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André-Quillen homology of commutative algebras

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ABSTRACT. These notes are an introduction to basic properties of André-Quillen homology for commutative algebras. They are an expanded version of my lectures at the summer school. The aim is to give fairly complete proofs of characterizations of smooth homomorphisms and of locally complete intersection homomorphisms in terms of vanishing of André-Quillen homology. The choice of the material, and the point of view, are guided by these goals.

CONTENTS

1. Introduction

In the late 60's André and Quillen introduced a (co)-homology theory for commutative algebras that now goes by the name of André-Quillen (co)-homology. This is the subject of these notes. They are no substitute for either the panoramic view that [**22**] provides, or the detailed exposition in [**23**] and [**2**].

My objective is to provide complete proofs of characterizations of two important classes of homomorphisms of noetherian rings: regular homomorphisms and locally complete intersection homomorphisms, in terms of André-Quillen homology. However, I have chosen to treat only the case when the homomorphism is essentially of finite type; this notion is recalled a few paragraphs below. One reason for this

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choice is that it is this class of homomorphisms which is of principal interest from the point of view of algebraic geometry.

The main reason is that there are technical hurdles, even at the level of definitions and which have nothing to do with André-Quillen homology, that have to be crossed in dealing with general homomorphisms, and delving into those aspects would be too much of a digression. The problem is intrinsic: There are many results for homomorphisms essentially of finite type (notably, those involving completions and localizations) that are simply not true in general, and require additional hypotheses. Some of these issues are discussed in the text.

André-Quillen homology is also discussed in Paul Goerss' notes in this volume. There it appears as the derived functor of abelianization, while here it viewed as the derived functor (in a non-abelian context) of Kähler differentials. Another difference is that in the former, as in Quillen's approach, simplicial resolutions are treated in the general context of cofibrant replacements in model categories. Here I have described, as André does, an explicit procedure for building simplicial resolutions. This approach is ad hoc, but it does allow one to construct resolutions in the main cases of interest in these notes. In any event, it was useful and entertaining to work with 'concrete' simplicial algebras and modules. However, when it comes to establishing the basic properties of André-Quillen homology, I have followed Quillen's more homotopy theoretic treatment, for I believe that it is easier to grasp.

A few words now on the exposition: Keeping in line with the aim of the summer school, and the composition of its participants, I have written these notes for an audience of homotopy theorists and (commutative) algebraists. Consequently, I have taken for granted material that will be familiar to mathematicians of either persuasion, but have attempted to treat with some care topics that may be unfamiliar to one or the other. For instance, I have not hesitated to work with the homotopy category of complexes of modules, and even its structure as a triangulated category, but I do discuss in detail simplicial resolutions (presumably for algebraists), and Kähler differentials (presumably for homotopy theorists).

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Notation. The rings in the paper are commutative.

1.1. *Complexes.* For these notes, the principal reference for homological algebra of complexes is Weibel's book [**26**], and sometimes also the article [**6**], by Avramov and Foxby. Complexes of modules will be graded homologically:

$$\cdots \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots .$$

When necessary, the differential of a complex M is denoted ∂^M. The *suspension* of M, denoted ΣM, is the complex with

$$(\Sigma M)_n = M_{n-1} \quad \text{and} \quad \partial^{\Sigma M} = -\partial^M .$$

Given complexes of R-modules L and M, the notation $L \simeq M$ indicates that L and M are homotopy equivalent.

1.2. *Homomorphisms.* Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. One says that φ is *flat* if the R-module S is flat. If the R-algebra S is finitely

generated, then φ is of *finite type*; it is *essentially of finite type* if S is a localization, at a multiplicatively closed set, of a finitely generated R-algebra.

The notation $(R, \mathfrak{m}, k)$ denotes a (commutative, noetherian) local ring R, with maximal ideal $\mathfrak{m}$, and residue field $k = R/\mathfrak{m}$. A homomorphism of local rings $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ is *local* if $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

For every prime ideal $\mathfrak{p}$ in R, we set $k(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$; this is the residue field of R at $\mathfrak{p}$. The *fiber* of φ over $\mathfrak{p}$ is the $k(\mathfrak{p})$-algebra $S \otimes_R k(\mathfrak{p})$. Given a prime ideal $\mathfrak{q}$ in S, the induced local homomorphism $R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ is denoted $\varphi_{\mathfrak{q}}$.

For results in commutative ring theory, we usually refer to Matsumura [**19**].

2. Kähler differentials

Let $\varphi\colon R \to S$ be a homomorphism of commutative rings and N an S-module.

The ring S is commutative, so any S-module (be it a left module or a right module) is canonically an S-bimodule; for instance, when N is a left S-module, the right S-module structure is defined as follows: for $n \in N$ and $s \in S$, set

$$n \cdot s = sn$$

In what follows, it will be assumed tacitly that any S-module, in particular, N, is an S-bimodule, and hence also an R-bimodule, via φ.

2.1. *Derivations.* An *R-linear derivation* of S with coefficients in N is a homomorphism of R-modules $\delta\colon S \to N$ satisfying the Leibniz rule:

$$\delta(st) = \delta(s)t + s\delta(t) \quad \text{for } s, t \in S.$$

An alternative definition is that δ is a homomorphism of abelian groups satisfying the Leibniz rule, and such that $\delta\varphi = 0$. The set of R-linear derivations of S with coefficients in N is denoted $\operatorname{Der}_R(S; N)$. This is a subset of $\operatorname{Hom}_R(S, N)$, and even an S-submodule, with the induced action:

$$(s \cdot \delta)(t) = s\delta(t)$$

for $s, t \in S$ and $\delta \in \operatorname{Der}_R(S; N)$.

EXERCISE 2.1.1. Let M be an S-module. The homomorphism of S-modules

$$\operatorname{Hom}_S(M, N) \otimes_S \operatorname{Hom}_R(S, M) \longrightarrow \operatorname{Hom}_R(S, N)$$
$$\alpha \otimes \beta \mapsto \alpha\beta$$

restricts to a homomorphism of S-modules:

$$\operatorname{Hom}_S(M, N) \otimes_S \operatorname{Der}_R(S; M) \longrightarrow \operatorname{Der}_R(S; N)$$

In particular, for each derivation $\delta\colon S \to M$, composition induces a homomorphism of S-modules $\operatorname{Hom}_S(M, N) \to \operatorname{Der}_R(S; N)$.

2.2. *Kähler differentials.* It is not hard to verify that the map $N \mapsto \operatorname{Der}_R(S; N)$ is an additive functor on the category of S-modules. It turns out that this functor is representable, that is to say, there is an S-module Ω and an R-linear derivation $\delta\colon S \to \Omega$ such that, for each S-module N, the induced homomorphism

$$\operatorname{Hom}_S(\Omega, N) \longrightarrow \operatorname{Der}_R(S; N)$$

of S-modules, is bijective. Such a pair (Ω, δ) is unique up to isomorphism, in a suitable sense of the word; one calls Ω the module of *Kähler differentials* and δ the

universal derivation of φ. In these notes, they are denoted Ω_φ and δ^φ respectively; we sometimes follow established usage of writing $\Omega_{S|R}$ for Ω_φ.

In one case, the existence of such an Ω_φ is clear:

EXERCISE 2.2.1. Prove that when φ is surjective $\Omega_\varphi = 0$.

A homomorphism φ such that $\Omega_\varphi = 0$ is said to be *unramified*.

There are various constructions of the module of Kähler differentials and the universal derivation; see Matsumura [**19**, §9], and Exercise 2.6. The one presented below is better tailored to our needs:

We are in the world of commutative rings, so the product map

$$\mu_R^S \colon S \otimes_R S \longrightarrow S \quad \text{where } s \otimes t \mapsto st$$

is a homomorphism of rings. Set $I = \operatorname{Ker}(\mu_R^S)$. Via μ_R^S the $S \otimes_R S$-module I/I^2 acquires the structure of an S-module. Set

$$\Omega_\varphi = I/I^2 \quad \text{and} \quad \delta^\varphi \colon S \longrightarrow \Omega_\varphi \quad \text{with } \delta^\varphi(s) = (1 \otimes s - s \otimes 1).$$

As the notation suggests, $(\Omega_\varphi, \delta^\varphi)$ is the universal pair we seek. The first step in the verification of this claim is left as an

EXERCISE 2.2.2. The map δ^φ is an R-linear derivation.

By Exercise 2.1.1, the map δ^φ induces a homomorphism of S-modules

$$(*) \qquad \operatorname{Hom}_S(\Omega_\varphi, N) \longrightarrow \operatorname{Der}_R(S; N)$$

We prove that this map is bijective by constructing an explicit inverse.

Let $\delta \colon S \to N$ be an R-linear derivation. As δ is a homomorphism of R-modules, extension of scalars yields a homomorphism of S-modules

$$\delta' \colon S \otimes_R S \to N, \quad \text{where} \quad \delta'(s \otimes t) = s\delta(t)$$

Here we view $S \otimes_R S$ as an S-module via action on the left hand factor of the tensor product: $s \cdot (x \otimes y) = (sx \otimes y)$. One thus obtains, by restriction, a homomorphism of S-modules $I \to N$, also denoted δ'.

EXERCISE 2.2.3. Verify the following claims.

(1) $\delta'(I^2) = 0$, so δ' induces a homomorphism of S-modules $\widetilde{\delta} \colon \Omega_\varphi \to N$.
(2) The assignment $\delta \mapsto \widetilde{\delta}$ gives a homomorphism $\operatorname{Der}_R(S; N) \to \operatorname{Hom}_S(\Omega_\varphi, N)$ of S-modules, and it is an inverse to the map $(*)$ above.

This exercise justifies the claim that $\delta^\varphi \colon S \to \Omega_\varphi$ is a universal derivation.

The next goal is an explicit presentation for Ω_φ as an S-module, given the presentation of S as an R-algebra. The first step towards it is the following exercise describing the module of Kähler differentials for polynomial extensions of R. Solve it in two ways: by using the construction in paragraph 2.2 above; by proving directly that it has the desired universal property.

EXERCISE 2.3. Let $S = R[Y]$ be the polynomial algebra over R on a set of variables Y, and $\varphi \colon R \to S$ the inclusion map. Prove that

$$\Omega_\varphi = \bigoplus_{y \in Y} S dy \quad \text{and} \quad \delta^\varphi(r) = \sum_{y \in Y} \frac{\partial r}{\partial y} dy\,.$$

Here, Ω_φ is a free S-module on a basis $\{dy\}_{y \in Y}$, and $\partial(-)/\partial y$ denotes partial derivative with respect to y.

2.4. *Jacobi-Zariski sequence.* Let $Q \xrightarrow{\psi} R \xrightarrow{\varphi} S$ be a homomorphism of commutative rings. One has a natural exact sequence of S-modules:

$$\Omega_\psi \otimes_R S \xrightarrow{\alpha} \Omega_{\varphi\psi} \xrightarrow{\beta} \Omega_\varphi \longrightarrow 0\,. \tag{2.4.1}$$

The maps in question are defined are follows: by restriction, the R-linear derivation $\delta^\varphi\colon S \to \Omega_\varphi$ is also a Q-linear derivation, hence it induces the homomorphism $\beta\colon \Omega_{\varphi\psi} \to \Omega_\varphi$ such that $\beta \circ \delta^{\varphi\psi} = \delta^\varphi$. In the same vein, $\delta^{\varphi\psi}\varphi\colon R \to \Omega_{\varphi\psi}$ is a Q-linear derivation, so it induces an R-linear homomorphism $\alpha'\colon \Omega_\psi \to \Omega_{\varphi\psi}$; the map α is obtained by extension of scalars, for $\Omega_{\varphi\psi}$ is an S-module.

I leave it to you to verify that the sequence (2.4.1) is exact. It is sometimes called the Jacobi-Zariski sequence. One way to view André-Quillen homology is that it extends this exact sequence to a long exact sequence; that is to say, it is a 'left derived functor' of Ω_-, viewed as a functor of algebras; see 6.7.

When N is a S-module, applying $\operatorname{Hom}_S(-,N)$ to the exact sequence (2.4.1), and using the identification in 2.2, yields an exact sequence of S-modules

$$0 \longrightarrow \operatorname{Der}_R(S\,;N) \longrightarrow \operatorname{Der}_Q(S\,;N) \longrightarrow \operatorname{Der}_Q(R\,;N)\,.$$

One could just as well have deduced (2.4.1) from the naturality of this sequence.

EXERCISE 2.4.1. Interpret the maps in the exact sequence above.

The following exercise builds on Exercise 2.3.

EXERCISE 2.5. Let $\psi\colon R[Y] \to R[Z]$ be a homomorphism of R-algebras, where Y and Z are sets of variables. Verify that the map

$$\Omega_{\psi\,|\,R}\colon \Omega_{R[Y]\,|\,R} \longrightarrow \Omega_{R[Z]\,|\,R} \quad \text{is defined by} \quad y \mapsto \sum_{z\in Z} \frac{\partial y}{\partial z}dz\,.$$

Sequence (2.4.1) allows for a 'concrete' description of the Kähler differentials:

EXERCISE 2.6. Write $S = R[Y]/(\boldsymbol{r})$, where $R[Y]$ is the polynomial ring over R on a set of variables Y and $\boldsymbol{r} = \{r_\lambda\}$ is a set of polynomials in $R[Y]$ indexed by the set Λ. Note that such a presentation of S is always possible. The homomorphism φ is then the composition $R \hookrightarrow R[Y] \twoheadrightarrow S$.

