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Real topological Hochschild homology and the Segal conjecture

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ABSTRACT

We give a new proof, independent of Lin's theorem, of the Segal conjecture for the cyclic group of order two. The key input is a calculation, as a Hopf algebroid, of the Real topological Hochschild homology of $\mathbb{F}_2$. This determines the E_2-page of the descent spectral sequence for the map $\mathrm{N}\mathbb{F}_2 \to \mathbb{F}_2$, where $\mathrm{N}\mathbb{F}_2$ is the C_2-equivariant Hill–Hopkins–Ravenel norm of $\mathbb{F}_2$. The E_2-page represents a new upper bound on the $RO(C_2)$-graded homotopy of $\mathrm{N}\mathbb{F}_2$, from which the Segal conjecture is an immediate corollary.

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1. Introduction

The Segal conjecture, for the cyclic group C_2 of order 2, is an equivalence

$$\pi_s^0(\mathbb{RP}^\infty) \cong \widehat{A}(C_2)$$

of the stable cohomotopy of $\mathbb{RP}^\infty$ with the completion of the Burnside ring of C_2 at the augmentation ideal. The conjecture follows from the following stronger result of Lin [16]:

Theorem 1.1 *(Lin). Let γ denote the canonical line bundle over $\mathbb{RP}^\infty$, and for each integer $n > 0$ let $\mathbb{RP}^\infty_{-n}$ denote the Thom spectrum of $-n\cdot\gamma$. Then there is an equivalence of spectra*

$$\mathbb{RP}^\infty_{-\infty} = \mathrm{holim}_n \mathbb{RP}^\infty_{-n} \simeq (S^{-1})^\wedge_2.$$

The only known proof of Lin's theorem proceeds via calculation of a certain continuous Ext group

$$\widehat{\mathrm{Ext}}_{\mathcal{A}}\left(\mathrm{H}^*(\mathbb{RP}^\infty_{-\infty};\mathbb{F}_2), \mathbb{F}_2\right).$$

The calculation is elegant, and has been generalized through the development of the *Singer construction* [19,1,17]. However, the simplicity of Lin's proof is fundamentally limited by the complexity of the Steenrod algebra $\mathcal{A}$. The goal of this paper is to provide a new, less computational proof of Lin's theorem. We cannot avoid calculating a completed Ext group, but the Ext we calculate is over a polynomial coalgebra $\mathbb{F}_2[x]$ rather than the Steenrod algebra $\mathcal{A}$. We trust the reader will agree that this reduces the complexity of the homological algebra.

Remark 1.2. Just as the Steenrod algebra $\mathcal{A}$ arises as (the dual of) the homology of $\mathbb{F}_2$, the polynomial coalgebra $\mathbb{F}_2[x]$ appears as the *topological Hochschild homology* of $\mathbb{F}_2$.

To explain our methods, we must review how the Segal conjecture has been both restated and generalized via the language of C_2-equivariant stable homotopy theory.

Notation 1.3. In the C_2-equivariant stable homotopy category, we use the notation $S = S^0$ to denote the unit object. This is the C_2-equivariant sphere spectrum. We use S^σ to denote the 1-point compactification of the sign representation. Depending on context, we use $\mathbb{F}_2$ to denote either the field with 2 elements or the non-equivariant Eilenberg–Maclane spectrum $\mathrm{H}\mathbb{F}_2$.

Recollection 1.4. In the C_2-equivariant stable homotopy category, the morphism

$$a : S^{-\sigma} \to S^0$$

is adjoint to the inclusion of fixed points into the sign representation σ. The *Borel completion* of a C_2-spectrum X is the a-completion

$$X_a^\wedge := \operatorname{holim}\left(\cdots \to X/a^n \to X/a^{n-1} \to \cdots \to X/a\right).$$

One says that a C_2-spectrum X is *Borel complete* if the natural map $X \to X_a^\wedge$ is an equivalence.

Theorem 1.5 *(Lin's theorem, restated). The natural map $S \to S_a^\wedge$ is an equivalence after 2-completion.*

We will explain the equivalence of the two variants of Lin's theorem in Section 5. In the above form, Lin's theorem has received a substantial generalization.

Recollection 1.6. For any ordinary spectrum X, the Hill–Hopkins–Ravenel norm $NX = N_e^{C_2}X$ is a C_2-equivariant refinement of the smash product $X \wedge X$, with C_2-action given by swapping the two copies of X [11, §B.5].

A version of the following was first proved in [14] (cf. [17, Theorem 5.13]). As we will recall in Section 5, the statement in full generality is a consequence of [20, III.1.7].

Theorem 1.7 *(Segal conjecture, strong form). Let X denote any bounded below spectrum. Then the natural map*

$$\mathrm{N}X \to (\mathrm{N}X)_a^\wedge$$

is an equivalence after 2-completion.

