

THE \mathbb{Z}/p -EQUIVARIANT SPECTRUM $BP\mathbb{R}$ FOR AN ODD PRIME p

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ABSTRACT. In the present paper, we construct a \mathbb{Z}/p -equivariant analog of the $\mathbb{Z}/2$ -equivariant spectrum $BP\mathbb{R}$ previously constructed by Hu and Kriz. We prove that this spectrum has some of the properties conjectured by Hill, Hopkins, and Ravenel. Our main construction method is an \mathbb{Z}/p -equivariant analog of the Brown-Peterson tower of BP , based on a previous description of the \mathbb{Z}/p -equivariant Steenrod algebra with constant coefficients by the authors. We also describe several variants of our construction and comparisons with other known equivariant spectra.

1. INTRODUCTION

Hill, Hopkins, and Ravenel [7] conjectured that there exists a \mathbb{Z}/p -equivariant structure on $BP^{(p-1)}$ for $p > 2$ such that the geometric fixed point spectrum is $H\mathbb{Z}/p$. This equivariant spectrum is expected to have a number of properties. The purpose of this paper is to give a construction of such a spectrum and prove some of the properties. Our construction is based on a mild extension of the computation of the \mathbb{Z}/p -equivariant Steenrod algebra with respect to the constant Mackey functor $\underline{\mathbb{Z}/p}$ [9, 16]. Following Sankar-Wilson [16], we denote by T the second desuspension of the \mathbb{Z}/p -equivariant degree p map

$$S^\beta \rightarrow S^2$$

where β is the basic irreducible representation of \mathbb{Z}/p . Denoting by $\tilde{\mathcal{L}}_{p-1}$ the \mathbb{Z}/p -Mackey functor whose non-equivariant part is the reduced

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regular representation \mathcal{L}_{p-1} of \mathbb{Z}/p and the \mathbb{Z}/p -fixed part is 0, there is a map of \mathbb{Z}/p -equivariant spectra

$$(1) \quad H\underline{\mathbb{Z}} \rightarrow \Sigma^{2p^{n-1}+(p^n-p^{n-1}-1)\beta} H\underline{\mathbb{Z}} \wedge T$$

(which comes from the fact that after smashing with $H\underline{\mathbb{Z}}$, the right-hand side of (1) becomes a summand of the left-hand side as an $H\underline{\mathbb{Z}}$ -module. Now we also have a “connecting map”

$$H\underline{\mathbb{Z}} \wedge T \rightarrow \Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1}$$

(see the cofibration sequence (21) below). Composing with (1), we get a map

$$(2) \quad Q'_n : H\underline{\mathbb{Z}} \rightarrow \Sigma^{2p^{n-1}-1+(p^n-p^{n-1})\beta} H\tilde{\mathcal{L}}_{p-1}$$

(see formula (22) below). This map is a \mathbb{Z}/p -equivariant analog of the integral Q_n -elements which form the first k-invariant of the Brown-Peterson construction of BP [3].

The approach of our construction is to construct a \mathbb{Z}/p -equivariant spectrum $BP\mathbb{R}$ by mimicking, in a minimal way, the Brown-Peterson construction [3] in the category of \mathbb{Z}/p -equivariant spectra. It is important to note that the appearance of the non-constant Mackey functor occurring on the right-hand side of (2) is a reflection of the fact that in $(BP\mathbb{R}_{\{e\}})_*$, v_n lies in a copy of the representation \mathcal{L}_{p-1} . We denote the generator of this representation by r_n . In fact, in the equivariant analogue of the BP tower, a key feature is the interplay between the Mackey functors $\underline{\mathbb{Z}}$, $\tilde{\mathcal{L}}_{p-1}$ and the Mackey functor $\underline{\mathcal{L}}_p$, which is the principal projective Mackey functor on a fixed element (i.e. has the integral regular representation as the non-equivariant part, and \mathbb{Z} as the fixed points).

The non-equivariant spectrum underlying our “minimal” $BP\mathbb{R}$ following the Brown-Peterson construction is somewhat smaller than the $\Lambda_{p-1}BP$ conjectured by Hill, Hopkins, and Ravenel [7]. In fact, at each v_n , we are forced to put into the non-equivariant coefficients the representations

$$(3) \quad (\mathbb{Z} \oplus \mathcal{L}_{p-1}r_n \oplus r_n^2\underline{\mathcal{L}}_p[r_n]) \otimes \mathbb{Z}[Nr_n]$$

where $\underline{\mathcal{L}}_p$ is the integral regular \mathbb{Z}/p -representation and N denotes the multiplicative norm (see formula (9) below).

The non-equivariant spectrum conjectured by Hill, Hopkins, and Ravenel [7] (which we denote by $BP\mathbb{R}^{HHR}$) should have, instead of (3),

$$(4) \quad \text{Sym}(\mathcal{L}_{p-1} \cdot r_n).$$

This, in fact, coincides with (3) for $p = 3$. For $p \geq 5$, (4) is bigger, but one can construct a candidate for the spectrum BPR^{HHR} by forming a wedge of BPR with a wedge of even suspensions of copies of the multiplicative norm of BP from $\{e\}$ to \mathbb{Z}/p . (See Section 5 below.)

We should remark that we can currently only make the equivariant analogue of the Brown-Peterson construction in the category of Borel-complete spectra. In this category, the obstructions to continuing the construction are either non-equivariant (and vanish for the same reason as in [3], i.e. evenness), or are non-torsion with respect to multiplication by the class represented by the inclusion

$$S^0 \rightarrow S^\beta,$$

which can therefore be treated on the level of geometric fixed points. On geometric fixed points, however, the equivariant Brown-Peterson tower splits into a wedge equivalences on summands, which is why those obstructions also vanish.

The fact that we work in the Borel-complete category, however, should not be a substantial restriction, since the spectrum BPR should be Borel-complete anyway (similarly as for $p = 2$). In fact, we have a \mathbb{Z}/p -equivariant analog of the Borel cohomology spectral sequence [8], which we present in Section 3. This allows us to calculate the $RO(\mathbb{Z}/p)$ -graded coefficients BPR_* completely (Theorem 2 below). Not having a “genuine” version of equivariant Brown-Peterson construction amounts to not having, at the moment, an analogue of the slice spectral sequence.

In Section 5, we also discuss analogues of the BPR construction on Johnson-Wilson-type spectra. Using the obstruction theory of Robinson [15] and Baker [2], we construct a \mathbb{Z}/p -action related to BPR on a completion of the smash product of $(p - 1)$ copies of $E(n)$ (or its variants, such as E_n). We conjecture that there is an “orientation” map from BPR^{HHR} to these spectra (however, we do not yet have a ring structure on the source).

It is also known that \mathbb{Z}/p acts on E_n when $(p - 1) \mid n$ via the subgroup of the Morava stabilizer group. The case $n = p - 1$ has been especially studied (see [13, 17]). It is reasonable to conjecture that BPR^{HHR} maps into this equivariant E_n in a way which is identity on the \mathcal{L}_{p-1} containing v_n in the non-equivariant coefficients, for which we give evidence on the level of formal group laws.

The present paper incorporates numerous observations, suggestions, and references communicated to us by Guchuan Li.

2. PRELIMINARY COMPUTATIONS

We will use the integral trivial, regular resp. reduced regular representation $\mathcal{L}_1, \mathcal{L}_p, \mathcal{L}_{p-1}$ of \mathbb{Z}/p . Note that all are isomorphic to their duals integrally. In case of \mathcal{L}_{p-1} , we have an isomorphism

$$\mathbb{Z}^p / (1, 1, \dots, 1) \rightarrow \{(a_1, \dots, a_p) \in \mathbb{Z}^p \mid \sum a_i = 0\}$$

by sending

$$(1, 0, \dots, 0) \mapsto (1, -1, 0, \dots, 0).$$

We will work with integral Mackey functors here. We denote by $\underline{\mathbb{Z}} = \underline{\mathcal{L}}_1$ the constant Mackey functor, and by $\tilde{\mathcal{L}}_{p-1}$ the Mackey functor equal to the integral reduced regular representation on the free orbit and 0 on the fixed orbit. We also have the co-constant Mackey functor $\overline{\mathcal{L}}_1$ where again both the non-equivariant and fixed parts are \mathbb{Z} and the corestriction is 1 while the restriction is p . We have short exact sequences

$$(5) \quad 0 \rightarrow \underline{\mathcal{L}}_1 \rightarrow \underline{\mathcal{L}}_p \rightarrow \tilde{\mathcal{L}}_{p-1} \rightarrow 0$$

$$(6) \quad 0 \rightarrow \tilde{\mathcal{L}}_{p-1} \rightarrow \underline{\mathcal{L}}_p \rightarrow \overline{\mathcal{L}}_1 \rightarrow 0.$$

We also note that for a non-trivial irreducible representation β of \mathbb{Z}/p ($p > 2$), we have

$$(7) \quad H\overline{\mathcal{L}}_1 = \Sigma^{2-\beta} H\underline{\mathcal{L}}_1.$$

We also have

$$(8) \quad H\tilde{\mathcal{L}}_{p-1} \wedge_{H\underline{\mathbb{Z}}} H\tilde{\mathcal{L}}_{p-1} = H\overline{\mathcal{L}}_1 \vee \bigvee_{p-2} H\underline{\mathcal{L}}_p.$$

We will work p -locally, so whenever we say “ \mathbb{Z} ,” we mean $\mathbb{Z}_{(p)}$. Then all equivariant spectra are $(\beta - \beta')$ -periodic for non-trivial irreducible \mathbb{Z}/p -representations β, β' . This is due to the fact that there exists a \mathbb{Z}/p -equivariant map of spaces

$$S^\beta \rightarrow S^{\beta'}$$

of some degree $k \in \mathbb{Z}/p^\times$, which is a p -local equivalence (cf. [16]). Thus, instead of $RO(\mathbb{Z}/p)$ -grading, we can consider R -grading where $R = \mathbb{Z}\{1, \beta\}$ for a chosen non-trivial irreducible \mathbb{Z}/p -representation β . This is also true for ordinary \mathbb{Z}/p -equivariant homology with coefficients over a $\underline{\mathbb{Z}}$ -Mackey module for a different reason (see [9]).

