Stable Stems

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ABSTRACT. We present a detailed analysis of 2-complete stable homotopy groups, both in the classical context and in the motivic context over \mathbb{C} . We use the motivic May spectral sequence to compute the cohomology of the motivic Steenrod algebra over \mathbb{C} through the 70-stem. We then use the motivic Adams spectral sequence to obtain motivic stable homotopy groups through the 59-stem. In addition to finding all Adams differentials in this range, we also resolve all hidden extensions by 2, η , and ν , except for a few carefully enumerated exceptions that remain unknown. The analogous classical stable homotopy groups are easy consequences.

We also compute the motivic stable homotopy groups of the cofiber of the motivic element τ . This computation is essential for resolving hidden extensions in the Adams spectral sequence. We show that the homotopy groups of the cofiber of τ are the same as the E_2 -page of the classical Adams-Novikov spectral sequence. This allows us to compute the classical Adams-Novikov spectral sequence, including differentials and hidden extensions, in a larger range than was previously known.

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CHAPTER 1

Introduction

One of the fundamental problems of stable homotopy theory is to compute the stable homotopy groups of the sphere spectrum. One reason for computing these groups is that maps between spheres control the construction of finite cell complexes.

After choosing a prime p and focusing on the p-complete stable homotopy groups instead of the integral homotopy groups, the Adams spectral sequence and the Adams-Novikov spectral sequence have proven to be the most effective tools for carrying out such computations.

At odd primes, the Adams-Novikov spectral sequence has clear computational advantages over the Adams spectral sequence. (Nevertheless, the conventional wisdom, derived from Mark Mahowald, is that one should compute with both spectral sequences because they emphasize distinct aspects of the same calculation.)

Computations at the prime 2 are generally more difficult than computations at odd primes. In this case, the Adams spectral sequence and the Adams-Novikov spectral sequence seem to be of equal complexity. The purpose of this manuscript is to thoroughly explore the Adams spectral sequence at 2 in both the classical and motivic contexts.

Motivic techniques are essential to our analysis. Working motivically instead of classically has both advantages and disadvantages. The main disadvantage is that the computation is larger and proportionally more difficult. On the other hand, there are several advantages. First, the presence of more non-zero classes allows the detection of otherwise elusive phenomena. Second, the additional motivic weight grading can easily eliminate possibilities that appear plausible from a classical perspective.

The original motivation for this work was to provide input to the ρ -Bockstein spectral sequence for computing the cohomology of the motivic Steenrod algebra over \mathbb{R} . The analysis of the ρ -Bockstein spectral sequence, and the further analysis of the motivic Adams spectral sequence over \mathbb{R} , will appear in future work.

This manuscript is a natural sequel to [13], where the first computational properties of the motivic May spectral sequence, as well as of the motivic Adams spectral sequence, were established.

1.1. The Adams spectral sequence program

The Adams spectral sequence starts with the cohomology of the Steenrod algebra A, i.e., $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$. There are two ways of approaching this algebraic object. First, one can compute by machine. This has been carried out to over 200 stems [9] [36]. Machines can also compute the higher structure of products and Massey products.

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The second approach is to compute by hand with the May spectral sequence. This will be carried out to 70 stems in Chapter 2. See also [40] for the classical case. See [19] for a detailed Ext chart through the 70-stem.

The E_{∞} -page of the May spectral sequence is the graded object associated to a filtration on $\operatorname{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$, which can hide some of the multiplicative structure. One can resolve these hidden multiplicative extensions with indirect arguments involving higher structure such as Massey products or algebraic Steenrod operations in the sense of [**31**]. A critical ingredient here is May's Convergence Theorem [**30**, Theorem 4.1], which allows the computation of Massey products in $\operatorname{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ via the differentials in the May spectral sequence.

The cohomology of the Steenrod algebra is the E_2 -page of the Adams spectral sequence. The next step is to compute the Adams differentials. This will be carried out in Chapter 3. Techniques for establishing differentials include:

- (1) Use knowledge of the image of J [2] to deduce differentials.
- (2) Compare to the completely understood Adams spectral sequence for the topological modular forms spectrum tmf [15].
- (3) Use the relationship between algebraic Steenrod operations and Adams differentials [11, VI.1].
- (4) Exploit Toda brackets to deduce relations in the stable homotopy ring, which then imply Adams differentials.

We have assembled all previously published results about the Adams differentials in Table 18.

The E_{∞} -page of the Adams spectral sequence is the graded object associated to a filtration on the stable homotopy groups, which can hide some of the multiplicative structure. The final step is to resolve these hidden multiplicative extensions. This will be carried out in Chapter 4. Analogously to the extensions that are hidden in the May spectral sequence, this generally involves indirect arguments with Toda brackets. We have assembled previously published results about these hidden extensions in Table 24.

The detailed analysis of the Adams spectral sequence requires substantial technical work with Toda brackets. A critical ingredient for computing Toda brackets is Moss's Convergence Theorem [35], which allows the computation of Toda brackets via the Adams differentials. We remind the reader to be cautious about indeterminacies in Massey products and Toda brackets.

1.2. Motivic homotopy theory

The formal construction of motivic homotopy theory requires the heavy machinery of simplicial presheaves and model categories [34] [23] [12]. We give a more intuitive description of motivic homotopy theory that will suffice for our purposes.

Motivic homotopy theory is a homotopy theory for algebraic varieties. Start with the category of smooth schemes over a field k (in this manuscript, k always equals \mathbb{C}). This category is inadequate for homotopical purposes because it does not possess enough gluing constructions, i.e., homotopy colimits.

In order to fix this problem, we can formally adjoin homotopy colimits. This takes us to the category of simplicial presheaves.

The next step is to restore some desired relations. If $\{U, V\}$ is a Zariski cover of a smooth scheme X, then X is the colimit of the diagram

$$(1.1) U \longleftarrow U \cap V \longrightarrow V$$

in the category of smooth schemes. However, when we formally adjoined homotopy colimits, we created a new object, distinct from X, that served as the homotopy pushout of Diagram 1.1. This is undesirable, so we formally declare that X is the homotopy pushout of Diagram 1.1, from which we obtain the local homotopy theory of simplicial presheaves. This homotopy theory has some convenient properties such as Mayer-Vietoris sequences.

In fact, one needs to work not with Zariski covers but with Nisnevich covers. See [34] for details on this technical point.

The final step is to formally declare that each projection map $X \times \mathbb{A}^1 \to X$ is a weak equivalence. This gives the unstable motivic homotopy category.

In unstable motivic homotopy theory, there are two distinct objects that play the role of circles:

- (1) $S^{1,0}$ is the usual simplicial circle.
- (2) $S^{1,1}$ is the punctured affine line $\mathbb{A}^1 0$.

For $p \ge q$, the unstable sphere $S^{p,q}$ is the appropriate smash product of copies of $S^{1,0}$ and $S^{1,1}$, so we have a bigraded family of spheres.

Stable motivic homotopy theory is the stabilization of unstable motivic homotopy theory with respect to this bigraded family of spheres. As a consequence, calculations such as motivic cohomology and motivic stable homotopy groups are bigraded.

Motivic homotopy theory over \mathbb{C} comes with a realization functor to ordinary homotopy theory. Given a complex scheme X, there is an associated topological space $X(\mathbb{C})$ of \mathbb{C} -valued points. This construction extends to a well-behaved functor between unstable and stable homotopy theories.

We will explain at the beginning of Chapter 3 that we have very good calculational control over this realization functor. We will use this relationship in both directions: to deduce motivic facts from classical results, and to deduce classical facts from motivic results.

One important difference between the classical case and the motivic case is that not every motivic spectrum is built out of spheres, i.e., not every motivic spectrum is cellular. Stable cellular motivic homotopy theory is more tractable than the full motivic homotopy theory, and many motivic spectra of particular interest, such as the Eilenberg-Mac Lane spectrum $H\mathbb{F}_2$, the algebraic K-theory spectrum KGL, and the algebraic cobordism spectrum MGL, are cellular. Stable motivic homotopy group calculations are fundamental to cellular motivic homotopy theory. However, the part of motivic homotopy theory that is not cellular is essentially invisible from the perspective of stable motivic homotopy groups.

Although one can study motivic homotopy theory over any base field (or even more general base schemes), we will work only over \mathbb{C} , or any algebraically closed field of characteristic 0. Even in this simplest case, we find a wealth of exotic phenomena that have no classical analogues.

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1.3. The motivic Steenrod algebra

The starting point for our Adams spectral sequence work is the description of the motivic Steenrod algebra over \mathbb{C} at the prime 2, which is a variation on the classical Steenrod algebra. First, the motivic cohomology of a point is $\mathbb{M}_2 = \mathbb{F}_2[\tau]$, where τ has degree (0, 1) [44].

The (dual) motivic Steenrod algebra over \mathbb{C} is [45] [43] [7, Section 5.2]

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]}{\tau_i^2 = \tau \xi_{i+1}}.$$

The reduced coproduct is determined by

$$\tilde{\phi}_{*}(\tau_{k}) = \xi_{k} \otimes \tau_{0} + \xi_{k-1}^{2} \otimes \tau_{1} + \dots + \xi_{k-i}^{2^{i}} \otimes \tau_{i} + \dots + \xi_{1}^{2^{k-1}} \otimes \tau_{k-1}$$
$$\tilde{\phi}_{*}(\xi_{k}) = \xi_{k-1}^{2} \otimes \xi_{1} + \xi_{k-2}^{4} \otimes \xi_{2} + \dots + \xi_{k-i}^{2^{i}} \otimes \xi_{i} + \dots + \xi_{1}^{2^{k-1}} \otimes \xi_{k-1}$$

The dual motivic Steenrod algebra has a few interesting features. First, if we invert τ , then we obtain a polynomial algebra that is essentially the same as the classical dual Steenrod algebra. This is a general feature. We will explain at the beginning of Chapter 3 that one recovers classical calculations from motivic calculations by inverting τ . This fact is useful in both directions: to deduce motivic facts from classical ones, and to deduce classical facts from motivic ones.

Second, if we set $\tau = 0$, we obtain a "p = 2 version" of the classical odd primary dual Steenrod algebra, with a family of exterior generators and another family of polynomial generators. This observation suggests that various classical techniques that are well-suited for odd primes may also work motivically at the prime 2.

1.4. Relationship between motivic and classical calculations

As a consequence of our detailed analysis of the motivic Adams spectral sequence, we recover the analysis of the classical Adams spectral sequence by inverting τ .

We will use known results about the classical Adams spectral sequence from [3], [4], [8], [27], and [41]. We have carefully collected these results in Tables 18 and 24.

A few of our calculations are inconsistent with calculations in [24] and [25], and we are unable to understand the exact sources of the discrepancies. For this reason, we have found it prudent to avoid relying directly on the calculations in [24] and [25]. However, we will follow [25] in establishing one particularly difficult Adams differential in Section 3.3.4.

Here is a summary of our calculations that are inconsistent with [24] and [25]:

- (1) There is a classical differential $d_3(Q_2) = gt$. This means that classical π_{56} has order 2, not order 4; and that classical π_{57} has order 8, not order 16.
- (2) The element h_1g_2 in the 45-stem does not support a hidden η extension to N.
- (3) The element C of the 50-stem does not support a hidden η extension to gn.
- (4) [24] claims that there is a hidden ν extension from $h_2h_5d_0$ to gn and that there is no hidden 2 extension on $h_0h_3g_2$. These two claims are incompatible; either both hidden extensions occur, or neither occur. (See Lemma 4.2.31.)

The proof of the non-existence of the hidden η extension on h_1g_2 is particularly interesting because it relies inherently on a motivic calculation. We know of no way to establish this result only with classical tools.

We draw particular attention to the Adams differential $d_2(D_1) = h_0^2 h_3 g_2$ in the 51-stem. Mark Mahowald privately communicated an argument for the presence of this differential to the author. However, this argument fails because of the calculation of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$ in Lemma 4.2.91, which was unknown to Mahowald. Zhouli Xu and the author discovered an independent proof, which is included in this manuscript as Lemma 3.3.13. This settles the order of π_{51} but not its group structure. It is possible that π_{51} contains an element of order 8. See [22] for a more complete discussion.

Related to the misunderstanding concerning the bracket $\langle \theta_4, 2, \sigma^2 \rangle$, the published literature contains incorrect proofs that θ_4^2 equals zero. Zhouli Xu has found the first correct proof of this relation [46]. This has implications for the strong Kervaire problem. Xu used the calculation of θ_4^2 to simplify the argument given in [3] that establishes the existence of the Kervaire class θ_5 .

We also remark on the hidden 2 extension in the 62-stem from $E_1 + C_0$ to R indicated in [25]. We cannot be absolutely certain of the status of this extension because it lies outside the range of our thorough analysis. However, it appears implausible from the motivic perspective. (For entirely different reasons related to v_2 -periodic homotopy groups, Mark Mahowald communicated privately to the author that he was also skeptical of this hidden extension.)

1.5. Relationship to the Adams-Novikov spectral sequence

We will describe a rigid relationship between the motivic Adams spectral sequence and the motivic Adams-Novikov spectral sequence in Chapter 6. In short, the E_2 -page of the classical Adams-Novikov spectral sequence is isomorphic to the bigraded homotopy groups $\pi_{*,*}(C\tau)$ of the cofiber of τ . Here τ is the element of the motivic stable homotopy group $\pi_{0,-1}$ that is detected by the element τ of \mathbb{M}_2 . Moreover, the classical Adams-Novikov spectral sequence is identical to the τ -Bockstein spectral sequence converging to stable motivic homotopy groups!

In Chapter 5, we will extensively compute $\pi_{*,*}(C\tau)$. In Chapter 6, we will apply this information to obtain information about the classical Adams-Novikov spectral sequence in previously unknown stems.

However, there are two places in earlier chapters where we use specific calculations from the classical Adams-Novikov spectral sequence. We would prefer arguments that are internal to the Adams spectral sequence, but they have so far eluded us. The specific calculations that we need are:

- (1) Lemma 4.2.7 shows that a certain possible hidden τ extension does not occur in the 57-stem. See also Remark 4.1.12. For this, we use that $\beta_{12/6}$ is the only element in the Adams-Novikov spectral sequence in the 58-stem with filtration 2 that is not divisible by α_1 [38].
- (2) Lemma 4.2.35 establishes a hidden 2 extension in the 54-stem. See also Remark 4.1.18. For this, we use that $\beta_{10/2}$ is the only element of the Adams-Novikov spectral sequence in the 54-stem with filtration 2 that is not divisible by α_1 , and that this element maps to $\Delta^2 h_2^2$ in the Adams-Novikov spectral sequence for tmf [5] [38].

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The E_2 -page of the motivic (or classical) Adams spectral sequence is readily computable by machine. On the other hand, there seem to be real obstructions to practical machine computation of the E_2 -page of the classical Adams-Novikov spectral sequence.

On the other hand, let us suppose that we did have machine computed data on the E_2 -page of the classical Adams-Novikov spectral sequence. The rigid relationship between motivic stable homotopy groups and the classical Adams-Novikov spectral sequence could be exploited to great effect to determine the pattern of differentials in both the Adams-Novikov and the Adams spectral sequences. We anticipate that all differentials through the 60-stem would be easy to deduce, and we would expect to be able to compute well past the 60-stem. For this reason, we foresee that the next major breakthrough in computing stable stems will involve machine computation of the Adams-Novikov E_2 -page.

1.6. How to use this manuscript

The exposition of such a technical calculation creates some inherent challenges. In the end, the most important parts of this project are the Adams charts from [19], the Adams-Novikov charts from [21], and the tables in Chapter 7. These tables contain a wealth of detailed information in a concise form. They summarize the essential calculational facts that allow the computation to proceed. In fact, the rest of the manuscript merely consists of detailed arguments that support the claims in the tables.

For readers interested in specific calculational facts, the tables in Chapter 7 are the place to start. These tables include references to more detailed proofs given elsewhere in the manuscript. The index also provides references to miscellaneous remarks about specific elements.

We draw attention to the following charts from [19] and [21] that are of particular interest:

- (1) A classical Adams E_2 chart with differentials.
- (2) A classical Adams E_{∞} chart with hidden extensions by 2, η , and ν .
- (3) A motivic Adams E_2 chart.
- (4) A motivic Adams E_{∞} chart with hidden τ extensions.
- (5) A classical Adams-Novikov E_2 chart with differentials.
- (6) A classical Adams-Novikov E_{∞} chart with hidden extensions by 2, η , and ν .

In each of the charts, we have been careful to document explicitly the remaining uncertainties in our calculations.

We also draw attention to the following tables from Chapter 7 that are of particular interest:

- (1) Tables 8, 20, 21, and 22 give all of the Adams differentials.
- (2) Table 16 gives some Massey products in the cohomology of the motivic Steenrod algebra, including indeterminacies.
- (3) Table 18 summarizes previously known results about classical Adams differentials.
- (4) Table 19 summarizes previously known results about classical Toda brackets.
- (5) Table 23 gives some Toda brackets, including indeterminacies.

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- (6) Table 24 summarizes previously known results about hidden extensions in the classical stable homotopy groups.
- (7) Table 47 gives a correspondence between elements of the classical Adams and Adams-Novikov E_{∞} pages.

These tables include specific references to complete proofs of each fact.

1.7. Notation

By convention, we give degrees in the form (s, f, w), where s is the stem; f is the Adams filtration; and w is the motivic weight. An element of degree (s, f, w) will appear on a chart at coordinates (s, f).

We will use the following notation extensively:

- (1) \mathbb{M}_2 is the mod 2 motivic cohomology of \mathbb{C} .
- (2) A is the mod 2 motivic Steenrod algebra over \mathbb{C} .
- (3) A(2) is the \mathbb{M}_2 -subalgebra of A generated by Sq¹, Sq², and Sq⁴.
- (4) Ext is the trigraded ring $\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$.
- (5) $A_{\rm cl}$ is the classical mod 2 Steenrod algebra.
- (6) Ext_{cl} is the bigraded ring $\text{Ext}_{A_{cl}}(\mathbb{F}_2, \mathbb{F}_2)$.
- (7) $\pi_{*,*}$ is the 2-complete motivic stable homotopy ring over \mathbb{C} .
- (8) $E_r(S^{0,0})$ is the E_r -page of the motivic Adams spectral sequence converging to $\pi_{*,*}$. Note that $E_2(S^{0,0})$ equals Ext.
- (9) For x in $E_{\infty}(S^{0,0})$, write $\{x\}$ for the set of all elements of $\pi_{*,*}$ that are represented by x.
- (10) τ is both an element of \mathbb{M}_2 , as well as the element of $\pi_{0,-1}$ that it represents in the motivic Adams spectral sequence.
- (11) $C\tau$ is the cofiber of $\tau: S^{0,-1} \to S^{0,0}$.
- (12) $H^{*,*}(C\tau)$ is the mod 2 motivic cohomology of $C\tau$.
- (13) $\pi_{*,*}(C\tau)$ are the 2-complete motivic stable homotopy groups of $C\tau$, which form a $\pi_{*,*}$ -module.
- (14) $E_r(C\tau)$ is the E_r -page of the motivic Adams spectral sequence that converges to $\pi_{*,*}(C\tau)$. Note that $E_r(C\tau)$ is an $E_r(S^{0,0})$ -module, and $E_2(C\tau)$ is equal to $\operatorname{Ext}_A(H^{*,*}(C\tau), \mathbb{M}_2)$.
- (15) For x in $E_2(S^{0,0})$, write x again (or $x_{C\tau}$ when absolutely necessary for clarity) for the image of x under the map $E_2(S^{0,0}) \to E_2(C\tau)$ induced by the inclusion $S^{0,0} \to C\tau$ of the bottom cell.
- (16) For x in $E_2(S^{0,0})$ such that $\tau x = 0$, write \overline{x} for a pre-image of x under the map $E_2(C\tau) \to E_2(S^{0,0})$ induced by the projection $C\tau \to S^{1,-1}$ to the top cell. There may be some indeterminacy in the choice of \overline{x} . See Section 5.1.5 and Table 40 for further discussion about these choices.
- (17) $E_r(S^0; BP)$ is the E_r -page of the classical Adams-Novikov spectral sequence.
- (18) $E_r(S^{0,0}; BPL)$ is the E_r -page of the motivic Adams-Novikov spectral sequence converging to $\pi_{*,*}$.
- (19) $E_r(C\tau; BPL)$ is the E_r -page of the motivic Adams-Novikov spectral sequence converging to $\pi_{*,*}(C\tau)$.

Table 1 lists some traditional notation for specific elements of the motivic stable homotopy ring. We will use this notation whenever it is convenient. A few remarks about these elements are in order:

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- (1) See [17, p. 28] for a geometric construction of τ .
- (2) Over fields that do not contain $\sqrt{-1}$, the motivic stable homotopy group $\pi_{0,0}$ contains an element that is usually called ϵ . Our use of the symbol ϵ follows Toda [42]. This should cause no confusion since we are working only over \mathbb{C} .
- (3) The element η_4 is defined to be the element of $\{h_1h_4\}$ such that $\eta^3\eta_4$ is zero. (The other element of $\{h_1h_4\}$ supports infinitely many multiplications by η .)
- (4) Similarly, η_5 is defined to be the element of $\{h_1h_5\}$ such that $\eta^7\eta_5$ is zero.

The element $\theta_{4.5}$ deserves additional discussion. We have perhaps presumptuously adopted this notation for an element of $\{h_3^2h_5\} = \{h_4^3\}$. This element is called α in [3]. To construct $\theta_{4.5}$, first choose an element $\theta'_{4.5}$ in $\{h_3^2h_5\}$ such that $4\theta'_{4.5}$ is contained in $\{h_0h_5d_0\}$. If $\eta\theta'_{4.5}$ is contained in $\{h_1h_5d_0\}$, then add an element of $\{h_5d_0\}$ to $\theta'_{4.5}$ and obtain an element $\theta''_{4.5}$ such that $\eta\theta''_{4.5}$ is contained in $\{B_1\}$. Next, if $\sigma\theta''_{4.5}$ is contained in $\{\tau h_1h_3g_2\}$, then add an element of $\{\tau h_1g_2\}$ to $\theta''_{4.5}$ to obtain an element $\theta_{4.5}$ such that $\sigma\theta_{4.5}$ is detected in Adams filtration at least 8. Note that $\sigma\theta_{4.5}$ may in fact be zero.

This does not specify just a single element of $\{h_3^2h_5\}$. The indeterminacy in the definition contains even multiples of $\theta_{4.5}$ and the element $\{\tau w\}$, but this indeterminacy does not present much difficulty.

In addition, we do not know whether $\nu\theta_{4.5}$ is contained in $\{B_2\}$. We know from Lemma 4.2.73 that there is an element θ of $\{h_3^2h_5\}$ such that $\nu\theta$ is contained in $\{B_2\}$. It is possible that θ is of the form $\theta_{4.5} + \beta$, where β belongs to $\{h_5d_0\}$. We can conclude only that either $\nu\theta_{4.5}$ or $\nu(\theta_{4.5} + \beta)$ belongs to $\{B_2\}$.

For more details on the properties of $\theta_{4.5}$, see Examples 4.1.6 and 4.1.7, as well as Lemmas 4.2.48 and 4.2.73.

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CHAPTER 2

The cohomology of the motivic Steenrod algebra

This chapter applies the motivic May spectral sequence to obtain the cohomology of the motivic Steenrod algebra through the 70-stem. We will freely borrow results from the classical May spectral sequence, i.e., from [29] and [40]. We will also need some facts from the cohomology of the classical Steenrod algebra that have been verified only by machine [9] [10].

The Ext chart in [19] is an essential companion to this chapter.

Outline. We begin in Section 2.1 with a review of the basic facts about the motivic Steenrod algebra over \mathbb{C} , the motivic May spectral sequence over \mathbb{C} , and the cohomology of the motivic Steenrod algebra. A critical ingredient is May's Convergence Theorem [**30**, Theorem 4.1], which allows the computation of Massey products in $\text{Ext}_A(\mathbb{F}_2, \mathbb{F}_2)$ via the differentials in the May spectral sequence. We will thoroughly review this result in Section 2.2.

Next, in Section 2.3 we describe the main points in computing the motivic May spectral sequence through the 70-stem. We rely heavily on results of [29] and [40], but we must also compute several exotic differentials, i.e., differentials that do not occur in the classical situation.

Having obtained the E_{∞} -page of the motivic May spectral sequence, the next step is to consider hidden extensions. In Section 2.4, we are able to resolve every possible hidden extension by τ , h_0 , h_1 , and h_2 through the range that we are considering, i.e., up to the 70-stem. The primary tools here are:

- (1) shuffling relations among Massey products.
- (2) Steenrod operations on Ext groups in the sense of [31].
- (3) classical hidden extensions established by machine computation [9]

Chapter 7 contains a series of tables that are essential for bookkeeping throughout the computations:

- (1) Tables 2 and 3 describe the May E_2 -page in terms of generators and relations and give the values of the May d_2 differential.
- (2) Tables 4 through 7 describe the May differentials d_r for $r \ge 4$.
- (3) Table 8 lists the multiplicative generators of the cohomology of the motivic Steenrod algebra over C.
- (4) Table 9 lists multiplicative generators of the May E_{∞} -page that become decomposable in Ext by hidden relations.
- (5) Table 10 lists all examples of multiplicative generators of the May E_{∞} page that represent more than one element in Ext. See Section 2.3.6 for
 more explanation.
- (6) Tables 11 through 15 list all extensions by τ , 2, η , and ν that are hidden in the May spectral sequence. A few miscellaneous hidden extensions are included as well.

- (7) Table 16 summarizes some Massey products.
- (8) Table 17 summarizes some matric Massey products.

Table 16 deserves additional explanation. In all cases, we have been careful to describe the indeterminacies accurately. The fifth column refers to an argument for establishing the Massey product, in one of the following forms:

- (1) An explicit proof given elsewhere in this manuscript.
- (2) A May differential implies the Massey product via May's Convergence Theorem 2.2.1.

The last column of Table 16 lists the specific results that rely on each Massey product. Frequently, these results are just a Toda bracket from Table 23.

Some examples. In this section, we describe several of the computational intricacies that are established later in the chapter. We also present a few questions that deserve further study.

EXAMPLE 2.0.1. An obvious question, which already arose in [13], is to find elements that are killed by τ^n but not by τ^{n-1} , for various values of n.

The element h_2g^2 , which is multiplicatively indecomposable, is the first example of an element that is killed by τ^3 but not by τ^2 . This occurs because of a hidden extension $\tau \cdot \tau h_2g^2 = Ph_1^4h_5$. There is an analogous relation $\tau^2h_2g = Ph_4$ that is not hidden. We do not know if this generalizes to a family of relations of the form $\tau^2h_2g^{2^k} = Ph_1^{2^{k+2}-4}h_{k+4}$.

We will show in Chapter 3 that h_2g^2 represents an element in motivic stable homotopy that is killed by τ^3 but not by τ^2 . This requires an analysis of the motivic Adams spectral sequence. In the vicinity of g^{2^k} , one might hope to find elements that are killed by τ^n but not by τ^{n-1} , for large values of n.

EXAMPLE 2.0.2. Classically, there is a relation $h_3 \cdot e_0 = h_1 h_4 c_0$ in the 24-stem of the cohomology of the Steenrod algebra. This relation is hidden on the E_{∞} -page of the May spectral sequence. We now give a proof of this classical relation that uses the cohomology of the motivic Steenrod algebra.

Motivically, it turns out that $h_2^3 e_0$ is non-zero, even though it is zero classically. This follows from the hidden extension $h_0 \cdot h_2^2 g = h_1^3 h_4 c_0$ (see Lemma 2.4.9). The relation $h_2^3 = h_1^2 h_3$ then implies that $h_1^2 h_3 e_0$ is non-zero. Therefore, $h_3 e_0$ is non-zero as well, and the only possibility is that $h_3 e_0 = h_1 h_4 c_0$.

EXAMPLE 2.0.3. Notice the hidden extension $h_0 \cdot h_2^2 g^2 = h_1^7 h_5 c_0$ (and similarly, the hidden extension $h_0 \cdot h_2^2 g = h_1^3 h_4 c_0$ that we discussed above in Example 2.0.2).

The next example in this family is $h_0 \cdot h_2^2 g^3 = h_1^9 D_4$, which at first does not appear to fit a pattern. However, there is a hidden extension $c_0 \cdot i_1 = h_1^4 D_4$, so we have $h_0 \cdot h_2^2 g^3 = h_1^5 c_0 i_1$. Presumably, there is an infinitely family of hidden extensions in which $h_0 \cdot h_2^2 g^k$ equals some power of h_1 times c_0 times an element related to Sq⁰ of elements associated to the image of J.

It is curious that $c_0 \cdot i_1$ is divisible by h_1^4 . An obvious question for further study is to determine the h_1 -divisibility of c_0 times elements related to Sq⁰ of elements associated to the image of J. For example, what is the largest power of h_1 that divides $g^2 i_1$?

EXAMPLE 2.0.4. Beware that g^2 and g^3 are not actually elements of the 40stem and 60-stem respectively. Rather, it is only τg^2 and τg^3 that exist (similarly, g does not exist in the 20-stem, but τg does exist). The reason is that there are May differentials taking g^2 to $h_1^8 h_5$, and g^3 to $h_1^6 i_1$. In other words, τg^2 and τg^3 are multiplicatively indecomposable elements. More generally, we anticipate that the element g^k does not exist because it supports a May differential related to Sq⁰ of an element in the image of J.

EXAMPLE 2.0.5. There is an isomorphism from the cohomology of the classical Steenrod algebra to the cohomology of the motivic Steenrod algebra over \mathbb{C} concentrated in degrees of the form (2s + f, f, s + f). This isomorphism preserves all higher structure, including algebraic Steenrod operations and Massey products. See Section 2.1.3 for more details.

For example, the existence of the classical element Ph_2 immediately implies that h_3g must be non-zero in the motivic setting; no calculations are necessary.

Another example is that $h_1^{2^k-1}h_{k+2}$ is non-zero motivically for all $k \ge 1$, because $h_0^{2^k-1}h_{k+1}$ is non-zero classically.

EXAMPLE 2.0.6. Many elements are h_1 -local in the sense that they support infinitely many multiplications by h_1 . In fact, any product of the symbols h_1 , c_0 , P, d_0 , e_0 , and g, if it exists, is non-zero. This is detectable in the cohomology of motivic A(2) [18].

Moreover, the element B_1 in the 46-stem is h_1 -local, and any product of B_1 with elements in the previous paragraph is again h_1 -local. We explore h_1 -local elements in great detail in [14].

EXAMPLE 2.0.7. The motivic analogue of the "wedge" subalgebra [28] appears to be more complicated than the classical version. For example, none of the wedge elements support multiplications by h_0 in the classical case. Motivically, many wedge elements do support h_0 multiplications. The results in this chapter naturally call for further study of the structure of the motivic wedge.

2.1. The motivic May spectral sequence

The following two deep theorems of Voevodsky are the starting points of our calculations.

THEOREM 2.1.1 ([44]). \mathbb{M}_2 is the bigraded ring $\mathbb{F}_2[\tau]$, where τ has bidegree (0,1).

THEOREM 2.1.2 ([43] [45]). The motivic Steenrod algebra A is the \mathbb{M}_2 -algebra generated by elements Sq^{2k} and Sq^{2k-1} for all $k \geq 1$, of bidegrees (2k, k) and (2k-1, k-1) respectively, and satisfying the following relations for a < 2b:

$$\operatorname{Sq}^{a}\operatorname{Sq}^{b} = \sum_{c} {\binom{b-1-c}{a-2c}} \tau^{?}\operatorname{Sq}^{a+b-c}\operatorname{Sq}^{c}.$$

The symbol ? stands for either 0 or 1, depending on which value makes the formula balanced in weight. See [13] for a more detailed discussion of the motivic Adem relations.

The A-module structure on \mathbb{M}_2 is trivial, i.e., every Sq^k acts by zero. This follows for simple degree reasons.

It is often helpful to work with the dual motivic Steenrod algebra $A_{*,*}$ [7, Section 5.2] [43] [45], which equals

$$\frac{\mathbb{M}_2[\tau_0, \tau_1, \dots, \xi_1, \xi_2, \dots]}{\tau_i^2 = \tau \xi_{i+1}}.$$

The reduced coproduct in $A_{*,*}$ is determined by

$$\tilde{\phi}_{*}(\tau_{k}) = \xi_{k} \otimes \tau_{0} + \xi_{k-1}^{2} \otimes \tau_{1} + \dots + \xi_{k-i}^{2^{i}} \otimes \tau_{i} + \dots + \xi_{1}^{2^{k-1}} \otimes \tau_{k-1}$$
$$\tilde{\phi}_{*}(\xi_{k}) = \xi_{k-1}^{2} \otimes \xi_{1} + \xi_{k-2}^{4} \otimes \xi_{2} + \dots + \xi_{k-i}^{2^{i}} \otimes \xi_{i} + \dots + \xi_{1}^{2^{k-1}} \otimes \xi_{k-1}.$$

2.1.1. Ext groups. We are interested in computing $\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$, which we abbreviate as Ext. This is a trigraded object. We will consistently use degrees of the form (s, f, w), where:

- (1) f is the Adams filtration, i.e., the homological degree.
- (2) s + f is the internal degree, i.e., corresponds to the first coordinate in the bidegrees of A.
- (3) s is the stem, i.e., the internal degree minus the Adams filtration.
- (4) w is the weight.

Note that $\operatorname{Ext}^{*,0,*} = \operatorname{Hom}_{A}^{*,*}(\mathbb{M}_{2},\mathbb{M}_{2})$ is dual to \mathbb{M}_{2} . We will abuse notation and write \mathbb{M}_{2} for this dual. Beware that now τ , which is really the dual of the τ that we discussed earlier, has degree (0,0,-1). Since Ext is a module over $\operatorname{Ext}^{*,0,*}$, i.e., over \mathbb{M}_{2} , we will always describe Ext as an \mathbb{M}_{2} -module.

The following result is the key tool for comparing classical and motivic computations. The point is that the motivic and classical computations become the same after inverting τ .

PROPOSITION 2.1.3 ([13]). There is an isomorphism of rings

$$\operatorname{Ext} \otimes_{\mathbb{M}_2} \mathbb{M}_2[\tau^{-1}] \cong \operatorname{Ext}_{A_{\operatorname{cl}}} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}].$$

2.1.2. The motivic May spectral sequence. The classical May spectral sequence arises by filtering the classical Steenrod algebra by powers of the augmentation ideal. The same approach can be applied in the motivic setting to obtain the motivic May spectral sequence. Details appear in [13]. Next we review the main points.

The motivic May spectral sequence is quadruply graded. We will always use gradings of the form (m, s, f, w), where m is the May filtration, and the other coordinates are as explained in Section 2.1.1.

Let Gr(A) be the associated graded algebra of A with respect to powers of the augmentation ideal.

THEOREM 2.1.4. The motivic May spectral sequence takes the form

$$E_2 = \operatorname{Ext}_{\operatorname{Gr}(A)}^{(m,s,f,w)}(\mathbb{M}_2,\mathbb{M}_2) \Rightarrow \operatorname{Ext}_A^{(s,f,w)}(\mathbb{M}_2,\mathbb{M}_2).$$

REMARK 2.1.5. As in the classical May spectral sequence, the odd differentials must be trivial for degree reasons.

PROPOSITION 2.1.6. After inverting τ , there is an isomorphism of spectral sequences between the motivic May spectral sequence of Theorem 2.1.4 and the classical May spectral sequence, tensored over \mathbb{F}_2 with $\mathbb{F}_2[\tau, \tau^{-1}]$. PROOF. Start with the fact that $A[\tau^{-1}]$ is isomorphic to $A_{cl} \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau, \tau^{-1}]$, with the same May filtrations.

This proposition means that differentials in the motivic May spectral sequence must be compatible with the classical differentials. This fact is critical to the success of our computations.

2.1.3. Ext in degrees with s + f - 2w = 0.

DEFINITION 2.1.7. Let A' be the subquotient \mathbb{M}_2 -algebra of A generated by Sq^{2k} for all $k \geq 0$, subject to the relation $\tau = 0$.

LEMMA 2.1.8. There is an isomorphism $A_{cl} \to A'$ that takes Sq^k to Sq^{2k} .

The isomorphism takes elements of degree n to elements of bidegree (2n, n).

PROOF. Modulo τ , the motivic Adem relation for $\operatorname{Sq}^{2a} \operatorname{Sq}^{2b}$ takes the form

$$\operatorname{Sq}^{2a} \operatorname{Sq}^{2b} = \sum_{c} {\binom{2b-1-2c}{2a-4c}} \operatorname{Sq}^{2a+2b-2c} \operatorname{Sq}^{2} c.$$

A standard fact from combinatorics says that

$$\binom{2b-1-2c}{2a-4c} = \binom{b-1-c}{a-2c}$$

modulo 2.

REMARK 2.1.9. Dually, A' corresponds to the quotient $\mathbb{F}_2[\xi_1, \xi_2, \ldots]$ of $A_{*,*}$, where we have set τ and τ_0, τ_1, \ldots to be zero. The dual to A' is visibly isomorphic to the dual of the classical Steenrod algebra.

DEFINITION 2.1.10. Let M be a bigraded A-module. The Chow degree of an element m in degree (t, w) is equal to t - 2w.

The terminology arises from the fact that the Chow degree is fundamental in Bloch's higher Chow group perspective on motivic cohomology [6].

DEFINITION 2.1.11. Let M be an A-module. Define the A'-module $Ch_0(M)$ to be the subset of M consisting of elements of Chow degree zero, with A'-module structure induced from the A-module structure on M.

The A'-module structure on $Ch_0(M)$ is well-defined since Sq^{2k} preserves Chow degrees.

THEOREM 2.1.12. There is an isomorphism from $\text{Ext}_{A_{cl}}$ to the subalgebra of Ext consisting of elements in degrees (s, f, w) with s+f-2w = 0. This isomorphism takes classical elements of degree (s, f) to motivic elements of degree (2s+f, f, s+f), and it preserves all higher structure, including products, squaring operations, and Massey products.

PROOF. There is a natural transformation

 $\operatorname{Hom}_{A}(-, \mathbb{M}_{2}) \to \operatorname{Hom}_{A'}(\operatorname{Ch}_{0}(-), \mathbb{F}_{2}),$

since $\operatorname{Ch}_0(\mathbb{M}_2) = \mathbb{F}_2$. Since Ch_0 is an exact functor, the derived functor of the right side is $\operatorname{Ext}_{A'}(\operatorname{Ch}_0(-), \mathbb{F}_2)$. The universal property of derived functors gives a natural transformation $\operatorname{Ext}_A(-, \mathbb{M}_2) \to \operatorname{Ext}_{A'}(\operatorname{Ch}_0(-), \mathbb{F}_2)$. Apply this natural transformation to \mathbb{M}_2 to obtain $\operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \to \operatorname{Ext}_{A'}(\operatorname{Ch}_0(\mathbb{M}_2), \mathbb{F}_2)$. The left

side is Ext, and the right side is isomorphic to $\text{Ext}_{A_{\text{cl}}}$ since A' is isomorphic to A_{cl} by Lemma 2.1.8.

We have now obtained a map $\operatorname{Ext} \to \operatorname{Ext}_{A_{cl}}$. We will verify that this map is an isomorphism on the part of Ext in degrees (s, f, w) with s + f - 2w = 0. Compare the classical May spectral sequence with the part of the motivic May spectral sequence in degrees (m, s, f, w) with s + f - 2w = 0. By direct inspection, the motivic E_1 -page in these degrees is the polynomial algebra over \mathbb{F}_2 generated by h_{ij} for i > 0 and j > 0. This is isomorphic to the classical E_1 -page, where the motivic element h_{ij} corresponds to the classical element $h_{i,j-1}$.

REMARK 2.1.13. Similar methods show that Ext is concentrated in degree (s, f, w) with $s + f - 2w \ge 0$. The map Ext \rightarrow Ext_{Acl} constructed in the proof annihilates elements in degrees (s, f, w) with s + f - 2w > 0. Thus, Ext_{Acl} is isomorphic to the quotient of Ext by elements of degree (s, f, w) with s + f - 2w > 0.

2.2. Massey products in the motivic May spectral sequence

We will frequently compute Massey products in Ext in order to resolve hidden extensions and to determine May differentials. The absolutely essential tool for computing such Massey products is May's Convergence Theorem [**30**, Theorem 4.1]. The point of this theorem is that under certain hypotheses, Massey products in Ext can be computed in the E_r -page of the motivic May spectral sequence. For the reader's convenience, we will state the theorem in the specific forms that we will use. We have slightly generalized the result of [**30**, Theorem 4.1] to allow for brackets that are not strictly defined. In order to avoid unnecessarily heavy notation, we have intentionally avoided the most general possible statements. The interested reader is encouraged to carry out these generalizations.

THEOREM 2.2.1 (May's Convergence Theorem). Let α_0 , α_1 , and α_2 be elements of Ext such that the Massey product $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. For each *i*, let a_i be a permanent cycle on the May E_r -page that detects α_i . Suppose further that:

- (1) there exist elements a_{01} and a_{12} on the May E_r -page such that $d_r(a_{01}) = a_0a_1$ and $d_r(a_{12}) = a_1a_2$.
- (2) if (m, s, f, w) is the degree of either a_{01} or a_{12} ; $m' \ge m$; and m'-t < m-r; then every May differential $d_t : E_t^{(m',s,f,w)} \to E_t^{(m'-t+1,s-1,f+1,w)}$ is zero.

Then $a_0a_{12} + a_{01}a_2$ detects an element of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ in Ext.

The point of condition (1) is that the bracket $\langle a_0, a_1, a_2 \rangle$ is defined in the differential graded algebra (E_r, d_r) . Condition (2) is an equivalent reformulation of condition (*) in [**30**, Theorem 4.1]. When computing $\langle a_0, a_1, a_2 \rangle$, one uses a differential $d_r : E_r^{(m,s,f,w)} \to E_r^{(m-r+1,s-1,f+1,w)}$. The idea of condition (2) is that there are no later "crossing" differentials d_t whose source has higher May filtration and whose target has strictly lower May filtration.

The proof of May's Convergence Theorem 2.2.1 is exactly the same as in [30] because every threefold Massey product is strictly defined in the sense that its subbrackets have no indeterminacy.

THEOREM 2.2.2 (May's Convergence Theorem). Let α_0 , α_1 , α_2 , and α_3 be elements of Ext such that the Massey product $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined. For each *i*, let a_i be a permanent cycle on the E_r -page that detects α_i . Suppose further that:

- (1) there are elements a_{01} , a_{12} , and a_{23} on the E_r -page such that $d_r(a_{01}) = a_0 a_1$, $d_r(a_{12}) = a_1 a_2$, and $d_r(a_{23}) = a_2 a_3$.
- (2) there are elements a_{02} and a_{13} on the E_r -page such that $d_r(a_{02}) = a_0a_{12} + a_{01}a_2$ and $d_r(a_{13}) = a_1a_{23} + a_{12}a_3$.
- (3) if (m, s, f, w) is the degree of a_{01} , a_{12} , a_{23} , a_{02} , or a_{13} ; and $m' \ge m$; and m' t < m r; then every differential

$$d_t: E_t^{(m',s,f,w)} \to E_t^{(m-t+1,s-1,f+1,w)}$$

is zero.

- (4) The subbracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ has no indeterminacy.
- (5) the indeterminacy of $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is generated by elements of the form $\alpha_1\beta$ and $\gamma\alpha_3$, where β and γ are detected in May filtrations strictly lower than the May filtrations of a_{23} and a_{12} respectively.

Then $a_0a_{13} + a_{01}a_{23} + a_{02}a_3$ detects an element of $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ in Ext.

The point of conditions (1) and (2) is that the bracket $\langle a_0, a_1, a_2, a_3 \rangle$ is defined in the differential graded algebra (E_r, d_r) . Condition (3) is an equivalent reformulation of condition (*) in [**30**, Theorem 4.1]. The point of this condition is that there are no later "crossing" differentials whose source has higher May filtration and whose target has strictly lower May filtration.

Condition (5) does not appear in [30], which only deals with strictly defined brackets. Of course, the theorem has a symmetric version in which the bracket $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ has no indeterminacy. It is probably possible to state a version of the theorem in which both threefold subbrackets have non-zero indeterminacy. However, additional conditions are required for such a fourfold bracket to be well-defined [20].

PROOF. Let *C* be the cobar resolution of the motivic Steenrod algebra whose homology is Ext. Let $\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, and $\tilde{\alpha}_3$ be explicit cycles in *C* representing α_0 , α_1 , α_2 , and α_3 . As in the proof of [**30**, Theorem 4.1], we may choose an element $\tilde{\alpha}_{01}$ of *C* such that $d(\tilde{\alpha}_{01}) = \tilde{\alpha}_0 \tilde{\alpha}_1$, and $\tilde{\alpha}_{01}$ is detected by a_{01} in the May E_r -page. We may similarly choose $\tilde{\alpha}_{12}$ and $\tilde{\alpha}_{23}$ whose boundaries are $\tilde{\alpha}_1 \tilde{\alpha}_2$ and $\tilde{\alpha}_2 \tilde{\alpha}_3$ and that are detected by a_{12} and a_{23} .

Next, we want to choose $\tilde{\alpha}_{13}$ in C whose boundary is $\tilde{\alpha}_1 \tilde{\alpha}_{23} + \tilde{\alpha}_{12} \tilde{\alpha}_3$ and that is detected by a_{13} . Because of the possible indeterminacy in $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, the cycle $\tilde{\alpha}_1 \tilde{\alpha}_{23} + \tilde{\alpha}_{12} \tilde{\alpha}_3$ may not be a boundary in C. However, since we are assuming that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ is defined, we can add cycles to $\tilde{\alpha}_{12}$ and $\tilde{\alpha}_{23}$ to ensure that $\tilde{\alpha}_1 \tilde{\alpha}_{23} + \tilde{\alpha}_{12} \tilde{\alpha}_3$ is a boundary. When we do this, condition (5) guarantees that $\tilde{\alpha}_{12}$ and $\tilde{\alpha}_{23}$ are still detected by a_{12} and a_{23} . Then we may choose $\tilde{\alpha}_{13}$ as in the proof of [**30**, Theorem 4.1].

Finally, we may choose $\tilde{\alpha}_{02}$ as in the proof of [**30**, Theorem 4.1]. Because $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ has no indeterminacy, we automatically know that $\tilde{\alpha}_0 \tilde{\alpha}_{12} + \tilde{\alpha}_{01} \tilde{\alpha}_2$ is a boundary in C.

For completeness, we will now also state May's Convergence Theorem for fivefold brackets. This result is used only in Lemma 2.4.24. The proof is essentially the same as the proof for fourfold brackets.

THEOREM 2.2.3 (May's Convergence Theorem). Let α_0 , α_1 , α_2 , α_3 , and α_4 be elements of Ext such that the Massey product $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is defined. For each i, let a_i be a permanent cycle on the E_r -page that detects α_i . Suppose further that:

- (1) there are elements a_{01} , a_{12} , a_{23} , and a_{34} in E_r such that $d_r(a_{01}) = a_0 a_1$, $d_r(a_{12}) = a_1 a_2$, $d_r(a_{23}) = a_2 a_3$, and $d_r(a_{34}) = a_3 a_4$.
- (2) there are elements a_{02} , a_{13} , and a_{24} in E_r such that $d_r(a_{02}) = a_0a_{12} + a_{01}a_2$, $d_r(a_{13}) = a_1a_{23} + a_{12}a_3$, and $d_r(a_{24}) = a_2a_{34} + a_{23}a_4$.
- (3) there are elements a_{03} and a_{14} in E_r such that $d_r(a_{03}) = a_0a_{13} + a_{01}a_{23} + a_{02}a_3$ and $d_r(a_{14}) = a_1a_{24} + a_{12}a_{34} + a_{13}a_4$.
- (4) if (m, s, f, w) is the degree of $a_{01}, a_{12}, a_{23}, a_{34}, a_{02}, a_{13}, a_{24}, a_{03}, or a_{14};$ $m' \ge m; and m' - t < m - r; then every differential <math>d_t : E_t^{(m', s, f, w)} \to E_t^{(m-t+1, s-1, f+1, w)}$ is zero.
- (5) the threefold subbrackets $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$, $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$, and $\langle \alpha_2, \alpha_3, \alpha_4 \rangle$ have no indeterminacy.
- (6) the subbracket $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ has no indeterminacy.
- (7) the indeterminacy of $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ is generated by elements contained in $\langle \beta, \alpha_3, \alpha_4 \rangle$, $\langle \alpha_1, \gamma, \alpha_4 \rangle$, and $\langle \alpha_1, \alpha_2, \delta \rangle$, where β , γ , and δ are detected in May filtrations strictly lower than the May filtrations of a_{12} , a_{23} , and a_{34} respectively.

Then $a_0a_{14} + a_{01}a_{24} + a_{02}a_{34} + a_{03}a_4$ detects an element of $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ in Ext.

Although we will use May's Convergence Theorem to compute most of the Massey For a few Massey products, we also need occasionally the following result [16] [1, Lemma 2.5.4].

PROPOSITION 2.2.4. Let x be an element of $\text{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$.

(1) If $h_0 x = 0$, then $\tau h_1 x$ belongs to $\langle h_0, x, h_0 \rangle$.

(2) If $n \ge 1$ and $h_n x = 0$, then $h_{n+1}x$ belongs to $\langle h_n, x, h_n \rangle$.

We will need the following results about shuffling higher brackets that are not strictly defined.

LEMMA 2.2.5. Suppose that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ are defined and that the indeterminacy of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ consists of multiples of α_2 . Then

 $\alpha_0 \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \subseteq \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4.$

PROOF. For each *i*, choose an element a_i that represents α_i . Let β be an element of $\alpha_0 \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$. There exist elements $a_{12}, a_{23}, a_{34}, a_{13}$, and a_{24} such that $d(a_{i,i+1}) = a_i a_{i+1}, d(a_{i,i+2}) = a_i a_{i+1,i+2} + a_{i,i+1} a_{i+2}$, and β is represented by

$$b = a_0 a_1 a_{24} + a_0 a_{12} a_{34} + a_0 a_{13} a_4.$$

By the assumption on the indeterminacy of $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$, we can then choose a_{01} and a_{02} such that $d(a_{01}) = a_0 a_1$ and $d(a_{02}) = a_0 a_{12} + a_{01} a_2$. Then

$$a_0a_{13}a_4 + a_{01}a_{23}a_4 + a_{02}a_3a_4$$

represents a class in $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$ that is homologous to b.

LEMMA 2.2.6. Suppose that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ are defined, and suppose that $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero. Then

$$\alpha_0 \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cap \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$$

is non-empty.

PROOF. Choose elements a_i that represent α_i . Choose a_{01} , a_{12} , and a_{02} such that $d(a_{01}) = a_0 a_1$, $d(a_{12}) = a_1 a_2$, and $d(a_{02}) = a_0 a_{12} + a_{01} a_2$. Also, choose a_{23} , a_{34} , and a_{24} such that $d(a_{23}) = a_2 a_3$, $d(a_{34}) = a_3 a_4$, and $d(a_{24}) = a_2 a_{34} + a_{23} a_4$. Since $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero, there exists a_{13} such that $d(a_{13}) = a_1 a_{23} + a_{12} a_3$. Then

 $a_0a_1a_{24} + a_0a_{12}a_{34} + a_0a_{13}a_4$

represents an element of $\alpha_0 \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$, and it is homologous to

$$a_0a_{13}a_4 + a_{01}a_{23}a_4 + a_{02}a_3a_4,$$

which represents an element of $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \alpha_4$.

LEMMA 2.2.7. Suppose that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle$ are defined and that $\langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ is strictly zero. Then

$$\alpha_0 \langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \rangle \subseteq \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \alpha_5.$$

PROOF. The proof is essentially the same as the proof of Lemma 2.2.5.

2.3. The May differentials

2.3.1. The May E_1 -page. The E_2 -page of the May spectral sequence is the cohomology of a differential graded algebra. In other words, the May spectral sequence really starts with an E_1 -page. As described in [13], the motivic E_1 -page is essentially the same as the classical E_1 -page. Specifically, the motivic E_1 -page is a polynomial algebra over \mathbb{M}_2 with generators h_{ij} for all i > 0 and $j \ge 0$, where:

- (1) h_{i0} has degree $(i, 2^i 2, 1, 2^{i-1} 1)$.
- (2) h_{ij} has degree $(i, 2^j(2^i 1) 1, 1, 2^{j-1}(2^i 1))$ for j > 0.

The d_1 -differential is described by the formula:

$$d_1(h_{ij}) = \sum_{0 < k < i} h_{kj} h_{i-k,k+j}.$$

2.3.2. The May E_2 -page. We now describe the E_2 -page of the motivic May spectral sequence. As explained in [13], it turns out that the motivic E_2 -page is essentially the same as the classical E_2 -page. The following proposition makes this precise.

Recall that Gr(A) is the associated graded object of the motivic Steenrod algebra with respect to powers of the augmentation ideal. Similarly, let $Gr(A_{cl})$ be the associated graded object of the classical Steenrod algebra with respect to powers of the augmentation ideal.

PROPOSITION 2.3.1 ([13]). There are graded ring isomorphisms (a) $\operatorname{Gr}(A) \cong \operatorname{Gr}(A_{\operatorname{cl}}) \otimes_{\mathbb{F}_2} \mathbb{F}_2[\tau].$

(b) $\operatorname{Ext}_{\operatorname{Gr}(A)}(\mathbb{M}_2, \mathbb{M}_2) \cong \operatorname{Ext}_{\operatorname{Gr}(A_{c1})}(\mathbb{F}_2, \mathbb{F}_2) \otimes_{\mathbb{F}_2} \mathbb{M}_2.$

In other words, explicit generators and relations for the E_2 -page can be lifted directly from the classical situation [40].

Moreover, because of Proposition 2.1.6, the values of the May d_2 differential can also be lifted from the classical situation, except that a few factors of τ show up to give the necessary weights. For example, classically we have the differential

$$d_2(b_{20}) = h_1^3 + h_0^2 h_2.$$

Motivically, this does not make sense, since b_{20} and $h_0^2 h_2$ have weight 2, while h_1^3 has weight 3. Therefore, the motivic differential must be

$$d_2(b_{20}) = \tau h_1^3 + h_0^2 h_2.$$

Table 2 lists the multiplicative generators of the E_2 -page through the 70-stem, and Table 3 lists a generating set of relations for the E_2 -page in the same range. Table 2 also gives the values of the May d_2 differential, all of which are easily deduced from the classical situation [40].

2.3.3. The May E_4 -page. Although the E_2 -page is quite large, the May d_2 differential is also very destructive. As a result, the E_4 -page becomes manageable. We obtain the E_4 -page by direct computation with the d_2 differential.

REMARK 2.3.2. As in [40], we use the notation $B = b_{30}b_{31} + b_{21}b_{40}$.

Having described the E_4 -page, it is now necessary to find the values of the May d_4 differential on the multiplicative generators. Most of the values of d_4 follow from comparison to the classical case [40], together with a few factors of τ to balance the weights. There is only one differential that is not classical.

LEMMA 2.3.3. $d_4(g) = h_1^4 h_4$.

PROOF. By the isomorphism of Theorem 2.1.12, we know that $h_1^4h_4$ cannot survive the motivic May spectral sequence because $h_0^4h_3$ is zero classically. There is only one possible differential that can kill $h_1^4h_4$.

See also [13] for a different proof of Lemma 2.3.3.

Table 4 lists the values of the d_4 differential on multiplicative generators of the E_4 -page.

2.3.4. The May E_6 -page. We can now obtain the E_6 -page by direct computation with the May d_4 differential and the Leibniz rule.

Having described the E_6 -page, it is now necessary to find the values of the May d_6 differential on the multiplicative generators. Most of these values follow from comparison to the classical case [40], together with a few factors of τ to balance the weights. There are only a few differentials that are not classical.

Lemma 2.3.4.

(1) $d_6(x_{56}) = h_1^2 h_5 c_0 d_0.$ (2) $d_6(Px_{56}) = Ph_1^2 h_5 c_0 d_0.$ (3) $d_6(B_{23}) = h_1^2 h_5 d_0 e_0.$

PROOF. We have the relation $h_1x_{56} = c_0\phi$. The d_6 differential on ϕ then implies that $d_6(h_1x_{56}) = h_1^3h_5c_0d_0$, from which it follows that $d_6(x_{56}) = h_1^2h_5c_0d_0$.

The arguments for the other two differentials are similar, using the relations $h_1 \cdot Px_{56} = Pc_0 \cdot \phi$ and $h_1B_{23} = e_0\phi$.

LEMMA 2.3.5. $d_6(c_0g^3) = h_1^{10}D_4$.

PROOF. Lemma 2.4.24 shows that $c_0 \cdot i_1 = h_1^4 D_4$. Since $h_1^6 i_1 = 0$, we conclude that $h_1^{10} D_4$ must be zero in Ext. There is only one possible differential that can hit $h_1^{10} D_4$.

REMARK 2.3.6. The value of $d_6(\Delta h_0^2 Y)$ given in [40, Proposition 4.37(c)] is incorrect because it is inconsistent with machine computations of $\operatorname{Ext}_{A_{cl}}[9]$. The value for $d_6(\Delta h_0^2 Y)$ given in Table 5 is the only possibility that is consistent with the machine computations.

Table 5 lists the values of the May d_6 differential on multiplicative generators of the E_6 -page.

2.3.5. The May E_8 -page. We can now obtain the May E_8 -page by direct computation with the May d_6 differential and the Leibniz rule. Once we reach the E_8 -page, we are nearly done. There are just a few more higher differentials to deal with.

Having described the E_8 -page, it is now necessary to find the values of the May d_8 differential on the multiplicative generators. Once again, most of these values follow from comparison to the classical case [40], together with a few factors of τ to balance the weights. There are only a few differentials that are not classical.

Lemma 2.3.7.

(1)
$$d_8(q^2) = h_1^8 h_5$$
.

- (1) $a_8(g) = h_1^{5,1}$ (2) $d_8(w) = Ph_1^5h_5$. (3) $d_8(\Delta c_0 g) = Ph_1^4h_5c_0$. (4) $d_8(Q_3) = h_1^4h_5^2$.

PROOF. It follows from Theorem 2.1.12 that $h_1^8 h_5$ must be zero in Ext, since $h_0^8 h_4$ is zero classically. There is only one differential that can possibly hit $h_1^8 h_5$.

We now know that $Ph_1^9h_5 = 0$ in Ext since $h_1^8h_5 = 0$. There is only one differential that can hit this. This shows that $d_8(w) = Ph_1^5h_5$.

Using the relation $c_0 w = h_1 \cdot \Delta c_0 g$, it follows that $d_8(h_1 \cdot \Delta c_0 g) = P h_1^5 h_5 c_0$, and then that $d_8(\Delta c_0 g) = Ph_1^4 h_5 c_0$.

Since $h_0^4 h_4^2$ is zero classically, it follows from Theorem 2.1.12 that $h_1^4 h_5^2$ must be zero in Ext. There is only one differential that can possibly hit $h_1^4 h_5^2$. \square

Table 6 lists the values of the May d_8 differential on multiplicative generators of the E_8 -page.

2.3.6. The May E_{∞} -page. Most of the higher May differentials are zero through the 70-stem. The exceptions are the May d_{12} differential, the May d_{16} differential, and the May d_{32} differential. All of the non-zero values of these differentials are easily deduced by comparison to the classical case [40].

Table 7 lists the values of these higher differentials on multiplicative generators of the higher pages. There are no more differentials to consider in our range, and we have determined the May E_{∞} -page.

The multiplicative generators for the E_{∞} -page through the 70-stem break into two groups. The first group consists of generators that are still multiplicative generators in Ext after hidden extensions have been considered; these are listed in Table 8. The second group consists of multiplicative generators of the E_{∞} -page that become decomposable in Ext because of a hidden extension; these are listed in Table 9.

It is traditional to use the same symbols for elements of the E_{∞} -page and for the elements of Ext that they represent. Generally, there is no ambiguity with this abuse of notation, but there are several exceptions. These exceptions occur when a multiplicative generator for the E_{∞} -page lies in the same degree as another element of the E_{∞} -page with lower May filtration.

The first such example occurs in the 18-stem, where the element f_0 of the E_{∞} page represents two elements of Ext because of the presence of the element $\tau h_1^3 h_4$ of lower May filtration. This particular example does not cause much difficulty.
Just arbitrarily choose one of these elements to be the generator of Ext. The
element disappears quickly from further analysis because f_0 supports an Adams d_2 differential.

However, later examples involve more subtlety and call for a careful distinction between the possibilities. There are no wrong choices, but it is important to be consistent with the notation in different arguments. For example, the element u' of the E_{∞} -page represents two elements of Ext because of the presence of $\tau d_0 l$. One of these elements is killed by τ , while the other element is killed by h_0 . Sloppy notation might lead to the false conclusion that there is a multiplicative generator of Ext in that degree that is killed by both τ and by h_0 .

Table 10 lists all such examples of multiplicative generators of the E_{∞} -page that represent more than one element in Ext. In many of these examples, we have given an algebraic specification of one element of Ext to serve as the multiplicative generator, sometimes by comparing to $\operatorname{Ext}_{A(2)}$ [18]. In some examples, we have not given a definition because an algebraic description is not readily available, and also because it does not seem to matter for later analysis. The reader is strongly warned to be cautious when working with these undefined elements.

The example τQ_3 deserves an additional remark. Here we have defined the element in terms of an Adams differential. This is merely a matter of convenience for later work with the Adams spectral sequence in Chapter 3.

2.4. Hidden May extensions

In order to pass from the E_{∞} -page to Ext, we must resolve some hidden extensions. In this section, we deal with all possible hidden extensions by τ , h_0 , h_1 , and h_2 . We will use several different tools, including:

- (1) Classical hidden extensions [9].
- (2) Shuffle relations with Massey products.
- (3) Steenrod operations in the sense of [**31**].
- (4) Theorem 2.1.12 for hidden extensions among elements in degrees (s, f, w) with s + f 2w = 0.

2.4.1. Hidden May τ **extensions.** By exhaustive search, the following results give all of the hidden τ extensions.

PROPOSITION 2.4.1. Table 11 lists all of the hidden τ extensions through the 70-stem.

PROOF. Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_0 \cdot e_0 g = h_0^4 x$. This implies that $\tau^2 \cdot h_0 e_0 g = h_0^4 x$ motivically.

