A MOTIVIC FILTRATION ON THE TOPOLOGICAL CYCLIC HOMOLOGY OF COMMUTATIVE RING SPECTRA

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Abstract. For a prime number p and a p-quasisyntomic commutative ring R, Bhatt–Morrow–Scholze defined motivic filtrations on the p-completions of $\mathrm{THH}(R), \mathrm{TC}^{-}(R), \mathrm{TP}(R)$, and $\mathrm{TC}(R)$, with the associated graded objects for $\mathrm{TP}(R)$ and $\mathrm{TC}(R)$ recovering the prismatic and syntomic cohomology of R, respectively. We give an alternate construction of these filtrations that applies also when R is a well-behaved commutative ring spectrum; for example, we can take R to be $\mathbb{S}$, MU, ku, ko, or tmf. We compute the mod (p, v_1) syntomic cohomology of the Adams summand ℓ and observe that, when $p \geq 3$, the motivic spectral sequence for $V(1)_*\mathrm{TC}(\ell)$ collapses at the E_2-page.

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§1. Introduction

Topological Hochschild homology is an invariant that to an associative ring spectrum R assigns a cyclotomic spectrum $\mathrm{THH}(R)$, closely related to the algebraic K-theory of R. In [BMS19], Bhatt–Morrow–Scholze fix a prime number p and study p-completed topological Hochschild homology $\mathrm{THH}(R)^{\wedge}_p$ for a class of discrete commutative rings R that they call *quasisyntomic*. In particular, they construct a natural *motivic filtration* on $\mathrm{THH}(R)^{\wedge}_p$ for such R. After accounting for the cyclotomic structure on $\mathrm{THH}(R)^{\wedge}_p$, the BMS motivic filtration allows one to define both the prismatic cohomology[1] and syntomic cohomology of R. The construction of prismatic and syntomic cohomology has in turn led to an explosion of new results in algebraic K-theory and mixed characteristic algebraic geometry [Bha21].

In the ICM address [Rog14, Conjecture 5.1], Rognes explains his long-standing conjecture that a similar motivic filtration exists on the mod (p, v_1) algebraic K-theory of the p-completed Adams summand $\ell^{\wedge}_p$, which is a commutative but not discrete ring spectrum. Specifically, the calculation by Ausoni and Rognes of $\mathrm{K}^{\mathrm{alg}}(\ell^{\wedge}_p)/(p, v_1)$ strongly suggests that there is a motivic filtration on $\mathrm{THH}(\ell)$ that is compatible with cyclotomic structure [AR02]. In [HW20, §6.1], the first and third authors studied a filtration on $\mathrm{THH}(\ell)$, and more generally on the THH of truncated Brown–Peterson spectra, that behaves much like the conjectured motivic filtration. Our construction of this filtration was very much ad-hoc, and it was unclear in what generality it could be defined. Furthermore, while the filtration was obviously S^1-equivariant, it did not

[1] Throughout this paper, the term "prismatic cohomology" will be used to refer to what is more properly called "Nygaard-completed absolute prismatic cohomology."

obviously interact well with the cyclotomic Frobenius. In other words, we were able to define the Nygaard-filtered prismatic cohomology of ℓ, but not the syntomic cohomology of ℓ.

Here, we offer and attempt to justify the following proposal:

Proposal 1.0.1. *There is a functorial filtration defined on all* $\mathbb{E}_\infty$*-rings that we call the* even filtration*. For a class of* $\mathbb{E}_\infty$*-rings* R *that we call* chromatically quasisyntomic*, the even filtration on* $\mathrm{THH}(R)$ *is the motivic filtration on* $\mathrm{THH}(R)$*, and it is compatible with all cyclotomic structure.*

The even filtration has a straightforward definition, and can be applied to $\mathbb{E}_\infty$ rings that do not arise from THH constructions. In addition to recovering the BMS motivic filtration, we prove that it can be used to recover the Adams–Novikov filtration on the sphere, the "synthetic analog" functor ν into Morel–Voevodsky's $\mathcal{SH}(\mathbb{C})$, the Hochschild–Kostant–Rosenberg filtration, and the global motivic filtration of Morin and Bhatt–Lurie.

For chromatically quasisyntomic $\mathbb{E}_\infty$-rings R, such as $R = \mathbb{S}$, MU, ℓ, ku, ko, or tmf, the even/motivic filtration on $\mathrm{THH}(R)$ can be calculated using explicit presentations: we find a map of cyclotomic $\mathbb{E}_\infty$-rings $\mathrm{THH}(R) \to B$ such that $\mathrm{fil}^n_{\mathrm{mot}} \mathrm{THH}(R) \simeq \lim_\Delta(\tau_{\geq 2n}(B^{\otimes_{\mathrm{THH}(R)} \bullet+1}))$. Readers familiar with [BMS19] may appreciate the analogy where R is a discrete p-quasisyntomic ring and B is THH of a quasiregular semiperfectoid cover (and everything is p-completed). Such presentations in particular allow us to compute $V(1)_*\mathrm{TC}(\ell)$ at the prime $p = 3$, which was not previously known.

§1.1. The even filtration and comparison theorems.

For motivation, let us recall how Bhatt–Morrow–Scholze defined their motivic filtration, which we denote here by $\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p$. First, they focus on a certain class of discrete commutative rings, which they call quasisyntomic and which from here forwards we will call p-complete p-quasisyntomic. Given a p-complete p-quasisyntomic ring R, they prove that there exists another such ring S and a map $R \to S$ with the following two properties:

1. the canonical map $\mathrm{THH}(R)^\wedge_p \to \lim_\Delta(\mathrm{THH}(S^{\otimes_R \bullet+1})^\wedge_p)$ is an equivalence;
2. for each $n \geq 0$, the spectrum $\mathrm{THH}(S^{\otimes_R n+1})^\wedge_p$ is *even*, i.e. its homotopy groups are concentrated in even degrees.

They then define $\mathrm{fil}^k_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p$ to be $\lim_\Delta(\tau_{\geq 2k}(\mathrm{THH}(S^{\otimes_R \bullet+1})^\wedge_p))$, showing that this is independent of the choice of S using the formalism of Grothendieck topologies.

More generally, the strategy of computing $\mathrm{THH}(R)$ by realizing it as a totalization of even ring spectra has been broadly applied to great effect [BMS19; LW20; HW20; KN19; AKN22; Lee22]. It inspires the following construction:

Definition 1.1.1. An $\mathbb{E}_\infty$-ring B is *even* if its homotopy groups π_*B are concentrated in even degrees. For any $\mathbb{E}_\infty$-ring A, we define $\mathrm{fil}^n_{\mathrm{ev}} A$ to be the limit, over all maps $A \to B$ with B even, of $\tau_{\geq 2n}B$. Together, the $\mathrm{fil}^n_{\mathrm{ev}} A$ assemble to define a filtered $\mathbb{E}_\infty$-ring $\mathrm{fil}^\star_{\mathrm{ev}} A$.

Remark 1.1.2. We make the above definition precise by considering the category CAlg of $\mathbb{E}_\infty$-rings, the full subcategory $\mathrm{CAlg}^{\mathrm{ev}}$ of even $\mathbb{E}_\infty$-rings, and the category $\mathrm{FilCAlg}_{\kappa_2}$ of (possibly large) filtered $\mathbb{E}_\infty$-rings.[2] Then $\mathrm{fil}^\star_{\mathrm{ev}}(-)$ is the right Kan extension, along the inclusion $\mathrm{CAlg}^{\mathrm{ev}} \hookrightarrow \mathrm{CAlg}$, of the double-speed Postnikov filtration functor $\tau_{\geq 2*} : \mathrm{CAlg}^{\mathrm{ev}} \to \mathrm{FilCAlg}$.

Suppose now that R is a discrete commutative ring with bounded p-power torsion for all primes p and such that the algebraic cotangent complex $\mathrm{L}^{\mathrm{alg}}_R$ has Tor-amplitude contained in

[2] For the definition of "filtered $\mathbb{E}_\infty$-ring" and comments on set-theoretic issues, see §1.5.

$[0,1]$. For example, R might be a polynomial ring over $\mathbb{Z}$, or the quotient of a polynomial ring by a regular sequence. For each prime number p, the p-completion $R^\wedge_p$ is p-complete p-quasisyntomic, so we may speak of $\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p = \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R^\wedge_p)^\wedge_p$. By gluing together the BMS filtration at all primes p, as well as rational information, Morin [Mor20] and Bhatt–Lurie [BL22, §6.4] constructed a *global motivic filtration* $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R)$. Here, we prove that this motivic filtration is the even filtration:

Theorem 1.1.3. *Let R be a discrete commutative ring with bounded p-power torsion for all primes p and such that the algebraic cotangent complex $\mathrm{L}^{\mathrm{alg}}_R$ has Tor-amplitude contained in $[0,1]$. Then there is a canonical equivalence*

$$\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R) \simeq \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(R),$$

where $\mathrm{fil}^\star_{\mathrm{mot}}$ denotes the global motivic filtration of Morin and Bhatt–Lurie.

Remark 1.1.4. Bhatt–Lurie further define the global motivic filtration on $\mathrm{THH}(R)$ for all animated commutative rings R, by left Kan extension from the case when R is a polynomial $\mathbb{Z}$-algebra. Since polynomial $\mathbb{Z}$-algebras satisfy the conditions of Theorem 1.1.3, one may use $\mathrm{fil}^\star_{\mathrm{ev}}$ and left Kan extension to recover $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R)$ for any animated commutative ring R. By p-completion, one may then recover $\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p$ for any p-complete p-quasisyntomic R; this can be also recovered directly from a p-complete variant of the even filtration, as will be discussed further below.

In light of the above theorem and remark, it is fair to say that the even filtration provides an alternate construction of the motivic filtration on the THH of animated commutative rings. Notably, the construction is inherently global, and avoids mention of perfectoid rings and the quasisyntomic site. Even more notably, the even filtration is defined on *any* $\mathbb{E}_\infty$-ring, not only $\mathbb{E}_\infty$-rings that arise as the THH of discrete commutative rings. For example, we may take the even filtration of the sphere spectrum: it turns out that the result is the décalage of the Adams–Novikov filtration, which features heavily in Morel–Voevodsky's theory of $\mathbb{C}$-motivic stable homotopy theory [MV99]. More generally, we have the following result:

Theorem 1.1.5. *For any $\mathbb{E}_\infty$-ring A,*

$$\mathrm{fil}^\star_{\mathrm{ev}} A \simeq \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}}(A \otimes \mathrm{MU}^{\otimes \bullet+1})),$$

where the limit is taken in the category of filtered $\mathbb{E}_\infty$-rings.

The corollary below then follows by the work of Gheorghe–Isaksen–Krause–Ricka [Ghe+18] (cf. [Pst18; GWX21]):

Corollary 1.1.6. *Fix a prime number p. Then, for every $\mathbb{E}_\infty$-ring A, the p-completion of $\mathrm{fil}^\star_{\mathrm{ev}} A$ is naturally an $\mathbb{E}_\infty$-algebra object in the category of $\mathbb{C}$-motivic spectra. Moreover, if A is bounded below and $\mathrm{MU}_* A$ is even, then, after p-completion,*

$$\mathrm{fil}^\star_{\mathrm{ev}} A \simeq \nu(A),$$

where ν is the synthetic analog functor from spectra to the p-completed cellular subcategory of $\mathcal{SH}(\mathbb{C})$.

Remark 1.1.7. In §2 we extend the notion of the even filtration to modules over $\mathbb{E}_\infty$-rings and so define a functor $(A, M) \mapsto \mathrm{fil}^\star_{\mathrm{ev}/A} M$, which recovers the above definition when $M = A$. With this notation it follows from the results in §2 and [Ghe+18] that

$$\mathrm{fil}^\star_{\mathrm{ev}/\mathbb{S}} M \simeq \nu(M),$$

for any bounded below spectrum M (or, more generally, for any MU-complete spectrum M). It is then special to the scenario where $\mathrm{MU}_* A$ is even that we have the middle equivalence in the string

$$\mathrm{fil}^\star_{\mathrm{ev}} A = \mathrm{fil}^\star_{\mathrm{ev}/A} A \simeq \mathrm{fil}^\star_{\mathrm{ev}/\mathbb{S}} A \simeq \nu(A).$$

In general, filtered modules over $\mathrm{fil}^\star_{\mathrm{ev}} A$ can be viewed as a deformation of the category of A-modules, and the functor $\mathrm{fil}^\star_{\mathrm{ev}/A} : \mathrm{Mod}_A \to \mathrm{FilMod}_{\mathrm{fil}^\star_{\mathrm{ev}} A}$ associates a natural deformation to any A-module.

Returning to topological Hochschild homology, we recall that the THH of an $\mathbb{E}_\infty$-ring is naturally a cyclotomic $\mathbb{E}_\infty$-ring [NS18]. In other words, there is an S^1-action, allowing us to form $\mathrm{TC}^-(R) = \mathrm{THH}(R)^{\mathrm{hS}^1}$ and $\mathrm{TP}(R) = \mathrm{THH}(R)^{\mathrm{tS}^1}$, together with a Frobenius φ at each prime p, which Nikolaus–Scholze observed allows one to define p-completed $\mathrm{TC}(R)$ by the formula

$$\mathrm{TC}(R)^\wedge_p \simeq \mathrm{fib}\left(\mathrm{TC}^-(R)^\wedge_p \xrightarrow{\varphi^{\mathrm{hS}^1} - \mathrm{can}} \mathrm{TP}(R)^\wedge_p\right),$$

at least when R is connective [NS18]. For R a p-complete p-quasisyntomic discrete ring, Bhatt–Morrow–Scholze defined motivic filtrations on not just $\mathrm{THH}(R)^\wedge_p$, but also $\mathrm{TC}^-(R)^\wedge_p$, $\mathrm{TP}(R)^\wedge_p$, and $\mathrm{TC}(R)^\wedge_p$. In §5, we prove that variants of the even filtration can be used to recover all of these BMS filtrations. The reader should look to §5 for a complete list of our comparison results; here, we explain as an example the result for p-completed TC:

Definition 1.1.8. We say that a cyclotomic $\mathbb{E}_\infty$-ring is *even* if its underlying $\mathbb{E}_\infty$-ring is even. Given any connective cyclotomic $\mathbb{E}_\infty$-ring R, we define $\mathrm{fil}^\star_{\mathrm{ev},p,\mathrm{TC}} \mathrm{THH}(R)$ to be the limit, over all cyclotomic $\mathbb{E}_\infty$-ring maps $\mathrm{THH}(R) \to B$ for which B is even and p-complete, of

$$\mathrm{fib}\left(\tau_{\geq 2\star}(B^{\mathrm{hS}^1})^\wedge_p \xrightarrow{\varphi^{\mathrm{hS}^1} - \mathrm{can}} \tau_{\geq 2\star}(B^{\mathrm{tS}^1})^\wedge_p\right).$$

Theorem 1.1.9. *Let R be a p-complete p-quasisyntomic commutative ring. Then*

$$\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TC}(R)^\wedge_p \simeq \mathrm{fil}^\star_{\mathrm{ev},p,\mathrm{TC}} \mathrm{THH}(R).$$

Finally, the even filtration may also be used to recover constructions in classical algebra. Namely, for $k \to R$ a map of discrete commutative rings, the Hochschild–Kostant–Rosenberg (HKR) filtration on the relative Hochschild homology $\mathrm{HH}(R/k)$ is a classical analog of the BMS filtration on $\mathrm{THH}(R)$. In the quasi-lci setting, the HKR filtration also turns out to be a special case of the even filtration:

Theorem 1.1.10. *Let $k \to R$ be a map of discrete commutative rings such that the algebraic cotangent complex $\mathrm{L}^{\mathrm{alg}}_{R/k}$ has Tor-amplitude contained in $[0,1]$. Then*

$$\mathrm{fil}^\star_{\mathrm{ev}} \mathrm{HH}(R/k) \simeq \mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k).$$

Our proofs of the above comparison theorems rely on descent properties of $\mathrm{fil}^\star_{\mathrm{ev}}$, studied in §2. In particular, we identify certain maps of $\mathbb{E}_\infty$-rings $A \to B$, which we call *evenly faithfully flat* (*eff*), along which there is an identification $\mathrm{fil}^\star_{\mathrm{ev}}(A) \simeq \lim_\Delta(\mathrm{fil}^\star_{\mathrm{ev}}(B^{\otimes_A \bullet+1}))$.

§1.2. THH of chromatically quasisyntomic rings. So far, we have discussed theorems showing that the even filtration recovers known filtrations. The ubiquity of these results suggests that there may be broader, unstudied contexts in which the even filtration is a useful tool. Here, we introduce the following class of $\mathbb{E}_\infty$-rings R for which the even filtration on $\mathrm{THH}(R)$ can be controlled:

Definition 1.2.1. A connective $\mathbb{E}_\infty$-ring R is *chromatically quasisyntomic* if MU_*R is even, has bounded p-power torsion for all primes p, and (when considered as an ungraded commutative ring) has algebraic cotangent complex $\mathrm{L}^{\mathrm{alg}}_{\mathrm{MU}_*R}$ with Tor-amplitude contained in $[0,1]$.

Example 1.2.2. If R is a discrete commutative ring with bounded p-power torsion for each prime p and such that the cotangent complex $\mathrm{L}^{\mathrm{alg}}_R$ has Tor-amplitude contained in $[0,1]$, then R is chromatically quasisyntomic. This is because MU_*R is a polynomial algebra $R[b_1, b_2, \cdots]$, by complex orientation theory.

Example 1.2.3. The $\mathbb{E}_\infty$-ring spectra $\mathbb{S}$, MU, ku, ko, and tmf are all chromatically quasisyntomic. Indeed, if R is any of these ring spectra, then MU_*R is a polynomial $\mathbb{Z}$-algebra concentrated in even degrees. The Adams summand ℓ is also chromatically quasisyntomic, with $\mathrm{MU}_*\ell$ a polynomial $\mathbb{Z}_{(p)}$-algebra.

Definition 1.2.4. For R a chromatically quasisyntomic $\mathbb{E}_\infty$-ring, we define:

- $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R)$ to be $\mathrm{fil}^\star_{\mathrm{ev}}\,\mathrm{THH}(R)$;
- $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(R)$ to be the limit, over all S^1-equivariant $\mathbb{E}_\infty$-ring[3] maps $\mathrm{THH}(R) \to B$ such that the non-equivariant ring underlying B is even, of $\tau_{\geq 2\star}(B^{\mathrm{hS}^1})$;
- $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R)$ to be the limit, over all S^1-equivariant $\mathbb{E}_\infty$-ring maps $\mathrm{THH}(R) \to B$ such that the non-equivariant ring underlying B is even, of $\tau_{\geq 2\star}(B^{\mathrm{tS}^1})$;
- $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ to be $\mathrm{fil}^\star_{\mathrm{ev},p,\mathrm{TC}}\,\mathrm{THH}(R)$ (as in Definition 1.1.8).

These *motivic filtrations* lead to *motivic spectral sequences* for $\mathrm{THH}_*(R)$, $\mathrm{TC}^-_*(R)$, $\mathrm{TP}_*(R)$, and $\pi_*\left(\mathrm{TC}(R)^\wedge_p\right)$, respectively. In analogy with the discrete case, we call the associated graded of $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R)$ the *prismatic cohomology* of R, and we call the associated graded of $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ the *syntomic cohomology* of R.[4]

In this context, we prove the following results:

Theorem 1.2.5. *The motivic filtrations of Definition 1.2.4 converge: that is, for R a chromatically quasisyntomic $\mathbb{E}_\infty$-ring, the colimits of the filtered diagrams* $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R)$*,* $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(R)$*,* $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R)$*, and* $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ *are* $\mathrm{THH}(R)$*,* $\mathrm{TC}^-(R)$*,* $\mathrm{TP}(R)$*, and* $\mathrm{TC}(R)^\wedge_p$*, respectively.*

Theorem 1.2.6. *Let R be a chromatically quasisyntomic $\mathbb{E}_\infty$-ring. Then, for each prime number p, the Nikolaus–Scholze Frobenius*

$$\varphi : \mathrm{TC}^-(R)^\wedge_p \to \mathrm{TP}(R)^\wedge_p$$

refines to a natural map

$$\varphi : \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(R)^\wedge_p \to \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R)^\wedge_p.$$

The same is true of the canonical map between the same objects, and $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(R)^\wedge_p$ *can be computed as the equalizer of the filtered Frobenius and canonical maps.*

[3]Here "S^1-equivariant $\mathbb{E}_\infty$-ring" refers to a local system of $\mathbb{E}_\infty$-rings over BS^1, not a more sophisticated notion of genuine equivariant homotopy theory.

[4]Beware that, in our terminology, the syntomic cohomology of R depends only on the p-completion of R (for the implicit prime p). When R is discrete, this recovers what is called the "syntomic cohomology of $\mathrm{Spf}(R)$" in [BL22, §7.4], as opposed to the "syntomic cohomology of $\mathrm{Spec}(R)$" defined in [BL22, §8.4].

Remark 1.2.7. Our proof of Theorem 1.2.6 is not formal, and in particular crucially uses the hypothesis that R is chromatically quasisyntomic. A key ingredient is a result of Steve Wilson, which states, for each positive integer i and even ring spectrum A, that $A_*\Omega^\infty\Sigma^{2i}\mathrm{MU}$ is an even, polynomial A_*-algebra. In §3, we enhance Wilson's result by proving that, for at least some even $\mathbb{E}_\infty$-rings A, THH relative to $A \otimes \Sigma^\infty_+\Omega^\infty\Sigma^{2i}\mathrm{MU}$ carries a cyclotomic structure. In other words, there is a dashed arrow and commutative diagram of S^1-equivariant $\mathbb{E}_\infty$-rings

$$\begin{tikzcd} \mathrm{THH}(A \otimes \Sigma^\infty_+\Omega^\infty\Sigma^{2i}\mathrm{MU}) \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathrm{THH}(A \otimes \Sigma^\infty_+\Omega^\infty\Sigma^{2i}\mathrm{MU})^{\mathrm{tC}_p} \arrow[d, "\pi^{\mathrm{tC}_p}"] \\ A \otimes \Sigma^\infty_+\Omega^\infty\Sigma^{2i}\mathrm{MU} \arrow[r, dashed] & (A \otimes \Sigma^\infty_+\Omega^\infty\Sigma^{2i}\mathrm{MU})^{\mathrm{tC}_p}. \end{tikzcd}$$

It would be interesting to know if any analog of Theorem 1.2.6 holds for a broader class of $\mathbb{E}_\infty$-rings R.

Remark 1.2.8. Concretely, given a chromatically quasisyntomic $\mathbb{E}_\infty$-ring R, one can compute the motivic spectral sequence for $\mathrm{THH}_*(R)$ by finding an $\mathbb{E}_\infty$-MU-algebra map $S \to \mathrm{MU} \otimes R$ with the following properties:

(1) π_*S is a polynomial MU_*-algebra, with polynomial generators in even degrees.

(2) The map $\pi_*S \to \mathrm{MU}_*R$ is surjective.

