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The η-local motivic sphere ☆

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ABSTRACT

We compute the h_1-localized cohomology of the motivic Steenrod algebra over $\mathbb{C}$. This serves as the input to an Adams spectral sequence that computes the motivic stable homotopy groups of the η-local motivic sphere. We compute some of the Adams differentials, and we state a conjecture about the remaining differentials.

1. Introduction

Consider the Hopf map $\eta : \mathbb{A}^2 \setminus \{0\} \longrightarrow \mathbb{P}^1$ that takes (x, y) to $[x : y]$. In motivic homotopy theory, $\mathbb{A}^2 \setminus \{0\}$ and $\mathbb{P}^1$ are models for the motivic spheres $S^{3,2}$ and $S^{2,1}$ respectively. Therefore, η represents an element of the stable motivic homotopy group $\pi_{1,1}$.

Computations of motivic stable homotopy groups share many similarities to the classical computations, but the motivic computations also exhibit "exotic" non-classical phenomena. One of the first examples is that η is not nilpotent, i.e., η^k is non-zero for all k [11].

Working over $\mathbb{C}$ (or any algebraically closed field of characteristic zero), we have an Adams spectral sequence for computing motivic stable 2-complete homotopy groups with good convergence properties [12, 6,7]. The E_2-page of this spectral sequence is the cohomology Ext_A of the motivic Steenrod algebra A. In the Adams spectral sequence, η is detected by the element h_1 of Ext_A. The failure of η to be nilpotent is detected by the fact that h_1^k is a non-zero permanent cycle for all k.

A further investigation of the motivic Adams E_2-page reveals a number of other classes that survive h_1-localization, i.e., classes x such that $h_1^k x$ is non-zero for all k. The first few are c_0 in the 8-stem, Ph_1 in

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the 9-stem, d_0 in the 14-stem, and e_0 in the 17-stem. A fairly predictable pattern emerges, involving classes that all map non-trivially to the cohomology of motivic $A(2)$ [8].

However, in the 46-stem, a surprise occurs. The class B_1 is h_1-local. This element is not detected by the cohomology of motivic $A(2)$. At this point, it has become clear that an algebraic computation of the h_1-localized cohomology $\mathrm{Ext}_A[h_1^{-1}]$ is an interesting and non-trivial problem.

The first goal of this article is calculate $\mathrm{Ext}_A[h_1^{-1}]$. We will show that it is a polynomial algebra over $\mathbb{F}_2[h_1^{\pm 1}]$ on infinitely many generators.

Theorem 1.1. *The h_1-localized algebra $\mathrm{Ext}_A[h_1^{-1}]$ is a polynomial algebra over $\mathbb{F}_2[h_1^{\pm 1}]$ on generators v_1^4 and v_n for $n \geq 2$, where:*

(1) *v_1^4 is in the 8-stem and has Adams filtration 4 and weight 4.*
(2) *v_n is in the $(2^{n+1}-2)$-stem and has Adams filtration 1 and weight $2^n - 1$.*

Although it is simple to state, this is actually a surprising answer. The elements d_0, e_0, and e_0g are indecomposable elements of Ext_A that are all h_1-local. From consideration of the May spectral sequence, or from the cohomology of $A(2)$, one might expect to have the relation $e_0^3 + d_0 \cdot e_0g = 0$.

In the terms of Theorem 1.1, $e_0^3 + d_0 \cdot e_0g$ corresponds to $h_1^9 v_3^3 + h_1^9 v_2^2 v_4$. Theorem 1.1 says that this expression is non-zero after h_1-localization, so it is non-zero before localization as well. The only possibility is that $e_0^3 + d_0 \cdot e_0g$ equals $h_1^5 B_1$.

The relation $e_0^3 + d_0 \cdot e_0g = h_1^5 B_1$ in Ext_A is hidden in the motivic May spectral sequence. As described in [9], it is tightly connected to the classical Adams differential $d_3(h_1h_5e_0) = h_1^2 B_1$ [3, Corollary 3.6].

Our h_1-local calculation surely implies other similarly exotic relations in higher stems. We do not yet possess a sufficiently detailed understanding of Ext_A in that range, so we cannot identify any explicit examples with certainty. However, we expect to see a hidden relation $e_0(e_0g)^2 + d_0 \cdot e_0g^3 = h_1^5 \cdot g^2 B_1$ in Ext_A in the 91-stem. Here, we anticipate that e_0g^3 and g^2B_1 are indecomposable elements.

So far, we have only discussed the entirely algebraic question of computing $\mathrm{Ext}_A[h_1^{-1}]$, which informs us about the structure of Ext_A. But $\mathrm{Ext}_A[h_1^{-1}]$ is also the E_2-page of an Adams spectral sequence that converges to the 2-complete motivic stable homotopy groups of the η-local motivic sphere $S^{0,0}[\eta^{-1}]$, which is the homotopy colimit of the sequence

$$S^{0,0} \xrightarrow{\eta} S^{-1,-1} \xrightarrow{\eta} S^{-2,-2} \xrightarrow{\eta} \cdots.$$

In order to compute $\pi_{*,*}S^{0,0}[\eta^{-1}]$, we thus only need to compute Adams differentials on $\mathrm{Ext}_A[h_1^{-1}]$. There are indeed non-trivial differentials. In the non-local case, we know that $d_2(e_0) = h_1^2 d_0$ and $d_2(e_0g) = h_1^2 e_0^2$ [9]. This implies the analogous h_1-local differentials $d_2(v_3) = h_1 v_2^2$ and $d_2(v_4) = h_1 v_3^2$.

Unfortunately, we have not been able to identify all of the Adams differentials. We expect the answer to turn out as stated in Conjecture 1.2.

Conjecture 1.2.[1] *For all $k \geq 3$, there is an Adams differential $d_2(v_k) = h_1 v_{k-1}^2$.*

Conjecture 1.2 has the following immediate consequences.

[1] After the first version of this manuscript appeared, Conjecture 1.2 was proved by Andrews and Miller [2].

Conjecture 1.3.[2]

(1) *The E_∞-page of the h_1-local Adams spectral sequence is*

$$\mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2]/v_2^2.$$

(2) *The 2-complete motivic stable homotopy groups of the η-local motivic sphere are*

$$\pi_{*,*}(S^{0,0}[\eta^{-1}]) \cong \mathbb{F}_2[\eta^{\pm 1}, \mu, \epsilon]/\epsilon^2,$$

where η has degree $(1,1)$; μ has degree $(9,5)$; and ϵ has degree $(8,5)$.

Proof. Given the Adams differentials proposed in Conjecture 1.2, we can compute that the h_1-local E_3-page is equal to $\mathbb{F}_2[h_1^{\pm 1}][v_1^4, v_2]/v_2^2$. For degree reasons, there are no possible higher differentials, so this expression is also equal to the h_1-local E_∞-page.

There are no possible hidden extensions in E_∞, so we obtain $\pi_{*,*}(S^{0,0}[\eta^{-1}])$ immediately. □

Remark 1.4. The h_1-localization of the element Ph_1 of Ext_A is $h_1 v_1^4$, and μ is the standard notation for the element of $\pi_{9,5}$ detected by Ph_1. The h_1-localization of the element c_0 of Ext_A is $h_1^2 v_2$, and ϵ is the standard notation for the element of $\pi_{8,5}$ detected by c_0.

We present one more consequence of Conjecture 1.2.

Theorem 1.5.[3] *The following are equivalent:*

(1) *Conjecture 1.2 holds.*
(2) *At $p = 2$, the α_1-localization of the classical Adams–Novikov spectral sequence E_2-page is a free $\mathbb{F}_2[\alpha_1^{\pm 1}]$-module with basis consisting of elements of the form $\alpha_{k/b}$ for $k = 1$ and $k \geq 3$.*

Here b is an integer that depends on k. The point is that $\alpha_{k/b}$ is the generator of the classical Adams–Novikov E_2-page in degree $(2k - 1, 1)$.

Proof. Recall that τ is an element of $\pi_{0,-1}$. Because $\tau\eta^4$ is zero, we know that the η-localization $C\tau[\eta^{-1}]$ of the cofiber $C\tau$ of τ splits as $S^{0,0}[\eta^{-1}] \vee S^{1,-1}[\eta^{-1}]$. Therefore, $\pi_{*,*}(C\tau[\eta^{-1}])$ is the same as two copies of $\pi_{*,*}(S^{0,0}[\eta^{-1}])$.

We explain in [9, Proposition 6.2.5] that $\pi_{*,*}(C\tau)$ is equal to the classical Adams–Novikov E_2-page. Therefore, $\pi_{*,*}(C\tau[\eta^{-1}])$ is the same as the α_1-localization of the Adams–Novikov E_2-page. □

The origin of this work lies in the second author's attempt to analyze the Adams spectral sequence beyond the 45-stem. The h_1-local calculations discussed in this article are a helpful tool in the analysis of Adams differentials. We expect that the h_1-local calculations will continue to be a useful tool in the further analysis of Adams differentials. For example, our work leads us to anticipate an Adams differential $d_2(e_0 g^3) = h_1^2 (e_0 g)^2$ in the 77-stem.

[2] Because Conjecture 1.2 is settled, so is Conjecture 1.3.
[3] Andrews and Miller [2] show that part (2) of Theorem 1.5 holds, thereby establishing Conjecture 1.2.

In this article, we are working exclusively in motivic homotopy theory over $\mathbb{C}$. A natural extension is to consider h_1-local and η-local calculations over other fields. Preliminary calculations over $\mathbb{R}$ show that the picture is significantly more complicated. We do not yet understand how η-local motivic calculations depend on the base field. We plan to explore this in more detail in future work.

The questions studied in this article become trivial in the classical situation, where h_1^4 is zero in the cohomology of the classical Steenrod algebra, and η^4 is zero in the classical 4-stem. This is consistent with the principle that τ-localization corresponds to passage from the motivic to the classical situations [9], and that τh_1^4 and $\tau\eta^4$ are both zero motivically.

However, η is not nilpotent in $\mathbb{Z}/2$-equivariant stable homotopy groups. The equivariant analogues of our calculations are interesting open questions.

