commutes in $D$.

(8.5.10)



**Constraint 0-by-0:** For each pair $j, k \in \{1,\ldots,n\}$ with $j \neq k$, and for objects $\langle x_i \rangle_{i \neq j,k} \in \prod_{i \neq j,k} C_i$, the following two morphisms are equal:

(8.5.11) $$e \overset{P_j^0}{\underset{P_k^0}{\rightrightarrows}} P\big( \langle x_i \rangle_{i \neq j,k} \bullet_k e \bullet_j e \big).$$

These are the component of $P_j^0$ with $x_k = $ e and the component of $P_k^0$ with $x_j = $ e.

This finishes the definition of a lax $n$-linear functor.

Moreover, we define the following.

- A *lax multilinear functor* is a lax $n$-linear functor for some $n \geq 0$.
- A lax $n$-linear functor $(P, \{P_j^2\}, \{P_j^0\})$ is
  - *strongly unital* if each $P_j^0$ is a natural isomorphism,
  - *strictly unital* if each $P_j^0$ is an identity,
  - *strongly unital and strongly monoidal* if each $P_j^0$ and each $P_j^2$ is a natural isomorphism,
  - *strictly unital and strong* if each $P_j^0$ is an identity and each $P_j^2$ is a natural isomorphism, and
  - *strict* if each $P_j^0$ and each $P_j^2$ is an identity natural transformation.   ◇

**Explanation 8.5.12** (Multilinearity and Strict Units). In the context of Definition 8.5.6 above, note that $(P, \{P_j^2\}, \{P_j^0\})$ is strictly unital if and only if $(P, \{P_j^2\})$ is an $n$-linear functor in the sense of Definition 1.4.2. When each $P_j^0$ is an identity, then the lax unity diagrams correspond to the constraint unity axiom (1.4.5) in the case $i = j$, and the constraint 0-by-2 diagrams (8.5.10) correspond to the constraint unity axiom (1.4.5) in the cases $i \neq j$. The other strictly unital variants above can be identified with corresponding variants at the end of Definition 1.4.2.   ◇

**Remark 8.5.13.** In the context of Definition 8.5.6 above, commutativity of each lax unity diagram (8.5.9) follows from the other, by the constraint symmetry axiom (1.4.7) for $P_j^2$ together with the equalities

$$\tilde{\zeta}_{e,?} = 1_? = \tilde{\zeta}_{?,e}$$

in any permutative category.   ◇

**Lemma 8.5.14.** *In the context of Definition 8.5.6, there is a bijection between multifunctors*

$$F : \otimes_{i=1}^n \mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{End}(\mathsf{D})$$

*and lax n-linear functors*

$$P : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \mathsf{D}.$$

*Proof.* Recall from Explanation 1.1.14 that a multifunctor out of a tensor product is characterized by (i) multifunctoriality in each variable separately and (ii) the interchange equality (1.1.17). In the context of Definition 8.5.6, this means that a multifunctor

$$(8.5.15) \qquad\qquad F : \bigotimes_{i=1}^n \mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{End}(\mathsf{D})$$

is characterized by an underlying functor of categories

$$(8.5.16) \qquad\qquad P : \prod_{i=1}^n \mathsf{C}_i \longrightarrow \mathsf{D}$$

such that the following conditions hold.

(i) In each variable separately, $F$ determines a multifunctor

$$\mathsf{End}(\mathsf{C}_i) \longrightarrow \mathsf{End}(\mathsf{D}).$$

(ii) The interchange equality

(8.5.17) 
$$F\big( \otimes_i \phi_i \big) = F\big( \otimes_i^t \phi_i \big) \cdot \xi^\otimes$$

holds for multimorphisms $\phi_i \in \mathsf{End}(C_i)\big( \langle x_i \rangle ; y_i \big)$, where $\xi^\otimes$ denotes the bijection (1.1.11) induced by transposition of tensor products.

To explain how these conditions relate to the data and axioms of lax multilinearity for $P$, note that each multimorphism

$$\phi_i \in \mathsf{End}(C_i)\big( \langle x_i \rangle ; y_i \big) = C_i\big( \oplus_\ell x_{i,\ell} , y_i \big)$$

with input profile

$$\langle x_i \rangle = \langle x_{i,\ell} \rangle_\ell = (x_{i,1}, \ldots, x_{i,n_i}) \in \mathsf{Prof}(C_i)$$

is determined uniquely as the composite of a characteristic multimorphism

$$\iota_{\langle x_i \rangle} \in \mathsf{End}(C_i)\big( \langle x_i \rangle ; \oplus_\ell x_{i,\ell} \big),$$

given by the identity morphism $1_{\oplus_\ell x_{i,\ell}}$ in $C_i$, and a morphism

$$\phi_i : \oplus_\ell x_{i,\ell} \longrightarrow y_i \quad \text{in} \quad C_i.$$

Furthermore, by associativity of the monoidal sum in $C_i$, each characteristic multimorphism $\iota_{\langle x_i \rangle}$ decomposes as an iterated composite in $\mathsf{End}(C_i)$ of multimorphisms $\iota_{\langle \rangle}$, $\iota_{x_{i,\ell}} = 1_{x_{i,\ell}}$, and $\iota_{(x_{i,\ell}, x_{i,\ell+1})}$. Of those cases, the first two are necessary when $\langle x_i \rangle$ has length zero or one.

For each $j \in \{1, \ldots, n\}$, each tuple

$$\langle x_j \rangle = \langle x_{j,\ell} \rangle_\ell = (x_{j,1}, \ldots, x_{j,n_j}) \in \mathsf{Prof}(C_j),$$

and each $n-1$ tuple of objects

$$x_i \in C_i \quad \text{with} \quad i \neq j,$$

let

$$\iota^j_{\langle x_j \rangle} = 1_{x_1} \otimes \cdots \otimes 1_{x_{j-1}} \otimes \iota_{\langle x_j \rangle} \otimes 1_{x_{j+1}} \otimes \cdots \otimes 1_{x_n}.$$

The two cases of interest are where $\langle x_j \rangle$ has length zero or two. For a pair of objects $x_j, x'_j \in C_j$, define the following:

$$P^0_j = F\iota^j_{\langle \rangle} \quad \text{and}$$

$$P^2_j = F\iota^j_{(x_j, x'_j)}.$$

With this notation, the reasoning above shows that the assignment of $F$ on multimorphisms $\otimes_i \phi_i$ is determined by

- its underlying functor $P$,
- the multimorphisms $P^0_j$ and $P^2_j$, and
- multifunctoriality of $F$.

The remainder of this proof identifies how the multifunctoriality of $F$ corresponds to the six multilinearity axioms of Definition 8.5.6.

Condition (i), multifunctoriality in each variable separately, is equivalent to the condition that the data $\{P^0_j\}$ and $\{P^2_j\}$ are natural with respect to the variables $x_i$ for $i \neq j$ and satisfy the axioms for lax unity (8.5.9), constraint associativity (1.4.6), and constraint symmetry (1.4.7).

For condition ([ii](#)), the interchange equality (8.5.17) holds in general if and only if it holds when, for each pair $j \neq k$ in $\{1, \ldots, n\}$, the multimorphisms $\phi_j$ and $\phi_k$ are characteristic multimorphisms:

$$\phi_j = \iota_{\langle\rangle} \quad \text{or} \quad \phi_j = \iota_{(x_j, x'_j)},$$

$$\phi_k = \iota_{\langle\rangle} \quad \text{or} \quad \phi_k = \iota_{(x_k, x'_k)},$$

and each other $\phi_i$ is a colored unit. By symmetry in $j$ and $k$, the reduction above results in three distinct cases. The interchange equality ([ii](#)) for those three cases corresponds to the constraint axioms 2-by-2 (1.4.8), 0-by-2 (8.5.10), and 0-by-0 (8.5.11) for $P$.

Thus, a multifunctor $F$ in (8.5.15) determines and is uniquely determined by an underlying functor $P$ in (8.5.16) with linearity and unit constraints

$$P_j^2 = F\iota_{(x_j, x'_j)}^j \quad \text{and} \quad P_j^0 = F\iota_{\langle\rangle}^j$$

satisfying the six multilinearity axioms of Definition 8.5.6. This completes the proof. $\qquad\square$

The following generalizes Definition 1.4.10.

**Definition 8.5.18.** Suppose $P, Q$ are lax $n$-linear functors as displayed below.

(8.5.19)

$$\prod_{j=1}^{n} C_j \xLongrightarrow{\theta} D$$
$$(P, \{P_j^2\}, \{P_j^0\})$$
$$(Q, \{Q_j^2\}, \{Q_j^0\})$$

A *lax $n$-linear transformation* $\theta : P \longrightarrow Q$ is a natural transformation of underlying functors that satisfies the constraint compatibility condition (1.4.13) together with the following lax unity axiom.

**Lax Unity:**

(8.5.20)

$$\begin{array}{ccc} & \xrightarrow{P_j^0} & P\big(\langle x_i \rangle_{i \neq j} \bullet_j e\big) \\ e & & \downarrow \theta \\ & \xrightarrow{Q_j^0} & Q\big(\langle x_i \rangle_{i \neq j} \bullet_j e\big) \end{array}$$

This finishes the definition of a lax $n$-linear transformation. Moreover, we define the following.

- A *lax multilinear transformation* is a lax $n$-linear transformation for some $n \geq 0$.
- Identities and compositions of lax multilinear transformations are defined componentwise.

Note that a lax $n$-linear transformation between strictly unital lax $n$-linear functors is the same as an $n$-linear transformation in the sense of Definition 1.4.10. $\qquad\diamond$

**Definition 8.5.21** (Multimorphism Categories)**.** We define the following categories of lax $n$-linear functors and transformations.

- $\mathsf{PermCat}(\langle C \rangle ; D)$ is the category with
  - lax $n$-linear functors $\langle C \rangle \longrightarrow D$ as objects and
  - lax $n$-linear transformations between them as morphisms.

- $\mathsf{PermCat}^{\mathsf{sg}}\big(\langle C \rangle ; D\big)$ is the full subcategory of strongly unital and strongly monoidal $n$-linear functors. ◇

Combined with the multimorphism categories of Definition 1.4.15, these fit into the following commutative diagram of full subcategory inclusions.

$$
(8.5.22) \qquad \mathsf{PermCat}^{\mathsf{st}}\big(\langle C \rangle ; D\big)
$$

$$
\mathsf{PermCat}^{\mathsf{sg}}\big(\langle C \rangle ; D\big) \longrightarrow \mathsf{PermCat}\big(\langle C \rangle ; D\big)
$$

$$
\mathsf{PermCat}^{\mathsf{sus}}\big(\langle C \rangle ; D\big) \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\big(\langle C \rangle ; D\big)
$$

The following generalizes Definition 1.4.16.

**Definition 8.5.23** (Symmetric Group Action). Suppose given lax $n$-linear functors $P$ and $Q$ together with a lax $n$-linear transformation $\theta$ as displayed below.

$$
(8.5.24) \qquad
\prod_{j=1}^{n} C_j
\overset{(P,\{P_j^2\},\{P_j^0\})}{\underset{(Q,\{Q_j^2\}),\{Q_j^0\})}{\theta \Downarrow}}
D
$$

For a permutation $\sigma \in \Sigma_n$, the symmetric group action

$$
(8.5.25) \qquad \mathsf{PermCat}\big(\langle C \rangle ; D\big) \xrightarrow[\ \cong\ ]{\ \sigma\ } \mathsf{PermCat}\big(\langle C \rangle \sigma ; D\big)
$$

sends the data (8.5.24) to the following composites and whiskerings, where $\sigma$ permutes the coordinates according to $\sigma$.

$$
(8.5.26) \qquad
\prod_{j=1}^{n} C_{\sigma(j)} \xrightarrow{\ \sigma\ } \prod_{j=1}^{n} C_j
\overset{(P,\{P_j^2\},\{P_j^0\})}{\underset{(Q,\{Q_j^2\},\{Q_j^0\})}{\theta \Downarrow}}
D
$$

For objects

$$
\langle a \rangle = \langle a_j \rangle_{j=1}^{n} \in \prod_{j=1}^{n} C_{\sigma(j)} \quad \text{and} \quad a_j' \in C_{\sigma(j)},
$$

the $j$-th linearity constraint of $P^{\sigma} = P \circ \sigma$ has component given as in (1.4.20):

$$
(P^{\sigma})_j^2 = P_{\sigma(j)}^2.
$$

The $j$-th unit constraint is likewise given by

$$
(P^{\sigma})_j^0 = P_{\sigma(j)}^0.
$$

As with Definition 1.4.16, each of the variants in Definition 8.5.6 is preserved by this action. ◇

The following generalizes Definition 1.4.21.

**Definition 8.5.27** (Multicategorical Composition). Suppose given, for each $j \in \{1, \ldots, n\}$,

- permutative categories $\langle B_j \rangle = \langle B_{j,i} \rangle_{i=1}^{k_j}$,
- lax $k_j$-linear functors $P_j', Q_j' : \langle B_j \rangle \longrightarrow C_j$, and

- a lax $k_j$-linear transformation $\theta_j : P'_j \longrightarrow Q'_j$ as follows.

(8.5.28)
$$\prod_{i=1}^{k_j} B_{j,i} \underset{Q'_j}{\overset{P'_j}{\rightrightarrows}} \quad \theta_j \Downarrow \quad C_j$$

With $\langle B \rangle = \langle \langle B_j \rangle \rangle_{j=1}^{n}$, the multicategorical composition functor

(8.5.29)        $\mathsf{PermCat}(\langle C \rangle ; D) \times \prod_{j=1}^{n} \mathsf{PermCat}(\langle B_j \rangle ; C_j) \xrightarrow{\ \gamma\ } \mathsf{PermCat}(\langle B \rangle ; D)$

sends the data (8.5.24) and (8.5.28) to the composites

(8.5.30)
$$\prod_{j=1}^{n} \prod_{i=1}^{k_j} B_{j,i} \underset{Q \circ \prod_j Q'_j}{\overset{P \circ \prod_j P'_j}{\rightrightarrows}} \quad \theta \otimes (\prod_j \theta_j) \Downarrow \quad D$$

defined as follows.

**Composite Lax Multilinear Functor:** The underlying functor $(P \circ \prod_j P'_j)$ and linearity constraints $(P \circ \prod_j P'_j)^2_\ell$ are defined as in Definition 1.4.21. The unit constraints $(P \circ \prod_j P'_j)^0_\ell$ are defined as follows.

Suppose
$$\ell = k_1 + \cdots + k_{a-1} + b,$$
for some $a \in \{1, \ldots, n\}$ and $b \in \{1, \ldots, k_a\}$, and suppose given

(8.5.31)    $\langle w_j \rangle = \langle w_{j,i} \rangle_{i=1}^{k_j} \in \prod_{i=1}^{k_j} B_{j,i}$    for    $j \in \{1, \ldots, n\}$    with    $j \neq a$    and
$w_{a,i} \in B_{a,i}$                for    $i \in \{1, \ldots, k_a\}$    with    $i \neq b$.

Then there is the tuple

(8.5.32)
$\langle P'_j \langle w_j \rangle \rangle_{j \neq a} = \big( P'_1 \langle w_1 \rangle, \ldots, P'_{a-1} \langle w_{a-1} \rangle, P'_{a+1} \langle w_{a+1} \rangle, \ldots, P'_n \langle w_n \rangle \big)$   in   $\prod_{j \neq a} C_j$.

The unit constraint $(P \circ \prod_j P'_j)^0_\ell$ is the following composite in D:

(8.5.33)
$$e \xrightarrow{\ P_a^0\ } P\big( \langle P'_j w_j \rangle_{j \neq a} \bullet_a e \big) \xrightarrow{\ P(\langle 1 \bullet_a (P'_a)^0_b \rangle)\ } P\big( \langle P'_j w_j \rangle_{j \neq a} \bullet_a P'_a (\langle w_{a,i} \rangle_{i \neq b} \bullet_b e) \big).$$

If the monoidal constraints $P_a^0$ and $(P'_a)^0_b$ are isomorphisms, respectively identities, then so is the composite (8.5.33). Therefore, as with Definition 1.4.21, each of the variants in Definition 8.5.6 is preserved by composition.

**Composite Lax Multinatural Transformation:** The lax $n$-linear transformation $\theta \otimes (\prod_j \theta_j)$ in (8.5.30) is the horizontal composite of the natural transformations $\prod_j \theta_j$ and $\theta$, as in (1.4.28).

The finishes the definition of the multicategorical composition in PermCat.

Verifying the lax multilinearity axioms for the composite $P \circ \prod_j P'_j$ is similar to that of the strictly unital case in [**JY∞**, Section 6.6]. In particular, the axioms (8.5.9)

through (8.5.11) hold by the corresponding axioms for $P$ and $P'_j$, along with natu-
rality and multifunctoriality of the data involved. For example, in the constraint
0-by-2 axiom (8.5.10) for $\left(P \circ \prod_j P'_j\right)^0_\ell$ and $\left(P \circ \prod_j P'_j\right)^2_m$ with

$$\ell = k_1 + \cdots + k_{a-1} + b \qquad \text{for} \quad a \in \{1,\ldots,n\} \quad \text{and} \quad b \in \{1,\ldots,k_a\}$$
$$m = k_1 + \cdots + k_{c-1} + d \qquad \text{for} \quad c \in \{1,\ldots,n\} \quad \text{and} \quad d \in \{1,\ldots,k_c\},$$

there are two cases to consider. If $c \neq a$, then verifying this case uses (8.5.10) for $P^0_a$
along with naturality of $P^2_c$, naturality of $P^0_a$, and multifunctoriality of $P$. If $c = a$
and $b \neq d$, then verifying this case uses (8.5.9) for $P^0_a$ and (8.5.10) for $(P'_a)^0_b$ and
$(P'_a)^2_d$, along with naturality of $P^2_a$ and multifunctoriality of $P$.

Similarly, the lax unity axiom (8.5.20) for $\theta \otimes \left(\prod_j \theta_j\right)$ follows from that of $\theta$ and
$\theta_j$ individually.                                                                                      $\diamond$

**Proposition 8.5.34.** *For small permutative categories $\langle C_j \rangle_{j=1}^n$ and $D$, the 2-functor*

$$\mathsf{End} : \mathsf{PermCat} \longrightarrow \mathsf{Multicat}$$

*in Proposition C.3.6 induces an isomorphism of categories*

$$\mathsf{PermCat}\big(\langle C \rangle; D\big) \xrightarrow[\cong]{\mathsf{End}} \mathsf{Multicat}\big(\langle \mathsf{End}(C) \rangle; \mathsf{End}(D)\big)$$
$$= \mathsf{Multicat}\big( \otimes_{i=1}^n \mathsf{End}(C_i), \mathsf{End}(D)\big)$$

*between*

- *the category of lax n-linear functors and transformations $\langle C \rangle \longrightarrow D$ and*
- *the category of multifunctors*

$$\otimes_{i=1}^n \mathsf{End}(C_i) \longrightarrow \mathsf{End}(D)$$

  *and multinatural transformations.*

*Proof.* This proof is an unpointed analog of Explanation 1.4.32. The bijection be-
tween lax multilinear functors from $\prod_i C_i$ to $D$ and multifunctors from $\otimes_i \mathsf{End}(C_i)$
to $\mathsf{End}(D)$ is given by Lemma 8.5.14.

The argument that $\mathsf{End}$ induces a bijection between lax $n$-linear transforma-
tions (Definition 8.5.18)

$$\theta : P \longrightarrow Q \quad \text{in} \quad \mathsf{PermCat}\big(\langle C \rangle; D\big)$$

and multinatural transformations (Definition C.1.25)

$$\omega : \mathsf{End}(P) \longrightarrow \mathsf{End}(Q) \quad \text{in} \quad \mathsf{Multicat}\big( \otimes_{i=1}^n \mathsf{End}(C_i), \mathsf{End}(D)\big)$$

is similar. Indeed, for

$$\theta : P \longrightarrow Q \quad \text{in} \quad \mathsf{PermCat}\big(\langle C \rangle; D\big),$$

the multinatural transformation

$$\mathsf{End}(\theta) : \mathsf{End}(P) \longrightarrow \mathsf{End}(Q)$$

is uniquely determined by the component morphisms

$$(8.5.35) \qquad\qquad \theta_{\langle c \rangle} : P\langle c \rangle \longrightarrow Q\langle c \rangle \quad \text{in} \quad D,$$

for $\langle c \rangle \in \prod_{i=1}^n C_i$. Using the same reduction to characteristic multimorphisms in the
proof of Lemma 8.5.14, the multinaturality axioms of $\omega = \mathsf{End}(\theta)$ determine and
are uniquely determined by the two lax multilinearity axioms of $\theta$.                    $\square$

The following generalizes Theorem 1.4.29.

**Theorem 8.5.36.** *There is a* Cat-*multicategory*

$$\mathsf{PermCat}$$

*defined by the following data.*

- *The objects are small permutative categories.*
- *The multimorphism categories are in Definition 8.5.21.*
- *The colored units are identity symmetric monoidal functors.*
- *The symmetric group action is in Definition 8.5.23.*
- *The multicategorical composition is in Definition 8.5.27.*

*Moreover, the multimorphism categories (8.5.22) give the following commutative diagram of sub-*Cat*-multicategories.*

(8.5.37)

$$
\begin{array}{ccc}
 & \mathsf{PermCat}^{\mathsf{sg}} & \longhookrightarrow & \mathsf{PermCat} \\
\nearrow & \uparrow & & \uparrow \\
\mathsf{PermCat}^{\mathsf{st}} & & & \\
\searrow & \downarrow & & \downarrow \\
 & \mathsf{PermCat}^{\mathsf{sus}} & \longhookrightarrow & \mathsf{PermCat}^{\mathsf{su}}
\end{array}
$$

The remainder of this section describes the closed structure for PermCat, PermCat$^{\mathsf{sg}}$, and PermCat$^{\mathsf{sus}}$.

**Internal Hom Objects.** The following generalizes the monoidal sum of multilinear functors.

**Explanation 8.5.38** (Monoidal Sum of Lax Multilinear Functors)**.** Suppose given small permutative categories D and $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ for $n \geq 0$. The monoidal sum on PermCat$^{\mathsf{su}}\big(\langle C \rangle; D\big)$, from Definition 8.2.1, generalizes to a monoidal sum on PermCat$\big(\langle C \rangle; D\big)$.

To explain this, first note that

$$\oplus : D \times D \longrightarrow D$$

is a strictly unital strong symmetric monoidal functor. Details for this appear in the proof of Lemma 6.4.11 statement (i), taking $F = G = 1_D$. The monoidal constraint of $\oplus$ is given by

(8.5.39)          $$(a \oplus b) \oplus (c \oplus d) \xrightarrow{\ 1 \oplus \xi \oplus 1\ } (a \oplus c) \oplus (b \oplus d)$$

for each quadruple $a, b, c, d \in D$. The associativity and braiding axioms for $\oplus$ both follow from coherence for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].

For multilinear functors $P$ and $Q$ in PermCat$^{\mathsf{su}}\big(\langle C \rangle; D\big)$, the underlying functor and linearity constraints of $P \oplus Q$ are given by those of the composite $\oplus \circ (P \times Q)$. This same definition applies more generally to lax multilinear functors $P$ and $Q$ in PermCat$\big(\langle C \rangle; D\big)$. Since $\oplus$ is strictly unital, the unit constraints of the sum are given by

(8.5.40)          $$(P \oplus Q)_j^0 = P_j^0 \oplus Q_j^0.$$

The linearity constraints are given as in (8.2.6).

Lax multilinearity of $P \oplus Q$ is a special case of general composition of lax multilinear functors, but can also be verified directly. The proof of Lemma 8.2.13 verifies the axioms (1.4.6) through (1.4.8) for $P \oplus Q$. The remaining axioms, (8.5.9)

through (8.5.11), are similar. In each case, one makes use of the corresponding ax-
ioms for $P$ and $Q$, together with strictness of the unit e. For (8.5.9) and (8.5.10), one
also uses naturality of $\xi$ and the identities

$$\xi_{?,e} = 1_? = \xi_{e,?}$$

that hold in any permutative category.                                                    ⋄

The monoidal sum from Explanation 8.5.38 gives the following generalization
of Definition 8.2.1.

**Definition 8.5.41.** For small permutative categories D and $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ for $n \geq 0$,
the *internal hom permutative category*

$$(8.5.42) \qquad \qquad \left( \underline{\mathsf{PermCat}}(\langle C \rangle; D), \oplus, \underline{e}, \underline{\xi} \right)$$

is given by the underlying categories

$$\mathsf{PermCat}(\langle C \rangle; D).$$

The monoidal sum $\oplus$ is given on objects (lax multilinear functors) by composition
with the strictly unital strong symmetric monoidal functor

$$(8.5.43) \qquad \qquad \oplus : D \times D \longrightarrow D$$

as described in Explanation 8.5.38. The monoidal sum on morphisms (lax multi-
linear transformations) is given by whiskering with $\oplus$. The monoidal unit $\underline{e}$ and
braiding $\underline{\xi}$ from Definition 8.2.1 also provide a monoidal unit and braiding for
$\mathsf{PermCat}(\langle C \rangle; D)$. Verification that this defines a permutative category (8.5.42) is
similar to the proof of Lemma 8.2.13, extended to the case of not-necessarily-strict
unit constraints.

This finishes the definition of the permutative structure on $\underline{\mathsf{PermCat}}(\langle C \rangle; D)$.
Moreover, because $\oplus$ (8.5.43) is a strictly unital strong symmetric monoidal functor,
composition with $\oplus$ also defines permutative structures in the strong and strictly
unital strong cases. These are denoted

$$(8.5.44) \quad \left( \underline{\mathsf{PermCat}^{\mathsf{sg}}}(\langle C \rangle; D), \oplus, \underline{e}, \underline{\xi} \right) \quad \text{and} \quad \left( \underline{\mathsf{PermCat}^{\mathsf{sus}}}(\langle C \rangle; D), \oplus, \underline{e}, \underline{\xi} \right)$$

respectively.                                                                             ⋄

**Remark 8.5.45.** The permutative structure described above does not specialize to
strict multilinear functors

$$P, Q \in \mathsf{PermCat}^{\mathsf{st}}\left( \langle C \rangle; D \right).$$

This is because, when $P$ and $Q$ are strict, the monoidal sum $P \oplus Q$ generally has
nontrivial monoidal constraint determined by that of $\oplus : D \times D \longrightarrow D$ (8.5.39).
As noted in Explanation 8.5.38, the latter is an isomorphism—generally not an
identity—determined by the symmetry isomorphism of D.                                     ⋄

**Symmetric Group Action on Internal Hom.** The following generalizes Defi-
nition 8.2.14.

**Definition 8.5.46.** For small permutative categories D and $\langle C \rangle = \langle C_i \rangle_{i=1}^n$ for $n \geq 0$
and a permutation $\sigma \in \Sigma_n$, we define the functor

$$(8.5.47) \qquad \qquad \underline{\mathsf{PermCat}}(\langle C \rangle; D) \xrightarrow[\cong]{\sigma} \underline{\mathsf{PermCat}}(\langle C \rangle \sigma; D)$$

as the isomorphism of underlying categories

$$\mathsf{PermCat}\big(\langle C\rangle\,;\,D\big) \xrightarrow[\cong]{\ \sigma\ } \mathsf{PermCat}\big(\langle C\rangle\sigma\,;\,D\big)$$

that is given, as in (1.4.18), by precomposition and whiskering with the permutation of factors

$$\textstyle\prod_{i=1}^{n} C_{\sigma(i)} \xrightarrow{\ \sigma\ } \prod_{i=1}^{n} C_i. \hspace{3cm} \diamond$$

An argument similar to that of Lemma 8.2.16 shows that (8.5.47) is a strict symmetric monoidal isomorphism and that the equivariance axioms for internal hom objects, (8.1.5) and (8.1.6), hold.

This describes the symmetric group action for $\underline{\mathsf{PermCat}}$. Moreover, the same action induces a symmetric group action in the strong and strictly unital strong cases (8.5.44) above. The same arguments verify the equivariance axioms in these cases.

**Multicategorical Evaluation.** The following generalizes Definition 8.3.1.

**Definition 8.5.48.** For small permutative categories $D$ and $\langle C\rangle = \langle C_i\rangle_{i=1}^{n}$ with $n \geq 0$, we define the data of a lax $(n+1)$-linear functor

(8.5.49) $$\underline{\mathsf{PermCat}}\big(\langle C\rangle\,;\,D\big) \times \textstyle\prod_{i=1}^{n} C_i \xrightarrow{\ \mathsf{ev}_{\langle C\rangle;D}\ } D$$

as follows.

**Underlying Functor:** The underlying functor is that of (8.3.3) and (8.3.4):

$$\mathsf{ev}_{\langle C\rangle;D}\big(P,\langle x\rangle\big) = P\langle x\rangle \quad\text{and}\quad \mathsf{ev}_{\langle C\rangle;D}\big(\theta,\langle f\rangle\big) = Q(f) \circ \theta_{\langle x\rangle} = \theta_{\langle y\rangle} \circ P(f).$$

**Linearity Constraints:** The linearity constraints are given by (8.3.5) and (8.3.6):

$$(\mathsf{ev}_{\langle C\rangle;D})_1^2 = 1 \quad\text{and}\quad (\mathsf{ev}_{\langle C\rangle;D})_{i+1}^2 = P_i^2 \quad\text{for}\quad i \in \{1,\dots,n\}.$$

**Unit Constraints:** The unit constraints are given by

$$(\mathsf{ev}_{\langle C\rangle;D})_1^0 = 1 \quad\text{and}\quad (\mathsf{ev}_{\langle C\rangle;D})_{i+1}^0 = P_i^0 \quad\text{for}\quad i \in \{1,\dots,n\}.$$

This finishes the definition of $\mathsf{ev}_{\langle C\rangle;D}$. Verification that these data define a lax $(n+1)$-linear functor is similar to the proof of Lemma 8.3.8. The axioms (8.5.9) through (8.5.11) for lax unity, constraint 0-by-2, and constraint 0-by-0, hold by those for $P$ in the cases $j > 0$ and $k > 0$. For the cases $j = 0$ or $k = 0$, these axioms hold because the constant functor $\underline{e}$ is strictly monoidal and because $(P \oplus Q)_j^0 = P_j^0 \oplus Q_j^0$ in (8.5.40).

To verify the evaluation bijection axiom (8.1.7), Explanation 8.5.50 below describes how the definitions of $\chi$ and $\Psi$ in Explanation 8.3.16 and Lemma 8.4.1 generalize to the lax multilinear case. The equivariance axiom (8.1.8) for evaluation bijection follows from the same argument in the proof of Lemma 8.4.9. Verification that the corresponding unit constraints are equal uses the same permutation of indices as in (8.4.13) and (8.4.14), with the definitions (8.5.52) and (8.5.53) from (8.1.7) below.

Because the unit and linearity constraints of $\mathsf{ev}$ depend on those of its first argument, $P$, the same definition specializes to the strong and strictly unital strong cases (8.5.44) above. The same arguments verify the evaluation bijection and equivariance axioms in these cases too. $\hspace{1cm}\diamond$

The following generalizes the functions $\chi$ and $\Psi$ from Explanation 8.3.16 and Lemma 8.4.1, respectively.

**Explanation 8.5.50** (The Functions $\chi$ and $\Psi$). For small permutative categories B, $\langle C \rangle = \langle C_i \rangle_{i=1}^n$, and $\langle D \rangle = \langle D_j \rangle_{j=1}^p$, there are inverse functions

$$(8.5.51) \qquad \mathsf{PermCat}\big(\langle C \rangle\,;\,\underline{\mathsf{PermCat}}(\langle D \rangle\,;\,B)\big) \underset{\Psi}{\overset{\chi}{\rightleftarrows}} \mathsf{PermCat}\big(\langle C \rangle, \langle D \rangle\,;\,B\big)$$

generalizing those of Explanation 8.3.16 and Lemma 8.4.1.
For

$$P = \big(P, \{P_i^2\}, \{P_i^0\}\big) \quad \text{in} \quad \mathsf{PermCat}\big(\langle C \rangle\,;\,\underline{\mathsf{PermCat}}(\langle D \rangle\,;\,B)\big),$$

$\chi P$ has linearity constraints given by (8.3.21) and (8.3.22). The unit constraints of $\chi P$ are given by corresponding components of $P_i^0$ and $(P\langle w \rangle)_{n+j}^0$ as shown in the following displays, with $\langle w \rangle \in \prod_k C_k$, $\langle y \rangle \in \prod_k D_k$, $i \in \{1, \ldots, n\}$, and $j \in \{1, \ldots, p\}$.

$$(8.5.52)$$

$$
\begin{array}{ccc}
\mathsf{e} & \xrightarrow{((\chi P)_i^0)_{\langle w_k \rangle_{k \neq i}, \langle y \rangle}} & (\chi P)\big((\langle w_k \rangle_{k \neq i} \bullet_i \mathsf{e})\,, \langle y \rangle\big) \\
\Big\| & & \Big\| \\
\mathsf{e} & \xrightarrow{((P_i^0)_{\langle w_k \rangle_{k \neq i}})_{\langle y \rangle}} & P(\langle w_k \rangle_{k \neq i} \bullet_i \mathsf{e})\langle y \rangle
\end{array}
$$

$$(8.5.53)$$

$$
\begin{array}{ccc}
\mathsf{e} & \xrightarrow{((\chi P)_{n+j}^0)_{\langle w \rangle, \langle y_k \rangle_{k \neq j}}} & (\chi P)\big(\langle w \rangle\,, (\langle y_k \rangle_{k \neq j} \bullet_j \mathsf{e})\big) \\
\Big\| & & \Big\| \\
\mathsf{e} & \xrightarrow{(P\langle w \rangle_j^0)_{\langle y_k \rangle_{k \neq j}}} & P\langle w \rangle(\langle y_k \rangle_{k \neq j} \bullet_j \mathsf{e})
\end{array}
$$

Each of the first three lax multilinearity axioms for $\chi P$, (1.4.6) through (1.4.8), follows as in Explanation 8.3.16. Verifying the other lax multilinearity axioms, (8.5.9) through (8.5.11), is similar, using the corresponding axioms for $\chi P$. Verification of the axioms involving indices $i \in \{1, \ldots, n\}$ and $n + j \in \{n+1, \ldots, n+p\}$ uses the lax multinaturality axioms (1.4.13) and (8.5.20) for $P_i^0$ and $P_i^2$.
For

$$R = \big(R, \{R_r^2\}, \{R_r^0\}\big) \quad \text{in} \quad \mathsf{PermCat}\big(\langle C \rangle, \langle D \rangle\,;\,\langle B \rangle\big),$$

and for each $\langle w \rangle \in \prod_k C_k$, the linearity constraints of $\Psi R\langle w \rangle$ and $\Psi R$ are given by (8.4.6) and (8.4.8), respectively. The unit constraints of $\Psi R\langle w \rangle$ and $\Psi R$ are determined by corresponding components of $R^0$ as shown in the following displays, with $\langle y \rangle$, $i$, and $j$ as above.

$$(8.5.54)$$

$$
\begin{array}{ccc}
\mathsf{e} & \xrightarrow{((\Psi R)\langle w \rangle_j^0)_{\langle y_k \rangle_{k \neq j}}} & (\Psi R)\langle w \rangle(\langle y_k \rangle_{k \neq j} \bullet_j \mathsf{e}) \\
\Big\| & & \Big\| \\
\mathsf{e} & \xrightarrow{(R_{n+j}^0)_{\langle w \rangle, \langle y_k \rangle_{k \neq j}}} & R\big(\langle w \rangle, (\langle y_k \rangle_{k \neq j} \bullet_j \mathsf{e})\big)
\end{array}
$$

$$(8.5.55)$$

$$
\begin{array}{ccc}
\mathsf{e} & \xrightarrow{\big(((\Psi R)_i^0)_{\langle w_k \rangle_{k \neq i}}\big)_{\langle y \rangle}} & (\Psi R)(\langle w_k \rangle_{k \neq i} \bullet_i \mathsf{e})\langle y \rangle \\
\Big\| & & \Big\| \\
\mathsf{e} & \xrightarrow{(R_i^0)_{\langle w_k \rangle_{k \neq i}, \langle y \rangle}} & R\big((\langle w_k \rangle_{k \neq i} \bullet_i \mathsf{e}), \langle y \rangle\big)
\end{array}
$$

Verification of the necessary axioms follows the same structure as in the proof of Lemma 8.4.1. For $\langle w \rangle \in \prod_k C_k$, each of the lax $p$-multilinearity axioms for $\Psi R\langle w \rangle$, $(\Psi R\langle w \rangle)_j^0$, and $(\Psi R\langle w \rangle)_j^2$ with $j \in \{1, \ldots, p\}$, (1.4.6) through (1.4.8) and (8.5.9)

through (8.5.11), follows from the corresponding axiom for $R$, $R^0_{n+j}$, and $R^2_{n+j}$. For $i \in \{1, \ldots, n\}$, the lax multinaturality axioms for $(\Psi R)^0_i$ and $(\Psi R)^2_i$ hold by the 2-by-2, 0-by-2, and 0-by-0 axioms for $R$, with indices $i$ and $n + j \in \{n+1, \ldots, n+p\}$. The remaining lax multilinearity axioms for $\Psi R$ then follow from the corresponding axioms for $R$.

Lastly, it remains to verify that $\chi$ and $\Psi$ are inverse functions. This follows the same argument given in the proof of Lemma 8.4.1, using (8.5.52) through (8.5.55) along with (8.3.5), (8.3.6), (8.3.21), and (8.3.22). $\diamond$

With these extensions to lax multilinear functors, the following result generalizes Theorem 8.4.15.

**Theorem 8.5.56.** *There is a closed multicategory*

$$\big(\mathsf{PermCat}, \underline{\mathsf{PermCat}}, \mathsf{ev}\big)$$

*consisting of the following data.*

- *The underlying multicategory is* $\mathsf{PermCat}$ *in Theorem 8.5.36.*
- *The internal hom objects are the permutative categories* $\underline{\mathsf{PermCat}}\big(\langle C \rangle; D\big)$ *in Definition 8.5.41.*
- *The symmetric group action on internal hom is given in Definition 8.5.46.*
- *The multicategorical evaluations are the lax multilinear functors*

$$\underline{\mathsf{PermCat}}\big(\langle C \rangle; D\big) \times \prod_{i=1}^n C_i \xrightarrow{\ \mathsf{ev}_{\langle C \rangle; D}\ } D$$

*in Definition 8.5.48.*

*Furthermore, the data above specialize to define closed multicategory structures*

$$\big(\mathsf{PermCat}^{\mathsf{sg}}, \underline{\mathsf{PermCat}}^{\mathsf{sg}}, \mathsf{ev}\big) \quad and \quad \big(\mathsf{PermCat}^{\mathsf{sus}}, \underline{\mathsf{PermCat}}^{\mathsf{sus}}, \mathsf{ev}\big).$$

**Remark 8.5.57.** Recall from Remark 8.5.45 that the monoidal sum of strict multilinear functors is generally not strict, and therefore the internal hom structures in Definition 8.5.41 do not specialize to $\mathsf{PermCat}^{\mathsf{st}}$. The same obstruction (generally nontrivial symmetry in the target, $D$) also prevents Theorem 8.5.56 from specializing to $\mathsf{PermCat}^{\mathsf{st}}$. $\diamond$

# Self-Enrichment and Standard Enrichment of Closed Multicategories

This chapter develops the definitions and basic theory of

- self-enrichment for closed multicategories and
- standard enrichment for multifunctors between closed multicategories.

For a non-symmetric multifunctor

$$\left(\mathsf{M}, \underline{\mathsf{M}}, \mathrm{ev}^{\mathsf{M}}\right) \xrightarrow{\ F\ } \left(\mathsf{N}, \underline{\mathsf{N}}, \mathrm{ev}^{\mathsf{N}}\right),$$

Definition 9.1.1 and Theorem 9.1.7 describe corresponding self-enriched categories for M and N. Definition 9.2.1 and Theorem 9.2.12 describe the induced N-functor

$$\widehat{F} : \mathsf{M}_F \longrightarrow \mathsf{N},$$

where $\mathsf{M}_F$ is the N-category obtained by applying the change-of-enrichment 2-functor (Proposition 7.1.9)

$$(-)_F : \mathsf{M}\text{-Cat} \longrightarrow \mathsf{N}\text{-Cat}$$

to M as an M-category. The definition of $\widehat{F}$ is given on objects by $F$ and on hom objects via $F$ together with the evaluation $\mathrm{ev}^{\mathsf{M}}$ from the closed structure of M.

Compositionality of the standard enrichment construction is treated in Section 9.3. For a composable pair of non-symmetric multifunctors between closed non-symmetric multicategories,

$$\left(\mathsf{M}, \underline{\mathsf{M}}, \mathrm{ev}^{\mathsf{M}}\right) \xrightarrow{\ F\ } \left(\mathsf{N}, \underline{\mathsf{N}}, \mathrm{ev}^{\mathsf{N}}\right) \xrightarrow{\ G\ } \left(\mathsf{P}, \underline{\mathsf{P}}, \mathrm{ev}^{\mathsf{P}}\right),$$

Theorem 9.3.6 shows that the following diagram of P-functors commutes.

(9.0.1)
$$
\begin{array}{ccc}
\mathsf{M}_{GF} & \xrightarrow{\ \widehat{GF}\ } & \mathsf{P} \\
\Big\| & & \Big\uparrow{\widehat{G}} \\
(\mathsf{M}_F)_G & \xrightarrow{\ \widehat{F}_G\ } & \mathsf{N}_G
\end{array}
$$

Section 9.4 applies this to the factorization of Elmendorf-Mandell $K$-theory, $\mathsf{K}^{\mathsf{EM}}$ from (2.5.8).

(9.0.2)
$$
\begin{array}{ccccc}
\mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\quad\quad \mathsf{K}^{\mathsf{EM}} \quad\quad} & & & \mathsf{Sp} \\
\Big\downarrow{\mathrm{End}_{\mathcal{M}_1}} & & & & \Big\uparrow{\mathsf{K}^{\mathcal{G}}} \\
\mathsf{Mod}^{\mathcal{M}_1} & \xrightarrow{\ J^{\mathcal{T}}\ } & \mathcal{G}_*\text{-Cat} & \xrightarrow{\ \mathrm{Ner}_*\ } & \mathcal{G}_*\text{-sSet}
\end{array}
$$

The introduction of Section 9.4 gives further review of the closed multicategories and multifunctors above. The factorization above yields a corresponding factorization of the standard enrichment $\widehat{\mathsf{K}^{\mathsf{EM}}}$ into four spectrally-enriched functors. Theorem 9.4.2 gives a precise statement, and the remainder of Section 9.4 gives further details.

**Connection with Other Chapters.** The diagram change of enrichment theory in Chapter 10 depends on the self-enrichments and standard enrichments developed here. That is then used to study the homotopy theory of enriched diagram categories, in Chapters 11 and 12. The standard enrichment of $\mathsf{K}^{\mathsf{EM}}$ and its factorization are used in Sections 10.5 and 10.6 for the development of corresponding spectral Mackey functors.

**Background.** The development of self-enrichment and standard enrichment in Sections 9.1 through 9.3 depends on the theory of multicategorical enrichment and closed multicategories, from Chapters 6 through 8. Section 9.4 uses the definition and factorization of Elmendorf-Mandell $K$-theory, $\mathsf{K}^{\mathsf{EM}}$, from Section 2.5.

**Chapter Summary.** Section 9.1 defines the self-enrichment of non-symmetric closed multicategories. Section 9.2 defines the standard enrichment of non-symmetric multifunctors between non-symmetric closed multicategories. Section 9.3 shows that the standard enrichment construction respects composition of non-symmetric multifunctors. This is applied in Section 9.4 to factor the standard enrichment of the Elmendorf-Mandell $K$-theory functor, $\mathsf{K}^{\mathsf{EM}}$, into four spectrally enriched functors. Here is a summary table.

| | |
|---|---|
| self-enrichment of a closed multicategory | 9.1.1 and 9.1.7 |
| application to permutative categories | 9.1.8 |
| application to symmetric monoidal closed categories | 9.1.9 |
| standard enrichment of a multifunctor | 9.2.1 and 9.2.12 |
| examples of standard enrichment | 9.2.14 and 9.2.17 |
| compositionality of standard enrichment | 9.3.6 |
| application to Elmendorf-Mandell $K$-theory | 9.4.2, 9.4.5, 9.4.9, 9.4.14, and 9.4.17 |

We remind the reader of Convention A.1.2 about universes and Convention A.1.30 about left normalized bracketing for iterated products.

## 9.1. Self-Enrichment of Closed Multicategories

In this section we observe that each non-symmetric closed multicategory is enriched in itself; see Theorem 9.1.7. In this context of self-enrichment, we prove two consistency results.

(1) In Proposition 9.1.8 we observe that, for the closed multicategory $\mathsf{PermCat}^{\mathsf{su}}$ of small permutative categories, the self-enrichment obtained from the closed multicategory structure coincides with the one in Theorem 6.4.20.
(2) For each symmetric monoidal closed category $\mathsf{V}$, the self-enrichment is the same whether $\mathsf{V}$ is regarded as a symmetric monoidal closed category or as a closed multicategory; see Proposition 9.1.9.

The self-enrichment of a non-symmetric closed multicategory is an integral part of several key constructions later in this work, including:

- the standard enrichment of a multifunctor (Definition 9.2.1),
- diagrams enriched in a multicategory (10.1.2), and
- Mackey functors enriched in a multicategory (10.1.3).

In short, self-enrichment is one of the key features of non-symmetric closed multicategories, and we will make full use of it.

**Canonical Self-Enrichment.** Recall M-categories for a non-symmetric multicategory M (Definition 6.1.1). For a non-symmetric closed multicategory M (Definition 8.1.1), recall that two multimorphisms are called *partners* if they correspond to each other under the evaluation bijection (8.1.7).

$$\mathsf{M}\left(\langle x\rangle\,;\,\underline{\mathsf{M}}\big(\langle y\rangle\,;\,z\big)\right) \xrightarrow[\cong]{\chi_{\langle x\rangle;\langle y\rangle;z}} \mathsf{M}\big(\langle x\rangle, \langle y\rangle\,;\,z\big)$$

$$f \longmapsto \gamma\big(\mathsf{ev}_{\langle y\rangle;z}\,;\,f, \langle 1_{y_j}\rangle_{j=1}^{p}\big)$$

Now we define the canonical self-enrichment.

**Definition 9.1.1.** Suppose $(\mathsf{M}, \underline{\mathsf{M}}, \mathsf{ev})$ is a non-symmetric closed multicategory. We define the data of an M-category

$$(\mathsf{M}, \circ, i),$$

which is called the *canonical self-enrichment of* M, as follows.

**Objects:** The objects are those of M.
**Hom Objects:** For each pair of objects $x, y \in \mathsf{M}$, the morphism object is the unary internal hom object $\underline{\mathsf{M}}(x\,;\,y)$ in M. This is the $n = 1$ case of (8.1.2).
**Identities:** For each object $x$ in M, the identity

$$(9.1.2) \qquad\qquad i_x : \langle\rangle \longrightarrow \underline{\mathsf{M}}(x\,;\,x),$$

which is a nullary multimorphism in M, is defined as the partner of the $x$-colored unit $1_x \in \mathsf{M}(x\,;\,x)$.
**Composition:** For objects $x, y, z \in \mathsf{M}$, the composition

$$(9.1.3) \qquad\qquad \big(\underline{\mathsf{M}}(y\,;\,z), \underline{\mathsf{M}}(x\,;\,y)\big) \xrightarrow{\ \circ\ } \underline{\mathsf{M}}(x\,;\,z),$$

which is a binary multimorphism in M, is defined as the partner of the following 3-ary multimorphism.

$$(9.1.4) \qquad \big(\underline{\mathsf{M}}(y\,;\,z), \underline{\mathsf{M}}(x\,;\,y), x\big) \xrightarrow{(1, \mathsf{ev}_{x;y})} \big(\underline{\mathsf{M}}(y\,;\,z), y\big) \xrightarrow{\mathsf{ev}_{y;z}} z$$

This finishes the definition of the canonical self-enrichment of M. Theorem 9.1.7 proves that it is an M-category.                                                          ◇

**Explanation 9.1.5** (Canonical Self-Enrichment). Definition 9.1.1 does not use anything about symmetric group action, either on M or on the internal hom objects. Thus it makes sense for a *non-symmetric* closed multicategory.

By definition (8.1.7), the composition $\circ$ in (9.1.3) and the identity $i_x$ in (9.1.2) are the *unique* binary, respectively, nullary, multimorphisms in M that make the

following two diagrams in M commute.

$$(9.1.6) \quad \begin{array}{ccc} \left(\underline{M}(y\,;z)\,,\,\underline{M}(x\,;y)\,,\,x\right) & \xrightarrow{(\circ\,,\,1_x)} & \left(\underline{M}(x\,;z)\,,\,x\right) \\ {\scriptstyle(1\,,\,\mathrm{ev}_{x;y})}\Big\downarrow & & \Big\downarrow {\scriptstyle\mathrm{ev}_{x;z}} \\ \left(\underline{M}(y\,;z)\,,\,y\right) & \xrightarrow{\quad \mathrm{ev}_{y;z} \quad} & z \end{array} \qquad \begin{array}{ccc} (\langle\rangle\,,\,x) & \xrightarrow{(i_x\,,\,1_x)} & \left(\underline{M}(x\,;x)\,,\,x\right) \\ & {\scriptstyle 1_x}\searrow & \Big\downarrow {\scriptstyle\mathrm{ev}_{x;x}} \\ & & x \end{array}$$

We call these the *associativity diagram* and the *unity diagram*, respectively, of the canonical self-enrichment of M. These diagrams are the analogs of those in (B.3.9) for a symmetric monoidal closed category. ◇

We now show that the canonical self-enrichment is well defined.

**Theorem 9.1.7.** *For each non-symmetric closed multicategory* (M, $\underline{M}$, ev)*, the canonical self-enrichment of* M *in Definition 9.1.1 is an* M*-category.*

*Proof.* We need to prove the associativity axiom (6.1.4) and the unity axiom (6.1.5) for the canonical self-enrichment of M.

*Associativity (6.1.4).* Consider objects $w, x, y, z \in$ M. Since taking partners is a bijection (8.1.7), it suffices to show that the two composites in the associativity diagram (6.1.4) for M have the same partners. These two partners are the left and right boundary composites of the following diagram in M, which we want to show is commutative. We abbreviate $\underline{M}(x\,;y)$ to $\underline{M}_{x;y}$.



The following statements hold for the diagram above.

- The sub-region labeled ✩ commutes by definition.
- The other four sub-regions commute by the associativity diagram in (9.1.6).

This proves the associativity axiom (6.1.4) for the canonical self-enrichment of M.

*Unity (6.1.5).* Similar to associativity, it suffices to show that the partners of the composites in the unity diagram (6.1.5) are equal. These partners are the boundary composites and the middle $\mathrm{ev}_{x;y}$ in the following diagram in M, which we want to

show is commutative.



The following statements hold for the diagram above.

- The two sub-regions labeled $\boxed{a}$ commute by the associativity diagram in (9.1.6).
- The two sub-regions labeled $\boxed{u}$ commute by the unity diagram in (9.1.6).
- The sub-regions labeled $\boxed{ru}$ and $\boxed{lu}$ commute by, respectively, the right unity (C.1.9) and left unity (C.1.10) of M.
- The remaining unlabeled quadrilateral commutes by definition.

This proves the unity axiom (6.1.5) for the canonical self-enrichment of M. $\qquad\square$

**Self-Enrichment of** $\mathsf{PermCat^{su}}$**.** Consider the closed multicategory $\mathsf{P^{su}} = \mathsf{PermCat^{su}}$ of small permutative categories (Theorem 8.4.15). By Theorem 9.1.7 $\mathsf{P^{su}}$ has a canonical self-enrichment. In other words, $\mathsf{P^{su}}$ has the structure of a $\mathsf{P^{su}}$-category. Moreover, we previously established a $\mathsf{P^{su}}$-category structure on $\mathsf{P^{su}}$ in Theorem 6.4.20. Now we observe that these $\mathsf{P^{su}}$-categories are the same.

**Proposition 9.1.8.** *For the closed multicategory* $\mathsf{PermCat^{su}}$,

- *the self-enrichment in Theorem 6.4.20 and*
- *the canonical self-enrichment in Theorem 9.1.7*

*are equal as* $\mathsf{P^{su}}$-*categories.*

*Proof.* The two $\mathsf{P^{su}}$-categories in question are the same for the following reasons.

(i) In each of Theorems 6.4.20 and 9.1.7, the $\mathsf{P^{su}}$-category has small permutative categories as objects.

(ii) For small permutative categories $\mathsf{C}$ and $\mathsf{D}$, the hom object in Definition 9.1.1 is the small permutative category $\underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D})$ in Definition 8.2.1. This is the same as the permutative category $\mathsf{P^{su}}(\mathsf{C},\mathsf{D})$ in Lemma 6.4.11, as we pointed out in Explanation 8.2.12.

(iii) The bilinear evaluation $\mathsf{ev}_{\mathsf{C};\mathsf{D}}$ in Lemma 8.3.8 coincides with the bilinear evaluation $\mathsf{ev}_{\mathsf{C},\mathsf{D}}$ in Proposition 6.5.7, as we pointed out in Explanation 8.3.7.

(iv) The identity of a small permutative category $\mathsf{C}$ in the sense of (9.1.2) is the unique 0-linear functor

$$i_{\mathsf{C}} : \mathbf{1} \longrightarrow \underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{C}),$$

which means a strictly unital symmetric monoidal functor $\mathsf{C} \longrightarrow \mathsf{C}$, that makes the unity diagram in (9.1.6) commutative. By Definition 8.3.1 for

$\mathrm{ev}_{C;C}$, the identity symmetric monoidal functor $1_C$ makes the unity diagram in (9.1.6) commutative. Thus uniqueness implies that $i_C$ is given by $1_C$. This is the same as the identity in Definition 6.4.19.

(v) For small permutative categories B, C, and D, the composition $\circ$ in (9.1.3) is the unique bilinear functor

$$\underline{P}^{\mathrm{su}}(C;D) \times \underline{P}^{\mathrm{su}}(B;C) \xrightarrow{\ \circ\ } \underline{P}^{\mathrm{su}}(B;D)$$

that makes the associativity diagram in (9.1.6) commutative. By Proposition 6.5.8 the composition bilinear functor $\mathrm{m}_{B,C,D}$ in Lemma 6.4.17 also makes the associativity diagram in (9.1.6) commutative. Thus uniqueness implies

$$\circ = \mathrm{m}_{B,C,D},$$

which is the composition in Definition 6.4.19.

This finishes the proof.                                                          □

**Self-Enrichment of Symmetric Monoidal Closed Categories.** For a symmetric monoidal closed category V, by Proposition 8.1.16 there is an endomorphism closed multicategory

$$\big(\mathrm{End}\,V, \underline{\mathrm{End}\,V}, \mathrm{ev}\big)$$

with internal hom objects and evaluation induced by those of V. There are two self-enrichment constructions in this context:

(1) V has a canonical self-enrichment (Theorem B.3.7).
(2) By Theorem 9.1.7 the endomorphism multicategory $\mathrm{End}\,V$ has a canonical self-enrichment, which is an $(\mathrm{End}\,V)$-category. By Proposition 6.2.1 enrichment in V and in $\mathrm{End}\,V$ are the same thing. Thus we may also regard the canonical self-enrichment of $\mathrm{End}\,V$ as a V-category (Definition B.1.1).

Now we observe that these two self-enrichment constructions are the same.

**Proposition 9.1.9.** *For a symmetric monoidal closed category* $\big(V, \otimes, 1, [,]\big)$,

- *the canonical self-enrichment of* V *in Theorem B.3.7 and*
- *the canonical self-enrichment of* $\mathrm{End}\,V$ *in Theorem 9.1.7*

*are equal as* V-*categories.*

*Proof.* We compare Definition B.3.4 for the canonical self-enrichment of V and Definition 9.1.1 for the canonical self-enrichment of $\mathrm{End}\,V$.

*Objects and Morphism Objects.* The objects of $\mathrm{End}\,V$ are those of V. For objects $x, y \in \mathrm{End}\,V$, the morphism object in $\mathrm{End}\,V$ is the internal hom object

$$\underline{\mathrm{End}\,V}\big(x\,;y\big) = [x,y] \quad \text{in} \quad V.$$

This is the same hom object as in Definition B.3.4.

*Identities.* The identity of an object $x \in \mathrm{End}\,V$ is the nullary multimorphism (9.1.2)

$$i_x \in (\mathrm{End}\,V)\big(\langle\rangle\,; \underline{\mathrm{End}\,V}(x;x)\big) = V\big(1, [x,x]\big)$$

that makes the left diagram in $\mathsf{End}\,\mathsf{V}$ below commutative.



The left diagram in $\mathsf{End}\,\mathsf{V}$ above means the right commutative diagram in $\mathsf{V}$, with $\lambda$ denoting the left unit isomorphism. Comparing the right diagram above with the right diagram in (B.3.9), the uniqueness of adjoints implies that $i_x$ is equal to the identity of $x$ in the canonical self-enrichment of $\mathsf{V}$.

*Composition.* For objects $x, y, z \in \mathsf{End}\,\mathsf{V}$, the composition in $\mathsf{End}\,\mathsf{V}$ is the binary multimorphism (9.1.3)

$$\circ \in (\mathsf{End}\,\mathsf{V})\Big(\underline{\mathsf{End}\,\mathsf{V}}(y;z)\,,\,\underline{\mathsf{End}\,\mathsf{V}}(x;y)\,;\,\underline{\mathsf{End}\,\mathsf{V}}(x;z)\Big)$$
$$= \mathsf{V}\big([y,z] \otimes [x,y]\,,\,[x,z]\big)$$

that makes the left diagram in $\mathsf{End}\,\mathsf{V}$ below commutative.



The left diagram in $\mathsf{End}\,\mathsf{V}$ above means the right commutative diagram in $\mathsf{V}$, with $\alpha$ denoting the associativity isomorphism. Comparing the right diagram above with the left diagram in (B.3.9), the uniqueness of adjoints implies that $\circ$ is equal to the composition $\mathsf{m}$ in the canonical self-enrichment of $\mathsf{V}$. $\square$

## 9.2. Standard Enrichment of a Multifunctor

In Theorem 9.1.7 we showed that each non-symmetric closed multicategory $\mathsf{M}$ has a canonical self-enrichment. In this section we use the canonical self-enrichment to show that each non-symmetric multifunctor $F$ between non-symmetric closed multicategories induces a multicategorically enriched functor $\widehat{F}$, called the standard enrichment of $F$. We discuss further functoriality properties and an application to $K$-theory in Sections 9.3 and 9.4. In subsequent chapters, the standard enrichment is one of the two key constructions for change of enrichment for enriched diagrams (10.2.3) and enriched Mackey functors (10.2.4).

Here is an outline of this section.

- The standard enrichment is constructed in Definition 9.2.1 and verified in Theorem 9.2.12.
- As an illustration of the construction, Example 9.2.14 describes the standard enrichment of the non-symmetric multifunctor

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$

in Theorem 5.2.6.

- As a consistency check, in Proposition 9.2.17 we prove that, for a monoidal functor between symmetric monoidal closed categories, the two standard enrichment constructions in Proposition B.4.17 and Theorem 9.2.12 are the same.

**Standard Enrichment.** Recall from Definition 8.1.1 that, in a non-symmetric closed multicategory, two multimorphisms that correspond to each other under the evaluation bijection (8.1.7) are called *partners*. The partner of a multimorphism $f$ is denoted $f^\#$.

**Definition 9.2.1** (Standard Enrichment). For a non-symmetric multifunctor (Definition C.1.19) between non-symmetric closed multicategories

$$F : \left(\mathsf{M}, \underline{\mathsf{M}}, \mathrm{ev}^{\mathsf{M}}\right) \longrightarrow \left(\mathsf{N}, \underline{\mathsf{N}}, \mathrm{ev}^{\mathsf{N}}\right),$$

we define the data of an N-functor (Definition 6.1.7)

(9.2.2)
$$\widehat{F} : \mathsf{M}_F \longrightarrow \mathsf{N},$$

which is called the *standard enrichment of F*, as follows.

**Domain:** The domain of $\widehat{F}$ is the N-category $\mathsf{M}_F$ obtained from the canonical self-enrichment of M (Theorem 9.1.7), which is an M-category, by applying the change-of-enrichment 2-functor (Proposition 7.1.9)

$$(-)_F : \mathsf{M}\text{-Cat} \longrightarrow \mathsf{N}\text{-Cat}.$$

**Codomain:** The codomain of $\widehat{F}$ is the canonical self-enrichment of N (Theorem 9.1.7), which is an N-category.

**Object Assignment:** $\widehat{F}$ has the same object assignment as $F$.

**Component Morphisms:** For each pair of objects $x, y \in \mathsf{M}$, the $(x, y)$-component unary multimorphism

(9.2.3)
$$\widehat{F}_{x,y} = \left(F(\mathrm{ev}^{\mathsf{M}}_{x;y})\right)^\# : F\underline{\mathsf{M}}(x; y) \longrightarrow \underline{\mathsf{N}}(Fx; Fy) \quad \text{in} \quad \mathsf{N}$$

is defined as the partner of the binary multimorphism

(9.2.4)
$$F(\mathrm{ev}^{\mathsf{M}}_{x;y}) : \left(F\underline{\mathsf{M}}(x; y), Fx\right) \longrightarrow Fy \quad \text{in} \quad \mathsf{N}.$$

This is the image under $F$ of the evaluation binary multimorphism (8.1.4)

$$\mathrm{ev}^{\mathsf{M}}_{x;y} : \left(\underline{\mathsf{M}}(x; y), x\right) \longrightarrow y$$

at $(x; y)$, which is part of the closed multicategory structure of M.

This finishes the definition of the standard enrichment $\widehat{F}$. We verify that $\widehat{F}$ is an N-functor in Theorem 9.2.12 below.                                        ◇

Before we show that $\widehat{F}$ is an N-functor, let us explain some aspects of the standard enrichment construction.

**Explanation 9.2.5** (Symmetry is Not Required). In Definition 9.2.1, even if M and N are closed multicategories—as opposed to non-symmetric ones—$F$ is *not* required to preserve the symmetric group action on (i) M and N (C.1.20) and (ii) their internal hom objects (8.1.3). This is possible because the change-of-enrichment 2-functor $(-)_F$ (Proposition 7.1.9) and the canonical self-enrichment (Theorem 9.1.7)

do not require symmetry. This point is important for our applications in Chapter 12 where we consider the non-symmetric multifunctors $\mathsf{F}_\bullet$ and $\mathsf{F}_{\mathcal{M}1}$ in Theorem 5.2.6 and (5.5.2); see Theorems 12.1.6 and 12.4.6. We discuss the standard enrichment of $\mathsf{F}_\bullet$ in Example 9.2.14 below. ◇

**Explanation 9.2.6** (Domain of $\widehat{F}$). Interpreting the change of enrichment (Definition 7.1.1) for the canonical self-enrichment of M (Theorem 9.1.7), we unpack the N-category $\mathsf{M}_F$, which is the domain of the standard enrichment $\widehat{F}$, as follows.

- $\mathsf{M}_F$ has the same objects as M.
- For each pair of objects $x, y \in \mathsf{M}_F$, the hom object is

$$(\mathsf{M}_F)(x, y) = F\underline{\mathsf{M}}(x; y) \quad \text{in} \quad \mathsf{N}.$$

- The identity of an object $x \in \mathsf{M}_F$ is the nullary multimorphism

(9.2.7)
$$\langle\rangle \xrightarrow{F(i_x)} F\underline{\mathsf{M}}(x; x) \quad \text{in} \quad \mathsf{N}$$

given by applying $F$ to the identity (9.1.2)

$$\langle\rangle \xrightarrow{i_x} \underline{\mathsf{M}}(x; x) \quad \text{in} \quad \mathsf{M}.$$

- For objects $x, y, z \in \mathsf{M}_F$, the composition binary multimorphism

(9.2.8)
$$\left(F\underline{\mathsf{M}}(y; z), F\underline{\mathsf{M}}(x; y)\right) \xrightarrow{F(\circ)} F\underline{\mathsf{M}}(x; z) \quad \text{in} \quad \mathsf{N}$$

is given by applying $F$ to the composition (9.1.3)

$$\left(\underline{\mathsf{M}}(y; z), \underline{\mathsf{M}}(x; y)\right) \xrightarrow{\circ} \underline{\mathsf{M}}(x; z) \quad \text{in} \quad \mathsf{M}.$$

Moreover, applying the non-symmetric multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ to the associativity and unity diagrams in (9.1.6) yields the following commutative diagrams in N.

(9.2.9)



We emphasize that the two commutative diagrams in (9.2.9) use the fact that $F$ preserves colored units (C.1.21) and composition (C.1.22), but they do *not* require $F$ to preserve the symmetric group action even if M and N are multicategories. ◇

**Explanation 9.2.10** (Component Morphisms of $\widehat{F}$). By definition (9.2.3), for objects $x, y \in \mathsf{M}$, the $(x, y)$-component unary multimorphism $\widehat{F}_{x,y}$ is the partner of $F\left(\mathrm{ev}_{x;y}^{\mathsf{M}}\right)$. By the definition of the evaluation bijection (8.1.7), this means that $\widehat{F}_{x,y}$ is the *unique* unary multimorphism that makes the following diagram in N commutative.

(9.2.11)



The diagram (9.2.11) is the analog of the first diagram in Explanation B.4.18 in the context of the standard enrichment of a monoidal functor between symmetric

monoidal closed categories. We make this connection precise in Proposition 9.2.17 below.                                                                                    ◇

Now we show that the standard enrichment is a well-defined enriched functor (Definition 6.1.7).

**Theorem 9.2.12.** *For each non-symmetric multifunctor between non-symmetric closed multicategories*

$$F : \left( \mathsf{M}, \underline{\mathsf{M}}, \mathsf{ev}^{\mathsf{M}} \right) \longrightarrow \left( \mathsf{N}, \underline{\mathsf{N}}, \mathsf{ev}^{\mathsf{N}} \right),$$

*the standard enrichment in Definition 9.2.1*

$$\widehat{F} : \mathsf{M}_F \longrightarrow \mathsf{N}$$

*is an* N*-functor.*

*Proof.* We need to show that $\widehat{F}$ preserves composition and identities in the sense of (6.1.9). For $\widehat{F}$ those diagrams are the diagrams in N in (9.2.13) below for objects $x, y, z \in \mathsf{M}$. We use (9.2.7) and (9.2.8) for the identities and composition of $\mathsf{M}_F$, and we denote $\underline{\mathsf{M}}(x; y)$ by $\underline{\mathsf{M}}_{x;y}$.

(9.2.13)



Since taking partners is a bijection (8.1.7), it suffices to show that, in each diagram in (9.2.13), the partners of the two composites are equal.

*Preservation of Composition.* For the left diagram in (9.2.13), the partners of the two composites are the two boundary composites in the following diagram in N, which we want to show is commutative.



The diagram above is commutative for the following reasons.

- The triangle labeled ♣ is commutative by definition.
- The sub-regions labeled ♠ and ♦ are commutative by the left diagrams in, respectively, (9.1.6) and (9.2.9).
- The remaining three sub-regions are commutative by (9.2.11).

This proves that the left diagram in (9.2.13) is commutative.

*Preservation of Identities.* For the right diagram in (9.2.13), the partners of the two composites are the two boundary composites in the following diagram in N.

$$
\begin{array}{ccc}
(\langle\rangle, Fx) & \xrightarrow{\ (F(i_x),\,1)\ } & (F\underline{M}_{x;x}, Fx) \\
\end{array}
$$

From left to right, the three sub-regions in the diagram above are commutative by, respectively, the unity diagram in (9.1.6), the right diagram in (9.2.9), and (9.2.11). This proves that the right diagram in (9.2.13) is commutative. □

**Examples of Standard Enrichment.**

**Example 9.2.14** (Standard Enrichment of F.). As an illustration of Definition 9.2.1, consider the non-symmetric multifunctor (Theorem 5.2.6)

$$\mathsf{F}_{\bullet} : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$$

and its standard enrichment PermCat$^{\mathsf{su}}$-functor

(9.2.15) $$\widehat{\mathsf{F}}_{\bullet} : (\mathsf{Multicat}_*)_{\mathsf{F}_{\bullet}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}.$$

For the context, first recall the following.

- Multicat$_*$ = M$_*$ is a symmetric monoidal closed category (Theorem 1.2.8).
  - Its objects are small pointed multicategories (Definition C.4.1).
  - Its monoidal product is the smash product, $\wedge$, in (1.2.3).
  - Its internal hom is the pointed internal hom, Hom$_*$, in (1.2.5).
  
  Via its endomorphism multicategory, M$_*$ is a closed multicategory (Proposition 8.1.16). It is enriched in itself (Proposition 9.1.9).
- PermCat$^{\mathsf{su}}$ = P$^{\mathsf{su}}$ is a closed multicategory (Theorem 8.4.15).
  - Its objects are small permutative categories (Definition A.1.14).
  - Its multicategory structure is discussed in Section 1.4.
  - Its internal hom objects, their symmetric group action, and multicategorical evaluation are constructed in, respectively, Definitions 8.2.1, 8.2.14, and 8.3.1
  - We proved its evaluation bijection axiom in Lemma 8.4.1. In particular, the inverse of the function $\chi$ is the function $\Psi$ in (8.4.2).
  
  As a closed multicategory, P$^{\mathsf{su}}$ is enriched in itself (Proposition 9.1.8).
- F$_{\bullet}$ is a non-symmetric Cat-multifunctor, hence also a non-symmetric multifunctor. It is genuinely non-symmetric because its construction (5.2.4) involves F$_{\bullet}^n$ in (5.1.2), which is induced by F$^n$ in Definition 3.4.14. As we mentioned in Explanation 3.4.33 (1), F$^n$ is not compatible with permutations. So F$_{\bullet}^n$ is also not compatible with permutations, leading to the non-symmetry of F$_{\bullet}$.
- As we discussed in detail in Explanation 7.1.12, the non-symmetric multifunctor F$_{\bullet}$ has an associated change-of-enrichment 2-functor

$$(-)_{\mathsf{F}_{\bullet}} : \mathsf{Multicat}_*\text{-}\mathsf{Cat} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}.$$

  When we apply $(-)_{\mathsf{F}_{\bullet}}$ to the M$_*$-category M$_*$, as in Explanation 9.2.6, we obtain the P$^{\mathsf{su}}$-category $(\mathsf{M}_*)_{\mathsf{F}_{\bullet}}$, which is the domain of $\widehat{\mathsf{F}}_{\bullet}$.

The object assignment of the standard enrichment $\widehat{\mathsf{F}}_\bullet$ is the same as that of $\mathsf{F}_\bullet$ in Definition 4.1.11. For each pair of small pointed multicategories $\mathsf{X}$ and $\mathsf{Y}$, the $(\mathsf{X}, \mathsf{Y})$-component is the strictly unital symmetric monoidal functor

(9.2.16) $\qquad \widehat{\mathsf{F}}_{\bullet \mathsf{X}, \mathsf{Y}} = \big(\mathsf{F}_\bullet(\mathrm{ev}^{M_*}_{\mathsf{X}, \mathsf{Y}})\big)^{\#} : \mathsf{F}_\bullet \mathrm{Hom}_*(\mathsf{X}, \mathsf{Y}) \longrightarrow \mathsf{P}^{\mathsf{su}}(\mathsf{F}_\bullet \mathsf{X}, \mathsf{F}_\bullet \mathsf{Y}).$

In other words, $\widehat{\mathsf{F}}_{\bullet \mathsf{X}, \mathsf{Y}}$ is the image under $\Psi$ (8.4.2) of the bilinear functor

$$\mathsf{F}_\bullet(\mathrm{ev}^{M_*}_{\mathsf{X}, \mathsf{Y}}) : \mathsf{F}_\bullet \mathrm{Hom}_*(\mathsf{X}, \mathsf{Y}) \times \mathsf{F}_\bullet \mathsf{X} \longrightarrow \mathsf{F}_\bullet \mathsf{Y}.$$

This bilinear functor is the image under $\mathsf{F}_\bullet$ (5.2.3) of the evaluation pointed multifunctor

$$\mathrm{ev}^{M_*}_{\mathsf{X}, \mathsf{Y}} : \mathrm{Hom}_*(\mathsf{X}, \mathsf{Y}) \wedge \mathsf{X} \longrightarrow \mathsf{Y}$$

in (B.3.2), which is regarded as a binary multimorphism in $M_*$. $\qquad\qquad \diamond$

Recall from (C.3.3) that each monoidal functor induces a non-symmetric multifunctor via the endomorphism construction. The next observation is a consistency result. It says that for a monoidal functor between symmetric monoidal closed categories, the standard enrichment in Proposition B.4.17 and Theorem 9.2.12 are the same.