Proofs for the more subtle cases are given below.

Lemma 2.4.2.

(1) $\tau \cdot \tau h_2 g^2 = P h_1^4 h_5.$ (2) $\tau \cdot \tau h_0 g^3 = P h_1^4 h_5 e_0.$ PROOF. Start with the relation $h_1 \cdot \tau g + h_2 f_0 = 0$, and apply the squaring operation Sq⁴. One needs that Sq³(τg) = $Ph_1^2h_5$ [10]. The result is the first hidden extension.

For the second, multiply the first hidden extension by e_0 .

Lemma 2.4.3.

(1) $\tau \cdot B_8 = Ph_5d_0.$ (2) $\tau \cdot h_1^2 B_{21} = Ph_5c_0d_0.$ (3) $\tau \cdot B_8d_0 = h_0^4 X_3.$

PROOF. There is a classical hidden extension $c_0 \cdot B_1 = Ph_1h_5d_0$ [9]. Motivically, there is a non-hidden relation $c_0 \cdot B_1 = h_1B_8$. It follows that $\tau \cdot h_1B_8 = Ph_1h_5d_0$ motivically.

For the second hidden extension, multiply the first hidden extension by c_0 . Note that $c_0B_8 = h_1^2B_{21}$ is detected in the E_{∞} -page of the May spectral sequence.

For the third hidden extension, multiply the first hidden extension by d_0 , and observe that $Ph_5d_0^2 = h_0^4X_3$, which is detected in the E_{∞} -page of the May spectral sequence.

Lemma 2.4.4.

 $\begin{array}{l} (1) \ \tau \cdot Pu' = h_0^5 R_1. \\ (2) \ \tau \cdot P^2 u' = h_0^9 R. \\ (3) \ \tau \cdot P^3 u' = h_0^6 R_1'. \end{array}$

PROOF. We first compute that $\langle \tau, u', h_0^3 \rangle = \{Q', Q' + \tau Pu\}$. One might try to apply May's Convergence Theorem 2.2.1 with the May differential $d_2(b_{20}b_{30}^3h_0(1)) = \tau u'$, but condition (2) of the theorem is not satisfied because of the "crossing" May differential $d_4(P\Delta h_0h_4) = P^2h_0h_4^2$.

Instead, note that $h_0 \cdot u' = \tau h_0 d_0 l$ by comparison to $\operatorname{Ext}_{A(2)}$ [18], so we have that $\langle \tau, u', h_0^3 \rangle = \langle \tau, \tau h_0 d_0 l, h_0^2 \rangle$. The latter bracket is given in Table 16.

Next, Table 16 shows that $Pu' = \langle u', h_0^3, h_0 h_3 \rangle$, with no indeterminacy. Use the previous paragraph and a shuffle to get that $\tau \cdot Pu' = h_0 h_3 Q'$. Finally, there is a classical hidden extension $h_3 \cdot Q' = h_0^4 R_1$ [9], which implies that the same formula holds motivically.

The argument for the second hidden extension is similar, using the shuffle

$$\tau \cdot P^2 u' = \tau \langle u', h_0^3, h_0^5 h_4 \rangle = \langle \tau, u', h_0^3 \rangle h_0^5 h_4 = h_0^5 h_4 Q'.$$

The first equality comes from Table 16. Also, we need the classical hidden extension $h_4 \cdot Q' = h_0^4 R$ [9], which implies that the same formula holds motivically.

The argument for the third hidden extension is also similar, using the shuffle

$$\tau \cdot P^{3}u' = \tau \langle u', h_{0}^{3}, h_{0}^{3}i \rangle = \langle \tau, u', h_{0}^{3} \rangle h_{0}^{3}i = h_{0}^{3}iQ'.$$

The first equality comes from Table 16. Also, we need the classical hidden extension $i \cdot Q' = h_0^3 R'_1$ [9].

LEMMA 2.4.5. $\tau \cdot k_1 = h_2 h_5 n$.

PROOF. First, Table 16 shows that $k = \langle d_0, h_3, h_0^2 h_3 \rangle$, with no indeterminacy. It follows from [**32**] that Sq⁰ $k = \langle Sq^0 d_0, Sq^0 h_3, Sq^0 h_0^2 h_3 \rangle$, with no indeterminacy. In other words, Sq⁰ $k = \langle \tau^2 d_1, h_4, \tau^2 h_1^2 h_4 \rangle$. From the classical calculation [**10**], Sq⁰ k also equals $\tau^3 h_2 h_5 n$.

On the other hand, Table 16 show that $k_1 = \langle d_1, h_4, h_1^2 h_4 \rangle$, with no indeterminacy.

This shows that $\tau^4 \cdot k_1 = \tau^3 h_2 h_5 n$ in Ext, from which it follows that $\tau \cdot k_1 = h_2 h_5 n$.

REMARK 2.4.6. In the 46-stem, $\tau \cdot u'$ does not equal $\tau^2 d_0 l$. Similarly, in the 49-stem, $\tau \cdot v'$ does not equal $\tau^2 e_0 l$. This is true by definition; see Table 10.

2.4.2. Hidden May h_0 extensions. By exhaustive search, the following results give all of the hidden h_0 extensions.

PROPOSITION 2.4.7. Table 12 lists all of the hidden h_0 extensions through the 70-stem.

PROOF. Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_0 \cdot r = s$. This implies that $h_0 \cdot r = s$ motivically as well.

Several other extensions are implied by the hidden τ extensions established in Section 2.4.1. For example, the extensions $\tau \cdot Pu' = h_0^4 S_1$ and $\tau \cdot \tau h_0 d_0^2 j = h_0^5 S_1$ imply that $h_0 \cdot Pu' = \tau h_0 d_0^2 j$.

Proofs for the more subtle cases are given below.

Lemma 2.4.8.

(1) $h_0 \cdot u' = \tau h_0 d_0 l.$ (2) $h_0 \cdot v' = \tau h_0 e_0 l.$ (3) $h_0 \cdot Pv' = \tau h_0 d_0^2 k.$ (4) $h_0 \cdot P^2 v' = \tau h_0 d_0^3 i$

PROOF. These follow by comparison to $\text{Ext}_{A(2)}$ [18].

Lemma 2.4.9.

 $\begin{array}{ll} (1) & h_0 \cdot h_2^2 g = h_1^3 h_4 c_0. \\ (2) & h_0 \cdot h_2^2 g^2 = h_1^7 h_5 c_0. \\ (3) & h_0 \cdot h_2^2 g^3 = h_1^9 D_4. \end{array}$

PROOF. For the first hidden extension, use the shuffle

$$h_1^3 h_4 \langle h_1, h_0, h_2^2 \rangle = \langle h_1^3 h_4, h_1, h_0 \rangle h_2^2.$$

Similarly, for the second hidden section, use the shuffle

$$h_1^7 h_5 \langle h_1, h_0, h_2^2 \rangle = \langle h_1^7 h_5, h_1, h_0 \rangle h_2^2.$$

For the third hidden extension, there is a hidden extension $c_0 \cdot i_1 = h_1^4 D_4$ that will be established in Lemma 2.4.24. Use this relation to compute that

$$h_1^9 D_4 = h_1^5 i_1 \langle h_1, h_2^2, h_0 \rangle = \langle h_1^5 i_1, h_1, h_2^2 \rangle h_0.$$

Finally, Table 16 shows that $h_2^2 g^3 = \langle h_1^5 i_1, h_1, h_2^2 \rangle$.

Lemma 2.4.10.

(1) $h_0 \cdot gr = Ph_1^3h_5c_0.$ (2) $h_0 \cdot lm = h_1^6X_1.$ (3) $h_0 \cdot m^2 = h_1^5c_0Q_2.$

REMARK 2.4.11. The three parts may seem unrelated, but note that $lm = e_0 gr$ and $m^2 = g^2 r$ on the E_8 -page of the May spectral sequence.

PROOF. Table 16 shows that $e_0 r = \langle \tau^2 g^2, h_2^2, h_0 \rangle$. Next observe that

$$h_2 \cdot e_0 r = \langle \tau^2 g^2, h_2^2, h_0 \rangle h_2 = \langle \tau^2 g^2, h_2^2, h_0 h_2 \rangle = \langle \tau^2 h_2 g^2, h_2, h_0 h_2 \rangle.$$

None of these brackets have indeterminacy.

Use the relation $Ph_1^4h_5 = \tau^2h_2g^2$ from Lemma 2.4.2 to write

$$h_2 \cdot e_0 r = \langle Ph_1^4 h_5, h_2, h_0 h_2 \rangle = Ph_1^3 h_5 \langle h_1, h_2, h_0 h_2 \rangle = Ph_1^3 h_5 c_0.$$

The last step is to show that $h_2 \cdot e_0 r = h_0 \cdot gr$. This follows from the calculation

 $h_0 \cdot gr = h_0 \langle h_1^3 h_4, h_1, r \rangle = \langle h_0, h_1^3 h_4, h_1 \rangle r = h_2 e_0 \cdot r,$

where the brackets are given in Table 16. This finishes the proof of part (1).

For part (2), we will prove below in Lemma 2.4.14 that $h_1^3 X_1 = Ph_5 c_0 e_0$. So we wish to show that $h_0 \cdot lm = Ph_1^3 h_5 c_0 e_0$. This follows immediately from part (1), using that $lm = e_0 gr$.

The proof of part (3) is similar to the proof of part (1). First, lm equals $\langle \tau^2 g^3, h_2^2, h_0 \rangle$. As above, this implies that $h_2 lm = \langle \tau^2 h_2 g^3, h_2, h_0 h_2 \rangle$. Now use the (not hidden) relation $\tau^2 h_2 g^3 = h_1^6 Q_2$ to deduce that $h_2 lm = h_1^5 c_0 Q_2$. The desired formula now follows since $h_2 l = h_0 m$.

LEMMA 2.4.12. $h_0 \cdot h_0^2 B_{22} = P h_1 h_5 c_0 d_0$.

PROOF. This follows from the hidden τ extension $\tau \cdot h_1^3 B_{21} = Ph_1h_5c_0d_0$ that follows from Lemma 2.4.3, together with the relation $\tau h_1^3 = h_0^2h_2$.

2.4.3. Hidden May h_1 extensions. By exhaustive search, the following results give all of the hidden h_1 extensions.

PROPOSITION 2.4.13. Table 13 lists all of the hidden h_1 extensions through the 70-stem.

PROOF. Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_1 \cdot x = h_2^2 d_1$. This implies that there is a motivic hidden extension $h_1 \cdot x = \tau h_2^2 d_1$.

Proofs for the more subtle cases are given below.

Lemma 2.4.14.

(1) $h_1 \cdot \tau h_1 G = h_5 c_0 e_0.$ (2) $h_1 \cdot h_1 B_3 = h_5 d_0 e_0.$ (3) $h_1 \cdot \tau P h_1 G = P h_5 c_0 e_0.$

(4) $h_1 \cdot h_1^2 X_3 = h_5 c_0 d_0 e_0.$

PROOF. Table 16 shows that $\tau h_1 G = \langle h_5, h_2 g, h_0^2 \rangle$. Shuffle to obtain

$$h_1 \cdot \tau h_1 G = \langle h_5, h_2 g, h_0^2 \rangle h_1 = h_5 \langle h_2 g, h_0^2, h_1 \rangle.$$

Finally, Table 16 shows that $c_0 e_0 = \langle h_2 g, h_0^2, h_1 \rangle$. This establishes the first hidden extension.

For the second hidden extension, Table 16 shows that $\langle h_5 c_0 e_0, h_0, h_2^2 \rangle$ equals $h_1 h_5 d_0 e_0$. From part (1), this equals $\langle \tau h_1^2 G, h_0, h_2^2 \rangle$, which equals $h_1^2 \langle \tau G, h_0, h_2^2 \rangle$ because there is no indeterminacy. This shows that $h_5 d_0 e_0$ is divisible by h_1 . The only possible hidden extension is $h_1 \cdot h_1 B_3 = h_5 d_0 e_0$.

For the third hidden extension, start with the relation $h_1 \cdot \tau PG = Ph_1 \cdot \tau G$ because there is no possible hidden relation. Therefore, using part (1),

$$h_1^3 \cdot \tau PG = Ph_1 \cdot h_1^2 \cdot \tau G = Ph_1h_5c_0e_0.$$

It follows that $h_1 \cdot \tau P h_1 G = P h_5 c_0 e_0$.

For the fourth hidden extension, use part (1) to conclude that $h_5c_0d_0e_0$ is divisible by h_1 . The only possibility is that $h_1 \cdot h_1^2 X_3 = h_5c_0d_0e_0$.

LEMMA 2.4.15. $h_1 \cdot h_1^2 B_6 = \tau h_2^2 d_1 g.$

PROOF. Table 16 shows that $d_1g = \langle d_1, h_1^3, h_1h_4 \rangle$. Using the hidden extension $h_1 \cdot x = \tau h_2^2 d_1$ [9], it follows that $\tau h_2^2 d_1 g = \langle h_1 x, h_1^3, h_1h_4 \rangle$, which equals $h_1 \langle x, h_1^3, h_1h_4 \rangle$ because there is no indeterminacy. Therefore, $\tau h_2^2 d_1 g$ is divisible by h_1 , and the only possibility is that $h_1 \cdot h_1^2 B_6 = \tau h_2^2 d_1 g$.

LEMMA 2.4.16. $h_1 \cdot h_1 D_{11} = \tau^2 c_1 g^2$.

PROOF. Begin by computing that $h_1D_{11} = \langle y, h_1^2, h_1^2h_4 \rangle$, using May's Convergence Theorem 2.2.1, the May differential $d_4(g) = h_1^4h_4$, and the relation $\Delta h_3^2 g = \Delta h_1^2 d_1$. Also recall the hidden extension $h_1 \cdot y = \tau^2 c_1 g$, which follows by comparison to the classical case [9].

It follows that

$$h_1^2 D_{11} = \langle h_1 y, h_1^4, h_4 \rangle = \langle \tau^2 c_1 g, h_1^4, h_4 \rangle$$

because there is no indeterminacy. Finally, Table 16 shows that $\tau^2 c_1 g^2$ equals $\langle \tau^2 c_1 g, h_1^2, h_1^2 h_4 \rangle$.

LEMMA 2.4.17. $h_1 \cdot C_0 = 0$.

PROOF. The only other possibility is that $h_1 \cdot C_0$ equals $h_0 h_5 l$.

First compute that C_0 belongs to $\langle h_0 h_3^2, h_0, h_1, \tau h_1 g_2 \rangle$ using May's Convergence Theorem 2.2.2 and the May differentials $d_4(\nu) = h_0^2 h_3^2$ and $d_4(x_{47}) = \tau h_1^2 g_2$. The subbracket $\langle h_0 h_3^2, h_0, h_1 \rangle$ is strictly zero. On the other hand, the subbracket $\langle h_0, h_1, \tau h_1 g_2 \rangle$ equals $\{0, \tau h_0 h_2 g_2\}$. Condition (5) of May's Convergence Theorem 2.2.2 is satisfied because the May filtration of $\tau h_2 g_2$ is less than the May filtration of x_{47} .

Because $\langle h_1, h_0 h_3^2, h_0 \rangle$ is zero, the hypothesis of Lemma 2.2.5 is satisfied. This implies that $h_1 \cdot C_0$ belongs to $\langle h_1, h_0 h_3^2, h_0, h_1 \rangle \tau h_1 g_2$. For degree reasons, the bracket $\langle h_1, h_0 h_3^2, h_0, h_1 \rangle$ consists of elements spanned by f_0 and $\tau h_1^3 h_4$. But the products $h_1 \cdot f_0$ and $h_1 \cdot \tau h_1^3 h_4$ are both zero, so $\langle h_1, h_0 h_3^2, h_0, h_1 \rangle \tau h_1 g_2$ must be zero. Therefore, $h_1 \cdot C_0$ is zero.

LEMMA 2.4.18. $h_1 \cdot r_1 = s_1$.

PROOF. This follows immediately from Theorem 2.1.12 and the classical relation $h_0 \cdot r = s$.

LEMMA 2.4.19. $h_1 \cdot h_1^2 q_1 = h_0^4 X_3$.

PROOF. Apply Sq⁶ to the relation $h_2r = h_1q$ to obtain that $h_3r^2 = h_1^2$ Sq⁵(q). Next, observe that $Sq^5(q) = h_1q_1$ by comparison to the classical case [10].

By comparison to the classical case, there is a relation $h_3r = h_0^2 x + \tau h_2^2 n$, so $h_3r^2 = h_0^2rx$. Finally, use the hidden extension $h_0 \cdot r = s$ and the non-hidden relation $sx = h_0^3 X_3$.

2.4.4. Hidden May h_2 extensions. By exhaustive search, the following results give all of the hidden h_2 extensions.

PROPOSITION 2.4.20. Table 14 lists all of the hidden h_2 extensions through the 70-stem.

PROOF. Many of the extensions follow by comparison to the classical case as described in [9]. For example, there is a classical hidden extension $h_2 \cdot Q_2 = h_5 k$. This implies that the same formula holds motivically.

Also, many extensions are implied by hidden h_0 extensions that we already established in Section 2.4.2. For example, there is a hidden extension $h_0 \cdot h_2^2 g = h_1^3 h_4 c_0$. This implies that there is also a hidden extension $h_2 \cdot h_0 h_2 g = h_1^3 h_4 c_0$.

Proofs for the more subtle cases are given below.

REMARK 2.4.21. We established the extensions

- (1) $h_2 \cdot e_0 r = P h_1^3 h_5 c_0$
- (2) $h_2 \cdot lm = h_1^5 c_0 Q_2$

in the proof of Lemma 2.4.10. The extension $h_2 \cdot km = h_1^6 X_1$ follows from Lemma 2.4.10 and the relation $h_2 k = h_0 l$.

LEMMA 2.4.22. $h_2 \cdot h_2 B_2 = h_1 h_5 c_0 d_0$.

PROOF. Table 16 shows that $h_2B_2 = \langle g_2, h_0^3, h_2^2 \rangle$, with no indeterminacy. Then $h_2 \cdot h_2B_2$ equals $\langle g_2, h_0^3, h_2^3 \rangle$, because there is no indeterminacy. This bracket equals $\langle g_2, h_0^3, h_1^2 h_3 \rangle$, which equals $\langle g_2, h_0^3, h_1 \rangle h_1 h_3$ since there is no indeterminacy. Table 16 also shows that the bracket $\langle g_2, h_0^3, h_1 \rangle$ equals B_1 .

We have now shown that $h_2 \cdot h_2 B_2$ equals $h_1 h_3 \cdot B_1$. It remains to show that there is a hidden extension $h_3 \cdot B_1 = h_5 c_0 d_0$. First observe that $B_1 \cdot h_1^2 d_0 = h_1^3 B_{21}$ by a non-hidden relation. This implies that $B_1 \cdot \tau h_1^2 d_0 = P h_1 h_5 c_0 d_0$ by Lemma 2.4.3.

Now there is a hidden extension $h_3 \cdot Ph_1 = \tau h_1^2 d_0$, so $B_1 \cdot h_3 \cdot Ph_1 = Ph_1 h_5 c_0 d_0$. The only possibility is that $h_3 \cdot B_1 = h_5 c_0 d_0$.

LEMMA 2.4.23. $h_2 \cdot B_6 = \tau e_1 g$.

PROOF. Table 16 shows that $\langle \tau, B_6, h_1^2 h_3 \rangle = h_2 C_0$ with no indeterminacy. This means that $\langle \tau, B_6, h_2^3 \rangle = h_2 C_0$. If $h_2 \cdot B_6$ were zero, then this would imply that $\langle \tau, B_6, h_2 \rangle h_2^2 = h_2 C_0$. However, $h_2 C_0$ cannot be divisible by h_2^2 .

2.4.5. Other hidden May extensions. We collect here a few miscellaneous extensions that are needed for various arguments.

LEMMA 2.4.24.

(1)
$$c_0 \cdot i_1 = h_1^4 D_4$$
.

(2)
$$Ph_1 \cdot i_1 = h_1^5 Q_2$$

(3) $c_0 \cdot Q_2 = PD_4$.

PROOF. Start by computing that $h_1^2 D_4$ belongs to $\langle c_0, h_4^2, h_3, h_1^3, h_1 h_3 \rangle$; we will not need to worry about the indeterminacy. One can use May's Convergence Theorem 2.2.3 and the May d_2 differential to make this computation. All of the threefold subbrackets are strictly zero, and one of the fourfold subbrackets is also strictly zero. However, $\langle c_0, h_4^2, h_3, h_1^3 \rangle$ equals $\{0, h_1^2 h_5 e_0\}$. Condition (7) of May's Convergence

Theorem 2.2.3 is satisfied because $h_1^2 h_5 e_0 = \langle h_5 c_0, h_3, h_1^3 \rangle$, and the May filtration of $h_5 c_0$ is less than the May filtration of $h_1 h_0(1,3)$.

The hypothesis of Lemma 2.2.7 is satisfied because $\langle h_3, h_1^3, h_1h_3, h_1^2 \rangle$ is strictly zero. Therefore, $h_1^4 D_4$ is contained in $c_0 \langle h_4^2, h_3, h_1^3, h_1h_3, h_1^2 \rangle$. The main point is that $h_1^4 D_4$ is divisible by c_0 . The only possibility is that $c_0 \cdot i_1 = h_1^4 D_4$. This establishes the first formula.

For the second formula, compute that h_1Q_2 equals $\langle h_4, h_1^2h_4, h_4, Ph_1 \rangle$ with no indeterminacy, using May's Convergence Theorem 2.2.2 and the May differentials $d_4(\nu_1) = h_1^2h_4^2$ and $d_4(\Delta h_1) = Ph_1h_4$. The subbracket $\langle h_1^2h_4, h_4, Ph_1 \rangle$ equals $\{0, Ph_1^3h_5\}$. Condition (5) of May's Convergence Theorem 2.2.2 is satisfied because the May filtration of $h_1^2h_5$ is less than the May filtration of ν_1 .

Next, compute that $i_1 = \langle h_1^4, h_4, h_1^2h_4, h_4 \rangle$ with no indeterminacy, using May's Convergence Theorem 2.2.2 and the May differentials $d_4(g) = h_1^4h_4$ and $d_4(\nu_1) = h_1^2h_4^2$. The subbracket $\langle h_1^4, h_4, h_1^2h_4 \rangle$ equals $\{0, h_1^6h_5\}$. Condition (5) of May's Convergence Theorem 2.2.2 is satisfied because the May filtration of $h_1^2h_5$ is less than the May filtration of ν_1 .

The hypothesis of Lemma 2.2.6 is satisfied because $\langle h_4, h_1^2 h_4, h_4 \rangle$ is strictly zero. Therefore,

$$h_1^4 \langle h_4, h_1^2 h_4, h_4, Ph_1 \rangle = \langle h_1^4, h_4, h_1^2 h_4, h_4 \rangle Ph_1,$$

and $h_1^5 Q_2 = Ph_1 \cdot i_1$. This establishes the second formula.

The third formula now follows easily. Compute that $Ph_1 \cdot c_0 \cdot i_1$ equals $Ph_1 \cdot h_1^4 D_4$ and also $c_0 \cdot h_1^5 Q_2$.

REMARK 2.4.25. Part (3) of Lemma 2.4.24 shows that the multiplicative generator PD_4 of the E_{∞} -page becomes decomposable in Ext by a hidden extension.

LEMMA 2.4.26. $c_0 \cdot B_6 = h_1^3 B_3$.

PROOF. Table 16 shows that $h_1^3Q_2 = \langle \tau, B_6, h_1^4 \rangle$. This bracket has no indeterminacy. It follows that $h_1^3c_0Q_2 = \langle \tau, B_6 \cdot c_0, h_1^4 \rangle$, since this bracket also has no indeterminacy.

The element $h_1^3 c_0 Q_2$ is non-zero by part (3) of Lemma 2.4.24. Therefore, $\langle h_1^4, B_6 \cdot c_0, \tau \rangle$ is not zero, so $B_6 \cdot c_0$ is non-zero. The only possibility is that it equals $h_1^3 B_3$.

LEMMA 2.4.27. $c_0 \cdot G_3 = Ph_1^3h_5e_0$.

PROOF. Start with the relation $h_1^2G_3 = h_2gr$. This implies that $h_1^2d_0G_3 = h_2d_0gr$, which equals $h_1^6X_1$ by Table 14. Therefore, $c_0^2G_3$ is non-zero, which means that c_0G_3 is also. The only possibility is that c_0G_3 equals $Ph_1^3h_5e_0$.

LEMMA 2.4.28. $h_0^2 B_4 + \tau h_1 B_{21} = g'_2$.

PROOF. On the E_{∞} -page, there is a relation $h_0^2 B_4 + \tau h_1 B_{21} = 0$. The hidden extension follows from the analogous classical hidden relation [9].

REMARK 2.4.29. Through the 70-stem, Lemma 2.4.28 is the only example of a hidden relation of the form $h_0 \cdot x + h_1 \cdot y$, $h_0 \cdot x + h_2 \cdot y$, or $h_1 \cdot x + h_2 \cdot y$.

CHAPTER 3

Differentials in the Adams spectral sequence

The main goal of this chapter is to compute the differentials in the motivic Adams spectral sequence. We will rely heavily on the computation of the Adams E_2 -page carried out in Chapter 2. We will borrow results from the classical Adams spectral sequence where necessary. Tables 18 and 19 summarize previously established results about the classical Adams spectral sequence, including differentials and Toda brackets. The tables give specific references to proofs. The main sources are [3], [4], [8], [27], [41], and [42].

The Adams charts in [19] are essential companions to this chapter.

The motivic Adams spectral sequence. We refer to [13], [17], and [33] for background on the construction and convergence of the motivic Adams spectral sequence over \mathbb{C} . In this section, we review just enough to proceed with our computations in later sections.

THEOREM 3.0.1 ([13] [17] [33]). The motivic Adams spectral sequence takes the form

$$E_2^{s,f,w} = \operatorname{Ext}_A^{s,f,w}(\mathbb{M}_2,\mathbb{M}_2) \Rightarrow \pi_{s,w},$$

with differentials of the form $d_r: E_r^{s,f,w} \to E_r^{s-1,f+r,w}$.

We will need to compare the motivic Adams spectral sequence to the classical Adams spectral sequence. The following proposition is implicit in [13, Sections 3.2 and 3.4].

PROPOSITION 3.0.2. After inverting τ , the motivic Adams spectral sequence becomes isomorphic to the classical Adams spectral sequence tensored over \mathbb{F}_2 with $\mathbb{M}_2[\tau^{-1}]$.

In particular, Proposition 3.0.2 implies that motivic differentials and motivic hidden extensions must be compatible with their classical analogues. This comparison will be a key tool.

Outline. A critical ingredient is Moss's Convergence Theorem [35], which allows the computation of Toda brackets in $\pi_{*,*}$ via the differentials in the Adams spectral sequence. We will thoroughly review this result in Section 3.1.

Section 3.2 describes the main points in establishing the Adams differentials. We postpone the numerous technical lemmas to Section 3.3.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 8, 20, 21, and 22 give the values of the motivic Adams differentials. The fourth columns of these tables refer to one argument that establishes each differential, which is not necessarily the first known proof. This takes one of the following forms:

(1) An explicit proof given elsewhere in this manuscript.

3. DIFFERENTIALS IN THE ADAMS SPECTRAL SEQUENCE

- (2) "image of J" means that the differential is easily deducible from the structure of the image of J [2]. indexJ@J!image of
- (3) "*tmf*" means that the differential can be detected in the Adams spectral sequence for *tmf* [15].
- (4) "Table 18" means that the differential is easily deduced from the analogous classical result.
- (5) "[11, VI.1]" means that the differential can be computed using the relationship between algebraic Steenrod operations and Adams differentials.

Table 23 summarizes some calculations of Toda brackets. In all cases, we have been careful to describe the indeterminacies accurately. The fifth column refers to an argument for establishing this differential, in one of the following forms:

- (1) An explicit proof given elsewhere in this manuscript.
- (2) A Massey product (which appears in Table 16) implies the Toda bracket via Moss's Convergence Theorem 3.1.1 with r = 2.
- (3) An Adams differential implies the Toda bracket via Moss's Convergence Theorem 3.1.1 with r > 2.

The last column of Table 23 lists the specific results that rely on each Toda bracket.

3.1. Toda brackets in the motivic Adams spectral sequence

We will frequently compute Toda brackets in the motivic stable homotopy groups in order to resolve hidden extensions and to determine Adams differentials. The absolutely essential tool for computing such Toda brackets is Moss's Convergence Theorem [35, Theorem 1.2]. The point of this theorem is that under certain hypotheses, Toda brackets can be computed via Massey products in the E_r -page of the motivic Adams spectral sequence. For the reader's convenience, we will state the Convergence Theorem in the specific forms that we will use.

The E_2 -page of the motivic Adams spectral sequence possesses Massey products, since it equals the cohomology of the motivic Steenrod algebra. Moreover, since (E_r, d_r) is a differential graded algebra for $r \ge 2$, the E_{r+1} -page of the motivic Adams spectral sequence also possesses Massey products that are computed with the Adams d_r differential. When necessary for clarity, we will use the notation $\langle a_0, \ldots, a_n \rangle_{E_{r+1}}$ to refer to Massey products in the E_{r+1} -page in this sense. Similarly, $\langle a_0, \ldots, a_n \rangle_{E_2}$ indicates a Massey product in Ext.

THEOREM 3.1.1 (Moss's Convergence Theorem). Let α_0 , α_1 , and α_2 be elements of the motivic stable homotopy groups such that the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$ is defined. Let a_i be a permanent cycle on the Adams E_r -page that detects α_i for each *i*. Suppose further that:

- (1) the Massey product $\langle a_0, a_1, a_2 \rangle_{E_r}$ is defined (in Ext when r = 2, or using the Adams d_{r-1} differential when $r \geq 3$).
- (2) if (s, f, w) is the degree of either a_0a_1 or a_1a_2 ; f' < f r + 1; f'' > f; and t = f'' f'; then every Adams differential $d_t : E_t^{(s+1,f',w)} \to E_t^{(s,f'',w)}$ is zero.

Then $\langle a_0, a_1, a_2 \rangle_{E_r}$ contains a permanent cycle that detects an element of the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.

Condition (2) is an equivalent reformulation of condition (1.3) in [**35**, Theorem 1.2]. When computing $\langle a_0, a_1, a_2 \rangle$, one uses a differential $d_{r-1} : E_r^{(s-1,f-r+1,w)} \to E_r^{(s,f,w)}$. The idea of condition (2) is that there are no later "crossing" differentials d_t whose source has strictly lower Adams filtration and whose target has strictly higher Adams filtration.

EXAMPLE 3.1.2. Consider the differential $d_2(h_4) = h_0 h_3^2$. This shows that $\langle \eta, 2, \sigma^2 \rangle$ intersects $\{h_1 h_4\}$. In fact, Table 23 shows that the bracket equals $\{h_1 h_4\} = \{\eta_4, \eta_4 + \eta \rho_{15}\}$.

EXAMPLE 3.1.3. Consider the Massey product $\langle h_2, h_3, h_0^2 h_3 \rangle$. Using the May differential $d_4(\nu) = h_0^2 h_3^2$ and May's Convergence Theorem 2.2.1, this Massey product contains f_0 with indeterminacy $\tau h_1^3 h_4$. However, this calculation tells us nothing about the Toda bracket $\langle \nu, \sigma, 4\sigma \rangle$. The presence of the later Adams differential $d_3(h_0h_4) = h_0d_0$ means that condition (2) of Moss's Convergence Theorem 3.1.1 is not satisfied.

EXAMPLE 3.1.4. Consider the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$. The relation $h_4^3 + h_3^2 h_5 = 0$ and the Adams differentials $d_2(h_5) = h_0 h_4^2$ and $d_2(h_4) = h_0 h_3^2$ show that the expression $\langle h_4^2, h_0, h_3^2 \rangle_{E_3}$ is zero. This implies that $\langle \theta_4, 2, \sigma^2 \rangle$ consists entirely of elements of Adams filtration strictly greater than 3. In particular, the Toda bracket is disjoint from $\{h_3^2h_5\}$. See Lemma 4.2.91 for more discussion of this Toda bracket.

One case of Moss's Convergence Theorem 3.1.1 says that Massey products in $\operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$ are compatible with Toda brackets in $\pi_{*,*}$, assuming that there are no interfering Adams differentials. Thus, we will use many Massey products in $\operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$, most of which are computed using May's Convergence Theorem 2.2.1.

We will also need the following lemma.

LEMMA 3.1.5. If 2α is zero, then $\tau\eta\alpha$ belongs to $\langle 2, \alpha, 2 \rangle$.

PROOF. The motivic case follows immediately from the classical case, which is proved in [42]. $\hfill \Box$

3.1.1. Toda brackets and cofibers. The purpose of this section is to establish a relationship between Toda brackets of the form $\langle \alpha_0, \ldots, \alpha_n \rangle$ and properties of the stable homotopy groups of the cofiber $C\alpha_0$ of α_0 . This relationship is well-known to those who use it. See [42, Proposition 1.8] for essentially the same result.

Suppose given a map $\alpha_0: S^{p,q} \to S^{0,0}$. Then we have a cofiber sequence

 $S^{p,q} \xrightarrow{\alpha_0} S^{0,0} \xrightarrow{j} C\alpha_0 \xrightarrow{q} S^{p+1,q} \xrightarrow{\alpha_0} S^{1,0}$

where j is the inclusion of the bottom cell, and q is projection onto the top cell. Note that $\pi_{*,*}(C\alpha_0)$ is a $\pi_{*,*}$ -module.

PROPOSITION 3.1.6. Let α_0 , α_1 , and α_2 be elements of $\pi_{*,*}$ such that $\alpha_0\alpha_1$ and $\alpha_1\alpha_2$ are zero. Let $\overline{\alpha_1}$ be an element of $\pi_{*,*}(C\alpha_0)$ such that $q_*(\overline{\alpha_1}) = \alpha_1$. In $\pi_{*,*}(C\alpha_0)$, the element $\overline{\alpha_1} \cdot \alpha_2$ belongs to $j_*(\langle \alpha_0, \alpha_1, \alpha_2 \rangle)$.

PROOF. The proof is described by the following diagram. The composition $\overline{\alpha_1}\alpha_2$ can be lifted to $S^{0,0}$ because $\alpha_1\alpha_2$ was assumed to be zero. This shows that

 $\overline{\alpha_1} \cdot \alpha_2$ is equal to $j_*(\beta)$. Finally, β is one possible definition of the Toda bracket $\langle \alpha_0, \alpha_1, \alpha_2 \rangle$.



REMARK 3.1.7. We have presented Proposition 3.1.6 in the context of stable motivic homotopy groups, but the proof works in the much greater generality of a stable model category. For example, the same result holds for Massey products, where one works in the derived category of a graded algebra A, and maps correspond to elements of Ext groups over A.

REMARK 3.1.8. Proposition 3.1.6 can be generalized to higher compositions. Suppose that $\langle \alpha_0, \ldots, \alpha_n \rangle$ is defined. Then the bracket $\langle \overline{\alpha_1}, \alpha_2, \ldots, \alpha_n \rangle$ is contained in $j_*(\langle \alpha_0, \ldots, \alpha_n \rangle)$. The proof is similar to the proof of Proposition 3.1.6, using the definition of higher Toda brackets [**39**, Appendix A].

3.2. Adams differentials

The E_2 -page of the motivic Adams spectral sequence is described in Chapter 2 (see also [13]). See [19] for a chart of the E_2 -page through the 70-stem. A list of multiplicative generators for the E_2 -page is given in Table 8.

Our next task is to compute the Adams differentials. The main point is to compute the Adams d_r differentials on the multiplicative generators of the E_r -page. Then one can compute the entire Adams d_r differential using that d_r is a derivation.

3.2.1. Adams d_2 differentials. Most of the Adams d_2 differentials are lifted directly from the classical situation, in the sense of Proposition 3.0.2. We provide a few representative examples of this phenomenon.

EXAMPLE 3.2.1. The classical differential $d_2(h_4) = h_0 h_3^2$ immediately implies that there is a motivic differential $d_2(h_4) = h_0 h_3^2$.

EXAMPLE 3.2.2. Unlike the classical situation, the elements $h_1^k d_0$ and $h_1^k e_0$ are non-zero in the E_2 -page for all $k \ge 0$. The classical differential $d_2(e_0) = h_1^2 d_0$ implies that there is a motivic differential $d_2(e_0) = h_1^2 d_0$, from which it follows that $d_2(h_1^k e_0) = h_1^{k+2} d_0$ for all $k \ge 0$. Technically, these are "exotic" differentials, although we will soon see subtler examples.

EXAMPLE 3.2.3. Consider the classical differential $d_2(h_0c_2) = h_1^2e_1$. Motivically, this formula does not make sense because the weights of h_0c_2 and $h_1^2e_1$ are 22 and 23 respectively. It follows that there is a motivic differential $d_2(h_0c_2) = \tau h_1^2e_1$. Then $h_1^2e_1$ is non-zero on the E_3 -page.

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PROPOSITION 3.2.4. Table 8 lists some values of the motivic Adams d_2 differential. The motivic Adams d_2 differential is zero on all other multiplicative generators of the E_2 -page, through the 70-stem.

PROOF. Table 8 cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the E_2 -page. These arguments break into several types:

- (1) Some differentials are consequences of the image of J calculation [2].
- (2) Some differentials follow by comparison to the Adams spectral sequence for *tmf* [15].
- (3) Some differentials follow by comparison to an analogous classical result.
- (4) One differential follows from the relationship between Adams differentials and algebraic Steenrod operations [11, VI.1].
- (5) The remaining differentials are proved in Section 3.3.1.

For the differentials whose values are zero, Section 3.3.1 includes proofs for the cases that are not obvious. $\hfill \Box$

In order to maintain the flow of the narrative, we have collected the technical computations of miscellaneous d_2 differentials in Section 3.3.1.

The E_2 chart in [19] indicates the Adams d_2 differentials, all of which are implied by the calculations in Table 8.

REMARK 3.2.5. Lemma 3.3.3 establishes three differentials $d_2(h_3g) = h_0h_2^2g$, $d_2(h_3g^2) = h_0h_2^2g^2$, and $d_2(h_3g^3) = h_0h_2^2g^3$. Presumably there is an infinite family of exotic differentials of the form

$$d_2(h_3g^k) = h_0 h_2^2 g^k.$$

REMARK 3.2.6. The differential $d_2(X_1) = h_0^2 B_4 + \tau h_1 B_{21}$ is inconsistent with the results of [25].

REMARK 3.2.7. In the 51-stem, we draw particular attention to the Adams differential $d_2(D_1) = h_0^2 h_3 g_2$. Mark Mahowald privately communicated an argument for the presence of this differential to the author. However, this argument fails because of the calculation of the Toda bracket $\langle \theta_4, 2, \sigma^2 \rangle$ in Lemma 4.2.91, which was unknown to Mahowald. Zhouli Xu and the author discovered an independent proof, which is included in Lemma 3.3.13. See [22] for a full discussion.

REMARK 3.2.8. As noted in Table 10, the element τQ_3 is defined in Ext such that $d_2(\tau Q_3) = 0$.

REMARK 3.2.9. Quite a few of the d_2 differentials in this section follow by comparison to the Adams spectral sequence for tmf, i.e., the Adams spectral sequence whose E_2 -page is the cohomology of the subalgebra A(2) of the Steenrod algebra. See [15] for detailed computations with this spectral sequence.

Presumably, there is a "motivic modular forms" spectrum that is the motivic analogue of *tmf*. If such a motivic spectrum existed, then the E_2 -page of its Adams spectral sequence would be the cohomology of motivic A(2), as described in [18]. Such a spectral sequence would help significantly in calculating the differentials in the motivic Adams spectral sequence for $S^{0,0}$ that we are considering here. **3.2.2.** Adams d_3 differentials. See [19] for a chart of the E_3 -page. This chart is complete through the 70-stem; however, the Adams d_3 differentials are complete only through the 65-stem.

The next step is to compute the Adams d_3 differential on the multiplicative generators of the E_3 -page.

PROPOSITION 3.2.10. Table 20 lists some values of the motivic Adams d_3 differential. The motivic Adams d_3 differential is zero on all other multiplicative generators of the E_3 -page, through the 65-stem, except that $d_3(D_3)$ might equal B_3 .

PROOF. Table 20 cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the E_3 -page. These arguments break into several types:

- (1) Some differentials are consequences of the image of J calculation [2].
- (2) Some differentials follow by comparison to the Adams spectral sequence for *tmf* [15].
- (3) Some differentials follow by comparison to an analogous classical result.
- (4) The remaining differentials are proved in Section 3.3.2.

For the differentials whose values are zero, Section 3.3.2 includes proofs for the cases that are not obvious. $\hfill\square$

In order to maintain the flow of the narrative, we have collected the technical computations of miscellaneous d_3 differentials in Section 3.3.2.

The E_3 chart in [19] indicates the Adams d_3 differentials, all of which are implied by the calculations in Table 20. The differentials are complete only through the 65-stem. Beyond the 65-stem, there are a number of unknown differentials.

REMARK 3.2.11. The chart in [25] indicates a differential $d_3(D_3) = B_3$. However, we have been unable to independently verify this differential. Because of the relation $h_1B_3 = h_4B_1$ and because $\{B_1\}$ contains $\eta\theta_{4.5}$, we know that h_1B_3 detects $\langle \eta\theta_{4.5}, \sigma^2, 2 \rangle$, as shown in Table 23. It follows that B_3 detects $\langle \theta_{4.5}, \sigma^2, 2 \rangle$ and that h_1B_3 detects $\eta_4\theta_{4.5}$. We have so far been unable to show that either $\langle \theta_{4.5}, \sigma^2, 2 \rangle$ or $\eta_4\theta_{4.5}$ is zero.

REMARK 3.2.12. We draw attention to the differential $d_3(h_1h_5e_0) = h_1^2B_1$. This can be derived from its classical analogue, which is carefully proved in [3]. Lemma 3.3.30 provides an independent proof. This proof originates from an algebraic hidden extension in the h_1 -local cohomology of the motivic Steenrod algebra [14].

REMARK 3.2.13. The differential $d_3(Q_2) = \tau^2 gt$ given in Lemma 3.3.37 is inconsistent with the chart in [25]. We do not understand the source of this discrepancy.

REMARK 3.2.14. We claim that $d_3(r_1)$ is zero; this is tentative because our analysis is incomplete in the relevant range. The only other possibility is that $d_3(r_1)$ equals $h_1^2 X_2$. However, we show in Lemma 4.2.12 that $h_1^2 X_2$ supports a hidden τ extension and must therefore be non-zero on the E_{∞} -page.

3.2.3. Adams d_4 differentials. See [19] for a chart of the E_4 -page. This chart is complete through the 65-stem. Beyond the 65-stem, because of unknown earlier differentials, the actual E_4 -page is a subquotient of what is shown in the chart.

The next step is to compute the Adams d_4 differentials on the multiplicative generators of the E_4 -page.

PROPOSITION 3.2.15. Table 21 lists some values of the motivic Adams d_4 differential. The motivic Adams d_4 differential is zero on all other multiplicative generators of the E_4 -page, through the 65-stem, with the possible exceptions that:

(1) $d_4(\tau h_1 X_1)$ or $d_4(R)$ might equal $\tau^2 d_0 e_0 r$.

(2) $d_4(C')$ or $d_4(\tau X_2)$ might equal h_2B_{21} or τh_2B_{21} respectively.

PROOF. Table 21 cites one possible argument (but not necessarily the earliest published result) for each non-zero differential on a multiplicative generator of the E_4 -page. These arguments break into several types:

- (1) Some differentials are consequences of the image of J calculation [2].
- (2) Some differentials follow by comparison to the Adams spectral sequence for *tmf* [15].
- (3) Some differentials follow by comparison to an analogous classical result.
- (4) The remaining differentials are proved in Section 3.3.3.

For the differentials whose values are zero, Section 3.3.3 includes proofs for the cases that are not obvious. $\hfill\square$

The E_4 chart in [19] indicates the Adams d_4 differentials, all of which are implied by the calculations in Table 21. The differentials are complete only through the 65-stem. Beyond the 65-stem, there are a number of unknown differentials.

REMARK 3.2.16. The chart in [25] indicates a classical differential $d_4(h_1X_1) = d_0e_0r$. However, we have been unable to independently verify this differential.

Because of the differential $d_5(\tau Ph_5e_0) = \tau d_0 z$ from Lemma 3.3.55, we strongly suspect that $\tau^2 d_0 e_0 r$ is hit by some differential, but there is more than one possibility.

Note that $\tau^2 d_0 e_0 r$ detects $\tau^2 \eta \overline{\kappa}^3$.

REMARK 3.2.17. The chart in [25] indicates a classical differential $d_4(C') = h_2 B_{21}$. However, we have been unable to independently verify this differential.

Because B_{21} detects $\kappa \theta_{4.5}$, we know that $h_2 B_{21}$ detects $\nu \kappa \theta_{4.5}$. If we could show that $\nu \kappa \theta_{4.5}$ is zero, then we could conclude that there is a differential $d_4(C') = h_2 B_{21}$.

3.2.4. Adams d_5 differentials. Because the d_4 differentials are relatively sparse, [19] does not provide a separate chart for the E_5 -page.

The next step is to compute the Adams d_5 differentials on the multiplicative generators of the E_5 -page.

PROPOSITION 3.2.18. Table 22 lists some values of the motivic Adams d_5 differential. The motivic Adams d_5 differential is zero on all other multiplicative generators of the E_5 -page, through the 65-stem, with the possible exceptions that:

(1) $d_5(A')$ might equal $\tau h_1 B_{21}$.

(2) $d_5(\tau h_1 H_1)$ might equal $\tau h_2 B_{21}$.

(3) $d_5(\tau h_1^2 X_1)$ might equal $\tau^3 d_0^2 e_0^2$.

PROOF. The differential $d_5(h_0^{22}h_6) = P^6 d_0$ follows from the calculation of the image of J [2]. The differential $d_5(h_1h_6) = 0$ follows from the existence of the classical element η_6 [26].

The remaining cases are computed in Section 3.3.4.

The chart of the E_4 -page in [19] indicates the very few d_5 differentials along with the d_4 differentials.

REMARK 3.2.19. The chart in [25] indicates a classical differential $d_5(A') = h_1 B_{21}$. However, we have been unable to independently verify this differential. Because B_{21} detects $\kappa \theta_{4.5}$, we know that $h_1 B_{21}$ detects $\eta \kappa \theta_{4.5}$. We have so far been unable to show that $\eta \kappa \theta_{4.5}$ is zero.

REMARK 3.2.20. We suspect that $d_5(\tau h_1 H_1)$ equals zero, not $\tau h_2 B_{21}$. This would follow immediately if we knew that $d_4(C') = h_2 B_{21}$ (see Proposition 3.2.15 and Remark 3.2.17).

REMARK 3.2.21. We show in Lemma 3.3.58 that $\tau^3 d_0^2 e_0^2$ is hit by some differential. We suspect that $d_5(\tau h_1^2 X_1)$ equals $\tau^3 d_0^2 e_0^2$. The other possibilities are $d_9(\tau X_2)$ and $d_{10}(\tau h_1 H_1)$.

3.2.5. Higher Adams differentials. At this point, we are almost done.

PROPOSITION 3.2.22. Through the 59-stem, the E_6 -page equals the E_{∞} -page.

PROOF. The only possible higher differential is that $d_6(h_5c_1)$ might equal $Ph_1^2h_5c_0$. However, we will show in the proof of Lemma 3.3.45 that $Ph_1^2h_5c_0$ cannot be hit by a differential.

The calculations of Adams differentials lead immediately to our main theorem.

THEOREM 3.2.23. The E_{∞} -page of the motivic Adams spectral sequence over \mathbb{C} is depicted in the chart in [19] through the 59-stem. Beyond the 59-stem, the actual E_{∞} -page is a subquotient of what is shown in the chart.

3.3. Adams differentials computations

In this section, we collect the technical computations that establish the Adams differentials discussed in Section 3.2.

3.3.1. Adams d_2 differentials computations. The first two lemmas establish well-known facts from the classical situation. However, explicit proofs are not readily available in the literature, so we supply them here.

LEMMA 3.3.1. $d_2(P^k e_0) = h_1^2 P^k d_0$.

PROOF. Because of the relation $2\kappa = 0$, there must be a differential $d_2(\beta) = h_0 d_0$ in the Adams spectral sequence for *tmf*. Here β is the class in the 15-stem as labeled in [15]. Then $d_2(h_2\beta) = h_0^2 e_0$.

Now f_0 maps to $h_2\beta$, so it follows that $d_2(f_0) = h_0^2 e_0$ in the classical Adams spectral sequence for the sphere. The same formula must hold motivically.

The relation $h_0 f_0 = \tau h_1 e_0$ then implies that $d_2(e_0) = h_1^2 d_0$. This establishes the formula for k = 0.

The argument for larger values of k is similar, using that $d_2(P^k h_2\beta) = P^k h_0^2 e_0$ in the Adams spectral sequence for tmf; $P^k h_0 j$ maps to $P^{k+1} h_2\beta$; and $P^k h_0^2 j = \tau P^{k+1} h_1 e_0$.

LEMMA 3.3.2. $d_2(l) = h_0 d_0 e_0$.

PROOF. The differential $d_2(k) = h_0 d_0^2$ follows by comparison to the Adams spectral sequence for *tmf*. The relation $h_2k = h_0l$ then implies that $d_2(l) = h_0d_0e_0$.

Lemma 3.3.3.

(1) $d_2(h_3g) = h_0h_2^2g.$ (2) $d_2(h_3g^2) = h_0h_2^2g^2.$ (3) $d_2(h_3g^3) = h_0h_2^2g^3.$

PROOF. Table 33 indicates that $\sigma \eta_4$ is contained in $\{h_4c_0\}$. Therefore, $h_1^3h_4c_0$ detects $\eta^3\sigma\eta_4$. However, $\eta^3\eta_4$ is zero.

This means that $h_1^3 h_4 c_0$ must be zero on the E_{∞} -page. The only possible differential is $d_2(h_3g) = h_1^3 h_4 c_0$. Finally, note that $h_1^3 h_4 c_0 = h_0 h_2^2 g$ in the E_2 -page. This establishes the first differential.

The argument for the second differential is essentially the same. The product $\eta^6 \epsilon \eta_5$ is detected by $h_1^7 h_5 c_0$. Since $\eta^3 \sigma = \eta^2 \epsilon$, we get that $\eta^7 \sigma \eta_5$ is also detected by $h_1^7 h_5 c_0$. However, $\eta^7 \eta_5$ is zero, so $h_1^7 h_5 c_0$ must be hit by some differential.

For the third differential, Table 15 shows that $c_0i_1 = h_1^4 D_4$ on the E_2 -page. This implies that $\eta^5 \epsilon\{i_1\}$ is contained in $\{h_0 h_2^2 g^3\}$ in $\pi_{66,40}$. Using that $\eta^3 \sigma = \eta^2 \epsilon$, we get that $\eta^6 \sigma\{i_1\}$ is contained in $\{h_0 h_2^2 g^3\}$. However, $\eta^6\{i_1\}$ equals zero, so some differential must hit $h_0 h_2^2 g^3$.

LEMMA 3.3.4. $d_2(e_0g) = h_1^2 e_0^2$.

PROOF. First note that $Ph_1 \cdot e_0g = h_1d_0^2e_0 + h_1^4v$; this is true in the May E_{∞} -page. Now apply d_2 to this formula to get

$$Ph_1 \cdot d_2(e_0g) = h_1^3 d_0^3 + h_1^6 u.$$

In particular, it follows that $d_2(e_0g)$ is non-zero. The only possibility is that $d_2(e_0g) = h_1^2 e_0^2$.

Lemma 3.3.5.

 $\begin{array}{ll} (1) \ d_2(u') = \tau h_0 d_0^2 e_0. \\ (2) \ d_2(Pu') = \tau P h_0 d_0^2 e_0. \\ (3) \ d_2(P^2u') = \tau P^2 h_0 d_0^2 e_0. \\ (4) \ d_2(P^3u') = \tau P^3 h_0 d_0^2 e_0. \\ (5) \ d_2(v') = h_1^2 u' + \tau h_0 d_0 e_0^2. \\ (6) \ d_2(Pv') = P h_1^2 u' + \tau h_0 d_0^4. \\ (7) \ d_2(P^2v') = P^2 h_1^2 u' + \tau P h_0 d_0^4. \end{array}$

PROOF. The first four formulas follow easily from the relations $h_0 u' = \tau h_0 d_0 l$, $h_0 \cdot P u' = \tau d_0^2 j$, $h_0 \cdot P^2 u' = \tau P h_0 d_0^2 j$, and $h_0 \cdot P^3 u' = \tau P^2 h_0 d_0^2 j$.

For the fifth formula, start with the relation $c_0v = h_1v'$, which holds already in the May E_{∞} -page. Apply d_2 to obtain $h_1^2c_0u = h_1d_2(v')$. We have $c_0u = h_1u'$ (also from the May E_{∞} -page), so $h_1d_2(v') = h_1^3u'$. It follows that $d_2(v')$ equals either h_1^2u' or $h_1^2u' + \tau h_0d_0e_0^2$. Because of the relation $h_0v' = \tau h_0e_0l$, it must be the latter.

The proofs of the sixth and seventh formulas are essentially the same, using the relations $Ph_1 \cdot v' = h_1 \cdot Pv'$, $P^2h_1 \cdot v' = h_1 \cdot P^2v'$, $h_0 \cdot Pv' = \tau h_0 d_0^2 k$, and $h_0 \cdot P^2v' = \tau h_0 d_0^3 i$.

LEMMA 3.3.6. $d_2(G_3) = h_0 gr$.

PROOF. The argument is similar to the proof of Lemma 3.3.3.

Let α be an element of $\{Ph_1h_5\}$ such that $\eta^3 \alpha$ is contained in $\nu\{\tau^2 g^2\}$. Now $\epsilon \alpha$ is contained in $\{Ph_1h_5c_0\}$. Using that $\eta^3 \sigma = \eta^2 \epsilon$ from Table 34, we get that $\eta^2 \epsilon \alpha = \eta^3 \sigma \alpha$, which is contained in $\nu \sigma\{\tau^2 g^2\}$. This is zero, since $\nu \sigma$ is zero.

This means that $Ph_1^3h_5c_0 = h_0gr$ must be zero on the E_{∞} -page of the Adams spectral sequence, but there are several possible differentials. We cannot have $d_2(\tau gn) = h_0gr$, since $\tau g \cdot n$ is the product of two permanent cycles. We cannot have $d_3(h_2B_2) = h_0gr$, since we will show later in Lemma 3.3.29 that B_2 does not support a d_3 differential. We cannot have $d_4(h_0^2h_3g_2) = h_0gr$, $d_5(h_0h_3g_2) = h_0gr$, or $d_6(h_3g_2) = h_0gr$, since we will show later in Lemma 3.3.50 that g_2 is a permanent cycle.

There is just one remaining possibility, so we conclude that $d_2(G_3) = h_0 gr$. \Box

LEMMA 3.3.7. $d_2(B_6) = 0$.

PROOF. The only other possibility is that $d_2(B_6)$ equals $h_1h_5c_0d_0$. If this were the case, then $d_2(\overline{B_6})$ would equal $h_1h_5 \cdot \overline{c_0d_0}$ in the motivic Adams spectral sequence for the cofiber of τ analyzed in Chapter 5. This is impossible because $h_1^4 \cdot \overline{B_6} = 0$ while $h_1^5h_5 \cdot \overline{c_0d_0}$ is non-zero.

LEMMA 3.3.8. $d_2(i_1) = 0$.

PROOF. The only other possibility is that $d_2(i_1) = h_1^4 h_5 e_0$. However, we will see below in Lemma 3.3.30 that $h_1^4 h_5 e_0$ must survive to the E_3 -page.

LEMMA 3.3.9. $d_2(gm) = h_0 e_0^2 g$.

PROOF. This follows easily from the relation $h_0gm = h_2e_0m$ and the differential $d_2(m) = h_0e_0^2$.

Lemma 3.3.10.

(1) $d_2(Q_1) = \tau h_1^2 x'.$ (2) $d_2(U) = P h_1^2 x'.$ (3) $d_2(R_2) = h_0 U.$ (4) $d_2(G_{11}) = h_0 d_0 x'.$

PROOF. First note that $d_2(R_1) = h_0^2 x'$, which follows from the classical case as shown in Table 18. Then the relation $h_2 R_1 = h_1 Q_1$ implies that $d_2(Q_1) = \tau h_1^2 x'$. This establishes the first formula.

Next, there is a relation $\tau h_1 U = P h_1 Q_1$, which is not hidden in the motivic May spectral sequence. Therefore, $\tau h_1 d_2(U)$ equals $\tau P h_1^3 x'$. It follows that $d_2(U)$ equals $P h_1^2 x'$. This establishes the second formula.

For the third formula, start with the relation $h_0^2 R_2 = \tau h_1 U$. This implies that $h_0^2 d_2(R_2)$ equals $\tau P h_1^3 x'$, which equals $h_0^3 U$. Therefore, $d_2(R_2)$ equals $h_0 U$.

For the fourth formula, start with the relation $h_0G_{11} = h_2R_2$. This implies that $h_0d_2(G_{11})$ equals h_0h_2U , which equals $h_0^2d_0x'$. Therefore, $d_2(G_{11})$ equals h_0d_0x' .

LEMMA 3.3.11.

(1) $d_2(H_1) = B_7.$ (2) $d_2(D_4) = h_1 B_6.$ PROOF. First note that classically $h_3^2H_1$ equals h_4A' [9]. Therefore, $h_3^2d_2(H_1)$ equals $h_0h_3^2A' + h_4d_2(A')$ classically, which equals $h_0h_3^2A'$ because $h_4d_2(A')$ must be zero. This implies that $d_2(H_1)$ is non-zero classically. The only motivic possibility is that $d_2(H_1)$ equals B_7 . This establishes the first formula.

Next, consider the relation $h_1^2 H_1 = h_3 D_4$, which is not hidden in the motivic May spectral sequence. It follows that $h_3 \cdot d_2(D_4) = h_1^2 B_7$. The only possibilities are that $d_2(D_4)$ equals $h_1 B_6$ or $h_1 B_6 + \tau h_1^2 G$.

Table 15 gives the hidden extension $c_0 \cdot i_1 = h_1^4 D_4$. Since $d_2(i_1) = 0$ from Lemma 3.3.8, it follows that $d_2(h_1^4 D_4) = 0$. Then $d_2(D_4)$ cannot equal $h_1 B_6 + \tau h_1^2 G$ since $h_1^4 \cdot \tau h_1^2 G$ is non-zero. This establishes the second formula.

Lemma 3.3.12.

(1) $d_2(X_1) = h_0^2 B_4 + \tau h_1 B_{21}.$ (2) $d_2(G_{21}) = h_0 X_3.$

(2) $d_2(\sigma_{21}) = h_0 \Lambda_3$. (3) $d_2(\tau G) = h_5 c_0 d_0$.

PROOF. First consider the relation $h_3R_1 = h_0^2X_1$ [9]. Table 18 shows that $d_2(R_1) = h_0^2x'$, so $h_0^2d_2(X_1)$ equals $h_0^2h_3x'$. There is another relation $h_0^2h_3x' = h_0^4B_4$, which is not hidden in the May spectral sequence. It follows that $d_2(X_1)$ equals either $h_0^2B_4$ or $h_0^2B_4 + \tau h_1B_{21}$.

Next consider the relation $h_1^2 X_1 = h_3 Q_1$ [9]. We know from Lemma 3.3.10 that $d_2(Q_1)$ equals $\tau h_1^2 x'$, so $h_1^2 d_2(X_1)$ equals $\tau h_1^2 h_3 x'$. There is another relation $\tau h_1^2 h_3 x' = \tau h_1^3 B_{21}$, which is not hidden in the May spectral sequence. It follows that $d_2(X_1)$ equals either $\tau h_1 B_{21}$ or $\tau h_1 B_{21} + h_0^2 B_4$.

Now combine the previous two paragraphs to obtain the first formula.

For the second formula, start with the relation $h_0^2 G_{21} = h_3 X_1 + \tau e_1 r$ from [9]. Then $d_2(h_0^2 G_{21})$ equals $h_3 d_2(X_1) = h_0^2 h_3 B_4$, which equals $h_0^3 X_3$ [9]. The second formula follows.

For the third formula, Table 13 gives the relation $Ph_1 \cdot \tau G = h_1^2 X_1$. The first formula implies that $Ph_1 \cdot d_2(\tau G) = \tau h_1^3 B_{21}$, which equals $Ph_1h_5c_0d_0$ by Table 11.

LEMMA 3.3.13. $d_2(D_1) = h_0^2 h_3 g_2$.

PROOF. This proof is due to Z. Xu [22].

Start with the Massey product $\tau G = \langle h_1, h_0, D_1 \rangle$. The higher Leibniz rule [**35**, Theorem 1.1] then implies that $d_2(\tau G) = \langle h_1, h_0, d_2(D_1) \rangle$ because there is no possible indeterminacy. We showed in Lemma 3.3.12 that $d_2(\tau G)$ equals $h_5 c_0 d_0$. This means that $d_2(D_1)$ is non-zero, and the only possibility is that $d_2(D_1)$ equals $h_0^2 h_{3g_2}$.

In fact, note that $h_0^2 h_3 g_2 = h_2^2 h_5 d_0$ and that $h_5 c_0 d_0 = \langle h_1, h_0, h_2^2 h_5 d_0 \rangle$, but this is not essential for the proof.

REMARK 3.3.14. The proof of Lemma 3.3.13 relies on the Massey product $\tau G = \langle h_1, h_0, D_1 \rangle$. One might attempt to prove this with May's Convergence Theorem 2.2.1 and the May differential $d_2(h_2b_{22}b_{40}) = h_0D_1$. However, there is a later differential $d_4(\Delta_1h_1) = h_1h_3g_2 + h_1h_5g$, so the hypotheses of May's Convergence Theorem 2.2.1 are not satisfied.

This bracket can be computed via the lambda algebra [22]. Moreover, it has been verified by computer calculation.

Lemma 3.3.15.