Our work raises the question of why Nishida's nilpotence theorem [14] fails in motivic homotopy theory. In addition to η, there are other non-nilpotent elements, such as the element μ of $\pi_{9,5}$ detected by Ph_1. More generally, the elements μ_{8k+1} of $\pi_{8k+1,4k+1}$ detected by $P^k h_1$ are also non-nilpotent. Starting in the 32-stem, there are other families of elements that are not nilpotent. These elements will be the subject of future work.

1.1. Organization

Section 2 contains a review of the motivic Steenrod algebra and sets our notation. Section 3 computes $\mathrm{Ext}_A[h_1^{-1}]$, as stated in Theorem 1.1. For completeness, we also discuss $\mathrm{Ext}_{A(2)}[h_1^{-1}]$.

Section 4 discusses the same computation from the point of view of the motivic May spectral sequence. The point of this section is that it allows us to analyze in Section 5 the localization map $\mathrm{Ext}_A \longrightarrow \mathrm{Ext}_A[h_1^{-1}]$ in detail through a range. This leads to some hidden relations in Ext_A that are needed in [9]. The localization map is essential for deducing information about Adams differentials in Ext_A from Adams differentials in $\mathrm{Ext}_A[h_1^{-1}]$. We also consider the May spectral sequence and the localization map for $A(2)$ in Sections 4 and 5. These sections are intended to be read in conjunction with the charts in Section 8.

Section 6 gives some computations of Adams differentials in support of Conjecture 1.2. We also discuss the role of a speculative "motivic modular forms" spectrum.

Much of the data for our computations, especially regarding the May spectral sequence, is given in tables to be found in Section 7.

2. Background

We continue with notation from [9] as follows:

(1) $\mathbb{M}_2$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients.
(2) A is the mod 2 motivic Steenrod algebra over $\mathbb{C}$, and $A_{*,*}$ is its dual.
(3) $A(n)$ is the $\mathbb{M}_2$-subalgebra of A generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$, and $A(n)_*$ is its dual.
(4) Ext_A is the trigraded ring $\mathrm{Ext}_A(\mathbb{M}_2, \mathbb{M}_2)$.
(5) More generally, Ext_B is the trigraded ring $\mathrm{Ext}_B(\mathbb{M}_2, \mathbb{M}_2)$ for any Hopf algebra B over $\mathbb{M}_2$.
(6) A_{cl} is the mod 2 classical Steenrod algebra, and A_*^{cl} is its dual.
(7) $A(n)_{\mathrm{cl}}$ is the $\mathbb{M}_2$-subalgebra of A_{cl} generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^4, \ldots, \mathrm{Sq}^{2^n}$, and $A(n)_*^{\mathrm{cl}}$ is its dual.
(8) $\mathrm{Ext}_{A_{\mathrm{cl}}}$ is the bigraded ring $\mathrm{Ext}_{A_{\mathrm{cl}}}(\mathbb{F}_2, \mathbb{F}_2)$.
(9) More generally, Ext_B is the trigraded ring $\mathrm{Ext}_B(\mathbb{F}_2, \mathbb{F}_2)$ for any Hopf algebra B over $\mathbb{F}_2$.

The following two theorems of Voevodsky are the starting points of our calculations.

Theorem 2.1. *(See [17].)* $\mathbb{M}_2$ *is the bigraded ring* $\mathbb{F}_2[\tau]$*, where* τ *has bidegree* $(0, 1)$.

Our main object of study will be a localization of Ext_A. It will be more convenient for us to work with the dual $A_{*,*} = \mathrm{Hom}_{\mathbb{M}_2}(A, \mathbb{M}_2)$.

Theorem 2.2. *(See [18; 19, Theorem 12.6].) The dual motivic Steenrod algebra $A_{*,*}$ is generated as an $\mathbb{M}_2$-algebra by ξ_i and τ_i, of degrees $(2(2^i-1), 2^i-1)$ and $(2^{i+1}-1, 2^i-1)$ respectively, subject to the relations*

$$\tau_i^2 = \tau\xi_{i+1}.$$

The coproduct is given on the generators by the following formulae, in which $\xi_0 = 1$:

$$\Delta(\tau_k) = \tau_k \otimes 1 + \sum_i \xi_{k-i}^{2^i} \otimes \tau_i$$

$$\Delta(\xi_k) = \sum_i \xi_{k-i}^{2^i} \otimes \xi_i.$$

Remark 2.3. The quotient $A_{*,*}/\tau = A_{*,*} \otimes_{\mathbb{M}_2} \mathbb{F}_2$ is analogous to the odd-primary classical dual Steenrod algebra, in the sense that there is an infinite family of exterior generators τ_i and an infinite family of polynomial generators ξ_i. On the other hand, the localization $A_{*,*}[\tau^{-1}]$ is analogous to the mod 2 classical dual Steenrod algebra, which has only polynomial generators τ_i.

2.1. Ext *groups*

We are interested in computing a localization of Ext_A. Before localization, this is a trigraded object. In [9], classes in Ext_A are described in degrees of the form (s, f, w), where:

(1) f is the Adams filtration, i.e., the homological degree.
(2) $s + f$ is the internal degree, corresponding to the first coordinate in the bidegrees of A.
(3) s is the stem, i.e., the internal degree minus the Adams filtration.
(4) w is the motivic weight.

In the cobar complex, ξ_1 represents an element h_1 of Ext_A in degree $(1, 1, 1)$. Because we will invert h_1, it is convenient to choose a new grading that is more h_1-invariant. Except where otherwise noted, we will use the grading (t, f, c), where:

(1) $t = s - w$ is the Milnor–Witt stem.
(2) f is the Adams filtration.
(3) $c = s + f - 2w$ is the Chow degree [9], which turns out to be a convenient grading for calculational purposes.

The terminology "Milnor–Witt stem" arises from the work of Morel [11], which describes the motivic stable homotopy groups $\pi_{s,w}$ with $s - w = 0$ in terms of Milnor–Witt K-theory.

The terminology "Chow degree" arises from the fact that the grading $s + f - 2w$ is a natural index from the higher Chow group perspective on motivic cohomology [4].

3. The h_1-local cohomology of A

The goal of this section is to compute $\mathrm{Ext}_A[h_1^{-1}]$ explicitly. We will accomplish this by expressing $A_{*,*}$ as a series of extensions of smaller Hopf algebras.

Definition 3.1. For each $i \geq -1$, let B_i be the subalgebra of $A_{*,*}$ generated by the elements ξ_k and also by $\tau_0, \ldots, \tau_i$.

In particular, B_{-1} is a polynomial $\mathbb{M}_2$-algebra on the elements ξ_k.

Lemma 3.2. $\mathrm{Ext}_{B_{-1}}[h_1^{-1}]$ *is isomorphic to* $\mathbb{M}_2[h_1^{\pm 1}]$.

Proof. We have an isomorphism $B_{-1} \longrightarrow A_*^{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2$ of Hopf algebras. Under this mapping, the element h_1 in $\mathrm{Ext}_{B_{-1}}$ corresponds to h_0 in $\mathrm{Ext}_{A_{\mathrm{cl}}}$. Adams's vanishing line of slope 1 [1] implies that $\mathrm{Ext}_{A_{\mathrm{cl}}}[h_0^{-1}]$ is isomorphic to $\mathbb{F}_2[h_0^{\pm 1}]$. □

We proceed to compute $\mathrm{Ext}_{B_n}[h_1^{-1}]$ inductively via a Cartan–Eilenberg spectral sequence [5, §XVI.6], [16, A1.3.14] for the extension of Hopf algebras

$$B_{n-1} \longrightarrow B_n \longrightarrow E(\tau_n),$$

where $E(\tau_n)$ is an exterior algebra on the generator τ_n. The extension is cocentral, so the spectral sequence takes the form

$$E_2 \cong \mathrm{Ext}_{B_{n-1}} \otimes \mathrm{Ext}_{E(\tau_n)} \cong \mathrm{Ext}_{B_{n-1}}[v_n] \Rightarrow \mathrm{Ext}_{B_n}.$$

The E_2-page of this spectral sequence has four gradings: three from Ext_{B_n} and one additional Cartan–Eilenberg grading associated with the filtration involved in construction of the spectral sequence. However, we will suppress the Cartan–Eilenberg grading because we won't need it for bookkeeping purposes.

The class v_n has degree $(2^n - 1, 1, 1)$ in the E_2-term. The differentials take the form

$$d_r : E_r^{(t,f,c)} \longrightarrow E_r^{(t-1,f+1,c)}.$$

Proposition 3.3.

(1) $\mathrm{Ext}_{B_0}[h_1^{-1}] \cong \mathbb{M}_2[h_1^{\pm 1}, v_0]$.
(2) $\mathrm{Ext}_{B_1}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4]$.
(3) $\mathrm{Ext}_{B_n}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4, v_2, \ldots, v_n]$ *for all* $n \geq 2$.

In each case, v_i *has degree* $(2^i - 1, 1, 1)$.

Proof. When $n = 0$, Lemma 3.2 says that the Cartan–Eilenberg spectral sequence takes the form

$$E_2 \cong \mathbb{M}_2[h_1^{\pm 1}][v_0] \Rightarrow \mathrm{Ext}_{B_0}[h_1^{-1}].$$

Since τ_0 is primitive, the spectral sequence collapses at E_2. We conclude that $\mathrm{Ext}_{B_0}[h_1^{-1}] \cong \mathbb{M}_2[h_1^{\pm 1}, v_0]$ with v_0 in degree $(0, 1, 1)$.