**Proposition 9.2.17.** *For each monoidal functor between symmetric monoidal closed categories*

$$(U, U^2, U^0) : \big(\mathsf{V}, \otimes, [,]\big) \longrightarrow \big(\mathsf{W}, \otimes, [,]\big),$$

*the following two $\mathsf{W}$-functors are the same:*

- *The standard enrichment*

$$\widehat{U} : \mathsf{V}_U \longrightarrow \mathsf{W}$$

  *of $U$ in Proposition B.4.17.*
- *The standard enrichment*

$$\widehat{\mathrm{End}\, U} : (\mathrm{End}\, \mathsf{V})_{\mathrm{End}\, U} \longrightarrow \mathrm{End}\, \mathsf{W}$$

  *in Theorem 9.2.12 of the non-symmetric multifunctor*

$$\mathrm{End}\, U : \mathrm{End}\, \mathsf{V} \longrightarrow \mathrm{End}\, \mathsf{W}.$$

*Proof.* This assertion follows by combining the following facts.

- By Proposition 6.2.1 enrichment in the symmetric monoidal category $\mathsf{V}$ and in the multicategory $\mathrm{End}\, \mathsf{V}$ are the same thing.
- By Proposition 8.1.16 $\mathrm{End}\, \mathsf{V}$ is a closed multicategory, with internal hom objects and evaluation induced by those of $(\mathsf{V}, [,])$.
- By Proposition 9.1.9 the canonical self-enrichment of the symmetric monoidal closed category $\mathsf{V}$ is equal to the canonical self-enrichment of the closed multicategory $\mathrm{End}\, \mathsf{V}$. These statements also hold for $\mathsf{W}$.
- By Proposition 7.3.1 the change-of-enrichment 2-functors $(-)_U$ and $(-)_{\mathrm{End}\, U}$ are equal. This yields an equality of $\mathsf{W}$-categories

$$\mathsf{V}_U = (\mathrm{End}\, \mathsf{V})_{\mathrm{End}\, U}.$$

  So the $\mathsf{W}$-functors in question, $\widehat{U}$ and $\widehat{\mathrm{End}\, U}$, have the same domain and the same codomain.
- Both $\widehat{U}$ and $\widehat{\mathrm{End}\, U}$ have the same object assignment as $U$.

It remains to show that $\widehat{U}$ and $\widehat{\operatorname{End} U}$ have the same component morphisms. For each pair of objects $x, y \in \mathsf{V}$, the diagram (9.2.11) for $F = \operatorname{End} U$ is the diagram

(9.2.18)

$$
\begin{array}{ccc}
U[x,y] \otimes Ux & \xrightarrow{\ \widehat{\operatorname{End} U}_{x,y} \otimes 1\ } & [Ux, Uy] \otimes Ux \\
{\scriptstyle U^2}\Big\downarrow & & \Big\downarrow{\scriptstyle \operatorname{ev}} \\
U\big([x,y] \otimes x\big) & \xrightarrow{\ U(\operatorname{ev})\ } & Uy
\end{array}
$$

in $\mathsf{W}$ because, by (C.3.4),

$$
(\operatorname{End} U)(\operatorname{ev}) = U(\operatorname{ev}) \circ U^2.
$$

The diagram (9.2.18) coincides with the first diagram in Explanation B.4.18, which has $\widehat{U}_{x,y}$ in place of $\widehat{\operatorname{End} U}_{x,y}$. Since (9.2.11) uniquely defines $\widehat{\operatorname{End} U}_{x,y}$, we conclude that it is equal to $\widehat{U}_{x,y}$. This proves that $\widehat{U}$ and $\widehat{\operatorname{End} U}$ have the same component morphisms. $\qquad\square$

## 9.3. Compositionality of Standard Enrichment

For each non-symmetric multifunctor

$$
F : \mathsf{M} \longrightarrow \mathsf{N}
$$

between non-symmetric closed multicategories, in Theorem 9.2.12 we constructed its standard enrichment N-functor

$$
\widehat{F} : \mathsf{M}_F \longrightarrow \mathsf{N}.
$$

In this section we show that the standard enrichment construction, $F \longmapsto \widehat{F}$, respects composition of non-symmetric multifunctors; see Theorem 9.3.6. In particular, this compositionality property holds for monoidal functors between symmetric monoidal closed categories (Example 9.3.12). In Section 9.4 we apply Theorem 9.3.6 to factor the standard enrichment of Elmendorf-Mandell $K$-theory, $\widehat{\mathsf{K}^{\mathsf{EM}}}$, into four spectrally enriched functors.

**Context of Compositionality.** For the set up, consider

(9.3.1) $$\mathsf{M} \xrightarrow{\ F\ } \mathsf{N} \xrightarrow{\ G\ } \mathsf{P}$$

consisting of

- non-symmetric closed multicategories M, N, and P (Definition 8.1.1) and
- non-symmetric multifunctors $F$ and $G$ (Definition C.1.19).

By Proposition 7.4.1 the following diagram of change-of-enrichment 2-functors is commutative.

(9.3.2)

$$
\begin{array}{ccc}
& \xrightarrow{\quad\quad (-)_{GF} \quad\quad} & \\
\mathsf{M}\text{-Cat} & \xrightarrow{\ (-)_F\ } \mathsf{N}\text{-Cat} \xrightarrow{\ (-)_G\ } & \mathsf{P}\text{-Cat}
\end{array}
$$

We consider the following three P-functors.

(i) The standard enrichment of $G$ in (9.2.2) is the P-functor

$$
\widehat{G} : \mathsf{N}_G \longrightarrow \mathsf{P}.
$$

(ii) The standard enrichment of the composite non-symmetric multifunctor $GF$ in (9.3.1) is the P-functor

$$\widehat{GF} : \mathsf{M}_{GF} \longrightarrow \mathsf{P}.$$

(iii) Using (9.3.2) and applying the change-of-enrichment 2-functor $(-)_G$ to the standard enrichment $\widehat{F} : \mathsf{M}_F \longrightarrow \mathsf{N}$ of $F$ in (9.2.2) yield the P-functor

$$\widehat{F}_G : \mathsf{M}_{GF} = (\mathsf{M}_F)_G \longrightarrow \mathsf{N}_G.$$

**Explanation 9.3.3** (Change of Enrichment of Standard Enrichment). In the context of (9.3.1) and (9.3.2), the P-functor

$$\widehat{F}_G : \mathsf{M}_{GF} = (\mathsf{M}_F)_G \longrightarrow \mathsf{N}_G$$

has the same object assignment as $F : \mathsf{M} \longrightarrow \mathsf{N}$. For each pair of objects $x, y \in \mathsf{M}$, the $(x, y)$-component of $\widehat{F}$ is the unary multimorphism (9.2.3)

$$\widehat{F}_{x,y} = \left(F(\mathrm{ev}^{\mathsf{M}}_{x;y})\right)^{\#} : F\underline{\mathsf{M}}(x;y) \longrightarrow \underline{\mathsf{N}}(Fx; Fy) \quad \text{in} \quad \mathsf{N}.$$

The $(x, y)$-component of $\widehat{F}_G$ is the unary multimorphism

(9.3.4)        $(\widehat{F}_G)_{x,y} = G\widehat{F}_{x,y} : GF\underline{\mathsf{M}}(x;y) \longrightarrow G\underline{\mathsf{N}}(Fx; Fy) \quad \text{in} \quad \mathsf{P}$

obtained by applying $G$ to $\widehat{F}_{x,y}$. In particular, applying the non-symmetric multifunctor $G$ to the commutative diagram (9.2.11) yields the following commutative diagram in P.

(9.3.5)

$$\left(GF\underline{\mathsf{M}}(x;y), GFx\right) \xrightarrow{\left(G\widehat{F}_{x,y}, \, 1_{GFx}\right)} \left(G\underline{\mathsf{N}}(Fx; Fy), GFx\right)$$

$$GF(\mathrm{ev}^{\mathsf{M}}_{x;y}) \searrow \qquad \downarrow G(\mathrm{ev}^{\mathsf{N}}_{Fx; Fy})$$

$$GFy$$

This uses the fact that $G$, as a non-symmetric multifunctor, preserves composition and colored units.                                                                    ◇

The main observation of this section is that the P-functors $\widehat{F}_G$, $\widehat{G}$, and $\widehat{GF}$ are related as follows.

**Theorem 9.3.6.** *For composable non-symmetric multifunctors between non-symmetric closed multicategories*

$$\left(\mathsf{M}, \underline{\mathsf{M}}, \mathrm{ev}^{\mathsf{M}}\right) \xrightarrow{\ F\ } \left(\mathsf{N}, \underline{\mathsf{N}}, \mathrm{ev}^{\mathsf{N}}\right) \xrightarrow{\ G\ } \left(\mathsf{P}, \underline{\mathsf{P}}, \mathrm{ev}^{\mathsf{P}}\right),$$

*the following diagram of P-functors commutes.*

(9.3.7)

$$
\begin{array}{ccc}
\mathsf{M}_{GF} & \xrightarrow{\ \widehat{GF}\ } & \mathsf{P} \\
\| & & \uparrow{\scriptstyle \widehat{G}} \\
(\mathsf{M}_F)_G & \xrightarrow{\ \widehat{F}_G\ } & \mathsf{N}_G
\end{array}
$$

*Proof.* Each of the two composites in (9.3.7) has the same object assignment as $GF$. It remains to show that the two composites have the same component morphisms.

For objects $x, y \in \mathsf{M}$, the $(x, y)$-component of $\widehat{GF}$ is, by definition (9.2.3), the unary multimorphism

(9.3.8)        $\widehat{GF}_{x,y} = \left(GF(\mathrm{ev}^{\mathsf{M}}_{x;y})\right)^{\#} : GF\underline{\mathsf{M}}(x;y) \longrightarrow \underline{\mathsf{P}}(GFx; GFy) \quad \text{in} \quad \mathsf{P}.$

As the partner of $GF(\mathrm{ev}^{\mathsf{M}}_{x;y})$, it is *uniquely* determined by the following commutative diagram in P, which is (9.2.11) applied to $GF$.

(9.3.9)

$$
\begin{array}{ccc}
\big(GF\underline{\mathsf{M}}(x;y),\,GFx\big) & \xrightarrow{\;(\widehat{GF}_{x,y},\,1_{GFx})\;} & \big(\underline{\mathsf{P}}(GFx;GFy),\,GFx\big) \\
& \searrow\raisebox{1ex}{$\scriptstyle GF(\mathrm{ev}^{\mathsf{M}}_{x;y})$} & \Big\downarrow\raisebox{0ex}{$\scriptstyle \mathrm{ev}^{\mathsf{P}}_{GFx;GFy}$} \\
& & GFy
\end{array}
$$

On the other hand, the $(x,y)$-component of $\widehat{G}\circ\widehat{F}_G$ is the following composite in P, with $G\widehat{F}_{x,y}$ as in (9.3.4).

(9.3.10)
$$
GF\underline{\mathsf{M}}(x;y) \xrightarrow{\;G\widehat{F}_{x,y}\;} G\underline{\mathsf{N}}(Fx;Fy) \xrightarrow{\;\widehat{G}_{Fx,Fy}\;} \underline{\mathsf{P}}(GFx;GFy)
$$

We want to show that (9.3.8) and (9.3.10) are equal. By uniqueness of partners, it suffices to show that the composite in (9.3.10) also makes the diagram (9.3.9) commutative.

To check this, we consider the following diagram in P, whose boundary is obtained from (9.3.9) by replacing $\widehat{GF}_{x,y}$ with the composite in (9.3.10).

(9.3.11)

$$
\begin{array}{ccc}
 & \big(G\underline{\mathsf{N}}(Fx;Fy),\,GFx\big) & \\
\raisebox{-1ex}{$\scriptstyle (G\widehat{F}_{x,y},\,1_{GFx})$}\nearrow & \Big\downarrow\raisebox{0ex}{$\scriptstyle G(\mathrm{ev}^{\mathsf{N}}_{Fx,Fy})$} & \searrow\raisebox{-1ex}{$\scriptstyle (\widehat{G}_{Fx,Fy},\,1_{GFx})$} \\
\big(GF\underline{\mathsf{M}}(x;y),\,GFx\big) & & \big(\underline{\mathsf{P}}(GFx;GFy),\,GFx\big) \\
\searrow\raisebox{-1ex}{$\scriptstyle GF(\mathrm{ev}^{\mathsf{M}}_{x;y})$} & GFy & \swarrow\raisebox{-1ex}{$\scriptstyle \mathrm{ev}^{\mathsf{P}}_{GFx;GFy}$}
\end{array}
$$

- The left sub-region in (9.3.11) is the commutative diagram (9.3.5).
- The right sub-region in (9.3.11) is the commutative diagram (9.2.11) for $\widehat{G}_{Fx,Fy}$.

Thus the composite in (9.3.10) also makes the diagram (9.3.9) commutative. This proves that the unary multimorphisms in (9.3.8) and (9.3.10) are equal. $\qquad\square$

**Example 9.3.12** (Monoidal Functors). Consider monoidal functors between symmetric monoidal closed categories

$$
\mathsf{V} \xrightarrow{\;T\;} \mathsf{W} \xrightarrow{\;U\;} \mathsf{X}.
$$

Passing to the endomorphism non-symmetric multifunctors (C.3.3)

$$
\mathsf{End}\,\mathsf{V} \xrightarrow{\;\mathsf{End}\,T\;} \mathsf{End}\,\mathsf{W} \xrightarrow{\;\mathsf{End}\,U\;} \mathsf{End}\,\mathsf{X},
$$

Theorem 9.3.6 yields the following commutative diagram of $(\mathsf{End}\,\mathsf{X})$-functors.

(9.3.13)

$$
\begin{array}{ccc}
(\mathsf{End}\,\mathsf{V})_{(\mathsf{End}\,U)(\mathsf{End}\,T)} & \xrightarrow{\;(\mathsf{End}\,\widehat{U})(\widehat{\mathsf{End}}\,T)\;} & \mathsf{End}\,\mathsf{X} \\
\big\| & & \Big\uparrow\raisebox{0ex}{$\scriptstyle \widehat{\mathsf{End}\,U}$} \\
\big((\mathsf{End}\,\mathsf{V})_{\mathsf{End}\,T}\big)_{\mathsf{End}\,U} & \xrightarrow{\;\widehat{\mathsf{End}\,T}_{\mathsf{End}\,U}\;} & (\mathsf{End}\,\mathsf{W})_{\mathsf{End}\,U}
\end{array}
$$

By Propositions 7.3.1, 9.2.17, and C.3.6, the commutative diagram (9.3.13) is equal to the following diagram of X-functors.

(9.3.14)

$$
\begin{array}{ccc}
\mathsf{V}_{UT} & \xrightarrow{\widehat{UT}} & \mathsf{X} \\
\Big\| & & \Big\uparrow{\widehat{U}} \\
(\mathsf{V}_T)_U & \xrightarrow{\widehat{T}_U} & \mathsf{W}_U
\end{array}
$$

The diagram (9.3.14) uses the canonical self-enrichment, change of enrichment, and standard enrichment in Theorem B.3.7 and Propositions B.4.6 and B.4.17 in the context of monoidal functors.                                    ◇

## 9.4. Factorization of $K$-Theory Standard Enrichment

In this section we illustrate Theorem 9.3.6 by applying it to Elmendorf-Mandell $K$-theory $\mathsf{K}^{\mathsf{EM}}$. The result is a factorization of the standard enrichment $\widehat{\mathsf{K}^{\mathsf{EM}}}$ into four spectrally enriched functors; see Theorem 9.4.2. For the relationship between the standard enrichment $\widehat{\mathsf{K}^{\mathsf{EM}}}$ and the work of Bohmann-Osorno [**BO15**], see Remark 9.4.4. Each Sp-functor in the factorization of $\widehat{\mathsf{K}^{\mathsf{EM}}}$ in Theorem 9.4.2 is either a standard enrichment functor (Theorem 9.2.12) or the change of enrichment of a standard enrichment functor. We explain these Sp-functors further in Explanations 9.4.5, 9.4.8, 9.4.9, 9.4.14, and 9.4.17.

**Context.** First recall from (2.5.8) that $\mathsf{K}^{\mathsf{EM}}$ factors into four multifunctors between closed multicategories as follows.

(9.4.1)

$$
\begin{array}{ccccccc}
\mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\hspace{3cm} \mathsf{K}^{\mathsf{EM}} \hspace{3cm}} & & & \mathsf{Sp} \\
{\scriptstyle \mathsf{End}_{\mathcal{M}\underline{1}}}\Big\downarrow & & & & \Big\uparrow{\scriptstyle \mathsf{K}^{\mathcal{G}}} \\
\mathsf{Mod}^{\mathcal{M}\underline{1}} & \xrightarrow{\mathsf{J}^{\mathcal{T}}} & \mathcal{G}_*\text{-}\mathsf{Cat} & \xrightarrow{\mathsf{Ner}_*} & \mathcal{G}_*\text{-}\mathsf{sSet}
\end{array}
$$

*Closed Multicategories*

- $\mathsf{PermCat}^{\mathsf{su}}$ is a closed multicategory by Theorem 8.4.15.
- Each of the other four multicategories in (9.4.1) is induced by a symmetric monoidal closed structure. See
  - Proposition 1.3.17 (7) for $\mathsf{Mod}^{\mathcal{M}\underline{1}}$,
  - (2.4.12) for $\mathcal{G}_*\text{-}\mathsf{Cat}$ and $\mathcal{G}_*\text{-}\mathsf{sSet}$, and
  - (2.5.2) for $\mathsf{Sp}$.
  By Proposition 8.1.16 and Example C.3.1, each of their endomorphism multicategories is a closed multicategory, which we denote by the same symbol.

*Multifunctors*

- $\mathsf{End}_{\mathcal{M}\underline{1}}$ is the Cat-multifunctor in Explanation 1.4.41.
- $\mathsf{J}^{\mathcal{T}}$ is the symmetric monoidal Cat-functor in (2.5.9).
- $\mathsf{Ner}_*$ is the symmetric monoidal sSet-functor in (2.5.11) induced by the nerve functor.
- $\mathsf{K}^{\mathcal{G}}$ is the symmetric monoidal sSet-functor in (2.5.12).

Each of the symmetric monoidal functors $\mathsf{J}^{\mathcal{T}}$, $\mathsf{Ner}_*$, and $\mathsf{K}^{\mathcal{G}}$ induces a multifunctor by the endomorphism construction (C.3.3).

**Theorem 9.4.2.** *The factorization (9.4.1) of multifunctors between closed multicategories*

$$\mathsf{K}^{\mathsf{EM}} = \mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}$$

*induces the following factorization of the standard enrichment* $\widehat{\mathsf{K}^{\mathsf{EM}}}$ *into four* Sp-*functors.*

(9.4.3)

$$
\begin{array}{ccc}
(\mathsf{PermCat}^{\mathsf{su}})_{\mathsf{K}^{\mathsf{EM}}} & \xrightarrow{\ \widehat{\mathsf{K}^{\mathsf{EM}}}\ } & \mathsf{Sp} \\
\left(\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}\right)_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} \Big\downarrow & & \Big\uparrow \widehat{\mathsf{K}^{\mathcal{G}}} \\
(\mathsf{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} & & (\mathcal{G}_*\text{-}\mathsf{sSet})_{\mathsf{K}^{\mathcal{G}}} \\
(\widehat{\mathsf{J}^{\mathcal{T}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} \searrow & & \nearrow (\widehat{\mathsf{Ner}_*})_{\mathsf{K}^{\mathcal{G}}} \\
& (\mathcal{G}_*\text{-}\mathsf{Cat})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} &
\end{array}
$$

*Proof.* The desired factorization of $\widehat{\mathsf{K}^{\mathsf{EM}}}$ in (9.4.3) is obtained by applying Theorem 9.3.6 three times to the factorization (9.4.1) of $\mathsf{K}^{\mathsf{EM}}$. More precisely, we compute as follows, where we denote some standard enrichment $\widehat{?}$ by $?^\wedge$ to improve readability.

$$
\begin{aligned}
\widehat{\mathsf{K}^{\mathsf{EM}}} &= \left(\mathsf{K}^{\mathcal{G}} \circ \mathsf{Ner}_* \circ \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}\right)^\wedge \\
&= \widehat{\mathsf{K}^{\mathcal{G}}} \circ \left(\mathsf{Ner}_* \circ \mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}\right)^\wedge_{\mathsf{K}^{\mathcal{G}}} \\
&= \widehat{\mathsf{K}^{\mathcal{G}}} \circ (\widehat{\mathsf{Ner}_*})_{\mathsf{K}^{\mathcal{G}}} \circ \left(\mathsf{J}^{\mathcal{T}} \circ \mathsf{End}_{\mathcal{M}\underline{1}}\right)^\wedge_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} \\
&= \widehat{\mathsf{K}^{\mathcal{G}}} \circ (\widehat{\mathsf{Ner}_*})_{\mathsf{K}^{\mathcal{G}}} \circ (\widehat{\mathsf{J}^{\mathcal{T}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} \circ (\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}
\end{aligned}
$$

For the last two equalities above, we also use the compositionality of change-of-enrichment 2-functors in Proposition 7.4.1. $\qquad\square$

We explain the Sp-functors in (9.4.3) in more detail after the following remark.

**Remark 9.4.4** (Work of Bohmann-Osorno). In [**BO15**, Theorem 6.2] there is a Sp-functor $\Phi$ that is categorically similar to the standard enrichment $\widehat{\mathsf{K}^{\mathsf{EM}}}$ in (9.4.3). The important difference is that, while $\widehat{\mathsf{K}^{\mathsf{EM}}}$ is constructed from Elmendorf-Mandell *K*-theory $\mathsf{K}^{\mathsf{EM}}$, the Bohmann-Osorno Sp-functor $\Phi$ is the standard enrichment of the *K*-theory non-symmetric multifunctor $\mathbb{K}$ in the Guillou-May Theorem 0.3.9 [**GM22, GMMO23**]. As far as the authors know, there is no known multiplicative comparison between $\mathsf{K}^{\mathsf{EM}}$ and $\mathbb{K}$. Thus we also do not know how $\widehat{\mathsf{K}^{\mathsf{EM}}}$ is related to $\Phi$. $\qquad\diamond$

The rest of this section explains the Sp-functors in (9.4.3) in more detail. We use the shortened notation

$$\mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}} \quad \text{and} \quad \underline{\mathsf{P}^{\mathsf{su}}} = \underline{\mathsf{PermCat}^{\mathsf{su}}}.$$

**Explanation 9.4.5** (The Sp-functor $\widehat{\mathsf{K}^{\mathsf{EM}}}$). Specifying to $\mathsf{K}^{\mathsf{EM}}$ (2.5.8), Theorem 9.2.12 says that the standard enrichment of $\mathsf{K}^{\mathsf{EM}}$ is the Sp-functor

(9.4.6) $$\widehat{\mathsf{K}^{\mathsf{EM}}} : (\mathsf{P}^{\mathsf{su}})_{\mathsf{K}^{\mathsf{EM}}} \longrightarrow \mathsf{Sp}.$$

Next we describe its object assignment and component morphisms.

*Object Assignment.* The standard enrichment $\widehat{\mathsf{K}^{\mathsf{EM}}}$ has the same object assignment as $\mathsf{K}^{\mathsf{EM}}$. In other words, it sends each small permutative category $\mathsf{C}$ to the connective symmetric spectrum $\mathsf{K}^{\mathsf{EM}}\mathsf{C}$.

*Components*. For each pair of small permutative categories $\mathsf{C}$ and $\mathsf{D}$, by definition (9.2.3) and (9.2.4), the component morphism

$$\widehat{\mathsf{K}^{\mathsf{EM}}}_{\mathsf{C},\mathsf{D}} : \mathsf{K}^{\mathsf{EM}}\underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D}) \longrightarrow \underline{\mathsf{Sp}}(\mathsf{K}^{\mathsf{EM}}\mathsf{C};\mathsf{K}^{\mathsf{EM}}\mathsf{D}) \quad \text{in} \quad \mathsf{Sp}$$

is the adjoint of the morphism

$$\mathsf{K}^{\mathsf{EM}}(\mathrm{ev}_{\mathsf{C};\mathsf{D}}) : \mathsf{K}^{\mathsf{EM}}\underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D}) \wedge \mathsf{K}^{\mathsf{EM}}\mathsf{C} \longrightarrow \mathsf{K}^{\mathsf{EM}}\mathsf{D}.$$

- $\underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D})$ is the small permutative category in Lemma 8.2.13. Since the domain has length 1, $\underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D})$ is equal to the hom object $\mathsf{P}^{\mathsf{su}}(\mathsf{C},\mathsf{D})$ in Definition 6.4.19.
- The evaluation is the bilinear functor (8.3.2)

$$\mathrm{ev}_{\mathsf{C};\mathsf{D}} : \underline{\mathsf{P}^{\mathsf{su}}}(\mathsf{C};\mathsf{D}) \times \mathsf{C} \longrightarrow \mathsf{D}.$$

  This is equal to the evaluation $\mathrm{ev}_{\mathsf{C},\mathsf{D}}$ in (6.5.2).
- $\wedge$ is the smash product of symmetric spectra [**JY∞**, 7.6.1].

By uniqueness of adjoints, $\widehat{\mathsf{K}^{\mathsf{EM}}}_{\mathsf{C},\mathsf{D}}$ is characterized by the following commutative diagram in $\mathsf{Sp}$.

(9.4.7)



This is the commutative diagram (9.2.11) for $F = \mathsf{K}^{\mathsf{EM}}$. ◇

**Explanation 9.4.8** (The Sp-Functor $\widehat{\mathsf{K}^{\mathcal{G}}}$)**.** In the diagram (9.4.3), the Sp-functor

$$\widehat{\mathsf{K}^{\mathcal{G}}} : (\mathcal{G}_*\text{-sSet})_{\mathsf{K}^{\mathcal{G}}} \longrightarrow \mathsf{Sp}$$

is the standard enrichment of the symmetric monoidal functor in (2.5.12),

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-sSet} \longrightarrow \mathsf{Sp}.$$

The standard enrichment Sp-functor $\widehat{\mathsf{K}^{\mathcal{G}}}$ exists by either Proposition B.4.17 or Theorem 9.2.12, which yield the same Sp-functor by Proposition 9.2.17. ◇

The other three constituent Sp-functors in (9.4.3) are the following.

$$(\widehat{\overline{\mathsf{End}_{\mathcal{M}\underline{1}}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} : (\mathsf{P}^{\mathsf{su}})_{\mathsf{K}^{\mathsf{EM}}} \longrightarrow (\mathsf{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$$

$$(\widehat{\mathsf{J}^{\mathcal{T}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} : (\mathsf{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} \longrightarrow (\mathcal{G}_*\text{-Cat})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*}$$

$$(\widehat{\mathsf{Ner}_*})_{\mathsf{K}^{\mathcal{G}}} : (\mathcal{G}_*\text{-Cat})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*} \longrightarrow (\mathcal{G}_*\text{-sSet})_{\mathsf{K}^{\mathcal{G}}}$$

Each of these three Sp-functors is obtained from the indicated standard enrichment $\widehat{?}$ by applying the change of enrichment in the subscript, as in Explanation 9.3.3. We describe them more explicitly in Explanations 9.4.9, 9.4.14, and 9.4.17 below.

**Explanation 9.4.9** (The Sp-Functor $(\widehat{\overline{\mathsf{End}_{\mathcal{M}\underline{1}}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$)**.** We obtain an explicit description of the Sp-functor

$$(\widehat{\overline{\mathsf{End}_{\mathcal{M}\underline{1}}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} : (\mathsf{P}^{\mathsf{su}})_{\mathsf{K}^{\mathsf{EM}}} \longrightarrow (\mathsf{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$$

in (9.4.3) by interpreting Explanation 9.3.3 with the multifunctors between closed multicategories

$$F = \mathsf{End}_{\mathcal{M}\underline{1}} : \mathsf{P^{su}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}} \quad \text{and}$$

$$G = \mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_* \,\mathsf{J}^{\mathcal{T}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathsf{Sp}$$

in Explanation 1.4.41 and (2.5.1), respectively. Note that $\mathsf{K^{EM}} = GF$ by definition (2.5.8).

*Object Assignment.* $(\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$ sends each small permutative category C to

$$(\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}(\mathsf{C}) = \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C} \quad \text{in} \quad \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

This is the endomorphism left $\mathcal{M}\underline{1}$-module in Example 1.3.15.

*Standard Enrichment of* $\mathsf{End}_{\mathcal{M}\underline{1}}$. For the component morphisms, we first consider the standard enrichment $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-functor of $\mathsf{End}_{\mathcal{M}\underline{1}}$ (Theorem 9.2.12),

(9.4.10) $$\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}} : (\mathsf{P^{su}})_{\mathsf{End}_{\mathcal{M}\underline{1}}} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

For small permutative categories C and D, the (C, D)-component of $\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}$ is the following morphism in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.

(9.4.11) $$\left(\mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{ev}_{\mathsf{C};\mathsf{D}})\right)^{\#} : \mathsf{End}_{\mathcal{M}\underline{1}}\underline{\mathsf{P^{su}}}(\mathsf{C};\mathsf{D}) \longrightarrow \mathsf{Hom}_*\left(\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C};\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{D}\right)$$

This is adjoint to the following morphism in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.

$$\mathsf{End}_{\mathcal{M}\underline{1}}(\mathsf{ev}_{\mathsf{C};\mathsf{D}}) : \mathsf{End}_{\mathcal{M}\underline{1}}\underline{\mathsf{P^{su}}}(\mathsf{C};\mathsf{D}) \wedge \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C} \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{D}$$

Here

$$\mathsf{ev}_{\mathsf{C};\mathsf{D}} : \underline{\mathsf{P^{su}}}(\mathsf{C};\mathsf{D}) \times \mathsf{C} \longrightarrow \mathsf{D}$$

is the evaluation bilinear functor in (8.3.2), which is the same as $\mathsf{ev}_{\mathsf{C},\mathsf{D}}$ in (6.5.2). The smash product, $\wedge$, and the internal hom, $\mathsf{Hom}_*$, are part of the symmetric monoidal closed structure of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ in Proposition 1.3.17 (7).

*Components of* $(\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$. The composite symmetric monoidal functor $\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}$ induces a change-of-enrichment 2-functor

(9.4.12) $$(-)_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} : \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-Cat} \longrightarrow \mathsf{Sp}\text{-Cat}.$$

This change of enrichment exists by either Proposition B.4.6 or Proposition 7.1.9, which yield the same 2-functor by Proposition 7.3.1.

Applying the change of enrichment (9.4.12) to the standard enrichment $\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}$ in (9.4.10) yields the Sp-functor in the upper left of the diagram (9.4.3):

(9.4.13)

$$(\mathsf{P^{su}})_{\mathsf{K^{EM}}} = \!\!= \left((\mathsf{P^{su}})_{\mathsf{End}_{\mathcal{M}\underline{1}}}\right)_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$$
$$(\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}} \Big\downarrow$$
$$(\mathsf{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{K}^{\mathcal{G}}\,\mathsf{Ner}_*\,\mathsf{J}^{\mathcal{T}}}$$

The equality of Sp-categories at the top of (9.4.13) follows from Proposition 7.4.1 applied to the factorization (9.4.1) of $\mathsf{K^{EM}}$. For small permutative categories C and

D, the $(C, D)$-component of $(\widehat{End_{\mathcal{M}\underline{1}}})_{K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}}}$ is the following morphism in Sp.

$$K^{EM} \underline{P}^{su}(C; D) \Longrightarrow K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}} End_{\mathcal{M}\underline{1}} \underline{P}^{su}(C; D)$$

$$K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}} \big( End_{\mathcal{M}\underline{1}}(ev_{C;D}) \big)^{\#} \Big\downarrow$$

$$K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}} Hom_* \big( End_{\mathcal{M}\underline{1}} C; End_{\mathcal{M}\underline{1}} D \big)$$

This is obtained from the morphism $\big( \widehat{End_{\mathcal{M}\underline{1}}} \big)_{C,D}$ in (9.4.11) by applying the functor $K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}}$. ◇

**Explanation 9.4.14** (The Sp-Functor $(\widehat{J^{\mathcal{T}}})_{K^{\mathcal{G}} \, Ner_*}$). The symmetric monoidal functor in (2.5.9),

$$J^{\mathcal{T}} : Mod^{\mathcal{M}\underline{1}} \longrightarrow \mathcal{G}_*\text{-Cat},$$

has a standard enrichment $(\mathcal{G}_*\text{-Cat})$-functor

$$\widehat{J^{\mathcal{T}}} : (Mod^{\mathcal{M}\underline{1}})_{J^{\mathcal{T}}} \longrightarrow \mathcal{G}_*\text{-Cat}$$

by Propositions 9.2.17 and B.4.17 and Theorem 9.2.12. The symmetric monoidal functor

$$K^{\mathcal{G}} \, Ner_* : \mathcal{G}_*\text{-Cat} \longrightarrow \mathcal{G}_*\text{-sSet} \longrightarrow Sp$$

induces a change-of-enrichment 2-functor

$$(-)_{K^{\mathcal{G}} \, Ner_*} : (\mathcal{G}_*\text{-Cat})\text{-Cat} \longrightarrow Sp\text{-Cat}$$

by Propositions 7.1.9, 7.3.1, and B.4.6. Applying this change of enrichment to the standard enrichment $\widehat{J^{\mathcal{T}}}$ yields the following Sp-functor in (9.4.3).

$$(Mod^{\mathcal{M}\underline{1}})_{K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}}} = \big( (Mod^{\mathcal{M}\underline{1}})_{J^{\mathcal{T}}} \big)_{K^{\mathcal{G}} \, Ner_*} \xrightarrow{(\widehat{J^{\mathcal{T}}})_{K^{\mathcal{G}} \, Ner_*}} (\mathcal{G}_*\text{-Cat})_{K^{\mathcal{G}} \, Ner_*}$$

On objects it sends each left $\mathcal{M}\underline{1}$-module N to

$$(\widehat{J^{\mathcal{T}}})_{K^{\mathcal{G}} \, Ner_*} (N) = J^{\mathcal{T}} N \quad \text{in} \quad \mathcal{G}_*\text{-Cat}.$$

For left $\mathcal{M}\underline{1}$-modules N and P, the $(N, P)$-component morphism in Sp

$$(9.4.15) \qquad K^{\mathcal{G}} \, Ner_* \, J^{\mathcal{T}} Hom_*(N, P) \xrightarrow{\big( (\widehat{J^{\mathcal{T}}})_{K^{\mathcal{G}} \, Ner_*} \big)_{N,P}} K^{\mathcal{G}} \, Ner_* \, Hom_{\mathcal{G}_*} \big( J^{\mathcal{T}} N, J^{\mathcal{T}} P \big)$$

is obtained by applying the functor $K^{\mathcal{G}} \, Ner_*$ to the adjoint—taken in the symmetric monoidal closed category $\big( \mathcal{G}_*\text{-Cat}, \wedge, Hom_{\mathcal{G}_*} \big)$ in (2.4.12)—of the following composite morphism.

$$(9.4.16) \qquad J^{\mathcal{T}} Hom_*(N, P) \wedge J^{\mathcal{T}} N \xrightarrow{(J^{\mathcal{T}})^2} J^{\mathcal{T}} \big( Hom_*(N, P) \wedge N \big) \xrightarrow{J^{\mathcal{T}}(ev)} J^{\mathcal{T}} P$$

- In (9.4.15) and (9.4.16), $\big( Mod^{\mathcal{M}\underline{1}}, \wedge, Hom_* \big)$ is the symmetric monoidal closed structure on $Mod^{\mathcal{M}\underline{1}}$ in Proposition 1.3.17 (7).
- $(J^{\mathcal{T}})^2$ is the monoidal constraint of the symmetric monoidal functor $J^{\mathcal{T}}$. See [**JY**∞, 10.3.11] for a detailed description.
- ev is the evaluation (B.3.2) in $(Mod^{\mathcal{M}\underline{1}}, \wedge, Hom_*)$.
- The first $\wedge$ in (9.4.16) and $Hom_{\mathcal{G}_*}$ in (9.4.15) are the pointed Day convolution and pointed hom for $\mathcal{G}_*$-categories in (2.4.14) and (2.4.15). ◇

**Explanation 9.4.17** (The Sp-Functor $(\widehat{\mathrm{Ner}_*})_{\mathsf{K}^{\mathcal{G}}}$). The symmetric monoidal functor in (2.5.11),

$$\mathrm{Ner}_* : \mathcal{G}_*\text{-Cat} \longrightarrow \mathcal{G}_*\text{-sSet},$$

is induced levelwise by the nerve functor, Ner. Its standard enrichment $(\mathcal{G}_*\text{-sSet})$-functor

$$\widehat{\mathrm{Ner}_*} : (\mathcal{G}_*\text{-Cat})_{\mathrm{Ner}_*} \longrightarrow \mathcal{G}_*\text{-sSet}$$

exists by Propositions 9.2.17 and B.4.17 and Theorem 9.2.12. The symmetric monoidal functor

$$\mathsf{K}^{\mathcal{G}} : \mathcal{G}_*\text{-sSet} \longrightarrow \mathsf{Sp}$$

in (2.5.12) induces a change-of-enrichment 2-functor

$$(-)_{\mathsf{K}^{\mathcal{G}}} : (\mathcal{G}_*\text{-sSet})\text{-Cat} \longrightarrow \mathsf{Sp}\text{-Cat}$$

by Propositions 7.1.9, 7.3.1, and B.4.6. Applying this change of enrichment to the standard enrichment $\widehat{\mathrm{Ner}_*}$ yields the following Sp-functor in (9.4.3).

$$(\mathcal{G}_*\text{-Cat})_{\mathsf{K}^{\mathcal{G}}\,\mathrm{Ner}_*} = \left((\mathcal{G}_*\text{-Cat})_{\mathrm{Ner}_*}\right)_{\mathsf{K}^{\mathcal{G}}} \xrightarrow{\ (\widehat{\mathrm{Ner}_*})_{\mathsf{K}^{\mathcal{G}}}\ } (\mathcal{G}_*\text{-sSet})_{\mathsf{K}^{\mathcal{G}}}$$

On objects it sends each $\mathcal{G}_*$-category $A$ to

$$(\widehat{\mathrm{Ner}_*})_{\mathsf{K}^{\mathcal{G}}}(A) = \mathrm{Ner}_* A \quad \text{in} \quad \mathcal{G}_*\text{-sSet}.$$

For $\mathcal{G}_*$-categories $A$ and $B$, the $(A, B)$-component morphism in Sp

$$(9.4.18) \qquad \mathsf{K}^{\mathcal{G}}\,\mathrm{Ner}_*\,\mathrm{Hom}_{\mathcal{G}_*}(A, B) \xrightarrow{\ \left((\widehat{\mathrm{Ner}_*})_{\mathsf{K}^{\mathcal{G}}}\right)_{A,B}\ } \mathsf{K}^{\mathcal{G}}\,\mathrm{Hom}_{\mathcal{G}_*}(\mathrm{Ner}_* A, \mathrm{Ner}_* B)$$

is obtained by applying the functor $\mathsf{K}^{\mathcal{G}}$ to the adjoint—taken in the symmetric monoidal closed category $(\mathcal{G}_*\text{-sSet}, \wedge, \mathrm{Hom}_{\mathcal{G}_*})$ in (2.4.12)—of the following composite morphism.

$$(9.4.19) \qquad \mathrm{Ner}_*\,\mathrm{Hom}_{\mathcal{G}_*}(A, B) \wedge \mathrm{Ner}_* A \xrightarrow{\ \mathrm{Ner}_*^2\ } \mathrm{Ner}_*\left(\mathrm{Hom}_{\mathcal{G}_*}(A, B) \wedge A\right) \xrightarrow{\ \mathrm{Ner}_*(\mathrm{ev})\ } \mathrm{Ner}_* B$$

- In (9.4.18) $\mathrm{Hom}_{\mathcal{G}_*}$ in the domain is the pointed hom in $\mathcal{G}_*$-Cat (2.4.15). The $\mathrm{Hom}_{\mathcal{G}_*}$ in the codomain is the one in $\mathcal{G}_*$-sSet.
- In (9.4.19) $\mathrm{Ner}_*^2$ is the monoidal constraint of the symmetric monoidal functor $\mathrm{Ner}_*$. See [**JY**∞, 3.8.4] for a detailed description.
- ev is the evaluation (B.3.2) in $(\mathcal{G}_*\text{-Cat}, \wedge, \mathrm{Hom}_{\mathcal{G}_*})$. ◇

# Enriched Diagrams and Mackey Functors of Closed Multicategories

This chapter studies categories of enriched functors

$$\mathsf{M\text{-}Cat}(\mathsf{C},\mathsf{M}) \quad \text{and} \quad \mathsf{M\text{-}Cat}(\mathsf{C}^{\mathsf{op}},\mathsf{M}),$$

where C is an M-category. At left, M is assumed to be a non-symmetric closed multicategory. At right, M is assumed to have the additional symmetric structure of a closed multicategory, so that $\mathsf{C}^{\mathsf{op}}$ is defined (Definition 6.6.1).

For a non-symmetric multifunctor between non-symmetric closed multicategories

$$F : \mathsf{M} \longrightarrow \mathsf{N},$$

the change-of-enrichment 2-functors from Proposition 7.1.9 induce diagram change of enrichment (Theorem 10.3.1)

$$F_\star : \mathsf{M\text{-}Cat}(\mathsf{C},\mathsf{M}) \longrightarrow \mathsf{N\text{-}Cat}\big(\mathsf{C}_F,\mathsf{N}\big).$$

When $F$ is a multifunctor between closed multicategories, there is a presheaf change of enrichment (Proposition 7.2.1 and Theorem 10.3.4)

$$F_\star : \mathsf{M\text{-}Cat}(\mathsf{C}^{\mathsf{op}},\mathsf{M}) \longrightarrow \mathsf{N\text{-}Cat}\big((\mathsf{C}_F)^{\mathsf{op}},\mathsf{N}\big).$$

These results apply to the following diagram of $K$-theory multifunctors from (2.5.1); see Example 10.3.5.

(10.0.1)



Each arrow in this diagram is a multifunctor between closed multicategories.

The compositionality of change-of-diagram and change-of-presheaf involves that of both the change of enrichment (Section 7.4) and the standard enrichment (Section 9.3). The resulting theory is explained in Section 10.4.

The main application in this chapter is Theorem 10.6.2, which shows that the factorization of $\mathsf{K}^{\mathsf{EM}}$ induces the following factorization of the diagram change-of-enrichment functor $\mathsf{K}^{\mathsf{EM}}_\star$, where $\mathsf{C}$ is a small $\mathsf{PermCat}^{\mathsf{su}}$-category.

$$(10.0.2)$$

$$
\begin{array}{ccc}
\mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big) & \xrightarrow{\ \mathsf{K}^{\mathsf{EM}}_\star\ } & \mathsf{Sp}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{K}^{\mathsf{EM}}}, \mathsf{Sp}\big) \\
{\scriptstyle (\mathsf{End}_{\mathcal{M}_1})_\star} \downarrow & & \uparrow {\scriptstyle \mathsf{K}^{\mathcal{G}}_\star} \\
\mathsf{Mod}^{\mathcal{M}_1}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}_1}}, \mathsf{Mod}^{\mathcal{M}_1}\big) & \quad (\mathcal{G}_*\text{-}\mathsf{sSet})\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{Ner}_* \mathsf{J}^{\mathcal{T}}\mathsf{End}_{\mathcal{M}_1}}, \mathcal{G}_*\text{-}\mathsf{sSet}\big) \\
& {\scriptstyle \mathsf{J}^{\mathcal{T}}_\star} \searrow \qquad\qquad \nearrow {\scriptstyle (\mathsf{Ner}_*)_\star} & \\
& (\mathcal{G}_*\text{-}\mathsf{Cat})\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{J}^{\mathcal{T}}\mathsf{End}_{\mathcal{M}_1}}, \mathcal{G}_*\text{-}\mathsf{Cat}\big) &
\end{array}
$$

A similar factorization holds with $\mathsf{C}$ and each $\mathsf{C}_?$ above replaced by $\mathsf{C}^{\mathsf{op}}$ and $(\mathsf{C}_?)^{\mathsf{op}}$, respectively.

**Connection with Other Chapters.** Chapter 11 develops homotopical properties of the enriched diagram and Mackey functor change-of-enrichment functors $F_\star$. Chapter 12 then gives the corresponding applications to *K*-theory multifunctors.

**Background.** The material in this chapter is a culmination of the multifunctorial enrichment and change of enrichment from Chapters 6 through 9. In particular, the applications to *K*-theory in Sections 10.5 and 10.6 depend on the descriptions in Section 9.4.

**Chapter Summary.** Section 10.1 gives the basic definitions of enriched diagrams and enriched Mackey functors. Section 10.2 defines change-of-enrichment functors for enriched diagrams. Section 10.3 contains the proofs that these are functorial, along with key *K*-theoretic applications. In Section 10.4 we show that the diagram change-of-enrichment construction respects composition. Section 10.5 applies the general results from Section 10.3 to the Elmendorf-Mandell *K*-theory functor $\mathsf{K}^{\mathsf{EM}}$. Section 10.6 applies the general results from Section 10.4 to factor the change of enrichment given by $\mathsf{K}^{\mathsf{EM}}$. Here is a summary table.

| | |
|---|---|
| enriched diagrams and enriched Mackey functors | 10.1.1 |
| diagram change of enrichment definition | 10.2.1, 10.2.5, and 10.2.13 |
| diagram change of enrichment functoriality | 10.3.1 and 10.3.4 |
| diagram change of enrichment compositions | 10.4.1 and 10.4.5 |
| application to Elmendorf-Mandell *K*-theory | 10.5.1, 10.5.8, and 10.5.11 |
| factorization of Elmendorf-Mandell change of enrichment | 10.6.2 and 10.6.5 |

## 10.1. Enriched Diagrams and Mackey Functors as Modules

In this section we introduce categories of enriched diagrams and enriched presheaves, also called Mackey functors, with respect to a (non-symmetric) closed multicategory.

- Diagrams and presheaves enriched in a (non-symmetric) closed multicategory are in Definition 10.1.1.

- We characterize the objects, morphisms, and composition in these categories in terms of partners in Propositions 10.1.8, 10.1.17, and 10.1.22. The upshot is that we may consider these categories as categories of modules over an enriched category.
- In Examples 10.1.26 and 10.1.27 we discuss enriched presheaves in the work of
  - **[SS03]** about stable model categories and
  - **[GM22]** about genuine equivariant spectra.

We discuss change of enrichment of enriched diagram and presheaf categories in the remaining sections of this chapter.

### Defining Enriched Diagrams and Mackey Functors.

**Definition 10.1.1.** Suppose $(M, \underline{M}, \mathrm{ev})$ is a non-symmetric closed multicategory (Definition 8.1.1). We also regard M as an M-category with the canonical self-enrichment $(M, \circ, i)$ (Theorem 9.1.7). Suppose $(C, m, i)$ is an M-category (Definition 6.1.1).

- The category

$$(10.1.2) \qquad\qquad \mathsf{M\text{-}Cat}(C, M)$$

  is called the C-*diagram category* of M. An object in this category is called a C-*diagram* enriched in M.
- Suppose, in addition, M is a closed multicategory, and $C^{\mathsf{op}}$ is the opposite M-category (Proposition 6.6.7). The $C^{\mathsf{op}}$-diagram category

$$(10.1.3) \qquad\qquad \mathsf{M\text{-}Cat}(C^{\mathsf{op}}, M)$$

  is also called the C-*presheaf category* of M and the C-*Mackey functor category* of M. An object in this category is also called a C-*presheaf* and a C-*Mackey functor* enriched in M.

This finishes the definition.                                                    ◇

Using Propositions 6.6.7 and 7.2.1, the discussion below about C-diagrams also applies to C-Mackey functors.

**Explanation 10.1.4** (Size). To define the category $\mathsf{M\text{-}Cat}(C, M)$, technically we need C and M to be small. We can deal with this issue in one of two ways.

(1) As in Convention A.1.2, if necessary we can move to a larger universe where C and M are small.
(2) We observe that all of our proofs and assertions are about M-functors $C \longrightarrow M$ and M-natural transformations between them. These notions are defined without assuming that C and M are small. We refer to the category $\mathsf{M\text{-}Cat}(C, M)$ simply because it provides a convenient context to phrase functorial and naturality properties. Thus, C and M do not need to be small.                                                    ◇

**Diagrams as Modules.** Recall that in a (non-symmetric) closed multicategory, taking partner, denoted $f \longmapsto f^{\#}$, is a bijection (8.1.7). In the rest of this section, we characterize the objects, morphisms, and composition in the category $\mathsf{M\text{-}Cat}(C, M)$ in terms of partners.

**Explanation 10.1.5** (Unpacking C-Diagrams). In (10.1.2) M-Cat(C, M) is a hom category in the 2-category M-Cat (Theorem 6.1.27). An object in M-Cat(C, M) is an M-functor $A : C \longrightarrow M$ (Definition 6.1.7). Such an M-functor has an object assignment

$$(10.1.6) \qquad\qquad A : \mathrm{Ob}\, C \longrightarrow \mathrm{Ob}\, M.$$

For objects $x, y \in C$, its $(x, y)$-component is a unary multimorphism

$$A_{x,y} : C(x, y) \longrightarrow \underline{M}(Ax; Ay) \quad \text{in} \quad M.$$

Its partner (8.1.7) is a binary multimorphism

$$(10.1.7) \qquad\qquad A^{\#}_{x,y} : \big(C(x, y), Ax\big) \longrightarrow Ay \quad \text{in} \quad M.$$

In Proposition 10.1.8 below, we interpret the M-functor axioms (6.1.9) for $A$ in terms of these componentwise partners. $\qquad\qquad\diamond$

The following observation allows us to regard a C-diagram in M as a left C-module.

**Proposition 10.1.8.** *In the context of Definition 10.1.1, an* M*-functor* $A : C \longrightarrow M$ *is uniquely determined by*

- *an object assignment as in (10.1.6) and*
- *component binary multimorphisms* $\big\{A^{\#}_{x,y}\big\}_{x,y\in C}$ *as in (10.1.7)*

*such that the following two diagrams in* M *commute for all objects* $x, y, z \in C$, *with* $C(x, y)$ *abbreviated to* $C_{x,y}$.

$$(10.1.9)$$



*Proof.* Since taking partner is a bijection (8.1.7), it suffices to show that the diagrams in (10.1.9) are the partners of the diagrams in (6.1.9), which are the axioms for an M-functor $C \longrightarrow M$. We abbreviate $\underline{M}(?; ?)$ and $\mathrm{ev}^M$ to $\underline{M}_{?;?}$ and ev, respectively. For objects $x, y \in C$, the component $A_{x,y}$ and its partner $A^{\#}_{x,y}$ determine each other via the following commutative diagram in M.

$$(10.1.10)$$

To see that the left diagram in (6.1.9) yields the left diagram in (10.1.9) upon taking partners, we consider the following diagram in M.

(10.1.11)



The sub-region labeled ☆, when composed with the lower right ev, yields the partners (8.1.7) of the two composites in the left diagram in (6.1.9) for $A : \mathsf{C} \longrightarrow \mathsf{M}$. The boundary of (10.1.11) is the left diagram in (10.1.9). Thus it suffices to show that the other four sub-regions in (10.1.11) commute.

- The middle diamond region commutes by the definition of ∘ in the canonical self-enrichment of M (9.1.6).
- The other three sub-regions commute by (10.1.10).

To see that the right diagram in (6.1.9) yields the right diagram in (10.1.9) upon taking partners, we consider the following diagram in M.

(10.1.12)



The top triangle, when composed with ev, yields the partners of the two composites in the right diagram in (6.1.9) for $A : \mathsf{C} \longrightarrow \mathsf{M}$. The boundary of (10.1.12) is the right diagram in (10.1.9). Thus it suffices to show that the other two sub-regions in (10.1.12) commute.

- The left sub-region commutes by the definition of $i_{Ax}$ in the canonical self-enrichment of M (9.1.6).
- The right sub-region commutes by (10.1.10).

This finishes the proof.                                                                 □

**Mackey Functors as Modules.** Recall that each category C enriched in a multicategory has an opposite $(\mathsf{C}^{\mathsf{op}}, \mathsf{m}^{\mathsf{op}}, i)$ (Proposition 6.6.7). The following observation is Proposition 10.1.8 applied to $\mathsf{C}^{\mathsf{op}}$.

**Proposition 10.1.13.** *In the context of (10.1.3), an M-functor $A : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{M}$ is uniquely determined by*

- *an object assignment $A : \mathrm{Ob}\,\mathsf{C} \longrightarrow \mathrm{Ob}\,\mathsf{M}$ and*

- *for each pair of objects $x, y \in \mathsf{C}$, a component binary multimorphism*

$$A_{x,y}^{\#} : \left(\mathsf{C}_{y,x}, Ax\right) \longrightarrow Ay \quad in \quad \mathsf{M}$$

*such that the following two diagrams in $\mathsf{M}$ commute for all objects $x, y, z \in \mathsf{C}$, with $\tau \in \Sigma_2$ the nonidentity permutation.*

(10.1.14)

$$
\begin{array}{ccc}
& \left(\mathsf{C}_{y,x}, \mathsf{C}_{z,y}, Ax\right) & \\
{\scriptstyle(\tau, 1)}\nearrow & & \searrow{\scriptstyle(\mathsf{m}, 1)} \\
\left(\mathsf{C}_{z,y}, \mathsf{C}_{y,x}, Ax\right) & & \left(\mathsf{C}_{z,x}, Ax\right) \\
{\scriptstyle(1, A_{x,y}^{\#})}\Big\downarrow & & \Big\downarrow{\scriptstyle A_{x,z}^{\#}} \\
\left(\mathsf{C}_{z,y}, Ay\right) & \xrightarrow{\ A_{y,z}^{\#}\ } & Az
\end{array}
\qquad
\begin{array}{ccc}
\left(\langle\rangle, Ax\right) & \xrightarrow{\ (i_x, 1)\ } & \left(\mathsf{C}_{x,x}, Ax\right) \\
\Big\| & & \Big\downarrow{\scriptstyle A_{x,x}^{\#}} \\
Ax & \xrightarrow{\ \ 1\ \ } & Ax
\end{array}
$$

### Diagram Morphisms as Module Morphisms.

**Explanation 10.1.15.** A morphism in $\mathsf{M}\text{-Cat}(\mathsf{C}, \mathsf{M})$ is an $\mathsf{M}$-natural transformation between $\mathsf{M}$-functors (Definition 6.1.14) as follows.

$$
\mathsf{C} \underset{B}{\overset{A}{\Longrightarrow\!\!\!\theta\!\!\!\Longrightarrow}} \mathsf{M}
$$

Such an $\mathsf{M}$-natural transformation consists of, for each object $x$ in $\mathsf{C}$, a component nullary multimorphism

$$\theta_x : \langle\rangle \longrightarrow \underline{\mathsf{M}}(Ax; Bx) \quad in \quad \mathsf{M}$$

that satisfies the naturality axiom (6.1.16).

- Identity morphisms in $\mathsf{M}\text{-Cat}(\mathsf{C}, \mathsf{M})$ are identity $\mathsf{M}$-natural transformations (6.1.17), where each component $\theta_x$ is the identity $i_{Ax}$ (9.1.2).
- Composition is given by vertical composition of $\mathsf{M}$-natural transformations (Definition 6.1.18). We characterize composition in Proposition 10.1.22 below.

The partner (8.1.7) of $\theta_x$ is a unary multimorphism

(10.1.16)                     $$\theta_x^{\#} : Ax \longrightarrow Bx \quad in \quad \mathsf{M}.$$

In Proposition 10.1.17 below, we interpret the naturality axiom (6.1.16) for $\theta$ in terms of these componentwise partners and those of $A$ and $B$ in (10.1.7).          ⋄

Proposition 10.1.8 above interprets an object in $\mathsf{M}\text{-Cat}(\mathsf{C}, \mathsf{M})$ as a left $\mathsf{C}$-module. The following observation interprets a morphism in $\mathsf{M}\text{-Cat}(\mathsf{C}, \mathsf{M})$ as a morphism of left $\mathsf{C}$-modules.

**Proposition 10.1.17.** *In the context of Definition 10.1.1, an $\mathsf{M}$-natural transformation $\theta : A \longrightarrow B$ is uniquely determined by component unary multimorphisms as in (10.1.16)*

$$\left\{\theta_x^{\#} : Ax \longrightarrow Bx\right\}_{x \in \mathsf{C}}$$

*such that the following diagram in* $\mathsf{M}$ *commutes for all objects* $x, y \in \mathsf{C}$, *with* $A_{x,y}^{\#}$ *and* $B_{x,y}^{\#}$ *as in (10.1.7).*

(10.1.18)

$$
\begin{array}{ccc}
\big(\mathsf{C}(x,y)\,,\, Ax\big) & \xrightarrow{\quad A_{x,y}^{\#} \quad} & Ay \\
{\scriptstyle (1\,,\, \theta_x^{\#})}\big\downarrow & & \big\downarrow{\scriptstyle \theta_y^{\#}} \\
\big(\mathsf{C}(x,y)\,,\, Bx\big) & \xrightarrow{\quad B_{x,y}^{\#} \quad} & By
\end{array}
$$

*Proof.* Since taking partner is a bijection (8.1.7), it suffices to show that the naturality diagram (6.1.16) for $\theta : A \longrightarrow B$ yields the diagram (10.1.18) upon taking partners. We use the same abbreviations as in the proof of Proposition 10.1.8, so

(10.1.19) $\qquad \mathsf{C}(?,?) = \mathsf{C}_{?,?}, \quad \underline{\mathsf{M}}(?;?) = \underline{\mathsf{M}}_{?;?}, \quad \text{and} \quad \mathsf{ev} = \mathsf{ev}^{\mathsf{M}}.$

For each object $x$ in $\mathsf{C}$, the $x$-component $\theta_x$ and its partner $\theta_x^{\#}$ determine each other via the following commutative diagram in $\mathsf{M}$.

(10.1.20)

$$
\begin{array}{ccc}
\big(\langle\rangle\,,\, Ax\big) & \xrightarrow{\quad\theta_x^{\#}\quad} & \\
{\scriptstyle (\theta_x\,,\, 1)}\big\downarrow & & \\
\big(\underline{\mathsf{M}}_{Ax;Bx}\,,\, Ax\big) & \xrightarrow{\quad\mathsf{ev}\quad} & Bx
\end{array}
$$

The boundary of the following diagram in $\mathsf{M}$ is (10.1.18).

(10.1.21)



The sub-region labeled ☆, when composed with the lower middle $\mathsf{ev}$, yields the partners (8.1.7) of the two composites in the naturality diagram (6.1.16) for $\theta : A \longrightarrow B$. Thus it suffices to show that the other sub-regions in (10.1.21) are commutative.

- The top left and bottom right sub-regions commute by (10.1.20).
- The top right and bottom left sub-regions commute by (10.1.10).
- The two remaining sub-regions commute by the definition of $\circ$ in the canonical self-enrichment of $\mathsf{M}$ (9.1.6).

This finishes the proof. $\qquad\qquad\square$

**Partner Characterization of Composition.** Proposition 10.1.22 below characterizes vertical composition of $\mathsf{M}$-natural transformations (Definition 6.1.18) in the category $\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{M})$ in terms of partners. It says that, at each component, composition commutes with taking partners.

**Proposition 10.1.22.** *In the context of Definition 10.1.1, suppose $\theta$ and $\psi$ are vertically composable* M*-natural transformations as follows.*

$$\begin{array}{c} F \\ \mathsf{C} \xrightarrow[H]{\begin{array}{c} \Downarrow\theta \\ \Downarrow\psi \end{array}} \mathsf{M} \\ \end{array}$$

*Then for each object $x$ in* C, *there is an equality of unary multimorphisms*

$$(10.1.23) \qquad (\psi\theta)_x^{\#} = \gamma^{\mathsf{M}}\big(\psi_x^{\#};\theta_x^{\#}\big) : Fx \longrightarrow Hx \quad in \quad \mathsf{M}.$$

*Proof.* We use the abbreviations in (10.1.19). By definition (6.1.20) the $x$-component of $\psi\theta$ is the following composite nullary multimorphism in M.

$$(10.1.24) \qquad \overbrace{\langle\rangle = \big(\langle\rangle,\langle\rangle\big) \xrightarrow{(\psi_x,\,\theta_x)} \big(\underline{\mathsf{M}}_{Gx;Hx},\underline{\mathsf{M}}_{Fx;Gx}\big) \xrightarrow{\circ} \underline{\mathsf{M}}_{Fx;Hx}}^{(\psi\theta)_x}$$

Its partner (10.1.20) is the left-bottom composite in the diagram (10.1.25) in M below. The right-hand side of the desired equality (10.1.23) is the top-right composite in (10.1.25).

$$(10.1.25)$$

$$\begin{array}{ccc}
\big(\langle\rangle,\langle\rangle,Fx\big) = Fx & \xrightarrow{\theta_x^{\#}} & Gx = \big(\langle\rangle,Gx\big) \\
{\scriptstyle(\psi_x,\theta_x,1)}\downarrow & & \downarrow{\scriptstyle(\psi_x,1)} \quad \psi_x^{\#} \\
\big(\underline{\mathsf{M}}_{Gx;Hx},\underline{\mathsf{M}}_{Fx;Gx},Fx\big) & \xrightarrow{(1,\,\mathsf{ev})} & \big(\underline{\mathsf{M}}_{Gx;Hx},Gx\big) \\
{\scriptstyle(\circ,1)}\downarrow & & \downarrow{\scriptstyle\mathsf{ev}} \\
\big(\underline{\mathsf{M}}_{Fx;Hx},Fx\big) & \xrightarrow{\mathsf{ev}} & Hx
\end{array}$$

The three sub-regions in (10.1.25) are commutative for the following reasons.

- The bottom left sub-region commutes by the definition of $\circ$ in (9.1.6).
- The top left sub-region commutes by the definition of $\theta_x^{\#}$ (10.1.20).
- The right sub-region commutes by the definition of $\psi_x^{\#}$ (10.1.20).

The commutative diagram (10.1.25) proves the desired equality (10.1.23). $\qquad\square$

### Examples of Mackey Functor Categories.

**Example 10.1.26** (Stable Model Categories)**.** Suppose M is a simplicial, cofibrantly generated, proper, and stable model category (Definition 0.4.1). The Schwede-Shipley Characterization Theorem 0.4.3 shows that, if $P$ is a set of compact generators of M, then there is a chain of simplicial Quillen equivalences

$$\mathsf{M} \simeq_Q \mathsf{Sp}\text{-}\mathsf{Cat}\big(\mathcal{E}(P)^{\mathsf{op}},\mathsf{Sp}\big)$$

between M and the $\mathcal{E}(P)$-presheaf category of Sp in the sense of (10.1.3). On the right-hand side, $\mathcal{E}(P)$ is the spectral endomorphism category (Definition 0.4.2) and $\mathcal{E}(P)^{\mathsf{op}}$ is its opposite Sp-category as in Proposition 6.6.7. $\qquad\diamond$

**Example 10.1.27** (Genuine Equivariant Spectra). Recall from Definition 0.3.5 the Burnside 2-category $G\mathcal{E}$ for a finite group $G$. The Guillou-May Theorem 0.3.9 gives a zigzag of Quillen equivalences

$$(10.1.28) \qquad G\text{-}\mathsf{Sp} \simeq_Q \mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}, \mathsf{Sp}\big)$$

between the category of genuine equivariant $G$-spectra, $G$-$\mathsf{Sp}$, and the category of spectral Mackey functors for $\mathbb{K}$ (Definition 0.3.8). As in Remark 9.4.4, $\mathbb{K}$ denotes the $K$-theory non-symmetric multifunctor in **[GM22, GMMO23]**.

We emphasize that in (10.1.28) the opposite in $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$ is taken in $\mathsf{Sp}$-$\mathsf{Cat}$ (Proposition 6.6.7) *after* the change of enrichment along $\mathbb{K}$. In Remark 10.5.5 we further discuss the relationship between

- the Guillou-May Quillen equivalence (10.1.28),
- the work of Bohmann-Osorno **[BO15]**, and
- our diagram change-of-enrichment functor in Theorem 10.3.1.

There, we note and discuss the nontrivial distinction between the $\mathsf{Sp}$-categories $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$ and $(G\mathcal{E}^{\mathsf{op}})_{\mathbb{K}}$. $\diamond$

## 10.2. Change of Enrichment of Enriched Diagrams and Mackey Functors

In this section we construct change-of-enrichment functors on enriched diagram and Mackey functor categories associated to (non-symmetric) multifunctors between (non-symmetric) closed multicategories (Definitions 8.1.1, 10.1.1, and C.1.19).

- The change-of-enrichment construction is in Definition 10.2.1.
- Explanations 10.2.5 and 10.2.13 unpack the change-of-enrichment construction on objects and morphisms.

We defer the proof that change of enrichment is a functor to Section 10.3; see Theorems 10.3.1 and 10.3.4. In Section 10.4 we show that these change-of-enrichment functors are compatible with composition of (non-symmetric) multifunctors.

**Defining $F_\star$.**

**Definition 10.2.1.** Suppose given a non-symmetric multifunctor between non-symmetric closed multicategories

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

and a small M-category C (Definition 6.1.1). We define the data of a functor

$$(10.2.2) \qquad F_\star : \mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{M}) \longrightarrow \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}_F, \mathsf{N}),$$

called the *diagram change of enrichment* of $F$ at C, as follows.

**Domain:** The domain of $F_\star$ is the C-diagram category of M in (10.1.2).

**Codomain:** $\mathsf{C}_F$ is the N-category obtained from C by applying the change-of-enrichment 2-functor in Proposition 7.1.9

$$(-)_F : \mathsf{M}\text{-}\mathsf{Cat} \longrightarrow \mathsf{N}\text{-}\mathsf{Cat}.$$

The codomain of $F_\star$ is the $\mathsf{C}_F$-diagram category of N in (10.1.2).

**Object and Morphism Assignments:** Suppose given

- M-functors $A$ and $B$ and
- an M-natural transformation $\psi : A \longrightarrow B$ in M-$\mathsf{Cat}(\mathsf{C}, \mathsf{M})$.

as in the left diagram below.

(10.2.3)

$$C \underset{B}{\overset{A}{\Downarrow\psi}} M \quad\overset{F_\star}{\longmapsto}\quad C_F \underset{B_F}{\overset{A_F}{\Downarrow\psi_F}} M_F \xrightarrow{\widehat{F}} N$$

Then $F_\star$ sends $A$, $B$, and $\psi$ to the composites and whiskering as in the right diagram in (10.2.3), with $\widehat{F}$ the standard enrichment of $F$ in Theorem 9.2.12.

This finishes the definition of $F_\star$. We also call $F_\star$ a *diagram change-of-enrichment functor*.

Moreover, we define the following.

- If we want to emphasize C, then we write $F_\star^C$ instead of $F_\star$.
- Suppose $F$ is a multifunctor between closed multicategories and $C^{op}$ is the opposite M-category of C (Proposition 6.6.7). Using Proposition 7.2.1 to identify the N-categories $(C^{op})_F$ and $(C_F)^{op}$, we call

(10.2.4)              $F_\star : \mathsf{M}\text{-}\mathsf{Cat}(C^{op}, M) \longrightarrow \mathsf{N}\text{-}\mathsf{Cat}\big((C_F)^{op}, N\big)$

the *presheaf change of enrichment* of $F$. We also call this $F_\star$ a *presheaf change-of-enrichment functor*.

This finishes the definition. Theorems 10.3.1 and 10.3.4 prove that $F_\star$ in (10.2.2) and (10.2.4) are functors.                                                                                    ◇

The assignment $F_\star$ in (10.2.3) is

- the change of enrichment $(-)_F$ (Proposition 7.1.9) followed by
- the standard enrichment $\widehat{F}$ (Theorem 9.2.12).

We describe $F_\star$ in more detail in Explanations 10.2.5 and 10.2.13 below.

**Unpacking $F_\star$.**

**Explanation 10.2.5** ($F_\star$ on Objects). For an M-functor $A : C \longrightarrow M$, in (10.2.3) the N-functor

(10.2.6)                    $F_\star A : C_F \xrightarrow{A_F} M_F \xrightarrow{\widehat{F}} N$

is the composite of

- $A_F$, which is the image of $A$ under the change of enrichment $(-)_F$, and
- the standard enrichment $\widehat{F}$ of $F$ in Theorem 9.2.12.

*Object Assignment.* More explicitly, the N-functor $A_F$ has the same object assignment as $A : C \longrightarrow M$. The standard enrichment $\widehat{F}$ has the same object assignment as $F$. Thus the object assignment of the N-functor $F_\star A$ is given by

(10.2.7)                    $(F_\star A)(x) = F\big(A(x)\big) \in N \quad \text{for} \quad x \in C.$

*Components.* To describe $F_\star A$ on hom objects, suppose given a pair of objects $x, y \in C$. The component $(F_\star A)_{x,y}$ is the following composite in N of unary multimorphisms.

(10.2.8)      $$FC(x,y) \xrightarrow{\ \ FA_{x,y}\ \ } F\underline{M}\big(Ax\,;\,Ay\big) \xrightarrow{\ \ \widehat{F}_{Ax,Ay}\ \ } \underline{N}\big(FAx\,;\,FAy\big)$$

with the overbrace labeled $(F_\star A)_{x,y}$.

The two constituent arrows in (10.2.8) are as follows.

- The unary multimorphism

(10.2.9) $$A_{x,y} : \mathsf{C}(x,y) \longrightarrow \underline{\mathsf{M}}(Ax\,;\,Ay) \quad \text{in} \quad \mathsf{M}$$

is the $(x,y)$-component of the M-functor $A$ (Definition 6.1.7). The left arrow in (10.2.8) is the image of $A_{x,y}$ under $F$, which is a unary multimorphism in N.
- The right arrow in (10.2.8) is the $(Ax, Ay)$-component unary multimorphism of the standard enrichment $\widehat{F}$ in (9.2.3). Its construction uses the closed structure on both M and N.

*Partner Characterization.* We can characterize $(F_\star A)_{x,y}$ in terms of its partner (8.1.7), using the following commutative diagram in N.

(10.2.10)

$$
\begin{array}{c}
\big((F_\star A)_{x,y}, 1_{FAx}\big) \\
\big(FA_{x,y}, 1_{FAx}\big) \\
\big(\widehat{F}_{Ax,Ay}, 1_{FAx}\big)
\end{array}
\quad
\begin{array}{c}
\big(F\mathsf{C}(x,y), FAx\big) \xrightarrow{\quad F(A^{\#}_{x,y}) \quad} \\
\big(F\underline{\mathsf{M}}(Ax\,;\,Ay), FAx\big) \xrightarrow{F(\mathrm{ev}^{\mathsf{M}}_{Ax;Ay})} FAy \\
\big(\underline{\mathsf{N}}(FAx\,;\,FAy), FAx\big) \xrightarrow{\quad \mathrm{ev}^{\mathsf{N}}_{FAx;FAy}}
\end{array}
$$

The diagram (10.2.10) commutes for the following reasons.

- $A^{\#}_{x,y}$ is the partner (8.1.7) of $A_{x,y}$ in (10.2.9). By definition it is the following composite in M.

(10.2.11)

$$
\begin{array}{c}
\big(\mathsf{C}(x,y), Ax\big) \xrightarrow{\quad A^{\#}_{x,y} \quad} \\
\big(A_{x,y}, 1_{Ax}\big) \downarrow \\
\big(\underline{\mathsf{M}}(Ax\,;\,Ay), Ax\big) \xrightarrow{\mathrm{ev}^{\mathsf{M}}_{Ax;Ay}} Ay
\end{array}
$$

The upper right region in (10.2.10) commutes because it is the image under $F$ of (10.2.11). This uses the fact that $F$, as a non-symmetric multifunctor, preserves colored units and composition.
- The bottom right region in (10.2.10) commutes by the definition of $\widehat{F}_{Ax,Ay}$; see (9.2.11).
- The left region in (10.2.10) commutes by (10.2.8).

In (10.2.10) the left-bottom composite is, by definition, the partner of $(F_\star A)_{x,y}$. Thus (10.2.10) yields the equality of binary multimorphisms

(10.2.12) $$(F_\star A)^{\#}_{x,y} = F(A^{\#}_{x,y}) : \big(F\mathsf{C}(x,y), FAx\big) \longrightarrow FAy \quad \text{in} \quad \mathsf{N}.$$

So the partner of $(F_\star A)_{x,y}$ is the image under $F$ of the partner of $A_{x,y}$.    ◇

**Explanation 10.2.13** ($F_\star$ on Morphisms)**.** Suppose $\psi$ is an M-natural transformation as follows.

$$\mathsf{C} \underset{B}{\overset{A}{\Longrightarrow}} \mathsf{M}$$

In the definition (10.2.3),

(10.2.14) $$F_\star \psi = 1_{\widehat{F}} * \psi_F : F_\star A = \widehat{F} \circ A_F \longrightarrow F_\star B = \widehat{F} \circ B_F$$

is the horizontal composite (Definition 6.1.22) below.

$$\mathsf{C}_F \underset{B_F}{\overset{A_F}{\Downarrow\psi_F}} \mathsf{M}_F \underset{\widehat{F}}{\overset{\widehat{F}}{\Downarrow 1_{\widehat{F}}}} \mathsf{N}$$

- The $\mathsf{N}$-natural transformation $\psi_F$ is the image of $\psi$ under the change of enrichment $(-)_F$.
- $1_{\widehat{F}}$ is the identity $\mathsf{N}$-natural transformation of the standard enrichment $\widehat{F}$ (6.1.17).