The motivic filtration will then be given by descent along the composite

$$\mathrm{THH}(R) \to \mathrm{THH}(R) \otimes \mathrm{MU} \simeq \mathrm{THH}(R \otimes \mathrm{MU}/\mathrm{MU}) \to \mathrm{THH}(R \otimes \mathrm{MU}/S).$$

Remark 1.2.9. Suppose that R is a chromatically quasisyntomic $\mathbb{E}_\infty$-ring spectrum such that $\pi_0 R$ is also chromatically quasisyntomic. Given any filtration on $\mathrm{K}^{\mathrm{alg}}(\pi_0 R)^\wedge_p$ compatible with the motivic filtration on $\mathrm{TC}(\pi_0 R)^\wedge_p$, one can define a filtration on $\mathrm{K}^{\mathrm{alg}}(R)^\wedge_p$ by pullback [DGM13, Theorem 0.0.2]. We thus expect that, by mixing Voevodsky's filtration on the algebraic K-theory of discrete rings with our motivic filtration on TC of $\mathbb{E}_\infty$-rings, one may obtain a motivic spectral sequence for the algebraic K-theory of many chromatically quasisyntomic ring spectra.

Remark 1.2.10. Let $\mathcal{O}_K$ be the ring of integers in a local number field, and fix a choice of uniformizer π. The works [BMS19; LW20; KN19; AKN22] study $\mathrm{THH}(\mathcal{O}_K)^\wedge_p$ and $\mathrm{TC}(\mathcal{O}_K)^\wedge_p$ by (p-completed) descent along the map

$$\mathrm{THH}(\mathcal{O}_K) \to \mathrm{THH}(\mathcal{O}_K/\mathbb{S}[\pi]),$$

which is a map of cyclotomic $\mathbb{E}_\infty$-ring spectra. It follows from our work here (specifically, from the fact that $\mathrm{THH}(\Sigma^\infty_+\mathbb{N}) \to \Sigma^\infty_+\mathbb{N}$ is evenly free) that the descent filtration so obtained agrees with the even filtration on $\mathrm{TC}(\mathcal{O}_K)^\wedge_p$, and hence with the BMS filtration on $\mathrm{TC}(\mathcal{O}_K)^\wedge_p$. A comparison of the descent and BMS filtrations was independently obtained by Antieau–Krause–Nikolaus, and will appear in their announced work [AKN22].

The work of Liu–Wang referenced in Remark 1.2.10 shows that the motivic filtration allows for the practical computation of TC of number rings [LW20]. Additionally, works such as [AKN22; Sul21] and [Mat22, §10] show that the motivic filtration can be used to make precise calculations of the algebraic K-theories of commutative rings with lci singularities.

Similarly, we expect the motivic filtration to be a useful computational tool in higher chromatic contexts. In [Lee22] David Jongwon Lee completely computes $\mathrm{THH}_*(\mathrm{ku})$ by use of the motivic spectral sequence. No complete computation of $\mathrm{THH}_*(\mathrm{ku})$ previously existed in the literature, though tremendous progress was achieved by Ausoni [Aus05] and Angeltveit–Hill–Lawson [AHL10], building on work of McClure–Staffeldt [MS93] and Angeltveit–Rognes [AR05]. In the final section of the paper, we explain another computational application of the motivic spectral sequence.

§1.3. The motivic spectral sequence for $V(1)_\star\mathrm{TC}(\ell)$. In §6 we will study the connective Adams summand ℓ of $\mathrm{ku}_{(p)}$ at an arbitrary prime $p \geq 2$. In seminal work, Ausoni and Rognes computed $V(1)_*\mathrm{TC}(\ell) = \pi_*(\mathrm{TC}(\ell)/(p, v_1))$ for primes $p \geq 5$ [AR02]. We demonstrate the computability of syntomic cohomology by giving an independent proof of their result. Furthermore, our methods just as easily compute $V(1)_*\mathrm{TC}(\ell)$ at the prime $p = 3$, which had not previously been computed. Our main theorem is the following:

Theorem 1.3.1. *For any prime $p \geq 2$, the mod (p, v_1) syntomic cohomology of ℓ is a free $\mathbb{F}_p[v_2]$-module on finitely many generators. A complete list of module generators is given by:*

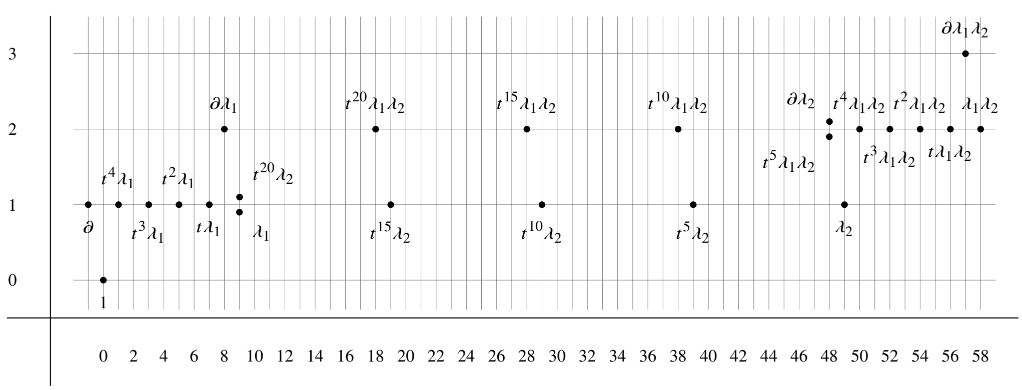
1. $\{1\}$*, in Adams weight* 0 *and degree* 0.
2. $\{\partial, t^d\lambda_1, t^{dp}\lambda_2 \mid 0 \leq d < p\}$*, in Adams weight* 1*. Here,* $|\partial| = -1$*,* $|t^d\lambda_1| = 2p - 2d - 1$*, and* $|t^{dp}\lambda_2| = 2p^2 - 2dp - 1$.
3. $\{t^d\lambda_1\lambda_2, t^{dp}\lambda_1\lambda_2, \partial\lambda_1, \partial\lambda_2 \mid 0 \leq d < p\}$*, in Adams weight* 2*. Here,* $|t^d\lambda_1\lambda_2| = 2p^2 - 2p - 2d - 2$*,* $|t^{dp}\lambda_1\lambda_2| = 2p^2 - 2p - 2dp - 2$*,* $|\partial\lambda_1| = 2p - 2$*, and* $|\partial\lambda_2| = 2p^2 - 2$.
4. $\{\partial\lambda_1\lambda_2\}$*, in Adams weight* 3 *and degree* $2p^2 + 2p - 3$.

Here we have used the following convention:

Definition 1.3.2. [Adams weight] If $M^\star$ is a graded object, we say that an element of $\pi_n M^a$ has *Adams weight* $2a - n$ and degree n.

When $p \geq 3$, so that $V(1) = \mathbb{S}/(p, v_1)$ exists as a spectrum, the mod (p, v_1) syntomic cohomology described by Theorem 1.3.1 is the E_2-page of a motivic spectral sequence converging to $V(1)_*\mathrm{TC}(\ell)$. Below, we draw a picture of the E_2-page of this spectral sequence for $p = 5$, with the horizontal axis recording degree and the vertical axis recording Adams weight. The Adams grading convention means that d_r differentials decrease degree by 1 and increase Adams weight by r.

$\mathbb{F}_5[v_2]$-module generators of the E_2-page of the motivic spectral sequence converging to $V(1)_*\mathrm{TC}(\ell)$, $p = 5$



The reader may notice the similarity between the picture of the motivic spectral sequence for $V(1)_*\mathrm{TC}(\ell)$, which we rigorously define here, and Rognes' conjectured picture for a conjectured motivic spectral sequence converging to $V(1)_*\mathrm{K}^{\mathrm{alg}}(\ell_p^\wedge)$ [Rog14, Example 5.2]. The rotational symmetry present in the above picture is indicative of Rognes' conjectured Tate–Poitou duality, which we will explore in future work.

The E_2-page of the motivic spectral sequence for $V(1)_*\mathrm{TC}(\ell)$ is concentrated on four horizontal lines, with classes of odd degree on the 1- and 3-lines and classes of even degree on the 0- and 2-lines. It follows by parity considerations that the only possible differentials are d_3's from the 0-line to the 3-line. On the 0-line is a copy of $\mathbb{F}_p[v_2]$, where the degree of v_2 is $2p^2-2$. A d_3 differential off of the 0-line would therefore have target of degree 1 less than a multiple of $2p^2-2$. However, the 3-line is concentrated in degrees $2p-1$ more than a multiple of $2p^2-2$. No differentials are possible, and we conclude the following corollary:

Corollary 1.3.3. *Let $p \geq 3$, so that $V(1) = \mathbb{S}/(p, v_1)$ exists as a spectrum. Then the motivic spectral sequence for $V(1)_*\mathrm{TC}(\ell)$ degenerates at the E_2-page.*

At primes $p \geq 5$, v_2 is a self map of $V(1)$ and the motivic spectral sequence for $V(1)_*\mathrm{TC}(\ell)$ is one of $\mathbb{F}_p[v_2]$-modules. We note that there are no possible $\mathbb{F}_p[v_2]$-module extension problems, and conclude that $V(1)_*\mathrm{TC}(\ell)$ is a free $\mathbb{F}_p[v_2]$-module on generators with the same names as those in Theorem 1.3.1. At the prime $p = 3$, v_2^9 is a self map of $V(1)$ [BP04], and we may similarly conclude that $V(1)_*\mathrm{TC}(\ell)$ is a free $\mathbb{F}_3[v_2^9]$-module.

Remark 1.3.4. It would be excellent to see the results of [AK+22] or [AR12] similarly recovered via motivic spectral sequences. These works compute the topological cyclic homologies of non-commutative ring spectra, which presents an obvious complication, but see Example 4.2.4.

Finally, we explain in §6.6 how Theorem 1.3.1 allows us to deduce the following two qualitative results:

Theorem 1.3.5. *For any prime $p \geq 2$ and type 3 p-local finite complex F, $F_*\mathrm{TC}(\ell)$ is finite. In particular,*

$$\mathrm{TC}(\ell)_{(p)} \to L_2^f\,\mathrm{TC}(\ell)_{(p)}$$

is a π_*-iso for $* \gg 0$.

Theorem 1.3.6. *The telescope conjecture is true of $\mathrm{TC}(\ell)$. In other words, the natural map*

$$L_2^f\,\mathrm{TC}(\ell) \to L_2\mathrm{TC}(\ell)$$

is an equivalence.

Theorem 1.3.5 was previously proved in [HW20], while Theorem 1.3.6 is new to this paper. Of course, a natural next question is to compute the syntomic cohomology of ℓ without modding out by p or v_1. The authors and Mike Hopkins will have more to say about the prismatic and syntomic cohomologies of ℓ, ku, ko, and MU in future work. We note that the prismatic cohomologies of $\mathbb{E}_\infty$-ring spectra may be of interest even to those studying prismatic cohomology of discrete rings. Specifically, the sequence of $\mathbb{E}_\infty$-ring maps $\mathbb{S} \to \mathrm{MU} \to \mathrm{ku} \to \mathbb{Z}$ suggests a natural factorization of known connections between WCart and the moduli stack of formal groups.

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§1.5. Conventions.

(1) Spc denotes the (∞-)category of spaces; Spt denotes the category of spectra; CAlg denotes the category of $\mathbb{E}_\infty$-rings; Mod denotes the category of pairs (A, M) where A is an $\mathbb{E}_\infty$-ring and M is an A-module.

(2) For a prime number p, CycSpt_p denotes the category of p-typical cyclotomic spectra (following the same conventions as in [HW20, Definition 3.2.1], in particular including the hypothesis of p-completeness); $\mathrm{CycCAlg}_p$ denotes the category of p-typical cyclotomic $\mathbb{E}_\infty$-rings, i.e. commutative algebra objects in CycSpt_p; and CycMod_p denotes the category of pairs (A, M) where A is a p-typical cyclotomic $\mathbb{E}_\infty$-ring and M is an A-module in CycSpt_p.

(3) FilSpt denotes the category of filtered spectra, i.e. the category of functors from the poset $(\mathbb{Z}, \geq)$ to Spt; we will generally denote a filtered spectrum by a symbol of the form $\mathrm{fil}^\star_? X$, referring to a diagram of spectra

$$\cdots \to \mathrm{fil}^2_? X \to \mathrm{fil}^1_? X \to \mathrm{fil}^0_? X \to \mathrm{fil}^{-1}_? X \to \mathrm{fil}^{-2}_? X \to \cdots.$$

Given a filtered spectrum $\mathrm{fil}^\star_? X$, we refer to $\mathrm{colim}_{n\to\infty} \mathrm{fil}^{-n}_? X$ as its *underlying object*, and we say $\mathrm{fil}^\star_? X$ is *complete* if $\lim_{n\to\infty} \mathrm{fil}^n_? X \simeq 0$. The category BiFilSpt of bifiltered spectra is defined similarly, using instead the product poset $(\mathbb{Z}, \geq) \times (\mathbb{Z}, \geq)$; we will generally denote a bifiltered spectrum by a symbol of the form $\mathrm{fil}^{\blacksquare}_{??} \mathrm{fil}^\star_? X$, where again the $\blacksquare$ and $\star$ stand for the two integer indices.

(4) FilCAlg denotes the category of filtered $\mathbb{E}_\infty$-rings, i.e. commutative algebra objects in FilSpt with respect to the Day convolution symmetric monoidal structure (determined by addition on $\mathbb{Z}$). For $\mathrm{fil}^\star_? A$ a filtered $\mathbb{E}_\infty$-ring, $\mathrm{FilMod}_{\mathrm{fil}^\star_? A}$ denotes the category of modules over $\mathrm{fil}^\star_? A$ in FilSpt.

(5) GrSpt denotes the category of graded spectra, i.e. the category of functors from the discrete set $\mathbb{Z}$ to Spt. A filtered spectrum $\mathrm{fil}^\star_? X$ has an *associated graded* object $\mathrm{gr}^\star_? X = \{\mathrm{gr}^n_? X\}_{n\in\mathbb{Z}}$, given by the formula

$$\mathrm{gr}^n_? X \simeq \mathrm{cofib}(\mathrm{fil}^{n+1}_? X \to \mathrm{fil}^n_? X).$$

We recall that a map of complete filtered spectra is an equivalence if and only if the induced map of associated graded objects is.

(6) Let κ_1, κ_2 denote the two smallest strongly inaccessible cardinals.[5] A mathematical object is *small* (resp. *large*) if it resides in the universe of sets of rank $< \kappa_1$ (resp. $< \kappa_2$). By default, all objects besides categories are assumed to be small. For example, in the notation set above, Spc is the large ∞-category of small spaces. When we allow large objects into the discussion, we will use a κ_2 to denote the corresponding category; e.g. Spc_{κ_2} denotes the ∞-category of possibly large spaces.

[5]We assume that these cardinals exist. This could be avoided without affecting the main results of the paper by making minor modifications to the constructions that depend on this assumption.

§2. The even filtration

In this section, we define the even filtration and its many variants. We then formulate and prove flat descent properties for each of these variants. The arguments here are not complicated, but must be carefully repeated in slightly different contexts (p-complete, S^1-equivariant, cyclotomic, etc.).

§2.1. Defining the filtrations. The most basic version of the even filtration construction was formulated in Definition 1.1.1. It will be convenient for some of our purposes to set things up here in slightly greater generality, namely for modules over $\mathbb{E}_\infty$-rings rather than simply $\mathbb{E}_\infty$-rings.

Notation 2.1.1. Let $\mathrm{CAlg}^{\mathrm{ev}}$ denote the full subcategory of CAlg spanned by the even $\mathbb{E}_\infty$-rings. Let $\mathrm{Mod}^{\mathrm{ev}}$ denote the full subcategory of Mod spanned by pairs (A, M) where A is even (but M need not be). We denote by

$$U_{\mathrm{Alg}} : \mathrm{Mod} \to \mathrm{CAlg} \qquad\qquad (A, M) \mapsto A$$
$$U_{\mathrm{Mod}} : \mathrm{Mod} \to \mathrm{Spt} \qquad\qquad (A, M) \mapsto M$$

the two forgetful functors.

Construction 2.1.2. Recall that there is a functor $\tau_{\geq 2\star} : \mathrm{Spt} \to \mathrm{FilSpt}$ sending a spectrum X to its double-speed Postnikov filtration

$$\cdots \to \tau_{\geq 4}(X) \to \tau_{\geq 2}(X) \to \tau_{\geq 0}(X) \to \tau_{\geq -2}(X) \to \tau_{\geq -4}(X) \to \cdots.$$

We denote by $(A, M) \mapsto \mathrm{fil}^\star_{\mathrm{ev}/A} M$ the functor $\mathrm{Mod} \to \mathrm{FilSpt}_{\kappa_2}$ given by the right Kan extension of the composition

$$\mathrm{Mod}^{\mathrm{ev}} \xrightarrow{U_{\mathrm{Mod}}} \mathrm{Spt} \xrightarrow{\tau_{\geq 2\star}} \mathrm{FilSpt}_{\kappa_2}$$

along the inclusion $\mathrm{Mod}^{\mathrm{ev}} \subseteq \mathrm{Mod}$. We refer to this construction as the *even filtration*.

Remark 2.1.3. If $A \to B$ is a map of $\mathbb{E}_\infty$-rings and M is an A-module, then $(B, M \otimes_A B)$ is initial among maps $(A, M) \to (B, N)$ in Mod lying over $A \to B$. It follows that the even filtration of Construction 2.1.2 is given by the following limit expression:

$$\mathrm{fil}^\star_{\mathrm{ev}/A} M \simeq \lim_{A \to B,\, B \in \mathrm{CAlg}^{\mathrm{ev}}} \tau_{\geq 2\star}(M \otimes_A B).$$

Remark 2.1.4. The functor $\tau_{\geq 2\star} : \mathrm{Spt} \to \mathrm{FilSpt}$ has a canonical lax symmetric monoidal structure, from which the even filtration functor $\mathrm{fil}^\star_{\mathrm{ev}/(-)}$ obtains the same. It follows that, for A an $\mathbb{E}_\infty$-ring and A' an $\mathbb{E}_\infty$-A-algebra, $\mathrm{fil}^\star_{\mathrm{ev}/A} A'$ is canonically a filtered $\mathbb{E}_\infty$-ring; in the case $A' = A$, we abbreviate by denoting the filtered $\mathbb{E}_\infty$-ring $\mathrm{fil}^\star_{\mathrm{ev}/A} A$ by $\mathrm{fil}^\star_{\mathrm{ev}} A$. The construction $A \mapsto \mathrm{fil}^\star_{\mathrm{ev}} A$ gives a functor

$$\mathrm{fil}^\star_{\mathrm{ev}} : \mathrm{CAlg} \to \mathrm{FilCAlg}_{\kappa_2},$$

and for A an $\mathbb{E}_\infty$-ring, $\mathrm{fil}^\star_{\mathrm{ev}/A}$ lifts to a functor

$$\mathrm{fil}^\star_{\mathrm{ev}/A} : \mathrm{Mod}_A \to \mathrm{FilMod}_{\mathrm{fil}^\star_{\mathrm{ev}} A}.$$

Remark 2.1.5. In Construction 2.1.2, we use the category $\mathrm{FilSpt}_{\kappa_2}$ with possibly large objects to ensure that the right Kan extension exists (cf. §1.5). However, our attention will be on $\mathbb{E}_\infty$-rings A for which we can prove that $\mathrm{fil}^\star_{\mathrm{ev}/A} M$ lies in the subcategory $\mathrm{FilSpt} \subset \mathrm{FilSpt}_{\kappa_2}$ for any A-module M. This is in particular true of any even A.

Remark 2.1.6. For any spectrum X, the double speed Postnikov filtration $\tau_{\geq 2\star}(X)$ is complete (that is, $\lim_{n\to\infty} \tau_{\geq 2n}(X) \simeq 0$). The collection of complete filtered spectra is closed under limits, so for any $\mathbb{E}_\infty$-ring A and A-module M, the even filtration $\mathrm{fil}^\star_{\mathrm{ev}/A} M$ is complete.

Variant 2.1.7. Fix a prime number p. Let Mod_p denote the full subcategory of Mod spanned by the pairs (A, M) where A and M are p-complete and let $\mathrm{Mod}_p^{\mathrm{ev}} := \mathrm{Mod}_p \cap \mathrm{Mod}^{\mathrm{ev}}$. Define

$$\mathrm{fil}^\star_{\mathrm{ev}/(-),p} : \mathrm{Mod}_p \to \mathrm{FilSpt}_{\kappa_2}$$

to be the right Kan extension of

$$\tau_{\geq 2*}(U_{\mathrm{Mod}}) : \mathrm{Mod}_p^{\mathrm{ev}} \to \mathrm{FilSpt}$$

along the inclusion $\mathrm{Mod}_p^{\mathrm{ev}} \subseteq \mathrm{Mod}_p$. We refer to this as the *p-complete even filtration*, and again set $\mathrm{fil}^\star_{\mathrm{ev},p} A := \mathrm{fil}^\star_{\mathrm{ev}/A,p} A$ for A a p-complete $\mathbb{E}_\infty$-ring.

Variant 2.1.8. Say that an $\mathbb{E}_\infty$-ring with S^1-action is *even* if its underlying $\mathbb{E}_\infty$-ring is even. Then $(\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1}$ (resp. $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1}$) is the full subcategory of $\mathrm{Mod}^{\mathrm{BS}^1}$ (resp. $\mathrm{Mod}_p^{\mathrm{BS}^1}$) spanned by pairs (A, M) where A is even. Define

$$\begin{aligned}
&\mathrm{fil}^\star_{\mathrm{ev}/(-),\mathrm{hS}^1} : \mathrm{Mod}^{\mathrm{BS}^1} \to \mathrm{FilSpt}_{\kappa_2}, && \mathrm{fil}^\star_{\mathrm{ev}/(-),p,\mathrm{hS}^1} : \mathrm{Mod}_p^{\mathrm{BS}^1} \to \mathrm{FilSpt}_{\kappa_2},\\
&\mathrm{fil}^\star_{\mathrm{ev}/(-),\mathrm{tS}^1} : \mathrm{Mod}^{\mathrm{BS}^1} \to \mathrm{FilSpt}_{\kappa_2}, && \mathrm{fil}^\star_{\mathrm{ev}/(-),p,\mathrm{tS}^1} : \mathrm{Mod}_p^{\mathrm{BS}^1} \to \mathrm{FilSpt}_{\kappa_2},\\
&\mathrm{fil}^\blacksquare_+ \mathrm{fil}^\star_{\mathrm{ev}/(-),\mathrm{hS}^1} : \mathrm{Mod}^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}_{\kappa_2}, && \mathrm{fil}^\blacksquare_+ \mathrm{fil}^\star_{\mathrm{ev}/(-),p,\mathrm{hS}^1} : \mathrm{Mod}_p^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}_{\kappa_2},\\
&\mathrm{fil}^\blacksquare_+ \mathrm{fil}^\star_{\mathrm{ev}/(-),\mathrm{tS}^1} : \mathrm{Mod}^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}_{\kappa_2}, && \mathrm{fil}^\blacksquare_+ \mathrm{fil}^\star_{\mathrm{ev}/(-),p,\mathrm{tS}^1} : \mathrm{Mod}_p^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}_{\kappa_2},
\end{aligned}$$

to be the right Kan extensions of

$$\begin{aligned}
&\tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{hS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{FilSpt}, && \tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{hS}^1}) : (\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{FilSpt},\\
&\tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{tS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{FilSpt}, && \tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{tS}^1})^\wedge_p : (\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{FilSpt},\\
&\tau_{\geq 2*}(\tau_{\geq\blacksquare}(U_{\mathrm{Mod}})^{\mathrm{hS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}, && \tau_{\geq 2*}(\tau_{\geq\blacksquare}(U_{\mathrm{Mod}})^{\mathrm{hS}^1}) : (\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{BiFilSpt},\\
&\tau_{\geq 2*}(\tau_{\geq\blacksquare}(U_{\mathrm{Mod}})^{\mathrm{tS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}, && \tau_{\geq 2*}(\tau_{\geq\blacksquare}(U_{\mathrm{Mod}})^{\mathrm{tS}^1})^\wedge_p : (\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{BiFilSpt},
\end{aligned}$$

respectively, along the inclusions $(\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \subseteq \mathrm{Mod}^{\mathrm{BS}^1}$ and $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1} \subseteq \mathrm{Mod}_p^{\mathrm{BS}^1}$. Again, in the case $M = A$ we omit the subcript indicating the ring.