Taking now $n = 1$, the computation of $\mathrm{Ext}_{B_0}[h_1^{-1}]$ in the previous paragraph tells us that the spectral sequence takes the form

$$E_2 \cong \mathbb{M}_2[h_1^{\pm 1}, v_0, v_1] \Rightarrow \mathrm{Ext}_{B_1}[h_1^{-1}],$$

with v_0 in degree $(0, 1, 1)$ and v_1 in degree $(1, 1, 1)$. The coproduct formula

$$\Delta(\tau_1) = \tau_1 \otimes 1 + \xi_1 \otimes \tau_0 + 1 \otimes \tau_1$$

gives rise to the differential $d_2(v_1) = h_1 v_0$. It follows that $E_3 \cong \mathbb{M}_2[h_1^{\pm 1}, v_1^2]$. There is next a differential $d_3(v_1^2) = \tau h_1^3$, which can be verified by the cobar complex calculation

$$d(\tau_1|\tau_1|\xi_1 + \xi_1|\tau_0\tau_1|\xi_1 + \tau_1\xi_1|\tau_0|\xi_1 + \xi_1^2|\tau_0|(\tau_1 + \tau_0\xi_1)) = \tau\xi_1|\xi_1|\xi_1|\xi_1.$$

The class v_1^4 in degree $(4, 4, 4)$ cannot support any higher differentials for degree reasons, and we have

$$\mathrm{Ext}_{B_1}[h_1^{-1}] \cong E_\infty = E_4 \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4].$$

For $n \geq 2$, the argument is by induction, using a Cartan–Eilenberg spectral sequence at every turn. Each of these spectral sequences collapses at E_2 since there are no possible values for differentials on v_n. □

Theorem 3.4.

$$\mathrm{Ext}_A[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, v_1^4, v_2, v_3, \ldots],$$

where h_1 has degree $(0, 1, 0)$; v_1^4 has degree $(4, 4, 4)$; and v_n has degree $(2^n - 1, 1, 1)$ for $n \geq 2$.

Proof. Since A is $\operatorname{colim} B_n$, Ext_A equals $\operatorname{colim} \mathrm{Ext}_{B_n}$. The calculation follows from part (3) of Proposition 3.3. □

Remark 3.5. Theorem 3.4 implies that the part of $\mathrm{Ext}_A[h_1^{-1}]$ in Chow degree zero is equal to $\mathbb{F}_2[h_1^{\pm 1}]$. Another more direct argument for this observation uses the isomorphism between $\mathrm{Ext}_{A_{\mathrm{cl}}}$ and the Chow degree zero part of Ext_A [9]. This isomorphism takes classical elements of degree (s, f) to motivic elements of degree $(s, f, 0)$. The classical calculation $\mathrm{Ext}_{A_{\mathrm{cl}}}[h_0^{-1}] = \mathbb{F}_2[h_0^{\pm 1}]$ corresponds to the Chow degree zero part of $\mathrm{Ext}_A[h_1^{-1}]$.

3.1. The h_1-local cohomology of $A(2)$

For completeness, we will also explicitly calculate $\mathrm{Ext}_{A(2)}[h_1^{-1}]$, where $A(2)$ is the $\mathbb{M}_2$-subalgebra of A generated by Sq^1, Sq^2, and Sq^4.

Dual to the inclusion $A(2) \hookrightarrow A$ is a quotient map

$$A_{*,*} \longrightarrow A(2)_{*,*} \cong \mathbb{M}_2[\xi_1, \xi_2, \tau_0, \tau_1, \tau_2]/(\xi_1^4, \xi_2^2, \tau_0^2 = \tau\xi_1, \tau_1^2 = \tau\xi_2, \tau_2^2).$$

We filter $A(2)_{*,*}$ by sub-Hopf algebras

$$B_{-1}(2) \subseteq B_0(2) \subseteq B_1(2) \subseteq A(2)_{*,*}$$

where $B_{-1}(2)$ is the subalgebra generated by ξ_1 and ξ_2; $B_0(2)$ is generated by ξ_1, ξ_2, and τ_0; and $B_1(2)$ is generated by ξ_1, ξ_2, τ_0, and τ_1. The notation is analogous to the notation in Definition 3.1.

Lemma 3.6. $\mathrm{Ext}_{B_{-1}(2)}[h_1^{-1}]$ *is isomorphic to* $\mathbb{M}_2[h_1^{\pm 1}, a_1]$*, where a_1 has degree $(4, 3, 0)$.*

Proof. We have an isomorphism $B_{-1}(2) \longrightarrow A(1)_*^{\mathrm{cl}} \otimes_{\mathbb{F}_2} \mathbb{M}_2$. Under this mapping, the element h_1 in $\mathrm{Ext}_{B_{-1}(2)}$ corresponds to h_0 in $\mathrm{Ext}_{A(1)_{\mathrm{cl}}}$. It is well-known that $\mathrm{Ext}_{A(1)_{\mathrm{cl}}}[h_0^{-1}]$ is isomorphic to $\mathbb{F}_2[h_0^{\pm 1}, a^{\mathrm{cl}}]$, where a^{cl} has degree $(4, 3)$ (for example, see [16, Theorem 3.1.25]). □

Proposition 3.7.

(1) $\mathrm{Ext}_{B_0(2)}[h_1^{-1}] \cong \mathbb{M}_2[h_1^{\pm 1}, a_1, v_0]$.
(2) $\mathrm{Ext}_{B_1(2)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, a_1, v_1^4]$.
(3) $\mathrm{Ext}_{A(2)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, a_1, v_1^4, v_2]$.

In each case, a_1 has degree $(4, 3, 0)$; v_0 has degree $(0, 1, 1)$; v_1^4 has degree $(4, 4, 4)$; and v_2 has degree $(3, 1, 1)$.

Proof. The proof uses a series of Cartan–Eilenberg spectral sequences as in the proof of Proposition 3.3, given Lemma 3.6 as the starting point. □

Remark 3.8. The classes a_1, v_1^4, and $h_1^2 v_2$ correspond respectively to the classes u, P, and c in [8, Theorem 4.13].

Remark 3.9. Using the structure of $\mathrm{Ext}_{A(2)_{\mathrm{cl}}}[h_0^{-1}]$, similar arguments show that

$$\mathrm{Ext}_{A(3)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, g, b_{31}, v_1^4, v_2, v_3],$$

where g has degree $(8, 4, 0)$ and b_{31} has degree $(12, 2, 0)$. Using the structure of $\mathrm{Ext}_{A(3)_{\mathrm{cl}}}[h_0^{-1}]$,

$$\mathrm{Ext}_{A(4)}[h_1^{-1}] \cong \mathbb{F}_2[h_1^{\pm 1}, g^2, \Delta_1, b_{41}, v_1^4, v_2, v_3, v_4],$$

where g^2 has degree $(16, 8, 0)$, Δ_1 has degree $(24, 4, 0)$, and b_{41} has degree $(28, 2, 0)$.

4. The h_1-local motivic May spectral sequence

Although Theorem 3.4 gives a complete description of $\mathrm{Ext}_A[h_1^{-1}]$, it unfortunately tells us very little about the localization map $\mathrm{Ext}_A \to \mathrm{Ext}_A[h_1^{-1}]$. The problem is that the proof of Theorem 3.4 is incompatible with the motivic May spectral sequence approach to Ext_A, as carried out in [9].

A detailed understanding of the localization map allows for the transfer of information from the well-understood $\mathrm{Ext}_A[h_1^{-1}]$ to the much more complicated Ext_A. In this section we carry out the computation of the h_1-localized motivic May spectral sequence. This will allow us to obtain information about the localization map in Section 5.

We recall the details of the motivic May spectral sequence from [6]. This spectral sequence has four gradings: three from Ext_A and one additional May grading associated with the filtration involved in construction of the spectral sequence. We will grade this spectral sequence in the form $(m - f, t, f, c)$, where m is the May grading, f is the Adams filtration, $t = s - w$ is the Milnor–Witt stem, and $c = s + f - 2w$ is the Chow degree.

The E_1-page is a polynomial algebra over $\mathbb{M}_2$ on generators h_{ij} for $i \geq 1$, $j \geq 0$, where

(1) h_{i0} has degree $(i - 1, 2^{i-1} - 1, 1, 1)$.
(2) h_{ij} has degree $(i - 1, 2^{j-1}(2^i - 1) - 1, 1, 0)$ for $j > 0$.

Note in particular that h_{i0} has Chow degree 1, while h_{ij} has Chow degree 0 for $j > 0$. In a sense, this wrinkle in the gradings is the primary source of “exotic” motivic phenomena that do not appear in the classical situation.

The d_1-differential is given by the classical formula

$$d_1(h_{ij}) = \sum_{0<k<i} h_{kj} h_{i-k,j+k}.$$

The May d_r-differential takes the form

$$d_r : E_r^{m-f,t,f,c} \longrightarrow E_r^{m-f-r,t-1,f+1,c}.$$

4.1. The h_1-local E_1-term

The elements h_{1n} are typically abbreviated as h_n. Consider the h_1-localization $E_1[h_1^{-1}]$ of the May E_1-page. In order to simplify the calculation, we introduce the following notation.

Definition 4.1. In $E_1[h_1^{-1}]$, define

(1) h'_{n0} to be $h_{n0} + h_1^{-1} h_{20} h_{n-1,1}$.
(2) $h'_{n-1,2}$ to be $h_{n-1,2} + h_1^{-1} \sum_{k=2}^{n-1} h_{k,1} h_{n-k,k}$.

We may replace the algebra generators h_{n0} and $h_{n-1,2}$ by h'_{n0} and $h'_{n-1,2}$ to obtain another set of algebra generators for $E_1[h_1^{-1}]$ that will turn out to be calculationally convenient.

Definition 4.2.

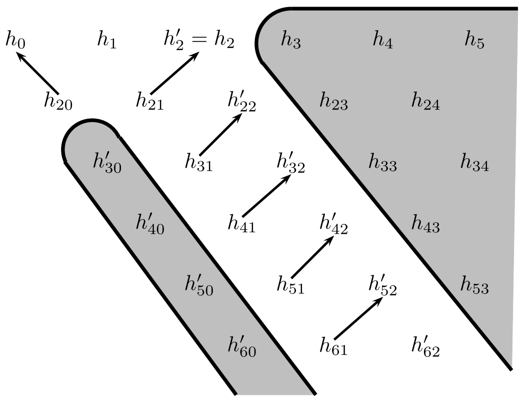
(1) Let F_1 be the $\mathbb{M}_2[h_1^{\pm 1}]$-subalgebra of $E_1[h_1^{-1}]$ on polynomial generators h_0, h_{20}, h_{n1} for all $n \geq 2$, and h'_{n2} for all $n \geq 1$.
(2) Let G_1 be the $\mathbb{F}_2$-subalgebra of $E_1[h_1^{-1}]$ on polynomial generators h'_{n0} for $n \geq 3$ and h_{ij} for $i \geq 1$ and $j \geq 3$.

Note that F_1 is a differential graded subalgebra of $E_1[h_1^{-1}]$ since $d_1(h_{20}) = h_1 h_0$ and $d_1(h_{n1}) = h_1 h'_{n-1,2}$. The generators of F_1 are indicated in the figure below as the elements that are outside of the shaded region.