*Components.* More explicitly, for each object $x \in \mathsf{C}$, the $x$-component of $\psi :$ $A \longrightarrow B$ is a nullary multimorphism

(10.2.15)                          $\psi_x : \langle\rangle \longrightarrow \underline{\mathsf{M}}(Ax\,;Bx)$   in   $\mathsf{M}$,

with $\langle\rangle$ denoting the empty sequence. After the change of enrichment $(-)_F$, the $x$-component of $\psi_F : A_F \longrightarrow B_F$ is the nullary multimorphism

$$(\psi_F)_x = F\psi_x : \langle\rangle \longrightarrow F\underline{\mathsf{M}}(Ax\,;Bx)   \text{in}   \mathsf{N}.$$

The $x$-component of $F_\star\psi$ is the nullary multimorphism given by the following composite in $\mathsf{N}$.

(10.2.16)            $$\langle\rangle \xrightarrow{\overset{(F_\star\psi)_x}{F\psi_x}} F\underline{\mathsf{M}}(Ax\,;Bx) \xrightarrow{\widehat{F}_{Ax,Bx}} \underline{\mathsf{N}}(FAx\,;FBx)$$

The right arrow in (10.2.16) is the $(Ax, Bx)$-component unary multimorphism of the standard enrichment $\widehat{F}$ in (9.2.3).

*Partner Characterization.* We can describe the component $(F_\star\psi)_x$ in terms of its partner (8.1.7), using the following commutative diagram in $\mathsf{N}$.

(10.2.17)

$$
\begin{array}{c}
((F_\star\psi)_x\,,\,1_{FAx}) \left\{ \begin{array}{c}
(\langle\rangle\,,\,FAx) \xrightarrow{\quad F(\psi_x^\#)\quad} \\
\downarrow{\scriptstyle(F\psi_x\,,\,1_{FAx})} \\
(F\underline{\mathsf{M}}(Ax\,;Bx)\,,\,FAx) \xrightarrow{F(\mathsf{ev}^\mathsf{M}_{Ax;Bx})} FBx \\
\downarrow{\scriptstyle(\widehat{F}_{Ax,Bx}\,,\,1_{FAx})} \\
(\underline{\mathsf{N}}(FAx\,;FBx)\,,\,FAx) \xrightarrow{\mathsf{ev}^\mathsf{N}_{FAx;FBx}}
\end{array}\right.
\end{array}
$$

The diagram (10.2.17) commutes for the following reasons.

- $\psi_x^\#$ is the partner (8.1.7) of $\psi_x$ in (10.2.15). By definition it is the following composite in $\mathsf{M}$.

(10.2.18)

$$
\begin{array}{c}
(\langle\rangle\,,\,Ax) \xrightarrow{\quad\psi_x^\#\quad} \\
\downarrow{\scriptstyle(\psi_x\,,\,1_{Ax})} \\
(\underline{\mathsf{M}}(Ax\,;Bx)\,,\,Ax) \xrightarrow{\mathsf{ev}^\mathsf{M}_{Ax;Bx}} Bx
\end{array}
$$

  The upper right region in (10.2.17) commutes because it is the image under $F$ of (10.2.18). This uses the fact that $F$, as a non-symmetric multifunctor, preserves colored units and composition.

- The bottom right region in (10.2.17) commutes by the definition of $\widehat{F}_{Ax,Bx}$; see (9.2.11).
- The left region in (10.2.17) commutes by (10.2.16).

In (10.2.17) the left-bottom composite is, by definition, the partner of $(F_\star \psi)_x$. Thus (10.2.17) yields the equality of unary multimorphisms

$$(10.2.19) \qquad (F_\star \psi)_x^\# = F(\psi_x^\#) : FAx \longrightarrow FBx \quad \text{in} \quad \mathsf{N}.$$

So the partner of $(F_\star \psi)_x$ is the image under $F$ of the partner of $\psi_x$. $\diamond$

## 10.3. Diagram and Mackey Functor Change-of-Enrichment Functors

This section has two purposes.

(1) We show that the diagram and Mackey functor change of enrichment in (10.2.2) and (10.2.4) are functors (Theorems 10.3.1 and 10.3.4).
(2) We illustrate them with $K$-theoretic functors in Examples 10.3.2 and 10.3.5.

### Diagram Change of Enrichment is a Functor.

**Theorem 10.3.1.** *For each non-symmetric multifunctor between non-symmetric closed multicategories*

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

*and each small $\mathsf{M}$-category $\mathsf{C}$, the diagram change of enrichment*

$$F_\star : \mathsf{M}\text{-Cat}(\mathsf{C}, \mathsf{M}) \longrightarrow \mathsf{N}\text{-Cat}(\mathsf{C}_F, \mathsf{N})$$

*in (10.2.2) is a functor.*

*Proof.* The assignments of $F_\star$

- on objects

$$A \longmapsto F_\star A = \widehat{F} \circ A_F$$

in (10.2.6) and
- on morphisms

$$\psi \longmapsto F_\star \psi = 1_{\widehat{F}} * \psi_F$$

in (10.2.14)

are well defined because they are given by composition of $\mathsf{N}$-functors and horizontal composition of $\mathsf{N}$-natural transformations, respectively.

*Preservation of Identity Morphisms.* Suppose $\psi = 1_A$ is the identity $\mathsf{M}$-natural transformation of an $\mathsf{M}$-functor $A : \mathsf{C} \longrightarrow \mathsf{M}$. Then the 2-functoriality of the change of enrichment $(-)_F$ (Proposition 7.1.9) implies the following equalities.

$$F_\star 1_A = 1_{\widehat{F}} * (1_A)_F = 1_{\widehat{F}} * 1_{(A_F)} = 1_{F_\star A}$$

*Preservation of Composition.* Suppose given $\mathsf{M}$-functors $A, B, D : \mathsf{C} \longrightarrow \mathsf{M}$ and vertically composable $\mathsf{M}$-natural transformations

$$A \xrightarrow{\psi} B \xrightarrow{\phi} D.$$

The following computation, using the 2-functoriality of $(-)_F$ in the second equality, shows that $F_\star$ preserves composition.

$$\begin{aligned}
F_\star(\phi\psi) &= 1_{\widehat{F}} * (\phi\psi)_F \\
&= 1_{\widehat{F}} * (\phi_F\psi_F) \\
&= \left(1_{\widehat{F}} * \phi_F\right)\left(1_{\widehat{F}} * \psi_F\right) \\
&= (F_\star\phi)(F_\star\psi)
\end{aligned}$$

This finishes the proof.                                                        $\square$

**Example 10.3.2** (Inverse $K$-Theory and Free Permutative Categories). Theorem 10.3.1 is applicable to the non-symmetric multifunctors in the following diagram.

(10.3.3)

$$\begin{array}{c}
\text{Multicat} \\
\xrightarrow{\quad F \quad} \\
\text{Multicat}_* \xrightarrow{\quad F_\bullet \quad} \text{PermCat}^{\mathsf{su}} \xleftarrow{\quad \mathcal{P} \quad} \Gamma\text{-Cat} \\
\xrightarrow{\quad F_{\mathcal{M}_{\underline{1}}} \quad} \\
\text{Mod}^{\mathcal{M}_{\underline{1}}}
\end{array}$$

- PermCat$^{\mathsf{su}}$ is a closed multicategory by Theorem 8.4.15.
- Multicat, Multicat$_*$, Mod$^{\mathcal{M}_{\underline{1}}}$, and $\Gamma$-Cat are symmetric monoidal closed categories by Theorems 1.1.26 and 1.2.8, Proposition 1.3.17, and (2.3.3). Thus they are closed multicategories by Proposition 8.1.16.
- $\mathcal{P}$ is a non-symmetric multifunctor by [**JY22b**, 1.3].
- F, F$_\bullet$, and F$_{\mathcal{M}_{\underline{1}}}$ are non-symmetric multifunctors by Theorems 3.4.31 and 5.2.6 and (5.5.2).

For example, Theorem 10.3.1 applied to inverse $K$-theory $\mathcal{P}$ says that, for each small ($\Gamma$-Cat)-category C (Definitions 6.1.1 and B.1.1), there is a diagram change-of-enrichment functor

$$\mathcal{P}_\star : (\Gamma\text{-Cat})\text{-Cat}\big(\mathsf{C}, \Gamma\text{-Cat}\big) \longrightarrow \text{PermCat}^{\mathsf{su}}\text{-Cat}\big(\mathsf{C}_{\mathcal{P}}, \text{PermCat}^{\mathsf{su}}\big)$$

defined as in (10.2.3). More explicitly, the functor $\mathcal{P}_\star$ sends each ($\Gamma$-Cat)-functor (Definitions 6.1.7 and B.1.8)

$$A : \mathsf{C} \longrightarrow \Gamma\text{-Cat}$$

to the composite PermCat$^{\mathsf{su}}$-functor

$$\mathcal{P}_\star A : \mathsf{C}_{\mathcal{P}} \xrightarrow{\;A_{\mathcal{P}}\;} (\Gamma\text{-Cat})_{\mathcal{P}} \xrightarrow{\;\widehat{\mathcal{P}}\;} \text{PermCat}^{\mathsf{su}}.$$

- $A_{\mathcal{P}}$ is the image of $A$ under the change-of-enrichment 2-functor (Proposition 7.1.9)

$$(-)_{\mathcal{P}} : (\Gamma\text{-Cat})\text{-Cat} \longrightarrow \text{PermCat}^{\mathsf{su}}\text{-Cat}$$

  along $\mathcal{P}$.
- The PermCat$^{\mathsf{su}}$-functor

$$\widehat{\mathcal{P}} : (\Gamma\text{-Cat})_{\mathcal{P}} \longrightarrow \text{PermCat}^{\mathsf{su}}$$

  is the standard enrichment of $\mathcal{P}$ (Theorem 9.2.12).

The morphism assignment of $\mathcal{P}_\star$ sends a ($\Gamma$-Cat)-natural transformation (Definitions 6.1.14 and B.1.10) $\psi$ to the whiskering

$$\mathcal{P}_\star \psi = 1_{\widehat{\mathcal{P}}} * \psi_\mathcal{P}$$

as in (10.2.14). The diagram change-of-enrichment functors $F_\star$, $(F_\bullet)_\star$, and $(F_{\mathcal{M}\underline{1}})_\star$ admit analogous description. ◇

We provide further examples and applications of Theorem 10.3.1 in Example 10.3.5 and Chapters 11 and 12. In particular, in Chapter 12 we use $(F_\bullet)_\star$ and $(F_{\mathcal{M}\underline{1}})_\star$ to construct equivalences of homotopy theories from diagrams and presheaves enriched in Multicat$_*$ and Mod$^{\mathcal{M}\underline{1}}$ to those enriched in PermCat$^{\mathsf{su}}$.

**Mackey Functor Change of Enrichment is a Functor.**

**Theorem 10.3.4.** *For each multifunctor between closed multicategories*

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

*and each small* M-*category* C, *the presheaf change of enrichment*

$$F_\star : \mathsf{M}\text{-Cat}(\mathsf{C}^{\mathsf{op}}, \mathsf{M}) \longrightarrow \mathsf{N}\text{-Cat}\big((\mathsf{C}_F)^{\mathsf{op}}, \mathsf{N}\big)$$

*in (10.2.4) is a functor.*

*Proof.* This is Theorem 10.3.1 applied to the opposite M-category C$^{\mathsf{op}}$. We use Proposition 7.2.1 to obtain the equality

$$(\mathsf{C}^{\mathsf{op}})_F = (\mathsf{C}_F)^{\mathsf{op}}$$

of N-categories. □

We stress that Theorem 10.3.4 does *not* apply to non-symmetric multifunctors. For the equality in its proof to hold, $F$ needs to preserve the symmetric group action as in the second equality in (7.2.2).

**Example 10.3.5** (*K*-Theory Multifunctors). Theorems 10.3.1 and 10.3.4 are applicable to the following multifunctors in (2.5.1).

(10.3.6)



Each arrow in (10.3.6) is a multifunctor between closed multicategories.

- PermCat$^{\mathsf{su}}$, $\Gamma$-Cat, and $\Gamma$-sSet are closed multicategories by (2.3.3), Proposition 8.1.16, and Theorem 8.4.15, as discussed in Example 10.3.2.
- Mod$^{\mathcal{M}\underline{1}}$, $\mathcal{G}_*$-Cat, $\mathcal{G}_*$-sSet, and Sp are symmetric monoidal closed categories (Proposition 1.3.17 and (2.4.12) and (2.5.2)), hence also closed multicategories (Proposition 8.1.16).
- $J^{\mathsf{EM}}$, End$_{\mathcal{M}\underline{1}}$, and K$^{\mathsf{EM}}$ are multifunctors.
- The other arrows in (10.3.6) are symmetric monoidal functors, hence also multifunctors via the endomorphism construction (C.3.3).

We discuss the case for K$^{\mathsf{EM}}$ in more detail in Sections 10.5 and 10.6. We emphasize that Theorem 10.3.4 does *not* apply to the arrows in (10.3.3)—namely, $\mathcal{P}$, F, F$_\bullet$, and F$_{\mathcal{M}\underline{1}}$—because those are *non-symmetric* multifunctors. ◇

### 10.4. Composition of Diagram Change-of-Enrichment Functors

In Theorem 10.3.1 we observe that there is a diagram change-of-enrichment functor

$$F_\star = F_\star^{\mathsf{C}} : \mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{M}) \longrightarrow \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}_F, \mathsf{N})$$

for each

- non-symmetric multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ between non-symmetric closed multicategories and
- small M-category C.

In this section we show that the construction, $F \longmapsto F_\star$, respects composition of non-symmetric multifunctors; see Theorem 10.4.1. The version for enriched Mackey functors is Theorem 10.4.5. We discuss applications of Theorems 10.4.1 and 10.4.5 to Elmendorf-Mandell $K$-theory in Section 10.6.

**Theorem 10.4.1.** *Suppose given non-symmetric multifunctors between non-symmetric closed multicategories*

$$\mathsf{M} \xrightarrow{\ F\ } \mathsf{N} \xrightarrow{\ G\ } \mathsf{P}$$

*and a small* M-*category* C*. Then the following diagram of functors commutes.*

(10.4.2)

$$\begin{array}{ccc}
\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{M}) & \xrightarrow{\ (GF)_\star^{\mathsf{C}}\ } & \mathsf{P}\text{-}\mathsf{Cat}(\mathsf{C}_{GF}, \mathsf{P}) \\
& {\scriptstyle F_\star^{\mathsf{C}}} \searrow \qquad \nearrow {\scriptstyle G_\star^{\mathsf{C}_F}} & \\
& \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}_F, \mathsf{N}) &
\end{array}$$

*Proof.* By Proposition 7.4.1 the following diagram of change-of-enrichment 2-functors commutes.

(10.4.3)

$$\begin{array}{ccc}
& \overset{(-)_{GF}}{\frown} & \\
\mathsf{M}\text{-}\mathsf{Cat} \xrightarrow{\ (-)_F\ } & \mathsf{N}\text{-}\mathsf{Cat} \xrightarrow{\ (-)_G\ } & \mathsf{P}\text{-}\mathsf{Cat}
\end{array}$$

This gives an equality of P-categories

$$(\mathsf{C}_F)_G = \mathsf{C}_{GF},$$

so the arrow $G_\star^{\mathsf{C}_F}$ in (10.4.2) is well defined.

To prove that (10.4.2) is commutative, suppose $A : \mathsf{C} \longrightarrow \mathsf{M}$ is an M-functor. By definition (10.2.6) $F_\star A$ is the composite N-functor

$$F_\star A : \mathsf{C}_F \xrightarrow{\ A_F\ } \mathsf{M}_F \xrightarrow{\ \widehat{F}\ } \mathsf{N}$$

with

- $(-)_F$ the change of enrichment in (10.4.3) and
- $\widehat{F}$ the standard enrichment of $F$ (Theorem 9.2.12).

Applying the change-of-enrichment 2-functor $(-)_G$ to the above composite and composing with the standard enrichment $\widehat{G}$, we obtain the composite P-functor along the top of the following diagram.

(10.4.4)

$$\mathsf{C}_{GF} \xrightarrow{\ A_{GF}\ } \mathsf{M}_{GF} = (\mathsf{M}_F)_G \xrightarrow{\ \widehat{F}_G\ } \mathsf{N}_G \xrightarrow{\ \widehat{G}\ } \mathsf{P}$$

with overbrace $(F_\star A)_G$ and underbrace $\widehat{GF}$.

Theorem 9.3.6 gives the equality of P-functors

$$\widehat{GF} = \widehat{G} \circ \widehat{F}_G : \mathsf{M}_{GF} \longrightarrow \mathsf{P}.$$

In the commutative diagram (10.4.4),

- the composite along the top is $G_\star F_\star(A)$, and
- the composite along the bottom is $(GF)_\star(A)$.

This proves that the diagram (10.4.2) is commutative on objects.

Replacing $A$ by an M-natural transformation in M-Cat(C,M), the previous paragraph also proves that the diagram (10.4.2) is commutative on morphisms. $\square$

The following result shows that presheaf change-of-enrichment functors are closed under composition. Recall from Definition C.1.19 that multifunctors are required to preserve the colored units, composition, and symmetric group action.

**Theorem 10.4.5.** *Suppose given multifunctors between closed multicategories*

$$\mathsf{M} \xrightarrow{\ F\ } \mathsf{N} \xrightarrow{\ G\ } \mathsf{P}$$

*and a small* M-*category* C. *Then the following diagram of presheaf change-of-enrichment functors commutes.*

(10.4.6)

$$
\begin{array}{ccc}
\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}^{\mathsf{op}},\mathsf{M}) & \xrightarrow{\ (GF)_\star^{\mathsf{C}^{\mathsf{op}}}\ } & \mathsf{P}\text{-}\mathsf{Cat}\big((\mathsf{C}_{GF})^{\mathsf{op}},\mathsf{P}\big) \\
& \searrow{\scriptstyle F_\star^{\mathsf{C}^{\mathsf{op}}}} \quad \nearrow{\scriptstyle G_\star^{(\mathsf{C}_F)^{\mathsf{op}}}} & \\
& \mathsf{N}\text{-}\mathsf{Cat}\big((\mathsf{C}_F)^{\mathsf{op}},\mathsf{N}\big) &
\end{array}
$$

*Proof.* This is Theorem 10.4.1 applied to the opposite M-category $\mathsf{C}^{\mathsf{op}}$ (Proposition 6.6.7). The equality of N-categories

$$(\mathsf{C}^{\mathsf{op}})_F = (\mathsf{C}_F)^{\mathsf{op}}$$

and the equalities of P-categories

$$\big((\mathsf{C}_F)^{\mathsf{op}}\big)_G = \big((\mathsf{C}_F)_G\big)^{\mathsf{op}} = (\mathsf{C}_{GF})^{\mathsf{op}} = (\mathsf{C}^{\mathsf{op}})_{GF}$$

are from Propositions 7.2.1 and 7.4.1. $\square$

We emphasize that Theorem 10.4.5 does *not* apply to non-symmetric multifunctors because the equalities in the proof above require that $F$ and $G$ preserve the symmetric group action.

## 10.5. Spectral Mackey Functors from *K*-Theory

Recall from (2.5.8) that Elmendorf-Mandell *K*-theory

$$\mathsf{K}^{\mathsf{EM}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}$$

is a multifunctor in the sSet-enriched sense. In this section we apply our general results about diagram and presheaf change of enrichment, Theorems 10.3.1 and 10.3.4, to obtain spectrally enriched diagrams and Mackey functors via Elmendorf-Mandell *K*-theory.

**Theorem 10.5.1.** *Suppose* $\mathsf{C}$ *is a small* $\mathsf{PermCat^{su}}$*-category. Then* $\mathsf{K^{EM}}$ *induces a diagram change-of-enrichment functor*

$$(10.5.2) \qquad \mathsf{PermCat^{su}\text{-}Cat}\big(\mathsf{C}, \mathsf{PermCat^{su}}\big) \xrightarrow{\mathsf{K_\star^{EM}}} \mathsf{Sp\text{-}Cat}\big(\mathsf{C_{K^{EM}}}, \mathsf{Sp}\big)$$

*and a presheaf change-of-enrichment functor*

$$(10.5.3) \qquad \mathsf{PermCat^{su}\text{-}Cat}\big(\mathsf{C^{op}}, \mathsf{PermCat^{su}}\big) \xrightarrow{\mathsf{K_\star^{EM}}} \mathsf{Sp\text{-}Cat}\big((\mathsf{C_{K^{EM}}})^{op}, \mathsf{Sp}\big).$$

*Proof.* The existence of the functor $\mathsf{K_\star^{EM}}$ in (10.5.2) is Theorem 10.3.1 applied to the multifunctor $F = \mathsf{K^{EM}}$ between closed multicategories.

- $\mathsf{PermCat^{su}}$ is a closed multicategory by Theorem 8.4.15.
- The symmetric monoidal closed category $\mathsf{Sp}$ of symmetric spectra is a closed multicategory by Proposition 8.1.16.

The existence of the functor in (10.5.3) is Theorem 10.3.4 applied to $\mathsf{K^{EM}}$. □

We describe the functor $\mathsf{K_\star^{EM}}$ in more detail in Explanations 10.5.8 and 10.5.11 after Remarks 10.5.4 and 10.5.5. In Section 10.6 we factor the functors in (10.5.2) and (10.5.3) through enriched diagram and Mackey functor categories defined on left $\mathcal{M}\underline{1}$-modules, $\mathcal{G}_*$-categories, and $\mathcal{G}_*$-simplicial sets.

**Symmetry and Opposite.**

**Remark 10.5.4** (Symmetry of $\mathsf{K^{EM}}$)**.** For the presheaf change-of-enrichment functor $\mathsf{K_\star^{EM}}$ in (10.5.3), it is crucial that $\mathsf{K^{EM}}$ is a multifunctor in the *symmetric* sense in order to identify the $\mathsf{Sp}$-categories

$$(\mathsf{C^{op}})_{\mathsf{K^{EM}}} \quad \text{and} \quad (\mathsf{C_{K^{EM}}})^{op}$$

using Proposition 7.2.1. Without this symmetry property of $\mathsf{K^{EM}}$, we would have to use $(\mathsf{C^{op}})_{\mathsf{K^{EM}}}$ in the codomain of $\mathsf{K_\star^{EM}}$, which is (10.5.2) for $\mathsf{C^{op}}$. Thus, the fact that the codomain of $\mathsf{K_\star^{EM}}$ in (10.5.3) is a category of spectrally enriched *presheaves*—as opposed to enriched diagrams—depends on the symmetry of $\mathsf{K^{EM}}$. ⋄

**Remark 10.5.5** (Symmetric and Non-Symmetric $K$-Theory Multifunctors)**.** The diagram change-of-enrichment functor $\mathsf{K_\star^{EM}}$ in (10.5.2) is the $\mathsf{K^{EM}}$ variant of the main result in [**BO15**, Theorem 7.5], which states the following using our notation:

Let $G$ be a finite group. Then there is a functor

$$(10.5.6) \qquad \mathbb{K}_\star : \mathsf{PermCat^{su}\text{-}Cat}\big(G\mathcal{E}^{op}, \mathsf{PermCat^{su}}\big) \longrightarrow \mathsf{Sp\text{-}Cat}\big((G\mathcal{E}^{op})_{\mathbb{K}}, \mathsf{Sp}\big).$$

In [**BO15**] $\mathbb{K}_\star$, $\mathsf{PermCat^{su}}$, and $\mathsf{Sp}$ are denoted $K_G$, *Perm*, and *Spec*, respectively. As in Theorem 0.3.9, Remark 9.4.4, and Example 10.1.27, $\mathbb{K}$ is the $K$-theory *non-symmetric* multifunctor in [**GM22**, **GMMO23**], and $G\mathcal{E}$ is the permutative Burnside category (Definition 0.3.5). The diagram change-of-enrichment functor $\mathbb{K}_\star$ in (10.5.6) exists by Theorem 10.3.1 applied to

- the non-symmetric multifunctor $\mathbb{K}$ and
- the $\mathsf{PermCat^{su}}$-category $G\mathcal{E}^{op}$, which is the opposite $\mathsf{PermCat^{su}}$-category of $G\mathcal{E}$ (Proposition 6.6.7).

We emphasize the following regarding the diagram change-of-enrichment functor $\mathbb{K}_\star$ in (10.5.6).

(i) Unlike Elmendorf-Mandell $K$-theory, $\mathbb{K}$ does not preserve symmetry [**GMMO23**, Theorem 8.12]. Thus we *cannot* use Proposition 7.2.1 to identify the $\mathsf{Sp}$-categories

- $(G\mathcal{E}^{op})_{\mathbb{K}}$, with opposite taken in PermCat$^{su}$-Cat, and
- $(G\mathcal{E}_{\mathbb{K}})^{op}$, with opposite taken in Sp-Cat.

(ii) Recall from Remark 0.3.7 that the PermCat$^{su}$-category $G\mathcal{E}$ and its opposite $G\mathcal{E}^{op}$ are *not* known to be equivalent as PermCat$^{su}$-categories. The assignment that sends a span of *G*-sets $(f, g)$ to $(g, f)$ as in (0.2.7) does not define a PermCat$^{su}$-functor

$$G\mathcal{E} \longrightarrow G\mathcal{E}^{op}$$

because it does not preserve composition as in (6.1.9).
(iii) The Sp-category $G\mathcal{E}_{\mathbb{K}}$ and its opposite $(G\mathcal{E}_{\mathbb{K}})^{op}$ are also *not* known to be equivalent.

Thus the codomain of $\mathbb{K}_{\star}$ has to be $(G\mathcal{E}^{op})_{\mathbb{K}}$ and *not* $(G\mathcal{E}_{\mathbb{K}})^{op}$ as stated in [**BO15**, 7.5], where $(G\mathcal{E}_{\mathbb{K}})^{op}$ is denoted $G\mathcal{B}^{op}$. As far as the authors know, the codomain of $\mathbb{K}_{\star}$ is a category of spectrally enriched diagrams but *not* spectrally enriched presheaves as in the Guillou-May Quillen equivalence (10.1.28). ◇

**Description of** $\mathsf{K}_{\star}^{EM}$. The rest of this section describes the functor $\mathsf{K}_{\star}^{EM}$ in (10.5.2) in detail. The same discussion also applies to $\mathsf{C}^{op}$ and $(\mathsf{C}^{op})_{\mathsf{K}^{EM}}$ = $(\mathsf{C}_{\mathsf{K}^{EM}})^{op}$; see Remark 10.5.4. As defined in (10.2.3), the diagram change-of-enrichment functor $\mathsf{K}_{\star}^{EM}$

- first applies the change-of-enrichment 2-functor (Proposition 7.1.9)

(10.5.7)          $(-)_{\mathsf{K}^{EM}} : \mathsf{PermCat}^{su}\text{-}\mathsf{Cat} \longrightarrow \mathsf{Sp}\text{-}\mathsf{Cat}.$

along $\mathsf{K}^{EM}$ and then
- composes or whiskers with the standard enrichment Sp-functor $\widehat{\mathsf{K}^{EM}}$ (Explanation 9.4.5).

**Explanation 10.5.8** ($\mathsf{K}_{\star}^{EM}$ on Objects). Consider a PermCat$^{su}$-functor

$$A : \mathsf{C} \longrightarrow \mathsf{PermCat}^{su}.$$

Applying Explanation 10.2.5 to the context of Theorem 10.5.1, the Sp-functor $\mathsf{K}_{\star}^{EM} A$ is the following composite.

(10.5.9)          $\mathsf{C}_{\mathsf{K}^{EM}} \xrightarrow{\ A_{\mathsf{K}^{EM}}\ } (\mathsf{PermCat}^{su})_{\mathsf{K}^{EM}} \xrightarrow{\ \widehat{\mathsf{K}^{EM}}\ } \mathsf{Sp}$

$$\overbrace{\hspace{6cm}}^{\mathsf{K}_{\star}^{EM} A}$$

- $A_{\mathsf{K}^{EM}}$ is the image of $A$ under the change-of-enrichment 2-functor $(-)_{\mathsf{K}^{EM}}$ in (10.5.7).
- $\widehat{\mathsf{K}^{EM}}$ is the standard enrichment of $\mathsf{K}^{EM}$ in Explanation 9.4.5.

Next we describe its object assignment and component morphisms.

*Object Assignment*. For an object $x \in \mathsf{C}$, the object assignment is

$$(\mathsf{K}_{\star}^{EM} A)(x) = \mathsf{K}^{EM}(Ax) \quad \text{in} \quad \mathsf{Sp}.$$

This is the Elmendorf-Mandell *K*-theory of the small permutative category $Ax$.

*Components.* For objects $x, y \in \mathsf{C}$, the component $(\mathsf{K}^{\mathsf{EM}}_{\star} A)_{x,y}$ is the following composite morphism in $\mathsf{Sp}$.

$$
\begin{array}{ccc}
\mathsf{K}^{\mathsf{EM}}\mathsf{C}(x,y) & \xrightarrow{(\mathsf{K}^{\mathsf{EM}}_{\star} A)_{x,y}} & \underline{\mathsf{Sp}}\big(\mathsf{K}^{\mathsf{EM}}(Ax); \mathsf{K}^{\mathsf{EM}}(Ay)\big) \\
{\scriptstyle \mathsf{K}^{\mathsf{EM}}(A_{x,y})} \downarrow & & \\
\mathsf{K}^{\mathsf{EM}}\underline{\mathsf{PermCat}^{\mathsf{su}}}(Ax; Ay) & \nearrow {\scriptstyle \widehat{\mathsf{K}^{\mathsf{EM}}}_{Ax,Ay}} &
\end{array}
$$

The adjoint of $(\mathsf{K}^{\mathsf{EM}}_{\star} A)_{x,y}$ in $\mathsf{Sp}$ is the following composite morphism.

(10.5.10)
$$
\begin{array}{ccc}
\mathsf{K}^{\mathsf{EM}}\mathsf{C}(x,y) \wedge \mathsf{K}^{\mathsf{EM}}(Ax) & \xrightarrow{\mathsf{K}^{\mathsf{EM}}(A^{\#}_{x,y})} & \mathsf{K}^{\mathsf{EM}}(Ay) \\
{\scriptstyle \mathsf{K}^{\mathsf{EM}}(A_{x,y}) \wedge 1} \downarrow & & \\
\mathsf{K}^{\mathsf{EM}}\underline{\mathsf{PermCat}^{\mathsf{su}}}(Ax; Ay) \wedge \mathsf{K}^{\mathsf{EM}}Ax & \nearrow {\scriptstyle \mathsf{K}^{\mathsf{EM}}(\mathrm{ev}_{Ax; Ay})} &
\end{array}
$$

Here $A^{\#}_{x,y}$ is the partner of $A_{x,y}$. It is defined in (10.2.11).          ◇

**Explanation 10.5.11** ($\mathsf{K}^{\mathsf{EM}}_{\star}$ on Morphisms)**.** Consider a $\mathsf{PermCat}^{\mathsf{su}}$-natural transformation (Explanation 6.3.16) $\psi$ as follows.

$$
\mathsf{C} \overset{A}{\underset{B}{\rightrightarrows}} \Downarrow\psi \quad \mathsf{PermCat}^{\mathsf{su}}
$$

For each object $x \in \mathsf{C}$, the $x$-component of $\psi$ is a nullary multimorphism

$$
\psi_x : \langle\rangle \longrightarrow \underline{\mathsf{PermCat}^{\mathsf{su}}}\big(Ax; Bx\big) \quad \text{in} \quad \mathsf{PermCat}^{\mathsf{su}}
$$

with $\langle\rangle$ denoting the empty sequence. This means a choice of an object in $\underline{\mathsf{PermCat}^{\mathsf{su}}}\big(Ax; Bx\big)$. In other words, $\psi_x$ is a strictly unital symmetric monoidal functor

(10.5.12)          $\psi_x : Ax \longrightarrow Bx \quad \text{in} \quad \mathsf{PermCat}^{\mathsf{su}} .$

Applying Explanation 10.2.13 to the context of Theorem 10.5.1, the image of $\psi$ under the diagram change-of-enrichment functor $\mathsf{K}^{\mathsf{EM}}_{\star}$,

$$
\mathsf{K}^{\mathsf{EM}}_{\star}\psi : \mathsf{K}^{\mathsf{EM}}_{\star} A = \widehat{\mathsf{K}^{\mathsf{EM}}} \circ A_{\mathsf{K}^{\mathsf{EM}}} \longrightarrow \mathsf{K}^{\mathsf{EM}}_{\star} B = \widehat{\mathsf{K}^{\mathsf{EM}}} \circ B_{\mathsf{K}^{\mathsf{EM}}},
$$

is the following whiskering of the $\mathsf{Sp}$-natural transformation $\psi_{\mathsf{K}^{\mathsf{EM}}}$ with the $\mathsf{Sp}$-functor $\widehat{\mathsf{K}^{\mathsf{EM}}}$ in Explanation 9.4.5.

$$
\mathsf{C}_{\mathsf{K}^{\mathsf{EM}}} \overset{A_{\mathsf{K}^{\mathsf{EM}}}}{\underset{B_{\mathsf{K}^{\mathsf{EM}}}}{\rightrightarrows}} \Downarrow\psi_{\mathsf{K}^{\mathsf{EM}}} \quad (\mathsf{PermCat}^{\mathsf{su}})_{\mathsf{K}^{\mathsf{EM}}} \xrightarrow{\widehat{\mathsf{K}^{\mathsf{EM}}}} \mathsf{Sp}
$$

For each object $x \in \mathsf{C}$, the $x$-component of $\mathsf{K}^{\mathsf{EM}}_{\star}\psi$ is a nullary multimorphism

$$
(\mathsf{K}^{\mathsf{EM}}_{\star}\psi)_x : \langle\rangle \longrightarrow \underline{\mathsf{Sp}}\big(\mathsf{K}^{\mathsf{EM}}(Ax); \mathsf{K}^{\mathsf{EM}}(Bx)\big) \quad \text{in} \quad \mathsf{Sp} .
$$

Since the multicategory structure on $\mathsf{Sp}$ is induced by a symmetric monoidal structure, this nullary multimorphism is a morphism

$$
(\mathsf{K}^{\mathsf{EM}}_{\star}\psi)_x : S \longrightarrow \underline{\mathsf{Sp}}\big(\mathsf{K}^{\mathsf{EM}}(Ax); \mathsf{K}^{\mathsf{EM}}(Bx)\big) \quad \text{in} \quad \mathsf{Sp},
$$

where $S$ denotes the symmetric sphere spectrum [**JY∞**, 7.4.1]. The adjoint of this morphism in $\mathsf{Sp}$ is the composite

(10.5.13)
$$S \wedge \mathsf{K}^{\mathsf{EM}}(Ax) \xrightarrow{\;\cong\;} \mathsf{K}^{\mathsf{EM}}(Ax) \xrightarrow{\;\mathsf{K}^{\mathsf{EM}}(\psi_x)\;} \mathsf{K}^{\mathsf{EM}}(Bx)$$

with the top arrow labeled $(\mathsf{K}^{\mathsf{EM}}_{\star}\psi)^{\#}_x$

with $\psi_x$ in (10.5.12). $\diamond$

## 10.6. Spectral Mackey Functors from Multicategorical Mackey Functors

In this section we apply Theorems 10.4.1 and 10.4.5 along with the factorization of Elmendorf-Mandell $K$-theory to factor the diagram and presheaf change-of-enrichment functors $\mathsf{K}^{\mathsf{EM}}_{\star}$ in Theorem 10.5.1. For the context, recall from (2.5.8) that Elmendorf-Mandell $K$-theory $\mathsf{K}^{\mathsf{EM}}$ is the following composite.

(10.6.1)
$$\begin{array}{ccc}
\mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\;\mathsf{K}^{\mathsf{EM}}\;} & \mathsf{Sp} \\
{\scriptstyle \mathsf{End}_{\mathcal{M}_1}}\downarrow & & \uparrow{\scriptstyle \mathsf{K}^{\mathcal{G}}} \\
\mathsf{Mod}^{\mathcal{M}_1} \xrightarrow{\;J^{\mathcal{T}}\;} \mathcal{G}_*\text{-}\mathsf{Cat} & \xrightarrow{\;\mathsf{Ner}_*\;} & \mathcal{G}_*\text{-}\mathsf{sSet}
\end{array}$$

As we explain under (9.4.1), the commutative diagram (10.6.1) consists of multifunctors between closed multicategories.

**Theorem 10.6.2.** *Suppose* $\mathsf{C}$ *is a small* $\mathsf{PermCat}^{\mathsf{su}}$*-category. Then the factorization (10.6.1) of* $\mathsf{K}^{\mathsf{EM}}$ *induces the following factorization of the diagram change-of-enrichment functor* $\mathsf{K}^{\mathsf{EM}}_{\star}$ *in (10.5.2).*

(10.6.3)
$$\begin{array}{ccc}
\mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big) & \xrightarrow{\;\mathsf{K}^{\mathsf{EM}}_{\star}\;} & \mathsf{Sp}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{K}^{\mathsf{EM}}}, \mathsf{Sp}\big) \\
{\scriptstyle (\mathsf{End}_{\mathcal{M}_1})_{\star}}\downarrow & & \uparrow{\scriptstyle \mathsf{K}^{\mathcal{G}}_{\star}} \\
\mathsf{Mod}^{\mathcal{M}_1}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}_1}}, \mathsf{Mod}^{\mathcal{M}_1}\big) & & (\mathcal{G}_*\text{-}\mathsf{sSet})\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{Ner}_* J^{\mathcal{T}}\mathsf{End}_{\mathcal{M}_1}}, \mathcal{G}_*\text{-}\mathsf{sSet}\big) \\
{\scriptstyle J^{\mathcal{T}}_{\star}}\searrow & & \nearrow{\scriptstyle (\mathsf{Ner}_*)_{\star}} \\
& (\mathcal{G}_*\text{-}\mathsf{Cat})\text{-}\mathsf{Cat}\big(\mathsf{C}_{J^{\mathcal{T}}\mathsf{End}_{\mathcal{M}_1}}, \mathcal{G}_*\text{-}\mathsf{Cat}\big) &
\end{array}$$

*Moreover, there is an analogous factorization of the presheaf change-of-enrichment functor* $\mathsf{K}^{\mathsf{EM}}_{\star}$ *in (10.5.3) with* $\mathsf{C}$ *and each* $\mathsf{C}_?$ *in (10.6.3) replaced by* $\mathsf{C}^{\mathsf{op}}$ *and* $(\mathsf{C}_?)^{\mathsf{op}}$*, respectively.*

*Proof.* The factorization (10.6.3) is the result of applying Theorem 10.4.1 three times to the commutative diagram (10.6.1). The second assertion about $\mathsf{C}^{\mathsf{op}}$ follows similarly from Theorem 10.4.5. $\square$

**Explanation 10.6.4** (Diagram Change of Enrichment). By definition (10.2.3) each diagram change-of-enrichment functor $F_{\star}$ is given by

- applying the change-of-enrichment 2-functor $(-)_F$ (Proposition 7.1.9) and then
- composing or whiskering the result with the standard enrichment $\widehat{F}$ in Theorem 9.2.12.

In Explanations 10.5.8 and 10.5.11 we describe the diagram change-of-enrichment functor $\mathsf{K}^{\mathsf{EM}}_\star$. The other four diagram change-of-enrichment functors in (10.6.3) admit analogous description. Just like Sp, each of the closed multicategories

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}, \quad \mathcal{G}_*\text{-}\mathsf{Cat}, \quad \text{and} \quad \mathcal{G}_*\text{-}\mathsf{sSet}$$

is a symmetric monoidal closed category by Proposition 1.3.17 and (2.4.12). In each case we can use the adjunction between the monoidal product and the closed structure to obtain the analogs of the adjoint description in (10.5.10) and (10.5.13).

◇

**Explanation 10.6.5** (Multicategorical Mackey Functors)**.** The second assertion of Theorem 10.6.2 gives, for each small $\mathsf{PermCat}^{\mathsf{su}}$-category C, the following factorization of the presheaf change-of-enrichment functor $\mathsf{K}^{\mathsf{EM}}_\star$ in (10.5.3).

$$(10.6.6)$$

Via the composite $\mathsf{K}^{\mathcal{G}}_\star \circ (\mathsf{Ner}_*)_\star \circ \mathsf{J}^{\mathcal{T}}_\star$, we obtain Sp-enriched Mackey functors from $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-enriched Mackey functors. This is what the title of this section refers to.   ◇

**Explanation 10.6.7** (Equivalences of Homotopy Theories)**.** In Theorem 12.4.6 we observe that $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ in each of (10.6.3) and (10.6.6) is an equivalence of homotopy theories. This implies that the homotopy theories of modules in $\mathsf{PermCat}^{\mathsf{su}}$ and in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are equivalent via $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$. An analogous equivalence of homotopy theories also holds with $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ replaced by $\mathsf{Multicat}_*$; see Theorem 12.1.6. These equivalences of homotopy theories are all instances of the much more general Theorems 11.4.14 and 11.4.24, which hold at the level of (non-symmetric) multifunctors between (non-symmetric) closed multicategories. We do *not* know whether $\mathsf{K}^{\mathsf{EM}}_\star$, $\mathsf{J}^{\mathcal{T}}_\star$, $(\mathsf{Ner}_*)_\star$, $\mathsf{K}^{\mathcal{G}}_\star$, or any of their composites are equivalences of homotopy theories or not. See Question D.3.   ◇

**Part 4**

# Homotopy Theory of Enriched Diagrams and Mackey Functors

# Homotopy Equivalences between Enriched Diagram and Mackey Functor Categories

Throughout this chapter we suppose given a pair of non-symmetric multifunctors between non-symmetric closed multicategories

$$F : \mathsf{M} \rightleftarrows \mathsf{N} : E,$$

together with multinatural transformations as follows.

$$\mathsf{M} \underset{EF}{\overset{1_{\mathsf{M}}}{\rightrightarrows}} {\Downarrow \kappa}\; \mathsf{M} \qquad\qquad \mathsf{N} \underset{FE}{\overset{1_{\mathsf{N}}}{\rightrightarrows}} {\Downarrow \xi}\; \mathsf{N}$$

The main results of this chapter identify conditions under which these data will induce, for each small N-category C, inverse equivalences of homotopy theories between enriched diagram categories (Theorem 11.4.14)

$$\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star}{\overset{F_\star^\xi}{\rightleftarrows}} \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{N})$$

and enriched Mackey functor categories (Theorem 11.4.24)

$$\mathsf{M}\text{-}\mathsf{Cat}((\mathsf{C}_E)^{\mathsf{op}}, \mathsf{M}) \underset{E_\star}{\overset{F_\star^\xi}{\rightleftarrows}} \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}^{\mathsf{op}}, \mathsf{N}),$$

with the latter requiring that $E$ be a multifunctor, not merely a non-symmetric multifunctor. The functors $E_\star$ are the diagram change of enrichment for $E$ (Theorem 10.3.1). The functors $F_\star^\xi$ are similar, and are described in Section 11.1.

It is important to note that, while $E$ is required to satisfy the additional symmetric group action axiom of a multifunctor in Theorem 11.4.24, $F$ is not. In the applications, Theorems 12.1.6 and 12.4.6 below, $E$ is an endomorphism multifunctor and $F$ is a corresponding free non-symmetric multifunctor.

**Connection with Other Chapters.** The results in this chapter provide a general approach to three main results in Chapter 12, as follows.

- The application $(F, E) = (\mathsf{F}_\bullet, \mathsf{E}_\bullet)$ and $(\kappa, \xi) = (\eta^\bullet, \varrho^\bullet)$ is described in Sections 12.1 through 12.3.
- The application $(F, E) = (\mathsf{F}_{\mathcal{M}\underline{1}}, \mathsf{E}_{\mathcal{M}\underline{1}})$ and $(\kappa, \xi) = (\eta^{\mathcal{M}\underline{1}}, \varrho^{\mathcal{M}\underline{1}})$ is described in Sections 12.4 and 12.5.
- The application $(F, E) = (\mathcal{M}\underline{1} \wedge -, \mathsf{U}_{\mathcal{M}\underline{1}})$ and $(\kappa, \xi) = (\hat{\eta}, \hat{\varepsilon}^{-1})$ is described in Sections 12.6 and 12.7.

For each application, we recall further background and context in the respective sections of Chapter 12.

**Background.** Section 10.1 describes enriched diagram categories and enriched Mackey functor categories. Sections 10.2 through 10.4 give the definitions and basic properties of the functors $E_\star$ and $F_\star$ for diagram change of enrichment. That material, and many of the details later in this chapter, depends on the basic theory of (self-)enrichment in a non-symmetric multicategory, from Chapters 6, 7, and 9.

**Chapter Summary.** The precise context and assumptions for this chapter are given in Definition 11.1.1. The remainder of Section 11.1 gives the definition and further explanations of the functor $F_\star^\xi$. Sections 11.2 and 11.3 construct two natural transformations, denoted $\kappa^\star$ and $\zeta^\star$, that compare the composites $E_\star F_\star^\xi$ and $F_\star^\xi E_\star$ to the respective identity functors. The main result, that $F_\star^\xi$ and $E_\star$ are inverse equivalences of homotopy theories in the sense of Definition 2.1.8, is in Section 11.4; see Theorem 11.4.14. With the further assumption that $E$ is a multifunctor (not merely a non-symmetric multifunctor), Theorem 11.4.24 yields inverse equivalences of homotopy theories between enriched Mackey functor categories. Here is a summary table.

| | |
|---|---|
| definition and explanations of $F_\star^\xi$ | 11.1.1, 11.1.7, and 11.1.14 |
| $\kappa^\star : 1 \longrightarrow E_\star F_\star^\xi$ | 11.2.1, 11.2.19, and 11.2.24 |
| $\zeta^\star : 1 \longrightarrow F_\star^\xi E_\star$ | 11.3.1, 11.3.18, and 11.3.23 |
| componentwise stable equivalences for enriched diagrams | 11.4.1, 11.4.7, 11.4.4, and 11.4.13 |
| $\left( F_\star^\xi, E_\star \right)$ inverse equivalence of homotopy theories | 11.4.14 and 11.4.24 |
| $\left( F, E \right)$ inverse equivalence of homotopy theories | 11.4.25 |

## 11.1. Comparing Enriched Diagram and Mackey Functor Categories

In this section we define a pair of functors

$$\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star}{\overset{F_\star^\xi}{\rightleftarrows}} \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$$

that compare two categories of enriched diagrams in the sense of Definition 10.1.1. The functor $E_\star$ is the diagram change of enrichment of $E : \mathsf{N} \longrightarrow \mathsf{M}$ (Theorem 10.3.1). The functor $F_\star^\xi$ and the context for this chapter are discussed in Definition 11.1.1. After that definition we unravel the functor $F_\star^\xi$.

- Explanation 11.1.7 describes $F_\star^\xi$ on objects.
- Explanation 11.1.14 describes $F_\star^\xi$ on morphisms.

**Defining the Functor $F_\star^\xi$.**

**Definition 11.1.1.** Suppose given the data (i) through (v) below.

- (i) $\left( \mathsf{M}, \underline{\mathsf{M}}, \mathrm{ev}^\mathsf{M} \right)$ and $\left( \mathsf{N}, \underline{\mathsf{N}}, \mathrm{ev}^\mathsf{N} \right)$ are non-symmetric closed multicategories (Definition 8.1.1).
- (ii) $(\mathsf{C}, m, i)$ is a small $\mathsf{N}$-category (Definition 6.1.1).
- (iii) $F$ and $E$ are non-symmetric multifunctors (Definition C.1.19) as follows.

$$F : \mathsf{M} \rightleftarrows \mathsf{N} : E$$

(iv) $\kappa$ and $\zeta$ are multinatural transformations (Definition C.1.25) as follows.

$$\mathsf{M} \underset{EF}{\overset{1_\mathsf{M}}{\rightrightarrows}} \Downarrow\kappa \;\; \mathsf{M} \qquad\qquad \mathsf{N} \underset{FE}{\overset{1_\mathsf{N}}{\rightrightarrows}} \Downarrow\zeta \;\; \mathsf{N}$$

(v) For each pair of objects $x, y \in \mathsf{C}$, the two unary multimorphisms in $\mathsf{M}$

(11.1.2)
$$EC(x,y) \xrightarrow[\overset{\kappa_{EC(x,y)}}{E\zeta_{C(x,y)}}]{} EFEC(x,y)$$

are equal, where $\mathsf{C}(x,y) \in \mathrm{Ob}\,\mathsf{N}$ is a morphism object of $\mathsf{C}$.

We define the functor $F_\star^\zeta$ as the composite

(11.1.3)
$$\overbrace{\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \xrightarrow{F_\star} \mathsf{N\text{-}Cat}(\mathsf{C}_{FE}, \mathsf{N}) \xrightarrow{\mathsf{C}_\zeta^*} \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})}^{F_\star^\zeta}$$

involving the following data.

- Each of $\mathsf{M}$ and $\mathsf{N}$ is equipped with the canonical self-enrichment (Theorem 9.1.7).
- The change-of-enrichment 2-functors (Propositions 7.1.9 and 7.4.1)

(11.1.4)
$$\mathsf{M\text{-}Cat} \underset{(-)_E}{\overset{(-)_F}{\rightleftarrows}} \mathsf{N\text{-}Cat} \circlearrowright \quad (-)_{FE} = (-)_F \circ (-)_E$$

are induced by $F$, $E$, and $FE$. The $\mathsf{M}$-category $\mathsf{C}_E$ and the $\mathsf{N}$-category $\mathsf{C}_{FE}$ are obtained from $\mathsf{C}$ by applying, respectively, $(-)_E$ and $(-)_{FE}$.
- $F_\star$ in (11.1.3) is the diagram change-of-enrichment functor of $F$ at the $\mathsf{M}$-category $\mathsf{C}_E$ (Theorem 10.3.1).
- The $\mathsf{N}$-functor

(11.1.5)
$$\mathsf{C}_\zeta : \mathsf{C} \longrightarrow \mathsf{C}_{FE}$$

is the $\mathsf{C}$-component (7.5.3) of the 2-natural transformation

$$\mathsf{N\text{-}Cat} \underset{(-)_{FE}}{\overset{1_{\mathsf{N\text{-}Cat}}}{\rightrightarrows}} \Downarrow(-)_\zeta \;\; \mathsf{N\text{-}Cat}$$

induced by $\zeta : 1_\mathsf{N} \longrightarrow FE$ (Proposition 7.5.5).
- The functor $\mathsf{C}_\zeta^*$ in (11.1.3) is given by pre-composing and whiskering with the $\mathsf{N}$-functor $\mathsf{C}_\zeta$ in (11.1.5).

This finishes the definition of the functor $F_\star^\zeta$. $\qquad\qquad\diamond$

**Remark 11.1.6.** The functor $F_\star^\zeta$ in (11.1.3) does *not* use the multinatural transformation $\kappa$ in Definition 11.1.1 (iv) and condition (v). For the discussion below, it is more convenient to state $\kappa$ and $\zeta$ together in one place. We use $\kappa$ in Definition 11.2.1 below. We use condition (v) in (11.2.23); see also Remark 11.3.22. Instances of condition (v) include

- Lemma 4.6.13, which is used in the proof of Theorem 12.1.6;
- Lemma 5.5.11, which is used in the proof of Theorem 12.4.6; and
- the right triangle identity of the 2-adjunction $(\mathcal{M}\underline{1} \wedge -, \mathsf{U}_{\mathcal{M}\underline{1}})$ in (12.6.8). $\quad\diamond$

In Explanations 11.1.7 and 11.1.14, we explicitly describe the object and morphism assignments of the functor

$$F_\star^\xi : \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \xrightarrow{F_\star} \mathsf{N\text{-}Cat}(\mathsf{C}_{FE}, \mathsf{N}) \xrightarrow{\mathsf{C}_\xi^*} \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$$

in (11.1.3).

**Unraveling the Functor $F_\star^\xi$.**

**Explanation 11.1.7** ($F_\star^\xi$ on Objects). For an M-functor $A : \mathsf{C}_E \longrightarrow \mathsf{M}$ (Definition 6.1.7), the N-functor

$$(11.1.8) \qquad \overbrace{\mathsf{C}_{FE} = (\mathsf{C}_E)_F \xrightarrow{A_F} \mathsf{M}_F \xrightarrow{\widehat{F}}}^{F_\star A} \mathsf{N}$$

is, by Definition 10.2.1, the composite of two N-functors.

- $A_F$ is the image of $A$ under the change-of-enrichment 2-functor $(-)_F$ (Proposition 7.1.9).
- $\widehat{F}$ is the standard enrichment of $F : \mathsf{M} \longrightarrow \mathsf{N}$ (Theorem 9.2.12).

Its object assignment is given by

$$(F_\star A)x = FAx \quad \text{for} \quad x \in \mathsf{C}.$$

For objects $x, y \in \mathsf{C}$, the $(x, y)$-component of $F_\star A$ is the composite unary multimorphism in N

$$(11.1.9) \qquad \overbrace{FE\mathsf{C}(x,y) \xrightarrow{FA_{x,y}} F\underline{\mathsf{M}}(Ax\,;\,Ay) \xrightarrow{\widehat{F}_{Ax,Ay}}}^{(F_\star A)_{x,y}} \underline{\mathsf{N}}(FAx\,;\,FAy).$$

By definition (9.2.3),

$$(11.1.10) \qquad \widehat{F}_{Ax,Ay} = \left(F(\mathsf{ev}_{Ax\,;\,Ay}^{\mathsf{M}})\right)^{\#}$$

is the partner of the binary multimorphism

$$F\left(\mathsf{ev}_{Ax\,;\,Ay}^{\mathsf{M}}\right) : \left(F\underline{\mathsf{M}}(Ax\,;\,Ay), FAx\right) \longrightarrow FAy \quad \text{in} \quad \mathsf{N}.$$

By definition (11.1.3) the N-functor

$$(11.1.11) \qquad \overbrace{\mathsf{C} \xrightarrow{\mathsf{C}_\xi} \mathsf{C}_{FE} = (\mathsf{C}_E)_F \xrightarrow{A_F} \mathsf{M}_F \xrightarrow{\widehat{F}}}^{F_\star^\xi A} \mathsf{N}$$

is the composite of $\mathsf{C}_\xi$ in (11.1.5) and $F_\star A$ in (11.1.8). Since $\mathsf{C}_\xi$ is the identity on objects, the object assignment of $F_\star^\xi A$ is given by

$$(11.1.12) \qquad (F_\star^\xi A)x = FAx \quad \text{for} \quad x \in \mathsf{C}.$$

For objects $x, y \in \mathsf{C}$, the $(x, y)$-component of $F_\star^\xi A$ is the composite unary multimorphism in N

$$(11.1.13)$$

$$
\begin{array}{ccc}
\mathsf{C}(x,y) & \xrightarrow{\quad (F_\star^\xi A)_{x,y} \quad} & \underline{\mathsf{N}}(FAx\,;\,FAy) \\
{\scriptstyle \xi_{\mathsf{C}(x,y)}} \searrow & & \nearrow {\scriptstyle \widehat{F}_{Ax,Ay}} \\
& FE\mathsf{C}(x,y) \xrightarrow{\quad FA_{x,y} \quad} F\underline{\mathsf{M}}(Ax\,;\,Ay) &
\end{array}
$$

of the $\mathsf{C}(x,y)$-component of $\xi$ and $(F_\star A)_{x,y}$ in (11.1.9). $\diamond$

**Explanation 11.1.14** ($F_\star^\xi$ on Morphisms)**.** Consider an M-natural transformation $\psi$ (Definition 6.1.14) as in the left diagram below.

$$(11.1.15) \qquad \mathsf{C}_E \underset{B}{\overset{A}{\underset{\Downarrow\psi}{\longrightarrow}}} \mathsf{M} \qquad\qquad \mathsf{C} \underset{F_\star^\xi B}{\overset{F_\star^\xi A}{\underset{\Downarrow F_\star^\xi \psi}{\longrightarrow}}} \mathsf{N}$$

Then the N-natural transformation $F_\star^\xi \psi$, as in the right diagram in (11.1.15), is given by the following whiskering, where $\psi_F$ is the image of $\psi$ under the change of enrichment $(-)_F$ (Proposition 7.1.9).

$$(11.1.16) \qquad \mathsf{C} \xrightarrow{\ \mathsf{C}_\xi\ } \mathsf{C}_{FE} = (\mathsf{C}_E)_F \underset{B_F}{\overset{A_F}{\underset{\Downarrow\psi_F}{\longrightarrow}}} \mathsf{M}_F \xrightarrow{\ \widehat{F}\ } \mathsf{N}$$

This diagram is obtained from (11.1.11) by replacing $A$ by $\psi$.

For each object $x \in \mathsf{C}$, since $\mathsf{C}_\xi$ is the identity on objects, the $x$-component of $F_\star^\xi \psi$ in (11.1.16) is the nullary multimorphism in N given by the composite

$$(11.1.17) \qquad \langle\rangle \xrightarrow{\ F\psi_x\ } F\underline{\mathsf{M}}(Ax; Bx) \xrightarrow{\ \widehat{F}_{Ax,Bx}\ } \underline{\mathsf{N}}(FAx; FBx)$$

with the composite labelled $(F_\star^\xi \psi)_x$.

precisely as in (10.2.16).

- $F\psi_x$ is the image under $F$ of the $x$-component of $\psi$, which is a nullary multimorphism

$$\langle\rangle \xrightarrow{\ \psi_x\ } \underline{\mathsf{M}}(Ax; Bx) \quad \text{in} \quad \mathsf{M}.$$

- The unary multimorphism

$$\widehat{F}_{Ax,Bx} = \big(F(\mathrm{ev}^{\mathsf{M}}_{Ax;Bx})\big)^\#$$

  is the partner of the binary multimorphism

$$F\big(\mathrm{ev}^{\mathsf{M}}_{Ax;Bx}\big) : \big(F\underline{\mathsf{M}}(Ax; Bx), FAx\big) \longrightarrow FBx \quad \text{in} \quad \mathsf{N}.$$

As in (10.2.19), the partner of $(F_\star^\xi \psi)_x$ in (11.1.17) is the unary multimorphism

$$(11.1.18) \qquad (F_\star^\xi \psi)_x^\# = F(\psi_x^\#) : FAx \longrightarrow FBx \quad \text{in} \quad \mathsf{N},$$

where $\psi_x^\# : Ax \longrightarrow Bx$ is the partner of $\psi_x$ (10.2.18). $\diamond$

In the context of Definition 11.1.1, there are two functors

$$(11.1.19) \qquad \mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star}{\overset{F_\star^\xi}{\rightleftarrows}} \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{N})$$

as follows.

- $F_\star^\xi = \mathsf{C}_\xi^* F_\star$ is the functor in (11.1.3).
- $E_\star$ is the diagram change-of-enrichment functor of $E : \mathsf{N} \longrightarrow \mathsf{M}$ at the N-category C (Theorem 10.3.1).

In Sections 11.2 and 11.3 we relate the composite functors $E_\star F_\star^\xi$ and $F_\star^\xi E_\star$ to the respective identity functors using the multinatural transformations $\kappa$ and $\xi$.

## 11.2. Comparing $E_\star F_\star^\xi$ and the Identity

Throughout this section we assume the same context as in Definition 11.1.1. In this section we extend the multinatural transformation in Definition 11.1.1 (iv)

$$\mathsf{M} \underset{EF}{\overset{1_\mathsf{M}}{\Longrightarrow}} \mathsf{M}$$

to a natural transformation $\kappa^\star$ comparing $E_\star F_\star^\xi$ in (11.1.19) and the identity functor. This section is organized as follows.

- $\kappa^\star$ is in Definition 11.2.1.
- To show that $\kappa^\star$ has the desired naturality properties, in Explanations 11.2.5 and 11.2.6 and Lemma 11.2.10 we discuss the codomain $E_\star F_\star^\xi$ of $\kappa^\star$ and its object assignment.
- We describe $E_\star F_\star^\xi$ on morphisms in Explanation 11.2.14 and Lemma 11.2.18.
- We show that $\kappa^\star$ is a natural transformation in Lemmas 11.2.19 and 11.2.24.

**The Natural Transformation $\kappa^\star$.**

**Definition 11.2.1.** In the context of Definition 11.1.1 and (11.1.19), we define the data of a natural transformation

$$(11.2.2) \qquad \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star F_\star^\xi}{\overset{1}{\Longrightarrow}} \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M})$$

as follows. For an M-functor $A : \mathsf{C}_E \longrightarrow \mathsf{M}$ (Definition 6.1.7), the $A$-component of $\kappa^\star$ is the M-natural transformation (Definition 6.1.14)

$$(11.2.3) \qquad \mathsf{C}_E \underset{E_\star F_\star^\xi A}{\overset{A}{\Longrightarrow}} \mathsf{M}$$

with, for each object $x \in \mathsf{C}$, $x$-component given by the nullary multimorphism

$$(11.2.4) \qquad (\kappa_A^\star)_x = \kappa_{Ax}^\# : \langle\rangle \longrightarrow \underline{\mathsf{M}}(Ax; EFAx) \quad \text{in} \quad \mathsf{M}.$$

This is the partner (8.1.11) of the $(Ax)$-component

$$\kappa_{Ax} : Ax \longrightarrow EFAx$$

of the multinatural transformation $\kappa : 1_\mathsf{M} \longrightarrow EF$, which is a unary multimorphism in M. This finishes the definition of $\kappa^\star$. We check that

- $\kappa_A^\star$ is an M-natural transformation in Lemma 11.2.19 and
- $\kappa^\star$ is a natural transformation in Lemma 11.2.24.                    $\diamond$

Before we prove the M-naturality of $\kappa_A^\star$ and the naturality of $\kappa^\star$, we first discuss the codomain $E_\star F_\star^\xi$ of $\kappa^\star$ in detail in Explanations 11.2.5, 11.2.6, and 11.2.14.

**Explanation 11.2.5** (Codomain of $\kappa^\star$)**.** In the context of Definitions 11.1.1 and 11.2.1, by (10.2.3) and (11.1.11) the codomain of $\kappa^\star$ in (11.2.2) is the composite of the functors

$$F_\star^\xi = \widehat{F} \circ (-)_F \circ \mathsf{C}_\xi \quad \text{and}$$
$$E_\star = \widehat{E} \circ (-)_E$$

involving the following.

- The change-of-enrichment 2-functors $(-)_E$ and $(-)_F$ are as in (11.1.4).
- $C_\xi : C \longrightarrow C_{FE}$ is the N-functor in (11.1.5).
- The standard enrichment

$$\widehat{F} : M_F \longrightarrow N \quad \text{and} \quad \widehat{E} : N_E \longrightarrow M$$

are from in Theorem 9.2.12. $\diamond$

**Explanation 11.2.6** ($E_\star F_\star^\xi$ on Objects). For an M-functor $A : C_E \longrightarrow M$ (Definition 6.1.7), the functoriality of $(-)_E$ implies that the codomain of $\kappa_A^\star$ in (11.2.3) is the following M-functor.

$$\begin{aligned}
E_\star F_\star^\xi A &= \widehat{E} \circ (F_\star^\xi A)_E \\
&= \widehat{E} \circ \big(\widehat{F} \circ A_F \circ C_\xi\big)_E \\
&= \widehat{E} \circ \widehat{F}_E \circ (A_F)_E \circ (C_\xi)_E : C_E \longrightarrow M
\end{aligned}$$

By Proposition 7.4.1, $E_\star F_\star^\xi A$ is the following composite M-functor.

(11.2.7)

$$\begin{array}{ccc}
C_E & \xrightarrow{\quad E_\star F_\star^\xi A \quad} & M \\
{\scriptstyle (C_\xi)_E}\downarrow & & \uparrow{\scriptstyle \widehat{E}} \\
C_{EFE} & \xrightarrow{A_{EF}} M_{EF} \xrightarrow{\widehat{F}_E} & N_E
\end{array}$$

We explain its object assignment and components in (11.2.8) and (11.2.9), respectively.

*Object Assignment*. By (10.2.7) applied to $E_\star$ and (11.1.12), the object assignment of $E_\star F_\star^\xi A$ in (11.2.7) is given by

(11.2.8) $$(E_\star F_\star^\xi A)x = EFAx \quad \text{for} \quad x \in C.$$

This implies that the $x$-component $(\kappa_A^\star)_x$ in (11.2.4) is well defined.

*Components*. By (11.2.7), for objects $x, y \in C$, the $(x, y)$-component of $E_\star F_\star^\xi A$ is the composite unary multimorphism along the boundary of the following diagram in M.

(11.2.9)

$$\begin{array}{c}
E C(x, y) \xrightarrow{\quad (E_\star F_\star^\xi A)_{x,y} \quad} \underline{M}\big(EFAx\,;\, EFAy\big) \\
{\scriptstyle E\xi_{C(x,y)}}\swarrow \quad {\scriptstyle (EF(A_{x,y}^\#))^\#} \qquad \qquad \nwarrow {\scriptstyle \widehat{E}_{FAx, FAy}} \\
EFEC(x, y) \qquad {\scriptstyle (EF(\mathrm{ev}_{Ax;Ay}^M))^\#} \qquad E\underline{N}\big(FAx\,;\, FAy\big) \\
{\scriptstyle EFA_{x,y}}\searrow \qquad \nearrow \qquad \nearrow {\scriptstyle E\widehat{F}_{Ax,Ay}} \\
EF\underline{M}\big(Ax\,;\, Ay\big)
\end{array}$$

By definition (9.2.3), in the lower right arrow, $\widehat{F}_{Ax,Ay}$ is as in (11.1.10). The upper right arrow

$$\widehat{E}_{FAx,FAy} = \big(E(\mathrm{ev}_{FAx;FAy}^N)\big)^\#$$

is the partner of the binary multimorphism

$$E\big(\mathrm{ev}_{FAx;FAy}^N\big) : \big(E\underline{N}\big(FAx\,;\, FAy\big), EFAx\big) \longrightarrow EFAy$$

in M. In Lemma 11.2.10 we discuss the two interior arrows in (11.2.9). $\diamond$

**Lemma 11.2.10.** *The diagram (11.2.9) is commutative.*

*Proof.* Since the boundary of the diagram (11.2.9) commutes by (11.2.7), it suffices to prove the following two equalities regarding its two interior arrows.

(11.2.11)
$$\gamma^{\mathsf{M}}\big(\widehat{E}_{FAx,FAy}; E\widehat{F}_{Ax,Ay}\big) = \big(EF(\mathrm{ev}^{\mathsf{M}}_{Ax;Ay})\big)^{\#}$$
$$\gamma^{\mathsf{M}}\big(\big(EF(\mathrm{ev}^{\mathsf{M}}_{Ax;Ay})\big)^{\#}; EFA_{x,y}\big) = \big(EF(A^{\#}_{x,y})\big)^{\#}$$

To prove these equalities, we consider the following diagram in M.

(11.2.12)
$$
\begin{array}{ccc}
\big(EFE\mathsf{C}(x,y), EFAx\big) & \xrightarrow{\ EF(A^{\#}_{x,y})\ } & EFAy \\
\ \downarrow{\scriptstyle (EFA_{x,y},\,1)} & & \parallel \\
\big(EF\underline{\mathsf{M}}(Ax;Ay), EFAx\big) & \xrightarrow{\ EF(\mathrm{ev}^{\mathsf{M}}_{Ax;Ay})\ } & EFAy \\
\ \downarrow{\scriptstyle (E\widehat{F}_{Ax,Ay},\,1)} & & \parallel \\
\big(E\underline{\mathsf{N}}(FAx;FAy), EFAx\big) & \xrightarrow{\ E(\mathrm{ev}^{\mathsf{N}}_{FAx;FAy})\ } & EFAy \\
\ \downarrow{\scriptstyle (\widehat{E}_{FAx,FAy},\,1)} & & \parallel \\
\big(\underline{\mathsf{M}}(EFAx;EFAy), EFAx\big) & \xrightarrow{\ \mathrm{ev}^{\mathsf{M}}_{EFAx;EFAy}\ } & EFAy
\end{array}
$$

The three sub-regions in (11.2.12) are commutative for the following reasons.

- In the top sub-region, $A^{\#}_{x,y}$ is the partner (8.1.7) of $A_{x,y}$. By definition the following diagram in M commutes.

(11.2.13)
$$
\begin{array}{ccc}
\big(E\mathsf{C}(x,y), Ax\big) & & \\
\ \downarrow{\scriptstyle (A_{x,y},\,1)} & \searrow{\scriptstyle A^{\#}_{x,y}} & \\
\big(\underline{\mathsf{M}}(Ax;Ay), Ax\big) & \xrightarrow[\ \mathrm{ev}^{\mathsf{M}}_{Ax;Ay}\ ]{} & Ay
\end{array}
$$

     Applying the non-symmetric multifunctor $EF$ to this commutative diagram yields the top sub-region in (11.2.12).
- The middle sub-region in (11.2.12) is obtained from the commutative diagram (9.2.11) defining $\widehat{F}_{Ax,Ay}$ by applying the non-symmetric multifunctor $E$.
- The bottom sub-region in (11.2.12) is the commutative diagram (9.2.11) that defines $\widehat{E}_{FAx,FAy}$.

Since taking partner is a bijection (8.1.7), the commutativity of the bottom two sub-regions in (11.2.12) proves the first desired equality in (11.2.11). The second equality in (11.2.11) follows from the first equality and the boundary of the commutative diagram (11.2.12).      □

**Explanation 11.2.14** ($E_\star F^{\xi}_\star$ on Morphisms)**.** Suppose $\psi : A \longrightarrow B$ is an M-natural transformation (Definition 6.1.14) as in the left diagram below.

$$
\mathsf{C}_E \ \underset{B}{\overset{A}{\rightrightarrows^{\Downarrow\psi}}} \ \mathsf{M}
\qquad\qquad
\mathsf{C}_E \ \underset{E_\star F^{\xi}_\star B}{\overset{E_\star F^{\xi}_\star A}{\rightrightarrows^{\Downarrow E_\star F^{\xi}_\star \psi}}} \ \mathsf{M}
$$

By Explanation 11.2.5 the M-natural transformation $E_\star F_\star^\xi \psi$, as in the right diagram above, is the following whiskering.

$$(11.2.15) \qquad C_E \xrightarrow{(C_\xi)_E} C_{EFE} \overset{A_{EF}}{\underset{B_{EF}}{\Downarrow \psi_{EF}}} M_{EF} = (M_F)_E \xrightarrow{\widehat{F}_E} N_E \xrightarrow{\widehat{E}} M$$

Since $(C_\xi)_E$ has the identity object assignment, by (7.1.8) and (11.2.15), for each object $x \in C$ the $x$-component of $E_\star F_\star^\xi \psi$ is the following composite nullary multimorphism in M.

$$(11.2.16) \qquad \begin{array}{ccc} \langle\rangle & \xrightarrow{(E_\star F_\star^\xi \psi)_x} & \underline{M}(EFAx; EFBx) \\[2pt] {\scriptstyle EF(\psi_x)} \downarrow & & \uparrow {\scriptstyle \widehat{E}_{FAx,FBx}} \\[2pt] EF\underline{M}(Ax; Bx) & \xrightarrow{E\widehat{F}_{Ax,Bx}} & E\underline{N}(FAx; FBx) \end{array}$$

In (11.2.16),

- $\psi_x : \langle\rangle \longrightarrow \underline{M}(Ax; Bx)$ is the $x$-component of $\psi$ (6.1.15), and
- the component morphisms $\widehat{F}_{Ax,Bx}$ and $\widehat{E}_{FAx,FBx}$ are as in (9.2.3).

Taking the partner (8.1.7) of $(E_\star F_\star^\xi \psi)_x$ yields the top unary multimorphism in M below.

$$(11.2.17) \qquad EFAx \mathrel{\substack{\xrightarrow{(E_\star F_\star^\xi \psi)_x^\#} \\ \xrightarrow{EF(\psi_x^\#)}}} EFBx$$

The bottom arrow is the image under $EF$ of the unary multimorphism

$$\psi_x^\# : Ax \longrightarrow Bx \quad \text{in} \quad M,$$

which is the partner of $\psi_x$. Lemma 11.2.18 proves that they are the same. $\diamond$

**Lemma 11.2.18.** *The two unary multimorphisms in (11.2.17) are equal.*

*Proof.* Since $(E_\star F_\star^\xi \psi)_x$ is the composite in (11.2.16), its partner is, by definition (8.1.7), the left-bottom composite in the following diagram in M.

$$\begin{array}{ccc} (\langle\rangle, EFAx) & \xrightarrow{EF(\psi_x^\#)} & EFBx \\[2pt] {\scriptstyle (EF(\psi_x),1)} \downarrow & & \Big\| \\[2pt] \big(EF\underline{M}(Ax; Bx), EFAx\big) & \xrightarrow{EF(\mathrm{ev}^M_{Ax;Bx})} & EFBx \\[2pt] {\scriptstyle (E\widehat{F}_{Ax,Bx},1)} \downarrow & & \Big\| \\[2pt] \big(E\underline{N}(FAx; FBx), EFAx\big) & \xrightarrow{E(\mathrm{ev}^N_{FAx;FBx})} & EFBx \\[2pt] {\scriptstyle (\widehat{E}_{FAx,FBx},1)} \downarrow & & \Big\| \\[2pt] \big(\underline{M}(EFAx; EFBx), EFAx\big) & \xrightarrow{\mathrm{ev}^M_{EFAx;EFBx}} & EFBx \end{array}$$

The three sub-regions in the above diagram are commutative for the following reasons.