Prove that the module of Kähler differentials of φ is presented by

$$\bigoplus_{\lambda\in\Lambda} Se_\lambda \xrightarrow{\partial} \bigoplus_{y\in Y} Sdy \longrightarrow \Omega_\varphi \longrightarrow 0\,, \quad \text{where}$$

$$\partial(e_\lambda) = \sum_{y\in Y} \frac{\partial(r_\lambda)}{\partial y}dy\,.$$

The matrix representing ∂ is the Jacobian matrix of the polynomials $\boldsymbol{r}$.

Here is an exercise to give you a feel for the procedure outlined above:

EXERCISE 2.7. Let k be a field, $S = k[y]$, the polynomial ring in the variable y, and let R be the subring $k[y^2, y^3]$. Find a presentation for the module of Kähler differentials for the inclusion $R \hookrightarrow S$.

The relevance of the following exercise should be obvious.

EXERCISE 2.8. Let S and T be R-algebras. Prove that there is a natural homomorphism of $(S \otimes_R T)$-modules

$$(S \otimes_R \Omega_{T|R}) \bigoplus (\Omega_{S|R} \otimes_R T) \longrightarrow \Omega_{(S \otimes_R T)|R}\,,$$

and that this map is bijective.

In a special case, one can readily extend (2.4.1) one step further to the left:

2.9. *The conormal sequence.* Suppose $S = R/I$, where I is an ideal in R, and $\varphi\colon R \to S$ the canonical surjection; in particular, $\Omega_\varphi = 0$. Let $\psi\colon Q \to R$ be a homomorphism of rings. The exact sequence (2.4.1) extends to an exact sequence

$$I/I^2 \xrightarrow{\ \zeta\ } \Omega_\psi \otimes_R S \xrightarrow{\ \alpha\ } \Omega_{\varphi\psi} \longrightarrow 0$$

of S-modules. The map ζ is defined as follows: restricting the universal derivation δ^ψ gives a Q-linear derivation $I \to \Omega_\psi$ and hence, by composition, a Q-linear derivation $\delta\colon I \to (\Omega_\psi \otimes_R S) = \Omega_\psi/I\Omega_\psi$. Keeping in mind that δ is a derivation it is easy to verify that $\delta(I^2) = 0$, so it factors through I/I^2; this is the map ζ. It is also elementary to check that ζ is S-linear.

3. Simplicial algebras

This section is a short recap on simplicial algebras and simplicial modules. The aim is to introduce enough structure, terminology, and notation to be able to work with simplicial algebras and their resolutions, and construct cotangent complexes, the topics of forthcoming sections. The reader may refer to [**13**] and [**20**] for in-depth treatments of things simplicial.

To begin with, let me try to explain what we are trying to do here.

3.1. *Computing derived functors.* I remarked during the discussion on Jacobi-Zariski sequence (2.4.1) that André-Quillen homology may be viewed as a derived functor of $\Omega_{\_}$. In order to understand the problem, and its solution, let us revisit the process of deriving a more familiar functor.

As before, let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let $\mathcal{M}(S)$ be the category of S-modules. Consider the functor

$$S \otimes_R -\colon \mathcal{M}(S) \longrightarrow \mathcal{M}(S) \quad \text{where } N \mapsto S \otimes_R N.$$

Each exact sequence of S-modules $0 \to N \to N' \to N'' \to 0$ gives rise to an exact sequence of S-modules

$$(*) \qquad S \otimes_R N \longrightarrow S \otimes_R N' \longrightarrow S \otimes_R N'' \longrightarrow 0$$

However, the homomorphism on the left is not injective, unless S is flat as an R-module; in short, the functor $S \otimes_R -$ is left-exact, but it is not exact. This lack of exactness is compensated by extending the sequence above to a long exact sequence. There are three steps involved in this process:

Step 1. Construct a projective resolution F of S over R.

Step 2. Show that F is unique up to homotopy of complexes of R-modules.

Step 3. Set $\mathsf{T}_\varphi = F \otimes_R S$, and for each S-module N set

$$\mathrm{H}^\varphi_n(N) = \mathrm{H}_n(\mathsf{T}_\varphi \otimes_S N)$$

One then has $\mathrm{H}_0^{\varphi}(N) = S \otimes_R N$, and the functors $\{\mathrm{H}_n^{\varphi}(-)\}_{n\geqslant 0}$ extend $(*)$ above to a long exact sequence of S-modules, which is what one wants. We have not discovered anything new here: $\mathrm{H}_n^{\varphi}(N) = \operatorname{Tor}_n^R(S, N)$.

Note that the complex T_{φ} is well-defined in the homotopy category of complexes of S-modules; this follows from Step 2. It is this property that dictates the kind of resolution we pick. For instance, flat resolutions, although a natural choice, would not work, for they are not unique, even up to homotopy. It is another matter that they can be used to compute $\mathrm{H}_n^{\varphi}(N)$.

We turn now to the functor of interest Ω_-. Taking a cue from the preceding discussion, the plan is to attach a complex of projective S-modules called the *cotangent complex of* φ, which I denote $\mathcal{L}_{\varphi}$, with

$$\mathrm{H}_0(\mathcal{L}_{\varphi} \otimes_S N) = \Omega_{\varphi} \otimes_S N$$

such that it extends the sequence (2.4.1) to a long exact sequence of S-modules. The functor Ω_- is non-linear: it takes into account the structure of S as an R-algebra, rather than as an R-module. Keeping this in mind, one should pick a suitable category of R-algebras, and a notion of homotopy for morphisms in that category, such that resolutions have the following properties:

(a) They must reflect the structure of S as an R algebra.
(b) They should be unique up to homotopy.
(c) The functor Ω_- must preserve homotopies, in a suitable sense of the word.

It turns out that simplicial algebras provide the right context for obtaining such resolutions; confer [**11**] for a discussion about why this is so.

QUESTION. Why is the category of differential graded R-algebras not suitable for the purpose on hand?

3.2. *Simplicial modules and algebras.* As usual, let R be a commutative ring.

A *simplicial R-module* is a simplicial object in the category of R-modules, that is to say, a collection $V = \{V_n\}_{n\geqslant 0}$ of R-modules such that, for each non-negative integer n, there are homomorphisms of R-modules:

$$d_i \colon V_n \to V_{n-1} \quad \text{and} \quad s_j \colon V_n \to V_{n+1} \qquad \text{for } 0 \leq i, j \leq n.$$

called *face maps* and *degeneracies*, respectively, satisfying the identities:

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{when } i < j \\ d_i s_j &= \begin{cases} s_{j-1} d_i & \text{when } i < j \\ 1 & \text{when } i = j, j+1 \\ s_j d_{i-1} & \text{when } i > j+1 \end{cases} \\ s_i s_j &= s_{j+1} s_i \quad \text{when } i \leq j \end{aligned} \tag{3.2.1}$$

Prescribing this data is equivalent to defining a contravariant functor from the ordinal number category to the category of R-modules; see [**11**, §(1.5)]

A *simplicial R-algebra* is a simplicial object in the category of R-algebras; thus, it is a simplicial R-module A where each A_n has the structure of an R-algebra, and the face and degeneracies are homomorphisms of R-algebras. A *simplicial module* over a simplicial R-algebra A is a simplicial R-module V where each V_n is an A_n-module and the face maps and degeneracies are compatible with those on A.

EXAMPLE 3.3. Given an R-module N, it is not hard to verify that the graded module $\mathsf{s}(N)$, with

$$\mathsf{s}(N)_n = N \quad \text{and} \quad d_i = \mathrm{id}^N = s_j \qquad \text{for } 0 \leq i, j \leq n.$$

is a simplicial R-module. For any R-algebra S, it is evident that $\mathsf{s}(S)$ is a simplicial R-algebra.

3.4. *Normalization.* Let V be a simplicial R-module. The *normalization* of V is the complex of R-modules $\mathsf{N}(V)$, defined by

$$\mathsf{N}(V)_n = \bigcap_{i=1}^{n} \operatorname{Ker}(d_i) \quad \text{with differential}$$
$$\partial_n = d_0 \colon \mathsf{N}(V)_n \longrightarrow \mathsf{N}(V)_{n-1}.$$

That ∂_n is a differential follows from (3.2.1). The *nth homotopy module* of V is the R-module

$$\pi_n(V) = \mathrm{H}_n(\mathsf{N}(V)).$$

This is not the 'right' way to introduce homotopy, but will serve the purpose here, see [**11**, (2.15)].

There is another way to pass from simplicial modules to complexes: The face maps on V give the graded R-module underlying V the structure of a complex of R-modules, with differential:

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_i \colon V_n \to V_{n-1}.$$

This complex is also denoted V; this could cause confusion, but will not, for the structure involved is usually clear from the context. Fortunately, the homology of this complex is the same as the homotopy, see, for instance, [**13**, Chapter III, (2.7)].

EXERCISE 3.5. Let S be an R-algebra. Verify that $\mathsf{N}(\mathsf{s}(S)) = S$.

EXERCISE 3.6. Let A be a simplicial R-algebra and V a simplicial A-module.

Verify that for each n, the R-submodules $\mathsf{N}(V)_n$, $\operatorname{Ker}(\partial_n)$, and $\operatorname{Im}(\partial_{n+1})$ of V_n are stable under the action of A_n, that is to say, they are A_n-submodules of V_n.

In particular, $\pi_0(A)$ is an R-algebra. Moreover, when V_n is a noetherian A_n-module, so is $\pi_n(V)$.

NOTES 3.7. For each simplicial R-algebra A, the graded module $\pi_*(A)$ is a commutative $\pi_0(A)$-algebra with divided powers, see [**12**, (2.3)]. Moreover, if V is a simplicial A-module, $\pi_*(V)$ is a graded $\pi_*(A)$-module. These structures play no role in this write-up, but they are an important facet of the theory; see [**4**] and [**7**].

NOTES 3.8. The functor $\mathsf{N}(-)$ from simplicial R-modules to complexes of R-modules is an equivalence of categories; this is the content of the Dold-Kan theorem, see [**11**, (4.1)].

EXAMPLE 3.9. Let A be a simplicial R-algebra and N an $\pi_0(A)$-module. Then $\mathsf{s}(N)$ is a simplicial A-module, where, for each non-negative integer n, the A_n-module structure on $\mathsf{s}(N)_n$ is induced via the composed homomorphism of rings

$$A_n \xrightarrow{d_{i_n}} A_{n-1} \xrightarrow{d_{i_{n-1}}} \cdots \xrightarrow{d_{i_1}} A_0.$$

It is an exercise to check that the choices of indices $i_n, \dots, i_1$ is irrelevant.

3.10. *Morphisms.* A *morphism* $\Phi\colon A \to B$ of simplicial R-algebras is a collection of homomorphisms of R-algebras $\Phi_n\colon A_n \to B_n$, one for each $n \geq 0$, commuting with both face maps and degeneracies. Such a Φ induces a homomorphism of R-modules

$$\pi_*(\Phi)\colon \pi_*(A) \to \pi_*(B)$$

One says that Φ is a *weak equivalence* if $\pi_*(\Phi)$ is bijective. I will leave it to you to formulate the definition of a morphism of simplicial modules.

EXAMPLE 3.11. Let A be a simplicial R-algebra. Following Example 3.9, it is not hard to verify that any homomorphism of R-algebras $\phi\colon \pi_0(A) \to S$ induces a morphism of simplicial algebras, $\Phi\colon A \to \mathsf{s}(S)$. Given Example 3.3 it is clear that

$$\pi_n(\Phi) = \begin{cases} \phi & \text{when } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus, Φ is a weak equivalence if and only if ϕ is bijective and $\pi_n(A) = 0$ for $n \geq 1$.

To summarize Examples 3.3 and 3.11: The functor $\mathsf{s}(-)$ is a faithful embedding of the category of R-algebras into the category of simplicial R-algebras, and $\pi_0(-)$ is a left adjoint to this embedding.

EXERCISE 3.11.1. Prove that the embedding is also full.

3.12. *Tensor products.* The *tensor product* of simplicial A-modules V and W is the simplicial A-module denoted $V \otimes_A W$, with

$$(V \otimes_A W)_n = V_n \otimes_{A_n} W_n \quad \text{for each } n \geq 0,$$

and face maps and degeneracies induced from those on V and W. When N is an $\pi_0(A)$-module, it is customary to write $V \otimes_A N$ for $V \otimes_A \mathsf{s}(N)$.

Various standard properties of tensor products (for example: associativity and commutativity) carry over to this context.

4. Simplicial resolutions

This section discusses simplicial resolutions. The first step is to introduce free extensions, which are analogues in simplicial algebra of bounded-below complexes of free modules in the homological algebra of complexes over rings.