Proposition 1. *1. The R -graded coefficients of $H\underline{\mathcal{L}}_1$ are \mathbb{Z} in degrees $2k - k\beta$, $k \geq 0$ and \mathbb{Z}/p in degrees $2k - \ell\beta$ with $0 \leq k < \ell$ (this is called the good wedge) and \mathbb{Z} in degrees $2k - k\beta$, $k < 0$ and \mathbb{Z}/p in degrees $-1 - 2k + \ell\beta$ where $2 < 2k + 1 < 2\ell$ (this is called the derived wedge).*

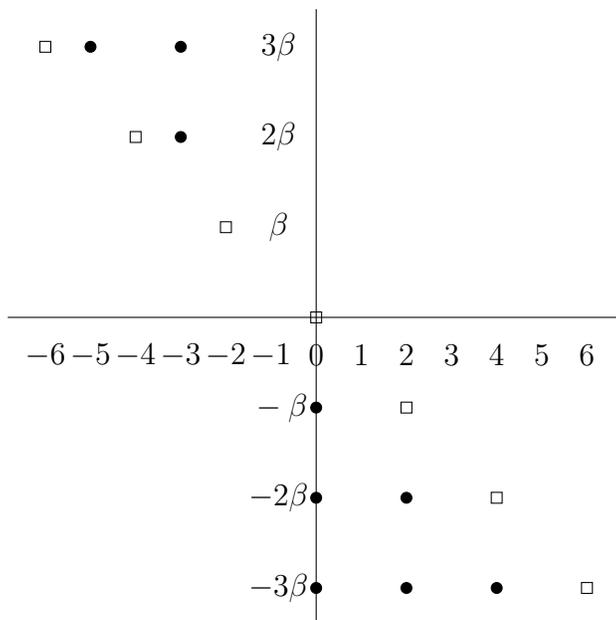


FIGURE 1. The coefficients $H\underline{\mathbb{Z}}_*$

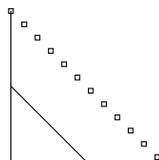


FIGURE 2. Shorthand notation for $H\underline{\mathbb{Z}}_*$

2. The R -graded coefficients of $H\tilde{\mathcal{L}}_{p-1}$ are \mathbb{Z}/p in degrees $2k + 1 - \ell\beta$ where $0 < 2k + 1 < 2\ell$ (this is the good wedge) and \mathbb{Z}/p in degrees $-2k + \ell\beta$ where $0 < k \leq \ell$ (this is the derived wedge).
 (In the remaining R -degrees, the coefficients are 0.)

□

The pattern of the R -graded coefficients of $H\underline{\mathcal{L}}_1 = H\underline{\mathbb{Z}}$ is shown in Figure 1 below (where a solid dot means a copy of \mathbb{Z}/p and an empty square means a copy of \mathbb{Z}).

In our tower illustration, we shall use the shortcut for this pattern shown in Figure 2.

Figure 3 shows the pattern of the R -graded coefficients of $\Sigma^{\beta-1}H\tilde{\mathcal{L}}_{p-1}$. The shift is so that the corner of the “good wedge” is at the $(0, 0)$ -point.

Figure 4 shows the shortcut we use for this pattern.

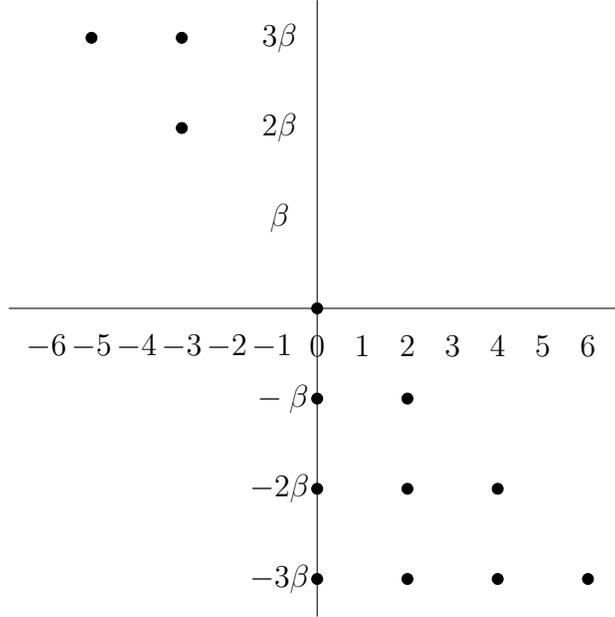


FIGURE 3. The coefficients $(\Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1})_*$

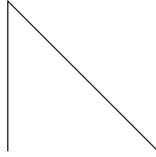


FIGURE 4. Shorthand notation for $(\Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1})_*$

Figure 5 shows the R -graded coefficients of $H\underline{\mathbb{Z}/p}$.

Our shortcut for this pattern is shown in Figure 6.

Figure 7 shows the R -graded coefficients of $\Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1} = \Sigma^{\beta-1} H(\tilde{\mathcal{L}}_{p-1}/p)$.

Our shorthand for this pattern is shown in Figure 8.

3. THE BOREL COHOMOLOGY SPECTRAL SEQUENCE

The Borel cohomology $F(E\mathbb{Z}/p_+, X)$ for a \mathbb{Z}/p -equivariant spectrum X will be denoted by X^c . Coefficients of $H\underline{\mathcal{L}}_1^c$, $H\tilde{\mathcal{L}}_{p-1}^c$ are obtained by inverting σ^{-2} in the good wedge in Proposition 1 (which acts by isomorphism wherever dimensionally possible).

Homotopy classes in a summand of the non-equivariant homotopy groups of a \mathbb{Z}/p -equivariant spectrum which is isomorphic to \mathcal{L}_p is called *negligible*. Such classes cannot receive or support differentials in the

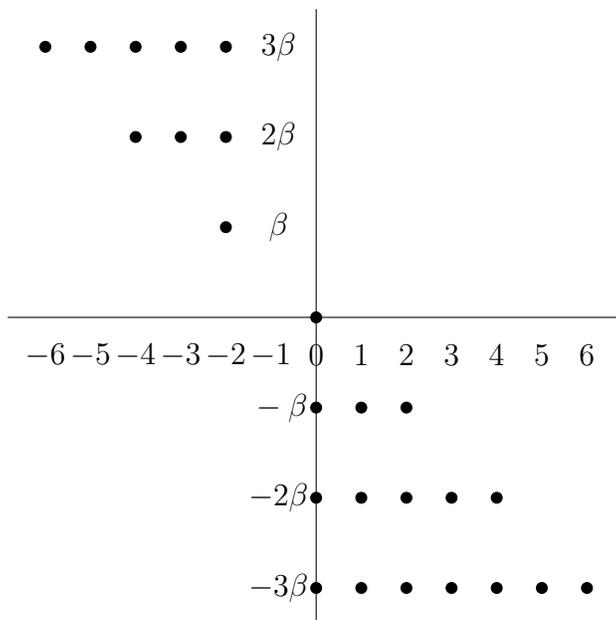


FIGURE 5. The coefficients $\underline{H\mathbb{Z}/p}_*$

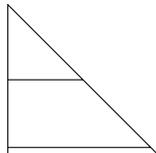


FIGURE 6. Shorthand notation for $\underline{H\mathbb{Z}/p}_*$

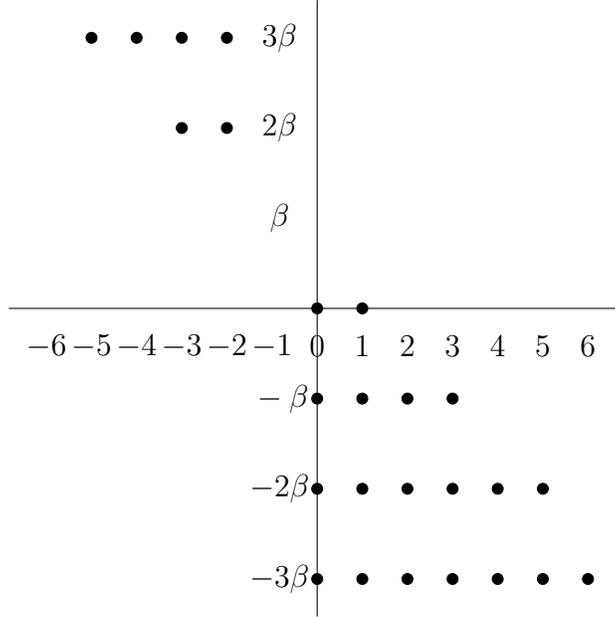
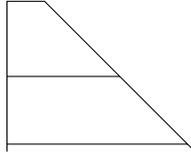
Borel cohomology spectral sequence because of the theory of Mackey functors.