Theorem 1.5 follows from Theorem 1.7 by setting X to be the sphere spectrum. As explained in [20, III.1.7], Theorem 1.7 follows in general from the case $X = \mathbb{F}_2$. In other words, since $\mathrm{N}\mathbb{F}_2$ is 2-complete, all statements of Lin's theorem are consequences of the following result:

Theorem 1.8. *The C_2-spectrum $\mathrm{N}\mathbb{F}_2$ is Borel complete.*

Theorem 1.8 is the form in which we will prove the Segal conjecture. It is important to note that, while Theorem 1.8 tells us that the spectra $\mathrm{N}\mathbb{F}_2$ and $(\mathrm{N}\mathbb{F}_2)_a^\wedge$ coincide, it does not shed light on the homotopy type of either one. As we now explain, our main theorem provides a computable upper bound on the homotopy groups of these spectra, and in this sense our results are stronger than the Segal conjecture.

We prove the following Theorem and Corollary independently of the Segal conjecture. From here on, all Hopf algebras, comodules, and homotopy groups will be indexed over $RO(C_2)$, the virtual representation ring of C_2. The functor Ext is then defined using relative injective resolutions in this category of $RO(C_2)$-graded comodules, just as in [18, §A.1].

Theorem A. *Let $\mathbb{F}_2[x]$ be the Hopf algebra with x primitive of degree $1+\sigma$, and let $\mathbb{F}_2[a,u]$ be the comodule algebra where the class a is primitive in degree $-\sigma$, u is in degree $1-\sigma$, and the coaction is determined by:*

$$u \mapsto u \otimes 1 + a^2 \otimes x.$$

Then there is a spectral sequence

$$E_2 = \widehat{\mathrm{Ext}}^{s,k+\ell\sigma}_{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2[a,u^{\pm 1}]) \Rightarrow \pi_{(k-s)+\ell\sigma}(\mathrm{N}\mathbb{F}_2)^{\wedge}_a.$$

Explicitly, the completed Ext *appearing in this E_2-page may be calculated as*

$$E_2 = \lim_n \mathrm{Ext}^{s,k+\ell\sigma}_{\mathbb{F}_2[x]/x^{2^n}}(\mathbb{F}_2, \mathbb{F}_2[a,u^{\pm 1}]).$$

Corollary B. *Let p and q denote integers such that $p+q<0$. Then*

$$\pi_{p+q\sigma}(\mathrm{N}\mathbb{F}_2)^{\wedge}_a = \begin{cases} 0 & p \neq 0 \\ \mathbb{F}_2\{a^{-q}\} & p = 0. \end{cases}$$

Corollary B follows from straightforward computation of the Ext groups appearing in Theorem A. As we will explain in Section 5, it immediately implies Theorem 1.8 and hence Theorem 1.7.

Remark 1.9. Our proof of Theorem A arises by considering the descent spectral sequence for the C_2-equivariant norm map

$$\mathrm{N}\mathbb{F}_2 \to \underline{\mathbb{F}_2},$$

where we use $\underline{\mathbb{F}_2}$ to denote the C_2-equivariant Eilenberg–Maclane spectrum of the constant Mackey functor, $\mathrm{H}\underline{\mathbb{F}_2}$. This norm map is a C_2-equivariant refinement of the usual multiplication map $\mathbb{F}_2 \otimes \mathbb{F}_2 \to \mathbb{F}_2$, which arises from the fact that $\underline{\mathbb{F}_2}$ is a C_2-commutative ring in C_2-spectra in the sense of §2 (see, e.g., [21]). The basic descent datum is the $RO(C_2)$-graded homotopy of

$$\underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2},$$

which is known as the *Real topological Hochschild homology* of $\underline{\mathbb{F}_2}$. These $RO(C_2)$-graded homotopy groups were computed as an algebra in [8]. We will need to know them as a *Hopf algebroid*, and not just as an algebra. Our computation of the Hopf algebroid structure maps is likely of independent interest, and appears in Section 2.

Remark 1.10. By the Segal conjecture, and the fact that $\mathrm{N}\mathbb{F}_2$ is 2-complete, the fixed points spectrum $\mathrm{N}\mathbb{F}_2^{C_2}$ is identified with the more classical object $(\mathbb{F}_2 \wedge \mathbb{F}_2)^{hC_2}$. There has been some interest in computing the homotopy groups of these fixed points, and we give a brief discussion in Section 6.

Outline

The spectral sequence in the main theorem is obtained by taking the a-completion of the relative Adams spectral sequence for the map $\mathrm{N}\mathbb{F}_2 \to \underline{\mathbb{F}_2}$. The E_2-term of this spectral sequence is governed by the Hopf algebroid structure on $\pi_\star(\underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2})$, otherwise known as Real topological Hochschild homology (cf. [8]). We determine this structure in §2 by comparison with underlying and geometric fixed points. In §3 we identify the E_2-page for the Borel completion with the indicated limit of Ext groups. In §4 we compute these Ext groups, and extract a vanishing result which implies the Segal conjecture for the group C_2. We give the proof of the Segal conjecture in §5. Finally, in §6 we indicate a computation of some low dimensional integer stems, and leave the reader with a few questions of interest.