- The top sub-region is $EF$ applied to the commutative diagram in $\mathsf{M}$

$$
\begin{array}{ccc}
(\langle\rangle,\, Ax) & \xrightarrow{\hspace{2cm}\psi_x^{\#}} & \\
\ \Big\downarrow{\scriptstyle(\psi_x,\,1)} & & \\
\big(\underline{\mathsf{M}}(Ax;\, Bx),\, Ax\big) & \xrightarrow{\ \mathrm{ev}^{\mathsf{M}}_{Ax;\, Bx}\ } & Bx
\end{array}
$$

  that defines the partner $\psi_x^{\#}$.
- The middle sub-region is $E$ applied to the commutative diagram (9.2.11) that defines $\widehat{F}_{Ax,Bx}$.
- The bottom sub-region is the commutative diagram (9.2.11) that defines $\widehat{E}_{FAx,FBx}$.

This proves that $(E_\star F_\star^\xi \psi)_x^{\#}$ is equal to $EF(\psi_x^{\#})$.                    $\square$


**Naturality of $\kappa^\star$.** To prove that $\kappa^\star$ is a natural transformation, we first check that its components are well defined.

**Lemma 11.2.19.** *In the context of Definitions 11.1.1 and 11.2.1, for each $\mathsf{M}$-functor $A :$*
$\mathsf{C}_E \longrightarrow \mathsf{M}$,

$$
\mathsf{C}_E \underset{E_\star F_\star^\xi A}{\overset{A}{\Longrightarrow\kappa_A^\star}} \mathsf{M}
$$

*in (11.2.3) is an $\mathsf{M}$-natural transformation.*


*Proof.* We must prove that the naturality diagram (6.1.16) for $\kappa_A^\star$, which is the diagram in $\mathsf{M}$ below for objects $x, y \in \mathsf{C}$, is commutative.

$$
(11.2.20)\quad
\begin{array}{ccc}
\big(\langle\rangle,\, \mathsf{C}_E(x,y)\big) & \xrightarrow{\ \big((\kappa_A^\star)_y,\, A_{x,y}\big)\ } & \big(\underline{\mathsf{M}}(Ay; EFAy),\, \underline{\mathsf{M}}(Ax; Ay)\big) \\
\ \big\| & & \Big\downarrow{\scriptstyle\circ} \\
\big(\mathsf{C}_E(x,y),\, \langle\rangle\big) & & \\
\ \big\downarrow{\scriptstyle\big((E_\star F_\star^\xi A)_{x,y},\, (\kappa_A^\star)_x\big)} & & \\
\big(\underline{\mathsf{M}}(EFAx; EFAy),\, \underline{\mathsf{M}}(Ax; EFAx)\big) & \xrightarrow{\hspace{1.5cm}\circ\hspace{1.5cm}} & \underline{\mathsf{M}}(Ax; EFAy)
\end{array}
$$

Since taking partner in $\mathsf{M}$ is a bijection (8.1.7), it suffices to show that the two composites in (11.2.20) have the same partner. We compute these partners in (11.2.21) and (11.2.22). Then we observe that they are equal to finish the proof.

*Top-Right Composite.* The partner of the top-right composite in (11.2.20) is, by definition, the left-bottom composite in the following diagram in M.

(11.2.21)

$$
\begin{array}{c}
(EC(x,y)\,,\,Ax) \xrightarrow{\ \left(\kappa_{EC(x,y)}\,,\,\kappa_{Ax}\right)\ } (EFEC(x,y)\,,\,EFAx) \\
\end{array}
$$

The four sub-regions in (11.2.21) are commutative for the following reasons.

- By Definition 7.1.1, there is an equality of objects in M

$$\mathsf{C}_E(x,y) = EC(x,y).$$

  The top trapezoid commutes by the naturality (C.1.26) of the multinatural transformation $\kappa : 1_{\mathsf{M}} \longrightarrow EF$.

- The top left triangle commutes by the definition of $A_{x,y}^{\#}$ in (11.2.13) and the definition of the $y$-component of $\kappa_A^{\star}$ in (11.2.4),

$$\left(\kappa_A^{\star}\right)_y = \kappa_{Ay}^{\#} : \langle\rangle \longrightarrow \underline{\mathsf{M}}\bigl(Ay\,;\,EFAy\bigr),$$

  as the partner of the $(Ay)$-component of $\kappa$.

- The bottom rectangle commutes by the definition of the composition $\circ$ in the canonical self-enrichment of M in (9.1.6).

- The right sub-region commutes by the definition of $\kappa_{Ay}^{\#}$ as the partner of $\kappa_{Ay}$ (8.1.7).

This proves that the diagram (11.2.21) is commutative.

*Left-Bottom Composite.* The partner of the left-bottom composite in (11.2.20) is the left-bottom composite in the following diagram in M.

(11.2.22)

The five sub-regions in (11.2.22) are commutative for the following reasons.

- The top sub-region commutes by the unity properties, (C.1.9) and (C.1.10), in M.
- The bottom triangle commutes by the definition of $\circ$ in (9.1.6).
- The sub-region labeled $\boxdot$ commutes by the definition of $(\kappa_A^\star)_x$ as the partner of $\kappa_{Ax}$ (11.2.4).
- The triangle labeled $\diamondsuit$ commutes by the definition of $(E_\star F_\star^\xi A)_{x,y}^\#$ as the partner of $(E_\star F_\star^\xi A)_{x,y}$ (8.1.7).
- The triangle labeled $\star$ is the boundary of the following diagram.

$$
\begin{array}{c}
\big(E\mathsf{C}(x,y)\,,\,EFAx\big) \xrightarrow{\quad (E_\star F_\star^\xi A)_{x,y}^\#\quad} EFAy \\[2em]
\big((E_\star F_\star^\xi A)_{x,y}\,,\,1\big) \qquad \mathrm{ev}^\mathsf{M} \\[1em]
\big(\underline{\mathsf{M}}(EFAx;EFAy)\,,\,EFAx\big) \\[1em]
\big(E\xi_{\mathsf{C}(x,y)}\,,\,1\big) \qquad \big((EF(A_{x,y}^\#))^\#\,,\,1\big) \qquad EF(A_{x,y}^\#) \\[1em]
\big(EFE\mathsf{C}(x,y)\,,\,EFAx\big)
\end{array}
$$

This diagram is commutative for the following reasons.
  - The top triangle commutes by the definition of $(E_\star F_\star^\xi A)_{x,y}^\#$ as the partner of $(E_\star F_\star^\xi A)_{x,y}$ (8.1.7).
  - The right sub-region commutes by the definition of $(EF(A_{x,y}^\#))^\#$ as the partner of $EF(A_{x,y}^\#)$ (8.1.7).
  - The left sub-region commutes by the top commutative triangle in (11.2.9), which is proved in Lemma 11.2.10.

This proves that the diagram (11.2.22) is commutative.

*Comparing Partners.* By the commutative diagrams (11.2.21) and (11.2.22), the partners of the top-right composite and of the left-bottom composite in the desired diagram (11.2.20) are the following binary multimorphisms in M.

(11.2.23)
$$
\begin{aligned}
&\gamma^\mathsf{M}\big(EF(A_{x,y}^\#)\,;\,\kappa_{E\mathsf{C}(x,y)}\,,\,\kappa_{Ax}\big) \\
&\gamma^\mathsf{M}\big(EF(A_{x,y}^\#)\,;\,E\xi_{\mathsf{C}(x,y)}\,,\,\kappa_{Ax}\big)
\end{aligned}
$$

The two binary multimorphisms in (11.2.23) are equal by Definition 11.1.1 (v), which assumes the equality

$$
\kappa_{E\mathsf{C}(x,y)} = E\xi_{\mathsf{C}(x,y)} : E\mathsf{C}(x,y) \longrightarrow EFE\mathsf{C}(x,y)
$$

for all objects $x, y \in \mathsf{C}$. Since taking partner is a bijection (8.1.7), we conclude that the diagram (11.2.20) is commutative. $\qquad\square$

**Lemma 11.2.24.** *In the context of Definitions 11.1.1 and 11.2.1,*

$$
\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star F_\star^\xi}{\overset{1}{\rightrightarrows}} {\Downarrow\kappa^\star}\ \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M})
$$

*in (11.2.2) is a natural transformation.*

*Proof.* Lemma 11.2.19 proves that each component of $\kappa^\star$ is a well-defined M-natural transformation. Naturality for $\kappa^\star$ means that, for each M-natural transformation $\psi$ as in

$$\mathsf{C}_E \overset{A}{\underset{B}{\Longrightarrow \psi}} \mathsf{M}$$

the following diagram of M-natural transformations in M-Cat$(\mathsf{C}_E, \mathsf{M})$ commutes.

(11.2.25)
$$
\begin{array}{ccc}
A & \xrightarrow{\ \kappa_A^\star\ } & E_\star F_\star^\xi A \\
\psi \downarrow & & \downarrow E_\star F_\star^\xi \psi \\
B & \xrightarrow{\ \kappa_B^\star\ } & E_\star F_\star^\xi B
\end{array}
$$

Since each M-natural transformation is determined by its components, it suffices to show that, for each object $x \in \mathsf{C}$, the two vertical composites in (11.2.25) have the same $x$-components. By Definition 6.1.18 these two $x$-components are the following two composite nullary multimorphisms in M.

(11.2.26)
$$
\begin{array}{ccc}
(\langle\rangle, \langle\rangle) & \xrightarrow{((E_\star F_\star^\xi \psi)_x,\, (\kappa_A^\star)_x)} & (\underline{\mathsf{M}}(EFAx; EFBx), \underline{\mathsf{M}}(Ax; EFAx)) \\
((\kappa_B^\star)_x,\, \psi_x) \downarrow & & \downarrow \circ \\
(\underline{\mathsf{M}}(Bx; EFBx), \underline{\mathsf{M}}(Ax; Bx)) & \xrightarrow{\quad \circ \quad} & \underline{\mathsf{M}}(Ax; EFBx)
\end{array}
$$

Since taking partner is a bijection (8.1.7), it suffices to show that the two composites in (11.2.26) have the same partner. We compute these partners in (11.2.27) and (11.2.28). Then we observe that they are equal to finish the proof.

*Top-Right Composite.* The partner of the top-right composite in (11.2.26) is the left-bottom composite unary multimorphism in the following diagram in M.

(11.2.27)
$$
\begin{array}{ccc}
(\langle\rangle, \langle\rangle, Ax) = Ax & \xrightarrow{\ \kappa_{Ax}\ } & (\langle\rangle, EFAx) \\
((E_\star F_\star^\xi \psi)_x, (\kappa_A^\star)_x, 1) \downarrow & ((EF(\psi_x^\#))^\#, 1) \downarrow & \\
(\underline{\mathsf{M}}(EFAx; EFBx), \underline{\mathsf{M}}(Ax; EFAx), Ax) & & EF(\psi_x^\#) \\
(\circ, 1) \downarrow & \xrightarrow{(1, \mathrm{ev}^\mathsf{M})} (\underline{\mathsf{M}}(EFAx; EFBx), EFAx) & \\
& \mathrm{ev}^\mathsf{M} \downarrow & \\
(\underline{\mathsf{M}}(Ax; EFBx), Ax) & \xrightarrow{\mathrm{ev}^\mathsf{M}} EFBx \longleftarrow &
\end{array}
$$

The three sub-regions in (11.2.27) are commutative for the following reasons.

- The bottom left sub-region commutes by the definition of $\circ$ in (9.1.6).
- The top left sub-region commutes by
  - the definition of $(\kappa_A^\star)_x$ as the partner of $\kappa_{Ax}$ in (11.2.4) and
  - Lemma 11.2.18, which implies that $(E_\star F_\star^\xi \psi)_x$ is the partner of $EF(\psi_x^\#)$.
- The right sub-region commutes by the definition of $(EF(\psi_x^\#))^\#$ as the partner of $EF(\psi_x^\#)$.

This proves that the diagram (11.2.27) is commutative.

*Left-Bottom Composite.* The partner of the left-bottom composite in (11.2.26) is the left-bottom composite unary multimorphism in the following diagram in M.

(11.2.28)

$$
\begin{array}{ccc}
\big(\langle\rangle, \langle\rangle, Ax\big) = Ax & \xrightarrow{\;\psi_x^{\#}\;} & \big(\langle\rangle, Bx\big) \\[2ex]
{\scriptstyle((\kappa_B^{\star})_x,\, \psi_x,\, 1)}\Big\downarrow & & {\scriptstyle(\kappa_{Bx}^{\#},\, 1)}\Big\downarrow \\[2ex]
\big(\underline{M}(Bx; EFBx), \underline{M}(Ax; Bx), Ax\big) & & \\[2ex]
{\scriptstyle(\circ,\,1)}\Big\downarrow & \xrightarrow{\;(1,\, \mathrm{ev}^{\mathsf{M}})\;} & \big(\underline{M}(Bx; EFBx), Bx\big) \\[2ex]
& & {\scriptstyle \mathrm{ev}^{\mathsf{M}}}\Big\downarrow \\[2ex]
\big(\underline{M}(Ax; EFBx), Ax\big) & \xrightarrow{\;\mathrm{ev}^{\mathsf{M}}\;} & EFBx
\end{array}
$$

The three sub-regions in (11.2.28) are commutative for the following reasons.

- The bottom left sub-region commutes by the definition of $\circ$ in (9.1.6).
- The top left sub-region commutes by
  - the definition of $(\kappa_B^{\star})_x$ as the partner of $\kappa_{Bx}$ in (11.2.4) and
  - the definition of $\psi_x^{\#}$ as the partner of $\psi_x$.
- The right sub-region commutes by the definition of $\kappa_{Bx}^{\#}$ as the partner of $\kappa_{Bx}$.

This proves that the diagram (11.2.28) is commutative.

*Comparing Partners.* By the commutative diagrams (11.2.27) and (11.2.28), the partners of the two composites in the desired diagram (11.2.26) are the composites in M as follows.

$$
\begin{array}{ccc}
Ax & \xrightarrow{\;\kappa_{Ax}\;} & EFAx \\[1ex]
{\scriptstyle\psi_x^{\#}}\Big\downarrow & & \Big\downarrow{\scriptstyle EF(\psi_x^{\#})} \\[1ex]
Bx & \xrightarrow{\;\kappa_{Bx}\;} & EFBx
\end{array}
$$

This diagram commutes by the naturality condition (C.1.26) of the multinatural transformation $\kappa : 1_{\mathsf{M}} \longrightarrow EF$. $\qquad\square$

## 11.3. Comparing $F_{\star}^{\xi} E_{\star}$ and the Identity

Throughout this section we assume the same context as in Definition 11.1.1 and consider the functors in (11.1.19):

$$
\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}_E, \mathsf{M}) \mathrel{\mathop{\rightleftarrows}^{F_{\star}^{\xi}}_{E_{\star}}} \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{N}).
$$

In this section we extend the multinatural transformation in Definition 11.1.1 (iv)

$$
\mathsf{N} \underset{FE}{\overset{1_{\mathsf{N}}}{\rightrightarrows}} \; {\Downarrow \xi} \; \mathsf{N}
$$

to a natural transformation $\xi^{\star}$ comparing $F_{\star}^{\xi} E_{\star}$ and the identity functor. This section is organized as follows.

- $\xi^{\star}$ is in Definition 11.3.1.

- To show that $\xi^\star$ has the desired naturality properties, in Explanation 11.3.5 and Lemma 11.3.9 we describe the codomain $F_\star^\xi E_\star$ of $\xi^\star$ on objects.
- We describe $F_\star^\xi E_\star$ on morphisms in Explanation 11.3.13 and Lemma 11.3.17.
- We show that $\xi^\star$ is a natural transformation in Lemmas 11.3.18 and 11.3.23.

**The Natural Transformation $\xi^\star$.**

**Definition 11.3.1.** In the context of Definition 11.1.1 and (11.1.19), we define the data of a natural transformation

$$(11.3.2) \qquad \text{N-Cat}(C, N) \underset{F_\star^\xi E_\star}{\overset{1}{\rightrightarrows}} \Downarrow \xi^\star \quad \text{N-Cat}(C, N)$$

as follows. For an N-functor $P : C \longrightarrow N$, the $P$-component of $\xi^\star$ is the N-natural transformation

$$(11.3.3) \qquad C \underset{F_\star^\xi E_\star P}{\overset{P}{\rightrightarrows}} \Downarrow \xi_P^\star \quad N$$

with, for each object $x \in C$, $x$-component given by the nullary multimorphism

$$(11.3.4) \qquad (\xi_P^\star)_x = \xi_{Px}^\# : \langle\rangle \longrightarrow \underline{N}(Px; FEPx) \quad \text{in} \quad N.$$

This is the partner (8.1.11) of the $(Px)$-component

$$\xi_{Px} : Px \longrightarrow FEPx$$

of the multinatural transformation $\xi : 1_N \longrightarrow FE$, which is a unary multimorphism in N. This finishes the definition of $\xi^\star$. We check that

- $\xi_P^\star$ is an N-natural transformation in Lemma 11.3.18 and
- $\xi^\star$ is a natural transformation in Lemma 11.3.23.  ⬦

Before we prove the N-naturality of $\xi_P^\star$ and the naturality of $\xi^\star$, we first discuss the codomain $F_\star^\xi E_\star$ of $\xi^\star$ in detail in Explanations 11.3.5 and 11.3.13.

**Explanation 11.3.5** ($F_\star^\xi E_\star$ on Objects)**.** The codomain of $\xi^\star$ in (11.3.2) is the composite of the functors

$$E_\star = \widehat{E} \circ (-)_E \quad \text{and}$$
$$F_\star^\xi = \widehat{F} \circ (-)_F \circ C_\xi$$

as in Explanation 11.2.5. For an N-functor $P : C \longrightarrow N$ (Definition 6.1.7), the codomain of $\xi_P^\star$ in (11.3.3) is the following composite N-functor.

$$(11.3.6)$$

$$\begin{array}{ccc}
C & \xrightarrow{\quad F_\star^\xi E_\star P \quad} & N \\
{\scriptstyle C_\xi} \downarrow & & \uparrow {\scriptstyle \widehat{F}} \\
C_{FE} & \xrightarrow[P_{FE}]{} N_{FE} = (N_E)_F \xrightarrow[\widehat{E}_F]{} & M_F
\end{array}$$

*Object Assignment.* Since $C_\xi$ (11.1.5) has the identity object assignment, by (7.1.6) and Definition 9.2.1 the object assignment of $F_\star^\xi E_\star P$ in (11.3.6) is given by

$$(11.3.7) \qquad (F_\star^\xi E_\star P)x = FEPx \quad \text{for} \quad x \in C.$$

This implies that the $x$-component $(\zeta_P^\star)_x$ in (11.3.4) is well defined.

*Components.* By (11.3.6), for objects $x, y \in \mathsf{C}$, the $(x, y)$-component of $F_\star^\zeta E_\star P$ is the composite unary multimorphism along the boundary of the following diagram in $\mathsf{N}$.

(11.3.8)

$$
\begin{array}{ccc}
\mathsf{C}(x,y) & \xrightarrow{\;(F_\star^\zeta E_\star P)_{x,y}\;} & \underline{\mathsf{N}}(FEPx\,;FEPy) \\
\end{array}
$$



In Lemma 11.3.9 we discuss the two interior arrows in (11.3.8).                    $\diamond$

**Lemma 11.3.9.** *The diagram (11.3.8) is commutative.*

*Proof.* Since the boundary of the diagram (11.3.8) commutes by (11.3.6), it suffices to prove the following two equalities regarding its two interior arrows.

(11.3.10)
$$
\gamma^\mathsf{N}\big(\widehat{F}_{EPx,EPy}\,;\,F\widehat{E}_{Px,Py}\big) = \big(FE(\mathsf{ev}^\mathsf{N}_{Px;Py})\big)^\#
$$
$$
\gamma^\mathsf{N}\left(\big(FE(\mathsf{ev}^\mathsf{N}_{Px;Py})\big)^\#\,;\,FEP_{x,y}\right) = \big(FE(P^\#_{x,y})\big)^\#
$$

To prove these equalities, we consider the following diagram in $\mathsf{N}$.

(11.3.11)



The three sub-regions in (11.3.11) are commutative for the following reasons.

- In the top sub-region, $P^\#_{x,y}$ is the partner (8.1.7) of $P_{x,y}$. By definition the following diagram in $\mathsf{N}$ commutes.

(11.3.12)



  Applying the non-symmetric multifunctor $FE$ to this commutative diagram yields the top sub-region in (11.3.11).
- The middle sub-region in (11.3.11) is obtained from the commutative diagram (9.2.11) defining $\widehat{E}_{Px,Py}$ by applying the non-symmetric multifunctor $F$.

- The bottom sub-region in (11.3.11) is the commutative diagram (9.2.11) that defines $\widehat{F}_{EPx,EPy}$.

Since taking partner is a bijection (8.1.7), the commutativity of the bottom two sub-regions in (11.3.11) proves the first desired equality in (11.3.10). The second equality in (11.3.10) follows from the first equality and the boundary of the commutative diagram (11.3.11). $\qquad\square$

**Explanation 11.3.13** ($F_\star^\xi E_\star$ on Morphisms). Suppose $\theta : P \longrightarrow Q$ is an N-natural transformation (Definition 6.1.14) as in the left diagram below.

$$
\mathsf{C} \xRightarrow[\;Q\;]{\;P\;} \Downarrow\theta \; \mathsf{N} \qquad\qquad \mathsf{C} \xRightarrow[\;F_\star^\xi E_\star Q\;]{\;F_\star^\xi E_\star P\;} \Downarrow F_\star^\xi E_\star\theta \; \mathsf{N}
$$

By Explanation 11.3.5 the N-natural transformation $F_\star^\xi E_\star\theta$, as in the right diagram above, is the following whiskering.

$$
(11.3.14) \qquad \mathsf{C} \xrightarrow{\;C_\xi\;} \mathsf{C}_{FE} \xRightarrow[\;Q_{FE}\;]{\;P_{FE}\;} \Downarrow\theta_{FE} \; \mathsf{N}_{FE} = (\mathsf{N}_E)_F \xrightarrow{\;\widehat{E}_F\;} \mathsf{M}_F \xrightarrow{\;\widehat{F}\;} \mathsf{N}
$$

Since $\mathsf{C}_\xi$ has the identity object assignment, by (7.1.8) and (11.3.14), for each object $x \in \mathsf{C}$ the $x$-component of $F_\star^\xi E_\star\theta$ is the following composite nullary multimorphism in N.

$$
(11.3.15) \qquad
\begin{array}{ccc}
\langle\rangle & \xrightarrow{\;(F_\star^\xi E_\star\theta)_x\;} & \underline{\mathsf{N}}\big(FEPx\,;FEQx\big) \\
{\scriptstyle FE(\theta_x)}\downarrow & & \uparrow{\scriptstyle \widehat{F}_{EPx,EQx}} \\
FE\underline{\mathsf{N}}(Px\,;Qx) & \xrightarrow{\;F\widehat{E}_{Px,Qx}\;} & F\underline{\mathsf{M}}\big(EPx\,;EQx\big)
\end{array}
$$

In (11.3.15),

- $\theta_x : \langle\rangle \longrightarrow \underline{\mathsf{N}}(Px\,;Qx)$ is the $x$-component of $\theta$ (6.1.15), and
- the component morphisms $\widehat{E}_{Px,Qx}$ and $\widehat{F}_{EPx,EQx}$ are as in (9.2.3).

Taking the partner (8.1.7) of $(F_\star^\xi E_\star\theta)_x$ yields the top unary multimorphism in N below.

$$
(11.3.16) \qquad FEPx \xrightarrow[\;FE(\theta_x^\#)\;]{\;(F_\star^\xi E_\star\theta)_x^\#\;} FEQx
$$

The bottom arrow is the image under $FE$ of the unary multimorphism

$$
\theta_x^\# : Px \longrightarrow Qx \quad\text{in}\quad \mathsf{N},
$$

which is the partner of $\theta_x$. Lemma 11.3.17 proves that they are the same. $\qquad\diamond$

**Lemma 11.3.17.** *The two unary multimorphisms in (11.3.16) are equal.*

*Proof.* Since $(F_\star^\zeta E_\star \theta)_x$ is the composite in (11.3.15), its partner is, by definition (8.1.7), the left-bottom composite in the following diagram in N.

$$
\begin{array}{ccc}
(\langle\rangle, FEPx) & \xrightarrow{\;\;FE(\theta_x^\#)\;\;} & FEQx \\[2pt]
{\scriptstyle(FE(\theta_x),\,1)}\Big\downarrow & & \Big\| \\[2pt]
\big(FE\underline{N}(Px;Qx), FEPx\big) & \xrightarrow{\;FE(\mathsf{ev}^{\mathsf N}_{Px;Qx})\;} & FEQx \\[2pt]
{\scriptstyle(F\widehat{E}_{Px,Qx},\,1)}\Big\downarrow & & \Big\| \\[2pt]
\big(F\underline{M}(EPx;EQx), FEPx\big) & \xrightarrow{\;F(\mathsf{ev}^{\mathsf M}_{EPx;EQx})\;} & FEQx \\[2pt]
{\scriptstyle(\widehat{F}_{EPx,EQx},\,1)}\Big\downarrow & & \Big\| \\[2pt]
\big(\underline{N}(FEPx;FEQx), FEPx\big) & \xrightarrow{\;\mathsf{ev}^{\mathsf N}_{FEPx;FEQx}\;} & FEQx
\end{array}
$$

The three sub-regions in the above diagram are commutative for the following reasons.

- The top sub-region is $FE$ applied to the commutative diagram in N

$$
\begin{array}{ccc}
(\langle\rangle, Px) & & \\
{\scriptstyle(\theta_x,\,1)}\Big\downarrow & \searrow^{\theta_x^\#} & \\
\big(\underline{N}(Px;Qx), Px\big) & \xrightarrow{\;\;\mathsf{ev}^{\mathsf N}_{Px;Qx}\;\;} & Qx
\end{array}
$$

  that defines the partner $\theta_x^\#$.
- The middle sub-region is $F$ applied to the commutative diagram (9.2.11) that defines $\widehat{E}_{Px,Qx}$.
- The bottom sub-region is the commutative diagram (9.2.11) that defines $\widehat{F}_{EPx,EQx}$.

This proves that $(F_\star^\zeta E_\star \theta)_x^\#$ is equal to $FE(\theta_x^\#)$. $\hfill\square$

**Naturality of $\xi^\star$.** To prove that $\xi^\star$ is a natural transformation, we first check that its components are well defined.

**Lemma 11.3.18.** *In the context of Definitions 11.1.1 and 11.3.1, for each* N*-functor* $P :$ C $\longrightarrow$ N,

$$
\mathsf{C} \underset{F_\star^\zeta E_\star P}{\overset{P}{\rightrightarrows}} \Big\Downarrow \xi_P^\star\; \mathsf{N}
$$

*in (11.3.3) is an* N*-natural transformation.*

*Proof.* We must prove that the naturality diagram (6.1.16) for $\xi_P^\star$, which is the diagram in N below for objects $x, y \in$ C, is commutative.

$$
\begin{array}{ccc}
(\langle\rangle, \mathsf{C}(x,y)) & \xrightarrow{\;\;((\xi_P^\star)_y,\,P_{x,y})\;\;} & \big(\underline{N}(Py;FEPy), \underline{N}(Px;Py)\big) \\[4pt]
\Big\| & & \\[2pt]
(\mathsf{C}(x,y), \langle\rangle) & & \Big\downarrow{\scriptstyle\circ} \\[4pt]
{\scriptstyle((F_\star^\zeta E_\star P)_{x,y},\,(\xi_P^\star)_x)}\Big\downarrow & & \\[4pt]
\big(\underline{N}(FEPx;FEPy), \underline{N}(Px;FEPx)\big) & \xrightarrow{\quad\circ\quad} & \underline{N}(Px;FEPy)
\end{array}
$$

(11.3.19)

Since taking partner in N is a bijection (8.1.7), it suffices to show that the two composites in (11.3.19) have the same partner. We compute these partners in (11.3.20) and (11.3.21). Then we observe that they are equal to finish the proof.

*Top-Right Composite.* The partner of the top-right composite in (11.3.19) is, by definition, the left-bottom composite in the following diagram in N.

$$
(11.3.20)
$$

$$
\begin{array}{c}
(\mathsf{C}(x,y),\,Px) \xrightarrow{\;(\xi_{\mathsf{C}(x,y)},\,\xi_{Px})\;} (FE\mathsf{C}(x,y),\,FEPx) \\
\parallel \quad\; {}^{P_{x,y}^{\#}} \searrow \qquad\qquad\qquad \downarrow {}^{FE(P_{x,y}^{\#})} \\
(\langle\rangle,\,\mathsf{C}(x,y),\,Px) \qquad (\langle\rangle,\,Py) = Py \xrightarrow{\;\xi_{Py}\;} FEPy \\
{}_{((\xi_P^\star)_y,\,P_{x,y},\,1)}\downarrow \qquad\qquad\qquad\qquad {}^{(\xi_{Py}^{\#},\,1)}\searrow \\
(\underline{\mathsf{N}}(Py;FEPy),\,\underline{\mathsf{N}}(Px;Py),\,Px) \xrightarrow{\;(1,\,\mathrm{ev}^{\mathsf{N}})\;} (\underline{\mathsf{N}}(Py;FEPy),\,Py) \quad {}_{1} \\
{}_{(\circ,\,1)}\downarrow \qquad\qquad\qquad\qquad \downarrow {}^{\mathrm{ev}^{\mathsf{N}}} \\
(\underline{\mathsf{N}}(Px;FEPy),\,Px) \xrightarrow{\qquad \mathrm{ev}^{\mathsf{N}} \qquad} FEPy
\end{array}
$$

The four sub-regions in (11.3.20) are commutative for the following reasons.

- The top trapezoid commutes by the naturality (C.1.26) of the multinatural transformation $\xi : 1_{\mathsf{N}} \longrightarrow FE$.
- The top left triangle commutes by the definition of $P_{x,y}^{\#}$ in (11.3.12) and the definition of the $y$-component of $\xi_P^\star$ in (11.3.4),

$$
(\xi_P^\star)_y = \xi_{Py}^{\#} : \langle\rangle \longrightarrow \underline{\mathsf{N}}\big(Py;\,FEPy\big),
$$

as the partner of the $(Py)$-component of $\xi$.
- The bottom rectangle commutes by the definition of the composition $\circ$ in the canonical self-enrichment of N in (9.1.6).
- The right sub-region commutes by the definition of $\xi_{Py}^{\#}$ as the partner of $\xi_{Py}$ (8.1.7).

This proves that the diagram (11.3.20) is commutative.

*Left-Bottom Composite.* The partner of the left-bottom composite in (11.3.19) is the left-bottom composite in the following diagram in N.

$$
(11.3.21)
$$

$$
\begin{array}{c}
\qquad\qquad\qquad\qquad (\xi_{\mathsf{C}(x,y)},\,\xi_{Px}) \\
(\mathsf{C}(x,y),\,\langle\rangle,\,Px) \xrightarrow{\;(1,\,\xi_{Px})\;} (\mathsf{C}(x,y),\,FEPx) \\
\qquad\qquad\qquad\qquad\qquad (\xi_{\mathsf{C}(x,y)},\,1) \\
{}_{((F_\star^\xi E_\star P)_{x,y},\,(\xi_P^\star)_x,\,1)}\downarrow \quad \square \quad {}_{((F_\star^\xi E_\star P)_{x,y},\,1)}\downarrow \qquad\qquad\qquad (FE\mathsf{C}(x,y),\,FEPx) \\
(\underline{\mathsf{N}}(FEPx;FEPy),\,\underline{\mathsf{N}}(Px;FEPx),\,Px) \\
{}^{(1,\,\mathrm{ev}^{\mathsf{N}})}\searrow \\
{}_{(\circ,\,1)}\downarrow \qquad\qquad (\underline{\mathsf{N}}(FEPx;FEPy),\,FEPx) \qquad \diamondsuit \qquad \star \qquad \downarrow {}^{FE(P_{x,y}^{\#})} \\
\qquad\qquad\qquad\qquad\qquad {}_{\mathrm{ev}^{\mathsf{N}}}\searrow \\
(\underline{\mathsf{N}}(Px;FEPy),\,Px) \xrightarrow{\qquad\qquad \mathrm{ev}^{\mathsf{N}} \qquad\qquad} FEPy
\end{array}
$$

The five sub-regions in (11.3.21) are commutative for the following reasons.

- The top sub-region commutes by the unity properties, (C.1.9) and (C.1.10), in $\mathsf{M}$.
- The bottom triangle commutes by the definition of $\circ$ in (9.1.6).
- The sub-region labeled $\boxdot$ commutes by the definition of $(\xi_P^\star)_x$ as the partner of $\xi_{Px}$ (11.3.4).
- The triangle labeled $\diamondsuit$ commutes by the definition of $(F_\star^\xi E_\star P)_{x,y}^\#$ as the partner of $(F_\star^\xi E_\star P)_{x,y}$ (8.1.7).
- The triangle labeled $\star$ is the boundary of the following diagram.



This diagram is commutative for the following reasons.

- – The top triangle commutes by the definition of $(F_\star^\xi E_\star P)_{x,y}^\#$ as the partner of $(F_\star^\xi E_\star P)_{x,y}$ (8.1.7).
- – The right sub-region commutes by the definition of $(FE(P_{x,y}^\#))^\#$ as the partner of $FE(P_{x,y}^\#)$ (8.1.7).
- – The left sub-region commutes by the top commutative triangle in (11.3.8), which is proved in Lemma 11.3.9.

This proves that the diagram (11.3.21) is commutative.

*Comparing Partners.* By the commutative diagrams (11.3.20) and (11.3.21), the partner of each composite in the desired diagram (11.3.19) is the following binary multimorphism in $\mathsf{N}$.

$$\gamma^{\mathsf{N}}\big(FE(P_{x,y}^\#)\,;\,\xi_{\mathsf{C}(x,y)}\,,\,\xi_{Px}\big)$$

Since taking partner is a bijection (8.1.7), we conclude that the diagram (11.3.19) is commutative. $\square$

**Remark 11.3.22** (Difference with $\kappa_A^\star$)**.** We structure the proofs of Lemmas 11.2.19 and 11.3.18 in a way that highlights their conceptual similarity. There is, however, one nontrivial difference between these two proofs of enriched naturality. In the last paragraph of the proof of Lemma 11.2.19, to conclude that the two binary multimorphisms in (11.2.23) are the same, we need to use the assumption in Definition 11.1.1 (v). On the other hand, in the last paragraph of the proof of Lemma 11.3.18, no such assumption is needed. $\diamond$

**Lemma 11.3.23.** *In the context of Definitions 11.1.1 and 11.3.1,*

$$\mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C},\mathsf{N}) \underset{F_\star^\xi E_\star}{\overset{1}{\rightrightarrows}} \Downarrow\xi^\star \; \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C},\mathsf{N})$$

*in (11.3.2) is a natural transformation.*

*Proof.* Lemma 11.3.18 proves that each component of $\xi^\star$ is a well-defined N-natural transformation. Naturality for $\xi^\star$ means that, for each N-natural transformation $\theta$ as in

$$C \underset{Q}{\overset{P}{\rightrightarrows}} \Downarrow\theta \; N$$

the following diagram of N-natural transformations in N-Cat$(C, N)$ commutes.

(11.3.24)
$$
\begin{array}{ccc}
P & \xrightarrow{\;\xi_P^\star\;} & F_\star^\xi E_\star P \\
{\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle F_\star^\xi E_\star \theta} \\
Q & \xrightarrow{\;\xi_Q^\star\;} & F_\star^\xi E_\star Q
\end{array}
$$

Since each N-natural transformation is determined by its components, it suffices to show that, for each object $x \in C$, the two vertical composites in (11.3.24) have the same $x$-components. By Definition 6.1.18 these two $x$-components are the following two composite nullary multimorphisms in N.

(11.3.25)
$$
\begin{array}{ccc}
(\langle\rangle, \langle\rangle) & \xrightarrow{\;\left((F_\star^\xi E_\star\theta)_x,\, (\xi_P^\star)_x\right)\;} & \left(\underline{N}(FEPx\,;\,FEQx),\, \underline{N}(Px\,;\,FEPx)\right) \\
{\scriptstyle\left((\xi_Q^\star)_x,\, \theta_x\right)}\downarrow & & \downarrow{\scriptstyle\circ} \\
\left(\underline{N}(Qx\,;\,FEQx),\, \underline{N}(Px\,;\,Qx)\right) & \xrightarrow{\;\;\circ\;\;} & \underline{N}(Px\,;\,FEQx)
\end{array}
$$

Since taking partner is a bijection (8.1.7), it suffices to show that the two composites in (11.3.25) have the same partner. We compute these partners in (11.3.26) and (11.3.27). Then we observe that they are equal to finish the proof.

*Top-Right Composite.* The partner of the top-right composite in (11.3.25) is the left-bottom composite unary multimorphism in the following diagram in N.

(11.3.26)
$$
\begin{array}{ccc}
(\langle\rangle, \langle\rangle, Px) = Px & \xrightarrow{\;\xi_{Px}\;} & (\langle\rangle, FEPx) \longrightarrow \\
{\scriptstyle\left((F_\star^\xi E_\star\theta)_x,\, (\xi_P^\star)_x,\, 1\right)}\downarrow & {\scriptstyle\left((FE(\theta_x^\#))^\#,\, 1\right)}\downarrow & \\
\left(\underline{N}(FEPx\,;\,FEQx),\, \underline{N}(Px\,;\,FEPx),\, Px\right) & & {\scriptstyle FE(\theta_x^\#)} \\
\phantom{x} & {\scriptstyle(1,\, \mathrm{ev}^N)}\searrow & \\
{\scriptstyle(\circ,\, 1)}\downarrow & \left(\underline{N}(FEPx\,;\,FEQx),\, FEPx\right) & \\
& \downarrow{\scriptstyle\mathrm{ev}^N} & \\
\left(\underline{N}(Px\,;\,FEQx),\, Px\right) & \xrightarrow{\;\mathrm{ev}^N\;} & FEQx \longleftarrow
\end{array}
$$

The three sub-regions in (11.3.26) are commutative for the following reasons.

- The bottom left sub-region commutes by the definition of $\circ$ in (9.1.6).
- The top left sub-region commutes by
    - the definition of $(\xi_P^\star)_x$ as the partner of $\xi_{Px}$ in (11.3.4) and
    - Lemma 11.3.17, which implies that $(F_\star^\xi E_\star\theta)_x$ is the partner of $FE(\theta_x^\#)$.
- The right sub-region commutes by the definition of $(FE(\theta_x^\#))^\#$ as the partner of $FE(\theta_x^\#)$.

This proves that the diagram (11.3.26) is commutative.

*Left-Bottom Composite.* The partner of the left-bottom composite in (11.3.25) is the left-bottom composite unary multimorphism in the following diagram in $\mathsf{N}$.

(11.3.27)

$$
\begin{array}{ccc}
\big(\langle\rangle,\langle\rangle,Px\big)=Px & \xrightarrow{\;\theta_x^{\#}\;} & \big(\langle\rangle,Qx\big) \\[2pt]
{\scriptstyle((\xi_Q^{\star})_x,\,\theta_x,\,1)}\big\downarrow & & {\scriptstyle(\xi_{Qx}^{\#},\,1)}\big\downarrow \\[2pt]
\big(\underline{\mathsf{N}}(Qx;FEQx),\underline{\mathsf{N}}(Px;Qx),Px\big) & & \\[2pt]
& \searrow{\scriptstyle(1,\,\mathrm{ev}^{\mathsf{N}})} & \\[2pt]
{\scriptstyle(\circ,\,1)}\big\downarrow & \big(\underline{\mathsf{N}}(Qx;FEQx),Qx\big) & \\[2pt]
& {\scriptstyle\mathrm{ev}^{\mathsf{N}}}\big\downarrow & \\[2pt]
\big(\underline{\mathsf{N}}(Px;FEQx),Px\big) & \xrightarrow{\;\mathrm{ev}^{\mathsf{N}}\;} & FEQx
\end{array}
$$

with $\xi_{Qx}$ along the right-hand outer arc down to $FEQx$.

The three sub-regions in (11.3.27) are commutative for the following reasons.

- The bottom left sub-region commutes by the definition of $\circ$ in (9.1.6).
- The top left sub-region commutes by
  - the definition of $(\xi_Q^{\star})_x$ as the partner of $\xi_{Qx}$ in (11.3.4) and
  - the definition of $\theta_x^{\#}$ as the partner of $\theta_x$.
- The right sub-region commutes by the definition of $\xi_{Qx}^{\#}$ as the partner of $\xi_{Qx}$.

This proves that the diagram (11.3.27) is commutative.

*Comparing Partners.* By the commutative diagrams (11.3.26) and (11.3.27), the partners of the two composites in the desired diagram (11.3.25) are the composites in $\mathsf{N}$ as follows.

$$
\begin{array}{ccc}
Px & \xrightarrow{\;\xi_{Px}\;} & FEPx \\[2pt]
{\scriptstyle\theta_x^{\#}}\big\downarrow & & \big\downarrow{\scriptstyle FE(\theta_x^{\#})} \\[2pt]
Qx & \xrightarrow{\;\xi_{Qx}\;} & FEQx
\end{array}
$$

This diagram commutes by the naturality condition (C.1.26) of the multinatural transformation $\xi : 1_{\mathsf{N}} \longrightarrow FE$. $\qquad\square$

## 11.4. Homotopy Equivalent Enriched Diagram and Mackey Functor Categories

In this section we apply the results in previous sections to prove that, if the data $(F, E, \kappa, \xi)$ in Definition 11.1.1 are inverse equivalences of homotopy theories (Definition 2.1.8), then they lift to inverse equivalences of homotopy theories between the enriched diagram categories in (11.1.19)

$$
\mathsf{M}\text{-}\mathsf{Cat}(\mathsf{C}_E, \mathsf{M}) \xrightleftharpoons[E_{\star}]{F_{\star}^{\xi}} \mathsf{N}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{N}).
$$

This section is organized as follows.

- Stable equivalences in categories of enriched diagrams are defined componentwise. We make this precise in Definitions 11.4.1 and 11.4.7. Lemmas 11.4.4 and 11.4.13 contain some basic properties of these componentwise stable equivalences.

- Theorem 11.4.14 is the main result of this chapter. It states that, under suitable conditions, $F_\star^{\tilde{\xi}}$ and $E_\star$ are inverse equivalences of homotopy theories.
- Theorem 11.4.24 is the variant that involves an opposite N-category $\mathsf{C}^{\mathrm{op}}$.
- Proposition 11.4.25 shows that, under the assumptions of Theorem 11.4.14, $F$ and $E$ are inverse equivalences of homotopy theories between the underlying categories. Therefore, the inverse equivalences of homotopy theories $F_\star^{\xi}$ and $E_\star$ are, in fact, lifted from the underlying categories.

**Componentwise Relative Structure on Enriched Diagram Categories.** To consider the homotopy theory of an enriched diagram category, we first define its relative category structure (Definition 2.1.6). Recall that a multicategory has an *underlying category* (Example C.1.16), which we denote by the same symbol. Recall that a subcategory is *wide* if it contains all the objects of the larger category. The next definition is an adaptation of Definition 3.5.4 to the current setting of enriched diagram categories.

**Definition 11.4.1.** Suppose P is a non-symmetric closed multicategory, and D is a P-category (Definitions 6.1.1 and 8.1.1). Suppose the underlying category of P is equipped with the structure of a relative category $(\mathsf{P}, \mathcal{W})$. For the category P-Cat(D, P) in (10.1.2), we define a wide subcategory

$$(11.4.2) \qquad \qquad \mathcal{W}_\blacktriangle \subset \mathsf{P\text{-}Cat}(\mathsf{D}, \mathsf{P})$$

as follows. A P-natural transformation $\theta$ (Definition 6.1.14) as in

$$\mathsf{D} \xrightarrow[B]{\overset{A}{\Downarrow \theta}} \mathsf{P}$$

is in $\mathcal{W}_\blacktriangle$ if, for each object $x$ in D, the unary multimorphism

$$(11.4.3) \qquad \qquad \theta_x^\# : Ax \longrightarrow Bx \quad \text{is in } \mathcal{W}.$$

Here $\theta_x^\#$ is the partner (8.1.11) of the $x$-component of $\theta$,

$$\theta_x : \langle\rangle \longrightarrow \underline{\mathsf{P}}(Ax; Bx),$$

which is a nullary multimorphism in P. We regard the pair

$$\big(\mathsf{P\text{-}Cat}(\mathsf{D}, \mathsf{P}), \mathcal{W}_\blacktriangle\big)$$

as a relative category. $\diamond$

In other words, $\theta$ is in $\mathcal{W}_\blacktriangle$ if each component of $\theta$ has its partner in $\mathcal{W}$.

Recall that a *category with weak equivalences* (Definition 2.1.6 (6)) is a relative category $(\mathsf{P}, \mathcal{W})$ such that $\mathcal{W}$ contains all the isomorphisms in P and has the 2-out-of-3 property.

**Lemma 11.4.4.** *In the context of Definition 11.4.1, statements (i) through (iv) below hold.*

(i) *The subcategory $\mathcal{W}_\blacktriangle$ in (11.4.2) is well defined.*
(ii) *If $\mathcal{W}$ has the 2-out-of-3 property, then so does $\mathcal{W}_\blacktriangle$.*
(iii) *If $\mathcal{W}$ contains all the isomorphisms, then so does $\mathcal{W}_\blacktriangle$.*
(iv) *If $(\mathsf{P}, \mathcal{W})$ is a category with weak equivalences, then so is $\big(\mathsf{P\text{-}Cat}(\mathsf{D}, \mathsf{P}), \mathcal{W}_\blacktriangle\big)$.*

*Proof.* Suppose $\theta$ and $\psi$ are P-natural transformations as in the left diagram below.

$$
\mathsf{D} \underset{C}{\overset{A}{\rightrightarrows}} \mathsf{P} \qquad \qquad \mathsf{D} \underset{C}{\overset{A}{\rightrightarrows}} \mathsf{P}
$$

For each object $x$ in D, the $x$-component of the vertical composite $\psi\theta$, as in the right diagram above, has partner (8.1.7) given by

$$
(11.4.5) \qquad\qquad (\psi\theta)^{\#}_{x} = \gamma^{\mathsf{P}}\left(\psi^{\#}_{x}; \theta^{\#}_{x}\right) : Ax \longrightarrow Cx
$$

by Proposition 10.1.22. Thus if $\theta$ and $\psi$ are in $\mathcal{W}_{\blacktriangle}$, then so is $\psi\theta$, proving statement (i). The equality (11.4.5) also shows that $\mathcal{W}_{\blacktriangle}$ has the 2-out-of-3 property whenever $\mathcal{W}$ does, proving statement (ii).

   To prove statement (iii), suppose $\theta$ and $\psi$ are inverses of each other. By (9.1.2) this means that, for each object $x$ in D, there are equalities

$$
(11.4.6) \qquad\qquad (\psi\theta)^{\#}_{x} = 1_{Ax} \quad \text{and} \quad (\theta\psi)^{\#}_{x} = 1_{Bx}.
$$

Together with (11.4.5) and the variant for $(\theta\psi)^{\#}_{x}$, the equalities in (11.4.6) imply that $\psi^{\#}_{x}$ and $\theta^{\#}_{x}$ are mutually inverse isomorphisms in the underlying category of P. This proves statement (iii).

   Statement (iv) follows from statements (i) through (iii).  □

   Next we apply Definition 11.4.1 to the context of the previous sections. We consider the underlying functor of the non-symmetric multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ (Example C.1.24), which we denote by the same symbol.

**Definition 11.4.7.** In the context of Definition 11.1.1, suppose, in addition, the underlying category of N is equipped with the structure of a category with weak equivalences $(\mathsf{N}, \mathcal{X})$ (Definition 2.1.6 (6)). We define the following.

   • We define the wide subcategory

$$
(11.4.8) \qquad\qquad F^{-1}\mathcal{X} \subset \mathsf{M}
$$

   as the preimage of $\mathcal{X}$ under the underlying functor of $F : \mathsf{M} \longrightarrow \mathsf{N}$. We refer to morphisms in $F^{-1}\mathcal{X}$ as *F-stable equivalences*. We regard the pair

$$
(11.4.9) \qquad\qquad \left(\mathsf{M}, F^{-1}\mathcal{X}\right)
$$

   as a relative category.
   • Applying Definition 11.4.1 with $(\mathsf{D}, \mathsf{P}, \mathcal{W})$ given by
       – $\left(\mathsf{C}_E, \mathsf{M}, F^{-1}\mathcal{X}\right)$ and
       – $(\mathsf{C}, \mathsf{N}, \mathcal{X})$,
   we obtain the following two relative categories.

$$
(11.4.10) \qquad \left(\mathsf{M}\text{-Cat}(\mathsf{C}_E, \mathsf{M}), (F^{-1}\mathcal{X})_{\blacktriangle}\right) \qquad \left(\mathsf{N}\text{-Cat}(\mathsf{C}, \mathsf{N}), \mathcal{X}_{\blacktriangle}\right)
$$

This finishes the definition.                                                        ◇

**Explanation 11.4.11** (Unpacking $(F^{-1}\mathcal{X})_{\blacktriangle}$). An M-natural transformation

$$
\mathsf{C}_E \underset{B}{\overset{A}{\rightrightarrows}} \mathsf{M}
$$

is in the subcategory $(F^{-1}\mathcal{X})_\blacktriangle$ if, for each object $x$ in $\mathsf{C}$, the unary multimorphism

$$\psi_x^\# : Ax \longrightarrow Bx \quad \text{is in } F^{-1}\mathcal{X} \subset \mathsf{M}.$$

This means that the unary multimorphism

(11.4.12) $\qquad\qquad F(\psi_x^\#) : FAx \longrightarrow FBx \quad \text{is in } \mathcal{X} \subset \mathsf{N}.$ ◇

**Lemma 11.4.13.** *The relative categories in (11.4.9) and (11.4.10) are categories with weak equivalences.*

*Proof.* The wide subcategory $F^{-1}\mathcal{X} \subset \mathsf{M}$ in (11.4.8) contains all the isomorphisms and has the 2-out-of-3 property for the following two reasons.

- By assumption $\mathcal{X} \subset \mathsf{N}$ has these two properties.
- The underlying functor of $F : \mathsf{M} \longrightarrow \mathsf{N}$ preserves identity morphisms and composition of morphisms.

The relative categories in (11.4.10) are categories with weak equivalences by Lemma 11.4.4 (iv) applied to $(\mathsf{C}_E, \mathsf{M}, F^{-1}\mathcal{X})$ and $(\mathsf{C}, \mathsf{N}, \mathcal{X})$. □

**Equivalences of Homotopy Theories.** Recall that *inverse equivalences of homotopy theories* are two relative functors in opposite direction such that each composite is connected to the identity functor by a zigzag of relative natural transformations (Definitions 2.1.6 and 2.1.8). Such functors are equivalences of homotopy theories by Proposition 2.1.9. Theorem 11.4.14 below is the main result of this chapter. It provides checkable criteria that guarantee that inverse equivalences of homotopy theories (Proposition 11.4.25)

$$\left(\mathsf{M}, F^{-1}\mathcal{X}\right) \underset{E}{\overset{F}{\underset{\sim}{\rightleftarrows}}} (\mathsf{N}, \mathcal{X})$$

lift to the categories of enriched diagrams in (11.4.10). Explanation 11.4.23 summarizes all the hypotheses in Theorem 11.4.14. The variant for the opposite N-category $\mathsf{C}^{\mathsf{op}}$ is Theorem 11.4.24. In Chapter 12 we apply Theorem 11.4.14 to the (non-symmetric) multifunctors connecting $\mathsf{PermCat}^{\mathsf{su}}$, $\mathsf{Multicat}_*$, and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$; see Theorems 12.1.6, 12.4.6, and 12.6.6 and Question D.4.

**Theorem 11.4.14.** *In the context of Definitions 11.1.1 and 11.4.7, suppose, in addition, the components of the multinatural transformations*

$$\mathsf{M} \underset{EF}{\overset{1_\mathsf{M}}{\Rightarrow}} \mathsf{M} \qquad and \qquad \mathsf{N} \underset{FE}{\overset{1_\mathsf{N}}{\Rightarrow}} \mathsf{N}$$

*are in $F^{-1}\mathcal{X}$ and $\mathcal{X}$, respectively. Then the functors in (11.1.19)*

(11.4.15) $\qquad \left(\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}), (F^{-1}\mathcal{X})_\blacktriangle\right) \underset{E_\star}{\overset{F_\star^\xi}{\underset{\sim}{\rightleftarrows}}} \left(\mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N}), \mathcal{X}_\blacktriangle\right)$

*are inverse equivalences of homotopy theories.*

*Proof.* The two relative categories in (11.4.15) are defined in (11.4.10). They are categories with weak equivalences by Lemma 11.4.13.

We prove statements (i) through (iii) below. By Definition 2.1.8 these statements imply that $F_\star^\xi$ and $E_\star$ are inverse equivalences of homotopy theories.

(i) $F_\star^\xi$ and $E_\star$ in (11.4.15) are relative functors.

(ii) The natural transformation in (11.2.2)

$$\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \overset{1}{\underset{E_\star F_\star^\xi}{\rightrightarrows}} \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M})$$

$\Downarrow \kappa^\star$

has each component in $(F^{-1}\mathcal{X})_\blacktriangle$.

(iii) The natural transformation in (11.3.2)

$$\mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N}) \overset{1}{\underset{F_\star^\xi E_\star}{\rightrightarrows}} \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$$

$\Downarrow \tilde{\xi}^\star$

has each component in $\mathcal{X}_\blacktriangle$.

*Statement (i): $F_\star^\xi$ is a Relative Functor.* This means that, for each M-natural transformation $\psi \in (F^{-1}\mathcal{X})_\blacktriangle$ as in (11.4.16), the N-natural transformation $F_\star^\xi \psi$ is in $\mathcal{X}_\blacktriangle$.

$$(11.4.16) \qquad \mathsf{C}_E \overset{A}{\underset{B}{\rightrightarrows}} \mathsf{M} \qquad\qquad \mathsf{C} \overset{F_\star^\xi A}{\underset{F_\star^\xi B}{\rightrightarrows}} \mathsf{N}$$

$\Downarrow \psi$                    $\Downarrow F_\star^\xi \psi$

By (11.1.18) and (11.4.3), the desired condition $F_\star^\xi \psi \in \mathcal{X}_\blacktriangle$ means that, for each object $x$ in C, the unary multimorphism

$$(11.4.17) \qquad\qquad (F_\star^\xi \psi)_x^\# = F(\psi_x^\#) \quad \text{is in } \mathcal{X}.$$

This is true by (11.4.12). Thus $F_\star^\xi$ is a relative functor.

*Statement (i): $E_\star$ is a Relative Functor.* This means that, for each N-natural transformation $\theta \in \mathcal{X}_\blacktriangle$ as in (11.4.18), the M-natural transformation $E_\star \theta$ is in $(F^{-1}\mathcal{X})_\blacktriangle$.

$$(11.4.18) \qquad \mathsf{C} \overset{P}{\underset{Q}{\rightrightarrows}} \mathsf{N} \qquad\qquad \mathsf{C}_E \overset{E_\star P}{\underset{E_\star Q}{\rightrightarrows}} \mathsf{M}$$

$\Downarrow \theta$                    $\Downarrow E_\star \theta$

By (11.4.12) and (10.2.19) applied to $E_\star \theta$, the desired condition $E_\star \theta \in (F^{-1}\mathcal{X})_\blacktriangle$ means that, for each object $x$ in C, the unary multimorphism

$$(11.4.19) \qquad F\big((E_\star \theta)_x^\#\big) = FE(\theta_x^\#) : FEPx \longrightarrow FEQx \quad \text{is in } \mathcal{X}.$$

To prove (11.4.19), we use the naturality of $\xi : 1_\mathsf{N} \longrightarrow FE$ (C.1.26) to obtain the following commutative diagram in N.

$$(11.4.20) \qquad \begin{array}{ccc} Px & \overset{\xi_{Px}}{\longrightarrow} & FEPx \\ {\scriptstyle \theta_x^\#}\downarrow & & \downarrow{\scriptstyle FE(\theta_x^\#)} \\ Qx & \underset{\xi_{Qx}}{\longrightarrow} & FEQx \end{array}$$

- The assumption $\theta \in \mathcal{X}_\blacktriangle$ means that each $\theta_x^\#$ is in $\mathcal{X}$.
- The components $\xi_{Px}$ and $\xi_{Qx}$ are in $\mathcal{X}$ by the assumption on $\xi$.

The commutative diagram (11.4.20) and the 2-out-of-3 property of $\mathcal{X}$ imply that $FE(\theta_x^\#)$ is in $\mathcal{X}$, proving the desired condition (11.4.19). Thus $E_\star$ is a relative functor. This finishes the proof of statement (i).

*Statement (ii).* Suppose $A : \mathsf{C}_E \longrightarrow \mathsf{M}$ is an M-functor. We want to show that the M-natural transformation in (11.2.3)

$$\mathsf{C}_E \underset{E_\star F_\star^\xi A}{\overset{A}{\rightrightarrows}} \Downarrow \kappa_A^\star \quad \mathsf{M}$$

is in $(F^{-1}\mathcal{X})_\blacktriangle$. By (11.2.4) and (11.4.12), the desired condition $\kappa_A^\star \in (F^{-1}\mathcal{X})_\blacktriangle$ means that, for each object $x$ in C, the unary multimorphism

(11.4.21) $\qquad F\big((\kappa_A^\star)_x^\#\big) = F(\kappa_{Ax}) : FAx \longrightarrow FEFAx \quad$ is in $\mathcal{X}$.

This is true because each component of $\kappa : 1_\mathsf{M} \longrightarrow EF$ is in $F^{-1}\mathcal{X}$ by assumption. This proves statement (ii).

*Statement (iii).* Suppose $P : \mathsf{C} \longrightarrow \mathsf{N}$ is an N-functor. We want to show that the N-natural transformation in (11.3.3)

$$\mathsf{C} \underset{F_\star^\xi E_\star P}{\overset{P}{\rightrightarrows}} \Downarrow \xi_P^\star \quad \mathsf{N}$$

is in $\mathcal{X}_\blacktriangle$. By (11.3.4) and (11.4.3), the desired condition $\xi_P^\star \in \mathcal{X}_\blacktriangle$ means that, for each object $x$ in C, the unary multimorphism

(11.4.22) $\qquad (\xi_P^\star)_x^\# = \xi_{Px} : Px \longrightarrow FEPx \quad$ is in $\mathcal{X}$.

This is true by the assumption that each component of $\xi$ is in $\mathcal{X}$. This proves statement (iii). $\qquad\square$

**Explanation 11.4.23** (Summary). We summarize the assumptions for Theorem 11.4.14 in (1) through (3) below.

(1) We assume (i) through (v) in Definition 11.1.1. These assumptions are multicategorical in nature. They do not involve relative category structures. Using these assumptions we construct
- the functors

$$\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star}{\overset{F_\star^\xi}{\rightleftarrows}} \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$$

in Theorem 10.3.1 (applied to $E_\star$) and (11.1.3) and
- the natural transformations

$$\mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M}) \underset{E_\star F_\star^\xi}{\overset{1}{\rightrightarrows}} \Downarrow \kappa^\star \quad \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M})$$

$$\mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N}) \underset{F_\star^\xi E_\star}{\overset{1}{\rightrightarrows}} \Downarrow \xi^\star \quad \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$$

in (11.2.2) and (11.3.2).
(2) We assume that $(\mathsf{N}, \mathcal{X})$ is a category with weak equivalences (Definition 11.4.7). Using this assumption we define the wide subcategories
- $\mathcal{X}_\blacktriangle \subset \mathsf{N\text{-}Cat}(\mathsf{C}, \mathsf{N})$ in (11.4.2),
- $F^{-1}\mathcal{X} \subset \mathsf{M}$ in (11.4.8), and

- $(F^{-1}\mathcal{X})_{\blacktriangle} \subset \mathsf{M\text{-}Cat}(\mathsf{C}_E, \mathsf{M})$ in (11.4.10).

The relative categories in (11.4.15) are defined using $(F^{-1}\mathcal{X})_{\blacktriangle}$ and $\mathcal{X}_{\blacktriangle}$.

(3) We assume that

- each component of $\kappa : 1_{\mathsf{M}} \longrightarrow EF$ is in $F^{-1}\mathcal{X}$ and
- each component of $\xi : 1_{\mathsf{N}} \longrightarrow FE$ is in $\mathcal{X}$.

In the proof of Theorem 11.4.14, we use the assumption about $\kappa$ to prove that $\kappa^{\star}$ is a relative natural transformation (11.4.21). We use the assumption about $\xi$ to prove that

- $E_{\star}$ is a relative functor (11.4.20) and
- $\xi^{\star}$ is a relative natural transformation (11.4.22).

As we explain in (11.4.17), the relative functoriality of $F_{\star}^{\xi}$ is a consequence of the definitions of $(F^{-1}\mathcal{X})_{\blacktriangle}$ and $F_{\star}^{\xi}\psi$. ⋄

Recall that each category C enriched in a multicategory N has an opposite N-category $\mathsf{C}^{\mathsf{op}}$ (Proposition 6.6.7), whose composition involves the symmetric group action on N. In the next result, we assume that $E$ is a multifunctor (Definition C.1.19), so it strictly preserves the symmetric group action. This result gives an equivalence of homotopy theories between enriched Mackey functor categories.

**Theorem 11.4.24.** *In the context of Theorem 11.4.14, suppose, furthermore, that $E$ : N $\longrightarrow$ M is a multifunctor between multicategories. Then the functors in (11.1.19) applied to $\mathsf{C}^{\mathsf{op}}$,*

$$\Big(\mathsf{M\text{-}Cat}\big((\mathsf{C}_E)^{\mathsf{op}}, \mathsf{M}\big), (F^{-1}\mathcal{X})_{\blacktriangle}\Big) \xrightleftharpoons[\;E_{\star}\;]{\;\;F_{\star}^{\xi}\;\;} \Big(\mathsf{N\text{-}Cat}(\mathsf{C}^{\mathsf{op}}, \mathsf{N}), \mathcal{X}_{\blacktriangle}\Big),$$

*are inverse equivalences of homotopy theories.*

*Proof.* This is Theorem 11.4.14 applied to the opposite N-category $\mathsf{C}^{\mathsf{op}}$. By Proposition 7.2.1 the multifunctoriality of $E$ yields the equality

$$(\mathsf{C}^{\mathsf{op}})_E = (\mathsf{C}_E)^{\mathsf{op}}$$

of M-categories. □

For completeness we end this section with the following observation that says that the functors $F$ and $E$ are inverse equivalences of homotopy theories.

**Proposition 11.4.25.** *Under the assumptions of Theorem 11.4.14, the functors*

$$\big(\mathsf{M}, F^{-1}\mathcal{X}\big) \xrightleftharpoons[\;E\;]{\;\;F\;\;} (\mathsf{N}, \mathcal{X})$$

*are inverse equivalences of homotopy theories.*

*Proof.* This is a much simpler variant of the proof of Theorem 11.4.14.

- $F$ is a relative functor by the definition of $F^{-1}\mathcal{X}$ in (11.4.8).
- To see that $E$ is a relative functor, suppose $f : a \longrightarrow b$ is a morphism in $\mathcal{X}$. We want to show that $Ef$ is in $F^{-1}\mathcal{X}$, which means $FEf \in \mathcal{X}$. The

naturality of $\xi$ yields the following commutative diagram in N.

$$
\begin{array}{ccc}
a & \xrightarrow{\ \xi_a\ } & FEa \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle FEf} \\
b & \xrightarrow{\ \xi_b\ } & FEb
\end{array}
$$

Similar to (11.4.20),
  – the assumption $f \in \mathcal{X}$,
  – the assumption that each component of $\xi$ is in $\mathcal{X}$, and
  – the 2-out-of-3 property of $\mathcal{X}$
imply that $FEf$ is in $\mathcal{X}$.
• The components of $\kappa$ and $\xi$ are in $F^{-1}\mathcal{X} \subset$ M and $\mathcal{X} \subset$ N, respectively, by assumption. So $\kappa$ and $\xi$ are relative natural transformations.

Thus, by Definition 2.1.8, the functors $F$ and $E$ are inverse equivalences of homotopy theories. $\square$

CHAPTER 12

# Applications to Multicategories and Permutative Categories

This chapter develops three main applications from Chapter 11. These are summarized below, where C is a small PermCat$^{su}$-category, D is a small Mod$^{\mathcal{M}\underline{1}}$-category, and each wide subcategory of stable equivalences is created as in (11.4.2). In each case, we state the equivalence of homotopy theories between enriched diagram categories that follows from Theorem 11.4.14. There are also variants for enriched Mackey functors that follow from Theorem 11.4.24 because, in each of the three applications, the reverse functor $E$ is a multifunctor in the symmetric sense (Definition C.1.19). The more precise statements in the body of this chapter give further details.

**Pointed Multicategories and Permutative Categories.** The first application concerns the following data.

$$\text{Multicat}_* \overset{\text{F}_{\bullet}}{\underset{\text{End}_{\bullet}}{\rightleftarrows}} \text{PermCat}^{su}$$

(12.0.1)

$$\text{Multicat}_* \overset{1}{\underset{\text{End}_{\bullet}\text{F}_{\bullet}}{\Downarrow\eta^{\bullet}}} \text{Multicat}_* \qquad \text{PermCat}^{su} \overset{1}{\underset{\text{F}_{\bullet}\text{End}_{\bullet}}{\Downarrow\varrho^{\bullet}}} \text{PermCat}^{su}$$

Theorem 12.1.6 shows that these induce an inverse equivalence of homotopy theories between enriched diagram categories

$$\left(\text{Multicat}_*\text{-Cat}\big(C_{\text{End}_{\bullet}}, \text{Multicat}_*\big), (\mathcal{S}_{\bullet})_{\blacktriangle}\right) \overset{(\text{F}_{\bullet})^{\varrho^{\bullet}}_{\star}}{\underset{\sim}{\overset{}{\underset{(\text{End}_{\bullet})_{\star}}{\rightleftarrows}}}} \left(\text{PermCat}^{su}\text{-Cat}\big(C, \text{PermCat}^{su}\big), \mathcal{S}_{\blacktriangle}\right).$$

$\mathcal{M}\underline{1}$**-Modules and Permutative Categories.** The second application concerns the following data.

$$\text{Mod}^{\mathcal{M}\underline{1}} \overset{\text{F}_{\mathcal{M}\underline{1}}}{\underset{\text{End}_{\mathcal{M}\underline{1}}}{\rightleftarrows}} \text{PermCat}^{su}$$

(12.0.2)

$$\text{Mod}^{\mathcal{M}\underline{1}} \overset{1}{\underset{\text{End}_{\mathcal{M}\underline{1}}\text{F}_{\mathcal{M}\underline{1}}}{\Downarrow\eta^{\mathcal{M}\underline{1}}}} \text{Mod}^{\mathcal{M}\underline{1}} \qquad \text{PermCat}^{su} \overset{1}{\underset{\text{F}_{\mathcal{M}\underline{1}}\text{End}_{\mathcal{M}\underline{1}}}{\Downarrow\varrho^{\mathcal{M}\underline{1}}}} \text{PermCat}^{su}$$

Theorem 12.4.6 shows that these induce an inverse equivalence of homotopy theories between enriched diagram categories

$$\left(\text{Mod}^{\mathcal{M}\underline{1}}\text{-Cat}\big(C_{\text{End}_{\mathcal{M}\underline{1}}}, \text{Mod}^{\mathcal{M}\underline{1}}\big), \mathcal{S}^{\mathcal{M}\underline{1}}_{\blacktriangle}\right) \overset{(\text{F}_{\mathcal{M}\underline{1}})^{\varrho^{\mathcal{M}\underline{1}}}_{\star}}{\underset{\sim}{\overset{}{\underset{(\text{End}_{\mathcal{M}\underline{1}})_{\star}}{\rightleftarrows}}}} \left(\text{PermCat}^{su}\text{-Cat}\big(C, \text{PermCat}^{su}\big), \mathcal{S}_{\blacktriangle}\right).$$

**Pointed Multicategories and $\mathcal{M}\underline{1}$-Modules.** The third application concerns the following data.

$$\mathsf{Multicat}_* \xrightarrow[U_{\mathcal{M}\underline{1}}]{\mathcal{M}\underline{1} \wedge -} \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

(12.0.3)

$$\mathsf{Multicat}_* \underset{U_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1} \wedge -)}{\overset{1}{\Longrightarrow \Downarrow \hat{\eta}}} \mathsf{Multicat}_* \qquad\qquad \mathsf{Mod}^{\mathcal{M}\underline{1}} \underset{(\mathcal{M}\underline{1} \wedge -)U_{\mathcal{M}\underline{1}}}{\overset{1}{\Longrightarrow \Downarrow \hat{\varepsilon}^{-1}}} \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

Theorem 12.6.6 shows that these induce an inverse equivalence of homotopy theories between enriched diagram categories

$$\left(\mathsf{Multicat}_*\text{-}\mathsf{Cat}\big(D_{U_{\mathcal{M}\underline{1}}}, \mathsf{Multicat}_*\big), (\mathcal{S}_\bullet)_\blacktriangle\right) \xrightarrow[(U_{\mathcal{M}\underline{1}})_\star]{\overset{(\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}}{\underset{\sim}{\longrightarrow}}} \left(\mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\big(D, \mathsf{Mod}^{\mathcal{M}\underline{1}}\big), \mathcal{S}_\blacktriangle^{\mathcal{M}\underline{1}}\right).$$

**Connection with Other Chapters.** The results in this chapter relate to broader work in homotopy theory of diagram spectra and spectral Mackey functors via the constructions in Sections 10.5 and 10.6. Those sections develop spectral Mackey functors from enriched Mackey functors in $\mathsf{PermCat}^{\mathsf{su}}$ and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.

**Background.** In addition to the results of Chapter 11, the main applications in this chapter depend on the following context. The underlying inverse equivalences of homotopy theories,

$$\big(\mathsf{F}_\bullet, \mathsf{End}_\bullet\big), \quad \big(\mathsf{F}_{\mathcal{M}\underline{1}}, \mathsf{End}_{\mathcal{M}\underline{1}}\big), \quad \text{and} \quad \big(\mathcal{M}\underline{1} \wedge -, U_{\mathcal{M}\underline{1}}\big)$$

are developed in Chapters 4 and 5. The closed multicategory structure for $\mathsf{PermCat}^{\mathsf{su}}$ is developed in Chapter 8. The corresponding structures for $\mathsf{Multicat}_*$ and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ follow from the general discussion in Section 8.1 about closed multicategorical structure on symmetric monoidal closed categories.

**Chapter Summary.** Section 12.1 describes the context and application for the data in (12.0.1). Sections 12.2 and 12.3 further unpack and explain the details of the functors involved.

Section 12.4 describes the context and application for the data in (12.0.2). Section 12.5 further unpacks and explains the details of the functors involved.

Section 12.6 describes the context and application for the data in (12.0.3). Section 12.7 further unpacks and explains the details of the functors involved.

Here is a summary table.

| | |
|---|---|
| $\big((\mathsf{F}_\bullet)_\star^{\varrho^\bullet}, (\mathsf{End}_\bullet)_\star\big)$ inverse equivalence of homotopy theories | 12.1.6 |
| explanations of $(\mathsf{End}_\bullet)_\star$ | 12.2.2 and 12.2.16 |
| explanations of $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$ | 12.3.1, 12.3.5, and 12.3.11 |
| $\big((\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}, (\mathsf{End}_{\mathcal{M}\underline{1}})_\star\big)$ inverse equivalence of homotopy theories | 12.4.6 |
| explanations of $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ | 12.5.1, 12.5.4, and 12.5.6 |
| explanations of $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ | 12.5.8, 12.5.12, and 12.5.18 |
| $\big((\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}, (U_{\mathcal{M}\underline{1}})_\star\big)$ inverse equivalence of homotopy theories | 12.6.6 |
| explanation of $(U_{\mathcal{M}\underline{1}})_\star$ | 12.7.1 |
| explanations of $(\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}$ | 12.7.3 and 12.7.7 |

## 12.1. Homotopy Equivalent Multicategorical and Permutative Enriched Diagrams

In this section we apply Theorems 11.4.14 and 11.4.24 to show that the categories of enriched diagrams and Mackey functors in pointed multicategories and permutative categories are connected by inverse equivalences of homotopy theories. See Theorem 12.1.6. As a result, left modules in $\mathsf{PermCat}^{\mathsf{su}}$ and left modules in $\mathsf{Multicat}_*$ have equivalent homotopy theories; see Explanation 12.1.8. In Sections 12.2 and 12.3 we explain in detail the functors that constitute this pair of inverse equivalences of homotopy theories.

**Context.** For the context first recall the diagram

$$(12.1.1) \qquad \mathsf{Multicat}_* \underset{\mathsf{End}_\bullet}{\overset{\mathsf{F}_\bullet}{\rightleftarrows}} \mathsf{PermCat}^{\mathsf{su}}$$

in (5.3.1) consisting of

- the Cat-multicategory $\mathsf{Multicat}_*$ in Explanation 1.2.9,
- the Cat-multicategory $\mathsf{PermCat}^{\mathsf{su}}$ in Theorem 1.4.29,
- the Cat-multifunctor $\mathsf{End}_\bullet$ in Explanation 1.4.32, and
- the non-symmetric Cat-multifunctor $\mathsf{F}_\bullet$ in Theorem 5.2.6.