Note also that G_1 is a differential graded subalgebra of $E_1[h_1^{-1}]$ because $d_1(h'_{n,0}) = \sum_{j=3}^{n-1} h_{n-j,j} h'_{j,0}$ if $n \geq 3$. The generators of G_1 are indicated in the figure below as the elements in the shaded regions.

Proposition 4.3. *$E_1[h_1^{-1}]$ splits as a tensor product $F_1 \otimes_{\mathbb{F}_2} G_1$.*

Proof. This follows immediately from the definitions, using that h'_{n0} and h'_{n2} can be used as algebra generators in place of h_{n0} and h_{n2}. □



We will now show that the perhaps obscurely defined subalgebra G_1 is isomorphic to the familiar classical May E_1-page.

Proposition 4.4. *Let E_1^{cl} be the E_1-page of the classical May spectral sequence. Consider the algebra map $S : E_1^{\mathrm{cl}} \longrightarrow G_1$ determined by*

(1) $S(h_{n0}) = h'_{n+2,0}$ *for all* $n \geq 1$.
(2) $S(h_{nk}) = h_{n,k+2}$ *for all* $n \geq 1$ *and* $k \geq 1$.

The map S is an isomorphism of differential graded algebras.

Proof. We need to check that S preserves the May d_1 differential. This is a straightforward computation. □

4.2. The h_1-local E_2-term

We now have a good understanding of $E_1[h_1^{-1}]$ from Propositions 4.3 and 4.4. Next we analyze the h_1-localization $E_2[h_1^{-1}]$ of the motivic May E_2-page. Since localization is exact, $E_2[h_1^{-1}]$ is isomorphic to the cohomology of the differential graded algebra $E_1[h_1^{-1}]$.

Proposition 4.5. *$E_2[h_1^{-1}]$ is isomorphic to*

$$\mathbb{M}_2[h_1^{\pm}][b_{20}, b_{21}, b_{31}, b_{41}, \ldots] \otimes_{\mathbb{F}_2} G_2,$$

where $b_{ij} = h_{ij}^2$ and G_2 is the cohomology of the differential graded algebra G_1 from Definition 4.2.

Proof. This follows from the splitting given in Proposition 4.3. The calculation of the cohomology of the subalgebra F_1 from Definition 4.2 is straightforward. □

Because of Proposition 4.4, we know that G_2 is isomorphic to the classical May E_2-page. May's original calculation [10] of the classical E_2-term in stems below 156 gives us complete understanding of G_2 in a much larger range because the map S from Proposition 4.4 approximately quadruples degrees.

Generators and relations for $E_2[h_1^{-1}]$ up to the Milnor–Witt 66-stem can be found in Tables 1 and 2, where we use the following notation. We remind the reader that all tables may be found in Section 7.

Notation 4.6. For an element x in the classical May spectral sequence, let $\mathbf{x}$ be the element $S(x)$ of the h_1-localized motivic May spectral sequence from Proposition 4.4.

According to this notation, the classical element $c_0 = h_1 h_0(1)$ may be written as $c_0 = h_1^2 \mathbf{h_0}$, so that the elements c_0 and $\mathbf{h_0}$ are practically interchangeable. As the primary goal of our computation is to relate the answer to Ext_A, we will most often choose to work with c_0. However, we will opt instead to use $\mathbf{h_0}$ when it illuminates the structure of the h_1-localized motivic May spectral sequence, especially in Section 4.3.

See also Table 3 for additional notation used on later pages. The names of many classes in the May spectral sequence have been chosen to agree with the notation of [9], and we denote the remaining new classes by y_n.

4.3. The h_1-local May differentials

We now understand $E_2[h_1^{-1}]$ in a very large range of dimensions. The next step is to compute the higher differentials to obtain $E_\infty[h_1^{-1}]$.

Proposition 4.7. *Table 1 gives the values of the May d_2 differential on the multiplicative generators of $E_2[h_1^{-1}]$ through the Milnor–Witt 66-stem.*

Proof. As discussed in [6, §5], the d_2 differential of the motivic May spectral sequence is easy to determine from the classical d_2 differential; the formulas are the same, except that powers of τ must sometimes be inserted to balance the weights. This, combined with the fact that h_1-localization kills the classes h_0 and h_2, leads to the values in Table 1. □

The values of the May d_2 differential given in Table 1 allow us to compute $E_4[h_1^{-1}]$ directly. A chart of $E_4[h_1^{-1}]$ through the Milnor–Witt 66-stem is given in Section 8.

Remark 4.8. The element $e_0 = b_{21}h_0(1)$, or $h_1\mathbf{h_0}b_{21}$, is a permanent cycle. The relation $e_0^2 = h_1^2\mathbf{h_0}^2 g = d_0 \cdot g = h_1^{-2}c_0^2 \cdot g$ leads to similar relations $e_0 \cdot e_0 g = h_1^{-2}c_0^2 g^2$, $e_0 g \cdot e_0 g = h_1^{-2}c_0^2 g^3$, etc. Therefore, the elements e_0^2, $e_0 \cdot e_0 g$, and $(e_0 g)^2$ are represented in c_0-towers in Section 8.

Remark 4.9. We do not attempt to list all relations in the May E_r pages for $r \geq 4$. Many of the relations that we use or that appear in the charts in Section 8 are simple consequences of E_2-relations and May differentials. One exception is the relation $c_0 \cdot y_{45} = 0$ in the E_4-page. The class y_{45} is decomposable, equal to

$$y_{45} = g\boldsymbol{\nu} + \mathbf{h_0^2 h_3}\Delta_1.$$

The differential

$$d_2(\mathbf{h_0}^2 b_{31}b_{41} + \mathbf{b_{20}}b_{21}b_{41} + \mathbf{b_{30}}b_{21}b_{31}) = g\boldsymbol{\nu} + \mathbf{h_0}^2\mathbf{h_3}\Delta_1 + \mathbf{h_1}(\mathbf{b_{20}}b_{41} + \mathbf{b_{30}}b_{31}),$$

combined with the E_2 relation $\mathbf{h_0 h_1} = 0$, shows that $\mathbf{h_0}y_{45} = 0$ in E_4.

Now we proceed to the higher May differentials and the higher h_1-localized pages of the motivic May spectral sequence.

Proposition 4.10. *For $r \geq 4$, some values for the May d_r differentials are given in Tables 4–10. The May d_r differentials are zero on all other multiplicative generators of $E_r[h_1^{-1}]$ through the Milnor–Witt 66-stem.*

Proof. Most of the differentials are forced by the known structure of $\mathrm{Ext}_A[h_1^{-1}]$ given in Theorem 3.4. For example, Theorem 3.4 implies that $h_1^k v_3 v_4$ are the only non-zero elements in the Milnor–Witt 22-stem with Chow degree 2. We know that $h_1^k e_0 \cdot e_0 g$ survive the May spectral sequence to detect these elements. However, the element ϕ in the May E_4-page is also in the Milnor–Witt 22-stem with Chow degree 2. Therefore, it cannot survive the May spectral sequence. The only possibility is that $d_6(\phi)$ equals $h_1 c_0^2 h_5$.

There are a handful of more difficult cases, which are handled individually in the following lemmas. □

Lemma 4.11.

(1) $d_4(\mathbf{P}) = \mathbf{h_0^4 h_3}$.
(2) $d_4(\boldsymbol{\Delta}) = \mathbf{h_4 P}$.

Proof. For the first formula, we use Nakamura's squaring operations [13] in the May spectral sequence to compute that

$$d_4(\mathbf{P}) = d_4(\mathrm{Sq}_0(\mathbf{b_{20}})) = \mathrm{Sq}_1 d_2(\mathbf{b_{20}}) = \mathrm{Sq}_1(\mathbf{h_0^2 h_2})$$
$$= \mathrm{Sq}_1(\mathbf{h_0^2})\mathrm{Sq}_0(\mathbf{h_2}) + \mathrm{Sq}_0(\mathbf{h_0^2})\mathrm{Sq}_1(\mathbf{h_2}) = 0 + \mathbf{h_0^4 h_3}.$$

The proof for the second formula is similar:

$$d_4(\boldsymbol{\Delta}) = d_4(\mathrm{Sq}_0(\mathbf{b_{30}})) = \mathrm{Sq}_1 d_2(\mathbf{b_{30}}) = \mathrm{Sq}_1(\mathbf{h_3 b_{20}})$$
$$= \mathrm{Sq}_1(\mathbf{h_3})\mathrm{Sq}_0(\mathbf{b_{20}}) + \mathrm{Sq}_0(\mathbf{h_3})\mathrm{Sq}_1(\mathbf{b_{20}}) = \mathbf{h_4 P} + 0. \qquad \square$$

Lemma 4.12. $d_6(c_0 g\phi\Delta_1) = h_1^7 D_3'\Delta_1$.

Proof. The class $D_3'\Delta_1$ cannot survive by Theorem 3.4. There are no classes for it to hit, and the only other differential that could possibly hit $h_1^7 D_3'\Delta_1$ is $d_{10}(h_1^{-8} c_0 g^4\phi)$. But $d_{10}(g^4\phi) = h_1^{13} g y_{45}$, so $c_0 g^4\phi$ is a d_{10}-cycle. $\square$

Lemma 4.13.

(1) $d_8(\Delta_1\mathbf{P}) = h_1^{-6} c_0^2 e_0^2 h_6$.
(2) $d_8(\mathbf{P^2}) = \mathbf{h_0^8 h_4}$.

Proof. Start with the relation $c_0^2 \cdot \Delta_1\phi = h_1^2 e_0 \cdot \Delta_1 B_1$. We will show in Lemma 4.14 that $\Delta_1 B_1$ is a permanent cycle. Therefore, $c_0^2\Delta_1\phi$ is a permanent cycle. However, $\Delta_1\phi$ cannot survive by Theorem 3.4, so we must have $d_{10}(\Delta_1\phi) = h_1^3 e_0^2 h_6$. Since $d_{10}(c_0^2\Delta_1\phi)$ must be zero, it follows that $c_0^2 e_0^2 h_6$ must be zero in $E_{10}[h_1^{-1}]$. The only possibility is that $d_8(\Delta_1\mathbf{P}) = h_1^{-6} c_0^2 e_0^2 h_6$. This establishes the first formula.