We will completely calculate the Borel cohomology spectral sequence of BPR at an odd prime p modulo negligible BP -summands in $BPR_{\{e\}}$. Modulo negligible BP -summands, $(BPR_{\{e\}})_*$ can be written as

$$(9) \quad B = \bigotimes_{n>0} (\mathcal{L}_1 \oplus \mathcal{L}_{p-1} \cdot r_n \oplus r_n^2 \mathcal{L}_p[r_n])[\Phi(v_n)]$$

where $\Phi(v_n)$ has the same dimension as v_n^p and r_n is a generator of a \mathcal{L}_{p-1} -subrepresentation which contains v_n .

Recall that in [8], Hu and Kriz started with the E^1 -term of the Borel cohomology spectral sequence and included d^1 in the differential pattern. This has the advantage of allowing to treat v_0 on the same level as the other v_n 's, which makes the pattern more natural.

FIGURE 7. The coefficients $(\Sigma^{\beta-1}H(\tilde{\mathcal{L}}_{p-1}/p))_*$ FIGURE 8. Shorthand notation for $(\Sigma^{\beta-1}H(\tilde{\mathcal{L}}_{p-1}/p))_*$

To describe the appropriate analog for p odd, we preview the slightly more complicated assortment of elements and differentials (14), (15), (16) we will encounter. From the point of view of this pattern, the most natural way to start is to put

$$(10) \quad p = \Phi(v_0),$$

and include the second, but not the first, differential of (14) for $n = 0$. Thus, the elements $v_0, \sigma^{\frac{2}{p}-2}$ of fractional degrees

$$|v_0| = \frac{2}{p} - 1 - \frac{\beta}{p}, \quad |\sigma^{\frac{2}{p}-2}| = \beta\left(\frac{1}{p} - 1\right) - \frac{2}{p} + 2$$

are not present in our spectral sequence, but the exterior generator

$$v_0\sigma^{\frac{2}{p}-2}$$

of degree

$$|v_0\sigma^{\frac{2}{p}-2}| = 1 - \beta$$

is included, making the E^1 -term

$$B[\sigma^2, \sigma^{-2}][b] \otimes \Lambda(v_0\sigma^{\frac{2}{p}-2}).$$

The d^1 differential has

$$d^1(v_0\sigma^{\frac{2}{p}-2}) = pb,$$

but also takes group cohomology on (9). This gives E^2 -term

$$(11) \quad E^2 = B'[\sigma^2, \sigma^{-2}][b]/pb$$

where if we put

$$(12) \quad v_I = \prod v_n^{i_n}, \quad I = (i_1, i_2, \dots), \quad i_n \in \{0, 1\},$$

(note that v_n^2 is negligible), and

$$(13) \quad |I| = \sum_n i_n,$$

then

$$B' = (\mathbb{Z}\{v_I : I \text{ even}\} \oplus \mathbb{Z}/p\{v_I : I \text{ odd}\})[\Phi(v_n)].$$

The geometric fixed points of BPR are $H\mathbb{Z}/p$, so we need a resolution with respect to the dual of the polynomial algebra $\mathbb{Z}[\sigma^{-2}]$, which is a divided power polynomial algebra. This implies two sets of differentials

$$(14) \quad \sigma^{-2p^{n-1}} \mapsto v_n b^?, \quad v_n \sigma^{-2(p-1)p^{n-1}} \mapsto \Phi(v_n) b^?.$$

To figure out the exact power of b in (14), we need to figure out the exact equivariant degree of v_n . The \mathbb{Z} -graded component must be $2p^{n-1} - 1$ in order for the differential (14) to decrease degree by 1. Given the fact that v_n sits in a $\tilde{\mathcal{L}}_{p-1}$, and given Proposition 1, its equivariant degree should be of the form $k + \ell\beta$ where $k + 2\ell = |v_n| - 1 = 2p^n - 3$. Thus, we conclude that

$$(15) \quad |v_n| = 2p^{n-1} - 1 + (p^n - p^{n-1} - 1)\beta.$$

Similarly, given the fact that $\Phi(v_n)$ sits in a copy of $\underline{\mathcal{L}}_1$, we have

$$(16) \quad |\Phi(v_n)| = 2p^n - 2 + (p^n - 1)(p - 1)\beta.$$

From this, we conclude that the differential pattern is

$$(17) \quad \sigma^{-2p^{n-1}} \mapsto v_n b^{p^{n-1}}, \quad v_n \sigma^{-2(p-1)p^{n-1}} \mapsto \Phi(v_n) b^{(p^n-1)(p-1)+1}.$$

(The differentials on $\sigma^{-2kp^{n-1}}$ for $2 \leq k \leq p-1$ are determined by the Leibniz rule.) This is called the *even differential pattern*.

The reason this does not tell the whole story is that we have (8), so v_I sits in a $\sigma^{2\ell}$ -shifted copy of $\tilde{\mathcal{L}}_{p-1}$ resp. $\underline{\mathcal{L}}_1$ depending on whether $|I|$

is even resp. odd. In fact, working out the shifts precisely, one gets that

$$(18) \quad |v_I| = \sum_{i_n=1} (2p^{n-1} - 1) + \left(\sum_{i_n=1} (p^n - p^{n-1}) - \lceil \frac{|I|}{2} \rceil \right) \beta.$$

When multiplying (17) by v_I with $i_n = 0$ and $|I|$ odd, the σ^{-2} -power will be in a copy of $\tilde{\mathcal{L}}_{p-1}$, (so non-equivariantly, its dimension goes down by 1), while the v_n will be in a copy of $\underline{\mathcal{L}}_1$, so non-equivariantly, its dimension goes up by 1. Since the non-equivariant dimension of b is -2 , we then obtain the *odd differential pattern*

$$(19) \quad \sigma^{-2p^{n-1}} \mapsto v_n b^{p^n}, \quad v_n \sigma^{-2(p-1)p^{n-1}} \mapsto \Phi(v_n) b^{(p^n-1)(p-1)}.$$

Of course, in the spectral sequence, the differential of every monomial has multiple summands, and we must select the summand which generates lowest b -torsion. Since, luckily,

$$p^n < (p^n - 1)(p - 1)$$

and

$$(p^n - 1)(p - 1) + 1 < p^{n+1} - 1,$$

the lowest b -torsion is always generated by v_n or $\Phi(v_n)$ with the lowest n . For this reason, the Tate spectral sequence (obtained by inverting b) converges to $\mathbb{Z}/p[b, b^{-1}]$ and similarly as in the case of $p = 2$, writing monomials

$$v_I \Phi(v_J)$$

where I is as above and $J = (j_1, j_2, \dots)$, $j_n \in \mathbb{N}_0$,

$$\Phi(v_J) = \prod_n \Phi(v_n)^{j_n},$$

letting $r = r(I)$ resp. $s = s(J)$ be the lowest n for which $i_n \neq 0$ resp. $j_n \neq 0$, (set to ∞ when not applicable), we obtain the following

Theorem 2. *Modulo negligible $BP_*[\sigma^2, \sigma^{-2}]$ -summands, the R -graded coefficients of $BP\mathbb{R}$ are a sum of*

$$\mathbb{Z}_{(p)}[b],$$

“even-pattern summands” where $|I|$ is odd and $r \leq s$:

$$\bigoplus_{\ell \equiv -1 \pmod p} \mathbb{Z}/p \cdot v_I \Phi(v_J) \cdot \sigma^{2p^{r-1}\ell} [b] / (b^{p^r-1})$$

or $|I|$ is even, $I \neq 0$ and $r > s$:

$$\bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}_{(p)} \cdot v_I \Phi(v_J) \cdot \sigma^{2p^s \ell} [b] / (b^{(p^s-1)(p-1)+1}, pb)$$

and “odd-pattern summands” where $|I|$ is even and $I \neq 0$ and $r \leq s$:

$$\bigoplus_{\ell \equiv -1 \pmod p} \mathbb{Z}_{(p)} \cdot v_I \Phi(v_J) \cdot \Sigma^{2p^{r-1}\ell}[b]/(b^{p^r}, pb)$$

or $|I|$ is odd and $r > s$:

$$\bigoplus_{\ell \in \mathbb{Z}} \mathbb{Z}/p \cdot v_I \Phi(v_J) \cdot \sigma^{2p^s \ell}[b]/(b^{(p^s-1)(p-1)}).$$

□

4. CONSTRUCTION

In this section, we will construct BPR at an odd prime p by realizing spectrally a \mathbb{Z}/p -equivariant analog of the Brown-Peterson resolution. Throughout this section, we will work in Borel cohomology, and omit this from the notation. With this convention, the integral homology of BPR , modulo negligible copies of

$$H\mathbb{Z}_*BP[\sigma^2, \sigma^{-2}],$$

is

$$H\mathbb{Z}[\theta_1, \theta_2, \dots] \wedge_{H\mathbb{Z}} \Lambda_{H\tilde{\mathcal{L}}_{p-1}}[\widehat{\xi}_1, \widehat{\xi}_2, \dots].$$

(The second term denotes the exterior algebra.) The k -invariants are defined as cohomological operations (in Borel cohomology), and our job is to prove that they can be realized spectrally (i.e. on the realization of the partial resolution at each step).