(1) $d_2(D_2) = h_0Q_2.$ (2) $d_2(A) = h_0B_3.$ (3) $d_2(A'') = h_0X_2.$

PROOF. There is a classical relation $e_0D_2 = h_0h_3G_{21}$ [9]. Since $d_2(G_{21}) = h_0X_3$ by Lemma 3.3.12, it follows that $e_0d_2(D_2)$ equals $h_0^2h_3X_3$, which is non-zero. The only possibilities are that $d_2(D_2)$ equals either h_0Q_2 or h_5j .

Next, there is a classical relation $iD_2 = 0$ [9]. It follows that $id_2(D_2)$ equals $Ph_0d_0D_2$, which is non-zero. The only possibilities are that $d_2(D_2)$ equals either h_0Q_2 or $h_0Q_2 + h_5j$.

We obtain a classical differential $d_2(D_2) = h_0Q_2$ by combining the previous two paragraphs. The same formula must hold motivically. This establishes the first claim.

For the second claim, use the first claim together with the relations $h_2D_2 = h_0A$ and $h_2Q_2 = h_0B_3$.

For the third claim, use the second claim together with the relations $h_0 A'' = h_2(A + A')$ and $h_2 B_3 = h_0 X_2$.

Lemma 3.3.16.

(1) $d_2(B_4) = h_0 B_{21}.$ (2) $d_2(B_{22}) = h_1^2 B_{21}.$

PROOF. There is a relation $Ph_2B_4 = iB_2$, which is not hidden in the May spectral sequence. It follows that $Ph_2d_2(B_4)$ equals $Ph_0d_0B_2$, which equals $Ph_0h_2B_{21}$. Therefore, $d_2(B_4)$ equals h_0B_{21} . This establishes the first formula.

Now consider the relation $h_0h_2B_4 = \tau h_1B_{22}$, which is not hidden in the May spectral sequence. This implies that $\tau h_1d_2(B_{22})$ equals $h_0^2h_2B_{21}$, which equals $\tau h_1^3B_{21}$. It follows that $d_2(B_{22})$ equals $h_1^2B_{21}$.

LEMMA 3.3.17. $d_2(C') = 0.$

PROOF. First note that $h_1C' = \tau d_1^2$ is a permanent cycle. Therefore, $h_1d_2(C')$ must equal zero, so $d_2(C')$ does not equal $h_1^2B_3$.

Lemma 3.3.18.

(1) $d_2(X_2) = h_1^2 B_3.$ (2) $d_2(D'_3) = h_1 X_3.$

PROOF. Note that $h_1B_3 = h_4B_1$. We will show in Lemma 4.2.48 that $\{B_1\}$ contains $\eta \theta_{4.5}$. As shown in Table 23, $\{h_1B_3\}$ intersects the bracket $\langle \eta \theta_{4.5}, 2, \sigma^2 \rangle$. In fact, $\langle \eta \theta_{4.5}, 2, \sigma^2 \rangle$ is contained in $\{h_1B_3\}$ because all of the possible indeterminacy is in strictly higher Adams filtration. This shows that $\theta_{4.5}\langle \eta, 2, \sigma^2 \rangle$ intersects $\{h_1B_3\}$.

For the first formula, note that $\{h_1^3B_3\}$ intersects $\langle \eta^2\theta_{4.5}, \eta, 2\rangle\sigma^2$. This last expression must be zero for degree reasons. Therefore, $h_1^3B_3$ must be killed by some differential. The only possibility is that $d_2(h_1X_2) = h_1^3B_3$, which implies that $d_2(X_2) = h_1^2B_3$.

For the second formula, note that $h_1^2 X_3 = h_1 c_0 B_3 = h_4 c_0 B_1$. Lemma 4.2.83 says that $h_4 c_0$ detects $\sigma \eta_4$. Therefore, $h_1^2 X_3$ detects $\eta \sigma \eta_4 \theta_{4.5}$. The Adams filtration of $\eta_4 \theta_{4.5}$ is at least 8; the Adams filtration of $\eta \eta_4 \theta_{4.5}$ is at least 11; and the Adams filtration of $\eta \sigma \eta_4 \theta_{4.5}$ is at least 12. Since the Adams filtration of $h_1^2 X_3$ is 11, it

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follows that $h_1^2 X_3$ must be hit by some differential. The only possibility is that $d_2(h_1 D'_3)$ equals $h_1^2 X_3$.

LEMMA 3.3.19.

(1) $d_2(\tau G_0) = h_2 C_0 + h_1 h_3 Q_2.$ (2) $d_2(h_2 G_0) = h_1 C''.$

PROOF. We work in the motivic Adams spectral sequence for the cofiber of τ of Chapter 5, where we have the relation $h_1h_3 \cdot \overline{D_4} = \tau G_0$. From Table 39, we know that $d_2(\overline{D_4}) = h_1 \cdot \overline{B_6} + Q_2$. It follows that $d_2(h_1h_3 \cdot \overline{D_4}) = h_1^2h_3 \cdot \overline{B_6} + h_1h_3Q_2$. Finally, observe that $h_1^2h_3 \cdot \overline{B_6} = h_2^2 \cdot \overline{B_6} = h_2C_0$. This establishes the first formula.

For the second formula, use the first formula together with the relations $\tau \cdot h_2 G_0 = h_2 \cdot \tau G_0$ and $h_2^2 C_0 = \tau h_1 C''$.

LEMMA 3.3.20.

(1) $d_2(\tau B_5) = \tau h_0^2 B_{23}.$ (2) $d_2(D'_2) = \tau^2 h_0^2 B_{23}.$ (3) $d_2(P(A + A')) = \tau^2 h_0 h_2 B_{23}.$

PROOF. Classically, there is a relation $iB_5 = 0$ [9]. Using that $d_2(i) = Ph_0d_0$, we get that $id_2(B_5)$ equals $Ph_0d_0B_5$ classically, which is non-zero. The only possibility is that there is a motivic differential $d_2(\tau B_5) = \tau h_0^2 B_{23}$. Note that the Ph_5j term is eliminated because of the motivic weight. This establishes the first formula.

Classically, there is a relation $iD'_2 = 0$ [9]. As in the previous paragraph, we get that $id_2(D'_2)$ equals $Ph_0d_0D'_2$ classically, which is non-zero. However, this time the motivic weights allow for two possibilities. It follows that $d_2(D'_2)$ equals either $\tau^2 h_0^2 B_{23}$ or $\tau^2 h_0^2 B_{23} + Ph_5 j$.

We know from [10] that classically, $\operatorname{Sq}^4(q)$ is non-zero, and $\operatorname{Sq}^5(q)$ is a multiple of h_1 . From [11, VI.1], we have that $d_2(\operatorname{Sq}^4(q)) = h_0 \operatorname{Sq}^5(q)$, which is zero. From the previous two paragraphs, it follows that $\operatorname{Sq}^4(q)$ must be $B_5 + D'_2$ classically, and $d_2(D'_2)$ must be $h_0^2 B_{23}$. The motivic formula $d_2(D'_2) = \tau^2 h_0^2 B_{23}$ follows immediately. This establishes the second formula.

For the third formula, there is a classical relation iP(A+A') = 0 [9]. As before, we get that $id_2(P(A + A'))$ equals $P^2h_0d_0(A + A')$, which is non-zero. It follows that $d_2(P(A + A'))$ equals $\tau^2h_0h_2B_{23}$ or $h_0^4G_{21}$. The relation $h_2D'_2 = h_0P(A + A')$ and the calculation of $d_2(D'_2)$ in the previous paragraph imply that $d_2(P(A + A'))$ equals $\tau^2h_0h_2B_{23}$.

LEMMA 3.3.21. $d_2(P^3v) = P^3h_1^2u$.

PROOF. We will show in Lemma 3.3.40 that $d_3(\tau^2 P^2 d_0 m) = P^3 h_1 u$. Since $h_1 \cdot \tau^2 P^2 d_0 m$ is zero, $h_1 \cdot P^3 h_1 u$ must be zero on the E_3 -page. Therefore, some d_2 differential must hit it. The only possibility is that $d_2(P^3 v) = P^3 h_1^2 u$. \Box

LEMMA 3.3.22. $d_2(X_3) = 0.$

PROOF. Start with the relation $h_1X_3 = B_3c_0$. This shows that $h_1d_2(X_3)$ is zero. Therefore, $d_2(X_3)$ cannot equal $h_1c_0Q_2$.

LEMMA 3.3.23. $d_2(R'_1) = P^2 h_0 x'$.

PROOF. First, there is a relation $h_0^6 R'_1 = \tau P^3 u'$, as shown in Table 11. Lemma 3.3.5 says that $d_2(P^3u') = \tau P^3 h_0 d_0^2 e_0$, so $h_0^6 d_2(R'_1)$ equals $\tau^2 P^3 h_0 d_0^2 e_0$. There is another relation $\tau^2 P^3 h_0 d_0^2 e_0 = P^2 h_0^7 x'$, as shown in Table 11. It follows that $d_2(R'_1)$ equals $P^2 h_0 x'$.

The next lemma computes a few d_2 differentials on decomposable elements. In principle, these differentials are consequences of the previous lemmas. However, the results of the calculations are unexpected because of some extensions that are hidden in the motivic May spectral sequence.

Lemma 3.3.24.

(1) $d_2(e_0^2g) = h_1^7 B_1.$ (2) $d_2(c_0e_0^2g) = h_1^8 B_8.$ (3) $d_2(e_0v) = h_1^5 x'.$ (4) $d_2(e_0v') = h_1^4 c_0 x' + \tau h_0 d_0 e_0^3.$

PROOF. In the first formula, we have $d_2(e_0 \cdot e_0 g) = h_1^2 d_0 \cdot e_0 g + e_0 \cdot h_1^2 e_0^2$. This simplifies to $h_1^7 B_1$, as shown in [14].

The second formula follows immediately from the first formula, using that $B_1c_0 = h_1B_8$.

For the third formula, start with the relation $Ph_1 \cdot B_1 = h_1^2 x'$. Since $h_1^7 B_1$ is hit by a d_2 differential, it follows that $h_1^9 x'$ must also be hit by a d_2 differential. The only possibility is that $d_2(e_0 v) = h_1^5 x'$.

For the fourth formula, $h_1^5 c_0 x'$ must be hit by the d_2 differential, since $h_1^5 x'$ is hit by the d_2 differential. The only possibility is that $d_2(h_1 e_0 v') = h_1^5 c_0 x'$. Therefore, $d_2(e_0 v')$ equals either $h_1^4 c_0 x'$ or $h_1^4 c_0 x' + \tau h_0 d_0 e_0^3$. The extension $h_0 \cdot e_0 v' = \tau h_0 d_0 e_0 m$ implies that the second possibility is correct.

3.3.2. Adams d_3 differentials computations.

LEMMA 3.3.25. $d_3(h_4c_0) = 0.$

PROOF. The only other possibility is that $d_3(h_4c_0)$ equals c_0d_0 . Table 39 shows that Pd_0 is hit by a differential in the Adams spectral sequence for the cofiber $C\tau$ of τ . Therefore, $\{Pd_0\}$ must be divisible by τ in the homotopy groups of $S^{0,0}$. The only possibility is that c_0d_0 is a non-zero permanent cycle and that $\tau \cdot \{c_0d_0\} = \{Pd_0\}$.

LEMMA 3.3.26.

(1) $d_3(\tau e_0 g) = c_0 d_0^2$. (2) $d_3(\tau d_0 v) = P h_1 u'$. (3) $d_3(\tau^2 g m) = h_1 d_0 u$. (4) $d_3(\tau e_0 g^2) = c_0 d_0 e_0^2$. (5) $d_3(\tau g v) = h_1 d_0 u'$. (6) $d_3(\tau P d_0 v) = P^2 h_1 u'$.

PROOF. For the first formula, there is a classical differential $d_4(e_0g) = Pd_0^2$ given in Table 18. Motivically, there must be a differential $d_4(\tau^2 e_0g) = Pd_0^2$. This shows that τe_0g cannot survive to E_4 .

The arguments for the remaining formulas are similar, using the existence of the classical differentials $d_4(d_0v) = P^2u$, $d_4(gm) = d_0^2j + h_0^5R_1$, $d_4(e_0g^2) = d_0^4$, $d_4(gv) = Pd_0u$, and $d_4(Pd_0v) = P^3u$. All of these classical differentials can be

detected in the Adams spectral sequence for tmf [15], except for the third one, which is an easy consequence of $d_4(e_0g) = Pd_0^2$.

LEMMA 3.3.27. (1) $d_3(\tau P d_0 e_0) = P^2 c_0 d_0$. (2) $d_3(\tau P^2 d_0 e_0) = P^3 c_0 d_0$. (3) $d_3(\tau P^3 d_0 e_0) = P^4 c_0 d_0$. (4) $d_3(\tau P^4 d_0 e_0) = P^5 c_0 d_0$.

PROOF. For the first formula, we know that $d_3(\tau d_0 e_0) = Pc_0 d_0$ by comparison to the classical case. Therefore, $d_3(\tau Ph_1 d_0 e_0) = P^2 h_1 c_0 d_0$. The desired formula follows immediately. The arguments for the second, third, and fourth formulas are essentially the same.

LEMMA 3.3.28. $d_3(Ph_5c_0) = 0.$

PROOF. The only other possibility is that $d_3(Ph_5c_0) = \tau d_0 l + u'$. However, $c_0(\tau d_0 l + u') = h_1 d_0 u$ is non-zero, while $Ph_5c_0^2 = Ph_1^2h_5d_0 = 0$ since $Ph_5d_0 = \tau B_8$ by Table 11.

LEMMA 3.3.29. $d_3(B_2) = 0.$

PROOF. First, B_{21} cannot support a d_3 differential, so $h_2B_{21} = d_0B_2$ cannot support a d_3 differential. This implies that $d_3(B_2)$ cannot equal e_0r , since d_0e_0r is non-zero on the E_3 -page.

Lemma 3.3.30.

$$\begin{array}{l} (1) \ d_3(h_1h_5e_0) = h_1^2B_1. \\ (2) \ d_3(h_5c_0e_0) = h_1^2B_8. \\ (3) \ d_3(Ph_5e_0) = h_1^2x'. \\ (4) \ d_3(h_1X_1 + \tau B_{22}) = c_0x'. \end{array}$$

PROOF. We pass to the motivic Adams spectral sequence for the cofiber of τ , as discussed in Chapter 5. Note that $h_1^6 h_5 \cdot \overline{h_1^2 e_0} = \tau e_0 g^2$ in the E_2 -page for the cofiber of τ . Also, $h_1^6 \cdot \overline{h_1^3 B_1} = c_0 d_0 e_0^2$ in the E_3 -page for the cofiber of τ . Now $d_3(\tau e_0 g^2) = \underline{c_0 d_0 e_0^2}$ on the E_3 -page for $S^{0,0}$, as shown in Lemma 3.3.26.

Now $d_3(\tau e_0 g^2) = c_0 d_0 e_0^2$ on the E_3 -page for $S^{0,0}$, as shown in Lemma 3.3.26. It follows that $d_3(h_5 \cdot \overline{h_1^2 e_0}) = \overline{h_1^3 B_1}$ on the E_3 -page for the cofiber of τ , and then $d_3(h_1^2 h_5 e_0) = h_1^3 B_1$ for $S^{0,0}$ as well. This establishes the first formula.

After multiplying by h_1 , the second and third formulas follow easily from the first.

For the fourth formula, start with the relation $Ph_5c_0e_0 = h_1^3X_1$ from Table 13. Multiply the third formula by c_0 to obtain the desired formula.

Lemma 3.3.31.

(1)
$$d_3(gr) = \tau h_1 d_0 e_0^2$$
.

(2) $d_3(m^2) = \tau h_1 e_0^4$.

PROOF. We have $d_3(\tau gr) = \tau^2 h_1 d_0 e_0^2$ because $d_3(r) = \tau h_1 d_0^2$. The first formula follows immediately.

For the second formula, multiply the first formula by τg and use multiplicative relations from [9] to obtain that $d_3(\tau m^2) = \tau^2 h_1 e_0^4$. The second formula follows immediately.

LEMMA 3.3.32. $d_3(\tau^2 G) = \tau B_8$.

PROOF. From Table 25, the product $\tau\epsilon\kappa$ belongs to $\{Pd_0\}$ in $\pi_{22,12}$. Since $h_1h_5c_0d_0 = 0$ in the E_{∞} -page by Lemma 3.3.12, we know that $\eta_5\epsilon\kappa$ is either zero or represented in E_{∞} in higher filtration. It follows that $\tau\eta_5\epsilon\kappa = \eta_5\{Pd_0\}$ is either zero or represented in E_{∞} in higher filtration. Now $Ph_1h_5d_0 = \tau h_1B_8$ by Table 11, so τh_1B_8 must be hit by some differential. The only possibility is that $d_3(\tau^2 G) = \tau B_8$.

LEMMA 3.3.33.

(1) $d_3(e_1g) = h_1gt.$ (2) $d_3(B_6) = \tau h_2gn.$ (3) $d_3(gt) = 0.$

PROOF. Start with the differential $d_3(e_1) = h_1 t$ from Table 18. Then $d_3(\tau e_1 g)$ equals $\tau h_1 t g$, which implies the first formula. The second formula follows easily, using that $\tau e_1 g = h_2 B_6$ from Table 14.

For the third formula, we know that $d_3(\tau gt) = 0$ because $d_3(\tau g) = 0$ and $d_3(t) = 0$. Therefore, $d_3(gt)$ cannot equal $\tau h_1 e_0^2 g$.

LEMMA 3.3.34.

(1) $d_3(h_5i) = h_0 x'.$ (2) $d_3(h_5j) = h_2 x'.$

PROOF. We proved in Lemma 3.3.30 that $d_3(h_5c_0e_0) = h_1B_8$. This implies that $d_3(\overline{h_5c_0e_0}) = \overline{h_1^2B_8}$ in the motivic Adams spectral sequence for the cofiber of τ , which is discussed in Chapter 5. The hidden extensions $h_0 \cdot \overline{h_5c_0e_0} = h_5j$ and $h_0 \cdot \overline{h_1^2B_8} = h_2x'$ then imply that $d_3(h_5j) = h_2x'$ for the cofiber of τ , which means that the same formula must hold for $S^{0,0}$. This establishes the second formula.

The first formula now follows easily, using the relation $h_2h_5i = h_0h_5j$.

LEMMA 3.3.35. $d_3(B_3) = 0$.

PROOF. The only other possibility is that $d_3(B_3) = B_{21}$. On the E_3 -page, h_2B_3 is zero while h_2B_{21} is non-zero.

LEMMA 3.3.36. $d_3(\tau g^3) = h_1^6 B_8$.

PROOF. Start with the hidden extension $\tau \eta^2 \cdot \{\tau g^2\} = \{d_0^3\}$, which follows from the analogous classical extension given in Table 24. This implies that $\tau \eta^2 \{\tau g^2\}\overline{\kappa} = \{\tau d_0^2 e_0^2\}$. In particular, $\eta^2 \{\tau^2 g^3\}$ must be non-zero.

Either τg^3 or $\tau g^3 + h_1^4 h_5 c_0 e_0$ survives the motivic Adams spectral sequence. In the first case, there is no possible non-zero value for a hidden extension of the form $\eta^2 \{\tau g^3\}$. The only remaining possibility is that $\tau g^3 + h_1^4 h_5 c_0 e_0$ survives, in which case $\eta^2 \{\tau g^3 + h_1^4 h_5 c_0 e_0\} = \{h_1^6 h_5 c_0 e_0\}$ is a non-hidden extension.

LEMMA 3.3.37.

(1) $d_3(C_0) = nr.$ (2) $d_3(E_1) = nr.$ (3) $d_3(Q_2) = \tau^2 gt.$ (4) $d_3(C'') = nm.$ PROOF. There are relations $C_0 = h_2^2 \cdot \overline{B_6}$ and $nr = h_2^2 \cdot \overline{\tau h_2 gn}$ in the motivic Adams spectral sequence for the cofiber of τ , as discussed in Chapter 5. From Lemma 3.3.33, we know that $d_3(\overline{B_6}) = \overline{\tau h_2 gn}$. The first formula now follows easily.

For the second formula, note that $gE_1 = gC_0$ classically, and that gnr is nonzero on the E_3 -page [9]. We already know that $d_3(gC_0) = gnr$ classically, so it follows that $d_3(E_1)$ also equals nr classically. The motivic formula is an immediate consequence.

For the third formula, there are classical relations $wQ_2 = g^2C_0$ and $wgt = g^2nr$ [9]. We already know that $d_3(g^2C_0) = g^2nr$, so it follows that $wd_3(Q_2) = w \cdot gt$. The desired formula follows immediately.

For the fourth formula, there is a classical relation $gC'' = rQ_2$ [9]. The d_3 differentials on r and Q_2 imply that $d_3(gC'') = grt$ classically, which equals gnm [9]. The desired formula follows immediately.

LEMMA 3.3.38.

(1)
$$d_3(\tau h_1 X_1) = 0.$$

(2) $d_3(R) = 0.$

PROOF. We have classical relations $h_1rX_1 = 0$ and $h_1^2d_0^2X_1 = 0$ [9]. Therefore, $rd_3(h_1X_1) = 0$ classically. On the other hand, rc_0x' is non-zero on the E_3 -page [9]. This shows that $d_3(h_1X_1)$ cannot equal c_0x' classically, which establishes the first formula.

An identical argument works for the second formula, using that rR = 0 and $h_1 d_0^2 R = 0$.

LEMMA 3.3.39. $d_3(\tau gw) = h_1^3 c_0 x'$.

PROOF. There is a hidden extension $\eta \cdot \{\tau w\} = \{\tau d_0 l + u'\}$, which follows from the analogous classical extension given in Table 24. This implies that $\eta \{\tau^2 gw\} = \{\tau^2 d_0 e_0 m\}$. If τgw were a permanent cycle, then $\eta \cdot \{\tau gw\}$ would be a non-zero hidden extension. But there is no possible value for this hidden extension. \Box

LEMMA 3.3.40. $d_3(\tau^2 P^2 d_0 m) = P^3 h_1 u.$

PROOF. Note that $P^2 d_0 m$ supports a d_4 differential in the Adams spectral sequence for tmf[15]. However, $P^2 d_0 m$ cannot support a d_4 differential in the classical Adams spectral sequence for the sphere. Therefore, $P^2 d_0 m$ cannot surve to the E_4 -page. The only possibility is that there is a classical differential $d_3(P^2 d_0 m) = P^3 h_1 u$, from which the motivic analogue follows immediately.

LEMMA 3.3.41. $d_3(\tau^2 B_5 + D'_2) = 0.$

PROOF. Classically, $(B_5 + D'_2)d_0$ is zero while d_0gw is non-zero on the E_3 -page [9]. Therefore $d_3(\tau^2 B_5 + D'_2)$ cannot equal $\tau^3 gw$.

LEMMA 3.3.42. $d_3(X_3) = 0.$

PROOF. The only other possibility is that $d_3(X_3)$ equals τnm . However, gnm is non-zero on the classical E_3 -page, while gX_3 is zero [9].

LEMMA 3.3.43. $d_3(h_2B_{23}) = 0.$

PROOF. This follows easily from the facts that $d_3(\tau B_{23}) = 0$ and that $h_2 \cdot \tau B_{23} = \tau \cdot h_2 B_{23}$.

LEMMA 3.3.44. (1) $d_3(h_2B_5) = h_1B_8d_0$.

(2) $d_3(\tau e_0 x') = Pc_0 x'.$

PROOF. We will show in Lemma 3.3.48 that $d_4(\tau h_2 B_5) = h_1 d_0 x'$. This means that $h_2 B_5$ cannot survive to E_4 . The only possibility is that $d_3(h_2 B_5) = h_1 B_8 d_0$. This establishes the first formula.

The proof of the second formula is similar. We will show in Lemma 3.3.48 that $d_4(\tau^2 e_0 x') = P^2 x'$, so $\tau e_0 x'$ cannot survive to E_4 . The only possibility is that $d_3(\tau e_0 x') = Pc_0 x'$.

3.3.3. Adams d_4 differentials computations.

LEMMA 3.3.45. $d_4(C) = 0.$

PROOF. The other possibility is that $d_4(C)$ equals $Ph_1^2h_5c_0$. We will show that $Ph_1^2h_5c_0$ survives and is non-zero in the E_{∞} -page.

Let α be an element of $\{Ph_1h_5c_0\}$. From Table 23, the bracket $\langle \eta^2, \alpha, \epsilon \rangle$ contains the element $\{Ph_1^3h_5e_0\}$. In order to compute this bracket, we need the relation $c_0 \cdot G_3 = Ph_1^3h_5e_0$ from Table 15. Note that the bracket has indeterminacy generated by $\eta^2 \{D_{11}\}$.

If $Ph_1^2h_5c_0$ were hit by a differential, then $\eta\alpha$ would be zero. Then $\eta\langle\eta,\alpha,\epsilon\rangle$ would equal $\langle\eta^2,\alpha,\epsilon\rangle$. But $\{Ph_1^3h_5e_0\}$ cannot be divisible by η . By contradiction, $Ph_1^2h_5c_0$ cannot be hit by a differential.

Lemma 3.3.46.

(1) $d_4(h_0h_5i) = 0.$ (2) $d_4(C_{11}) = 0.$

PROOF. The only non-zero possibility for $d_4(h_0h_5i)$ is τd_0u . However, d_0u survives to a non-zero homotopy class in the Adams spectral sequence for tmf [15]. This implies that τd_0u survives to a non-zero homotopy class in the motivic Adams spectral sequence. This establishes the first formula.

The proof of the second formula is similar. The only non-zero possibility for $d_4(C_{11})$ is $\tau^3 d_0 e_0 m$, but $d_0 e_0 m$ survives to a non-zero homotopy class in the Adams spectral sequence for tmf [15].

LEMMA 3.3.47. $d_4(\tau^2 e_0 g^2) = d_0^4$.

PROOF. First note that $d_4(\tau^2 e_0 g) = Pd_0^2$, which follows from its classical analogue given in Table 18. Multiply this formula by $h_1d_0^2$ to obtain that $d_4(\tau^2 h_1d_0e_0^3)$ equals $Ph_1d_0^4$. Finally, note that $\tau^2 h_1d_0e_0^3$ equals $Ph_1\cdot\tau^2 e_0g^2$. The desired formula follows.

LEMMA 3.3.48. (1) $d_4(\tau^2 h_1 B_{22}) = Ph_1 x'.$ (2) $d_4(\tau h_2 B_5) = h_1 d_0 x'.$ (3) $d_4(\tau^2 e_0 x') = P^2 x'.$

PROOF. In the classical situation, $d_0 \cdot Ph_1x'$ is non-zero on the E_4 -page, and $d_0 \cdot h_1B_{22} = (d_0e_0 + h_0^7h_5) \cdot B_1$ [9]. Using that $d_4(d_0e_0 + h_0^7h_5) = P^2d_0$, it follows that there is a classical differential $d_4(h_1B_{22}) = Ph_1x'$. The motivic differential follows immediately.

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The arguments for the second and third differentials are similar. For the second, use that $Pd_0 \cdot h_1d_0x'$ is non-zero on the E_4 -page; $Pd_0 \cdot h_2B_5 = h_1x' \cdot e_0g$ [9]; and $d_4(e_0g) = Pd_0^2$ classically.

For the third formula, use that $h_1d_0 \cdot P^2x'$ is non-zero on the E_4 -page; $h_1d_0 \cdot e_0x' = Pd_0 \cdot h_1B_{22}$ [9]; and $d_4(h_1B_{22}) = Ph_1x'$ classically from the first part of the lemma.

LEMMA 3.3.49. $d_4(\tau^2 m^2) = d_0^2 z$.

PROOF. First note that $d_4(\tau^2 gr) = ij$, which follows by comparison to $tmf[\mathbf{15}]$. Multiply by τg to obtain that $d_4(\tau^3 m^2) = \tau d_0^2 z$, using multiplicative relations from [9]. The desired formula follows.

3.3.4. Adams d_5 differentials computations.

LEMMA 3.3.50. $d_5(g_2) = 0$.

PROOF. The only non-zero possibility is that $d_5(g_2)$ equals $\tau^2 h_2 g^2$. However, $e_0 \cdot g_2$ is zero in the E_5 -page, while $e_0 \cdot \tau^2 h_2 g^2$ is non-zero in the E_5 -page. \Box

LEMMA 3.3.51. $d_5(B_2) = 0.$

PROOF. The only other possibility is that $d_5(B_2)$ equals h_1u' . Recall from Table 24 that there is a classical hidden extension $\eta\{d_0l\} = \{Pu\}$. This implies that $\eta\{\tau d_0l+u'\}$ is non-zero motivically. Therefore, h_1u' cannot be zero in E_{∞} . \Box

Our next goal is to show that $d_5(\tau Ph_5e_0) = \tau d_0 z$. We will need a few preliminary lemmas. This approach follows [25, Theorem 2.2], but we have corrected and clarified the details in that argument.

LEMMA 3.3.52. $\langle \{q\}, 2, 8\sigma \rangle = \{0, 2\tau \overline{\kappa}^2\}.$

PROOF. Table 16 shows that $\langle q, h_0, h_0^3 h_3 \rangle$ equals $\tau h_1 u$. Then Moss's Convergence Theorem 3.1.1 implies that $\langle \{q\}, 2, 8\sigma \rangle$ contains $\{\tau h_1 u\}$. Table 27 shows that $\{\tau h_1 u\}$ equals $2\tau \overline{\kappa}^2$. Finally, use Lemma 4.2.87 to show that $2\tau \overline{\kappa}^2 = \tau \epsilon \{q\}$ is in the indeterminacy of the bracket.

LEMMA 3.3.53. The bracket $\langle 2, 8\sigma, 2, \sigma^2 \rangle$ contains $\tau \nu \overline{\kappa}$.

PROOF. The subbracket $\langle 2, 8\sigma, 2 \rangle$ is strictly zero, as shown in Table 23. We will next show that the subbracket $\langle 8\sigma, 2, \sigma^2 \rangle$ is also strictly zero. First, the shuffle

$$\langle 8\sigma, 2, \sigma^2 \rangle \eta = 8\sigma \langle 2, \sigma^2, \eta \rangle$$

implies that the subbracket is annihilated by η . This rules out Pd_0 . Moss's Convergence Theorem 3.1.1 with the Adams differential $d_2(h_4) = h_0 h_3^2$ implies that the subbracket is detected in Adams filtration strictly greater than 5. This rules out $\tau h_2 c_1$. The only remaining possibility is that the subbracket contains zero, and there is no possible indeterminacy.

We will work in the motivic Adams spectral sequence for the cofiber C2 of 2. We write $E_2(C2)$ for $\operatorname{Ext}_A(H^{*,*}(C2), \mathbb{M}_2)$, i.e., the E_2 -page of the motivic Adams spectral sequence for C2. The cofiber sequence

$$S^{0,0} \xrightarrow{2} S^{0,0} \xrightarrow{j} C2 \xrightarrow{q} S^{1,0}$$

induces a map $q_*: E_2(C2) \to \Sigma^{1,0}E_2$. Let $\overline{h_0^3h_3}$ be an element of $E_2(C2)$ such that $q_*(\overline{h_0^3h_3})$ equals $h_0^3h_3$, and let $\{\overline{h_0^3h_3}\}$ be the corresponding element in $\pi_{8,4}(C2)$. Then $j_*\langle 2, 8\sigma, 2, \sigma^2 \rangle$ equals $\langle \{\overline{h_0^3h_3}\}, 2, \sigma^2 \rangle$ in $\pi_{23,12}(C2)$ by Remark 3.1.8.

Because of the Adams differential $d_2(h_4) = h_0 h_3^2$, we know that $\langle \{\overline{h_0^3 h_3}\}, 2, \sigma^2 \rangle$ is detected by $h_4 \cdot \overline{h_0^3 h_3}$ in $E_{\infty}(C2)$. Here we are using a slight generalization of Moss's Convergence Theorem 3.1.1, in which one considers Toda brackets of maps between different objects (see [**35**] for the classical case).

Finally, we need to compute $h_4 \cdot \overline{h_0^3 h_3}$ in $E_2(C2)$. This equals $j_* \langle h_4, h_0^3 h_3, h_0 \rangle$ by Remark 3.1.7 (see also Proposition 5.0.1 for an analogous result). Table 16 shows that $\langle h_4, h_0^3 h_3, h_0 \rangle$ equals $\tau^2 h_2 g$.

LEMMA 3.3.54. $\langle \epsilon, 2, \sigma^2 \rangle = \{ \sigma \eta_4, \sigma \eta_4 + 4\nu \overline{\kappa} \}.$

PROOF. Using the Adams differential $d_2(h_4) = h_0 h_3^2$ and Moss's Convergence Theorem 3.1.1, we know that $\langle \epsilon, 2, \sigma^2 \rangle$ intersects $\{h_4 c_0\}$.

Lemma 4.2.83 shows that $\sigma\eta_4$ is contained in $\{h_4c_0\}$. The indeterminacy of $\{h_4c_0\}$ is generated by τh_2g , τh_0h_2g , and Ph_1d_0 . By Lemma 4.2.17, the indeterminacy consists of multiples of $\nu\overline{\kappa}$. Therefore, $\{h_4c_0\}$ consists of elements of the form $\sigma\eta_4 + k\nu\overline{\kappa}$ for $0 \le k \le 7$.

Note that $\langle \epsilon, 2, \sigma^2 \rangle 2$ equals $\epsilon \langle 2, \sigma^2, 2 \rangle$, which is zero because $\langle 2, \sigma^2, 2 \rangle$ contains 0 by Table 23. Therefore, if $\sigma \eta_4 + k \nu \overline{\kappa}$ belongs to $\langle \epsilon, 2, \sigma^2 \rangle$, then k equals 0 or 4.

We now know that either $\sigma\eta_4$ or $\sigma\eta_4 + 4\nu\overline{\kappa}$ belongs to $\langle \sigma^2, 2, \epsilon \rangle$. But $\tau\eta\epsilon\kappa$ equals $4\nu\overline{\kappa}$ by Lemma 4.2.17 and the hidden τ extension from $h_1c_0d_0$ to Ph_1d_0 given in Table 25, so $4\nu\overline{\kappa}$ belongs to the indeterminacy of the bracket. It follows that both $\sigma\eta_4$ and $\sigma\eta_4 + 4\nu\overline{\kappa}$ belong to the bracket.

LEMMA 3.3.55. $d_5(\tau Ph_5e_0) = \tau d_0 z$.

PROOF. By Lemma 4.2.55, $\tau\eta\kappa\overline{\kappa}^2$ is detected by d_0z . On the other hand, $\nu\{q\}\overline{\kappa}$ equals $\tau\eta\kappa\overline{\kappa}^2$, by Table 24. We will show that $\tau\nu\{q\}\overline{\kappa}$ must be zero. It will follow that some differential must hit τd_0z , and there is just one possibility.

From Lemma 3.3.53, we know that $\tau\nu\{q\}\overline{\kappa}$ is contained in $\{q\}\langle 2, 8\sigma, 2, \sigma^2\rangle$, which is contained in $\langle \alpha, 2, \sigma^2 \rangle$ for some element α in $\langle \{q\}, 2, 8\sigma\rangle$. By Lemma 3.3.52, the two possible values for α are 0 and $2\tau\overline{\kappa}^2$.

First suppose that α is zero. Then $\tau \nu \{q\} \overline{\kappa}$ is contained in $\langle 0, 2, \sigma^2 \rangle$, which is strictly zero.

Next suppose that α is $2\tau \overline{\kappa}^2$. By Lemma 4.2.87, we know that $\epsilon\{q\}$ equals $\{h_1u\}$, which equals $2\overline{\kappa}^2$ by Table 27. Therefore, the element $\tau\nu\{q\}\overline{\kappa}$ is contained in $\langle \tau\epsilon\{q\}, 2, \sigma^2 \rangle$. This bracket has no indeterminacy, so it equals $\tau\{q\}\langle\epsilon, 2, \sigma^2\rangle$. Using that $4\nu\overline{\kappa}\cdot\tau\{q\}$ is zero, Lemma 3.3.54 implies that $\tau\nu\{q\}\overline{\kappa}$ equals $\sigma\eta_4\cdot\tau\{q\}$.

We will show in the proof of Lemma 4.2.87 that $\sigma\{q\}$ equals $\nu\{t\}$. So $\sigma\eta_4 \cdot \tau\{q\}$ equals $\tau \nu \eta_4\{t\}$, which is zero because $\nu \eta_4$ is zero.

LEMMA 3.3.56. $d_5(r_1) = 0$

PROOF. The only other possibility is that $d_5(r_1)$ equals h_2B_{22} . However, h_2r_1 is zero, while $h_2^2B_{22}$ is non-zero in the E_5 -page.

LEMMA 3.3.57. $d_5(\tau h_2 C') = 0.$

PROOF. We do not know whether $d_4(C')$ equals h_2B_{21} , so there are two situations to consider (see Proposition 3.2.15 and Remark 3.2.17).

In the first case, assume that $d_4(C') = 0$. Then C' survives to the E_5 -page, and $d_5(C')$ must equal zero because there are no other possibilities. It follows that $d_5(\tau h_2 C') = 0$.

In the second case, assume that $d_4(C')$ equals h_2B_{21} Then $d_4(h_2C') = h_0h_2B_{22}$, but $\tau h_2C'$ survives to the E_5 -page.

The only possible non-zero value for $d_5(\tau h_2 C')$ is $\tau^4 gw$. However, gw survives to a non-zero homotopy class in the classical Adams spectral sequence for tmf [15]. Therefore, gw survives to a non-zero homotopy class in the classical Adams spectral sequence for S^0 . This implies that $\tau^4 gw$ cannot be hit by a motivic differential. \Box

Our next goal is to show that the element $\tau^3 d_0^2 e_0^2$ is hit by some differential. We include it in this section because it is likely hit by $d_5(\tau h_1^2 X_1)$.

LEMMA 3.3.58. The element $\tau^3 d_0^2 e_0^2$ is hit by some differential.

PROOF. Classically, $\eta^2 \overline{\kappa}^2$ is detected by d_0^3 because of the hidden η extension from g^2 to z and from z to d_0^3 shown in Table 24. This implies that motivically, $\tau^2 \eta^2 \overline{\kappa}^3$ is detected by $\tau^3 d_0^2 e_0^2$. Therefore, we need to show that $\tau^2 \eta^2 \overline{\kappa}^3$ is zero.

Compute that $\langle \kappa, 2\nu, \nu \rangle$ equals $\{\eta \overline{\kappa}, \eta \overline{\kappa} + \nu \nu_4\}$, using Moss's Convergence Theorem 3.1.1 and the Adams differential $d_2(f_0) = h_0^2 e_0$. It follows that $\langle \kappa, 2, \nu^2 \rangle$ equals either $\eta \overline{\kappa}$ or $\eta \overline{\kappa} + \nu \nu_4$. Using Moss's Convergence Theorem 3.1.1 and the Adams differential $d_3(h_0h_4) = h_0d_0$, we get that $\langle \kappa, 2, \nu^2 \rangle$ is detected in Adams filtration strictly greater than 4. Therefore, $\langle \kappa, 2, \nu^2 \rangle$ equals $\eta \overline{\kappa}$.

strictly greater than 4. Therefore, $\langle \kappa, 2, \nu^2 \rangle$ equals $\eta \overline{\kappa}$. This means that $\tau^2 \eta^2 \overline{\kappa}^3$ equals $\tau^2 \eta \overline{\kappa}^2 \langle \kappa, 2, \nu^2 \rangle$. We showed in the proof of Lemma 3.3.55 that $\tau^2 \eta \kappa \overline{\kappa}^2$ is zero. Therefore, $\tau^2 \eta^2 \overline{\kappa}^3$ is contained in $\langle 0, 2, \nu^2 \rangle$, which is strictly zero.

CHAPTER 4

Hidden extensions in the Adams spectral sequence

The main goal of this chapter is to compute hidden extensions in the Adams spectral sequence. We rely on the computation of the Adams E_{∞} -page carried out in Chapter 3. We will borrow results from the classical Adams spectral sequence where necessary. Table 24 summarizes previously established hidden extensions in the classical Adams spectral sequence. The table gives specific references to proofs. The main sources are [3], [4], and [27].

The Adams E_{∞} chart in [19] is an essential companion to this chapter.

Outline. Section 4.1 describes the main points in establishing the hidden extensions. We postpone the numerous technical proofs to Section 4.2.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 25, 27, 29, and 31 list the hidden extensions by τ , 2, η , and ν . The fourth columns of these tables refer to one argument that establishes each hidden extension, which is not necessarily the first known proof. This takes one of the following forms:

- (1) An explicit proof given elsewhere in this manuscript.
- (2) "image of J" means that the hidden extension is easily deducible from the structure of the image of J [2].
- (3) "cofiber of τ " means that the hidden extension is easily deduced from the structure of the homotopy groups of the cofiber of τ , as described in Chapter 5.
- (4) "Table 24" means that the hidden extension is easily deduced from an analogous classical hidden extension.

Tables 33 and 34 give some additional miscellaneous hidden extensions, again with references to a proof.

Tables 26, 28, 30, and 32 give partial information about hidden extensions in stems 59 through 70. These results should be taken as tentative, since the analysis of Adams differentials in this range is incomplete.

4.1. Hidden Adams extensions

4.1.1. The definition of a hidden extension. First we will be precise about the exact nature of a hidden extension. The most naive notion of a hidden extension is a non-zero product $\alpha\beta$ in $\pi_{*,*}$ such that α and β are detected in the E_{∞} -page by a and b respectively and ab = 0 in the E_{∞} -page. However, this notion is too general, as the following example illustrates.

EXAMPLE 4.1.1. Consider $\{h_3^2\}$ in $\pi_{14,8}$, which consists of the two elements σ^2 and $\sigma^2 + \kappa$. We have $h_1 h_3^2 = 0$ in E_{∞} , but $\eta(\sigma^2 + \kappa)$ is non-zero in $\pi_{15,9}$.

This type of situation is not usually considered a hidden extension. Because of the non-zero product h_1d_0 in E_{∞} , one can see immediately that there exists an element β of $\{h_3^2\}$ such that $\eta\beta$ is non-zero.

However, it is not immediately clear whether β is σ^2 or $\sigma^2 + \kappa$. Distinguishing these possibilities requires further analysis. In fact, $\eta \sigma^2 = 0$ [42].

In order to avoid an abundance of not very interesting situations similar to Example 4.1.1, we make the following formal definition of a hidden extension.

DEFINITION 4.1.2. Let α be an element of $\pi_{*,*}$ that is detected by an element a of the E_{∞} -page of the motivic Adams spectral sequence. A hidden extension by α is a pair of elements b and c of E_{∞} such that:

- (1) ab = 0 in the E_{∞} -page.
- (2) There exists an element β of $\{b\}$ such that $\alpha\beta$ is contained in $\{c\}$.
- (3) If there exists an element β' of $\{b'\}$ such that $\alpha\beta'$ is contained in $\{c\}$, then the Adams filtration of b' is less than or equal to the Adams filtration of b.

In other words, b is the element of highest filtration such that there is an α multiplication from $\{b\}$ into $\{c\}$. Consider the situation of Example 4.1.1. Because $\eta\{d_0\}$ is contained in $\{h_1d_0\}$ and the Adams filtration of d_0 is greater than the Adams filtration of h_3^2 , condition (3) of Definition 4.1.2 implies that there is not a hidden η extension from h_3^2 to h_1d_0 .

REMARK 4.1.3. Condition (3) of Definition 4.1.2 implies that b' is not divisible by a in E_{∞} . This allows one to easily reduce the number of cases that must be checked when searching for hidden extensions.

LEMMA 4.1.4. Let α be an element of $\pi_{*,*}$. Let b be an element of the E_{∞} -page of the motivic Adams spectral sequence, and suppose that there exists an element β of $\{b\}$ such that $\alpha\beta$ is zero. Then there is no hidden α extension on b.

PROOF. Suppose that there exists some element β' of $\{b\}$ such that $\alpha\beta'$ is in $\{c\}$. Then $\alpha(\beta + \beta')$ is also in $\{c\}$, and the Adams filtration of $\beta + \beta'$ is strictly greater than the Adams filtration of β' . This implies that there is not a hidden α extension from b to c.

LEMMA 4.1.5. Let α be an element of $\pi_{*,*}$ that is detected by an element a of the E_{∞} -page of the motivic Adams spectral sequence. Suppose that b and c are elements of E_{∞} such that:

(1) ab = 0 in the E_{∞} -page.

(2) α {b} is contained in {c}.

Then there is a hidden α extension from b to c.

PROOF. Let β be any element of $\{b\}$, so $\alpha\beta$ is contained in $\{c\}$. Let b' be an element of the E_{∞} -page, and let β' in $\{b'\}$ be an element such that $\alpha\beta'$ is also contained in $\{c\}$.

Since both $\alpha\beta$ and $\alpha\beta'$ are contained in $\{c\}$, their sum $\alpha(\beta + \beta')$ is detected in Adams filtration strictly greater than the Adams filtration of c. Therefore, $\beta + \beta'$ is not an element of $\{b\}$, which means that the Adams filtration of b' must be less than or equal to the Adams filtration of b.

EXAMPLE 4.1.6. The conditions of Lemma 4.1.5 are not quite equivalent to the conditions of Definition 4.1.2. The difference is well illustrated by an example. We will show in Lemma 4.2.48 that there is a hidden η extension from $h_3^2h_5$ to B_1 , so there exists an element β of $\{h_3^2h_5\}$ such that $\eta\beta$ is contained in $\{B_1\}$. Now let β' be an element of $\{h_5d_0\}$, so $\beta + \beta'$ is an element of $\{h_3^2h_5\}$ because the Adams filtration of h_5d_0 is greater than the Adams filtration of $h_3^2h_5$.

Note that $\eta\beta'$ is contained in $\{h_1h_5d_0\}$. The Adams filtration of $h_1h_5d_0$ is less than the Adams filtration of B_1 , so $\eta(\beta + \beta')$ is contained in $\{h_1h_5d_0\}$. This shows that the conditions of Lemma 4.1.5 are not satisfied.

The difference between Definition 4.1.2 and Lemma 4.1.5 occurs precisely when there are "crossing" α extensions. In the chart of the E_{∞} -page in [19], the straight line from $h_3^2h_5$ to B_1 crosses the straight line from h_5d_0 to $h_1h_5d_0$.

EXAMPLE 4.1.7. Through the 59-stem, the issue of "crossing" extensions occurs in only two other places. First, we will show in Lemma 4.2.73 that there is a hidden extension from $h_3^2h_5$ to B_2 . In the E_{∞} chart in [19], the straight line from $h_3^2h_5$ to B_2 crosses the straight line from h_5d_0 to $h_2h_5d_0$. Therefore there exists an element β of $\{h_3^2h_5\}$ such that $\nu\beta$ is not contained in $\{B_2\}$.

Second, we will show in Lemma 4.2.46 that there is a hidden extension from h_1f_1 to τh_2c_1g . In the E_{∞} chart in [19], the straight line from h_1f_1 to τh_2c_1g crosses the straight line from $h_1^2h_5c_0$ to $h_1^3h_5c_0$. Therefore, there exists an element β of $\{h_1f_1\}$ such that $\eta\beta$ is not contained in $\{\tau h_2c_1g\}$.

We will thoroughly explore hidden extensions in the sense of Definition 4.1.2. However, such hidden extensions do not completely determine the multiplicative structure of $\pi_{*,*}$. For example, the relation $\eta\sigma^2 = 0$ discussed in Example 4.1.1 does not fit into this formal framework.

Something even more complicated occurs with the relation $h_2^3 + h_1^2 h_3 = 0$ in the E_{∞} -page. There is a hidden relation here, in the sense that $\nu^3 + \eta^2 \sigma$ does not equal zero; rather, it equals $\eta \epsilon$ [42]. We do not attempt to systematically address these types of compound relations.

4.1.2. Hidden Adams τ extensions. For hidden τ extensions, the key tool is the homotopy of the cofiber $C\tau$ of τ . This calculation is fully explored in Chapter 5. Let α be an element of $\pi_{*,*}$. Then α maps to zero under the inclusion $S^{0,0} \to C\tau$ of the bottom cell if and only if α is divisible by τ in $\pi_{*,*}$.

PROPOSITION 4.1.8. Table 25 shows some hidden τ extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden τ extensions in this range, with the possible exceptions that there might be hidden τ extensions:

- (1) from h_1i_1 to h_1B_8 .
- (2) from j_1 to B_{21} .

PROOF. Table 25 cites one possible argument for each hidden τ extension. These arguments break into two types:

- (1) In many cases, we know from Chapter 5 that an element of $\{b'\}$ maps to zero in $\pi_{*,*}(C\tau)$, where $C\tau$ is the cofiber of τ . Therefore, this element of $\{b'\}$ is divisible by τ in $\pi_{*,*}$, which implies that there must be a hidden τ extension. Usually there is just one possible hidden τ extension.
- (2) Other more difficult cases are proved in Section 4.2.1.

For many of the possible hidden τ extensions from b to b', we know from Chapter 5 that none of the elements of $\{b'\}$ map to zero in $\pi_{*,*}(C\tau)$. Therefore, none of the elements of $\{b'\}$ is divisible by τ , so none of these possible hidden τ extensions are actual hidden τ extensions. A number of more difficult non-existence proofs are given in Section 4.2.1.

In order to maintain the flow of the narrative, we have collected the technical computations of hidden extensions in Section 4.2.1.

REMARK 4.1.9. Table 26 shows some additional hidden τ extensions in stems 60 through 69. These results are tentative because the analysis of the E_{∞} -page is incomplete in this range. Tentative proofs in Section 4.2.1 are clearly indicated.

REMARK 4.1.10. We show in Lemma 4.2.5 that there is no hidden τ extension on $h_1^2g_2$. This contradicts the claim in [24] that there is a classical hidden η extension from h_1g_2 to N. We do not understand the source of this discrepancy. See also Remark 4.1.22.

REMARK 4.1.11. There may be a hidden τ extension from h_1i_1 to h_1B_8 . This extension occurs if and only if $d_3(\overline{h_1i_1})$ equals h_1B_8 in the Adams spectral sequence for the cofiber of τ (see Proposition 5.2.11). If this extension occurs, then it implies that there is a hidden relation $\nu\{C\} + \tau\{i_1\} = \{B_8\}$.

REMARK 4.1.12. We show in Lemma 4.2.7 that there is no hidden τ extension on D_{11} . This proof is different in spirit from the rest of this manuscript because it uses specific calculations in the classical Adams-Novikov spectral sequence. This is especially relevant since Chapter 6 uses the calculations here to derive Adams-Novikov calculations, so there is some danger of circular arguments. We would prefer to have a proof that is internal to the motivic Adams spectral sequence.

REMARK 4.1.13. Remark 3.2.17 explains that the following three claims are equivalent:

- (1) there is a hidden τ extension from j_1 to B_{21} .
- (2) $d_4(j_1) = B_{21}$ in the motivic Adams spectral sequence for the cofiber of τ .

(3) $d_4(C') = h_2 B_{21}$ in the motivic Adams spectral sequence for the sphere.

4.1.3. Hidden Adams 2 extensions.

PROPOSITION 4.1.14. Table 27 shows some hidden 2 extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden 2 extensions in this range, with the possible exceptions that there might be hidden 2 extensions:

(1) from $h_0h_3g_2$ to τgn .

(2) from j_1 to $\tau^2 c_1 g^2$.

PROOF. Table 27 cites one possible argument (but not necessarily the earliest published result) for each hidden 2 extension. One extension follows from its classical analogue given in Table 24. The remaining cases are proved in Section 4.2.2.

A number of non-existence proofs are given in Section 4.2.2.

In order to maintain the flow of the narrative, we have collected the technical computations of various hidden 2 extensions in Section 4.2.2.

REMARK 4.1.15. Table 28 shows some additional hidden 2 extensions in stems 60 through 69. These results are tentative because the analysis of the E_{∞} -page is incomplete in this range. Tentative proofs in Section 4.2.2 are clearly indicated.

REMARK 4.1.16. Recall from Table 24 that there is a hidden 4 extension from $h_3^2h_5$ to $h_0h_5d_0$. It is tempting to consider this as a hidden 2 extension from $h_0h_3^2h_5$ to $h_0h_5d_0$, but this is not consistent with Definition 4.1.2.

REMARK 4.1.17. There is a possible hidden 2 extension from $h_0h_3g_2$ to τgn . We show in Lemma 4.2.31 that this hidden extension occurs if and only if there is a hidden ν extension from $h_2h_5d_0$ to τgn . Lemma 4.2.31 is inconsistent with results of [24], which indicates the hidden ν extension but not the hidden 2 extension. We do not understand the source of this discrepancy.

REMARK 4.1.18. We show in Lemma 4.2.35 that there is a hidden 2 extension from h_0h_5i to $\tau^3e_0^2g$. This proof is different in spirit from the rest of this manuscript because it uses specific calculations in the classical Adams-Novikov spectral sequence. This is especially relevant since Chapter 6 uses the calculations here to derive Adams-Novikov calculations, so there is some danger of circular arguments. We would prefer to have a proof that is internal to the motivic Adams spectral sequence.

We point out one other remarkable property of this hidden extension. Up to the 59-stem, it is the only example of a 2 extension that is hidden in both the Adams spectral sequence and the Adams-Novikov spectral sequence. (There are several η extensions and ν extensions that are hidden in both spectral sequences.)

4.1.4. Hidden Adams η extensions.

PROPOSITION 4.1.19. Table 29 shows some hidden η extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden η extensions in this range, with the possible exceptions that there might be a hidden η extension from $\tau h_1 Q_2$ to τB_{21} .

PROOF. Table 29 cites one possible argument (but not necessarily the earliest published result) for each hidden η extension. These arguments break into three types:

- (1) Some hidden extensions follow from the calculation of the image of J [2].
- (2) Some hidden extensions follow from their classical analogues given in Table 24.
- (3) The remaining more difficult cases are proved in Section 4.2.3.

A number of non-existence proofs are given in Section 4.2.3.

In order to maintain the flow of the narrative, we collect the technical results establishing various hidden η extensions in Section 4.2.3.

REMARK 4.1.20. Table 30 shows some additional hidden η extensions in stems 60 through 69. These results are tentative because the analysis of the E_{∞} -page is incomplete in this range. Tentative proofs in Section 4.2.3 are clearly indicated.

REMARK 4.1.21. We show in Lemma 4.2.47 that there is no hidden η extension on $\tau h_1 g_2$. This contradicts the claim in [24] that there is a classical hidden η extension from $h_1 g_2$ to N. We do not understand the source of this discrepancy. REMARK 4.1.22. The element $\eta^2 \{g_2\}$ is considered in [3, Lemma 4.3], where it is shown to be equal to $\sigma^2 \{d_1\}$. Our results indicate that both are zero classically; this is consistent with a careful reading of [3, Lemma 4.3].

Motivically, $\eta^2 \{g_2\} = \sigma^2 \{d_1\}$ is non-zero because they are detected by $h_1^2 g_2 = h_3^2 d_1$. However, Lemma 4.2.47 implies that $\tau \eta^2 \{g_2\}$ and $\tau \sigma^2 \{d_1\}$ are both zero.

REMARK 4.1.23. We show in Lemma 4.2.52 that there is no hidden η extension on C. This contradicts the claim in [24] that there is a classical hidden η extension from C to gn. We do not understand the source of this discrepancy.

4.1.5. Hidden Adams ν extensions.

PROPOSITION 4.1.24. Table 31 shows some hidden ν extensions in $\pi_{*,*}$, through the 59-stem. These are the only hidden ν extensions in this range, with the possible exceptions that there might be hidden ν extensions:

- (1) from $h_2h_5d_0$ to τgn .
- (2) from i_1 to gt.

PROOF. Table 31 cites one possible argument (but not necessarily the earliest published result) for each hidden ν extension. These arguments break into two types:

(1) Some hidden extensions follow from their classical analogues given in Table 24.

- (2) The remaining more difficult cases are proved in Section 4.2.4.
- A number of non-existence proofs are given in Section 4.2.4.

In order to maintain the flow of the narrative, we have collected the technical results establishing various hidden ν extensions in Section 4.2.4.

REMARK 4.1.25. Table 32 shows some additional hidden ν extensions in stems 60 through 69. These results are tentative because the analysis of the E_{∞} -page is incomplete in this range. Tentative proofs in Section 4.2.4 are clearly indicated.

REMARK 4.1.26. We draw the reader's attention to the curious hidden ν extensions on h_2c_1 , h_2c_1g , and N. These are "exotic" extensions that have no classical analogues. The hidden extension on N contradicts the claim in [24] that there is a hidden η extension from h_1g_2 to N. In addition to the proof provided in Lemma 4.2.63, one can also establish these hidden extensions by computing in the motivic Adams spectral sequence for the cofiber of ν . One can show that $\{h_1^2h_4c_0\}$, $\{Ph_1^2h_5c_0\}$, and $\{h_1^6h_5c_0\}$ all map to zero in the cofiber of ν , which implies that they are divisible by ν . indexh4c0@ h_4c_0

REMARK 4.1.27. There is a possible hidden ν extension from $h_2h_5d_0$ to τgn . We show in Lemma 4.2.31 that this hidden extension occurs if and only if there is a hidden 2 extension from $h_0h_3g_2$ to τgn . Lemma 4.2.31 is inconsistent with results of [24], which indicates the hidden ν extension but not the hidden 2 extension. We do not understand the source of this discrepancy.

4.2. Hidden Adams extensions computations

In this section, we collect the technical computations that establish the hidden extensions discussed in Section 4.1.

4.2.1. Hidden Adams τ extensions computations.

LEMMA 4.2.1.

- (1) There is a hidden τ extension from h_1h_3g to d_0^2 .
- (2) There is a hidden τ extension from $h_1h_3g^2$ to $d_0e_0^2$.

PROOF. We will show in Lemma 4.2.85 that $\epsilon \overline{\kappa} = \kappa^2$ in $\pi_{28,16}$. Therefore, κ^2 is contained in $\overline{\kappa} \langle 2, \nu^2, \eta \rangle$.

Let $C\tau$ be the cofiber of τ , whose homotopy is studied thoroughly in Chapter 5. Let $\overline{\kappa}_{C\tau}$ be the image of $\overline{\kappa}$ in $\pi_{20,11}(C\tau)$. Then the image of κ^2 in $\pi_{28,16}(C\tau)$ is contained in $\overline{\kappa}_{C\tau}\langle 2, \nu^2, \eta \rangle$. Because $\overline{\kappa}_{C\tau} \cdot 2$ is zero, we can shuffle to obtain $\langle \overline{\kappa}_{C\tau}, 2, \nu^2 \rangle \eta$.

Now $\pi_{27,15}(C\tau)$ consists only of the element $\{P^3h_1^3\}$. However, this element cannot belong to $\langle \overline{\kappa}_{C\tau}, 2, \nu^2 \rangle$ because $\{P^3h_1^3\}$ supports infinitely many multiplications by η , while the elements in the bracket cannot. Therefore, $\langle \overline{\kappa}_{C\tau}, 2, \nu^2 \rangle$ is zero, and the image of κ^2 in $\pi_{28,16}(C\tau)$ is zero.

Therefore, κ^2 in $\pi_{28,14}$ is divisible by τ , and there is just one possible hidden τ extension. This completes the proof of the first claim.

The proof for the second claim is analogous, using that $\epsilon\{\tau g^2\} = \kappa\{e_0^2\}$ from Lemma 4.2.85. The bracket $\langle\{\tau g^2\}_{C\tau}, 2, \nu^2\rangle$ in $\pi_{47,27}(C\tau)$ must be zero because there are no other possibilities.

Lemma 4.2.2.

- (1) There is no hidden τ extension on h_1d_1 .
- (2) There is no hidden τ extension on h_1d_1g .

PROOF. For the first formula, the only other possibility is that there is a hidden τ extension from h_1d_1 to h_1q . We will show that this is impossible.

Proposition 6.2.5 shows that the element $\{d_1\}$ of $\pi_{32,18}$ is detected in Adams-Novikov filtration 4. Therefore, $\{d_1\}$ realizes to zero in $\pi_{32} tmf[\mathbf{5}]$, so $\tau \eta\{d_1\}$ also realizes to zero in $\pi_{33} tmf$.

On the other hand, $\{h_1e_0^2\}$ realizes to a non-zero element of $\pi_{35}tmf$. The classical hidden extension $\nu\{q\} = \{h_1e_0^2\}$ given in Table 24 then implies that $\{q\}$ realizes to a non-zero element of $\pi_{32}tmf$. Then $\{h_1q\}$ also realizes to a non-zero element of $\pi_{33}tmf$.

This shows that $\tau \eta\{d_1\}$ cannot belong to $\{h_1q\}$, so it must be zero. Now Lemma 4.1.4 establishes the first claim.

For the second claim, Table 23 shows that $\{d_1g\} = \langle \{d_1\}, \eta^3, \eta_4 \rangle$, again with no indeterminacy. Now shuffle to obtain $\tau \eta \{d_1g\} = \langle \tau \eta, \{d_1\}, \eta^3 \rangle \eta_4$. The element $\{\tau h_2 e_0^2\}$ is the only non-zero element that could possibly be contained in $\langle \tau \eta, \{d_1\}, \eta^3 \rangle$. In any case, $\langle \tau \eta, \{d_1\}, \eta^3 \rangle \eta_4$ is zero. This shows that $\tau \eta \{d_1g\}$ is zero. Lemma 4.1.4 establishes the second claim.

LEMMA 4.2.3. There is a hidden τ extension from $\tau h_0 g^2$ to $h_1 u$.

PROOF. Classically, there is a hidden 2 extension from g^2 to $h_1 u$ given in Table 24. This implies that there is a motivic hidden 2 extension from $\tau^2 g^2$ to $h_1 u$. The desired hidden τ extension follows.

Lemma 4.2.4.

(1) There is a hidden τ extension from $\tau h_1 g^2$ to z.

(2) There is a hidden τ extension from $\tau h_1 e_0^2 g$ to $d_0 z$.

PROOF. There is a classical hidden η extension from g^2 to z given in Table 24. It follows that there is a motivic hidden η extension from $\tau^3 g^2$ to z. The first claim follows immediately.

For the second claim, multiply the first hidden extension by d_0 .

LEMMA 4.2.5. There is no hidden τ extension on $h_1^2g_2$.

PROOF. We will show in Lemma 4.2.47 that there is no hidden η extension on $\tau h_1 g_2$. This implies that there is no hidden τ extension on $h_1^2 g_2$.

LEMMA 4.2.6. There is no hidden τ extension on $\tau h_2 d_1 g$.