4.1. *Free simplicial extensions.* Let A be a simplicial R-algebra. We call a *free*[1] simplicial extension of A on a graded set $X = \{X_n\}_{n\geqslant 0}$ of indeterminates a simplicial R-algebra, denoted $A[X]$, satisfying the following conditions:

(i) $A[X]_n = A_n[X_n]$, the polynomial ring over A_n on the variables X_n;
(ii) $s_j(X_n) \subseteq X_{n+1}$ for each j, n;
(iii) The inclusion $A \hookrightarrow A[X]$ is a morphism of simplicial R-algebras.

Note that there is no restriction on the face maps.

For instance, if $S = R[Y]$ is a polynomial ring over R, then the simplicial algebra $\mathsf{s}(S)$ is a free extension of $\mathsf{s}(R)$, with $X_n = Y$ for each n.

4.2. *Base change.* If $A[X]$ is a free extension of A, and $\Phi\colon A \to B$ is a morphism of simplicial algebras, then $B \otimes_A A[X]$ is a free extension of B.

[1]See footnote for Definition 4.20 of [**11**]

4.3. *Existence of resolutions.* Let $\phi\colon A \to B$ be a morphism of simplicial R-algebras. A *simplicial resolution* of B over A is a factorization of ϕ as a diagram

$$A \xrightarrowtail{} A[X] \xrightarrow{\Phi} B$$

of morphisms of simplicial algebra, with $A \to A[X]$ a free extension and Φ a surjective weak equivalence. Usually, one refers to $A[X]$ itself as a simplicial resolution of B over A. Simplicial resolutions exist; one procedure for constructing them is described in the paragraphs below; see 4.12.

4.4. *Existence of lifting.* Given a commutative diagram of simplicial algebras

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \overset{\kappa}{\nearrow} & \downarrow{\scriptstyle \simeq}\,\Phi \\ A[X] & \longrightarrow & C \end{array}$$

where Φ is surjective and a weak equivalence, there exists a morphism κ that preserves the commutativity of the diagram.

For a proof of the lifting property, see [**11**, (5.4)]. The morphism κ in the diagram above is unique up to homotopy, in a sense described below. The definition may appear to come out of the blue, but it is a special case of a notion of homotopy in model categories. Much of this following discussion is best viewed in that general context; see Dwyer and Spalinski [**10**, (4.1)], or [**11**, (2.2)].

4.5. *Homotopy.* Let $A \to A[X]$ be a free extension. For each integer n, one has the product morphism

$$\mu_n = A_n[X_n] \otimes_{A_n} A_n[X_n] \longrightarrow A_n[X_n]\,.$$

They form a morphism of simplicial A-algebras

$$\mu\colon A[X] \otimes_A A[X] \to A[X]\,.$$

It is convenient to write $A[X,X]$ for $A[X] \otimes_A A[X]$.

The simplicial algebra $A[X,X]$ has the functorial properties one expects of a product. Namely, given morphisms $\Phi, \Psi\colon A[X] \to B$ of simplicial A-algebras, there is an induced morphism of simplicial algebras

$$\Phi \odot \Psi\colon A[X,X] \longrightarrow B \quad \text{with} \quad (\Phi \odot \Psi)_n(x \otimes x') = \Phi(x)\Psi(x')\,.$$

Let $A[X,X,Y]$ be a simplicial resolution of $A[X]$ over $A[X,X]$; it is called a *cylinder object* for the A-algebra $A[X]$, see [**11**, (2.4)]. The morphisms Φ and Ψ are *homotopic* if $\Phi \odot \Psi$ extends to a cylinder object, that is to say, there is a commutative diagram of morphisms of simplicial A-algebras

$$\begin{array}{ccc} A[X,X] & \longrightarrow & A[X,X,Y] \\ & {\scriptstyle \Phi\odot\Psi}\searrow & \downarrow \\ & & B \end{array}$$

Given the lifting property of free extensions 4.4, it is easy to check that the notion of homotopy does not depend on the choice of a cylinder object, and that homotopy is an equivalence relation on morphisms of A-algebras, see [**10**, (4.7)].

4.6. *Uniqueness of lifting.* Using the lifting property 4.4 of free extensions a formal argument shows that the lifting map κ in 4.4 is unique up to homotopy of simplicial A-algebras, see [**10**, (4.9)].

4.7. *Uniqueness of resolutions.* A standard argument using lifting properties of free extensions 4.4, 4.6 yields that simplicial resolutions are unique up to homotopy of simplicial A-algebras.

Given an R-algebra S, it is accepted usage to speak of a simplicial resolution of the R-algebra S, meaning a simplicial resolution of $\mathsf{s}(S)$ over $\mathsf{s}(R)$.

REMARK 4.8. Let S be an R-algebra and M an R-module. Let $R[X]$ be a simplicial resolution of the R-algebra S. The complex underlying $R[X]$ is an R-free resolution of S, so for each integer n, one has

$$\pi_n(R[X] \otimes_R M) = \operatorname{Tor}_n^R(S, M)\,.$$

Next we outline a procedure for constructing simplicial resolutions. The strategy is the one used to obtain free resolutions:

4.9. *Resolutions of modules over rings.* Let M be an R-module. A free resolution of M over R may be built as follows: One constructs a sequence of complexes of free R-modules $0 \subset F^{(0)} \subseteq F^{(1)} \subseteq \cdots$ such that $F^{(0)}$ is a free module mapping onto M, and for each $d \geq 1$ one has

$$\mathrm{H}_i(F^{(d)}) = \begin{cases} M & \text{for } i = 0\,, \\ 0 & \text{for } 1 \leq i \leq d-1\,. \end{cases}$$

Given $F^{(d-1)}$ one builds $F^{(d)}$ by killing cycles in $F_{d-1}^{(d-1)}$ that are not boundaries. In detail: choose a set of cycles $\{z_\lambda\}_{\lambda\in\Lambda}$ which generate $\mathrm{H}_{d-1}(F^{(d-1)})$, and set

$$F_d = \bigoplus_{\lambda\in\Lambda} Re_\lambda$$

$$F^{(d)} = F^{(d-1)} \bigoplus \Sigma^d F_d \quad \text{with} \quad \partial(e_\lambda) = z_\lambda\,.$$

The homology of $F^{(d)}$ is readily computed from the short exact sequence of complexes $0 \to F^{(d-1)} \to F^{(d)} \to \Sigma^d F_d \to 0$.

Then the complex $\cup_{d\geqslant 0} F^{(d)}$ is the desired free resolution of M.

A similar procedure can be used to construct simplicial resolutions. The crucial step then is a method for killing cycles. In the category of modules, to kill a cycle in degree $d-1$ we attached a free module, F_d, in degree d. In the category of simplicial algebras, we have to attach a (polynomial) variable in degree d; however, the simplicial identities (3.2.1) (notably, $d_i s_j = 1$ for $i = j, j+1$) force us to then attach a whole slew of variables in higher degrees.

4.10. *Killing cycles.* Let A be a simplicial R-algebra, d a positive integer, and let $w \in A_{d-1}$ be a cycle in $\mathsf{N}(A)_{d-1}$, the normalized chain complex of A, see 3.4.

The goal is to construct a free extension of A in which the cycle w becomes a boundary; I write $A[\{x\} \mid \partial(x) = w]$, or just $A[\{x\}]$ when the cycle being killed is understood, for the resulting simplicial algebra. It has the following properties:

(a) For each integer n, the A_n-algebra $A[\{x\}]_n$ is free on a set X_n of finite cardinality. In particular, if the ring A_n is noetherian, so is the ring $A[\{x\}]_n$.

(b) The inclusion $A \hookrightarrow A[\{x\}]$ induces isomorphisms

$$\pi_n(A) \cong \pi_n(A[\{x\}]) \qquad \text{for } n \leq d-2.$$

(c) One has an exact sequence of $\pi_0(A)$-modules

$$0 \to A_{d-1}\mathrm{cls}(w) \to \pi_{d-1}(A) \to \pi_{d-1}(A[\{x\}]) \to 0\,.$$

where $\mathrm{cls}(w)$ is the class of the cycle w in $\pi_{d-1}(A)$. Note that the ideal $A_{d-1}w \subseteq A_{d-1}$ consists of cycles.

The construction of $A[\{x\}]$ is as follows:

The set X. For each positive integer n, set

$$X_n = \{x_t \mid t\colon [n] \to [d] \text{ is surjective and monotone.}\}$$

Clearly, $\mathrm{card}(X_n)$ is finite, as claimed.

The face and degeneracies on $A[\{x\}]$ extend those on A, so to define them, it suffices to specify their action the set X. This process involves the co-face and co-degeneracy maps, see [**11**, (1.10)].

Degeneracies. For each $x_t \in X_n$, set

$$s_j(x_t) = x_{t\circ s^j} \quad \text{for } 0 \leq j \leq n.$$

Here $s^j\colon [n+1] \to [n]$ is the jth co-degeneracy operator.

Face maps. The set X_d is a singleton: $\{x_{\mathrm{id}}\}$. Set

$$d_i(x_{\mathrm{id}}) = \begin{cases} w & i = 0 \\ 0 & 1 \leq i \leq d \end{cases}$$

It remains to define face maps on X_n for $n \geq d+1$. Fix such an n and a surjective monotone map $t\colon [n] \to [d]$. If for a co-face map $d^i\colon [n-1] \to [n]$ the composed map $t \circ d^i\colon [n-1] \to [d]$ is not surjective, then one has a commutative diagram

$$\begin{array}{ccc} [n] & \xrightarrow{\ t\ } & [d] \\ {\scriptstyle d^i}\uparrow & & \uparrow{\scriptstyle d^j} \\ [n-1] & \xrightarrow[\ u\ ]{} & [d-1] \end{array}$$

where u is surjective and monotone. The face operators on X_n is defined as follows:

$$d_i(x_t) = \begin{cases} x_{t\circ d^i} & \text{if } t \circ d^i \text{ is surjective,} \\ u_*(d_j(w)) & \text{otherwise.} \end{cases}$$

where $u_*\colon A_{d-1} \to A_{n-1}$ is the map induced by u. Here we are viewing A as defining a contravariant functor from the ordinal number category to the category of R-algebras; u_* is the image of u under this functor, see [**11**, (1.10)].

EXERCISE 4.10.1. Prove that, with the prescription above, $A[\{x\}]$ is a simplicial R-algebra, and a free extension of A.

As to the properties of $A[\{x\}]$: given 4.10(a), it is clear that 4.10(b) holds.

EXERCISE 4.10.2. Prove that property 4.10.(c) holds. Hint: use 3.4.

4.11. *Generating cycles.* The preceding construction can also be used to create cycles in degree d: take $w = 0$.

Using 4.10 and 4.11, and taking limits one arrives at the conclusion below. This is a good place to point out that this construction of resolutions is best viewed in the context of a general technique called the 'small object argument', due to Quillen, see [**11**, (3.5)].

4.12. *Resolutions exist.* Given a morphism $\Phi\colon A \to B$ of simplicial R-algebras, a simplicial resolution of B over A exists. As noted before, see 4.7, such a resolution is unique up to homotopy of simplicial A-algebras.

The following result is clear from the properties of the construction in 4.10.

PROPOSITION. *When R is noetherian and the R-algebra S is finitely generated, S admits a simplicial resolution $R[X]$ with* $\operatorname{card}(X_n)$ *finite for each n.* □

Next I will describe an explicit resolution of the homomorphism $R[y] \twoheadrightarrow R$ where $y \mapsto 0$. It serves both as an elementary example of a simplicial resolution, and as a way to construct resolutions of complete intersections; see 4.16.

CONSTRUCTION 4.13. Let $R[y]$ be the polynomial ring over R, in the variable y, and let $\lambda\colon R[y] \to R$ be the homomorphism of R-algebras defined by $\lambda(y) = 0$.

What is described below is the bar construction of the R-algebra $R[y]$ with coefficients in R; see [**16**, Chapter X, §2]. It is built as follows: For each $n \geq 0$, set

$$B_n = R[y] \otimes_R R[y]^{\otimes n}$$

It is convenient to write $b[b_1|\cdots|b_n]$ for the element $b \otimes b_1 \otimes \cdots \otimes b_n$ in B_n.

Evidently, B_n is a polynomial algebra over $R[y]$ over the set of n indeterminates $X_n = \{x_{nk}\}_{k=0}^{n-1}$, where

$$x_{nk} = [1|\cdots|y|\cdots|1] \quad \text{with } y \text{ the } (k+1)\text{st tensor.}$$

For each $0 \leq i \leq n$, extend the mapping

$$d_i([b_1|\dots|b_n]) = \begin{cases} b_1[b_2|\dots|b_n] & \text{for } i = 0 \\ [b_1|\dots|b_ib_{i+1}|\dots|b_n] & \text{for } 1 \leq i \leq n-1 \\ [b_1|\dots|b_{n-1}]\lambda(b_n) & \text{for } i = n \end{cases}$$

to a homomorphism of $R[y]$-algebras $d_i\colon B_n \to B_{n-1}$. In the same vein, for each $0 \leq j \leq n$ extend the mapping

$$s_j([b_1|\dots|b_n]) = \begin{cases} [1|b_1|\dots|b_n] & \text{for } j = 0 \\ [b_1|1|\dots|b_jb_{j+1}|\dots|b_n] & \text{for } 1 \leq j \leq n-1 \\ [b_1|\dots|b_n|1] & \text{for } j = n \end{cases}$$

to a homomorphism of $R[y]$-algebras $s_j\colon B_n \to B_{n+1}$.