Conventions

We assume the reader is acquainted with equivariant homotopy theory at the level of [11, §2,§3]. If (A, Γ) is a Hopf algebroid and M is a comodule, we will abbreviate $\mathrm{Ext}_\Gamma(A, M)$ as $\mathrm{Ext}_\Gamma(M)$.

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2. The real topological Hochschild homology of $\underline{\mathbb{F}_2}$

Let CAlg^{C_2} denote the (∞-)category of C_2-commutative algebras in genuine C_2-spectra.[1] Recall from [8] that, if $R \in \mathsf{CAlg}^{C_2}$ is a C_2-commutative ring spectrum,[2] then the *Real topological Hochschild homology* of R is the C_2-spectrum[3]

[1] As an explicit model, one could take the ∞-category underlying the model category structure on commutative algebras in G-spectra constructed in [11, §B.7]. We warn the reader that a C_2-commutative algebra has more structure than an $\mathbb{E}_\infty$-algebra object in the ∞-category of genuine C_2-spectra.

[2] More generally, this definition makes sense if R is an $\mathbb{E}_\sigma$-ring in the sense of [10, §2.2].

[3] This spectrum is denoted $\mathrm{THR}(R)$ in [8].

$$\mathrm{THH}^\sigma(R) := R \otimes_{\mathrm{N}R} R.$$

Here we are using that part of the structure of a C_2-commutative algebra is a ring map $\mathrm{N}R \to R$, which makes R into a module over $\mathrm{N}R$.

The underlying spectrum is the topological Hochschild homology of R (viewed as an ordinary ring spectrum). The geometric fixed points are given by

$$(\mathrm{THH}^\sigma(R))^{\Phi C_2} \simeq R^{\Phi C_2} \otimes_R R^{\Phi C_2}.$$

Dotto–Moi–Patchkoria–Reeh computed the Real topological Hochschild homology of $\underline{\mathbb{F}_2}$ in [8, Theorem 5.18]:

Theorem 2.1. $\mathrm{THH}^\sigma(\underline{\mathbb{F}_2})$ *is the free* $\mathbb{E}_1$-$\underline{\mathbb{F}_2}$*-algebra on a generator* x *in degree* ρ*. In particular, there is an isomorphism of* $RO(C_2)$*-graded rings:*

$$\mathrm{THH}^\sigma(\underline{\mathbb{F}_2})_\star \cong (\underline{\mathbb{F}_2})_\star[x], \quad |x| = \rho,$$

where $\rho = 1 + \sigma$ *is the regular representation.*

If $\mathrm{THH}^\sigma(R)_\star$ is flat over $R_\star$ then the pair $(R_\star, \mathrm{THH}^\sigma(R)_\star)$ forms a Hopf algebroid in the usual way, since we may identify it with the Hopf algebroid associated to the relative Adams spectral sequence for the map

$$\mathrm{N}R \to R,$$

as in Baker-Lazarev [2].

Remark 2.2. In the classical setting, the left and right units for $\mathrm{THH}(R)$ are always homotopic, and, when the relevant flatness hypothesis is satisfied, the associated Hopf algebroid is always a Hopf algebra. This is no longer true for real Hochschild homology, as we will see below. The reason is that the inclusions of the two different fixed points into S^σ are not *equivariantly* homotopic.

Before stating the structure theorem, we recall ([9,13]) that the homotopy groups of $\underline{\mathbb{F}_2}$ are given by

$$\pi_\star \underline{\mathbb{F}_2} = \mathbb{F}_2[a, u] \oplus \frac{\mathbb{F}_2(a, u)}{\mathbb{F}_2[a, u]}\{\theta\}$$

where:

- $a : S^{-\sigma} \to \underline{\mathbb{F}}_2$ is the Hurewicz image of the map $S^{-\sigma} \to S^0$ adjoint to the inclusion of fixed points.

- $u : S^{1-\sigma} \to \underline{\mathbb{F}_2}$ is the unique homotopy class extending the underlying unit $C_{2+} \to \underline{\mathbb{F}_2}$ along the map $C_{2+} \to S^{1-\sigma}$.
- $\theta : S^{2\sigma} \to S^2$ is the degree 2 cover.

Theorem 2.3. *The Hopf algebroid structure on* $((\underline{\mathbb{F}_2})_\star, \mathrm{THH}^\sigma(\underline{\mathbb{F}_2})_\star)$ *is given as follows:*

- *The left units on generators are, for* $i, j \geq 0$*:*

$$\begin{aligned} \eta_L(a) &= a, \\ \eta_L(\theta) &= \theta, \\ \eta_L(u) &= u, \\ \eta_L(\theta a^{-i} u^{-j}) &= \theta a^{-i} u^{-j}. \end{aligned}$$

- *The right units on generators are, for* $i, j \geq 0$*:*

$$\begin{aligned} \eta_R(a) &= a, \\ \eta_R(\theta) &= \theta, \\ \eta_R(u) &= u + a^2 x, \\ \eta_R(\theta a^{-i} u^{-j}) &= (u + a^2 x)^{-j} \frac{\theta}{a^i}. \end{aligned}$$

  (Note that the apparently infinite sum in the last formula is finite because $\frac{\theta}{a^i}$ *is* a*-torsion.)*
- *The comultiplication is determined by*

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

In particular, the elements a *and* θ *are primitive.*

Proof. Since $\mathrm{N}S^0 = S^0$ the Hopf algebroid structure on $\mathrm{THH}^\sigma(S^0)$ is trivial. Both θ and a lie in the Hurewicz image of $S^0 \to \underline{\mathbb{F}_2}$, so we conclude that they are primitive as indicated.