The two composites in (12.1.1) are connected to the respective identity functors via the following Cat-multinatural transformations from Lemmas 5.3.2 and 5.3.3.

$$(12.1.2) \qquad \mathsf{Multicat}_* \underset{\mathsf{End}_\bullet\mathsf{F}_\bullet}{\overset{1}{\underset{\Downarrow\eta^\bullet}{\rightrightarrows}}} \mathsf{Multicat}_* \qquad \mathsf{PermCat}^{\mathsf{su}} \underset{\mathsf{F}_\bullet\mathsf{End}_\bullet}{\overset{1}{\underset{\Downarrow\varrho^\bullet}{\rightrightarrows}}} \mathsf{PermCat}^{\mathsf{su}}$$

The underlying categories of $\mathsf{Multicat}_*$ and $\mathsf{PermCat}^{\mathsf{su}}$ are equipped with the relative category structures

$$(12.1.3) \qquad \left(\mathsf{Multicat}_*, \mathcal{S}_\bullet\right) \quad \text{and} \quad \left(\mathsf{PermCat}^{\mathsf{su}}, \mathcal{S}\right)$$

in (4.7.2) and (2.5.14).

- The wide subcategory of stable equivalences

$$\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$$

is created by Segal $K$-theory $\mathsf{K}^{\mathsf{Se}}$ (2.5.3). So a strictly unital symmetric monoidal functor $P$ is in $\mathcal{S}$ if and only if $\mathsf{K}^{\mathsf{Se}}P$ is a stable equivalence of symmetric spectra. For a small $\mathsf{PermCat}^{\mathsf{su}}$-category $\mathsf{C}$, the wide subcategory in (12.1.7) below

$$(12.1.4) \qquad \mathcal{S}_{\blacktriangle} \subset \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\left(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\right)$$

is defined as in (11.4.2) using $\mathcal{S}$.

- The wide subcategory of $\mathsf{F}_\bullet$-stable equivalences

$$\mathcal{S}_\bullet = \mathsf{F}_\bullet^{-1}(\mathcal{S}) \subset \mathsf{Multicat}_*$$

is created by $\mathsf{F}_\bullet$. The wide subcategory in (12.1.7) below

$$(12.1.5) \qquad (\mathcal{S}_\bullet)_{\blacktriangle} \subset \mathsf{Multicat}_*\text{-}\mathsf{Cat}\left(\mathsf{C}_{\mathsf{End}_\bullet}, \mathsf{Multicat}_*\right)$$

is defined as in (11.4.2) using $\mathcal{S}_\bullet$.

**Equivalences of Homotopy Theories.** In Theorem 5.4.1 we observe that the pair $(\mathsf{F}_\bullet, \mathsf{End}_\bullet)$ induces inverse equivalences of homotopy theories between the respective categories of non-symmetric Q-algebras for each small non-symmetric Cat-multicategory Q. The following observation extends the inverse equivalences of homotopy theories $(\mathsf{F}_\bullet, \mathsf{End}_\bullet)$ to categories of enriched diagrams and Mackey functors. In Sections 12.2 and 12.3 we further explain the functors $(\mathsf{End}_\bullet)_\star$ and $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$.

**Theorem 12.1.6.** *Suppose* C *is a small* $\mathsf{PermCat}^{\mathsf{su}}$-*category. Then the functors*

$$(12.1.7) \quad \left( \mathsf{Multicat}_\star\text{-}\mathsf{Cat}(\mathsf{C}_{\mathsf{End}_\bullet}, \mathsf{Multicat}_\star), (\mathcal{S}_\bullet)_\blacktriangle \right) \xrightleftharpoons[\substack{\sim \\ (\mathsf{End}_\bullet)_\star}]{(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}} \left( \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}), \mathcal{S}_\blacktriangle \right),$$

*defined by the data in (12.1.1) through (12.1.5), are inverse equivalences of homotopy theories.*

*Moreover, the variant with* $(\mathsf{C}_{\mathsf{End}_\bullet})^{\mathsf{op}}$ *and* $\mathsf{C}^{\mathsf{op}}$ *replacing, respectively,* $\mathsf{C}_{\mathsf{End}_\bullet}$ *and* C *is also true.*

*Proof.* The first assertion is an instance of Theorem 11.4.14, which is applicable in the current setting as we now explain. Following the summary in Explanation 11.4.23, first we verify that Definition 11.1.1 (i) through (v) are satisfied in the current context.

(i) $\mathsf{M} = \mathsf{Multicat}_\star$ is a closed multicategory by
  - Proposition 8.1.16 and
  - the fact that it is a symmetric monoidal closed category (Theorem 1.2.8).

  By Theorem 8.4.15, $\mathsf{N} = \mathsf{PermCat}^{\mathsf{su}}$ is a closed multicategory.

(ii) C is, by assumption, a small $\mathsf{PermCat}^{\mathsf{su}}$-category.

(iii) $F = \mathsf{F}_\bullet$ in (12.1.1) is a non-symmetric multifunctor by Theorem 5.2.6, and $E = \mathsf{End}_\bullet$ is a multifunctor by Proposition 1.4.31.

(iv) $\kappa = \eta^\bullet$ and $\xi = \varrho^\bullet$ in (12.1.2) are multinatural transformations by Lemmas 5.3.2 and 5.3.3, respectively.

(v) In the current setting, the condition (11.1.2) is the equality of the following two pointed multifunctors for each pair of objects $x, y \in$ C.

$$\mathsf{End}_\bullet\mathsf{C}(x,y) \xrightleftharpoons[\mathsf{End}_\bullet\varrho^\bullet_{\mathsf{C}(x,y)}]{\eta^\bullet_{\mathsf{End}_\bullet\mathsf{C}(x,y)}} \mathsf{End}_\bullet\mathsf{F}_\bullet\mathsf{End}_\bullet\mathsf{C}(x,y)$$

  This equality holds by Lemma 4.6.13 because each hom object $\mathsf{C}(x,y)$ is a small permutative category.

Thus Definition 11.1.1 (i) through (v) hold in the context of (12.1.1) through (12.1.5). Next, the only assumption in Definition 11.4.7 is that the relative category

$$(\mathsf{N}, \mathcal{X}) = \left( \mathsf{PermCat}^{\mathsf{su}}, \mathcal{S} \right)$$

is a category with weak equivalences (Definition 2.1.6 (6)). This is true for the following two reasons.

- The wide subcategory $\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$ is created by a functor, namely, Segal $K$-theory (2.5.14)

$$\mathsf{K}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0}.$$

- The class of stable equivalences in $\mathsf{Sp}_{\geq 0}$ contains all the isomorphisms and has the 2-out-of-3 property.

In the current setting there are equalities of wide subcategories

$$F^{-1}\mathcal{X} = \mathsf{F}_{\bullet}^{-1}(\mathcal{S}) = \mathcal{S}_{\bullet} \subset \mathsf{Multicat}_*.$$

The data in (12.1.7) are those in (11.4.15) in the current context.

Finally, each component of $\varrho^{\bullet}$ is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$ by Remark 2.5.15 (2) because it admits a left adjoint by Proposition 4.6.6. Moreover, in the proof of Theorem 4.7.3 we explain that each component of $\eta^{\bullet}$ is an $\mathsf{F}_{\bullet}$-stable equivalence in $\mathsf{Multicat}_*$. Thus Theorem 11.4.14 is applicable in the current setting, proving the first assertion.

The second assertion about $(\mathsf{C}_{\mathsf{End}_{\bullet}})^{\mathsf{op}}$ and $\mathsf{C}^{\mathsf{op}}$ is an instance of Theorem 11.4.24. It is applicable because $\mathsf{End}_{\bullet}$ is a multifunctor (Proposition 1.4.31). □

**Explanation 12.1.8** (Homotopy Equivalent Categories of Modules). By Propositions 10.1.8 and 10.1.17, for each small $\mathsf{PermCat}^{\mathsf{su}}$-category $\mathsf{C}$, the functors in (12.1.7) are inverse equivalences of homotopy theories between

- left $\mathsf{C}$-modules in $\mathsf{PermCat}^{\mathsf{su}}$ and
- left $\mathsf{C}_{\mathsf{End}_{\bullet}}$-modules in $\mathsf{Multicat}_*$.

We explain the functors $(\mathsf{End}_{\bullet})_{\star}$ and $(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}$ in detail in Sections 12.2 and 12.3. ◇

**Remark 12.1.9** (Non-Existence of Unpointed Version). We do not know of any analog of Theorem 12.1.6 for the symmetric monoidal closed category $\mathsf{Multicat}$ (Theorem 1.1.26), whose objects are small multicategories. The reason is that the multinatural transformation $\varrho^{\bullet}$ in (12.1.2) is necessary to define the functor $(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}$. As we discuss in Remark 3.3.13, for each permutative category $\mathsf{C}$, the symmetric monoidal functor $\varrho_{\mathsf{C}}$ is *not* strictly unital. Thus we *cannot* use $\varrho$ to define a multinatural transformation $1_{\mathsf{PermCat}^{\mathsf{su}}} \longrightarrow \mathsf{F}\,\mathsf{End}$. ◇

## 12.2. Permutative to Multicategorical Enriched Diagrams

In this section we explain in detail the equivalence of homotopy theories in (12.1.7)

$$\mathsf{Multicat}_*\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\bullet}}, \mathsf{Multicat}_*\big) \xleftarrow[\sim]{(\mathsf{End}_{\bullet})_{\star}} \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big)$$

that produces pointed multicategorical enriched diagrams from permutative enriched diagrams. To simplify the notation, we use the following abbreviations throughout this section.

(12.2.1) $\qquad\qquad \mathsf{M}_* = \mathsf{Multicat}_* \qquad\qquad \mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}$

This section is organized as follows.

- Explanation 12.2.2 describes $(\mathsf{End}_{\bullet})_{\star}$ in terms of $(-)_{\mathsf{End}_{\bullet}}$ and $\widehat{\mathsf{End}_{\bullet}}$.
- Explanation 12.2.4 describes the 2-functor $(-)_{\mathsf{End}_{\bullet}}$.
- Explanation 12.2.9 describes the standard enrichment $\widehat{\mathsf{End}_{\bullet}}$.
- Explanation 12.2.16 summarizes Explanations 12.2.2, 12.2.4, and 12.2.9.

**Explanation 12.2.2** (Unpacking $(\mathsf{End}_{\bullet})_{\star}$). The diagram change-of-enrichment functor in (12.1.7)

$$(\mathsf{End}_{\bullet})_{\star} : \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{P}^{\mathsf{su}}) \longrightarrow \mathsf{M}_*\text{-}\mathsf{Cat}(\mathsf{C}_{\mathsf{End}_{\bullet}}, \mathsf{M}_*)$$

is defined in (10.2.3) and verified in Theorem 10.3.1. To understand its assignments on objects and morphisms, consider

- $\mathsf{P}^{\mathsf{su}}$-functors $A, B : \mathsf{C} \longrightarrow \mathsf{P}^{\mathsf{su}}$ (Explanation 6.3.12) and
- a $\mathsf{P}^{\mathsf{su}}$-natural transformation $\psi : A \longrightarrow B$ (Explanation 6.3.16)

as in the left diagram below.

$$(12.2.3) \qquad \mathsf{C} \underset{B}{\overset{A}{\Longrightarrow \psi}} \mathsf{P}^{\mathsf{su}} \quad \xmapsto{\;(\mathsf{End}_{\bullet})_{\star}\;} \quad \mathsf{C}_{\mathsf{End}_{\bullet}} \underset{B_{\mathsf{End}_{\bullet}}}{\overset{A_{\mathsf{End}_{\bullet}}}{\Downarrow \psi_{\mathsf{End}_{\bullet}}}} (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_{\bullet}} \xrightarrow{\;\widehat{\mathsf{End}_{\bullet}}\;} \mathsf{M}_{*}$$

Then $(\mathsf{End}_{\bullet})_{\star}$ sends $A$, $B$, and $\psi$ to the composites and whiskering as in the right diagram in (12.2.3). In other words, the functor $(\mathsf{End}_{\bullet})_{\star}$

- first applies the change of enrichment $(-)_{\mathsf{End}_{\bullet}}$ and then
- composes or whiskers with the standard enrichment $\widehat{\mathsf{End}_{\bullet}}$.

We describe $(-)_{\mathsf{End}_{\bullet}}$ and $\widehat{\mathsf{End}_{\bullet}}$ further in Explanations 12.2.4 and 12.2.9 below. Then we summarize the discussion in Explanation 12.2.16.                                    ⬦

**Explanation 12.2.4** (Unpacking $(-)_{\mathsf{End}_{\bullet}}$). We describe the change-of-enrichment 2-functor in (12.2.3)

$$(-)_{\mathsf{End}_{\bullet}} : \mathsf{P}^{\mathsf{su}}\text{-Cat} \longrightarrow \mathsf{M}_{*}\text{-Cat}$$

by interpreting Definition 7.1.1 for the multifunctor (Explanation 1.4.32)

$$\mathsf{End}_{\bullet} : \mathsf{P}^{\mathsf{su}} \longrightarrow \mathsf{M}_{*}.$$

The existence of $(-)_{\mathsf{End}_{\bullet}}$ is an instance of Proposition 7.1.9.

*Objects.* First we consider a $\mathsf{P}^{\mathsf{su}}$-category $(\mathsf{D}, \mathsf{m}^{\mathsf{D}})$ (Explanation 6.3.2).

- The $\mathsf{M}_{*}$-category $\mathsf{D}_{\mathsf{End}_{\bullet}}$ has the same objects as $\mathsf{D}$. So $(\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_{\bullet}}$ has small permutative categories as objects.
- For each pair of objects $a, b \in \mathsf{D}$, its hom object is

$$(\mathsf{D}_{\mathsf{End}_{\bullet}})(a, b) = \mathsf{End}_{\bullet}\mathsf{D}(a, b) \quad \text{in} \quad \mathsf{M}_{*}.$$

So for small permutative categories $\mathsf{X}$ and $\mathsf{Y}$, by Theorem 6.4.20 there is a hom object

$$(\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_{\bullet}}(\mathsf{X}, \mathsf{Y}) = \mathsf{End}_{\bullet}\mathsf{P}^{\mathsf{su}}(\mathsf{X}, \mathsf{Y}) \quad \text{in} \quad \mathsf{M}_{*}.$$

  - In $\mathsf{P}^{\mathsf{su}}(\mathsf{X}, \mathsf{Y})$ the objects are strictly unital symmetric monoidal functors $\mathsf{X} \longrightarrow \mathsf{Y}$.
  - The morphisms are monoidal natural transformations.
  - The monoidal structure is defined pointwise in the codomain $\mathsf{Y}$.
  $\mathsf{End}_{\bullet}\mathsf{P}^{\mathsf{su}}(\mathsf{X}, \mathsf{Y})$ is the pointed multicategory associated to the permutative category $\mathsf{P}^{\mathsf{su}}(\mathsf{X}, \mathsf{Y})$.
- For objects $a, b, c \in \mathsf{D}$, the composition binary multimorphism in $\mathsf{M}_{*}$

$$\mathsf{End}_{\bullet}\big(\mathsf{m}^{\mathsf{D}}_{a,b,c}\big) : \big(\mathsf{End}_{\bullet}\mathsf{D}(b,c)\,,\,\mathsf{End}_{\bullet}\mathsf{D}(a,b)\big) \longrightarrow \mathsf{End}_{\bullet}\mathsf{D}(a,c)$$

is the image under $\mathsf{End}_{\bullet}$ (Proposition 1.4.31) of the composition in $\mathsf{D}$,

$$\mathsf{m}^{\mathsf{D}}_{a,b,c} : \mathsf{D}(b,c) \times \mathsf{D}(a,b) \longrightarrow \mathsf{D}(a,c).$$

The latter is a bilinear functor of permutative categories (Definition 1.4.2). The composition in $\mathsf{P}^{\mathsf{su}}$ as a $\mathsf{P}^{\mathsf{su}}$-category (Definition 6.4.19) extends composition of strictly unital symmetric monoidal functors and is a bilinear functor.

*1-Cells.* For a $\mathsf{P}^{\mathsf{su}}$-functor $A : \mathsf{C} \longrightarrow \mathsf{P}^{\mathsf{su}}$ as in (12.2.3), the $\mathsf{M}_*$-functor

$$A_{\mathsf{End}_\bullet} : \mathsf{C}_{\mathsf{End}_\bullet} \longrightarrow (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_\bullet}$$

has the same object assignment as $A$. In other words, for each object $x$ in $\mathsf{C}$,

(12.2.5) $$(A_{\mathsf{End}_\bullet})x = Ax \quad \text{in} \quad \mathsf{P}^{\mathsf{su}}.$$

For objects $x, y \in \mathsf{C}$, the $(x, y)$-component pointed multifunctor

(12.2.6) $$(A_{\mathsf{End}_\bullet})_{x,y} = \mathsf{End}_\bullet(A_{x,y}) : \mathsf{End}_\bullet\mathsf{C}(x, y) \longrightarrow \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Ay)$$

is the image under $\mathsf{End}_\bullet$ of the $(x, y)$-component strictly unital symmetric monoidal functor

$$\left(A_{x,y}, A_{x,y}^2\right) : \mathsf{C}(x, y) \longrightarrow \mathsf{P}^{\mathsf{su}}(Ax, Ay),$$

as defined in Example C.4.8 (i). The same explanation also applies to the $\mathsf{P}^{\mathsf{su}}$-functor $B : \mathsf{C} \longrightarrow \mathsf{P}^{\mathsf{su}}$.

*2-Cells.* A $\mathsf{P}^{\mathsf{su}}$-natural transformation $\psi : A \longrightarrow B$ as in (12.2.3) is determined by its components. For each object $x \in \mathsf{C}$, the $x$-component is a nullary multimorphism

$$\psi_x : \langle\rangle \longrightarrow \mathsf{P}^{\mathsf{su}}(Ax, Bx) \quad \text{in} \quad \mathsf{P}^{\mathsf{su}}.$$

By Definitions 1.4.2 and 1.4.15, such a nullary multimorphism $\psi_x$ is a 0-linear functor to $\mathsf{P}^{\mathsf{su}}(Ax, Bx)$. This, in turn, means a choice of an object in the permutative category $\mathsf{P}^{\mathsf{su}}(Ax, Bx)$. In other words, each component

(12.2.7) $$\psi_x : Ax \longrightarrow Bx$$

is a strictly unital symmetric monoidal functor.

Under the change of enrichment $(-)_{\mathsf{End}_\bullet}$, the $\mathsf{M}_*$-natural transformation in (12.2.3)

$$\psi_{\mathsf{End}_\bullet} : A_{\mathsf{End}_\bullet} \longrightarrow B_{\mathsf{End}_\bullet}$$

has, for each object $x$ in $\mathsf{C}$, $x$-component nullary multimorphism

(12.2.8) $$(\psi_{\mathsf{End}_\bullet})_x : \langle\rangle \longrightarrow \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Bx) \quad \text{in} \quad \mathsf{M}_*.$$

Since the multicategory structure on $\mathsf{M}_*$ is induced by its symmetric monoidal structure, such a nullary multimorphism is a pointed multifunctor

$$(\psi_{\mathsf{End}_\bullet})_x : \mathsf{I}_+ = \mathsf{I} \amalg \mathsf{T} \longrightarrow \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Bx)$$

from the smash unit $\mathsf{I}_+$ in (1.2.4). Preservation of basepoints and $\mathsf{I}$ being the initial operad imply that $(\psi_{\mathsf{End}_\bullet})_x$ is a choice of an object in $\mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Bx)$, which means an object in $\mathsf{P}^{\mathsf{su}}(Ax, Bx)$. This, in turn, means a strictly unital symmetric monoidal functor $Ax \longrightarrow Bx$, which is given by $\psi_x$ in (12.2.7).

In summary, the components of the $\mathsf{M}_*$-natural transformation $\psi_{\mathsf{End}_\bullet}$ are the components of the $\mathsf{P}^{\mathsf{su}}$-natural transformation $\psi$. $\diamond$

**Explanation 12.2.9** (Unpacking $\widehat{\mathsf{End}_\bullet}$). We describe the last arrow in (12.2.3)

$$\widehat{\mathsf{End}_\bullet} : (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_\bullet} \longrightarrow \mathsf{M}_*.$$

This is the standard enrichment of $\mathsf{End}_\bullet$, which is an $\mathsf{M}_*$-functor (Theorem 9.2.12). Its object assignment is the same as that of $\mathsf{End}_\bullet$ (Example C.4.8). In other words,

$$(12.2.10) \qquad\qquad \widehat{\mathsf{End}_\bullet}\,\mathsf{X} = \mathsf{End}_\bullet\mathsf{X} \quad \text{in} \quad \mathsf{M}_*$$

for each small permutative category $\mathsf{X}$.

For small permutative categories $\mathsf{X}$ and $\mathsf{Y}$, the $(\mathsf{X},\mathsf{Y})$-component of $\widehat{\mathsf{End}_\bullet}$ is a pointed multifunctor

$$(12.2.11) \qquad\qquad (\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}} : \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y}) \longrightarrow \mathsf{Hom}_*(\mathsf{End}_\bullet\mathsf{X}, \mathsf{End}_\bullet\mathsf{Y}).$$

In the codomain, $\mathsf{Hom}_*$ is the pointed internal hom (1.2.5) in $\mathsf{M}_*$.

- The objects of $\mathsf{Hom}_*(?,?)$ are pointed multifunctors.
- Its multimorphisms are pointed transformations (Explanation 1.2.6).

Next we describe the component pointed multifunctor $(\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}}$ in two equivalent ways.

First, $(\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}}$ is uniquely determined by its adjoint, which is the arrow $\mathsf{End}_\bullet(\mathsf{ev}_{\mathsf{X},\mathsf{Y}})$ in the commutative diagram (12.2.12) in $\mathsf{M}_*$.

$$(12.2.12)$$

$$\mathsf{Hom}_*(\mathsf{End}_\bullet\mathsf{X}, \mathsf{End}_\bullet\mathsf{Y}) \wedge \mathsf{End}_\bullet\mathsf{X}$$
$$(\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}} \wedge 1 \qquad\qquad \mathsf{ev}^{\mathsf{M}_*}$$
$$\mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y}) \wedge \mathsf{End}_\bullet\mathsf{X} \xrightarrow{\ \mathsf{End}_\bullet(\mathsf{ev}_{\mathsf{X},\mathsf{Y}})\ } \mathsf{End}_\bullet\mathsf{Y}$$

The diagram (12.2.12) is the diagram (9.2.11) for $\widehat{\mathsf{End}_\bullet}$.

- In (12.2.12) $\wedge$ is the smash product (1.2.3) in $\mathsf{M}_*$.
- The evaluation for permutative categories

$$\mathsf{ev}_{\mathsf{X},\mathsf{Y}} : \mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y}) \times \mathsf{X} \longrightarrow \mathsf{Y}$$

  is the bilinear functor in (6.5.2).
- $\mathsf{ev}^{\mathsf{M}_*}$ is the evaluation (B.3.2) in the symmetric monoidal closed category $\mathsf{M}_*$ (Theorem 1.2.8).

Alternatively, we obtain from (12.2.12) a direct description of the pointed multifunctor $(\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}}$ as follows. An object in $\mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y})$ is an object in $\mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y})$, which is a strictly unital symmetric monoidal functor

$$(Q, Q^2) : \mathsf{X} \longrightarrow \mathsf{Y}.$$

Its image under $(\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}}$ is the pointed multifunctor

$$(12.2.13) \qquad\qquad (\widehat{\mathsf{End}_\bullet})_{\mathsf{X},\mathsf{Y}}(Q, Q^2) = \mathsf{End}_\bullet(Q, Q^2) : \mathsf{End}_\bullet\mathsf{X} \longrightarrow \mathsf{End}_\bullet\mathsf{Y}$$

obtained from $(Q, Q^2)$ by applying $\mathsf{End}_\bullet$ (Example C.4.8 (i)).

For $n \geq 0$ an $n$-ary multimorphism

$$\theta \in \left(\mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y})\right)\left(\left\langle(Q_i, Q_i^2)\right\rangle_{i=1}^n ; (Q, Q^2)\right)$$
$$= \mathsf{P}^{\mathsf{su}}(\mathsf{X},\mathsf{Y})\left(\oplus_{i=1}^n (Q_i, Q_i^2), (Q, Q^2)\right)$$

is a monoidal natural transformation as follows.

$$\begin{array}{ccc} & \overset{\oplus_{i=1}^n Q_i}{\overbrace{\phantom{XXXXX}}} & \\ X & \underset{Q}{\overset{\Downarrow \theta}{\longrightarrow}} & Y \end{array}$$

The domain of $\theta$ is the strictly unital symmetric monoidal functor

$$\Big( \oplus_{i=1}^n Q_i , \, \big(\oplus_{i=1}^n Q_i\big)^2 \Big) : X \longrightarrow Y.$$

- As a functor the sum is taken pointwise in $Y$, so

$$\big( \oplus_{i=1}^n Q_i \big) x = \oplus_{i=1}^n (Q_i x) \quad \text{for} \quad x \in X.$$

- $\oplus_{i=1}^n Q_i$ is strictly unital because each $Q_i$ is so.
- For objects $x, x' \in X$, the $(x, x')$-component of its monoidal constraint is the following composite in $Y$, with the isomorphism permuting the objects using the braiding in $Y$.

$$\big( \oplus_{i=1}^n Q_i x \big) \oplus \big( \oplus_{i=1}^n Q_i x' \big) \xrightarrow{(\oplus_{i=1}^n Q_i)^2_{x,x'}} \oplus_{i=1}^n Q_i (x \oplus x')$$

$$\cong \searrow \qquad \qquad \nearrow \oplus_{i=1}^n (Q_i^2)_{x,x'}$$

$$\oplus_{i=1}^n \big( Q_i x \oplus Q_i x' \big)$$

For each object $x \in X$, the $x$-component of $\theta$ is a morphism

(12.2.14) $$\theta_x : \oplus_{i=1}^n Q_i x \longrightarrow Qx \quad \text{in} \quad Y.$$

Applying $(\widehat{\mathsf{End}\mathord{\cdot}})_{X,Y}$ to $\theta$ yields the $n$-ary pointed transformation (Explanation 1.2.6)

$$(\widehat{\mathsf{End}\mathord{\cdot}})_{X,Y}(\theta) \in \mathsf{Hom}_*(\mathsf{End}\mathord{\cdot}X, \mathsf{End}\mathord{\cdot}Y)\big( \langle \mathsf{End}\mathord{\cdot}Q_i \rangle_{i=1}^n \, ; \, \mathsf{End}\mathord{\cdot}Q \big).$$

For each object $x \in \mathsf{End}\mathord{\cdot}X$, meaning $x \in X$, the $x$-component of $(\widehat{\mathsf{End}\mathord{\cdot}})_{X,Y}(\theta)$ is the $n$-ary multimorphism

(12.2.15)
$$(\widehat{\mathsf{End}\mathord{\cdot}})_{X,Y}(\theta)_x \in (\mathsf{End}\mathord{\cdot}Y)\Big( \big\langle (\mathsf{End}\mathord{\cdot}Q_i)x \big\rangle_{i=1}^n \, ; \, (\mathsf{End}\mathord{\cdot}Q)x \Big)$$
$$= Y\big( \oplus_{i=1}^n Q_i x \, , \, Qx \big)$$

given by the $x$-component $\theta_x$ in (12.2.14). In summary, the components of $(\widehat{\mathsf{End}\mathord{\cdot}})_{X,Y}(\theta)$ are the components of $\theta$. $\diamond$

**Explanation 12.2.16** (Back to $(\mathsf{End}\mathord{\cdot})_\star$). We summarize Explanations 12.2.2, 12.2.4, and 12.2.9. For a $\mathsf{P}^{\mathsf{su}}$-functor $A : C \longrightarrow \mathsf{P}^{\mathsf{su}}$ as in (12.2.3), the composite $\mathsf{M}_*$-functor

$$\begin{array}{ccc} & \overset{(\mathsf{End}\mathord{\cdot})_\star A}{\overbrace{\phantom{XXXXXXXXXXXXX}}} & \\ C_{\mathsf{End}\mathord{\cdot}} \xrightarrow{A_{\mathsf{End}\mathord{\cdot}}} (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}\mathord{\cdot}} \xrightarrow{\widehat{\mathsf{End}\mathord{\cdot}}} \mathsf{M}_* \end{array}$$

has, for each object $x \in C$, object assignment

$$\big( (\mathsf{End}\mathord{\cdot})_\star A \big) x = \mathsf{End}\mathord{\cdot}(Ax) \quad \text{in} \quad \mathsf{M}_*$$

by (12.2.5) and (12.2.10). For objects $x, y \in \mathsf{C}$, the $(x, y)$-component pointed multifunctor of $(\mathsf{End}_\bullet)_\star A$ is the composite

(12.2.17)

$$
\begin{array}{ccc}
\mathsf{End}_\bullet\mathsf{C}(x, y) & \xrightarrow{\;\left((\mathsf{End}_\bullet)_\star A\right)_{x,y}\;} & \mathsf{Hom}_*\left(\mathsf{End}_\bullet(Ax), \mathsf{End}_\bullet(Ay)\right) \\
{\scriptstyle \mathsf{End}_\bullet(A_{x,y})} \searrow & & \nearrow {\scriptstyle (\widehat{\mathsf{End}_\bullet})_{Ax, Ay}} \\
& \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Ay) &
\end{array}
$$

by (12.2.6) and (12.2.11).

For a $\mathsf{P}^{\mathsf{su}}$-natural transformation $\psi : A \longrightarrow B$ as in (12.2.3), the $\mathsf{M}_*$-natural transformation $(\mathsf{End}_\bullet)_\star \psi$ is the whiskering below.

$$
\mathsf{C}_{\mathsf{End}_\bullet} \underset{B_{\mathsf{End}_\bullet}}{\overset{A_{\mathsf{End}_\bullet}}{\Longrightarrow \Downarrow \psi_{\mathsf{End}_\bullet}}} (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_\bullet} \xrightarrow{\;\widehat{\mathsf{End}_\bullet}\;} \mathsf{M}_*
$$

For each object $x \in \mathsf{C}$, its $x$-component nullary multimorphism is the following composite in $\mathsf{M}_*$.

$$
\begin{array}{ccc}
\langle\rangle & \xrightarrow{\;\left((\mathsf{End}_\bullet)_\star \psi\right)_x\;} & \mathsf{Hom}_*\left(\mathsf{End}_\bullet(Ax), \mathsf{End}_\bullet(Bx)\right) \\
{\scriptstyle (\psi_{\mathsf{End}_\bullet})_x} \downarrow & & \nearrow {\scriptstyle (\widehat{\mathsf{End}_\bullet})_{Ax, Bx}} \\
& \mathsf{End}_\bullet\mathsf{P}^{\mathsf{su}}(Ax, Bx) &
\end{array}
$$

By (12.2.8) and (12.2.13), this is given by the pointed multifunctor

(12.2.18) $$\left((\mathsf{End}_\bullet)_\star \psi\right)_x = \mathsf{End}_\bullet(\psi_x) : \mathsf{End}_\bullet(Ax) \longrightarrow \mathsf{End}_\bullet(Bx)$$

obtained from the strictly unital symmetric monoidal functor $\psi_x$ in (12.2.7) by applying $\mathsf{End}_\bullet$. $\diamond$

## 12.3. Multicategorical to Permutative Enriched Diagrams

We continue to use the abbreviations in (12.2.1), so

$$\mathsf{M}_* = \mathsf{Multicat}_* \quad \text{and} \quad \mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.$$

In this section we explain in detail the equivalence of homotopy theories in (12.1.7)

$$\mathsf{M}_*\text{-}\mathsf{Cat}\left(\mathsf{C}_{\mathsf{End}_\bullet}, \mathsf{M}_*\right) \xrightarrow[\sim]{\;(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}\;} \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}\left(\mathsf{C}, \mathsf{P}^{\mathsf{su}}\right)$$

that produces permutative enriched diagrams from pointed multicategorical enriched diagrams. This section is organized as follows.

- Explanation 12.3.1 describes $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$ in terms of $(\mathsf{F}_\bullet)_\star$ and $\mathsf{C}_{\varrho^\bullet}^*$.
- Explanation 12.3.5 describes $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$ on objects.
- Explanation 12.3.11 describes $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$ on morphisms.

**Explanation 12.3.1** (Unpacking $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$). The functor $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$ is an instance of the functor $F_\star^\xi$ (11.1.3) defined with the non-symmetric multifunctor (Theorem 5.2.6)

$$F = \mathsf{F}_\bullet : \mathsf{M}_* \longrightarrow \mathsf{P}^{\mathsf{su}}$$

and the multinatural transformation $\xi = \varrho^{\bullet}$ given by

$$\mathsf{P}^{\mathsf{su}} \overset{1}{\underset{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}}{\rightrightarrows}} \Downarrow\varrho^{\bullet} \; \mathsf{P}^{\mathsf{su}}$$

in Lemma 5.3.3.

- $\mathsf{F}_{\bullet}$ is defined on objects and multimorphisms in Definitions 4.1.11 and 5.2.2, respectively.
- The components of $\varrho^{\bullet}$ are defined in (4.6.2).

By definition (11.1.3) the functor $(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}$ is the following composite.

$$(12.3.2)$$

$$\mathsf{M}_{\star}\text{-}\mathsf{Cat}(\mathsf{C}_{\mathsf{End}_{\bullet}}, \mathsf{M}_{\star}) \xrightarrow{\;(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}\;} \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}(\mathsf{C}, \mathsf{P}^{\mathsf{su}})$$

$$(\mathsf{F}_{\bullet})_{\star} \searrow \qquad \nearrow \mathsf{C}_{\varrho^{\bullet}}^{\ast}$$

$$\mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}(\mathsf{C}_{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}}, \mathsf{P}^{\mathsf{su}})$$

The two constituent functors in (12.3.2) are as follows.

- $(\mathsf{F}_{\bullet})_{\star}$ is the diagram change-of-enrichment functor (Theorem 10.3.1) of $\mathsf{F}_{\bullet}$ at the $\mathsf{M}_{\star}$-category $\mathsf{C}_{\mathsf{End}_{\bullet}}$. The latter is the image of the $\mathsf{P}^{\mathsf{su}}$-category $\mathsf{C}$ under the change of enrichment $(-)_{\mathsf{End}_{\bullet}}$ (Explanation 12.2.4).
- $\mathsf{C}_{\varrho^{\bullet}}^{\ast}$ is defined by pre-composition and whiskering with the $\mathsf{P}^{\mathsf{su}}$-functor

$$(12.3.3) \qquad \mathsf{C}_{\varrho^{\bullet}} : \mathsf{C} \longrightarrow \mathsf{C}_{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}} = (\mathsf{C}_{\mathsf{End}_{\bullet}})_{\mathsf{F}_{\bullet}}.$$

This $\mathsf{P}^{\mathsf{su}}$-functor is the $\mathsf{C}$-component of the 2-natural transformation (Proposition 7.5.5)

$$\mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat} \overset{1}{\underset{(-)_{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}} = (-)_{\mathsf{F}_{\bullet}}(-)_{\mathsf{End}_{\bullet}}}{\rightrightarrows}} \Downarrow(-)_{\varrho^{\bullet}} \; \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}$$

induced by the multinatural transformation $\varrho^{\bullet} : 1_{\mathsf{P}^{\mathsf{su}}} \longrightarrow \mathsf{F}_{\bullet}\mathsf{End}_{\bullet}$. Here $(-)_{\mathsf{F}_{\bullet}}$ is the change-of-enrichment 2-functor (Explanation 7.1.12)

$$(12.3.4) \qquad (-)_{\mathsf{F}_{\bullet}} : \mathsf{M}_{\star}\text{-}\mathsf{Cat} \longrightarrow \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}$$

induced by the non-symmetric multifunctor $\mathsf{F}_{\bullet} : \mathsf{M}_{\star} \longrightarrow \mathsf{P}^{\mathsf{su}}$.

We describe $(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}$ on objects and morphisms in Explanations 12.3.5 and 12.3.11, respectively. $\diamond$

**Explanation 12.3.5** $((\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}}$ on Objects). Consider an $\mathsf{M}_{\star}$-functor (Definition B.1.8)

$$A : \mathsf{C}_{\mathsf{End}_{\bullet}} \longrightarrow \mathsf{M}_{\star}.$$

By (11.1.11) the $\mathsf{P}^{\mathsf{su}}$-functor $(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}} A$ is the following composite.

$$(12.3.6) \qquad \overset{(\mathsf{F}_{\bullet})_{\star}^{\varrho^{\bullet}} A}{\overbrace{\mathsf{C} \xrightarrow{\;\mathsf{C}_{\varrho^{\bullet}}\;} \mathsf{C}_{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}} = (\mathsf{C}_{\mathsf{End}_{\bullet}})_{\mathsf{F}_{\bullet}} \xrightarrow{\;A_{\mathsf{F}_{\bullet}}\;} (\mathsf{M}_{\star})_{\mathsf{F}_{\bullet}} \xrightarrow{\;\widehat{\mathsf{F}_{\bullet}}\;} \mathsf{P}^{\mathsf{su}}}}$$

The constituent $\mathsf{P}^{\mathsf{su}}$-functors in (12.3.6) are as follows.

(i) $\mathsf{C}_{\varrho^{\bullet}}$ is the $\mathsf{P}^{\mathsf{su}}$-functor in (12.3.3).
   - Its object assignment is the identity function.

- For objects $x, y \in \mathsf{C}$, its $(x, y)$-component is the strictly unital symmetric monoidal functor

(12.3.7) $$\varrho^{\bullet}_{\mathsf{C}(x,y)} : \mathsf{C}(x, y) \longrightarrow \mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C}(x, y)$$

  given by the $\mathsf{C}(x, y)$-component of $\varrho^{\bullet}$ (4.6.2).

(ii) $A_{\mathsf{F}_{\bullet}}$ is the change of enrichment of $A$ under $(-)_{\mathsf{F}_{\bullet}}$ (Explanation 7.1.12).
- Its object assignment is the same as that of $A$.
- For objects $x, y \in \mathsf{C}$, its $(x, y)$-component is the strict symmetric monoidal functor (Definition 4.1.12)

(12.3.8) $$\mathsf{F}_{\bullet}A_{x,y} : \mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C}(x, y) \longrightarrow \mathsf{F}_{\bullet}\mathsf{Hom}_{*}(Ax, Ay)$$

  given by applying $\mathsf{F}_{\bullet}$ to the $(x, y)$-component pointed multifunctor of $A$. Here $\mathsf{Hom}_{*}$ is the pointed internal hom for small pointed multicategories (1.2.5).

(iii) $\widehat{\mathsf{F}_{\bullet}}$ is the standard enrichment $\mathsf{P}^{\mathsf{su}}$-functor of $\mathsf{F}_{\bullet} : \mathsf{M}_{*} \longrightarrow \mathsf{P}^{\mathsf{su}}$ in (9.2.15).
- Its object assignment is the same as that of $\mathsf{F}_{\bullet}$.
- For small pointed multicategories $\mathsf{X}$ and $\mathsf{Y}$, its $(\mathsf{X}, \mathsf{Y})$-component is the strictly unital symmetric monoidal functor

(12.3.9) $$\widehat{\mathsf{F}_{\bullet}}_{\mathsf{X},\mathsf{Y}} = \big(\mathsf{F}_{\bullet}(\mathsf{ev}^{\mathsf{M}_*}_{\mathsf{X},\mathsf{Y}})\big)^{\#} : \mathsf{F}_{\bullet}\mathsf{Hom}_{*}(\mathsf{X}, \mathsf{Y}) \longrightarrow \mathsf{P}^{\mathsf{su}}(\mathsf{F}_{\bullet}\mathsf{X}, \mathsf{F}_{\bullet}\mathsf{Y}).$$

Combining (12.3.6) through (12.3.9), the object assignment of $(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star} A$ is given by, for each object $x \in \mathsf{C}$,

$$\big((\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star} A\big)x = \mathsf{F}_{\bullet}(Ax) \quad \text{in} \quad \mathsf{P}^{\mathsf{su}}.$$

For objects $x, y \in \mathsf{C}$, its $(x, y)$-component strictly unital symmetric monoidal functor is the following composite.

(12.3.10)
$$
\begin{array}{ccc}
\mathsf{C}(x,y) & \xrightarrow{\ ((\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star} A)_{x,y}\ } & \mathsf{P}^{\mathsf{su}}\big(\mathsf{F}_{\bullet}(Ax), \mathsf{F}_{\bullet}(Ay)\big) \\
\scriptstyle\varrho^{\bullet}_{\mathsf{C}(x,y)} \downarrow\ \ & & \ \ \nearrow \scriptstyle \widehat{\mathsf{F}_{\bullet}}_{Ax,Ay} \\
\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C}(x,y) & \xrightarrow{\ \mathsf{F}_{\bullet}A_{x,y}\ } & \mathsf{F}_{\bullet}\mathsf{Hom}_{*}(Ax, Ay)
\end{array}
$$

The diagram (12.3.10) is the diagram (11.1.13) in the current context.          ◇

**Explanation 12.3.11** $((\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}$ on Morphisms). Consider an $\mathsf{M}_{*}$-natural transformation $\psi$ (Definition B.1.10) as in the left diagram below.

$$
\mathsf{C}_{\mathsf{End}_{\bullet}} \underset{B}{\overset{A}{\Rightarrow\!\!\!\Downarrow\psi\!\!\!\Rightarrow}} \mathsf{M}_{*}
\qquad\qquad
\mathsf{C} \underset{(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}B}{\overset{(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}A}{\Rightarrow\!\!\!\Downarrow(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}\psi\!\!\!\Rightarrow}} \mathsf{P}^{\mathsf{su}}
$$

The $\mathsf{P}^{\mathsf{su}}$-natural transformation $(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star} \psi$, as in the right diagram above, is the following whiskering.

(12.3.12)
$$
\mathsf{C} \xrightarrow{\ \mathsf{C}_{\varrho^{\bullet}}\ } \mathsf{C}_{\mathsf{F}_{\bullet}\mathsf{End}_{\bullet}} = (\mathsf{C}_{\mathsf{End}_{\bullet}})_{\mathsf{F}_{\bullet}} \underset{B_{\mathsf{F}_{\bullet}}}{\overset{A_{\mathsf{F}_{\bullet}}}{\Rightarrow\!\!\!\Downarrow\psi_{\mathsf{F}_{\bullet}}\!\!\!\Rightarrow}} (\mathsf{M}_{*})_{\mathsf{F}_{\bullet}} \xrightarrow{\ \widehat{\mathsf{F}_{\bullet}}\ } \mathsf{P}^{\mathsf{su}}
$$

The whiskering (12.3.12) is the one in (11.1.16) in the current context.

- The $\mathsf{P}^{\mathsf{su}}$-functors $\mathsf{C}_{\varrho^{\bullet}}$ and $\widehat{\mathsf{F}_{\bullet}}$ are as in (12.3.6).
- $\psi_{\mathsf{F}_{\bullet}}$ is the change of enrichment of $\psi$ under $(-)_{\mathsf{F}_{\bullet}}$ in Explanation 7.1.12.

Next we describe the components of $(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}\psi$ explicitly. For each object $x \in \mathsf{C}$, the $x$-component of $\psi$ is a pointed multifunctor

$$\psi_x : \mathsf{I}_+ = \mathsf{I} \coprod \mathsf{T} \longrightarrow \mathsf{Hom}_*(Ax, Bx)$$

from the smash unit $\mathsf{I}_+$ in (1.2.4). Preservation of basepoint and $\mathsf{I}$ being the initial operad imply that $\psi_x$ is equivalent to a choice of an object in $\mathsf{Hom}_*(Ax, Bx)$. This means a pointed multifunctor

$$(12.3.13) \qquad\qquad\qquad \psi_x : Ax \longrightarrow Bx.$$

The $x$-component of $(\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}\psi$ is a nullary multimorphism

$$\left((\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}\psi\right)_x : \langle\rangle \longrightarrow \mathsf{P}^{\mathsf{su}}\left(\mathsf{F}_{\bullet}(Ax), \mathsf{F}_{\bullet}(Bx)\right) \quad \text{in} \quad \mathsf{P}^{\mathsf{su}}.$$

This means a choice of an object in $\mathsf{P}^{\mathsf{su}}\left(\mathsf{F}_{\bullet}(Ax), \mathsf{F}_{\bullet}(Bx)\right)$. This, in turn, means a strictly unital symmetric monoidal functor $\mathsf{F}_{\bullet}(Ax) \longrightarrow \mathsf{F}_{\bullet}(Bx)$. Using (11.1.18) in the current context, we obtain the $x$-component

$$\left((\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}\psi\right)_x = \mathsf{F}_{\bullet}(\psi_x) : \mathsf{F}_{\bullet}(Ax) \longrightarrow \mathsf{F}_{\bullet}(Bx)$$

by applying $\mathsf{F}_{\bullet}$ to $\psi_x$ in (12.3.13). By Definition 4.1.12 $\mathsf{F}_{\bullet}(\psi_x)$ is a *strict* symmetric monoidal functor. $\diamond$

## 12.4. Homotopy Equivalent $\mathcal{M}\underline{1}$-Modules and Permutative Enriched Diagrams

In this section we apply Theorems 11.4.14 and 11.4.24 to show that the categories of enriched diagrams and Mackey functors in left $\mathcal{M}\underline{1}$-modules and permutative categories are connected by inverse equivalences of homotopy theories. See Theorem 12.4.6. Explanation 12.4.8 summarizes Theorems 12.1.6 and 12.4.6 in terms of left modules in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, $\mathsf{Multicat}_*$, and $\mathsf{PermCat}^{\mathsf{su}}$. We explain these equivalences of homotopy theories further in Section 12.5.

**Context.** For the context first recall the diagram

$$(12.4.1) \qquad\qquad \mathsf{Mod}^{\mathcal{M}\underline{1}} \underset{\mathsf{End}_{\mathcal{M}\underline{1}}}{\overset{\mathsf{F}_{\mathcal{M}\underline{1}}}{\rightleftarrows}} \mathsf{PermCat}^{\mathsf{su}}$$

in (5.5.1) consisting of

- the Cat-multicategory $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ in Explanation 1.3.24,
- the Cat-multicategory $\mathsf{PermCat}^{\mathsf{su}}$ in Theorem 1.4.29,
- the Cat-multifunctor $\mathsf{End}_{\mathcal{M}\underline{1}}$ in Explanation 1.4.41, and
- the non-symmetric Cat-multifunctor $\mathsf{F}_{\mathcal{M}\underline{1}} = \mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}$ in (5.5.2).

The two composites in (12.4.1) are connected to the respective identity functors via the following Cat-multinatural transformations from Definitions 5.5.5 and 5.5.8.

$$(12.4.2) \qquad \mathsf{Mod}^{\mathcal{M}\underline{1}} \underset{\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}}{\overset{1}{\underset{\Downarrow\eta^{\mathcal{M}\underline{1}}}{\rightrightarrows}}} \mathsf{Mod}^{\mathcal{M}\underline{1}} \qquad \mathsf{PermCat}^{\mathsf{su}} \underset{\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}}{\overset{1}{\underset{\Downarrow\varrho^{\mathcal{M}\underline{1}}}{\rightrightarrows}}} \mathsf{PermCat}^{\mathsf{su}}$$

The underlying categories of $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and $\mathsf{PermCat}^{\mathsf{su}}$ are equipped with the relative category structures

$$(12.4.3) \qquad\qquad \left(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}}\right) \quad \text{and} \quad \left(\mathsf{PermCat}^{\mathsf{su}}, \mathcal{S}\right)$$

in (4.7.2) and (2.5.14).

- The wide subcategory of stable equivalences

$$\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$$

  is created by Segal $K$-theory $\mathsf{K}^{\mathsf{Se}}$ (2.5.3). For a small $\mathsf{PermCat}^{\mathsf{su}}$-category $\mathsf{C}$, the wide subcategory in (12.4.7) below

(12.4.4) $$\mathcal{S}_{\blacktriangle} \subset \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big)$$

  is defined as in (11.4.2) using $\mathcal{S}$.
- The wide subcategory of $\mathsf{F}_{\mathcal{M}1}$-stable equivalences

$$\mathcal{S}^{\mathcal{M}1} = \mathsf{F}_{\mathcal{M}1}^{-1}(\mathcal{S}) \subset \mathsf{Mod}^{\mathcal{M}1}$$

  is created by $\mathsf{F}_{\mathcal{M}1}$. The wide subcategory in (12.4.7) below

(12.4.5) $$\mathcal{S}_{\blacktriangle}^{\mathcal{M}1} \subset \mathsf{Mod}^{\mathcal{M}1}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}1}}, \mathsf{Mod}^{\mathcal{M}1}\big)$$

  is defined as in (11.4.2) using $\mathcal{S}^{\mathcal{M}1}$.

**Equivalences of Homotopy Theories.** In Theorem 5.5.12 we observe that the pair $(\mathsf{F}_{\mathcal{M}1}, \mathsf{End}_{\mathcal{M}1})$ induces inverse equivalences of homotopy theories between the respective categories of non-symmetric Q-algebras for each small non-symmetric Cat-multicategories Q. The following observation extends the inverse equivalences of homotopy theories $(\mathsf{F}_{\mathcal{M}1}, \mathsf{End}_{\mathcal{M}1})$ to categories of enriched diagrams and Mackey functors. It is the $\mathsf{Mod}^{\mathcal{M}1}$ analog of Theorem 12.1.6.

**Theorem 12.4.6.** *Suppose* $\mathsf{C}$ *is a small* $\mathsf{PermCat}^{\mathsf{su}}$-*category. Then the functors*

(12.4.7) $$\Big(\mathsf{Mod}^{\mathcal{M}1}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}1}}, \mathsf{Mod}^{\mathcal{M}1}\big), \mathcal{S}_{\blacktriangle}^{\mathcal{M}1}\Big) \underset{(\mathsf{End}_{\mathcal{M}1})_{\star}}{\overset{(\mathsf{F}_{\mathcal{M}1})_{\star}^{\varrho^{\mathcal{M}1}}}{\underset{\sim}{\rightleftarrows}}} \Big(\mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big), \mathcal{S}_{\blacktriangle}\Big),$$

*defined by the data in (12.4.1) through (12.4.5), are inverse equivalences of homotopy theories.*

*Moreover, the variant with* $(\mathsf{C}_{\mathsf{End}_{\mathcal{M}1}})^{\mathsf{op}}$ *and* $\mathsf{C}^{\mathsf{op}}$ *replacing, respectively,* $\mathsf{C}_{\mathsf{End}_{\mathcal{M}1}}$ *and* $\mathsf{C}$ *is also true.*

*Proof.* The first assertion is an instance of Theorem 11.4.14, which is applicable in the current setting as we now explain. Following the summary in Explanation 11.4.23, first we verify that Definition 11.1.1 (i) through (v) are satisfied in the current context.

(i) $\mathsf{M} = \mathsf{Mod}^{\mathcal{M}1}$ is a closed multicategory by
  - Proposition 8.1.16 and
  - the fact that it is a symmetric monoidal closed category (Proposition 1.3.17 (7)).

  By Theorem 8.4.15, $\mathsf{N} = \mathsf{PermCat}^{\mathsf{su}}$ is a closed multicategory.
(ii) $\mathsf{C}$ is, by assumption, a small $\mathsf{PermCat}^{\mathsf{su}}$-category.
(iii) $F = \mathsf{F}_{\mathcal{M}1}$ in (12.4.1) is a non-symmetric multifunctor by definition (5.5.2), and $E = \mathsf{End}_{\mathcal{M}1}$ is a multifunctor by Theorem 1.4.38.
(iv) $\kappa = \eta^{\mathcal{M}1}$ and $\bar{\xi} = \varrho^{\mathcal{M}1}$ in (12.4.2) are multinatural transformations by
  - Explanation 5.5.7 for $\eta^{\mathcal{M}1}$ and
  - Lemma 5.3.3 and Definition 5.5.8 for $\varrho^{\mathcal{M}1}$.

(v) In the current setting, the condition (11.1.2) is the equality of the following two left $\mathcal{M}\underline{1}$-module morphisms for each pair of objects $x, y \in C$.

$$\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}(x,y) \xrightarrow[\mathsf{End}_{\mathcal{M}\underline{1}}\varrho^{\mathcal{M}\underline{1}}_{\mathsf{C}(x,y)}]{\eta^{\mathcal{M}\underline{1}}_{\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}(x,y)}} \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{F}_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}(x,y)$$

This equality holds by Lemma 5.5.11 because each hom object $\mathsf{C}(x,y)$ is a small permutative category.

Thus Definition 11.1.1 (i) through (v) hold in the context of (12.4.1) through (12.4.5). Next, the only assumption in Definition 11.4.7 is that the relative category

$$(\mathsf{N}, \mathcal{X}) = \left( \mathsf{PermCat}^{\mathsf{su}}, \mathcal{S} \right)$$

is a category with weak equivalences (Definition 2.1.6 (6)). This is true for the following two reasons.

- The wide subcategory $\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$ is created by a functor, namely, Segal $K$-theory (2.5.14)

$$\mathsf{K}^{\mathsf{Se}} : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}_{\geq 0}.$$

- The class of stable equivalences in $\mathsf{Sp}_{\geq 0}$ contains all the isomorphisms and has the 2-out-of-3 property.

In the current setting there are equalities of wide subcategories

$$F^{-1}\mathcal{X} = \mathsf{F}^{-1}_{\mathcal{M}\underline{1}}(\mathcal{S}) = \mathcal{S}^{\mathcal{M}\underline{1}} \subset \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

The data in (12.4.7) are those in (11.4.15) in the current context.

Finally, the components of $\eta^{\mathcal{M}\underline{1}}$ are $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalences, as explained in the proof of Theorem 4.8.3. Moreover, by definition (5.5.9) each component of $\varrho^{\mathcal{M}\underline{1}}$ is a component of $\varrho^{\bullet}$. The latter is a stable equivalence in $\mathsf{PermCat}^{\mathsf{su}}$ by Remark 2.5.15 (2) because it has a left adjoint by Proposition 4.6.6. Thus Theorem 11.4.14 is applicable in the current setting, proving the first assertion.

The second assertion about $(\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}})^{\mathsf{op}}$ and $\mathsf{C}^{\mathsf{op}}$ is an instance of Theorem 11.4.24. It is applicable because $\mathsf{End}_{\mathcal{M}\underline{1}}$ is a multifunctor (Theorem 1.4.38). $\square$

**Explanation 12.4.8** (Homotopy Equivalent Categories of Modules). The categories of enriched diagrams in (12.1.7) and (12.4.7) are categories of left modules by Propositions 10.1.8 and 10.1.17. Thus, Theorems 12.1.6 and 12.4.6 together assert that, for each small $\mathsf{PermCat}^{\mathsf{su}}$-category $\mathsf{C}$, the functors in the diagram

(12.4.9)
$$\left( \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}}, \mathsf{Mod}^{\mathcal{M}\underline{1}}\big), \mathcal{S}^{\mathcal{M}\underline{1}}_{\blacktriangle} \right)$$
$$(\mathsf{End}_{\mathcal{M}\underline{1}})_{\star} \uparrow \wr \downarrow (\mathsf{F}_{\mathcal{M}\underline{1}})^{\varrho^{\mathcal{M}\underline{1}}}_{\star}$$
$$\left( \mathsf{PermCat}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{PermCat}^{\mathsf{su}}\big), \mathcal{S}_{\blacktriangle} \right)$$
$$(\mathsf{End}_{\bullet})_{\star} \downarrow \wr \uparrow (\mathsf{F}_{\bullet})^{\varrho^{\bullet}}_{\star}$$
$$\left( \mathsf{Multicat}_{\ast}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\bullet}}, \mathsf{Multicat}_{\ast}\big), (\mathcal{S}_{\bullet})_{\blacktriangle} \right)$$

are equivalences of homotopy theories between

- left $\mathsf{C}$-modules in $\mathsf{PermCat}^{\mathsf{su}}$,
- left $\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}}$-modules in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$, and
- left $\mathsf{C}_{\mathsf{End}_{\bullet}}$-modules in $\mathsf{Multicat}_{\ast}$.

Furthermore, these equivalences of homotopy theories still hold if $\mathsf{C}$, $\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}}$, and $\mathsf{C}_{\mathsf{End}_\bullet}$ are replaced by $\mathsf{C}^{\mathsf{op}}$, $(\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}})^{\mathsf{op}}$, and $(\mathsf{C}_{\mathsf{End}_\bullet})^{\mathsf{op}}$, respectively.      $\diamond$

## 12.5. Explanation of the Equivalences of Homotopy Theories

In this section we

(1) explain in detail the inverse equivalences of homotopy theories $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ and $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ in (12.4.9) and
(2) compare them with the inverse equivalences of homotopy theories $(\mathsf{End}_\bullet)_\star$ and $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$.

This section is organized as follows.

- Explanation 12.5.1 describes how $(\mathsf{End}_\bullet)_\star$ factors through $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$.
- Explanations 12.5.4 and 12.5.6 describe $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ on objects and morphisms, respectively.
- Explanation 12.5.8 describes how $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ factors through $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$.
- Explanations 12.5.12 and 12.5.18 describe $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ on objects and morphisms, respectively.

Throughout this section we use the shortened notation in (12.2.1), so

$$\mathsf{M}_* = \mathsf{Multicat}_* \quad \text{and} \quad \mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.$$

**The Functor** $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$.

**Explanation 12.5.1** (Factoring $(\mathsf{End}_\bullet)_\star$ Through $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$). By (1.4.39) there is a factorization of multifunctors

$$(12.5.2) \qquad \mathsf{End}_\bullet : \mathsf{P}^{\mathsf{su}} \xrightarrow{\ \mathsf{End}_{\mathcal{M}\underline{1}}\ } \mathsf{Mod}^{\mathcal{M}\underline{1}} \xrightarrow{\ \mathsf{U}_{\mathcal{M}\underline{1}}\ } \mathsf{M}_*.$$

By Theorem 10.4.1 this factorization yields the following commutative diagram.

$$(12.5.3)$$

$$
\begin{array}{c}
\mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}}, \mathsf{Mod}^{\mathcal{M}\underline{1}}\big) \\
\\
\xleftarrow{\quad (\mathsf{End}_{\mathcal{M}\underline{1}})_\star} \\
(\mathsf{U}_{\mathcal{M}\underline{1}})_\star \downarrow \qquad\qquad \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(\mathsf{C}, \mathsf{P}^{\mathsf{su}}\big) \\
\\
\xrightarrow{\quad (\mathsf{End}_\bullet)_\star} \\
\mathsf{M}_*\text{-}\mathsf{Cat}\big(\mathsf{C}_{\mathsf{End}_\bullet}, \mathsf{M}_*\big)
\end{array}
$$

In (12.5.3) each arrow is a diagram change-of-enrichment functor (Theorem 10.3.1). We describe $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ on objects and morphisms in Explanations 12.5.4 and 12.5.6 below.      $\diamond$

**Explanation 12.5.4** $((\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ on Objects$)$. The functor $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ in (12.5.3) has an analogous description as $(\mathsf{End}_\bullet)_\star$ in Explanation 12.2.16. More explicitly, the functor $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ sends a $\mathsf{P}^{\mathsf{su}}$-functor $A : \mathsf{C} \longrightarrow \mathsf{P}^{\mathsf{su}}$ to the following composite $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-functor.

$$(12.5.5) \qquad \mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}} \xrightarrow{\ A_{\mathsf{End}_{\mathcal{M}\underline{1}}}\ } (\mathsf{P}^{\mathsf{su}})_{\mathsf{End}_{\mathcal{M}\underline{1}}} \xrightarrow{\ \widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}\ } \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

with the top composite labeled $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star A$.

- $A_{\mathsf{End}_{\mathcal{M}\underline{1}}}$ is the image of $A$ under the change-of-enrichment 2-functor

$$(-)_{\mathsf{End}_{\mathcal{M}\underline{1}}} : \mathsf{P^{su}\text{-}Cat} \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}$$

  along $\mathsf{End}_{\mathcal{M}\underline{1}}$ (Explanation 1.4.41 and Proposition 7.1.9).
- $\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}$ is the standard enrichment of $\mathsf{End}_{\mathcal{M}\underline{1}}$ (Theorem 9.2.12).

The $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-functor $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star A$ in (12.5.5) sends each object $x \in \mathsf{C}$ to

$$\big((\mathsf{End}_{\mathcal{M}\underline{1}})_\star A\big)x = \mathsf{End}_{\mathcal{M}\underline{1}}(Ax) \quad \text{in} \quad \mathsf{Mod}^{\mathcal{M}\underline{1}}.$$

For objects $x, y \in \mathsf{C}$, the $(x,y)$-component left $\mathcal{M}\underline{1}$-module morphism of $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star A$ is the following composite, where $\mathsf{Hom}_\star$ is the internal hom in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ (Proposition 1.3.17 (7)).

$$\begin{array}{ccc}
\mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{C}(x,y) & \xrightarrow{\ \big((\mathsf{End}_{\mathcal{M}\underline{1}})_\star A\big)_{x,y}\ } & \mathsf{Hom}_\star\big(\mathsf{End}_{\mathcal{M}\underline{1}}(Ax),\, \mathsf{End}_{\mathcal{M}\underline{1}}(Ay)\big) \\
{\scriptstyle \mathsf{End}_{\mathcal{M}\underline{1}}(A_{x,y})} \searrow & & \nearrow {\scriptstyle (\widehat{\mathsf{End}_{\mathcal{M}\underline{1}}})_{Ax,Ay}} \\
& \mathsf{End}_{\mathcal{M}\underline{1}}\mathsf{P^{su}}(Ax, Ay) &
\end{array}$$

The underlying pointed multifunctor of this left $\mathcal{M}\underline{1}$-module morphism is equal to the one in (12.2.17) by Proposition 1.3.17 (7) and the factorization (12.5.2). $\qquad \diamond$

**Explanation 12.5.6** (($(\mathsf{End}_{\mathcal{M}\underline{1}})_\star$ on Morphisms). For a $\mathsf{P^{su}}$-natural transformation $\psi : A \longrightarrow B$ as in (12.2.3), the $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-natural transformation $(\mathsf{End}_{\mathcal{M}\underline{1}})_\star \psi$ is the whiskering below.

$$\mathsf{C}_{\mathsf{End}_{\mathcal{M}\underline{1}}} \underset{B_{\mathsf{End}_{\mathcal{M}\underline{1}}}}{\overset{A_{\mathsf{End}_{\mathcal{M}\underline{1}}}}{\underset{\Downarrow \psi_{\mathsf{End}_{\mathcal{M}\underline{1}}}}{\rightrightarrows}}} (\mathsf{P^{su}})_{\mathsf{End}_{\mathcal{M}\underline{1}}} \xrightarrow{\ \widehat{\mathsf{End}_{\mathcal{M}\underline{1}}}\ } \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

For each object $x \in \mathsf{C}$, its $x$-component is given by the left $\mathcal{M}\underline{1}$-module morphism

$$(12.5.7) \qquad \big((\mathsf{End}_{\mathcal{M}\underline{1}})_\star \psi\big)_x = \mathsf{End}_{\mathcal{M}\underline{1}}(\psi_x) : \mathsf{End}_{\mathcal{M}\underline{1}}(Ax) \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}(Bx).$$

This is obtained from the strictly unital symmetric monoidal functor in (12.2.7)

$$\psi_x : Ax \longrightarrow Bx$$

by applying $\mathsf{End}_{\mathcal{M}\underline{1}}$. The underlying pointed multifunctor of $\mathsf{End}_{\mathcal{M}\underline{1}}(\psi_x)$ in (12.5.7) is equal to $\mathsf{End}_\bullet(\psi_x)$ in (12.2.18) by the factorization (12.5.2). $\qquad \diamond$

**The Functor $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$.**

**Explanation 12.5.8** (Factoring $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ Through $(\mathsf{F}_\bullet)_\star^{\varrho^\bullet}$). By definition (11.1.3) the functor $(\mathsf{F}_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ is the composite along the top of the following diagram.

$$(12.5.9)$$

The two triangles in (12.5.9) are commutative for the following reasons.

(1) The diagram change of enrichment $(U_{\mathcal{M}1})_\star$ is well defined by the factorization (12.5.2)

$$\mathsf{End}_{\bullet} = U_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}$$

and the functoriality of change of enrichment with respect to composition (Proposition 7.4.1). The diagram change of enrichment $(F_{\bullet})_\star$ is well defined by the equalities

$$F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1} = F_{\bullet}U_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1} = F_{\bullet}\mathsf{End}_{\bullet},$$

with the first equality given by the definition (5.5.2) of $F_{\mathcal{M}1}$ as the composite

$$F_{\mathcal{M}1} : \mathsf{Mod}^{\mathcal{M}1} \xrightarrow{\ U_{\mathcal{M}1}\ } M_* \xrightarrow{\ F_{\bullet}\ } \mathsf{P}^{\mathsf{su}}.$$

The left triangle in (12.5.9) commutes by the functoriality of diagram change of enrichment with respect to composition (Theorem 10.4.1).

(2) The $\mathsf{P}^{\mathsf{su}}$-functor

$$(12.5.10) \qquad\qquad C_{\varrho\mathcal{M}1} : C \longrightarrow C_{F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}}$$

is the C-component of the 2-natural transformation (Proposition 7.5.5)

$$\mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat} \overbrace{\underset{(-)_{F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}} = (-)_{F_{\mathcal{M}1}}(-)_{\mathsf{End}_{\mathcal{M}1}}}{\overset{1}{\underset{\Downarrow (-)_{\varrho\mathcal{M}1}}{\rightrightarrows}}}} \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}$$

induced by the multinatural transformation (5.5.9)

$$\varrho^{\mathcal{M}1} : 1_{\mathsf{P}^{\mathsf{su}}} \longrightarrow F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1} = F_{\bullet}\mathsf{End}_{\bullet}.$$

There is an equality of $\mathsf{P}^{\mathsf{su}}$-functors

$$(12.5.11) \qquad\qquad C_{\varrho\mathcal{M}1} = C_{\varrho\bullet} : C \longrightarrow C_{F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}} = C_{F_{\bullet}\mathsf{End}_{\bullet}},$$

with $C_{\varrho\bullet}$ the $\mathsf{P}^{\mathsf{su}}$-functor in (12.3.3), for the following two reasons.

(i) Both $C_{\varrho\mathcal{M}1}$ and $C_{\varrho\bullet}$ are the identity function on objects.

(ii) For objects $x, y \in C$, the $(x, y)$-component strictly unital symmetric monoidal functors

$$C(x,y) \underset{\varrho^{\bullet}_{C(x,y)}}{\overset{\varrho^{\mathcal{M}1}_{C(x,y)}}{\rightrightarrows}} F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}C(x,y) = F_{\bullet}\mathsf{End}_{\bullet}C(x,y)$$

are equal by the definition (5.5.9) of $\varrho^{\mathcal{M}1}$.

The equality (12.5.11) implies the equality of functors

$$C^*_{\varrho\mathcal{M}1} = C^*_{\varrho\bullet} : \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}\big(C_{F_{\mathcal{M}1}\mathsf{End}_{\mathcal{M}1}}, \mathsf{P}^{\mathsf{su}}\big) \longrightarrow \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}(C, \mathsf{P}^{\mathsf{su}}).$$

Thus the right triangle in (12.5.9) commutes by the definition (12.3.2) of $(F_{\bullet})^{\varrho\bullet}_\star$.

In summary, the diagram (12.5.9) factors $(F_{\mathcal{M}1})^{\varrho\mathcal{M}1}_\star$ through $(F_{\bullet})^{\varrho\bullet}_\star$.    ◇

**Explanation 12.5.12** $((F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ on Objects$)$**.** The functor $(F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ in (12.5.9) has an analogous description as $(F_\bullet)_\star^{\varrho^\bullet}$ in Explanations 12.3.5 and 12.3.11 by replacing $\mathrm{End}_\bullet$, $F_\bullet$, $\varrho^\bullet$, and $M_\star$ with $\mathrm{End}_{\mathcal{M}\underline{1}}$, $F_{\mathcal{M}\underline{1}}$, $\varrho^{\mathcal{M}\underline{1}}$, and $\mathrm{Mod}^{\mathcal{M}\underline{1}}$, respectively. More explicitly, $(F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ sends a $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functor

$$(12.5.13) \qquad\qquad A : \mathsf{C}_{\mathrm{End}_{\mathcal{M}\underline{1}}} \longrightarrow \mathrm{Mod}^{\mathcal{M}\underline{1}}$$

to the following composite $\mathsf{P}^{\mathsf{su}}$-functor.

$$(12.5.14)$$

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\quad (F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} A \quad} & \mathsf{P}^{\mathsf{su}} \\
{\scriptstyle \mathsf{C}_{\varrho^{\mathcal{M}\underline{1}}}}\Big\downarrow & & \Big\uparrow{\scriptstyle \widehat{F_{\mathcal{M}\underline{1}}}} \\
\mathsf{C}_{F_{\mathcal{M}\underline{1}}\mathrm{End}_{\mathcal{M}\underline{1}}} = (\mathsf{C}_{\mathrm{End}_{\mathcal{M}\underline{1}}})_{F_{\mathcal{M}\underline{1}}} & \xrightarrow{\quad A_{F_{\mathcal{M}\underline{1}}} \quad} & (\mathrm{Mod}^{\mathcal{M}\underline{1}})_{F_{\mathcal{M}\underline{1}}}
\end{array}
$$

- $\mathsf{C}_{\varrho^{\mathcal{M}\underline{1}}}$ is the $\mathsf{P}^{\mathsf{su}}$-functor in (12.5.10).
- $A_{F_{\mathcal{M}\underline{1}}}$ is the image of $A$ under the change-of-enrichment 2-functor

$$(12.5.15) \qquad\qquad (-)_{F_{\mathcal{M}\underline{1}}} : \mathrm{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat} \longrightarrow \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}$$

  along $F_{\mathcal{M}\underline{1}}$ (Theorem 4.4.1 and Proposition 7.1.9).
- $\widehat{F_{\mathcal{M}\underline{1}}}$ is the standard enrichment $\mathsf{P}^{\mathsf{su}}$-functor of $F_{\mathcal{M}\underline{1}}$ (Theorem 9.2.12).

The $\mathsf{P}^{\mathsf{su}}$-functor in (12.5.14) sends each object $x \in \mathsf{C}$ to

$$(12.5.16) \qquad\qquad \big((F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} A\big)x = F_{\mathcal{M}\underline{1}}(Ax) = F_\bullet U_{\mathcal{M}\underline{1}}(Ax) \quad \text{in} \quad \mathsf{P}^{\mathsf{su}},$$

with the second equality from the definition (5.5.2) of $F_{\mathcal{M}\underline{1}}$.

For objects $x, y \in \mathsf{C}$, the $(x,y)$-component of the $\mathsf{P}^{\mathsf{su}}$-functor in (12.5.14) is the following composite strictly unital symmetric monoidal functor.

$$(12.5.17)$$

$$
\begin{array}{ccc}
\mathsf{C}(x,y) & \xrightarrow{\quad ((F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} A)_{x,y} \quad} & \mathsf{P}^{\mathsf{su}}\big(F_{\mathcal{M}\underline{1}}(Ax), F_{\mathcal{M}\underline{1}}(Ay)\big) \\
{\scriptstyle \varrho^{\mathcal{M}\underline{1}}_{\mathsf{C}(x,y)}}\Big\downarrow & & \Big\uparrow{\scriptstyle (\widehat{F_{\mathcal{M}\underline{1}}})_{Ax,Ay}} \\
F_{\mathcal{M}\underline{1}}\mathrm{End}_{\mathcal{M}\underline{1}}\mathsf{C}(x,y) & \xrightarrow{\quad F_{\mathcal{M}\underline{1}} A_{x,y} \quad} & F_{\mathcal{M}\underline{1}}\mathrm{Hom}_\star(Ax, Ay)
\end{array}
$$

By the factorization (12.5.9), there is an equality of strictly unital symmetric monoidal functors

$$\big((F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} A\big)_{x,y} = \big((F_\bullet)_\star^{\varrho^\bullet}((U_{\mathcal{M}\underline{1}})_\star A)\big)_{x,y}$$

with the left-hand side from (12.5.17) and the right-hand side from (12.3.10). ◇

**Explanation 12.5.18** $((F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}}$ on Morphisms$)$**.** Consider a $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-natural transformation $\psi$ (Definition B.1.10) as in the left diagram below.

$$
\mathsf{C}_{\mathrm{End}_{\mathcal{M}\underline{1}}} \xRightarrow[\;B\;]{\;A\;}{\Downarrow\psi} \mathrm{Mod}^{\mathcal{M}\underline{1}} \qquad\qquad \mathsf{C} \xRightarrow[\;(F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} B\;]{\;(F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} A\;}{\Downarrow (F_{\mathcal{M}\underline{1}})_\star^{\varrho^{\mathcal{M}\underline{1}}} \psi} \mathsf{P}^{\mathsf{su}}
$$

The $\mathsf{P}^{\mathsf{su}}$-natural transformation $(\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}\psi$, as in the right diagram above, is the following whiskering.

$$(12.5.19) \qquad \mathsf{C} \xrightarrow{\mathsf{C}_{\varrho^{\mathcal{M}\underline{1}}}} \mathsf{C}_{\mathsf{F}_{\mathcal{M}\underline{1}}\mathrm{End}_{\mathcal{M}\underline{1}}} = (\mathsf{C}_{\mathrm{End}_{\mathcal{M}\underline{1}}})_{\mathsf{F}_{\mathcal{M}\underline{1}}} \overset{A_{\mathsf{F}_{\mathcal{M}\underline{1}}}}{\underset{B_{\mathsf{F}_{\mathcal{M}\underline{1}}}}{\Longrightarrow}} \Downarrow\psi_{\mathsf{F}_{\mathcal{M}\underline{1}}} (\mathrm{Mod}^{\mathcal{M}\underline{1}})_{\mathsf{F}_{\mathcal{M}\underline{1}}} \xrightarrow{\widehat{\mathsf{F}_{\mathcal{M}\underline{1}}}} \mathsf{P}^{\mathsf{su}}$$

The whiskering (12.5.19) is the one in (11.1.16) in the current context and is an analog of (12.3.12).