EXERCISE 4.13.1. The $R[y]$-algebra $B = \{B_n\}$, with face and degeneracy operators defined above, is a free simplicial extension of $R[y]$.

Note that the homomorphism λ consists of killing the cycle y in $R[y]$.

EXERCISE 4.13.2. Reconcile the construction given in this paragraph with the free extension $R[y][\{x\} \mid \partial(x) = y]$ obtained from 4.10.

The relevant properties of this free extension are as follows.

LEMMA 4.14. *The canonical surjection $\epsilon\colon B \to R$ is a weak equivalence, and hence a simplicial resolution of R over $R[y]$.*

Let K denote the complex of $R[y]$-modules $0 \to R[y] \xrightarrow{y} R[y] \to 0$ concentrated in degrees 0 and 1. The morphism of complexes $\nu\colon K \to B$ defined by $\nu_0 = \mathrm{id}^{R[y]}$ and $\nu_1(r) = r \otimes y$, is a homotopy equivalence.

SKETCH OF A PROOF. One way to prove this result is as follows: It can be checked directly that ν is compatible with the surjections $\epsilon\colon B \to R$ and $\pi\colon K \to R$. Moreover, both ϵ and π are quasi-isomorphisms; this is clear for π, and is not hard to prove for ϵ, see [**16**, Chapter X, §2]. Thus, since B and K are both bounded below complexes of free $R[y]$-modules, it follows that ν is a homotopy equivalence.

Another method is to prove first that ν is a homotopy equivalence, and so deduce that ϵ is a weak equivalence. I will leave it to you to construct a homotopy inverse to ν. □

Given the preceding construction, it is easy to get a simplicial resolution of R over $R[y_1, \dots, y_d]$. The underlying idea is explained in the following exercise.

EXERCISE 4.15. Let K be a commutative ring and let $R' \to S'$ and $R'' \to S''$ be homomorphism of commutative K-algebras, such that R' and S' are flat as K-modules.

If B' and B'' are simplicial resolutions of S' over R', and of S'' over R'', respectively, then $B' \otimes_K B''$ is a simplicial resolution of $S' \otimes_K S''$ over $R' \otimes_K R''$.

Building on the Construction 4.13, I will describe a simplicial resolution of the R-algebra $R/(r)$, when r is a nonzerodivisor on R.

CONSTRUCTION 4.16. Let $S = R/(r)$, and $\varphi\colon R \to S$ the canonical surjection.

Let $R[y]$ denote the polynomial ring over R, in the variable y. Let $\lambda\colon R[y] \to R$ and $\psi\colon R[y] \to R$ be homomorphisms of rings defined by $\lambda(y) = 0$ and $\psi(y) = r$, respectively. Consider the diagram of homomorphisms of commutative rings:

$$\begin{array}{ccc} R[y] & \xrightarrow{\lambda} & R \\ {\scriptstyle\psi}\downarrow & & \downarrow \\ R & \xrightarrow{\varphi} & S = R^{\psi} \otimes_{R[y]} {}^{\epsilon}R \end{array}$$

Let $R[y] \to B \to R$ be the simplicial resolution of λ; see Construction 4.13 and Lemma 4.14. Set

$$A = R^{\psi} \otimes_{R[y]} B\,.$$

This is a free extension of R. Base change along along ψ yields a diagram $R \to A \to S$ of morphisms of simplicial algebras. Since the complex of $R[y]$-modules B is homotopy equivalent to the complex K, defined in Lemma 4.14, one obtains that the complex of R-modules A is homotopy equivalent to the complex $R^{\psi} \otimes_{R[y]} K$, that is to say, to the complex:

$$0 \longrightarrow R \xrightarrow{r} R \longrightarrow 0$$

Therefore, we conclude:

$$\pi_n(A) = \begin{cases} S & \text{if } n = 0 \\ (0\colon {}_R r) & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

Given this calculation, the proof of the following result is clear.

LEMMA. *If r is a nonzerodivisor on R, then A is a simplicial resolution of the R-algebra S.* □

The modules and face and degeneracy maps in A are described completely by the data in 4.13.

EXERCISE 4.17. Suppose that $r_1, \dots, r_c$ is an R-regular sequence, see 8.1. Mimic the proof of the preceding lemma to construct a simplicial resolution of $R/(\boldsymbol{r})$ over R. Hint: use Exercise 4.15 and Remark 8.2.

5. The cotangent complex

We are now prepared to introduce the protagonist of these notes: the cotangent complex of a homomorphism. This section describes one construction of the cotangent complex, and a discussion of its basic properties.

5.1. *Kähler differentials.* Since $\Omega_{-|R}$ is a functor on the category of R-algebras, it extends to a functor on the category of simplicial R-algebras: Given a simplicial R-algebra A, one obtains a simplicial A-module $\Omega_{A|R}$, with

$$(\Omega_{A|R})_n = \Omega_{A_n|R}$$

for each n, and face maps and degeneracies induced by those on A. Moreover, each morphism $\Phi\colon A \to B$ of simplicial R-algebras induces a morphism of simplicial R-modules $\Omega_{\Phi|R}\colon \Omega_{A|R} \to \Omega_{B|R}$. All this is clear from the properties of $\Omega_{-|R}$ discussed in Section 2.

5.2. *The cotangent complex.* Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let A be a simplicial resolution of S over R, and set

$$\mathcal{L}_\varphi = \Omega_{A|R} \otimes_A S\,.$$

Thus, $\mathcal{L}_\varphi$ is a simplicial S-module; the associated complex of S-modules is called the *cotangent complex* of S over R; more precisely, of φ. This too we denote $\mathcal{L}_\varphi$.

Simplicial resolutions are unique up to homotopy, see 4.12, and $\Omega_{-|R}$ transforms homotopy equivalent morphisms of simplicial algebras into homotopy equivalent morphisms of simplicial modules, so the complex $\mathcal{L}_\varphi$ is well defined in the homotopy category of complexes of S-modules; this is explained in the next paragraph. It is in this sense that one speaks of *the* cotangent complex.

The crucial point is the following.

5.3. *Weak equivalences and differentials.* If a morphism of free simplicial R-algebras $\Phi\colon R[X] \to R[Y]$ is a weak equivalence, then the induced morphism of simplicial R-modules $\Omega_{R[X]|R} \to \Omega_{R[Y]|R}$ is also a weak equivalence.

The idea is that Φ admits a homotopy inverse, and hence there is a homotopy inverse also to $\Omega_{\Phi|R}$. Perhaps the best way to formalize this argument is to use the model category structures on the categories of simplicial R-algebras and on simplicial R-modules, see [**22**, §1].

5.4. *Homotopies.* If $\Phi, \Psi\colon R[X] \to B$ are homotopic morphisms of simplicial R-algebras, then the induced morphisms of simplicial R-modules $\Omega_{\Phi\,|\,R}$ and $\Omega_{\Psi\,|\,R}$, from $\Omega_{R[X]\,|\,R}$ to $\Omega_{B\,|\,R}$, are homotopic.

Indeed, suppose $R[X, X, Y]$ is a cylinder object for the R-algebra $R[X]$, see 4.5. Thus, there is diagram of simplicial R-algebras

$$R[X, X] \rightarrowtail R[X, X, Y] \xrightarrow{\simeq} R[X]$$

where the composed is the product map. Applying $\Omega_{-\,|\,R}$ yields a diagram of simplicial R-modules

$$\Omega_{R[X,X]\,|\,R} \rightarrowtail \Omega_{R[X,X,Y]\,|\,R} \xrightarrow{\simeq} \Omega_{R[X]\,|\,R}$$

The crucial information in the diagram is that the arrow on the right is a weak equivalence; this is by 5.3. Concatenating this diagram with the natural morphism of simplicial R-modules

$$\Omega_{R[X]\,|\,R} \bigoplus \Omega_{R[X]\,|\,R} \rightarrowtail \left(R[X] \otimes_R \Omega_{R[X]\,|\,R}\right) \bigoplus \left(\Omega_{R[X]\,|\,R} \otimes_R R[X]\right) \cong \Omega_{R[X,X]\,|\,R}$$

where the isomorphism is by Exercise 2.8, one obtains a diagram

$$\Omega_{R[X]\,|\,R} \bigoplus \Omega_{R[X]\,|\,R} \rightarrowtail \Omega_{R[X,X,Y]\,|\,R} \xrightarrow{\simeq} \Omega_{R[X]\,|\,R}$$

of simplicial R-modules. It is easy to check the composed map is: $(a, b) \mapsto a + b$. The diagram above is tantamount to the statement that $\Omega_{R[X,X,Y]\,|\,R}$ is a cylinder object for the simplicial R-module $\Omega_{R[X]\,|\,R}$, see [**11**, (2.4)].

Now, applying $\Omega_{-\,|\,R}$ to the diagram defining a homotopy between Φ and Ψ, see 4.5, one obtains a commutative diagram of simplicial R-modules

$$\begin{array}{ccc} \Omega_{R[X]\,|\,R} \bigoplus \Omega_{R[X]\,|\,R} & \longrightarrow & \Omega_{R[X,X,Y]\,|\,R} \\ & \searrow^{\Omega_{\Phi\,|\,R}+\Omega_{\Psi\,|\,R}} & \downarrow \simeq \\ & & \Omega_{B\,|\,R} \end{array}$$

Since $\Omega_{R[X,X,Y]\,|\,R}$ is a cylinder object for $\Omega_{R[X]\,|\,R}$, the diagram above means that $\Omega_{\Phi\,|\,R}$ and $\Omega_{\Psi\,|\,R}$ are homotopic, see [**11**, (2.7)].

EXERCISE 5.5. Let $A \xrightarrow{\epsilon} S$ be a simplicial resolution, as above. Then $A \otimes_R S$ is a simplicial R-algebra. Let J be the kernel of the morphism of simplicial S-algebras $A \otimes_R S \to \mathsf{s}(S)$, where $\epsilon(a \otimes s) = \epsilon(a)s$. Note that J is a simplicial ideal in $A \otimes_R S$; that is to say, J is a simplicial $(A \otimes_R S)$-submodule of $A \otimes_R S$.

Prove that one has an isomorphism of simplicial S-modules:

$$\mathcal{L}_\varphi \cong J/J^2\,.$$

Here J^2 is the simplicial ideal in $A \otimes_R S$ with $(J^2)_n = (J_n)^2$.

NOTES 5.6. The gist of the preceding exercise is that one may view the cotangent complex as 'derived indecomposables'. There are other interpretations of the cotangent complex: as the derived functor of the abelianization functor, see [**11**, (4.24)]; as cotriple homology, see [**26**, §8.8].

In [**3**], André introduces cotangent complex as in 5.2, but by using a canonical resolution of the R-algebra S. This has the benefit that one does have to worry that that it is well-defined, and so avoids, in particular, the discussion in 5.4. However,

to establish any substantial property of cotangent complexes, one would have to prove that they can be obtained from any simplicial resolution, and so he does.

REMARK 5.7. Let $A = R[X]$ be a simplicial resolution of the R-algebra S. For each integer n, one has $A_n = R[X_n]$, so Exercise 2.3 yields: $\Omega_{A_n|R}$ is a free A_n-module, and hence $(\mathcal{L}_\varphi)_n$ is a free S-module, on a basis of cardinality $\operatorname{card}(X_n)$.

5.8. *André-Quillen homology and cohomology.* The cotangent complex of φ is well-defined up to homotopy of complexes of S-modules, so for each S-module N and integer n, the following S-modules are well-defined:

$$\mathrm{D}_n(S|R;N) = \mathrm{H}_n(\mathcal{L}_\varphi \otimes_S N) \quad \text{and} \quad \mathrm{D}^n(S|R;N) = \mathrm{H}_{-n}(\operatorname{Hom}_S(\mathcal{L}_\varphi, N))$$

These are the nth *André-Quillen homology*, respectively, *André-Quillen cohomology*, of S over R with coefficients in N.

The cotangent complex is a complex of free S-modules concentrated in non-negative degrees, therefore

$$\mathrm{D}_n(S|R;N) = \operatorname{Tor}_n^S(\mathcal{L}_\varphi, N) \quad \text{and} \quad \mathrm{D}^n(S|R;N) = \operatorname{Ext}_S^n(\mathcal{L}_\varphi, N)\,.$$

Given this interpretation, a standard argument in the homological algebra of complexes, see, for instance, [**6**, (2.4P), (2.4F)], yields the result below. For any complex L of S-modules, $\operatorname{fd}_S L$ is the *flat dimension* of L; thus, $\operatorname{fd}_S L \le n$ means that L is quasi-isomorphic to a complex $0 \to F_n \to \cdots \to F_i \to 0$ of flat S-modules. The number $\operatorname{pd}_S L$ is the *projective dimension* of L.

PROPOSITION 5.9. *Let n be a non-negative integer.*
One has $\mathrm{D}_i(S|R;-) = 0$ for $i \ge n+1$ if and only if $\operatorname{fd}_S(\mathcal{L}_\varphi) \le n$.
One has $\mathrm{D}^i(S|R;-) = 0$ for $i \ge n+1$ if and only if $\operatorname{pd}_S(\mathcal{L}_\varphi) \le n$. □

Next I describe the cotangent complex in two cases of interest; it turns out that these are essentially the only contexts in which one has information in closed form on the cotangent complex.