PROOF. The only other possibility is that there is a hidden τ extension from $\tau h_2 d_1 g$ to $d_0 z$. However, we showed in Lemma 4.2.4 that there is a hidden τ extension from $\tau h_1 e_0^2 g$ to $d_0 z$. Since $\{\tau h_1 e_0^2 g\}$ is contained in the indeterminacy of $\{\tau h_2 d_1 g\}$, there exists an element of $\{\tau h_2 d_1 g\}$ that is annihilated by τ . Lemma 4.1.4 finishes the proof.

LEMMA 4.2.7. There is no hidden τ extension on D_{11} .

PROOF. This proof is different in spirit from the rest of the manuscript because it relies on specific calculations in the classical Adams-Novikov spectral sequence.

There is an element $\beta_{12/6}$ in the Adams-Novikov spectral sequence in the 58stem with filtration 2 [38]. Using Proposition 6.2.5, if this class survives, then it would correspond to an element of $\pi_{58,30}$ that is not divisible by τ . By inspection of the E_{∞} -page of the motivic Adams spectral sequence, there is no such element in $\pi_{58,30}$. Therefore, $\beta_{12/6}$ must support a differential in the Adams-Novikov spectral sequence.

Using the framework of Chapter 6, an Adams-Novikov d_{2r+1} differential on $\beta_{12/6}$ would correspond to an element of $\pi_{57,r+30}$ that is not divisible by τ ; that is not killed by τ^{r-1} ; and that is annihilated by τ^r . By inspection of the E_{∞} -page of the motivic Adams spectral sequence, the only possibility is that r = 1, and the corresponding element of $\pi_{57,31}$ is detected by D_{11} .

LEMMA 4.2.8. There is no hidden τ extension on $h_3^2 g_2$.

PROOF. The only other possibility is that there is a hidden τ extension from $h_3^2g_2$ to h_1Q_2 . However, $\eta\{h_1Q_2\}$ equals $\{h_1^2Q_2\}$, which is non-zero. On the other hand, $\{h_3^2g_2\}$ contains the element $\sigma^2\{g_2\}$. This is annihilated by η because $\eta\sigma^2 = 0$ [42]. It follows that $\tau\{h_3^2g_2\}$ cannot intersect $\{h_1Q_2\}$.

LEMMA 4.2.9. There is no hidden τ extension on h_3d_1g .

PROOF. The only other possibility is that there is a hidden τ extension from h_3d_1g to $Ph_1^3h_5e_0$. We will argue that this cannot occur.

Let α be an element of $\{h_3d_1g\}$. Note that α equals either $\sigma\{d_1g\}$ or $\sigma\{d_1g\} + \nu\{gt\}$. In either case, $\eta\alpha$ equals $\eta\sigma\{d_1g\}$, which is a non-zero element of $\{h_1h_3d_1g\}$. We know from Lemma 4.2.2 that $\tau\eta\{d_1g\}$ is zero, so $\tau\eta\alpha$ is zero.

On the other hand, $\eta\{Ph_1^3h_5e_0\}$ equals $\{\tau^2h_0g^3\}$, which is non-zero. Therefore, $\tau\alpha$ cannot equal $\{Ph_1^3h_5e_0\}$.

LEMMA 4.2.10. There is a hidden τ extension from $Ph_1^3h_5e_0$ to τd_0w .

PROOF. Table 23 shows that $\langle \tau, \nu \overline{\kappa}^2, \eta \rangle$ in $\pi_{45,24}$ contains the element $\{\tau w\}$. This bracket has indeterminacy generated by $\tau \eta \{g_2\}$. Table 23 also shows that the bracket $\langle \nu \overline{\kappa}^2, \eta, \eta \kappa \rangle$ equals $\{Ph_1^4h_5e_0\}$, with no indeterminacy.

Now use the shuffle $\tau \langle \nu \overline{\kappa}^2, \eta, \eta \kappa \rangle = \langle \tau, \nu \overline{\kappa}^2, \eta \rangle \eta \kappa$ to conclude that $\tau \{Ph_1^4h_5e_0\}$ equals $\eta \kappa \{\tau w\}$.

There is a classical extension $\eta\{w\} = \{d_0l\}$, as shown in Table 24. It follows that there is a motivic relation $\eta\kappa\{\tau w\} = \{\tau d_0^2 l + d_0 u'\}$; in particular, it is non-zero.

We have shown that $\tau\{Ph_1^4h_5e_0\}$ is non-zero. But this equals $\tau\eta\{Ph_1^3h_5e_0\}$, so $\tau\{Ph_1^3h_5e_0\}$ is also non-zero. There is just one possible non-zero value.

LEMMA 4.2.11. There is no hidden τ extension on $\tau^2 c_1 g^2$.

PROOF. The only other possibility is that there is a hidden τ extension from $\tau^2 c_1 g^2$ to $\tau d_0 w$. We showed in Lemma 4.2.10 that there is a hidden τ extension from $Ph_1^3h_5e_0$ to τd_0w . Since $\{Ph_1^3h_5e_0\}$ is contained in the indeterminacy of $\{\tau^2 c_1 g^2\}$, there exists an element of $\{\tau^2 c_1 g^2\}$ that is annihilated by τ .

LEMMA 4.2.12. Tentatively, there is a hidden τ extension from $h_1^2 X_2$ to τB_{23} .

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

There exists an element of $\{\tau B_{23}\}$ that maps to zero in the homotopy groups of the cofiber of τ , which is described in Chapter 5. Therefore, this element of $\{\tau B_{23}\}$ is divisible by τ . The only possibility is that there is a hidden τ extension from $h_1^2 X_2$ to τB_{23} .

LEMMA 4.2.13. Tentatively, there is a hidden τ extension from $h_1^4 X_2$ to $B_8 d_0$.

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows from the hidden τ extension from $h_1^2 X_2$ to τB_{23} given in Lemma 4.2.12 and the hidden η extension from $\tau h_1 B_{23}$ to $B_8 d_0$ given in Lemma 4.2.60. \Box

LEMMA 4.2.14. Tentatively, there is a hidden τ extension from B_8d_0 to d_0x' .

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows immediately from the hidden τ extension from B_8 to x' given in Table 25.

4.2.2. Hidden Adams 2 extensions computations.

LEMMA 4.2.15. There is no hidden 2 extension on $h_2^2h_4$.

PROOF. The only other possibility is that there is a hidden 2 extension from $h_2^2h_4$ to τh_1g . However, we will show later in Lemma 4.2.39 that there is a hidden η extension on τh_1g .

LEMMA 4.2.16. There is no hidden 2 extension on h_4c_0 .

PROOF. We showed in Lemma 4.2.83 that $\sigma\eta_4$ belongs to $\{h_4c_0\}$, and $2\eta_4$ equals zero. Now use Lemma 4.1.4 to finish the claim.

Lemma 4.2.17.

(1) There is a hidden 2 extension from h_0h_2g to $h_1c_0d_0$.

- (2) There is a hidden 2 extension from $\tau h_0 h_2 g$ to $Ph_1 d_0$.
- (3) There is a hidden 2 extension from $h_0h_2g^2$ to $h_1c_0e_0^2$.
- (4) There is a hidden 2 extension from $\tau h_0 h_2 g^2$ to $h_1 d_0^3$.

PROOF. Recall that there is a classical hidden 2 extension from h_0h_2g to Ph_1d_0 , as shown in Table 24. This immediately implies the second claim. The first formula now follows from the second, using the hidden τ extension from $h_1c_0d_0$ to Ph_1d_0 given in Table 25.

For the last two formulas, recall that there is a classical hidden extension $\eta^2 \overline{\kappa}^2 = \{d_0^3\}$, as shown in Table 24. This implies that there is a motivic hidden extension $\tau \eta^3 \{\tau g^2\} = \{h_1 d_0^3\}$. Use the relation $\tau \eta^3 = 4\nu$ to deduce the fourth formula.

The third formula follows from the fourth, using the hidden τ extension from $h_1 c_0 e_0^2$ to $h_1 d_0^3$ given in Table 25.

Lemma 4.2.18.

(1) There is no hidden 2 extension on h_1h_5 .

(2) There is no hidden 2 extension on $h_1h_3h_5$.

PROOF. For the first claim, the only other possibility is that there is a hidden 2 extension from h_1h_5 to τd_1 . Table 23 shows that the Toda bracket $\langle \eta, 2, \theta_4 \rangle$ intersects $\{h_1h_5\}$. Shuffle to obtain

$$2\langle \eta, 2, \theta_4 \rangle = \langle 2, \eta, 2 \rangle \theta_4.$$

This expression equals $\tau \eta^2 \theta_4$ by Table 23, which must be zero. Lemma 4.1.4 now finishes the first claim.

The second claim follows easily since $\sigma\eta_5$ is contained in $\{h_1h_3h_5\}$.

LEMMA 4.2.19. There is no hidden 2 extension on p.

PROOF. The only other possibility is that there is a hidden 2 extension from p to h_1q . Table 24 shows that $\nu\theta_4$ is contained in $\{p\}$. Also, $2\theta_4$ is zero. Lemma 4.1.4 now finishes the proof.

Lemma 4.2.20.

- (1) There is no hidden 2 extension on h_2d_1 .
- (2) There is no hidden 2 extension on h_3d_1 .
- (3) There is no hidden 2 extension on h_2d_1g .
- (4) There is no hidden 2 extension on h_3d_1g .

PROOF. These follow immediately from Lemma 4.1.4, together with the facts that $\nu\{d_1\}$ is contained in $\{h_2d_1\}$; $\sigma\{d_1\}$ is contained in $\{h_3d_1\}$; $\nu\{d_1g\}$ is contained in $\{h_2d_1g\}$; $\sigma\{d_1g\}$ is contained in $\{h_3d_1g\}$; and $2\{d_1\}$ and $2\{d_1g\}$ are both zero.

LEMMA 4.2.21. There is no hidden 2 extension on h_5c_0 .

PROOF. The only other possibility is that there is a hidden 2 extension from h_5c_0 to τ^2c_1g . Table 23 shows that $\langle \epsilon, 2, \theta_4 \rangle$ intersects $\{h_5c_0\}$. Now shuffle to obtain that

$$2\langle \epsilon, 2, \theta_4 \rangle = \langle 2, \epsilon, 2 \rangle \theta_4.$$

By Table 23, this equals $\tau \eta \epsilon \theta_4$, which must be zero. Lemma 4.1.4 now finishes the argument.

LEMMA 4.2.22. There is no hidden 2 extension on Ph_1h_5 .

PROOF. The other possibilities are hidden 2 extensions to $\tau^3 g^2$ or $h_1 u$. We will show that neither can occur.

We already know from Table 24 that there is a hidden 2 extension from $\tau^3 g^2$ to $h_1 u$. Therefore, there cannot be a hidden 2 extension from Ph_1h_5 to $h_1 u$.

Table 29 shows a hidden η extension from $\tau^3 g^2$ to z. This implies that $\tau^3 g^2$ cannot be the target of a hidden 2 extension.

LEMMA 4.2.23. There is no hidden 2 extension on $h_0^2 h_5 d_0$.

PROOF. As shown in Table 24, τw supports a hidden η extension. Therefore, it cannot be the target of a hidden 2 extension.

LEMMA 4.2.24. There is no hidden 2 extension on h_2g_2 .

PROOF. From the relation $h_2g_2 + h_3f_1 = 0$ in the E_2 -page, we know that $\{h_2g_2\}$ contains $\sigma\{f_1\}$. Also, $\{f_1\}$ contains an element that is annihilated by 2, so $\{h_2g_2\}$ contains an element that is annihilated by 2. Lemma 4.1.4 finishes the argument.

LEMMA 4.2.25. There is no hidden 2 extension on Ph_5c_0 .

PROOF. The element Ph_5c_0 detects $\rho_{15}\eta_5$ [41, Lemma 2.5]. Also, $2\eta_5$ is zero, so $\{Ph_5c_0\}$ contains an element that is annihilated by 2. Lemma 4.1.4 finishes the proof.

Lemma 4.2.26.

(1) There is a hidden 2 extension from e_0r to h_1u' .

(2) There is a hidden 2 extension from $\tau e_0 r$ to Pu.

PROOF. Table 24 shows that there is a hidden η extension from τw to $\tau d_0 l + u'$. Also, from Table 25, there is a hidden τ extension from $h_1 u'$ to Pu. Therefore, $\tau \eta^2 \{\tau w\} = \{Pu\}.$

Recall from Table 23 that $\tau \eta^2 = \langle 2, \eta, 2 \rangle$. Since $2\{\tau w\}$ is zero, we can shuffle to obtain

$$\tau \eta^2 \{\tau w\} = \langle 2, \eta, 2 \rangle \{\tau w\} = 2 \langle \eta, 2, \{\tau w\} \rangle.$$

This shows that $\{Pu\}$ is divisible by 2.

By Lemmas 4.2.24 and 4.2.25, the only possibility is that there is a hidden 2 extension from $\tau e_0 r$ to Pu. This establishes the second claim.

The first claim now follows from the second, using the hidden τ extension from h_1u' to Pu given in Table 25.

LEMMA 4.2.27. There is no hidden 2 extension on $h_2h_5d_0$.

PROOF. There is an element of $2\{h_5d_0\}$ that is divisible by 4, as shown in Table 24. Therefore, there is an element of $2\nu\{h_5d_0\}$ that is divisible by 4. However, zero is the only element of $\pi_{48,26}$ that is divisible by 4.

LEMMA 4.2.28. There is no hidden 2 extension on h_0B_2 .

PROOF. We will show later in Lemma 4.2.73 that there exists an element α of $\{h_3^2h_5\}$ such that $\nu\alpha$ belongs to $\{B_2\}$. (We do not know whether α equals $\theta_{4.5}$, but that does not matter here. See Section 1.7 for further discussion.) Therefore, $2\nu\alpha$ belongs to $\{h_0B_2\}$.

Now $2 \cdot 2\nu\alpha$ equals $\tau \eta^3 \alpha$. There is a classical relation $\eta^3 \alpha = 0$ [3, Lemma 3.5], which implies that $\eta^3 \alpha$ equals zero motivically as well.

LEMMA 4.2.29.

(1) There is no hidden 2 extension on h_5c_1 .

(2) There is no hidden 2 extension on $h_2h_5c_1$.

PROOF. Table 23 show that $\langle \overline{\sigma}, 2, \theta_4 \rangle$ intersects $\{h_5c_1\}$. Now shuffle to obtain

$$2\langle \overline{\sigma}, 2, \theta_4 \rangle = \langle 2, \overline{\sigma}, 2 \rangle \theta_4.$$

Table 23 shows that $\langle 2, \overline{\sigma}, 2 \rangle$ consists of multiples of 2, and $2\theta_4$ is zero. Therefore, $2\langle \overline{\sigma}, 2, \theta_4 \rangle$ is zero. Lemma 4.1.4 now establishes the first claim.

The second claim follows immediately from the first because $h_2h_5c_1$ contains $\nu\{h_5c_1\}$.

LEMMA 4.2.30. There is no hidden 2 extension from $h_0h_3g_2$ to h_2B_2 .

PROOF. The element $h_0h_3g_2$ detects $2\sigma\{g_2\}$. We will show later in Lemma 4.2.75 that h_2B_2 supports a hidden ν extension. Therefore, none of the elements of $\{h_2B_2\}$ are divisible by σ .

LEMMA 4.2.31. There is a hidden 2 extension on $h_0h_3g_2$ if and only if there is a hidden ν extension on $h_2h_5d_0$.

PROOF. Let β be an element of $\{h_5d_0\}$. Table 23 shows that $\langle 2, \eta, \eta\beta \rangle$ intersects $\{h_2h_5d_0\}$, and $\langle \eta, \eta\beta, \nu \rangle$ intersects $\{h_0h_3g_2\}$. Now consider the shuffle

$$2\langle \eta, \eta\beta, \nu\rangle = \langle 2, \eta, \eta\beta\rangle\nu.$$

The indeterminacy here is zero.

LEMMA 4.2.32. There is no hidden 2 extension on i_1 .

PROOF. The only other possibility is that there is a hidden 2 extension from i_1 to $h_1^2G_3$. Because of the hidden τ extension from $h_1^2G_3$ to d_0u given in Table 25, this would imply a hidden 2 extension from τi_1 to d_0u .

However, τi_1 detects $\nu\{C\}$, and $2\{C\}$ is zero. Therefore, $\{\tau i_1\}$ contains an element that is annihilated by 2. Lemma 4.1.4 implies that there cannot be a hidden 2 extension on τi_1 .

Lemma 4.2.33.

(1) There is no hidden 2 extension on B_8 .

(2) There is no hidden 2 extension on x'.

PROOF. We will show in Lemma 4.2.88 that B_8 detects $\epsilon \theta_{4.5}$. Since 2ϵ is zero, it follows that $\{B_8\}$ contains an element that is annihilated by 2. Lemma 4.1.4 establishes the first claim.

The second claim follows from the first, using the hidden τ extension from B_8 to x' given in Table 25.

LEMMA 4.2.34. There is no hidden 2 extension on h_2gn .

PROOF. Note that $\nu\{gn\}$ is contained in $\{h_2gn\}$, and $2\{gn\}$ is zero. Therefore, $\{h_2gn\}$ contains an element that is annihilated by 2, and Lemma 4.1.4 finishes the proof.

LEMMA 4.2.35. There is a hidden 2 extension from h_0h_5i to $\tau^4e_0^2g$.

PROOF. This proof is different in spirit from the rest of the manuscript because it relies on specific calculations in the classical Adams-Novikov spectral sequence.

The class h_0h_5i detects an element of $\pi_{54,28}$ that is not divisible by τ . By Proposition 6.2.5, this corresponds to an element in the classical Adams-Novikov spectral sequence in the 54-stem with filtration 2. The only possibility is the element $\beta_{10/2}$ [38].

The image of $\beta_{10/2}$ in the Adams-Novikov spectral sequence for $tmf[\mathbf{5}]$ is $\Delta^2 h_2^2$. Since there is no filtration shift, this is detectable in the chromatic spectral sequence. In the Adams-Novikov spectral sequence for tmf, there is a hidden 2 extension from $\Delta^2 h_2^2$ to the class that detects $\kappa \overline{\kappa}^2$. Therefore, in the Adams-Novikov spectral sequence for the sphere, there must also be a hidden 2 extension from $\beta_{10/2}$ to the class that detects $\kappa \overline{\kappa}^2$.

Since $\beta_{10/2}$ corresponds to h_0h_5i , it follows that in the Adams spectral sequence, there is a hidden 2 extension from h_0h_5i to $\tau^4e_0^2g$.

LEMMA 4.2.36. There is no hidden 2 extension on B_{21} .

PROOF. We showed in Lemma 4.2.93 that B_{21} detects a multiple of κ . Since 2κ is zero, it follows that $\{B_{21}\}$ contains an element that is annihilated by 2. Lemma 4.1.4 finishes the proof.

Lemma 4.2.37.

- (1) Tentatively, there is a hidden 2 extension from $\tau^3 g^3$ to $d_0 u' + \tau d_0^2 l$.
- (2) Tentatively, there is a hidden 2 extension from $\tau h_0 h_2 g^3$ to $h_1 d_0^2 e_0^2$.
- (3) Tentatively, there is a hidden 2 extension from $\tau e_0 gr$ to $d_0^2 u$.

PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The first formula follows immediately from the hidden τ extension from $\tau^2 h_0 g^3$ to $d_0 u' + \tau d_0^2 l$ given in Table 26. The second formula follows immediately from the hidden τ extension from $h_0^2 h_2 g^3$ to $h_1 d_0^2 e_0^2$ given in Table 26. The third formula follows immediately from the hidden τ extension from $h_0 e_0 gr$ to $d_0^2 u$ given in Table 26.

4.2.3. Hidden Adams η extensions computations.

LEMMA 4.2.38. There is no hidden η extension on c_1 .

PROOF. The only other possibility is that there is a hidden η extension from c_1 to $h_0^2 g$. We will show in Lemma 4.2.62 that there is a hidden ν extension on $h_0^2 g$. Therefore, it cannot be the target of a hidden η extension.

LEMMA 4.2.39.

- (1) There is a hidden η extension from $\tau h_1 g$ to $c_0 d_0$.
- (2) There is a hidden η extension from $\tau^2 h_1 g$ to Pd_0 .
- (3) There is a hidden η extension from $\tau h_1 g^2$ to $c_0 e_0^2$.
- (4) There is a hidden η extension from z to τd_0^3 .

PROOF. There is a classical hidden η extension from h_1g to Pd_0 , as shown in Table 24. This implies that there is a motivic hidden η extension from $\tau^2 h_1 g$ to Pd_0 . This establishes the second claim.

The first claim follows from the second claim, using the hidden τ extension from $c_0 d_0$ to $P d_0$ given in Table 25.

Next, there is a classical hidden η extension from z to d_0^3 , as shown in Table 24. This implies that there is a motivic hidden η extension from z to τd_0^3 . This establishes the fourth claim.

The third claim follows from the fourth, using the hidden τ extensions from $\tau^2 h_1 g^2$ to z and from $c_0 e_0^2$ to d_0^3 given in Table 25.

LEMMA 4.2.40. There is no hidden η extension on p.

PROOF. Classically, $\nu \theta_4$ belongs to $\{p\}$, as shown in Table 24, so the same formula holds motivically. Therefore, $\{p\}$ contains an element that is annihilated by η . Lemma 4.1.4 finishes the argument.

LEMMA 4.2.41. There is a hidden η extension from $h_0^2 h_3 h_5$ to $\tau^2 c_1 g$.

PROOF. First, $\tau^2 c_1 g$ equals $h_1 y$ on the E_2 -page [9]. Use Moss's Convergence Theorem 3.1.1 together with the Adams differential $d_2(y) = h_0^3 x$ to conclude that $\{\tau^2 c_1 g\}$ intersects $\langle \eta, 2, \alpha \rangle$, where α is any element of $\{h_0^2 x\}$.

However, the later Adams differential $d_4(h_0h_3h_5) = h_0^2 x$ implies that 0 belongs to $\{h_0^2x\}$. Therefore, $\{\tau^2c_1g\}$ intersects $\langle\eta, 2, 0\rangle$. In other words, there exists an element of $\{\tau^2c_1g\}$ that is a multiple of η . The only possibility is that there is a hidden η extension from $h_0^2h_3h_5$ to τ^2c_1g .

REMARK 4.2.42. Lemma 4.2.41 shows that $\eta\{h_0^2h_3h_5\}$ is an element of $\{\tau^2c_1g\}$. However, $\{\tau^2c_1g\}$ contains two elements because u is in higher Adams filtration. The sum $\eta\{h_0^2h_3h_5\} + \tau \overline{\sigma\kappa}$ is either zero or equal to $\{u\}$. Both $\{h_0^2h_3h_5\}$ and $\overline{\sigma}$ map to zero in $\pi_{*,*}(tmf)$, while $\{u\}$ is non-zero in $\pi_{*,*}(tmf)$. Therefore, $\eta\{h_0^2h_3h_5\} + \tau \overline{\sigma\kappa}$ must be zero. We will need this observation in Lemmas 5.3.4 and 5.3.8.

LEMMA 4.2.43. There is no hidden η extension on $\tau h_3 d_1$.

PROOF. We know that $\sigma\{\tau d_1\}$ is contained in $\{\tau h_3 d_1\}$, and there exists an element of $\{\tau d_1\}$ that is annihilated by η . Therefore, $\{\tau h_3 d_1\}$ contains an element that is annihilated by η . Lemma 4.1.4 finishes the proof.

LEMMA 4.2.44. There is no hidden η extension on c_1g .

PROOF. Since $\nu\{t\}$ is contained in $\{\tau c_1 g\}$, Lemma 4.1.4 implies that there is no hidden η extension on $\tau c_1 g$. In particular, there cannot be a hidden η extension from $\tau c_1 g$ to $\tau h_0^2 g^2$. Therefore, there cannot be a hidden η extension from $c_1 g$ to $h_0^2 g^2$.

LEMMA 4.2.45. There is no hidden η extension on $\tau h_1 h_5 c_0$.

PROOF. Table 29 shows a hidden η extension from $\tau^3 g^2$ to z. Therefore, there cannot be a hidden η extension from $\tau h_1 h_5 c_0$ to z.

LEMMA 4.2.46. There is a hidden η extension from $h_1 f_1$ to $\tau h_2 c_1 g$.

PROOF. Note that $\{\tau h_2 c_1 g\}$ contains $\nu^2 \{t\}$. Table 23 shows that $\nu^2 = \langle \eta, \nu, \eta \rangle$. Shuffle to compute that

$$\nu^{2}\{t\} = \langle \eta, \nu, \eta \rangle \{t\} = \eta \langle \nu, \eta, \{t\} \rangle,$$

so $\nu^2{t}$ is divisible by η . The only possibility is that there is a hidden η extension from $h_1 f_1$ to $\tau h_2 c_1 g$.

LEMMA 4.2.47. There is no hidden η extension on $\tau h_1 g_2$.

PROOF. We will show in Lemma 4.2.63 that N supports a hidden ν extension. Therefore, N cannot be the target of a hidden η extension.

For degree reasons, $\eta^3 \{g_2\}$ must be zero. Therefore, $\tau \eta^3 \{g_2\}$ must be zero. This implies that the target of a hidden η extension on $\tau h_1 g_2$ cannot support an h_1 multiplication. Hence, there cannot be a hidden η extension from $\tau h_1 g_2$ to B_1 or to $\tau d_0 l + u'$.

LEMMA 4.2.48. There is a hidden η extension from $h_3^2h_5$ to B_1 .

PROOF. This follows immediately from the analogous classical hidden extension given in Table 24, but we repeat the interesting proof from [41] here for completeness.

First, Table 16 shows that $B_1 = \langle h_1, h_0, h_0^2 g_2 \rangle$. Then Moss's Convergence Theorem 3.1.1 implies that $\{B_1\}$ intersects $\langle \eta, 2, \{h_0^2 g_2\} \rangle$.

Next, the classical product $\sigma\theta_4$ belongs to $\{x\}$, as shown in Table 24. Since $h_3x = h_0^2g_2$ on the E_2 -page [9], it follows that $\sigma^2\theta_4$ equals $\{h_0^2g_2\}$. The same formula holds motivically.

Now $\langle \eta, 2, \sigma^2 \theta_4 \rangle$ is contained in $\langle \eta, 2\sigma^2, \theta_4 \rangle$, which equals $\langle \eta, 0, \theta_4 \rangle$. Therefore, $\{B_1\}$ contains an element of the form $\theta_4 \alpha + \eta \beta$.

The possible non-zero values for α are η_4 or $\eta_{\rho_{15}}$. In the first case, $\theta_4 \alpha$ equals $\theta_4 \langle 2, \sigma^2, \eta \rangle$, which equals $\langle \theta_4, 2, \sigma^2 \rangle \eta$. Therefore, in either case, $\theta_4 \alpha$ is a multiple of η , so we can assume that α is zero.

We have now shown that $\{B_1\}$ contains a multiple of η . Because of Lemma 4.2.47, the only possibility is that there is a hidden η extension from $h_3^2 h_5$ to B_1 . \Box

Lemma 4.2.49.

(1) There is no hidden η extension on $h_1h_5d_0$.

(2) There is no hidden η extension on N.

PROOF. We showed in Lemma 4.2.26 that there is a hidden 2 extension on e_0r . Therefore, e_0r cannot be the target of a hidden η extension.

LEMMA 4.2.50. There is no hidden η extension on h_1B_1 .

PROOF. Classically, there is no hidden η extension on h_1B_1 [3, Theorem 3.1(i) and Lemma 3.5]. Therefore, there cannot be a motivic hidden η extension from h_1B_1 to $\tau d_0 e_0^2$.

LEMMA 4.2.51. There is no hidden η extension on h_5c_1 .

PROOF. Table 23 shows that $\{h_5c_1\}$ is contained in $\langle \nu, \sigma, \sigma\eta_5 \rangle$. Next compute that

$$\eta \langle \nu, \sigma, \sigma \eta_5 \rangle = \langle \eta, \nu, \sigma \rangle \eta_5,$$

which equals zero because $\langle \eta, \nu, \sigma \rangle$ is zero. Therefore, $\{h_5c_1\}$ contains an element that is annihilated by η , so Lemma 4.1.4 says that there cannot be a hidden η extension on h_5c_1 .

LEMMA 4.2.52. There is no hidden η extension on C.

PROOF. Table 23 shows that $\{C\}$ equals $\langle \nu, \eta, \tau \eta \alpha \rangle$, where α is any element of $\{g_2\}$. Compute that

$$\eta \langle \nu, \eta, \tau \eta \alpha \rangle = \langle \eta, \nu, \eta \rangle \tau \eta \alpha = \nu^2 \cdot \tau \eta \alpha = 0.$$

This shows that $\{C\}$ contains an element that is annihilated by η , so Lemma 4.1.4 implies that there cannot be a hidden η extension on C.

LEMMA 4.2.53. There is a hidden η extension from $\tau^2 e_0 m$ to $d_0 u$.

PROOF. This follows immediately from the hidden τ extensions from h_1G_3 to $\tau^2 e_0 m$ and from $h_1^2 G_3$ to $d_0 u$ given in Table 25.

LEMMA 4.2.54. There is no hidden η extension on τi_1 .

PROOF. Suppose that there exists an element α of $\{\tau i_1\}$ such that $\eta \alpha$ belongs to $\{\tau^2 e_0^2 g\}$. Using the hidden τ extension from $\tau^2 h_1 e_0^2 g$ to $d_0 z$ given in Table 25, this would imply that $\tau \eta^2 \alpha$ equals $\{d_0 z\}$.

Recall from Table 23 that $\tau \eta^2 = \langle 2, \eta, 2 \rangle$. Then the shuffle

$$\tau \eta^2 \alpha = \langle 2, \eta, 2 \rangle \{\tau i_1\} = 2 \langle \eta, 2, \alpha \rangle$$

shows that $\{d_0z\}$ is divisible by 2. However, this is not possible.

LEMMA 4.2.55. There is a hidden η extension from $\tau^3 e_0^2 g$ to $d_0 z$.

PROOF. This follows immediately from the hidden τ extension from $\tau^2 h_1 e_0^2 g$ to $d_0 z$ given in Table 25.

LEMMA 4.2.56. There is no hidden η extension on h_1x' .

PROOF. We already showed in Lemma 4.2.55 that there is a hidden η extension from $\tau^3 e_0^2 g$ to $d_0 z$. Therefore, there cannot be a hidden η extension from $h_1 x'$ to $d_0 z$.

LEMMA 4.2.57. There is no hidden η extension on $h_3^2 g_2$.

PROOF. Note that $\sigma^2\{g_2\}$ is contained in $\{h_3^2g_2\}$, and $\eta\sigma^2$ is zero [42]. Therefore, $\{h_3^2g_2\}$ contains an element that is annihilated by η , and Lemma 4.1.4 implies that there cannot be a hidden η extension on $h_3^2g_2$.

LEMMA 4.2.58. There is no hidden η extension from $\tau h_1 Q_2$ to $\tau^2 d_0 w$.

PROOF. Classically, d_0w maps to a non-zero element in the E_{∞} -page of the Adams spectral sequence for tmf [15]. In tmf, this class cannot be the target of an η extension.

Lemma 4.2.59.

(1) Tentatively, there is a hidden η extension from $\tau d_0 w$ to $d_0 u' + \tau d_0^2 l$.

(2) Tentatively, there is a hidden η extension from $\tau^3 g^3$ to $d_0 e_0 r$.

- (3) Tentatively, there is a hidden η extension from $\tau^2 h_1 g^3$ to $d_0^2 e_0^2$.
- (4) Tentatively, there is a hidden η extension from d_0e_0r to $\tau^2 d_0^2 e_0^2$.
- (5) Tentatively, there is a hidden η extension from $\tau^2 g w$ to $\tau^2 d_0 e_0 m$.
- (6) Tentatively, there is a hidden η extension from $\tau^2 d_0 e_0 m$ to $d_0^2 u$.

PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

For the first formula, use the hidden τ extensions from $Ph_1^3h_5e_0$ to τd_0w given in Lemma 4.2.10 and from $\tau^2h_0g^3$ to $d_0u' + \tau d_0^2l$ given in Table 26. The second formula follows immediately from the hidden τ extension from $\tau^2h_1g^3$ to d_0e_0r given in Table 26. The third formula follows immediately from the hidden τ extension from $h_1^6h_5c_0e_0$ to $d_0^2e_0^2$ given in Table 26. The fourth formula follows immediately from the hidden τ extensions from $\tau^2h_1g^3$ to d_0e_0r and from $h_1^6h_5c_0e_0$ to $d_0^2e_0^2$ given in Table 26. The fifth formula follows immediately from the hidden τ extension from $h_1^5X_1$ to $\tau^2d_0e_0m$ given in Table 26. The sixth formula follows immediately from the hidden τ extensions from $h_1^5X_1$ to $\tau^2d_0e_0m$ and from h_0e_0gr to d_0^2u given in Table 26.

Lemma 4.2.60.

- (1) Tentatively, there is a hidden η extension from $\tau h_1 B_{23}$ to $B_8 d_0$.
- (2) Tentatively, there is a hidden η extension from $\tau^2 h_1 B_{23}$ to $d_0 x'$.

PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The first formula follows from the hidden η extension from $\tau h_1 g$ to $c_0 d_0$ given in Lemma 4.2.39, using that $\theta_{4.5}\{\tau h_1 g\}$ is contained in $\{\tau h_1 B_{23}\}$ by Lemma 4.2.94 and that $\theta_{4.5}\{c_0 d_0\}$ is contained in $\{B_8 d_0\}$ by Lemma 4.2.88.

The second formula follows from the first, using the hidden τ extension from B_8d_0 to d_0x' given in Lemma 4.2.14.

4.2.4. Hidden Adams ν extensions computations.

LEMMA 4.2.61. There is no hidden ν extension on $h_0h_2h_4$.

PROOF. This follows immediately from Lemma 4.2.15, where we showed that there is no hidden 2 extension on $h_2^2h_4$.

LEMMA 4.2.62.

- (1) There is a hidden ν extension from $h_0^2 g$ to $h_1 c_0 d_0$.
- (2) There is a hidden ν extension from $\tau h_0^2 g$ to $Ph_1 d_0$.
- (3) There is a hidden ν extension from $h_0^2 g^2$ to $h_1 c_0 e_0^2$.
- (4) There is a hidden ν extension from $\tau h_0^2 g^2$ to $h_1 d_0^3$.

PROOF. These follow immediately from the hidden 2 extensions established in Lemma 4.2.17. $\hfill \Box$

Lemma 4.2.63.

- (1) There is a hidden ν extension from h_2c_1 to $h_1^2h_4c_0$.
- (2) There is a hidden ν extension from h_2c_1g to $h_1^6h_5c_0$.
- (3) There is a hidden ν extension from N to $Ph_1^2h_5c_0$.

PROOF. Table 16 shows that $\langle h_2, h_2c_1, h_1 \rangle$ equals h_3g . This Massey product contains no permanent cycles because h_3g supports an Adams differential by Lemma 3.3.3. Therefore, (the contrapositive of) Moss's Convergence Theorem 3.1.1 implies that the Toda bracket $\langle \nu, \nu \overline{\sigma}, \eta \rangle$ is not well-defined. The only possibility is that $\nu^2 \overline{\sigma}$ is non-zero. This implies that there is a hidden ν extension on h_2c_1 , and the only possible target for this hidden extension is $h_1^2 h_4 c_0$. This finishes the first claim. The proof of the second claim is similar. Table 16 show that $\langle h_2, h_2c_1g, h_1 \rangle$ equals h_3g^2 . Since h_3g^2 supports a differential by Lemma 3.3.3, Moss's Convergence Theorem 3.1.1 implies that the Toda bracket $\langle \nu, \alpha, \eta \rangle$ is not well-defined for any α in $\{h_2c_1g\}$. This implies that there is a hidden ν extension on h_2c_1g , and the only possible target is $h_1^6h_5c_0$. This finishes the second claim.

For the third claim, we will first compute $\langle h_2, N, h_1 \rangle$ on the E_2 -page. The May differential $d_2(\Delta b_{21}h_1(1)) = h_2N$ and May's Convergence Theorem 2.2.1 imply that $\langle h_2, N, h_1 \rangle$ equals an element that is detected by G_3 in the E_{∞} -page of the May spectral sequence. Because of the presence of τgn in lower May filtration, the bracket equals either G_3 or $G_3 + \tau gn$. In any case, both of these elements support an Adams d_2 differential by Lemma 3.3.6 because τgn is a product of permanent cycles. Moss's Convergence Theorem 3.1.1 then implies that the Toda bracket $\langle \nu, \alpha, \eta \rangle$ is not well-defined for any α in $\{N\}$. This implies that there is a hidden ν extension on N, and the only possible target is $Ph_1^2h_5c_0$. This finishes the third claim. \Box

Lemma 4.2.64.

- (1) There is a hidden ν extension from $\tau h_2^2 g$ to $h_1 d_0^2$.
- (2) There is a hidden ν extension from $\tau h_2^2 g^2$ to $h_1 d_0 e_0^2$.

PROOF. These follow from the hidden τ extensions from $h_1^2 h_3 g$ to $h_1 d_0^2$ and from $h_1^2 h_3 g^2$ to $h_1 d_0 e_0^2$ given in Table 25.

LEMMA 4.2.65. There is no hidden ν extension on h_1h_5 .

PROOF. The only other possibility is that there is a hidden ν extension from h_1h_5 to $\tau^2h_1e_0^2$. We know from Table 24 that $\{\tau^2h_1e_0^2\}$ contains $\nu\{q\}$. Since $\{q\}$ belongs to the indeterminacy of $\{h_1h_5\}$, there exists an element of $\{h_1h_5\}$ that is annihilated by ν . Lemma 4.1.4 finishes the proof.

LEMMA 4.2.66. There is no hidden ν extension on p.

PROOF. Recall that there is a hidden ν extension from h_4^2 to p, as shown in Table 24. If there were a hidden ν extension from p to t, then $\nu^4 \theta_4$ would belong to $\{\tau h_2 c_1 g\}$. This is impossible since ν^4 is zero.

LEMMA 4.2.67. There is no hidden ν extension on x.

PROOF. Recall from Table 24 that there is a classical hidden σ extension from h_4^2 to x. Therefore, $\sigma \theta_4$ belongs to $\{x\}$ motivically as well, so x cannot support a hidden ν extension.

LEMMA 4.2.68. There is no hidden ν extension on $h_0^2 h_3 h_5$.

PROOF. The only other possibility is that there is a hidden ν extension from $h_0^2h_3h_5$ to $Ph_1^2h_5$ or to z. However, from Lemma 4.2.39, both $\{Ph_1^2h_5\}$ and $\{z\}$ support multiplications by η . Therefore, neither $Ph_1^2h_5$ nor z can be the target of a hidden ν extension.

Lemma 4.2.69.

- (1) There is no hidden ν extension on $h_1h_3h_5$.
- (2) There is no hidden ν extension on h_3d_1 .
- (3) There is no hidden ν extension on $\tau^2 c_1 g$.
- (4) There is no hidden ν extension on h_3g_2 .
PROOF. In each case, the possible source of the hidden extension detects an element that is divisible by σ . Therefore, each possible source cannot support a hidden ν extension.

Note that $h_3q = \tau^2 c_1 g$ on the E_2 -page [9].

LEMMA 4.2.70. There is no hidden ν extension on h_5c_0 .

PROOF. First, Table 23 shows that $\{h_5c_0\}$ contains $\langle \epsilon, 2, \theta_4 \rangle$. Then shuffle to obtain

$$\nu \langle \epsilon, 2, \theta_4 \rangle = \langle \nu, \epsilon, 2 \rangle \theta_4.$$

Since $\langle \nu, \epsilon, 2 \rangle$ is zero, there is an element of $\{h_5 c_0\}$ that is annihilated by ν . Lemma 4.1.4 finishes the proof. \square

LEMMA 4.2.71.

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- There is a hidden ν extension from u to τd₀³.
 There is a hidden ν extension from τw to τ²d₀e₀².

PROOF. First shuffle to compute that

$$\nu \langle \eta, \nu, \{\tau^2 e_0^2\} \rangle = \langle \nu, \eta, \nu \rangle \{\tau^2 e_0^2\} = (\epsilon + \eta \sigma) \{\tau^2 e_0^2\}.$$

This last expression equals $\tau^2 \{c_0 e_0^2\}$, which equals $\{\tau d_0^3\}$ because of the hidden τ extension from $c_0 e_0^2$ to d_0^3 given in Table 25.

Therefore, $\{\tau d_0^3\}$ is divisible by ν . Lemmas 4.2.69 and 4.2.70 eliminate most of the possibilities. The only remaining possibility is that there is a hidden ν extension on u. This establishes the first claim.

The proof of the second claim is similar. Shuffle to compute that

$$\nu \langle \eta, \nu, \tau \overline{\kappa}^2 \rangle = \langle \nu, \eta, \nu \rangle \tau \overline{\kappa}^2 = (\epsilon + \eta \sigma) \tau \overline{\kappa}^2 = \tau \epsilon \overline{\kappa}^2.$$

By Lemma 4.2.85, this last expression is detected by $\tau^2 d_0 e_0^2$.

Therefore, $\{\tau^2 d_0 e_0^2\}$ is divisible by ν . Because of Lemmas 4.2.72 and 4.2.73, the only possibility is that there is a hidden ν extension from τw to $\tau^2 d_0 e_0^2$.

Lemma 4.2.72.

- (1) There is no hidden ν extension on $Ph_0h_2h_5$.
- (2) There is no hidden ν extension on h_0q_2 .

(3) There is no hidden ν extension on $h_0h_5d_0$.

PROOF. These follow immediately from Lemmas 4.2.23, 4.2.24, and 4.2.27. \Box

Lemma 4.2.73.

- (1) There is a hidden ν extension from $h_3^2h_5$ to B_2 .
- (2) There is a hidden ν extension from $h_0h_3^2h_5$ to h_0B_2 .

PROOF. The proof is similar in spirit to the proof of Lemma 4.2.48. Table 16 shows that $\langle h_2, h_0^2 g_2, h_0 \rangle$ equals $\{B_2, B_2 + h_0^2 h_5 e_0\}$. Then Moss's Convergence Theorem 3.1.1 implies that $\langle \nu, \sigma^2 \theta_4, 2 \rangle$ intersects $\{B_2\}$. Here we are using that $\sigma^2 \theta_4$ belongs to $\{h_0^2g_2\}$, as shown in the proof of Lemma 4.2.48.

This bracket contains $\langle \nu, \sigma^2, 2\theta_4 \rangle$, which contains zero since $2\theta_4$ is zero. It follows that $\{B_2\}$ contains an element in the indeterminacy of $\langle \nu, \sigma^2 \theta_4, 2 \rangle$. The only possibility is that there is a hidden ν extension from $h_3^2 h_5$ to B_2 . This finishes the proof of the first hidden extension.

The second hidden extension follows immediately from the first.

LEMMA 4.2.74. There is no hidden ν extension on B_1 .

PROOF. We showed in Lemma 4.2.48 that $\{B_1\}$ contains an element that is divisible by η . Therefore, B_1 cannot support a hidden ν extension.

Lemma 4.2.75.

- (1) There is a hidden ν extension from h_2B_2 to h_1B_8 .
- (2) There is a hidden ν extension from $\tau h_2 B_2$ to $h_1 x'$.

PROOF. As discussed in Section 1.7, $\sigma\theta_{4.5}$ is detected in Adams filtration greater than 6. Thus, $\eta^2 \sigma \theta_{4.5}$ is zero, even though $\sigma \theta_{4.5}$ itself could possibly be detected by $\tau^2 d_1 g$ or $\tau^2 e_0 m$.

Recall from Table 34 that $\eta^2 \sigma + \nu^3 = \eta \epsilon$. Therefore, $\nu^3 \theta_{4.5}$ equals $\eta \epsilon \theta_{4.5}$. Lemma 4.2.88 implies that $\eta \epsilon \theta_{4.5}$ is detected by $h_1 B_8$, so $\{h_1 B_8\}$ contains an element that is divisible by ν . The only possibility is that there must be a hidden ν extension from $h_2 B_2$ to $h_1 B_8$. This establishes the first claim.

The second claim follows easily from the first, using the hidden τ extension from h_1B_8 to h_1x' given in Table 25.

Lemma 4.2.76.

- (1) There is a hidden ν extension from h_1G_3 to $\tau^2 h_1 e_0^2 g$.
- (2) There is a hidden ν extension from $\tau^2 e_0 m$ to $d_0 z$.

PROOF. Table 23 shows that $\langle \{q\}, \eta^3, \eta_4 \rangle$ equals $\{h_1G_3\}$, and $\langle \{\tau^2h_1e_0^2\}, \eta^3, \eta_4 \rangle$ equals $\{\tau^2h_1e_0^2g\}$. Neither Toda bracket has indeterminacy; for the second bracket, one needs that $\eta_4\{t\}$ is contained in

$$\langle \eta, \sigma^2, 2 \rangle \{t\} = \eta \langle \sigma^2, 2, \{t\} \rangle,$$

which must be zero.

Now compute that

$$\nu\{h_1G_3\} = \nu\langle\{q\}, \eta^3, \eta_4\rangle = \langle\nu\{q\}, \eta^3, \eta_4\rangle = \langle\{\tau^2h_1e_0^2\}, \eta^3, \eta_4\rangle = \{\tau^2h_1e_0^2g\}.$$

Here we are using that none of the Toda brackets has indeterminacy, and we are using Table 24 to identify $\nu\{q\}$ with $\{\tau^2 h_1 e_0^2\}$. This establishes the first claim.

The second claim follows easily from the first, using the hidden τ extensions from h_1G_3 to $\tau^2 e_0 m$ and from $\tau^2 h_1 e_0^2 g$ to $d_0 z$ given in Table 25.

LEMMA 4.2.77. There is no hidden ν extension on $\tau^2 d_1 g$.

PROOF. We showed in Lemma 4.2.76 that there is a hidden ν extension from $\tau^2 e_0 m$ to $d_0 z$. Therefore, there cannot be a hidden ν extension from $\tau^2 d_1 g$ to $d_0 z$.

LEMMA 4.2.78. There is a hidden ν extension from $h_1^6 h_5 e_0$ to $h_2 e_0^2 g$.

PROOF. This follows immediately from the hidden τ extension from $h_1^6 h_5 e_0$ to $\tau e_0^2 g$ given in Table 25.

LEMMA 4.2.79. Tentatively, there is a hidden ν extension from $h_0h_2h_5i$ to $\tau^2 d_0^2 l$.

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

As explained in the proof of Lemma 4.2.35, the class h_0h_5i dectects an element of classical π_{54} that maps to an element of π_{54} tmf that is detected by $\Delta^2 h_2^2$ in the Adams-Novikov spectral sequence for tmf. Then $h_0h_2h_5i$ detects an element in π_{57} that maps that maps to an element of π_{57} tmf that is detected by $\Delta^2 h_2^3$ in the Adams-Novikov spectral sequence for tmf.

In the classical Adams-Novikov spectral sequence for tmf, there is a hidden ν extension from $\Delta^2 h_2^3$ to $2g^3$ [5]. Therefore, the corresponding hidden extension must occur in the motivic Adams spectral sequence as well.

Lemma 4.2.80.

- (1) Tentatively, there is a hidden ν extension from $Ph_1^3h_5e_0$ to $\tau d_0^2e_0^2$.
- (2) Tentatively, there is a hidden ν extension from $\tau d_0 w$ to $\tau^2 d_0^2 e_0^2$.
- (3) Tentatively, there is a hidden ν extension from $\tau gw + h_1^4 X_1$ to $\tau^2 e_0^4$.

PROOF. The claims are tentative because our analysis of Adams differentials is incomplete in the relevant range.

The second formula follows from the hidden ν extension from τw to $\tau^2 d_0 e_0^2$ given in Lemma 4.2.71. The first formula then follows using the hidden τ extension from $Ph_1^3h_5e_0$ given in Lemma 4.2.10.

For the third formula, start with the hidden ν extension from τw to $\tau^2 d_0 e_0^2$. Multiply by τg to obtain a hidden ν extension from $\tau^2 g w$ to $\tau^3 e_0^4$. The third formula follows immediately.

LEMMA 4.2.81. Tentatively, there is a hidden ν extension from $\tau h_0^2 g^3$ to $h_1 d_0^2 e_0^2$.

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

This follows immediately from the hidden τ extension from $h_0^2 h_2 g^3$ to $h_1 d_0^2 e_0^2$ given in Table 26.

LEMMA 4.2.82. Tentatively, there is a hidden ν extension from $h_2c_1g^2$ to $h_1^8D_4$.

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

The argument is essentially the same as the proof of Lemma 4.2.63. Table 16 shows that $\langle h_2, h_2c_1g^2, h_1 \rangle$ equals h_3g^3 . Since h_3g^3 supports a differential by Lemma 3.3.3, Moss's Convergence Theorem 3.1.1 implies that the Toda bracket $\langle \nu, \alpha, \eta \rangle$ is not well-defined for any α in $\{h_2c_1g^2\}$. This implies that there is a hidden ν extension on $h_2c_1g^2$, and the only possible target is $h_1^8D_4$.

4.2.5. Miscellaneous Adams hidden extensions. In this section, we include some miscellaneous hidden extensions. They are needed at various points for technical arguments, but they are interesting for their own sakes as well.

LEMMA 4.2.83. There is a hidden σ extension from h_1h_4 to h_4c_0 .

PROOF. The product $\eta \epsilon \eta_4$ is contained in $\{h_1^2 h_4 c_0\}$. Now recall the hidden relation $\eta \epsilon = \eta^2 \sigma + \nu^3$ from Table 34. Also $\nu \eta_4$ is zero because there is no other possibility. Therefore, $\eta^2 \sigma \eta_4$ is contained in $\{h_1^2 h_4 c_0\}$. It follows that $\sigma \eta_4$ is contained in $\{h_4 c_0\}$.

LEMMA 4.2.84. There is no hidden σ extension on $h_1h_3h_5$.

PROOF. The element $\sigma\eta_5$ belongs to $\{h_1h_3h_5\}$. We will show that $\sigma^2\eta_5$ is zero and then apply Lemma 4.1.4.

Table 23 shows that η_5 belongs to $\langle \eta, 2, \theta_4 \rangle$. Then $\sigma^2 \eta_5$ belongs to

$$\sigma^2 \langle \eta, 2, \theta_4 \rangle = \langle \sigma^2, \eta, 2 \rangle \theta_4.$$

Finally, we must show that $\langle \sigma^2, \eta, 2 \rangle$ is zero in $\pi_{16,9}$. First shuffle to obtain

$$\langle \sigma^2, \eta, 2 \rangle \eta = \sigma^2 \langle \eta, 2, \eta \rangle.$$

Table 23 shows that $\langle \eta, 2, \eta \rangle$ equals $\{2\nu, 6\nu\}$, so $\sigma^2 \langle \eta, 2, \eta \rangle$ is zero. Since multiplication by η is injective on $\pi_{16,9}$, this shows that $\langle \sigma^2, \eta, 2 \rangle$ is zero.

LEMMA 4.2.85.

(1) There is a hidden ϵ extension from τg to d_0^2 .

(2) There is a hidden ϵ extension from τg^2 to $d_0 e_0^2$.

PROOF. Table 23 shows that ϵ is contained in $\langle 2\nu, \nu, \eta \rangle$. Therefore, $\eta \epsilon \overline{\kappa}$ equals $\langle 2\nu, \nu, \eta \rangle \eta \overline{\kappa}$ with no indeterminacy. This expression equals $\langle 2\nu, \nu, \eta^2 \overline{\kappa} \rangle$ because the latter still has no indeterminacy.

Lemma 4.2.39 tells us that we can rewrite this bracket as $\langle 2\nu, \nu, \{c_0d_0\}\rangle$, which equals $\langle 2\nu, \nu, \epsilon \rangle \kappa$. Table 23 shows that $\langle 2\nu, \nu, \epsilon \rangle$ equals $\eta \kappa$.

Therefore, $\eta \epsilon \overline{\kappa}$ equals $\eta \kappa^2$. It follows that $\epsilon \overline{\kappa}$ equals κ^2 . This establishes the first claim.

The argument for the second claim is essentially the same. Start with $\eta \epsilon \{\tau g^2\} = \langle 2\nu, \nu, \eta \rangle \eta \{\tau g^2\}$. This equals $\langle 2\nu, \nu, \{c_0 e_0^2\} \rangle$, which is the same as $\{h_1 d_0 e_0^2\}$. This shows that $\eta \epsilon \{\tau g^2\}$ equals $\eta \{d_0 e_0^2\}$, so $\epsilon \{\tau g^2\}$ equals $\{d_0 e_0^2\}$.

REMARK 4.2.86. Based on the calculations in Lemma 4.2.85, one might expect that there is a hidden ϵ extension from $\tau g^3 + h_1^4 h_5 c_0 e_0$ to e_0^4 .

LEMMA 4.2.87. There is a hidden ϵ extension from q to h_1u .

PROOF. This proof follows the argument of the proof of [25, Lemma 2.1], which we include for completeness.

First, recall from Table 23 that $\epsilon + \eta \sigma$ equals $\langle \nu, \eta, \nu \rangle$. Then $(\epsilon + \eta \sigma)\{q\}$ equals $\langle \nu, \eta, \nu \rangle \{q\}$, which is contained in $\langle \nu, \eta, \nu \{q\} \rangle$. It follows from Table 24 that $\nu \{q\}$ equals $\tau \eta \kappa \overline{\kappa}$, so $(\epsilon + \eta \sigma)\{q\}$ belongs to $\langle \nu, \eta, \tau \eta \kappa \overline{\kappa} \rangle$.

On the other hand, this bracket contains $\langle \nu, \eta, \tau \eta \kappa \rangle \overline{\kappa}$. Table 23 shows that $\langle \nu, \eta, \tau \eta \kappa \rangle$ equals $\{\tau h_0 g\} = \{2\overline{\kappa}, 6\overline{\kappa}\}$, and $4\overline{\kappa}^2$ is zero. Therefore, $\langle \nu, \eta, \tau \eta \kappa \rangle \overline{\kappa}$ equals $2\overline{\kappa}^2$.

This shows that the difference $(\epsilon + \eta \sigma) \{q\} - 2\overline{\kappa}^2$ is contained in the indeterminacy of the bracket $\langle \nu, \eta, \tau \eta \kappa \overline{\kappa} \rangle$. The indeterminacy of this bracket consists of multiples of ν .

Each of the terms in $(\epsilon + \eta \sigma) \{q\} - 2\overline{\kappa}^2$ is in Adams filtration at least 9, and there are no multiples of ν in those filtrations. Therefore, $(\epsilon + \eta \sigma) \{q\}$ equals $2\overline{\kappa}^2$.

We now need to show that $\eta\sigma\{q\}$ is zero. Because $h_3q = h_2t$ in Ext, we know that $\sigma\{q\} + \nu\{t\}$ either equals zero or $\{u\}$. Note that $\kappa(\sigma\{q\} + \nu\{t\})$ is zero, while $\kappa\{u\} = \{d_0u\}$ is non-zero. Therefore, $\sigma\{q\} + \nu\{t\}$ equals zero, and $\eta\sigma\{q\}$ is zero as well.

LEMMA 4.2.88. There is a hidden ϵ extension from $h_3^2 h_5$ to B_8 .

PROOF. First, there is a relation $h_1B_8 = c_0B_1$ on the E_2 -page, which is not hidden in the May spectral sequence. Since B_1 detects $\eta\theta_{4.5}$ by definition of $\theta_{4.5}$ (see Section 1.7), we get that h_1B_8 detects $\eta\epsilon\theta_{4.5}$ and that B_8 detects $\epsilon\theta_{4.5}$.

On the E_{∞} -page, we have the relation $h_2^3 h_5 = h_1^2 h_3 h_5$ in the 40-stem. We will next show that this relation gives rise to a compound hidden extension that is analogous to Toda's relation $\nu^3 + \eta^2 \sigma = \eta \epsilon$ (see Table 34). the Note that the element $\epsilon \eta_5$ is detected by $h_1 h_5 c_0$, whose Adams filtration is higher than the Adams filtration of $h_2^3 h_5 = h_1^2 h_3 h_5$.

LEMMA 4.2.89. $\nu \{h_2^2 h_5\} + \eta \sigma \eta_5$ equals $\epsilon \eta_5$.

PROOF. Table 23 shows that $\langle \nu^2, 2, \theta_4 \rangle$ equals $\{h_2^2 h_5\}$. Note that $\{x\}$ belongs to the indeterminacy, since there is a hidden σ extension from h_4^2 to x as shown in Table 24.

Similarly, $\langle \nu^3, 2, \theta_4 \rangle$ intersects $\{h_2^3 h_5\}$, with no indeterminacy. In order to compute the indeterminacy, we need to know that $\eta \mu_9 \theta_4$ is zero. This follows from the calculation

$$\eta \theta_4 \langle \eta, 2, 8\sigma \rangle = \langle \eta \theta_4, \eta, 2 \rangle 8\sigma = 0.$$

Table 23 also shows that $\langle \eta, 2, \theta_4 \rangle$ equals $\{\eta_5, \eta_5 + \eta \rho_{31}\}$. With these tools, compute that

$$\nu\{h_2^2h_5\} = \nu\langle\nu^2, 2, \theta_4\rangle = \langle\nu^3, 2, \theta_4\rangle$$

because there is no indeterminacy in the last bracket. This equals $\langle \eta^2 \sigma + \eta \epsilon, 2, \theta_4 \rangle$, which equals $(\eta \sigma + \epsilon) \langle \eta, 2, \theta_4 \rangle$, again because there is no indeterminacy. Finally, this last expression equals $\eta \sigma \eta_5 + \epsilon \eta_5$.

LEMMA 4.2.90. There is a hidden ν_4 extension from h_4^2 to $h_2h_5d_0$.

PROOF. Table 23 shows that $\langle \sigma, \nu, \sigma \rangle$ consists of a single element α contained in $\{h_2h_4\}$. Then α must be of the form $k\nu_4$ or $k\nu_4 + \eta\mu_{17}$ where k is odd. Since $2\theta_4$ and $\eta\mu_{17}\theta_4$ are both zero, we conclude that $\langle \sigma, \nu, \sigma \rangle \theta_4$ equals $\nu_4\theta_4$.

Table 24 shows that $\sigma \theta_4$ equals $\{x\}$. Therefore, $\nu_4 \theta_4$ is contained in $\langle \sigma, \nu, \{x\} \rangle$. Next compute that $h_2 h_5 d_0 = \langle h_3, h_2, x \rangle$ with no indeterminacy. This follows from the shuffle

$$h_2\langle h_3, h_2, x \rangle = \langle h_2, h_3, h_2 \rangle x = h_3^2 x = h_2^2 h_5 d_0.$$

Then Moss's Convergence Theorem 3.1.1 implies that the Toda bracket $\langle \sigma, \nu, \{x\} \rangle$ intersects $\{h_2h_5d_0\}$. The indeterminacy in $\langle \sigma, \nu, \{x\} \rangle$ is concentrated in Adams filtration strictly greater than 6, so $\langle \sigma, \nu, \{x\} \rangle$ is contained in $\{h_2h_5d_0\}$. This shows that $\nu_4\theta_4$ is contained in $\{h_2h_5d_0\}$.

LEMMA 4.2.91. $\langle \theta_4, 2, \sigma^2 \rangle$ is contained in $\{h_0h_3^2h_5\}$, with indeterminacy generated by $\rho_{15}\theta_4$ in $\{h_0^2h_5d_0\}$.

PROOF. Table 23 shows that ν_4 is contained in $\langle 2, \sigma^2, \nu \rangle$. Therefore, $\nu_4 \theta_4$ is contained in $\langle \theta_4, 2, \sigma^2 \rangle \nu$. On the other hand, Lemma 4.2.90 says that $\nu_4 \theta_4$ is contained in $\{h_2 h_5 d_0\}$.

We have now shown that $\langle \theta_4, 2, \sigma^2 \rangle$ contains an element α such that $\nu \alpha$ belongs to $\{h_2h_5d_0\}$. In particular, α has Adams filtration at most 5. In addition, we know that 2α is zero because of the shuffle

$$\langle \theta_4, 2, \sigma^2 \rangle 2 = \theta_4 \langle 2, \sigma^2, 2 \rangle = 0$$

Here we have used Table 23 for the bracket $\langle 2, \sigma^2, 2 \rangle$. The only possibility is that α belongs to $\{h_0h_3^2h_5\}$.

The indeterminacy follows immediately from [41, Corollary 2.8].

LEMMA 4.2.92. There is a hidden η_4 extension from h_4^2 to $h_1h_5d_0$.

PROOF. Table 23 shows that η_4 belongs to the Toda bracket $\langle \eta, \sigma^2, 2 \rangle$. Then $\eta_4 \theta_4$ belongs to $\eta \langle \sigma^2, 2, \theta_4 \rangle$.

Recall from the proof of Lemma 4.2.91 that $\langle \sigma^2, 2, \theta_4 \rangle$ consists of elements α in $\{h_0h_3^2h_5\}$ of order 2. Table 24 shows that there is a hidden 4 extension from $h_3^2h_5$ to $h_0h_5d_0$. It follows that each α must be of the form $2\gamma - \beta$, where γ belongs to $\{h_3^2h_5\}$ and β belongs to $\{h_5d_0\}$. Then $\eta\alpha = \eta\beta$ must belong to $\{h_1h_5d_0\}$. This shows that $\eta_4\theta_4$ belongs to $\{h_1h_5d_0\}$.

LEMMA 4.2.93. There is a hidden κ extension from either $h_3^2h_5$ or h_5d_0 to B_{21} .

PROOF. The element $\tau h_1 B_{21}$ may be the target of an Adams differential. Regardless, the element $h_1 B_{21}$ is non-zero on the E_{∞} -page. Note that $h_1 B_{21}$ equals $d_0 B_1$ on the E_2 -page [9]. Since B_1 detects $\eta \theta_{4.5}$ by definition of $\theta_{4.5}$ (see Section 1.7), $d_0 B_{21}$ detects $\eta \kappa \theta_{4.5}$. This implies that B_{21} detects $\kappa \theta_{4.5}$.

Therefore, B_{21} must be the target of a hidden κ extension. The possible sources of this hidden extension are $h_3^2 h_5$ or $h_5 d_0$.