The element x is primitive for degree reasons: the only elements in $\pi_\rho(\mathrm{THH}^\sigma(\underline{\mathbb{F}_2}) \otimes_{\underline{\mathbb{F}_2}} \mathrm{THH}^\sigma(\underline{\mathbb{F}_2}))$ are $x \otimes 1$ and $1 \otimes x$.

Now we compute the right unit on u. Observe that, as a vector space, $\pi_{1-\sigma}\underline{\mathbb{F}_2}[x] = \mathbb{F}_2\{u, a^2 x\}$ so we must have $\eta_R(u) = \alpha u + \beta a^2 x$ for some numbers α and β in $\mathbb{F}_2$. On underlying homotopy we have $\mathrm{res}(u) = 1$ so that $\alpha = 1$. Now observe that the map

$$\mathrm{N}\mathbb{F}_2 \to \underline{\mathbb{F}_2}$$

becomes, upon taking geometric fixed points, the map

$$\mathbb{F}_2 \to \mathbb{F}_2[t]$$

where $t = u/a$.

The descent Hopf algebroid for this map has left and right units $\eta_L, \eta_R : \mathbb{F}_2[t] \to \mathbb{F}_2[t] \otimes_{\mathbb{F}_2} \mathbb{F}_2[t]$ given by $t \mapsto 1 \otimes t$ and $t \mapsto t \otimes 1$. In particular, $\eta_L - \eta_R$ is nonzero on geometric fixed points, so β must be nonzero, completing the proof of the claim.

It remains to compute the right unit on elements of the form $\theta a^{-i}u^{-j}$. For any C_2-spectrum X, consider the diagram where each square is a homotopy pullback and ∂ is the induced map:

$$\begin{tikzcd}
\Sigma^{-1}X^{\wedge}_a[a^{-1}] \arrow[rr] \arrow[dr, "\partial"] \arrow[dd] & & 0 \arrow[d] \\
& X \arrow[r] \arrow[d] & X[a^{-1}] \arrow[d] \\
0 \arrow[r] & X^{\wedge}_a \arrow[r] & X^{\wedge}_a[a^{-1}]
\end{tikzcd}$$

If X is a homotopy ring, then each object has a canonical X-module structure, and maps in the squares are maps of X-modules; hence ∂ can be given the structure of a map of X-modules.

In particular, we have a commutative diagram:

$$\begin{tikzcd}
\Sigma^{-1}(\underline{\mathbb{F}_2})^{\wedge}_a[a^{-1}] \arrow[r, "\eta_R"] \arrow[d, "\partial"'] & \Sigma^{-1}(\underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2})^{\wedge}_a[a^{-1}] \arrow[d, "\partial"] \\
\underline{\mathbb{F}_2} \arrow[r, "\eta_R"'] & \underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2}
\end{tikzcd}$$

The construction $X \mapsto X^{\wedge}_a[a^{-1}]$ is lax symmetric monoidal, so the map

$$(\underline{\mathbb{F}_2})^{\wedge}_a[a^{-1}] \to \left(\underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2}\right)^{\wedge}_a [a^{-1}]$$

is still a ring map and hence

$$\eta_R(a^{-i-1}u^{-j-1}) = a^{-i-1}\eta_R(u)^{-j-1} = a^{-i-1}(u + a^2x)^{-j-1}.$$

From the Mayer-Vietoris sequence of the arithmetic square, we see that $\partial(a^{-1}u^{-1})$ must be nonzero and therefore equal to θ. It follows that $\partial(a^{-i-1}u^{-j-1}) = \theta a^{-i}u^{-j}$, since there is a unique element in that degree which, when multiplied by $a^i u^j$, is equal to θ. The result follows. $\square$

3. The construction of the spectral sequence

Theorem 3.1. *There is a spectral sequence*

$$E_2 = \lim_n \operatorname{Ext}^{s,k+\ell\sigma}_{\mathbb{F}_2[x]/x^{2^n}}(\mathbb{F}_2[a,u^{\pm 1}]) \Rightarrow \pi_{(k-s)+\ell\sigma}(\mathrm{N}\mathbb{F}_2)^\wedge_a.$$

Proof. Since $\mathrm{N}\mathbb{F}_2$ is connective and 2-complete, there is an identification[4]:

$$\mathrm{N}\mathbb{F}_2 \simeq \operatorname{holim}_\Delta \underline{\mathbb{F}_2}^{\otimes_{\mathrm{N}\mathbb{F}_2} \bullet+1}.$$