Variant 2.1.9. Say that a p-typical cyclotomic $\mathbb{E}_\infty$-ring is *even* if its underlying $\mathbb{E}_\infty$-ring is even. Let $\mathrm{CycMod}_p^{\mathrm{ev}}$ denote the full subcategory of CycMod_p spanned by the pairs (A, M) where A is even. We define

$$\mathrm{fil}^\star_{\mathrm{ev}/(-),p,\mathrm{TC}}(-) : \mathrm{CycMod}_p \to \mathrm{FilSpt}_{\kappa_2},$$

to be the right Kan extension of

$$\mathrm{fib}(\varphi - \mathrm{can} : \tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{hS}^1})^\wedge_p \to \tau_{\geq 2*}(U_{\mathrm{Mod}}^{\mathrm{tS}^1})^\wedge_p) : \mathrm{CycMod}_p^{\mathrm{ev}} \to \mathrm{FilSpt},$$

along the inclusion $\mathrm{CycMod}_p^{\mathrm{ev}} \subseteq \mathrm{CycMod}_p$. Yet again, in the case $M = A$ we omit the subcript indicating the ring.

§2.2. Descent properties of the filtrations. The key to computing these even filtrations in our cases of interest is a simple flat descent property, which we formulate and prove in this subsection.

Definition 2.2.1. Following [BMS19; BL22], for a fixed prime number p, we say that a map $A \to B$ of (discrete) commutative rings is *p-completely flat* if $(A/p) \otimes_A^{\mathrm{L}} B$ is a flat A/p-module concentrated in homological degree 0. The map is furthermore said to be *p-completely faithfully flat* if $(A/p) \otimes_A^{\mathrm{L}} B$ is a faithfully flat A/p-module.

We introduce one further definition here: we say that a map $A \to B$ of commutative rings is *discretely p-completely faithfully flat* if, for every commutative ring C and map $A \to C$, the p-completed pushout $(B \otimes_A^{\mathrm{L}} C)_p^\wedge$ is discrete and p-completely faithfully flat over C.

Remark 2.2.2. It follows from [Lur18, Proposition 2.7.3.2(c)] that a map $A \to B$ of discrete commutative rings is p-completely flat if and only if the induced map $A \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Z}/p \to B \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Z}/p$ is faithfully flat in the sense of [Lur18, Definition D.4.4.1]. The reader may thus replace the underived construction A/p with the derived construction $A \otimes_{\mathbb{Z}}^{\mathrm{L}} \mathbb{Z}/p$ in the above definition.

Example 2.2.3. Let $A \to B$ be a map of p-complete commutative rings. Suppose that there is an A-module M which is free on a nonempty set of generators and an isomorphism of A-modules $B \simeq M_p^\wedge$. Then B is discretely p-completely faithfully flat over A.

We now generalize and define the relevant notions of flatness in the setting of even $\mathbb{E}_\infty$-rings.

Definition 2.2.4. We say that a map of even $\mathbb{E}_\infty$-rings $f : A \to B$ is *faithfully flat* (resp. *p-completely faithfully flat*) if the induced map of (ungraded) commutative rings $\pi_*(f) : \pi_*(A) \to \pi_*(B)$ is faithfully flat (resp. p-completely faithfully flat).

We say that a map of even $\mathbb{E}_\infty$-rings $f : A \to B$ is *discretely p-completely faithfully flat* if the induced map of (ungraded) commutative rings $\pi_*(f) : \pi_*(A) \to \pi_*(B)$ is discretely p-completely faithfully flat.

Warning 2.2.5. The notion of faithful flatness in Definition 2.2.4 is distinct from that in [Lur18, Definition D.4.4.1]. It is the former that is used throughout this paper (except where explicitly stated otherwise).

Proposition 2.2.6. *Let $A \to B$ be a discretely p-completely faithfully flat map of even $\mathbb{E}_\infty$-rings. Then for any even $\mathbb{E}_\infty$-ring C and map $A \to C$, the p-completed pushout $(B \otimes_A C)_p^\wedge$ is even, and has homotopy groups given by $(\pi_*(B) \otimes_{\pi_*(A)}^{\mathrm{L}} \pi_*(C))_p^\wedge$.*

Proof. The filtered object $(\tau_{\geq \star}(B) \otimes_{\tau_{\geq \star}(A)} \tau_{\geq \star}(C))_p^\wedge$ is complete, with underlying object $(B \otimes_A C)_p^\wedge$ and associated graded object $\Sigma^*(\pi_*(B) \otimes_{\pi_*(A)}^{\mathrm{L}} \pi_*(C))_p^\wedge$. The spectral sequence associated to this filtered object collapses to give the desired result. $\square$

As usual, the above notions of flatness give rise to Grothendieck topologies.

Definition 2.2.7. We say that a sieve on $(A, M) \in (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{op}}$ is a *flat covering sieve* (resp. *p-completely flat covering sieve*) if it contains a finite collection of maps $\{(A, M) \to (B_i, M_i)\}_{1 \leq i \leq n}$ such that the map $A \to \prod_i B_i$ is faithfully flat (resp. discretely p-completely faithfully flat) and each of the morphisms $(A, M) \to (B_i, M_i)$ induce an equivalence $M \otimes_A B_i \simeq M_i$ (resp. $(M \otimes_A B_i)_p^\wedge \simeq M_i$).

Proposition 2.2.8. *The flat covering families of Definition 2.2.7 define a Grothendieck topology on $(\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{op}}$ and the p-completely flat covering families define a Grothendieck topology on $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{op}}$.*

For a category $\mathcal{C}$ admitting small limits, a functor $F : \mathrm{Mod}^{\mathrm{ev}} \to \mathcal{C}$ is a sheaf for the flat topology if and only if the following conditions are satisfied:

(1) *F preserves finite products.*

(2) *For every $A \to B$ which is faithfully flat, the map*

$$F(A, M) \to \lim_{\Delta} F(B^{\otimes_A \bullet+1}, M \otimes_A B^{\otimes_A \bullet+1})$$

is an equivalence.

The analogous claim holds for the discretely p-completely flat topology.

Proof. The proofs of [Lur18, A.3.2.1, A.3.3.1] carry over verbatim (the only pushouts required to exist are those along faithfully flat or discretely p-completely faithfully flat maps, which do indeed exist.) $\square$

Definition 2.2.9. We refer to the Grothendieck topologies of Proposition 2.2.8 as the *flat topology* on $\mathrm{Mod}^{\mathrm{ev}}$ and the *p-completely flat topology* on $\mathrm{Mod}_p^{\mathrm{ev}}$.

Since pushouts in $(\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1}$ and $\mathrm{CycMod}_p^{\mathrm{ev}}$ are computed in $\mathrm{Mod}^{\mathrm{ev}}$ and $\mathrm{Mod}_p^{\mathrm{ev}}$, respectively, the above induce topologies on $(\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1}$ and $\mathrm{CycMod}_p^{\mathrm{ev}}$, which we call by the same names.

We now turn to studying the descent properties of our various filtrations. We will need the following lemma, which is well-known.

Lemma 2.2.10. *Let A be an even $\mathbb{E}_\infty$-ring with S^1-action. Then, for any S^1-equivariant A-module M, the canonical maps*

$$M^{\mathrm{hS}^1} \otimes_{A^{\mathrm{hS}^1}} A \to M,$$

$$M^{\mathrm{hS}^1} \otimes_{A^{\mathrm{hS}^1}} A^{\mathrm{tS}^1} \to M^{\mathrm{tS}^1}$$

are equivalences. Moreover, M^{hS^1} is A-complete as an A^{hS^1}-module.

Proof. We argue as in [NS18, Lemma IV.4.12]. We may replace A by $\tau_{\geq 0}A$ and reduce to the case when A is connective. Since A is even the homotopy fixed point spectral sequence collapses and there is a non-canonical isomorphism $\pi_*(A^{\mathrm{hS}^1}) \cong \pi_*(A)[\![t]\!]$. It follows that $A \simeq A^{\mathrm{hS}^1}/t$ is a perfect A^{hS^1}-module and that $(-) \otimes_{A^{\mathrm{hS}^1}} A$ commutes with all limits and colimits. Using the equivalences

$$M^{\mathrm{hS}^1} \simeq \mathrm{colim}(\tau_{\geq n} M)^{\mathrm{hS}^1}$$

$$M^{\mathrm{hS}^1} \simeq \lim(\tau_{\leq m} M)^{\mathrm{hS}^1}$$

we may reduce the first claim to the case where M is discrete and the claim follows by direct calculation.

To prove the second claim it suffices by [NS18, Theorem I.4.1(iii)] to prove that the fiber of $M^{\mathrm{hS}^1} \to M^{\mathrm{hS}^1}[t^{-1}]$ commutes with all colimits in the variable M. The fiber is given, up to suspension, by the colimit of the functors $M \mapsto M^{\mathrm{hS}^1}/t^n$. When $n = 1$ this functor commutes with colimits by the first claim, and by induction we see that each functor commutes with colimits. The result follows.

Finally we show that M^{hS^1} is A-complete or, equivalently, is t-complete. Since $M^{\mathrm{hS}^1} \simeq \lim(\tau_{\leq n} M)^{\mathrm{hS}^1}$ we may reduce to the case where M is bounded above. Then the terms $\Sigma^{-2n} M^{\mathrm{hS}^1}$ become increasingly coconnective and hence

$$\lim(\cdots \Sigma^{-2n} M^{\mathrm{hS}^1} \xrightarrow{t} \Sigma^{-2n+2} M^{\mathrm{hS}^1} \to \cdots \to M^{\mathrm{hS}^1}) \simeq 0. \qquad \square$$

Lemma 2.2.11. (1) *The functor $\pi_*(U_{\mathrm{Mod}}) : \mathrm{Mod}^{\mathrm{ev}} \to \mathrm{GrSpt}$ is a sheaf for the flat topology and restricts to a sheaf for the p-completely flat topology on $\mathrm{Mod}_p^{\mathrm{ev}}$.*

(2) *The functors* $\pi_*(U_{\mathrm{Mod}}^{\mathrm{hS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{hS}^1} \to \mathrm{GrSpt}$ *and* $\pi_*(U_{\mathrm{Mod}}^{\mathrm{tS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{tS}^1} \to \mathrm{GrSpt}$ *are sheaves for the flat topology and restrict to sheaves for the* p*-completely flat topology on* $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1}$.

Proof. We will prove the p-complete statements; the integral statements can be addressed similarly.

(1) If $A \to B$ is discretely p-completely faithfully flat, then we need to prove that

$$\pi_* M \to \lim_{\Delta} (\pi_*(M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p)$$

is an equivalence. Equivalently, using Proposition 2.2.6, that

$$\pi_* M \to \lim_{\Delta} (\pi_*(M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p) = \lim_{\Delta} (\pi_*(M) \otimes^{\mathrm{L}}_{\pi_*(A)} (\pi_*(B))^{\otimes^{\mathrm{L}}_{\pi_*(A)} \bullet+1}))^{\wedge}_p$$

is an equivalence. Both sides being p-complete, it suffices to show this map is an equivalence after (derived) base change along $\mathbb{Z} \to \mathbb{Z}/p$. But

$$\pi_*(A) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p \to \pi_*(B) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p$$

is faithfully flat in the sense of [Lur18, Definition D.4.4.1] (after forgetting the gradings). The claim now follows from [Lur18, Theorem D.6.3.1].

(2) If $A \to B$ is discretely p-completely faithfully flat, then we need to prove that

$$\pi_*(M^{\mathrm{hS}^1}) \to \lim_{\Delta} (\pi_*((M \otimes_A B^{\otimes_A \bullet+1})^{\mathrm{hS}^1})^{\wedge}_p)$$

is an equivalence. Since A is even, it is complex-orientable and there is a non-canonical isomorphism $\pi_*(A^{\mathrm{hS}^1}) \cong \pi_*(A)[\![t]\!]$. Arguing as in Proposition 2.2.6, we have that

$$\pi_*((M \otimes_A B^{\otimes_A \bullet+1})^{\mathrm{hS}^1})^{\wedge}_p \simeq (\pi_*(M^{\mathrm{hS}^1}) \otimes^{\mathrm{L}}_{\pi_*(A^{\mathrm{hS}^1})} \pi_*(B^{\mathrm{hS}^1})^{\otimes^{\mathrm{L}}_{\pi_*(A^{\mathrm{hS}^1})} \bullet+1})^{\wedge}_{p,t}.$$

As M^{hS^1} is both (derived) p and t-complete by the previous lemma, it suffices to prove the claim after taking the (derived) base-change along $\mathbb{Z}[t] \to \mathbb{Z}/p$. But for any even ring R we have

$$\pi_*(R^{\mathrm{hS}^1}) \otimes^{\mathrm{L}}_{\mathbb{Z}[t]} \mathbb{Z}/p \simeq \pi_*(R) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p.$$

Again, since $\pi_*(A) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p \to \pi_*(B) \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathbb{Z}/p$ is faithfully flat in the sense of [Lur18, Definition D.4.4.1] the result follows from [Lur18, Theorem D.6.3.1]. Since

$$\pi_*(M^{\mathrm{hS}^1})[t^{-1}] \simeq \pi_*(M^{\mathrm{tS}^1}),$$

by the previous lemma, we deduce the second claim in (2). □

Proposition 2.2.12. (1) *The functor* $\tau_{\geq 2\star}(U_{\mathrm{Mod}}) : \mathrm{Mod}^{\mathrm{ev}} \to \mathrm{FilSpt}$ *is a sheaf for the flat topology and restricts to a sheaf for the* p*-completely flat topology on* $\mathrm{Mod}_p^{\mathrm{ev}}$.

(2) *The functors* $\tau_{\geq 2\star}(U_{\mathrm{Mod}}^{\mathrm{hS}^1}), \tau_{\geq 2\star}(U_{\mathrm{Mod}}^{\mathrm{tS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{FilSpt}$ *are sheaves for the flat topology and restrict to sheaves for the* p*-completely flat topology on* $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1}$.

(3) *The functors* $\tau_{\geq 2\star}(\tau_{\geq \blacksquare}(U_{\mathrm{Mod}})^{\mathrm{hS}^1}), \tau_{\geq 2\star}(\tau_{\geq \blacksquare}(U_{\mathrm{Mod}})^{\mathrm{tS}^1}) : (\mathrm{Mod}^{\mathrm{ev}})^{\mathrm{BS}^1} \to \mathrm{BiFilSpt}$ *are sheaves for the flat topology when restricted to the full subcategory of* (A, M) *where* A *is connective, and restrict to sheaves for the* p*-completely flat topology on the analogous subcategory of* $(\mathrm{Mod}_p^{\mathrm{ev}})^{\mathrm{BS}^1}$.

(4) *The functor*

$$\mathrm{fib}(\mathrm{id} - \varphi : \tau_{\geq 2\star}(U^{\mathrm{hS}^1}_{\mathrm{Mod}}) \to \tau_{\geq 2\star}(U^{\mathrm{tS}^1}_{\mathrm{Mod}}))^{\wedge}_p : \mathrm{CycMod}^{\mathrm{ev}}_p \to \mathrm{FilSpt}$$

is a sheaf for the p-completely flat topology.

Proof. Claim (3) follows from claim (2) by replacing a module with its $\blacksquare$-connective cover (which remains a module since we assumed the ring was connective in claim (3)). Claim (4) follows from claim (2). For claims (1) and (2) it suffices to prove the analogous statement for $\tau_{\geq\star}$ replacing $\tau_{\geq 2\star}$ since the functor doubling the speed of a filtration preserves limits. As these filtrations are complete, it suffices to check the claim after passage to associated graded. But then the claims follow by the preceding lemma. □

The upshot of Proposition 2.2.12 is a collection of descent statements for the even filtration and its variants, which we now enumerate to end this subsection.

Definition 2.2.13. A map of $\mathbb{E}_\infty$-rings $A \to B$ is *eff* (evenly faithfully flat) if for any even $\mathbb{E}_\infty$-ring C and map of $\mathbb{E}_\infty$-rings $A \to C$, the pushout $B \otimes_A C$ is even and faithfully flat over C.

A map of $\mathbb{E}_\infty$-rings $A \to B$ is *discretely p-completely eff* if for any even $\mathbb{E}_\infty$-ring C and map of $\mathbb{E}_\infty$-rings $A \to C$, the p-completed pushout $(B \otimes_A C)^{\wedge}_p$ is even and discretely p-completely faithfully flat over C.

A map of $\mathbb{E}_\infty$-rings with S^1-action or a map of p-typical cyclotomic $\mathbb{E}_\infty$-rings is said to be eff (resp. discretely p-completely eff) if the underlying map of $\mathbb{E}_\infty$-rings is.

Corollary 2.2.14. (1) *For $A \to B$ an eff map of $\mathbb{E}_\infty$-rings and M an A-module, the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}/A}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1}}(M \otimes_A B^{\otimes_A \bullet+1}))$$

is an equivalence. For $A \to B$ a discretely p-completely eff map of p-complete $\mathbb{E}_\infty$-rings and M a p-complete module, the canonical map

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},p}((M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p))$$

is an equivalence.

(2) *For $A \to B$ an eff map of $\mathbb{E}_\infty$-rings with S^1-action and M an S^1-equivariant module, the canonical maps*

$$\mathrm{fil}^\star_{\mathrm{ev}/A,\mathrm{hS}^1}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},\mathrm{hS}^1}(M \otimes_A B^{\otimes_A \bullet+1})),$$
$$\mathrm{fil}^\star_{\mathrm{ev}/A,\mathrm{tS}^1}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},\mathrm{tS}^1}(M \otimes_A B^{\otimes_A \bullet+1}))$$

are equivalences. For $A \to B$ a discretely p-completely eff map of p-complete $\mathbb{E}_\infty$-rings with S^1-action and M an S^1-equivariant module, the canonical maps

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{hS}^1}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},p,\mathrm{hS}^1}((M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p)),$$
$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{tS}^1}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},p,\mathrm{tS}^1}((M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p)),$$

are equivalences.

(3) *For $A \to B$ a discretely p-completely eff map of p-typical cyclotomic $\mathbb{E}_\infty$-rings and M a cyclotomic A-module, the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{TC}}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/B^{\otimes_A \bullet+1},p,\mathrm{TC}}((M \otimes_A B^{\otimes_A \bullet+1})^{\wedge}_p))$$

is an equivalence.

Corollary 2.2.15. *Let $A \to B$ be a discretely p-completely eff map of p-typical cyclotomic $\mathbb{E}_\infty$-rings where B is even. Let M be a cyclotomic A-module. Then the cyclotomic Frobenius and canonical maps $\varphi, \mathrm{can} : M^{\mathrm{hS}^1} \to (M^{\mathrm{tS}^1})^\wedge_p$ refine to maps $\varphi, \mathrm{can} : \mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{hS}^1} M \to \mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{tS}^1} M$, and there is a canonical equivalence*

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{TC}}(M) \simeq \mathrm{fib}(\varphi - \mathrm{can} : \mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{hS}^1} M \to \mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{tS}^1} M).$$

Proof. Let us temporarily denote by G and H the right Kan extension of the functors $(A, M) \mapsto \tau_{\geq 2*}(M^{\mathrm{hS}^1})$ and $(A, M) \mapsto \tau_{\geq 2*}((M^{\mathrm{tS}^1})^\wedge_p)$, respectively, along the inclusion $\mathrm{CycMod}^{\mathrm{ev}}_p \subseteq \mathrm{CycMod}_p$. Then, tautologically, both $\tau_{\geq 2*}(\varphi)$ and $\tau_{\geq 2*}(\mathrm{can})$ extend to maps of filtered objects and we have a fiber sequence

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{TC}}(M) \simeq \mathrm{fib}(\varphi - \mathrm{can} : G \to H).$$

It remains to identify $G(A, M)$ with $\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{hS}^1}(M)$ and $H(A, M)$ with $\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{tS}^1}(M)$. But this follows from (3) in the previous corollary, and the fact that $B^{\otimes_A \bullet+1}$ is even. ◻

Remark 2.2.16. In general, if $\mathcal{T}$ is a subcanonical site, then one says that a sheaf X on $\mathcal{T}$ is a geometric stack if it admits an effective epimorphism $h_T \to X$ from a representable sheaf and the diagonal $X \to X \times X$ is representable, i.e. whenever $h_S, h_{S'} \to X$ are maps from representables, then $h_S \times_X h_{S'}$ is representable. (This notion is heavily dependent on the choice of site presenting the topos $\mathrm{Shv}(\mathcal{T})$.) Those $\mathbb{E}_\infty$-rings A with an eff map to an even $\mathbb{E}_\infty$-ring give examples of geometric stacks on the site $(\mathrm{CAlg}^{\mathrm{ev}})^{\mathrm{op}}$, and similarly in the equivariant and cyclotomic examples. The good behavior guaranteed in the preceding Corollary 2.2.15 is part of a general paradigm where properties of geometric stacks more closely resemble those of "affines".

Corollary 2.2.17. [Novikov descent] *The following statements hold:*

(1) *For A an $\mathbb{E}_\infty$-ring and M an A-module, the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}/A}(M) \to \lim_\Delta \left(\mathrm{fil}^\star_{\mathrm{ev}/A\otimes\mathrm{MU}^{\bullet+1}}(M \otimes \mathrm{MU}^{\otimes\bullet+1})\right)$$

is an equivalence. For A a p-complete $\mathbb{E}_\infty$-ring and M a p-complete A-module, the canonical map

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p}(M) \to \lim_\Delta \left(\mathrm{fil}^\star_{\mathrm{ev}/(A\otimes\mathrm{MU}^{\bullet+1})^\wedge_p,p}((M \otimes \mathrm{MU}^{\otimes\bullet+1})^\wedge_p)\right)$$

is an equivalence.