While introducing some negligible terms is necessary, the corresponding non-equivariant homology will remain even-degree and non-torsion, and there is therefore no obstruction to spectral realization of the negligible k -invariants, by Brown-Peterson’s “even-odd argument” [3]. On the other hand, the non-negligible k -invariants are non-torsion either rationally or non-torsion with respect to b and hence the obstruction to spectral realization is 0 because it vanishes rationally, resp. on geometric (i.e., in our case, Tate) fixed points.

The Borel cohomology version of the \mathbb{Z}/p -equivariant Brown-Peterson resolution is essentially described as a product over $H\mathbb{Z}$ of the following sequences:

Starting with $H\mathbb{Z}$, we have

$$(20) \quad \begin{aligned} & H\mathbb{Z} \wedge H\mathbb{Z} = \\ & \Lambda_{n \geq 1}(H\mathbb{Z} \vee (H\mathbb{Z} \wedge T\{\widehat{\xi}_n \xi_n^k \mid k = 0, \dots, p-2\}) \vee H\mathbb{Z}/p\{\widehat{\xi}_n \xi_n^{p-1}\})[\theta_n]. \end{aligned}$$

The $H\underline{\mathbb{Z}}/p$ arises, in the notation of [9], as two copies of $H\underline{\mathbb{Z}}$ stuck together by p -multiplication, in other words from the fact that

$$\beta(\underline{\mu}_n) = \underline{\theta}_n.$$

Now one has a cofibration sequence

$$(21) \quad \Sigma^{\beta-2} H\tilde{\mathcal{L}}_{p-1} \rightarrow H\underline{\mathbb{Z}} \wedge T \rightarrow \Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1}$$

(split on R -graded coefficients). Then, there is an operation

$$(22) \quad Q'_n : H\underline{\mathbb{Z}} \rightarrow \Sigma^\beta H\tilde{\mathcal{L}}_{p-1}\{v_n\}$$

which applies (21) to the $\underline{\xi}_n$ -wedge summand of (20). (Here the notation v_n refers to the R -graded indexing of the previous section.)

Comment: The operations Q'_n form the first stage of our equivariant version of the $BP\mathbb{R}$ tower. One may wonder how these elements are identified. To this end, it is beneficial to consider what happens on the level of geometrical fixed points of homology. We have

$$(23) \quad \begin{aligned} & \Phi^{\mathbb{Z}/p}(H\underline{\mathbb{Z}} \wedge BP\mathbb{R})_* = \\ & \Phi^{\mathbb{Z}/p}(H\underline{\mathbb{Z}}) \wedge \Phi^{\mathbb{Z}/p}(BP\mathbb{R})_*[b, b^{-1}] = \\ & A_*[\sigma^{-2}][b, b^{-1}]. \end{aligned}$$

On the other hand, we have

$$(24) \quad \begin{aligned} & \Phi^{\mathbb{Z}/p}(H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}})_* = \\ & \Phi^{\mathbb{Z}/p}(H\underline{\mathbb{Z}}) \wedge \Phi^{\mathbb{Z}/p}(H\underline{\mathbb{Z}})_*[b, b^{-1}] = \\ & A_*[\sigma^{-2}, \rho^{-2}][b, b^{-1}]. \end{aligned}$$

Thus, to the eyes of $\text{mod } p$ homology, to get from $H\underline{\mathbb{Z}}$ to $BP\mathbb{R}$ on geometrical fixed points, we must kill the ρ^{-2} -powers in (24), while preserving the other generators.

To describe how this is done, let us recall briefly the $p = 2$ case. As remarked in [8], for $p = 2$, a BP-tower construction of $BP\mathbb{R}$ on the level of geometric fixed points is modelled on the cobar complex of $\mathbb{Z}/2[\rho^{-2}]$, which is the same thing as the cohomological Koszul complex of the divided polynomial power algebra $\mathbb{Z}/2[\rho^{-2}]^\vee$, which is an exterior algebra. Thus, the generators are in total degrees $2^i - 1$, $i = 1, 2, \dots$, which is correct, since the dimension of v_i is $(2^i - 1)(1 + \alpha)$, and the α may be disregarded on the level of geometrical fixed points. In particular, for $p = 2$, on geometrical fixed points, the $BP\mathbb{R}$ tower is split (i.e. the maps are equivalences of cancelling wedge summands), and will work for any choice of Q'_n which sends $\rho^{-2^n} \mapsto 1$ (up to multiplying by a power of a). We also note that modulo possible error terms in the derived wedge,

this condition, together with its dimension, essentially determines Q'_n , since on the good wedge, inverting a is injective.

For $p > 2$, the construction is precisely analogous (the only difference being that for $p = 2$, error terms in the derived wedge do not matter, since we have an a priori model of BPR coming from $M\mathbb{R}$ via spectral algebra, while for $p > 2$, no a priori geometric model is known, so we circumvent the difficulty by working in Borel cohomology).

On the level of geometric fixed points, however, again, the BPR -tower is modelled on the cohomological Koszul complex of the divided polynomial power algebra

$$(25) \quad \mathbb{Z}/p[\rho^{-2}]^\vee.$$

Now (25) is a tensor product of truncated polynomial algebras of the form

$$(26) \quad \mathbb{Z}/p[x]/x^p$$

where x is the dual of

$$(27) \quad \rho^{-2p^{n-1}}, \quad n = 1, 2, \dots$$

This means that our primary operation corresponds to an exterior generator of cohomological degree 1 and topological degree $2p^{n-1}$. (The total degree $2p^{n-1} - 1$ corresponds to the geometric fixed point version of v_n .) Then there is a secondary generator in cohomological degree 2 and topological degree $2p^n$, which gives total degree $2p^n - 2$, corresponding to the geometric fixed point version of $\Phi(v_n)$.

Recalling the short exact sequence of $\underline{\mathbb{Z}}$ -modules

$$0 \rightarrow \underline{\mathbb{Z}} \rightarrow \underline{\mathcal{L}}_p \rightarrow \tilde{\mathcal{L}}_{p-1} \rightarrow 0,$$

when taking geometric fixed points (i.e. inverting b), $H\tilde{\mathcal{L}}_{p-1}$ becomes identified with $\Sigma H\underline{\mathbb{Z}}$, so we see that ignoring the β -part, our primary operation is at least of the right degree $2p^{n-1}$.

To assert that we actually have the right element, we need to show how Q'_n cancels the element (25). To this end, we recall the computation in [9] in Borel cohomology:

$$\begin{aligned} \xi_{\underline{1}} &= (\sigma^{-2} - \rho^{-2})b^{1-p} \\ \xi_{\underline{n+1}} &= (\sigma^{2p^{n-1}-2p^n}\xi_{\underline{n}} - \rho^{-2}\xi_n)b^{p^n-p^{n+1}}. \end{aligned}$$

The first equation eliminates the term ρ^{-2} modulo the allowed error terms in (23), while the first term of the second equation can be similarly used by induction to eliminate $\rho^{-2p^{n-1}}$.

We will see below how the transpotence also corresponds to $\Phi(v_n)$. However, since it is defined by canceling the homology element corresponding to the ‘‘Kudo element’’ $x^{p-1}y$ where x is as in (26) and y is its Koszul dual, we have no statement on the level of the dual Steenrod algebra in that case. Similarly as in the $p = 2$ case, again, on the level of geometric fixed points, our BPR -tower splits, i.e. is described as a wedge of equivalences of wedge summands.

We also note that the Borel cohomology pieces we add to our tower match (9). Therefore, by the calculation of Section 3, adding Borel cohomology pieces, on the Tate level, we recover the b -non-torsion part of the coefficients, which matches the above proposed geometric fixed points. This shows that both constructions correspond correctly.

Now to continue our construction, smashing the cofibration sequence

$$S^{\beta-2} \rightarrow S^0 \rightarrow T$$

with $H\tilde{\mathcal{L}}_{p-1}$, we have a cofibration sequence

$$(28) \quad H\tilde{\mathcal{L}}_{p-1} \rightarrow H\tilde{\mathcal{L}}_{p-1} \wedge T \rightarrow \Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1}.$$

The ξ_n -multiple of Q'_n kills on the R -graded coefficients of the middle term of the $|\xi_n|$ -suspension of (28) everything except the $|\xi_n|$ -suspension of a copy of $H\mathcal{L}_p$ coming from the first term of (28). Thus, we obtain a k -invariant

$$(29) \quad \overline{Q}_n : H\tilde{\mathcal{L}}_{p-1}\{v_n\} \rightarrow H\mathcal{L}_p\{v_n^2\}.$$

On the other hand, the fiber F of (28) contains a copy of a fiber of

$$(30) \quad H\underline{\mathbb{Z}} \cdot \underline{\mu}_n \rightarrow H\tilde{\mathcal{L}}_{p-1} \wedge T \cdot \widehat{\xi}_n \xi_n^{\beta-2}$$

which has a factor of $H\underline{\mathbb{Z}}$. We define a k -invariant

$$(31) \quad Q''_n : F \rightarrow H\underline{\mathbb{Z}} \cdot \Phi(v_n)$$

isomorphically on that factor.