Since a-completion preserves homotopy limits, we have

$$(\mathrm{N}\mathbb{F}_2)^\wedge_a \simeq \operatorname{holim}_\Delta \left(\underline{\mathbb{F}_2}^{\otimes_{\mathrm{N}\mathbb{F}_2} \bullet+1}\right)^\wedge_a.$$

Thus we get a spectral sequence with E_1-term given by

$$E_1^{s,\star+s} = \pi_\star \left(\underline{\mathbb{F}_2}^{\otimes_{\mathrm{N}\mathbb{F}_2} s+1}\right)^\wedge_a.$$

Since $\underline{\mathbb{F}_2} \otimes_{\mathrm{N}\mathbb{F}_2} \underline{\mathbb{F}_2}$ is free as an $\underline{\mathbb{F}_2}$-module, the same is true for each term $\underline{\mathbb{F}_2}^{\otimes_{\mathrm{N}\mathbb{F}_2} s+1}$. The a-completion of $\underline{\mathbb{F}_2}$ has $RO(C_2)$-graded homotopy groups

$$\pi_\star(\underline{\mathbb{F}_2})^\wedge_a \cong \mathbb{F}_2[a,u^{\pm 1}].$$

Combined with the computation in the previous section, we may then identify the sth term of the E_1-page with

$$\mathbb{F}_2[a,u^{\pm 1},x_1,...,x_s]^\wedge_a,$$

and the d_1-differentials are determined by the coaction $u \mapsto u + a^2 x$. Here observe that the a-completion is taken in the *graded* sense. Since the underlying degree of a is -1, the underlying degree of u is zero, and the underlying degrees of the x_i are 2, we find that, in a fixed degree, any element which is highly divisible by a must also be highly divisible by the ideal $(x_1,...,x_n)$, and vice versa. Therefore, we may rewrite the sth term as:

$$\begin{aligned}\mathbb{F}_2[a,u^{\pm 1},x_1,...,x_s]^\wedge_a &\cong \mathbb{F}_2[a,u^{\pm 1},x_1,...,x_s]^\wedge_{(x_1,...,x_s)} \\ &= \lim_n \mathbb{F}_2[a,u^{\pm 1}] \otimes \left(\mathbb{F}_2[x]/(x^{2^n})\right)^{\otimes s},\end{aligned}$$

where the completions and limit are understood in the graded setting. In other words, we may identify the E_1-term with the limit of the cobar complexes[5]:

[4] The proof is that of [5, Theorem 6.6], where one replaces the Postnikov tower with the C_2-equivariant Postnikov tower.

[5] For a refresher on the cobar complex, see [18, A1.2.11].

$$E_1 = \lim_n C^*_{\mathbb{F}_2[x]/(x^{2^n})}(\mathbb{F}_2[a, u^{\pm 1}]).$$

By the Milnor exact sequence (as in, e.g., [22, Theorem 3.5.8]), this gives the desired computation of the E_2-term modulo a possible $\lim^1$ contribution. But for fixed n and tridegree, these groups are finite-dimensional vector spaces over $\mathbb{F}_2$, so the $\lim^1$ vanishes and the result follows. □

4. Computation of the E_2-page

In this section we compute some information about the E_2-page of the spectral sequence from the previous section. Our principal aim will be to prove Corollary B from the Introduction.

Write $\{E_r^{(n)}\}$ for the x-adic spectral sequence (as in [18, A1.3.9]):

$$E_1^{(n)} = \mathbb{F}_2[a, u, y_0, ..., y_{n-1}] \Rightarrow \mathrm{Ext}_{\mathbb{F}_2[x]/(x^{2^n})}(\mathbb{F}_2[a, u]),$$

where y_i is represented by $[x^{2^i}]$ in the cobar complex ([18, A1.2.11]), and write $\{E_r\}$ or $\{E_r^{(\infty)}\}$ for the x-adic spectral sequence

$$E_1 = \mathbb{F}_2[a, u, y_i : i \geq 0] \Rightarrow \widehat{\mathrm{Ext}}_{\mathbb{F}_2[x]}(\mathbb{F}_2[a, u]).$$

These spectral sequences are obtained by filtering the cobar complex as in [18, §A.2.3].

Theorem 4.1. *We have ring isomorphisms*

$$E_\infty = \mathbb{F}_2[a, u^{2^{r+1}m} y_r : m, r \geq 0]/(a^{2^{r+1}} u^{2^{r+1}m} y_r).$$
$$E_\infty^{(n)} = \mathbb{F}_2[a, u^{2^n}, u^{2^{r+1}m} y_r : m \geq 0, 0 \leq r \leq n-1]/(a^{2^{r+1}} u^{2^{r+1}m} y_r).$$

Moreover, there are no nontrivial $\mathbb{F}_2[a]$-module extensions.

The proof will require the following lemma.