PROPOSITION 5.10. *If $S = R[Y]$, a polynomial ring over R in variables Y, then the S-module $\Omega_{S|R}$ is free, and*

$$\mathcal{L}_\varphi \simeq \Omega_{S|R}$$

as complexes of S-modules. Thus, $\mathrm{D}_n(S|R;N) = 0 = \mathrm{D}^n(S|R;N)$ for $n \ge 1$.

PROOF. The freeness of $\Omega_{S|R}$ is the content of Exercise 2.3. Note that the simplicial R-algebra $\mathsf{s}(S)$ is itself a simplicial resolution of S over R. Therefore, one has the first isomorphism below

$$\mathcal{L}_\varphi \cong \Omega_{\mathsf{s}(S)|R} \otimes_{\mathsf{s}(S)} S \cong \mathsf{s}(\Omega_{S|R})\,.$$

The second isomorphism is verified by inspection. Thus, as a complex of S-modules $\mathcal{L}_\varphi$, is isomorphic to

$$\cdots \longrightarrow \Omega_{S|R} \xrightarrow{0} \Omega_{S|R} \xrightarrow{1} \Omega_{S|R} \xrightarrow{0} \Omega_{S|R} \to 0\,.$$

Hence, $\mathcal{L}_\varphi$ is homotopy equivalent to $\Omega_{S|R}$. The remaining assertions now follow, since the S-module $\Omega_{S|R}$ is free. □

PROPOSITION 5.11. *If $S = R/(r)$, where r is a nonzerodivisor in R, and $\varphi\colon R \to S$ is the surjection, then*

$$\mathcal{L}_\varphi \simeq \Sigma S\,.$$

PROOF. The proof uses the notation in 4.13 and 4.16.

It is clear that $\mathcal{L}_\varphi$, which equals $\Omega_{A|R} \otimes_R S$, is a complex of free S-modules beginning in degree 1, and with

$$(\mathcal{L}_\varphi)_n = \bigoplus_{i=0}^{n-1} Sx_{ni} \qquad \text{for each } n \geq 1.$$

In describing the differential on $\mathcal{L}_\varphi$, it is useful to introduce the following symbol: for each integer n, set

$$\epsilon(l,m) = \sum_{k=l}^{m} (-1)^k = \begin{cases} 0 & \text{if } m-l \text{ is even;} \\ -1 & \text{if } m-l \text{ is odd.} \end{cases}$$

With this notation, using the description of A ensuing from 4.13 and Exercise 2.5, one finds that the differential on $(\mathcal{L}_\varphi)_n$ is given by

$$\partial(x_{ni}) = \begin{cases} \epsilon(1,n)x_{n-1,0} & \text{for } i = 0; \\ \epsilon(0,i)x_{n-1,i-1} + \epsilon(i+1,n)x_{n-1,i} & \text{for } 1 \leq i \leq n-2; \\ \epsilon(0,n-1)x_{n-1,n-2} & \text{for } i = n-1. \end{cases}$$

The entries of the matrix representing the differential

$$\partial_n \colon (\mathcal{L}_\varphi)_n \to (\mathcal{L}_\varphi)_{n-1}$$

are either 0 or 1. I claim that $\mathcal{L}_\varphi$ is homotopy equivalent to ΣS.

Indeed, since the matrices representing the differentials consist of zeros and ones, it suffices to verify this assertion when $S = \mathbb{Z}$ (why?); in particular, we may assume S is noetherian. A routine calculation establishes that for any homomorphism $S \to l$, where l is a field, one has

$$\operatorname{rank}_l(\partial_n \otimes_S l) = \begin{cases} \frac{n-2}{2} & \text{if } n \geq 2 \text{ is even;} \\ \frac{n+1}{2} & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

Therefore, for each integer $n \geq 2$, one has that

$$n = \operatorname{rank}_l(\partial_n \otimes_S l) + \operatorname{rank}_l(\partial_{n+1} \otimes_S l)$$

It now remains to do Exercise 5.12 below, noting that $\mathrm{H}_1(\mathcal{L}_\varphi) = S$. □

EXERCISE 5.12. Let S be a noetherian ring and $L = \{L_n\}_{n \geqslant 1}$ a complex of finite free S-modules such that for each prime ideal $\mathfrak{q}$ in S, one has

$$\operatorname{rank}_S(L_n) = \operatorname{rank}_l(\partial_n \otimes_S k(\mathfrak{q})) + \operatorname{rank}_l(\partial_{n+1} \otimes_S k(\mathfrak{q})) \quad \text{for } n \geq 2.$$

Prove that $\mathrm{H}_n(L) = 0$ for $n \geq 2$, and that L is homotopy equivalent to $\Sigma \mathrm{H}_1(L)$.

NOTES 5.13. With better machinery one can give more efficient proofs of Proposition 5.11. For instance, writing Δ^1 for the standard 1-simplex, it is easy to verify that $\mathcal{L}_\varphi$ is the free the simplicial S-module on the simplicial set $\Delta^1/\partial\Delta^1$, which implies the desired statement about its homotopy, see [**11**, (1.15)].

Alternatively, one could note that $\mathcal{L}_\varphi$ is the simplicial complex corresponding to the chain complex with S in degree 1 (and so zero differential) under the Dold-Kan correspondence [**11**, (4.1)], so its homotopy is S.

Exercise 6.10 outlines a third approach. The argument presented above was intended to show that, sometimes, one can work directly with simplicial resolutions and compute cotangent complexes. Unfortunately, this is perhaps the only instance when this is possible, unless one is in characteristic zero, see [**22**, (9.5)].

6. Basic properties

This section is a precis of basic properties of the cotangent complex; usually, they are accompanied by corresponding statements concerning the André-Quillen homology modules. The analogues for cohomology, which are easy to guess, are generally omitted.

As before, let $\varphi\colon R \to S$ be a homomorphism of rings.

6.1. *Functoriality.* The functor $\mathcal{L}_\varphi \otimes_S -$, defined on the homotopy category of complexes of S-modules, is exact. Therefore, the sequence $\{\mathrm{D}_n(S|R;-)\}_{n\in\mathbb{Z}}$ is a homological functor on the category of S-modules.

6.2. *Normalization.* There are isomorphisms of functors

$$\mathrm{D}_0(S|R;-) \cong \Omega_{S|R} \otimes_S - \qquad \text{and} \qquad \mathrm{D}_n(S|R;-) = 0 \quad \text{for each} \quad n < 0\,,$$

where $\Omega_{S|R}$ denotes the S-module of Kähler differentials of S over R.

Indeed, this is immediate from the right-exactness of $\Omega_{-|R}$; see (2.4.1).

6.3. *Base change.* Consider a commutative diagram

$$\begin{array}{ccc} R' & \xrightarrow{\varphi'} & S' \\ {\scriptstyle\rho}\downarrow & & \downarrow \\ R & \xrightarrow[\varphi'\otimes_{R'}R=\varphi]{} & (S'\otimes_{R'}R) \cong S \end{array}$$

of homomorphisms of rings. It induces a morphism of complexes of S-modules:

$$\mathcal{L}_{\varphi'} \otimes_{R'} R \longrightarrow \mathcal{L}_\varphi$$

which is well defined up to homotopy. This morphism is a homotopy equivalence when $\operatorname{Tor}_n^{R'}(S', R) = 0$ for $n \geq 1$; for instance, when either φ' or ρ is flat. In this case, one has isomorphisms of functors

$$\mathrm{D}_n(S|R;-) \cong \mathrm{D}_n(S'|R';-) \quad \text{for each } n \in \mathbb{Z},$$

where S-modules are viewed as S'-modules via the homomorphism $S'\otimes_{R'}\rho\colon S' \to S$.

Indeed, let $A' \to S'$ be a simplicial resolution of S' over R'. This induces a morphism of simplicial R-algebras:

$$A' \otimes_{R'} R \longrightarrow S' \otimes_{R'} R = S$$

Evidently, $A' \otimes_{R'} R$ is a free simplicial extension of R. Thus, if A is a simplicial resolution of S over R, the lifting property yields a morphism of simplicial R-algebra $A'\otimes_{R'} R \to A$, well defined up to homotopy, see 4.4 and 4.6. By functoriality, $\Omega_{-|R}$ induces a morphism of complexes of S-modules:

$$\Omega_{(A'\otimes_{R'}R)|R} \otimes_{(A'\otimes_{R'}R)} S \longrightarrow \Omega_{A|R} \otimes_A S = \mathcal{L}_\varphi\,,$$

well-defined up to homotopy of complexes of S-modules. It remains to identify the complex on the left, and this is accomplished below:

$$\begin{aligned} \Omega_{(A'\otimes_{R'}R)|R} \otimes_{(A'\otimes_{R'}R)} S &\cong \left(\Omega_{A'|R'} \otimes_{A'} R\right) \otimes_{(A'\otimes_{R'}R)} (S' \otimes_{R'} R) \\ &\cong \left(\Omega_{A'|R'} \otimes_{A'} S'\right) \otimes_{R'} R \\ &= \mathcal{L}_{\varphi'} \otimes_{R'} R \end{aligned}$$

The isomorphisms are all verified directly.

Suppose $\operatorname{Tor}_n^{R'}(S', R) = 0$ for $n \geq 1$. As noted in 4.8, one has an isomorphism

$$\pi_n(A' \otimes_{R'} R) \cong \operatorname{Tor}_n^{R'}(S, R) \quad \text{for each } n.$$

Therefore, the augmentation $A' \otimes_{R'} R \to S$ is a weak equivalence, and hence $A' \otimes_{R'} R$ is a simplicial resolution of S over R. Thus, the morphism $A' \otimes_{R'} R \to A$ is a homotopy equivalence, and hence so is the induced morphism $\mathcal{L}_{\varphi'} \otimes_{R'} R \to \mathcal{L}_\varphi$.

Here is a beautiful application, due to André, of the preceding property:

PROPOSITION 6.4. *Let U be a multiplicatively closed subset of R, let $S = U^{-1}R$, and let $\varphi\colon R \to S$ be the localization map. Then $\mathcal{L}_\varphi \simeq 0$.*

PROOF. The complex $\mathcal{L}_\varphi$ consists of S-modules, and the functor $- \otimes_R S$ is the identity on the category of S-modules, so one obtains the isomorphism below

$$\mathcal{L}_\varphi \cong \mathcal{L}_\varphi \otimes_R S \simeq \mathcal{L}_{\varphi \otimes_R S} \simeq \mathcal{L}_{\mathrm{id}^S} \simeq 0$$

The first homotopy equivalence holds by Property 6.3, since the homomorphism φ is flat, the second one holds because the homomorphism $\varphi \otimes_R S\colon S \otimes_R S \to S$ is the identity, while the last one follows, for example, from Proposition 5.10. □

6.5. *Localization.* Fix a prime ideal $\mathfrak{q}$ in S, set $\mathfrak{p} = R \cap \mathfrak{q}$, and denote $\varphi_\mathfrak{q}\colon R_\mathfrak{p} \to S_\mathfrak{q}$ the localization of φ at $\mathfrak{q}$. One has a homotopy equivalence

$$\mathcal{L}_{\varphi_\mathfrak{q}} \simeq S_\mathfrak{q} \otimes_S \mathcal{L}_\varphi$$

of complexes of $S_\mathfrak{q}$-modules. In particular, for each $n \in \mathbb{Z}$, there is an isomorphism of functors of S-modules

$$\mathrm{D}_n(S|R; -)_\mathfrak{q} \cong \mathrm{D}_n(S_\mathfrak{q}|R; -_\mathfrak{q}) \cong \mathrm{D}_n(S_\mathfrak{q}|R_\mathfrak{p}; -_\mathfrak{q})$$

See [**2**] for a proof of these assertions. Alternatively:

EXERCISE 6.5.1. Prove the assertions above.

In this context, one has the following useful remark which permits one to reduce many problems concerning the vanishing of André-Quillen homology to the case of homomorphisms of local rings.

PROPOSITION 6.6. *Let $\varphi\colon R \to S$ be a homomorphism of rings. For each integer n, the following conditions are equivalent:*

(a) $\mathrm{D}_n(S|R; -) = 0$ *on the category of S-modules;*
(b) $\mathrm{D}_n(S_\mathfrak{q}|R_{\mathfrak{q}\cap R}; -) = 0$ *on the category of $S_\mathfrak{q}$-modules, for each $\mathfrak{q} \in \operatorname{Spec} S$.* □

The proof of this result is elementary, given property 6.5. Under an additional hypothesis on φ, there is a significant improvement to the preceding result; see Proposition 8.7.

6.7. *Transitivity.* Each homomorphism of rings $\psi\colon Q \to R$ induces the following exact triangle in the homotopy category of complexes of S-modules:

$$(S \otimes_R \mathcal{L}_\psi) \longrightarrow \mathcal{L}_{\varphi\circ\psi} \longrightarrow \mathcal{L}_\varphi \longrightarrow \Sigma(S \otimes_R \mathcal{L}_\psi)$$

This induces an exact sequence of functors of S-modules

$$\cdots \longrightarrow \mathrm{D}_{n+1}(S|R; -) \longrightarrow \mathrm{D}_n(R|Q; -) \longrightarrow \mathrm{D}_n(S|Q; -) \longrightarrow \mathrm{D}_n(S|R; -) \longrightarrow \cdots.$$

It is called the *Jacobi-Zariski sequence* arising from the diagram $Q \to R \to S$. It extends (2.4.1) to a long exact sequence of S-modules.