In more detail, we have a cofibration sequence

$$(32) \quad H\underline{\mathbb{Z}} \rightarrow HT \rightarrow \Sigma^{\beta-1} H\underline{\mathbb{Z}}$$

and thus also

$$(33) \quad H\tilde{\mathcal{L}}_{p-1} \rightarrow HT \wedge_{H\underline{\mathbb{Z}}} H\tilde{\mathcal{L}}_{p-1} \rightarrow \Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1}.$$

On the other hand, we have a cofibration sequence

$$(34) \quad H\tilde{\mathcal{L}}_{p-1} \rightarrow H\mathcal{L}_p \rightarrow \Sigma^{2-\beta} H\underline{\mathbb{Z}},$$

or

$$(35) \quad H\underline{\mathbb{Z}} \rightarrow \Sigma^{\beta-1} H\tilde{\mathcal{L}}_{p-1} \rightarrow \Sigma^{-1} H\mathcal{L}_p.$$

Taking the derived pullback of (33) via the first map (35), we obtain a cofibration sequence

$$(36) \quad H\tilde{\mathcal{L}}_{p-1} \rightarrow HX \rightarrow H\underline{\mathbb{Z}}.$$

However, (36) must split since there is no essential $H\underline{\mathbb{Z}}$ -module map

$$H\underline{\mathbb{Z}} \rightarrow \Sigma H\tilde{\mathcal{L}}_{p-1}$$

(by dimensional reasons).

This is the *even pattern* in the Borel cohomology \mathbb{Z}/p -equivariant Brown-Peterson resolution. An illustration is given in Figure 9 of the Appendix.

The *odd pattern* is obtained essentially by smashing this over $H\underline{\mathbb{Z}}$ with $H\tilde{\mathcal{L}}_{p-1}$, but the negligible term appears at a slightly different place.

The first k-invariant comes from composing the “connecting map”

$$\tilde{\mathcal{L}}_{p-1} \rightarrow \underline{\mathcal{L}}_p$$

with the second map (28), applied to the $\widehat{\xi}_n$ -summand of (20) smashed over $H\underline{\mathbb{Z}}$ with $H\tilde{\mathcal{L}}_{p-1}$. This gives a negligible k-invariant

$$\overline{Q} : H\tilde{\mathcal{L}}_{p-1} \rightarrow H\underline{\mathcal{L}}_p \{v_n\}$$

and its fiber F' has a quotient of $H\underline{\mathbb{Z}} \cdot \underline{\xi}_n$. We obtain a k-invariant

$$(37) \quad Q'_n : F' \rightarrow H\underline{\mathbb{Z}} \cdot v_n$$

by applying an isomorphism on this quotient. The argument follows (32)-(36).

On the other hand, we can apply the second map (21) to the $\underline{\xi}_n^{p-2} \cdot \widehat{\xi}_n$ copy in (20) to obtain the k-invariant

$$Q''_n : F'' \rightarrow H\tilde{\mathcal{L}}_{p-1} \cdot \Phi(v_n)$$

where F'' is the fiber of (37).

An illustration of the odd pattern is given in Figure 10 of the Appendix.

5. COMPARISON WITH JOHNSON-WILSON SPECTRA

In this section, we shall investigate Johnson-Wilson spectra analogs of BPR . Denote by I_n the ideal $(p, v_1, \dots, v_{n-1}) \subset BP_*$.

Proposition 3. *The spectrum $(E(n) \wedge \cdots \wedge E(n))_{I_n}^\wedge$ is an A_∞ -ring spectrum and the space of its A_∞ -selfmaps is homotopically discrete. (The same statement holds for other “flavors” of $E(n)$, notably E_n .)*

Proof. We follow the method of Robinson [15] and Baker [2]. We first recall that the ideal generated by I_n by inclusions

$$BP_* \rightarrow (BP \wedge \cdots \wedge BP)_*$$

via one of the factors does not depend on the factor. (See [14], Theorem A2.2.6.)

Next, let

$$E(n, k) = \bigwedge_k E(n),$$

$$\Sigma(n, k)_* = (\bigwedge_k E(n))_* / I_n.$$

One writes $\Sigma(n)_* = \Sigma(n, 1)_*$. The method of Robinson [15], Baker [2] formula (3.10) reduces this task to proving a vanishing result for a certain $\mathcal{E}xt$ -group

$$(38) \quad \mathcal{E}xt_{\Sigma(n, 2k)}^{s>0}(\Sigma(n, k), \Sigma(n, k)) = 0$$

where s denotes the cohomological degree.

To define the $\mathcal{E}xt$ -groups concerned, we denote by E the smash of k copies of $E(n)$, the ring structure on E_*E is given by the composition

$$(39) \quad (E \wedge E)_* \otimes (E \wedge E)_* \rightarrow (E \wedge E \wedge E \wedge E)_* \rightarrow (E \wedge E)_*$$

where the first map is give by sending the first two coordinate to the first and last coordinate in the target, and the last two coordinates into the middle two coordinates (without changing order internally), and the second map is given by multiplication in the first two and the last two coordinates. The tensor product is over \mathbb{Z} . The left E_*E -module structure on E_* is defined analogously, replacing the last two coordinates $E \wedge E$ in the source and middle two coordinates in the target of (39) by E . It is customary to refer to this construction as ‘‘Hochschild cohomology’’ ([15]), even though this is not correct in full generality.

To prove (38), we recall from [15] that

$$(40) \quad \Sigma(n)_* = \left(\bigotimes_{i \geq 1} \right)_{K(n)_*} \left(K(n)_* \prod \frac{K(n)_*[t_i]}{(v_n t_i^{p^n-1} - v_n^{p^i})} \right)$$

In our present setting, we have duplicate copies of the coordinates t_i , but the essential point is that the $E_*E/I_n = \Sigma(n, 2k)$ forms a direct limit of étale extensions of $K(n)_*$, and thus the given $\mathcal{E}xt^{>0} = 0$.

□

We claim that for all n , there is a \mathbb{Z}/p -action on $(\bigwedge_{p-1} E(n))_{I_n}^\wedge$ compatible with the \mathbb{Z}/p -action on the non-equivariant spectrum underlying the BPR we constructed above. To this end, we need to make a couple of remarks. First of all, the conjecture of Hill, Hopkins and Ravenel concerned the existence of a \mathbb{Z}/p -equivariant spectrum, which we denote by BPR^{HHR} , which would satisfy

$$(41) \quad BPR_{\{e\}}^{HHR} = \bigwedge_{p-1} BP.$$

Our construction gives the pattern (9) above. We see [1] that this gives the right answer at the prime $p = 3$, but is smaller than the coefficients of (41) for $p \geq 5$. One notes however that BPR^{HHR} can be obtained from BPR by taking a wedge with even suspensions of copies of the additive norm $\mathbb{Z}/p_+ \wedge BP$.

Now we claim that our construction determines a \mathbb{Z}/p -action on $(\bigwedge_{p-1} BP)_*$ which sends the universal formal group law to strictly isomorphic formal group laws (recall that a strict isomorphism between isomorphic formal group laws on a torsion-free ring is uniquely determined). In a graded ring $R \supseteq BP_*$, maps $BP_* \rightarrow R$ which carry the universal formal group law to an isomorphic formal group law are characterized by elements $\bar{t}_i \in R$ of the same degree as t_i , which, when substituted for t_i into $\eta_R(v_n)$, give the image of v_n .

In fact, we do not need the whole construction of BPR , its first k -invariant (22) determines the information we need. Non-equivariantly, this gives an operation

$$(42) \quad H\mathbb{Z} \rightarrow \bigvee_{p-1} \Sigma^{2p^n-1} H\mathbb{Z}.$$

We know that this operation is just the integral Q_n landing in an $H\mathbb{Z}$ wedge summand of the right hand side. It is, further, invariant under \mathbb{Z}/p -action. This identifies the wedge summand which is supported by v_n . In homotopy groups, on \mathcal{L}_{p-1} , it is an element which reduces modulo p to an element of L_{p-1} which is annihilated by $1 - \gamma$ where γ is the generator of \mathbb{Z}/p .

Since, however, we also know from [3] that v_n is only determined modulo I_n , we see that the action on v_n given by the \mathbb{Z}/p -action on (42) necessarily sends the universal FGL to isomorphic FGL's.

Along with (3), this then implies

Theorem 4. *There exists a strict action of \mathbb{Z}/p (in the “naive” sense) on*

$$\left(\bigwedge_{p-1} E(n)\right)_{I_n}^\wedge$$

(and its I_n -complete variants, replacing, for example, $E(n)$ by E_n) by morphisms of A_∞ -ring spectra, which on coefficients coincides with the action on compositions of $(p-2)$ strict isomorphisms of FGL's given by the first k -invariant (42).

□

It is worth noting that while these considerations produce many equivalent actions on chains of $(p-2)$ strict isomorphisms of FGL's, for $p > 2$, there does not appear to be one canonical construction such as the $-i_F(x)$ -series associated with $BP\mathbb{R}$ for $p = 2$.

It makes sense to conjecture that there exists an A_∞ -structure on $BP\mathbb{R}^{HHR}$ and an A_∞ -map from $BP\mathbb{R}^{HHR}$ into the \mathbb{Z}/p -equivariant Borel-complete spectrum defined by Theorem 4. At the moment, however, several ingredients are missing toward proving this, the first of which is the proof of the existence of an A_∞ -ring structure (or even a non-rigid ring structure) on our construction of $BP\mathbb{R}^{HHR}$.