Lemma 4.2. *The elements $u^{2^{r+1}m} y_r \in E_1^{(n)}(\mathbb{F}_2[a, u])$ are permanent cycles for all $m \geq 0$ and $0 \leq r \leq n-1$.*

Proof. Let δ denote the Bockstein

$$\delta : \mathrm{Ext}^0(\mathbb{F}_2[a, u]/(a^{2^{r+1}})) \to \mathrm{Ext}^1(\mathbb{F}_2[a, u]).$$

In the cobar complex computing $\mathrm{Ext}_{\mathbb{F}_2[x]}(\mathbb{F}_2[a, u])$ we have

$$d(u^{2^r(2m+1)}) = \eta_L(u^{2^r(2m+1)}) - \eta_R(u^{2^r(2m+1)}) \equiv u^{2^{r+1}m} a^{2^{r+1}} [x^{2^r}] \mod a^{2^{r+1}+1} x^{2^r+1}.$$

It follows that $u^{2^r(2m+1)}$ is primitive in $\mathbb{F}_2[a,u]/(a^{2^{r+1}})$ and that $\delta(u^{2^r(2m+1)})$ is represented by $u^{2^{r+1}m}[x^{2^r}]$ modulo terms of higher filtration. This provides a lift of the element $u^{2^{r+1}m}y_r$ to a cocycle in the cobar complex, which completes the proof. $\square$

Proof of Theorem 4.1. We will prove by induction on $t \le n$ that $E^{(n)}_{2^{t-1}+1} = E^{(n)}_{2^t}$, and

$$E^{(n)}_{2^t} = \mathbb{F}_2[a, u^{2^t}, u^{2^{r+1}m}y_r : m \ge 0, 0 \le r \le n-1]/(a^{2^{r+1}}u^{2^{r+1}m}y_r : r \le t-1).$$

Note that, in this case, $E^{(n)}_{2^n}$ is generated by permanent cycles so the spectral sequence stops at this page, which is also the advertised answer.

The base case is trivial, so we assume the result holds for t and turn to the inductive step. Let $I = (i_0, i_1, ..., i_{n-1})$ be a tuple of nonnegative integers and denote by y_I the monomial $y_0^{i_0}y_1^{i_1}\cdots y_{n-1}^{i_{n-1}}$. Given such a monomial, denote by $m(I)$ the minimal nonzero index in I. Then the elements

$$a^m u^k y_I$$

with

- $m(I) \le t-1$, $m \le 2^{m(I)+1}-1$, and k divisible by $2^{m(I)+1}$; or
- $m(I) \ge t$ and k divisible by 2^t

form an $\mathbb{F}_2$-basis for $E^{(n)}_{2^t}$. If $m(I) \le t-1$ then this element is a product of permanent cycles by the previous lemma. Otherwise, using the cobar differential $d(u^{2^t}) = a^{2^{t+1}}[x^{2^t}]$, we see that

$$d_{2^t}(a^m u^{2^t\ell}y_I) = a^m y_I(\ell(u^{2^t})^{\ell-1}a^{2^{t+1}}y_t),$$

and so

$$E^{(n)}_{2^t+1} = \mathbb{F}_2[a, u^{2^{t+1}}, u^{2^{r+1}m}y_r]/(a^{2^{r+1}}u^{2^{r+1}m}y_r : r \le t).$$

In the cobar complex we have $d(u^{2^{t+1}}) = a^{2^{t+2}}[x^{2^{t+1}}]$, so $u^{2^{t+1}}$ survives to $E_{2^{t+1}}$ in the x-adic spectral sequence, and the other algebra generators are permanent cycles. This completes the induction and the theorem follows modulo extension problems. With notation as in the previous lemma, we note that $\delta(u^{2^r(2m+1)})$ provides a lift of $u^{2^{r+1}m}y_r$ which is automatically annihilated by $a^{2^{r+1}}$. This resolves the $\mathbb{F}_2[a]$-module extension problem.

The case of $n = \infty$ is essentially the same (or could also be deduced from the above computation). $\square$

We are now ready to deduce Corollary B from the Introduction:

Corollary 4.3. *Let p and q denote integers such that $p+q<0$. Then*

$$\pi_{p+q\sigma}(\mathrm{N}\mathbb{F}_2)^\wedge_a = \begin{cases} 0 & p \neq 0 \\ \mathbb{F}_2\{a^{-q}\} & p = 0. \end{cases}$$

Proof. It suffices, using the spectral sequence from Theorem A, to prove that, when $k-s+\ell<0$, the groups

$$\widehat{\mathrm{Ext}}^{s,k+\ell\sigma}_{\mathbb{F}_2[x]}(\mathbb{F}_2, \mathbb{F}_2[a, u^{\pm 1}])$$

vanish for $k-s \neq 0$ and are given by $\mathbb{F}_2\{a^{-\ell}\}$ when $k-s=0$. Indeed, once this result is known on the E_2-page, we see that the classes $a^{-\ell}$ must be permanent cycles since their potential targets lie in the vanishing range. They cannot be the target of differentials since they lie in filtration 0. From now on we write $k-s=p$ and $\ell=q$, for consistency with the statement of the corollary.

We will show that, in positive filtration, the E_2-term vanishes when $p+q<0$; the filtration zero contribution is easily seen to be just $\mathbb{F}_2\{a^{-q}\}$. It further suffices to verify this vanishing for each group $\mathrm{Ext}_{\mathbb{F}_2[x]/(x^{2^n})}(\mathbb{F}_2[a, u^{\pm 1}])$ appearing in the limit defining the E_2-page.