LEMMA 4.2.94. Tentatively, there is a hidden $\overline{\kappa}$ extension from either $h_3^2h_5$ or h_5d_0 to τB_{23} .

PROOF. The claim is tentative because our analysis of Adams differentials is incomplete in the relevant range.

The element $\tau h_1 B_{23}$ equals $\tau g B_1$ on the E_2 -page [9]. Since B_1 detects $\eta \theta_{4.5}$ by definition of $\theta_{4.5}$ (see Section 1.7), $\tau g B_1$ detects $\eta \overline{\kappa} \theta_{4.5}$. Therefore, τB_{23} detects $\overline{\kappa} \theta_{4.5}$.

It follows that τB_{23} is the target of a hidden $\overline{\kappa}$ extension. The possible sources for this hidden extension are $h_3^2 h_5$ or $h_5 d_0$.

REMARK 4.2.95. Lemma 4.2.94 is tentative because there are unknown Adams differentials in the relevant range.

CHAPTER 5

The cofiber of τ

The purpose of this chapter is to compute the motivic stable homotopy groups of the cofiber $C\tau$ of τ . We obtain nearly complete results up to the 63-stem, and we have partial results up to the 70-stem. The Adams charts for $C\tau$ in [19] are essential companions to this chapter.

The element τ realizes to 1 in the classical stable homotopy groups. Therefore, $C\tau$ is an "entirely exotic" object in motivic stable homotopy, since it realizes classically to the trivial spectrum.

There are two main motivations for this calculation. First, it is the key to resolving hidden τ extensions that were discussed in Section 4.1.2. Second, we will show in Proposition 6.2.5 that the motivic homotopy groups of $C\tau$ are isomorphic to the classical Adams-Novikov E_2 -page. Thus the calculations in this chapter will allow us to reverse-engineer the classical Adams-Novikov spectral sequence.

The computational method will be the motivic Adams spectral sequence [13] [17] [33] for $C\tau$, which takes the form

$$E_2 = \operatorname{Ext}_A(H^{*,*}(C\tau); \mathbb{M}_2) \Rightarrow \pi_{*,*}(C\tau).$$

We write $E_2(C\tau)$ for this E_2 -page $\operatorname{Ext}_A(H^{*,*}(C\tau); \mathbb{M}_2)$. See [17] for convergence properties of this spectral sequence.

Outline. The first step in executing the motivic Adams spectral sequence for $C\tau$ is to algebraically compute the E_2 -page, i.e., $\operatorname{Ext}_A(H^{*,*}(C\tau), \mathbb{M}_2)$. We carry this out in Section 5.1 using the long exact sequence

$$\longrightarrow \operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \xrightarrow{\tau} \operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2) \longrightarrow \operatorname{Ext}_A(H^{*,*}(C\tau), \mathbb{M}_2) \longrightarrow .$$

Some additional work is required in resolving hidden extensions for the action of $\operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$ on $E_2(C\tau)$.

The next step is to compute the Adams differentials. In Section 5.2, we use a variety of methods to obtain these computations. The most important is to borrow results about differentials in the motivic Adams spectral sequence for $S^{0,0}$ from Tables 8, 20, 21, and 22. In addition, there are several computations that require analyses of brackets and hidden extensions.

The complete understanding of the Adams differentials allows for the computation of the E_{∞} -page of the motivic Adams spectral sequence for $C\tau$. The final step, carried out in Section 5.3, is to resolve hidden extensions by 2, η , and ν in $\pi_{*,*}(C\tau)$.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 35, 36, and 37 give extensions by h_0 , h_1 , and h_2 in $E_2(C\tau)$ that are hidden in the long exact sequence that computes $E_2(C\tau)$. The fourth columns of these tables refer to one argument that establishes each hidden extension. This takes one of the following forms:

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- (1) An explicit proof given elsewhere in this manuscript.
- (2) A May differential that computes a Massey product of the form $\langle h_i, x, \tau \rangle$ via May's Convergence Theorem 2.2.1. This Massey product implies the hidden extension in $E_2(C\tau)$ by Proposition 5.0.1.

Table 38 gives some additional miscellaneous hidden extensions in $E_2(C\tau)$, again with references to a proof.

Table 39 lists the generators of $E_2(C\tau)$ as a module over $\operatorname{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$. Table 40 lists all examples of generators of $E_2(C\tau)$ for which there is some ambiguity. See Section 5.1.5 for more explanation.

Tables 39 and 41 provide the values of d_2 and d_3 differentials in the Adams spectral sequence for $C\tau$. The fourth columns of these tables refer to one argument that establishes each differential. This takes one of the following forms:

- (1) An explicit proof given elsewhere in this manuscript.
- (2) "top cell" means that the differential is detected by projection $E_r(C\tau) \rightarrow E_r(S^{0,0})$ to the top cell.
- (3) Some differentials can be established with an algebraic relation to another differential that is detected by the inclusion $E_r(S^{0,0}) \to E_r(C\tau)$ of the bottom cell.

Table 42 describes the part of the projection $\pi_{*,*}(C\tau) \to \pi_{*,*}$ to the top cell that are hidden by the map $E_{\infty}(C\tau) \to E_{\infty}(S^{0,0})$ of Adams E_{∞} -pages. See Proposition 5.2.26 for more explanation.

Table 43 gives the extensions by 2, η , and ν in $\pi_{*,*}(C\tau)$ that are hidden in $E_{\infty}(C\tau)$. The fourth column refers to one argument that establishes each hidden extension. This takes one of the following forms:

- (1) An explicit proof given elsewhere in this manuscript.
- (2) "top cell" means that the hidden extension is detected by the projection $\pi_{*,*}(C\tau) \to \pi_{*,*}(S^{0,0})$ to the top cell.
- (3) "bottom cell" means that the hidden extension is detected by the inclusion $\pi_{*,*} \to \pi_{*,*}(C\tau)$ of the bottom cell.

Massey products and cofibers. We will rely heavily on Massey products and Toda brackets, using the well-known relationship between Toda brackets and hidden extensions in the homotopy groups of a cofiber. See Proposition 3.1.6 for an explicit statement. We will also need a similar result for Massey products.

PROPOSITION 5.0.1. Let y and z belong to $E_2(S^{0,0})$ such that τy and zy are both zero. In $E_2(C\tau)$, there is a hidden extension

$$z \cdot \overline{y} \in \langle z, y, \tau \rangle,$$

where the Massey product is computed in $E_2(S^{0,0})$ and then pushed forward along the map $E_2(S^{0,0}) \to E_2(C\tau)$.

PROOF. The proof is identical to the proof of Proposition 3.1.6, except that we work in the derived category of chain complexes of A-modules instead of the motivic stable homotopy category. In this derived category, the cofiber of $\tau : \mathbb{M}_2 \to \mathbb{M}_2$ is $H^{*,*}(C\tau)$.

5.1. The Adams E_2 -page for the cofiber of τ

The main tool for computing $E_2(C\tau) = \operatorname{Ext}_A(H^{*,*}(C\tau), \mathbb{M}_2)$ is the long exact sequence

$$\cdots \longrightarrow E_2(S^{0,0}) \xrightarrow{\tau} E_2(S^{0,0}) \longrightarrow E_2(C\tau) \longrightarrow \cdots$$

associated to the cofiber sequence

 $S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1}.$

This yields a short exact sequence

$$0 \longrightarrow \operatorname{coker}(\tau) \longrightarrow E_2(C\tau) \longrightarrow \operatorname{ker}(\tau) \longrightarrow 0.$$

The desired $E_2(C\tau)$ is almost completely described by the previous short exact sequence. It only remains to compute some hidden extensions.

5.1.1. Hidden extensions in the Adams E_2 -page for the cofiber of τ . We will resolve all possible hidden extensions by h_0 , h_1 , and h_2 through the 70stem. The reader should refer to the charts in [19] in order to make sense of the following results.

THEOREM 5.1.1. Tables 35, 36, and 37 give some hidden extensions by h_0 , h_1 , and h_2 in $E_2(C\tau)$. Through the 70-stem, all other possible hidden extensions by $h_0, h_1, and h_2$ are either zero or are easily implied by extensions in the tables, with the possible exceptions that:

- (1) $h_2 \cdot \overline{\tau^2 h_1 g^2}$ might equal τw . (2) $h_0 \cdot \overline{c_0 Q_2}$ and $h_2 \cdot \overline{c_0 Q_2}$ are either both zero, or equal D'_2 and P(A + A')respectively.
- (3) $h_1^3 c_0 \cdot \overline{D_4}$ equals either $h_2 B_5$ or $h_2 B_5 + h_1^2 X_3$.

EXAMPLE 5.1.2. In the 14-stem, there is a hidden extension $h_2 \cdot h_1^2 c_0 = h_0 d_0$, which does not appear in Table 37. This is easily implied by the hidden extension $h_0 \cdot h_1^2 c_0 = P h_2$, which does appear in Table 35.

PROOF. Most of these hidden extensions are established with Proposition 5.0.1, so we just need to compute Massey products of the form $\langle h_i, x, \tau \rangle$ in $E_2(S^{0,0})$. Most of these Massey products are computed using May's Convergence Theorem 2.2.1. The fourth columns of Tables 35, 36, and 37 indicate which May differentials are relevant for computing each bracket.

A few hidden extensions require more complicated proofs. These proofs are given in the following lemmas.

5.1.2. Hidden h_0 extensions in the Adams E_2 -page for the cofiber of au.

LEMMA 5.1.3.

- (1) $h_0 \cdot \overline{c_0 e_0} = j$. (2) $h_0 \cdot \overline{P^k c_0 e_0} = P^k j.$
- (3) $h_0 \cdot \overline{c_0 e_0 q} = d_0 l$.

PROOF. We prove the first formula. The proofs for the other formulas are essentially the same.

By Proposition 5.0.1, we must compute $\langle h_0, c_0 e_0, \tau \rangle$ in $E_2(S^{0,0})$. We may attempt to compute this bracket using May's Convergence Theorem 2.2.1 with the May differential $d_2(b_{30}h_0(1)^2) = \tau c_0 e_0$. However, the hypothesis of May's Convergence Theorem 2.2.1 is not satisfied because of the later May differential $d_4(\Delta h_1^2) = Ph_1^2h_4$.

Instead, note that $h_2^2 \langle h_0, c_0 e_0, \tau \rangle$ equals $\langle h_2^2, h_0, c_0 e_0 \rangle \tau$. Table 16 shows that the last bracket equals $h_1 d_0 e_0$.

Therefore, $h_2^2 \langle h_0, c_0 e_0, \tau \rangle$ equals $\tau h_1 d_0 e_0$. It follows that $\langle h_0, c_0 e_0, \tau \rangle$ equals j.

LEMMA 5.1.4. $h_0 \cdot \overline{h_1 d_1 g} = h_1 h_5 c_0 d_0$.

PROOF. By Proposition 5.0.1, we must compute $\langle h_0, h_1 d_1 g, \tau \rangle$ in $E_2(S^{0,0})$. Because there is no indeterminacy, we have

$$\langle h_0, h_1 d_1 g, \tau \rangle = \langle h_0, d_1, \tau h_1 g \rangle = \langle h_0, d_1, h_2 f_0 \rangle = \langle h_0, d_1, f_0 \rangle h_2.$$

Table 16 shows that $h_2B_2 = \langle h_0, d_1, f_0 \rangle$. Finally, use that $h_2 \cdot h_2B_2 = h_1h_5c_0d_0$ from Table 14.

LEMMA 5.1.5. $h_0 \cdot \overline{h_1^2 B_8} = h_2 x'.$

PROOF. By Proposition 5.0.1, we must compute the bracket $\langle h_0, h_1^2 B_8, \tau \rangle$, which equals $\langle h_0, h_1, \tau h_1 B_8 \rangle$ because there is no indeterminacy. Table 16 shows that $\langle h_0, h_1, \tau h_1 B_8 \rangle$ equals $h_2 x'$. Note that $\tau h_1 B_8 = P h_1 h_5 d_0$ from Table 11. \Box

5.1.3. Hidden h_1 extensions in the Adams E_2 -page for the cofiber of τ .

LEMMA 5.1.6. $h_1 \cdot \overline{\tau h_0 e_0^3} = d_0 u.$

PROOF. Using Proposition 5.0.1, we wish to compute the bracket $\langle h_1, \tau h_0 e_0^3, \tau \rangle$ in $E_2(S^{0,0})$. We may attempt to use May's Convergence Theorem 2.2.1 with the May differential $d_4(\Delta d_0^2) = \tau^2 h_0 e_0^3$. However, the conditions of May's Convergence Theorem 2.2.1 are not satisfied because of the later May differential $d_8(\Delta^2 h_1^4) = P^2 h_1^4 h_5$.

Instead, Table 36 shows that $h_1 \cdot \overline{\tau h_0 d_0 e_0^2}$ equals Pv. Next, observe that $d_0 \cdot \overline{\tau h_0 e_0^3} + e_0 \cdot \overline{\tau h_0 d_0 e_0^2}$ is either zero or $h_1^2 U$. In either case, $h_1 d_0 \cdot \overline{\tau h_0 e_0^3}$ must be non-zero. It follows that $h_1 \cdot \overline{\tau h_0 e_0^3}$ is also non-zero, and there is just one possible non-zero value.

LEMMA 5.1.7. $h_1^2 h_5 \cdot \overline{c_0 d_0} = P h_5 e_0.$

PROOF. Table 38 shows that

$$h_1^2 \cdot \overline{c_0 d_0} + d_0 \cdot \overline{h_1^2 c_0} = P e_0,$$

which means that

$$h_1^5 h_5 \cdot \overline{c_0 d_0} + h_1^3 h_5 d_0 \cdot \overline{h_1^2 c_0} = P h_1^3 h_5 e_0.$$

But $h_1^3 h_5 d_0 = 0$, so $h_1^5 h_5 \cdot \overline{c_0 d_0} = P h_1^3 h_5 e_0$, from which the desired formula follows.

LEMMA 5.1.8. $h_1^2 \cdot \overline{h_5 d_0 e_0} = \tau B_{23} + c_0 Q_2.$

PROOF. Because of Proposition 5.0.1, we wish to compute the Massey product $\langle h_1, h_1h_5d_0e_0, \tau \rangle$ in $E_2(S^{0,0})$. We may attempt to use May's Convergence Theorem 2.2.1 with the May differential $d_6(B_{23}) = h_1^2h_5d_0e_0$.

However, there is a subtlety here. The element τB_{23} belongs to the May E_{∞} page for $E_2(S^{0,0})$. It represents two elements in $E_2(S^{0,0})$ because of the presence
of PD_4 with lower May filtration. Thus, we have only determined so far that $h_1^2 \cdot \overline{h_5 d_0 e_0}$ equals either τB_{23} or $\tau B_{23} + c_0 Q_2$.

This ambiguity is resolved essentially by definition. In Table 10, the element τB_{23} in $E_2(S^{0,0})$ is defined such that $\langle h_1, h_1 h_5 d_0 e_0, \tau \rangle$ equals $\tau B_{23} + c_0 Q_2$. \Box

LEMMA 5.1.9. $h_1^5 \cdot \overline{h_1^2 Q_2} = \tau g w + h_1^4 X_1.$

PROOF. Because of Proposition 5.0.1, we wish to compute the Massey product $\langle h_1^5, h_1^2Q_2, \tau \rangle$. We may attempt to use May's Convergence Theorem 2.2.1 with the May differential $d_4(\Delta h_1g^2) = h_1^7Q_2$.

As in the proof of Lemma 5.1.8, there is a subtlety here. The element τgw belongs to the May E_{∞} -page for $E_2(S^{0,0})$. It represents two elements in $E_2(S^{0,0})$ because of the presence of $Ph_1h_5c_0e_0$ with lower May filtration. Recall that Table 10 defines τgw to be the element of $E_2(S^{0,0})$ such that $h_1 \cdot \tau gw = 0$.

We have determined so far that $h_1^5 \cdot h_1^2 Q_2$ equals either $\tau g w$ or $\tau g w + h_1^4 X_1$. Table 20 gives a non-zero value for the Adams differential $d_3(\tau g w)$. On the other hand, $d_3(\overline{h_1^2}Q_2)$ is zero. Therefore, $h_1^5 \cdot \overline{h_1^2}Q_2$ cannot equal $\tau g w$.

REMARK 5.1.10. The proof of Lemma 5.1.9 is not entirely algebraic in the sense that it relies on Adams differentials. We would prefer a purely algebraic proof, but it has so far eluded us.

LEMMA 5.1.11. $h_1^3 c_0 \cdot \overline{D_4}$ equals either $h_2 B_5$ or $h_2 B_5 + h_1^2 X_3$.

PROOF. Because of Proposition 5.0.1, we wish to compute $\langle h_1^3 c_0, D_4, \tau \rangle$. We may attempt to use May's Convergence Theorem 2.2.1 with the May differential $d_4(\phi g) = h_1^5 X_2$.

As in the proof of Lemma 5.1.8, there is a subtlety here. The element h_2B_5 belongs to the May E_{∞} -page for $E_2(S^{0,0})$. It represents two elements in $E_2(S^{0,0})$ because of the presence of $h_1^2X_3$ with lower May filtration (see Table 10).

5.1.4. Other extensions in the Adams E_2 -page for the cofiber of τ . We finish this section with some additional miscellaneous hidden extensions.

LEMMA 5.1.12. $h_1^3 \cdot \overline{B_6} + h_2 \cdot \overline{\tau h_2 d_1 g} = h_1^2 Q_2.$

PROOF. Table 13 gives the hidden extension $h_1 \cdot h_1^2 B_6 = \tau h_2^2 d_1 g$ in $E_2(S^{0,0})$. This means that $h_1^3 \cdot \overline{B_6} + h_2 \cdot \overline{\tau h_2 d_1 g}$ belongs to the image of $E_2(S^{0,0}) \to E_2(C\tau)$.

Next, compute that $h_1^3Q_2 = \langle h_1^4, B_6, \tau \rangle$ using May's Convergence Theorem 2.2.1 with the May differentials $d_2(b_{30}b_{40}h_1(1)) = \tau B_6$ and $d_2(h_1^2b_{21}^2b_{30}b_{31} + h_1^2b_{21}^3b_{40}) = h_1^4B_6$. Therefore, $h_1^4 \cdot \overline{B_6} = h_1^3Q_2$ by Proposition 5.0.1. The desired formula now follows.

REMARK 5.1.13. Through the 70-stem, Lemma 5.1.12 is the only example of a hidden relation of the form $h_0 \cdot \overline{x} + h_1 \cdot \overline{y}$, $h_0 \cdot \overline{x} + h_2 \cdot \overline{y}$, or $h_1 \cdot \overline{x} + h_2 \cdot \overline{y}$ in $E_2(C\tau)$.

LEMMA 5.1.14. $Ph_1 \cdot \overline{B_6} = h_1 q_1$.

5. THE COFIBER OF τ

PROOF. Compute that h_1q_1 is contained in the Massey product $\langle Ph_1, B_6, \tau \rangle$ in $E_2(S^{0,0})$, using May's Convergence Theorem 2.2.1 with the May differentials $d_2(b_{30}b_{40}h_1(1)) = \tau B_6$ and $d_2(\Delta Bh_1^3) = Ph_1 \cdot B_6$. The bracket has indeterminacy generated by $\tau^2 h_0 B_{23}$, so it equals $\{h_1 q_1, h_1 q_1 + \tau^2 h_0 B_{23}\}.$

Push forward this bracket into $E_2(C\tau)$, where it collapses to the single element h_1q_1 since $\tau^2 h_0 B_{23}$ maps to zero in $E_2(C\tau)$. Proposition 5.0.1 now gives the desired result.

Lemma 5.1.15.

 $\begin{array}{l} (1) \quad h_1^2 \cdot \overline{c_0 d_0} + d_0 \cdot \overline{h_1^2 c_0} = P e_0. \\ (2) \quad c_0 \cdot \overline{h_1^2 e_0} + e_0 \cdot \overline{h_1^2 c_0} = d_0^2. \\ (3) \quad h_1^2 \cdot \overline{h_1 d_0 u} + d_0 \cdot \overline{h_1^3 u} = P v'. \end{array}$

PROOF. These formulas have essentially the same proof. We prove only the first formula.

Table 17 shows that there is a matric bracket

$$Pe_0 = \left\langle \begin{bmatrix} h_1^2 & d_0 \end{bmatrix}, \begin{bmatrix} c_0 d_0 \\ h_1^2 c_0 \end{bmatrix}, \tau \right\rangle$$

A matric version of Proposition 5.0.1 gives the desired hidden extension.

Before considering the next hidden extension, we need a bracket computation.

LEMMA 5.1.16. $h_1^2 d_0^2 = \langle c_0 e_0, \tau, h_1^4 \rangle.$

PROOF. The bracket cannot be computed directly with May's Convergence Theorem 2.2.1 because of the the later May differential $d_4(\Delta h_1^2) = Ph_1^2h_4$. Therefore, we must follow a more complicated route.

Begin with the computation $h_1 c_0 e_0 = \langle d_0, h_3, h_1^4 \rangle$ from Table 16. Therefore,

$$\langle h_1 c_0 e_0, \tau, h_1^4 \rangle = \langle \langle d_0, h_3, h_1^4 \rangle, \tau, h_1^4 \rangle$$

which equals $d_0(h_3, h_1^4, \tau, h_1^4)$ by a standard formal property of Massey products since there are no indeterminacies.

Next, compute that $h_1^3 d_0 = \langle h_3, h_1^4, \tau, h_1^4 \rangle$ using May's Convergence Theorem 2.2.2 with the May differentials $d_2(h_1b_{20}) = \tau h_1^4$, $d_2(h_1^2b_{21}) = h_1^4h_3$, and $d_2(h_1b_{30}) = h_1^4h_3$ $\tau h_1^2 b_{21} + h_1 h_3 b_{20}$. Note that both subbrackets $\langle h_1^4, \tau, h_1^4 \rangle$ and $\langle h_3, h_1^4, \tau \rangle$ are strictly zero.

We have now shown that $\langle h_1 c_0 e_0, \tau, h_1^4 \rangle$ equals $h_1^3 d_0^2$. The desired formula now follows immediately.

Lemma 5.1.17.

(1)
$$h_1^2 \cdot \overline{c_0 e_0} + e_0 \cdot \overline{h_1^2 c_0} = d_0^2.$$

(2) $d_0 \cdot \overline{c_0 e_0} + e_0 \cdot \overline{c_0 d_0} = h_1 u.$

PROOF. For the first formula, by a matric version of Proposition 5.0.1, we wish to compute that

$$d_0^2 = \left\langle \left[\begin{array}{cc} h_1^2 & e_0 \end{array} \right], \left[\begin{array}{c} c_0 e_0 \\ h_1^2 c_0 \end{array} \right], \tau \right\rangle.$$

One might attempt to compute this with a matric version of May's Convergence Theorem 2.2.1. However, the hypotheses of May's Convergence Theorem 2.2.1 do not apply because of the presence of the later May differential $d_4(\Delta h_1^2) = Ph_1^2h_4$.

Instead, we will show that

$$\left\langle \left[\begin{array}{cc} h_1^2 & e_0 \end{array} \right], \left[\begin{array}{c} c_0 e_0 \\ h_1^2 c_0 \end{array} \right], \tau \right\rangle h_1^4$$

equals $h_1^4 d_0^2$, from which the desired bracket follows immediately. Shuffle to obtain

$$h_1^2 \langle c_0 e_0, \tau, h_1^4 \rangle + e_0 \langle h_1^2 c_0, \tau, h_1^4 \rangle.$$

By Table 16, the expression equals $h_1^4 d_0^2$ as desired. This completes the proof of the first formula.

The proof of the second formula is similar. We wish to compute that

$$h_1 u = \left\langle \begin{bmatrix} d_0 & e_0 \end{bmatrix}, \begin{bmatrix} c_0 e_0 \\ c_0 d_0 \end{bmatrix}, \tau \right\rangle$$

Again, the hypotheses of May's Convergence Theorem 2.2.1 do not apply.

Instead, we will show that

$$\left\langle \begin{bmatrix} d_0 & e_0 \end{bmatrix}, \begin{bmatrix} c_0 e_0 \\ c_0 d_0 \end{bmatrix}, \tau \right\rangle h_1^4$$

equals $h_1^5 u$, from which the desired bracket follows immediately. Shuffle to obtain

$$d_0\langle c_0e_0,\tau,h_1^4\rangle + e_0\langle c_0d_0,\tau,h_1^4\rangle.$$

By Table 16, this expression equals $h_1^2 d_0^3 + h_1^2 e_0 \cdot P e_0$. Note that $e_0 \cdot P e_0$ equals $d_0^3 + h_1^3 u$; this is already true in the May E_{∞} -page. Therefore, $h_1^2 d_0^3 + h_1^2 e_0 \cdot P e_0$ equals $h_1^5 u$, as desired.

LEMMA 5.1.18.
$$h_1^2 e_0^2 \cdot \overline{h_1^2 e_0} + d_0 e_0 g \cdot \overline{h_1^4} + h_1^6 \cdot \overline{h_1^3 B_1} = c_0 d_0 e_0^2.$$

PROOF. The relation $e_0^3 + d_0 \cdot e_0 g = h_1^5 B_1$ is hidden in the May spectral sequence [14].

By Proposition 5.0.1, we wish to compute that

$$c_0 d_0 e_0^2 = \left\langle \begin{bmatrix} h_1^2 e_0^2 & d_0 e_0 g & h_1^4 \end{bmatrix}, \begin{bmatrix} h_1^2 e_0 \\ h_1^4 \\ h_1^3 B_1 \end{bmatrix}, \tau \right\rangle.$$

This will follow if we can show that $h_1^4 c_0 d_0 e_0^2$ equals

$$\left\langle \left[\begin{array}{ccc} h_1^2 e_0^2 & d_0 e_0 g & h_1^4 \end{array} \right], \left[\begin{array}{c} h_1^2 e_0 \\ h_1^4 \\ h_1^3 B_1 \end{array} \right], \tau \right\rangle h_1^4.$$

This expression equals

$$h_1^2 e_0^2 \langle h_1^2 e_0, \tau, h_1^4 \rangle + d_0 e_0 g \langle h_1^4, \tau, h_1^4 \rangle + h_1^4 \langle h_1^3 B_1, \tau, h_1^4 \rangle.$$

The first two terms can be computed with Table 16. The possible non-zero values for the third bracket are multiples of h_0 , which means that the third term is zero in any case.

The desired formula now follows.

Lemma 5.1.19.

5.1.5. The Adams E_2 -page for the cofiber of τ . Having resolved hidden extensions, we can now state our main theorem about $E_2(C\tau)$.

THEOREM 5.1.20. The E_2 -page of the Adams spectral sequence for $C\tau$ is depicted in [19] through the 70-stem. Table 39 lists the $E_2(S^{0,0})$ -module generators of $E_2(C\tau)$ through the 70-stem.

For most of the generators in Table 39, the notation \overline{x} is unambiguous. In other words, in each relevant degree, there is just a single element \overline{x} of $E_2(C\tau)$ that projects to x in $E_2(S^{0,0})$.

However, there are several cases in which there is a choice of representative for \overline{x} because of the presence of an element in the same degree in the image of the map $E_2(S^{0,0}) \to E_2(C\tau)$. One such example occurs in the 56-stem with $\tau h_0 gm$. The presence of $h_2 x'$ means that there are actually two possible choices for $\overline{\tau h_0 gm}$.

Table 40 lists all such examples of $E_2(S^{0,0})$ -module generators of $E_2(C\tau)$ for which there is some ambiguity. In some cases, we have given an algebraic specification of one element of $E_2(C\tau)$ to serve as the generator. These choices are essentially arbitrary, but it is important to be consistent with the notation between different arguments.

In some cases, we have not given a definition because an algebraic description is not readily available, and also because it does not seem to matter for later analysis. The reader is strongly warned to be cautious when working with these undefined elements.

The generator $\overline{h_1 i_1}$ deserves one additional comment. In this case, the presence of $\tau h_1 G$ and B_6 means that there are four possible choices for this generator. We have given two algebraic specifications for $h_1 i_1$, which determines a unique element from these four.

5.2. Adams differentials for the cofiber of τ

We have now computed the E_2 -page of the Adams spectral sequence for $C\tau$. See [19] for a chart of $E_2(C\tau)$ through the 70-stem.

The next step is to compute the Adams differentials. The main point is to compute the Adams d_r differentials on the $E_r(S^{0,0})$ -module generators of $E_r(C\tau)$. Then one can compute the Adams d_r differential on any element, using the Adams d_r differentials for $E_r(S^{0,0})$ given in Tables 8, 20, 21, and 22.

5.2.1. Adams d_2 differentials for the cofiber of τ .

PROPOSITION 5.2.1. Table 39 lists some values of the motivic Adams d_2 differential for $C\tau$. The motivic Adams d_2 differential is zero on all other $E_2(S^{0,0})$ module generators of $E_2(C\tau)$, through the 70-stem, with the possible exceptions that:

- (1) $d_2(\overline{h_1i_1})$ might equal $h_1h_5c_0d_0$. (2) $d_2(\overline{h_1r_1})$ might equal τh_1G_0 .

PROOF. We use several different approaches to establish the Adams d_2 differentials:

(1) From an Adams differential $d_2(x) = y$ in $E_2(S^{0,0})$, push forward along the inclusion $S^{0,0} \to C\tau$ of the bottom cell to obtain the same formula in $E_2(C\tau).$

- (2) From an Adams differential $d_2(x) = y$ in $E_2(S^{0,0})$, use the projection $C\tau \to S^{1,-1}$ and pull back to $d_2(\overline{x}) = \overline{y}$ in $E_2(C\tau)$, up to a possible error term that belongs to the image of the inclusion $E_2(S^{0,0}) \to E_2(C\tau)$ of the bottom cell.
- (3) Push forward a differential from $E_2(S^{0,0})$ as in (1), and then use a hidden extension in $E_2(C\tau)$. For example, $d_2(c_0d_0) = Pd_0$ because $h_0 \cdot c_0d_0 = i$ in $E_2(C\tau)$ and $d_2(i) = Ph_0d_0$ in $E_2(S^{0,0})$.
- (4) Work h_1 -locally. For example, consider the hidden extensions $h_1^2 \cdot \overline{c_0 e_0} + e_0 \cdot \overline{h_1^2 c_0} = d_0^2$ and $h_1^2 \cdot \overline{c_0 d_0} + d_0 \cdot \overline{h_1^2 c_0} = Pe_0$ from Table 38. It follows that $d_2(\overline{c_0 e_0}) = h_1^2 \cdot \overline{c_0 d_0} + Pe_0$.

Most of the differentials are computed with straightforward applications of these techniques. The remaining cases are computed in the following lemmas. \Box

The chart of $E_2(C\tau)$ in [19] indicates the Adams d_2 differentials, all of which are implied by the calculations in Tables 8 and 39.

LEMMA 5.2.2. $d_2(\overline{h_1^2 e_0 g}) = h_1^2 e_0 \cdot \overline{h_1^2 e_0} + c_0 d_0 e_0.$

PROOF. Table 8 gives the differential $d_2(h_1^2e_0g) = h_1^4e_0^2$ in $E_2(S^{0,0})$ Therefore, $d_2(\overline{h_1^2e_0g})$ is either $h_1^2e_0 \cdot \overline{h_1^2e_0}$ or $h_1^2e_0 \cdot \overline{h_1^2e_0} + c_0d_0e_0$. However, $d_2(h_1^2e_0 \cdot \overline{h_1^2e_0}) = h_1^2c_0d_0^2$, so $h_1^2e_0 \cdot \overline{h_1^2e_0}$ cannot be the target of a d_2 differential. \Box

LEMMA 5.2.3. $d_2(\overline{\tau^2 h_1 g^2}) = z$.

PROOF. We will argue that z must be zero in $E_{\infty}(C\tau)$. There is only one possible differential that can kill it.

Table 24 gives a classical extension $\eta \cdot \{g^2\} = \{z\}$ in π_{41} . This implies that there must be a hidden relation $\tau \cdot \{\tau^2 h_1 g^2\} = \{z\}$ in $\pi_{41,22}$. In particular, $\{z\}$ is divisible by τ in $\pi_{*,*}$. This means that $\{z\}$ maps to zero in $\pi_{41,22}(C\tau)$.

LEMMA 5.2.4.

(1)
$$d_2(\overline{h_1d_0u}) = Pu'.$$

$$(2) \ d_2(Ph_1d_0u) = P^2u'$$

PROOF. Table 39 implies that $d_2(d_0 \cdot \overline{h_1 v}) = d_0 \cdot \overline{h_1^3 u}$. By Lemma 5.1.19, this equals $h_1^2 \cdot \overline{h_1 d_0 u} + Pv'$.

Therefore, $h_1^2 \cdot d_2(\overline{h_1 d_0 u})$ equals $d_2(Pv')$. By Table 8, $d_2(Pv')$ equals $Ph_1^2u' + \tau h_0 d_0^4$ in $E_2(S^{0,0})$. Therefore, $d_2(Pv')$ equals Ph_1^2u' in $E_2(C\tau)$. It follows that $d_2(\overline{h_1 d_0 u})$ must equal Pu'. This establishes the first formula.

The second formula follows by multiplying the first formula by Ph_1 .

LEMMA 5.2.5. $d_2(\overline{D_4}) = h_1 \cdot \overline{B_6} + Q_2.$

PROOF. Pull back the differential $d_2(D_4) = h_1 B_6$ from $E_2(S^{0,0})$ to conclude that $d_2(\overline{D_4}) = h_1 \cdot \overline{B_6}$ modulo a possible error term that comes from pushing forward from $E_2(S^{0,0})$. To establish the error term, use that $h_0 \cdot \overline{D_4} = D_2$ and that $d_2(D_2) = h_0 Q_2$.

LEMMA 5.2.6. $d_2(\overline{h_1c_0x'}) = Ph_1x'.$

PROOF. Table 39 implies that $d_2(e_0 \cdot \overline{v'})$ equals $h_1^2 e_0 \cdot \overline{u'} + h_1^2 d_0 \cdot \overline{v'} + e_0 \cdot \overline{\tau} h_0 d_0 e_0^2$. Recall from [14] the relation $e_0 u' + d_0 v' = h_1^2 c_0 x'$, which is hidden in the May spectral sequence. This implies that $d_2(e_0 \cdot \overline{v'})$ equals $h_1^3 \cdot \overline{h_1 c_0 x'} + e_0 \cdot \overline{\tau} h_0 d_0 e_0^2$. There is a hidden extension $h_1 e_0 \cdot \overline{\tau h_0 d_0 e_0^2} = P e_0 v$. Therefore, $d_2(h_1 e_0 \cdot \overline{v'})$ equals $h_1^4 \cdot \overline{h_1 c_0 x'} + P e_0 v$, so $h_1^4 \cdot d_2(\overline{h_1 c_0 x'})$ must equal $d_2(P e_0 v)$.

By Table 8, $d_2(Pe_0v) = Ph_1^2d_0v + Ph_1^2e_0u$ in $E_2(S^{0,0})$. This equals Ph_1^5x' by [14]. It follows that $d_2(\overline{h_1c_0x'})$ equals Ph_1x' .

LEMMA 5.2.7. $d_2(\overline{c_0Q_2}) = 0.$

PROOF. Start with the relation $h_1 \cdot \overline{c_0 Q_2} = Ph_1 \cdot \overline{D_4}$, which follows from Lemma 2.4.24. Using Lemma 5.2.5, it follows that $h_1 \cdot d_2(c_0 \overline{Q_2}) = Ph_1^2 \cdot \overline{B_6} + Ph_1 Q_2$. We know from [9] that $Ph_1Q_2 = h_1^2q_1$, and we know from Lemma 5.1.14 that $Ph_1^2 \cdot \overline{B_6} = h_1^2q_1$.

REMARK 5.2.8. We emphasize the calculation $d_2(e_0g \cdot \overline{h_1^2 e_0}) = h_1^6 \cdot \overline{h_1^3 B_1} + c_0 d_0 e_0^2$, which follows from the Leibniz rule and Lemma 5.1.18. This implies that $h_1^6 \cdot \overline{h_1^3 B_1}$ equals $c_0 d_0 e_0^2$ in $E_3(C\tau)$. This formula is critical for later Adams differentials.

5.2.2. Adams d_3 differentials for the cofiber of τ . See [19] for a chart of $E_3(C\tau)$. This chart is complete through the 70-stem; however, the Adams d_3 differentials are complete only through the 64-stem.

REMARK 5.2.9. There are a number of classes in $E_2(S^{0,0})$ that do not survive to $E_3(S^{0,0})$, but their images in $E_2(C\tau)$ do survive to $E_3(C\tau)$. The first few examples of this phenomenon are h_0y , h_0c_2 , and h_0^4Q' . These elements give rise to $E_3(S^{0,0})$ -module generators of $E_3(C\tau)$.

REMARK 5.2.10. Note the class in the 55-stem labeled "?". This class is either $\overline{h_1 i_1}$ or $\overline{h_1 i_1} + \tau h_1 G$, depending on whether $d_2(\overline{h_1 i_1})$ is zero or non-zero. In the first case, we have that $h_1^5 \cdot \overline{h_1 i_1} = \tau g^3$, from which $d_3(\overline{h_1 i_1})$ would equal $h_1 B_8$. In the second case, we have that $h_1^5 \cdot (\overline{h_1 i_1} + \tau h_1 G) = \tau g^3 + h_1^4 h_5 c_0 e_0$, from which $d_3(\overline{h_1 i_1} + \tau h_1 G)$ would equal zero.

The next step is to compute Adams d_3 differentials on the $E_3(S^{0,0})$ -module generators of $E_3(C\tau)$.

PROPOSITION 5.2.11. Table 41 lists some values of the motivic Adams d_3 differential for $C\tau$. The motivic Adams d_3 differential is zero on all other $E_3(S^{0,0})$ module generators of $E_3(C\tau)$, through the 65-stem, with the possible exception that $d_3(\overline{h_1i_1})$ equals h_1B_8 , if $\overline{h_1i_1}$ survives to $E_3(C\tau)$.

PROOF. The techniques for establishing these differentials are the same as in the proof of Proposition 5.2.1 for d_2 differentials, except that the h_1 -local calculations are no longer useful. The few remaining cases are computed in the following lemmas.

The chart of $E_3(C\tau)$ in [19] indicates the Adams d_3 differentials, all of which are implied by the calculations in Tables 20 and 41. The differentials are complete only through the 64-stem. Beyond the 64-stem, there are a number of unknown differentials.

LEMMA 5.2.12. (1) $d_2(\overline{h_1 h_2 a}) = a$

(1) $d_3(\overline{h_1h_3g}) = d_0^2.$ (2) $d_3(\overline{h_1h_3g^2}) = d_0e_0^2.$

PROOF. We showed in Lemma 4.2.1 that both $\{d_0^2\}$ and $\{d_0e_0^2\}$ are divisible by τ in $\pi_{*,*}$. Therefore, the classes d_0^2 and $d_0e_0^2$ of $E_{\infty}(S^{0,0})$ must map to zero in $E_{\infty}(C\tau)$. For each element, there is just one possible differential that can hit it.

LEMMA 5.2.13. $d_3(\overline{h_1^2 g_2}) = 0.$

PROOF. The only other possibility is that $d_3(\overline{h_1^2 g_2})$ equals N. We showed in Lemma 4.2.5 that the elements of $\{N\}$ are not divisible by τ in $\pi_{*,*}$. Therefore, $\{N\}$ maps to $\pi_{*,*}(C\tau)$ non-trivially. The only possibility is that N is non-zero in $E_{\infty}(C\tau)$.

LEMMA 5.2.14. $d_3(\overline{h_1G_3}) = \overline{\tau h_0 e_0^3}.$

PROOF. Table 16 shows that $h_1G_3 = \langle h_3, h_1^3, Ph_1^2h_5 \rangle$. It follows that $c_0 \cdot h_1G_3 = \langle c_0, h_3, h_1^3 \rangle Ph_1^2h_5$. Table 16 shows that $\langle c_0, h_3, h_1^3 \rangle = h_1^2e_0$.

We have now shown that $c_0 \cdot h_1 G_3 = Ph_1^4 h_5 e_0$. It follows that either $c_0 \cdot \overline{h_1 G_3} = h_1^2 \cdot \overline{Ph_1^2 h_5 e_0}$ or $c_0 \cdot \overline{h_1 G_3} = h_1^2 \cdot \overline{Ph_1^2 h_5 e_0} + h_1^2 B_{21}$. In either case, $c_0 \cdot \overline{h_1 G_3} = h_1^2 \cdot \overline{Ph_1^2 h_5 e_0}$ in $E_3(C\tau)$ since $h_1^2 B_{21}$ is hit by an Adams d_2 differential.

Since $d_3(h_1^2 \cdot \overline{Ph_1^2h_5e_0}) = d_0u'$ is non-zero, we conclude that $d_3(\overline{h_1G_3})$ is also non-zero, and there is just one possible non-zero value.

LEMMA 5.2.15. $d_3(\overline{h_1d_1g}) = 0.$

PROOF. The only other possibility is that $d_3(\overline{h_1d_1g}) = h_1^2G_3$. If this were the case, then $\{h_1^2G_3\}$ in $\pi_{53,30}$ would be divisible by τ . If $\{h_1^2G_3\}$ were divisible by τ , then the only possibility would be that $\tau\{h_1d_1g\} = \{h_1^2G_3\}$. However, $\tau\{h_1d_1g\}$ is zero by Lemma 4.2.2.

LEMMA 5.2.16. $d_3(\overline{h_1^3 D_4}) = h_1 B_{21}$.

PROOF. Recall from Lemma 5.1.11 that $h_1^3 c_0 \cdot \overline{D_4}$ equals either $h_2 B_5$ or $h_2 B_5 + h_1^2 X_3$. It follows that $c_0 \cdot \overline{h_1^3 D_4}$ equals either $h_2 B_5$ or $h_2 B_5 + h_1^2 X_3$. However, these two elements are equal in $E_3(C\tau)$ since $h_1^2 X_3$ is the target of an Adams d_2 differential.

We know that $d_3(h_2B_5) = h_1B_8d_0$ by Table 20. It follows that $d_3(\overline{h_1^3D_4})$ is non-zero, and there is just one possibility.

LEMMA 5.2.17. $d_3(\overline{Ph_5c_0e_0}) = \overline{h_1^2c_0x'} + U.$

PROOF. First note that either $h_1 \cdot \overline{Ph_5c_0e_0} = Ph_1 \cdot \overline{h_5c_0e_0}$ or $h_1 \cdot \overline{Ph_5c_0e_0} = \underline{Ph_1} \cdot \overline{h_5c_0e_0} = \underline{Ph_1} \cdot \overline{h_5c_0e_0} + h_1^2q_1$. In either case, $d_3(h_1 \cdot \overline{Ph_5c_0e_0}) = Ph_1 \cdot h_1^2B_8$ since $d_3(\overline{h_5c_0e_0}) = \overline{h_1^2B_8}$ and $d_3(h_1^2q_1) = 0$.

Finally, we must compute that $Ph_1 \cdot \overline{h_1^2 B_8} = h_1 \cdot \overline{h_1^2 c_0 x'} + h_1 U$. Because of the relation $B_8 \cdot Ph_1 = c_0 x'$, either $Ph_1 \cdot \overline{h_1^2 B_8} = h_1^2 \cdot \overline{h_1 c_0 x'}$ or $Ph_1 \cdot \overline{h_1^2 B_8} = h_1^2 \cdot \overline{h_1 c_0 x'} + h_1 U$. The second case must be correct because this is the element that survives to $E_3(C\tau)$.

5.2.3. Adams d_4 differentials for the cofiber of τ . See [19] for a chart of $E_4(C\tau)$. This chart is complete through the 64-stem. Beyond the 64-stem, because of unknown earlier differentials, the actual E_4 -page is a subquotient of what is shown in the chart.

The next step is to compute Adams d_4 differentials on the $E_4(S^{0,0})$ -module generators of $E_4(C\tau)$.

PROPOSITION 5.2.18. The motivic Adams d_4 differential for the cofiber of τ is zero on all $E_4(S^{0,0})$ -module generators of $E_4(C\tau)$ through the 63-stem, except that:

(1) $d_4(h_0^{15}h_6) = \overline{\tau P^2 h_0 d_0^2 e_0}.$ (2) $d_4(\overline{j_1})$ might equal $B_{21}.$

PROOF. For degree reasons, there are very few possible differentials. The only difficult cases are addressed in Lemmas 5.2.21 and 5.2.22. \Box

The chart of $E_4(C\tau)$ in [19] indicates the Adams d_4 differentials, all of which are implied by the calculations in Proposition 5.2.18 and Table 21. The differentials are complete only through the 63-stem. Beyond the 63-stem, there are a number of unknown differentials.

REMARK 5.2.19. Recall that $h_0^{15}h_6$ does not survive to $E_4(S^{0,0})$, so this element is an $E_4(S^{0,0})$ -module generator of $E_4(C\tau)$. This is the reason that the formula for $d_4(h_0^{15}h_6)$ appears in the statement of Proposition 5.2.18.

REMARK 5.2.20. The possible differential $d_4(C') = h_2 B_{21}$ in $E_4(S^{0,0})$ mentioned in Proposition 3.2.15 occurs if and only if $d_4(\overline{j_1}) = B_{21}$ in $E_4(C\tau)$. This follows immediately from the relation $h_2 \cdot \overline{j_1} = C'$.

LEMMA 5.2.21. $d_4(h_0D_2) = 0.$

PROOF. We showed in Lemma 4.2.7 that $h_0h_2h_5i$ detects an element α of $\pi_{57,30}$ that is not divisible by τ . Therefore, α maps to a non-zero element of $\pi_{57,30}(C\tau)$. The only possibility is that this element of $\pi_{57,30}(C\tau)$ is detected by h_1Q_1 . In particular, h_1Q_1 cannot equal $d_4(h_0D_2)$.

LEMMA 5.2.22. $d_4(\overline{h_3d_1g}) = 0.$

PROOF. The only other possibility is that $d_4(\overline{h_3d_1g})$ equals $Ph_1^3h_5e_0$. We showed in Lemma 4.2.9 that the element $\{Ph_1^3h_5e_0\}$ of $\pi_{59,33}$ is not divisible by τ . Therefore, $Ph_1^3h_5e_0$ is not hit by a differential in the Adams spectral sequence for $C\tau$.

5.2.4. Higher Adams differentials for the cofiber of τ . At this point, we are nearly done. There is just one more differential to compute.

LEMMA 5.2.23. $d_5(h_2h_5) = 0.$

PROOF. The only other possibility is that $d_5(h_2h_5)$ equals h_1q . We showed in Lemma 4.2.2 that the element $\{h_1q\}$ of $\pi_{33,18}$ is not divisible by τ . Therefore, h_1q cannot be hit by a differential in the Adams spectral sequence for the cofiber of τ .

The $E_4(C\tau)$ chart in [19] indicates the very few d_5 differentials along with the d_4 differentials.

5.2.5. The Adams E_{∞} -page for the cofiber of τ . Using the Adams differentials given in Table 39, Table 41, and Proposition 5.2.18, as well as the Adams differentials for $S^{0,0}$ given in Tables 8, 20, 21, and 22, we can now directly compute the E_{∞} -page of the Adams spectral sequence for $C\tau$.

THEOREM 5.2.24. The E_{∞} -page of the Adams spectral sequence for $C\tau$ is depicted in [19]. This chart is complete through the 63-stem. Beyond the 63-stem, $E_{\infty}(C\tau)$ is a subquotient of what is shown in the chart.

Through the 63-stem, all unknown differentials are indicated as dashed lines. Beyond the 63-stem, there are a number of unknown differentials.

In a range, we now have a complete understanding of $E_{\infty}(C\tau)$, which is the associated graded object of $\pi_{*,*}(C\tau)$ with respect to the Adams filtration. In order to better understand $\pi_{*,*}(C\tau)$ itself, we would like to compute the maps of homotopy groups induced by the inclusion $j: S^{0,0} \to C\tau$ of the bottom cell and the projection $q: C\tau \to S^{1,-1}$ to the top cell.

PROPOSITION 5.2.25. The map $j_*: \pi_{*,*} \to \pi_{*,*}C\tau$ induced by the inclusion of the bottom cell is described as follows, through the 59-stem. Let α be an element of $\pi_{*,*}$ detected by a in $E_{\infty}(S^{0,0})$.

- (1) If a does not equal $h_0h_2h_5i$, then $j_*(\alpha)$ is detected by $j_*(a)$ in $E_{\infty}(C\tau)$.
- (2) If a equals $h_0h_2h_5i$, then $j_*(\alpha)$ is detected by h_1Q_1 in $E_{\infty}(C\tau)$.

PROOF. This is a straightforward calculation, using that there is an induced map $E_{\infty}(S^{0,0}) \to E_{\infty}(C\tau)$.

It is curious that the Adams filtration hides so little about the map j_* .

PROPOSITION 5.2.26. The map $q_*: \pi_{*,*}C\tau \to \pi_{*-1,*+1}$ induced by the projection to the top cell is described as follows, through the 59-stem.

- (1) An element of $\pi_{*,*}(C\tau)$ in the image of $j_*: \pi_{*,*} \to \pi_{*,*}(C\tau)$ (as described by Proposition 5.2.25) maps to 0 in $\pi_{*-1,*+1}$.
- (2) An element of $\pi_{*,*}(C\tau)$ detected by \overline{x} in $E_{\infty}(C\tau)$ maps to an element of $\pi_{*-1,*+1}$ detected by x in $E_{\infty}(S^{0,0})$.
- (3) The remaining possibilities are described in Table 42.

PROOF. The part of q_* that is not hidden by the Adams filtration is described in (1) and (2). The part of q_* that is hidden by the Adams filtration is described in Table 42. These are the only possible values that are compatible with the long exact sequence

$$\cdots \longrightarrow \pi_{*,*+1} \longrightarrow \pi_{*,*} \longrightarrow \pi_{*,*}(C\tau) \longrightarrow \pi_{*-1,*+1} \longrightarrow \cdots$$

5.3. Hidden Adams extensions for the cofiber of τ

Finally, we will consider hidden extensions by 2, η , and ν in the motivic stable homotopy groups $\pi_{*,*}(C\tau)$ of the cofiber of τ . We will show in Lemma 6.2.4 that there are no hidden τ extensions in $\pi_{*,*}(C\tau)$.

Recall from Proposition 3.1.6 that a hidden extension by α in $\pi_{*,*}(C\tau)$ is the same as a Toda bracket in $\pi_{*,*}$ of the form $\langle \tau, \beta, \alpha \rangle$. Many such Toda brackets are detected in Ext by a corresponding Massey product of the form $\langle \tau, b, a \rangle$. In this circumstance, the extension by α is already detected in $E_{\infty}(C\tau)$.

However, there are some Toda brackets of the form $\langle \tau, \beta, \alpha \rangle$ that are not detected by Massey products in Ext. In this section, we will study such Toda brackets methodically.

PROPOSITION 5.3.1. Table 43 shows some hidden extensions by 2, η , and ν in $\pi_{*,*}(C\tau)$. Through the 59-stem, there are no other hidden extensions by 2, η , and ν , except that:

- (1) there might be a hidden 2 extension from $\overline{h_1d_1g}$ to h_1B_8 .
- (2) there might be a hidden 2 extension from Q_2 to h_1Q_1 .
- (3) there might be a hidden 2 extension from $h_1^2 D_4$ to $Ph_1^3 h_5 e_0$.
- (4) there might be a hidden η extension from $h_1^2 g_2$ to $h_0 B_2$.
- (5) there might be a hidden ν extension from B_6 to h_1D_{11} .
- (6) there might be a hidden ν extension from $\overline{\tau h_2 d_1 g}$ to B_{21} .
- (7) if $\overline{h_1 i_1} + \tau h_1 G$ survives to $E_{\infty}(C\tau)$, then there might be a hidden ν extension from $\overline{h_1 i_1} + \tau h_1 G$ to $h_1 D_{11}$.
- (8) if $\overline{j_1}$ survives to $E_{\infty}(C\tau)$, then there might be a hidden 2 extension from $\overline{j_1}$ to $h_1 \cdot \overline{h_3G_3}$.

PROOF. Some of the extensions are detected by the projection $q: C\tau \to S^{1,-1}$ to the top cell, and some of the extensions are detected by the inclusion $j: S^{0,0} \to C\tau$ of the bottom cell. The remaining cases are established in the following lemmas.

REMARK 5.3.2. The possible hidden η extension on $h_1^2 g_2$ is connected to some of the other uncertainties in our calculations. Suppose that there is a hidden τ extension from $h_1 i_1$ to $h_1 B_8$ in $\pi_{*,*}$ (see Remark 4.1.11). Then $\nu\{C\} + \tau\{i_1\}$ is detected by B_8 , and there is a hidden ν extension in $\pi_{*,*}(C\tau)$ from C to B_8 . If $\{\overline{h_1^2 g_2}\}\eta$ were zero, then we could further compute that

$$\{B_8\} = \{\overline{h_1^2 g_2}\}\nu^2 = \{\overline{h_1^2 g_2}\}\langle\eta,\nu,\eta\rangle = \langle\{\overline{h_1^2 g_2}\},\eta,\nu\rangle\eta$$

in $\pi_{*,*}(C\tau)$. However, $\{B_8\}$ cannot be divisible by η in $\pi_{*,*}(C\tau)$. Therefore, $\{\overline{h_1^2g_2}\}\eta$ would be non-zero in $\pi_{*,*}(C\tau)$.

LEMMA 5.3.3. There is no hidden ν extension on $\overline{h_1^4 h_5}$.

PROOF. The only other possibility is that there is a hidden ν extension from $\overline{h_1^4 h_5}$ to u. We will show that the Toda bracket $\langle \tau, \eta^3 \eta_5, \nu \rangle$ does not contain $\{u\}$.

The bracket contains $\langle \tau \eta^3, \eta_5, \nu \rangle$, which equals $\langle 4\nu, \eta_5, \nu \rangle$. This bracket contains $4\langle \nu, \eta_5, \nu \rangle$. Note that $\langle \nu, \eta_5, \nu \rangle$ intersects $\{h_1h_3h_5\}$, but $4\langle \nu, \eta_5, \nu \rangle$ is zero.

Finally, the bracket $\langle \tau, \eta^3 \eta_5, \nu \rangle$ has indeterminacy generated by $\tau\{h_3 d_1\}$ and $\tau^2\{c_1g\}$. Therefore, $\{u\}$ is not in the bracket.

LEMMA 5.3.4. There is a hidden η extension from $h_0 y$ to u.

PROOF. Table 42 shows that projection to the top cell maps $\{h_0y\}$ to $\{\tau h_2 e_0^2\}$ in $\pi_{37,21}$. The bracket $\langle \tau, \{\tau h_2 e_0^2\}, \eta \rangle$ contains $\langle \{\tau^2 e_0^2\}, \nu, \eta \rangle$ which contains $\{u\}$ by Table 23.

The indeterminacy of $\langle \tau, \{\tau h_2 e_0^2\}, \eta \rangle$ is generated by $\tau \sigma \{d_1\}$, and $\eta \{h_0^2 h_3 h_5\}$. Note that $\tau \overline{\sigma \kappa}$ is equal to $\eta \{h_0^2 h_3 h_5\}$, as shown in Remark 4.2.42. Since $\{u\}$ is not in the indeterminacy, the bracket does not contain zero.

LEMMA 5.3.5. There is no hidden ν extension on h_0c_2 .

PROOF. According to Table 42, the projection to the top cell takes the elements of $\pi_{41,22}(C\tau)$ detected by h_0c_2 to elements of $\pi_{40,23}$ that are detected by $h_1h_3d_1$. These elements of $\pi_{40,23}$ must also be annihilated by τ , so they must be $\eta\sigma\{d_1\}$ and $\eta \sigma \{d_1\} + \{\tau h_0^2 g^2\}.$

It remains to compute the Toda bracket $\langle \tau, \eta \sigma \{ d_1 \}, \nu \rangle$. This bracket contains $\langle \tau, \eta \{ d_1 \}, 0 \rangle$, which equals zero.

LEMMA 5.3.6. There is no hidden 2 extension on h_0c_2 .

PROOF. We showed in Lemma 5.3.5 that there is no hidden ν extension on h_0c_2 . Therefore, there cannot be a hidden 2 extension from h_0c_2 to $\tau h_0^2 g^2$.

There are no other possible hidden 2 extensions on h_0c_2 .

LEMMA 5.3.7. There is no hidden 2 extension on $h_3 \cdot \overline{h_3^2 q}$.

PROOF. The projection to the top cell detects that $h_3 \cdot \overline{h_3^2 g}$ is the target of a hidden η extension from h_0c_2 . Therefore, $h_3 \cdot \overline{h_3^2g}$ cannot support a hidden 2 extension.

LEMMA 5.3.8. There is no hidden η extension on $\overline{\tau h_2 c_1 g}$.

PROOF. The projection to the top cell takes the element $\{\overline{\tau h_2 c_1 g}\}$ of $\pi_{43,23}(C\tau)$ to an element of $\pi_{42,24}$ that is detected by $\tau h_2 c_1 g$. Two elements of $\pi_{42,24}$ are detected by $\tau h_2 c_1 g$, but only one element is killed by τ . The relation $\eta \{h_0^2 h_3 h_5\} =$ $\tau \overline{\sigma \kappa}$ from Remark 4.2.42 implies that $\nu \overline{\sigma \kappa}$ is the element of $\pi_{42,24}$ that is killed by τ and detected by $\tau h_2 c_1 g$. Therefore, the top cell detects that there is no hidden η extension on $\overline{\tau h_2 c_1 g}$.

LEMMA 5.3.9. There is a hidden ν extension from d_0r to h_1u' .

PROOF. The inclusion of the bottom cell shows that there is a hidden 2 extension from e_0r to h_1u' in $\pi_{47,26}(C\tau)$. The hidden ν extension on d_0r is an immediate consequence.

CHAPTER 6

Reverse engineering the Adams-Novikov spectral sequence

In this chapter, we will show that the classical Adams-Novikov E_2 -page is identical to the the motivic stable homotopy groups $\pi_{*,*}(C\tau)$ of the cofiber of τ computed in Chapter 5. Moreover, the classical Adams-Novikov differentials and hidden extensions can also be deduced from prior knowledge of motivic stable homotopy groups. We will apply this program to provide detailed computational information about the classical Adams-Novikov spectral sequence in previously unknown stems.

In fact, the classical Adams-Novikov spectral sequence appears to be identical to the τ -Bockstein spectral sequence converging to stable motivic homotopy groups. We have only a computational understanding of this curious phenomenon. Our work calls for a more conceptual study of this relationship.

The simple pattern of weights in the motivic Adams-Novikov spectral sequence is the key idea that allows this program to proceed. See Theorem 6.1.4 for more explanation. For example, for simple degree reasons, there can be no hidden τ extensions in the motivic Adams-Novikov spectral sequence. Also for simple degree reasons, there are no "exotic" Adams-Novikov differentials; each non-zero motivic differential corresponds to a classical non-zero analogue.

Outline. Section 6.1 describes the motivic Adams-Novikov spectral sequence in general terms. Section 6.2 deals with specific properties of the motivic Adams-Novikov spectral sequence for the cofiber of τ . The main point is that this spectral sequence collapses. Section 6.3 carries out the translation of information about $\pi_{*,*}(C\tau)$ into information about the classical Adams-Novikov spectral sequence.

Chapter 7 contains a series of tables that summarize the essential computational facts in a concise form. Tables 44, 45, and 46 list the extensions by 2, η , and ν that are hidden in the Adams-Novikov spectral sequence.

Table 47 gives a correspondence between elements of the classical Adams E_{∞} page and elements of the classical Adams-Novikov E_{∞} -page. When possible, the
table also gives an element of π_* that is detected by these E_{∞} elements.

Tables 48 and 49 list the classical Adams-Novikov elements that are boundaries and that support differentials respectively. The tables list the corresponding elements of $\pi_{*,*}(C\tau)$.

Classical Adams-Novikov inputs. The point of this chapter is to deduce information about the Adams-Novikov spectral sequence from prior knowledge of the motivic stable homotopy groups obtained in Chapters 3, 4, and 5. To avoid circularity, Chapters 3, 4, and 5 intentionally avoid use of the Adams-Novikov spectral sequence whenever possible. However, we need a few computational facts about the Adams-Novikov spectral sequence in Chapter 4:

- (1) Lemma 4.2.7 shows that a certain possible hidden τ extension does not occur in the 57-stem. See also Remark 4.1.12. For this, we use that $\beta_{12/6}$ is the only element in the Adams-Novikov spectral sequence in the 58-stem with filtration 2 that is not divisible by α_1 [38].
- (2) Lemma 4.2.35 establishes a hidden 2 extension in the 54-stem. See also Remark 4.1.18. For this, we use that $\beta_{10/2}$ is the only element of the Adams-Novikov spectral sequence in the 54-stem with filtration 2 that is not divisible by α_1 , and that this element maps to $\Delta^2 h_2^2$ in the Adams-Novikov spectral sequence for tmf [5] [38].

Some examples.

EXAMPLE 6.0.1. Consider the element $\{h_1^2h_3g\}$ of $\pi_{29,18}$. This element is killed by τ^2 but not by τ .

The Adams-Novikov element $\alpha_1 z_{28}$ detects $\{h_1^2 h_3 g\}$ (see the charts in [21]). Therefore, $\tau^2 \alpha_1 z_{28}$ must be hit by some Adams-Novikov differential. This implies that there is a classical Adams-Novikov d_5 differential from the 30-stem to the 29-stem. This differential is well-known [37].

EXAMPLE 6.0.2. Consider the element $\{h_1^7 h_5 e_0\}$ of $\pi_{55,33}$. This element is killed by τ^4 but not by τ^3 .

The Adams-Novikov element $\alpha_1 z_{54,10}$ detects $\{h_1^7 h_5 e_0\}$ (see the charts in [21]). Therefore, $\tau^4 \alpha_1 z_{54,10}$ must be hit by some Adams-Novikov differential. This implies that there is a classical Adams-Novikov d_9 differential from the 56-stem to the 55-stem. This differential lies far beyond previous calculations.

6.1. The motivic Adams-Novikov spectral sequence

We adopt the following notation for the classical Adams-Novikov spectral sequence.

DEFINITION 6.1.1. Let $E_r(S^0; BP)$ (and $E_{\infty}(S^0; BP)$) be the pages of the classical Adams-Novikov spectral sequence for S^0 . We write $E_r^{s,f}(S^0; BP)$ for the part of $E_r(S^0; BP)$ in stem s and filtration f.