- The $\mathsf{P}^{\mathsf{su}}$-functors $\mathsf{C}_{\varrho^{\mathcal{M}\underline{1}}}$ and $\widehat{\mathsf{F}_{\mathcal{M}\underline{1}}}$ are as in (12.5.14).
- $\psi_{\mathsf{F}_{\mathcal{M}\underline{1}}}$ is the change of enrichment of $\psi$ under $(-)_{\mathsf{F}_{\mathcal{M}\underline{1}}}$ in (12.5.15).

Next we describe the components of $(\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}\psi$ explicitly. For each object $x \in$ $\mathsf{C}$, the $x$-component of $\psi$ is a left $\mathcal{M}\underline{1}$-module morphism

$$\psi_x : \mathcal{M}\underline{1} \longrightarrow \mathrm{Hom}_*(Ax, Bx).$$

By the $\wedge$-$\mathrm{Hom}_*$ adjunction in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ (Proposition 1.3.17 (7)), $\psi_x$ is uniquely determined by its adjoint, which is a left $\mathcal{M}\underline{1}$-module morphism that we also denote by

$$(12.5.20) \qquad\qquad\qquad \psi_x : Ax \longrightarrow Bx.$$

On the other hand, by (12.5.16) the $x$-component of $(\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}\psi$ is a nullary multimorphism

$$\left((\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}\psi\right)_x : \langle\rangle \longrightarrow \mathsf{P}^{\mathsf{su}}\left(\mathsf{F}_{\mathcal{M}\underline{1}}(Ax), \mathsf{F}_{\mathcal{M}\underline{1}}(Bx)\right) \quad \mathrm{in} \quad \mathsf{P}^{\mathsf{su}}.$$

This means a choice of an object in $\mathsf{P}^{\mathsf{su}}\left(\mathsf{F}_{\mathcal{M}\underline{1}}(Ax), \mathsf{F}_{\mathcal{M}\underline{1}}(Bx)\right)$. This, in turn, means a strictly unital symmetric monoidal functor $\mathsf{F}_{\mathcal{M}\underline{1}}(Ax) \longrightarrow \mathsf{F}_{\mathcal{M}\underline{1}}(Bx)$. Using (11.1.18) in the current context, we obtain the $x$-component

$$\begin{aligned}\left((\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}\psi\right)_x &= \mathsf{F}_{\mathcal{M}\underline{1}}(\psi_x)\\ &= \mathsf{F}_\bullet\mathsf{U}_{\mathcal{M}\underline{1}}(\psi_x) : \mathsf{F}_{\mathcal{M}\underline{1}}(Ax) \longrightarrow \mathsf{F}_{\mathcal{M}\underline{1}}(Bx)\end{aligned}$$

by applying $\mathsf{F}_{\mathcal{M}\underline{1}} = \mathsf{F}_\bullet\mathsf{U}_{\mathcal{M}\underline{1}}$ to $\psi_x$ in (12.5.20). By Definition 4.1.12 $\mathsf{F}_\bullet\mathsf{U}_{\mathcal{M}\underline{1}}(\psi_x)$ is a *strict* symmetric monoidal functor.                            ◇

## 12.6. Homotopy Equivalent Multicategorical and $\mathcal{M}\underline{1}$-Modules Enriched Diagrams

As we discuss in (12.4.9), enriched diagrams in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ and $\mathrm{Multicat}_*$ are connected by two zigzags of equivalences of homotopy theories:

$$\left((\mathsf{F}_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}}, (\mathsf{F}_\bullet)_{\star}^{\varrho^\bullet}\right) \quad \mathrm{and} \quad \left((\mathrm{End}_{\mathcal{M}\underline{1}})_\star, (\mathrm{End}_\bullet)_\star\right).$$

Each of these two zigzags goes through enriched diagrams in $\mathrm{PermCat}^{\mathsf{su}}$. In this section we apply Theorems 11.4.14 and 11.4.24 to show that the categories of enriched diagrams and Mackey functors in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ and $\mathrm{Multicat}_*$ are directly connected by inverse equivalences of homotopy theories. See Theorem 12.6.6 and the summary in Explanation 12.6.9. We explain these functors further in Section 12.7.

**Context.** For the context first recall the diagram

$$(12.6.1) \qquad \mathsf{Multicat}_* \underset{\mathsf{U}_{\mathcal{M}\underline{1}}}{\overset{\mathcal{M}\underline{1} \wedge -}{\rightleftarrows}} \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

consisting of

- the Cat-multicategory $\mathsf{Multicat}_*$ in Explanation 1.2.9,
- the Cat-multicategory $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ in Explanation 1.3.24,
- the Cat-multifunctor $\mathcal{M}\underline{1} \wedge -$ in Proposition 1.3.26.
- the Cat-multifunctor $\mathsf{U}_{\mathcal{M}\underline{1}}$ in Explanation 1.3.29, and

The two composite functors in (12.6.1) are connected to the respective identity functors via the following Cat-multinatural transformations from Proposition 1.3.31, where $\hat{\varepsilon}^{-1}$ denotes the inverse of $\hat{\varepsilon}$.

$$(12.6.2) \qquad \mathsf{Multicat}_* \overset{1}{\underset{\mathsf{U}_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1} \wedge -)}{\Downarrow \hat{\eta}}} \mathsf{Multicat}_* \qquad \mathsf{Mod}^{\mathcal{M}\underline{1}} \overset{1}{\underset{(\mathcal{M}\underline{1} \wedge -)\mathsf{U}_{\mathcal{M}\underline{1}}}{\Downarrow \hat{\varepsilon}^{-1}}} \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

The underlying categories of $\mathsf{Multicat}_*$ and $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ are equipped with the relative category structures in (4.7.2):

$$(12.6.3) \qquad \left(\mathsf{Multicat}_*, \mathcal{S}_\bullet\right) \quad \text{and} \quad \left(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}}\right).$$

By Theorem 4.8.1 the functors in (12.6.1) are inverse equivalences of homotopy theories. The algebra version is Theorem 5.5.14.

- The wide subcategory of $\mathsf{F}_\bullet$-stable equivalences

$$\mathcal{S}_\bullet = \mathsf{F}_\bullet^{-1}(\mathcal{S}) \subset \mathsf{Multicat}_*$$

is created by the functor

$$\mathsf{F}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}.$$

The wide subcategory $\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$ is created by Segal $K$-theory $\mathsf{K}^{\mathsf{Se}}$; see (2.5.14). For a small $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-category D, the wide subcategory in (12.6.7) below

$$(12.6.4) \qquad (\mathcal{S}_\bullet)_\blacktriangle \subset \mathsf{Multicat}_*\text{-}\mathsf{Cat}\left(\mathsf{D}_{\mathsf{U}_{\mathcal{M}\underline{1}}}, \mathsf{Multicat}_*\right)$$

is defined as in (11.4.2) using $\mathcal{S}_\bullet$.

- The wide subcategory of $\mathsf{F}_{\mathcal{M}\underline{1}}$-stable equivalences

$$\mathcal{S}^{\mathcal{M}\underline{1}} = \mathsf{F}_{\mathcal{M}\underline{1}}^{-1}(\mathcal{S}) \subset \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

is created by the functor

$$\mathsf{F}_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}.$$

The wide subcategory in (12.6.7) below

$$(12.6.5) \qquad \mathcal{S}^{\mathcal{M}\underline{1}}_\blacktriangle \subset \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\left(\mathsf{D}, \mathsf{Mod}^{\mathcal{M}\underline{1}}\right)$$

is defined as in (11.4.2) using $\mathcal{S}^{\mathcal{M}\underline{1}}$.

**Equivalences of Homotopy Theories.** In Theorem 4.8.1 we observe that the pair $(\mathcal{M}\underline{1} \wedge -, U_{\mathcal{M}\underline{1}})$ is an adjoint equivalence of homotopy theories. The following observation extends these equivalences of homotopy theories to categories of enriched diagrams and Mackey functors.

**Theorem 12.6.6.** *Suppose* D *is a small* $\mathsf{Mod}^{\mathcal{M}\underline{1}}$*-category. Then the functors*

$$(12.6.7) \quad \left(\mathsf{Multicat}_*\text{-}\mathsf{Cat}\big(\mathsf{D}_{U_{\mathcal{M}\underline{1}}}, \mathsf{Multicat}_*\big), (\mathcal{S}_{\textbf{.}})_{\blacktriangle}\right) \xrightleftharpoons[\;(U_{\mathcal{M}\underline{1}})_\star\;]{\;(\mathcal{M}\underline{1}\wedge-)^{\hat{\varepsilon}^{-1}}_{\star}\;}_{\sim} \left(\mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\big(\mathsf{D}, \mathsf{Mod}^{\mathcal{M}\underline{1}}\big), \mathcal{S}^{\mathcal{M}\underline{1}}_{\blacktriangle}\right),$$

*defined by the data in (12.6.1) through (12.6.5), are inverse equivalences of homotopy theories.*

*Moreover, the variant with* $(\mathsf{D}_{U_{\mathcal{M}\underline{1}}})^{\mathsf{op}}$ *and* $\mathsf{D}^{\mathsf{op}}$ *replacing, respectively,* $\mathsf{D}_{U_{\mathcal{M}\underline{1}}}$ *and* D *is also true.*

*Proof.* The first assertion is an instance of Theorem 11.4.14, which is applicable in the current setting as we now explain. Following the summary in Explanation 11.4.23, first we verify that Definition 11.1.1 (i) through (v) are satisfied in the current context.

  (i) $\mathsf{M} = \mathsf{Multicat}_*$ and $\mathsf{N} = \mathsf{Mod}^{\mathcal{M}\underline{1}}$ are closed multicategories by
      • Proposition 8.1.16 and
      • the fact that they are symmetric monoidal closed categories (Theorem 1.2.8 and Proposition 1.3.17).
 (ii) D is, by assumption, a small $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-category.
(iii) $F = \mathcal{M}\underline{1} \wedge -$ in (12.6.1) is a multifunctor by Proposition 1.3.26, and $E = U_{\mathcal{M}\underline{1}}$ is a multifunctor by Explanation 1.3.29.
(iv) $\kappa = \hat{\eta}$ and $\xi = \hat{\varepsilon}^{-1}$ in (12.6.2) are multinatural transformations by Proposition 1.3.31.
 (v) In the current setting, the condition (11.1.2) is the equality of the following two pointed multifunctors for each pair of objects $x, y \in \mathsf{D}$.

$$U_{\mathcal{M}\underline{1}}\mathsf{D}(x,y) \xrightleftharpoons[\;U_{\mathcal{M}\underline{1}}\hat{\varepsilon}^{-1}_{\mathsf{D}(x,y)}\;]{\;\hat{\eta}_{U_{\mathcal{M}\underline{1}}\mathsf{D}(x,y)}\;} U_{\mathcal{M}\underline{1}}(\mathcal{M}\underline{1} \wedge -)U_{\mathcal{M}\underline{1}}\mathsf{D}(x,y)$$

      Since $\mathsf{D}(x,y)$ is a left $\mathcal{M}\underline{1}$-module, the equality of these two arrows follows from the right triangle identity (A.2.12) of the 2-adjunction

$$(12.6.8) \qquad \big((\mathcal{M}\underline{1} \wedge -), U_{\mathcal{M}\underline{1}}, \hat{\eta}, \hat{\varepsilon}\big) : \mathsf{Multicat}_* \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$$

      in Proposition 1.3.19.

Thus Definition 11.1.1 (i) through (v) hold in the context of (12.6.1) through (12.6.5). Next, the only assumption in Definition 11.4.7 is that the relative category

$$(\mathsf{N}, \mathcal{X}) = \big(\mathsf{Mod}^{\mathcal{M}\underline{1}}, \mathcal{S}^{\mathcal{M}\underline{1}}\big)$$

is a category with weak equivalences (Definition 2.1.6 (6)). This is true for the following two reasons.

  • By Definition 4.7.1 the wide subcategory $\mathcal{S}^{\mathcal{M}\underline{1}} \subset \mathsf{Mod}^{\mathcal{M}\underline{1}}$ is created by the functor

$$\mathsf{F}_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}.$$

  • The class of stable equivalences $\mathcal{S} \subset \mathsf{PermCat}^{\mathsf{su}}$ contains all the isomorphisms and has the 2-out-of-3 property.

By (4.4.5) and (4.7.2) there are equalities of wide subcategories as follows.

$$
\begin{aligned}
F^{-1}\mathcal{X} &= (\mathcal{M}\underline{1} \wedge -)^{-1}(\mathcal{S}^{\mathcal{M}\underline{1}}) \\
&= (\mathcal{M}\underline{1} \wedge -)^{-1} F_{\mathcal{M}\underline{1}}^{-1}(\mathcal{S}) \\
&= F_{\bullet}^{-1}(\mathcal{S}) \\
&= \mathcal{S}_{\bullet} \subset \mathsf{Multicat}_*
\end{aligned}
$$

The data in (12.6.7) are those in (11.4.15) in the current context.

Finally, we observe that the components of $\hat{\eta}$ and $\hat{\varepsilon}^{-1}$ are stable equivalences.

- Each component of $\hat{\varepsilon}^{-1}$ (1.3.22) is an isomorphism, hence also an $F_{\mathcal{M}\underline{1}}$-stable equivalence in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$.
- The left triangle identity (A.2.12) for the 2-adjunction (12.6.8) and the 2-out-of-3 property imply that each component of $\hat{\eta}$ is an $F_{\bullet}$-stable equivalence in $\mathsf{Multicat}_*$.

Thus Theorem 11.4.14 is applicable in the current setting, proving the first assertion.

The second assertion about $(D_{U_{\mathcal{M}\underline{1}}})^{\mathsf{op}}$ and $D^{\mathsf{op}}$ is an instance of Theorem 11.4.24. It is applicable because $U_{\mathcal{M}\underline{1}}$ is a multifunctor (Explanation 1.3.29). $\qquad\square$

**Explanation 12.6.9** (Homotopy Equivalent Categories of Modules). The categories of enriched diagrams in (12.6.7) are categories of left modules by Propositions 10.1.8 and 10.1.17. Thus, Theorem 12.6.6 asserts that, for each small $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-category D, the functors in (12.6.7) are equivalences of homotopy theories between

- left D-modules in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ and
- left $D_{U_{\mathcal{M}\underline{1}}}$-modules in $\mathsf{Multicat}_*$.

Furthermore, these equivalences of homotopy theories still hold if D and $D_{U_{\mathcal{M}\underline{1}}}$ are replaced by $D^{\mathsf{op}}$ and $(D_{U_{\mathcal{M}\underline{1}}})^{\mathsf{op}}$, respectively. We explain the functors $(U_{\mathcal{M}\underline{1}})_{\star}$ and $(\mathcal{M}\underline{1} \wedge -)_{\star}^{\hat{\varepsilon}^{-1}}$ in (12.6.7) in more detail in Section 12.7 below.

We use the abbreviations in (12.2.1), so

$$
\mathsf{M}_* = \mathsf{Multicat}_* \quad \text{and} \quad \mathsf{P}^{\mathsf{su}} = \mathsf{PermCat}^{\mathsf{su}}.
$$

Theorems 12.1.6, 12.4.6, and 12.6.6 together yield the following diagram of inverse equivalences of homotopy theories for each small $\mathsf{PermCat}^{\mathsf{su}}$-category C.

(12.6.10)

$$
\left( \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\left(C_{\mathsf{End}_{\mathcal{M}\underline{1}}}, \mathsf{Mod}^{\mathcal{M}\underline{1}}\right), \mathcal{S}_{\blacktriangle}^{\mathcal{M}\underline{1}} \right)
$$

$$
(\mathcal{M}\underline{1} \wedge -)_{\star}^{\hat{\varepsilon}^{-1}} \Big\Vert (U_{\mathcal{M}\underline{1}})_{\star} \qquad
\begin{array}{c} (\mathsf{End}_{\mathcal{M}\underline{1}})_{\star} \searrow \quad (F_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}} \\ \left( \mathsf{P}^{\mathsf{su}}\text{-}\mathsf{Cat}\left(C, \mathsf{P}^{\mathsf{su}}\right), \mathcal{S}_{\blacktriangle} \right) \\ (\mathsf{End}_{\bullet})_{\star} \searrow \quad (F_{\bullet})_{\star}^{\varrho^{\bullet}} \end{array}
$$

$$
\left( \mathsf{M}_*\text{-}\mathsf{Cat}\left(C_{\mathsf{End}_{\bullet}}, \mathsf{M}_*\right), (\mathcal{S}_{\bullet})_{\blacktriangle} \right)
$$

Moreover, by (12.5.3) and (12.5.9), the factorizations

$$
(\mathsf{End}_{\bullet})_{\star} = (U_{\mathcal{M}\underline{1}})_{\star}(\mathsf{End}_{\mathcal{M}\underline{1}})_{\star} \quad \text{and}
$$

$$
(F_{\mathcal{M}\underline{1}})_{\star}^{\varrho^{\mathcal{M}\underline{1}}} = (F_{\bullet})_{\star}^{\varrho^{\bullet}}(U_{\mathcal{M}\underline{1}})_{\star}
$$

hold in (12.6.10). $\qquad\diamond$

## 12.7. Explanation of the Equivalences of Homotopy Theories

In this section we explain in detail the inverse equivalences of homotopy theories $(U_{\mathcal{M}\underline{1}})_\star$ and $(\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}$ in (12.6.7). We use the abbreviation

$$M_* = \mathsf{Multicat}_*$$

and denote by D a small $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-category (Definition B.1.1). This section is organized as follows.

- Explanation 12.7.1 describes the functor $(U_{\mathcal{M}\underline{1}})_\star$.
- Explanation 12.7.3 describes $(\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}$ on objects.
- Explanation 12.7.7 describes $(\mathcal{M}\underline{1} \wedge -)_\star^{\hat{\varepsilon}^{-1}}$ on morphisms.

**Explanation 12.7.1** (The Functor $(U_{\mathcal{M}\underline{1}})_\star$). The diagram change-of-enrichment functor

$$\mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat}\big(D, \mathsf{Mod}^{\mathcal{M}\underline{1}}\big) \xrightarrow{\;(U_{\mathcal{M}\underline{1}})_\star\;} M_*\text{-}\mathsf{Cat}\big(D_{U_{\mathcal{M}\underline{1}}}, M_*\big)$$

is defined in (10.2.3) and verified in Theorem 10.3.1. To understand its assignments on objects and morphisms, consider

- $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-functors $A, B : D \longrightarrow \mathsf{Mod}^{\mathcal{M}\underline{1}}$ (Definition B.1.8) and
- a $\mathsf{Mod}^{\mathcal{M}\underline{1}}$-natural transformation $\psi : A \longrightarrow B$ (Definition B.1.10)

as in the left diagram below.

$$(12.7.2) \qquad D \overset{A}{\underset{B}{\Downarrow \psi}} \mathsf{Mod}^{\mathcal{M}\underline{1}} \;\longmapsto\; D_{U_{\mathcal{M}\underline{1}}} \overset{A_{U_{\mathcal{M}\underline{1}}}}{\underset{B_{U_{\mathcal{M}\underline{1}}}}{\Downarrow \psi_{U_{\mathcal{M}\underline{1}}}}} (\mathsf{Mod}^{\mathcal{M}\underline{1}})_{U_{\mathcal{M}\underline{1}}} \xrightarrow{\;\widehat{U_{\mathcal{M}\underline{1}}}\;} M_*$$

Then $(U_{\mathcal{M}\underline{1}})_\star$ sends $A$, $B$, and $\psi$ to the composites and whiskering as in the right diagram in (12.7.2). In other words, the functor $(U_{\mathcal{M}\underline{1}})_\star$

- first applies the change-of-enrichment 2-functor along the multifunctor $U_{\mathcal{M}\underline{1}}$ ((1.3.30) and Proposition 7.1.9)

$$(-)_{U_{\mathcal{M}\underline{1}}} : \mathsf{Mod}^{\mathcal{M}\underline{1}}\text{-}\mathsf{Cat} \longrightarrow M_*\text{-}\mathsf{Cat}$$

  and then
- composes or whiskers with the standard enrichment $\widehat{U_{\mathcal{M}\underline{1}}}$ (Theorem 9.2.12).

Next we describe the object assignment and components of $(U_{\mathcal{M}\underline{1}})_\star A$. For each object $x \in D$, the object assignment of $(U_{\mathcal{M}\underline{1}})_\star A$ is

$$\big((U_{\mathcal{M}\underline{1}})_\star A\big)x = U_{\mathcal{M}\underline{1}}(Ax) \quad \text{in} \quad M_*.$$

For objects $x, y \in D$, the $(x, y)$-component of $A$ is a left $\mathcal{M}\underline{1}$-module morphism

$$A_{x,y} : D(x, y) \longrightarrow \mathsf{Hom}_*(Ax, Ay).$$

Its adjoint in $\mathsf{Mod}^{\mathcal{M}\underline{1}}$ is a morphism

$$A_{x,y}^\# : D(x, y) \wedge Ax \longrightarrow Ay.$$

The $(x,y)$-component of $(U_{\mathcal{M}\underline{1}})_\star A$ is the following composite in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$.

$$U_{\mathcal{M}\underline{1}}D(x,y) \xrightarrow{\;\;((U_{\mathcal{M}\underline{1}})_\star A)_{x,y}\;\;} \mathrm{Hom}_*\big(U_{\mathcal{M}\underline{1}}(Ax),\, U_{\mathcal{M}\underline{1}}(Ay)\big)$$

with $U_{\mathcal{M}\underline{1}}(A_{x,y})$ and $U_{\mathcal{M}\underline{1}}\mathrm{Hom}_*(Ax,Ay)$, $(\widehat{U_{\mathcal{M}\underline{1}}})_{Ax,Ay}$.

Its adjoint in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is the following composite, with $U^2_{\mathcal{M}\underline{1}}$ the monoidal constraint of $U_{\mathcal{M}\underline{1}}$, which is the identity by Explanation 1.3.27.

$$U_{\mathcal{M}\underline{1}}D(x,y) \wedge U_{\mathcal{M}\underline{1}}(Ax) \xrightarrow{\;\;((U_{\mathcal{M}\underline{1}})_\star A)^\#_{x,y}\;\;} U_{\mathcal{M}\underline{1}}(Ay)$$

with $U^2_{\mathcal{M}\underline{1}}$ and $U_{\mathcal{M}\underline{1}}\big(D(x,y) \wedge Ax\big)$, $U_{\mathcal{M}\underline{1}}(A^\#_{x,y})$.

The $x$-component of $(U_{\mathcal{M}\underline{1}})_\star \psi$ is the morphism

$$\big((U_{\mathcal{M}\underline{1}})_\star \psi\big)_x = U_{\mathcal{M}\underline{1}}(\psi_x) : U_{\mathcal{M}\underline{1}}(Ax) \longrightarrow U_{\mathcal{M}\underline{1}}(Bx) \quad \text{in} \quad M_*.$$

This is obtained from the $x$-component of $\psi$ by applying $U_{\mathcal{M}\underline{1}}$.              ◇

**Explanation 12.7.3** $((\mathcal{M}\underline{1} \wedge -)^{\hat{\varepsilon}^{-1}}_\star$ on Objects$)$**.** We abbreviate the top functor in (12.6.7) to

(12.7.4)      $H = (\mathcal{M}\underline{1} \wedge -)^{\hat{\varepsilon}^{-1}}_\star : M_*\text{-Cat}\big(D_{U_{\mathcal{M}\underline{1}}}, M_*\big) \longrightarrow \mathrm{Mod}^{\mathcal{M}\underline{1}}\text{-Cat}(D, \mathrm{Mod}^{\mathcal{M}\underline{1}}).$

The functor $H$ sends an $M_*$-functor

$$P : D_{U_{\mathcal{M}\underline{1}}} \longrightarrow M_*$$

to the following composite $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functor.

(12.7.5)
$$\begin{array}{ccc}
D & \xrightarrow{\;\;HP\;\;} & \mathrm{Mod}^{\mathcal{M}\underline{1}} \\
{\scriptstyle D_{\hat{\varepsilon}^{-1}}}\downarrow & & \uparrow{\scriptstyle \widehat{\mathcal{M}\underline{1}\wedge -}} \\
D_{(\mathcal{M}\underline{1}\wedge -)U_{\mathcal{M}\underline{1}}} = (D_{U_{\mathcal{M}\underline{1}}})_{\mathcal{M}\underline{1}\wedge -} & \xrightarrow{\;P_{\mathcal{M}\underline{1}\wedge -}\;} & (M_*)_{\mathcal{M}\underline{1}\wedge -}
\end{array}$$

- The $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functor $D_{\hat{\varepsilon}^{-1}}$ is the $D$-component of the 2-natural transformation (Proposition 7.5.5)

$$\mathrm{Mod}^{\mathcal{M}\underline{1}}\text{-Cat} \underset{(-)_{(\mathcal{M}\underline{1}\wedge -)U_{\mathcal{M}\underline{1}}} = (-)_{\mathcal{M}\underline{1}\wedge -}(-)_{U_{\mathcal{M}\underline{1}}}}{\overset{1}{\rightrightarrows}} \mathrm{Mod}^{\mathcal{M}\underline{1}}\text{-Cat}$$

with $\Downarrow (-)_{\hat{\varepsilon}^{-1}}$, induced by the multinatural transformation (Proposition 1.3.31)

$$\hat{\varepsilon}^{-1} : 1_{\mathrm{Mod}^{\mathcal{M}\underline{1}}} \longrightarrow (\mathcal{M}\underline{1}\wedge -)U_{\mathcal{M}\underline{1}}.$$

- $P_{\mathcal{M}\underline{1}\wedge -}$ is the image of $P$ under the change-of-enrichment 2-functor

(12.7.6)              $(-)_{\mathcal{M}\underline{1}\wedge -} : M_*\text{-Cat} \longrightarrow \mathrm{Mod}^{\mathcal{M}\underline{1}}\text{-Cat}$

  along the multifunctor $\mathcal{M}\underline{1} \wedge -$ (Propositions 1.3.26 and 7.1.9).
- $\widehat{\mathcal{M}\underline{1}\wedge -}$ is the standard enrichment $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functor of $\mathcal{M}\underline{1} \wedge -$ (Theorem 9.2.12).

The $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functor $\mathsf{H}P$ in (12.7.5) sends each object $x \in \mathsf{D}$ to
$$(\mathsf{H}P)x = \mathcal{M}\underline{1} \wedge Px \quad \text{in} \quad \mathrm{Mod}^{\mathcal{M}\underline{1}}.$$

For objects $x, y \in \mathsf{D}$, the $(x,y)$-component of $P$ is a pointed multifunctor
$$P_{x,y} : \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{D}(x,y) \longrightarrow \mathrm{Hom}_*(Px, Py).$$

Its adjoint in $\mathsf{M}_*$ is a pointed multifunctor
$$P^{\#}_{x,y} : \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{D}(x,y) \wedge Px \longrightarrow Py.$$

The $(x,y)$-component of $\mathsf{H}P$ is the following composite morphism in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$.

$$
\begin{array}{ccc}
\mathsf{D}(x,y) & \xrightarrow{\;\;(\mathsf{H}P)_{x,y}\;\;} & \mathrm{Hom}_*(\mathcal{M}\underline{1} \wedge Px, \mathcal{M}\underline{1} \wedge Py) \\[2pt]
{\scriptstyle \hat{\varepsilon}^{-1}_{\mathsf{D}(x,y)}} \downarrow {\scriptstyle \cong} & & \uparrow {\scriptstyle (\widehat{\mathcal{M}\underline{1} \wedge -})_{Px,Py}} \\[2pt]
\mathcal{M}\underline{1} \wedge \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{D}(x,y) & \xrightarrow[\;\;\mathcal{M}\underline{1} \wedge P_{x,y}\;\;]{} & \mathcal{M}\underline{1} \wedge \mathrm{Hom}_*(Px, Py)
\end{array}
$$

The adjoint of $(\mathsf{H}P)_{x,y}$ in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is the following composite, with $\prod_{\underline{1},\underline{1}}$ the partition product in (1.3.7).

$$
\begin{array}{ccc}
\mathsf{D}(x,y) \wedge (\mathcal{M}\underline{1} \wedge Px) & \xrightarrow{\;\;(\mathsf{H}P)^{\#}_{x,y}\;\;} & \mathcal{M}\underline{1} \wedge Py \\[2pt]
{\scriptstyle \hat{\varepsilon}^{-1}_{\mathsf{D}(x,y)} \wedge 1} \downarrow {\scriptstyle \cong} & & \uparrow {\scriptstyle \prod_{\underline{1},\underline{1}} \wedge P^{\#}_{x,y}} \\[2pt]
\big(\mathcal{M}\underline{1} \wedge \mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{D}(x,y)\big) \wedge (\mathcal{M}\underline{1} \wedge Px) & \xrightarrow[\;\;\text{permute}\;\;]{\cong} & (\mathcal{M}\underline{1} \wedge \mathcal{M}\underline{1}) \wedge \big(\mathsf{U}_{\mathcal{M}\underline{1}}\mathsf{D}(x,y) \wedge Px\big)
\end{array}
$$

The bottom horizontal isomorphism permutes the middle two factors. $\diamond$

**Explanation 12.7.7** $((\mathcal{M}\underline{1} \wedge -)^{\hat{\varepsilon}^{-1}}_{\star}$ on Morphisms). Consider an $\mathsf{M}_*$-natural transformation $\psi$ (Definition B.1.10) as in the left diagram below.

$$
\mathsf{D}_{\mathsf{U}_{\mathcal{M}\underline{1}}} \underset{Q}{\overset{P}{\rightrightarrows}} \Downarrow\psi \; \mathsf{M}_* \qquad\qquad \mathsf{D} \underset{\mathsf{H}Q}{\overset{\mathsf{H}P}{\rightrightarrows}} \Downarrow\mathsf{H}\psi \; \mathrm{Mod}^{\mathcal{M}\underline{1}}
$$

With the notation in (12.7.4), the $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-natural transformation $\mathsf{H}\psi$, as in the right diagram above, is the following whiskering.

(12.7.8)
$$
\begin{array}{ccc}
\mathsf{D} & & \mathrm{Mod}^{\mathcal{M}\underline{1}} \\[2pt]
{\scriptstyle \mathsf{D}_{\hat{\varepsilon}^{-1}}} \downarrow & & \uparrow {\scriptstyle \widehat{\mathcal{M}\underline{1} \wedge -}} \\[2pt]
\mathsf{D}_{(\mathcal{M}\underline{1} \wedge -)\mathsf{U}_{\mathcal{M}\underline{1}}} = (\mathsf{D}_{\mathsf{U}_{\mathcal{M}\underline{1}}})_{\mathcal{M}\underline{1} \wedge -} & \underset{Q_{\mathcal{M}\underline{1} \wedge -}}{\overset{P_{\mathcal{M}\underline{1} \wedge -}}{\rightrightarrows}} \Downarrow\psi_{\mathcal{M}\underline{1} \wedge -} & (\mathsf{M}_*)_{\mathcal{M}\underline{1} \wedge -}
\end{array}
$$

The whiskering (12.7.8) is the one in (11.1.16) in the current context. It is obtained from (12.7.5) by replacing $P$ with $\psi$.

- The $\mathrm{Mod}^{\mathcal{M}\underline{1}}$-functors $\mathsf{D}_{\hat{\varepsilon}^{-1}}$ and $\widehat{\mathcal{M}\underline{1} \wedge -}$ are as in (12.7.5).
- $\psi_{\mathcal{M}\underline{1} \wedge -}$ is the change of enrichment of $\psi$ under $(-)_{\mathcal{M}\underline{1} \wedge -}$ in (12.7.6).

For each object $x \in \mathsf{D}$, the $x$-component of $\psi$ is a pointed multifunctor
$$\psi_x : Px \longrightarrow Qx.$$

The $x$-component of $\mathsf{H}\psi$ is the following morphism in $\mathrm{Mod}^{\mathcal{M}\underline{1}}$.
$$(\mathsf{H}\psi)_x = \mathcal{M}\underline{1} \wedge \psi_x : \mathcal{M}\underline{1} \wedge Px \longrightarrow \mathcal{M}\underline{1} \wedge Qx$$

This is obtained from $\psi_x$ by applying $\mathcal{M}\underline{1} \wedge -$ (Proposition 1.3.19).                    $\diamond$

# Appendices

# Categories

In this appendix we review monoidal categories and 2-categories. The following table summarizes the main content in this appendix.

References for Appendices A.1 and A.2 are [**JS93, ML98, Yau∞a, Yau∞b**] and [**JY21**], respectively.

## A.1. Monoidal Categories

In this section we review Grothendieck universes, monoidal categories, monoidal functors, and monoidal natural transformations.

**Definition A.1.1.** A *universe* is a set $\mathcal{U}$ that satisfies (i) through (iv) below:

  (i) If $a \in \mathcal{U}$ and $b \in a$, then $b \in \mathcal{U}$.
 (ii) If $a \in \mathcal{U}$, then $\mathcal{P}(a) \in \mathcal{U}$, where $\mathcal{P}(a)$ is the set of subsets of $a$.
(iii) If $a \in \mathcal{U}$ and $x_j \in \mathcal{U}$ for each $j \in a$, then the union $\bigcup_{j \in a} x_j \in \mathcal{U}$.
 (iv) $\mathbb{N} \in \mathcal{U}$, where $\mathbb{N}$ is the set of finite ordinals.            ◇

**Convention A.1.2** (Universe)**.** We assume Grothendieck's *Axiom of Universes*:

> Every set belongs to some universe.

We fix a universe $\mathcal{U}$. An element in $\mathcal{U}$ is called a *set*. A subset of $\mathcal{U}$ is called a *class*. A categorical structure is called *small* if it has a set of objects. We automatically replace $\mathcal{U}$ by a larger universe $\mathcal{V}$ in which $\mathcal{U}$ is a set whenever necessary. For more discussion of universes, see [**JY21**, Section 1.1] and [**AGV72, ML69**].            ◇

**Definition A.1.3.** A *monoidal category* is a sextuple

$$(\mathsf{C}, \otimes, 1, \alpha, \lambda, \rho)$$

consisting of the following data.

- $\mathsf{C}$ is a category.
- $\otimes : \mathsf{C} \times \mathsf{C} \longrightarrow \mathsf{C}$ is a functor, which is called the *monoidal product*.

- $1 \in \mathsf{C}$ is an object, which is called the *monoidal unit*.
- $\alpha$, $\lambda$, and $\rho$ are natural isomorphisms with the following components for objects $x, y, z \in \mathsf{C}$.

$$(x \otimes y) \otimes z \xrightarrow[\cong]{\alpha_{x,y,z}} x \otimes (y \otimes z)$$

$$1 \otimes x \xrightarrow[\cong]{\lambda_x} x \xleftarrow[\cong]{\rho_x} x \otimes 1$$

They are called the *associativity isomorphism*, the *left unit isomorphism*, and the *right unit isomorphism*, respectively.

The data above are required to make the following *middle unity* and *pentagon* diagrams commute for objects $w, x, y, z \in \mathsf{C}$, where $\otimes$ is omitted to save space.

(A.1.4)



This finishes the definition of a monoidal category. Moreover, we define the following.

- We call a monoidal category *strict* if $\alpha$, $\lambda$, and $\rho$ are identity natural transformations.
- We also call a (monoidal) category a *(monoidal) 1-category*.    ⋄

**Remark A.1.5** (Unity Properties)**.** In each monoidal category, the unit isomorphisms agree at the monoidal unit:

(A.1.6)                              $\lambda_1 = \rho_1 : 1 \otimes 1 \longrightarrow 1.$

Moreover, the following unity diagrams commute.

(A.1.7)



See [**JY21**, Section 2.2] for the proofs.    ⋄

**Definition A.1.8.** A *monoid* in a monoidal category $\mathsf{C}$ is a triple $(x, \mathsf{m}, i)$ consisting of the following data.

- $x \in \mathsf{C}$ is an object.
- $\mathsf{m} : x \otimes x \longrightarrow x$ is a morphism, which is called the *multiplication*.
- $i : 1 \longrightarrow x$ is a morphism, which is called the *unit*.

The data above are required to make the following associativity and unity diagrams commute.

A *morphism* of monoids

$$(x, \mathsf{m}^x, i^x) \xrightarrow{\quad f \quad} (y, \mathsf{m}^y, i^y)$$

is a morphism $f : x \longrightarrow y$ in C such that the diagrams

$$
\begin{array}{ccc}
x \otimes x & \xrightarrow{\ f \otimes f\ } & y \otimes y \\
{\scriptstyle \mathsf{m}^x} \downarrow & & \downarrow {\scriptstyle \mathsf{m}^y} \\
x & \xrightarrow{\quad f \quad} & y
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{\ i^x\ } & x \\
& {\scriptstyle i^y} \searrow & \downarrow {\scriptstyle f} \\
& & y
\end{array}
$$

commute.                                                                    ◇

**Definition A.1.9.** Suppose $(x, \mathsf{m}, i)$ is a monoid in a monoidal category $(\mathsf{C}, \otimes, 1)$.

(1) A *left $x$-module* is a pair $(a, \mu)$ consisting of the following data.
   - $a$ is an object in C.
   - $\mu : x \otimes a \longrightarrow a$ is a morphism, called the *structure morphism*.
   
   The data above are required to make the following associativity and unity diagrams commute.

$$
\begin{array}{ccc}
(x \otimes x) \otimes a & \xrightarrow{\ \alpha\ } & x \otimes (x \otimes a) \\
{\scriptstyle \mathsf{m} \otimes 1_a} \downarrow & & \downarrow {\scriptstyle 1_x \otimes \mu} \\
& & x \otimes a \\
& & \downarrow {\scriptstyle \mu} \\
x \otimes a & \xrightarrow{\quad \mu \quad} & a
\end{array}
\qquad
\begin{array}{ccc}
1 \otimes a & & \\
{\scriptstyle i \otimes 1_a} \downarrow & \searrow {\scriptstyle \lambda_a} & \\
x \otimes a & \xrightarrow{\ \mu\ } & a
\end{array}
$$

(2) A *morphism* of left $x$-modules

$$(a, \mu^a) \xrightarrow{\quad f \quad} (b, \mu^b)$$

is a morphism $f : a \longrightarrow b$ in C such that the following diagram commutes.

$$
\begin{array}{ccc}
x \otimes a & \xrightarrow{\ 1_x \otimes f\ } & x \otimes b \\
{\scriptstyle \mu^a} \downarrow & & \downarrow {\scriptstyle \mu^b} \\
a & \xrightarrow{\quad f \quad} & b
\end{array}
$$

(3) *Right $x$-modules*, with structure morphisms $a \otimes x \longrightarrow a$, and their morphisms are defined similarly.                                    ◇

**Definition A.1.10.** A *braided monoidal category* is a pair $(\mathsf{C}, \xi)$ consisting of the following data.

- C is a monoidal category (Definition A.1.3).
- $\xi$ is a natural isomorphism, which is called the *braiding*, with components

$$x \otimes y \xrightarrow[\cong]{\ \xi_{x,y}\ } y \otimes x \quad \text{for} \quad x, y \in \mathsf{C}.$$

The hexagon diagrams

$$(A.1.11)$$

$$
\begin{array}{ccc}
(y \otimes x) \otimes z & \xrightarrow{\;\alpha\;} & y \otimes (x \otimes z) \\
{}^{\zeta^{\otimes}_{x,y} \otimes 1_z} \nearrow & & \searrow {}^{1_y \otimes \zeta^{\otimes}_{x,z}} \\
(x \otimes y) \otimes z & & y \otimes (z \otimes x) \\
{}^{\alpha} \searrow & & \nearrow {}^{\alpha} \\
x \otimes (y \otimes z) & \xrightarrow{\;\zeta^{\otimes}_{x,y \otimes z}\;} & (y \otimes z) \otimes x
\end{array}
\qquad
\begin{array}{ccc}
x \otimes (z \otimes y) & \xrightarrow{\;\alpha^{-1}\;} & (x \otimes z) \otimes y \\
{}^{1_x \otimes \zeta^{\otimes}_{y,z}} \nearrow & & \searrow {}^{\zeta^{\otimes}_{x,z} \otimes 1_y} \\
x \otimes (y \otimes z) & & (z \otimes x) \otimes y \\
{}^{\alpha^{-1}} \searrow & & \nearrow {}^{\alpha^{-1}} \\
(x \otimes y) \otimes z & \xrightarrow{\;\zeta_{x \otimes y,z}\;} & z \otimes (x \otimes y)
\end{array}
$$

are required to commute for objects $x, y, z \in \mathsf{C}$.                    ◇

**Remark A.1.12** (Unity Properties). In each braided monoidal category, the following unity diagrams commute for each object $x$.

$$(A.1.13)$$

$$
\begin{array}{ccc}
x \otimes 1 & \xrightarrow{\;\zeta_{x,1}\;} & 1 \otimes x \\
{}_{\rho_x} \searrow & & \swarrow {}_{\lambda_x} \\
 & x &
\end{array}
\qquad
\begin{array}{ccc}
1 \otimes x & \xrightarrow{\;\zeta_{1,x}\;} & x \otimes 1 \\
{}_{\lambda_x} \searrow & & \swarrow {}_{\rho_x} \\
 & x &
\end{array}
$$

See [**Yau∞b**, 1.3.21] for the proof.                    ◇

**Definition A.1.14.** A *symmetric monoidal category* is a pair $(\mathsf{C}, \xi)$ consisting of the following data.

- $\mathsf{C}$ is a monoidal category (Definition A.1.3).
- $\xi$ is a natural isomorphism, which is called the *symmetry isomorphism* or the *braiding*, with components

$$
x \otimes y \xrightarrow[\cong]{\;\xi_{x,y}\;} y \otimes x \quad \text{for} \quad x, y \in \mathsf{C}.
$$

The data above are required to make the following symmetry and hexagon diagrams commute for objects $x, y, z \in \mathsf{C}$.

$$(A.1.15)$$

$$
\begin{array}{ccc}
x \otimes y & \xrightarrow{\;1_{x \otimes y}\;} & x \otimes y \\
{}_{\xi_{x,y}} \searrow & & \nearrow {}_{\xi_{y,x}} \\
 & y \otimes x &
\end{array}
\qquad
\begin{array}{ccc}
(y \otimes x) \otimes z & \xrightarrow{\;\alpha\;} & y \otimes (x \otimes z) \\
{}^{\xi_{x,y} \otimes 1_z} \nearrow & & \searrow {}^{1_y \otimes \xi_{x,z}} \\
(x \otimes y) \otimes z & & y \otimes (z \otimes x) \\
{}^{\alpha} \searrow & & \nearrow {}^{\alpha} \\
x \otimes (y \otimes z) & \xrightarrow{\;\xi_{x,y \otimes z}\;} & (y \otimes z) \otimes x
\end{array}
$$

A *permutative category* is a strict symmetric monoidal category. For a generic permutative category, we often write its monoidal product and monoidal unit as $\oplus$ and $e$, respectively.                    ◇

**Remark A.1.16** (Symmetry Implies Braided). The symmetry axiom, $\xi_{y,x} \xi_{x,y} = 1$, implies that the hexagons (A.1.11) are equivalent. Thus, a symmetric monoidal category is precisely a braided monoidal category that satisfies the symmetry axiom. As a result, the unity diagrams (A.1.13) commute in each symmetric monoidal category.                    ◇

The following permutative category plays an important role in both Segal and Elmendorf-Mandell $K$-theory (Sections 2.3 and 2.4).

**Definition A.1.17** (Pointed Finite Sets)**.** We define the permutative category

$$\left(\mathcal{F}, \wedge, \underline{1}, \xi\right)$$

as follows.

**Objects:** The objects in $\mathcal{F}$ are the pointed finite sets

$$\underline{n} = \{0, \dots, n\} \quad \text{for} \quad n \geq 0$$

with basepoint 0.

**Morphisms:** For $m, n \geq 0$, the set of morphisms $\mathcal{F}(\underline{m}, \underline{n})$ is the set of pointed functions $\underline{m} \longrightarrow \underline{n}$, that is, functions that preserve the basepoint 0.

**Monoidal Product:** It is given on objects by the smash product of pointed finite sets and the lexicographic ordering,

$$\underline{m} \wedge \underline{n} = \underline{mn}.$$

It is given explicitly by the identification

$$\underline{m} \wedge \underline{n} \ni (i, j) \longmapsto \begin{cases} 0 \in \underline{mn} & \text{if either } i \text{ or } j \text{ is } 0, \text{ and} \\ (i-1)n + j \in \underline{mn} & \text{if } i, j > 0. \end{cases}$$

This extends to pointed functions by functoriality of the smash product of pointed finite sets. The monoidal product is strictly associative.

**Monoidal Unit:** The strict monoidal unit is $\underline{1} = \{0, 1\}$.

**Braiding:** Its component at $\underline{m}, \underline{n}$ is the pointed bijection

$$\underline{m} \wedge \underline{n} \xrightarrow[\cong]{\xi_{\underline{m},\underline{n}}} \underline{n} \wedge \underline{m}$$

given by

$$\underline{m} \wedge \underline{n} \ni (i, j) \longmapsto \begin{cases} 0 & \text{if either } i \text{ or } j \text{ is } 0, \text{ and} \\ (j, i) & \text{if } i, j > 0. \end{cases}$$

This finishes the definition of $\mathcal{F}$. ◇

**Definition A.1.18.** In a symmetric monoidal category $(\mathsf{C}, \xi)$, a *commutative monoid* is a monoid $(x, \mathsf{m}, i)$ as in Definition A.1.8 such that the diagram

$$x \otimes x \xrightarrow{\xi_{x,x}} x \otimes x$$

$$\mathsf{m} \searrow \quad \swarrow \mathsf{m}$$

$$x$$

commutes. ◇

**Definition A.1.19.** A symmetric monoidal category $(\mathsf{C}, \otimes)$ is *closed* if, for each object $x \in \mathsf{C}$, the functor

$$- \otimes x : \mathsf{C} \longrightarrow \mathsf{C}$$

admits a right adjoint, which is called an *internal hom*. A right adjoint of $- \otimes x$ is denoted by $\mathsf{Hom}(x, -)$ or $[x, -]$. ◇

**Definition A.1.20** (Diagrams)**.** For a small category $\mathcal{B}$ and a category $\mathsf{C}$, the *diagram category* $\mathcal{B}$-$\mathsf{C}$ is defined by the following data.

- Its objects are functors $\mathcal{B} \longrightarrow \mathsf{C}$.
- Its morphisms are natural transformations between such functors.
- Identity morphisms are identity natural transformations.

- Composition is vertical composition of natural transformations.

Moreover, we define the following.

- A *$\mathcal{B}$-diagram in* C is a functor $\mathcal{B} \longrightarrow$ C.
- For the category Cat of small categories and functors, a functor $\mathcal{B} \longrightarrow$ Cat is called a *$\mathcal{B}$-indexed category*.                                          ◇

**Example A.1.21** (Small Categories). $(\mathsf{Cat}, \times, \mathbf{1}, [,])$ is a symmetric monoidal closed category, where Cat is the category of small categories and functors.

- The monoidal product is the Cartesian product, denoted $\times$.
- The monoidal unit is the terminal category **1** with only one object $*$ and its identity morphism $1_*$.
- The closed structure $[,]$ is given by diagram categories (Definition A.1.20).

The category Cat is both complete and cocomplete. For an elementary proof of its cocompleteness, see [**Yau20b**, Section 1.4].                                          ◇

**Monoidal Functors and Natural Transformations.**

**Definition A.1.22.** Suppose C and D are monoidal categories. A *monoidal functor*

$$(F, F^2, F^0) : \mathsf{C} \longrightarrow \mathsf{D}$$

is a triple consisting of the following data.

- $F : \mathsf{C} \longrightarrow \mathsf{D}$ is a functor.
- $F^0 : 1 \longrightarrow F1$ is a morphism in D, which is called the *unit constraint*.
- $F^2$ is a natural transformation, which is called the *monoidal constraint*, with components

$$Fx \otimes Fy \xrightarrow{\ F^2_{x,y}\ } F(x \otimes y) \quad \text{for} \quad x, y \in \mathsf{C}.$$

The data above are required to make the following unity and associativity diagrams commute for objects $x, y, z \in \mathsf{C}$.

(A.1.23)
$$
\begin{array}{ccc}
1 \otimes Fx & \xrightarrow{\ \lambda_{Fx}\ } & Fx \\
{\scriptstyle F^0 \otimes 1_{Fx}}\downarrow & & \uparrow{\scriptstyle F\lambda_x} \\
F1 \otimes Fx & \xrightarrow{\ F^2\ } & F(1 \otimes x)
\end{array}
\qquad
\begin{array}{ccc}
Fx \otimes 1 & \xrightarrow{\ \rho_{Fx}\ } & Fx \\
{\scriptstyle 1_{Fx} \otimes F^0}\downarrow & & \uparrow{\scriptstyle F\rho_x} \\
Fx \otimes F1 & \xrightarrow{\ F^2\ } & F(x \otimes 1)
\end{array}
$$

(A.1.24)
$$
\begin{array}{ccc}
(Fx \otimes Fy) \otimes Fz & \xrightarrow{\ \alpha\ } & Fx \otimes (Fy \otimes Fz) \\
{\scriptstyle F^2 \otimes 1_{Fz}}\downarrow & & \downarrow{\scriptstyle 1_{Fx} \otimes F^2} \\
F(x \otimes y) \otimes Fz & & Fx \otimes F(y \otimes z) \\
{\scriptstyle F^2}\downarrow & & \downarrow{\scriptstyle F^2} \\
F((x \otimes y) \otimes z) & \xrightarrow{\ F\alpha\ } & F(x \otimes (y \otimes z))
\end{array}
$$

A monoidal functor $(F, F^2, F^0)$ is

- *strictly unital* if $F^0$ is the identity morphism;
- *strong* if $F^0$ and $F^2$ are isomorphisms; and
- *strict* if $F^0$ and $F^2$ are identities.

An *identity monoidal functor* has $F$, $F^0$, and $F^2$ all given by identities.

Moreover, a monoidal functor $(F, F^2, F^0)$ between braided monoidal categories C and D (Definition A.1.10) is a *braided monoidal functor* if the following diagram commutes for objects $x, y \in$ C.

(A.1.25)
$$
\begin{array}{ccc}
Fx \otimes Fy & \xrightarrow{\underset{\cong}{\xi_{Fx,Fy}}} & Fy \otimes Fx \\
{\scriptstyle F^2}\downarrow & & \downarrow{\scriptstyle F^2} \\
F(x \otimes y) & \xrightarrow[\cong]{F\xi_{x,y}} & F(y \otimes z)
\end{array}
$$

A *symmetric monoidal functor* is a braided monoidal functor between symmetric monoidal categories (Definition A.1.14). ◇

**Definition A.1.26.** Suppose given monoidal functors
$$
\mathsf{C} \xrightarrow{(F, F^2, F^0)} \mathsf{D} \xrightarrow{(G, G^2, G^0)} \mathsf{E}.
$$

The *composite monoidal functor*
$$
\mathsf{C} \xrightarrow{\left(GF, (GF)^2, (GF)^0\right)} \mathsf{E}
$$

has unit constraint given by the composite
$$
1 \xrightarrow{G^0} G1 \xrightarrow{G(F^0)} GF1
$$

and monoidal constraint given by the composite
$$
GFx \otimes GFy \xrightarrow{G^2_{Fx,Fy}} G(Fx \otimes Fy) \xrightarrow{G(F^2_{x,y})} GF(x \otimes y)
$$

for objects $x, y \in$ C. ◇

**Definition A.1.27.** Suppose
$$
(F, F^2, F^0) \quad \text{and} \quad (G, G^2, G^0) : \mathsf{C} \longrightarrow \mathsf{D}
$$

are monoidal functors between monoidal categories C and D. A *monoidal natural transformation* $\theta : F \longrightarrow G$ is a natural transformation of the underlying functors such that the following unit constraint and monoidal constraint diagrams in D commute for objects $x, y \in$ C.

(A.1.28)
$$
\begin{array}{ccc}
1 \xrightarrow{F^0} F1 & \qquad & Fx \otimes Fy \xrightarrow{\theta_x \otimes \theta_y} Gx \otimes Gy \\
\phantom{x}\searrow_{G^0} \quad \downarrow_{\theta_1} & & \phantom{x}\downarrow_{F^2} \qquad\qquad \downarrow_{G^2} \\
\phantom{xxxx} G1 & & F(x \otimes y) \xrightarrow{\theta_{x \otimes y}} G(x \otimes y)
\end{array}
$$

A (monoidal) natural transformation is also denoted by

(A.1.29)
$$
\mathsf{C} \underset{G}{\overset{F}{\rightrightarrows}} {\scriptstyle \Downarrow \theta} \; \mathsf{D}
$$

and is called a *2-cell*. ◇

Detailed discussion of pasting diagrams involving 2-cells is in [**JY21**, Ch. 3].

**Convention A.1.30** (Left Normalized Bracketing)**.** An iterated monoidal product is *left normalized* with the left half of each pair of parentheses at the far left, unless a different bracketing is specified. An empty monoidal product is the monoidal unit.                                                                                          ◇

**Explanation A.1.31.** As an example of Convention A.1.30, we denote

$$w \otimes x \otimes y \otimes z = \big((w \otimes x) \otimes y\big) \otimes z.$$

We omit parentheses for iterated monoidal products and tacitly insert the necessary associativity and unit isomorphisms. This is justified by Mac Lane's Strictification Theorem [**ML98**, XI.3.1]: Each monoidal category $\mathsf{C}$ admits a canonical strong monoidal adjoint equivalence $\mathsf{C} \rightleftarrows \mathsf{C}_{\mathsf{st}}$ with $\mathsf{C}_{\mathsf{st}}$ a strict monoidal category. The analogous braided and symmetric strictification theorems are in [**Yau22**, 21.3.1 and 21.6.1]. In each case, since an equivalence is full and faithful, the strict diagrams commute if and only if their preimages in $\mathsf{C}$ commute.                                     ◇

## A.2. 2-Categories

In this section we review

- 2-categories, 2-functors, 2-natural transformations, 2-adjunctions, and
- the 2-categories of small categories, small permutative categories, and small 2-categories.

**Definition A.2.1.** A *2-category* A consists of the following data.

**Objects:** It is equipped with a class $\mathsf{A}_0$ of *objects*.

**1-Cells:** For each pair of objects $a, b \in \mathsf{A}_0$, it is equipped with a class

$$\mathsf{A}_1(a, b)$$

of *1-cells* from $a$ to $b$. Such a 1-cell is denoted $a \longrightarrow b$.

**2-Cells:** For 1-cells $f, f' \in \mathsf{A}_1(a, b)$, it is equipped with a set

$$\mathsf{A}_2(f, f')$$

of *2-cells* from $f$ to $f'$. Such a 2-cell is denoted $f \longrightarrow f'$.

**Identities:** A is equipped with

- an *identity 1-cell*
$$1_a \in \mathsf{A}_1(a, a)$$

for each object $a$ and
- an *identity 2-cell*
$$1_f \in \mathsf{A}_2(f, f)$$

for each 1-cell $f \in \mathsf{A}_1(a, b)$.

**Compositions:** In the following compositions, $a$, $b$, and $c$ denote objects in A.

- For 1-cells $f, f', f'' \in \mathsf{A}_1(a, b)$, it is equipped with an assignment

$$\mathsf{A}_2(f', f'') \times \mathsf{A}_2(f, f') \xrightarrow{\ v\ } \mathsf{A}_2(f, f'') \ , \qquad v(\theta', \theta) = \theta'\theta$$

called the *vertical composition* of 2-cells.

- It is equipped with an assignment

$$\mathsf{A}_1(b, c) \times \mathsf{A}_1(a, b) \xrightarrow{\ h_1\ } \mathsf{A}_1(a, c) \ , \qquad h_1(g, f) = gf$$

called the *horizontal composition* of 1-cells.

- For 1-cells $f, f' \in A_1(a,b)$ and $g, g' \in A_1(b,c)$, it is equipped with an assignment

$$A_2(g,g') \times A_2(f,f') \xrightarrow{h_2} A_2(gf, g'f') \ , \quad h_2(\phi, \theta) = \phi * \theta$$

  called the *horizontal composition* of 2-cells.

The data above are required to satisfy (i) through (iv) below:

  (i) Vertical composition is associative and unital for identity 2-cells.
 (ii) Horizontal composition preserves vertical composition and identity 2-cells.
(iii) Horizontal composition of 1-cells is associative and unital for identity 1-cells.
 (iv) Horizontal composition of 2-cells is associative and unital for identity 2-cells of identity 1-cells.

This finishes the definition of a 2-category.

Moreover, we define the following.

- For objects $a$ and $b$ in a 2-category A, the *hom category* $A(a,b)$ is the category defined by the following data.
  – Objects are 1-cells from $a$ to $b$.
  – Morphisms are 2-cells between 1-cells $a \longrightarrow b$.
  – Composition is vertical composition of 2-cells.
  – Identities are identity 2-cells.
- A 2-category is *locally small* if each hom category is small.
- A 2-category is *small* if it has a set of objects and is locally small.
- The *underlying 1-category* of a 2-category is defined as follows.
  – It has the same class of objects.
  – Morphisms are 1-cells.
  – Composition is horizontal composition of 1-cells.
  – Identity morphisms are identity 1-cells.

We also use the 2-cell notation (A.1.29) for 2-cells in a 2-category.          ◇

**Example A.2.2** (Small Categories). The category Cat in Example A.1.21 is the underlying 1-category of a 2-category, in which the 2-cells are natural transformations. Horizontal, respectively vertical, composition of 2-cells in Cat is the same as that of natural transformations.          ◇

**Definition A.2.3** (Small Permutative Categories). Denote by

$$\mathsf{PermCat}$$

the 2-category with

- small permutative categories as objects,
- symmetric monoidal functors as 1-cells, and
- monoidal natural transformations as 2-cells.

Moreover, we define the following locally-full sub-2-categories of PermCat with the same objects but restricting the 1-cells.

- $\mathsf{PermCat}^{\mathsf{su}}$ has 1-cells given by strictly unital symmetric monoidal functors.
- $\mathsf{PermCat}^{\mathsf{sg}}$ has 1-cells given by strong symmetric monoidal functors.

- PermCat$^{\mathsf{sus}}$ has 1-cells given by strictly unital strong symmetric monoidal functors.
- PermCat$^{\mathsf{st}}$ has 1-cells given by strict symmetric monoidal functors.

In each case the 2-cells are given by monoidal natural transformations.                    ◇

The extensions of PermCat$^{\mathsf{su}}$, PermCat$^{\mathsf{sus}}$, and PermCat$^{\mathsf{st}}$ to Cat-multicategories are in Theorem 1.4.29.

**Definition A.2.4.** For 2-categories A and B, a *2-functor $F : \mathsf{A} \longrightarrow \mathsf{B}$* consists of the following data.

**Object Assignment:** It is equipped with a function
$$\mathsf{A}_0 \xrightarrow{\ F_0\ } \mathsf{B}_0.$$

**1-Cell Assignment:** For each pair of objects $a, b$ in A, it is equipped with a function
$$\mathsf{A}_1(a, b) \xrightarrow{\ F_1\ } \mathsf{B}_1(F_0 a, F_0 b).$$

**2-Cell Assignment:** For each pair of objects $a, b$ in A and 1-cells $f, f' \in \mathsf{A}_1(a, b)$, it is equipped with a function
$$\mathsf{A}_2(f, f') \xrightarrow{\ F_2\ } \mathsf{B}_2(F_1 f, F_1 f').$$

The data above are required to satisfy (i) through (iii) below, with each of $F_0$, $F_1$, and $F_2$ abbreviated to $F$:

(i) The object and 1-cell assignments of $F$ form a functor from the underlying 1-category of A to the underlying 1-category of B.

(ii) For each pair of objects $a, b$ in A, the 1-cell and 2-cell assignments of $F$ form a functor between hom categories:
$$\mathsf{A}(a, b) \xrightarrow{\ F\ } \mathsf{B}(Fa, Fb).$$

(iii) $F$ preserves horizontal composition of 2-cells.

This finishes the definition of a 2-functor.

Moreover, we define the following.

- The *identity 2-functor* $1_{\mathsf{A}} : \mathsf{A} \longrightarrow \mathsf{A}$ is defined by the identity assignments on objects, 1-cells, and 2-cells.
- Given a 2-functor $G : \mathsf{B} \longrightarrow \mathsf{C}$, the *composite $GF$* is the 2-functor
$$\mathsf{A} \xrightarrow{\ GF\ } \mathsf{C}$$
defined by separately composing the assignments on objects, 1-cells, and 2-cells.                    ◇

**Example A.2.5.** In the context of Definition A.2.3, there are inclusion 2-functors as follows.

(A.2.6)



Each 2-functor in (A.2.6) is the identity on

- objects, which are small permutative categories, and
- 2-cells between each pair of 1-cells, which are monoidal natural transformations. ◇

**Definition A.2.7.** For 2-functors $F, G : A \longrightarrow B$ between 2-categories A and B, a *2-natural transformation* $\varphi : F \longrightarrow G$ consists of, for each object $a$ in A, a *component 1-cell*

$$Fa \xrightarrow{\varphi_a} Ga \quad \text{in} \quad B$$

such that the axioms (A.2.8) and (A.2.9) below hold:

**1-Cell Naturality:** For each 1-cell $f : a \longrightarrow b$ in A, the following two composite 1-cells in $B(Fa, Gb)$ are equal.

(A.2.8)
$$
\begin{array}{ccc}
Fa & \xrightarrow{\varphi_a} & Ga \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
Fb & \xrightarrow{\varphi_b} & Gb
\end{array}
$$

**2-Cell Naturality:** For each 2-cell $\theta : f \longrightarrow g$ in $A(a, b)$, the two whiskered 2-cells in the following diagram are equal.

(A.2.9)
$$
Ff\left(\underset{\xRightarrow{F\theta}}{}\right)Fg \qquad Gf\left(\underset{\xRightarrow{G\theta}}{}\right)Gg
$$

with
$$
\begin{array}{ccc}
Fa & \xrightarrow{\varphi_a} & Ga \\
Fb & \xrightarrow{\varphi_b} & Gb
\end{array}
$$

The axiom (A.2.9) means the following 2-cell equality in $B(Fa, Gb)$:

$$G\theta * 1_{\varphi_a} = 1_{\varphi_b} * F\theta.$$

This finishes the definition of a 2-natural transformation.

Moreover, we define the following.

- We extend the 2-cell notation (A.1.29) to 2-natural transformations.
- A *2-natural isomorphism* is a 2-natural transformation such that each component 1-cell is an isomorphism in the underlying 1-category.
- Identity 2-natural transformations, horizontal composition of 2-natural transformations, and vertical composition of 2-natural transformations are defined componentwise. ◇

**Example A.2.10** (Small 2-Categories). Analogous to Example A.2.2, there is a 2-category 2Cat with

- small 2-categories as objects,
- 2-functors as 1-cells, and
- 2-natural transformations as 2-cells.

Horizontal, respectively vertical, composition of 2-cells in 2Cat is given by that of 2-natural transformations. We also use the notation 2Cat for its underlying 1-category. ◇

**Definition A.2.11.** Suppose A and B are 2-categories. A *2-adjunction* from A to B is a quadruple

$$(F, G, \eta, \varepsilon) : A \longrightarrow B$$

consisting of the following data.

- $F : A \longrightarrow B$ is a 2-functor, which is called the *left adjoint*.

- $G : B \longrightarrow A$ is a 2-functor, which is called the *right adjoint*.
- $\eta : 1_A \longrightarrow GF$ is a 2-natural transformation, which is called the *unit*.
- $\varepsilon : FG \longrightarrow 1_B$ is a 2-natural transformation, which is called the *counit*.

The data above are required to make the following two diagrams commute, where $*$ denotes horizontal composition of 2-natural transformations.

(A.2.12)

$$
\begin{array}{ccc}
F \xrightarrow{\ 1_F * \eta\ } FGF & \qquad & G \xrightarrow{\ \eta * 1_G\ } GFG \\
\phantom{1_F}\searrow_{1_F} \ \downarrow{\varepsilon * 1_F} & & \phantom{1_G}\searrow_{1_G} \ \downarrow{1_G * \varepsilon} \\
\phantom{xx} F & & \phantom{xx} G
\end{array}
$$

The two diagrams in (A.2.12) are called, respectively, the *left triangle identity* and the *right triangle identity*. Such a 2-adjunction is also denoted $F \dashv G$.                          ⋄

# Enriched Category Theory

In this appendix we review elements of enriched category theory. The following table summarizes the main content in this appendix.

| B.1. Enriched Categories | |
|---|---|
| enriched categories, functors, and natural transformations | B.1.1, B.1.8, and B.1.10 |
| unit and opposite V-categories | B.1.6 and B.1.13 |
| Cat-categories as locally small 2-categories | B.1.7 |
| 2-category V-Cat | B.1.12 |
| **B.2. Enriched Monoidal Categories** | |
| tensor products and monoidal category of V-categories | B.2.1 and B.2.6 |
| monoidal V-categories (braided, symmetric) | B.2.7 (B.2.13, B.2.16) |
| monoidal Cat-category (V-Cat, ⊗) of V-categories | B.2.6 and B.2.18 |
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| **B.3. Self-Enriched Symmetric Monoidal Categories** | |
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| change-of-enrichment 2-functor $(-)_U$ | B.4.1 and B.4.6 |
| composition of change-of-enrichment 2-functors | B.4.7 |
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| standard enrichment of a symmetric monoidal functor | B.4.13, B.4.17, and B.4.18 |

The material in this chapter is adapted from [**JY∞**]; see also [**Kel05**]. We remind the reader of Conventions A.1.2 and A.1.30.

## B.1. Enriched Categories

In this section we review enriched variants of categories, functors, natural transformations, and opposite categories. Suppose

$$(\mathsf{V}, \otimes, 1, \alpha, \lambda, \rho)$$

is a monoidal category (Definition A.1.3). A braiding on $V$ is not needed until Definition B.1.13 of the opposite $V$-category. The material in this section is adapted from [**JY∞**, Sections 1.1 and 1.2].

**Definition B.1.1.** A $V$-*category* $C$, which is also called a *category enriched in* $V$, consists of the following data.

**Objects:** It is equipped with a class $\mathsf{Ob}(C)$, whose elements are called *objects*.
**Hom Objects:** Each pair of objects $x, y$ in $C$ is equipped with a *hom object*

$$C(x, y) \in V.$$

**Composition:** For objects $x, y, z$ in $C$, it is equipped with a morphism in $V$

$$(B.1.2) \qquad C(y, z) \otimes C(x, y) \xrightarrow{\ m_{x,y,z}\ } C(x, z)$$

called the *composition*.
**Identities:** Each object $x$ in $C$ is equipped with a morphism in $V$

$$(B.1.3) \qquad 1 \xrightarrow{\ i_x\ } C(x, x)$$

called the *identity* of $x$.

The data above are required to make the following *associativity diagram* and *unity diagram* commute for objects $w, x, y, z$ in $C$.

$$(B.1.4)$$

$$
\begin{array}{ccc}
\big(C(y,z) \otimes C(x,y)\big) \otimes C(w,x) & \xrightarrow{\ \alpha\ } & C(y,z) \otimes \big(C(x,y) \otimes C(w,x)\big) \\
{\scriptstyle m \otimes 1}\Big\downarrow & & \Big\downarrow {\scriptstyle 1 \otimes m} \\
& & C(y,z) \otimes C(w,y) \\
& & \Big\downarrow {\scriptstyle m} \\
C(x,z) \otimes C(w,x) & \xrightarrow{\quad m \quad} & C(w,z)
\end{array}
$$

$$(B.1.5)$$

$$
\begin{array}{ccc}
1 \otimes C(x,y) \xrightarrow{\ \lambda\ } & & \xleftarrow{\ \rho\ } C(x,y) \otimes 1 \\
{\scriptstyle i_y \otimes 1}\Big\downarrow & & \Big\downarrow {\scriptstyle 1 \otimes i_x} \\
C(y,y) \otimes C(x,y) \xrightarrow{\ m\ } C(x,y) & \xleftarrow{\ m\ } & C(x,y) \otimes C(x,x)
\end{array}
$$

This finishes the definition of a $V$-category. Moreover, a $V$-category $C$ is *small* if $\mathsf{Ob}(C)$ is a set. $\diamond$

**Definition B.1.6.** The *unit* $V$-*category*, $\mathbb{I}$, is the one-object $V$-category whose unique hom object is the monoidal unit, $1$, of $V$. The composition and identity structure morphisms are given, respectively, by the left unit isomorphism $\lambda_1$ and the identity morphism $1_1$. $\diamond$

**Proposition B.1.7.** *Regarding* $(\mathsf{Cat}, \times, \mathbf{1})$ *as a monoidal category, a locally small 2-category is precisely a* $\mathsf{Cat}$-*category.*

**Definition B.1.8.** A $V$-*functor*

$$F : C \longrightarrow D$$

between $V$-categories $C$ and $D$ consists of the following data.

**Object Assignment:** It is equipped with a function

$$F : \mathsf{Ob}(C) \longrightarrow \mathsf{Ob}(D).$$

**Component Morphisms:** It is equipped with, for each pair of objects $x, y$ in C, a morphism

$$C(x, y) \xrightarrow{\ F_{x,y}\ } D(Fx, Fy) \quad \text{in} \quad V.$$

The data above is required to make the following two diagrams commute for objects $x, y, z$ in C.

(B.1.9)

$$
\begin{array}{ccc}
C(y,z) \otimes C(x,y) & \xrightarrow{\ m\ } & C(x,z) \\
{\scriptstyle F \otimes F} \downarrow & & \downarrow {\scriptstyle F} \\
D(Fy,Fz) \otimes D(Fx,Fy) & \xrightarrow{\ m\ } & D(Fx,Fz)
\end{array}
\qquad
\begin{array}{ccc}
1 & \xrightarrow{\ i_x\ } & C(x,x) \\
{\scriptstyle i_{Fx}} \searrow & & \downarrow {\scriptstyle F} \\
& & D(Fx,Fx)
\end{array}
$$

Moreover, we define the following.

- An identity V-functor is given by the identity object assignment and identity component morphisms.
- Composition of V-functors is defined by separately composing the object assignments and the component morphisms. ◇

**Definition B.1.10.** For V-functors $F, G : C \longrightarrow D$ between V-categories C and D, a V-*natural transformation* $\theta : F \longrightarrow G$ consists of, for each object $x$ in C, a morphism

$$1 \xrightarrow{\ \theta_x\ } D(Fx, Gx) \quad \text{in} \quad V,$$

called the *x-component* of $\theta$. The following *naturality diagram* in V is required to commute for objects $x, y$ in C.

(B.1.11)

$$
\begin{array}{ccc}
& 1 \otimes C(x,y) & \xrightarrow{\ \theta_y \otimes F\ } & D(Fy,Gy) \otimes D(Fx,Fy) \\
{\scriptstyle \lambda^{-1}} \nearrow {\scriptstyle \cong} & & & \searrow {\scriptstyle m} \\
C(x,y) & & & D(Fx,Gy) \\
{\scriptstyle \rho^{-1}} \searrow {\scriptstyle \cong} & & & \nearrow {\scriptstyle m} \\
& C(x,y) \otimes 1 & \xrightarrow{\ G \otimes \theta_x\ } & D(Gx,Gy) \otimes D(Fx,Gx)
\end{array}
$$

Moreover, each of the following notions is defined componentwise:

- identity V-natural transformations,
- horizontal composition of V-natural transformations, and
- vertical composition of V-natural transformations.

We use the 2-cell notation (A.1.29) for V-natural transformations. ◇

**Example B.1.12** (Small Enriched Categories). Each monoidal category V has an associated 2-category V-Cat defined by the following data.

- Objects are small V-categories.
- 1-cells are V-functors.
- 2-cells are V-natural transformations.

The 2-category Cat in Example A.2.2 is the special case for $V = (\mathsf{Set}, \times, *)$. ◇

For the following definition, we assume that V is a braided monoidal category (Definition A.1.10). The next definition is [**JY∞**, 1.2.16].

**Definition B.1.13.** Suppose C is a V-category with $(V, \otimes, 1, \xi)$ a braided monoidal category. The *opposite* V-*category*, $C^{\mathsf{op}}$, is defined as follows.