Since u^{2^n} is $\mathbb{F}_2[x]/(x^{2^n})$-primitive in $\mathbb{F}_2[a,u]$, we have that

$$\mathrm{Ext}_{\mathbb{F}_2[x]/(x^{2^n})}(\mathbb{F}_2[a, u^{\pm 1}]) = \mathrm{Ext}_{\mathbb{F}_2[x]/(x^{2^n})}(\mathbb{F}_2[a,u])[(u^{2^n})^{-1}].$$

Since u has underlying topological degree 0, we can verify the vanishing claim before inverting u. But there it follows immediately from Theorem 4.1, since each multiplicative generator of the associated graded, in positive filtration, satisfies $p+q \geq 0$. $\square$

5. The Segal conjecture

In this section, we prove the Segal conjecture in the following form:

Theorem 5.1. *Let X denote any bounded below spectrum. Then the natural map*

$$\mathrm{N}X \to (\mathrm{N}X)^\wedge_a$$

is an equivalence after 2-completion.

The key point is the following standard observation:

Lemma 5.2. *Let X be a bounded below spectrum. Then, to prove Theorem 5.1, it suffices to show that*

$$(\mathrm{N}X)[a^{-1}] \to (\mathrm{N}X)^\wedge_a[a^{-1}]$$

is an equivalence after 2*-completion. This in turn is equivalent to the claim that the Tate diagonal*

$$X \to (X \wedge X)^{tC_2}$$

is an equivalence after 2*-completion.*

Proof. The first part of the lemma follows from the pullback fracture square

$$\begin{tikzcd} \mathrm{N}X \arrow[r] \arrow[d] & (\mathrm{N}X)[a^{-1}] \arrow[d] \\ (\mathrm{N}X)^{\wedge}_{a} \arrow[r] & (\mathrm{N}X)^{\wedge}_{a}[a^{-1}] \end{tikzcd}$$

Since this is a pullback, the left hand vertical map is an equivalence after 2-completion if and only if the right hand vertical map is an equivalence after 2-completion. To obtain the second part of the lemma, note that the non-equivariant map underlying a is nullhomotopic. Thus, the non-equivariant spectra underlying $(\mathrm{N}X)[a^{-1}]$ and $(\mathrm{N}X)^{\wedge}_{a}[a^{-1}]$ are trivial. This means that it suffices to check that the map on C_2-fixed points

$$\left((\mathrm{N}X)[a^{-1}]\right)^{C_2} \to \left((\mathrm{N}X)^{\wedge}_{a}[a^{-1}]\right)^{C_2}$$

is an equivalence after 2-completion. The above map is identified with the Tate diagonal via [4, §2] and [20, §III.1.5]. □

We are now ready to prove Theorem 5.1.

Proof of Theorem 5.1. By an argument of Nikolaus–Scholze [20, proof of III.1.7], the Tate diagonal

$$X \to (X \wedge X)^{tC_2}$$

is an equivalence after 2-completion, for all bounded below X, if it is when $X = \mathbb{F}_2$. Since $((\mathrm{N}\mathbb{F}_2)[a^{-1}])^{C_2} = \mathbb{F}_2$ ([11, Prop. 2.57]), we are reduced to proving that

$$(\mathrm{N}\mathbb{F}_2)[a^{-1}] \to (\mathrm{N}\mathbb{F}_2)^{\wedge}_{a}[a^{-1}]$$

is an equivalence.

Observe that, when a acts invertibly on a C_2-spectrum Y, we have

$$\cdot a^q : \pi_{p+q\sigma} Y \xrightarrow{\cong} \pi_p(Y^{C_2}).$$

Since $(\mathrm{N}\mathbb{F}_2[a^{-1}])^{C_2} = \mathbb{F}_2$, we deduce that $\pi_\star \mathrm{N}\mathbb{F}_2[a^{-1}] = \mathbb{F}_2[a^{\pm 1}]$. So it suffices to show that:

$$\pi_\star(\mathrm{N}\mathbb{F}_2)^\wedge_a[a^{-1}] = \mathbb{F}_2[a^{\pm 1}].$$

Multiplication by a decreases underlying topological dimension, so it further suffices to show that, when $p+q<0$, we have:

$$\pi_{p+q\sigma}(\mathrm{N}\mathbb{F}_2)^\wedge_a = \begin{cases} 0 & p \neq 0 \\ \mathbb{F}_2\{a^{-q}\} & p = 0 \end{cases}$$

This is the statement of Corollary B. □

Remark 5.3. Setting $X = S^0$, it follows from the above that

$$(S^0)^\wedge_2 \simeq (S^0 \wedge S^0)^{tC_2} \simeq (S^0)^{tC_2}.$$

After identifying $(S^0)^{tC_2}$ with $\Sigma\mathbb{RP}^\infty_{-\infty}$, the original version of Lin's theorem follows.