For the second formula, we use Nakamura's squaring operations [13] as in the proof of Lemma 4.11 to compute

$$d_8(\mathbf{P^2}) = d_8(\mathrm{Sq}_0(\mathbf{P})) = \mathrm{Sq}_1 d_4(\mathbf{P}) = \mathrm{Sq}_1(\mathbf{h_0^4 h_3})$$
$$= \mathbf{h_0^8 h_4} + \mathrm{Sq}_1(\mathbf{h_0^4})\mathbf{h_3^2} = \mathbf{h_0^8 h_4}. \qquad \square$$

Lemma 4.14. $d_{10}(\Delta_1 B_1) = 0$.

Proof. We know that $d_{16}(e_0 g^4) = h_1^{16} e_0 h_6$, yet $c_0^2 \cdot e_0 g^4 = h_1^2 e_0^2 \cdot e_0 g^3$ is a permanent cycle. So $c_0^2 e_0 h_6$ must be zero in $E_{16}[h_1^{-1}]$. The only possibilities are that $d_{10}(\Delta_1 B_1) = h_1 c_0^2 e_0 h_6$ or $d_{14}(g^3 B_1) = h_1^9 c_0^2 e_0 h_6$.

In the notation of Theorem 3.4, c_0, e_0, $e_0 g$ and $e_0 g^3$ correspond to $h_1^2 v_2$, $h_1^3 v_3$, $h_1^7 v_4$, and $h_1^{15} v_5$ respectively. Since these elements are algebraically independent, we conclude that the relation $h_1^2 e_0 (e_0 g)^2 + c_0^2 \cdot e_0 g^3 = 0$ in $E_\infty[h_1^{-1}]$ must be resolved by

$$h_1^2 e_0 (e_0 g)^2 + c_0^2 \cdot e_0 g^3 = h_1^7 g^2 B_1$$

in $\mathrm{Ext}_A[h_1^{-1}]$.

Similarly, the relation $e_0^2 \cdot e_0 g^3 + (e_0 g)^3 = 0$ in $E_\infty[h_1^{-1}]$ must be resolved in $\mathrm{Ext}_A[h_1^{-1}]$ by

$$e_0^2 \cdot e_0 g^3 + (e_0 g)^3 = x,$$

where x is either $h_1^{13}\Delta_1 B_1$ or $h_1^5 g^3 B_1$. Multiply the first hidden relation by e_0^2, multiply the second hidden relation by c_0^2, and add to obtain

$$h_1^2 e_0^3 (e_0 g)^2 + c_0^2 (e_0 g)^3 = h_1^7 e_0^2 \cdot g^2 B_1 + c_0^2 x.$$

Again using Theorem 3.4, the left side of this last relation is not zero, so the right side is also not zero. This implies that x cannot be $h_1^5 g^3 B_1$, since $h_1^7 e_0^2 \cdot g^2 B_1$ equals $h_1^5 c_0^2 \cdot g^3 B_1$ in the May spectral sequence with no possible hidden extension.

Therefore, x must be $h_1^{13} \Delta_1 B_1$, and $\Delta_1 B_1$ must survive the May spectral sequence. □

4.4. $E_\infty[h_1^{-1}]$ and $\mathrm{Ext}_A[h_1^{-1}]$

The May differentials given in Section 4.3 allow us to compute $E_\infty[h_1^{-1}]$ explicitly through the Milnor–Witt 66-stem. See Section 8 for a chart of this calculation.

The final step is to pass from $E_\infty[h_1^{-1}]$ to $\mathrm{Ext}_A[h_1^{-1}]$ by resolving hidden extensions.

Proposition 4.15. *Table 11 lists some relations in $\mathrm{Ext}_A[h_1^{-1}]$ that are hidden in $E_\infty[h_1^{-1}]$. Through the Milnor–Witt 66-stem, all other hidden relations are multiplicative consequences of these relations.*

Proof. Arguments for the relations involving $g^2 B_1$ and $\Delta_1 B_1$ were given already in the proof of Lemma 4.14. The other relations in Table 11 are established similarly. □

Finally, we have calculated $\mathrm{Ext}_A[h_1^{-1}]$ through the Milnor–Witt 66-stem with the May spectral sequence and obtained the same answer as in Theorem 3.4. Multiplicative generators for $\mathrm{Ext}_A[h_1^{-1}]$ through the Milnor–Witt 66-stem are given in Table 12.

4.5. The h_1-local May spectral sequence for $A(2)$

We sketch here the calculation of the h_1-localized May spectral sequence over $A(2)$. The E_1-term is a polynomial algebra on the generators h_0, h_1, h_2, h_{20}, h_{21}, and h_{30}.

Then $d_1(h_{20}) = h_0 h_1$ and $d_1(h_{21}) = h_1 h_2$. As in Section 4.1, we replace h_{30} by $h'_{30} = h_1^{-1} h_0(1) = h_{30} + h_1^{-1} h_{20} h_{21}$, so that $d_1(h'_{30}) = 0$.

The E_2-page is then the polynomial algebra $\mathbb{M}_2[h_1^{\pm 1}, b_{20}, b_{21}, h'_{30}]$, and the only differential is $d_2(b_{20}) = \tau h_1^3$.

It follows that E_3 is given by $\mathbb{F}_2[h_1^{\pm 1}, b_{20}^2, b_{21}, h'_{30}]$. No more differentials are possible, and $E_3 = E_\infty$.

Note that b_{20}^2, b_{21}, and h'_{30} correspond respectively to v_1^4, $h_1^{-1} a_1$, and v_2 in the notation of Proposition 3.7.

5. The localization map

The calculation of Ext_A is given in [9] up to the 70-stem. In this section, we will use the May spectral sequence analysis of $\mathrm{Ext}_A[h_1^{-1}]$ from Section 4 to determine the localization map

$$L : \mathrm{Ext}_A \longrightarrow \mathrm{Ext}_A[h_1^{-1}]$$

in the same range. A detailed understanding of the localization map is essential for transfer of information between the localized and non-localized situations.

Proposition 5.1. *Table 13 lists some values of the localization map $L : \mathrm{Ext}_A \to \mathrm{Ext}_A[h_1^{-1}]$ on multiplicative generators of Ext_A. Through the 70-stem, the localization map is zero on all generators of Ext_A not listed in Table 13.*

Proof. Note that $\mathrm{Ext}_A[h_1^{-1}]$ is concentrated in degrees (t, f, c) such that $t - c$ is even. Many of the generators of Ext_A are in degrees (t, f, c) such that $t - c$ is odd. Therefore, all of these generators must map to 0 in the localization.

The values of L on u, v, u', v', and U follow from applying the May E_4 relation $h_1^4\Delta = d_0^2 + Pg$ to the May descriptions of these classes.

The remaining values are again determined by their May descriptions, together with the value of $L(B_1)$, which follows from the relation $h_1^7 B_1 = h_1^2 e_0^3 + c_0^2 \cdot e_0 g$ established in Proposition 4.15. □

Table 13 gives values for the localization map in two forms. First, it uses the notation from Theorem 3.4 involving the elements v_n. Second, it uses a different notation for the generators of $\mathrm{Ext}_A[h_1^{-1}]$ given in Table 12 that is more compatible with the standard notation for Ext_A.

With a detailed understanding of the localization map in hand, we can establish some hidden relations in Ext_A that are needed in [9].

Corollary 5.2. *The following hidden extensions hold in* Ext_A*:*

(1) $e_0^3 + d_0 \cdot e_0 g = h_1^5 B_1$.
(2) $d_0 v + e_0 u = h_1^3 x'$.
(3) $e_0 u' + d_0 v' = h_1^2 c_0 x'$.

Proof. Table 13 says that $L(e_0^3 + d_0 \cdot e_0 g)$ equals $h_1^9 v_3^3 + h_1^9 v_2^2 v_4$, which is non-zero. It follows that $e_0^3 + d_0 \cdot e_0 g$ is non-zero in Ext_A. From the calculation in [9], the only possibility is that it equals $h_1^5 B_1$. This establishes the first formula.

The argument for the second formula is similar. Table 13 says that $L(d_0 v + e_0 u)$ equals $h_1^6 v_1^4 v_2^2 v_4 + h_1^6 v_1^4 v_3^3$, which is non-zero. It follows that $d_0 v + e_0 u$ is non-zero in Ext_A, and the only possibility is that it equals $h_1^3 x'$.

For the third formula, Table 13 says that $L(e_0 u' + d_0 v')$ equals $h_1^7 v_1^4 v_2 v_3^3 + h_1^7 v_1^4 v_2^3 v_4$, which is non-zero. It follows that $e_0 u' + d_0 v'$ is non-zero in Ext_A. There are several possible non-zero values for $e_0 u' + d_0 v'$. However, $e_0 u' + d_0 v'$ must be annihilated by τ because both u' and v' are. Then $h_1^2 c_0 x'$ is the only possible value. □

5.1. The localization map for $A(2)$

For completeness, we also describe the localization map

$$\mathrm{Ext}_{A(2)} \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}].$$

The calculation of $\mathrm{Ext}_{A(2)}$ is given in [8].

Proposition 5.3. *Table 14 lists some values of the localization map* $L : \mathrm{Ext}_{A(2)} \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$ *on multiplicative generators of* $\mathrm{Ext}_{A(2)}$*. The localization map is zero on all generators of* $\mathrm{Ext}_{A(2)}$ *not listed in Table 14.*

Proof. The generators for $\mathrm{Ext}_{A(2)}$ are given in [8, Table 7]. The values of L follow by comparison of the localized and unlocalized May spectral sequences for $A(2)$. □

Now consider the diagram

$$\begin{array}{ccc} \mathrm{Ext}_A & \longrightarrow & \mathrm{Ext}_A[h_1^{-1}] \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{A(2)} & \longrightarrow & \mathrm{Ext}_{A(2)}[h_1^{-1}] \end{array}$$

in which the horizontal maps are localizations and the vertical maps are induced by the inclusion $A(2) \longrightarrow A$. Given that $\mathrm{Ext}_A[h_1^{-1}]$ and $\mathrm{Ext}_{A(2)}[h_1^{-1}]$ are computed explicitly in Theorem 3.4 and Proposition 3.7, one might expect that the map $\mathrm{Ext}_A[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$ would be easy to determine. The obvious guess is that this map takes v_1^4 to v_1^4, takes v_2 to v_2, and takes v_n to 0 for $n \geq 3$. However, the Cartan–Eilenberg spectral sequences of Section 3 hide some of the values of this map.