**Objects:** $\mathsf{C}^{\mathsf{op}}$ has the same class of objects as $\mathsf{C}$.

**Hom Objects:** Each pair of objects $x, y$ in $\mathsf{C}^{\mathsf{op}}$ is equipped with the hom object

$$\mathsf{C}^{\mathsf{op}}(x, y) = \mathsf{C}(y, x).$$

**Composition:** The composition in $\mathsf{C}^{\mathsf{op}}$ is defined for each triple of objects $x, y, z$ in $\mathsf{C}^{\mathsf{op}}$ as the following composite in $\mathsf{V}$ using the braiding $\xi$ of $\mathsf{V}$ and the composition $\mathsf{m}$ of $\mathsf{C}$:

(B.1.14)

$$
\begin{array}{ccc}
\mathsf{C}^{\mathsf{op}}(y, z) \otimes \mathsf{C}^{\mathsf{op}}(x, y) & & \mathsf{C}^{\mathsf{op}}(x, z) \\
\| & & \| \\
\mathsf{C}(z, y) \otimes \mathsf{C}(y, x) \xrightarrow{\ \xi\ } \mathsf{C}(y, x) \otimes \mathsf{C}(z, y) \xrightarrow{\ \mathsf{m}\ } & & \mathsf{C}(z, x)
\end{array}
$$

**Identities:** The identity of each object $x$ in $\mathsf{C}^{\mathsf{op}}$ is the same as the identity of $x$ in $\mathsf{C}$:

$$1 \xrightarrow{\ i_x\ } \mathsf{C}(x, x) = \mathsf{C}^{\mathsf{op}}(x, x).$$

This finishes the definition of the opposite $\mathsf{V}$-category.

Moreover, we extend the opposite construction to $\mathsf{V}$-functors and $\mathsf{V}$-natural transformations as follows.

- For a $\mathsf{V}$-functor $F : \mathsf{C} \longrightarrow \mathsf{D}$, the *opposite* $\mathsf{V}$-*functor*

$$F^{\mathsf{op}} : \mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{D}^{\mathsf{op}}$$

  has
  - the same object assignment as $F$ and
  - the following component morphisms for objects $x, y$ in $\mathsf{C}^{\mathsf{op}}$:

$$F_{x,y}^{\mathsf{op}} = F_{y,x} : \mathsf{C}^{\mathsf{op}}(x, y) \longrightarrow \mathsf{D}^{\mathsf{op}}(Fx, Fy).$$

- For a $\mathsf{V}$-natural transformation $\theta : F \longrightarrow G$ with $F, G : \mathsf{C} \longrightarrow \mathsf{D}$ both $\mathsf{V}$-functors, the *opposite* $\mathsf{V}$-*natural transformation*

$$\theta^{\mathsf{op}} : G^{\mathsf{op}} \longrightarrow F^{\mathsf{op}}$$

  is defined by the component morphisms

$$\theta_x^{\mathsf{op}} = \theta_x : 1 \longrightarrow \mathsf{D}^{\mathsf{op}}(Gx, Fx)$$

  for objects $x$ in $\mathsf{C}^{\mathsf{op}}$.                                          $\diamond$

## B.2. Enriched Monoidal Categories

In this section we review monoidal categories, functors, and natural transformations enriched in a braided monoidal category $(\mathsf{V}, \otimes, 1, \alpha, \lambda, \rho, \xi)$ (Definition A.1.10). Whenever we need $\mathsf{V}$ to be symmetric monoidal, we state so explicitly. The material in this section is adapted from [**JY**$\infty$, Chapter 1].

**Tensor Product of Enriched Categories.**

**Definition B.2.1.** Suppose $\mathsf{C}$ and $\mathsf{D}$ are $\mathsf{V}$-categories. The *tensor product*, $\mathsf{C} \otimes \mathsf{D}$, is the $\mathsf{V}$-category defined by the following data.

**Objects:** Its class of objects is

$$\mathsf{Ob}(\mathsf{C} \otimes \mathsf{D}) = \mathsf{Ob}(\mathsf{C}) \times \mathsf{Ob}(\mathsf{D}).$$

Objects in $\mathsf{C} \otimes \mathsf{D}$ are denoted $x \otimes y$ for $x \in \mathsf{C}$ and $y \in \mathsf{D}$.

**Hom Objects:** For objects $x \otimes y$ and $x' \otimes y'$, the hom object is the monoidal product

$$(\mathsf{C} \otimes \mathsf{D})(x \otimes y, x' \otimes y') = \mathsf{C}(x, x') \otimes \mathsf{D}(y, y') \quad \text{in} \quad \mathsf{V}.$$

**Composition:** For objects $x \otimes y$, $x' \otimes y'$, and $x'' \otimes y''$, the composition is the following composite in $\mathsf{V}$, where $\xi_{\mathrm{mid}}$ interchanges the middle two factors using the associativity isomorphism and braiding in $\mathsf{V}$.

$$\big(\mathsf{C}(x',x'') \otimes \mathsf{D}(y',y'')\big) \otimes \big(\mathsf{C}(x,x') \otimes \mathsf{D}(y,y')\big) \qquad\qquad \mathsf{C}(x,x'') \otimes \mathsf{D}(y,y'')$$

$$\xi_{\mathrm{mid}} \searrow \qquad\qquad \nearrow \mathsf{m} \otimes \mathsf{m}$$

$$\big(\mathsf{C}(x',x'') \otimes \mathsf{C}(x,x')\big) \otimes \big(\mathsf{D}(y',y'') \otimes \mathsf{D}(y,y')\big)$$

**Identities:** The identity of an object $x \otimes y$ is the following composite in $\mathsf{V}$.

$$1 \xrightarrow[\cong]{\lambda^{-1}} 1 \otimes 1 \xrightarrow{i_x \otimes i_y} \mathsf{C}(x,x) \otimes \mathsf{D}(y,y) = (\mathsf{C} \otimes \mathsf{D})(x \otimes y, x \otimes y)$$

This finishes the definition of the $\mathsf{V}$-category $\mathsf{C} \otimes \mathsf{D}$. The tensor product $\otimes$ extends to $\mathsf{V}$-functors and $\mathsf{V}$-natural transformations componentwise. ◇

Recall from Example B.1.12 the 2-category $\mathsf{V}$-$\mathsf{Cat}$ of small $\mathsf{V}$-categories, $\mathsf{V}$-functors, and $\mathsf{V}$-natural transformations.

**Proposition B.2.2.** *The tensor product is a 2-functor*

$$\mathsf{V}\text{-}\mathsf{Cat} \times \mathsf{V}\text{-}\mathsf{Cat} \xrightarrow{\ \otimes\ } \mathsf{V}\text{-}\mathsf{Cat}.$$

Recall the unit $\mathsf{V}$-category $\mathbb{I}$ in Definition B.1.6. The tensor product on $\mathsf{V}$-$\mathsf{Cat}$ is part of a monoidal structure, with the following unit and associativity isomorphisms.

**Definition B.2.3.** We define the *left unitor* $\ell^\otimes$ and the *right unitor* $r^\otimes$ as the 2-natural isomorphisms



as follows. The unitor components at a $\mathsf{V}$-category $\mathsf{C}$ are the $\mathsf{V}$-functors

$$\mathbb{I} \otimes \mathsf{C} \xrightarrow{\ell_\mathsf{C}^\otimes} \mathsf{C} \xleftarrow{r_\mathsf{C}^\otimes} \mathsf{C} \otimes \mathbb{I}$$

given

- on objects by the unitors for the Cartesian product and
- on hom objects by the unit isomorphisms

$$1 \otimes \mathsf{C}(x,x') \xrightarrow{\ \lambda\ } \mathsf{C}(x,x') \xleftarrow{\ \rho\ } \mathsf{C}(x,x') \otimes 1 \quad \text{in} \quad \mathsf{V}$$

for objects $x, x' \in \mathsf{C}$. ◇

**Definition B.2.4.** We define the *associator* $a^\otimes$ as the 2-natural isomorphism

as follows. For small V-categories C, D, and E, the associator component is the V-functor

$$(C \otimes D) \otimes E \xrightarrow{\ a^\otimes_{C,D,E}\ } C \otimes (D \otimes E)$$

given

- on objects by the associativity isomorphism of the Cartesian product and
- on hom objects by the associativity isomorphism

$$\big(C(x,x') \otimes D(y,y')\big) \otimes E(z,z') \xrightarrow{\ \alpha\ } C(x,x') \otimes \big(D(y,y') \otimes E(z,z')\big)$$

in V for objects $x, x' \in C$, $y, y' \in D$, and $z, z' \in E$.                                      ◇

The Cartesian product on objects and the monoidal structure of V both satisfy the unity and pentagon axioms in (A.1.4), (A.1.6), and (A.1.7). Thus the data $(\otimes, \mathbb{I}, a^\otimes, \ell^\otimes, r^\otimes)$ in Definitions B.1.6, B.2.1, B.2.3, and B.2.4 also satisfy these axioms.

**Definition B.2.5.** Suppose that $(V, \xi)$ is a symmetric monoidal category. We define the *braiding* $\beta^\otimes$ as the 2-natural isomorphism



as follows, with $\tau$ permuting the two arguments. For small V-categories C and D, the braiding component is the V-functor

$$C \otimes D \xrightarrow{\ \beta^\otimes_{C,D}\ } D \otimes C$$

given

- on objects by the braiding of the Cartesian product and
- on hom objects by the braiding

$$C(x,x') \otimes D(y,y') \xrightarrow{\ \xi\ } D(y,y') \otimes C(x,x')$$

in V for objects $x, x' \in C$ and $y, y' \in D$.                                      ◇

The Cartesian product on objects and the symmetric monoidal structure of V both satisfy the hexagon, unity, and symmetry axioms in (A.1.11), (A.1.13), and (A.1.15). Thus the data $(\otimes, \mathbb{I}, a^\otimes, \ell^\otimes, r^\otimes, \beta^\otimes)$ also satisfy these axioms.

**Theorem B.2.6.** *Suppose* $V = (V, \otimes, \xi)$ *is a braided monoidal category. Then*

$$\big(V\text{-Cat}, \otimes, \mathbb{I}, a^\otimes, \ell^\otimes, r^\otimes\big)$$

*is a monoidal category. If* V *is symmetric monoidal, then so is* $(V\text{-Cat}, \beta^\otimes)$.

**The Monoidal** Cat**-Category of Enriched Categories.** Theorem B.2.6 is improved to (symmetric) monoidal Cat-categories in Theorem B.2.18 below. To make the necessary definitions, we

- abbreviate the tensor product of V-categories to juxtaposition and
- use superscript with a minus sign, $?^{-\otimes}$, to denote the inverse $(?^\otimes)^{-1}$.

**Definition B.2.7.** A *monoidal* $\mathsf{V}$*-category* is a sextuple

$$\left(\mathsf{K}, \boxtimes, I^{\boxtimes}, a^{\boxtimes}, \ell^{\boxtimes}, r^{\boxtimes}\right)$$

consisting of the following data.

**Base** $\mathsf{V}$**-category:** It has a $\mathsf{V}$-category $\mathsf{K}$, called the *base* $\mathsf{V}$*-category*.

**Monoidal Composition:** It has a $\mathsf{V}$-functor

$$\mathsf{K} \otimes \mathsf{K} \xrightarrow{\ \boxtimes\ } \mathsf{K}$$

called the *monoidal composition*.

**Monoidal Identity:** It has a $\mathsf{V}$-functor

$$\mathbb{I} \xrightarrow{\ I^{\boxtimes}\ } \mathsf{K}$$

called the *monoidal identity*. The image of the unique object in $\mathbb{I}$ is also denoted $I^{\boxtimes}$ and called the *identity object*.

**Monoidal Unitors:** It has $\mathsf{V}$-natural isomorphisms

(B.2.8)



called the *left monoidal unitor* and the *right monoidal unitor*, respectively. Their components at an object $x \in \mathsf{K}$ are, respectively,

$$1 \xrightarrow{\ \ell_x^{\boxtimes}\ } \mathsf{K}(I^{\boxtimes} \boxtimes x, x) \quad \text{and} \quad 1 \xrightarrow{\ r_x^{\boxtimes}\ } \mathsf{K}(x \boxtimes I^{\boxtimes}, x).$$

**Monoidal Associator:** It has a $\mathsf{V}$-natural isomorphism

(B.2.9)



called the *monoidal associator*. Its component at a triple of objects $x, y, z \in \mathsf{K}$ is a morphism in $\mathsf{V}$

$$1 \xrightarrow{\ a_{x,y,z}^{\boxtimes}\ } \mathsf{K}\big((x \boxtimes y) \boxtimes z, \, x \boxtimes (y \boxtimes z)\big).$$

These data are required to satisfy the following two axioms, with 1 denoting the identity $\mathsf{V}$-functor.

**Unity Axiom:** The composites of the following two *middle unity pasting diagrams* are equal.

$$
\begin{array}{c}
\text{(first diagram)} \\
(K\mathbb{I})K \xrightarrow{(1I^{\boxtimes})1} (K^2)K \xrightarrow{\boxtimes 1} K^2 \\
K^2 \xrightarrow{r^{-\otimes}1} \quad \star \quad \downarrow a^{\otimes} \qquad \downarrow a^{\otimes} \qquad a^{\boxtimes} \\
K^2 \xrightarrow{1\ell^{-\otimes}} K(\mathbb{I}K) \xrightarrow{1(I^{\boxtimes}1)} K(K^2) \qquad \downarrow \boxtimes \\
1\ell^{\boxtimes} \\
1^2 \longrightarrow K^2 \xrightarrow{\boxtimes} K
\end{array}
$$

(B.2.10)

$$\|$$

$$
\begin{array}{c}
(K\mathbb{I})K \xrightarrow{(1I^{\boxtimes})1} (K^2)K \xrightarrow{\boxtimes 1} K^2 \\
K^2 \xrightarrow{r^{-\otimes}1} \quad r^{\boxtimes}1 \\
1^2 \qquad \downarrow \boxtimes \\
1^2 \longrightarrow K^2 \xrightarrow{\boxtimes} K
\end{array}
$$

In the first diagram in (B.2.10), the unlabeled rectangle commutes by naturality of $a^{\otimes}$. The region labeled $\star$ commutes by the middle unity for $\ell^{\otimes}$ and $r^{\otimes}$.

**Pentagon Axiom:** The composites of the following two *pentagon pasting diagrams* are equal.

$$
\begin{array}{c}
(K(K^2))K \xleftarrow{a^{\otimes}1} ((K^2)K)K \\
\downarrow a^{\otimes} \qquad \star \qquad \downarrow a^{\otimes} \qquad (\boxtimes 1)1 \\
K((K^2)K) \qquad \qquad (K^2)K \\
1a^{\otimes} \quad a^{\otimes} \quad (K^2)(K^2) \xrightarrow{\boxtimes 1^2} K(K^2) \quad a^{\boxtimes} \quad \boxtimes 1 \\
K(K(K^2)) \qquad 1^2\boxtimes \qquad 1\boxtimes \qquad K^2 \\
1(1\boxtimes) \quad a^{\otimes} \quad (K^2)K \xrightarrow{\boxtimes 1} K^2 \quad a^{\boxtimes} \\
K(K^2) \xrightarrow{1\boxtimes} K^2 \xrightarrow{\boxtimes} K
\end{array}
$$

(B.2.11)

$$\|$$

$$
\begin{array}{c}
(K(K^2))K \xleftarrow{a^{\otimes}1} ((K^2)K)K \\
\downarrow a^{\otimes} \qquad (1\boxtimes)1 \qquad (\boxtimes 1)1 \\
K((K^2)K) \qquad a^{\boxtimes}1 \qquad (K^2)K \\
1a^{\otimes} \quad 1(\boxtimes 1) \quad a^{\otimes} \quad (K^2)K \\
K(K(K^2)) \quad K(K^2) \qquad a^{\boxtimes} \quad \boxtimes 1 \\
1(1\boxtimes) \quad 1a^{\boxtimes} \quad 1\boxtimes \qquad K^2 \\
K(K^2) \xrightarrow{1\boxtimes} K^2 \xrightarrow{\boxtimes} K
\end{array}
$$

The central square in the first diagram in (B.2.11) commutes by 2-functoriality of $\otimes$ in each variable. The other unmarked quadrilaterals in the two diagrams in (B.2.11) commute by 2-naturality of $a^{\otimes}$. The pentagon labeled $\star$ commutes by the pentagon axiom for $a^{\otimes}$.

This finishes the definition of a monoidal V-category. ◇

For the definition of a braided monoidal V-category, we use a mate of $a^{\boxtimes}$ similar to the mates of a pentagonator in [**JY21**, 12.1.4].

**Definition B.2.12.** For a monoidal V-category K, we denote by $a_1^{\boxtimes}$ the mate of $a^{\boxtimes}$ given by the inverse of $a^{\otimes}$, as shown below.



We denote by $a_1^{-\boxtimes}$ the inverse of $a_1^{\boxtimes}$. ◇

**Definition B.2.13.** For a symmetric monoidal category V, a *braided monoidal* V-*category* is a pair $(K, \beta^{\boxtimes})$ consisting of the following data.

- K is a monoidal V-category (Definition B.2.7).
- $\beta^{\boxtimes}$ is a V-natural isomorphism



   called the *braiding* of K.

These data are required to satisfy the following two axioms.

**Left Hexagon Axiom:** The composites of the following two *left hexagon pasting diagrams* are equal.

(B.2.14)



   In (B.2.14), the unlabeled quadrilateral commutes by 2-naturality of $\beta^{\otimes}$. The hexagon labeled $\star$ commutes by the left hexagon axiom for $\beta^{\otimes}$.

**Right Hexagon Axiom:** The composites of the following two *right hexagon pasting diagrams* are equal.

(B.2.15)

$$
\begin{array}{c}
\text{(right hexagon pasting diagrams)}
\end{array}
$$

In (B.2.15), the unlabeled quadrilateral commutes by 2-naturality of $\beta^\otimes$. The hexagon labeled $\star$ commutes by the right hexagon axiom for $\beta^\otimes$. The 2-cell isomorphism $a_1^{-\boxtimes}$ is the inverse of $a_1^\boxtimes$ (Definition B.2.12).

This finishes the definition of a braided monoidal V-category.                    ◇

**Definition B.2.16.** For a symmetric monoidal category V, a *symmetric monoidal V-category* is a braided monoidal V-category $(K, \beta^\boxtimes)$ that satisfies the following axiom.

**Symmetry Axiom:** The composites of the following two *symmetry pasting diagrams* are equal.

(B.2.17)

$$
\begin{array}{c}
\text{(symmetry pasting diagrams)}
\end{array}
$$

The right hand diagram in (B.2.17) is the identity V-natural transformation. In the left hand diagram, the region labeled $\star$ commutes by the symmetry axiom for $\beta^\otimes$. When this axiom holds, $\beta^\boxtimes$ is also called the *symmetry* of K.                    ◇

The following result uses the symmetric monoidal category $(\mathsf{Cat}, \times, \mathbf{1})$ in Example A.1.21.

**Theorem B.2.18.** *Suppose* $\mathsf{V} = (\mathsf{V}, \otimes, \xi)$ *is a braided monoidal category.*

   *(1) There is a monoidal* $\mathsf{Cat}$*-category (Definition B.2.7)*

$$\big(\mathsf{V}\text{-}\mathsf{Cat}, \otimes, \mathbb{I}, a^\otimes, \ell^\otimes, r^\otimes\big).$$

   *(2) If* $\mathsf{V}$ *is a symmetric monoidal category, then* $(\mathsf{V}\text{-}\mathsf{Cat}, \beta^\otimes)$ *is a symmetric monoidal* $\mathsf{Cat}$*-category (Definition B.2.16).*

### Enriched Monoidal Functors and Natural Transformations.

**Definition B.2.19.** Suppose K is a monoidal V-category with $(\mathsf{V}, \xi)$ a braided monoidal category. We denote by $\ell_1^\boxtimes$ and $r_1^\boxtimes$ the mates of $\ell^\boxtimes$ and $r^\boxtimes$ given, respectively,

by using $\ell^\otimes$ and $r^\otimes$ in place of their inverses, as shown below.

$$
\begin{array}{ccc}
\mathbb{I}K & \xrightarrow{\ I^\boxtimes 1_K\ } & K^2 \\
{\scriptstyle \ell^\otimes}\swarrow & \Downarrow \ell_1^\boxtimes & \downarrow{\scriptstyle \boxtimes} \\
K & \xrightarrow[\ 1_K\ ]{} & K
\end{array}
\qquad\qquad
\begin{array}{ccc}
K\mathbb{I} & \xrightarrow{\ 1_K I^\boxtimes\ } & K^2 \\
{\scriptstyle r^\otimes}\swarrow & \Downarrow r_1^\boxtimes & \downarrow{\scriptstyle \boxtimes} \\
K & \xrightarrow[\ 1_K\ ]{} & K
\end{array}
$$

$\diamond$

**Definition B.2.20.** Suppose $K$ and $L$ are monoidal $V$-categories with $V$ a braided monoidal category. A *monoidal $V$-functor*

$$(F, F^2, F^0) : K \longrightarrow L$$

is a triple consisting of the following data.

- $F : K \longrightarrow L$ is a $V$-functor.
- $F^2$ and $F^0$ are $V$-natural transformations as follows.

$$
\begin{array}{ccc}
K^2 & \xrightarrow{\ F\otimes F\ } & L^2 \\
{\scriptstyle \boxtimes}\downarrow & F^2\ \Swarrow & \downarrow{\scriptstyle \boxtimes} \\
K & \xrightarrow[\ F\ ]{} & L
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \mathbb{I} & \\
{\scriptstyle I^\boxtimes}\swarrow & F^0\Swarrow & \searrow{\scriptstyle I^\boxtimes} \\
K & \xrightarrow[\ F\ ]{} & L
\end{array}
$$

They are called the *monoidal constraint* and the *unit constraint*, respectively.

These data are required to satisfy the following associativity and unity axioms.

**Associativity:** The composites of the following two *associativity pasting diagrams* are equal.

(B.2.21)



In the right hand diagram in (B.2.21), the unlabeled parallelogram commutes by naturality of $a^\otimes$.

**Left Unity:** The composites of the following two *left unity pasting diagrams* are equal.

(B.2.22)



In the right hand diagram in (B.2.22), the lower unlabeled quadrilateral commutes by naturality of $\ell^\otimes$. The upper unlabeled region commutes by

2-functorality of $\otimes$. The 2-cell isomorphisms labeled $\ell_1^{\boxtimes}$ are each the mate of $\ell^{\boxtimes}$ (Definition B.2.19).

**Right Unity:** The composites of the following two *right unity pasting diagrams* are equal.



(B.2.23)

In the right hand diagram in (B.2.23), the lower unlabeled quadrilateral commutes by naturality of $r^{\otimes}$. The upper unlabeled region commutes by 2-functorality of $\otimes$. The 2-cell isomorphisms labeled $r_1^{\boxtimes}$ are each the mate of $r^{\boxtimes}$ (Definition B.2.19).

This finishes the definition of a monoidal V-functor.

Moreover, we define the following variants.

- A *unital monoidal* V-*functor* is one for which $F^0$ is invertible.
- A *strictly unital monoidal* V-*functor* is one for which $F^0$ is an identity.
- A *strong monoidal* V-*functor* is one for which both $F^0$ and $F^2$ are invertible.
- A *strict monoidal* V-*functor* is one for which both $F^0$ and $F^2$ are identities.

For each variant, composition of composable monoidal V-functors is defined by composing the V-functors, pasting the monoidal constraints, and pasting the unit constraint. ◇

**Definition B.2.24.** Suppose K and L are braided monoidal V-categories with V a symmetric monoidal category. A *braided monoidal* V-functor

$$(F, F^2, F^0) : K \longrightarrow L$$

is a monoidal V-functor that satisfies the following axiom.

**Braid Axiom:** The composites of the following two *braiding pasting diagrams* are equal.



(B.2.25)

In the left hand diagram in (B.2.25), the unlabeled quadrilateral commutes by naturality of $\beta^{\otimes}$. This finishes the definition of a braided monoidal V-functor.

If K and L are symmetric monoidal V-categories, then we say that $F$ is a *symmetric monoidal* V-*functor*. ◇

**Definition B.2.26.** Suppose $F, G : K \longrightarrow L$ are monoidal V-functors between monoidal V-categories with V a braided monoidal category. A *monoidal* V-*natural transformation*

$$\theta : F \longrightarrow G$$

is a V-natural transformation of underlying V-functors (Definition B.1.10) that satisfies the following two additional axioms.

**Monoidal Naturality:** The composites of the following two *monoidal naturality pasting diagrams* are equal.



**Unit Naturality:** The composites of the following two *unit naturality pasting diagrams* are equal.



This finishes the definition of a monoidal V-natural transformation. Identity and composites of monoidal V-natural transformations are defined via underlying V-natural transformations. ◇

**Theorem B.2.27.** *Suppose* $V = (V, \otimes, \xi)$ *is a braided monoidal category. For items* (2) *and* (3) *below, suppose that* V *is a symmetric monoidal category.*

(1) *There exists a 2-category* V-MCat *with small monoidal V-categories as objects, monoidal V-functors as 1-cells, and monoidal V-natural transformations as 2-cells.*

(2) *There exists a 2-category* V-BMCat *with small braided monoidal V-categories as objects, braided monoidal V-functors as 1-cells, and monoidal V-natural transformations as 2-cells.*

(3) *There exists a 2-category* V-SMCat *with small symmetric monoidal V-categories as objects, symmetric monoidal V-functors as 1-cells, and monoidal V-natural transformations as 2-cells.*

*Moreover, there exist forgetful 2-functors*

$$V\text{-SMCat} \longrightarrow V\text{-BMCat} \longrightarrow V\text{-MCat} \longrightarrow V\text{-Cat}.$$

## B.3. Self-Enriched Symmetric Monoidal Categories

In this section we review the self-enriched symmetric monoidal structure of a symmetric monoidal closed category (Definition A.1.19). The material in this section is adapted from [**JY∞**, Chapter 3]. Throughout this section, we assume that

$$\left(V, \otimes, 1, \alpha, \lambda, \rho, \xi, [,]\right)$$

is a symmetric monoidal closed category.

**Definition B.3.1** (Evaluation and Coevaluation). Suppose $x$ is an object in V.

- The *evaluation at $x$* is the counit

(B.3.2) $$[x,-] \otimes x \xrightarrow{\mathrm{ev}_{x,-}} 1_\mathsf{V}.$$

- The *coevaluation at $x$* is the unit

(B.3.3) $$1_\mathsf{V} \xrightarrow{\mathrm{coev}_{x,-}} [x,- \otimes x].$$

These natural transformations refer to the adjunction

$$- \otimes x : \mathsf{V} \rightleftarrows \mathsf{V} : [x,-]$$

that is part of the closed structure of $\mathsf{V}$.                                    ◇

Recall the notion of a $\mathsf{V}$-category in Definition B.1.1.

**Definition B.3.4** (Canonical Self-Enrichment)**.**  We define the data of a $\mathsf{V}$-category $\underline{\mathsf{V}}$, called the *canonical self-enrichment* of $\mathsf{V}$, as follows.

**Objects:** $\mathrm{Ob}(\underline{\mathsf{V}}) = \mathrm{Ob}(\mathsf{V})$.
**Hom Objects:** Each pair of objects $x, y \in \underline{\mathsf{V}}$ is equipped with the hom object

$$\underline{\mathsf{V}}(x,y) = [x,y] \in \mathsf{V}.$$

**Composition:** For objects $x, y, z \in \underline{\mathsf{V}}$, the composition morphism

$$[y,z] \otimes [x,y] \xrightarrow{\mathrm{m}_{x,y,z}} [x,z]$$

is the adjoint of the following composite morphism in $\mathsf{V}$.

(B.3.5)

$$
\begin{array}{ccc}
([y,z] \otimes [x,y]) \otimes x & & z \\
{\scriptstyle \alpha} \downarrow {\scriptstyle \cong} & & \uparrow {\scriptstyle \mathrm{ev}} \\
[y,z] \otimes ([x,y] \otimes x) & \xrightarrow{\ 1 \otimes \mathrm{ev}\ } & [y,z] \otimes y
\end{array}
$$

**Identities:** The identity

$$1 \xrightarrow{\ i_x\ } [x,x] \quad \text{for} \quad x \in \underline{\mathsf{V}}$$

is adjoint to the left unit isomorphism

(B.3.6) $$1 \otimes x \xrightarrow[\cong]{\lambda} x \quad \text{in} \quad \mathsf{V}.$$

This finishes the definition of $\underline{\mathsf{V}}$. If there is no danger of confusion, we abbreviate $\underline{\mathsf{V}}$ to $\mathsf{V}$.                                    ◇

Recall from Definitions B.2.7, B.2.13, and B.2.16 the notion of a symmetric monoidal $\mathsf{V}$-category. The following result combines [**JY∞**, 3.1.11 and 3.3.2].

**Theorem B.3.7.** *Suppose $\mathsf{V}$ is a symmetric monoidal closed category. Then the following statements hold.*

*(1) $\underline{\mathsf{V}}$ in Definition B.3.4 is a $\mathsf{V}$-category.*
*(2) The symmetric monoidal structure on $\mathsf{V}$ extends to $\underline{\mathsf{V}}$ such that $\underline{\mathsf{V}}$ is a symmetric monoidal $\mathsf{V}$-category.*

**Explanation B.3.8** (Canonical Self-Enrichment). In the V-category $\underline{V}$, the uniqueness of adjoints implies that the composition m and the identity $i$ are uniquely characterized by the following two diagrams in V.

(B.3.9)

$$
\begin{array}{ccc}
\left([y,z]\otimes[x,y]\right)\otimes x & \xrightarrow{\;m\otimes 1\;} & [x,z]\otimes x \\
\alpha\downarrow\cong & & \\
[y,z]\otimes\left([x,y]\otimes x\right) & & \Big\downarrow\text{ev} \\
1\otimes\text{ev}\downarrow & & \\
[y,z]\otimes y & \xrightarrow{\quad\text{ev}\quad} & z
\end{array}
\qquad
\begin{array}{ccc}
1\otimes x & \xrightarrow{\;i\otimes 1\;} & [x,x]\otimes x \\
& \lambda\;\searrow\;\cong & \Big\downarrow\text{ev} \\
& & x
\end{array}
$$

We use these diagrams in Proposition 9.1.9. ◇

**Explanation B.3.10** (Symmetric V-Monoidal Structure). In Theorem B.3.7 (2), the symmetric monoidal V-category structure on the V-category $\underline{V}$ is given as follows.

**Monoidal Composition:** The V-functor

$$\boxtimes : \underline{V}\otimes\underline{V}\longrightarrow\underline{V}$$

has object assignment

$$x\boxtimes y = x\otimes y \quad\text{for}\quad x,y\in V.$$

We use the notation $\boxtimes$ to avoid confusion with the monoidal product $\otimes$ of V and the tensor product of V-categories.

For a pair of objects $(x,x'),(y,y')\in\underline{V}\otimes\underline{V}$, the morphism

$$\boxtimes_{(x,x'),(y,y')} : [x,y]\otimes[x',y']\longrightarrow[x\otimes x',y\otimes y']\quad\text{in}\quad V$$

is adjoint to each of the following two equal composites, with $\zeta_{\text{mid}}$ interchanging the middle two factors.

(B.3.11)

$$
\begin{array}{ccc}
\left([x,y]\otimes[x',y']\right)\otimes(x\otimes x') & \xrightarrow{\;\boxtimes\otimes 1\;} & [x\otimes x',y\otimes y']\otimes(x\otimes x') \\
\zeta_{\text{mid}}\downarrow & & \Big\downarrow\text{ev} \\
\left([x,y]\otimes x\right)\otimes\left([x',y']\otimes x'\right) & \xrightarrow{\;\text{ev}\otimes\text{ev}\;} & y\otimes y'
\end{array}
$$

**Monoidal Identity:** The V-functor

$$I : \mathbb{I}\longrightarrow\underline{V}$$

has object assignment

$$I(*) = 1.$$

The morphism between hom objects

$$I : 1\longrightarrow[1,1]\quad\text{in}\quad V$$

is adjoint to each of the following two equal composites.

(B.3.12)

$$
\begin{array}{ccc}
1\otimes 1 & \xrightarrow{\;I\otimes 1\;} & [1,1]\otimes 1 \\
& \lambda\;\searrow\;\cong & \Big\downarrow\text{ev} \\
& & 1
\end{array}
$$

**Other Structure:** The monoidal associator, monoidal unitors, and braiding have
component morphisms

(B.3.13)
$$\begin{cases} \alpha^{\perp}_{x,y,z} : 1 \longrightarrow \big[ (x \boxtimes y) \boxtimes z, x \boxtimes (y \boxtimes z) \big] \\ \lambda^{\perp}_x : 1 \longrightarrow [1 \boxtimes x, x] \\ \rho^{\perp}_x : 1 \longrightarrow [x \boxtimes 1, x] \\ \xi^{\perp}_{x,y} : 1 \longrightarrow [x \boxtimes y, y \boxtimes x] \end{cases}$$

adjoint to the composites of $\lambda$ with the corresponding components of $\alpha$,
$\lambda$, $\rho$, and $\xi$, respectively, in $\mathsf{V}$. In each case, the adjoint component is
equal to the composite $\mathrm{ev} \circ (?^{\perp} \otimes 1)$, similar to the top-right composites in
(B.3.11) and (B.3.12).

This finishes the description of the symmetric monoidal $\mathsf{V}$-category $\underline{\mathsf{V}}$.                    ⋄

## B.4. Change of Enrichment

Recall from Example B.1.12 that each monoidal category $\mathsf{V}$ has an associated
2-category $\mathsf{V}$-Cat of small $\mathsf{V}$-categories, $\mathsf{V}$-functors, and $\mathsf{V}$-natural transformations.
In this section we review properties of changing the enriching monoidal category
$\mathsf{V}$. Whenever we need the enriching monoidal category to be braided or symmetric
(Definitions A.1.10 and A.1.14), we state so explicitly. The material in this section
is adapted from [**JY∞**, Chapters 2 and 3].

**Definition B.4.1.** Suppose given a monoidal functor between monoidal categories
$$(U, U^2, U^0) : (\mathsf{V}, \otimes, 1) \longrightarrow (\mathsf{W}, \otimes, 1).$$
We define the data of a 2-functor
$$(-)_U : \mathsf{V}\text{-Cat} \longrightarrow \mathsf{W}\text{-Cat},$$
called the *change of enrichment*, as follows.

**Object Assignment:** For a $\mathsf{V}$-category $(\mathsf{C}, \mathsf{m}, i)$, the $\mathsf{W}$-category
$$\big( \mathsf{C}_U, \mathsf{m}_U, i_U \big)$$
has objects
$$\mathrm{Ob}(\mathsf{C}_U) = \mathrm{Ob}(\mathsf{C})$$
and hom objects
$$\mathsf{C}_U(x,y) = U\mathsf{C}(x,y) \in \mathsf{W} \quad \text{for} \quad x,y \in \mathsf{C}_U.$$
The composition in $\mathsf{C}_U$ is the following composite in $\mathsf{W}$ for $x,y,z \in \mathsf{C}_U$.

(B.4.2)
$$\begin{array}{ccc} U\mathsf{C}(y,z) \otimes U\mathsf{C}(x,y) & \xrightarrow{\ \mathsf{m}_U\ } & U\mathsf{C}(x,z) \\ {\scriptstyle U^2}\big\downarrow & \nearrow_{\scriptstyle U\mathsf{m}} & \\ U\big(\mathsf{C}(y,z) \otimes \mathsf{C}(x,y)\big) & & \end{array}$$

The identity of an object $x \in \mathsf{C}_U$ is the following composite.

(B.4.3)
$$1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ Ui_x\ } U\mathsf{C}(x,x)$$

with the top arc labeled $(i_U)_x$.

**1-Cell Assignment:** For a V-functor $F : C \longrightarrow D$, the W-functor

$$F_U : C_U \longrightarrow D_U$$

has the same object assignment as $F$. On hom objects it is the morphism

(B.4.4)        $(F_U)_{x,y} = U(F_{x,y}) : UC(x, y) \longrightarrow UD(Fx, Fy)$   in   W

for $x, y \in C_U$.

**2-Cell Assignment:** For a V-natural transformation $\theta$ as in the left diagram below



the W-natural transformation $\theta_U$, as in the right diagram above, has component morphism at $x \in C_U$ given by the following composite.

(B.4.5)

$$1 \xrightarrow{\; U^0 \;} U1 \xrightarrow{\; U\theta_x \;} UD(Fx, Gx)$$

with $(\theta_U)_x$ denoting the whole composite.

This finishes the definition of $(-)_U$.                                          ◇

The following is [**JY∞**, 2.1.2].

**Proposition B.4.6.** *In the context of Definition B.4.1,*

$$(-)_U : V\text{-Cat} \longrightarrow W\text{-Cat}$$

*is a 2-functor.*

Change of enrichment is compatible with composition of monoidal functors (Definition A.1.26), as in the following result from [**JY∞**, 2.2.4].

**Proposition B.4.7.** *Given monoidal functors between monoidal categories*

$$V_1 \xrightarrow{\; U_1 \;} V_2 \xrightarrow{\; U_2 \;} V_3,$$

*the following diagram of change-of-enrichment 2-functors commutes.*

$$V_1\text{-Cat} \xrightarrow{\; (-)_{U_1} \;} V_2\text{-Cat} \xrightarrow{\; (-)_{U_2} \;} V_3\text{-Cat}$$

with the top composite $(-)_{U_2 U_1}$.

**Compatibility with Enriched Tensor Product.** For a braided monoidal category $V$, $(V\text{-Cat}, \otimes)$ is a monoidal Cat-category, which is, furthermore, symmetric if $V$ is symmetric (Theorem B.2.18). Change of enrichment is compatible with the tensor product of enriched categories (Definition B.2.1), using the following definitions.

**Definition B.4.8.** Suppose given a braided monoidal functor between braided monoidal categories

$$(U, U^2, U^0) : V \longrightarrow W.$$

We define monoidal constraint $(-)_U^2$ and unit constraint $(-)_U^0$ for the change-of-enrichment 2-functor $(-)_U$ as follows.

**Monoidal Constraint:** Its component W-functor at small V-categories C and D,

$$(-)_U^2 : C_U \otimes D_U \longrightarrow (C \otimes D)_U,$$

has the identity object assignment. On hom objects it is given by the following morphism in W for $x, x' \in C$ and $y, y' \in D$.

$$
\begin{array}{ccc}
(C_U \otimes D_U)(x \otimes y, x' \otimes y') & & (C \otimes D)_U(x \otimes y, x' \otimes y') \\
\| & & \| \\
UC(x,x') \otimes UD(y,y') & \xrightarrow{\quad U^2 \quad} & U\big(C(x,x') \otimes D(y,y')\big)
\end{array}
$$

**Unit Constraint:** It is the W-functor

$$(-)_U^0 : \mathbb{I} \longrightarrow \mathbb{I}_U$$

given by the identity on the unique object and the morphism

$$1 \xrightarrow{\quad U^0 \quad} U1 \quad \text{in} \quad W$$

on the unique hom object.

This finishes the definition of $(-)_U^2$ and $(-)_U^0$.                                    ◇

The following is [**JY∞**, 2.3.7], which uses Theorem B.2.18 and Definitions B.2.20 and B.2.24.

**Theorem B.4.9.** *For each braided monoidal functor between braided monoidal categories*

$$(U, U^2, U^0) : V \longrightarrow W,$$

*the triple in Definitions B.4.1 and B.4.8*

$$\big((-)_U, (-)_U^2, (-)_U^0\big) : (V\text{-Cat}, \otimes) \longrightarrow (W\text{-Cat}, \otimes)$$

*is a monoidal* Cat*-functor. Moreover, if U is a symmetric monoidal functor between symmetric monoidal categories, then* $(-)_U$ *is a symmetric monoidal* Cat*-functor.*

**Compatibility with Enriched Monoidal Structure.** Change of enrichment preserves enriched monoidal structure, as in the following result from [**JY∞**, 2.4.10].

**Theorem B.4.10.** *Suppose given a braided monoidal functor between braided monoidal categories*

$$U : V \longrightarrow W.$$

*For* (1) *and* (2) *below, the braided and symmetric monoidal cases assume that U, V, and W are symmetric monoidal.*

(1) *If* K *is a (braided, respectively symmetric) monoidal* V*-category, then* $K_U$ *is a (braided, respectively symmetric) monoidal* W*-category.*

(2) *If* F : K ⟶ L *is a (braided, respectively symmetric) monoidal* V*-functor between (braided, respectively symmetric) monoidal* V*-categories, then*

$$F_U : K_U \longrightarrow L_U$$

*is a (braided, respectively symmetric) monoidal* W*-functor.*

(3) *If* θ : F ⟶ G *is a monoidal* V*-natural transformation between monoidal* V*-functors F and G, then*

$$\theta_U : F_U \longrightarrow G_U$$

*is a monoidal* W*-natural transformation.*

Theorem B.3.7 (2) and Theorem B.4.10 (1) yield the following.

**Corollary B.4.11.** *Suppose given a symmetric monoidal functor between symmetric monoidal categories*

$$(U, U^2, U^0) : \mathsf{V} \longrightarrow \mathsf{W}$$

*with* $(\mathsf{V}, [,])$ *closed. Then* $\underline{\mathsf{V}}_U$ *is a symmetric monoidal* $\mathsf{W}$*-category.*

**Explanation B.4.12.** In Corollary B.4.11 the symmetric monoidal $\mathsf{W}$-category $\underline{\mathsf{V}}_U$ is given explicitly as follows.

**Underlying $\mathsf{W}$-Category:** The $\mathsf{W}$-category

$$\big(\underline{\mathsf{V}}_U, \mathsf{m}_U, i_U\big)$$

is obtained from the $\mathsf{V}$-category $(\underline{\mathsf{V}}, \mathsf{m}, i)$ in Definition B.3.4 by applying the change-of-enrichment $(-)_U$ in Definition B.4.1. In other words, it has objects

$$\mathrm{Ob}(\underline{\mathsf{V}}_U) = \mathrm{Ob}(\underline{\mathsf{V}}) = \mathrm{Ob}(\mathsf{V})$$

and hom objects

$$\underline{\mathsf{V}}_U(x, y) = U\underline{\mathsf{V}}(x, y) = U[x, y] \in \mathsf{W} \quad \text{for} \quad x, y \in \mathsf{V}.$$

The composition is the following composite, with $\mathsf{m}$ the composition in $\underline{\mathsf{V}}$ in (B.3.9).

$$
\begin{array}{ccc}
U[y,z] \otimes U[x,y] & \xrightarrow{\ \mathsf{m}_U\ } & U[x,z] \\
{\scriptstyle U^2}\big\downarrow & \nearrow {\scriptstyle U\mathsf{m}} & \\
U\big([y,z] \otimes [x,y]\big) & &
\end{array}
$$

The identity of an object $x \in \underline{\mathsf{V}}_U$ is the following composite, with $i$ the identity in $\underline{\mathsf{V}}$ in (B.3.9).

$$
\begin{array}{ccc}
& \overbrace{\qquad\qquad (i_U)_x \qquad\qquad} & \\
1 \xrightarrow{\ U^0\ } & U1 \xrightarrow{\ Ui_x\ } & U[x,x]
\end{array}
$$

**Monoidal Composition:** The $\mathsf{W}$-functor

$$\underline{\mathsf{V}}_U \otimes \underline{\mathsf{V}}_U \xrightarrow{\ \boxtimes_U\ } \underline{\mathsf{V}}_U$$

is given on objects by

$$x \boxtimes_U y = x \otimes y \quad \text{for} \quad x, y \in \mathsf{V}.$$

On hom objects it is the composite

$$
\begin{array}{ccc}
U[x,y] \otimes U[x',y'] & \xrightarrow{\ \boxtimes_U\ } & U\big[x \otimes x', y \otimes y'\big] \\
{\scriptstyle U^2}\big\downarrow & \nearrow {\scriptstyle U\boxtimes} & \\
U\big([x,y] \otimes [x',y']\big) & &
\end{array}
$$

for $x \otimes x', y \otimes y' \in \underline{\mathsf{V}}_U \otimes \underline{\mathsf{V}}_U$, with $\boxtimes$ as in (B.3.11).

**Monoidal Unit:** The W-functor

$$I_U : \mathbb{I} \longrightarrow \underline{V}_U$$

has object assignment

$$I_U(*) = 1.$$

On hom objects it is the following composite, with I as in (B.3.12).

$$
\overbrace{1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ U(\mathrm{I})\ } U[1,1]}^{I_U}
$$

**Other Structure:** The monoidal associator, monoidal unitors, and braiding have component morphisms as follows, with $\alpha^\perp$, $\lambda^\perp$, $\rho^\perp$, and $\xi^\perp$ as in (B.3.13).

$$
\begin{cases}
1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ U\alpha^\perp_{x,y,z}\ } U\big[(x \boxtimes y) \boxtimes z, x \boxtimes (y \boxtimes z)\big] \\[6pt]
1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ U\lambda^\perp_x\ } U[1 \boxtimes x, x] \\[6pt]
1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ U\rho^\perp_x\ } U[x \boxtimes 1, x] \\[6pt]
1 \xrightarrow{\ U^0\ } U1 \xrightarrow{\ U\xi^\perp_{x,y}\ } U[x \boxtimes y, y \boxtimes x]
\end{cases}
$$

This finishes the description of the symmetric monoidal W-category $\underline{V}_U$.            ◇

**Standard Enrichment.** The following definition uses the symmetric monoidal W-category $\underline{V}_U$ in Explanation B.4.12.

**Definition B.4.13.** Suppose given a monoidal functor

$$(U, U^2, U^0) : (V, \otimes, 1, [,]) \longrightarrow (W, \otimes, 1, [,])$$

between symmetric monoidal closed categories. We define the data of a monoidal W-functor

$$\big(\widehat{U}, \widehat{U}^2, \widehat{U}^0\big) : \underline{V}_U \longrightarrow \underline{W},$$

called the *standard enrichment* of $U$, as follows.

**Object Assignment:** The object assignment of $\widehat{U}$ is the same as that of $U$.
**Component Morphisms:** For objects $x, y \in V$, the component morphism

$$\widehat{U}_{x,y} : U[x,y] \longrightarrow [Ux, Uy] \quad \text{in} \quad W$$

is adjoint to the composite

(B.4.14)        $$U[x,y] \otimes Ux \xrightarrow{\ U^2\ } U\big([x,y] \otimes x\big) \xrightarrow{\ U(\mathrm{ev})\ } Uy.$$

**Unit Constraint:** The morphism

$$\widehat{U}^0 : 1 \longrightarrow [1, U1] \quad \text{in} \quad W$$

is adjoint to the composite

(B.4.15)                              $$1 \otimes 1 \xrightarrow[\cong]{\ \lambda\ } 1 \xrightarrow{\ U^0\ } U1.$$

**Monoidal Constraint:** Its component morphism

$$\widehat{U}^2_{x \otimes x'} : 1 \longrightarrow \left[ Ux \otimes Ux', U(x \otimes x') \right] \quad \text{for} \quad x \otimes x' \in \mathsf{V} \otimes \mathsf{V}$$

is adjoint to the composite

(B.4.16) $$1 \otimes (Ux \otimes Ux') \xrightarrow[\cong]{\lambda} Ux \otimes Ux' \xrightarrow{U^2_{x,x'}} U(x \otimes x').$$

This finishes the definition of the standard enrichment of $U$. ◇

The following is [**JY**∞, 3.3.4].

**Proposition B.4.17.** *In the context of Definition B.4.13, the standard enrichment*

$$\left( \widehat{U}, \widehat{U}^2, \widehat{U}^0 \right) : \underline{\mathsf{V}}_U \longrightarrow \underline{\mathsf{W}}$$

*is a monoidal* W-*functor. Moreover, if $U$ is a symmetric monoidal functor, then $\widehat{U}$ is a symmetric monoidal* W-*functor.*

**Explanation B.4.18.** Each of $\widehat{U}_{x,y}$, $\widehat{U}^0$, and $\widehat{U}^2_{x \otimes x'}$ is defined by its adjoint in, respectively, (B.4.14) through (B.4.16). Thus the uniqueness of adjoints implies that these structure morphisms are uniquely characterized by the following commutative diagrams.

$$
\begin{array}{ccc}
U[x,y] \otimes Ux & \xrightarrow{\widehat{U}_{x,y} \otimes 1} & [Ux, Uy] \otimes Ux \\
{\scriptstyle U^2} \downarrow & & \downarrow {\scriptstyle \mathrm{ev}} \\
U([x,y] \otimes x) & \xrightarrow{U(\mathrm{ev})} & Uy
\end{array}
\qquad
\begin{array}{ccc}
1 \otimes 1 & \xrightarrow{\widehat{U}^0 \otimes 1} & [1, U1] \otimes 1 \\
{\scriptstyle \lambda} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \mathrm{ev}} \\
1 & \xrightarrow{U^0} & U1
\end{array}
$$

$$
\begin{array}{ccc}
1 \otimes (Ux \otimes Ux') & \xrightarrow{\widehat{U}^2_{x \otimes x'} \otimes 1} & [Ux \otimes Ux', U(x \otimes x')] \otimes (Ux \otimes Ux') \\
{\scriptstyle \lambda} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \mathrm{ev}} \\
Ux \otimes Ux' & \xrightarrow{U^2_{x,x'}} & U(x \otimes x')
\end{array}
$$

We use this adjoint characterization of $\widehat{U}_{x,y}$ in Proposition 9.2.17. ◇

# Multicategories

In this appendix we review basic elements of multicategory theory. The following table summarizes the main content in this appendix.

| C.1. Enriched Multicategories to C.3. Endomorphism Multicategories | |
|---|---|
| V-multicategories (multifunctors, multinatural transformations) | C.1.3 (C.1.19, C.1.25) |
| underlying V-category | C.1.16 |
| 2-category V-Multicat of V-multicategories | C.1.33 |
| Cat-multinatural transformations | C.2.2 |
| endomorphism multicategories End (enriched) | C.3.1 (C.3.8) |
| C.4. Pointed Multicategories | |
| pointed multicategories, multifunctors, and multinatural transformations | C.4.1 |
| pointed endomorphism multicategories End. | C.4.8 |
| 2-category Multicat$_*$ of pointed multicategories | C.4.9 |
| free-forgetful 2-adjunction $(-)_+ : \mathsf{Multicat} \rightleftarrows \mathsf{Multicat}_* : \mathsf{U}.$ | C.4.16 |

The main reference for this chapter is [**JY∞**]; see also [**Yau16**]. Conventions A.1.2 and A.1.30 are still in effect.

## C.1. Enriched Multicategories

In this section we review enriched multicategories, multifunctors, and multinatural transformations. The material in this section is adapted from [**JY∞**, Section 6.1]. Throughout this section we assume that

$$\left(\mathsf{V}, \otimes, 1, \alpha, \lambda, \rho, \xi\right)$$

is a symmetric monoidal category (Definition A.1.14). We use the following notation for finite tuples of objects.

**Definition C.1.1.** Suppose $S$ is a class.

**Profiles:** The class of finite tuples in $S$ is denoted by

$$\mathsf{Prof}(S) = \coprod_{k \geq 0} S^k.$$

- An element in $\mathsf{Prof}(S)$ is called an *S-profile*.
- An $S$-profile of length $n = \mathrm{len}\langle x \rangle$ is denoted by

$$\langle x \rangle = (x_1, \ldots, x_n) = \langle x_j \rangle_{j=1}^n \in S^n.$$

  The empty $S$-profile is denoted by $\langle \rangle$.
- An element in $\mathsf{Prof}(S) \times S$ is denoted by $(\langle x \rangle ; y)$ with $\langle x \rangle \in \mathsf{Prof}(S)$ and $y \in S$.

**Concatenation:** For two $S$-profiles $\langle x \rangle = \langle x_i \rangle_{i=1}^m$ and $\langle y \rangle = \langle y_j \rangle_{j=1}^n$, their *concatenation* is the $S$-profile

(C.1.2)                         $$\langle x \rangle \oplus \langle y \rangle = \left( x_1, \ldots, x_m, y_1, \ldots, y_n \right).$$

Concatenation is associative with the empty tuple $\langle \rangle$ as the strict unit.   ⋄

The symmetric group on $n$ letters is denoted $\Sigma_n$.

**Definition C.1.3.** A $V$-*multicategory* is a triple

$$(M, \gamma, 1)$$

consisting of the following data.

**Objects:** It is equipped with a class $\mathrm{Ob}\,M$, whose elements are called *objects*. We abbreviate $\mathrm{Prof}(\mathrm{Ob}\,M)$ to $\mathrm{Prof}(M)$.

**Multimorphisms:** For $(\langle x \rangle ; x') \in \mathrm{Prof}(M) \times \mathrm{Ob}\,M$ with $\langle x \rangle = \langle x_j \rangle_{j=1}^n$, it is equipped with an object in $V$

$$M\left( \langle x \rangle ; x' \right) = M\left( x_1, \ldots, x_n ; x' \right).$$

It is called the *n-ary operation object* or *n-ary multimorphism object* with *input profile* $\langle x \rangle$ and *output* $x'$.
  - We also say *nullary* for 0-ary, *unary* for 1-ary, and *binary* for 2-ary.
  - If objects in $V$ have underlying sets (for example, if $V$ is Set or Cat), then an element in $M\left( \langle x \rangle ; x' \right)$ is called an *n-ary multimorphism* or *n-ary operation* and denoted

$$\langle x \rangle = (x_1, \ldots, x_n) \longrightarrow x'.$$

**Symmetric Group Action:** For $(\langle x \rangle ; x') \in \mathrm{Prof}(M) \times \mathrm{Ob}\,M$ and a permutation $\sigma \in \Sigma_n$, $M$ is equipped with an isomorphism in $V$

(C.1.4)                         $$M\left( \langle x \rangle ; x' \right) \xrightarrow[\cong]{\ \sigma\ } M\left( \langle x \rangle \sigma ; x' \right),$$

called the *right $\sigma$-action* or the *symmetric group action*, where

$$\langle x \rangle \sigma = \left( x_{\sigma(1)}, \ldots, x_{\sigma(n)} \right) = \langle x_{\sigma(j)} \rangle_{j=1}^n \in \mathrm{Prof}(M)$$

is the right permutation of $\langle x \rangle$ by $\sigma$.

**Units:** Each object $x$ in $M$ is equipped with a morphism

(C.1.5)                         $$1 \xrightarrow{\ 1_x\ } M\left( x ; x \right),$$

called the *x-colored unit*.

**Composition:** Suppose given
  - $\left( \langle x' \rangle ; x'' \right) \in \mathrm{Prof}(M) \times \mathrm{Ob}\,M$ with $\langle x' \rangle = \langle x'_j \rangle_{j=1}^n \in \mathrm{Prof}(M)$ and
  - $\langle x_j \rangle = \langle x_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Prof}(M)$ for each $j \in \{1, \ldots, n\}$ with $\langle x \rangle = \oplus_{j=1}^n \langle x_j \rangle$.

Then $M$ is equipped with a morphism in $V$

(C.1.6)        $$M\left( \langle x' \rangle ; x'' \right) \otimes \bigotimes_{j=1}^n M\left( \langle x_j \rangle ; x'_j \right) \xrightarrow{\ \gamma\ } M\left( \langle x \rangle ; x'' \right),$$

called the *composition* or *multicategorical composition*. If objects in $V$ have underlying sets, then we also denote composition diagrammatically by

$$\left( \langle x_1 \rangle, \ldots, \langle x_n \rangle \right) \xrightarrow{\ (f_1, \ldots, f_n)\ } \langle x' \rangle \xrightarrow{\ f\ } x''$$

for multimorphisms

$$f \in \mathsf{M}(\langle x' \rangle ; x'') \quad \text{and} \quad f_j \in \mathsf{M}(\langle x_j \rangle ; x'_j).$$

The data above are required to satisfy the following axioms.

**Symmetric Group Action:** The identity in $\Sigma_n$ acts as the identity morphism on $\mathsf{M}(\langle x \rangle ; x')$ with $n = \mathrm{len}\langle x \rangle$. Moreover, for $\sigma, \tau \in \Sigma_n$, the following diagram in $\mathsf{V}$ commutes.

(C.1.7)
$$
\begin{array}{ccc}
\mathsf{M}(\langle x \rangle ; x') & \xrightarrow{\ \sigma\ } & \mathsf{M}(\langle x \rangle \sigma ; x') \\
& \searrow{\scriptstyle \sigma\tau} & \ \downarrow{\scriptstyle \tau} \\
& & \mathsf{M}(\langle x \rangle \sigma\tau ; x')
\end{array}
$$

**Associativity:** Suppose given
- $(\langle x'' \rangle ; x''') \in \mathrm{Prof}(\mathsf{M}) \times \mathrm{Ob}\,\mathsf{M}$ with $\langle x'' \rangle = \langle x''_j \rangle_{j=1}^n \in \mathrm{Prof}(\mathsf{M})$,
- $\langle x'_j \rangle = \langle x'_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Prof}(\mathsf{M})$ for each $j \in \{1,\dots,n\}$ with $\langle x' \rangle = \oplus_{j=1}^n \langle x'_j \rangle$ and $k_j > 0$ for at least one $j$, and
- $\langle x_{j,i} \rangle = \langle x_{j,i,p} \rangle_{p=1}^{\ell_{j,i}} \in \mathrm{Prof}(\mathsf{M})$ for each $j \in \{1,\dots,n\}$ and each $i \in \{1,\dots,k_j\}$ with $\langle x_j \rangle = \oplus_{i=1}^{k_j} \langle x_{j,i} \rangle$ and $\langle x \rangle = \oplus_{j=1}^n \langle x_j \rangle$.

Then the *associativity diagram* below commutes.

(C.1.8)


**Unity:** Suppose $(\langle x \rangle ; x') \in \mathrm{Prof}(\mathsf{M}) \times \mathrm{Ob}\,\mathsf{M}$ with $\langle x \rangle = \langle x_j \rangle_{j=1}^n \in \mathrm{Prof}(\mathsf{M})$.

(1) If $n \geq 1$, then the following *right unity diagram* commutes.

(C.1.9)
$$
\begin{array}{ccc}
\mathsf{M}(\langle x \rangle ; x') \otimes 1^{\otimes n} & & \\
{\scriptstyle 1 \otimes (\otimes_j 1_{x_j})}\big\downarrow & \searrow{\scriptstyle \rho} & \\
\mathsf{M}(\langle x \rangle ; x') \otimes \otimes_{j=1}^n \mathsf{M}(x_j ; x_j) & \xrightarrow{\ \gamma\ } & \mathsf{M}(\langle x \rangle ; x')
\end{array}
$$

(2) The *left unity diagram* below commutes.

(C.1.10)

$$
\begin{array}{ccc}
1 \otimes \mathsf{M}(\langle x \rangle; x') & \xrightarrow{\quad\lambda\quad} & \\
{\scriptstyle 1_{x'} \otimes 1} \downarrow & & \\
\mathsf{M}(x'; x') \otimes \mathsf{M}(\langle x \rangle; x') & \xrightarrow{\;\gamma\;} & \mathsf{M}(\langle x \rangle; x')
\end{array}
$$

**Equivariance:** Suppose $\mathrm{len}\langle x_j \rangle = k_j \geq 0$ in the definition of $\gamma$ (C.1.6).

(1) For each $\sigma \in \Sigma_n$, the following *top equivariance diagram* commutes.

(C.1.11)

$$
\begin{array}{ccc}
\mathsf{M}(\langle x' \rangle; x'') \otimes \otimes_{j=1}^{n} \mathsf{M}(\langle x_j \rangle; x'_j) & \xrightarrow{(\sigma, \sigma^{-1})} & \mathsf{M}(\langle x' \rangle \sigma; x'') \otimes \otimes_{j=1}^{n} \mathsf{M}(\langle x_{\sigma(j)} \rangle; x'_{\sigma(j)}) \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mathsf{M}(\langle x_1 \rangle, \ldots, \langle x_n \rangle; x'') & \xrightarrow{\sigma \langle k_{\sigma(1)}, \ldots, k_{\sigma(n)} \rangle} & \mathsf{M}(\langle x_{\sigma(1)} \rangle, \ldots, \langle x_{\sigma(n)} \rangle; x'')
\end{array}
$$

In (C.1.11) the block permutation

(C.1.12)

$$
\sigma \langle k_{\sigma(1)}, \ldots, k_{\sigma(n)} \rangle \in \Sigma_{k_1 + \cdots + k_n}
$$

permutes $n$ consecutive intervals of lengths $k_{\sigma(1)}$, ..., $k_{\sigma(n)}$, respectively, as $\sigma$ permutes $\{1, \ldots, n\}$, without changing the order within each interval.

(2) Given permutations $\tau_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$, the following *bottom equivariance diagram* commutes.

(C.1.13)

$$
\begin{array}{ccc}
\mathsf{M}(\langle x' \rangle; x'') \otimes \otimes_{j=1}^{n} \mathsf{M}(\langle x_j \rangle; x'_j) & \xrightarrow{(1, \otimes_j \tau_j)} & \mathsf{M}(\langle x' \rangle; x'') \otimes \otimes_{j=1}^{n} \mathsf{M}(\langle x_j \rangle \tau_j; x'_j) \\
{\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\
\mathsf{M}(\langle x_1 \rangle, \ldots, \langle x_n \rangle; x'') & \xrightarrow{\tau_1 \times \cdots \times \tau_n} & \mathsf{M}(\langle x_1 \rangle \tau_1, \ldots, \langle x_n \rangle \tau_n; x'')
\end{array}
$$

In (C.1.13) the block sum

(C.1.14)

$$
\tau_1 \times \cdots \times \tau_n \in \Sigma_{k_1 + \cdots + k_n}
$$

is the image of $(\tau_1, \ldots, \tau_n)$ under the canonical inclusion

$$
\Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \longrightarrow \Sigma_{k_1 + \cdots + k_n}.
$$

This finishes the definition of a V-multicategory. A V-multicategory is *small* if it has a set of objects.

Moreover, we define the following variants.

- A *non-symmetric* V-*multicategory* is defined in the same way as a V-multicategory by omitting the symmetric group action and the axioms (C.1.7), (C.1.11), and (C.1.13) involving the symmetric group action.
- A *(non-symmetric) multicategory* is a (non-symmetric) Set-multicategory, where $(\mathsf{Set}, \times, *)$ is the symmetric monoidal category of sets and functions with the Cartesian product as the monoidal product.
- A V-*operad* is a V-multicategory with one object. If M is a V-operad, then its $n$-ary multimorphism object is denoted by $\mathsf{M}_n \in \mathsf{V}$.
- An *operad* is a Set-operad, that is, a multicategory with one object.          ⋄

**Remark C.1.15** (Related Concepts)**.**

(1) In the literature, including [**Lam69**] where this concept originated, a *multicategory* sometimes means a non-symmetric multicategory in the sense of Definition C.1.3. Our convention is to include the symmetric group action by default. We always include the word *non-symmetric* if we are referring to the variant without the symmetric group action.
(2) There are more conceptual ways to define enriched multicategories as (i) monoids in a certain monoidal category and (ii) algebras over some monad. See [**Yau20a**, Ch. 4].
(3) There are variants of enriched multicategories whose equivariant structure is parametrized by groups different from the symmetric groups, such as the braid groups. See [**Yau22**].
(4) There are many different but related generalizations of enriched multicategories whose operation objects have input and output profiles of arbitrary lengths. See [**JY21**, Section 2.5] for one such variant called *polycategories* and [**YJ15**] for other variants. ⋄

**Example C.1.16** (Underlying V-Categories)**.** Each non-symmetric V-multicategory $(\mathsf{M}, \gamma, 1)$ has an underlying V-category (Definition B.1.1) defined as follows.

- It has the same class of objects as $\mathsf{M}$.
- For objects $x, y \in \mathsf{M}$, the hom object is $\mathsf{M}(x ; y)$.
- The identities are the colored units in $\mathsf{M}$.
- The composition is given by

$$\mathsf{M}(y ; z) \otimes \mathsf{M}(x ; y) \xrightarrow{\ \gamma\ } \mathsf{M}(x ; z)$$

for objects $x, y, z \in \mathsf{M}$.

The associativity and unity diagrams, (B.1.4) and (B.1.5), of a V-category are the 1-ary restrictions of, respectively, the associativity and unity diagrams, (C.1.8) through (C.1.10), of a V-multicategory.

Suppose V is the symmetric monoidal category $(\mathsf{Cat}, \times, \mathbf{1})$ of small categories and functors with the Cartesian product. Then each non-symmetric Cat-multicategory has an underlying 2-category by Proposition B.1.7. ⋄

**Example C.1.17.** With $\mathsf{V} = (\mathsf{Set}, \times, *)$, the *terminal multicategory* $\mathsf{T}$ consists of a single object $*$ and a single $n$-ary operation $\iota_n$ for each $n \geq 0$. ⋄

**Example C.1.18** (Endomorphism Operad)**.** For each V-multicategory $\mathsf{M}$ and $x \in \mathsf{Ob}\,\mathsf{M}$, the *endomorphism* V-*operad* $\mathsf{End}(x)$ consists of the single object $x$ and $n$-ary multimorphism object

$$\mathsf{End}(x)_n = \mathsf{M}(\langle x \rangle ; x) \in \mathsf{V},$$

with $\langle x \rangle$ the $n$-tuple of copies of $x$. Its multicategory structure is inherited from $\mathsf{M}$. ⋄

**The 2-Category of Enriched Multicategories.**

**Definition C.1.19.** Suppose $\mathsf{M}$ and $\mathsf{N}$ are V-multicategories. A V-*multifunctor*

$$F : \mathsf{M} \longrightarrow \mathsf{N}$$

consists of the following data.

**Object Assignment:** It is equipped with a function

$$F : \mathsf{Ob}\,\mathsf{M} \longrightarrow \mathsf{Ob}\,\mathsf{N}.$$

**Component Morphisms:** For $(\langle x \rangle; y) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ with $\langle x \rangle = \langle x_j \rangle_{j=1}^n$, it is
equipped with a morphism

$$\mathsf{M}(\langle x \rangle; y) \xrightarrow{\ F\ } \mathsf{N}(F\langle x \rangle; Fy) \quad \text{in} \quad \mathsf{V}$$

where $F\langle x \rangle = \langle Fx_j \rangle_{j=1}^n$.

The data above are required to satisfy the following axioms.

**Symmetric Group Action:** For $(\langle x \rangle; y) \in \mathsf{Prof}(\mathsf{M}) \times \mathsf{Ob}\,\mathsf{M}$ and $\sigma \in \Sigma_n$, the follow-
ing diagram in $\mathsf{V}$ commutes.

(C.1.20)
$$\begin{array}{ccc}
\mathsf{M}(\langle x \rangle; y) & \xrightarrow{\ F\ } & \mathsf{N}(F\langle x \rangle; Fy) \\
{\scriptstyle\sigma}\big\downarrow{\scriptstyle\cong} & & {\scriptstyle\sigma}\big\downarrow{\scriptstyle\cong} \\
\mathsf{M}(\langle x \rangle\sigma; y) & \xrightarrow{\ F\ } & \mathsf{N}(F\langle x \rangle\sigma; y)
\end{array}$$

**Units:** For each $x \in \mathsf{Ob}\,\mathsf{M}$, the following diagram in $\mathsf{V}$ commutes.

(C.1.21)
$$\begin{array}{ccc}
 & \xrightarrow{\ 1_x\ } & \mathsf{M}(x; x) \\
1 & & \big\downarrow{\scriptstyle F} \\
 & \xrightarrow[\ 1_{Fx}\ ]{} & \mathsf{N}(Fx; Fx)
\end{array}$$

**Composition:** For $x''$, $\langle x' \rangle$, and $\langle x \rangle = \oplus_{j=1}^n \langle x_j \rangle$ as in (C.1.6), the following diagram
in $\mathsf{V}$ commutes.

(C.1.22)
$$\begin{array}{ccc}
\mathsf{M}(\langle x' \rangle; x'') \otimes \bigotimes_{j=1}^n \mathsf{M}(\langle x_j \rangle; x_j') & \xrightarrow{\ (F, \otimes_j F)\ } & \mathsf{N}(F\langle x' \rangle; Fx'') \otimes \bigotimes_{j=1}^n \mathsf{N}(F\langle x_j \rangle; Fx_j') \\
{\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\
\mathsf{M}(\langle x \rangle; x'') & \xrightarrow[\ F\ ]{} & \mathsf{N}(F\langle x \rangle; Fx'')
\end{array}$$

This finishes the definition of a $\mathsf{V}$-multifunctor.

Moreover, we define the following.

- For a $\mathsf{V}$-multifunctor $G : \mathsf{N} \longrightarrow \mathsf{P}$, the *composition*

$$GF : \mathsf{M} \longrightarrow \mathsf{P}$$

  is the $\mathsf{V}$-multifunctor with object assignment given by the composite
  function

$$\mathsf{Ob}\,\mathsf{M} \xrightarrow{\ F\ } \mathsf{Ob}\,\mathsf{N} \xrightarrow{\ G\ } \mathsf{Ob}\,\mathsf{P}$$

  and component morphisms given by the composites

(C.1.23)     $$\mathsf{M}(\langle x \rangle; y) \xrightarrow{\ F\ } \mathsf{N}(F\langle x \rangle; Fy) \xrightarrow{\ G\ } \mathsf{P}(GF\langle x \rangle; GFy).$$

- The *identity $\mathsf{V}$-multifunctor* $1_\mathsf{M} : \mathsf{M} \longrightarrow \mathsf{M}$ has the identity object assign-
  ment and identity component morphisms.
- A $\mathsf{V}$-*operad morphism* is a $\mathsf{V}$-multifunctor between $\mathsf{V}$-multicategories with
  one object.
- A *non-symmetric $\mathsf{V}$-multifunctor* $F : \mathsf{M} \longrightarrow \mathsf{N}$ between non-symmetric $\mathsf{V}$-
  multicategories is defined in the same way as a $\mathsf{V}$-multifunctor but with-
  out the symmetric group action axiom (C.1.20). Composition and identi-
  ties are defined as above.

- A *(non-symmetric) multifunctor* is a (non-symmetric) Set-multifunctor. ◇

**Example C.1.24** (Underlying V-Functors). Continuing Example C.1.16, each (non-symmetric) V-multifunctor (Definition C.1.19) restricts to a V-functor between the underlying V-categories. The compatibility diagrams (B.1.9) of a V-functor are the unit diagram (C.1.21) and the unary restriction of the composition diagram (C.1.22). ◇

**Definition C.1.25.** Suppose $F, G : \mathsf{M} \longrightarrow \mathsf{N}$ are V-multifunctors. A V-*multinatural transformation* $\theta : F \longrightarrow G$ consists of, for each $x \in \mathrm{Ob}\,\mathsf{M}$, a component morphism

$$1 \xrightarrow{\ \theta_x\ } \mathsf{N}(Fx\,;\,Gx) \quad \text{in} \quad \mathsf{V}$$

such that the following V-*naturality diagram* commutes for $(\langle x \rangle\,;\,y) \in \mathrm{Prof}(\mathsf{M}) \times \mathrm{Ob}\,\mathsf{M}$ with $\langle x \rangle = \langle x_j \rangle_{j=1}^n$.

(C.1.26)

$$
\begin{array}{ccc}
1 \otimes \mathsf{M}(\langle x \rangle\,;\,y) & \xrightarrow{\ \theta_y \otimes F\ } & \mathsf{N}(Fy\,;\,Gy) \otimes \mathsf{N}(F\langle x \rangle\,;\,Fy) \\
{\scriptstyle \lambda^{-1}} \Big\uparrow & & \Big\downarrow {\scriptstyle \gamma} \\
\mathsf{M}(\langle x \rangle\,;\,y) & & \mathsf{N}(F\langle x \rangle\,;\,Gy) \\
{\scriptstyle \rho^{-1}} \Big\downarrow & & \Big\uparrow {\scriptstyle \gamma} \\
\mathsf{M}(\langle x \rangle\,;\,y) \otimes \otimes_{j=1}^n 1 & \xrightarrow{\ G \otimes \otimes_{j=1}^n \theta_{x_j}\ } & \mathsf{N}(G\langle x \rangle\,;\,Gy) \otimes \otimes_{j=1}^n \mathsf{N}(Fx_j\,;\,Gx_j)
\end{array}
$$

This finishes the definition of a V-multinatural transformation.

Moreover, we define the following.

- The *identity* V-*multinatural transformation* $1_F : F \longrightarrow F$ has each component given by a colored unit:

$$(1_F)_x = 1_{Fx} \quad \text{for} \quad x \in \mathrm{Ob}\,\mathsf{M}.$$

- A *multinatural transformation* is a Set-multinatural transformation.
- A V-*multinatural transformation* $\theta : F \longrightarrow G$ between non-symmetric V-multifunctors $F, G : \mathsf{M} \longrightarrow \mathsf{N}$ is defined as above. In this case, we also call $\theta$ a *non-symmetric* V-*multinatural transformation* if we want to emphasize that its domain and codomain are non-symmetric V-multifunctors.

We use the 2-cell notation (A.1.29) for (non-symmetric) V-multinatural transformation. ◇

**Example C.1.27** (Underlying V-Natural Transformations). Continuing Examples C.1.16 and C.1.24, each V-multinatural transformation (Definition C.1.25) is also a V-natural transformation (Definition B.1.10) between the underlying V-functors between the underlying V-categories. The naturality diagram (B.1.11) of a V-natural transformation is the unary restriction of the naturality diagram (C.1.26) of a V-multinatural transformation. ◇

**Definition C.1.28.** Suppose M, N, and P are V-multicategories.

(1) Suppose $\theta$ and $\psi$ are V-multinatural transformations as in the left diagram below.

(C.1.29)

$$
\mathsf{M} \underset{H}{\overset{F}{\Longrightarrow}} \mathsf{N} \qquad\qquad \mathsf{M} \underset{H}{\overset{F}{\Longrightarrow}} \mathsf{N}
$$

The *vertical composition $\psi\theta$*, as in the right diagram above, is the V-multinatural transformation with component at each $x \in \operatorname{Ob}\mathsf{M}$ given by the following composite in V.