The even Adams-Novikov differentials d_{2r} are all zero, so we will only consider $E_r(S^0; BP)$ when r is odd (or is ∞).

We now describe the motivic Adams-Novikov spectral sequence. Recall that BPL is the motivic analogue of the classical Brown-Peterson spectrum BP.

DEFINITION 6.1.2. Let $E_r(S^{0,0}; BPL)$ (and $E_{\infty}(S^{0,0}; BPL)$) be the pages of the motivic Adams-Novikov spectral sequence for the motivic sphere $S^{0,0}$. We write $E_2^{s,f,w}(S^{0,0}; BPL)$ for the part of $E_2(S^{0,0}; BPL)$ in stem s, filtration f, and weight w.

Our goal is to describe the motivic Adams-Novikov spectral sequence in terms of the classical Adams-Novikov spectral sequence, as in [17, Theorem 8 and Section 4].

DEFINITION 6.1.3. Define the tri-graded object $\overline{E}_2(S^{0,0}; BPL)$ such that: (1) $\overline{E}_2^{s,f,\frac{s+f}{2}}(S^{0,0}; BPL)$ is isomorphic to $E_2^{s,f}(S^0; BP)$.

(2)
$$\overline{E}_2^{s,f,w}(S^{0,0};BPL)$$
 is zero if $w \neq \frac{s+f}{2}$.

The following theorem completely describes the motivic $E_2(S^{0,0}; BPL)$ -page in terms of the classical $E_2(S^0; BP)$ -page.

THEOREM 6.1.4. [17, Theorem 8 and Section 4] The $E_2(S^{0,0}; BPL)$ -page of the motivic Adams-Novikov spectral sequence is isomorphic to the tri-graded object $\overline{E}_2(S^{0,0}; BPL) \otimes_{\mathbb{Z}_2} \mathbb{Z}_2[\tau]$, where τ has degree (0, 0, -1).

In other words, in order to produce the motivic E_2 -page, start with the classical E_2 -page. At degree (s, f), replace each copy of \mathbb{Z}_2 or $\mathbb{Z}/2^n$ with a copy of $\mathbb{Z}_2[\tau]$ or $\mathbb{Z}/2^n[\tau]$, where the generator has weight $\frac{s+f}{2}$.

We will now compare the classical and motivic Adams-Novikov spectral sequences. As we have seen in earlier chapters, τ -localization corresponds to passage from the motivic to classical situations.

THEOREM 6.1.5. After inverting τ , the motivic Adams-Novikov spectral sequence is isomorphic to the classical Adams-Novikov spectral sequence tensored over \mathbb{Z}_2 with $\mathbb{Z}_2[\tau^{\pm 1}]$.

PROOF. The proof is analogous to the corresponding result for the motivic and classical Adams spectral sequences. See Proposition 3.0.2 and [13, Sections 3.2 and 3.4].

6.2. The motivic Adams-Novikov spectral sequence for the cofiber of τ

We will now study the motivic Adams-Novikov spectral sequence that computes the homotopy groups of the cofiber $C\tau$ of τ .

DEFINITION 6.2.1. Let $E_r(C\tau; BPL)$ (and $E_{\infty}(C\tau; BPL)$) be the pages of the motivic Adams-Novikov spectral sequence for $C\tau$. We write $E_2^{s,f,w}(C\tau; BPL)$ for the part of $E_2(C\tau; BPL)$ in stem s, filtration f, and weight w.

LEMMA 6.2.2. $E_2(C\tau; BPL)$ is isomorphic to $\overline{E}_2(S^{0,0}; BPL)$.

PROOF. The cofiber sequence

$$S^{0,-1} \xrightarrow{\tau} S^{0,0} \longrightarrow C\tau \longrightarrow S^{1,-1}$$

induces a long exact sequence

 $\cdots \longrightarrow E_2(S^{0,0}; BPL) \xrightarrow{\tau} E_2(S^{0,0}; BPL) \longrightarrow E_2(C\tau; BPL) \longrightarrow \cdots$

Theorem 6.1.4 tells us that the map $\tau : E_2(S^{0,0}; BPL) \to E_2(S^{0,0}; BPL)$ is injective, so $E_2(C\tau; BPL)$ is isomorphic to the cokernel of τ . Theorem 6.1.4 tells us that this cokernel is isomorphic to $\overline{E}_2(S^{0,0}; BPL)$.

LEMMA 6.2.3. There are no differentials in the motivic Adams-Novikov spectral sequence for τ .

PROOF. Lemma 6.2.2 tells us that $E_2(C\tau; BPL)$ is concentrated in tridegrees (s, f, w) where s + f - 2w equals zero. The Adams-Novikov d_r differential increases s + f - 2w by r - 1. Therefore, all differentials are zero.

LEMMA 6.2.4. There are no hidden τ extensions in $E_{\infty}(C\tau; BPL)$.

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PROOF. Let x and y be two elements of $E_{\infty}(C\tau; BPL)$ of degrees (s, f, w) and (s, f', w') with f' > f. Then w' > w since $w = \frac{s+f}{2}$ and $w' = \frac{s+f'}{2}$. For degree reasons, it is not possible that there is a hidden τ extension from x to y because τ has degree (0, -1).

PROPOSITION 6.2.5. There is an isomorphism $\pi_{*,*}(C\tau) \to E_2(S^0; BP)$ that takes the group $\pi_{s,w}(C\tau)$ into $E_2^{s,2w-s}(S^0;BP)$.

PROOF. Lemma 6.2.2 and Definition 6.1.3 say that $E_2(C\tau; BPL)$ is isomorphic to $E_2(S^0; BP)$. Lemma 6.2.3 implies that $E_{\infty}(C\tau; BPL)$ is also isomorphic to $E_2(S^0; BP)$. As in the proof of Lemma 6.2.4, for degree reasons there cannot be hidden extensions of any kind. Therefore, $\pi_{*,*}(C\tau)$ is also isomorphic to $E_2(S^0; BP).$

6.3. Adams-Novikov calculations

We will now provide explicit calculations of the classical Adams-Novikov spectral sequence. The charts in [21] are an essential companion to this section.

6.3.1. The classical Adams-Novikov *E*₂-page. We use the traditional notation for elements of the α family, as described in [37]. We draw particular to attention to α_1 in degree (1,1) and $\alpha_{2/2}$ in degree (3,1). These elements detect η and ν respectively.

For elements not in the α family, we have labelled decomposable elements as products whenever possible. For elements that are not known to be products, we use arbitrary symbols of the form $z_{s,f}$ and $z'_{s,f}$ for elements in the s-stem with filtration f. When there is no amibiguity, we simplify this to z_s and z'_s .

Our notation is unfortunately arbitrary and does not necessarily convey deeper structure. However, at least it allows us to give names to every element in the spectral sequence. Our notation is not compatible with the standard notation for elements of the Adams-Novikov spectral sequence [37].

EXAMPLE 6.3.1. Consider the elements in degree (46, 4) in the Adams-Novikov E_2 chart in [21]. From left to right, they are $\alpha_1^2 z_{44,2}$, $\alpha_1 z_{45}$, $\alpha_1 z_{45}'$, and $\alpha_{2/2} z_{43,3}$.

THEOREM 6.3.2. The E_2 -page of the classical Adams-Novikov spectral sequence is depicted through the 59-stem in the chart in [21]. The chart is complete except for the uncertainties described in Propositions 6.3.3 and 6.3.4, and the following:

- (1) $\alpha_1 z_{47,3}$ might equal $2\alpha_{2/2} z'_{45}$. (2) If $\alpha_1 z_8 z'_{45}$ is non-zero, then $2z_{54,6}$ might equal $\alpha_1 z_8 z'_{45}$.
- (3) $\alpha_{2/2}z_{53}$ might equal $\alpha_1^2 z_{54,6}$.
- (4) $2z_{57}$ might equal $\alpha_1 z_{56,2}$.
- (5) $\alpha_{2/2}z_{55}$ or $\alpha_{2/2}z'_{55}$ might equal $\alpha_1^2 z_{56,4}$.
- (6) $\alpha_{2/2} z_{56,4}$ might equal $z_{59,5}$.
- (7) If $z_{60,4}$ is non-zero, then $2z_{60,4}$ might equal $\alpha_1^2 z_{58,2}$.

PROOF. This follows immediately from Proposition 6.2.5 and the calculation of $\pi_{*,*}(C\tau)$ given in Chapter 5. The uncertainties are consequences of uncertainties in the structure of $\pi_{*,*}(C\tau)$. \square

PROPOSITION 6.3.3. Modulo elements of the form $\alpha_1^n \alpha_{k/b}$, in the 53-stem, 54stem, and 55-stem, either case (1) or case (2) occurs.

(1) $E_2^{54,6}(S^0; BP)$ has order four, containing two distinct non-zero elements $z_{54,6}$ and $\alpha_1 z_8 z'_{45} = \alpha_{2/2}^3 z'_{45}$; $E_2^{55,5}(S^0; BP)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ with generators z_{55} and z'_{55} ; $\alpha_1 z_{55}' = z_{56,6};$ and $\alpha_{2/2}^2 z_{47,3}$ is zero in $E_2^{53,5}(S^0; BP)$. (2) $E_2^{54,6}(S^0; BP)$ has order two; $E_2^{55,5}(S^0; BP)$ has order two; and $\alpha_{2/2}^2 z_{47,3} = z_8 z'_{45}$ in $E_2^{53,5}(S^0; BP)$.

PROOF. In the motivic Adams spectral sequence for $C\tau$, there is a possible d_3 differential hitting $h_1 B_8$ discussed in Proposition 5.2.11. Case (1) of the proposition corresponds to the possibility that this differential does not occur. Case (2) of the proposition corresponds to the possibility that this differential does occur.

PROPOSITION 6.3.4. Modulo elements of the form $\alpha_1^n \alpha_{k/b}$, in the 59-stem and 60-stem, either case (1) or case (2) occurs.

(1) $E_2^{59,5}(S^0; BP)$ is isomorphic to $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, with generators $\alpha_1^2 z_{57}$ and

and $E_2^{60,4}(S^0; BP)$ has order four, containing two distinct non-zero elements $z_{60,4}$ and $\alpha_{2/2}z_{57}$.

(2) the only non-zero element of $E_2^{59,5}(S^0; BP)$ is $\alpha_1^2 z_{57}$; and $E_2^{60,4}(S^0; BP)$ has order two.

PROOF. In the motivic Adams spectral sequence for $C\tau$, there is a possible d_4 differential hitting B_{21} discussed in Proposition 5.2.18. Case (1) of the proposition corresponds to the possibility that this differential does not occur. Case (2) of the proposition corresponds to the possibility that this differential does occur.

LEMMA 6.3.5. Assume that case (1) of Proposition 6.3.3 occurs. Then $\alpha_1 z_{47,3}$ equals $2\alpha_{2/2}z'_{45}$.

PROOF. Case (1) of Proposition 6.3.3 says that $\alpha_{2/2}^2 z_{47,3}$ is not divisible by α_1 . If $\alpha_1 z_{47,3}$ were zero, then we could shuffle Massey products to obtain

$$\alpha_{2/2}^2 z_{47,3} = \langle \alpha_1, \alpha_{2/2}, \alpha_1 \rangle z_{47,3} = \alpha_1 \langle \alpha_{2/2}, \alpha_1, z_{47,3} \rangle$$

Therefore, $\alpha_1 z_{47,3}$ must be non-zero.

Under the isomorphism of Proposition 6.2.5, the element $z_{47,3}$ of $E_2^{47,3}(S^0; BP)$ corresponds to the element $\overline{h_1^2 g_2}$ in $\pi_{47,25}(C\tau)$. We showed in Proposition 5.3.1 that the only possible hidden η extension on $\overline{h_1^2 g_2}$ takes the value $h_0 B_2$, which corresponds in $E_2^{48,26}(S^0; BP)$ to $2\alpha_{2/2} z'_{45}$.

6.3.2. Adams-Novikov differentials. Having obtained the Adams-Novikov E_2 -page, we next compute differentials.

THEOREM 6.3.6. The differentials in the classical Adams-Novikov spectral sequence are depicted through the 59-stem in the chart in [21]. The chart is complete except for the following:

- (1) if z'_{55} exists in $E_2^{55,5}(S^0; BP)$, then $d_3(z'_{55}) = \alpha_1 z_{53}$. (2) if $z_{60,4}$ exists in $E_2^{60,4}(S^0; BP)$, then $d_3(z_{60,4}) = z'_{59,7}$.

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PROOF. There is only one pattern of differentials in the motivic Adams-Novikov spectral sequence that will give the same answer for $\pi_{*,*}$ that was already obtained in Chapters 3 and 4. For example, $\{h_1^2h_4c_0\}$ in $\pi_{25,15}$ is annihilated by τ . Therefore, there must be a motivic Adams-Novikov differential hitting $\tau \alpha_1^2 \alpha_{4/4} z_{16}$. The only possibility is that $d_3(z_{26}) = \tau \alpha_1^2 \alpha_{4/4} z_{16}$.

Having obtained the motivic Adams-Novikov differentials in this way, the classical Adams-Novikov differentials follow immediately.

The uncertainties in the statement of the theorem are associated with the uncertainties in Propositions 6.3.3 and 6.3.4.

6.3.3. The classical Adams-Novikov E_{∞} -page.

THEOREM 6.3.7. The E_{∞} -page of the classical Adams-Novikov is depicted in the chart in [21] through the 59-stem. The chart includes all hidden extensions by 2, η , and ν . The chart is complete except for the uncertainties described in Propositions 6.3.9 and 6.3.10, and the following:

(1) There might be a hidden ν extension from $\alpha_{2/2}z_{45}$ to z_{51} .

(2) There might be a hidden 2 extension from $2\alpha_{4/4}z_{44}$ to z_{51} .

PROOF. The E_{∞} -page can be computed directly from Theorems 6.3.2 and 6.3.6 because we know the E_2 -page and all differentials up to some specified uncertainties.

The hidden extensions by 2, η , and ν all follow from extensions in $\pi_{*,*}(C\tau)$, as computed in Chapter 5.

Tables 44, 45, and 46 list all of the hidden extensions by 2, η , and ν in the motivic Adams-Novikov spectral sequence.

REMARK 6.3.8. From Lemma 4.2.31, the possible extension (1) in Theorem 6.3.7 occurs if and only if the possible extension (2) occurs.

PROPOSITION 6.3.9. In the 53-stem, 54-stem, and 55-stem, either case (1) or case (2) occurs.

(1) $\alpha_1 z_8 z''_{45} = \alpha_{2/2}^3 z''_{45}$ is a non-zero element of $E_{\infty}^{54,6}(S^0; BP)$; $E_{\infty}^{54,8}(S^0; BP)$ is zero; $\alpha_{2/2} z_{50}$ is zero in $E_{\infty}^{53,5}(S^0; BP)$; and there is a hidden ν extension from z_{50} to z_{53} . (2) $E_{\infty}^{54,6}(S^0; BP)$ is zero; $\alpha_1 z_{53}$ is a non-zero element of $E_{\infty}^{54,8}(S^0; BP)$; $\alpha_{2/2} z_{50} = z_8 z''_{45}$ in $E_{\infty}^{53,5}(S^0; BP)$; and there is a hidden ν extension from $\alpha_{2/2}^2 z'_{45}$ to $\alpha_1 z_{53}$.

PROOF. The two cases are associated with the two cases of Proposition 6.3.3. See also the first uncertainty in Theorem 6.3.6. $\hfill \Box$

PROPOSITION 6.3.10. In the 59-stem, either case (1) or case (2) occurs.

 z_{59,5} is the only non-zero element of E^{59,5}_∞(S⁰; BP); and z_{59,7} is the only non-zero element of E^{59,7}_∞(S⁰; BP).
 E^{59,5}_∞(S⁰; BP) is zero; and E^{59,7}(C⁰; BP) has two computing z and z'.

and $E_{\infty}^{59,7}(S^0; BP)$ has two generators $z_{59,7}$ and $z'_{59,7}$.

PROOF. The two cases are associated with the two cases of Proposition 6.3.4. See also the second uncertainty in Theorem 6.3.6. $\hfill \Box$

CHAPTER 7

Tables

Table	1:	Notation	for	$\pi_{*,*}$
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element	(s,w)	Ext	definition
τ	(0, -1)	au	
2	(0, 0)	h_0	
η	(1, 1)	h_1	
ν	(3,2)	h_2	
σ	(7, 4)	h_3	
ϵ	(8, 5)	c_0	
μ_{8k+1}	(1,1) + k(8,4)	$P^k h_1$	
ζ_{8k+3}	(3,2) + k(8,4)	$P^k h_2$	
κ	(14, 8)	d_0	
ρ_{15}	(15, 8)	$h_{0}^{3}h_{4}$	
η_4	(16, 9)	h_1h_4	$\eta^3 \cdot \eta_4 = 0$
$ u_4$	(18, 10)	h_2h_4	$ u_4 = \langle 2\sigma, \sigma, \nu angle $
$\overline{\sigma}$	(19, 11)	c_1	
$\overline{\kappa}$	(20, 11)	au g	
ρ_{23}	(23, 12)	$h_{0}^{2}i$	
$ heta_4$	(30, 16)	h_4^{2}	
ρ_{31}	(31, 16)	$h_0^{10} h_5$	
η_5	(32, 17)	h_1h_5	$\eta_5 \in \langle \eta, 2, \theta_4 \rangle, \eta^7 \cdot \eta_5 = 0$
$\theta_{4.5}$	(45, 24)	h_4^3	$4\theta_{4.5} \in \{h_0h_5d_0\}, \eta\theta_{4.5} \in \{B_1\}$
		-	$\sigma\theta_{4.5} \notin \{\tau h_1 h_3 g_2\}$

Table 2: May E_2 -page generators

	(m,s,f,w)	d_2	description
h_0	(1, 0, 1, 0)		h_{10}
h_1	(1, 1, 1, 1)		h_{11}
h_2	(1, 3, 1, 2)		h_{12}
b_{20}	(4, 4, 2, 2)	$ au h_1^3 + h_0^2 h_2$	h_{20}^2
h_3	(1, 7, 1, 4)	- •	h_{13}^{-1}
$h_0(1)$	(4, 7, 2, 4)	$h_0 h_2^2$	$h_{20}h_{21} + h_{11}h_{30}$
b_{21}	(4, 10, 2, 6)	$h_2^3 + h_1^2 h_3$	h_{21}^2
b_{30}	(6, 12, 2, 6)	$ au \tilde{h}_1 b_{21} + h_3 b_{20}$	$h_{30}^{\tilde{2}^{+}}$
h_4	(1, 15, 1, 8)		h_{14}

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Table 2: May E_2 -page generators

	(m,s,f,w)	d_2	description
$h_1(1)$	(4, 16, 2, 9)	$h_1 h_3^2$	$h_{21}h_{22} + h_{12}h_{31}$
b_{22}	(4, 22, 2, 12)	$h_3^3 + h_2^2 h_4$	h_{22}^2
b_{31}	(6, 26, 2, 14)	$h_4b_{21} + h_2b_{22}$	h_{31}^2
b_{40}	(8, 28, 2, 14)	$h_4b_{30} + \tau h_1b_{31}$	h_{40}^2
h_5	(1, 31, 1, 16)		h_{15}
$h_2(1)$	(4, 34, 2, 18)	$h_2 h_4^2$	$h_{22}h_{23} + h_{13}h_{32}$
$h_0(1,3)$	(7, 38, 3, 20)	$h_4^2 h_0(1) + h_0 h_2 h_2(1)$	$h_{50}h_{11}h_{13} + h_{40}h_{11}h_{23} +$
			$+h_{20}h_{41}h_{13}+h_{20}h_{31}h_{23}$
b_{23}	(4, 46, 2, 24)	$h_4^3 + h_3^2 h_5$	h_{23}^2
$h_0(1,2)$	(9, 46, 3, 24)	$h_3h_0(1,3)$	$h_{30}h_{31}h_{32} + h_{30}h_{41}h_{22} +$
			$+h_{40}h_{21}h_{32} + h_{40}h_{41}h_{12} +$
			$+h_{50}h_{21}h_{22}+h_{50}h_{31}h_{12}$
b_{32}	(6, 54, 2, 28)	$h_5b_{22} + h_3b_{23}$	h_{32}^2
b_{41}	(8, 58, 2, 30)	$h_5b_{31} + h_2b_{32}$	h_{41}^2
b_{50}	(10, 60, 2, 30)	$h_5b_{40} + \tau h_1b_{41}$	h_{50}^2
h_6	(1, 63, 1, 32)		h_{16}
$h_{3}(1)$	(4, 70, 2, 36)	$h_{3}h_{5}^{2}$	$h_{23}h_{24} + h_{14}h_{33}$

Table 3: May E_2 -page relations

relation	(m,s,f,w)
h_0h_1	(2, 1, 2, 1)
h_1h_2	(2, 4, 2, 3)
$h_2 b_{20} = h_0 h_0(1)$	(5, 7, 3, 4)
h_2h_3	(2, 10, 2, 6)
$h_2 h_0(1) = h_0 b_{21}$	(5, 10, 3, 6)
$h_3h_0(1)$	(5, 14, 3, 8)
$h_0(1)^2 = b_{20}b_{21} + h_1^2b_{30}$	(8, 14, 4, 8)
$h_0 h_1(1)$	(5, 16, 3, 9)
$h_3 b_{21} = h_1 h_1(1)$	(5, 17, 3, 10)
$b_{20}h_1(1) = h_1h_3b_{30}$	(8, 20, 4, 11)
h_3h_4	(2, 22, 2, 12)
$h_3h_1(1) = h_1b_{22}$	$\left(5,23,3,13\right)$
$h_0(1)h_1(1)$	$\left(8,23,4,13\right)$
$b_{20}b_{22} = h_0^2 b_{31} + h_3^2 b_{30}$	(8, 26, 4, 14)
$b_{22}h_0(1) = h_0h_2b_{31}$	(8, 29, 4, 16)
$h_4h_1(1)$	$\left(5, 31, 3, 17\right)$
$h_1(1)^2 = b_{21}b_{22} + h_2^2b_{31}$	$\left(8, 32, 4, 18\right)$
$h_1 h_2(1)$	$\left(5,35,3,19\right)$
$h_4 b_{22} = h_2 h_2(1)$	$\left(5,37,3,20\right)$
$b_{20}h_2(1) = h_0h_0(1,3)$	(8, 38, 4, 20)
$h_2h_0(1,3) = h_0h_4b_{31}$	(8, 41, 4, 22)
$h_0(1)h_2(1) = h_0h_4b_{31}$	(8, 41, 4, 22)

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Table 3: May E_2 -page relations

relation	(m,s,f,w)
$b_{21}h_2(1) = h_2h_4b_{31}$	(8, 44, 4, 24)
$h_0(1)h_0(1,3) = h_1^2 h_4 b_{40} + h_4 b_{20} b_{31}$	(11, 45, 5, 24)
h_4h_5	(2, 46, 2, 24)
$b_{30}h_2(1) = h_0h_0(1,2) + h_2h_4b_{40}$	(10, 46, 4, 24)
$b_{21}h_0(1,3) = h_1^2h_0(1,2) + h_4b_{31}h_0(1)$	(11, 48, 5, 26)
$h_4h_2(1) = h_2b_{23}$	(5, 49, 3, 26)
$h_1(1)h_2(1)$	(8, 50, 4, 27)
$b_{30}h_0(1,3) = b_{20}h_0(1,2) + h_4b_{40}h_0(1)$	(13, 50, 5, 26)
$b_{23}h_0(1) = h_4h_0(1,3)$	(8, 53, 4, 28)
$h_0(1)h_0(1,2) = h_4b_{40}b_{21} + h_4b_{30}b_{31}$	(13, 53, 5, 28)
$h_1(1)h_0(1,3) = h_1h_3h_0(1,2)$	(11, 54, 5, 29)
$b_{21}b_{23} = h_1^2 b_{32} + h_4^2 b_{31}$	(8, 56, 4, 30)
$b_{30}b_{23} = b_{20}b_{32} + h_0^2 b_{41} + h_4^2 b_{40}$	(10, 58, 4, 30)
$b_{22}h_0(1,3) = h_3^2h_0(1,2) + h_0b_{31}h_2(1)$	(11, 60, 5, 32)
$b_{32}h_0(1) = h_4h_0(1,2) + h_0h_2b_{41}$	(10, 61, 4, 32)
$b_{23}h_1(1) = h_1h_3b_{32}$	(8, 62, 4, 33)
$h_5h_2(1)$	(5, 65, 3, 34)
$b_{22}b_{23} = h_2(1)^2 + h_3^2 b_{32}$	(8, 68, 4, 36)
$h_5 h_0(1,3)$	(8, 69, 4, 36)

Table 4: The May d_4 differential

	(m,s,f,w)	description	d_4
P	(8, 8, 4, 4)	b_{20}^2	$h_0^4 h_3$
ν	(7, 15, 3, 8)	$h_2 b_{30}$	$h_0^2 h_3^2$
g	(8, 20, 4, 12)	b_{21}^2	$h_{1}^{4}h_{4}$
Δ	(12, 24, 4, 12)	b_{30}^2	$\tau^2 h_2 g + P h_4$
ν_1	$\left(7, 33, 3, 18\right)$	$h_{3}b_{31}$	$h_1^2 h_4^2$
x_{34}	(7, 34, 5, 18)	$h_0^3 h_2(1) + h_0 h_4^2 b_{20}$	
x_{35}	(10, 35, 4, 18)	$h_0 h_3 b_{40}$	x_{34}
x_{47}	(13, 47, 5, 25)	$h_2 b_{40} h_1(1)$	$ au h_1^2 g_2$
x_{49}	(10, 49, 4, 26)	$h_2 h_0(1,2)$	$h_0h_3c_2$
Δ_1	(12, 52, 4, 28)	b_{31}^2	$h_5g + h_3g_2$
Γ	(16, 56, 4, 28)	b_{40}^2	$\Delta h_5 + \tau^2 \Delta_1 h_2$
x_{59}	(19, 59, 7, 31)	$h_2 b_{30} b_{40} h_1(1)$	$\tau^2 e_1 g$
x_{63}	(20, 63, 8, 33)	$h_1 b_{20} b_{30} h_0(1,2) +$	$ au h_5 d_0 e_0$
		$+\tau h_1^2 b_{31} b_{40} h_0(1) +$	
		$+\tau b_{20}b_{31}^2h_0(1)$	
x_{65}	(20, 65, 8, 34)	$h_0h_3b_{20}b_{31}b_{40}$	$h_0^3 A^{\prime\prime}$
x_{68}	(16, 68, 6, 36)	$h_3^2 b_{31} b_{40} + \tau b_{31}^2 h_1(1)$	τs_1
ν_2	$\left(7,69,3,36\right)$	$h_4 b_{32}$	$h_2^2 h_5^2$
x_{69}	(13, 69, 5, 36)	$h_3b_{40}h_2(1)$	$h_0^2 d_2$

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Table 5: 7	The	Mav	d_6	differential
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	(m,s,f,w)	description	d_6
Pi	(23, 31, 11, 16)	$P^2 \nu$	$h_0^5 s$
Pr	(22, 38, 10, 20)	$P\Delta h_2^2$	$h_0^{\dot{6}}x$
Y	(16, 45, 6, 24)	$Bh_0(1)$	$h_0^3 g_2$
ϕ	(17, 49, 7, 27)	$Bh_{1}b_{21}$	$h_1^3 h_5 d_0$
X	(23, 54, 9, 28)	$Bh_0b_{20}b_{30}$	$Ph_0h_5d_0$
PQ'	(37, 55, 17, 28)	$P^2 \Delta h_0^2 \nu$	$h_0^9 X$
x_{56}	(21, 56, 9, 31)	$Bh_1b_{21}h_0(1)$	$h_1^2 h_5 c_0 d_0$
x'_{56}	(24, 56, 10, 30)	PBb_{21}	$Ph_1^2h_5d_0$
ϕ'	(23, 57, 9, 30)	$Bh_0b_{30}h_0(1)$	$Ph_0h_5e_0$
Px_{56}	(29, 64, 13, 35)		$Ph_1^2h_5c_0d_0$
Px'_{56}	(32, 64, 14, 34)		$P^2h_1^2h_5d_0$
B_{23}	(24, 65, 10, 36)	Yg	$h_1^2 h_5 d_0 e_0$
$P\phi'$	(31, 65, 13, 34)		$P^2h_0h_5e_0$
$c_0 g^3$	(29, 68, 15, 41)		$h_1^{10}D_4$
$\Delta h_0^2 Y$	(30, 69, 12, 36)		$\Delta h_0^5 g_2 + h_0 h_5 d_0 i$

Table 6: The May d_8 differential

-	(-
element	(m, s, f, w)	description	d_8
P^2	(16, 16, 8, 8)		$h_0^8 h_4$
Δh_3	(13, 31, 5, 16)		$h_0^{4}h_4^{2}$
g^2	(16, 40, 8, 24)		$h_{1}^{8}h_{5}$
w	(21, 45, 9, 25)	$\Delta h_1 g$	$Ph_1^5h_5$
Δ^2	(24, 48, 8, 24)		$P^2 h_5$
$\Delta c_0 g$	(25, 52, 11, 29)		$Ph_{1}^{4}h_{5}c_{0}$
Q_3	(13, 67, 5, 36)	$\Delta_1 h_4$	$h_{1}^{4}h_{5}^{2}$
$\Gamma h_0 h_3^2$	(19, 70, 7, 36)		$h_0^4 p'$

Table 7: Higher May differentials

1 (1	1
element	(m, s, f, w)	a_r	value
P^2Q'	(45, 63, 21, 32)	d_{12}	$Ph_{0}^{10}h_{5}i$
P^4	(32, 32, 16, 16)	d_{16}	$h_0^{16}h_5$
$\Delta^2 h_4$	(25, 63, 9, 32)	d_{16}	$h_0^8 h_5^2$
P^8	(64, 64, 32, 32)	d_{32}	$h_0^{32}h_6$

Table 8: Adams E_2 generators

element	(m,s,f,w)	May description	d_2	reference
h_0	(1, 0, 1, 0)			
h_1	(1, 1, 1, 1)			
h_2	(1, 3, 1, 2)			
h_3	(1, 7, 1, 4)			
c_0	(5,8,3,5)	$h_1 h_0(1)$		
Ph_1	(9,9,5,5)			
Ph_2	(9, 11, 5, 6)			
d_0	(8, 14, 4, 8)	$h_0(1)^2$		
h_4	(1, 15, 1, 8)		$h_0 h_3^2$	image of J
Pc_0	$\left(13,16,7,9\right)$			
e_0	(8, 17, 4, 10)	$b_{21}h_0(1)$	$h_{1}^{2}d_{0}$	Lemma $3.3.1$
P^2h_1	$\left(17,17,9,9\right)$			
f_0	(8, 18, 4, 10)	$h_2 \nu$	$h_0^2 e_0$	tmf
c_1	(5, 19, 3, 11)	$h_2h_1(1)$		
P^2h_2	(17, 19, 9, 10)	- 0		
au g	(8, 20, 4, 11)	$ au b_{21}^2$		
Pd_0	(16, 22, 8, 12)			
$h_2 g$	(9, 23, 5, 14)	_		
i	(15, 23, 7, 12)	$P\nu$	Ph_0d_0	tmf
$P^2 c_0$	(21, 24, 11, 13)			
Pe_0	(16, 25, 8, 14)		$Ph_1^2d_0$	Lemma 3.3.1
$P^{\circ}h_{1}$	(25, 25, 13, 13)	1 1 1 (1)9	D1	
j	(15, 26, 7, 14)	$h_0 b_{30} h_0(1)^2$	Ph_0e_0	tmf
h_3g	(9, 27, 5, 16)		$h_0h_2^2g$	Lemma 3.3.3
$P^{3}h_{2}$	(25, 27, 13, 14)	,	1 12	
ĸ	(15, 29, 7, 16)	$d_0\nu$	$h_0 d_0^2$	tmf
r P^{2} l	(14, 30, 6, 16)	Δh_2^2		
$P^2 d_0$	(24, 30, 12, 16)		1 12	·
h_5	(1, 31, 1, 16)	1 1 1 (1)	$h_0 h_4^2$	image of J
n	(11, 31, 5, 17)	$n_2 n_3 n_1(1)$		
a_1	(0, 32, 4, 10) (14, 22, 6, 17)	$n_1(1)^-$		
q	(14, 32, 0, 17) (15, 22, 7, 18)	$\Delta n_1 n_3$	hdo	Lomma 220
$D^3 a$	(10, 32, 7, 10) (20, 22, 15, 17)	$e_0\nu$	$n_0 u_0 e_0$	Lemma 5.5.2
Γc_0	(29, 32, 10, 17) (8, 32, 4, 18)	hau		
$p P^2 c_1$	(0, 33, 4, 10) (24, 22, 12, 18)	$n_0\nu_1$	$D^2 h^2 d_a$	Lomma 221
P^4h	(24, 33, 12, 18) (33, 33, 17, 17)		$I n_1 a_0$	Lemma 3.3.1
P_i	(33, 35, 11, 11) (23, 34, 11, 18)		$P^2h_{0}c_{0}$	tmf
1 J m	(15, 35, 7, 20)	<i>au</i>	h_0e^2	tmf
P^4h_2	$(33 \ 35 \ 17 \ 18)$	9°		5110
$\frac{1}{t}$	(12, 36, 6, 20)	$\tau b_{21}^2 h_1(1) + h_2^2 h_{22} h_{22}$		
\hat{x}	(11, 37, 5, 20)	$h_{21}h_{1}(1) + h_{1}h_{22}h_{30}$		
$\tilde{e}_0 q$	(16, 37, 8, 22)	2022030 + 702040	$h_1^2 e_0^2$	Lemma 3.3.4
e_1	(8, 38, 4, 21)	$b_{22}h_1(1)$		
$ \begin{array}{c} f_{0} \\ c_{1} \\ P^{2}h_{2} \\ \tau g \\ Pd_{0} \\ h_{2}g \\ i \\ P^{2}c_{0} \\ Pe_{0} \\ P^{3}h_{1} \\ j \\ h_{3}g \\ P^{3}h_{2} \\ k \\ r \\ P^{2}d_{0} \\ h_{5} \\ n \\ d_{1} \\ q \\ l \\ P^{3}c_{0} \\ p \\ P^{2}e_{0} \\ P^{4}h_{1} \\ Pj \\ m \\ P^{4}h_{2} \\ t \\ x \\ e_{0}g \\ e_{1} \end{array} $	$\begin{array}{l}(8,18,4,10)\\(5,19,3,11)\\(17,19,9,10)\\(8,20,4,11)\\(16,22,8,12)\\(9,23,5,14)\\(15,23,7,12)\\(21,24,11,13)\\(16,25,8,14)\\(25,25,13,13)\\(15,26,7,14)\\(9,27,5,16)\\(25,27,13,14)\\(15,29,7,16)\\(14,30,6,16)\\(24,30,12,16)\\(14,30,6,16)\\(24,30,12,16)\\(11,31,5,17)\\(8,32,4,18)\\(14,32,6,17)\\(15,32,7,18)\\(29,32,15,17)\\(8,33,4,18)\\(24,33,12,18)\\(33,33,17,17)\\(23,34,11,18)\\(15,35,7,20)\\(33,35,17,18)\\(12,36,6,20)\\(11,37,5,20)\\(16,37,8,22)\\(8,38,4,21)\end{array}$	$ \begin{array}{l} h_{2}\nu \\ h_{2}h_{1}(1) \\ \\ \tau b_{21}^{2} \\ \\ P\nu \\ \\ h_{0}b_{30}h_{0}(1)^{2} \\ \\ \\ \frac{d_{0}\nu}{\Delta h_{2}^{2}} \\ \\ h_{2}b_{30}h_{1}(1) \\ h_{1}(1)^{2} \\ \\ \Delta h_{1}h_{3} \\ e_{0}\nu \\ \\ h_{0}\nu_{1} \\ \\ \\ g\nu \\ \\ \\ \tau b_{21}^{2}h_{1}(1) + h_{1}^{2}b_{22}b_{30} \\ \\ h_{2}b_{22}b_{30} + h_{2}^{3}b_{40} \\ \\ b_{22}h_{1}(1) \end{array} $	$h_0^2 e_0$ $Ph_0 d_0$ $Ph_1^2 d_0$ $Ph_0 e_0$ $h_0 h_2^2 g$ $h_0 d_0^2$ $h_0 h_4^2$ $h_0 d_0 e_0$ $P^2 h_1^2 d_0$ $P^2 h_0 e_0$ $h_0 e_0^2$ $h_1^2 e_0^2$	tmf tmf Lemma 3.3.1 tmf Lemma 3.3.2 tmf image of J Lemma 3.3.2 Lemma 3.3.1 tmf tmf tmf

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Table 8: Adams E_2 generators

element	(m,s,f,w)	May description	d_2	reference
y	(14, 38, 6, 20)	Δh_3^2	$h_0^3 x$	Table 18
P^3d_0	(32, 38, 16, 20)			
c_1g	$\left(13,39,7,23\right)$			
u	$\left(21,39,9,21\right)$	$\Delta h_1 d_0$		
P^2i	(31, 39, 15, 20)		$P^{3}h_{0}d_{0}$	tmf
f_1	(8, 40, 4, 22)	$h_3 u_1$		
$ au g^2$	(16, 40, 8, 23)			
P^4c_0	(37, 40, 19, 21)			
c_2	(5, 41, 3, 22)	$h_3h_2(1)$	$h_0 f_1$	Table 18
z	(22, 41, 10, 22)	$\Delta h_0^2 e_0$		
P^3e_0	(32, 41, 16, 22)		$P^{3}h_{1}^{2}d_{0}$	Lemma $3.3.1$
P^5h_1	(41, 41, 21, 21)			
v	(21, 42, 9, 23)	$\Delta h_1 e_0$	$h_1^2 u$	Table 18
$P^2 j$	(31, 42, 15, 22)		$P^{3}h_{0}e_{0}$	tmf
h_2g^2	(17, 43, 9, 26)			
P^5h_2	(41, 43, 21, 22)			
g_2	(8, 44, 4, 24)	b_{22}^2		
au w	(21, 45, 9, 24)	$\tau \overline{\Delta} h_1 g$		
B_1	(17, 46, 7, 25)	Yh_1		
N	(18, 46, 8, 25)	$\Delta h_2 c_1$		
u'	(25, 46, 11, 25)	$\Delta c_0 d_0$	$ au h_0 d_0^2 e_0$	Lemma $3.3.5$
h_3g^2	(17, 47, 9, 28)		$h_0 h_2^2 g^2$	Lemma 3.3.3
$P^{4}d_{0}$	(40, 46, 20, 24)		20	
Q'	(29, 47, 13, 24)	$P\Delta h_0^2 \nu$	$h_0 i^2$	Table 18
Pu	(29, 47, 13, 25)	$P\Delta h_1 d_0$		
B_2	(17, 48, 7, 26)	Yh_2		
$P^{5}c_{0}$	(45, 48, 23, 25)			
v'	(25, 49, 11, 27)	$\Delta c_0 e_0$	$h_1^2 u' +$	Lemma $3.3.5$
	X • • • • /		$+\tau h_0 d_0 e_0^2$	
P^4e_0	(40, 49, 20, 26)		$P^4h_1^2d_0$	Lemma 3.3.1
$P^{6}h_{1}$	(49, 49, 25, 25)		1	
C	(14, 50, 6, 27)	$h_2 x_{47}$		
gr	(22, 50, 10, 28)	$\Delta h_2^2 q$		
$\tilde{P}v$	(29, 50, 13, 27)	$P\Delta h_1 e_0$	Ph_1^2u	Table 18
$P^3 j$	(39, 50, 19, 26)	- •	$P^4 \dot{h}_0 e_0$	tmf
G_3	(21, 51, 9, 28)	$\Delta h_3 q$	$h_0 qr$	Lemma 3.3.6
qn	(19, 51, 9, 29)	*0	•0	
\tilde{P}^6h_2	(49, 51, 25, 26)			
D_1	(11, 52, 5, 28)	$h_2 x_{49}$	$h_0^2 h_3 q_2$	Lemma 3.3.13
d_1q	(16, 52, 8, 30)	- 10	0 002	
i_1	(15, 53, 7, 30)	$q\nu_1$		Lemma 3.3.8
\tilde{B}_8	(21, 53, 9, 29)	Yc_0		-
x'	(24, 53, 10, 28)	PY		
τG	(14, 54, 6, 29)	$ au\Delta_1 h_1^2$	$h_5c_0d_0$	Lemma 3.3.12
R_1	(26, 54, 10, 28)	$\Delta^2 \hat{h}_2^2$	$h_0^2 x'$	Table 18

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Table 8: Adams E_2 generators

element	(m,s,f,w)	May description	d_2	reference
Pu'	(33, 54, 15, 29)	$P\Delta c_0 d_0$	$\tau Ph_0 d_0^2 e_0$	Lemma $3.3.5$
P^5d_0	(48, 54, 24, 28)			
B_6	(17, 55, 7, 30)	$Bh_1h_1(1)$		Lemma $3.3.7$
gm	(23, 55, 11, 32)	$g^2 \nu$	$h_0 e_0^2 g$	Lemma $3.3.9$
P^2u	(37, 55, 17, 29)			
P^4i	(47, 55, 23, 28)		$P^{5}h_{0}d_{0}$	tmf
gt	(20, 56, 10, 32)			
Q_1	(26, 56, 10, 29)	$\Delta^2 h_1 h_3$	$ au h_1^2 x'$	Lemma 3.3.10
$P^{6}c_{0}$	(53, 56, 27, 29)			
D_4	(14, 57, 6, 31)	$h_1 b_{21} h_0(1,2)$	h_1B_6	Lemma 3.3.11
Q_2	(19, 57, 7, 30)	$\Delta \nu_1$		
D_{11}	(21, 57, 9, 31)	$\Delta h_1 d_1$		_
Pv'	(33, 57, 15, 31)	$P\Delta c_0 e_0$	$Ph_{1}^{2}u'+$	Lemma 3.3.5
5	($+\tau h_0 d_0^4$	
$P^{5}e_{0}$	(48, 57, 24, 30)		$P^{5}h_{1}^{2}d_{0}$	Lemma 3.3.1
$P'h_1$	(57, 57, 29, 29)			
D_2	(16, 58, 6, 30)	$h_0 b_{30} h_0(1,2)$	h_0Q_2	Lemma 3.3.15
e_1g	(16, 58, 8, 33)		5 212	— 11 40
$P^2 v$	(37, 58, 17, 31)		$P^{2}h_{1}^{2}u$	Table 18
$P^{4}j$	(47, 58, 23, 30)		$P^{3}h_{0}e_{0}$	tmf
j_1	(15, 59, 7, 33)	$h_1b_{21}b_{22}b_{31}$		
B_{21}	(24, 59, 10, 32)	$Y d_0$		
$c_1 g^2$	(21, 59, 11, 35)			
$P'h_2$	(57, 59, 29, 30)	171		
B_3	(17, 60, 7, 32)	Yh_4	1 D	T 0.0.10
B_4	(23, 60, 9, 32)	Υν	$h_0 B_{21}$	Lemma 3.3.16
τg^{s}	(24, 60, 12, 35)			
$h_0 g^{\circ}$	(25, 60, 13, 36)	1 (1 0)		
D_3	(10, 61, 4, 32)	$h_4h_0(1,2)$	1 D	T 0.0.1F
A	(16, 61, 6, 32)	$h_2 b_{30} h_0(1,2)$	$h_0 B_3$	Lemma 3.3.15
A'	(10, 01, 0, 32)	$n_2 b_{30} n_0 (1, 2) +$		
D	$(17 \ c1 \ 7 \ 99)$	$+n_0n_3o_{31}o_{40}$		
B_7	(17, 01, 7, 33)	Bn_10_{22}	$L^2 D$	T
X_1	(23, 61, 9, 32)	Δx	$h_0^-B_4 + h_0^-B_4$	Lemma 3.3.12
77	(19 69 5 99)	L(1)L(1,0)	$+\tau n_1 B_{21}$	T
Π_1	(13, 02, 3, 33)	$n_1(1)n_0(1,2)$	B_7	Lemma 5.5.11
C_0	(20, 62, 8, 33)	$n_2 x_{59}$		
E_1	(20, 62, 8, 33)	Δe_1	12 D	T
B_{22}	(24, 02, 10, 34)	$Y e_0$	$n_{1}B_{21}$	Lemma 5.5.10
R P^2 /	(20, 62, 10, 32)	$\Delta^2 h_3^2$	$D^2 I I^2$	T 995
$P^- u$	(41, 62, 19, 33)		$\tau P^{-} n_0 d_0^{-} e_0$	Lemma 3.3.5
$P^a a_0$	(30, 02, 28, 32) (1, 62, 1, 20)		h h2	ima ma cf T
n_6	(1, 03, 1, 32)	$L = L (1)^{2}$	$n_0 n_{\overline{5}}$	image of J
U" V	(17, 63, 7, 34)	$n_2 b_{40} n_1 (1)^2$	12 D	Lemma 3.3.17
X_2	(17, 63, 7, 34)	$ au h_1 b_{21} b_{31}^2 +$	$h_{1}^{2}B_{3}$	Lemma 3.3.18

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Table 8: Adams E_2 generators

element	(m,s,f,w)	May description	d_2	reference
		$+h_1^2h_3b_{31}b_{40}$		
$h_2 g^3$	(25, 63, 13, 38)	1 0 01 10		
$P^{3}u$	(45, 63, 21, 33)			
$A^{\prime\prime}$	(14, 64, 6, 34)	$h_0 b_{31} h_0(1,3)$	$h_0 X_2$	Lemma 3.3.15
q_1	(26, 64, 10, 33)	$\Delta^2 h_1 h_4$	° -	
Ū	(34, 64, 14, 34)	$\Delta^2 h_1^2 d_0$	Ph_1^2x'	Lemma 3.3.10
$P^{7}c_{0}$	(61, 64, 31, 33)	1 0	1	
k_1	(15, 65, 7, 36)	$d_1\nu_1$		
τB_{23}	(24, 65, 10, 35)			
R_2	(33, 65, 13, 34)	$\Delta^2 h_0 e_0$	$h_0 U$	Lemma 3.3.10
$\tau q w$	(29, 65, 13, 36)	0 0	0	
P^2v'	(41, 65, 19, 35)		$P^2h_1^2u'+$	Lemma $3.3.5$
	() / / /		$+\tau P h_0 d_0^4$	
$P^{6}e_{0}$	(56, 65, 28, 34)		$P^6 h_1^2 d_0$	Lemma 3.3.1
$P^{8}h_{1}^{\circ}$	(65, 65, 33, 33)		1 0	
r_1	(14, 66, 6, 36)	$\Delta_1 h_2^2$		
τG_0	(17, 66, 7, 35)	$\tau b_{21}^2 h_0(1,2) +$	$h_2C_0 +$	Lemma 3.3.19
0	(.)) .))	$+h_1h_3b_{30}h_0(1,2)$	$+h_1h_2Q_2$	
$ au B_5$	(24, 66, 10, 35)	$\tau b_{21}^2 b_{20}^2 b_{22} +$	$\tau h_0^2 B_{23}$	Lemma 3.3.20
. = 0	(,,,)	$+\tau h_{2}^{2}b_{21}^{2}b_{30}b_{40}+$	0-23	
		$+h_1h_3h_{30}^3h_{20}$		
D'_2	(24, 66, 10, 34)	$PD_{2} + \Delta h_{0}h_{3}x_{35}$	$\tau^2 h_0^2 B_{23}$	Lemma 3.3.20
$P^{3}v$	(45, 66, 21, 35)	1 2 2 1 - 10003~33	$P^{3}h_{1}^{2}u$	Lemma 3.3.21
$P^5 i$	(55, 66, 27, 34)		$P^6h_0e_0$	tmf
$\frac{1}{n_1}$	(11, 67, 5, 36)	$h_{2}b_{21}h_{2}(1)$	h_0r_1	[11 . VI.1]
τQ_3	(13, 67, 5, 35)			
$h_0 Q_3$	(14, 67, 6, 36)			
C''	(21, 67, 9, 37)	ax_{47}		
X_3	(21, 67, 9, 36)	$\Delta_1 h_0^2 \nu + \tau \Delta_1 h_1 d_0$		Lemma 3.3.22
C_{11}	(29, 67, 11, 35)	$\Delta^2 c_1$		
$h_{3}a^{3}$	(25, 67, 13, 40)	_ 01	$h_0 h_0^2 a^3$	Lemma 3.3.3
P^8h_2	(65, 67, 33, 34)		101023	
d_2	(8, 68, 4, 36)	$h_{2}(1)^{2}$		
G_{21}	(20, 68, 8, 36)	Δa_2	$h_0 X_2$	Lemma 3.3.12
$h_2 B_{22}$	(25, 68, 11, 38)	-92		
G_{11}	$(33 \ 68 \ 13 \ 36)$	$\Lambda^2 h_2 e_0$	$h_0 d_0 x'$	Lemma 3 3 10
n'	(86, 694, 36)	$h_0\nu_2$	ri(a)a	Lonnia 0.0.10
D'_{2}	(18, 69, 8, 37)	$h_1 b_{20} b_{31} h_0 (1 \ 3) +$	$h_1 X_2$	Lemma 3 3 18
23	(10,00,0,01)	$+h_1^3b_{40}h_0(1,3)$	101319	Lomma 9.9.10
h_2G_0	(18, 69, 8, 38)		$h_1 C''$	Lemma 3.3.19
P(A+A')	(24, 69, 10, 36)		$ au^2 h_0 h_2 B_{23}$	Lemma 3.3.20
h_2B_5	(25, 69, 11, 38)			
τW_1	$\left(33,69,13,36\right)$	$ au\Delta^2 h_1 g$		
P^2x'	(40, 69, 18, 36)			
p_1	(8, 70, 4, 37)	$h_1\nu_2$		
Table 8: Adams E_2 generators

element	(m,s,f,w)	May description	d_2	reference
$ \begin{array}{c} h_2 Q_3 \\ R_1' \\ P^3 u' \\ P^7 d_0 \end{array} $	$\begin{array}{c}(14,70,6,38)\\(41,70,17,36)\\(49,70,23,37)\\(64,70,32,36)\end{array}$	$\Delta^2 P h_0 d_0$	$\frac{P^2h_0x'}{\tau P^3h_0d_0^2e_0}$	Lemma 3.3.23 Lemma 3.3.5

Table 9: Temporary May E_∞ generators

element	(m,s,f,w)	description
s	(13, 30, 7, 16)	$Ph_4h_0(1) + h_0^3b_{20}b_{31}$
P^2s	(29, 46, 15, 24)	
S_1	(25, 54, 11, 28)	$h_0^2 X$
g_2'	(23, 60, 11, 32)	$P\Delta_1 h_0^3$
τPG	(22, 62, 10, 33)	$\tau P \Delta_1 h_1^2$
Ph_5i	(24, 62, 12, 32)	
P^4s	(45, 62, 23, 32)	
PD_4	(22, 65, 10, 35)	
s_1	(13, 67, 7, 37)	$h_5 b_{21}^2 h_1(1) + h_1^3 b_{21} b_{32}$
Ph_5^2	(10, 70, 6, 36)	-
$\tau P^2 G$	(30, 70, 14, 37)	
P^2S_1	(41, 70, 19, 36)	

Table 10: Ambiguous Ext generators

element	(m,s,f,w)	ambiguity	definition
f_0	(8, 18, 4, 10)	$ au h_1^3 h_4$	
y	(14, 38, 6, 20)	$ au^2 \dot{h}_2^2 d_1$	
f_1	(8, 40, 4, 22)	$h_{1}^{2}h_{3}h_{5}$	
u'	(25, 46, 11, 25)	$ au d_0 l$	$\tau \cdot u' = 0$
B_2	(17, 48, 7, 26)	$h_0^2 h_5 e_0$	
v'	(25, 49, 11, 27)	$ au e_0 l$	$\tau \cdot v' = 0$
G_3	(21, 51, 9, 28)	au gn	$h_2 \cdot G_3 = 0$
R_1	(26, 54, 10, 28)	$\tau Ph_1h_5d_0$	
Pu'	(33, 54, 15, 29)	$ au d_0^2 j$	$\langle u',h_0^3,h_0h_3 angle$
B_6	(17, 55, 7, 30)	$\tau h_1 G$	$\tau \cdot B_6 = 0$
Q_1	(26, 56, 10, 29)	$ au^3 gt$	
Pv'	(33, 57, 15, 31)	$ au d_0^2 k$	$\tau \cdot Pv' = 0$
B_4	(23, 60, 9, 32)	h_0h_5k	
$ au g^3$	(24, 60, 12, 35)	$h_1^4 h_5 c_0 e_0$	$h_1^2 \cdot \tau g^3 = 0$
H_1	(13, 62, 5, 33)	h_1D_3	
R	(26, 62, 10, 32)	$\tau^2 B_{22}, \tau^2 PG$	$h_1 \cdot R = 0$
P^2u'	(41, 62, 19, 33)	$ au P d_0^2 j$	$\langle u', h_0^3, h_0^5 h_4 angle$
q_1	(26, 64, 10, 33)	$ au^2 h_1^2 E_1$	

Table 10: Ambiguous Ext generators

element	(m,s,f,w)	ambiguity	definition
U	(34, 64, 14, 34)	$\tau^2 km$	$\Delta^2 h_1^2 d_0$ in $\operatorname{Ext}_{A(2)}$
τB_{23}	(24, 65, 10, 35)	PD_4	$c_0Q_2 + \langle h_1, h_1h_5d_0e_0, \tau \rangle$
R_2	(33, 65, 13, 34)	$ au^3 g w$	0 in $\operatorname{Ext}_{A(2)}$
au g w	(29, 65, 13, 36)	$Ph_1h_5c_0e_0$	$h_1 \cdot \tau g w = 0$
P^2v'	(41, 65, 19, 35)	$ au d_0^3 i$	$\tau \cdot P^2 v' = 0$
τG_0	(17, 66, 7, 35)	$ au h_0 r_1$	$\langle au, h_1^2 H_1, h_1 angle$
n_1	(11, 67, 5, 36)	$h_{1}^{4}h_{6}$	$h_1 \cdot n_1 = 0$
$ au Q_3$	$\left(13,67,5,35\right)$	τn_1	Adams $d_2(\tau Q_3) = 0$
h_0Q_3	(14, 67, 6, 36)	h_0n_1	$\tau \cdot h_0 Q_3 = h_0 \cdot \tau Q_3$
C_{11}	(29, 67, 11, 35)	$ au h_0^2 X_3$	$h_0 \cdot C_{11} = 0$
G_{21}	(20, 68, 8, 36)	$\tau h_3 B_7$	
G_{11}	(33, 68, 13, 36)	$h_0^5 G_{21}$	
h_2B_5	(25, 69, 11, 38)	$h_{1}^{2}X_{3}$	$h_1 \cdot h_2 B_5 = 0$
$P^2 x'$	(40, 69, 18, 36)	$d_0^2 z$	0 in $\operatorname{Ext}_{A(2)}$
R'_1	(41, 70, 17, 36)	$ au^3 d_0^2 v$	$\langle \tau, Pc_0 x', h_0 \rangle$
P^3u'	(49, 70, 23, 37)	$ au P^2 d_0^2 j$	$\langle u', h_0^3, h_0^3 i angle$

Table 11: Hidden May τ extensions

(s, f, w)	x	$ au \cdot x$	reference
(30, 11, 16)	Pc_0d_0	$h_{0}^{4}s$	classical
(37, 9, 20)	$ au h_0 e_0 g$	$h_0^4 x$	classical
(37, 10, 20)	$ au h_0^2 e_0 g$	$h_0^5 x$	classical
(41, 5, 22)	$h_1 f_1$	$h_0^2 c_2$	classical
(43, 9, 24)	$ au h_2 g^2$	$Ph_1^4h_5$	Lemma $2.4.2$
(46, 19, 24)	$P^{3}c_{0}d_{0}$	$P^2 h_0^4 s$	classical
(53, 9, 28)	B_8	Ph_5d_0	Lemma $2.4.3$
(53, 17, 28)	$ au Ph_0 d_0^2 e_0$	$h_0^7 x'$	classical
(53, 18, 28)	$ au Ph_0^2 d_0^2 e_0$	$h_0^8 x'$	classical
(54, 10, 29)	h_1B_8	$Ph_1h_5d_0$	$\tau \cdot B_8$
(54, 15, 28)	Pu'	$h_{0}^{4}S_{1}$	Lemma $2.4.4$
(54, 16, 28)	$ au h_0 d_0^2 j$	$h_0^5 S_1$	classical
(54, 17, 28)	$ au^2 Ph_1 d_0^2 e_0$	$h_0^6 S_1$	classical
(58, 8, 30)	h_1Q_2	$h_0^2 D_2$	classical
(60, 13, 34)	$ au h_0 g^3$	$Ph_1^4h_5e_0$	Lemma $2.4.2$
(61, 12, 33)	$h_1^2 B_{21}$	$Ph_5c_0d_0$	Lemma $2.4.3$
(61, 13, 32)	$x'c_0$	$h_0^4 X_1$	classical
(62, 13, 34)	$h_1^3 B_{21}$	$Ph_1h_5c_0d_0$	$ au \cdot h_1^2 B_{21}$
(62, 19, 32)	P^2u'	$Ph_0^7h_5i$	Lemma $2.4.4$
(62, 20, 32)	$ au Ph_0 d_0^2 j$	$Ph_{0}^{8}h_{5}i$	classical
(62, 21, 32)	$ au^2 P^2 h_1 d_0^2 e_0$	$Ph_0^9h_5i$	classical
(62, 27, 32)	$P^5c_0d_0$	$P^{4}h_{0}^{4}s$	classical
(64, 8, 34)	$h_1 X_2$	$h_0^2 A^{\prime\prime}$	classical

(s, f, w)	x	$\tau \cdot x$	reference
(65, 7, 35)	k_1	h_2h_5n	Lemma $2.4.5$
(65, 9, 34)	$h_1h_3Q_2$	$h_0^2 h_3 D_2$	classical
(66, 12, 34)	$h_{1}^{2}q_{1}$	$h_0^2 D_2'$	classical
(67, 13, 36)	$B_8 d_0$	$h_0^4 X_3$	Lemma $2.4.3$
(68, 12, 36)	$\tau h_0 h_2 B_{23}$	$h_0^4 G_{21} + h_5 d_0 i$	classical
(69, 7, 36)	$ au h_1^2 Q_3$	$h_{0}^{3}p'$	classical
(69, 25, 36)	$ au P^3 h_0 d_0^2 e_0$	$P^{2}h_{0}^{7}x'$	classical
(69, 26, 36)	$ au P^3 h_0^2 d_0^2 e_0$	$P^2 h_0^8 x'$	classical
(70, 6, 36)	$ au h_2 Q_3$	Ph_5^2	classical
(70, 7, 36)	$ au h_0 h_2 Q_3$	$Ph_{0}h_{5}^{2}$	classical
(70, 23, 36)	P^3u'	$P^{2}h_{0}^{4}S_{1}$	Lemma $2.4.4$
(70, 24, 36)	$ au P^2 h_0 d_0^2 j$	$h_0^5 S_1$	classical
(70, 25, 36)	$ au P^2 h_0^2 d_0^2 j$	$h_0^6 S_1$	classical

Table 11: Hidden May τ extensions

Table 12: Hidden May $h_0 \ {\rm extensions}$

(s, f, w)	x	$h_0 \cdot x$	reference
(26, 7, 16)	$h_2^2 g$	$h_{1}^{3}h_{4}c_{0}$	Lemma 2.4.9
(30, 7, 16)	r	s	classical
(46, 11, 28)	$h_{2}^{2}g^{2}$	$h_{1}^{7}h_{5}c_{0}$	Lemma 2.4.9
(46, 12, 25)	$u^{\overline{\prime}}$	$ au ar{h}_0 d_0 l$	Lemma 2.4.8
(46, 15, 24)	i^2	P^2s	classical
(49, 12, 27)	v'	$ au h_0 e_0 l$	Lemma 2.4.8
(50, 11, 28)	gr	$Ph_{1}^{3}h_{5}c_{0}$	Lemma 2.4.10
(54, 16, 29)	Pu'	$ au h_0 d_0^2 j$	$\tau \cdot Pu'$
(54, 11, 28)	R_1	S_1	classical
(56, 11, 29)	Q_1	$ au h_2 x'$	classical
(57, 16, 31)	Pv'	$ au h_0 d_0 e_0 j$	Lemma 2.4.8
(60, 8, 32)	B_3	h_5k	classical
(60, 16, 33)	$d_0 u'$	$ au h_0 e_0^2 j$	$h_0 \cdot u'$
(62, 20, 33)	P^2u'	$\tau Ph_0 d_0^2 j$	$\tau \cdot P^2 u'$
(62, 12, 32)	$h_0 R$	Ph_5i	classical
(62, 13, 34)	$h_0^2 B_{22}$	$Ph_1h_5c_0d_0$	Lemma $2.4.12$
(62, 23, 32)	$\tilde{P^2}i^2$	P^4s	classical
(63, 8, 34)	X_2	$h_5 l$	classical
(63, 15, 38)	$h_0 h_2 g^3$	$h_1^7 h_5 c_0 e_0$	$h_0 \cdot h_2^2 g^2$
(63, 16, 35)	$e_0 u'$	$ au h_0 e_0^2 k$	$h_0 \cdot u'$
(64, 8, 34)	$h_2 A'$	$ au^{2}d_{1}^{2}$	classical
(64, 15, 34)	U	Ph_2x'	classical
(65, 20, 35)	P^2v'	$ au h_0 d_0^3 i$	Lemma 2.4.8
(66, 15, 40)	$h_{2}^{2}g^{3}$	$h_{1}^{9}D_{4}^{0}$	Lemma 2.4.9
(66, 16, 37)	$\bar{e_0v'}$	$ au ar{h}_0 e_0^2 l$	$h_0 \cdot v'$
(67, 14, 38)	lm	$h_1^6 X_1^{-1}$	Lemma 2.4.10