(2) *For A an $\mathbb{E}_\infty$-ring with S^1-action and M an S^1-equivariant A-module, the canonical maps*

$$\mathrm{fil}^\star_{\mathrm{ev}/A,\mathrm{hS}^1}(M) \to \lim_\Delta(\mathrm{fil}^\star_{\mathrm{ev}/A\otimes\mathrm{MU}^{\otimes\bullet+1},\mathrm{hS}^1}(M \otimes \mathrm{MU}^{\otimes\bullet+1})),$$

$$\mathrm{fil}^\star_{\mathrm{ev}/A,\mathrm{tS}^1}(M) \to \lim_\Delta(\mathrm{fil}^\star_{\mathrm{ev}/A\otimes\mathrm{MU}^{\otimes\bullet+1},\mathrm{tS}^1}(M \otimes \mathrm{MU}^{\otimes\bullet+1}))$$

are equivalences. For A a p-complete $\mathbb{E}_\infty$-ring with S^1-action and M a p-complete S^1-equivariant A-module, the canonical maps

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{hS}^1}(M) \to \lim_\Delta(\mathrm{fil}^\star_{\mathrm{ev}/(A\otimes\mathrm{MU}^{\otimes\bullet+1})^\wedge_p,p,\mathrm{hS}^1}((M \otimes \mathrm{MU}^{\otimes\bullet+1})^\wedge_p)),$$

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{tS}^1}(M) \to \lim_\Delta(\mathrm{fil}^\star_{\mathrm{ev}/(A\otimes\mathrm{MU}^{\otimes\bullet+1})^\wedge_p,p,\mathrm{tS}^1}((M \otimes \mathrm{MU}^{\otimes\bullet+1})^\wedge_p))$$

are equivalences, where MU is considered to have trivial S^1-action.

(3) *For A a p-typical cyclotomic $\mathbb{E}_\infty$-ring and M a cyclotomic A-module, the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}/A,p,\mathrm{TC}}(M) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{ev}/(A\otimes \mathrm{MU}^{\otimes\bullet+1})^\wedge_p,p,\mathrm{TC}}((M\otimes \mathrm{MU}^{\otimes\bullet+1})^\wedge_p))$$

is an equivalence, were MU *is considered to have trivial cyclotomic structure.*

Proof. It suffices by the closure of eff maps under pushouts together with Corollary 2.2.14 to show that $\mathbb{S}\to \mathrm{MU}$ is eff and discretely p-completely faithfully flat. If B is any even $\mathbb{E}_\infty$-ring then B is complex orientable and hence $\pi_*(B\otimes_{\mathbb{S}}\mathrm{MU})$ is isomorphic to a polynomial ring over $\pi_*(B)$ on a single generator in each even degree. In particular, this is even and both faithfully flat and discretely p-completely faithfully flat over $\pi_*(B)$. □

§2.3. Completion and rationalization. Let us also record some statements concerning the interaction of the even filtration with p-completion and rationalization.

Definition 2.3.1. We say that a map of $\mathbb{E}_\infty$-rings $A\to B$ is *evenly free* if for any nonzero even $\mathbb{E}_\infty$-ring C and map $A\to C$, the pushout $B\otimes_A C$ is equivalent as a C-module to a nonzero direct sum of even shifts of C.

Remark 2.3.2. Let $A\to B$ be an evenly free map of $\mathbb{E}_\infty$-rings. Then $A\to B$ is eff, and for any prime p, the induced map $A^\wedge_p\to B^\wedge_p$ is discretely p-completely eff (by Example 2.2.3).

Example 2.3.3. The proof of Corollary 2.2.17 shows that, for any $\mathbb{E}_\infty$-ring A, the map $A\to A\otimes\mathrm{MU}$ is evenly free.

Proposition 2.3.4. *Let A be an $\mathbb{E}_\infty$-ring. Suppose that there exists an even $\mathbb{E}_\infty$-ring B and a 1-connective, eff map $A\to B$. Then the canonical map*

$$\mathrm{fil}^\star_{\mathrm{ev}}(A)\otimes\mathbb{Q}\to \mathrm{fil}^\star_{\mathrm{ev}}(A\otimes\mathbb{Q})$$

is an equivalence.

Proof. Fix a map $A\to B$ as in the statement. Then $B\otimes\mathbb{Q}$ is also even and $A\otimes\mathbb{Q}\to B\otimes\mathbb{Q}$ is also eff. Thus, by Corollary 2.2.14, we have

$$\mathrm{fil}^\star_{\mathrm{ev}}(A)\simeq \lim_{\Delta}(\tau_{\geq 2\star}(B^{\otimes_A\bullet+1})),\quad \mathrm{fil}^\star_{\mathrm{ev}}(A\otimes\mathbb{Q})\simeq \lim_{\Delta}(\tau_{\geq 2\star}(B^{\otimes_A\bullet+1}\otimes\mathbb{Q})),$$

so it suffices to show that the canonical map

$$\lim_{\Delta}(\tau_{\geq 2\star}(B^{\otimes_A\bullet+1}))\otimes\mathbb{Q}\to \lim_{\Delta}(\tau_{\geq 2\star}(B^{\otimes_A\bullet+1}\otimes\mathbb{Q}))$$

is an equivalence. It follows from the 1-connectivity of the map $A\to B$ that the source of the map is a complete filtered object (by e.g. [HW20, Proposition C.2.5]), and the target evidently is too, so it suffices to show that the induced map on each associated graded piece

$$\lim_{\Delta}(\pi_{2n}(B^{\otimes_A\bullet+1}))\otimes\mathbb{Q}\to \lim_{\Delta}(\pi_{2n}(B^{\otimes_A\bullet+1}\otimes\mathbb{Q}))$$

is an equivalence. This follows from the fact that, for any cosimplicial abelian group $X^\bullet$, the canonical map $\pi^i(X^\bullet)\otimes\mathbb{Q}\to\pi^i(X^\bullet\otimes\mathbb{Q})$ is an isomorphism for each i. □

Proposition 2.3.5. *Let $A\to A'$ be a map of connective $\mathbb{E}_\infty$-rings such that the induced map $A^\wedge\otimes\mathbb{Q}\to(A')^\wedge\otimes\mathbb{Q}$ is an equivalence, where $(-)^\wedge$ denotes profinite completion. Suppose that there exists a connective $\mathbb{E}_\infty$-ring B and a 1-connective, evenly free map $A\to B$ such that $B^\wedge$ and $(A'\otimes_A B)^\wedge$ are even. Then the canonical map*

$$\left(\prod_p \mathrm{fil}^\star_{\mathrm{ev},p}(A^\wedge_p)\right)\otimes\mathbb{Q}\to\left(\prod_p \mathrm{fil}^\star_{\mathrm{ev},p}((A')^\wedge_p)\right)\otimes\mathbb{Q}$$

is an equivalence.

Proof. Fix a map $A \to B$ as in the statement. By Corollary 2.2.14 and Remark 2.3.2, we have

$$\prod_p \mathrm{fil}^\star_{\mathrm{ev},p}(A^\wedge_p) \simeq \prod_p \lim_\Delta(\tau_{\geq 2\star}((B^{\otimes_A \bullet+1})^\wedge_p)) \simeq \lim_\Delta(\tau_{\geq 2\star}((B^{\otimes_A \bullet+1})^\wedge)),$$

and as in the proof of Proposition 2.3.4, $A \to B$ also being 1-connective further implies that

$$\left(\prod_p \mathrm{fil}^\star_{\mathrm{ev},p}(A^\wedge_p)\right) \otimes \mathbb{Q} \simeq \lim_\Delta(\tau_{\geq 2\star}((B^{\otimes_A \bullet+1})^\wedge \otimes \mathbb{Q})).$$

Letting $B' := A' \otimes_A B$, the above also holds when A, B are replaced by A', B'. We may now conclude by observing that $A^\wedge \otimes \mathbb{Q} \to (A')^\wedge \otimes \mathbb{Q}$ being an equivalence implies that $(B^{\otimes_A \bullet+1})^\wedge \otimes \mathbb{Q} \to ((B')^{\otimes_{(A')} \bullet+1})^\wedge \otimes \mathbb{Q}$ is a (levelwise) equivalence. □

§3. Wilson spaces

The purpose of this section is to produce cyclotomic bases with excellent evenness properties. These will be used in the next section to show that, for certain $\mathbb{E}_\infty$-rings, syntomic cohomology is computable from prismatic cohomology.

§3.1. Preliminaries. We being by recording a few useful facts about the following homotopy types, which were first studied by Steve Wilson in his PhD thesis [Wil73]:

Definition 3.1.1. For each integer $i > 0$, the $(2i)$th *Wilson space* is defined to be

$$W_{2i} = \Omega^\infty \Sigma^{2i} \mathrm{MU}.$$

We define W_0 to be the pullback

$$\begin{tikzcd} W_0 \arrow[r] \arrow[d] & \Omega^\infty \mathrm{MU} \arrow[d] \\ \mathbb{N} \arrow[r] & \mathbb{Z}. \end{tikzcd}$$

Here, the right-hand vertical map is the canonical map $\Omega^\infty \mathrm{MU} \to \Omega^\infty \tau_{\leq 0}\mathrm{MU}$, and the bottom horizontal map is the inclusion of the non-negative integers into the integers.

Theorem 3.1.2. [Wilson] *Suppose R is an even $\mathbb{E}_\infty$-ring. Then, for each integer $i \geq 0$, $R_* W_{2i}$ is a polynomial R_* algebra, with polynomial generators in even degrees.*

Proof. In [Wil73, Theorem 3.3], Steve Wilson proved for every $i > 0$ that $\mathrm{H}_*(W_{2i}; \mathbb{Z})$ is a polynomial $\mathbb{Z}$-algebra, with polynomial generators in even degrees. It follows by the Atiyah–Hizerburch spectral sequence that the same is true of $R_* W_{2i}$.

It remains only to check the case $i = 0$. For this, we recall that Wilson proved that $\mathrm{H}_*(\Omega^\infty_0 \mathrm{MU}; \mathbb{Z})$ is an even polynomial $\mathbb{Z}$-algebra, where $\Omega^\infty_0\mathrm{MU}$ is a path component of $\Omega^\infty\mathrm{MU}$. To finish, it suffices to prove that $W_0 \simeq \mathbb{N} \times \Omega^\infty_0 \mathrm{MU}$ in the category of $\mathbb{E}_2$-spaces. In other words, it will suffice to show that the defining $\mathbb{E}_\infty$-space map $W_0 \to \mathbb{N}$ is split by an $\mathbb{E}_2$-space map $\mathbb{N} \to W_0$. Indeed, to construct such an $\mathbb{E}_2$-space map it suffices to produce a double loop map $\mathbb{Z} \to \Omega^\infty \mathrm{MU}$ such that the composite $\mathbb{Z} \to \Omega^\infty\mathrm{MU} \to \mathbb{Z}$ is the identity. Such a map can be obtained by taking Ω^2 of the canonical complex orientation $\mathbb{CP}^\infty \to \Omega^\infty \Sigma^2 \mathrm{MU}$. □

Since each W_{2i} is an $\mathbb{E}_\infty$-space, each $\Sigma^\infty_+ W_{2i}$ is an $\mathbb{E}_\infty$-ring spectrum. We will need the following fact about these $\mathbb{E}_\infty$-rings:

Proposition 3.1.3. *Suppose R is a connective, even $\mathbb{E}_\infty$-ring, and that $x \in \pi_{2i}R$ is an element in the homotopy groups of R. Then there exists an $\mathbb{E}_\infty$-ring map*

$$f : \Sigma^\infty_+ W_{2i} \to R$$

such that x is in the image of $\pi_{2i} f$.

Proposition 3.1.3 will be an immediate consequence of the following lemma, which gives an even cell decomposition of $\Sigma^\infty_+ W_{2i}$ in the category of $\mathbb{E}_\infty$-ring spectra.

Lemma 3.1.4. *There exists a sequence of $\mathbb{E}_\infty$-ring maps*

$$\mathrm{Free}_{\mathbb{E}_\infty}(S^{2i}) = Y_{2i} \to Y_{2i+2} \to Y_{2i+4} \to \cdots \to \Sigma^\infty_+ W_{2i}$$

such that:

(1) *The filtered colimit of the Y_{2k} is $\Sigma^\infty_+ W_{2i}$*
(2) *Each map $Y_{2k} \to Y_{2k+2}$ is the bottom arrow in a pushout of $\mathbb{E}_\infty$-ring spectra*

$$\begin{tikzcd}
\mathrm{Free}_{\mathbb{E}_\infty}(\bigoplus S^{2k+1}) \arrow[r, "\mathrm{Free}_{\mathbb{E}_\infty}(\bigoplus S^{2k+1} \to 0)"] \arrow[d] & \mathbb{S} \arrow[d] \\
Y_{2k} \arrow[r] & Y_{2k+2}.
\end{tikzcd}$$

Proof. If $i > 0$, this follows by applying $\Sigma^\infty_+ \Omega^\infty \Sigma^{2i}(-)$ to an even cell decomposition of MU in the category of spectra. When $i = 0$ we will need the fact that, if $A \to B$ is a map of $\mathbb{E}_\infty$-rings with 1-connective cofiber, and the $\mathbb{E}_\infty$-cotangent complex $\mathrm{L}^{\mathbb{E}_\infty}_{B/A}$ has a cell structure with cells $B \otimes Z_i$, each Z_i a sum of i-spheres, then the map factors as

$$A = A_{-1} \to A_0 \to A_1 \to \cdots \to B,$$

where $A_i \to A_{i+1} = A_i \otimes_{\mathrm{Free}(\Sigma^{-1} Z_i)} \mathbb{S}$. We apply this result with $B = \Sigma^\infty_+ W_0$ and A the free $\mathbb{E}_\infty$-ring on a degree 0 class, mapping to B via an isomorphism on π_0. □

Proof of Proposition 3.1.3. Let $f_{2i} : \mathrm{Free}_{\mathbb{E}_\infty}(S^{2i}) \to R$ denote the $\mathbb{E}_\infty$-ring map adjoint to the map $x : S^{2i} \to R$. Inductively assuming for $k \geq i$ that we have defined a map $f_{2k} : Y_{2k} \to R$, it suffices to prove that we may extend f_{2k} through a map $f_{2k+2} : Y_{2k+2} \to R$. Indeed, the obstructions to doing so lie in an odd homotopy group of R, which is trivial by assumption. □

Warning 3.1.5. The map guaranteed to exist by Proposition 3.1.3 is not canonical in any sense. In particular, we do not claim any sort of functorial construction. The same warning applies to many of the other constructions in this section, all made via obstruction theory.

Since each Wilson space W_{2i} is an $\mathbb{E}_\infty$-space, any finite product of Wilson spaces has a canonical $\mathbb{E}_\infty$-space structure. In the category of $\mathbb{E}_\infty$-spaces, a finite product is a finite coproduct, and we will more generally have occasion to consider infinite coproducts:

Definition 3.1.6. A *weak product of Wilson spaces* is a coproduct, in $\mathbb{E}_\infty$-spaces, of some collection $\{W_{2i_j}\}_{j \in J}$ of Wilson spaces.

Proposition 3.1.7. *Suppose that R is an even $\mathbb{E}_\infty$-ring and that W is a weak product of Wilson spaces. Then $R_* W$ is an even polynomial R_*-algebra.*

Proof. The coproduct in $\mathbb{E}_\infty$-rings is the tensor product. □

Proposition 3.1.8. *Suppose that R is a connective, even $\mathbb{E}_\infty$-ring. Then there exists a weak product of Wilson spaces W together with an $\mathbb{E}_\infty$-ring map*

$$f : \Sigma^\infty_+ W \to R$$

such that $\pi_* f$ is surjective.

Proof. For each element $x \in \pi_{2i} R$, Proposition 3.1.3 allows us to choose an $\mathbb{E}_\infty$-ring map $\Sigma^\infty_+ W_{2i} \to R$ whose image in π_* contains x. Taking the coproduct of such maps, over all elements x in the homotopy groups of R, yields the result. □

Finally, the following technical result is a key ingredient in our proof that motivic filtrations respect cyclotomic structure:

Theorem 3.1.9. *Let R be an even $\mathbb{E}_\infty$-ring with S^1-action. Then, for any $i \geq 0$ and any S^1-equivariant $\mathbb{E}_\infty$-ring map $\mathrm{THH}(\Sigma^\infty_+ W_{2i}) \to R$, there exists a factorization*

$$\begin{tikzcd} \mathrm{THH}(\Sigma^\infty_+ W_{2i}) \arrow[r] \arrow[d, "\pi"] & R \\ \Sigma^\infty_+ W_{2i}. \arrow[ur] & \end{tikzcd}$$

Here, π is the canonical S^1-equivariant projection $\mathrm{THH}(\Sigma^\infty_+ W_{2i}) \to \Sigma^\infty_+ W_{2i}$, where $\Sigma^\infty_+ W_{2i}$ has trivial S^1-action.

Proof. We prove by induction on $k \geq i$ that there exist factorizations

$$\begin{tikzcd} \mathrm{THH}(Y_{2k}) \arrow[r] \arrow[d] & R \\ Y_{2k}, \arrow[ur, dashed] & \end{tikzcd}$$

where the Y_{2k} are as in Lemma 3.1.4. First, we prove this for $k = i$, where the desired diagram is of the form

$$\begin{tikzcd} \mathrm{THH}\left(\mathrm{Free}_{\mathbb{E}_\infty}(S^{2i})\right) \arrow[r] \arrow[d] & R \\ \mathrm{Free}_{\mathbb{E}_\infty}(S^{2i}). \arrow[ur, dashed] & \end{tikzcd}$$

By the universal property of THH, this is asking whether a certain non-equivariant map $S^{2i} \to R$ factors through an S^1-equivariant map $S^{2i} \to R$. Since R is even, the homotopy fixed point spectral sequence for $\pi_*(R^{\mathrm{hS}^1})$ collapses, and in particular the map $\pi_*(R^{\mathrm{hS}^1}) \to \pi_*(R)$ is surjective.

To finish the induction, we need to produce a dashed arrow completing the following diagram:

$$\begin{tikzcd} \mathrm{THH}(Y_{2k}) \arrow[r] \arrow[d] & \mathrm{THH}(Y_{2k+2}) \arrow[r] \arrow[d] & R \\ Y_{2k} \arrow[r] \arrow[urr] & Y_{2k+2}. \arrow[ur, dashed] & \end{tikzcd}$$

Because the map $Y_{2k} \to Y_{2k+2}$ is a cell attachment, extensions of the given S^1-equivariant $\mathbb{E}_\infty$-ring map $Y_{2k} \to R$ through Y_{2k+2} are given by nullhomotopies of an S^1-equivariant spectrum

map
$$\bigoplus S^{2k+1} \to R.$$
We need to show that we can find such a nullhomotopy compatible with a given nullhomotopy of the underlying non-equivariant map
$$\bigoplus S^{2k+1} \to R.$$
Again, this comes down to the facts that R and R^{hS^1} are even, and in particular that $\pi_*(R^{\mathrm{hS}^1}) \to \pi_*(R)$ is surjective. □

Corollary 3.1.10. *Let R be an even $\mathbb{E}_\infty$-ring with S^1-action and W a weak product of Wilson spaces. Then, for any $\mathbb{E}_\infty$-ring map $\mathrm{THH}(\Sigma^\infty_+ W) \to R$, there exists a factorization*
$$\begin{tikzcd} \mathrm{THH}(\Sigma^\infty_+ W) \arrow[r] \arrow[d, "\pi"] & R \\ \Sigma^\infty_+ W. \arrow[ur] & \end{tikzcd}$$
Here, π is the canonical S^1-equivariant projection $\mathrm{THH}(\Sigma^\infty_+ W) \to \Sigma^\infty_+ W$, where $\Sigma^\infty_+ W$ has trivial S^1-action.

Proof. Since $\mathrm{THH}(\Sigma^\infty_+ -)$ and Σ^∞_+ both preserve coproducts of $\mathbb{E}_\infty$-spaces, this follows immediately from the preceding theorem. □

§3.2. A sufficient supply of cyclotomic bases. In order to understand cyclotomic structure on THH of discrete rings, it has proven extremely useful to understand cyclotomic structure on Hochschild homology relative to $\Sigma^\infty_+\mathbb{N}$ [LW20; AKN22]. As noted for example in [Law21], it is rarely the case that Hochschild homology relative to an $\mathbb{E}_\infty$-ring A carries cyclotomic structure. In such cases, we say that A is a cyclotomic base:

Definition 3.2.1. A *cyclotomic base* is an $\mathbb{E}_\infty$-ring A together with the data of a commutative diagram of S^1-equivariant $\mathbb{E}_\infty$-rings
$$\begin{tikzcd} \mathrm{THH}(A) \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathrm{THH}(A)^{\mathrm{tC}_p} \arrow[d, "\pi^{\mathrm{tC}_p}"] \\ A \arrow[r] & A^{\mathrm{tC}_p}, \end{tikzcd}$$
where $\pi : \mathrm{THH}(A) \to A$ is the canonical projection to A with trivial action.

We warn the reader that the above definition is not quite the same as the one adopted in [Ant+20, Appendix A], but has been discussed in [BMS19, §11.1] and [Law21, Remark 12.1]. If A is a cyclotomic base, and R is an $\mathbb{E}_\infty$-A-algebra, then we may use the formula
$$\mathrm{THH}(R/A) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(A)} A$$
to define the structure of a cyclotomic $\mathbb{E}_\infty$-ring on $\mathrm{THH}(R/A)$. By definition, this cyclotomic $\mathbb{E}_\infty$-ring structure is compatible with the map $\mathrm{THH}(R) \to \mathrm{THH}(R/A)$.

To date, practically all known examples of cyclotomic bases have been built out of $A = \mathbb{S}$ and $A = \Sigma^\infty_+\mathbb{N}$. In this section, our goal will be to use Wilson spaces to construct many additional examples of cyclotomic bases, by obstruction theory.

Definition 3.2.2. A connective $\mathbb{E}_\infty$-ring A is said to be *strongly even* if:

(1) There exists a homotopy commutative ring map $\mathrm{MU} \to A$, such that as an MU-module A splits as a direct sum of even suspensions of MU. After localization at any prime p, we further require that the composite $\pi_*\mathrm{BP} \to \pi_*\mathrm{MU}_{(p)} \to \pi_* A_{(p)}$ present $\pi_* A_{(p)}$ as a polynomial ring over $\pi_*\mathrm{BP}$.
(2) In the category of $\mathbb{E}_\infty$-ring spectra, A admits an even cell decomposition.