(C.1.30)

$$
\begin{array}{ccc}
1 & \xrightarrow{\;(\psi\theta)_x\;} & \mathsf{N}\big(Fx\,;Hx\big) \\[2mm]
{\scriptstyle\lambda^{-1}}\Big\downarrow & & \Big\uparrow{\scriptstyle\gamma} \\[2mm]
1\otimes 1 & \xrightarrow{\;\psi_x\otimes\theta_x\;} & \mathsf{N}\big(Gx\,;Hx\big)\otimes\mathsf{N}\big(Fx\,;Gx\big)
\end{array}
$$

(2) Suppose $\theta$ and $\theta'$ are V-multinatural transformations as in the left diagram below.

(C.1.31)

$$
\mathsf{M} \underset{G}{\overset{F}{\Longrightarrow}}{\Downarrow\theta}\; \mathsf{N} \underset{G'}{\overset{F'}{\Longrightarrow}}{\Downarrow\theta'}\; \mathsf{P} \qquad\qquad \mathsf{M} \underset{G'G}{\overset{F'F}{\Longrightarrow}}{\Downarrow\theta'*\theta}\; \mathsf{P}
$$

The *horizontal composition $\theta' * \theta$*, as in the right diagram above, is the V-multinatural transformation with component at each $x \in \operatorname{Ob}\mathsf{M}$ given by the following composite in V.

(C.1.32)

$$
\begin{array}{ccc}
1 & \xrightarrow{\;(\theta'*\theta)_x\;} & \mathsf{P}\big(F'Fx\,;G'Gx\big) \\[3mm]
& & \Big\uparrow{\scriptstyle\gamma} \\[2mm]
{\scriptstyle\lambda^{-1}}\Big\downarrow & & \mathsf{P}\big(F'Gx\,;G'Gx\big)\otimes\mathsf{P}\big(F'Fx\,;F'Gx\big) \\[3mm]
& & \Big\uparrow{\scriptstyle 1\otimes F'} \\[2mm]
1\otimes 1 & \xrightarrow{\;\theta'_{Gx}\otimes\theta_x\;} & \mathsf{P}\big(F'Gx\,;G'Gx\big)\otimes\mathsf{N}\big(Fx\,;Gx\big)
\end{array}
$$

Vertical and horizontal compositions of non-symmetric V-multinatural transformations are defined as above.                                                                               ◇

**Theorem C.1.33.** *Suppose* V *is a symmetric monoidal category.*

*(1) There is a 2-category*

$$\mathsf{V\text{-}Multicat}$$

*consisting of the following data.*
- *Its objects are small* V*-multicategories.*
- *For small* V*-multicategories* M *and* N*, the hom category*

$$\mathsf{V\text{-}Multicat}(\mathsf{M},\mathsf{N})$$

*is defined as follows.*
  - *Its objects are* V*-multifunctors* $\mathsf{M} \longrightarrow \mathsf{N}$.
  - *Its morphisms are* V*-multinatural transformations.*
  - *Identity morphisms are identity* V*-multinatural transformations.*
  - *Composition is vertical composition of* V*-multinatural transformations.*
- *The identity 1-cell* $1_\mathsf{M}$ *is the identity* V*-multifunctor* $1_\mathsf{M}$.
- *Horizontal composition of 1-cells is the composition of* V*-multifunctors.*
- *Horizontal composition of 2-cells is that of* V*-multinatural transformations.*

*(2) There is an analogous 2-category*

$$\mathsf{V\text{-}Multicat}^{\mathsf{ns}}$$

*with*

- *non-symmetric small* $\mathsf{V}$-*multicategories as objects,*
- *non-symmetric* $\mathsf{V}$-*multifunctors as 1-cells, and*
- *non-symmetric* $\mathsf{V}$-*multinatural transformations as 2-cells.*

*(3) Suppose, furthermore,* $\mathsf{V}$ *is a complete and cocomplete symmetric monoidal closed category. Then the underlying 1-categories of* $\mathsf{V}$-Multicat *and* $\mathsf{V}$-Multicat$^{\mathrm{ns}}$ *are complete and cocomplete.*

*Proof.* Assertions (1) and (3) for $\mathsf{V}$-Multicat are proved using essentially the same proofs as [**JY21**, 2.4.26] and [**JY∞**, 5.5.14], respectively, which deal with the case $\mathsf{V}$ = Set. The analogous statements for the non-symmetric case use the same proofs by ignoring the symmetric group action.                                                     □

We define

(C.1.34)
$$\mathsf{Multicat} = \mathsf{Set\text{-}Multicat} \quad \text{and}$$
$$\mathsf{Multicat}^{\mathrm{ns}} = \mathsf{Set\text{-}Multicat}^{\mathrm{ns}}$$

which are, respectively, $\mathsf{V}$-Multicat and $\mathsf{V}$-Multicat$^{\mathrm{ns}}$ with $(\mathsf{V}, \otimes, 1) = (\mathsf{Set}, \times, *)$. In Section 1.1 we extend the 2-category Multicat to

- a symmetric monoidal Cat-category (Theorem 1.1.19) and
- a Cat-multicategory (Explanation 1.1.20).

**Example C.1.35** (Initial and Terminal Objects)**.**

(i) With $(\mathsf{V}, \otimes) = (\mathsf{Set}, \times)$, the *initial operad* $\mathsf{I}$ has a single object $*$ and a single unit operation $1_* \in \mathsf{I}_1$.

(ii) The *initial* $\mathsf{V}$-*multicategory* has an empty set of objects.

(iii) If $\mathsf{V}$ has a terminal object $\mathsf{T}$, then a *terminal* $\mathsf{V}$-*multicategory* $\mathsf{T}$ has a single object $*$ and $n$-ary multimorphism object

$$\mathsf{T}_n = \mathsf{T}(\overbrace{*, \ldots, *}^{n \text{ terms}}; *) = \mathsf{T}$$

for each $n \geq 0$.                                                                                          ◇

## C.2. Categorically-Enriched Multicategories

In this section we review multicategories enriched in Cat. Recall from Example A.1.21 the symmetric monoidal closed category $(\mathsf{Cat}, \times, \mathbf{1})$ of small categories and functors with the monoidal product given by the Cartesian product.

**Definition C.2.1.** Suppose $\langle x \rangle = \langle x_j \rangle_{j=1}^n, y, z$ are objects in a Cat-multicategory $(\mathsf{M}, \gamma, 1)$. The category

$$\mathsf{M}\big(\langle x \rangle; y\big)$$

is called a *multimorphism category*.

- An object in $\mathsf{M}\big(\langle x \rangle; y\big)$ is called an *n-ary 1-cell* and is denoted $\langle x \rangle \longrightarrow y$.
- A morphism

$$\theta : f \longrightarrow g \quad \text{in} \quad \mathsf{M}\big(\langle x \rangle; y\big)$$

is called an *n-ary 2-cell*. We extend the 2-cell notation (A.1.29) to *n*-ary 2-cells.

The same terminology applies to non-symmetric Cat-multicategories.                     ◇

**Explanation C.2.2** (Cat-Multinatural Transformations). Suppose M and N are Cat-multicategories, and

$$F, G : M \longrightarrow N$$

are Cat-*multifunctors*. A Cat-*multinatural transformation* $\theta : F \longrightarrow G$ consists of, for each $x \in \mathrm{Ob}\,M$, a component 1-ary 1-cell

(C.2.3)                          $$Fx \xrightarrow{\ \theta_x\ } Gx \quad \text{in} \quad N$$

such that the following two Cat-*naturality conditions* hold:

**Objects:** For each $k$-ary 1-cell

$$p : \langle x \rangle \longrightarrow x' \quad \text{in} \quad M$$

with $\langle x \rangle = \langle x_j \rangle_{j=1}^k$, denote by

(C.2.4)
$$\begin{cases} F\langle x \rangle = \langle Fx_j \rangle_{j=1}^k \in (\mathrm{Ob}\,N)^k \\ \theta_{\langle x \rangle} = \langle \theta_{x_j} \rangle_{j=1}^k \in \prod_{j=1}^k N\big(Fx_j\,;\,Gx_j\big). \end{cases}$$

Then the following equality of $k$-ary 1-cells holds, where the composition is taken in N:

(C.2.5)              $$\gamma\big(Gp\,;\,\theta_{\langle x \rangle}\big) = \gamma\big(\theta_{x'}\,;\,Fp\big) \quad \text{in} \quad N\big(F\langle x \rangle\,;\,Gx'\big).$$

**Morphisms:** For each $k$-ary 2-cell

$$f : p \longrightarrow q \quad \text{in} \quad M\big(\langle x \rangle\,;\,x'\big),$$

the following equality of $k$-ary 2-cells holds, with $1_{\theta_{\langle x \rangle}} = \langle 1_{\theta_{x_j}} \rangle_{j=1}^k$:

(C.2.6)              $$\gamma\big(Gf\,;\,1_{\theta_{\langle x \rangle}}\big) = \gamma\big(1_{\theta_{x'}}\,;\,Ff\big) \quad \text{in} \quad N\big(F\langle x \rangle\,;\,Gx'\big).$$

The conditions (C.2.5) and (C.2.6) together comprise the V-naturality condition (C.1.26) with V = Cat.

The two sides of the object Cat-naturality condition (C.2.5) use the following two compositions in N on objects.

(C.2.7)

$$\begin{array}{ccc} N\big(G\langle x \rangle\,;\,Gx'\big) \times \prod_{j=1}^k N\big(Fx_j\,;\,Gx_j\big) & & N\big(Fx'\,;\,Gx'\big) \times N\big(F\langle x \rangle\,;\,Fx'\big) \\ & \searrow{\gamma} \qquad \gamma\swarrow & \\ & N\big(F\langle x \rangle\,;\,Gx'\big) & \end{array}$$

The object Cat-naturality condition (C.2.5) is the commutative diagram

(C.2.8)
$$\begin{array}{ccc} F\langle x \rangle & \xrightarrow{\ \theta_{\langle x \rangle}\ } & G\langle x \rangle \\ {\scriptstyle Fp}\big\downarrow & & \big\downarrow{\scriptstyle Gp} \\ Fx' & \xrightarrow{\ \theta_{x'}\ } & Gx' \end{array}$$

involving the compositions (C.2.7) in N.

The two sides of the morphism Cat-naturality condition (C.2.6) use the compositions in (C.2.7) on morphisms. The condition (C.2.6) is the equality of multicategorical pasting diagrams

(C.2.9)



involving the composition in N.

By the object Cat-naturality condition (C.2.5) for $\theta' : F' \longrightarrow G'$, there are two ways to express the $x$-component of the horizontal composite (C.1.31) for $x \in \mathsf{Ob}\,\mathsf{M}$:

(C.2.10)
$$
\begin{aligned}
(\theta' * \theta)_x &= \gamma\big(\theta'_{Gx}\,;\, F'\theta_x\big) \\
&= \gamma\big(G'\theta_x\,;\, \theta'_{Fx}\big).
\end{aligned}
$$

A non-symmetric Cat-multinatural transformation admits the same description as above.  ◇

## C.3. Endomorphism Multicategories

In this section we review the endomorphism construction that goes from symmetric monoidal categories to multicategories, which mean Set-multicategories. This construction defines a 2-functor; see Proposition C.3.6. The enriched variant is in Definition C.3.8. The material in this section is adapted from [**JY∞**, Section 6.3].

**Example C.3.1** (Endomorphism Multicategory). Suppose $(\mathsf{C}, \oplus, e, \xi)$ is a permutative category (Definition A.1.14). Then it has an associated *endomorphism multicategory* $\mathsf{End}(\mathsf{C})$ defined as follows.

- The object class is $\mathsf{Ob}\,\mathsf{C}$.
- The $n$-ary multimorphism set is

$$
\mathsf{End}(\mathsf{C})\big(\langle x \rangle\,;\, y\big) = \mathsf{C}\big(\oplus_{j=1}^n x_j\,,\, y\big)
$$

  for $(\langle x \rangle\,;\, y) \in \mathsf{Prof}(\mathsf{C}) \times \mathsf{C}$ with $\langle x \rangle = \langle x_j \rangle_{j=1}^n$. By definition, an empty $\oplus$ in $\mathsf{C}$ is the monoidal unit $e$.
- The symmetric group action is induced by the braiding $\xi$.
- For each object $x$ in $\mathsf{C}$, the $x$-colored unit is the identity morphism $1_x$.
- The multicategorical composition in $\mathsf{End}(\mathsf{C})$ is induced by $\oplus$ and composition in $\mathsf{C}$.

There are two variants of the above endomorphism construction.

(1) If $(\mathsf{C}, \otimes, 1, \xi)$ is a symmetric monoidal category that is not necessarily strict, then $\mathsf{End}(\mathsf{C})$ is still a multicategory.
  - The $n$-ary multimorphism set is

(C.3.2)
$$
\mathsf{End}(\mathsf{C})\big(\langle x \rangle\,;\, y\big) = \mathsf{C}\big(\otimes_{j=1}^n x_j\,,\, y\big)
$$

    with $\otimes_{j=1}^n x_j$ using Convention A.1.30 of left normalized bracketing. By definition, an empty $\otimes$ is the monoidal unit 1.
  - The symmetric group action is induced by the braiding $\xi$ and the associativity isomorphism $\alpha$ in $\mathsf{C}$.

- The composition in $\mathsf{End}(\mathsf{C})$ is induced by $\alpha$, $\otimes$, and composition in $\mathsf{C}$. If nullary multimorphisms are involved, then we also use the left and right unit isomorphisms in $\mathsf{C}$.

(2) If $(\mathsf{C}, \otimes, 1)$ is a monoidal category, then the above definitions, without the symmetric group action, yield a non-symmetric multicategory that we also denote by $\mathsf{End}(\mathsf{C})$.

Furthermore, the endomorphism multicategory extends to symmetric monoidal functors and monoidal natural transformations as follow. Each symmetric monoidal functor between symmetric monoidal categories

$$(P, P^2, P^0) : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a multifunctor

(C.3.3)                          $\mathsf{End}(P) : \mathsf{End}(\mathsf{C}) \longrightarrow \mathsf{End}(\mathsf{D})$

with the same object assignment as $P$. For an $n$-ary multimorphism

$$f \in \mathsf{End}(\mathsf{C})\big(\langle x \rangle ; y\big) = \mathsf{C}\big(\otimes_{j=1}^{n} x_j , y\big)$$

with $(\langle x \rangle ; y)$ as above, its image in $\mathsf{End}(\mathsf{D})\big(\langle Px \rangle ; Py\big)$ is the composite

(C.3.4)                  $\otimes_{j=1}^{n} Px_j \xrightarrow{\ P^2\ } P\big(\otimes_{j=1}^{n} x_j\big) \xrightarrow{\ Pf\ } Py$   in   $\mathsf{D}$.

The first morphism $P^2$ in (C.3.4) means

- the unit constraint $P^0 : 1 \longrightarrow P1$ if $n = 0$,
- the identity if $n = 1$, and
- a repeated application of the monoidal constraint $P^2$ if $n > 1$.

Suppose $\theta$ is a monoidal natural transformation between symmetric monoidal functors between symmetric monoidal categories, as in the left diagram below.

$$\mathsf{C} \underset{(Q, Q^2, Q^0)}{\overset{(P, P^2, P^0)}{\Longrightarrow \theta}} \mathsf{D} \qquad\qquad \mathsf{End}(\mathsf{C}) \underset{\mathsf{End}(Q)}{\overset{\mathsf{End}(P)}{\Longrightarrow \mathsf{End}(\theta)}} \mathsf{End}(\mathsf{D})$$

Then $\theta$ induces a multinatural transformation $\mathsf{End}(\theta)$, as in the right diagram above, with component morphisms

(C.3.5)                          $\mathsf{End}(\theta)_x = \theta_x$   for   $x \in \mathsf{C}$.

The non-symmetric variants of the above statements also hold.

(1) If $P$ is a monoidal functor between monoidal categories, then $\mathsf{End}(P)$ is a non-symmetric multifunctor.

(2) If $\theta$ is a monoidal natural transformation between monoidal functors between monoidal categories, then $\mathsf{End}(\theta)$ is a non-symmetric multinatural transformation.

If there is no danger of confusion, we abbreviate $\mathsf{End}(\mathsf{C})$ to $\mathsf{C}$ and similarly for $\mathsf{End}(P)$ and $\mathsf{End}(\theta)$. We extend this example to the pointed context in Example C.4.8 below.                                                                                   ◇

Recall the 2-categories

- Multicat of small multicategories, multifunctors, and multinatural transformations (Theorem C.1.33 with $\mathsf{V} = (\mathsf{Set}, \times, *)$) and

- PermCat of small permutative categories, symmetric monoidal functors, and monoidal natural transformations (Definition A.2.3).

**Proposition C.3.6.** *The endomorphism multicategory in Example C.3.1 defines a 2-functor*

$$\mathsf{End} : \mathsf{PermCat} \longrightarrow \mathsf{Multicat}.$$

We also denote by $\mathsf{End}$ the restriction of the 2-functor in Proposition C.3.6 to any one of the locally-full sub-2-categories of $\mathsf{PermCat}$ in Definition A.2.3, including $\mathsf{PermCat}^{\mathsf{su}}$ and $\mathsf{PermCat}^{\mathsf{st}}$. In (1.4.39) we discuss an extension of $\mathsf{End}$, with domain $\mathsf{PermCat}^{\mathsf{su}}$, to a $\mathsf{Cat}$-multifunctor.

**Enriched Endomorphism Multicategories.** For the rest of this section, we assume that $(\mathsf{V}, \otimes, 1, \xi)$ is a symmetric monoidal category. Next we review the $\mathsf{V}$-multicategory associated to a symmetric monoidal $\mathsf{V}$-category. The endomorphism multicategory in Example C.3.1 is the special case $(\mathsf{V}, \otimes) = (\mathsf{Set}, \times)$.

**Convention C.3.7** (Left Normalized Product)**.** Suppose $(\mathsf{K}, \boxtimes)$ is a monoidal $\mathsf{V}$-category (Definition B.2.7), and $\langle x \rangle = \langle x_j \rangle_{j=1}^n$ is an $n$-tuple of objects of $\mathsf{K}$ for some $n \geq 0$. We define the *left normalized product* as the object

$$\boxtimes \langle x \rangle = \overset{n}{\underset{j=1}{\boxtimes}}\, x_j = \big( \cdots \big( (x_1 \boxtimes x_2) \boxtimes x_3 \big) \cdots \big) \boxtimes x_n,$$

which is the identity object of $\mathsf{K}$ if $n = 0$. $\diamond$

**Definition C.3.8.** Suppose $(\mathsf{K}, \beta^{\boxtimes})$ is a symmetric monoidal $\mathsf{V}$-category (Definition B.2.16) with $\mathsf{V}$ a symmetric monoidal category. The *endomorphism $\mathsf{V}$-multicategory* of $\mathsf{K}$, denoted $\mathsf{End}(\mathsf{K})$, consists of the following data.

**Objects:** $\mathsf{Ob}(\mathsf{End}(\mathsf{K})) = \mathsf{Ob}\,\mathsf{K}$.

**Multimorphism Objects:** For an object $x' \in \mathsf{K}$ and a tuple $\langle x \rangle \in \mathsf{Prof}(\mathsf{K})$, we define the object

$$\mathsf{End}(\mathsf{K})\big( \langle x \rangle\,;\, x' \big) = \mathsf{K}\big( \boxtimes \langle x \rangle,\, x' \big) \quad \text{in} \quad \mathsf{V}$$

with $\boxtimes \langle x \rangle$ denoting the left normalized product in Convention C.3.7.

**Symmetric Group Action:** For objects $\langle x \rangle$ and $x'$ as above and a permutation $\sigma \in \Sigma_n$, the right $\sigma$-action is defined as the following composite.

$$
\begin{array}{ccc}
\mathsf{K}\big( \boxtimes\langle x \rangle, x' \big) & \xrightarrow{\quad\sigma\quad} & \mathsf{K}\big( \boxtimes\langle x \rangle\sigma, x' \big) \\[2mm]
{\scriptstyle \rho^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle m} \\[2mm]
\mathsf{K}\big( \boxtimes\langle x \rangle, x' \big) \otimes 1 & \xrightarrow{\ 1 \otimes \beta^{\boxtimes}_{\sigma}\ } & \mathsf{K}\big( \boxtimes\langle x \rangle, x' \big) \otimes \mathsf{K}\big( \boxtimes\langle x \rangle\sigma, \boxtimes\langle x \rangle \big)
\end{array}
$$

In the above diagram, $\beta^{\boxtimes}_{\sigma}$ denotes the $\mathsf{V}$-natural isomorphism that permutes coordinates according to the permutation $\sigma$.

**Units:** For each object $x \in \mathsf{K}$, the $x$-colored unit of $\mathsf{End}(\mathsf{K})$ is the identity of $x$ in $\mathsf{K}$,

$$1 \xrightarrow{\quad 1_x \quad} \mathsf{K}(x, x) = \mathsf{End}(\mathsf{K})\big( x\,;\, x \big).$$

**Composition:** The composition $\gamma$ in $\mathsf{End}(\mathsf{K})$ is defined as the following composite for tuples of objects

$$x'', \quad \langle x' \rangle = \langle x'_j \rangle_{j=1}^n, \quad \text{and} \quad \langle x_j \rangle = \langle x_{j,i} \rangle_{i=1}^{k_j}$$

with $j \in \{1, \ldots, n\}$ and $\langle x \rangle = \langle \langle x_j \rangle \rangle_{j=1}^n$.

$$
\begin{array}{ccc}
\mathsf{K}\big(\boxtimes\langle x'\rangle, x''\big) \otimes \overset{n}{\underset{j=1}{\bigotimes}} \mathsf{K}\big(\overset{k_j}{\underset{i=1}{\boxtimes}} x_{j,i}, x_j'\big) & \xrightarrow{\ \gamma\ } & \mathsf{K}\big(\boxtimes\langle x\rangle, x''\big) \\
{\scriptstyle 1 \otimes \boxtimes_{j=1}^{n-1}} \Big\downarrow & & \Big\uparrow {\scriptstyle \cong} \\
\mathsf{K}\big(\boxtimes\langle x'\rangle, x''\big) \otimes \mathsf{K}\big(\overset{n}{\underset{j=1}{\boxtimes}} \overset{k_j}{\underset{i=1}{\boxtimes}} x_{j,i}, \boxtimes\langle x'\rangle\big) & \xrightarrow{\ \ \mathsf{m}\ \ } & \mathsf{K}\big(\overset{n}{\underset{j=1}{\boxtimes}} \overset{k_j}{\underset{i=1}{\boxtimes}} x_{j,i}, x''\big)
\end{array}
$$

This finishes the definition of $\mathsf{End}(\mathsf{K})$. To simplify the notation, we also denote $\mathsf{End}(\mathsf{K})$ by $\mathsf{K}$.                                                                              ◇

The following result is [**JY**∞, 6.3.6].

**Proposition C.3.9.** *In the context of Definition C.3.8,* $\mathsf{End}(\mathsf{K})$ *is a* $\mathsf{V}$*-multicategory.*

If there is no danger of confusion, we abbreviate $\mathsf{End}(\mathsf{K})$ to $\mathsf{K}$.

## C.4. Pointed Multicategories

Recall that a *multicategory* means a Set-multicategory (Definition C.1.3), where $(\mathsf{Set}, \times, *)$ is the symmetric monoidal category of sets with the Cartesian product. In this section we review *pointed* multicategories.

- The 2-category $\mathsf{Multicat}_*$ of small pointed multicategories is in Theorem C.4.9.
- The free-forgetful 2-adjunction between $\mathsf{Multicat}$ and $\mathsf{Multicat}_*$ is in Proposition C.4.16.

The material in this section is adapted from [**JY**∞, Section 5.3]. Recall the terminal multicategory $\mathsf{T}$ in Example C.1.17. It has a single object $*$ and one $n$-ary operation $\iota_n$ for each $n \geq 0$.

**Definition C.4.1.** We define the following.

(1) A *pointed multicategory* $(\mathsf{M}, i)$ is a pair consisting of the following data.
- $\mathsf{M}$ is a multicategory (Definition C.1.3).
- $i : \mathsf{T} \longrightarrow \mathsf{M}$ is a multifunctor (Definition C.1.19), which is called the *pointed structure*.

We denote
- $i(*) \in \mathsf{Ob}\,\mathsf{M}$ by $*$, which is called the *basepoint object*, and
- $i(\iota_n) \in \mathsf{M}\big(\langle * \rangle_{j=1}^n ; *\big)$ by $\iota_n$ or $\iota^n$, which is called the *$n$-ary basepoint operation*, for each $n \geq 0$.

(2) For pointed multicategories $(\mathsf{M}, i^{\mathsf{M}})$ and $(\mathsf{N}, i^{\mathsf{N}})$, a *pointed multifunctor*

$$
F : (\mathsf{M}, i^{\mathsf{M}}) \longrightarrow (\mathsf{N}, i^{\mathsf{N}})
$$

is a multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ (Definition C.1.19) such that the following diagram of multifunctors commutes.

(C.4.2)
$$
\begin{array}{ccc}
 & \overset{i^{\mathsf{M}}}{\nearrow} & \mathsf{M} \\
\mathsf{T} & & \Big\downarrow {\scriptstyle F} \\
 & \underset{i^{\mathsf{N}}}{\searrow} & \mathsf{N}
\end{array}
$$

(3) A *pointed multinatural transformation*

$$
\theta : F \longrightarrow G : (\mathsf{M}, i^{\mathsf{M}}) \longrightarrow (\mathsf{N}, i^{\mathsf{N}})
$$

between pointed multifunctors $F$ and $G$ is a multinatural transformation (Definition C.1.25) such that the basepoint component

(C.4.3) $$\theta_* = 1_* \in \mathsf{N}\big(F(*)\,;\, G(*)\big) = \mathsf{N}\big(*\,;\,*\big),$$

which is the colored unit of the basepoint object $*$ in $\mathsf{N}$.

With these definitions, composites of pointed multifunctors and multinatural transformations are again pointed. $\diamond$

**Explanation C.4.4** (Pointed Structure). For a multicategory $(\mathsf{M}, \gamma, 1)$, a multifunctor $i : \mathsf{T} \longrightarrow \mathsf{M}$ (Definition C.1.19) is uniquely determined by

- a *basepoint object* $* \in \mathrm{Ob}\,\mathsf{M}$ and
- an *$n$-ary basepoint operation*

$$\iota_n \in \mathsf{M}\big(\langle *\rangle_{j=1}^n\,;\,*\big) \quad \text{for} \quad \geq 0$$

such that the following three conditions hold:

**Symmetry:** For each $n \geq 0$ and permutation $\sigma \in \Sigma_n$, there is an equality

(C.4.5) $$\iota_n \cdot \sigma = \iota_n.$$

**Unity:** There is an equality

(C.4.6) $$\iota_1 = 1_* \in \mathsf{M}(*\,;\,*),$$

which is the colored unit of $*$ in $\mathsf{M}$.

**Composition:** For $n \geq 1$ and $k_j \geq 0$ for $j \in \{1, \ldots, n\}$, there is an equality

(C.4.7) $$\gamma\big(\iota_n\,;\, \langle \iota_{k_j}\rangle_{j=1}^n\big) = \iota_{k_1 + \cdots + k_n}.$$

Moreover, for a pointed multifunctor

$$F : (\mathsf{M}, i^{\mathsf{M}}) \longrightarrow (\mathsf{N}, i^{\mathsf{N}}),$$

the commutative diagram (C.4.2) means

- $F(*) = *$ in $\mathrm{Ob}\,\mathsf{N}$ and
- $F(\iota_n) = \iota_n$ for each $n \geq 0$.

In other words, a pointed multifunctor is a multifunctor that preserves the basepoint object and the basepoint operations. $\diamond$

**Example C.4.8** (Pointed Endomorphism Multicategory). Each permutative category $(\mathsf{C}, \oplus, \mathrm{e}, \xi)$ has an associated pointed multicategory

$$\mathsf{End}_{\bullet}(\mathsf{C}) = \big(\mathsf{End}(\mathsf{C}), i\big)$$

defined as follows.

- $\mathsf{End}(\mathsf{C})$ is the endomorphism multicategory in Example C.3.1.
- Using Explanation C.4.4, the pointed structure is given by the multifunctor

$$i : \mathsf{T} \longrightarrow \mathsf{End}(\mathsf{C})$$

determined by
  - the basepoint object $\mathrm{e} \in \mathsf{C}$ and
  - $n$-ary basepoint operations

$$\iota^n = 1_{\mathrm{e}} : \bigoplus_{j=1}^n \mathrm{e} = \mathrm{e} \longrightarrow \mathrm{e} \quad \text{for} \quad n \geq 0.$$

As in Example C.3.1, the above definitions still yield a pointed multicategory $\mathsf{End}_\bullet(\mathsf{C})$ if $(\mathsf{C}, \otimes, 1)$ is a symmetric monoidal category that is not necessarily strict. In this case, the $n$-ary basepoint operation

$$\iota^n : \otimes_{j=1}^n 1 \overset{\cong}{\longrightarrow} 1$$

is

- the identity $1_1$ if $n = 0$ or if $n = 1$ and
- an iterate of the right unit isomorphism $\rho$ in $\mathsf{C}$ if $n > 1$.

The unity condition (C.4.6) holds by definition. The symmetry and composition conditions, (C.4.5) and (C.4.7), hold by the Coherence Theorem for symmetric monoidal categories [**ML98**, XI.1 Theorem 1].

Moreover, the following statements hold:

(i) Each *strictly unital* symmetric monoidal functor between symmetric monoidal categories

$$(P, P^2, P^0 = 1) : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a pointed multifunctor

$$\mathsf{End}_\bullet(P) : \mathsf{End}_\bullet(\mathsf{C}) \longrightarrow \mathsf{End}_\bullet(\mathsf{D})$$

given by the multifunctor $\mathsf{End}(P)$ in (C.3.3). Strict unity of $P$ ensures that $\mathsf{End}(P)$ is pointed as in Explanation C.4.4.

(ii) Each monoidal natural transformation between strictly unital symmetric monoidal functors between symmetric monoidal categories

$$\theta : (P, P^2, P^0 = 1) \longrightarrow (Q, Q^2, Q^0 = 1) : \mathsf{C} \longrightarrow \mathsf{D}$$

induces a pointed multinatural transformation

$$\mathsf{End}_\bullet(\theta) : \mathsf{End}_\bullet(P) \longrightarrow \mathsf{End}_\bullet(Q) : \mathsf{End}_\bullet(\mathsf{C}) \longrightarrow \mathsf{End}_\bullet(\mathsf{D})$$

with components as in (C.3.5):

$$\mathsf{End}_\bullet(\theta)_x = \theta_x \quad \text{for} \quad x \in \mathsf{C}.$$

The pointed condition

$$\mathsf{End}_\bullet(\theta)_1 = \theta_1 = 1_1 : 1 \longrightarrow 1 \quad \text{in} \quad \mathsf{D}$$

follows from

- the left diagram in (A.1.28) and
- the assumption that both unit constraints $P^0$ and $Q^0$ are the identities.

If there is no danger of confusion, we denote $\mathsf{End}_\bullet(\mathsf{C})$ by $\mathsf{C}$. We extend this example to left $\mathcal{M}\underline{1}$-modules in Example 1.3.15. ◇

Pointed multifunctors are multifunctors satisfying the extra property (C.4.2). Likewise, pointed multinatural transformations are multinatural transformations with the extra property (C.4.3). These extra properties are closed under the compositions of the 2-category Multicat (Theorem C.1.33). Therefore, the proof for the existence of the 2-category Multicat also yields the following.

**Theorem C.4.9.** *In the context of Definition C.4.1, there is a 2-category*

$$\mathsf{Multicat}_*$$

*defined by the following data.*

- *The objects are small pointed multicategories.*
- *The 1-cells are pointed multifunctors.*
- *The 2-cells are pointed multinatural transformations.*
- *The horizontal and vertical compositions and identity 1-cells and 2-cells are defined as in the 2-category* Multicat.

In Section 1.2 we discuss extensions of the 2-category $\mathsf{Multicat}_*$ to

- a symmetric monoidal Cat-category (Theorem 1.2.8) and
- a Cat-multicategory (Explanation 1.2.9).

**Proposition C.4.10.** *The pointed endomorphism multicategory in Example C.4.8 defines a 2-functor*

$$\mathsf{End}_\bullet : \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Multicat}_*$$

*with* $\mathsf{PermCat}^{\mathsf{su}}$ *the 2-category in Definition A.2.3.*

We also denote by $\mathsf{End}_\bullet$ the restriction of the 2-functor in Proposition C.4.10 to the locally-full sub-2-category $\mathsf{PermCat}^{\mathsf{st}}$ in Definition A.2.3. In (1.4.39) we discuss an extension of $\mathsf{End}_\bullet$ to a Cat-multifunctor.

Propositions C.4.11 and C.4.16 follow directly from the definitions.

**Proposition C.4.11.** *There is a forgetful 2-functor*

$$\mathsf{U}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{Multicat}$$

*that sends*

- *a small pointed multicategory* $(\mathsf{M}, i)$ *to the multicategory* $\mathsf{M}$,
- *a pointed multifunctor $F$ to the multifunctor $F$, and*
- *a pointed multinatural transformation $\theta$ to the multinatural transformation $\theta$.*

**Explanation C.4.12.** The 2-functors in Propositions C.3.6, C.4.10, and C.4.11 yield a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathsf{End} & & \\
 & \nearrow & & \searrow & \\
\mathsf{PermCat}^{\mathsf{su}} & \xrightarrow{\ \mathsf{End}_\bullet\ } & \mathsf{Multicat}_* & \xrightarrow{\ \mathsf{U}_\bullet\ } & \mathsf{Multicat}
\end{array}
$$

with End restricted to $\mathsf{PermCat}^{\mathsf{su}}$. ◇

The forgetful 2-functor $\mathsf{U}_\bullet$ admits a left 2-adjoint defined as follows.

**Definition C.4.13** (Adjoining a Basepoint). We define a 2-functor

(C.4.14)                    $(-)_+ : \mathsf{Multicat} \longrightarrow \mathsf{Multicat}_*$

together with 2-natural transformations

(C.4.15)
$$\eta^+ : 1_{\mathsf{Multicat}} \longrightarrow \mathsf{U}_\bullet \circ (-)_+ \quad \text{and}$$
$$\varepsilon^+ : (-)_+ \circ \mathsf{U}_\bullet \longrightarrow 1_{\mathsf{Multicat}_*}$$

as follows.

**The 2-Functor** $(-)_+$**:** This is defined by the following assignments.

- For a small multicategory $\mathsf{M}$, we define the pointed multicategory

$$\mathsf{M}_+ = \mathsf{M} \coprod \mathsf{T}$$

with pointed structure given by the $\mathsf{T}$ summand, where the coproduct is taken in Multicat.

- For a multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ between small multicategories, we define the pointed multifunctor

$$F_+ = F \amalg 1_\mathsf{T} : \mathsf{M}_+ \longrightarrow \mathsf{N}_+.$$

- Suppose $\theta : F \longrightarrow G$ is a multinatural transformation for multifunctors $F, G : \mathsf{M} \longrightarrow \mathsf{N}$ between small multicategories. The pointed multinatural transformation

$$\theta_+ : F_+ \longrightarrow G_+$$

is given by
  - the same components as $\theta$ for objects in $\mathsf{M}$ and
  - $1_*$ in $\mathsf{N}_+$ for the basepoint object $* \in \mathsf{M}_+$.

**Unit:** For a small multicategory $\mathsf{M}$, the unit $\eta^+$ has component multifunctor

$$\eta^+_\mathsf{M} : \mathsf{M} \longrightarrow \mathsf{U}_\bullet(\mathsf{M}_+) = \mathsf{M} \amalg \mathsf{T}$$

given by the inclusion of the $\mathsf{M}$ summand.

**Counit:** For a small pointed multicategory $(\mathsf{N}, i)$, the counit $\varepsilon^+$ has component pointed multifunctor

$$\varepsilon^+_\mathsf{N} = \big(1_\mathsf{N}, i\big) : \big(\mathsf{U}_\bullet(\mathsf{N}, i)\big)_+ = \mathsf{N} \amalg \mathsf{T} \longrightarrow \mathsf{N}.$$

This finishes the definition.                                                                 $\diamond$

Recall the notion of a 2-adjunction in Definition A.2.11.

**Proposition C.4.16.** *In the context of Definition C.4.13, there is a 2-adjunction*

$$\big((-)_+, \mathsf{U}_\bullet, \eta^+, \varepsilon^+\big) : \mathsf{Multicat} \longrightarrow \mathsf{Multicat}_*.$$

# Open Questions

In this chapter, we discuss open questions related to the topics of this work.

**Question D.1** (Diagrams and Presheaves on $G\mathcal{E}$). Considering the Burnside 2-category $G\mathcal{E}$ (Definition 0.3.5), Remark 0.3.7 notes that the assignment

$$(f, g) \longmapsto (g, f),$$

sending a span to its reverse, is not functorial on 1-cells. However, it may be a pseudofunctor. That possibility raises the following question.

Is there an equivalence of homotopy theories

$$\mathsf{PermCat^{su}\text{-}Cat}\big(G\mathcal{E}, \mathsf{PermCat^{su}}\big) \longrightarrow \mathsf{PermCat^{su}\text{-}Cat}\big(G\mathcal{E}^{op}, \mathsf{PermCat^{su}}\big)$$

with respect to the stable equivalences (2.5.14)? Such an equivalence, combined with a change of enrichment as in Theorem 10.5.1, would give another approach to $G$-spectra and would further inform the discussion in Remark 10.5.5.

The approach to equivalences of homotopy theories that we use throughout this work requires an underlying (1-)functor, as in Definition 2.1.8. Therefore, answering the above question in the affirmative appears to require new and likely interesting extensions of that basic approach. Note, moreover, that this question is a special case of Question D.4 below. ◇

**Question D.2** (The Forgetful $\mathsf{U}_\bullet$). Considering the stable equivalences

$$\mathcal{S}_\bullet \subset \mathsf{Multicat}_* \quad \text{and} \quad \mathcal{S}^\mathsf{F} \subset \mathsf{Multicat}$$

in (4.0.1), is the forgetful functor

$$\mathsf{U}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{Multicat}$$

a relative functor? If so, then the equality $\mathsf{End} = \mathsf{U}_\bullet \circ \mathsf{End}_\bullet$, together with the other results in Chapter 4, implies that $\mathsf{U}_\bullet$ is also an equivalence of homotopy theories. In definition of $\eta_\mathsf{M}^\bullet$, (4.3.2) suppresses the forgetful $\mathsf{U}_\bullet$. Including it, (4.3.2) defines $\eta_\mathsf{M}^\bullet$ as the pointed multifunctor such that

$$\mathsf{U}_\bullet \eta_\mathsf{M}^\bullet = \mathsf{End}(\mathsf{p}_\mathsf{M}) \circ \eta_{\mathsf{U}_\bullet \mathsf{M}}.$$

Observe, by Theorems 3.5.3 and 4.7.3, that

- $\eta_{\mathsf{U}_\bullet \mathsf{M}}$ and $\eta_\mathsf{M}^\bullet$ are stable equivalences, and
- $\mathsf{End}(\mathsf{p}_\mathsf{M})$ is an F-stable equivalence if and only if $\mathsf{p}_\mathsf{M}$ is a stable equivalence.

Therefore, it follows that $\mathsf{U}_\bullet$ is a relative functor if and only if $\mathsf{p}_\mathsf{M}$ is a stable equivalence for each pointed multicategory M.

For

$$H : \mathsf{M} \longrightarrow \mathsf{N} \in \mathsf{Multicat}_*,$$

there is a commutative diagram induced by naturality of $\mathsf{p}$ (Proposition 4.2.3).

$$
\begin{array}{ccccc}
\mathsf{FT} & \xrightarrow{\;\mathsf{F}\iota^{\mathsf{M}}\;} & \mathsf{FM} & \xrightarrow{\;\mathsf{p_M}\;} & \mathsf{F_\bullet M} \\
{\scriptstyle 1}\downarrow & & {\scriptstyle \mathsf{F}H}\downarrow & & {\scriptstyle \mathsf{F_\bullet}H}\downarrow \\
\mathsf{FT} & \xrightarrow{\;\mathsf{F}\iota^{\mathsf{N}}\;} & \mathsf{FN} & \xrightarrow{\;\mathsf{p_N}\;} & \mathsf{F_\bullet N}
\end{array}
$$

Since the empty tuple is an initial object in $\mathsf{FT}$, the nerve $\mathrm{Ner}(\mathsf{FT})$ is contractible. A quasifibration argument, such as Quillen's Theorem A [**Qui73**], might be one way to approach this question.

Another approach might try to show directly that $\mathsf{p_M}$ is a stable equivalence for each M. For this second approach, show that there is an equivalence of categories

$$\mathsf{F}(\mathsf{M}_+) \simeq \mathsf{FM}$$

given by deleting the (disjoint) basepoint objects and operations from each tuple of objects and morphisms in $\mathsf{F}(\mathsf{M}_+)$. This, combined with the isomorphism

$$\mathsf{F_\bullet}(\mathsf{M}_+) \cong \mathsf{FM}$$

from (4.3.14), implies that

$$\mathsf{p}_{(\mathsf{M}_+)} : \mathsf{F}(\mathsf{M}_+) \xrightarrow{\;\simeq\;} \mathsf{F_\bullet}(\mathsf{M}_+)$$

is an equivalence of categories. Further work will be needed to determine whether or not there is a stable equivalence between

$$\mathsf{F_\bullet}(\mathsf{M}) \quad \text{and} \quad \mathsf{F_\bullet}(\mathsf{M}_+)$$

for all small pointed multicategories M. The authors are not aware of either a proof or a counterexample to such an equivalence.     ⋄

**Question D.3** (*K*-Theoretic Equivalences of Homotopy Theories)**.** Are the diagram and presheaf change-of-enrichment functors $\mathsf{J}^{\mathcal{T}}_\star$, $(\mathrm{Ner}_*)_\star$, $\mathsf{K}^{\mathcal{G}}_\star$, and $\mathsf{K}^{\mathsf{EM}}_\star$ in (10.6.3) and (10.6.6) equivalences of homotopy theories (Definition 2.1.7)?     ⋄

**Question D.4** (Morita Theory for Closed Multicategories)**.** Develop Morita theory for diagram categories enriched in a non-symmetric closed multicategory M (Definition 10.1.1). In other words, for M-categories C and D, give criteria that guarantee that the categories

$$\mathsf{M\text{-}Cat}(\mathsf{C}, \mathsf{M}) \quad \text{and} \quad \mathsf{M\text{-}Cat}(\mathsf{D}, \mathsf{M})$$

are

   (i) equivalent or

  (ii) connected by equivalences of homotopy theories.

Moreover, for a closed multicategory M, we ask the same questions for the enriched presheaf categories

$$\mathsf{M\text{-}Cat}(\mathsf{C}^{\mathsf{op}}, \mathsf{M}) \quad \text{and} \quad \mathsf{M\text{-}Cat}(\mathsf{D}^{\mathsf{op}}, \mathsf{M})$$

with $\mathsf{C}^{\mathsf{op}}$ and $\mathsf{D}^{\mathsf{op}}$ the opposite M-categories of C and D (Proposition 6.6.7), respectively. There is a huge literature on Morita theory in many different contexts. See, for example,

- [**Coh03**, Sections 4.4 and 4.5] for Morita theory of modules over rings,

- **[Lin74, FPP75]** for Morita theory of categories enriched in a closed category, and
- **[SS03**, Section 4] and Example 10.1.26 for Morita theory of modules over symmetric ring spectra.

Our Theorems 11.4.14, 11.4.24, 12.1.6, 12.4.6, and 12.6.6 are a kind of Morita theory that involves a change of closed multicategories. ◇

# Back Matter

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# List of Main Facts

(1.3.16) $\mathrm{End}_{\mathcal{M}\underline{1}} : \mathrm{PermCat}^{\mathsf{su}} \longrightarrow \mathrm{Mod}^{\mathcal{M}\underline{1}}$ is a 2-functor.

(1.3.17) Each small pointed multicategory has at most one left $\mathcal{M}\underline{1}$-module structure, and the structure morphism is an isomorphism. $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is a full sub-2-category of $\mathrm{Multicat}_*$. $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is a complete and cocomplete symmetric monoidal closed category.

(1.3.19) There is a free-forgetful adjunction $\mathcal{M}\underline{1} \wedge - : \mathrm{Multicat}_* \rightleftarrows \mathrm{Mod}^{\mathcal{M}\underline{1}} : \mathrm{U}_{\mathcal{M}\underline{1}}$.

(1.3.22) The counit of $\left(\mathcal{M}\underline{1} \wedge -, \mathrm{U}_{\mathcal{M}\underline{1}}\right)$ is componentwise an isomorphism.

(1.3.23) $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is a symmetric monoidal Cat-category.

(1.3.24) $\mathrm{Mod}^{\mathcal{M}\underline{1}}$ is a Cat-multicategory.

(1.3.26) $\mathcal{M}\underline{1} \wedge -$ is a strong symmetric Cat-monoidal functor, hence also a Cat-multifunctor.

(1.3.27) $\mathrm{U}_{\mathcal{M}\underline{1}}$ is a symmetric monoidal Cat-functor.

(1.3.29) $\mathrm{U}_{\mathcal{M}\underline{1}}$ is a Cat-multifunctor.

(1.3.31) The unit and counit of $\left(\mathcal{M}\underline{1} \wedge -, \mathrm{U}_{\mathcal{M}\underline{1}}\right)$ are monoidal Cat-natural transformations, hence also Cat-multinatural transformations.

(1.4.9) A 1-linear functor is precisely a strictly unital symmetric monoidal functor.

(1.4.14) A 1-linear transformation is precisely a monoidal natural transformation.

(1.4.29) There are Cat-multicategories $\mathrm{PermCat}^{\mathsf{su}}$ and $\mathrm{PermCat}^{\mathsf{st}}$.

(1.4.31) $\mathrm{End}_\bullet$ induces an isomorphism between multimorphism categories.

(1.4.32) $\mathrm{End}_\bullet$ is a Cat-multifunctor.

(1.4.38) $\mathrm{End} = \mathrm{U}_\bullet \mathrm{End}_\bullet$ and $\mathrm{End}_\bullet = \mathrm{U}_{\mathcal{M}\underline{1}} \mathrm{End}_{\mathcal{M}\underline{1}}$.

(1.4.40) $\mathrm{End}$ is a Cat-multifunctor.

(1.4.41) $\mathrm{End}_{\mathcal{M}\underline{1}}$ is a Cat-multifunctor.

## **Chapter 2**. Infinite Loop Space Machines

(2.1.5) The complete Segal space model structure on bisimplicial sets is a simplicial model structure whose fibrant objects are precisely the complete Segal spaces.

(2.1.9) Inverse equivalences of homotopy theories are equivalences of homotopy theories.

(2.1.10) An adjoint equivalence of homotopy theories induces equivalences of homotopy theories.

(2.2.7) For a complete and cocomplete symmetric monoidal closed category C, the category $\mathrm{C}_*$ of pointed objects is also a complete and cocomplete symmetric monoidal closed category.

(2.2.19) $\mathcal{D}_*$-V is a complete and cocomplete symmetric monoidal closed category with the pointed Day convolution.

(2.2.21) $\mathcal{D}_*$-V is enriched and (co)tensored over $\mathrm{V}_*$. $\mathcal{D}_*$-V is a V-multicategory.

(2.3.3) $\Gamma$-V is a complete and cocomplete symmetric monoidal closed category.

(2.4.12) $\mathcal{G}_*$-V is a complete and cocomplete symmetric monoidal closed category.

(2.4.18) Length-one inclusion defines a pointed functor $i : \mathcal{F} \longrightarrow \mathcal{G}$.

(2.4.19) Smash product $\wedge : \mathcal{G} \longrightarrow \mathcal{F}$ is a strict symmetric monoidal pointed functor.

(2.5.1) Each functor in (2.5.1), except $\mathrm{J}^{\mathcal{T}}$ and $\mathrm{K}^{\mathcal{G}}$, is an equivalence of homotopy theories.

(2.5.3) Segal $K$-theory is the composite functor $\mathrm{K}^{\mathsf{Se}} = \mathrm{K}^{\mathcal{F}} \mathrm{Ner}_* \mathrm{J}^{\mathsf{Se}}$.

(2.5.4) Segal $J$-theory is not a multifunctor, so neither is $\mathrm{K}^{\mathsf{Se}}$.

(2.5.8) Elmendorf-Mandell $K$-theory is the multifunctor $\mathrm{K}^{\mathsf{EM}} = \mathrm{K}^{\mathcal{G}} \mathrm{Ner}_* \mathrm{J}^{\mathcal{T}} \mathrm{End}_{\mathcal{M}\underline{1}}$.

(2.5.16) Segal $K$-theory is an equivalence of homotopy theories.

(2.5.20) Elmendorf-Mandell $K$-theory is an equivalence of homotopy theories.

## Chapter 3. Homotopy Theory of Multicategories

(3.1.13) Each multicategory M has an associated free permutative category FM.

(3.1.14) FI is isomorphic to the permutation category.

(3.1.15) FT is isomorphic to the category of natural numbers and morphisms of finite sets.

(3.1.21) $\mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{st}}$ is a 2-functor.

(3.2.8) F is a left 2-adjoint of End.

(3.3.7) The counit $\varepsilon$ of (F, End) admits a componentwise right adjoint $\varrho$.

(3.3.12) $\varrho_{\mathsf{C}} : \mathsf{C} \longrightarrow \mathsf{FEnd}(\mathsf{C})$ is a symmetric monoidal functor.

(3.4.21) $\mathsf{F}^n$ is a strong $n$-linear functor that is 2-natural with respect to multifunctors and multinatural transformations.

(3.4.31) $\mathsf{F} : \mathsf{Multicat} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ is a non-symmetric Cat-multifunctor.

(3.4.34) The unit $\eta : 1 \longrightarrow \mathsf{EndF}$ is a non-symmetric Cat-multinatural transformation.

(3.5.3) $\mathsf{F} : \mathsf{Multicat} \rightleftarrows \mathsf{PermCat}^{\mathsf{st}} : \mathsf{End}$ is an adjoint equivalence of homotopy theories.

(3.5.5) For each small non-symmetric Cat-multicategory Q, $(\mathsf{F}^{\mathsf{Q}}, \mathsf{End}^{\mathsf{Q}})$ are inverse equivalences of homotopy theories between $\mathsf{Multicat}^{\mathsf{Q}}$ and $(\mathsf{PermCat}^{\mathsf{su}})^{\mathsf{Q}}$.

(3.5.7) $\mathsf{F} : \mathsf{Multicat} \rightleftarrows \mathsf{PermCat}^{\mathsf{su}} : \mathsf{End}$ are inverse equivalences of homotopy theories.

(3.5.9) Inclusion $I : \mathsf{PermCat}^{\mathsf{st}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ is an equivalence of homotopy theories.

## Part 2. Homotopy Theory of Pointed Multicategories, $\mathcal{M}\underline{1}$-Modules, and Permutative Categories

## Chapter 4. Pointed Multicategories and $\mathcal{M}\underline{1}$-Modules Model All Connective Spectra

(4.1.11) Each pointed multicategory M has an associated permutative category $\mathsf{F}_{\bullet}\mathsf{M}$.

(4.1.17) $\mathsf{F}_{\bullet} : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{st}}$ is a 2-functor.

(4.2.3) $\mathsf{p} : \mathsf{F} \longrightarrow \mathsf{F}_{\bullet}$ is a 2-natural transformation with each component a strict symmetric monoidal functor.

(4.2.5) $\mathsf{p}_{\mathsf{M}}$ is a 2-pushout of $\mathsf{FT} \longrightarrow \mathbf{1}$ in $\mathsf{PermCat}^{\mathsf{st}}$.

(4.3.5) For each small pointed multicategory M, $\eta_{\mathsf{M}}^{\bullet} : \mathsf{M} \longrightarrow \mathsf{End}_{\bullet}\mathsf{F}_{\bullet}\mathsf{M}$ is a pointed multifunctor that is 2-natural in M.

(4.3.6) For each small permutative category C, $\varepsilon_{\mathsf{C}}^{\bullet} : \mathsf{F}_{\bullet}\mathsf{End}_{\bullet}\mathsf{C} \longrightarrow \mathsf{C}$ is a strict symmetric monoidal functor.

(4.3.9) $\varepsilon_{\mathsf{C}}^{\bullet}$ is 2-natural in C.

(4.3.11) There is a 2-adjunction $\mathsf{F}_{\bullet} : \mathsf{Multicat}_* \rightleftarrows \mathsf{PermCat}^{\mathsf{st}} : \mathsf{End}_{\bullet}$.

(4.3.14) There is a 2-natural isomorphism $\mathsf{F} \cong \mathsf{F}_{\bullet} \circ (-)_+$.

(4.4.1) There is a 2-adjunction $\mathsf{F}_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \rightleftarrows \mathsf{PermCat}^{\mathsf{st}} : \mathsf{End}_{\mathcal{M}\underline{1}}$ with $\mathsf{F}_{\mathcal{M}\underline{1}} = \mathsf{F}_{\bullet}\mathsf{U}_{\mathcal{M}\underline{1}}$.

(4.4.5) There is a 2-natural isomorphism $\mathsf{F}_{\bullet} \cong \mathsf{F}_{\mathcal{M}\underline{1}} \circ (\mathcal{M}\underline{1} \wedge -)$.

(4.5.3) $\mathsf{F}_{\bullet}\mathsf{T} \cong \mathbf{1}$.

(4.5.7) A left $\mathcal{M}\underline{1}$-module structure on a small pointed multicategory M determines and is uniquely determined by binary operations $\pi_1^2(x)$ for objects $x \in \mathsf{M}$ that satisfy basepoint, unit, and interchange conditions.

(4.5.21) For each left $\mathcal{M}\underline{1}$-module M, each morphism in $F_{\mathcal{M}\underline{1}}M$ is represented by a length-one sequence.

(4.6.5) For each small permutative category C, $\varrho_C^\bullet : C \longrightarrow F_\bullet\mathsf{End}_\bullet C$ is a strictly unital symmetric monoidal functor.

(4.6.6) The adjunction $\varepsilon_C \dashv \varrho_C$ extends to an adjunction $\varepsilon_C^\bullet \dashv \varrho_C^\bullet$ in $\mathsf{PermCat}^{\mathsf{su}}$.

(4.6.13) $\eta_{\mathsf{End}_\bullet C}^\bullet = \mathsf{End}_\bullet \varrho_C^\bullet$.

(4.7.3) $F_\bullet : \mathsf{Multicat}_* \rightleftharpoons \mathsf{PermCat}^{\mathsf{st}} : \mathsf{End}_\bullet$ is an adjoint equivalence of homotopy theories.

(4.7.4) $(-)_+ : \mathsf{Multicat} \longrightarrow \mathsf{Multicat}_*$ is an equivalence of homotopy theories.

(4.8.1) $\mathcal{M}\underline{1} \wedge - : \mathsf{Multicat}_* \rightleftharpoons \mathsf{Mod}^{\mathcal{M}\underline{1}} : U_{\mathcal{M}\underline{1}}$ is an adjoint equivalence of homotopy theories.

(4.8.3) $F_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \rightleftharpoons \mathsf{PermCat}^{\mathsf{st}} : \mathsf{End}_{\mathcal{M}\underline{1}}$ is an adjoint equivalence of homotopy theories.

## **Chapter 5**. Multiplicative Homotopy Theory

(5.1.9) $F_\bullet^n$ is a strong $n$-linear functor.

(5.1.11) $F_\bullet^n$ is 2-natural with respect to pointed multifunctors and pointed multinatural transformations.

(5.2.6) $F_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ is a non-symmetric Cat-multifunctor.

(5.2.8) $p : FU_\bullet \longrightarrow F_\bullet$ is a non-symmetric Cat-multinatural transformation.

(5.3.2) $\eta^\bullet : 1 \longrightarrow \mathsf{End}_\bullet F_\bullet$ is a non-symmetric Cat-multinatural transformation.

(5.3.3) $\varrho^\bullet : 1 \longrightarrow F_\bullet\mathsf{End}_\bullet$ is a non-symmetric Cat-multinatural transformation.

(5.4.1) For each small non-symmetric Cat-multicategory Q, $(F_\bullet^Q, \mathsf{End}_\bullet^Q)$ are inverse equivalences of homotopy theories between $\mathsf{Multicat}_*^Q$ and $(\mathsf{PermCat}^{\mathsf{su}})^Q$.

(5.5.2) $F_{\mathcal{M}\underline{1}} : \mathsf{Mod}^{\mathcal{M}\underline{1}} \longrightarrow \mathsf{PermCat}^{\mathsf{su}}$ is a non-symmetric Cat-multifunctor.

(5.5.5) $\eta^{\mathcal{M}\underline{1}} : 1 \longrightarrow \mathsf{End}_{\mathcal{M}\underline{1}}F_{\mathcal{M}\underline{1}}$ is a non-symmetric Cat-multinatural transformation.

(5.5.8) $\varrho^{\mathcal{M}\underline{1}} : 1 \longrightarrow F_{\mathcal{M}\underline{1}}\mathsf{End}_{\mathcal{M}\underline{1}}$ is a non-symmetric Cat-multinatural transformation.

(5.5.11) $\eta_{\mathsf{End}_{\mathcal{M}\underline{1}}C}^{\mathcal{M}\underline{1}} = \mathsf{End}_{\mathcal{M}\underline{1}}\varrho_C^{\mathcal{M}\underline{1}}$.

(5.5.12) For each small non-symmetric Cat-multicategory Q, $(F_{\mathcal{M}\underline{1}}^Q, \mathsf{End}_{\mathcal{M}\underline{1}}^Q)$ are inverse equivalences of homotopy theories between $(\mathsf{Mod}^{\mathcal{M}\underline{1}})^Q$ and $(\mathsf{PermCat}^{\mathsf{su}})^Q$.

(5.5.14) For each small (non-)symmetric Cat-multicategory Q, $\big((\mathcal{M}\underline{1} \wedge -)^Q, U_{\mathcal{M}\underline{1}}^Q\big)$ are inverse equivalences of homotopy theories between $\mathsf{Multicat}_*^Q$ and $(\mathsf{Mod}^{\mathcal{M}\underline{1}})^Q$.

**Part 3**. Enrichment of Diagrams and Mackey Functors in Closed Multicategories

## **Chapter 6**. Multicategorically Enriched Categories

(6.1.27) For each non-symmetric multicategory M, there is a 2-category M-Cat of small M-categories, M-functors, and M-natural transformations.

(6.2.1) For a monoidal category V, V-Cat and $(\mathsf{End}\,V)$-Cat are the same 2-categories.

(6.4.11) For small permutative categories C and D, $P^{\mathsf{su}}(C, D)$ is a small permutative category.

(6.4.17) Composition $m_{B,C,D}$ is a bilinear functor.

(6.4.20) $P^{\mathsf{su}}$ is a $P^{\mathsf{su}}$-category.

(6.5.7) For small permutative categories C and D, $\mathsf{ev}_{C,D}$ is a bilinear functor.

(6.5.8) $m_{B,C,D}$ is compatible with evaluation.

(6.6.7) For a multicategory M and an M-category C, $C^{\mathsf{op}}$ is an M-category.

(6.6.8) For a symmetric monoidal category V, an opposite V-category is the same as an opposite (End V)-category.

## Chapter 7. Change of Multicategorical Enrichment

(7.1.9) Each non-symmetric multifunctor $F$ induces a change-of-enrichment 2-functor.

(7.2.1) The change-of-enrichment 2-functor of a multifunctor preserves opposite enriched categories.

(7.3.1) For a monoidal functor $U$, the change-of-enrichment 2-functors along $U$ and End $U$ are the same.

(7.4.1) Change-of-enrichment 2-functors of non-symmetric multifunctors are closed under composition.

(7.5.6) There is a 2-functor $\mathbf{E} : \mathsf{Multicat}^{\mathsf{ns}} \longrightarrow 2\mathsf{Cat}$ that sends a small non-symmetric multicategory M to M-Cat.

## Chapter 8. The Closed Multicategory of Permutative Categories

(8.1.1) A closed multicategory is a multicategory equipped with $n$-ary internal hom objects, symmetric group action on internal hom objects, and multicategorical evaluation that satisfy equivariance and evaluation bijection axioms.

(8.1.16) For each symmetric monoidal closed category, the endomorphism multicategory is closed.

(8.2.13) Each $\underline{\mathsf{P}^{\mathsf{su}}}(\langle \mathsf{C} \rangle; \mathsf{D})$ is a permutative category.

(8.2.16) The permutative categories $\underline{\mathsf{P}^{\mathsf{su}}}(\langle \mathsf{C} \rangle; \mathsf{D})$ admit symmetric group action that satisfies the equivariance axioms for internal hom objects.

(8.3.8) Each $\mathsf{ev}_{\langle \mathsf{C} \rangle; \mathsf{D}}$ is a multilinear functor.

(8.4.1) $\mathsf{P}^{\mathsf{su}}$ satisfies the evaluation bijection axiom.

(8.4.2) For $\mathsf{P}^{\mathsf{su}}$, the inverse of $\chi$ is $\Psi$.

(8.4.9) $\mathsf{P}^{\mathsf{su}}$ satisfies the equivariance axioms for evaluation bijection.

(8.4.15) $\mathsf{P}^{\mathsf{su}}$ is a closed multicategory.

## Chapter 9. Self-Enrichment and Standard Enrichment

(9.1.7) Each non-symmetric closed multicategory admits a canonical self-enrichment.

(9.1.8) For $\mathsf{P}^{\mathsf{su}}$, the self-enrichment coincides with the canonical self-enrichment.

(9.1.9) For a symmetric monoidal closed category V, the canonical self-enrichment of V coincides with the canonical self-enrichment of End V.

(9.2.12) Each non-symmetric multifunctor $F$ admits a standard enrichment $\widehat{F}$.

(9.2.17) For a monoidal functor $U$ between symmetric monoidal closed categories, the standard enrichment of $U$ coincides with the standard enrichment of End $U$.

(9.3.6) Standard enrichment functors are closed under composition in an appropriate sense.

(9.4.2) The standard enrichment of $\mathsf{K}^{\mathsf{EM}}$ factors into four spectral functors.

## Chapter 10. Enriched Mackey Functors of Closed Multicategories

(10.1.8) For a non-symmetric closed multicategory M and an M-category C, a C-diagram in M is precisely a left C-module.

(10.1.13) For a closed multicategory M and an M-category C, a C-Mackey functor in M is precisely a left $\mathsf{C}^{\mathsf{op}}$-module.

(10.1.17) An M-natural transformation between C-diagrams in M is precisely a left C-module morphism.

(10.1.22) For vertically composable M-natural transformations between C-diagrams in M, composition commutes with taking partners componentwise.

(10.1.26) Each simplicial, cofibrantly generated, proper, and stable model category is Quillen equivalent to a category of spectral Mackey functors.

(10.1.27) For each finite group $G$, the category of genuine $G$-equivariant spectra is Quillen equivalent to a spectral Mackey functor category associated to the permutative Burnside category.

(10.3.1) For each non-symmetric multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ between non-symmetric closed multicategories and a small M-category C, there is an induced diagram change-of-enrichment functor from M-Cat(C, M) to N-Cat($C_F$, N).

(10.3.4) For each multifunctor $F : \mathsf{M} \longrightarrow \mathsf{N}$ between closed multicategories and a small M-category C, there is an induced presheaf change-of-enrichment functor from M-Cat(C$^{\mathsf{op}}$, M) to N-Cat(($C_F$)$^{\mathsf{op}}$, N).

(10.4.1) Diagram change-of-enrichment functors are closed under composition.

(10.4.5) Presheaf change-of-enrichment functors are closed under composition.

(10.5.1) $\mathsf{K}^{\mathsf{EM}}$ induces diagram and presheaf change-of-enrichment functors.

(10.6.2) $\mathsf{K}^{\mathsf{EM}}_\star$ factors into four change-of-enrichment functors.

### Part 4. Homotopy Theory of Enriched Diagrams and Mackey Functors

### Chapter 11. Homotopy Equivalences between Enriched Diagram Categories

(11.1.1) For non-symmetric multifunctors $F : \mathsf{M} \rightleftarrows \mathsf{N} : E$ between non-symmetric closed multicategories, a small N-category C, and a multinatural transformation $\xi : 1_\mathsf{N} \longrightarrow FE$, there is an induced functor $F^\xi_\star$ from M-Cat($C_E$, M) to N-Cat(C, N).

(11.2.1) A multinatural transformation $\kappa : 1_\mathsf{M} \longrightarrow EF$ induces a natural transformation $\kappa^\star : 1 \longrightarrow E_\star F^\xi_\star$ on M-Cat($C_E$, M).

(11.3.1) A multinatural transformation $\xi : 1_\mathsf{N} \longrightarrow FE$ induces a natural transformation $\xi^\star : 1 \longrightarrow F^\xi_\star E_\star$ on N-Cat(C, N).

(11.4.1) For a non-symmetric closed multicategory P equipped with a relative category structure $\mathcal{W}$ and a P-category D, there is an induced relative category structure on P-Cat(D, P).

(11.4.14) Under appropriate assumptions, inverse equivalences of homotopy theories $(F, E)$ lift to inverse equivalences of homotopy theories $(F^\xi_\star, E_\star)$ between M-Cat($C_E$, M) and N-Cat(C, N).

(11.4.24) If, furthermore, $E$ is a multifunctor, then $(F^\xi_\star, E_\star)$ are inverse equivalences of homotopy theories between M-Cat(($C_E$)$^{\mathsf{op}}$, M) and N-Cat(C$^{\mathsf{op}}$, N).

### Chapter 12. Applications to Multicategories and Permutative Categories

(12.1.6) $\mathsf{F}_\bullet$ and $\mathsf{End}_\bullet$ induce inverse equivalences of homotopy theories between (i) $C_{\mathsf{End}_\bullet}$-diagrams in Multicat$_*$ and C-diagrams in PermCat$^{\mathsf{su}}$ and (ii) $C_{\mathsf{End}_\bullet}$-Mackey functors in Multicat$_*$ and C-Mackey functors in PermCat$^{\mathsf{su}}$.

(12.4.6) $\mathsf{F}_{\mathcal{M}1}$ and $\mathsf{End}_{\mathcal{M}1}$ induce inverse equivalences of homotopy theories between (i) $C_{\mathsf{End}_{\mathcal{M}1}}$-diagrams in Mod$^{\mathcal{M}1}$ and C-diagrams in PermCat$^{\mathsf{su}}$ and (ii) $C_{\mathsf{End}_{\mathcal{M}1}}$-Mackey functors in Mod$^{\mathcal{M}1}$ and C-Mackey functors in PermCat$^{\mathsf{su}}$.

(12.6.6) $\mathcal{M}\underline{1} \wedge -$ and $U_{\mathcal{M}\underline{1}}$ induce inverse equivalences of homotopy theories between (i) $D_{U_{\mathcal{M}\underline{1}}}$-diagrams in Multicat$_*$ and D-diagrams in Mod$^{\mathcal{M}\underline{1}}$ and (ii) $D_{U_{\mathcal{M}\underline{1}}}$-Mackey functors in Multicat$_*$ and D-Mackey functors in Mod$^{\mathcal{M}\underline{1}}$.

## Appendix A. Categories

(A.1.2) **Grothendieck's Axiom of Universes**. Every set belongs to some universe.
(A.1.6) Each monoidal category satisfies $\lambda_1 = \rho_1$.
(A.1.7) Each monoidal category satisfies the left and right unity properties.
(A.1.13) Each braided monoidal category satisfies $\rho = \lambda\xi_{-,1}$ and $\lambda = \rho\xi_{1,-}$
(A.1.16) A symmetric monoidal category is precisely a braided monoidal category that satisfies the symmetry axiom.
(A.1.21) Cat is a symmetric monoidal closed category.
(A.1.30) Iterated monoidal products are left normalized.
(A.2.2) Cat is a 2-category.
(A.2.3) PermCat, PermCat$^{\mathsf{st}}$, and PermCat$^{\mathsf{su}}$ are 2-categories.

## Appendix B. Enriched Category Theory

(B.1.7) A locally small 2-category is precisely a Cat-category.
(B.1.12) V-Cat is a 2-category.
(B.2.2) The tensor product is a 2-functor on V-Cat.
(B.2.6) V-Cat is a monoidal category if V is braided monoidal. It is symmetric monoidal if V is.
(B.2.18) V-Cat is a monoidal Cat-category if V is braided monoidal. It is a symmetric monoidal Cat-category if V is symmetric monoidal.
(B.2.27) If V is a braided monoidal category, then there is a 2-category of small monoidal V-categories. If V is a symmetric monoidal category, then there are a 2-category of small braided monoidal V-categories and a 2-category of small symmetric monoidal V-categories.
(B.3.2) Evaluation is the counit of an adjunction.
(B.3.7) For a symmetric monoidal closed category V, the canonical self-enrichment is a symmetric monoidal V-category.
(B.4.6) Each monoidal functor induces a change-of-enrichment 2-functor.
(B.4.7) Change-of-enrichment 2-functors are closed under composition.
(B.4.9) For a braided monoidal functor $U$, change of enrichment is a monoidal Cat-functor, which is symmetric if $U$ is.
(B.4.10) Change of enrichment preserves enriched monoidal structure.
(B.4.11) For a symmetric monoidal functor $U : V \longrightarrow W$ with V symmetric monoidal closed, $\underline{V}_U$ is a symmetric monoidal W-category.
(B.4.17) For a monoidal functor $U : V \longrightarrow W$ between symmetric monoidal closed categories, the standard enrichment is a monoidal W-functor, which is symmetric if $U$ is.

## Appendix C. Multicategories

(C.1.16) Each non-symmetric V-multicategory has an underlying V-category.
(C.1.17) The terminal multicategory T consists of a single object and a single $n$-ary operation for each $n$.
(C.1.18) Each object in a V-multicategory generates an endomorphism V-operad.
(C.1.24) Each V-multifunctor restricts to a V-functor.

(C.1.27) Each V-multinatural transformation restricts to a V-natural transformation.

(C.1.33) There is a 2-category with (non-symmetric) small V-multicategories as objects.

(C.1.35) The initial V-multicategory has an empty set of objects. The terminal V-multicategory has one object and each multimorphism object given by the terminal object in V.

(C.2.2) A Cat-multinatural transformation consists of component 1-ary 1-cells that satisfy two Cat-naturality conditions for objects and morphisms.

(C.3.1) Each symmetric monoidal category has an endomorphism multicategory. Each symmetric monoidal functor induces a multifunctor. Each monoidal natural transformation between symmetric monoidal functors induces a multinatural transformation.

(C.3.6) The endomorphism multicategory defines a 2-functor.

(C.3.9) Each symmetric monoidal V-category induces a V-multicategory.

(C.4.4) A pointed structure on a multicategory consists of a basepoint object and $n$-ary basepoint operations that satisfy symmetry, unity, and composition axioms.

(C.4.8) Each symmetric monoidal category induces a pointed endomorphism multicategory with the basepoint object given by the monoidal unit. Each strictly unital symmetric monoidal functor induces a pointed multifunctor.

(C.4.9) There is a 2-category with small pointed multicategories as objects.

(C.4.10) The pointed endomorphism multicategory defines a 2-functor.

(C.4.11) There is a forgetful 2-functor $\mathsf{U}_\bullet : \mathsf{Multicat}_* \longrightarrow \mathsf{Multicat}$.

(C.4.16) Adjoining a basepoint is a left 2-adjoint of $\mathsf{U}_\bullet$.

# List of Notations

| Standard Notations | Description |
|---|---|
| $\mathsf{Ob}(\mathsf{C})$, $\mathsf{Ob}\,\mathsf{C}$ | objects in a category $\mathsf{C}$ |
| $\mathsf{C}(X,Y)$, $\mathsf{C}(X;Y)$ | set of morphisms $X \longrightarrow Y$ |
| $1, 1_X$ | identity morphism |
| $\mathrm{dom}(f)$, $\mathrm{cod}(f)$ | domain and codomain of a morphism |
| $g \circ f$, $gf$ | composition of morphisms |
| $\cong$, $\xrightarrow{\cong}$ | an isomorphism |
| $\sim$, $\xrightarrow{\sim}$ | an equivalence |
| $F : \mathsf{C} \longrightarrow \mathsf{D}$ | a functor |
| $\mathrm{Id}_\mathsf{C}$, $1_\mathsf{C}$ | identity functor |
| $\mathbf{1}$ | terminal category |
| $(\mathsf{Set}, \times, *)$ | category of sets and functions |
| $\theta_X$ | a component of a natural transformation $\theta$ |
| $1_F$ | identity natural transformation |
| $\phi\theta$ | vertical composition of natural transformations |
| $\theta' * \theta$ | horizontal composition of natural transformations |
| $(L, R)$, $L \dashv R$ | an adjunction |
| $\eta, \varepsilon$ | unit and counit of an adjunction |
| $\varnothing$, $\varnothing^\mathsf{C}$ | an initial object |
| $\coprod$, $\amalg$ | a coproduct |
| $\prod$, $\pi$ | a product |
| $\Sigma_n$ | symmetric group on $n$ letters |

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