Table 12: Hidden May h_0 extensions

(s, f, w)	x	$h_0 \cdot x$	reference
$\begin{array}{c} (68,15,36) \\ (68,20,37) \\ (70,24,37) \\ (70,15,40) \\ (70,19,36) \end{array}$	$h_0 G_{11} \\ P d_0 u' \\ P^3 u' \\ m^2 \\ h_0 R'_1$	$ \begin{array}{c} \tau h_1 d_0 x' \\ \tau h_0 d_0^3 j \\ \tau P^2 h_0 d_0^2 j \\ h_1^5 c_0 Q_2 \\ P^2 S_1 \end{array} $	classical $h_0 \cdot u'$ $\tau \cdot P^3 u'$ Lemma 2.4.10 classical

Table 13: Hidden May h_1 extensions

(s, f, w)	x	$h_1 \cdot x$	reference
(38, 6, 21)	x	$\tau h_2^2 d_1$	classical
(39, 7, 21)	y	$\tau^2 c_1 g$	classical
(56, 8, 31)	$\tau h_1 G$	$h_5 c_0 e_0$	Lemma $2.4.14$
(58, 10, 33)	$h_{1}^{2}B_{6}$	$ au h_2^2 d_1 g$	Lemma $2.4.15$
(59, 11, 33)	$h_1 D_{11}$	$\tau^2 c_1 g^2$	Lemma $2.4.16$
(62, 9, 34)	h_1B_3	$h_5 d_0 e_0$	Lemma $2.4.14$
(62, 10, 33)	X_1	τPG	classical
(64, 8, 35)	C'	$ au d_1^2$	classical
(64, 12, 35)	τPh_1G	$Ph_5c_0e_0$	Lemma $2.4.14$
(67, 7, 37)	r_1	s_1	Lemma 2.4.18
(67, 13, 36)	$h_{1}^{2}q_{1}$	$h_0^4 X_3$	Lemma 2.4.19
(68, 10, 38)	C''	d_1t	classical
(70, 5, 37)	p'	$h_{5}^{2}c_{0}$	classical
(60, 12, 39)	$h_{1}^{2}X_{3}$	$h_5 c_0 d_0 e_0$	Lemma $2.4.14$
(70, 14, 37)	$ au W_1$	$ au P^2 G$	classical

Table 14: Hidden May h_2 extensions

(s, f, w)	x	$h_2 \cdot x$	reference
(26, 7, 16)	h_0h_2g	$h_{1}^{3}h_{4}c_{0}$	$h_0 \cdot h_2^2 g$
(46, 11, 28)	$h_0h_2g^2$	$h_{1}^{7}h_{5}c_{0}$	$h_0 \cdot h_2^2 g^2$
(49, 12, 27)	u'	$ au h_0 e_0 l$	$h_0 \cdot u'$
(50, 11, 28)	e_0r	$Ph_1^3h_5c_0$	Remark 2.4.21
(52, 12, 29)	v'	$ au h_0 e_0 m$	$h_0 \cdot v'$
(54, 9, 30)	h_2B_2	$h_1h_5c_0d_0$	Lemma $2.4.22$
(57, 16, 31)	Pu'	$ au h_0 d_0 e_0 j$	$h_0 \cdot Pu'$
(58, 8, 32)	B_6	$ au e_1 g$	Lemma $2.4.23$
(59, 11, 31)	Q_1	$ au h_0 B_{21}$	classical
(60, 8, 32)	Q_2	h_5k	classical
(60, 9, 32)	h_0Q_2	h_0h_5k	classical
(60, 16, 33)	Pv'	$ au h_0 e_0^2 j$	$h_0 \cdot Pv'$
(62, 13, 34)	$h_0^2 B_{21}$	$Ph_1h_5c_0d_0$	$h_0 \cdot h_0^2 B_{22}$
(63, 8, 34)	B_3	h_5l	classical

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(s, f, w)	x	$h_2 \cdot x$	reference
(63, 15, 38)	$h_0^2 g^3$	$h_1^7 h_5 c_0 e_0$	$h_0 \cdot h_0 h_2 g^3$
(63, 16, 35)	$d_0 u'$	$ au h_0 e_0^2 k$	$h_2 \cdot u'$
(64, 7, 34)	A + A'	$h_0 A^{\prime\prime}$	classical
(64, 8, 34)	$h_0(A+A')$	$h_0^2 A^{\prime\prime}$	classical
(64, 8, 35)	B_7	$ au d_1^2$	classical
(64, 14, 36)	km	$h_{1}^{6}X_{1}$	Remark 2.4.21
(65, 20, 35)	P^2u'	$ au ar{h}_0 d_0^3 i$	$h_0 \cdot P^2 u'$
(66, 14, 40)	$h_0h_2g^3$	$h_{1}^{9}D_{4}^{0}$	$h_0\cdot h_2^2g^3$
(66, 16, 37)	$e_0 u'$	$ au h_0 e_0^2 l$	$h_2 \cdot u'$
(67, 15, 36)	U	$h_0 d_0 x'$	classical
(68, 10, 37)	h_2C_0	$ au d_1 t$	classical
(68, 14, 36)	R_2	$ au h_1 d_0 x'$	classical
(68, 20, 37)	P^2v'	$ au h_0 d_0^3 j$	$h_0 \cdot P^2 v'$
(69, 16, 39)	$e_0 v'$	$ au h_0 e_0^{2} m$	$h_2 \cdot v'$
(70, 15, 40)	lm	$h_1^5 c_0 \check{Q}_2$	Remark $2.4.21$

Table 14: Hidden May h_2 extensions

Table 15: Some miscellaneous hidden May extensions

(s, f, w)	relation	reference
$\begin{array}{c} (59, 12, 33) \\ (60, 11, 32) \\ (61, 10, 35) \\ (62, 12, 35) \\ (63, 10, 35) \end{array}$	$c_{0} \cdot G_{3} = Ph_{1}^{3}h_{5}e_{0}$ $h_{0}^{2}B_{4} + \tau h_{1}B_{21} = g'_{2}$ $c_{0} \cdot i_{1} = h_{1}^{4}D_{4}$ $Ph_{1} \cdot i_{1} = h_{1}^{5}Q_{2}$ $c_{0} \cdot B_{c} = h_{2}^{3}B_{2}$	Lemma 2.4.27 Lemma 2.4.28 Lemma 2.4.24 Lemma 2.4.24 Lemma 2.4.26
(65, 10, 35)	$c_0 \cdot Q_2 = PD_4$	Lemma 2.4.24

Table 10. Some massey products in Ext

(s, f, w)	bracket	contains	indeterminacy	proof	used for
(2, 2, 1)	$\langle h_0, h_1, h_0 angle$	$ au h_1^2$		Proposition 2.2.4	$\langle 2, \eta, 2 \rangle$
(3, 2, 2)	$\langle h_1, h_0, h_1 \rangle$	$h_0 \dot{h}_2$		Proposition 2.2.4	$\langle \eta, 2, \eta \rangle$
(6, 2, 4)	$\langle h_1, h_2, h_1 \rangle$	h_2^2		Proposition 2.2.4	$\langle \eta, \nu, \eta \rangle$
(8, 3, 5)	$\langle h_1, h_2, h_0 h_2 \rangle$	c_0		$d_2(h_0(1)) = h_0 h_2^2$	Lemmas 2.4.9, 2.4.10
	$=\langle h_1, h_0, h_2^2 \rangle$				
(8, 5, 4)	$\langle h_0, h_0^3 h_3, h_0 angle$	0			$\langle 2, 8\sigma, 2 \rangle$
(9, 4, 5)	$\langle h_0, c_0, h_0 angle$	$ au h_1 c_0$		Proposition 2.2.4	$\langle 2, \epsilon, 2 \rangle$
(9, 7, 7)	$\langle h_1^4, au, h_1^4 angle$	0			Lemma $5.1.16$
(12, 4, 7)	$\langle au, h_1^4, h_3 angle$	0			Lemma $5.1.16$
(14, 2, 8)	$\langle h_2, h_3, h_2 angle$	h_{3}^{2}		Proposition 2.2.4	Lemma 4.2.90
(15, 5, 9)	$\langle h_0 h_2, h_2, c_0 angle$	$h_1 d_0$		$d_2(h_0(1)) = h_0 h_2^2$	$\langle 2\nu, \nu, \epsilon \rangle$
(15, 8, 10)	$\langle h_1^2 c_0, au, h_1^4 angle$	0			Lemma $5.1.17$
(16, 4, 9)	$\langle h_0 h_3^2, h_0, h_1 angle$	0			Lemma 2.4.17
(17, 7, 11)	$\langle h_1^4, au, h_1^4, h_3 angle$	$h_{1}^{3}d_{0}$		$d_2(h_1b_{20}) = \tau h_1^4$	Lemma $5.1.16$
				$d_2(h_1^2 b_{21}) = h_1^4 h_3$	
				$d_2(h_1b_{30}) = \tau h_1^2 b_{21} + h_1 h_3 b_{20}$	
(18, 2, 10)	$\langle h_3, h_2, h_3 \rangle$	h_2h_4		Proposition 2.2.4	$\langle \sigma, u, \sigma angle$
(19, 6, 12)	$\langle c_0, h_3, h_1^3 angle$	$h_{1}^{2}e_{0}$		$d_2(h_1b_{21}) = h_1^3h_3$	Lemma $5.2.14$
(20, 4, 11)	$\langle h_0, c_1, h_0 \rangle$	0		Proposition 2.2.4	$\langle 2, \overline{\sigma}, 2 \rangle$
(20, 5, 12)	$\langle h_0, h_1, h_1^3 h_4 \rangle$	$h_0 g$		$d_4(g) = h_1^4 h_4$	Lemmas 2.4.9, 2.4.10
(22, 6, 14)	$\langle h_3, h_1^3, h_1h_3, h_1^2 \rangle$	0			Lemma 2.4.24
(23, 5, 12)	$\langle h_4, h_0^3 h_3, h_0 angle$	$ au^2 h_2 g$		$d_4(P) = h_0^4 h_3$	Lemma 3.3.53
(24, 9, 15)	$\langle h_1^2 e_0, \tau, h_1^4 \rangle$	$h_1^2 c_0 d_0$		$d_2(h_1b_{30}h_0(1)) = \tau h_1^2 e_0$	Lemma $5.1.18$
				$d_2(h_1b_{20}) = \tau h_1^4$	
(25, 7, 15)	$\langle h_2 g, h_0^2, h_1 \rangle$	$c_0 e_0$		$d_2(h_0(1)^2b_{21}) = h_2^3h_0(1)^2$	Lemma 2.4.14
(26, 8, 16)	$\langle h_1^4, h_3, d_0 \rangle$	$h_1c_0e_0$		$d_2(h_1^2b_{21}) = h_1^4h_3$	Lemma $5.1.16$

(s, f, w)	bracket	contains	indeterminacy	proof	used for
(27, 5, 16)	$\langle h_2, h_2 c_1, h_1 \rangle$	h_3g		$d_2(b_{21}h_1(1)) = h_2^2 c_1$	Lemma 4.2.63
(27, 10, 16)	$\langle c_0 d_0, \tau, h_1^4 \rangle$	$Ph_1^2e_0$		$d_2(b_{20}b_{30}h_0(1)) = \tau c_0 d_0$	Lemma 5.1.17
(29, 7, 16)	$\langle d_0, h_3, h_0^2 h_3 \rangle$	k		$d_4(\nu) = h_0^2 h_3^2$	Lemma 2.4.5
(30, 10, 18)	$\langle c_0 e_0, au, h_1^4 \rangle$	$h_1^2 d_0^2$		Lemma 5.1.16	Lemma $5.1.17$
(32, 9, 19)	$\langle h_2^2, h_0, c_0 e_0 \rangle$	$h_1 d_0 e_0$		$d_2(h_0(1)) = h_0 h_2^2$	Lemma 5.1.3
(37, 7, 22)	$\langle h_1^4, h_4, h_1^2 h_4 \rangle$	0	$h_{1}^{6}h_{5}$		Lemma 2.4.24
(39, 3, 21)	$\langle h_2, h_1 h_5, h_2 \rangle$	$h_1h_3h_5$		Proposition 2.2.4	$\langle u, \eta_5, u angle$
(40, 9, 24)	$\langle h_1^7 h_5, h_1, h_0 angle$	$h_0 g^2$		$d_8(g^2) = h_1^8 h_5$	Lemma 2.4.9
(40, 10, 21)	$\langle q, h_0, h_0^3 h_3 angle$	$ au h_1 u$		$d_4(P) = h_0^4 h_3$	Lemma $3.3.52$
(42, 8, 23)	$\langle h_1^2 h_4, h_4, Ph_1 \rangle$	0	$Ph_1^3h_5$		Lemma 2.4.24
(45, 9, 24)	$\langle au, au^2 h_2 g^2, h_1 angle$	au w		$d_8(w) = Ph_1^5 h_5$	$\langle au, u \overline{\kappa}^2, \eta angle$
(46, 7, 25)	$\langle h_1, h_0, h_0^2 g_2 \rangle$	B_1		$d_6(Y) = h_0^3 g_2$	Lemmas $2.4.22, 4.2.48$
(46, 7, 25)	$\langle g_2, h_0^3, h_1 angle$	B_1		$d_6(Y) = h_0^3 g_2$	
(47, 6, 25)	$\langle h_0, h_1, \tau h_1 g_2 \rangle$	0	$ au h_0 h_2 g_2$		Lemma 2.4.17
(47, 9, 28)	$\langle h_2, h_2 c_1 g, h_1 \rangle$	h_3g^2		$d_2(b_{21}^3h_1(1)) = h_2^2c_1g$	Lemma 4.2.63
(47, 10, 26)	$\langle au^2 g^2, h_2^2, h_0 angle$	e_0r		$d_4(\Delta h_2 g) = \tau^2 h_2^2 g^2$	Lemma 2.4.10
(47, 13, 24)	$\langle au, u', h_0^3 angle$	Q'	$\tau P u$	Lemma 2.4.4	Lemma 2.4.4
(47, 13, 24)	$\langle au, au h_0 d_0 l, h_0^2 angle$	Q'	$\tau P u$	$d_4(P\Delta\nu) = \tau^2 h_0 d_0 l$	Lemma 2.4.4
(48, 4, 26)	$\langle h_4, h_1^2 h_4, h_4 \rangle$	0			Lemma 2.4.24
(48, 6, 26)	$\langle h_3, h_2, x \rangle$	$h_2h_5d_0$		Lemma 4.2.90	Lemma 4.2.90
(48, 7, 26)	$\langle h_2, h_0^2 g_2, h_0 angle$	B_2	$h_0^2 h_5 e_0$	$d_6(Y) = h_0^3 g_2$	Lemma 4.2.73
(50, 4, 27)	$\langle h_2, h_3, h_1 h_3 h_5 \rangle$	h_5c_1		$d_2(h_5h_1(1)) = h_1h_3^2h_5$	$\langle u, \sigma, \sigma \eta_5 angle$
(50, 6, 27)	$\langle h_2, h_1, \tau h_1 g_2 \rangle$	C		$d_4(x_{47}) = \tau h_1^2 g_2$	$\langle u, \eta, au\eta\{g_2\} angle$
(50, 7, 28)	$\langle h_5 c_0, h_3, h_1^3 angle$	$h_1^2 h_5 e_0$		$d_2(h_1b_{21}) = h_1^3h_3$	Lemma 2.4.24
(50, 7, 28)	$\langle c_0, h_4^2, h_3, h_1^3 \rangle$	0	$h_{1}^{2}h_{5}e_{0}$		Lemma 2.4.24
(50, 10, 28)	$\langle h_1^3 h_4, h_1, r angle$	gr		$d_4(g) = h_1^4 h_4$	Lemma 2.4.10

Table 16: Some Massey products in Ext

Table 16: Some Massey products in Ext

(s, f, w)	bracket	contains	indeterminacy	proof	used for
(51, 8, 28)	$\langle g_2, h_0^3, h_2^2 \rangle$	h_2B_2		$d_6(Y) = h_0^3 g_2$	Lemma 2.4.22
(51, 8, 28)	$\langle h_0, d_1, f_0 \rangle$	h_2B_2		$d_2(Bh_2b_{21}) = f_0d_1$	Lemma 5.1.4
(51, 9, 28)	$\langle h_2, N, h_1 \rangle$	G_3 or $G_3 + \tau gn$		$d_2(\Delta b_{21}h_1(1)) = h_2 N$	Lemma $4.2.63$
(52, 8, 30)	$\langle d_1, h_1^3, h_1 h_4 angle$	d_1g		$d_4(g) = h_1^4 h_4$	Lemma 2.4.15,
					$\langle \{d_1\}, \eta^3, \eta_4 angle$
(52, 10, 29)	$\langle q, h_1^3, h_1 h_4 \rangle$	h_1G_3		$d_4(g) = h_1^4 h_4$	$\langle \{q\}, \eta^3, \eta_4 \rangle$
(52, 10, 29)	$\langle h_3, h_1^3, Ph_1^2h_5 \rangle$	h_1G_3		$d_8(w) = Ph_1^5h_5$	Lemma $5.2.14$
(53, 7, 30)	$\langle h_4^2, h_3, h_1^3, h_1h_3, h_1^2 \rangle$	i_1	?	Lemma 2.4.24	Lemma $2.4.24$
(53, 7, 30)	$\langle h_1^4, h_4, h_1^2 h_4, h_4 \rangle$	i_1		Lemma 2.4.24	Lemma 2.4.24
(54, 6, 29)	$\langle h_1, h_0, D_1 \rangle$	τG		Remark 3.3.14	Lemma 3.3.13
(54, 15, 29)	$\langle u', h_0^3, h_0 h_3 \rangle$	Pu'		$d_4(P) = h_0^4 h_3$	Lemma 2.4.4
(55, 7, 30)	$\langle h_5, h_2g, h_0^2 \rangle$	$\tau h_1 G$		$d_4(\Delta_1 h_2) = h_2 h_5 g$	Lemma 2.4.14
(55, 13, 31)	$\langle \tau^2 h_1 e_0^2, h_1^3, h_1 h_4 \rangle$	$ au^2 h_1 e_0^2 g$		$d_4(g) = h_1^4 h_4$	$\langle \{\tau^2 h_1 e_0^2\}, \eta^3, \eta_4 \rangle$
(56, 11, 30)	$\langle h_0, h_1, \tau h_1 B_8 \rangle$	$h_2 x'$		$d_6(x'_{56}) = Ph_1^2h_5d_0$	Lemma 5.1.5
(57, 9, 32)	$\langle x, h_1^2, h_1^2 h_4 \rangle$	$h_{1}^{2}B_{6}$		Lemma 2.4.15	Lemma $2.4.15$
(58, 8, 31)	$\langle h_4, h_1^2 h_4, h_4, P h_1 \rangle$	h_1Q_2		Lemma 2.4.24	Lemma 2.4.24
(58, 10, 32)	$\langle y, h_1^2, h_1^2 h_4 \rangle$	$h_1 D_{11}$		$d_4(g) = h_1^4 h_4$	Lemma 2.4.16
(59, 8, 33)	$\langle c_0, h_4^2, h_3, h_1^3, h_1 h_3 \rangle$	$h_{1}^{2}D_{4}$?	Lemma 2.4.24	Lemma 2.4.24
(59, 11, 35)	$\langle c_1 g, h_1^2, h_1^2 h_4 \rangle$	$c_1 g^2$		$d_4(g) = h_1^4 h_4$	Lemma 2.4.16
(60, 10, 33)	$\langle \tau, B_6, h_1^4 \rangle$	$h_{1}^{3}Q_{2}$		$d_2(b_{30}b_{40}h_1(1)) = \tau B_6$	Lemma 2.4.26
				$d_2(h_1^2b_{21}^2b_{30}b_{31} + h_1^2b_{21}^3b_{40}) = h_1^4B_6$	
(61, 8, 33)	$\langle \tau G, h_0, h_2^2 \rangle$	h_1B_3		Lemma 2.4.14	Lemma $2.4.14$
(62, 8, 33)	$\langle h_0 h_3^2, h_0, h_1, \tau h_1 g_2 \rangle$	C_0	?	Lemma 2.4.17	Lemma 2.4.17
(62, 19, 33)	$\langle u', h_0^3, h_0^5 h_4 angle$	P^2u'		$d_8(P^2) = h_0^8 h_4$	Lemma 2.4.4
(63, 10, 35)	$\langle h_5 c_0 e_0, h_0, h_2^2 \rangle$	$h_1h_5d_0e_0$		$d_2(h_0(1)) = h_0 h_2^2$	Lemma 2.4.14
(65, 7, 36)	$\langle d_1, h_4, h_1^2 h_4 \rangle$	k_1		$d_4(u_1) = h_1^2 h_4^2$	Lemma 2.4.5

(s, f, w)	bracket	contains	indeterminacy	proof	used for
(65, 9, 35)	$\langle \tau, B_6, h_1^2 h_3 \rangle$	h_2C_0		$d_2(b_{30}b_{40}h_1(1)) = \tau B_6$	Lemma 2.4.23
(66, 14, 40)	$\langle h_{1}^{5}i_{1},h_{1},h_{2}^{2}\rangle$	$h_{2}^{2}g^{3}$		$d_2(Bh_1b_{21}h_1(1)) = h_1^2h_3B_6$ $d_4(g^3) = h_1^6i_1$	Lemma 2.4.9
(67, 13, 40)	$\langle h_2, h_2 c_1 g^2, h_1 \rangle$	$h_{3}g^{3}$		$d_2(b_{21}^5 h_1(1)) = h_2^2 c_1 g^2$	Lemma 4.2.82
(67, 14, 38)	$\langle au^2 g^3, h_2, h_0 h_2 angle$	lm		$d_4(\Delta h_2 g^2) = \tau^2 h_2 g^3$	Lemma 2.4.10
(70, 23, 37)	$\langle u',h_0^3,h_0^3i angle$	P^3u'		$d_4(P^3) = h_0^6 i$	Lemma 2.4.4

Table 16: Some Massey products in Ext

Table 17: Some matric Massey products in Ext

(s, f, w)	bracket	equals	proof	used in
(25, 8, 14)	$\left\langle \left[\begin{array}{cc} h_1^2 & d_0 \end{array} \right], \left[\begin{array}{c} c_0 d_0 \\ h_1^2 c_0 \end{array} \right], \tau \right\rangle$	Pe_0	$d_2(b_{20}h_0(1)) = \tau h_1^2 c_0 d_2(b_{20}b_{21}b_{30}) = \tau c_0 d_0$	Lemma 5.1.15
(28, 8, 16)	$\left\langle \left[\begin{array}{cc} c_0 & e_0 \end{array} ight], \left[\begin{array}{cc} h_1^2 e_0 \\ h_1^2 c_0 \end{array} ight], au ight angle$	d_{0}^{2}	$d_2(h_1b_{30}h_0(1)) = \tau h_1^2 e_0 d_2(b_{20}h_0(1)) = \tau h_1^2 c_0$	Lemma 5.1.15
(28, 8, 16)	$\left\langle \left[egin{array}{cc} h_1^2 & e_0 \end{array} ight], \left[egin{array}{cc} c_0 e_0 \ h_1^2 c_0 \end{array} ight], au ight angle$	d_0^2	Lemma 5.1.17	Lemma 5.1.17
(40, 10, 22)	$\left\langle \left[\begin{array}{cc} d_0 & e_0 \end{array}\right], \left[\begin{array}{c} c_0 e_0 \\ c_0 d_0 \end{array}\right], \tau \right\rangle$	$h_1 u$	Lemma 5.1.17	Lemma 5.1.17
(56, 15, 33)	$\left\langle \left[\begin{array}{ccc} h_1^2 e_0^2 & d_0 e_0 g & h_1^4 \end{array} \right], \left[\begin{array}{c} h_1^2 e_0 \\ h_1^4 \\ h_1^3 B_1 \end{array} \right], \tau \right\rangle$	$c_0 d_0 e_0^2$	Lemma 5.1.18	Lemma 5.1.18
(57, 15, 31)	$\left\langle \left[\begin{array}{cc} h_1^2 & d_0 \end{array} \right], \left[\begin{array}{cc} h_1 d_0 u \\ h_1^3 u \end{array} \right], \tau \right\rangle$	Pv'	$d_2(h_1b_{20}b_{30}^3h_0(1)^2) = \tau h_1d_0u$ $d_2(h_1b_{20}b_{30}^2h_0(1)^2) = \tau h_1^3u$	Lemma 5.1.15

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(s, f)	d_r	x	$d_r(x)$	reference
(30, 6)	d_3	r	$h_1 d_0^2$	[27 , Theorem 2.2.2]
(31, 8)	d_3	$d_0 e_0$	Pc_0d_0	[27 , Proposition 4.3.1]
(31, 8)	d_4	$d_0 e_0 + h_0^7 h_5$	$P^2 d_0$	[27 , Corollary 4.3.2]
(34, 2)	d_3	h_2h_5	$h_0 p$	[27 , Proposition 3.3.7]
(37, 8)	d_4	e_0g	Pd_0^2	[27 , Theorem 4.2.1]
(38, 2)	d_4	h_3h_5	$h_0 x$	[27 , Theorem 7.3.7]
(38, 4)	d_3	e_1	$h_1 t$	[8 , Theorem 4.1]
(38, 6)	d_2	y	$h_0^3 x$	[27 , Theorem 5.1.4]
(39, 12)	d_4	Pd_0e_0	P^3d_0	[27 , Corollary 4.3.4]
(41, 3)	d_2	c_2	$h_0 f_1$	[27 , Corollary 3.3.6]
(42, 9)	d_2	v	$h_1^2 u$	[27 , Proposition 6.1.5]
(44, 10)	d_3	d_0r	$h_1 d_0^3$	[27 , Corollary 4.4.2]
(45, 12)	d_4	$d_0^2 e_0$	$P^{2}d_{0}^{2}$	[27 , Theorem 4.2.3]
(46, 14)	d_3	i^2	$P^2 h_1 d_0^2$	[27 , Proposition 4.4.1]
(47, 13)	d_2	Q'_{-}	P^2h_0r	[3 , p. 540]
(47, 16)	d_4	$P^{2}d_{0}e_{0}$	$P^4 d_0$	[27 , Corollary 4.3.4]
(47, 18)	d_3	$h_0^5 Q'$	$P^{4}h_{0}d_{0}$	[3 , p. 540]
(49, 6)	d_3	$h_1h_5e_0$	$h_{1}^{2}B_{1}$	[3 , Corollary 3.6]
(49, 11)	d_3	d_0m	Ph_1u	[27 , Proposition 6.1.3]
(50, 13)	d_2	Pv	Ph_1^2u	[27 , Corollary 6.1.4]
(53, 16)	d_4	$Pd_{0}^{2}e_{0}$	$P^{3}d_{0}^{2}$	[27 , Corollary 4.3.3]
(54, 10)	d_2	R_1	$h_0^2 x'$	[27 , Proposition 5.2.3]
(55, 20)	d_4	$P^{3}d_{0}e_{0}$	$P^{5}d_{0}$	[27 , Corollary 4.3.4]
(56, 13)	d_4	$d_0 v$	$P^2 u$	[27 , Proposition 6.1.1]
(57, 15)	d_3	Pd_0m	P^2h_1u	[27 , Corollary 6.1.2]
(58, 17)	d_2	$P^2 v$	$P^{2}h_{1}^{2}u$	[27 , Corollary 6.1.2]
(61, 20)	d_4	$P^2 d_0^2 e_0$	$P^{4}d_{0}^{2}$	[27 , Corollary 4.3.3]
(64, 25)	d_4	$P^4h_1d_0e_0$	$P^{6}h_{1}d_{0}$	[27 , Corollary 4.3.5]
(69, 24)	d_4	$P^{3}d_{0}^{2}e_{0}$	$P^{5}d_{0}^{2}$	[27 , Corollary 4.3.3]

Table 18: Classical Adams differentials

s	bracket	contains	indeterminacy	reference
8	$\langle 2, \eta, \nu, \eta^2 \rangle$	$\epsilon = \{c_0\}$		[41 , Lemma 1.5]
8	$\langle u, \eta, u angle$	$\eta\sigma + \epsilon \in \{h_1h_3\}$		[42 , p. 189]
8	$\langle \nu^2, 2, \eta \rangle$	$\epsilon = \{c_0\}$	$\eta \sigma \in \{h_1 h_3\}$	[42 , p. 189]
9	$\langle 8\sigma, 2, \eta \rangle$	$\mu_9 = \{Ph_1\}$	$\eta^2 \sigma \in \{h_1^2 h_3\} \\ \eta \epsilon \in \{h_1 c_0\}$	[42 , p. 189]
11	$\langle 2\sigma, 8, \nu \rangle$	$\zeta_{11} \in \{Ph_2\}$		[42 , p. 189]
15	$\langle \epsilon, 2, \nu^2 \rangle$	$\eta \kappa \in \{h_1 d_0\}$		[42 , p. 189]
16	$\langle \eta, 2, \kappa angle$	0	$\eta \rho_{15} = \{ Pc_0 \}$	[3 , Lemma 2.4]
16	$\langle \sigma^2, 2, \eta \rangle$	$\eta_4 \in \{h_1h_4\}$	$\eta \rho_{15} = \{ Pc_0 \}$	[42 , p. 189]
17	$\langle \mu_9, 2, 8\sigma \rangle$	$\mu_{17} = \{P^2 h_1\}$	$\mu_9 \epsilon \in \{Ph_1c_0\}$	[42 , p. 189]
18	$\langle 2\sigma, \sigma, \nu \rangle$	$\nu_4 \in \{h_2 h_4\}$		[42 , p. 189]
18	$\langle \sigma, \nu, \sigma \rangle$	$7\nu_4 \in \{h_2h_4\}$		[42 , p. 189]
19	$\langle \zeta_{11}, 8, 2\sigma \rangle$	$\zeta_{19} \in \{P^2h_2\}$		[42 , p. 189]
19	$\langle u, \sigma, \eta \sigma angle$	$\overline{\sigma} \in \{c_1\}$		[42 , p. 189]
30	$\langle \sigma, 2\sigma, \sigma, 2\sigma \rangle$	$\theta_4 = \{h_4^2\}$		[27 , Theorem 8.1.1]
31	$\langle \nu, \sigma, \overline{\kappa} \rangle$	intersects $\{n\}$		[4, Proposition 3.1.1]
31	$\langle \sigma^2, 2, \eta_4 \rangle$	$\eta\theta_4 \in \{h_1h_4^2\}$		[4, Proposition 3.2.1]
32	$\langle \sigma^2, \eta, \sigma^2, \eta angle$	contained in $\{d_1\}$	$\eta \rho_{31} = \{P^3 c_0\}$	[4, Proposition 3.1.4]
32	$\langle \eta, 2, \theta_4 \rangle$	$\eta_5 \in \{h_1h_5\}$	$\eta \rho_{31} = \{ P^3 c_0 \}$	[4, Proposition 3.2.2]
32	$\langle \eta, \kappa^2, 2, \eta \rangle$	contains $\{q\}$	$\eta_5 \in \{h_1 h_5\} \\ \eta_{\beta_{31}} = \{P^3 c_0\}$	[4, Proposition 3.3.1]
33	$\langle \eta_4, \eta_4, 2 \rangle$	contained in $\{p\}$		[4, Proposition 3.3.3]
34	$\langle \sigma, \overline{\sigma}, \sigma \rangle$	$\nu\{n\} \in \{h_2n\}$		[8, Corollary 4.3]
34	$\langle \eta, 2, \eta_5 \rangle$	contained in $\{h_0h_2h_5\}$	$\eta^2 \eta_5 \in \{h_0^2 h_2 h_5\}$ $\eta \mu_{33} = \{P^4 h_1^2\}$	[4, Corollary 3.2.3]
35	$\langle \sigma, \overline{\kappa}, \sigma \rangle$	contained in $\{h_2d_1\}$		[4, Proposition 3.1.2]

Table 19: Some classical Toda brackets

Table 19: Some classical Toda brackets

s	bracket	contains	indeterminacy	reference
35	$\langle \sigma^2, \eta, \overline{\sigma} \rangle$	contained in $\{h_2d_1\}$		[4, Proposition 3.1.3]
36	$\langle \nu, \eta, \eta \theta_4 \rangle$	$\{t\}$		[8 , Corollary 4.3]
36	$\langle u, \eta_4, \eta_4 \rangle$	$\{t\}$		[8 , Corollary 4.3]
36	$\langle \overline{\sigma}, 2, \eta_4 \rangle$	0		[8 , Corollary 4.3]
36	$\langle \epsilon + \eta \sigma, \sigma, \overline{\kappa} \rangle$	$\{t\}$		[8 , Section 5]
36	$\langle \{n\}, \eta, \nu \rangle$	$\{t\}$		[4, Proposition 3.1.5]
37	$\langle \theta_4, 2, \nu^2 \rangle$	$\{h_2^2 h_5\}$	$\sigma\theta_4 = \{x\}$	[4 , Proposition 3.2.4]
38	$\langle \rho_{15}, \sigma, 2\sigma, \sigma \rangle$	contained in $\{h_0^2h_3h_5\}$?	[41 , Proposition 2.9]
39	$\langle \theta_4, 2, \epsilon \rangle$	contained in $\{h_5c_0\}$		[4 , Proposition 3.2.4]
39	$\langle \eta, \nu, \kappa \overline{\kappa} \rangle$	$\{u\}$	$\eta\{h_0^2h_3h_5\} \in \{c_1g\}$	[4 , Proposition 3.4.4]
40	$\langle \theta_4, 2, \mu_9 \rangle$	contained in $\{Ph_1h_5\}$	$\rho_{31}\mu_9 = \{P^4c_0\}$	[4 , Proposition 3.2.4]
44	$\langle u heta_4, u, \sigma angle$	${h_0g_2}$	$\sigma^2 \theta_4 = \{h_0^2 g_2\}$	[4 , Proposition 3.5.3]
45	$\langle 2, \theta_4, \kappa \rangle$	intersects $\{h_5d_0\}$	$2\{h_3^2h_5\} \subseteq \{h_0h_3^2h_5\}$	[4 , Section 4]
			$2k\{h_5d_0\} \subseteq \{h_0h_5d_0\}$	-

Table 20: Adams d_3 differentials

element	(s, f, w)	d_3	reference
h_0h_4	(15, 2, 8)	$h_0 d_0$	image of J
r	(30, 6, 16)	$ au h_{1} d_{0}^{2}$	Table 18
$h_{0}^{3}h_{5}$	(31, 4, 16)	h_0r	image of J
$ au d_0 e_0$	(31, 8, 17)	Pc_0d_0	Table 18
h_2h_5	(34, 2, 18)	$\tau h_1 d_1$	Table 18
$ au e_0 g$	(37, 8, 21)	$c_0 d_0^2$	Lemma 3.3.26
e_1	(38, 4, 21)	$h_1 t$	Table 18
$\tau P d_0 e_0$	(39, 12, 21)	$P^{2}c_{0}d_{0}$	Lemma 3.3.27
i^2	(46, 14, 24)	$\tau P^2 h_1 d_0^2$	Table 18
$\tau P^2 d_0 e_0$	(47, 16, 25)	$P^3c_0d_0$	Lemma 3.3.27
$h_0^5 Q'$	(47, 18, 24)	$P^{4}h_{0}d_{0}$	Table 18
$h_1h_5e_0$	(49, 6, 27)	$h_{1}^{2}B_{1}$	Lemma 3.3.30
$ au^2 d_0 m$	(49, 11, 26)	$\dot{Ph_1u}$	Table 18
gr	(50, 10, 28)	$\tau h_1 d_0 e_0^2$	Lemma 3.3.31
$\tau^2 G$	(54, 6, 28)	τB_8	Lemma 3.3.32
$h_5 i$	(54, 8, 28)	$h_0 x'$	Lemma 3.3.34
B_6	(55, 7, 30)	$\tau h_2 gn$	Lemma 3.3.33
$\tau^2 gm$	(55, 11, 30)	$h_1 d_0 u$	Lemma 3.3.26
$\tau P^3 d_0 e_0$	(55, 20, 29)	$P^{4}c_{0}d_{0}$	Lemma 3.3.27
$h_5 c_0 e_0$	(56, 8, 31)	$h_{1}^{2}B_{8}$	Lemma 3.3.30
Ph_5e_0	(56, 9, 30)	$h_{1}^{\frac{1}{2}}x'$	Lemma 3.3.30
$ au d_0 v$	(56, 13, 30)	$\dot{Ph_1u'}$	Lemma 3.3.26
Q_2	(57, 7, 30)	$\tau^2 g t$	Lemma 3.3.37
$h_{5}j$	(57, 8, 30)	$h_2 x'$	Lemma 3.3.34
$\tau e_0 g^2$	(57, 12, 33)	$c_0 d_0 e_0^2$	Lemma 3.3.26
$\tau^2 P d_0 m$	(57, 15, 30)	P^2h_1u	Table 18
e_1q	(58, 8, 33)	$h_1 q t$	Lemma 3.3.33
τq^3	(60, 12, 35)	$h_{1}^{6}B8$	Lemma 3.3.36
D_3	(61, 4, 32)	?	
$\tilde{C_0}$	(62, 8, 33)	nr	Lemma 3.3.37
$\tilde{E_1}$	(62, 8, 33)	nr	Lemma 3.3.37
$h_1 X_1 + \tau B_{22}$	(62, 10, 33)	$c_0 x'$	Lemma 3.3.30
$\tau q v$	(62, 13, 34)	$h_1 d_0 u'$	Lemma 3.3.26
$P^2 i^2$	(62, 22, 32)	$\tau P^4 h_1 d_0^2$	tmf
$h_0^7 h_6$	(63, 8, 32)	$h_0 R$	image of J
$\tau P^4 d_0 e_0$	(63, 24, 33)	$P^5c_0d_0$	Lemma 3.3.27
$\tau P d_0 v$	(64, 17, 34)	$P^2 h_1 u'$	Lemma 3.3.26
$\tau q w$	(65, 13, 36)	$h_{1}^{3}c_{0}x'$	Lemma 3.3.39
$ ilde{ au^2}P^2d_0m$	(65, 19, 34)	$P^{3}h_{1}u$	Lemma 3.3.40
C''	(67, 9, 37)	nm	Lemma 3.3.37
h_2B_5	(69, 11, 38)	$h_1 B_8 d_0$	Lemma 3.3.44
$ au W_1$	(69, 13, 36)	$\tau^4 e_0^4$	tmf
m^2	(70, 14, 40)	$\tau h_1 e_0^4$	Lemma 3.3.31
$ au e_0 x'$	(70, 14, 37)	Pc_0x'	Lemma 3.3.44

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element	(s, f, w)	d_4	reference
$\tau^2 d_0 e_0 + h_0^7 h_5$	(31, 8, 16)	$P^2 d_0$	Table 18
$ au^2 e_0 g$	(37, 8, 20)	Pd_0^2	Table 18
h_3h_5	(38, 2, 20)	$h_0 x$	Table 18
$ au^2 P d_0 e_0$	(39, 12, 20)	$P^{3}d_{0}$	Table 18
$ au^2 P^2 d_0 e_0$	(47, 16, 24)	P^4d_0	Table 18
$ au^2 gr$	(50, 10, 26)	ij	tmf
$ au^3 gm$	(55, 11, 29)	$Pu' + \tau d_0^2 j$	tmf
$ au^2 P^3 d_0 e_0$	(55, 20, 28)	P^5d_0	Table 18
$ au^2 d_0 v$	(56, 13, 29)	P^2u	Table 18
$ au^2 e_0 g^2$	(57, 12, 32)	d_0^4	Lemma $3.3.47$
$ au^2 d_0^2 r$	(58, 14, 30)	Pij	tmf
$\tau h_1 X_1$	(62, 10, 32)	?	
R	(62, 10, 32)	?	
$ au^2 g v$	(62, 13, 33)	Pd_0u	tmf
C'	(63, 7, 34)	?	
$ au X_2$	(63, 7, 33)	?	
$\tau^2 h_1 B_{22}$	(63, 11, 33)	Ph_1x'	Lemma $3.3.48$
$ au^3 d_0^2 m$	(63, 15, 33)	$P^2u' + \tau P d_0^2 j$	tmf
$h_0^{18}h_6$	(63, 19, 32)	$P^{2}h_{0}i^{2}$	image of J
$ au^2 P d_0 v$	(64, 17, 33)	P^3u	tmf
$ au^2 P d_0^2 r$	(66, 18, 34)	$P^2 i j$	tmf
$ au h_2 B_5$	(69, 11, 37)	$h_1 d_0 x'$	Lemma 3.3.48
$\tau^2 m^2$	(70, 14, 38)	$d_0^2 z$	Lemma 3.3.49
$ au^2 e_0 x'$	(70, 14, 36)	$\tilde{P^2}x'$	Lemma $3.3.48$

Table 21: Adams d_4 differentials

Table 22: Adams d_5 differentials

element	(s, f, w)	d_5	reference
τPh_5e_0	(56, 9, 29)	$\tau d_0 z$	Lemma 3.3.55
A'	(61, 6, 32)	?	
$\tau h_1 H_1$	(63, 6, 33)	?	
$\tau h_1^2 X_1$	(63, 11, 33)	?	
$h_0^{22}h_6$	(63, 23, 32)	P^6d_0	image of J

(s,w)	bracket	contains	indeterminacy	proof	used in
(2, 1)	$\langle 2,\eta,2\rangle$	$\tau\eta^2 = \{\tau h_1^2\}$		$\tau h_1^2 = \langle h_0, h_1, h_0 \rangle$	Lemmas 4.2.18, 4.2.26, 4.2.54
(3, 2)	$\langle \eta, 2, \eta \rangle$	$\{2\nu, 6\nu\} = \{h_0h_2\}$	$4\nu = \{\tau h_1^3\}$	$h_0h_2 = \langle h_1, h_0, h_1 \rangle$	Lemma $4.2.84$
(6, 4)	$\langle \eta, u, \eta angle$	$\nu^2 = \{h_2^2\}$		$h_2^2 = \langle h_1, h_2, h_1 \rangle$	Lemmas $4.2.46, 4.2.52$
(8, 4)	$\langle 2, 8\sigma, 2 \rangle$	0		$0 = \langle h_0, h_0^3 h_3, h_0 \rangle$	Lemma 3.3.53
(8, 5)	$\langle 2\nu, \nu, \eta \rangle$	$\epsilon = \{c_0\}$	$\eta \sigma \in \{h_1h_3\}$	$c_0 = \langle h_0 h_2, h_2, h_1 angle$	Lemmas $4.2.1, 4.2.85$
(8, 5)	$\langle u, \eta, u angle$	$\epsilon + \eta \sigma \in \{h_1 h_3\}$		Table 19	Lemmas $4.2.87, 5.3.4$
(9, 5)	$\langle \eta, 2, 8\sigma \rangle$	$\mu_9 = \{Ph_1\}$	$\tau \eta^2 \sigma \in \{\tau h_1^2 h_3\}$ $\tau \eta \epsilon \in \{\tau h_1 c_0\}$	$Ph_1 = \langle h_1, h_0, h_0^3 h_3 \rangle$	Lemma 4.2.89
(9, 5)	$\langle 2, \epsilon, 2 \rangle$	$\tau\eta\epsilon\in\{\tau h_1c_0\}$		$\tau h_1 c_0 = \langle h_0, c_0, h_0 \rangle$	Lemma 4.2.21
(12, 7)	$\langle \eta, \nu, \sigma \rangle$	0			Lemma 4.2.51
(12, 7)	$\langle \nu, \epsilon, 2 \rangle$	0			Lemma 4.2.70
(15, 8)	$\langle 2, \sigma^2, 2 \rangle$	0	$\{2k\rho_{15}\} = \{h_0^4h_4\}$	Lemma 3.1.5	Lemmas $3.3.54, 4.2.91$
(15, 9)	$\langle 2\nu, \nu, \epsilon \rangle$	$\eta \kappa = \{h_1 d_0\}$		$h_1 d_0 = \langle h_0 h_2, h_2, c_0 \rangle$	Lemma $4.2.85$
(16, 9)	$\langle \eta, 2, \sigma^2 \rangle$	$\eta_4 \in \{h_1 h_4\}$	$\eta \rho_{15} = \{ Pc_0 \}$	$d_2(h_4) = h_0 h_3^2$	Lemmas $3.3.18, 3.3.53,$
	$=\langle \eta, \sigma^2, 2 \rangle$				4.2.48, 4.2.76, 4.2.92
(16, 9)	$\langle \sigma^2, \eta, 2 \rangle$	0		Lemma $4.2.84$	Lemma $4.2.84$
(18, 10)	$\langle 2\sigma, \sigma, \nu \rangle$	$\nu_4 \in \{h_2 h_4\}$		$d_2(h_4) = h_0 h_3^2$	Lemma 4.2.91
(18, 10)	$\langle \sigma, \nu, \sigma \rangle$	intersects $\{h_2h_4\}$		$h_2h_4 = \langle h_3, h_2, h_3 \rangle$	Lemma 4.2.90
(20, 11)	$\langle 2, \overline{\sigma}, 2 \rangle$	0	$\{2k\overline{\kappa}\} = \{\tau h_0 g\}$	$0 = \langle h_0, c_1, h_0 \rangle$	Lemma 4.2.29
(20, 12)	$\langle u, \eta, \eta \kappa angle$	$\{h_0g\}$	$\nu^2\kappa = \{h_0^2g\}$	$d_2(e_0) = h_1^2 d_0$	Lemma $4.2.87$
(21, 12)	$\langle \kappa, 2, \nu^2 \rangle$	$\eta \overline{\kappa} \in \{\tau h_1 g\}$		Lemma $3.3.58$	Lemma $3.3.58$
(22, 12)	$\langle 8\sigma, 2, \sigma^2 \rangle$	0		Lemma 3.3.53	Lemma 3.3.53
(23, 12)	$\langle 2, 8\sigma, 2, \sigma^2 \rangle$	$\tau\nu\overline{\kappa}\in\{\tau^2h_2g\}$	$\{2k\rho_{23}\} = \{h_0^3i\}$ $\{2k\tau^2\nu\overline{\kappa}\} \subset \{\tau^2h_0h_2q\}$	Lemma 3.3.53	Lemma 3.3.53
			$\tau\sigma\eta_4 \in \{\tau h_4 c_0\}$		

Table 23: Some Toda brackets

Table	23:	Some	Toda	brackets

(s, w)	bracket	contains	indeterminacy	proof	used in
			?		
(23, 13)	$\langle \epsilon, 2, \sigma^2 \rangle$	$\sigma\eta_4 \in \{h_4c_0\}$	$4\nu\overline{\kappa} = \{Ph_1d_0\}$	Lemma 3.3.54	Lemma 3.3.54
(32, 17)	$\langle \eta, 2, \theta_4 \rangle$	$\eta_5 \in \{h_1h_5\}$	$\eta \rho_{31} = \{P^3 c_0\}$	$d_2(h_5) = h_0 h_4^2$	Lemmas 4.2.18,
	(1) / -/	1° (- °)	<i>n •</i> (•)		4.2.84, 4.2.89, 4.2.84
(37, 20)	$\langle \nu^2, 2, \theta_4 \rangle$	$\{h_2^2h_5\}$	$\sigma\theta_4 = \{x\}$	$d_2(h_5) = h_0 h_4^2$	Lemma 4.2.89
(39, 21)	$\langle \epsilon, 2, \theta_4 \rangle$	contained in $\{h_5c_0\}$		$d_2(h_5) = h_0 h_4^2$	Lemmas 4.2.21, 4.2.70
(39, 21)	$\langle \eta, \nu, \tau \kappa \overline{\kappa} \rangle$	$\{u\}$	$\eta\{h_0^2h_3h_5\} \in \{\tau^2 c_1g\}$	Lemmas 4.2.41, 4.2.71	Lemmas 4.2.71, 5.3.4
(39, 21)	$\langle u, \eta_5, u angle$	intersects $\{h_1h_3h_5\}$	$\tau\nu\{t\} \in \{\tau^2 c_1 g\}$	$h_1h_3h_5 = \langle h_2, h_1h_5, h_2 \rangle$	Lemma 5.3.3
(40, 21)	$\langle \{q\}, 2, 8\sigma \rangle$	0	$2\tau \overline{\kappa}^2 \in \{\tau h_1 u\}$	Lemma 3.3.52	Lemma 3.3.52
(41, 23)	$\langle \nu, \eta, \{t\} \rangle$	contained in $\{h_1f_1\}$		Lemma 4.2.46	Lemma 4.2.46
(45, 24)	$\langle \theta_4, 2, \sigma^2 \rangle$	contained in $\{h_0 h_3^2 h_5\}$	$ \rho_{15}\theta_4 \in \{h_0^2 h_5 d_0\} $	Lemma 4.2.91	Lemma 4.2.92
(45, 24)	$\langle \eta, \nu, \tau \overline{\kappa}^2 \rangle$	$\{\tau w\}$	$\tau\eta\{g_2\}\in\{\tau h_1g_2\}$	Lemma 4.2.71	Lemma 4.2.71
(45, 24)	$\langle au, u \overline{\kappa}^2, \eta angle$	$\{\tau w\}$	$\tau\eta\{g_2\}\in\{\tau h_1g_2\}$	$\tau w = \langle \tau, \tau^2 h_2 g^2, h_1 \rangle$	Lemma $4.2.10$
(47, 25)	$\langle \eta, 2, \{\tau w\} \rangle$	$\{\tau e_0 r\}$	$\eta\{\tau^2 d_0 l\} = \{Pu\}$	Lemma $4.2.26$	Lemma $4.2.26$
			$ au\eta^2 heta_{4.5}\in\{ au h_1B_1\}$		
(48, 26)	$\langle 2, \eta, \{h_1h_5d_0\}\rangle$	intersects $\{h_2h_5d_0\}$	$2\nu\theta_{4.5} \in \{h_0B_2\}$	$d_2(h_5 e_0) = h_1^2 h_5 d_0$	Lemma 4.2.31
(50, 27)	$\langle \overline{\sigma}, 2, \theta_4 \rangle$	contained in $\{h_5c_1\}$		$d_2(h_5) = h_0 h_4^2$	Lemma 4.2.29
(50, 27)	$\langle u, \sigma, \sigma \eta_5 angle$	contained in $\{h_5c_1\}$		$h_5c_1 = \langle h_2, h_3, h_1h_3h_5 \rangle$	Lemma $4.2.51$
				Lemma $4.2.84$	
(50, 27)	$\langle u, \eta, \tau \eta \{g_2\} angle$	$\{C\}$		$C = \langle h_2, h_1, \tau h_1 g_2 \rangle$	Lemma $4.2.52$
				Lemma 4.2.47	
(51, 28)	$\langle \eta, \{h_1h_5d_0\}, \nu angle$	intersects $\{h_0h_3g_2\}$	$\nu\{B_2\} \in \{h_2 B_2\}$	$d_2(h_5 e_0) = h_1^2 h_5 d_0$	Lemma 4.2.31
			?		
(52, 29)	$\langle \{q\}, \eta^3, \eta_4 \rangle$	$\{h_1G_3\}$		$h_1G_3 = \langle q, h_1^3, h_1h_4 \rangle$	Lemma 4.2.76
(52, 30)	$\langle \{d_1\}, \eta^3, \eta_4 \rangle$	$\{d_1g\}$		$d_1g = \langle d_1, h_1^3, h_1h_4 \rangle$	Lemma 4.2.2
(55, 31)	$\langle \{\tau^2 h_1 e_0^2\}, \eta^3, \eta_4 \rangle$	$\{\tau^2 h_1 e_0^2 g\}$		$\tau^2 h_1 e_0^2 g =$	Lemma 4.2.76

(s,w)	bracket	contains	indeterminacy	proof	used in
$\begin{array}{c} (59,33) \\ (60,32) \\ (60,34) \\ (61,33) \end{array}$	$ \begin{array}{l} \langle \eta^2, \{Ph_1h_5c_0\}, \epsilon \rangle \\ \langle \theta_{4.5}, \sigma^2, 2 \rangle \\ \langle \nu \overline{\kappa}^2, \eta, \eta \kappa \rangle \\ \langle \eta \theta_{4.5}, \sigma^2, 2 \rangle \end{array} $	$ \{Ph_1^3h_5e_0\} $ intersects $\{B_3\}$ $\{Ph_1^4h_5e_0\} $ intersects $\{h_1B_3\}$	$\eta^2 \{D_{11}\} \in \{\tau^2 c_1 g^2\}$?	$= \langle \tau^2 h_1 e_0^2, h_1^3, h_1 h_4 \rangle$ Lemma 4.2.76 $d_2(G_3) = P h_1^3 h_5 c_0$ Remark 3.2.11 $d_2(e_0) = h_1^2 d_0$ $d_2(h_4) = h_0 h_3^2$	Lemma 3.3.45 Remark 3.2.11 Lemma 4.2.10 Remark 3.2.11 Lemma 3.3.18

Table 23: Some Toda brackets

Table 24: Classical Adams hidden extensions

(s, f)	type	from	to	reference
(20, 6)	ν	$h_0^2 g$	Ph_1d_0	[27 , Theorem 2.1.1]
(21, 5)	η	h_1g	Pd_0	[27 , Theorem 2.1.1]
(21, 6)	2	h_0h_2g	Ph_1d_0	[27 , Theorem 2.1.1]
(30, 2)	ν	h_4^2	p	$[4, Proposition \ 3.3.5]$
(30, 2)	σ	h_4^2	x	[4 , Proposition 3.5.1]
(32, 6)	ν	q	$h_1 e_0^2$	[4, Proposition 3.3.1]
(40, 8)	2	g^2	$h_1 u$	[4 , Proposition 3.4.3]
(40, 8)	η	g^2	z	[4, Corollary 3.4.2]
(41, 10)	η	z	d_0^3	[4 , Proposition 3.4.1]
(45, 3)	4	h_4^3	$h_0 h_5 d_0$	[3, Theorem 3.3(i)]
(45, 3)	η	h_4^3	B_1	[3, Theorem 3.1(i)]
(45, 9)	η	w	$d_0 l$	[3 , Theorem 3.1(iv)]
(46, 11)	η	$d_0 l$	Pu	[3, Theorem 3.1(ii)]
(47, 10)	η	e_0r	$d_0 e_0^2$	[3 , Theorem 3.1(vi)]

Table 25: Hidden Adams τ extensions

(s, f, w)	from	to	reference
(22, 7, 13)	$c_0 d_0$	Pd_0	cofiber of τ
(23, 8, 14)	$h_1c_0d_0$	Ph_1d_0	cofiber of τ
(28, 6, 17)	h_1h_3g	d_0^2	Lemma $4.2.1$
(29, 7, 18)	$h_1^2 h_3 g$	$h_1 d_0^2$	Lemma $4.2.1$
(40, 9, 23)	$ au h_0 g^2$	$h_1 u$	Lemma $4.2.3$
(41, 9, 23)	$\tau^2 h_1 g^2$	z	Lemma $4.2.4$
(42, 11, 25)	$c_0 e_0^2$	d_{0}^{3}	cofiber of τ
(43, 12, 26)	$h_1 c_0 e_0^2$	$h_1 d_0^3$	cofiber of τ
(47, 12, 26)	$h_1 u'$	Pu	cofiber of τ
(48, 10, 29)	$h_1h_3g^2$	$d_0 e_0^2$	Lemma $4.2.1$
(49, 11, 30)	$h_{1}^{2}h_{3}g^{2}$	$h_1 d_0 e_0^2$	Lemma $4.2.1$
(52, 10, 29)	h_1G_3	$\tau^2 e_0 m$	cofiber of τ
(53, 9, 29)	B_8	x'	cofiber of τ
(53, 11, 30)	$h_{1}^{2}G_{3}$	$d_0 u$	cofiber of τ
(54, 8, 31)	$h_1 i_1$?	
(54, 10, 30)	h_1B_8	$h_1 x'$	cofiber of τ
(54, 11, 32)	$h_{1}^{6}h_{5}e_{0}$	$ au e_0^2 g$	cofiber of τ
(55, 12, 33)	$h_{1}^{7}h_{5}e_{0}$	$ au h_1 e_0^2 g$	cofiber of τ
(55, 13, 31)	$ au^2 h_1 e_0^2 g$	$d_0 z$	Lemma $4.2.4$
(59, 7, 33)	j_1	?	
(59, 12, 33)	$Ph_1^3h_5e_0$	$ au d_0 w$	Lemma $4.2.10$

7. TABLES

Table 26: Tentative hidden Adams τ extensions

(s, f, w)	from	to	reference
(60, 13, 34)	$ au^2 h_0 g^3$	$d_0 u' + \tau d_0^2 l$	cofiber of τ
(61, 13, 35)	$ au^2 h_1 g^3$	$d_0 e_0 r$	cofiber of τ
(62, 14, 37)	$h_1^6 h_5 c_0 e_0$	$d_0^2 e_0^2$	cofiber of τ
(63, 15, 38)	$h_0^2 h_2 g^3$	$h_1 d_0^2 e_0^2$	cofiber of τ
(65, 9, 36)	$h_1^2 X_2$	$ au B_{23}$	Lemma $4.2.12$
(66, 10, 37)	$h_{1}^{\bar{3}}X_{2}$	$\tau h_1 B_{23}$	Lemma $4.2.12$
(66, 14, 37)	$h_{1}^{5}X_{1}$	$\tau^2 d_0 e_0 m$	cofiber of τ
(67, 11, 38)	$h_{1}^{\bar{4}}X_{2}$	B_8d_0	Lemma $4.2.13$
(67, 13, 37)	$B_8 d_0$	$d_0 x'$	Lemma $4.2.14$
(67, 15, 38)	$h_0 e_0 gr$	$d_0^2 u$	cofiber of τ
(68, 14, 41)	$h_1h_3g^3$	e_0^4	cofiber of τ
(69, 15, 42)	$h_{1}^{2}h_{3}g^{3}$	$h_1 e_0^4$	cofiber of τ

Table 27: Hidden Adams 2 extensions

(s, f, w)	from	to	reference
(23, 6, 14)	h_0h_2q	$h_1 c_0 d_0$	Lemma 4.2.17
(23, 6, 13)	$\tau h_0 h_2 g$	Ph_1d_0	Lemma 4.2.17
(40, 8, 22)	$ au^2 g^2$	$h_1 u$	Table 24
(43, 10, 26)	$h_0h_2g^2$	$h_1 c_0 e_0^2$	Lemma $4.2.17$
(43, 10, 25)	$ au h_0 h_2 g^2$	$h_1 d_0^3$	Lemma $4.2.17$
(47, 10, 26)	e_0r	$h_1 u'$	Lemma $4.2.26$
(47, 10, 25)	$\tau e_0 r$	Pu	Lemma $4.2.26$
(51, 6, 28)	$h_0 h_3 g_2$?	
(54, 9, 28)	h_0h_5i	$ au^4 e_0^2 g$	Lemma $4.2.35$
(59, 7, 33)	j_1	?	

Table 28: Tentative hidden Adams 2 extensions

(s, f, w)	from	to	reference
$\begin{array}{c} (60, 12, 33) \\ (63, 14, 37) \\ (67, 14, 37) \end{array}$	$ au^3 g^3 \ au h_0 h_2 g^3 \ au e_0 g r$	$\begin{array}{c} d_{0}u' + \tau d_{0}^{2}l \\ h_{1}d_{0}^{2}e_{0}^{2} \\ d_{0}^{2}u \end{array}$	Lemma 4.2.37 Lemma 4.2.37 Lemma 4.2.37

Table 29: Hidden Adams η extensions

(s,f,w)	from	to	reference
(15, 4, 8) (21, 5, 12) (21, 5, 11) (23, 9, 12)	$egin{array}{l} h_0^3h_4 \ au h_1g \ au^2h_1g \ h_0^2i \end{array}$	$egin{array}{l} Pc_0 \ c_0 d_0 \ Pd_0 \ P^2 c_0 \end{array}$	image of J Lemma 4.2.39 Lemma 4.2.39 image of J

(s, f, w)	from	to	reference
(31, 11, 16)	$h_0^{10} h_5$	P^3c_0	image of J
(38, 4, 20)	$h_0^2 h_3 h_5$	$\tau^2 c_1 g$	Lemma $4.2.41$
(39, 17, 20)	$P^{2}h_{0}^{2}i$	P^4c_0	image of J
(40, 8, 21)	$ au^3 g^2$	z	Table 24
(41, 5, 23)	$h_1 f_1$	$ au h_2 c_1 g$	Lemma $4.2.46$
(41, 9, 24)	$ au h_1 g^2$	$c_0 e_0^2$	Lemma $4.2.39$
(41, 10, 22)	z	$ au d_0^3$	Lemma $4.2.39$
(45, 3, 24)	$h_{3}^{2}h_{5}$	B_1	Lemma $4.2.48$
(45, 9, 24)	au w	$\tau d_0 l + u'$	Table 24
(46, 11, 24)	$\tau^2 d_0 l$	Pu	Table 24
(47, 10, 26)	e_0r	$ au d_0 e_0^2$	Table 24
(47, 20, 24)	$h_0^7 Q'$	P^5c_0	image of J
(52, 11, 28)	$\tau^2 e_0 m$	$d_0 u$	Lemma $4.2.53$
(54, 12, 29)	$ au^3 e_0^2 g$	$d_0 z$	Lemma $4.2.55$
(55, 25, 28)	$P^{4}h_{0}^{2}i$	$P^{6}c_{0}$	image of J
(58, 8, 30)	$\tau h_1 Q_2$?	