For a proof of this assertion, see [**11**, (4.32)], or [**22**, (5.1)].

Use the transitivity sequence to solve the following exercises.

EXERCISE 6.8. Let $U \subset S$ be multiplicatively closed subset, and $\eta\colon S \to U^{-1}S$ the localization map. Prove that one has a homotopy equivalence

$$\mathcal{L}_{\eta\circ\varphi} \simeq U^{-1}S \otimes_S \mathcal{L}_\varphi$$

of complexes of $U^{-1}S$ modules.

EXERCISE 6.9. Let $\psi\colon S \to R$ be a homomorphism of rings such that the map $\varphi\psi\colon S \to S$ equals id^S; said otherwise, S is an algebra retract of R.

Prove that one has a homotopy equivalence of complexes of S-modules:

$$\mathcal{L}_\varphi \simeq \Sigma(\mathcal{L}_\psi \otimes_R S)\,.$$

In particular, $\mathrm{D}_n(S|R;-) \cong \mathrm{D}_{n-1}(R|S;-)$ as functors of S-modules.

EXERCISE 6.10. Use the preceding exercise, and the discussion in Construction 4.16, to prove Proposition 5.11.

6.11. *Finiteness.* Suppose R is noetherian and φ is essentially of finite type.

The complex $\mathcal{L}_\varphi$ is then homotopic to a complex

$$\cdots \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0\,,$$

where for each n, the S-module L_n is finitely generated and free. Thus, when the S-module N is finitely generated so are $\mathrm{D}_n(S|R;N)$ and $\mathrm{D}^n(S|R;N)$.

Indeed, by hypothesis φ admits a factorization

$$R \xrightarrow{\eta} U^{-1}R[Y] \xrightarrow{\varphi'} S$$

with $\mathrm{card}(Y)$ finite, U a multiplicatively closed subset of $R[Y]$, and φ' a surjective homomorphism of rings. Since η factors as $R \to R[U] \to U^{-1}R[Y]$, it follows from Proposition 5.10 and Exercise 6.8 that $\mathcal{L}_\eta$ is equivalent to a complex of finitely generated free $U^{-1}R[Y]$ modules. On the other hand, Proposition 4.12 and Remark 5.7 imply that the complex $\mathcal{L}_{\varphi'}$ consists of finitely generated free modules S-modules. Now the desired result is a consequence of 6.7, applied to the diagram above.

6.12. *Low degrees.* As usual, low degree cohomology modules admit alternative interpretations. First, a piece of notation: $S \ltimes N$ denotes the ring with being $S \oplus N$ the underlying abelian group and product given by $(s,x)(t,y) = (st, sy + tx)$.

To begin with, the S-module $\mathrm{D}^0(S|R;N)$, which is $\mathrm{Der}_R(S\,;N)$, is the set of R-algebra homomorphisms $\alpha\colon S \ltimes N \to S \ltimes N$ extending the identity map both on N and on S, that is to say, such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & S \ltimes N & \xrightarrow{\epsilon} & S & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \alpha} & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & S \ltimes N & \xrightarrow{\epsilon} & S & \longrightarrow & 0\,.
\end{array}$$

Here ϵ is the canonical surjection. This claim is not hard to verify; see [**19**, §25].

The S-module $\mathrm{D}^1(S|R;N)$ is the set of isomorphism classes of extensions of R-modules

$$0 \longrightarrow N \xrightarrow{\iota} \widetilde{S} \xrightarrow{\epsilon} S \longrightarrow 0\,,$$

where ϵ is a homomorphism of R-algebras with $\mathrm{Ker}(\epsilon)^2 = (0)$, and the given S module structure on N coincides with the one induced by ι; see [**2**, Chapter XVI] for a proof.

When $S = R/I$, one has $\mathrm{D}_0(S|R;N) = 0 = \mathrm{D}^0(S|R;N)$, see 6.2, and

$$\mathrm{D}_1(S|R;N) = (I/I^2) \otimes_S N \quad \text{and} \quad \mathrm{D}^1(S|R;N) = \mathrm{Hom}_S(I/I^2, N) \tag{6.12.1}$$

These claims are justified by Proposition 7.1.

7. André-Quillen homology and the Tor functor

In this section we discuss the relationship between the André-Quillen homology modules $\{\mathrm{D}_n(S|R;N)\}$, where N is an S-module, and the S-modules $\{\mathrm{Tor}^R_n(S,N)\}$.

Let $\epsilon\colon A \to S$ be a simplicial resolution of the R-algebra S, and let J denote the simplicial ideal $\mathrm{Ker}(A \otimes_R S \to \mathsf{s}(S))$; see Exercise 5.5.

One has an exact sequence of simplicial S-modules

$$0 \to J \to A \otimes_R S \to \mathsf{s}(S) \to 0\,.$$

Since $\mathsf{s}(S)_n = S$ for each n, applying $-\otimes_S N$ preserves the exactness of the sequence above, so passing to the homology long exact sequence yields

$$\pi_n(J \otimes_S N) = \begin{cases} \mathrm{Ker}(S \otimes_R N \to N) & \text{when } n = 0; \\ \mathrm{Tor}^R_n(S,N) & \text{when } n \geq 1. \end{cases}$$

Here one is using Remark 4.8.

The morphism $J \to J/J^2$ induces a morphism $J \otimes_S N \to (J/J^2) \otimes_S N$ of simplicial modules. In homology this yields, keeping in mind Exercise 5.5 and the preceding display, homomorphisms of S-modules:

$$\mathrm{Tor}^R_n(S,N) \longrightarrow \mathrm{D}_n(S|R;N) \quad \text{for } n \geq 1.$$

Naturally, the properties of this map are determined by those of the simplicial ideal J, which in turn reflects properties of the R-algebra structure of S. The following result, which justifies the claim in (6.12.1), is one manifestation of this phenomenon.

PROPOSITION 7.1. *Assume that φ is surjective, and set $I = \mathrm{Ker}(\varphi)$. One has natural isomorphisms of S-modules*

$$\mathrm{D}_1(S|R;N) \cong \mathrm{Tor}^R_1(S,N) \cong (I/I^2) \otimes_S N\,.$$

PROOF. Since φ is surjective, one may choose a simplicial resolution A of S with $A_0 = R$. Set $B = A \otimes_R S$. The crucial point in the proof is the following

Claim. $\mathrm{H}_0(J^2) = 0 = \mathrm{H}_1(J^2)$.

Indeed, by choice of A, one has $J_0 = 0$, which explains the first equality. Moreover, the cycles in $\mathsf{N}(J^2)_1$ equal J_1^2, and hence a sum of elements of the form xy, where x and y are in J_1. However, xy is a boundary: the element

$$w = s_0(xy) - s_0(x)s_1(y)$$

is an element in $\mathsf{N}(J^2)$ and $d_0(w) = xy$. Thus, $\mathrm{H}_1(J^2) = 0$.

Now, in the exact sequence $0 \to J^2 \to J \to J/J^2 \to 0$ of simplicial modules, for each integer n, the S-module $(J/J^2)_n$ is free, so one obtains an exact sequence

$$0 \to J^2 \otimes_S N \to J \otimes_S N \to (J/J^2) \otimes_S N \to 0\,.$$

Passing to homology and applying the claim above yields the first of the desired isomorphisms.

As to the second one: $\mathrm{Tor}^R_1(S,N) \cong (I/I^2) \otimes_S N$, consider the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow S \longrightarrow 0$$

and apply to it the functor $- \otimes_R N$. □

The next theorem was proved by Quillen, see [**22**, (6.12)], [**2**, Chapter XX, (24)]; it extends the proposition above, for when φ is surjective, the multiplication map $\mu_R^S\colon S \otimes_R S \to S$ is bijective.

THEOREM 7.2. *If μ_R^S is bijective, then $\mathrm{H}_i(J^n) = 0$ for $n \geq 1$ and $i \leq n-1$.* □

This result is a critical component in proving the convergence of a spectral sequence relating André-Quillen homology and the Tor functor:

7.3. *The fundamental spectral sequence.* Suppose that μ_R^S is bijective. The S-modules underlying the sub-quotients of the filtration $\cdots \subseteq J^2 \subseteq J \subseteq A$ are free, so one obtains a filtration

$$\cdots \subseteq (J^2 \otimes_S N) \subseteq (J \otimes_S N) \subseteq (A \otimes_S N)\,.$$

This induces a spectral sequence with

$${}^1\mathrm{E}_{p,q} = \left((J^q/J^{q+1}) \otimes_S N\right)_{p+q}$$

and abutting to $\mathrm{H}_{p+q}(A \otimes_S N) = \operatorname{Tor}^R_{p+q}(S,N)$. It follows from the connectivity theorem 7.2 that

$${}^2\mathrm{E}_{p,q} = \pi_{p+q}\left(\left(J^q/J^{q+1}\right) \otimes_S N\right) = 0 \quad \text{for } p \leq -1.$$

Thus, the spectral sequence converges. Given Exercise 5.5, the 5-term exact sequence arising from the edge homomorphisms of the spectral sequence yields

PROPOSITION 7.4. *If μ_R^S is bijective, there is an exact sequence of S-modules*

$$\operatorname{Tor}^R_3(S,N) \to \mathrm{D}_3(S|R;N) \to \wedge^2_S \operatorname{Tor}^R_1(S,S) \otimes_S N \to \cdots$$
$$\cdots \to \operatorname{Tor}^R_2(S,N) \to \mathrm{D}_2(S|R;N) \to 0\,.$$

This result will be used in the study of homomorphisms of noetherian rings, which is the topic of the next section.

8. Locally complete intersection homomorphisms

The remainder of this article concerns the role of André-Quillen homology in the study of homomorphisms of commutative rings. The section focuses on complete intersection homomorphisms, while the next one is dedicated to regular homomorphisms. Henceforth, the tacit assumption is that rings are noetherian. Recently, I learned of a new book by Majadas and Rodicio [**17**] aimed at providing a comprehensive treatment of the basic results in this topic.

8.1. *Regular sequences.* A sequence $\boldsymbol{r} = r_1, \dots, r_c$ of elements of R is said to be *regular* if $(\boldsymbol{r}) \neq R$ and r_i is a nonzerodivisor on $R/(r_1, \dots, r_{i-1})$ for $i = 1, \dots, c$.

For example, in the ring $R[y_1, \dots, y_c]$, the sequence $y_1, \dots, y_c$ is regular.

REMARK 8.2. Given an element r in R, write $\mathrm{K}[r\,;R]$ for the complex of R-modules $0 \to R \xrightarrow{r} R \to 0$, with non-zero modules situated in degrees 0 and 1. Given a sequence of elements $\boldsymbol{r} = r_1, \dots, r_c$ in R, set

$$\mathrm{K}[\boldsymbol{r}\,;R] = \mathrm{K}[r_1\,;R] \otimes_R \cdots \otimes_R \mathrm{K}[r_c\,;R]$$

This is the *Koszul complex* on the elements $\boldsymbol{r}$.

Koszul complexes were applied to the study of regular sequences by Auslander and Buchsbaum who proved: if $\boldsymbol{r}$ is a regular sequence, then $\mathrm{H}_n(\mathrm{K}[\boldsymbol{r}\,;R]) = 0$ for

$n \geq 1$, so the augmentation $\mathrm{K}[\boldsymbol{r}\,;R] \to R/\boldsymbol{r}R$ is a quasi-isomorphism. The converse holds when $\boldsymbol{r}$ is contained in the Jacobson radical of R, see [**19**, (16.5)].

8.3. *Locally complete intersection homomorphisms.* Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings.

When φ is surjective, it is *complete intersection* if the ideal $\operatorname{Ker}(\varphi)$ is generated by a regular sequence; it is *locally complete intersection* if for each prime ideal $\mathfrak{q}$ in S, the homomorphism $\varphi_{\mathfrak{q}}\colon R_{\mathfrak{q}\cap R} \to S_{\mathfrak{q}}$ is complete intersection.

When φ is a homomorphism essentially of finite type, it is *locally complete intersection* if in some factorization

$$R \to R' \xrightarrow{\varphi'} S$$

of φ where R' is of the form $U^{-1}R[Y]$, where U is a multiplicatively closed subset in $R[Y]$, and φ' is surjective, the homomorphism φ' is locally complete intersection. It is not too hard that this property is independent of the chosen factorization; it becomes easy, once Theorem 8.4 is on hand.

Avramov has introduced a notion of a complete intersection homomorphism *at a prime* $\mathfrak{q}$ *in* $\operatorname{Spec} S$, and of locally complete intersection homomorphisms, for arbitrary homomorphisms of noetherian rings. It is based on the theory of 'Cohen factorizations'; see [**4**, §1].

Vanishing of André-Quillen homology is linked to the locally complete intersection property by following result, which was proved by Lichtenbaum and Schlessinger, André, and Quillen in the case when φ is essentially of finite type, and by Avramov in the general case.