6. Epilogue

Integer stems

Over the last few years, there have been several attempts to understand the homotopy groups of the non-equivariant spectrum

$$(\mathrm{N}\mathbb{F}_2)^{C_2} = (\mathbb{F}_2 \wedge \mathbb{F}_2)^{hC_2}.$$

This seems especially interesting in light of forthcoming work of Mingcong Zeng and Lennart Meier, which uses the equivalence

$$\Phi^{C_2} N_{C_2}^{C_4}\mathrm{BP}\mathbf{R} \simeq \mathrm{N}\mathbb{F}_2$$

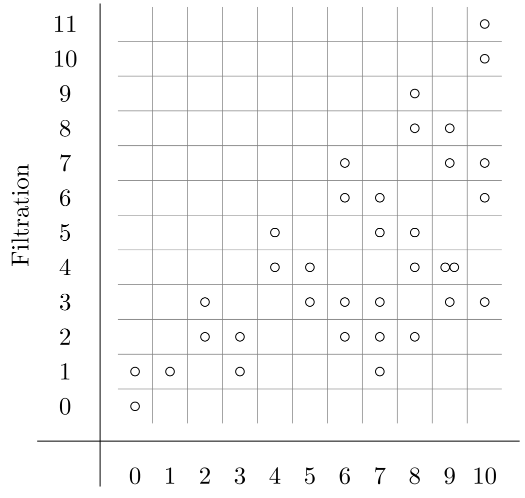
to relate these homotopy groups to the slice spectral sequence differentials studied by Hill-Shi-Wang-Xu [12]

The most straightforward approach to these homotopy groups is via the homotopy fixed point spectral sequence. However, even the E_2-page, given by the group cohomology $H^*(C_2; \mathcal{A}_*)$, is largely unknown at this time [7]. Another approach, pursued independently in unpublished work by J.D. Quigley and Tyler Lawson, is to use the non-equivariant $\mathbb{F}_2$-Adams spectral sequence. Quigley was able to use the Adams spectral sequence to obtain some results about $\pi_*(\mathrm{N}\mathbb{F}_2^{C_2})$ for $* < 10$. We suspect that the use of the equivariant $\mathbb{F}_2$-Adams spectral sequence for $\pi_\star\mathrm{N}\mathbb{F}_2$ would lead to similar complications as those encountered by Lawson and Quigley.

The *relative* Adams spectral sequence of this paper, restricted to integer stems $\pi_{p+0\sigma}\mathrm{N}\mathbb{F}_2$, provides yet another route to these homotopy groups. We draw the E_2-page below, with each circle representing a single copy of $\mathbb{F}_2$:

Remark 6.1. The E_2-page is easy to compute if one is only interested in integer stems. In this case, even on the E_1-page, the only possible contributions come from $\mathrm{Ext}_{\mathbb{F}_2[x]}(\mathbb{F}_2[a,u])$ so we may forego inverting u and the corresponding completion.

E_2 page of the spectral sequence in degrees contributing to $\pi_{p+0\sigma}\mathrm{N}\mathbb{F}_2$



Using a low-dimensional cell structure for $\mathrm{N}\mathbb{F}_2$, one can show that there is a nontrivial extension between the classes in degrees $(0,0)$ and $(0,1)$, and in particular that $\pi_0\mathrm{N}\mathbb{F}_2 \cong \mathbb{Z}/4\mathbb{Z}$. One can also prove that the class in $(1,1)$ detects η.

We believe, but have not verified, that there is a d_2 differential from the class in degree $(5,3)$ to the class in degree $(4,5)$. This should be a consequence of a whole family of d_2 differentials

$$d_2(y_{i+1}) = (ay_0)y_i^2,$$

connected to each other via power operations. We suspect that an equivariant analog of work of Kahn [15], as generalized by Bruner [6, §VI], could establish this family of differentials.

Further Questions

Question 6.2. Can the spectral sequence in this paper be used to recover any of the exotic differentials established by Hill-Shi-Wang-Xu in [12]? While the a_λ-inverted slice spectral sequence also converges to the $RO(C_2)$-graded homotopy of $\mathrm{N}\mathbb{F}_2$, the two spectral sequences differ greatly on the E_2-page. It is conceivable that they become much more similar after running the slice differentials from Hill-Hopkins-Ravenel, since these implement the a-torsion visible on our E_2-page.

Question 6.3. Can our method of proof be generalized to deduce the Segal conjecture for elementary p-groups? The Segal conjecture for elementary p-groups is the key computational input for the Segal conjecture in general [1]. For the group C_p, it seems that a study of $\underline{\mathbb{F}_p} \otimes_{N_e^{C_p}\mathbb{F}_p} \underline{\mathbb{F}_p}$ would be relevant.

Question 6.4. The groups $\mathrm{Ext}_{\mathbb{F}_2[x]}(\mathbb{F}_2[a,u])$ are much smaller than the version with u inverted. Real motivic homotopy theory provides a setting similar to C_2-equivariant homotopy theory in which the negative cone is not present (see, e.g., [3]). Is there a notion of motivic topological Hochschild homology of $\mathbb{F}_2$ whose homotopy groups are the Hopf algebroid $\mathbb{F}_2[a,u,x]$?

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