For each $i \geq 0$, the $\mathbb{E}_\infty$-ring $\Sigma^\infty_+ W_{2i}$ admits an even $\mathbb{E}_\infty$ cell decomposition; this is precisely the statement Lemma 3.1.4. However, for no i is $\Sigma^\infty_+ W_{2i}$ strongly even. Indeed, as a spectrum $\Sigma^\infty_+ W_{2i}$ contains a unit summand equivalent to $\mathbb{S}$, and so $\Sigma^\infty_+ W_{2i}$ cannot be a wedge of even suspensions of MU. At the end of this section, we will prove that at least one strongly even $\mathbb{E}_\infty$-ring exists, by direct construction. First, we list some of the properties enjoyed by strongly even $\mathbb{E}_\infty$-rings:

Proposition 3.2.3. *Suppose A is strongly even. Then the unit map $\mathbb{S} \to A$ is eff, and its p-completion is discretely p-completely eff.*

Proof. By [CM15, Theorem 1.2], we may find an $\mathbb{E}_2$-ring map $\mathrm{MU} \to A$ which makes A into a free MU-module. Now, suppose that B is any $\mathbb{E}_\infty$-ring. We calculate $B \otimes A \simeq (B \otimes \mathrm{MU}) \otimes_{\mathrm{MU}} A$, where the equivalence is as $\mathbb{E}_1$-ring spectra. Since $B \otimes \mathrm{MU}$ is a free B-module, and A is a free MU module, we learn that $B \otimes A$ is a free B-module, on even suspensions of B. $\square$

Proposition 3.2.4. *Suppose A is strongly even, that R is a p-local, even $\mathbb{E}_\infty$-ring spectrum, and that W is a weak product of Wilson spaces. Then the natural map*

$$\pi_*(R) \to \pi_*(R \otimes A \otimes W)$$

presents the codomain as a polynomial algebra over the domain, on even generators.

Proof. First, note that $\pi_*(\mathrm{BP} \otimes R)$ is a polynomial $\pi_* R$-algebra, by p-local complex orientation theory. Since R is p-local, $A \otimes R$ is equivalent to $A_{(p)} \otimes R$, and we have assumed $\pi_*\left(A_{(p)}\right)$ to be a polynomial $\pi_*\mathrm{BP}$-algebra. It follows that $\pi_*(R \otimes A)$ is a polynomial $\pi_* R$-algebra, and we finish by Theorem 3.1.2. $\square$

Proposition 3.2.5. *Suppose that A is strongly even, and that R is an even $\mathbb{E}_\infty$-ring with S^1-action. Then any S^1-equivariant $\mathbb{E}_\infty$-ring map $\mathrm{THH}(A) \to R$ factors through the projection $\pi : \mathrm{THH}(A) \to A$.*

Proof. This follows from the assumed even cell decomposition, exactly as in the proof of Theorem 3.1.9. $\square$

Theorem 3.2.6. *Suppose that A is a strongly even $\mathbb{E}_\infty$-ring spectrum and that W is a weak product of Wilson spaces. Then there exists a cyclotomic base with $A \otimes \Sigma^\infty_+ W$ as its underlying $\mathbb{E}_\infty$-ring.*

Proof. We need to produce a diagram of S^1-equivariant $\mathbb{E}_\infty$-ring spectra

$$\begin{tikzcd}
\mathrm{THH}(A \otimes \Sigma^\infty_+ W) \arrow[r] \arrow[d] & \mathrm{THH}(A \otimes \Sigma^\infty_+ W)^{\mathrm{tC}_p} \arrow[d] \\
A \otimes \Sigma^\infty_+ W \arrow[r, dashed] & (A \otimes \Sigma^\infty_+ W)^{\mathrm{tC}_p}.
\end{tikzcd}$$

Note that $(A \otimes \Sigma^\infty_+ W)^{\mathrm{tC}_p}$ is even, because $A \otimes \Sigma^\infty_+ W$ is a free MU-module concentrated in even degrees. Therefore, we may finish by combining Proposition 3.2.5 and Corollary 3.1.10. $\square$

Finally, we prove that at least one strongly even $\mathbb{E}_\infty$-ring exists, by direct construction:

Definition 3.2.7. Fix any map of spectra $\gamma : \mathrm{MU} \to \mathrm{ku}$ that is a surjection on π_*. We denote by MW the $\mathbb{E}_\infty$-ring that is the Thom spectrum associated to the map

$$\Omega^\infty \Sigma^2 \gamma : W_2 \to \mathrm{BU}$$

Theorem 3.2.8. *The* $\mathbb{E}_\infty$*-ring* MW *is strongly even.*

Proof. By Thomifying $\Omega^\infty \Sigma^2(-)$, applied to an even cell decomposition of the spectrum MU, we see that MW has an even cell decomposition as an $\mathbb{E}_\infty$-ring spectrum. It remains to check that:

(1) There exists a homotopy commutative ring homomorphism $\mathrm{MU} \to \mathrm{MW}$ that makes MW a free MU-module concentrated in even degrees.

(2) After p-localization, MW is a polynomial BP-algebra.

First, we claim that the infinite loop map $\Omega^\infty \Sigma^2 \gamma : W_2 \to \mathrm{BU}$ has a double loop map section. To construct this section, we may take Ω^2 of a section of the pointed space map $\Omega^\infty \Sigma^4 \gamma : W_4 \to \mathrm{BSU}$. To see that the latter section exists, we note that *any* map $\mathrm{BSU} \to \mathrm{BSU}$ lifts through $\Omega^\infty \Sigma^4 \gamma$. This is because the obstruction to such a lift is a map from BSU to a space that, by assumption, has only odd homotopy groups.

Now, the above section induces a splitting of W_2, as a double loop space, into the product of BU and $\Omega^\infty \mathrm{fib}(\Sigma^2 \gamma)$. It follows that MW is, as an $\mathbb{E}_2$-ring, the tensor product of MU and $\Sigma^\infty_+ \mathrm{fib}(\Omega^\infty \Sigma^2 \gamma)$. To prove (1), it therefore suffices to prove that $\mathrm{MU}_* \mathrm{fib}(\Omega^\infty \Sigma^2 \gamma)$ is a free MU_*-module concentrated in even degrees. By the Atiyah–Hizerburch spectral sequence, it suffices to check that $\mathrm{H}_*\left(\mathrm{fib}(\Omega^\infty \Sigma^2 \gamma); \mathbb{Z}\right)$ is a free $\mathbb{Z}$-module concentrated in even degrees. This is a submodule of $H_*(W_2; \mathbb{Z})$, by the above double loop space splitting, and we finish by noting that submodules of free $\mathbb{Z}$-modules are free $\mathbb{Z}$-modules.

To prove (2), it suffices to check that $\mathrm{BP}_*(\mathrm{fib}(\Omega^\infty \Sigma^2 \gamma))$ is a polynomial BP_*-algebra. Since $\mathrm{fib}(\Omega^\infty \Sigma^2 \gamma)$ is a retract of $\Omega^\infty \Sigma^2 \mathrm{MU}$, it must have finitely generated homotopy groups and $\mathbb{Z}_{(p)}$-homology groups [Wil73]. We may therefore conclude the result by [Wil73, Theorem 6.2]. ◻

§4. The motivic filtration

In this section, we define and prove basic properties of the motivic filtrations on $\mathrm{THH}(R)$, $\mathrm{TC}^-(R)$, $\mathrm{TP}(R)$, and $\mathrm{TC}(R)^\wedge_p$, whenever R is a well-behaved ring spectrum. We begin by specifying that well-behaved means *chromatically quasisyntomic*, which is meant to be a generalization of the main definition of [BMS19, §4] and [BL22, Appendix C] to the setting of not necessarily discrete $\mathbb{E}_\infty$-ring spectra.

§4.1. Chromatically quasisyntomic $\mathbb{E}_\infty$-rings.

Definition 4.1.1. Let $f : A \to B$ be a map of even $\mathbb{E}_\infty$-rings. Below, we regard $\pi_*(A)$ and $\pi_*(B)$ as ungraded commutative rings.

(1) The map f is a *quasiregular quotient* (resp. *p-quasiregular quotient*) if $\mathrm{L}^{\mathrm{alg}}_{\pi_*(B)/\pi_*(A)}$ has Tor-amplitude concentrated in degree 1 (resp. p-complete Tor-amplitude concentrated in degree 1).

(2) The map f is *quasi-lci* (resp. *p-quasi-lci*) if $\mathrm{L}^{\mathrm{alg}}_{\pi_*(B)/\pi_*(A)}$ is a $\pi_*(B)$-module with Tor-amplitude concentrated in $[0,1]$ (resp. p-complete Tor-amplitude concentrated in $[0,1]$).

Definition 4.1.2. We say that an even $\mathbb{E}_\infty$-ring A is *p-quasisyntomic* if $\pi_*(A)$ is a p-quasisyntomic commutative ring, i.e. the unit map $\mathbb{Z} \to \pi_*(A)$ is p-quasi-lci and $\pi_*(A)$ has bounded p-power torsion.

The terminology "quasi-lci" is partially motivated by the following proposition.

Proposition 4.1.3. *A map $k \to R$ of connective, even $\mathbb{E}_\infty$-rings is quasi-lci if and only if there exists a factorization*

$$k \to S \to R$$

where S is even and connective with $\pi_*(S)$ a polynomial $\pi_*(k)$-algebra and $S \to R$ is a quasiregular quotient. Moreover, if $k \to R$ is quasi-lci, then any factorization

$$k \to S \to R$$

wth $\pi_*(S)$ a polynomial $\pi_*(k)$-algebra and $\pi_*(S) \to \pi_*(R)$ surjective necessarily has $S \to R$ a quasiregular quotient.

Proof. First suppose such a factorization exists; then the claim is immediate from the fiber sequence

$$R_* \otimes^{\mathrm{L}}_{S_*} \mathrm{L}^{\mathrm{alg}}_{S_*/k_*} \to \mathrm{L}^{\mathrm{alg}}_{R_*/k_*} \to \mathrm{L}^{\mathrm{alg}}_{R_*/S_*}.$$

For the converse, we will require results from §3. By Proposition 3.1.8, we may choose an $\mathbb{E}_\infty$-map $\Sigma^\infty_+ W \to R$ from the suspension spectrum of a weak product of Wilson spaces which is surjective on homotopy groups. By Proposition 3.1.7, $S := k \otimes \Sigma^\infty_+ W$ is even and connective with homotopy groups polynomial over $\pi_*(k)$. From the fiber sequence

$$R_* \otimes^{\mathrm{L}}_{S_*} \mathrm{L}^{\mathrm{alg}}_{S_*/k_*} \to \mathrm{L}^{\mathrm{alg}}_{R_*/k_*} \to \mathrm{L}^{\mathrm{alg}}_{R_*/S_*}$$

we see that $\mathrm{L}^{\mathrm{alg}}_{R_*/S_*}$ has Tor-amplitude in $[0, 1]$. Moreover, since the map $S_* \to R_*$ is surjective and S_* is a polynomial algebra, we deduce that the map $R_* \otimes^{\mathrm{L}}_{S_*} \mathrm{L}^{\mathrm{alg}}_{S_*/k_*} \to \mathrm{L}^{\mathrm{alg}}_{R_*/k_*}$ is surjective on π_0 and hence also after any base change to a discrete R_*-module. It follows that $\mathrm{L}^{\mathrm{alg}}_{R_*/S_*}$ in fact has Tor-amplitude 1, as desired.

The final claim follows from the same argument. □

Definition 4.1.4. A map of $\mathbb{E}_\infty$-rings $k \to R$ is *chromatically quasi-lci* (resp. *chromatically p-quasi-lci*) if:

(1) both $k \otimes \mathrm{MU}$ and $R \otimes \mathrm{MU}$ are even, and

(2) the induced map $k \otimes \mathrm{MU} \to R \otimes \mathrm{MU}$ is quasi-lci (resp. p-quasi-lci).

An $\mathbb{E}_\infty$-ring R is *chromatically p-quasisyntomic* if $R \otimes \mathrm{MU}$ is even and p-quasisyntomic. An $\mathbb{E}_\infty$-ring R is *chromatically quasisyntomic* if $R \otimes \mathrm{MU}$ is even, the unit map $\mathbb{Z} \to \mathrm{MU}_*(R)$ is quasi-lci, and $\mathrm{MU}_*(R)$ has bounded p-power torsion for all primes p.

Example 4.1.5. If R is any of $\mathbb{S}$, ko, or tmf, then $\mathrm{MU}_* R$ is a polynomial $\mathbb{Z}$-algebra. Thus, all three of $\mathbb{S}$, ko, and tmf are chromatically quasisyntomic $\mathbb{E}_\infty$-rings.

In the case that k and R are both even $\mathbb{E}_\infty$-rings, the chromatic definitions collapse to more familiar notions:

Proposition 4.1.6. *Suppose $k \to R$ is a map of even $\mathbb{E}_\infty$-rings. Then it is a quasi-lci map if and only if it is a chromatically quasi-lci map. Similarly, a map of even $\mathbb{E}_\infty$-rings is p-quasi-lci if and only if chromatically p-quasi-lci.*

Proof. By complex-orientation theory, there are compatible isomorphisms $\mathrm{MU}_* k \cong k_*[b_1, b_2, \cdots]$ and $\mathrm{MU}_* R \cong R_*[b_1, b_2, \cdots]$. We calculate that

$$\mathrm{L}^{\mathrm{alg}}_{\mathrm{MU}_* R/\mathrm{MU}_* k} \simeq \mathbb{Z}[b_1, b_2, \cdots] \otimes^{\mathrm{L}}_{\mathbb{Z}} \mathrm{L}^{\mathrm{alg}}_{R_*/k_*} \simeq \mathrm{MU}_* R \otimes_{R_*} \mathrm{L}^{\mathrm{alg}}_{R_*/k_*},$$

since the algebraic cotangent complex is compatible with derived base-change, and finish by noting that $\mathrm{MU}_* R$ is a free R_*-module. □

Proposition 4.1.7. *Let A be an even $\mathbb{E}_\infty$-ring. Then A is chromatically quasisyntomic (resp. chromatically p-quasisyntomic) if and only if $\pi_* A$ is quasisyntomic (resp. p-quasisyntomic).*

Proof. By complex orientation theory, $\mathrm{MU}_* A \cong \pi_*(A)[b_1, b_2, \ldots]$, which is a free $\pi_*(A)$-module. □

The following result will be used in the proof of Theorem 4.2.10.

Proposition 4.1.8. *Let R be a chromatically p-quasisyntomic $\mathbb{E}_\infty$-ring and let A be a strongly even $\mathbb{E}_\infty$-ring (Definition 3.2.2). Then $R \otimes A$ is even and p-quasisyntomic.*

Proof. Evenness is immediate from $R \otimes \mathrm{MU}$ being even and A splitting as a direct sum of even suspensions of MU. This implies that $\pi_*(R \otimes A \otimes \mathrm{MU})$ is an even polynomial algebra over $\pi_*(R \otimes A)$, and hence $R \otimes A$ is p-quasisyntomic if and only if $R \otimes A \otimes \mathrm{MU}$ is p-quasisyntomic; we will thus be done if we can show the latter. Noting that a commutative ring B is p-quasisyntomic if and only if its p-localization $B_{(p)}$ is p-quasisyntomic, this follows from the hypothesis that $R \otimes \mathrm{MU}$ is p-quasisyntomic and the fact that $\pi_*(R \otimes \mathrm{MU} \otimes A)_{(p)}$ is an even polynomial algebra over $\pi_*(R \otimes \mathrm{MU})_{(p)}$ (Proposition 3.2.4). □

§4.2. Motivic filtrations. With the definitions and constructions of §2 and §4.1 in place, we now define our motivic filtrations on Hochschild homology and its associated invariants.

Definition 4.2.1. For a chromatically quasi-lci map of connective $\mathbb{E}_\infty$-rings $k \to R$ and $\mathrm{THH}(R/k)$-module M (assumed to be S^1-equivariant in the latter two cases), we define

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} M &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)} M, \\
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} M^{\mathrm{hS}^1} &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k),\mathrm{hS}^1} M, \\
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} M^{\mathrm{tS}^1} &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k),\mathrm{tS}^1} M.
\end{aligned}$$

In the case $M = \mathrm{THH}(R/k)$ we abbreviate:

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R/k) &:= \mathrm{fil}^\star_{\mathrm{ev}} \mathrm{THH}(R/k), \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(R/k) &:= \mathrm{fil}^\star_{\mathrm{ev},\mathrm{hS}^1} \mathrm{THH}(R/k), \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TP}(R/k) &:= \mathrm{fil}^\star_{\mathrm{ev},\mathrm{tS}^1} \mathrm{THH}(R/k).
\end{aligned}$$

Variant 4.2.2. For a prime number p and a chromatically p-quasi-lci map of connective $\mathbb{E}_\infty$-rings $k \to R$ and a p-complete $\mathrm{THH}(R/k)$-module (assumed to be S^1-equivariant in the latter two cases), we define

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} M &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)^\wedge_p, p} M, \\
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} M^{\mathrm{hS}^1} &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)^\wedge_p, p, \mathrm{hS}^1} M, \\
\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} (M^{\mathrm{tS}^1})^\wedge_p &:= \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)^\wedge_p, p, \mathrm{tS}^1} M.
\end{aligned}$$

When $M = \mathrm{THH}(R/k)^\wedge_p$ we abbreviate:

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R/k)^\wedge_p &:= \mathrm{fil}^\star_{\mathrm{ev},p} \mathrm{THH}(R/k)^\wedge_p, \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(R/k)^\wedge_p &:= \mathrm{fil}^\star_{\mathrm{ev},p,\mathrm{hS}^1} \mathrm{THH}(R/k)^\wedge_p, \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TP}(R/k)^\wedge_p &:= \mathrm{fil}^\star_{\mathrm{ev},p,\mathrm{tS}^1} \mathrm{THH}(R/k)^\wedge_p.
\end{aligned}$$

When k is a cyclotomic base[6], and M is a p-typical cyclotomic $\mathrm{THH}(R/k)$-module, we further define

$$\mathrm{fil}^\star_{\mathrm{mot}/(R|k)} \mathrm{TC}(M)^\wedge_p := \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)^\wedge_p,p,\mathrm{TC}} M,$$

and abbreviate:

$$\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}(R/k)^\wedge_p := \mathrm{fil}^\star_{\mathrm{ev}/\mathrm{THH}(R/k)^\wedge_p,p,\mathrm{TC}} \mathrm{THH}(R/k)^\wedge_p.$$

Example 4.2.3. We check that $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(\ell) \simeq \lim_\Delta \left(\tau_{\geq 2\star}\mathrm{THH}(\ell/\mathrm{MU}^{\otimes \bullet+1})\right)$, which is the descent studied by the first and third authors in [HW20, §6.1]. Since $\mathrm{THH}(\ell/\mathrm{MU}^{\otimes\bullet+1}) \simeq \mathrm{THH}(\ell) \otimes_{\mathrm{THH}(\mathrm{MU})} (\mathrm{MU})^{\otimes_{\mathrm{THH}(\mathrm{MU})}\bullet+1}$, it suffices to check that:

(1) $\mathrm{THH}(\ell/\mathrm{MU})$ is even.

(2) $\mathrm{THH}(\mathrm{MU}) \to \mathrm{MU}$ is eff.

The first of these points is [HW20, Theorem E]. To prove the second point, given any map of $\mathbb{E}_\infty$-rings $\mathrm{THH}(\mathrm{MU}) \to A$ where A is even, we must show that $A \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$ is even and that $A \to A \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$ is faithfully flat. In fact, we will show that $A \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}$ is a free A-module, equivalent to a direct sums of even suspensions of A.

Recall that $\mathrm{THH}_*(\mathrm{MU}) \cong \Lambda_{\mathrm{MU}_*}(\sigma b_1, \sigma b_2, \cdots)$ is an exterior algebra over MU_* generated by classes σb_i in odd degrees [BCS10; Rog20]. Because A is even, each of the classes σb_i must map to zero along the map $\mathrm{THH}_*(\mathrm{MU}) \to \pi_*(A)$. The Künneth spectral sequence for $\pi_*\left(A \otimes_{\mathrm{THH}(\mathrm{MU})} \mathrm{MU}\right)$ therefore has E_2-term given by $\pi_*(A) \otimes_{\mathrm{MU}_*} \Gamma_{\mathrm{MU}_*}(\sigma^2 b_i)$, where $\Gamma_{\mathrm{MU}_*}(\sigma^2 b_i)$ denotes a divided power algebra on even degree generators. The spectral sequence collapses for degree reasons, and the result is a free $\pi_*(A)$-module.

Example 4.2.4. With the same argument as above, we see that $\mathrm{fil}^\star_{\mathrm{mot}/\mathrm{MU}} \mathrm{THH}(\mathrm{BP}\langle n\rangle) \simeq \lim_\Delta \left(\tau_{\geq 2\star}\mathrm{THH}(\mathrm{BP}\langle n\rangle/\mathrm{MU}^{\otimes\bullet+1})\right)$, also studied by the first and third authors in [HW20, §6.1].

In the remainder of the section we set up some basic theory, culminating in proofs of Theorem 1.2.5 and Theorem 1.2.6 from the introduction. Roughly speaking, these theorems state that the motivic filtrations converge and that the motivic filtration on $\mathrm{TC}^\wedge_p$ is compatible with the motivic filtrations on TC^- and TP. For ease of exposition, we choose to focus on the case of rings rather than modules over their Hochschild homology. The necessary modifications for the module case are straightforward and left to the interested reader.

Lemma 4.2.5. *Let $k \to S$ be a map of connective, even $\mathbb{E}_\infty$-rings such that $\pi_*(S)$ is a polynomial $\pi_*(k)$-algebra. Then the map $\mathrm{THH}(S/k) \to S$ is evenly free, hence eff and discretely p-completely eff.*

Proof. Suppose $S = k[x_i]$, where the x_i are a (possibly infinite) collection of polynomial generators. Then we may calculate the homotopy groups of

$$\mathrm{THH}(S/k) = S \otimes_{S\otimes_k S} S \simeq k[x_i] \otimes_{k[x_i, x_i']} k[x_i]$$

[6]See Definition 3.2.1 for our definition of a cyclotomic base.

via a spectral sequence with associated graded

$$\mathrm{THH}_*(S_*/k_*) \cong k_*[x_i] \otimes_{k_*} \Lambda_{k_*}(dx_i).$$

This spectral sequence collapses because each multiplicative generator is a permanent cycle. Now, consider a pushout square of the form

$$\begin{tikzcd} \mathrm{THH}(S/k) \arrow[r] \arrow[d] & S \arrow[d] \\ C \arrow[r] & C \otimes_{\mathrm{THH}(S/k)} S, \end{tikzcd}$$

where C is a nonzero even $\mathbb{E}_\infty$-ring. The map $\mathrm{THH}_*(S/k) \to \pi_* C$ must send each class dx_i to 0, because each class dx_i is in odd degree. We may calculate $\pi_*(C \otimes_{\mathrm{THH}(S/k)} S)$ via a spectral sequence with associated graded

$$C_* \otimes^{\mathrm{L}}_{k_*[x_i] \otimes_{k_*} \Lambda_{k_*}(dx_i)} k_*[x_i] \cong C_* \otimes_{k_*} \Gamma_{k_*}(\sigma^2 x_i),$$

where $\Gamma_{k_*}(\sigma^2 x_i)$ is a divided power algebra on even degree classes. This spectral sequence must degenerate for degree reasons, and the result is a nonzero free C_*-module.