Lemma 5.4. *The map* $\mathrm{Ext}_A[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$ *takes* v_1^4, v_2, v_3, v_4, v_5, *and* v_6 *to* v_1^4, v_2, $h_1^{-3}a_1v_2$, $h_1^{-9}a_1^3v_2$, $h_1^{-21}a_1^7v_2$, *and* $h_1^{-45}a_1^{15}v_2$.

Proof. This follows from the May spectral sequence calculations of Section 4. The given values for $\mathrm{Ext}_A[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$ are apparent on the May E_∞-pages. We are using that a_1 is represented by $a_1 = h_1 b_{21}$. □

Lemma 5.4 suggests an obvious conjecture on the complete description of the h_1-localized restriction $\mathrm{Ext}_A[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$.

Conjecture 5.5. *The map* $\mathrm{Ext}_A[h_1^{-1}] \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$ *takes* v_1^4 *to* v_1^4 *and takes* v_n *to* $h_1^{-3(2^{n-2}-1)}a_1^{2^{n-2}-1}v_2$ *for* $n \geq 2$.

6. The Adams spectral sequence for $S[\eta^{-1}]$

Recall that Ext_A is the E_2-page for the motivic Adams spectral sequence that converges to the 2-complete motivic stable homotopy groups $\pi_{*,*}$ of the motivic sphere $S^{0,0}$. The element h_1 in Ext_A detects the motivic Hopf map η in $\pi_{1,1}$.

Definition 6.1. Let $S^{0,0}[\eta^{-1}]$ to be the homotopy colimit of the sequence

$$S^{0,0} \xrightarrow{\eta} S^{-1,-1} \xrightarrow{\eta} S^{-2,-2} \xrightarrow{\eta} \cdots.$$

The homotopy groups $\pi_{*,*}(S^{0,0}[\eta^{-1}])$ are then the target of an h_1-localized Adams spectral sequence whose E_2-page is $\mathrm{Ext}_A[h_1^{-1}]$. However, we must consider convergence. A priori, there could be an infinite family of homotopy classes linked together by infinitely many hidden η-multiplications. These classes would not be detected in $\mathrm{Ext}_A[h_1^{-1}]$. But this cannot occur, as the argument of [1] carries over readily to the motivic setting to establish a vanishing line of slope 1 in Ext_A.

In the (s, f, w)-grading, the Adams differentials behave according to

$$d_r : E_r^{s,f,w} \longrightarrow E_r^{s-1,f+r,w}.$$

In the h_1-invariant grading, this becomes

$$d_r : E_r^{t,f,c} \longrightarrow E_r^{t-1,f+r,c+r-1}.$$

Proposition 6.2. *The Adams d_2 differential for $S^{0,0}[\eta^{-1}]$ takes the following values.*

(1) $d_2(v_1^4) = 0$.
(2) $d_2(v_2) = 0$.
(3) $d_2(v_3) = h_1 v_2^2$.
(4) $d_2(v_4) = h_1 v_3^2$.

Proof. The first two formulas follow immediately because there are no possible non-zero values. The third formula follows from the Adams differential $d_2(e_0) = h_1^2 d_0$ in the unlocalized case [9], together with the fact that the localization map takes c_0 and e_0 to $h_1^2 v_2$ and $h_1^3 v_3$. The fourth formula follows from the Adams differential $d_2(e_0 g) = h_1^2 e_0^2$ in the unlocalized case [9], together with the fact that the localization map takes e_0 and $e_0 g$ to $h_1^3 v_3$ and $h_1^7 v_4$. □

Proposition 6.2 suggests an obvious conjecture for the values of the Adams d_2 differential on the rest of the generators of $\mathrm{Ext}_A[h_1^{-1}]$. See Conjecture 1.2 for an explicit statement.

6.1. Motivic modular forms and Adams differentials

In the classical case, the topological modular forms spectrum *tmf* is a spectrum whose $\mathbb{F}_2$-cohomology is equal to the quotient $A_{\mathrm{cl}}//A(2)_{\mathrm{cl}}$. This implies that $\mathrm{Ext}_{A(2)_{\mathrm{cl}}}$ is the E_2-page of the Adams spectral sequence converging to the 2-complete homotopy groups of *tmf*.

One might speculate that there is a motivic spectrum *mmf* (called "motivic modular forms") whose motivic $\mathbb{F}_2$-cohomology is isomorphic to $A//A(2)$. Then $\mathrm{Ext}_{A(2)}$ would be the E_2-page of the motivic Adams spectral sequence converging to the 2-complete motivic homotopy groups of *mmf*. However, no such motivic spectrum is known to exist. See [15] for one piece of the program for constructing *mmf*.

In any case, we assume for the rest of this section that *mmf* does exist, and we explore some of the computational consequences.

Lemma 6.3. *(See [8, §4.4].) Suppose that mmf exists. Then, in the Adams spectral sequence*

$$\mathrm{Ext}_{A(2)} \Rightarrow \pi_{*,*}(mmf),$$

there is an Adams differential $d_2(a_1) = h_1^2 c_0$.

Proof. Since $d_2(e_0) = h_1^2 d_0$ in the Adams spectral sequence for $S^{0,0}$, it follows that $d_2(e_0) = h_1^2 d_0$ in the Adams spectral sequence for *mmf* as well.

We have the relation $c_0 a_1 = h_1^2 e_0$ in $\mathrm{Ext}_{A(2)}$ [8]. Therefore, a_1 must support a differential, and $h_1^2 c_0$ is the only possible value.

Note that the element a_1 was called u in [8]. □

Proposition 6.4.[4] *Suppose that mmf exists. Then Conjecture 1.2 is equivalent to Conjecture 5.5.*

Proof. The existence of *mmf* ensures that the Adams d_2 differential is compatible with the map $r : \mathrm{Ext}_A \longrightarrow \mathrm{Ext}_{A(2)}$, so that $d_2(r(v_n)) = r(d_2(v_n))$. For degree reasons, $r(v_n)$ is either equal to $h_1^{-3(2^{n-2}-1)} a_1^{2^{n-2}-1} v_2$, or it is zero. Also for degree reasons, $d_2(v_n)$ is either equal to $h_1 v_{n-1}^2$, or it is zero.

Suppose that Conjecture 1.2 holds. Then $r(d_2(v_n))$ equals $h_1 r(v_{n-1})^2$. We may assume by induction that $r(v_{n-1}^2)$ equals $h_1^{-3(2^{n-2}-2)} a_1^{2^{n-2}-2} v_2^2$. In particular, this shows that $r(d_2(v_n))$ is non-zero. But $r(d_2(v_n))$ equals $d_2(r(v_n))$, so $r(v_n)$ must also be non-zero. This establishes Conjecture 5.5.

Now suppose that Conjecture 5.5 holds. Then $d_2(r(v_n))$ is equal to $h_1^{-3(2^{n-2}-2)+1} a_1^{2^{n-2}-2} v_2^2$ because of the differential $d_2(a_1) = h_1^2 c_0$ from Lemma 6.3. But $d_2(r(v_n))$ equals $r(d_2(v_n))$, so $d_2(v_n)$ must also be non-zero. This establishes Conjecture 1.2. □

[4] Conjecture 1.2 is settled, so Proposition 6.4 now says that if *mmf* exists, then Conjecture 5.5 holds.

7. Tables

Table 1
Generators for May $E_2[h_1^{-1}]$ in $t \leq 66$.

	Description	$(m-f,t,f,c)$	d_2
$\mathbf{h_0}$	$h_1^{-2}c_0 = h_{30} + h_1^{-1}h_{20}h_{21}$	$(2,3,1,1)$	
$\mathbf{h_1}$	h_3	$(0,3,1,0)$	
$\mathbf{h_2}$	h_4	$(0,7,1,0)$	
$\mathbf{b_{20}}$	$h_{40}^2 + h_1^{-2}h_{20}^2h_{31}^2$	$(6,14,2,2)$	$\mathbf{h_0^2h_2}$
$\mathbf{h_3}$	h_5	$(0,15,1,0)$	
$\mathbf{h_0(1)}$	$h_1^{-1}h_0(1,3) = h_{50}h_3 + h_1^{-1}h_{20}h_{31}h_{23} + h_{40}h_{23} + h_1^{-1}h_{20}h_{41}h_3$	$(4,18,2,1)$	$\mathbf{h_0h_2^2}$
$\mathbf{b_{21}}$	$b_{23} = h_{23}^2$	$(2,22,2,0)$	$\mathbf{h_1^2h_3 + h_2^3}$
$\mathbf{b_{30}}$	$h_{50}^2 + h_1^{-2}h_{20}^2h_{41}^2$	$(8,30,2,2)$	$\mathbf{h_3b_{20}}$
$\mathbf{h_4}$	h_6	$(0,31,1,0)$	
$\mathbf{h_1(1)}$	$h_3(1) = h_4h_{33} + h_{23}h_{24}$	$(2,34,2,0)$	$\mathbf{h_1h_3^2}$
$\mathbf{b_{22}}$	$b_{24} = h_{24}^2$	$(2,46,2,0)$	$\mathbf{h_2^2h_4 + h_3^3}$
$\mathbf{b_{31}}$	$b_{33} = h_{33}^2$	$(4,54,2,0)$	$\mathbf{h_4b_{21} + h_2b_{22}}$
$\mathbf{b_{40}}$	$h_{60}^2 + h_1^{-2}h_{20}^2h_{51}^2$	$(10,62,2,2)$	$\mathbf{h_4b_{30}}$
$\mathbf{h_5}$	h_7	$(0,63,1,0)$	
b_{20}	$b_{20} = h_{20}^2$	$(2,2,2,2)$	τh_1^3
b_{21}	$b_{21} = h_{21}^2$	$(2,4,2,0)$	$h_1^2\mathbf{h_1}$
b_{31}	$b_{31} = h_{31}^2$	$(4,12,2,0)$	$\mathbf{h_2}b_{21}$
b_{41}	$b_{41} = h_{41}^2$	$(6,28,2,0)$	$\mathbf{h_3}b_{31}$
b_{51}	$b_{51} = h_{51}^2$	$(8,60,2,0)$	$\mathbf{h_4}b_{41}$

Table 2
Relations for May $E_2[h_1^{-1}]$ in $t \leq 66$.