THEOREM 8.4. *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings. The following conditions are equivalent.*

(a) *$\varphi\colon R \to S$ is locally complete intersection.*
(b) *$\mathrm{D}_n(S|R;-) = 0$ for $n \geq 2$.*
(c) *$\mathrm{D}_2(S|R;-) = 0$.*

Condition (b) may be restated as: $\operatorname{fd}_S \mathcal{L}_\varphi \leq 1$; see Proposition 5.9.

We prove the theorem above when φ is essentially of finite type. In that case, the implication (c) $\implies$ (a) is reduced to the more general result below.

THEOREM 8.5. *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a local homomorphism, essentially of finite type. If $\mathrm{D}_2(S|R;l) = 0$, then φ is locally complete intersection.*

REMARK 8.6. The hypothesis that φ is essentially of finite type is needed: the local homomorphism $\zeta\colon (R,\mathfrak{m},k) \to (\widehat{R},\mathfrak{m}\widehat{R},k)$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R, is flat, so base change along $R \to k$ yields, by 6.3, the isomorphism below:

$$\mathrm{D}_n(\widehat{R}|R;k) \cong \mathrm{D}_n(k|k;k) = 0 \quad \text{for each } n.$$

However, ζ is locally complete intersection if and only if the formal fibers of R, that is to say, the fibres of the homomorphism $R \to \widehat{R}$, are locally complete intersection rings, in the sense of 8.13, and this is not always the case; see [**18**] and [**24**].

PROOF OF THEOREM 8.5. By hypothesis, φ can be factored as $R \xrightarrow{\eta} R' \xrightarrow{\varphi'} S$, where $R' = R[Y]_{\mathfrak{q}}$, with $\operatorname{card}(Y)$ finite, $\mathfrak{q}$ is a prime ideal in $R[Y]$, and φ' is a surjective homomorphism. Proposition 5.10 and Exercise 6.8 yield $\mathrm{D}_n(R'|R;-) = 0$ for $n \geq 2$, so the Jacobi-Zariski sequence 6.7 yields isomorphisms

$$\mathrm{D}_2(S|R';l) \cong \mathrm{D}_2(S|R;l) = 0\,.$$

Therefore, replacing R' by R, one may assume φ is surjective. In particular, $k = l$.

Suppose $\operatorname{Ker}(\varphi)$ is minimally generated by $\boldsymbol{r} = r_1, \dots, r_c$, so $S = R/(\boldsymbol{r})R$. We prove, by an induction on c, that the sequence $\boldsymbol{r}$ is regular.

When $c = 1$, so that $S = R/rR$, Proposition 7.4, specialized to $N = k$, yields an exact sequence

$$\to \wedge^2 \operatorname{Tor}_1^R(S,S) \otimes_S k \to \operatorname{Tor}_2^R(S,k) \to \operatorname{D}_2(S|R;k) \to 0\,.$$

Note that $\operatorname{Tor}_1^R(S,S) = (r)/(r^2)$, so $\operatorname{Tor}_1^R(S,S) \otimes_S k \cong k$, and hence

$$\wedge^2 \operatorname{Tor}_1^R(S,S) \otimes_S k \cong \wedge^2 k = 0\,.$$

Thus, since $\operatorname{D}_2(S|R;k) = 0$, the exact sequence above implies $\operatorname{Tor}_2^R(S,k) = 0$. The ring R is local and R-module S is finitely generated, so the last equality implies $\operatorname{pd}_R S \le 1$, see [**19**, §19, Lemma 1]. Since the complex

$$0 \to R \xrightarrow{r} R \to 0$$

is the beginning of a minimal resolution of S, one deduces that it is the minimal resolution. In particular, r is a nonzerodivisor on R, as required.

Suppose the result has been proved whenever $\operatorname{Ker}(\varphi)$ is minimally generated by $c-1$ elements. Set $R' = R/(r_1, \dots, r_{c-1})R$. The Jacobi-Zariski sequence 6.7 arising from the diagram $R \to R' \to S$ yields an exact sequence

$$\to \operatorname{D}_2(S|R;k) \to \operatorname{D}_2(S|R';k) \to \operatorname{D}_1(R'|R;k) \to \operatorname{D}_1(S|R;k) \to \operatorname{D}_1(S|R';k) \to 0$$

It follows from Proposition 7.1 that

$$\operatorname{D}_1(R'|R;k) \cong k^{c-1}, \quad \operatorname{D}_1(S|R;k) \cong k^c, \quad \text{and } \operatorname{D}_1(S|R';k) \cong k\,.$$

Thus, since $\operatorname{D}_2(S|R;k) = 0$, the exact sequence above yields an exact sequence

$$0 \to \operatorname{D}_2(S|R';k) \to k^{c-1} \to k^c \to k \to 0$$

Therefore, $\operatorname{D}_2(S|R';k) = 0$, and since $S = R'/r_cR'$ the basis of the induction implies r_c is a nonzerodivisor on R'. In particular, $\operatorname{D}_3(S|R';k) = 0$, by Proposition 5.11. Given that $\operatorname{D}_2(S|R;k) = 0$, the Jacobi-Zariski sequence 6.7 yields

$$\operatorname{D}_2(R'|R;l) \cong \operatorname{D}_2(S|R;l) = 0$$

Consequently, $\operatorname{D}_2(R'|R;k) = 0$. Thus, the induction hypothesis implies the sequence $r_1, \dots, r_{c-1}$ is regular on R. This is as desired, since r_c is regular on R'. □

Here is another simplification which results in the theory of André-Quillen homology when the homomorphism under consideration is essentially of finite type.

LEMMA 8.7. *Let $\varphi\colon R \to (S, \mathfrak{n}, l)$ be a local homomorphism essentially of finite type. The complex of S-modules $\mathcal{L}_\varphi$ is homotopic to a complex*

$$\cdots \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_1 \longrightarrow L_0 \longrightarrow 0$$

of finite free S-modules, and with $\partial(L) \subseteq \mathfrak{n}L$.

In particular, for each integer n, one has $\operatorname{rank}_S(L_n) = \operatorname{rank}_l \operatorname{D}_n(S|R;l)$, so that if $\operatorname{D}_n(S|R;l) = 0$, then $\operatorname{D}_n(S|R;-) = 0$ on the category of S-modules.

PROOF. One way to prove this result is to note that, since φ is essentially of finite type, $\mathcal{L}_\varphi$ is homotopy equivalent to a complex $L = \cdots \to L_1 \to L_0 \to 0$ of finite free S-modules, see 6.11. Since S is local, L is homotopic to one with $\partial(L) \subseteq \mathfrak{n}L$; for instance, see [**5**, (1.1.2)]. The desired claim is now clear. □

PROOF OF THEOREM *8.4*. We give the argument when φ is essentially of finite type; see [**4**] for the general case. All three conditions are local properties: condition (a) by inspection, and conditions (b) and (c) by Lemma 8.7. Thus, one may assume $\varphi\colon (R,\mathfrak{m},k)\to(S,\mathfrak{n},l)$ is a local homomorphism.

Now (c) $\implies$ (a) follows from Theorem 8.5, while (b) $\implies$ (c) is obvious.

(a) $\implies$ (b). Arguing as in the proof of Theorem 8.5, one may reduce to the case where φ is surjective. Suppose $\operatorname{Ker}(\varphi)$ is minimally generated by elements $\boldsymbol{r}=r_1,\dots,r_c$; thus $S=R/\boldsymbol{r}R$. An elementary induction on c, using Proposition 5.11 and Property 6.7, yields $\mathcal{L}_\varphi\simeq\Sigma S^c$. Therefore, $\mathrm{D}_n(S|R;-)=0$ for $n\geq 2$. □

Now the following exercise should not be too taxing.

EXERCISE 8.8. Suppose φ is essentially of finite type. Prove that when φ is locally complete intersection, in any factorization $R\to U^{-1}R[Y]\xrightarrow{\varphi'}S$ of φ, where φ' is surjective, the homomorphism φ' is locally complete intersection.

Here is an exercise which illustrates the flexibility afforded by the characterization in Theorem 8.4. To better appreciate it, try to solve it without using the machinery of André-Quillen homology.

EXERCISE 8.9. Let $\varphi\colon R\to S$ be a homomorphism of noetherian rings, essentially of finite type, and let $R\to R'$ be a flat homomorphism.

Prove that if φ is locally complete intersection, then so is the induced homomorphism $\varphi\otimes_R R'\colon R'\to(S\otimes_R R')$, and that the converse holds when $R\to R'$ is faithfully flat. Hint: for the converse, use the going-down theorem, see [**19**, (9.5)].

8.10. *Extensions of fields.* Let $\phi\colon k\to l$ be a homomorphism of fields.

EXERCISE 8.10.1. Prove that when the field l is finitely generated over k, the homomorphism ϕ is locally complete intersection.

It is easy to check that ϕ is locally complete intersection in general, in the sense of [**4**]. Thus, $\mathrm{D}_n(l|k;-)=0$ for $n\geq 2$, by Theorem 8.4. The l-vectorspace $\mathrm{D}_1(l|k;l)$ is called the *module of imperfection*, and denoted $\Gamma_{l|k}$; see [**19**, §26].

When $h\to k$ is another homomorphism of fields, the Jacobi-Zariski sequence 6.7 arising from the diagram $h\to k\to l$ yields an exact sequence of l-vectorspaces:

$$0\to\Gamma_{k|h}\otimes_k l\to\Gamma_{l|h}\to\Gamma_{l|k}\to\Omega_{k|h}\otimes_k l\to\Omega_{l|h}\to\Omega_{l|k}\to 0$$

Computing ranks one obtains the Cartier equality, see [**19**, (26.10)].

8.11. *Regular local rings.* A local ring $(R,\mathfrak{m},k)$ is *regular* if the ideal $\mathfrak{m}$ has a set of generators that form an R-regular sequence. This condition translates to: the surjection $R\to k$ is complete intersection, in the sense of 8.3. The following result is a corollary of Theorems 8.4 and 8.5.

PROPOSITION 8.12. *Let R be a local ring, with residue field k. The following conditions are equivalent.*

(a) *R is regular;*
(b) *$\mathrm{D}_n(k|R;-)=0$ for $n\geq 2$;*
(c) *$\mathrm{D}_2(k|R;k)=0$.*

When R is regular, $\mathfrak{m}$ its maximal ideal, and $\epsilon\colon R\to k$ is the canonical surjection, then $\mathcal{L}_\epsilon\simeq(\mathfrak{m}/\mathfrak{m}^2)$. □

This result is a homological characterization of the regularity property, akin to the one by Auslander, Buchsbaum, and Serre: R is regular iff every R-module has finite projective dimension iff k has finite projective dimension, see [**19**, §19].

8.13. *Complete intersections.* Let $(R, \mathfrak{m}, k)$ be a local ring, and let $\widehat{R}$ denote the $\mathfrak{m}$-adic completion of R. Cohen's structure theorem provides a surjection $\epsilon\colon Q \twoheadrightarrow \widehat{R}$ with Q a regular local ring; see [**19**, (29.4)]. Such a homomorphism ϵ is said to be a *Cohen presentation* of $\widehat{R}$.

The local ring R is *complete intersection* if in a Cohen presentation $\epsilon\colon Q \twoheadrightarrow \widehat{R}$, the ideal $\operatorname{Ker}(\epsilon)$ is generated by a regular sequence; that is to say, ϵ is a complete intersection homomorphism. It is known, and is implicit in the proof of the result below, that when R is complete intersection, any Cohen presentation of $\widehat{R}$ is a complete intersection homomorphism.

PROPOSITION 8.14. *Let R be a local ring, with residue field k. The following conditions are equivalent.*

(a) *R is complete intersection;*
(b) $\mathrm{D}_n(k|R;-) = 0$ *for* $n \geq 3$*;*
(c) $\mathrm{D}_3(k|R;k) = 0$.

PROOF. Since k is a field, and $\mathrm{D}_n(k|R;-)$ commutes with arbitrary direct sums (check this), condition (b) is equivalent to:

(b') $\mathrm{D}_n(k|R;k) = 0$ for $n \geq 3$.

The homomorphism $R \to \widehat{R}$ is flat, see [**19**, (8.8)], so base change along it yields isomorphisms

$$\mathrm{D}_n(k|\widehat{R};k) \cong \mathrm{D}_n(k|R;k) \quad \text{for } n \in \mathbb{Z}.$$

Therefore, we may assume that R is complete, and hence that there is a surjection $\epsilon\colon Q \to R$, where Q is a regular local ring, see 8.13.

Proposition 8.12 yields $\mathrm{D}_n(k|Q;k) = 0$ for $n \geq 2$, so the Jacobi-Zariski sequence 6.7 applied to the diagram $Q \to R \to k$ provides isomorphisms

$$\mathrm{D}_n(k|R;k) \cong \mathrm{D}_{n-1}(R|Q;k) \quad \text{for } n \geq 3.$$

Now, when R is complete intersection, there is a choice of ϵ which is complete intersection. Then Theorem 8.4 implies $\mathrm{D}_n(R|Q;-) = 0$ for $n \geq 2$; note that, since ϵ is surjective, we are applying the theorem in a case where it was proved. Thus, the isomorphisms above imply condition (b').