The above shows that the map $\mathrm{THH}(S/k) \to S$ is evenly free. By Remark 2.3.2, this implies that the map is eff and discretely p-completely eff. □

Lemma 4.2.6. (1) *Let $S \to R$ be a quasiregular quotient of connective, even $\mathbb{E}_\infty$-rings. Then* $\mathrm{THH}(R/S)$ *is even.*

(2) *Let $S \to R$ be a p-quasiregular quotient of connective, p-complete $\mathbb{E}_\infty$-rings and suppose that $\pi_*(R)$ has bounded p-power torsion. Then* $\mathrm{THH}(R/S)^\wedge_p$ *is even.*

Proof. (1) There is a spectral sequence converging to $\mathrm{THH}_*(R/S)$ with associated graded $\mathrm{THH}_*(R_*/S_*)$, so it will suffice to prove that the latter ring is even. Now, the HKR filtration gives a spectral sequence for $\mathrm{THH}_*(R_*/S_*)$ with associated graded $\mathrm{LSym}\left(\Sigma \mathrm{L}^{\mathrm{alg}}_{R_*/S_*}\right)$. Since $S \to R$ is a quasiregular quotient, $\Sigma^{-1}\mathrm{L}^{\mathrm{alg}}_{R_*/S_*}$ is a flat R_*-module concentrated in even internal degrees, and the result follows.

(2) By the same argument as above, we are reduced to showing that $\mathrm{LSym}\left(\Sigma \mathrm{L}^{\mathrm{alg}}_{R_*/S_*}\right)$ is concentrated in even degrees after p-completion. First observe that $(\Sigma^{-1}\mathrm{L}^{\mathrm{alg}}_{R_*/S_*})^\wedge_p$ is p-completely flat by hypothesis. Since the formation of (derived) divided powers commutes with base change, we deduce that $\Gamma^n_{R_*}((\Sigma^{-1}\mathrm{L}^{\mathrm{alg}}_{R_*/S_*})^\wedge_p)$ is also p-completely flat. By [BMS19, Lemma 4.7] and our assumption that R_* has bounded p^∞-torsion, we deduce that $\Gamma^n_{R_*}((\Sigma^{-1}\mathrm{L}^{\mathrm{alg}}_{R_*/S_*})^\wedge_p)$ is discrete after p-completion. By a theorem of Illusie (see [Lur18, Proposition 25.2.4.2]), we deduce that

$$\mathrm{LSym}^n\left(\Sigma \mathrm{L}^{\mathrm{alg}}_{R_*/S_*}\right) \cong \Sigma^{2n}\Gamma^n_{R_*}(\Sigma^{-1}\mathrm{L}^{\mathrm{alg}}_{R_*/S_*})$$

is concentrated in degree $2n$ after p-completion. This completes the proof. □

Corollary 4.2.7. *Let $k \to S \to R$ be maps of connective, even $\mathbb{E}_\infty$-rings such that $\pi_*(S)$ is a polynomial $\pi_*(k)$-algebra and $S \to R$ is a quasiregular quotient (resp. a p-quasiregular quotient with R_* having bounded p^∞-torsion). Then the map* $\mathrm{THH}(R/k) \to \mathrm{THH}(R/S)$ *(resp. the map* $\mathrm{THH}(R/k)^\wedge_p \to \mathrm{THH}(R/S)^\wedge_p$*) is eff (resp. p-completely eff).*

Proof. This follows from Lemmas 4.2.5 and 4.2.6, as the map $\mathrm{THH}(R/k) \to \mathrm{THH}(R/S)$ is obtained by pushing out the map $\mathrm{THH}(S/k) \to S$ along the map $\mathrm{THH}(S/k) \to \mathrm{THH}(R/k)$, and the same statement holds after p-completion. $\square$

Corollary 4.2.8. *Let $k \to R$ be a chromatically quasi-lci map of connective $\mathbb{E}_\infty$-rings and assume that R_* has bounded p^∞-torsion. Then*

$$(\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R/k))^\wedge_p \simeq \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R/k)^\wedge_p.$$

Proof. By Novikov descent we may replace R and k by $R \otimes \mathrm{MU}$ and $k \otimes \mathrm{MU}$ and thereby reduce to the case when $k \to R$ is a quasi-lci map of connective, even $\mathbb{E}_\infty$-rings. By Proposition 4.1.3 we may produce a factorization $k \to S \to R$ as in Corollary 4.2.7. The result follows from the conclusion of Corollary 4.2.7. $\square$

The following statements generalize both Theorem 1.2.5 and Theorem 1.2.6 from the introduction.

Theorem 4.2.9. *Suppose $k \to R$ is a chromatically quasi-lci map of connective $\mathbb{E}_\infty$-rings. Then*

- $\mathrm{fil}^0_{\mathrm{mot}} \mathrm{THH}(R/k) = \mathrm{THH}(R/k)$.
- $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(R/k)$ *has colimit* $\mathrm{TC}^-(R/k)$.
- $\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TP}(R/k)$ *has colimit* $\mathrm{TP}(R/k)$.

Moreover, the fiber of $\mathrm{fil}^i_{\mathrm{mot}} \mathrm{THH}(R/k) \to \mathrm{THH}(R/k)$ *(and hence also of* $\mathrm{fil}^i_{\mathrm{mot}} \mathrm{TC}^-(R/k) \to \mathrm{TC}^-(R/k)$*) is* $(2i-2)$*-truncated.*

Proof. Since $\mathrm{THH}(R/k)$ is connective the Adams-Novikov tower converges so that

$$\mathrm{THH}(R/k) = \lim_{\Delta} \mathrm{THH}(R/k) \otimes \mathrm{MU}^{\otimes \bullet+1}.$$

We may therefore replace R and k by $R \otimes \mathrm{MU}$ and $k \otimes \mathrm{MU}$ and thereby reduce to the case when $k \to R$ is a quasi-lci map of connective, even $\mathbb{E}_\infty$-rings. By Proposition 4.1.3, we may choose a factorization $k \to S \to R$ where S_* is an even polynomial ring over k_* and $S \to R$ is a quasiregular quotient. By Corollary 4.2.7, $\mathrm{THH}(R/S)$ is even and we have

$$\mathrm{fil}^\star \mathrm{THH}(R/k) = \lim_{\Delta} \tau_{\geq 2\star} \mathrm{THH}(R/S)^{\otimes_{\mathrm{THH}(R/k)} \bullet+1}.$$

It follows that

$$\mathrm{fil}^0 \mathrm{THH}(R/k) = \lim_{\Delta} \mathrm{THH}(R/S)^{\otimes_{\mathrm{THH}(R/k)} \bullet+1}.$$

On the other hand, for any map $A \to B$ of connective $\mathbb{E}_\infty$-rings with 1-connective fiber, the map

$$A \to \lim_{\Delta} B^{\otimes_A \bullet+1}$$

is an equivalence, so the first claim follows.

Now we turn to the remaining claims. Let us (for the purposes of this proof) say that a filtered spectrum $F^\star Y$ *strongly converges to* Y if there is a a map $F^\star Y \to Y$ to the constant filtered spectrum at Y such that the fibers of the maps

$$F^i Y \to Y$$

have is $(2i-2)$-truncated. Observe, in particular, that this implies $\mathrm{colim}\, F^\star Y = Y$, since the fiber of the map $\mathrm{colim}\, F^\star Y \to Y$, having arbitrary coconnectivity, is necessarily zero. Since spectra which are $\leq n$ are closed under arbitrary limits, the property of strong convergence is closed under limits Moreover, we always have that $\tau_{\geq 2\star} Y$ strongly converges to Y.

By the same argument as in the claim for $\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(R/k)$, we are then reduced to proving that, if $A \to B$ is a map of S^1-equivariant, connective $\mathbb{E}_\infty$-rings, with 1-connective fiber, then

$$A^{\mathrm{hS}^1} = \lim_{\Delta}(B^{\otimes_A \bullet+1})^{\mathrm{hS}^1},$$
$$A^{\mathrm{tS}^1} = \lim_{\Delta}(B^{\otimes_A \bullet+1})^{\mathrm{tS}^1}.$$

But both homotopy and Tate fixed points commute with limits of *connective* spectra, so the claim follows. □

Theorem 4.2.10. *Suppose $k \to R$ is a chromatically p-quasi-lci map of connective $\mathbb{E}_\infty$-rings, and assume that $\mathrm{MU}_* R$ has bounded p-power torsion. Then*

- $\mathrm{fil}^{0}_{\mathrm{mot}}\,\mathrm{THH}(R/k)^{\wedge}_{p}$ *has colimit* $\mathrm{THH}(R/k)^{\wedge}_{p}$.
- $\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(R/k)^{\wedge}_{p}$ *has colimit* $\mathrm{TC}^{-}(R/k)^{\wedge}_{p}$.
- $\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(R/k)^{\wedge}_{p}$ *has colimit* $\mathrm{TP}(R/k)^{\wedge}_{p}$.

Furthermore, if $k = \mathbb{S}$ there exist maps

$$\varphi : \mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(R)^{\wedge}_{p} \to \mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(R)^{\wedge}_{p},\ \textit{and}$$

$$\mathrm{can} : \mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(R)^{\wedge}_{p} \to \mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(R)^{\wedge}_{p},$$

converging to $\varphi : \mathrm{TC}^{-}(R)^{\wedge}_{p} \to \mathrm{TP}(R)^{\wedge}_{p}$ *and* $\mathrm{can} : \mathrm{TC}^{-}(R)^{\wedge}_{p} \to \mathrm{TP}(R)^{\wedge}_{p}$, *respectively, such that there is an equivalence*

$$\mathrm{fil}^{\star}_{\mathrm{mot}}\,\mathrm{TC}(R)^{\wedge}_{p} \simeq \mathrm{fib}\,(\varphi - \mathrm{can})\,.$$

Proof. The claims about the convergence of the motivic filtrations are proved exactly as in the previous theorem.

It remains to prove the existence of the filtered φ and the filtered can in the case $k = \mathbb{S}$. For this, we use the notation and theory developed in §3.2. Specifically, we choose a strongly even $\mathbb{E}_\infty$-ring A (e.g. $A = \mathrm{MW}$) and a weak product of Wilson spaces W with a surjective $\mathbb{E}_\infty$-A-algebra map $A \otimes \Sigma^{\infty}_{+} W \to A \otimes R$, noting that $A \otimes R$ is even and p-quasisyntomic by Proposition 4.1.8. Then the map

$$\mathrm{THH}(R)^{\wedge}_{p} \to \mathrm{THH}(A \otimes R / A \otimes \Sigma^{\infty}_{+} W)^{\wedge}_{p}$$

is a p-completely eff map of p-complete, connective, cyclotomic $\mathbb{E}_\infty$-rings by Theorem 3.2.6 and Lemma 4.2.5, and the target is even by Lemma 4.2.6. The claim now follows by Corollary 2.2.15. □

Remark 4.2.11. The filtered φ and can of Theorem 4.2.10 are natural in $\mathbb{E}_\infty$-ring maps between chromatically p-quasisyntomic $\mathbb{E}_\infty$-rings R (see the proof of Corollary 2.2.15).

Remark 4.2.12. It follows from the proof of the previous theorem that if R is a chromatically quasisyntomic then $\mathrm{THH}(R)$ admits an evenly free (and hence eff) map to an even cyclotomic ring. As indicated in Remark 2.2.16, the convenience of the motivic filtration should hold under this weaker requirement alone. It would be very interesting to have a checkable characterization of those R such that $\mathrm{THH}(R)$ admits an eff map to an even cyclotomic ring.

§5. Comparison theorems

In this section, we will compare the motivic filtrations defined above with filtrations defined previously in special cases.

Notation 5.0.1. (1) For $k \to R$ any map of commutative rings, we have the Hochschild–Kostant–Rosenberg (HKR) filtration on Hochschild homology, $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k)$ (see, for example, [BMS19, §2]). For a prime number p, we denote the p-completion of the HKR filtration by $\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k)^\wedge_p$.

(2) For $k \to R$ a quasi-lci map of commutative rings, we have Antieau's Beilinson filtrations on negative and periodic cyclic homology, $\mathrm{fil}^\star_{\mathrm{B}} \mathrm{HC}^-(R/k)$ and $\mathrm{fil}^\star_{\mathrm{B}} \mathrm{HP}(R/k)$ [Ant19].

(3) For p a prime number and $k \to R$ a p-quasi-lci map of p-quasisyntomic p-complete commutative rings, we have the Bhatt–Morrow–Scholze (BMS) filtrations on p-completed negative and periodic cyclic homology, $\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{HC}^-(R/k)^\wedge_p$ and $\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{HP}(R/k)^\wedge_p$ [BMS19, §5].

(4) For p a prime number and R a p-quasisyntomic p-complete commutative ring, we have the BMS filtrations on p-completed topological Hochschild, topological negative cyclic, topological periodic cyclic, and topological cyclic homology [BMS19, §7],
$$\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p, \quad \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TC}^-(R)^\wedge_p, \quad \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TP}(R/k)^\wedge_p, \quad \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TC}(R)^\wedge_p.$$

(5) For R any commutative ring, we have the Bhatt–Lurie filtration on topological Hochschild, topological negative cyclic, and topological periodic cyclic homology [BL22, §6.4],
$$\mathrm{fil}^\star_{\mathrm{BL}} \mathrm{THH}(R), \quad \mathrm{fil}^\star_{\mathrm{BL}} \mathrm{TC}^-(R), \quad \mathrm{fil}^\star_{\mathrm{BL}} \mathrm{TP}(R/k).$$

Theorem 5.0.2. *For $k \to R$ a quasi-lci map of discrete commutative rings, there are natural identifications*
$$\begin{aligned} \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HH}(R/k) &\simeq \mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k), \\ \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HC}^-(R/k) &\simeq \mathrm{fil}^\star_{\mathrm{B}} \mathrm{HC}^-(R/k), \\ \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HP}(R/k) &\simeq \mathrm{fil}^\star_{\mathrm{B}} \mathrm{HP}(R/k). \end{aligned}$$

For p a prime number and $k \to R$ a p-quasi-lci map of p-quasisyntomic p-complete commutative rings, there are natural identifications
$$\begin{aligned} \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HH}(R/k)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k)^\wedge_p, \\ \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HC}^-(R/k)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{HC}^-(R/k)^\wedge_p, \\ \mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HP}(R/k)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{HP}(R/k)^\wedge_p. \end{aligned}$$

Proof. We will establish the first identification; the rest can be established similarly. Let $k \to R$ be a quasi-lci map of commutative rings. Let S be the polynomial k-algebra with generators indexed by the elements of R, which comes equipped with a canonical surjection $S \to R$. Then we have
$$\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{HH}(R/k) \simeq \lim_{\Delta}(\tau_{\geq 2\star}(\mathrm{HH}(R/S^{\otimes_k \bullet+1}))) \simeq \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/S^{\otimes_k \bullet+1})),$$
where the first equivalence follows from Corollary 4.2.7 and the second equivalence follows from the identifications
$$\mathrm{gr}^i_{\mathrm{HKR}} \mathrm{HH}(R/S^{\otimes_k \bullet+1}) \simeq \Sigma^i \mathrm{L}\Lambda^i_R(\mathrm{L}^{\mathrm{alg}}_{R/S^{\otimes_k \bullet+1}})$$
and the fact that $\mathrm{L}^{\mathrm{alg}}_{R/S^{\otimes_k n+1}}$ has Tor-amplitude concentrated in degree 1. It thus suffices to show that the canonical map
$$\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/k) \to \lim_{\Delta}(\mathrm{fil}^\star_{\mathrm{HKR}} \mathrm{HH}(R/S^{\otimes_k \bullet+1}))$$

is an equivalence. We can check this after passing to associated graded objects, so it is enough to show that the canonical maps

$$\mathrm{L}\Lambda^i_R(\mathrm{L}^{\mathrm{alg}}_{R/k}) \to \lim_{\Delta}(\mathrm{L}\Lambda^i_R(\mathrm{L}^{\mathrm{alg}}_{R/S^{\otimes_k \bullet+1}}))$$

are equivalences.

Consider the commutative diagram of cosimplicial R-modules

$$\begin{tikzcd}
R \otimes_S \mathrm{L}^{\mathrm{alg}}_{S/k} \arrow[r] \arrow[d] & \mathrm{L}^{\mathrm{alg}}_{R/k} \arrow[r] \arrow[d] & \mathrm{L}^{\mathrm{alg}}_{R/S} \arrow[d] \\
R \otimes_S \mathrm{L}^{\mathrm{alg}}_{S/S^{\otimes_k \bullet+1}} \arrow[r] & \mathrm{L}^{\mathrm{alg}}_{R/S^{\otimes_k \bullet+1}} \arrow[r] & \mathrm{L}^{\mathrm{alg}}_{R/S}
\end{tikzcd}$$

in which the top row consists of constant cosimplicial objects, each row is a transitivity cofiber sequence, and the map between the two rows is induced by the map of cosimplicial commutative rings $k \to S^{\otimes_k \bullet+1}$. In [Bha12, Proof of Corollary 2.7], it is shown that the map $\mathrm{L}^{\mathrm{alg}}_{S/k} \to \mathrm{L}^{\mathrm{alg}}_{S/S^{\otimes_k \bullet+1}}$ is a homotopy equivalence of cosimplicial S-modules. Thus, the left-hand vertical map is a homotopy equivalence of cosimplicial R-modules, which implies that the same is true of the middle vertical map, since the right-hand vertical map is the identity map. It follows that the map $\mathrm{L}\Lambda^i_R(\mathrm{L}^{\mathrm{alg}}_{R/k}) \to \mathrm{L}\Lambda^i_R(\mathrm{L}^{\mathrm{alg}}_{R/S^{\otimes_k \bullet+1}})$ is a homotopy equivalence of cosimplicial R-modules for all i, implying the desired claim. □

Theorem 5.0.3. *For p a prime number and R a p-quasisyntomic p-complete commutative ring, there are natural identifications*

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{THH}(R)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p, \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}^-(R)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TC}^-(R)^\wedge_p, \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TP}(R)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TP}(R)^\wedge_p, \\
\mathrm{fil}^\star_{\mathrm{mot}} \mathrm{TC}(R)^\wedge_p &\simeq \mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{TC}(R)^\wedge_p.
\end{aligned}$$

Proof. [7] Let R' be the polynomial ring over $\mathbb{Z}$ on generators indexed by the set underlying R, so that we have a natural surjection $R' \twoheadrightarrow R$. Let S' be the ring obtained by adjoining all p-power roots of the polynomial generators of R', and form the p-completed pushout $S := (R \otimes_{R'} S')^\wedge_p$. Then S is quasiregular semiperfectoid and the map $R \to S$ is a p-quasisyntomic cover, so from [BMS19, §7] we have that $\mathrm{THH}(S)^\wedge_p$ is even and a natural equivalence

$$\mathrm{fil}^\star_{\mathrm{BMS}} \mathrm{THH}(R)^\wedge_p \simeq \lim_{\Delta}(\tau_{\geq 2\star}(\mathrm{THH}(S^{\otimes_R \bullet+1})^\wedge_p)),$$

and similarly for TC^-, TP, and TC. It now follows from Corollary 2.2.14 that to prove the claim, it suffices to show that the map $\mathrm{THH}(R)^\wedge_p \to \mathrm{THH}(S)^\wedge_p$ is discretely p-completely eff.

Let $\mathbb{S}_{R'}$ be the polynomial $\mathbb{E}_\infty$-ring over $\mathbb{S}$ on generators indexed by the set underlying R (i.e. the tensor product over this set of copies of the monoid ring $\mathbb{S}[\mathbb{N}]$) and let $\mathbb{S}_{S'}$ be the $\mathbb{E}_\infty$-ring obtained by adjoining all p-power roots of the polynomial generators of $\mathbb{S}_{R'}$ (i.e. a

[7] We thank Bhargav Bhatt for suggesting the argument written here (any error is our own responsibility). Our original argument used Theorem 5.0.2 to establish quasisyntomic descent for our motivic filtrations.

tensor product of copies of the monoid ring $\mathbb{S}[\mathbb{N}[1/p]]$). Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{THH}(\mathbb{S}_{R'}) \arrow[r] \arrow[d] & \mathbb{S}_{R'} \arrow[r] \arrow[d] & \mathbb{S}_{S'} \arrow[d] \\
\mathrm{THH}(R) \arrow[r] & \mathrm{THH}(R/\mathbb{S}_{R'}) \arrow[r] & \mathrm{THH}(S/\mathbb{S}_{S'}).
\end{tikzcd}$$

By [BMS19, Proposition 11.7], the map $\mathrm{THH}(S) \to \mathrm{THH}(S/\mathbb{S}_{S'})$ is an equivalence after p-completion. It is thus enough to show that each of the bottom horizonal maps is discretely p-completely eff. Since each square in the diagram is in fact a pushout square, it furthermore suffices to show that each of the top horizontal maps is discretely p-completely eff. This is clear for the right-hand map, as $\mathbb{S}_{S'}$ is free as a module over $\mathbb{S}_{R'}$. For the left-hand map, we may check after tensoring with MU (since $\mathbb{S} \to \mathrm{MU}$ is evenly free), and then we may use Lemma 4.2.5. ◻

Theorem 5.0.4. *For R a commutative ring with bounded p-power torsion for all primes p and with algebraic cotangent complex $\mathrm{L}_R^{\mathrm{alg}}$ having Tor amplitude contained in $[0,1]$, there are natural identifications*

$$\begin{aligned}
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R) &\simeq \mathrm{fil}^\star_{\mathrm{BL}}\,\mathrm{THH}(R),\\
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(R) &\simeq \mathrm{fil}^\star_{\mathrm{BL}}\,\mathrm{TC}^-(R),\\
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(R) &\simeq \mathrm{fil}^\star_{\mathrm{BL}}\,\mathrm{TP}(R).
\end{aligned}$$

Proof. Let us just establish the identification for THH. Let R be a commutative ring with bounded p-power torsion for all primes p and such that the algebraic cotangent complex L_R^{alg} has Tor amplitude contained in $[0,1]$. Then we have a commutative square

$$\begin{tikzcd}
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R) \arrow[r] \arrow[d] & \prod_p (\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R))^\wedge_p \arrow[d] \\
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R) \arrow[r] & \prod_p (\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R))^\wedge_p,
\end{tikzcd}$$

and we have a defining pullback square

$$\begin{tikzcd}
\mathrm{fil}^\star_{\mathrm{BL}}\,\mathrm{THH}(R) \arrow[r] \arrow[d] & \prod_p \mathrm{fil}^\star_{\mathrm{BMS}}\,\mathrm{THH}(R)^\wedge_p \arrow[d] \\
\mathrm{fil}^\star_{\mathrm{HKR}}\,\mathrm{HH}(R) \arrow[r] & \prod_p \mathrm{fil}^\star_{\mathrm{HKR}}\,\mathrm{HH}(R)^\wedge_p
\end{tikzcd}$$

From Theorem 5.0.2 and Corollary 4.2.8, we obtain an identification between the lower arrows of the two squares. Noting that the p-completion of R is a p-quasisyntomic discrete commutative ring and that $\mathrm{THH}(R)^\wedge_p \simeq \mathrm{THH}(R^\wedge_p)^\wedge_p$, Theorem 5.0.3 and Corollary 4.2.8 give an identification between the upper right objects of the two squares. The natural transformation $\mathrm{fil}^\star_{\mathrm{BMS}}\,\mathrm{THH}(-)^\wedge_p \to \mathrm{fil}^\star_{\mathrm{HKR}}\,\mathrm{HH}(-)^\wedge_p$ on p-quasisyntomic p-complete commutative rings is the unique natural map of filtered objects compatible with the canonical map $\mathrm{THH}(-)^\wedge_p \to \mathrm{HH}(-)^\wedge_p$, since the filtrations are descended from the double-speed Postnikov filtration for quasiregular semiperfectoid rings. It follows that the right-hand arrows of the squares identify as well. Thus, to finish the proof, it suffices to show that the first square is a pullback diagram.