It is important to note that there also exists a \mathbb{Z}/p -action on E_{p-1} induced from the \mathbb{Z}/p -subgroup of the Morava stabilizer group. (For simplicity, let us assume $p \geq 5$.) The \mathbb{Z}/p -action on coefficients is discussed in Nave [13], Symonds [17]. Essentially, from the representation-theoretical point of view, the non-negligible part is an exterior algebra on \mathcal{L}_{p-1} , tensored with a polynomial algebra on Nv_n . (This is in no contradiction with the homotopy-theoretical point of view, where we are dealing with the completion of a Laurent series ring.)

The generating \mathcal{L}_{p-1} contains all of

$$(43) \quad u, u_{n-1}u, \dots, u_1u$$

where u_i are the Lubin-Tate generators (i.e. are related to v_i by change of normalization) and u^{-1} is a root of v_n . In fact, it is proved in [13] that u generates this \mathcal{L}_{p-1} -representation and the elements (43), in the order listed, are related by applying $1 - \gamma$.

It was conjectured in [7] that there should be a \mathbb{Z}/p equivariant map from

$$(44) \quad BP\mathbb{R}^{HHR} \rightarrow (E_{p-1})_{\mathbb{Z}/p}$$

which would restrict to an isomorphism of the \mathcal{L}_{p-1} -representations between the \mathcal{L}_{p-1} containing v_1 and the $u^{-p^{p-1}+1}$ -multiple of (43). Such a morphism of representations certainly exists, but making this assignment implies that the non-equivariant elements v_i in our construction are sent to 0 for $i > 1$. This may seem absurd, since the element $u^{-p^{p-1}+1}$ is of type v_{p-1} , and is supposed to be inverted. The answer, however,

is of course that there is no real contradiction, since when dealing with a chain of $p - 2$ strict isomorphisms of the universal formal group law, there could be many elements of type v_{p-1} , including some which contain a summand which is a unit multiple of t_{p-1} . Thus, the element $u^{-p^{p-1}+1}$ could be an image of a generator of a separate copy of BP in $BPR_{\{e\}}$. In fact, on the level of non-equivariant coefficients with \mathbb{Z}/p -action, such maps are easily shown to exist. The existence of the comparison map (44) on the level of \mathbb{Z}/p -equivariant spectra is at present still open.

6. ODD PRIME REAL ORIENTATIONS AND E_∞ -CONSTRUCTIONS

One knows (see [11]) that MU is an E_∞ -ring spectrum. On the other hand, we have the complex orientation

$$(45) \quad \Sigma^{-2}\mathbb{C}P^\infty \rightarrow MU.$$

Thus, (after appropriate discussion of cofibrancy), the universal property allows us to extend (45) to an E_∞ -map

$$(46) \quad C_\bullet \Sigma^{-2}\mathbb{C}P^\infty \rightarrow MU$$

where $C_\bullet E$ denotes the unital free E_∞ -ring spectrum on a spectrum E with unit (meaning a morphism $S \rightarrow E$). We remark in fact that (46) is a retraction (i.e. that in the derived category of spectra, the target splits off as a direct summand, as an ordinary ring spectrum). This is due to the fact that $C_\bullet \Sigma^{-2}\mathbb{C}P^\infty$ is, by definition, complex-oriented, so the complex orientation gives a section of (46) in the category of ordinary commutative ring spectra.

A similar story is true, essentially without change, for Real-oriented $\mathbb{Z}/2$ -equivariant spectra: Denoting by \mathbb{S}^1 the unit sphere in \mathbb{C} with the $\mathbb{Z}/2$ -action of complex conjugation, we have a Real orientation

$$(47) \quad B\mathbb{S}^1 \rightarrow M\mathbb{R},$$

which gives rise to a $\mathbb{Z}/2$ -equivariant E_∞ -map

$$(48) \quad C_\bullet \Sigma^{-2}B\mathbb{S}^1 \rightarrow M\mathbb{R}$$

which, again, has a section in the category of ordinary $\mathbb{Z}/2$ -equivariant commutative ring spectra (and hence, in particular, splits in the category of $\mathbb{Z}/2$ -equivariant spectra).

This raises the question what happens for a prime $p > 2$. Hahn, Senger and Wilson [6] defined a version of Real orientations of \mathbb{Z}/p -equivariant spectra for odd primes p as follows. One denotes

$$\mathbb{T} = B_{\mathbb{Z}/p}\mathcal{L}_{p-1}.$$

The \mathbb{Z}/p -equivariant space \mathbb{T} can, indeed, be identified with the subset of $(S^1)^{\mathbb{Z}/p}$ consisting of all tuples (z_1, \dots, z_p) where

$$\prod_{j=1}^p z_j = 1.$$

Note that in particular,

$$(49) \quad \mathbb{T}^{\mathbb{Z}/p} = \mathbb{Z}/p.$$

Selecting once and for all any (non-equivariant path from 0 to 1 in (49) specifies a \mathbb{Z}/p -equivariant map

$$(50) \quad \widetilde{\mathbb{Z}/p} \rightarrow \mathbb{T},$$

which, in turn, by adjunction, gives a \mathbb{Z}/p -equivariant map

$$(51) \quad \iota : \Sigma \widetilde{\mathbb{Z}/p} \rightarrow B\mathbb{T}.$$

The right-hand side is, again, equivalent whether whether we take $B_{\mathbb{Z}/p}$ or the “naive” Schubert cell construction. In fact, we have the following

Proposition 5. *Let $\mathbb{T} = \{(z_1, \dots, z_p) \mid z_i \in S^1, \prod z_i = 1\}$ be the \mathbb{Z}/p -equivariant group so that \mathbb{Z}/p acts by permuting the coordinates. Then the equivariant classifying space $B_{\mathbb{Z}/p}\mathbb{T}$ is equivalent to the fiber of*

$$(52) \quad B((S^1)^{\mathbb{Z}/p}) \rightarrow BS^1.$$

In (52), B denotes the bar construction, $(S^1)^{\mathbb{Z}/p}$ is the $\{(z_1, \dots, z_p) \mid z_i \in S^1\}$ and the map is induced by multiplying the coordinates.

Comment: The bar construction is equivariant and also preserves the “multiplicative norm” on space level, so the source of (52) is a product of p copies of $\mathbb{C}P^\infty$ with action by permutation of coordinates, while the target of (52) is a fixed copy of $\mathbb{C}P^\infty$.

Proof. Notice that on fixed points, (52) is $p : BS^1 \rightarrow BS^1$, (meaning the bar construction applied to the p ’th power map on S^1), so it suffices to show that $(B_{\mathbb{Z}/p}\mathbb{T})^{\mathbb{Z}/p} \simeq B\mathbb{Z}/p$, as the underlying level is easy. By [10, Theorem 10],

$$(53) \quad (B_{\mathbb{Z}/p}\mathbb{T})^{\mathbb{Z}/p} \simeq \coprod_{[\Lambda]} BW_{\mathbb{T} \rtimes \mathbb{Z}/p}(\Lambda),$$

where the index is over subgroups $\Lambda \subset \mathbb{T} \rtimes \mathbb{Z}/p$ isomorphic to \mathbb{Z}/p and intersecting \mathbb{T} trivially, up to \mathbb{T} -conjugation. Write $\mathbb{Z}/p = \langle \gamma \rangle$ and take $x = ((z_1, \dots, z_p), \gamma) \in \mathbb{T} \rtimes \mathbb{Z}/p$. Then

$$x^p = ((\prod z_i, \dots, \prod z_i), \gamma^p) = e \text{ and } x^{-1} = ((z_2^{-1}, \dots, z_p^{-1}, z_1^{-1}), \gamma^{-1}).$$

This means any element x of $\mathbb{T} \rtimes \mathbb{Z}/p$ not contained in \mathbb{T} generates a Λ . However, these are all conjugate. Let $y = ((y_1, \dots, y_p), \gamma)$ to be determined. Setting

$$yxy^{-1} = ((y_1x_py_p^{-1}, y_2x_1y_1^{-1}, \dots, y_px_{p-1}y_{p-1}^{-1}), \gamma) = ((1, 1, \dots, 1), \gamma),$$

we obtain

$$\begin{aligned} y_p/y_1 &= x_p \\ y_1/y_2 &= x_1 \\ &\dots \\ y_{p-1}/y_p &= x_{p-1} \end{aligned}$$

It is not hard to see that this allows a solution with $\prod y_i = 1$. Now, using $\Lambda = \langle ((1, 1, \dots, 1), \gamma) \rangle$ without loss of generality, one obtains $(B_{\mathbb{Z}/p}\mathbb{T})^{\mathbb{Z}/p} \simeq B\mathbb{Z}/p$ from (53). \square

Note that in the proof, the only thing we used about S^1 is that it is an abelian compact Lie group. However, the same argument often extends to more general groups. For example, we can show that the fiber of

$$B(\mathbb{Z}^{\mathbb{Z}/p}) \rightarrow B\mathbb{Z}$$

is $B_{\mathbb{Z}/p}\mathcal{L}_{p-1}$, recalling $\mathcal{L}_{p-1} = \{(a_1, \dots, a_p) \mid z_i \in \mathbb{Z}, \sum a_i = 0\}$. In other words, $\mathbb{T} \simeq B_{\mathbb{Z}/p}\mathcal{L}_{p-1}$. Although the group \mathbb{Z} is non-compact, the formalism works in this case, since the family consists of finite subgroups. Instead of [10, Theorem 10], we use the fact that $H^1(\mathbb{Z}/p, \mathcal{L}_{p-1}) = \mathbb{Z}/p$, by the long exact sequence in cohomology induced by the short exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{L}_p \rightarrow \mathcal{L}_{p-1} \rightarrow 0.$$

Hahn Senger, and Wilson [6] define a \mathbb{Z}/p -equivariant commutative ring spectrum E to be *Real-oriented* if the following diagram can be completed:

$$(54) \quad \begin{array}{ccc} \Sigma \widetilde{\mathbb{Z}/p} & \longrightarrow & \Sigma \widetilde{\mathbb{Z}/p} \wedge E \\ \downarrow \iota & \nearrow & \\ B\mathbb{T} & & \end{array}$$

where the horizontal map is the smash of $\Sigma\widetilde{\mathbb{Z}/p}$ with the unit. If this happens, one can show that the Schubert-cell spectral sequence converging to $E^*B\mathbb{T}$ (see Proposition 5) collapses.