Conversely, given (c), one obtains $\mathrm{D}_n(R|Q;k) = 0$ for $n = 2$, by the displayed isomorphisms. Now Theorem 8.5 yields that ϵ is complete intersection. Hence, R is complete intersection. □

EXERCISE 8.15. Let $(R, \mathfrak{m}, k)$ be a local ring, and $\epsilon\colon Q \twoheadrightarrow R$ a surjective homomorphism with Q a regular local ring. Prove that the ring R is complete intersection if and only if the homomorphism ϵ is a complete intersection.

8.16. *The Quillen conjectures.* For homomorphisms of noetherian rings, and essentially of finite type, in [**22**, (5.6), (5.7)] Quillen made the following conjectures:

Conjecture I. If $\operatorname{fd}_S \mathcal{L}_\varphi$ and $\operatorname{fd}_R S$ are both finite, then the homomorphism φ is locally complete intersection.

Conjecture II. If $\operatorname{fd}_S \mathcal{L}_\varphi$ is finite, then $\operatorname{fd}_S \mathcal{L}_\varphi \leq 2$.

Recall that $\operatorname{fd}_S \mathcal{L}_\varphi \leq n$ if and only if $\mathrm{D}_i(S|R;-) = 0$ for $i \geq n+1$, see Proposition 5.9, so the Quillen conjectures can be phrased in terms of vanishing of André-Quillen homology functors.

Avramov [**4**] proved the following result, settling Conjecture I in the affirmative:

THEOREM 8.17. *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings. The following conditions are equivalent.*

(i) *φ is locally complete intersection.*
(ii) *$\mathrm{D}_n(S|R;-) = 0$ for $n \gg 0$ and $\operatorname{fd}_R S$ is locally finite.*

If S has characteristic 0, then they are also equivalent to

(iii) *$\mathrm{D}_m(S|R;-) = 0$ for some integer $m \geq 2$ and $\operatorname{fd}_R S$ is locally finite.* □

Jim Turner [**25**] gave a different proof of Quillen's conjecture I, in the special case when φ is essentially of finite type, and the residue fields of R are all of positive characteristic.

In [**7**], Conjecture II is settled for homomorphisms that admit algebra retracts:

THEOREM 8.18. *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings such that there exists a homomorphism $\psi\colon S \to R$ with $\varphi \circ \psi = \mathrm{id}^S$. The following conditions are equivalent.*

(i) *$\psi_{\mathfrak{p}}$ is complete intersection for each $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \supseteq \operatorname{Ker}(\varphi)$.*
(ii) *$\mathrm{D}_n(S|R;-) = 0$ for $n \gg 0$.*
(iii) *$\mathrm{D}_n(S|R;-) = 0$ for $n \geq 3$.*

If, in addition, S has characteristic 0, they are also equivalent to

(iv) *$\mathrm{D}_m(S|R;-) = 0$ for some integer $m \geq 3$.* □

The general case of Conjecture II remains open. I should like to note that these conjectures are about noetherian rings; they are false if one drops that hypothesis, see Planas-Vilanova [**21**], and also [**1**].

NOTES 8.19. The results in this section, and in the next, involve only the homology functors $\mathrm{D}_n(S|R;-)$. In view of Proposition 5.9, one can phrase many of them also in terms of the cohomology functors $\mathrm{D}^n(S|R;-)$.

9. Regular homomorphisms

In this section we turn to regular homomorphisms. Regular local rings have been encountered already in 8.11. A (not-necessarily local) noetherian ring S is said to be *regular* if the local $S_{\mathfrak{q}}$ is regular for each prime ideal $\mathfrak{q}$ in S.

A regular local ring is regular, because the regularity property localizes. This last result is immediate from the characterization of regularity by Auslander, Buchsbaum, and Serre referred to earlier, see [**19**, (19.3)].

A homomorphism $\varphi\colon R \to S$ of noetherian rings is *regular* if S is flat over R and the ring $S \otimes_R l$ is regular whenever $R \to l$ is a homomorphism essentially of finite type and l is a field. If in addition φ is essentially of finite type, then one says that φ is *smooth*; an alternative terminology is *geometrically regular.*

EXAMPLE 9.1. Let X be a finite set of variables. The inclusion $R \hookrightarrow R[X]$ is smooth, whereas the inclusion $R \hookrightarrow R[[X]]$ is regular.

EXAMPLE 9.2. An extension of fields $k \to l$ is regular if and only if it is separable; this is not too difficult to prove when l is finitely generated as a field over k. See [**2**, Chapter VII] for the argument in the general case.

The issue with separability is well-illustrated in the following example: when k is a field of characteristic p, and $a \in k$ does not have a pth root in k, the field extension $k \to l = k[x]/(x^p - a)$ is not geometrically regular: $l \otimes_k l$ is a local ring with nilpotents, and hence it is not regular.

REMARK 9.3. Note that the definition of a regular homomorphism has a different flavour when compared to that of a locally complete intersection homomorphism. The one for regularity is due to Grothendieck, and it is in line with his point of view that a homomorphism $\varphi\colon R \to S$ is deemed to have a certain property (regularity, complete intersection, Gorenstein, Cohen-Macaulay, et cetra), if the homomorphism is flat and its fibres have the *geometric* version of the corresponding property.

One does not define complete intersection homomorphisms this way for it would be too restrictive a notion; for instance, it would preclude surjective homomorphisms defined by regular sequences, because they are not flat.

The content of the next exercise is that a complete intersection homomorphism in the sense of Grothendieck is locally complete intersection, as defined in 8.3.

EXERCISE 9.4. Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings such that S is flat over R. Prove that φ is locally complete intersection if and only if for each prime ideal $\mathfrak{p}$ in R, the fibre ring $S \otimes_R \kappa(\mathfrak{p})$ is locally complete intersection.

The definitive criterion for regularity in terms of André-Quillen homology is due to André.

THEOREM 9.5. *Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings. The following conditions are equivalent.*

(a) *φ is regular.*
(b) $\mathrm{D}_n(S|R;-) = 0$ *for each* $n \geq 1$.
(c) $\mathrm{D}_1(S|R;-) = 0$.

Once again, I will provide a proof only in the case where φ is essentially of finite type: Under this hypothesis, arguing as in the proof of Theorem 8.4, one may deduce Theorem 9.5 from the following result.

THEOREM 9.6. *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a local homomorphism, essentially of finite type. The following conditions are equivalent.*

(a) *φ is smooth.*
(b) $\mathrm{D}_n(S|R;-) = 0$ *for each* $n \geq 1$, *and the S-module $\Omega_{S|R}$ is finite free.*
(c) $\mathrm{D}_1(S|R;l) = 0$.

Thus, when φ is smooth, one has that $\mathcal{L}_\varphi \simeq \Omega_{S|R}$.

PROOF. (a) $\Longrightarrow$ (b). The R-module S is flat, so base change of φ along the composed homomorphism $R \xrightarrow{\varphi} S \to l$ yields an isomorphism

$$\mathrm{D}_n(S|R;l) \cong \mathrm{D}_n(S \otimes_R l|l;l) \quad \text{for each } n.$$

The composed map $l \to (S \otimes_R l) \to l$ equals id^l, so Exercise 6.9 yields isomorphisms

$$\mathrm{D}_n(S \otimes_R l|l;l) \cong \mathrm{D}_{n+1}(l|S \otimes_R l;l)\,.$$

Since $R \to l$ is essentially of finite type, smoothness of φ implies the ring $S' = S \otimes_R l$ is regular. Let $\mathfrak{n}'$ be the maximal ideal of S' such that $S'/\mathfrak{n}' = l$. Then, the local ring $S'_{\mathfrak{n}'}$ is regular, so Corollary 8.11 implies the second isomorphism below

$$\mathrm{D}_n(l|S';l) = \mathrm{D}_n(l|S'_{\mathfrak{n}'};l) = 0 \quad \text{for } n \geq 2,$$

while the first one is by 6.5. Combining this with the preceding displays yields $\mathrm{D}_n(S|R;l) = 0$ for $n \geq 1$. It remains to recall Lemma 8.7.

Evidently, (b) $\implies$ (c).

(c) $\implies$ (a). Since $\mathrm{D}_1(S|R;l) = 0$, it follows from Lemma 8.7 that $\mathcal{L}_\varphi$ is homotopy equivalent to a complex of finite free S-modules L with $L_1 = 0$. Therefore, one has that

$$\mathrm{D}^1(S|R;S) = \mathrm{H}_{-1}(\mathrm{Hom}_S(L,S)) = 0$$

This is equivalent to the statement that any R-algebra extension of S by a square-zero ideal is split; see 6.12. According to a theorem of Grothendieck, this property characterizes the smoothness of S smooth over R; see [**14**]. □

9.7. *A local-global principle.* Let $\varphi\colon R \to S$ be a homomorphism of noetherian rings, $\mathfrak{q}$ a prime ideal in S, and set $\mathfrak{p} = \mathfrak{q} \cap R$. One says that φ is *regular at* $\mathfrak{q}$ if $\varphi_{\mathfrak{q}}$ is flat and the $k(\mathfrak{p})$-algebra $(S \otimes_R k(\mathfrak{p}))_{\mathfrak{q}}$ is geometrically regular.

The exercise below is an important local-global principle for regularity. In it, the hypothesis that φ is essentially of finite type is insurmountable; see Remark 8.6. There is an analogue for the complete intersection property; see [**4**, §5].

EXERCISE 9.8. Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be local homomorphism, essentially of finite type. Prove that if φ is regular at $\mathfrak{n}$, then φ is regular.

Given Theorems 8.4 and 9.5, it is not hard to prove the following result, which is a crucial step in the Hochschild-Kostant-Rosenberg theorem that calculates the Hochschild homology and cohomology of smooth algebras, see [**15**], [**8**, (1.1)].

THEOREM 9.9. *Let $\eta\colon K \to S$ be a homomorphism of noetherian rings essentially of finite type, such that S is flat as an K-module.*

Then η is smooth if and only if the product map $\mu^S_R\colon S \otimes_K S \to S$ is locally complete intersection.

PROOF. Set $S^e = S \otimes_K S$. Since η is essentially of finite type, the ring S^e is noetherian. Let $\psi = \eta \otimes_K S$; it is a homomorphism of rings $S \to S^e$, defined by $\psi(s) = 1 \otimes s$ for $s \in S$. Since S is flat over K, base change yields a homotopy equivalence of complexes of S^e-modules:

$$\mathcal{L}_\eta \otimes_K S \simeq \mathcal{L}_\psi\,.$$

The action of S^e on $\mathcal{L}_\eta \otimes_K S$ is given by $(s \otimes s')(l \otimes t) = (sl \otimes s't)$. The composition

$$S \xrightarrow{\psi} S^e \xrightarrow{\mu} S$$

is the identity on S, so Exercise 6.9 yields a homotopy equivalence of S-modules

$$\mathcal{L}_\mu \simeq \Sigma(\mathcal{L}_\psi \otimes_{S^e} S)\,.$$

Combining the two equivalences above, one gets the homotopy equivalence of S-modules in the following diagram

$$\mathcal{L}_\mu \simeq \Sigma\big((\mathcal{L}_\eta \otimes_K S) \otimes_{S^e} S\big) \cong \Sigma(\mathcal{L}_\eta \otimes_S S) = \Sigma\mathcal{L}_\eta\,.$$

The isomorphism is justified in the exercise below. Therefore, on the category of S-modules, one has isomorphisms

$$\mathrm{D}_n(S|S^e;-) \cong \mathrm{D}_{n+1}(S^e|S;-) \quad \text{for each } n.$$

Theorems 8.4 and 9.5 now provide the desired conclusion. □

EXERCISE 9.10. Let $K \to S$ be a homomorphism of rings, set $S^e = S \otimes_K S$, and let M and N be S-modules. As usual, $M \otimes_K N$ has a natural structure of a (right) S^e-module, with $(m \otimes n).(r \otimes s) = mr \otimes sn$. View S as an S^e module via the product map $\mu\colon S^e \to S$.

Prove that the natural homomorphism of S-modules below is bijective:

$$(M \otimes_K N) \otimes_{S^e} S \longrightarrow M \otimes_S N\,.$$

Extend this result to the case when M and N are complexes of S-modules. Caveat: take care of the signs.

NOTES 9.11. André-Quillen homology does not appear in the statement of Theorem 9.9. This situation is typical: André-Quillen theory provides streamlined proofs of many results concerning Hochschild homology, and is sometimes indispensable, see [**9**]. There is a mathematical reason for this, see [**22**, (8.1)].

9.12. *Étale homomorphisms.* A homomorphism $\varphi\colon R \to S$ of noetherian rings and essentially of finite type is said to be *étale* if it is smooth and unramified.

EXERCISE 9.13. Let k be a field, and R the polynomial ring $k[x_1, \dots, x_d]$. Let f be an element in R, and set $S = R/(f)$. Find necessary and sufficient conditions on f for the homomorphism $R \to S$ to be étale.

EXERCISE 9.14. Formulate and prove analogues of Theorems 9.5 and 9.9 for étale homomorphisms.

If you want to check whether you are on the right track, see [**22**, (5.4), (5.5)].

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