Consider the extended diagram

$$\begin{tikzcd}
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R) \arrow[r] \arrow[d] & \prod_p (\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(R))^\wedge_p \arrow[d] \\
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R) \arrow[r] \arrow[d] & \prod_p (\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R))^\wedge_p \arrow[d] \\
(\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R)) \otimes \mathbb{Q} \arrow[r] & (\prod_p (\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{HH}(R))^\wedge_p) \otimes \mathbb{Q}
\end{tikzcd}$$

The lower square is an arithmetic square, hence a pullback square, so it suffices to show that the outer square is a pullback square. In fact, the outer square is also an arithmetic square; this is a consequence of Propositions 2.3.4 and 2.3.5, using the following observations:

- The canonical maps $\mathrm{THH}(R) \otimes \mathbb{Q} \to \mathrm{HH}(R) \otimes \mathbb{Q}$ and $\mathrm{THH}(R)^\wedge \otimes \mathbb{Q}$ and $\mathrm{HH}(R)^\wedge \otimes \mathbb{Q}$ are equivalences (where $(-)^\wedge$ denotes profinite completion). For the latter, note that the canonical map $\mathrm{THH}(R)^\wedge \otimes_{\mathrm{THH}(\mathbb{Z})} \mathbb{Z} \to \mathrm{HH}(\mathbb{Z})^\wedge$ is an equivalence. All of this follows from the finiteness of $\pi_i(\mathbb{S})$ for $i > 0$ (cf. [BMS19, Lemma 2.5]).
- By Proposition 4.1.3, we may choose a connective, even $\mathbb{E}_\infty$-MU-algebra S where $\pi_*(S)$ is a polynomial algebra over $\pi_*(\mathrm{MU})$, together with a map of $\mathbb{E}_\infty$-MU-algebras $S \to R \otimes \mathrm{MU}$ such that the induced map $\pi_*(S) \to \pi_*(R \otimes \mathrm{MU})$ is a quasiregular quotient. Then the map $\mathrm{THH}(R) \to \mathrm{THH}(R \otimes \mathrm{MU}/S)$ is 1-connective and evenly free (Lemma 4.2.5 and Example 2.3.3), and both $\mathrm{THH}(R \otimes \mathrm{MU}/S)$ and $\mathrm{HH}(R) \otimes_{\mathrm{THH}(R)} \mathrm{HH}(R \otimes \mathrm{MU}/S) \simeq \mathrm{HH}(R \otimes \mathrm{MU}/\mathbb{Z} \otimes S)$ are even and have even profinite completions (Lemma 4.2.6). □

§6. The mod (p, v_1) syntomic cohomology of the Adams summand

In this section we compute the mod (p, v_1) syntomic cohomology of the connective Adams summand ℓ at a fixed prime p. In other words, we compute $(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}(\ell))/(p, v_1)$. We allow p to be any prime, including the primes 2 and 3 where the Smith Toda complex $V(1) = \mathbb{S}/(p, v_1)$ does not exist as a homotopy ring spectrum. At primes $p \geq 5$, where $V(1)$ does exist as a homotopy ring spectrum, this gives an independent proof of the seminal calculation of $V(1)_*\mathrm{TC}(\ell)$ by Ausoni–Rognes [AR02]. Before stating our main result, we precisely define the mod $(p, v_1, \cdots, v_k)$ reductions of prismatic and syntomic cohomology, even in the absence of a Smith–Toda complex:

Definition 6.0.1. By [Ghe+18; Pst18], the category of modules over $\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}$ has a t-structure with heart equivalent to the category of even $\mathrm{MU}_*\mathrm{MU}$-comodules. In particular, there is a commutative comodule algebra $\mathrm{MU}_*/(p, v_1, \cdots, v_k)$ for each $k \geq 0$, and we denote the corresponding $\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}$ algebra by $\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}/(p, \cdots, v_k)$. More generally, for any graded $\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}$-module M, we define

$$M/(p, \ldots, v_k) := M \otimes_{\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}} (\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}/(p, \cdots, v_k)) .$$

For any $\mathbb{E}_\infty$-ring R, $\mathrm{gr}^\star_{\mathrm{ev}}\,R/(p, \cdots, v_k)$ is a $\mathrm{gr}^\star_{\mathrm{ev}}\,\mathbb{S}$-algebra, and the same is true for the equivariant, p-complete, and cyclotomic variants of the even filtration.

Our main result is as follows (where we recall Definition 1.3.2 for the notion of Adams weight):

Theorem 6.0.2. *The mod (p, v_1, v_2) syntomic cohomology of ℓ is a finite $\mathbb{F}_p$-vector space. As a vector space, it is isomorphic to*

(1) $\mathbb{F}_p\{1\}$, *in Adams weight* 0 *and degree* 0.

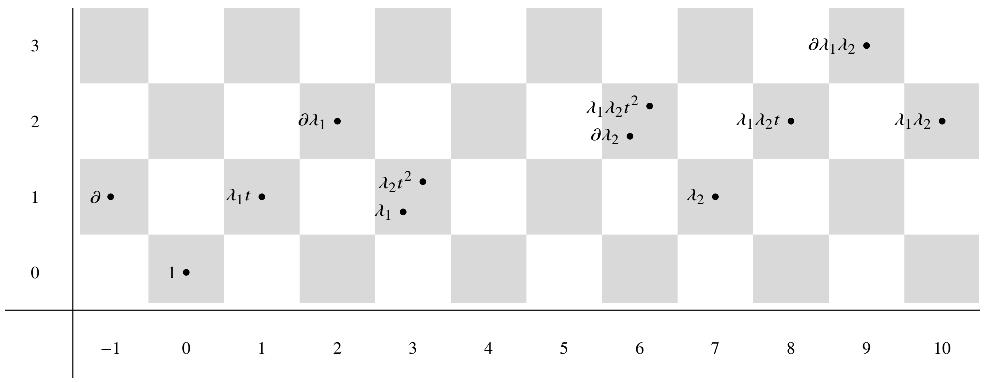
(2) $\mathbb{F}_p\{\partial, t^d\lambda_1, t^{dp}\lambda_2 \mid 0 \le d < p\}$, *in Adams weight* 1. *Here,* $|\partial| = -1$, $|t^d\lambda_1| = 2p - 2d - 1$, *and* $|t^{dp}\lambda_2| = 2p^2 - 2dp - 1$.

(3) $\mathbb{F}_p\{t^d\lambda_1\lambda_2, t^{dp}\lambda_1\lambda_2, \partial\lambda_1, \partial\lambda_2 \mid 0 \le d < p\}$, *in Adams weight* 2. *Here,* $|t^d\lambda_1\lambda_2| = 2p^2 - 2p - 2d - 2$, $|t^{dp}\lambda_1\lambda_2| = 2p^2 - 2p - 2dp - 2$, $|\partial\lambda_1| = 2p - 2$, *and* $|\partial\lambda_2| = 2p^2 - 2$.

(4) $\mathbb{F}_p\{\partial\lambda_1\lambda_2\}$, *in Adams weight* 3 *and degree* $2p^2 + 2p - 3$.

The v_2*-Bockstein spectral sequence (converging to the mod* (p, v_1) *syntomic cohomology of* ℓ *as an* $\mathbb{F}_p[v_2]$*-module) collapses with no differentials.*

The mod $(2, v_1, v_2)$-syntomic cohomology of $\ell = \mathrm{ku}$



We will spend most of the remainder of this section proving the first part of Theorem 6.0.2. The statement about the v_2-Bockstein spectral sequence is comparatively straightforward. Indeed, differentials in the v_2-Bockstein spectral sequence increase Adams weight by 1. Since the degree of v_2 is $2p^2 - 2$, no differentials are possible for bidegree reasons. There are no $\mathbb{F}_p[v_2]$-module extension problems to solve, and we conclude Theorem 1.3.1.

Remark 6.0.3. Suppose that $p \ge 3$, so that the Smith–Toda complex $V(1) = \mathbb{S}/(p, v_1)$ exists as a spectrum. Then the mod (p, v_1) syntomic cohomology of ℓ is the E_2-page of a spectral sequence converging to $V(1)_*\mathrm{TC}(R)$. This is the spectral sequence associated to the filtered spectrum $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}(\ell) \otimes_{\mathrm{fil}^\star_{\mathrm{ev}}\mathbb{S}} \mathrm{fil}^\star_{\mathrm{ev}/\mathbb{S}}(V(1))$.

§6.1. Recollections from previous work. Our computations will rely upon two fundamental facts about $\mathrm{THH}(\ell)$ that can be proven by non-motivic means. The first fact, due originally to McClure–Staffeldt at odd primes [MS93] and Angeltveit–Rognes at the prime 2 [AR05], is the calculation of $\mathrm{THH}_*(\ell)/(p, v_1)$:

Theorem 6.1.1. *There is an isomorphism of algebras*

$$\pi_*\left(\mathrm{THH}(\ell) \otimes_\ell \mathbb{F}_p\right) \cong \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu],$$

for classes λ_1 *in degree* $2p - 1$, λ_2 *in degree* $2p^2 - 1$, *and* μ *in degree* $2p^2$.

To state the second fact, note that the sequence of $\mathbb{E}_\infty$-ring maps

$$\ell \to \mathrm{THH}(\ell) \xrightarrow{\varphi} \mathrm{THH}(\ell)^{\mathrm{tC}_p}$$

allows us to view the cyclotomic Frobenius φ as a map of ℓ-algebras. We then have the following result, which is a version of the Segal conjecture:

Theorem 6.1.2. *The mod (p, v_1) Frobenius map $\pi_*(\mathrm{THH}(\ell) \otimes_\ell \mathbb{F}_p) \xrightarrow{\varphi} \pi_*\left(\mathrm{THH}(\ell)^{\mathrm{tC}_p} \otimes_\ell \mathbb{F}_p\right)$ is identified under the isomorphism of Theorem 6.1.1 with the ring map*

$$\Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu] \to \Lambda(\lambda_1, \lambda_2) \otimes \mathbb{F}_p[\mu^{\pm 1}]$$

that inverts the class μ.

Theorem 6.1.2 was first proved for primes $p \geq 5$ by Ausoni–Rognes [AR02, Theorem 5.5]. An argument at $p = 2$ is given in the thesis of Sverre Lunøe-Nielsen [Lun05]. The result can be deduced at all primes from the discussion of the first and third authors in [HW20, §4]. To recall that argument, note that filtering ℓ by its $\mathbb{F}_p$-Adams filtration gives a map of spectral sequences, converging to the mod (p, v_1) Frobenius, which on E_2-pages is π_* of

$$\mathrm{THH}(\mathbb{F}_p[v_0, v_1]) \otimes_{\mathbb{F}_p[v_0,v_1]} \mathbb{F}_p \xrightarrow{\varphi} \mathrm{THH}(\mathbb{F}_p[v_0, v_1])^{\mathrm{tC}_p} \otimes_{\mathbb{F}_p[v_0,v_1]} \mathbb{F}_p.$$

This map of E_2-pages can be identified with the ring map

$$\mathbb{F}_p[x] \otimes \Lambda(\sigma v_0, \sigma v_1) \to \mathbb{F}_p[x^{\pm}] \otimes \Lambda(\sigma v_0, \sigma v_1)$$

that inverts the degree 2 Bökstedt generator $x \in \mathrm{THH}_*(\mathbb{F}_p)$. In both spectral sequences, the class μ is detected by x^{p^2}. Upon checking that $\varphi(\mu)$ is invertible, and not merely invertible on the E_2-page, we can deduce Theorem 6.1.2 by the fact that it is true on associated graded.

It follows from the Leibniz rule that x^{-p^2} is a permanent cycle in the codomain spectral sequence, and therefore that $\varphi(\mu)$ has an inverse up to higher filtration. By completeness of the filtration on homotopy groups, $\varphi(\mu)$ is itself invertible.

Remark 6.1.3. A key step in the Ausoni–Rognes work, also featured in [AK+22], is the construction of elements in $V(0)_*\mathrm{TC}(\ell)$ lifting the classes λ_1 and λ_2 of Theorem 6.1.1. Our arguments below do not presuppose the existence of such lifts.

§6.2. A useful isomorphism. In order to calculate the mod (p, v_1, v_2) syntomic cohomology of ℓ we will first need to compute the mod (p, v_1, v_2) prismatic cohomology of ℓ, or in other words $(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell))/(p, v_1, v_2)$. A useful and perhaps surprising fact is that the mod (p, v_1, v_2) prismatic cohomology of ℓ has a second interpretation: as we explain in this brief section, it is canonically isomorphic to $\left(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$ (cf. [BM22, Construction 3.3]). For the purposes of this section we are defining $\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}$ via the pushout in filtered $\mathbb{E}_\infty$-rings:

$$\begin{tikzcd}
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell) \arrow[r, "\varphi"] \arrow[d] & \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell) \arrow[d] \\
\mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell) \arrow[r] & \mathrm{fil}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}.
\end{tikzcd}$$

To define the isomorphism, note that $v_2 = 0$ in $(\mathrm{gr}^\star_{\mathrm{ev}}\,\ell)/(p, v_1)$, so the sequence of algebra maps

$$\mathrm{gr}^\star_{\mathrm{ev}}\,\ell \to \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell) \xrightarrow{\varphi} \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}$$

imply that $v_2 = 0$ in $\left(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p, v_1)$. Thus, the natural map

$$\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p, v_1) \to \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p, v_1)$$

factors over a map

$$g : \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p, v_1, v_2) \to \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p, v_1)$$

Theorem 6.2.1. *The map g above is an isomorphism*

$$\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)\right)/(p,v_1,v_2) \xrightarrow{\cong} \left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}\right)/(p,v_1)$$

Proof. We can compute the motivic associated graded for $\mathrm{TP}(\ell)$ via the cobar complex with sth term given by $\pi_*\mathrm{TP}(\ell/\mathrm{MW}^{\otimes s+1})$. The domain of g is calculated by the complex obtained from modding out each term $\pi_*\mathrm{TP}(\ell/\mathrm{MW}^{\otimes s+1})$ by p, v_1 and v_2. The codomain is obtained from the complex $\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes \bullet+1})^{\mathrm{tC}_p}$ by levelwise killing p and v_1.

We first note that, for each value of $s \geq 0$, $v_2 = 0$ in $\left(\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1)$. This can be seen from the existence of the relative cyclotomic Frobenius map

$$\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})/(p,v_1) \to \pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})^{\mathrm{tC}_p}/(p,v_1),$$

because $v_2 \equiv 0$ modulo (p,v_1) in any $\pi_*\ell$-algebra such as $\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})$. It follows that g extends to a map of cobar complexes, which levelwise is of the form

$$g_s : \left(\pi_*\mathrm{TP}(\ell/\mathrm{MW}^{\otimes s+1})\right)/(p,v_1,v_2) \to \left(\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1).$$

To prove that g is an isomorphism, we will prove the stronger claim that g_s is an isomorphism for each $s \geq 0$.

To see that g_s is an isomorphism, it follows from the lemma below that the group $\left(\pi_*\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})^{\mathrm{tC}_p}\right)/(p,v_1)$ can be computed from $\left(\pi_*\mathrm{TP}(\ell/\mathrm{MW}^{\otimes s+1})\right)/(p,v_1)$ by killing $[p](t)$ for any complex orientation t, so it suffices to show that v_2 is a unit multiple of $[p](t)$. We know that v_2 is some multiple of $[p](t)$, because v_2 becomes 0 when $[p](t)$ is killed. Since $[p](t) = v_2 t^{p^2} + \mathcal{O}(t^{p^2+1})$, we must have $v_2 = t^{-p^2}([p](t))(1+\mathcal{O}(t))$, so v_2 is a unit multiple of $[p](t)$. □

Lemma 6.2.2. *Let $M \in \mathrm{Mod}_{\mathrm{MU}}^{\mathrm{BS}^1}$ be an* S^1*-equivariant* MU*-module. Then the map*

$$M^{\mathrm{tS}^1}/[p](t) = M^{\mathrm{tS}^1} \otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p} \to M^{\mathrm{tC}_p}$$

is an equivalence.

Proof. We argue exactly as in [NS18, p. IV.4.12]. Since $\mathrm{MU}^{\mathrm{tC}_p} = \mathrm{MU}^{\mathrm{tS}^1}/[p](t)$ is a perfect $\mathrm{MU}^{\mathrm{tS}^1}$-module, the functor $(-)\otimes_{\mathrm{MU}^{\mathrm{tS}^1}} \mathrm{MU}^{\mathrm{tC}_p}$ commutes with all limits and colimits. Using the equivalences $M^{tG} = \mathrm{colim}(\tau_{\geq n}M)^{tG}$ and $M^{tG} = \lim(\tau_{\leq n}M)^{tG}$ we are reduced to the case when M is bounded, and further to the case where M is discrete. In this case the S^1-action (and hence the C_p-action) must be trivial, and then one concludes by direct observation. □

Remark 6.2.3. We will study and compute $\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)\right)/(p,v_1,v_2)$ at any prime p. However, if the Smith–Toda complex $V(2) = \mathbb{S}/(p,v_1,v_2)$ does not exist, we cannot say that $\left(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)\right)/(p,v_1,v_2)$ is the E_2-page of a spectral sequence converging to $V(2)_\star\mathrm{TP}(\ell)$. In contrast, it is always the case that $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)^{\mathrm{tC}_p}/(p,v_1)$ is the E_2-page of a spectral sequence converging to $\mathrm{THH}(\ell)^{\mathrm{tC}_p}\otimes_\ell \mathbb{F}_p$. This spectral sequence is the one associated to the cosimplicial object $\mathrm{THH}(\ell/\mathrm{MW}^{\otimes\bullet+1})^{\mathrm{tC}_p}\otimes_\ell \mathbb{F}_p$, where the map from ℓ into $\mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})^{\mathrm{tC}_p}$ is given by the composite of the map $\ell \to \mathrm{THH}(\ell/\mathrm{MW}^{\otimes s+1})$ with the relative cyclotomic Frobenius.

§6.3. Naming conventions in the cobar complex. In the next sections we begin our explicit computations of the prismatic and syntomic cohomologies of ℓ. We make all of these computations via a convenient, specific resolution, which was also used in [HW20, Section 6].

Definition 6.3.1. The *cobar complex* computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$ is the E_1-page of the descent spectral sequence for

$$\mathrm{THH}(\ell) \to \mathrm{THH}(\ell/\mathrm{MU}).$$

This cobar complex has sth term given by $\pi_\star\mathrm{THH}(\ell/\mathrm{MU}^{\otimes s+1})$. Cocycles in the sth term represent elements in the sth Adams weight of $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$.

Similarly, we refer to $\pi_\star\left(\mathrm{THH}(\ell/\mathrm{MU}^{\otimes\bullet+1})^{\mathrm{hS}^1}\right)$ as the cobar complex computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$, and $\pi_\star\left(\mathrm{THH}(\ell/\mathrm{MU}^{\otimes\bullet+1})^{\mathrm{tS}^1}\right)$ as the cobar complex computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)$. There are canonical maps from the cobar complex computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$ to the cobar complexes computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)$ and $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$, respectively.

We emphasize again that the cobar complexes of Definition 6.3.1 are merely our preferred presentations for the much more canonical $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$, $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$, and $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)$. Using the Quillen map $\mathrm{BP} \to \mathrm{MU}_{(p)}$, as well as various unit maps such as $\mathrm{MU}_{(p)} \to \mathrm{TC}^-(\ell/\mathrm{MU}_{(p)}) = \mathrm{TC}^-(\ell/\mathrm{MU})$, we obtain a map from the Adams–Novikov E_1-page to the cobar complex computing $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$. In particular, any Adams–Novikov cocycle names a cocycle in the cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$, and, via the canonical maps, also names cocycles in the cobar complexes for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)$ and $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$.

When referring to elements of the Adams–Novikov E_1-page, we will use the (standard) conventions of [Wil82, §3]. For example, $t_1^2 + v_1t_1$, which is a cocycle at $p = 2$ representing $\nu \in \pi_*(\mathbb{S})$, also represents a class of Adams weight 1 in $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$.

Remark 6.3.2. We will need the fact that t_1 is a cocycle on the Adams–Novikov E_1-page, or in other words that $d(t_1) = 0$. This furthermore implies that $d(t_1^p) \equiv 0$ modulo p.

Definition 6.3.3. Using the unit map $\mathrm{BP}^{\mathrm{hS}^1} \to \mathrm{MU}_{(p)}^{\mathrm{hS}^1} \to \mathrm{TC}^-(\ell/\mathrm{MU})$, we send the standard complex orientation of BP from [Wil82, Section 3] to a class in the cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$ that we name t.

Remark 6.3.4. We have the standard formula from [Wil82, Lemma 3.14]:

$$\eta_R(t) = c(t +_{\mathbb{G}} t_1t^p +_{\mathbb{G}} t_2t^{p^2} + \cdots),$$

where $+_{\mathbb{G}}$ denotes addition using the BP_* formal group law and c denotes the conjugation action on $\mathrm{BP}_*\mathrm{BP}$, extended by the rule that $c(t) = t$. All we really need from this formula is the fact that $\eta_R(t) \equiv t + t_1t^p$ modulo p, v_1, and t^{p+2}. This implies that $\eta_R(t^p) \equiv t^p + t_1^pt^{p^2}$ modulo p, v_1, and t^{p^2+2p}.

Remark 6.3.5. Each term $\mathrm{THH}_\star(\ell/\mathrm{MU}^{\otimes s+1})$ in the cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$ is a free $\pi_*\ell$-module, and so has (p, v_1) as a regular sequence. By the mod (p, v_1) cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell)$ we will mean the complex obtained by modding out both p and v_1 levelwise. This complex computes $(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell))/(p, v_1)$, which is the E_2-term of the motivic spectral sequence for $\pi_\star(\mathrm{THH}(\ell) \otimes_\ell \mathbb{F}_p)$.

We may also speak of the mod (p, v_1) cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)$, which has sth term isomorphic to $\mathrm{THH}_\star(\ell/\mathrm{MU}^{\otimes s+1})[\![t]\!]/(p, v_1)$, and analogously the mod (p, v_1) cobar complex for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)$. At the prime 2, where $V(1) = \mathbb{S}/(2, v_1)$ does not exist, these cobar complexes do not necessarily compute the E_2-page of any topologically relevant spectral sequence.