Relation	$(m-f,t,f,c)$
$\mathbf{h_0h_1}$	$(2,6,2,1)$
$\mathbf{h_1h_2}$	$(0,10,2,0)$
$\mathbf{h_2b_{20} + h_0h_0(1)}$	$(6,21,3,2)$
$\mathbf{h_2h_3}$	$(0,22,2,0)$
$\mathbf{h_2h_0(1) + h_0b_{21}}$	$(4,25,3,1)$
$\mathbf{h_3h_0(1)}$	$(4,33,3,1)$
$\mathbf{b_{20}b_{21} + h_1^2b_{30} + h_0(1)^2}$	$(8,36,4,2)$
$\mathbf{h_0h_1(1)}$	$(4,37,3,1)$
$\mathbf{h_3b_{21} + h_1h_1(1)}$	$(2,37,3,0)$
$\mathbf{h_3h_4}$	$(0,46,2,0)$
$\mathbf{b_{20}h_1(1) + h_1h_3b_{30}}$	$(8,48,4,2)$
$\mathbf{h_3h_1(1) + h_1b_{22}}$	$(2,49,3,0)$
$\mathbf{h_0(1)h_1(1)}$	$(6,52,4,1)$
$\mathbf{b_{20}b_{22} + h_0^2b_{31} + h_3^2b_{30}}$	$(8,60,4,2)$
$\mathbf{b_{22}h_0(1) + h_0h_2b_{31}}$	$(6,64,4,1)$
$\mathbf{h_4h_1(1)}$	$(2,65,3,0)$

Table 3
Notation for the h_1-localized May spectral sequence.

	Description	$(m-f,t,f,c)$
P	b_{20}^2	$(4,4,4,4)$
e_0	$b_{21}h_0(1) = h_1\mathbf{h_0}b_{21}$	$(4,7,4,1)$
g	b_{21}^2	$(4,8,4,0)$
B	$\mathbf{b_{20}}b_{21} + \mathbf{h_0^2}b_{31}$	$(8,18,4,2)$
B_1	c_0B	$(10,21,7,3)$
ϕ	$h_1b_{21}B$	$(10,22,7,2)$
Δ_1	b_{31}^2	$(8,24,4,0)$
D_4	$g\mathbf{h_0(1)} + \mathbf{h_0h_2}b_{21}b_{31}$	$(8,26,6,1)$
$\mathbf{P}$	$\mathbf{b_{20}^2}$	$(12,28,4,4)$
s_1	$h_1h_4^2b_{21}b_{31} + h_1gb_{23} + h_1^{-1}h_3h_5b_{21}^3$	$(6,30,7,0)$
D_3'	$h_1^4\mathbf{b_{20}h_0(1)}$	$(10,32,8,3)$
y_{34}	$h_5{}^2b_{21} + h_1^2h_3(1)$	$(2,34,4,0)$
y_{35}	h_4b_{41}	$(6,35,3,0)$
$\mathbf{d_0}$	$\mathbf{h_0(1)^2}$	$(8,36,4,2)$

Table 3 (*continued*)

	Description	$(m-f,t,f,c)$
$\boldsymbol{\nu}$	$\mathbf{h_2 b_{30}}$	$(8,37,3,2)$
$\mathbf{e_0}$	$\mathbf{b_{20}h_0(1)}$	$(6,40,4,1)$
$\mathbf{g}$	$\mathbf{b_{21}^2}$	$(4,44,4,0)$
y_{45}	$g\boldsymbol{\nu} + \mathbf{h_0^2h_3}\Delta_1$	$(12,45,7,2)$
y_{60}	$h_1^8\mathbf{b_{21}b_{30}} + \mathbf{h_2^2b_{30}}b_{21}b_{31}g$	$(14,60,12,2)$
$\boldsymbol{\Delta}$	$\mathbf{b_{30}^2}$	$(16,60,4,4)$
y_{61}	$\mathbf{h_4h_0}(1)b_{31} + \mathbf{h_0b_{22}}b_{31} + \mathbf{h_0h_3^2}b_{41} + \mathbf{h_0b_{31}}b_{21}$	$(8,61,5,1)$
y_{64}	$\mathbf{b_{30}h_1(1)}$	$(10,64,4,2)$

Table 4
The h_1-localized May d_4 differential.

	$(m-f,t,f,c)$	d_4
g	$(4,8,4,0)$	$h_1^4h_4$
Δ_1	$(8,24,4,0)$	gh_5
$\mathbf{P}$	$(12,28,4,4)$	$\mathbf{h_0^4h_3}$
y_{35}	$(6,35,3,0)$	y_{34}
$\boldsymbol{\nu}$	$(8,37,3,2)$	$\mathbf{h_0^2h_3^2}$
$\boldsymbol{\Delta}$	$(16,60,4,4)$	$\mathbf{h_4P}$
$b_{41}\boldsymbol{\nu}$	$(14,65,5,2)$	$h_1^2y_{64}$

Table 5
The h_1-localized May d_6 differential.

	$(m-f,t,f,c)$	d_6
ϕ	$(10,22,7,2)$	$h_1c_0^2h_5$
c_0g^3	$(14,27,15,1)$	$h_1^{10}D_4$
h_4g^3	$(12,31,13,0)$	$h_1^7s_1$
$c_0g\phi$	$(16,33,14,3)$	$h_1^7D_3'$
$h_4g\phi$	$(14,37,12,2)$	$h_1^9\mathbf{d_0}$
h_4gD_4	$(12,41,11,1)$	$h_1^8\mathbf{e_0}$
h_4gs_1	$(10,45,12,0)$	$h_1^9\mathbf{g}$
$h_4g^3\Delta_1$	$(20,55,17,0)$	$h_1^7s_1\Delta_1$
$c_0g\phi\Delta_1$	$(24,57,18,3)$	$h_1^7D_3'\Delta_1$
g^2y_{45}	$(20,61,15,2)$	$h_1^4y_{60}$
$h_4g\phi\Delta_1$	$(22,61,16,2)$	$h_1^9\Delta_1\mathbf{d_0}$
$c_0g\phi\mathbf{P}$	$(28,61,18,7)$	$h_1^7D_3'\mathbf{P}$
$h_4g\phi b_{41}$	$(20,65,14,2)$	$h_1^9b_{41}\mathbf{d_0}$
$h_4gD_4\Delta_1$	$(20,65,15,1)$	$h_1^8\Delta_1\mathbf{e_0}$

Table 6
The h_1-localized May d_8 differential.

	$(m-f,t,f,c)$	d_8
g^2	$(8,16,8,0)$	$h_1^8h_5$
Δ_1^2	$(16,48,8,0)$	g^2h_6
$\Delta_1\mathbf{P}$	$(20,52,8,4)$	$h_1^{-6}{c_0}^2e_0^2h_6$
$\mathbf{P}^2$	$(24,56,8,8)$	$\mathbf{h_0^8h_4}$
$c_0\Delta_1y_{35}$	$(16,62,10,1)$	$h_1^6y_{61}$

Table 7
The h_1-localized May d_{10} differential.

	$(m-f,t,f,c)$	d_{10}
$\Delta_1\phi$	$(18,46,11,2)$	$h_1^3e_0^2h_6$
$\phi\mathbf{P}$	$(22,50,11,6)$	$h_1^{-7}c_0^6h_6$
c_0g^6	$(26,51,27,1)$	$h_1^{18}\Delta_1D_4$
$g^4\phi$	$(26,54,23,2)$	$h_1^{13}gy_{45}$
$g^2\phi^2$	$(28,60,22,4)$	$h_1^{12}c_0\mathbf{b_{30}}D_4$
$g^2\Delta_1\phi$	$(26,62,19,2)$	$h_1^{13}\Delta_1\boldsymbol{\nu}$

Table 8
The h_1-localized May d_{12} differential.

	$(m-f,t,f,c)$	d_{12}
h_5g^6	$(24,63,25,0)$	$h_1^{12}h_4h_6g^3$
ϕ^2	$(20,44,14,4)$	$h_1^2c_0^4h_6$
$c_0g^4\phi$	$(28,57,26,3)$	$h_1^{19}\mathbf{b_{30}}D_4$
$e_0g\Delta_1\phi$	$(26,61,19,3)$	$h_1^8h_6e_0\phi$
$c_0g^2\phi^2$	$(30,63,25,5)$	$h_1^{16}\mathbf{b_{30}}D_3'$
$g^2\Delta_1D_4$	$(24,66,18,1)$	$h_1^8h_6gD_4$

Table 9
The h_1-localized May d_{14} differential.

	$(m-f,t,f,c)$	d_{14}
$g^2\phi$	$(18,38,15,2)$	$h_1^9c_0^2h_6$
e_0g^6	$(28,55,28,1)$	$h_1^{21}b_{41}D_4$
$g^4e_0\phi$	$(30,61,27,3)$	$h_1^{18}b_{41}D_3'$

Table 10
The h_1-localized higher May differentials.

	$(m-f,t,f,c)$	d_r	Value
g^4	$(16,32,16,0)$	d_{16}	$h_1^{16}h_6$
$e_0^2g^5$	$(28,54,28,2)$	d_{18}	$h_1^{21}h_6\phi$
g^8	$(32,64,32,0)$	d_{32}	$h_1^{32}h_7$

Table 11
Hidden Relations in $\mathrm{Ext}_A[h_1^{-1}]$.

	(t,f,c)
$h_1^2e_0^3+c_0^2\cdot e_0g=h_1^7B_1$	$(21,14,3)$
$h_1^2e_0(e_0g)^2+c_0^2\cdot e_0g^3=h_1^7g^2B_1$	$(37,22,3)$
$h_1^4e_0^4\cdot e_0g+c_0^4\cdot e_0g^3=h_1^{14}B_1\phi$	$(43,28,5)$
$e_0^2\cdot e_0g^3+(e_0g)^3=h_1^{13}\Delta_1B_1$	$(45,24,3)$
$h_1^6e_0^7+h_1^4c_0^2e_0^4\cdot e_0g+h_1^2c_0^4e_0(e_0g)^2+c_0^6\cdot e_0g^3=h_1^{23}\mathbf{P}B_1$	$(49,34,7)$

Table 12
Generators for $\mathrm{Ext}_A[h_1^{-1}]$ in $t\le 66$.