In effect, we claim that for a Real-oriented \mathbb{Z}/p -equivariant spectrum E , modulo "negligible parts," i.e. copies of the additive norm from $\{e\}$ to \mathbb{Z}/p on $E_{\{e\}}$, we have

$$(55) \quad E \wedge B\mathbb{T} \sim \bigwedge_E (E \vee E \wedge \Sigma\widetilde{\mathbb{Z}/p}) \wedge_E E[Nx]$$

(we mean this up to homotopy, not in any coherent sense). In fact, the $E \wedge \Sigma\widetilde{\mathbb{Z}/p}$ occurs by the definition of \mathbb{Z}/p -equivariant Real orientation, while the class Nx is simply the multiplicative norm (on the level of spaces) of the complex orientation of $E_{\{e\}}$ (whose existence also follows from the definition of Real orientation by forgetting the \mathbb{Z}/p -action), restricted by the inclusion map

$$B\mathbb{T} \rightarrow B((S^1)^{\mathbb{Z}/p}).$$

By the observation, the ring $E^*[Nx]$ splits off as a canonical summand of $E^*B\mathbb{T}$. Multiplication on $B\mathbb{T}$ then gives rise to a p -valued formal group law in the sense of Buchstaber [4, 5], see Definition 1.2 of [5]. To recapitulate, Buchstaber defines a p -valued formal group law as a polynomial of the form

$$(56) \quad \Theta(x, y) = Z^p - \theta_1(x, y)Z^{p-1} + \cdots - \theta_p(x, y)$$

where $\theta_i(x, y)$ are power series which satisfies the conditions

$$\Theta(x, 0) = Z^n + \sum_{i=1}^n (-1)^i \binom{n}{i} x^i Z^{n-i},$$

$$(57) \quad \Theta(\Theta(x, y), z) = \Theta(x, \Theta(y, z)),$$

$$\Theta(x, y) = \Theta(y, x).$$

The associativity condition (57) requires some explanation.

If $F(x, y)$ is an ordinary formal group law, then a p -valued formal group law can be obtained as

$$\Theta(x, y) = \prod_{j=1}^n (Z - F_j(x, y))$$

where

$$F_j(x, y) = \exp_F((\log_F x)^{1/p} + \zeta_p^j (\log_F y)^{1/p})^p.$$

In the associativity condition, we can similarly write terms in three variables

$$(58) \quad F_{jk}(x, y, z) = \exp_F((\log_F x)^{1/p} + \zeta_p^j(\log_F y)^{1/p} + \zeta_p^k(\log_F z)^{1/p})^p$$

and the multivalued formal sum in three variables should be

$$(59) \quad \Theta(x, y, z) = \prod_{j,k=1}^n (Z - F_{jk}(x, y, z)).$$

This can be processed in two different ways into expressions involving only the series θ_j :

We can write (58) as

$$F_{jk}(x, y, z) = \exp_F((\log_F x)^{1/p} + \zeta_p^j((\log_F y)^{1/p} + \zeta_p^{k-j}(\log_F z)^{1/p})^p)^{1/p}$$

and thus express (59) in terms of

$$\theta_i(x, F_j(y, z)).$$

The resulting expression, however, is symmetrical in the $F_j(y, z)$, which allows them to be reduced to $\theta_s(y, z)$. Permuting the variables x, y, z cyclically and doing the same thing gives another expression. The equality of both expressions is what one means by (57).

For $p = 2$, Real-oriented spectra produce an ordinary formal group law due to the fact that the orientation class is a polynomial generator. As we saw in (55) (and as is, in some sense, the theme of this paper), however, for $p > 2$ the orientation class is only an “exterior generator” from the point of view of representation theory, which is why we have to restrict to the polynomial ring on the (space-level) multiplicative norm of the non-equivariant orientation class, which leads only to a p -valued formal group law. Note that the collapse (55) is necessary to assure that the ring $E^*[Nx]$ indeed survives as a canonical summand of the cell spectral sequence given by Proposition 5.

For our present purposes, we recall from [11] that the \mathbb{Z}/p -equivariant Spanier-Whitehead dual of $\widetilde{\mathbb{Z}/p}$ is given by

$$(60) \quad D\widetilde{\mathbb{Z}/p} = \Sigma^{-\beta}\widetilde{\mathbb{Z}/p}.$$

Thus,

$$\Sigma^{-1-\beta}\widetilde{\mathbb{Z}/p} \wedge B\mathbb{T}$$

becomes a unital spectrum via ι and the spectrum

$$(61) \quad C_\bullet \Sigma^{-1-\beta}\widetilde{\mathbb{Z}/p} \wedge B\mathbb{T}$$

becomes Real-oriented in the sense of [6]. It seems therefore reasonable to conjecture that after localizing at p , $BP\mathbb{R}$ will split off (61).

One also notes that since $(E_{p-1})_{\mathbb{Z}/p}$ is both E_∞ and Real-oriented (as proved in [6]), we do know that there is a \mathbb{Z}/p -equivariant E_∞ -map

$$C_\bullet \Sigma^{-1-\beta} \widetilde{\mathbb{Z}/p} \wedge B\mathbb{T} \rightarrow E_{p-1}.$$

7. APPENDIX: GRAPHICAL ILLUSTRATION OF THE CONSTRUCTION OF $BP\mathbb{R}$ AT $p > 2$

The purpose of this Appendix is to provide a graphical illustration of the construction given in Section 4. The even (resp. odd) pattern of the tower is depicted in Figure 9 (resp. Figure 10). Let us briefly recall the classical Brown-Peterson BP -tower [3]. Their tower is described by taking the homology $H\mathbb{Z}/p_* H\mathbb{Z}$ and killing those elements which are not supposed to be in $H\mathbb{Z}/p_* BP$. These are, of course, all the copies of $H\mathbb{Z}/p_* = \mathbb{Z}/p$ indexed by a generator $\xi_R \tau_E$ in Milnor's notation [12] where $E = (e_1, e_2, \dots) \neq 0$, $e_n \in \{0, 1\}$. In the first stage of the tower, we attach copies of $H\mathbb{Z}$ along the Milnor primitives Q_n , (the corresponding generators being thus labelled by v_n), which sends $\tau_n \mapsto 1$, thereby killing all these elements, creating, however, "error terms" due to the relations $\tau_n^2 = 0$. These are corrected in subsequent stages of the tower, thus making the generators v_n polynomial.

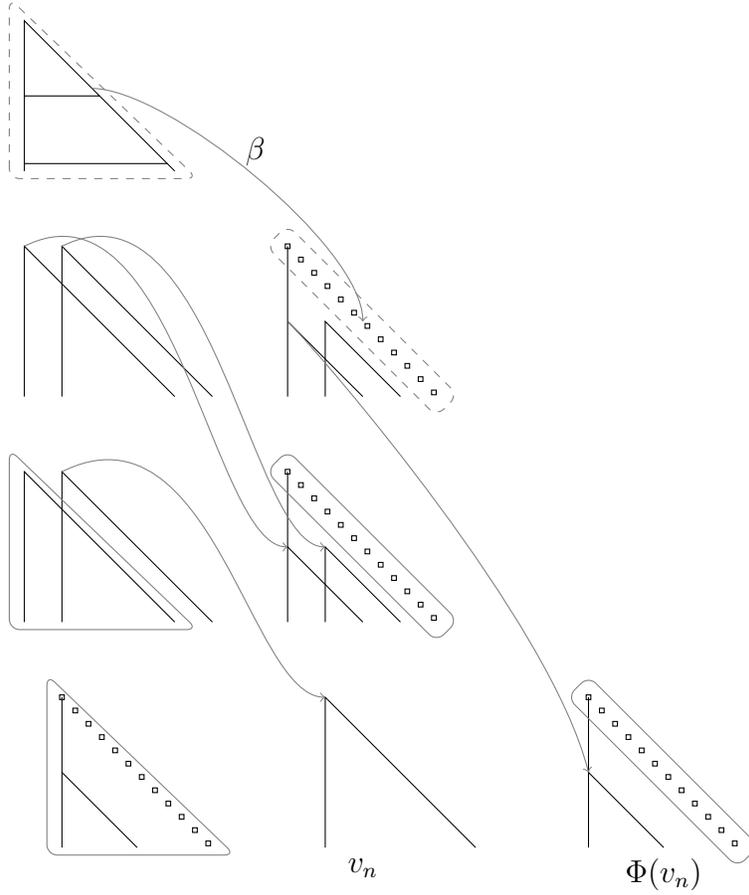
This construction has an exact analogue in constructing $BP\mathbb{R}$ for $p = 2$ [8]; while the relation for τ_n^2 is more complicated, it does not affect the multiplication rule of the v_n 's (this also occurs in the classical BP tower for $p = 2$).