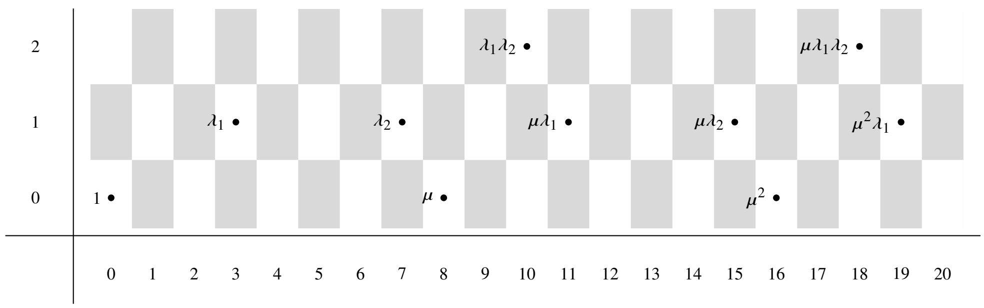
§6.4. Hochschild homology. The motivic spectral sequence for $\mathrm{THH}(\ell) \otimes_\ell \mathbb{F}_p$ was computed in [HW20, §6.1], and we recall the results below.

Lemma 6.4.1. [HW20] *In the mod (p, v_1) cobar complex for* $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}_{\star}(\ell)$*, the elements v_2 and t_1 are both divisible by t.*

Definition 6.4.2. We write $\sigma^2 v_2$ and $\sigma^2 t_1$ for the elements in the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell)$ defined by the relations $t\sigma^2 v_2 = v_2$ and $t\sigma^2 t_1 = t_1$, respectively. Using the canonical map between the cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell)$ and the cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)$, we may also speak of classes $\sigma^2 v_2$ and $\sigma^2 t_1$ in the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)$

Theorem 6.4.3. [HW20] *The motivic spectral sequence for $\pi_\star\left(\mathrm{THH}(\ell) \otimes_\ell \mathbb{F}_p\right)$ collapses. In the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)$, the elements $\sigma^2 v_2, \sigma^2 t_1$, and $(\sigma^2 t_1)^p$ are cocycles representing μ, λ_1, and λ_2, respectively.*

The first 20 stems of the $E_2 = E_\infty$ page of the motivic spectral sequence for $\pi_*\left(\mathrm{THH}(\mathrm{ku}) \otimes_{\mathrm{ku}} \mathbb{F}_2\right)$



Remark 6.4.4. The reader may find it enlightening to understand why $\sigma^2 v_2$, $\sigma^2 t_1$, and $(\sigma^2 t_1)^p$ are cocycles in the cobar complex for $(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell))/(p, v_1)$. First, in the mod (p, v_1, t^{p+2}) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell)$ we may make the calculations

$$0 = d(v_2) = d(t\sigma^2 v_2) = \eta_R(t)\eta_R(\sigma^2 v_2) - t\sigma^2 v_2 = (t + t^p t_1)\eta_R(\sigma^2 v_2) - t\sigma^2 v_2,$$

$$0 = d(t_1) = d(t\sigma^2 t_1) = t^p t_1 | \sigma^2 t_1 - t d(\sigma^2 t_1), \text{ and}$$

$$0 = d(t_1^p) = d(t^p (\sigma^2 t_1)^p) = -t^p d((\sigma^2 t_1)^p).$$

The above equations immediately imply that $td(\sigma^2 v_2) \equiv td(\sigma^2 t_1) \equiv 0$ modulo t^2 and $t^p d((\sigma^2 t_1)^p) \equiv 0$ modulo t^{p+1}. We may divide by powers of t, which are not zero divisors in any term of the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell)$, and learn that $d(\sigma^2 v_2) \equiv d(\sigma^2 t_1) \equiv d((\sigma^2 t_1)^p) \equiv 0$ modulo t. Finally, the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)$ is obtained from the mod (p, v_1) cobar complex for $gr^{\star}_{\mathrm{mot}}\mathrm{TC}^{-}(\ell)$ by killing the element t.

Remark 6.4.5. Each term $\mathrm{THH}_\star(\ell/\mathrm{MU}^{\otimes s+1})/(p, v_1)$ in the mod (p, v_1) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell)$ is a free $\mathbb{F}_p[\sigma^2 v_2]$-module. This means in particular that $(p, v_1, v_2 = t\sigma^2 v_2)$ is a regular sequence in $\mathrm{TC}^{-}(\ell/\mathrm{MU}^{\otimes s+1})$ and in $\mathrm{TP}(\ell/\mathrm{MU}^{\otimes s+1})$, so we may profitably speak of e.g. the mod (p, v_1, v_2) cobar complex for $\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell)$.

§6.5. Prismatic cohomology. In this section we will calculate $(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TP}(\ell))/(p, v_1, v_2)$ and $(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell))/(p, v_1, v_2)$. Our strategy will be to use the second filtration, $\mathrm{fil}^{\bullet}_{+}$, introduced in §2. We will call the resulting spectral sequences the *algebraic t-Bockstein spectral sequences*, and they have signature:

$$\pi_*(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{THH}(\ell))[t] \Rightarrow \pi_*(\mathrm{gr}^{\star}_{\mathrm{mot}}\,\mathrm{TC}^{-}(\ell))$$

$$\pi_*(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell))[t^{\pm 1}] \Rightarrow \pi_*(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell))$$

The map between the first and second spectral sequences which inverts t converges to the canonical map $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^- \to \mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}$.

The elements p and v_1 are detected by the likewise named elements in $\pi_*(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{THH}(\ell))$, and after killing these elements v_2 is detected by $t\mu$. We may therefore choose appropriately filtered lifts of p, v_1, and v_2 and take the cofiber by each element in turn to obtain spectral sequences with signatures:

$$\mathbb{F}_p[t,\mu]/(t\mu) \otimes \Lambda(\lambda_1,\lambda_2) \Rightarrow \pi_*(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\ell)/(p,v_1,v_2))$$

$$\mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda_1,\lambda_2) \Rightarrow \pi_*(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\ell)/(p,v_1,v_2))$$

We explain the behavior of the second spectral sequence in the following theorem.

Theorem 6.5.1. *The algebraic t-Bockstein spectral sequence for* $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))\,/(p,v_1,v_2)$ *has* E_1*-page given by* $\mathbb{F}_p[t^{\pm 1}] \otimes \Lambda(\lambda_1,\lambda_2)$. *The spectral sequence is determined by multiplicative structure together with the following facts:*

(1) *The classes* t^{p^2}, λ_1, *and* λ_2 *are permanent cycles.*
(2) *There is a* d_p *differential* $d_p(t) = t^{p+1}\lambda_1$.
(3) *There is a* d_{p^2} *differential* $d_{p^2}(t^p) = t^{p^2+p}\lambda_2$.

The E_∞*-page is* $\mathbb{F}_p[t^{\pm p^2}] \otimes \Lambda(\lambda_1,\lambda_2)$.

Proof. In the mod (p, v_1, v_2) cobar complex, we compute

$$\eta_R(t) \equiv t + t^p t_1 \text{ modulo } t^{p+2},$$

and then note that $t^p t_1 = t^{p+1}\sigma^2 t_1$. Since λ_1 is represented by $\sigma^2 t_1$ in the mod (p, v_1) cobar complex for $\mathrm{THH}(\ell)$, we conclude the claimed d_p differential. Taking pth powers, we compute

$$\eta_R(t^p) \equiv t^p + t^{p^2} t_1^p \text{ modulo } t^{p^2+2p},$$

and then note that $t^{p^2} t_1^p = t^{p^2+p}(\sigma^2 t_1)^p$. Since λ_2 is represented by $(\sigma^2 t_1)^p$ in the mod (p, v_1) cobar complex for $\mathrm{THH}(\ell)$, we conclude the claimed d_{p^2} differential.

It remains to see that λ_1, λ_2, and t^{p^2} are permanent cycles. First, we note that

$$\eta_R(t^{p^2}) \equiv t^{p^2} + t^{p^3} t_1^{p^2} = t^{p^2} + t^{p^3+p^2}(\sigma^2 t_1)^{p^2} \text{ modulo } t^{p^3+2p^2}.$$

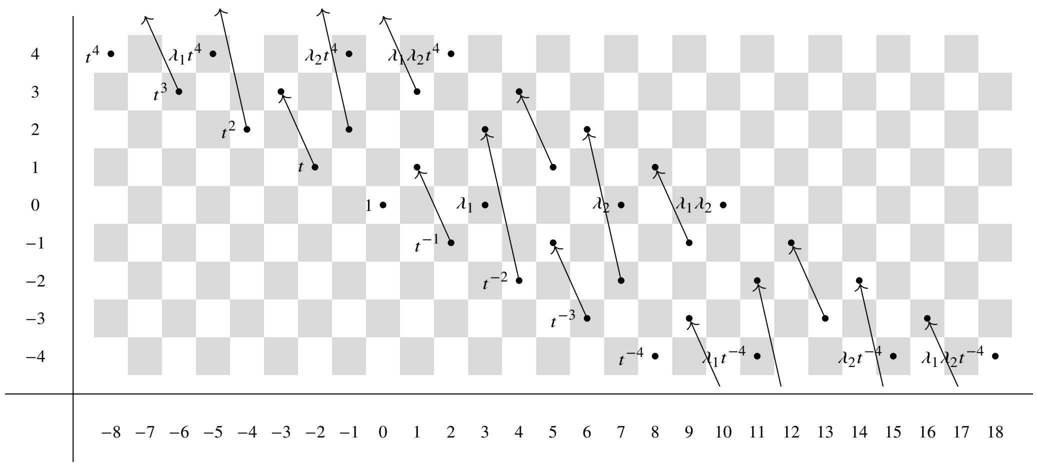
This is the same as t^{p^2} modulo $t^{p^3+p^2}$, and so t^{p^2} must survive to the E_{p^3} page of the spectral sequence. For sparsity reasons, it follows that t^{p^2} is a permanent cycle. To see that λ_2 is a permanent cycle, we make the following computation in the mod (p, v_1, v_2) cobar complex:

$$\begin{aligned} 0 &= d(t_1^p) \\ &= d(t^p(\sigma^2 t_1)^p) \\ &= t^{p^2} t_1^p | (\sigma^2 t_1)^p - t^p d((\sigma^2 t_1)^p) \text{ modulo } t^{p^2+2p} \\ &= t^{p^2+p}(\sigma^2 t_1)^p | (\sigma^2 t_1)^p - t^p d((\sigma^2 t_1)^p) \text{ modulo } t^{p^2+2p}. \end{aligned}$$

In particular, $d((\sigma^2 t_1)^p)$ is zero modulo t^{p^2}, so λ_2 survives to the E_{p^2}-page of the spectral sequence. For sparsity reasons it follows that λ_2 is a permanent cycle.

Finally, the only way λ_1 could fail to be a permanent cycle is via a differential $d_{p^2}(\lambda_1) \doteq \lambda_1\lambda_2 t^{p^2}$. If such a differential occurred, we would learn that $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))\,/(p,v_1,v_2)$ is trivial in degree $2p-1$. However, it is also the E_2-page of a spectral sequence converging to $\mathrm{THH}(\ell)^{tC_p} \otimes_\ell \mathbb{F}_p$, by Remark 6.2.3, and the latter object is nontrivial in degree $2p-1$. □

The algebraic t-Bockstein spectral sequence for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TP}(\mathrm{ku})/(2, v_1, v_2)$



Corollary 6.5.2. *The algebraic t-Bockstein spectral sequence for $(\mathrm{gr}^\star_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p, v_1, v_2)$ has E_1-page given by $\mathbb{F}_p[t,\mu]/(t\mu) \otimes \Lambda(\lambda_1, \lambda_2)$. The spectral sequence is determined by multiplicative structure together with the following facts:*

(1) *The classes $t^{p^2}, \lambda_1, \lambda_2$, and μ are permanent cycles.*

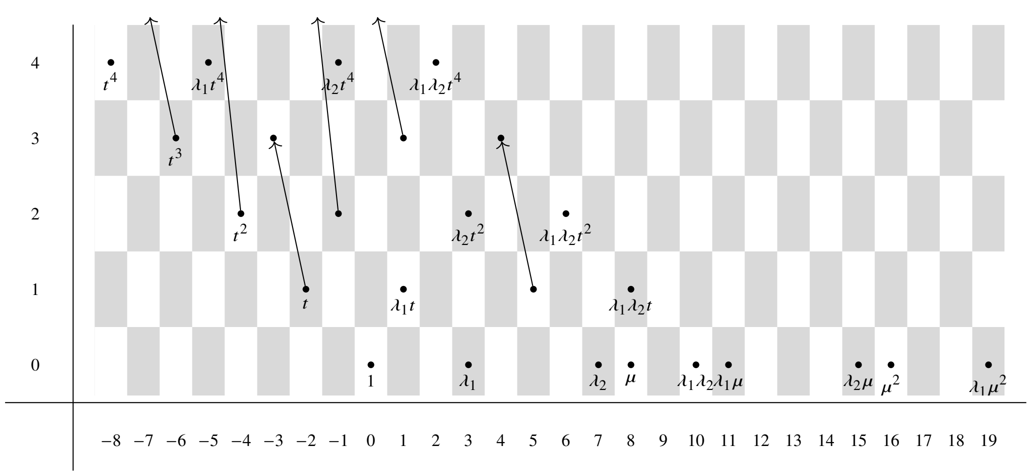
(2) *There is a d_p differential $d_p(t) = t^{p+1}\lambda_1$.*

(3) *There is a d_{p^2} differential $d_{p^2}(t^p) = t^{p^2+p}\lambda_2$.*

The E_∞-page is $\mathbb{F}_p[t^{p^2},\mu]/(t^{p^2}\mu) \otimes \Lambda(\lambda_1,\lambda_2) \oplus \mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0 < d < p\}$.

Proof. The canonical map from the cobar complex for $\mathrm{TC}^-(\ell)$ to the cobar complex for $\mathrm{TP}(\ell)$ induces a map of t-Bockstein spectral sequences, from which we can read off the claimed differentials and the facts that t^{p^2}, λ_1, and λ_2 are permanent cycles. No other differentials are possible, by sparsity. □

Stems -8 to 20 of the algebraic t-Bockstein spectral sequence for $\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}^-(\mathrm{ku})/(2, v_1, v_2)$



Corollary 6.5.3. *There are isomorphisms of $\mathbb{F}_p$ vector spaces*

$$(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1,v_2) \cong \mathbb{F}_p[t^{p^2},\mu]/(t^{p^2}\mu)\otimes\Lambda(\lambda_1,\lambda_2) \\ \oplus\mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0<d<p\},$$

$$(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2) \cong \mathbb{F}_p[t^{\pm p^2}]\otimes\Lambda(\lambda_1,\lambda_2)$$

The canonical map

$$(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1,v_2) \to (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2)$$

sends each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{kp^2}$ to the correspondingly named class in the target, where $\epsilon_1,\epsilon_2\in\{0,1\}$ and $k\geq 0$. It is zero on all other classes.

Proof. The only subtle point is to prove that classes not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{kp^2}$ all map to zero. The calculations above prove this to be the case after taking t-adic associated graded, and all non-zero classes in $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2)$ are in low enough t-adic filtrations that no filtration jumps are possible. □

§6.6. Syntomic cohomology. By Corollary 6.5.3, we understand the canonical map

$$\mathrm{can}: (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1,v_2) \to (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2).$$

To compute $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}(\ell))/(p,v_1,v_2)$, it remains to understand the Frobenius map

$$\varphi: (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1,v_2) \to (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2).$$

For this, we contemplate the following diagram:

$$\begin{tikzcd}
(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1) \arrow[r, "\varphi"] \arrow[d] & (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1) \arrow[d] \\
(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell))/(p,v_1) \arrow[r, "\varphi"] & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)^{\mathrm{t}C_p}\right)/(p,v_1)
\end{tikzcd}$$

Since $v_2=0$ in $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell))/(p,v_1)$, for example because $\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)$ is an algebra over $\mathrm{gr}^\star_{\mathrm{ev}}\,\ell$, the diagram factors through a square of the form

$$\begin{tikzcd}
(\mathrm{gr}^*_{\mathrm{mot}}\,\mathrm{TC}^-(\ell))/(p,v_1,v_2) \arrow[r, "\varphi"] \arrow[d, "f"] & (\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2) \arrow[d, "g"] \\
(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell))/(p,v_1) \arrow[r, "\varphi"] & \left(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell)^{\mathrm{t}C_p}\right)/(p,v_1)
\end{tikzcd}$$

Here, the map f is an isomorphism from the 0-line of the spectral sequence of Corollary 6.5.2 onto $(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{THH}(\ell))/(p,v_1)$. It is trivial on classes above the 0-line. The map g is the isomorphism of Theorem 6.2.1.

Corollary 6.6.1. *In terms of the isomorphisms*

$$(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TC}^-(\ell))/(p,v_1,v_2) \cong \mathbb{F}_p[t^{p^2},\mu]/(t^{p^2}\mu)\otimes\Lambda(\lambda_1,\lambda_2) \\ \oplus\mathbb{F}_p\{t^d\lambda_1, t^{pd}\lambda_2, t^d\lambda_1\lambda_2, t^{pd}\lambda_1\lambda_2 \mid 0<d<p\},$$

$$(\mathrm{gr}^*_{\mathrm{mot}}\mathrm{TP}(\ell))/(p,v_1,v_2) \cong \mathbb{F}_p[t^{\pm p^2}]\otimes\Lambda(\lambda_1,\lambda_2)$$

of Corollary 6.5.3, the Frobenius is trivial on classes not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ where $k \geq 0$ and $\epsilon_1, \epsilon_2 \in \{0,1\}$. On the other hand, the Frobenius sends each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ to an $\mathbb{F}_p^\times$ multiple of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$.

Proof. The map f is already trivial on every class not of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$. Theorem 6.1.2, together with the fact that g is an isomorphism, implies that each class of the form $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$ has non-trivial Frobenius image. The only non-trivial classes in the codomain, in the same degree as $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}\mu^k$, are $\mathbb{F}_p^\times$ multiples of the class named $\lambda_1^{\epsilon_1}\lambda_2^{\epsilon_2}t^{-p^2k}$. □

We can now deduce the main theorem of this section:

Proof of Theorem 6.0.2. We deduce the first part of Theorem 6.0.2 as an immediate consequence of the combination of Corollary 6.5.3 and Corollary 6.6.1, with the symbol ∂ decorating classes in $(\mathrm{gr}^\star_{\mathrm{mot}}\,\mathrm{TC}(\ell))/(p, v_1, v_2)$ that come from the cokernel of φ − can. The second part of Theorem 6.0.2, about the v_2-Bockstein spectral sequence, follows by the argument given immediately after the theorem statement, which relies on the elementary lemma below applied to $R = \mathrm{gr}^\star_{\mathrm{ev}}(\mathbb{S})/(p, v_1)$ and $M = \mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p, v_1)$. We remind the reader that v_2 lives in π_{2p^2-2} of the (p^2-1)'st graded piece of $\mathrm{gr}^\star_{\mathrm{ev}}(\mathbb{S})/(p, v_1)$, and that our convention for displaying spectral sequences is to draw a term from $\pi_n L^a$ of a graded object L in column n and row $2a - n$. □

Lemma 6.6.2. *Let R be a graded ring, M a graded R-module. If $L^\star$ is graded, write $\pi_{n,a}L$ for $\pi_n(L^a)$. Then the Bockstein spectral sequence for $\pi_{*,*}(M^\wedge_x)$ associated to an element $x \in \pi_{n,a}(R)$, with E_1-page $\pi_{*,*}(M/x)[x]$, has d_r differentials that send elements of bidegree (m, b) to elements of bidegree $(m + rn - 1, b + ra)$.*

To finish the paper, we record proofs of the final two theorems mentioned in the introduction.

Corollary 6.6.3. *For any prime number p and p-local type 3 complex M, $M_*\mathrm{TC}(\ell)$ is finite.*

Proof. By thick subcategory considerations, it suffices to prove this for M equal to a generalized Moore spectrum of the form $\mathbb{S}/(p^i, v_1^j, v_2^k)$, where $j \gg i$ and $k \gg j$ so that killing (p^i, v_1^j, v_2^k) is a well-defined operation in $\mathrm{MU}_*\mathrm{MU}$-comodules. There is then a spectral sequence converging to $M_*\mathrm{TC}(\ell)$ beginning with $\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p^i, v_1^j, v_2^k)$. The latter object may be resolved by finitely many copies of $\mathrm{gr}^\star_{\mathrm{mot}}(\mathrm{TC}(\ell))/(p, v_1, v_2)$, and so is finite. □

As explained in [HW20, §3], Corollary 6.6.3 implies that the map

$$\mathrm{TC}(\ell)_{(p)} \to L_2^f\,\mathrm{TC}(\ell)_{(p)}$$

is a π_*-iso in degrees $* \gg 0$, which can be seen as a telescopic analog of the Lichtenbaum–Quillen conjecture. In fact, one can localize at a wedge of Morava K-theories rather than a wedge of telescopes, which we record as our final result.

Theorem 6.6.4. *The telescope conjecture is true of $\mathrm{TC}(\ell)$. In other words, the natural map*

$$L_2^f\,\mathrm{TC}(\ell) \to L_2\mathrm{TC}(\ell)$$

is an equivalence.

Proof. We say that the height 2 telescope conjecture holds for a spectrum X if the natural map $L_2^f X \to L_2 X$ is an equivalence. First note that, since L_2 and L_2^f are smashing localizations, if the height 2 telescope conjecture holds for a ring R then it also holds for every R-module. We will prove the height 2 telescope conjecture for $\mathrm{TC}^-(\ell)$. Since $\mathrm{TP}(\ell)$ is a module over $\mathrm{TC}^-(\ell)$, we may conclude the height 2 telescope conjecture for $\mathrm{TP}(\ell)$ and then, by the Nikolaus–Scholze fiber sequence, for $\mathrm{TC}(\ell)$.

By the work of Mahowald and Miller [Mah82; Mil81], to prove the height 2 telescope conjecture for $\mathrm{TC}^-(\ell)$ it suffices to prove it for $F \otimes \mathrm{TC}^-(\ell)$, where F is a p-local finite type 2 complex. For this, consider the equivalence

$$F \otimes \mathrm{TC}^-(\ell) \simeq F \otimes \left(\lim_{\Delta} \mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})\right) \simeq \lim_{\Delta} \left(F \otimes \mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})\right),$$

where we may pass F inside of the totalization because it is finite. Now, the L_2^f localization of $F \otimes \mathrm{TC}^-(\ell)$ is given by $v_2^{-1} F \otimes \mathrm{TC}^-(\ell)$. A corollary of our work above is that the motivic spectral sequence for $F_* \mathrm{TC}^-(\ell)$ has, when displayed in Adams grading, an eventual horizontal vanishing line. It follows (e.g. from [CM21, Lemma 2.34]) that

$$v_2^{-1} F \otimes \mathrm{TC}^-(\ell) \simeq \lim_{\Delta} \left(v_2^{-1} F \otimes \mathrm{TC}^-(\ell/\mathrm{MU}^{\otimes \bullet+1})\right).$$

Each term inside the totalization is an L_2^f-local MU-module, and hence is L_2-local. Thus, the totalization is also L_2-local. ◻

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