Table 29: Hidden Adams η extensions

Table 30: Tentative hidden Adams η extensions

(s, f, w)	from	to	reference
$\begin{array}{c} (59,13,32)\\ (60,12,33)\\ (61,13,35)\\ (61,14,34)\\ (63,26,32)\\ (65,13,35)\\ (61,14,26)\\ (65,13,35)\\ (61,14,26)\\ (65,13,35)\\ (61,14,26)\\$	$ au d_0 w \\ au^3 g^3 \\ au^2 h_1 g^3 \\ d_0 e_0 r \\ h_0^{25} h_6 \\ au^2 g w \\ d_0 B$		Lemma 4.2.59 Lemma 4.2.59 Lemma 4.2.59 Lemma 4.2.59 image of J Lemma 4.2.59
(66, 11, 36) (66, 11, 35)	$ au h_1 B_{23} \ au^2 h_1 B_{23}$	$B_8 d_0 \\ d_0 x'$	Lemma 4.2.60 Lemma 4.2.60
(66, 15, 36)	$\tau^2 d_0 e_0 m$	$d_0^{2}u$	Lemma 4.2.59

Table 31: Hidden Adams ν extensions

(s, f, w)	from	to	reference
(20, 6, 12)	$h_0^2 g$	$h_1c_0d_0$	Lemma 4.2.62
(20, 6, 11)	$ au h_0^2 g$	Ph_1d_0	Lemma $4.2.62$
(22, 4, 13)	h_2c_1	$h_{1}^{2}h_{4}c_{0}$	Lemma $4.2.63$
(26, 6, 15)	$ au h_2^2 g$	$h_1 d_0^2$	Lemma $4.2.64$
(30, 2, 16)	h_4^2	p	Table 24
(32, 6, 17)	q	$ au^2 h_1 e_0^2$	Table 24
(39, 9, 21)	u	$ au d_0^3$	Lemma $4.2.71$
(40, 10, 24)	$h_{0}^{2}g^{2}$	$h_1 c_0 e_0^2$	Lemma $4.2.62$
(40, 10, 23)	$ au h_0^2 g^2$	$h_1 d_0^3$	Lemma $4.2.62$

Table 31: Hidden Adams ν extensions

(s,f,w)	from	to	reference
(42, 8, 25)	h_2c_1g	$h_{1}^{6}h_{5}c_{0}$	Lemma 4.2.63
(45, 3, 24)	$h_{3}^{2}h_{5}$	B_2	Lemma $4.2.73$
(45, 4, 24)	$h_0 h_3^2 h_5$	h_0B_2	Lemma $4.2.73$
(45, 9, 24)	τw	$ au^{2}d_{0}e_{0}^{2}$	Lemma $4.2.71$
(46, 7, 25)	N	$Ph_1^2h_5c_0$	Lemma $4.2.63$
(46, 10, 27)	$ au h_2^2 g^2$	$h_1 d_0 e_0^2$	Lemma $4.2.64$
(48, 6, 26)	$h_{2}h_{5}d_{0}$?	
(51, 8, 28)	h_2B_2	h_1B_8	Lemma $4.2.75$
(51, 8, 27)	$\tau h_2 B_2$	$h_1 x'$	Lemma $4.2.75$
(52, 10, 29)	h_1G_3	$ au^2 h_1 e_0^2 g$	Lemma $4.2.76$
(52, 11, 28)	$\tau^2 e_0 m$	$d_0 z$	Lemma $4.2.76$
(53, 7, 30)	i_1	?	
(54, 11, 32)	$h_{1}^{6}h_{5}e_{0}$	$h_2 e_0^2 g$	Lemma $4.2.78$

Table 32: Tentative hidden Adams ν extensions

(s, f, w)	from	to	reference
(57, 10, 30)	$h_0h_2h_5i$	$\tau^2 d_0^2 l$	Lemma 4.2.79
(59, 13, 32)	$ au d_0 w$	$ au^{2}d_{0}^{2}e_{0}^{2}$	Lemma $4.2.80$
(59, 12, 33)	$Ph_1^3h_5e_0$	$\tau d_0^2 e_0^2$	Lemma $4.2.80$
(60, 14, 35)	$ au h_0^2 g^3$	$h_1 d_0^2 e_0^2$	Lemma $4.2.81$
(62, 12, 37)	$h_2 c_1 g^2$	$h_{1}^{8}D_{4}$	Lemma $4.2.82$
(65, 13, 36)	$\tau gw + h_1^4 X_1$	$\tau^2 e_0^4$	Lemma $4.2.80$

Table 33: Some miscellaneous hidden Adams extensions

(s, f, w)	type	from	to	reference
(16, 2, 9)	σ	h_1h_4	h_4c_0	Lemma 4.2.83
(20, 4, 11)	ϵ	au g	d_0^2	Lemma $4.2.85$
(40, 8, 23)	ϵ	$ au g^2$	$d_0 e_0^2$	Lemma $4.2.85$
(32, 6, 17)	ϵ	q	$h_1 u$	Lemma $4.2.87$
(45, 3, 24)	ϵ	$h_{3}^{2}h_{5}$	B_8	Lemma $4.2.88$
(30, 2, 16)	ν_4	h_4^2	$h_{2}h_{5}d_{0}$	Lemma $4.2.90$
(30, 2, 16)	η_4	h_{4}^{2}	$h_1h_5d_0$	Lemma $4.2.92$
(45, ?, 24)	κ	$h_{3}^{2}h_{5} \text{ or } h_{5}d_{0}$	B_{21}	Lemma $4.2.93$
(45, ?, 24)	$\overline{\kappa}$	$h_{3}^{2}h_{5}$ or $h_{5}d_{0}$	τB_{23}	Lemma $4.2.94$
				(tentative)

Table 34: Some compound hidden Adams extensions

(s, w)	relation	reference
(9,6) (40,22)	$\nu^3 + \eta^2 \sigma = \eta \epsilon$ $\nu \{h_2^2 h_5\} + \eta \sigma \eta_5 = \epsilon \eta_5$	[42] Lemma 4.2.89

Table 35:	Hidden	h_0	extensions	in	$E_2($	$C\tau$)
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(s, f, w)	\overline{x}	$h_0 \cdot \overline{x}$	reference
(11, 4, 6)	$h_{1}^{2}c_{0}$	Ph_2	$d_2(b_{20}h_0(1)) = \tau h_1^2 c_0$
(11, 4, 6)	$\overline{P^k h_1^2 c_0}$	$P^{k+1}h_2$	$d_2(b_{20}^{2k+1}h_0(1)) = \tau P^k h_1^2 c_0$
+k(8,4,4)			
(23, 6, 12)	$\overline{c_0 d_0}$	i	$d_2(b_{20}b_{30}h_0(1)) = \tau c_0 d_0$
(26, 6, 14)	$c_0 e_0$	j	Lemma 5.1.3
(26, 6, 14)	$P^k c_0 e_0$	$P^k j$	Lemma 5.1.3
+k(8,4,4)		-2	- (-5 (-))
(39, 14, 20)	$P^2 c_0 d_0$	P^2i	$d_2(b_{20}^3b_{30}h_0(1)) = \tau P^2 c_0 d_0$
(41, 9, 22)	$h_0 \cdot \underline{\tau h_0 g^2}$	z	$d_4(\Delta h_0 e_0) = \tau^2 h_0^2 g^2$
(44, 9, 24)	$h_2 \cdot \tau h_0 g^2$	d_0r	$d_4(\Delta h_0 g) = \tau^2 h_0 h_2 g^2$
(46, 10, 26)	$\overline{c_0 e_0 g}$	$d_0 l$	Lemma 5.1.3
(46, 13, 24)	$\frac{h_0\cdot\tau h_0 d_0^2 e_0}{1-2}$	i^2	$d_4(P\Delta h_0 d_0) = \tau^2 h_0^2 d_0^2 e_0$
(47, 9, 26)	$\tau h_2^2 g^2$	e_0r	$d_4(\Delta h_2 g) = \tau^2 h_2^2 g^2$
(47, 12, 24)	$h_0^2 \cdot \underline{u'}$	Q'	$d_4(\Delta h_0 i) = \tau^2 h_0^2 d_0 l$
(49, 13, 26)	$h_0 \cdot \underline{\tau h_0 d_0 e_0^2}$	ij	$d_4(P\Delta h_0 e_0) = \tau^2 h_0^2 d_0 e_0^2$
(52, 13, 28)	$h_0\cdot au h_0 e_0^3$	ik	$d_4(P\Delta h_2 e_0) = \tau^2 h_0^2 e_0^3$
(54, 8, 30)	h_1d_1g	$h_1h_5c_0d_0$	Lemma 5.1.4
(55, 13, 30)	$h_0 \cdot au h_0 e_0^2 g$	il	$d_4(\Delta h_0 d_0 e_0) = \tau^2 h_0^2 e_0^2 g$
(55, 22, 28)	$P^4 c_0 d_0$	P^4i	$d_2(b_{20}^9b_{30}h_0(1)) = \tau P^4c_0d_0$
(56, 10, 30)	$h_1^2 B_8$	$h_2 x'$	Lemma 5.1.5
(57, 17, 30)	$h_0 \cdot \tau h_0 d_0^4$	Pij	$d_4(P^2\Delta h_0 e_0) = \tau^2 h_0^2 d_0^4$
(58, 5, 30)	D_4	D_2	$d_2(b_{30}h_0(1,2)) = \tau D_4$
(58, 13, 32)	$h_0 \cdot \tau h_0 e_0 g^2$	im	$d_4(\Delta h_0 e_0^2) = \tau^2 h_0^2 e_0 g^2$
(61, 13, 34)	$h_2 \cdot \tau h_0 e_0 g^2$	jm	$d_4(\Delta h_0 e_0 g) = \tau^2 h_0^2 g^3$
(62, 12, 34)	$h_2 \cdot \tau h_2 gm$	$ au e_0 w$	$d_4(\Delta h_2 g\nu) = \tau^2 h_2^2 gm$
(62, 21, 32)	$h_0 \cdot \tau P^2 h_0 d_0^2 e_0$	$P^2 i^2$	$d_4(P^3\Delta h_0 d_0) = \tau^2 P^2 h_0^2 d_0^2 e_0$
(65, 21, 34)	$h_0 \cdot \tau P h_0 d_0^4$	$P^2 i j$	$d_4(P^3\Delta h_0 e_0) = \tau^2 P h_0^2 d_0^4$
(66, 9, 34)	c_0Q_2	?	
(67, 13, 38)	$\tau h_2^2 g^3$	lm	$d_4(\Delta h_2 g^2) = \tau^2 h_2^2 g^3$
(68, 6, 36)	h_1r_1	$h_3(A+A')$	$d_4(x_{68}) = \tau h_1 r_1$
(68, 21, 36)	$h_0 \cdot \overline{\tau} P h_0 d_0^3 e_0$	P^2ik	$d_4(P^2\Delta h_0 d_0^2) = \tau^2 P h_0^2 d_0^3 e_0$
(69,9,36)	$h_1 \overline{X_3}$	P(A+A')	$d_2(b_{20}^2h_3b_{31}b_{40} + \tau h_1b_{20}b_{30}b_{31}^2)$
$(70 \ 16 \ 36)$	$\overline{Pc_0 x'}$	B'_{*}	$= \tau h_1 X_3$ $d_2 (P^2 B h_{22} h_{22}) - \tau P c_2 r'$

(s, f, w)	\overline{x}	$h_1 \cdot \overline{x}$	reference
(35, 5, 19)	$h_{1}^{2}d_{1}$	t	$d_2(h_1b_{30}b_{22}) = \tau h_1^2 d_1$
	Ĩ		$d_2(b_{21}^2h_1(1)) = h_1^3d_1$
(39, 7, 22)	$h_1^3h_5\cdot\overline{h_1^4}$	$ au g^2$	$d_8(g^2) = h_1^8 h_5$
(41, 8, 22)	$ au h_0 g^2$	v	$d_4(\Delta e_0) = \tau^2 h_0 g^2$
(44, 8, 23)	$\overline{\tau^2 h_2 g^2}$	au w	$d_8(w) = Ph_1^5h_5$
(46, 12, 24)	$ au h_0 d_0^2 e_0$	Pu	$d_4(P\Delta d_0) = \tau^2 h_0 d_0^2 e_0$
(49, 12, 26)	$ au h_0 d_0 e_0^2$	Pv	$d_4(\Delta P e_0) = \tau^2 h_0 d_0 e_0^2$
(52, 12, 28)	$ au h_0 e_0^3$	$d_0 u$	Lemma 5.1.6
(55, 8, 29)	$h_1h_5 \cdot \overline{c_0d_0}$	Ph_5e_0	Lemma 5.1.7
(55, 9, 31)	$h_1 \cdot \overline{h_1 d_1 g}$	gt	$d_2(h_1b_{30}b_{22}g) = \tau h_1^2 d_1g$
(_	$d_2(h_1(1)g^2) = h_1^3 d_1 g$
(56, 8, 30)	$\tau h_2 d_1 g$	D_{11}	$d_4(\Delta d_1) = \tau^2 h_2 d_1 g$
(56, 11, 32)	$h_1^5 h_5 \cdot h_1^2 e_0$	$ au e_0 g^2$	$d_8(e_0g^2) = h_1^8h_5e_0$
(57, 16, 30)	$\tau h_0 d_0^4$	P^2v	$d_4(P^2 \Delta e_0) = \tau^2 h_0 d_0^4$
(59, 11, 34)	$h_1^4 \cdot h_1 \underline{i_1}$	$ au g^3$	$d_4(g^3) = h_1^6 i_1$
(60, 10, 32)	$h_1h_3 \cdot \underline{G_3}$	nr	$d_4(\Delta t) = \tau^3 c_1 g^2$
(61, 12, 33)	$h_1^2 h_5 \cdot P h_1^2 e_0$	$ au e_0 w$	$d_8(e_0 w) = P h_1^5 h_5 e_0$
(62, 20, 32)	$\tau P^2 h_0 \underline{d_0^2} e_0$	P^3u	$d_4(P^3\Delta d_0) = \tau^2 P^2 h_0 d_0^2 e_0$
(64, 8, 34)	$h_1h_3 \cdot B_6$	h_2C_0	$d_2(Bh_1b_{21}h_1(1)) = h_1^3B_7$
	- 0	_	$d_2(h_1^2b_{30}b_{22}b_{40}) = \tau h_1^2 B_7$
(64, 9, 34)	$h_1^2 \cdot h_5 d_0 e_0$	$\tau B_{23} + c_0 Q_2$	Lemma 5.1.8
(64, 12, 35)	$h_1^4 \cdot \underline{h_1^2 Q_2}$	$\tau gw + h_1^4 X_1$	Lemma 5.1.9
(65, 6, 34)	$h_3 \cdot D_4$	$ au G_0$	$d_2(h_3b_{30}h_0(1,2)) = \tau h_1^2 H_1$
		- 9	$d_2(b_{21}^2 h_0(1,2)) = h_1^3 H_1$
(65, 20, 34)	$\tau Ph_0 d_{\underline{0}}^4$	P^3v	$d_4(P^3\Delta e_0) = \tau^2 P h_0 d_0^4$
(68, 8, 37)	$h_1h_3 \cdot j_1$	$h_0h_2G_0$	$d_2(h_1b_{21}^2b_{22}b_{31}) = h_1^2h_3j_1$
(00, 10, 07)	12 D		$d_2(h_1^2h_1(1)b_{22}b_{40}) = \tau h_1h_3j_1$
(68, 10, 37)	$n_1^2 c_0 \cdot D_4$	$h_2 B_5$ or $h_2 B_1 + h_2 V$	Lemma 5.1.11
(68, 12, 35)	$\overline{\tau B_8 d_0}$	$\frac{n_2 B_5 + n_1 X_3}{\tau W_1}$	$d_8(\tau\Delta^2 g) = \tau h_0^4 X_3$

Table 36: Hidden h_1 extensions in $E_2(C\tau)$

Table 37: Hidden h_2 extensions in $E_2(C\tau)$

(s, f, w)	\overline{x}	$h_2 \cdot \overline{x}$	reference
(28, 4, 15)	h_3g	n	$d_2(b_{30}h_1(1)) = \tau h_3 g$
(42, 8, 22)	$ au^2 h_1 g^2$?	
(43, 7, 23)	$\tau h_2 c_1 g$	N	$d_4(\Delta c_1) = \tau^2 h_2 c_1 g$
(44, 9, 24)	$h_2 \cdot \overline{\tau h_0 g^2}$	e_0r	$d_4(\Delta h_0 g) = \tau^2 h_0 h_2 g^2$
(47, 5, 25)	$\overline{h_1^2 g_2}$	C	$d_4(h_2b_{40}h_1(1)) = \tau h_1^2 g_2$
(48, 8, 27)	$\overline{h_3g^2}$	gn	$d_2(b_{21}^2b_{30}h_1(1)) = \tau h_3g^2$
(54, 8, 30)	$\overline{h_1d_1g}$	$h_{1}^{2}B_{6}$	$d_2(b_{21}^2b_{30}b_{22} + h_2^2b_{21}^2b_{40}) = \tau h_1 d_1 g$

(s, f, w)	\overline{x}	$h_2 \cdot \overline{x}$	reference
(58, 5, 30)	$\overline{D_4}$	A	$d_2(b_{30}h_0(1,2)) = \tau D_4$
(58, 10, 31)	$h_2 \cdot \overline{\tau h_2 g n}$	nr	$d_4(\Delta h_2 n) = \tau^2 h_2^2 g n$
(59, 5, 31)	$h_{3}^{2}g_{2}$	$h_5 n$	$d_2(h_5b_{30}h_1(1)) = \tau h_3h_5g$
(59, 7, 31)	$h_2 \cdot \overline{B_6}$	C_0	$d_4(x_{59}) = \tau^2 e_1 g$
(60, 6, 32)	$\overline{j_1}$	C'	$d_2(h_1(1)^2 b_{40}) = \tau j_1$
(61, 13, 34)	$h_2 e_0 \cdot \overline{\tau h_2 e_0 g}$	km	$d_4(\Delta h_0 e_0 g) = \tau^2 h_2^2 e_0^2 g$
(62, 11, 32)	$\overline{Ph_5c_0d_0}$	Ph_5j	$d_2(Ph_5b_{20}h_0(1)b_{30}) = \tau Ph_5c_0d_0$
(62, 12, 34)	$h_2 \cdot \overline{\tau h_2 gm}$	au g w	$d_4(\Delta h_2 g\nu) = \tau^2 h_2^2 gm$
(63, 11, 35)	$ au h_2 c_1 g^2$	nm	$d_4(\Delta c_1 g) = \tau^2 h_2 c_1 g^2$
(64, 13, 36)	$\overline{\tau h_0 h_2 g^3}$	lm	$d_4(\Delta h_0 g^2) = \tau^2 h_0 h_2 g^3$
(66, 8, 36)	$h_1 d_1^2$	$h_1 h_3 B_7$	$d_2(b_{30}b_{22}h_1(1)^2) = \tau h_1 d_1^2$
(66, 9, 34)	$\overline{c_0 Q_2}$?	-
(67, 13, 38)	$\overline{ au h_2^2 g^3}$	m^2	$d_4(\Delta h_2 g^2) = \tau^2 h_2^2 g^3$

Table 37: Hidden h_2 extensions in $E_2(C\tau)$

Table 38: Some miscellaneous hidden extensions in $E_2(C\tau)$

(s, f, w)	relation	reference
(25, 8, 14)	$h_1^2 \cdot \overline{c_0 d_0} + d_0 \cdot \overline{h_1^2 c_0} = P e_0$	Lemma 5.1.15
(28, 8, 16)	$c_0 \cdot \overline{h_1^2 e_0} + e_0 \cdot \overline{h_1^2 c_0} = d_0^2$	Lemma $5.1.15$
(28, 8, 16)	$h_1^2 \cdot \overline{c_0 e_0} + e_0 \cdot \overline{h_1^2 c_0} = d_0^2$	Lemma $5.1.17$
(40, 10, 22)	$d_0 \cdot \overline{c_0 e_0} + e_0 \cdot \overline{c_0 d_0} = h_1 u$	Lemma $5.1.17$
(56, 15, 33)	$h_1^2 e_0^2 \cdot \overline{h_1^2 e_0} + d_0 e_0 g \cdot \overline{h_1^4} + h_1^6 \cdot \overline{h_1^3 B_1} = c_0 d_0 e_0^2$	Lemma $5.1.18$
(57, 15, 31)	$h_1^2 \cdot \overline{h_1 d_0 u} + d_0 \cdot \overline{h_1^3 u} = Pv'$	Lemma $5.1.15$
(59, 9, 32)	$h_1^3 \cdot \overline{B_6} + h_2 \cdot \overline{\tau h_2 d_1 g} = h_1^2 Q_2$	Lemma $5.1.12$
(65, 11, 34)	$Ph_1 \cdot \overline{B_6} = h_1 q_1$	Lemma $5.1.14$

Table 39: $E_2(C\tau)$ generators

(s, f, w)	x	$d_2(x)$	reference
(5, 3, 3)	$\overline{h_1^4}$		
(11, 4, 6)	$\overline{h_1^2 c_0}$		
(5, 3, 3)	$\overline{P^k h_1^4}$		
+k(8,4,4)			
(11, 4, 6)	$P^k h_1^2 c_0$		
+k(8,4,4)			
(20, 5, 11)	$h_{1}^{2}e_{0}$	$d_0 \cdot h_1^4$	top $\underline{\text{cell}}$
(23, 6, 12)	$c_0 d_0$	Pd_0	$h_0 \cdot c_0 d_0 = i$
			$d_2(i)$ in bottom cell
(26, 6, 14)	$\overline{c_0 e_0}$	$h_1^2 \cdot c_0 d_0$	$h_0 \cdot \overline{c_0 e_0} = j$

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Table 39: $E_2(C\tau)$ generators

(s, f, w)	x	$d_2(x)$	reference
		$+Pe_0$	$d_2(j)$ in bottom cell $d_2(a,a_2)$ in top cell
(28, 4, 15)	haa	$h_1 h_4 \cdot \overline{h^2 c_0}$	$a_2(c_0c_0)$ in top cent
(20, 4, 10) (20, 5, 11)	$\frac{n_{39}}{P^k h_{1e0}^2}$	$P^k d_0 \cdot \overline{h_1^4}$	top cell
(20, 0, 11) + $k(8, 4, 4)$	1 1120	1 00 101	top cen
(26, 6, 14)	$\overline{P^k c_0 e_0}$	$P^k h_1^2 \cdot \overline{c_0 d_0}$	$h_0 \cdot \overline{P^k c_0 e_0} = P^k j$
+k(8,4,4)	0.0	$+P^{k+1}e_0$	$d_2(P^k j)$ in bottom cell
			$d_2(P^k c_0 e_0)$ in top cell
(39, 14, 20)	$\overline{P^2c_0d_0}$	P^3d_0	$h_0 \cdot \overline{P^2 c_0 d_0} = P^2 i$
	- 0	- 0 - 0	$d_2(P^2i)$ in bottom cell
(40, 9, 23)	$h_1^2 e_0 g$	$h_1^2 e_0 \cdot h_1^2 e_0$	Lemma 5.2.2
(41 0 00)	1 2	$+c_0 d_0 e_0$	1 1 2
(41, 8, 22)	$ au n_0 g^2$	$n_1 u$	$n_1 \cdot \tau n_0 g^2 \equiv v$ $d_2(v)$ in bottom coll
$(42 \ 8 \ 22)$	$\overline{\tau^2 h_1 a^2}$	~	Lemma 5.2.3
(43, 7, 23)	$\frac{\tau h_{1g}}{\tau h_{2}c_{1}a}$	~	Lemma 9.2.9
(43.11,23)	$\frac{h^2}{h^3 u}$		
(44, 8, 23)	$\frac{\pi m_{1}}{\tau^{2}h_{2}q^{2}}$		
(44, 9, 23)	$\overline{h_1v}^{=0}$	$\overline{h_1^3 u}$	top cell
(46, 10, 26)	$\overline{c_0 e_0 g}$	$h_1^{1/2}e_0\cdot\overline{c_0e_0}$	$h_0 \cdot \overline{c_0 e_0 g} = d_0 l$
		$+d_{0}^{2}e_{0}$	$d_2(d_0l)$ in bottom cell
			$d_2(c_0e_0g)$ in top cell
(46, 12, 24)	$\frac{\tau h_0 d_0^2 e_0}{2}$		
(47, 5, 25)	$\frac{h_1^2 g_2}{12}$		
(47, 9, 26)	$\frac{\tau h_2^2 g^2}{d}$		
(47, 10, 24)	$\frac{u'}{1}$	$\tau h_0 d_0^2 e_0$	top cell
(48, 8, 27)	$\frac{h_3g^2}{2}$	$h_1^5 h_5 \cdot h_1^2 c_0$	top <u>cell</u>
(49, 12, 26)	$ au h_0 d_0 e_0^2$	Ph_1u	$h_1 \cdot \tau h_0 d_0 e_0^2 = P v$
(50, 0, 07)	130		$a_2(Pv)$ in bottom cell
(50, 9, 27) (50, 10, 26)	$\frac{n_1^*}{w'}B_1$	$h^2 \cdot \overline{u'}$	top cell
(50, 10, 20)	U	$h_1 \cdot a + \tau h_0 d_0 e_0^2$	top cen
(51, 15, 27)	Ph_1^3u		
(52, 8, 27)	$\overline{G_3}$	$h_1h_5\cdot \overline{Ph_1^2c_0}$	top cell
(52, 12, 28)	$\overline{\tau h_0 e_0^3}$	10 110	±
(52, 13, 27)	$\frac{v}{Ph_1v}$	Ph_1^3u	top cell
(54, 8, 30)	$\overline{h_1d_1g}$	Ŧ	-
(55, 7, 30)	$\overline{h_1 i_1}$?	
(55, 9, 29)	$ au h_2 gn$		
(55, 13, 29)	$h_1 d_0 u$	Pu'	Lemma $5.2.4$
(55, 22, 28)	$P^{4}c_{0}d_{0}$	P^5d_0	$h_0 \cdot P^4 c_0 d_0 = P^4 i$
			$d_2(P^4i)$ in bottom cell

7. TABLES

Table 39: $E_2(C\tau)$ generators

(s, f, w)	x	$d_2(x)$	reference
(56, 6, 29)	$\overline{B_6}$		
(56, 8, 30)	$\overline{\tau h_2 d_1 g}$		
(56, 10, 30)	$h_{1}^{2}B_{8}$		
(56, 11, 30)	$\overline{\tau h_0 gm}$	$h_0 d_0 \cdot \overline{\tau h_0 g^2}$	top cell
(57, 16, 30)	$\tau h_0 d_0^4$	P^2h_1u	$h_1 \cdot \overline{\tau h_0 d_0^4} = P^2 v$
	Ũ		$d_2(P^2v)$ in bottom cell
(58, 5, 30)	$\overline{D_4}$	$h_1 \cdot \overline{B_6} + Q_2$	Lemma 5.2.5
(58, 14, 30)	Pv'	$Ph_1^2 \cdot \overline{u'} + \overline{\tau h_0 d_0^4}$	top cell
(59, 5, 31)	$h_{3}^{2}g_{2}$		
(59, 11, 32)	$ au h_2 gm$	$h_0 e_0 \cdot \overline{\tau h_0 g^2}$	top cell
(59, 19, 31)	$\overline{P^2h_1^3u}$		
(60, 6, 32)	$\overline{j_1}$		
(60, 8, 31)	$h_{1}^{2}Q_{2}$		
(60, 17, 31)	P^2h_1v	$\overline{P^2h_1^3u}$	top cell
(62, 11, 32)	$Ph_5c_0d_0$		
(62, 20, 32)	$\tau P^2 h_0 d_0^2 e_0$		
(63, 8, 33)	$h_5 d_0 e_0$		
(63, 11, 35)	$\tau h_2 c_1 g^2$		
(63, 13, 33)	h_1c_0x'	Ph_1x'	Lemma 5.2.6
(63, 17, 33)	$\frac{Ph_1d_0u}{2}u$	$P^2 u'$	Lemma 5.2.4
(65, 11, 35)	$h_1^2 B_{22}$	$B_{21} \cdot h_1^4$	top <u>cell</u>
(65, 20, 34)	$\tau Ph_0 d_0^4$	P^3h_1u	$h_1 \cdot \tau P h_0 d_0^4 = P^3 v$
			$d_2(P^3v)$ in bottom cell
(66, 9, 34)	$\frac{c_0Q_2}{\overline{D^2}}$		Lemma 5.2.7
(66, 18, 34)	$\frac{P^2v'}{12v}$	$\tau Ph_0d_0^4 + P^2h_1^2 \cdot u'$	top cell
(67, 13, 39)	$\frac{\tau h_2^2 g^3}{1000000000000000000000000000000000000$		
(67, 15, 35)	$\frac{h_1^2 U}{\pi^2 (1 + 2)^2}$	$x' \cdot Ph_1^4$	top cell
(67, 23, 35)	$\frac{P^{3}h_{1}^{3}u}{1}$	0	
(68, 6, 36)	$\frac{h_1r_1}{D}$?	
(68, 12, 35)	$\frac{\tau B_8 d_0}{1}$	$19\overline{D}$	4 11
(68, 12, 39)	$\frac{n_3g^3}{D^31}$	$\frac{n_1 \cdot D_4}{D^{3+3}}$	top cell
(68, 21, 35)	$\frac{P^{3}h_{1}v}{V}$	$P^{3}h_{1}^{3}u$	top cell
(69, 9, 36)	$\frac{n_1 \Lambda_3}{1 l^2}$		
(69, 17, 37)	$\frac{n_1 d_0^2 u}{D_2 m'}$	$D^2 m'$	$h \overline{D_{a}} m D'$
(10, 10, 30)	$PC_0 x'$	$\Gamma^{-}x$	$n_0 \cdot Pc_0 x = R_1$ $d_2(P')$ in bottom coll
			$u_2(n_1)$ in bottom cell

Table 40: Ambiguous $E_2(C\tau)$ generators

(s, f, w)	x	ambiguity	definition
(55, 7, 30)	$\overline{h_1 i_1}$	$\tau h_1 G$	$h_1^5 \cdot \overline{h_1 i_1} = \tau g^3$
		B_6	$c_0 \cdot \overline{h_1 i_1} = h_1^5 \cdot \overline{D_4}$
(56, 11, 30)	$\overline{\tau h_0 gm}$	$h_2 x'$	
(59, 11, 32)	$\tau h_2 gm$	$h_0 B_{21}$	
(62, 11, 32)	$Ph_5c_0d_0$	$h_0 R$	$h_0 \cdot \overline{Ph_5c_0d_0} = 0$
(68, 6, 36)	$\overline{h_1r_1}$	$\tau h_1 Q_3$	

Table 41: Adams d_3 differentials for $E_3(C\tau)$

(s, f, w)	x	$d_3(x)$	reference
(29, 5, 16)	$\overline{h_1h_3g}$	d_0^2	Lemma 5.2.12
(47, 17, 24)	$h_0^4 Q'$	P^4d_0	$d_3(h_0^5Q')$ in bottom cell
(49, 9, 28)	$\overline{h_1 h_3 g^2}$	$d_0 e_0^2$	Lemma 5.2.12
(51, 6, 27)	$\overline{h_1^2 h_5 e_0}$	$\overline{h_1^3 B_1}$	$h_1^6 h_5 \cdot \overline{h_1^2 e_0} = \tau e_0 g^2$
			$h_1^6 \cdot \overline{h_1^3 B_1} = c_0 d_0 e_0^2$ in $E_3(C\tau)$
			$d_3(\tau e_0 g^2)$ in bottom cell
(53, 9, 28)	$\overline{h_1G_3}$	$\overline{ au h_0 e_0^3}$	Lemma 5.2.14
(54, 7, 28)	$\overline{h_5 c_0 d_0}$	x'	$h_0 \cdot \overline{h_5 c_0 d_0} = h_5 i$
			$d_3(h_5 i)$ in bottom cell
(55, 7, 30)	$\overline{h_1 i_1}$ or	?	
	$\overline{h_1 i_1} + \tau h_1 G$		
(56, 6, 29)	B_6	$ au h_2 g n$	top cell
(57, 7, 30)	$\overline{h_5 c_0 e_0}$	$h_{1}^{2}B_{8}$	top cell
(59, 10, 31)	$Ph_1^2h_5e_0$	$x' \cdot \overline{h_1^4}$	top cell
(61, 8, 33)	$h_{1}^{3}D_{4}$	$h_1 B_{21}$	Lemma 5.2.16
(65, 11, 34)	$\overline{Ph_5c_0e_0}$	$h_1 \cdot \overline{h_1 c_0 x'} + U$	Lemma 5.2.17
(68, 10, 37)	$\overline{h_1^2 D_4 c_0}$	$B_8 d_0$	$h_1 \cdot \overline{h_1^2 D_4 c_0} = h_2 B_5$
			$d_3(h_2B_5)$ in bottom cell

Table 42: Projection to the top cell of $C\tau$

(s, f, w)	element of $E_{\infty}(C\tau)$	element of $E_{\infty}(S^{0,0})$
(30, 6, 16)	r	$h_1 d_0^2$
(34, 2, 18)	h_2h_5	h_1d_1
(38, 7, 20)	$h_0 y$	$ au h_2 e_0^2$
(41, 4, 22)	h_0c_2	$h_1h_3d_1$
(44, 10, 24)	d_0r	$h_1 d_0^3$
(50, 10, 28)	gr	$h_1 d_0 e_0^2$
(55, 7, 30)	B_6	h_2gn

Table 42: Projection to the top cell of $C\tau$

(s, f, w)	element of $E_{\infty}(C\tau)$	element of $E_{\infty}(S^{0,0})$
(56, 10, 29)	Q_1	$d_0 z$
(57, 7, 30)	Q_2	au gt
(58, 7, 30)	h_0D_2	D_{11}
(58, 11, 32)	$Ph_1^2h_5e_0$	$ au h_2 e_0^2 g$
(59, 8, 33)	$h_1^2 D_4$	$h_2^2 d_1 g$

Table 43: Hidden Adams extensions in $E_\infty(C\tau)$

(s, f, w)	type	from	to	reference
(35, 5, 19)	η	h_2h_5	$h_3^2 g$	top cell
(39, 9, 21)	η	$h_0 y$	u	Lemma $5.3.4$
(41, 9, 22)	ν	$h_0 y$	$\overline{\tau h_0^2 g^2}$	top cell
(42, 6, 23)	η	h_0c_2	$h_3 \cdot \overline{h_3^2 g}$	top cell
(47, 12, 26)	ν	d_0r	$h_1 u'$	Lemma $5.3.9$
(48, 8, 26)	η	$h_{1}^{2}g_{2}$?	
(54, 10, 30)	2	$\overline{h_1d_1g}$?	
(57, 11, 30)	2	Q_2	?	
(57, 11, 30)	u	h_0h_5i	h_1Q_1	bottom cell
(58, 10, 32)	ν	$\overline{h_1 i_1} + \tau h_1 G$?	
(58, 10, 32)	ν	B_6	?	
(59, 10, 32)	ν	$\overline{\tau h_2 d_1 g}$?	
(59, 9, 31)	η	h_0D_2	h_3G_3	top cell
(59, 12, 33)	2	$h_1^2 D_4$?	
(60, 10, 32)	ν	Q_2	$h_1 \cdot \overline{h_3 G_3}$	top cell
(60, 10, 32)	2	$\overline{j_1}$?	

Table 44: Hidden Adams-Novikov 2 extensions

(s,f,w)	from	to
(3, 1, 2)	$4\alpha_{2/2}$	$ au \alpha_1^3$
(11, 1, 6)	$4\alpha_{6/3}$	$ au lpha_1^2 lpha_5$
(18, 2, 10)	$2z_{18}$	$\tau \alpha_1^2 z_{16}$
(19, 1, 10)	$4\alpha_{10/3}$	$ au lpha_1^2 lpha_9$
(20, 2, 11)	$z_{20,2}$	$\tau z_{20,4}$
(27, 1, 14)	$4\alpha_{14/3}$	$ au lpha_1^2 lpha_{13}$
(34, 2, 18)	$2z_{34,2}$	$ au lpha_1^2 z'_{32,2}$
(35, 1, 18)	$4\alpha_{18/3}$	$\tau \alpha_1^2 \alpha_{17}$
(40, 6, 23)	$\alpha_1^4 z_{36}$	$ au z_{40,8}$
(42, 2, 22)	$4z_{42}$	$ au lpha_1^2 z_{40,2}$
(43, 1, 22)	$4\alpha_{22/3}$	$\tau \alpha_1^2 \alpha_{21}$

Table 44: Hidden Adams-Novikov 2 extensions

(s, f, w)	from	to
(51, 1, 26)	$2\alpha_{26/3}$	$\tau \alpha_1^2 \alpha_{25}$
(51, 5, 28)	$4\alpha_{4/4}z_{44,4}$?
(54, 2, 28)	$z_{54,2}$	$ au^4 z_{54,10}$
(59, 1, 30)	$4\alpha_{30/3}$	$ au lpha_1^2 lpha_{29}$
(59, 7, 33)	$z'_{59,7}$?

Table 45: Hidden Adams-Novikov η extensions

(s, f, w)	from	to
(37, 3, 20)	$\alpha_{4/4} z_{30}$	$ au \alpha_{2/2}^2 z_{32,4}$
(38, 2, 20)	z_{38}	$ au^{2}z_{39,7}$
(39, 3, 21)	$\alpha_1 z'_{38}$	$\tau^2 z_{40,8}$
(41, 5, 23)	$\alpha_1 z_{40,4}$	$ au lpha_{2/2} z_{39,7}$
(47, 5, 26)	$2z_{47,5}$	$ au^2 z_{48}$
(58, 6, 32)	$\alpha_1^2 z_{56,4}$	$ au^2 z_{59,11}$

Table 46: Hidden Adams-Novikov ν extensions

(s,f,w)	from	to
(0, 0, 0)	4	α_1^3
(20, 2, 11)	$z_{20,2}$	z_{23}
(32, 2, 17)	$z_{32,2}$	$\alpha_1 z_{34,6}$
(36, 4, 20)	$\alpha_1^2 z_{34,2}$	$z_{39,7}$
(39, 3, 21)	$\alpha_1 z'_{38}$	$\alpha_1^6 z_{36}$
(40, 6, 23)	$\alpha_{1}^{4}z_{36}$	$ au z_{43}$
(45, 3, 24)	$\alpha_1 z_{44,2}$	z_{48}
(48, 4, 26)	$\alpha_{2/2} z_{45}$?
(50, 4, 27)	z_{50}	?
(51, 5, 28)	$\alpha_{2/2}^2 z'_{45}$?
(52, 6, 29)	$z_{52,6}$	$\tau^2 \alpha_1 z_{54,10}$
(56, 8, 32)	$\alpha_1^2 z_{54,6}$	$ au z_{59,11}$

Table 47: Correspondence between classical Adams and Adams-Novikov E_∞

s	Adams	Adams-Novikov	detects
0	h_0^k	2^k	2^k
1	h_1	α_1	η
2	h_1^2	α_1^2	η^2
3	h_2	$\alpha_{2/2}$	u
3	h_0h_2	$2\alpha_{2/2}$	2ν

s	Adams	Adams-Novikov	detects
3	$h_0^2 h_2$	α_1^3	4ν
6	h_2^2	$\alpha_{2/2}^{2}$	$ u^2$
7	$h_0^{\overline{k}}h_3$	$2^{\tilde{k}}\alpha_{4/4}$	σ
8	h_1h_3	$\alpha_1 \alpha_{4/4}$	$\eta\sigma$
8	c_0	$z_8 + \alpha_1 \alpha_{4/4}$	ϵ
9	h_1c_0	$\alpha_1 z_8 + \alpha_1^2 \alpha_{4/4}$	$\eta\epsilon$
9	$h_{1}^{2}h_{3}$	$\alpha_1^2 \alpha_{4/4}$	$\eta^2 \sigma$
8k + 1	$P^k h_1$	α_{4k+1}	μ_{8k+1}
8k + 2	$P^k h_1^2$	$\alpha_1 \alpha_{4k+1}$	$\eta\mu_{8k+1}$
8k+3	$P^k h_2$	$2\alpha_{4k+2/3}$	ζ_{8k+3}
8k+3	$P^k h_0 h_2$	$4\alpha_{4k+2/3}$	$2\zeta_{8k+3}$
8k + 3	$P^k h_0^2 h_2$	$\alpha_1^2 \alpha_{4k+1}$	$4\zeta_{8k+3}$
14	h_{3}^{2}	$\alpha_{4/4}^2$	σ^2
14	d_0	z_{14}	κ
15	$h_0^{k+3}h_4$	$2^k \alpha_{8/5}$	$2^{k}\rho_{15}$
15	$h_1 d_0$	$\alpha_1 z_{14}$	$\eta\kappa$
16	h_1h_4	z_{16}	η_4
8k+8	$P^k c_0$	$\alpha_1 \alpha_{4k+4/b}$	$\eta \rho_{8k+7}$
17	$h_{1}^{2}h_{4}$	$\alpha_1 z_{16}$	$\eta\eta_4$
17	$h_2 d_0$	$\alpha_{2/2} z_{14}$	$ u\kappa$
8k + 9	$P^k h_1 c_0$	$\alpha_1^2 \alpha_{4k+4/b}$	$\eta^2 \rho_{8k+7}$
18	h_2h_4	z_{18}	$ u_4$
18	$h_0 h_2 h_4$	$2z_{18}$	$2\nu_4$
18	$h_{1}^{3}h_{4}$	$\alpha_1^2 z_{16}$	$4\nu_4$
19	c_1	z_{19}	$\overline{\sigma}$
20	g	$z_{20,2}$	$\overline{\kappa}$
20	$h_0 g$	$z_{20,4}$	$2\overline{\kappa}$
20	$h_0^2 g$	$2z_{20,4}$	$4\overline{\kappa}$
21	$h_{2}^{2}h_{4}$	$\alpha_{2/2} z_{18}$	$\nu \nu_4$
21	h_1g	$\alpha_1 z_{20,2}$	$\eta \overline{\kappa}$
22	h_2c_1	$\alpha_{2/2} z_{19}$	$\nu \overline{\sigma}$
22	Pd_0	$\alpha_1^2 z_{20,2}$	$\eta^2 \overline{\kappa}$
23	$h_4 c_0$	$\alpha_{4/4} z_{16}$	$\sigma \eta_4$
23	h_2g	z_{23}	$\nu \overline{\kappa}$
23	h_0h_2g	$2z_{23}$	$2\nu\kappa$
23	Ph_1d_0	$4z_{23}$	$4\nu\kappa$
23	$h_0^{n+2}i$	$2^{n} \alpha_{12/4}$	$2^{\kappa} \rho_{23}$
24	$n_1 n_4 c_0$	$\alpha_1 \alpha_{4/4} z_{16}$	$\eta \sigma \eta_4$
26	$n_{2}^{2}g_{12}$	$\alpha_{2/2} z_{23}$	$\nu^2\kappa$
28	d_{0}^{2}	z_{28}	κ- 0
3U 91	n_{4}^{-}	z_{30}	θ_4
31 91	$n_1 n_4^2$	$\alpha_1 z_{30}$	$\eta \theta_4$
31	n	z_{31}	

Table 47: Correspondence between classical Adams and Adams-Novikov E_∞

s	Adams	Adams-Novikov	detects
31	$h_0^{k+10} h_5$	$2^k \alpha_{16/6}$	$2^{k}\rho_{31}$
32	$h_1 h_5$	$z'_{32,2}$	η_5
32	d_1	$z_{32,4}$	
32	q	$z_{32,2}$	
33	$h_{1}^{2}h_{5}$	$\alpha_1 z'_{32,2}$	$\eta\eta_5$
33	p^{-}	$\alpha_2 z_{30}$	$ u heta_4$
33	h_1q	$\alpha_1 z_{32,2}$	
34	$h_0 h_2 h_5$	$2z_{34,2}$	
34	$h_{1}^{3}h_{5}$	$\alpha_1^2 z'_{32,2}$	$\eta^2 \eta_5$
34	h_2n	$\alpha_{2/2} z_{31}$	
34	e_{0}^{2}	$z_{34,6}$	$\kappa \overline{\kappa}$
35	h_2d_1	$\alpha_{2/2} z_{32,4}$	
35	$h_1 e_0^2$	$\alpha_1 z_{34,6}$	$\eta \kappa \overline{\kappa}$
36	t	$\alpha_1^2 z_{34,2}$	
37	$h_2^2 h_5$	$\alpha_{2/2} z_{34,2}$	
37	x	$\alpha_{4/4} z_{30}$	$\sigma heta_4$
38	$h_{0}^{2}h_{3}h_{5}$	z_{38}	
38	$h_{0}^{3}h_{3}h_{5}$	$2z_{38}$	
38	$h_{2}^{2}d_{1}$	$\alpha_{2/2}^2 z_{32,4}$	
39	$h_1h_3h_5$	$z_{39,3}$	
39	h_5c_0	$z'_{39,3}$	
39	h_3d_1	$\alpha_{4/4} z_{32,4}$	
39	$h_2 t$	$z_{39,7}$	
39	u	$\alpha_1 z'_{38}$	
39	$P^2 h_0^{k+2} i$	$2^k \alpha_{20/4}$	$2^{k}\rho_{39}$
40	$h_{1}^{2}h_{3}h_{5}$	$\alpha_1 z_{39,3}$	
40	f_1	$z_{40,4}$	
40	$h_1h_5c_0$	$\alpha_1 z'_{39,3}$	
40	Ph_1h_5	$z_{40,2}$	
40	g^2	$\alpha_{1}^{4}z_{36}$	$\overline{\kappa}^2$
40	$h_1 u$	$z_{40,8}$	$2\overline{\kappa}^2$
41	$h_1 f_1$	$\alpha_1 z_{40,4}$	
41	$Ph_1^2h_5$	$\alpha_1 z_{40,2}$	
41	z	$lpha_1^5 z_{36}$	$\eta \overline{\kappa}^2$
42	Ph_2h_5	$2z_{42}$	
42	$Ph_0h_2h_5$	$4z_{42}$	
42	$Ph_{1}^{3}h_{5}$	$\alpha_1^2 z_{40,2}$	0.0
42	d_0^3	$\alpha_1^6 z_{36}$	$\eta^2 \overline{\kappa}^2$
44	g_2	$z_{44,4}$	
44	h_0g_2	$2z_{44,4}$	
44	$h_0^2 g_2$	$4z_{44,4}$	
45	$h_{3}^{2}h_{5}$	z'_{45}	$\theta_{4.5}$
45	$h_0 h_3^2 h_5$	$2z'_{45}$	$2\theta_{4.5}$

Table 47: Correspondence between classical Adams and Adams-Novikov E_∞

s	Adams	Adams-Novikov	detects
45	$h_5 d_0$	$z_{45} + 2z'_{45}$	
45	$h_{1}g_{2}$	$\alpha_1 z_{44,4}$	
45	$h_0h_5d_0$	$4z'_{45}$	$4\theta_{4.5}$
45	$h_0^2 h_5 d_0$	$8z'_{45}$	$8\theta_{4.5}$
45	w	$\alpha_1 z_{44,2}$	
46	$h_1h_5d_0$	$\alpha_1 z_{45}$	
46	B_1	$\alpha_1 z'_{45}$	$\eta \theta_{4.5}$
46	N	$\alpha_{2/2} z_{43,3}$	
46	$d_0 l$	$\alpha_1^2 z_{44,2}$	
47	h_2g_2	$\alpha_{2/2} z_{44,4}$	
47	Ph_5c_0	$\alpha_{8/5} z'_{32,2}$	2
47	h_1B_1	$\alpha_1^2 z'_{45}$	$\eta^2 heta_{4.5}$
47	$e_0 r$	$2z_{47,5}$	
47	Pu	$4z_{47,5}$	
47	$h_0^{\kappa+\gamma}Q'$	$2^k \alpha_{24/5}$	$2^{k}\rho_{47}$
48	$h_2h_5d_0$	$\alpha_{2/2} z_{45}$	
48	B_2	$\alpha_{2/2} z'_{45}$	$ u \theta_{4.5}$
48	h_0B_2	$2\alpha_{2/2}z'_{45}$	$2\nu\theta_{4.5}$
48	$Ph_1h_5c_0$	$\alpha_1 \alpha_{8/5} z'_{32,2}$	0
48	$d_0 e_0^2$	z_{48}	$\kappa^2 \overline{\kappa}$
50	h_5c_1	z'_{50}	
50	C	z_{50}	
51	h_3g_2	$\alpha_{4/4} z_{44,4}$	
51	$h_0 h_3 g_2$	$2\alpha_{4/4}z_{44,4}$	2.0
51	h_2B_2	$\alpha_{2/2}^2 z'_{45}$	$\nu^2 \theta_{4.5}$
51	gn	z_{51}	
52	$h_1 h_3 g_2$	$\alpha_1 \alpha_{4/4} z_{44,4}$	
52	d_1g	$z_{52,8}$	
52	e_0m	$z_{52,6}$,	
53	$h_2h_5c_1$	$\alpha_{2/2} z'_{50}$	
53	h_2C	$z_8 z'_{45}$ or z_{53}	0
53	x'	$z_8 z'_{45}$ or z_{53}	$\epsilon \theta_{4.5}$
53	$d_0 u$	$\alpha_1 z_{52,6}$	
54	$h_0 h_5 i$	$z_{54,2}$	
54 54	$h_1 x'$	$\alpha_1 z_8 z'_{45}$ or $\alpha_1 z_{53,5}$	-2
54	$e_0^2 g$	$z_{54,10}$	$\kappa \kappa^2$
55	$P^{4}h_{0}^{\kappa+2}i$	$2^{\kappa} \alpha_{28/4}$	$2^{\kappa}\rho_{55}$
57	$h_0 h_2 h_5 i$	$\alpha_{2/2} z_{54,2}$	
58	h_1Q_2	$\alpha_1 z_{57}$	0
59 50	B_{21}	$z_{59,5}$ or $z'_{59,7}$	$\kappa \theta_{4.5}$
59	d_0w	$z_{59,7}$	

Table 47: Correspondence between classical Adams and Adams-Novikov E_∞

boundary $\pi_{*,*}(C\tau)$ (s, f) $\begin{array}{c} \alpha_1^{k+4} \\ \alpha_1^{k+3} \alpha_{4/4} \\ \alpha_1^{k+3} \alpha_5 \end{array}$ $\begin{array}{c} h_1^{k+4} \\ h_1^{k+2} c_0 \end{array}$ (4,4) + k(1,1)(10,4) + k(1,1) Ph_{1}^{k+4} (12,4) + k(1,1) $\alpha_1^{k+3} \alpha_{8/5}$ $h_1^{k+2}Pc_0$ (18,4) + k(1,1) $\alpha_1^{\bar{k}+3}\alpha_9$ (20,4) + k(1,1) $\begin{array}{c} \alpha_1 & \alpha_9 \\ \alpha_1^{k+3} \alpha_{12/4} \\ \alpha_1^{k+3} \alpha_{13} \\ \alpha_1^{k+3} \alpha_{16/6} \\ \alpha_1^{k+3} \alpha_{17} \\ \alpha_1^{k+3} \alpha_{20/4} \\ \ldots \\ \alpha_{k+3}^{k+3} \end{array}$ (26, 4) + k(1, 1) $\begin{array}{c} h_1 & I & c_0 \\ P^3 h_1^{k+4} \\ h_1^{k+2} P^3 c_0 \\ P^4 h_1^{k+4} \\ h_1^{k+2} P^4 c_0 \\ P^5 I^{k+4} \end{array}$ (28,4) + k(1,1)(34, 4) + k(1, 1)(36, 4) + k(1, 1)(42,4) + k(1,1) $P^{5}h_{1}^{k+4}$ $\begin{array}{c} \alpha_1^{k+3} \alpha_{21} \\ \alpha_1^{k+3} \alpha_{21} \\ \alpha_1^{k+3} \alpha_{24/5} \end{array}$ (44,4) + k(1,1) $\begin{array}{c} h_{1}^{k} h_{1}^{k+2} P^{5} c_{0} \\ P^{6} h_{1}^{k+4} \\ h_{1}^{k+2} P^{6} c_{0} \end{array}$ (50,4) + k(1,1) $\begin{array}{c} \alpha_1^{k} & \alpha_{24/5}^{k} \\ \alpha_1^{k+3} \alpha_{25} \\ \alpha_1^{k+3} \alpha_{28/4} \end{array}$ (54, 4) + k(1, 1)(58, 4) + k(1, 1) $\alpha_1^2 \alpha_{4/4} z_{16}$ $h_1^2 h_4 c_0$ (25, 5) $h_{1}^{2}h_{3}g$ (29, 7) $\alpha_1 z_{28}$ h_1d_1 (33, 5) $\alpha_1 z_{32,4}$ $\alpha_1^2 z_{32,4}$ $h_1^2 d_1$ (34, 6) $\begin{array}{c} \alpha_1^{3} z_{32,2}' \\ \alpha_1^{3} z_{32,2}' \\ \alpha_1^{4} z_{32,2}' \end{array}$ $h_{1}^{4}h_{5}$ (35, 5) $h_{1}^{5}h_{5}$ (36, 6) $\alpha_1^5 z_{32,2}'$ (37, 7) $h_{1}^{6}h_{5}$ $h_2 e_0^2$ (37, 7) $\alpha_{2/2} z_{34,6}$ $\alpha_1^{\tilde{6}} z'_{32,2}$ $h_{1}^{7}h_{5}$ (38, 8)(40, 6) $h_1h_3d_1$ $\alpha_1 \alpha_{4/4} z_{32,4}$ $2z_{40,8}$ $h_0^2 g^2$ (40, 8) $\alpha_{1}^{2} z'_{39,3}$ $h_{1}^{2}h_{5}c_{0}$ (41, 5) $\alpha_1^2 \alpha_{4/4} z_{32,4}$ $h_1^2 h_3 d_1$ (41, 7) $\alpha_1^{\bar{3}} z'_{39,3}$ (42, 6) $h_1^3 h_5 c_0$ (42, 8) $\substack{\alpha_{2/2} z_{39,7} \\ \alpha_1^4 z_{39,3}'}$ h_2c_1g $h_1^4 h_5 c_0$ (43,7) h_2g^2 (43, 9) $z_{43,9}$ $2z_{43,9}$ $h_0 h_2 g^2$ (43, 9) $h_1 c_0 e_0^2$ (43, 9) $4z_{43,9}$ $\alpha_1^5 z_{39,3}^\prime$ (44, 8) $h_1^5 h_5 c_0$ $\alpha_1^6 z'_{39,3}$ $h_1^6 h_5 c_0$ (45, 9) $h_1^2 g_2 \\ h_2^2 g^2$ (46, 6) $\alpha_1^2 z_{44,4}$ (46, 10) $\alpha_{2/2} z_{43,9}$ $\bar{Ph_1^2}h_5c_0$ (49, 5) $\alpha_1^2 \alpha_{8/5} z'_{32,2}$ $h_1^2 h_3 g^2$ (49, 11) $\alpha_1 z_{48}$ h_1d_1g (53, 9) $\alpha_1 z_{52,8}$? $\alpha_1 z_{53}$ (54, 8) $h_1 i_1$ (54, 8) $\substack{\alpha_{2/2} z_{51} \\ \alpha_1^2 z_{52,8}}$ h_2gn (54, 10) $h_{1}^{2}d_{1}g$

Table 48: Classical Adams-Novikov boundaries

(s,f)	boundary	$\pi_{*,*}(C\tau)$
(55, 9)	$\alpha_{2/2} z_{52,8}$	$h_2 d_1 g$
(55, 9)	$\alpha_1^2 z_{53}$	$h_{1}^{2}i_{1}$
(55, 11)	$\alpha_{1}z_{54,10}$	$h_1^7 h_5 e_0$
(56, 8)	$\alpha_1^2 z_{54,6}$	gt
(56, 10)	$lpha_1^3 z_{53}$	$h_{1}^{3}i_{1}$
(57, 5)	$\alpha_{1}z_{56,4}$	D_{11}
(57, 11)	$\alpha_{1}^{4}z_{53}$	$h_{1}^{4}i_{1}$
(57, 11)	$\alpha_{2/2} z_{54,10}$	$h_2 d_0 g^2$
(58, 6)	$\alpha_{4/4}^2 z_{44,4}$	$h_{3}^{2}g_{2}$
(58, 6)	$\alpha_1^2 z_{56,4}$	$h_1 D_{11}$
(58, 10)	$\alpha_{2/2}^2 z_{52,8}$	$h_2^2 d_1 g$
(58, 12)	$\alpha_{1}^{5'}z_{53}$	$h_{1}^{5}i_{1}$
(59, 5)	$\alpha_{1}^{2}z_{57}$	$h_1^2 Q_2$
(59, 7)	? $z'_{59,7}$	j_1
(59, 9)	$\alpha_{4/4} z_{52,8}$	h_3d_1g
(59, 11)	$z_{59,11}$	c_1g^2

Table 48: Classical Adams-Novikov boundaries

Table 49: Classical Adams-Novikov non-permanent classes

(s, f)	class	$\pi_{*,*}(C\tau)$
(5,1) + k(1,1)	$\alpha_1^k \alpha_3$	$h_1^k \cdot \overline{h_1^4}$
(11,1) + k(1,1)	$\alpha_1^k \alpha_{6/3}$	$h_1^k \cdot \overline{h_1^2 c_0}$
(13,1) + k(1,1)	$\alpha_1^k \alpha_7$	$h_1^k \cdot \overline{Ph_1^4}k$
(19,1) + k(1,1)	$\alpha_1^k \alpha_{10/3}$	$h_1^k \cdot \overline{Ph_1^2c_0}$
(21,1) + k(1,1)	$\alpha_1^k \alpha_{11}$	$h_1^k \cdot \overline{P^2 h_1^4}$
(27,1) + k(1,1)	$\alpha_1^k \alpha_{14/3}$	$h_1^k \cdot \overline{P^2 h_1^2 c_0}$
(29,1) + k(1,1)	$\alpha_1^k \alpha_{15}$	$h_1^k \cdot \overline{P^3 h_1^4}$
(35,1) + k(1,1)	$\alpha_1^k \alpha_{18/3}$	$h_1^k \cdot \overline{P^3 h_1^2 c_0}$
(37,1) + k(1,1)	$\alpha_1^k \alpha_{19}$	$h_1^k \cdot \overline{P^4 h_1^4}$
(43,1) + k(1,1)	$\alpha_1^k \alpha_{22/3}$	$h_1^k \cdot \overline{P^4 h_1^2 c_0}$
(45,1) + k(1,1)	$\alpha_1^k \alpha_{23}$	$h_1^k \cdot \overline{P^5 h_1^4}$
(51,1) + k(1,1)	$\alpha_1^k \alpha_{26/3}$	$h_1^k \cdot \overline{P^5 h_1^2 c_0}$
(53,1) + k(1,1)	$\alpha_1^k \alpha_{27}$	$h_1^k \cdot \overline{P^6 h_1^4}$
(59,1) + k(1,1)	$\alpha_1^k \alpha_{30/3}$	$h_1^k \cdot \overline{P^6 h_1^2 c_0}$
(26, 2)	z_{26}	$\overline{h_1^2 h_4 c_0}$
(30, 2)	z_{30}^{\prime}	r^{-}
(34, 2)	$z_{34,2}$	h_2h_5
(35, 3)	$\alpha_{1}z_{34,2}$	$h_{3}^{2}g$
(36, 2)	z_{36}	$h_1^4 h_5$
(37, 3)	$\alpha_1 z_{36}$	$h_1 \cdot \overline{h_1^4 h_5}$
(38, 2)	z'_{38}	$h_0 y$

(s,f)	class	$\pi_{*,*}(C\tau)$
(38, 4)	$\alpha_1^2 z_{36}$	$h_1^2 \cdot \overline{h_1^4 h_5}$
(39, 5)	$\alpha_{1}^{3}z_{36}$	$h_1^3 \cdot \overline{h_1^4 h_5}$
(41, 3)	z_{41}	h_0c_2
(41, 3)	$\alpha_{2/2} z'_{38}$	$\overline{\tau h_0^2 g^2}$
(42, 2)	z_{42}	$\overline{h_1^2 h_5 c_0}$
(42, 4)	$\alpha_1 z_{41}$	$h_3 \cdot \overline{h_3^2 g}$
(43, 3)	$\alpha_1 z_{42}$	$h_1 \cdot \overline{h_1^2 h_5 c_0}$
(43, 3)	$z_{43,3}$	$\overline{\tau h_2 c_1 g}$
(44, 2)	$z_{44,2}$	$\overline{ au^2 h_2 g^2}$
(44, 4)	$\alpha_{1}^{2}z_{42}$	$h_1^2 \cdot \overline{h_1^2 h_5 c_0}$
(44, 4)	$z'_{44,4}$	$\overline{\tau h_0 h_2 g^2}$
(44, 4)	$2z_{44,4}'$	d_0r
(45, 5)	$\alpha_1^3 z_{42}$	$h_1^3 \cdot \overline{h_1^2 h_5 c_0}$
(46, 6)	$\alpha_1^4 z_{42}$	$h_1^4 \cdot \overline{h_1^2 h_5 c_0}$
(47, 3)	$z_{47,3}$	$\overline{h_1^2 g_2}$
(47, 5)	$z_{47,5}$	$\overline{ au h_2^2 g^2}$
(50, 2)	$z_{50,2}$	$Ph_1^2h_5c_0$
(50, 6)	$\alpha_{2/2} z_{47,5}$	gr
(54, 6)	$z_{54,6}$	$\overline{h_1d_1g}$
(55, 5)	z_{55}	B_6
(55, 7)	$\alpha_1 z_{54,6}$	$h_1^2 d_1 g$
(55, 5)	? z'_{55}	$h_1 i_1 + \tau h_1 G$
(56, 2)	$z_{56,2}$	Q_1
(56, 4)	$z_{56,4}$	$\frac{\tau h_2 d_1 g}{\tau^2 h_2 d_1 g}$
(56, 6)	$z_{56,6}$	$h_1^2 i_1 + h_5 c_0 e_0$
(57, 3)	z_{57}	Q_2
(57,7)	$\alpha_1 z_{56,6}$	$h_1 \cdot h_1^2 i_1 + h_1 h_5 c_0 e_0$
(58, 2)	$z_{58,2}$	$h_0 D_2$ $D h^2 h$
(58, 6)	$z_{58,6}$	$Pn_{\bar{1}}n_{\bar{5}}e_{0}$
(58, 8)	$lpha_1^2 z_{56,6}$	$\frac{h_1^2 \cdot h_1^2 i_1 + h_1^2 h_5 c_0 e_0}{h_1^2}$
(59,3)	$z_{59,3}$	$\frac{h_3^2 g_2}{h_1 G}$
(59,3)	$\alpha_1 z_{58,2}$	h_3G_3
(09, 7)	$z_{59,7}$	$n_1 D_4$
(59, 9)	$lpha_1^{\star}z_{56,6}$	$\frac{n_{\tilde{1}} \cdot n_{\tilde{1}}}{120} n_{1} + n_{1} n_{5} c_{0} e_{0}$
(60, 2)	$z_{60,2}$	$\frac{n_{\tilde{1}}}{\dot{a}}Q_2$
(00, 4)	: <i>Z</i> 60,4	$\frac{J_1}{b_1 - \frac{1}{b_2 - C_2}}$
(00, 4)	$lpha_{2/2}z_{57}$	$\frac{n_1 \cdot n_3 G_3}{b d a}$
(00,0)	$^{\sim}60,6$	$n_3 a_1 g$

Table 49: Classical Adams-Novikov non-permanent classes
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