In the case of the $BP\mathbb{R}$ tower for $p > 2$ odd, we proceed similarly, with several complications. One is that unlike the non-equivariant case, where all the constituent pieces of the tower are suspensions of $H\mathbb{Z}$, or the $BP\mathbb{R}$ at $p = 2$, where the pieces are $RO(\mathbb{Z}/2)$ -graded suspensions of $H\mathbb{Z}$, in constructing $BP\mathbb{R}$ at $p > 2$, we encounter $RO(\mathbb{Z}/p)$ -graded suspensions of several different kinds of pieces, namely $H\mathbb{Z}$, $H\tilde{\mathcal{L}}_{p-1}$, $H\mathbb{Z}/p$, and $H\mathcal{L}_p$. Thus, the general plan of our pictures is to depict the homology of the generators involved, using the symbols introduced in Figures 2,4,6, while showing the k -invariants killing appropriate parts of their $RO(\mathbb{Z}/p)$ -graded coefficients (depicted by the curved arrows). We remind the reader that while generators of the form $H\mathcal{L}_p$ are necessary to complete the tower, such a generator represents a free spectrum, thus spawning a non-equivariant BP tower, which is considered negligible and is omitted from the picture. This corresponds to higher

powers of the v_n generators being negligible, thus making the picture necessarily incomplete. Finally, our pictures depict the good wedges of the genuine Eilenberg-Mac Lane generators. We see that the primary k-invariants wipe out the “odd part” of the good wedges of generators of the equivariant dual Steenrod algebra, as well as the new homology of the generators arising in the tower, completely. The reader should recall, however, that due to the presence of the derived wedges, this “slice version” of the tower at present remains conjectural, and we only have a Borel cohomology version. The reason we keep the genuine equivariant homology picture is to demonstrate in what exact degrees the generator classes reside.

7.1. Detailed description of the even pattern. In Figure 9, the illustration is for the prime $p = 3$. The generator in the bottom left corner is the bottom element $H\underline{\mathbb{Z}}$. The triangle it is enclosed in depicts the fact that this generator survives. The leftmost column of Figure 9 depicts what the first k-invariant does on the factor denoted by $T_\theta(t_i)$ in Theorem A of [16] (corresponding to the generator monomials in (4) of [9]). Going from the bottom, the second icon shows the element $\widehat{\xi}_n$ which survives (and hence is encircled in a triangle). The rest of this copy of $H\underline{\mathbb{Z}} \wedge T$ (in the notation of [9]) supports the k-invariant Q'_n going to the $H\widetilde{\mathcal{L}}_{p-1}$ corresponding to the generator v_n . (We can imagine this part of the tower occurring multiplied by a polynomial in $\Phi(v_m)$ for any $m \in \mathbb{N}$ and an even monomial in v_m , $m > n$, occurring in the first power.) Now for the higher terms of the first column, a different pattern occurs, since they are not supposed to contain any surviving coefficient elements. Accordingly, they are taken isomorphically by the k-invariant Q'_n to the corresponding copy of $H\underline{X}$ (the second icon, from the bottom, in the second column from the left of Figure 9). We observe that this leaves a copy of $H\underline{\mathcal{L}}_p$ in the target unattended, which supports a \overline{Q}_n into a negligible part of the tower (not depicted in Figure 9).

Since we are at $p = 3$, the fourth icon (from the bottom) in the first column of Figure 9 is already the top term (see (4) of [9]), supporting an $H\underline{\mathbb{Z}}/p$ according to Theorem A of [16]. This is enclosed in a dashed triangle, depicting two new phenomena, since only one generator in homology, denoted in [9] by $\underline{\theta}_n$, survives. This element, however, is integral, and accordingly, is taken by Q'_n by a Bockstein to its target, indicating that no additional negligible term is spawned in this dimension (since the negligible higher powers of v_n are already visible in the undepicted negligible part of our tower, starting with v_n^2). Also, we note that the remainder of third icon (from the bottom) of the second

FIGURE 9. The even pattern of the BPR -tower

column from the left of Figure 9 is also matched by Q'_n with half of the coefficients of the top $H\underline{\mathbb{Z}}/p$ term of the first column, thus leaving an integral surviving pattern.

Finally, Figure 9 depicts a Q''_n matching the top term of the second column counted from the left with the last column (which has only one icon). This is the transpotence k -invariant, supporting the $\Phi(v_n)$ generator. Note that, again, a copy of $H\underline{\mathcal{L}}_p$ is unattended, signifying another negligible generator in the tower.

This part of the tower is repeated by multiplication by powers of $\underline{\theta}_n$.

7.2. Detailed description of the odd pattern. We now turn to Figure 10. We can think of this as continuation of Figure 9 with n replaced by a larger number m , where the generator v_m , i.e. the surviving bottom icon of the second column of Figure 9 representing an $H\underline{\mathcal{L}}_{p-1}$,

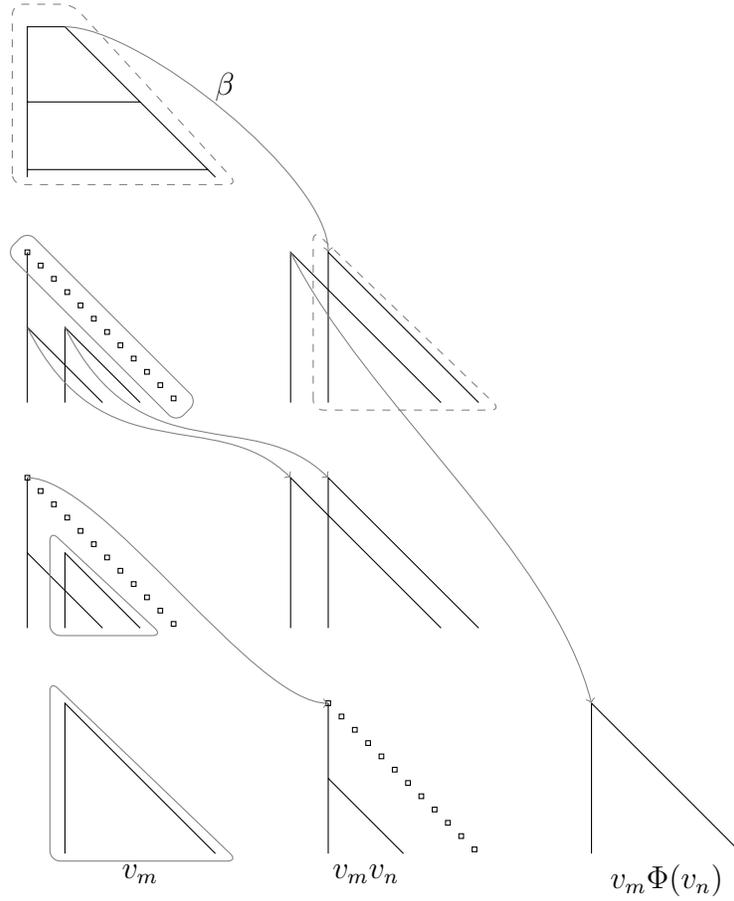


FIGURE 10. The odd pattern of the BPR -tower

is moved to the bottom left corner of Figure 10. The next two icons (from the bottom) of the first column of Figure 10 represent copies of HX (see Section 4). The differential Q'_n matches the coefficients of this term with the $H\mathbb{Z}$ in the bottom of the second column of Figure 10, leaving a copy of $H\tilde{\mathcal{L}}_{p-1}$ in the second icon of the first column, which survives in the homology of BPR . The third icon from the bottom in the first column of Figure 10 is again an HX , this time matched by Q'_n with a target $H\mathbb{Z} \wedge T$ represented by the second icon from the bottom of the second column from the left of Figure 10. This time, we see that there is a seemingly surviving copy of $H\mathcal{L}_p$ in the source, which, however, supports another negligible part of the BPR tower, via the k -invariant \overline{Q} .

At the top of the first column (counted from the left) of Figure 10, we have a copy of $H\tilde{\mathcal{L}}_{p-1}/p$. This partially survives into a copy of $H\tilde{\mathcal{L}}_{p-1}$

on the θ_n generator. To achieve this effect, there is a Bockstein into one $H\tilde{\mathcal{L}}_{p-1}$ part of the target $H\mathbb{Z} \wedge T$ in the third icon from the bottom of the second column of Figure 10, realized by the k-invariant Q'_n .

The remaining copy of $H\tilde{\mathcal{L}}_{p-1}$ is matched, again, by the “transposence” differential Q''_n with the icon in the third column of Figure 10, representing $v_m\Phi(v_n)$.

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