May name	Theorem 3.4 name	(t,f,c)	Adams d_2
c_0	$h_1^2v_2$	$(3,3,1)$	
P	v_1^4	$(4,4,4)$	
e_0	$h_1^3v_3$	$(7,4,1)$	c_0^2
e_0g	$h_1^7v_4$	$(15,8,1)$	$h_1^2e_0^2$
e_0g^3	$h_1^{15}v_5$	$(31,16,1)$	$h_1^2(e_0g)^2$
e_0g^7	$h_1^{31}v_6$	$(63,32,1)$	$h_1^2(e_0g^3)^2$

Table 13
The localization map $\mathrm{Ext}_A \longrightarrow \mathrm{Ext}_A[h_1^{-1}]$.

Element	May description	(s, f, w)	Theorem 3.4 value	Table 12 value
$P^k h_1$		$(1,1,1) + k(8,4,4)$	$h_1 v_1^{4k}$	$h_1 P^k$
$P^k c_0$		$(8,3,5) + k(8,4,4)$	$h_1^2 v_1^{4k} v_2$	$P^k c_0$
$P^k d_0$		$(14,4,8) + k(8,4,4)$	$h_1^2 v_1^{4k} v_2^2$	$h_1^{-2} P^k c_0^2$
$P^k e_0$		$(17,4,10) + k(8,4,4)$	$h_1^3 v_1^{4k} v_3$	$P^k e_0$
$e_0 g$		$(37,8,22)$	$h_1^7 v_4$	$e_0 g$
u	$\Delta h_1 d_0$	$(39,9,21)$	$h_1^3(v_1^4 v_3^2 + v_2^6)$	$h_1^{-3}(Pe_0^2 + c_0^6)$
$P^k u$		$(39,9,21) + k(8,4,4)$	$h_1^3 v_1^{4k}(v_1^4 v_3^2 + v_2^6)$	$h_1^{-3} P^k(Pe_0^2 + c_0^6)$
v	$\Delta h_1 e_0$	$(42,9,23)$	$h_1^4(v_1^4 v_4 + v_2^4 v_3)$	$h_1^{-3}(Pe_0 g + h_1^{-4} c_0^4 e_0)$
$P^k v$		$(42,9,23) + k(8,4,4)$	$h_1^4 v_1^4(v_1^4 v_4 + v_2^4 v_3)$	$h_1^{-3} P^k(Pe_0 g + h_1^{-4} c_0^4 e_0)$
B_1	$c_0 B$	$(46,7,25)$	$h_1^4(v_2^2 v_4 + v_3^3)$	$h_1^{-7} c_0^2 e_0 g + h_1^{-5} e_0^3$
u'	$\Delta c_0 d_0$	$(46,11,25)$	$h_1^4(v_1^4 v_2 v_3^2 + v_2^7)$	$h_1^{-4}(Pc_0 e_0^2 + h_1^{-6} c_0^7)$
$P^k u'$		$(46,11,25) + k(8,4,4)$	$h_1^4 v_1^{4k}(v_1^4 v_2 v_3^2 + v_2^7)$	$h_1^{-4} P^k(Pc_0 e_0^2 + h_1^{-6} c_0^7)$
v'	$\Delta c_0 e_0$	$(49,11,27)$	$h_1^5(v_1^4 v_2 v_4 + v_2^5 v_3)$	$h_1^{-4}(Pc_0 e_0 g + h_1^{-4} c_0^5 e_0)$
$P^k v'$		$(49,11,27) + k(8,4,4)$	$h_1^5 v_1^{4k}(v_1^4 v_2 v_4 + v_2^5 v_3)$	$h_1^{-4} P^k(Pc_0 e_0 g + h_1^{-4} c_0^5 e_0)$
B_8	$h_0(1)B_1$	$(53,9,29)$	$h_1^5(v_2^3 v_4 + v_2 v_3^3)$	$h_1^{-6} c_0(h_1^{-2} c_0^2 e_0 g + e_0^3)$
x'	$h_0(1)BP$	$(53,10,28)$	$h_1^3 v_1^4(v_2^2 v_4 + v_3^3)$	$h_1^{-6} P(h_1^{-2} c_0^2 e_0 g + e_0^3)$
B_{21}	$h_0(1)^3 B$	$(59,10,32)$	$h_1^5(v_2^4 v_4 + v_2^2 v_3^3)$	$h_1^{-8} c_0^2(h_1^{-2} c_0^2 e_0 g + e_0^3)$
B_{22}	$b_{21} d_0 B$	$(62,10,34)$	$h_1^6(v_2^2 v_3 v_4 + v_3^4)$	$h_1^{-6} e_0(h_1^{-2} c_0^2 e_0 g + e_0^3)$
U	$\Delta^2 h_1^2 d_0$	$(64,14,34)$	$h_1^4(v_1^8 v_3 v_4 + v_2^{10})$	$h_1^{-6}(P^2 e_0 e_0 g + h_1^{-10} c_0^{10})$
$P^2 x'$		$(69,18,36)$	$h_1^3 v_1^{12}(v_2^2 v_4 + v_3^3)$	$h_1^{-6} P^3(h_1^{-2} c_0^2 e_0 g + e_0^3)$

Table 14
The localization map for $\mathrm{Ext}_{A(2)} \longrightarrow \mathrm{Ext}_{A(2)}[h_1^{-1}]$.

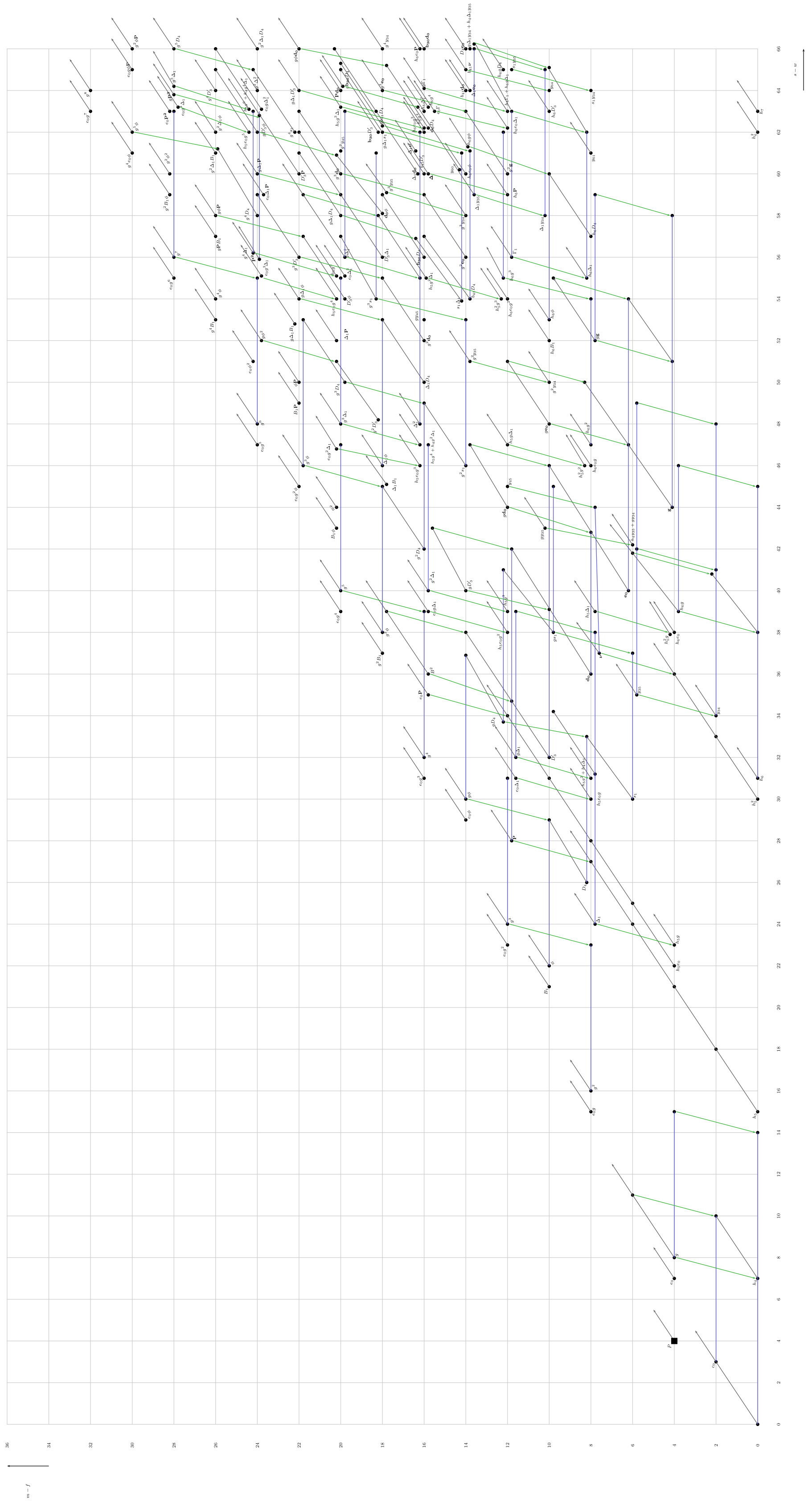
Element	May description	(s, f, w)	Value
P		$(8,4,4)$	v_1^4
c		$(8,3,5)$	$h_1 v_2$
u	$h_1 b_{21}$	$(11,3,7)$	a_1
d		$(14,4,8)$	$h_1^2 v_2^2$
e		$(17,4,10)$	$a_1 v_2$
g		$(20,4,12)$	$h_1^{-2} a_1^2$
Δh_1		$(25,5,13)$	$h_1^{-5} v_1^4 a_1^2 + h_1 v_2^4$
Δc		$(32,7,17)$	$h_1^{-5} v_1^4 v_2 a_1^2 + h_1 v_2^5$
Δu		$(35,7,19)$	$h_1^{-6} v_1^4 a_1^3 + v_2^4 a_1$
Δ^2		$(48,8,24)$	$h_1^{-12} v_1^8 a_1^4 + v_2^8$

8. h_1-localized motivic May spectral sequence charts

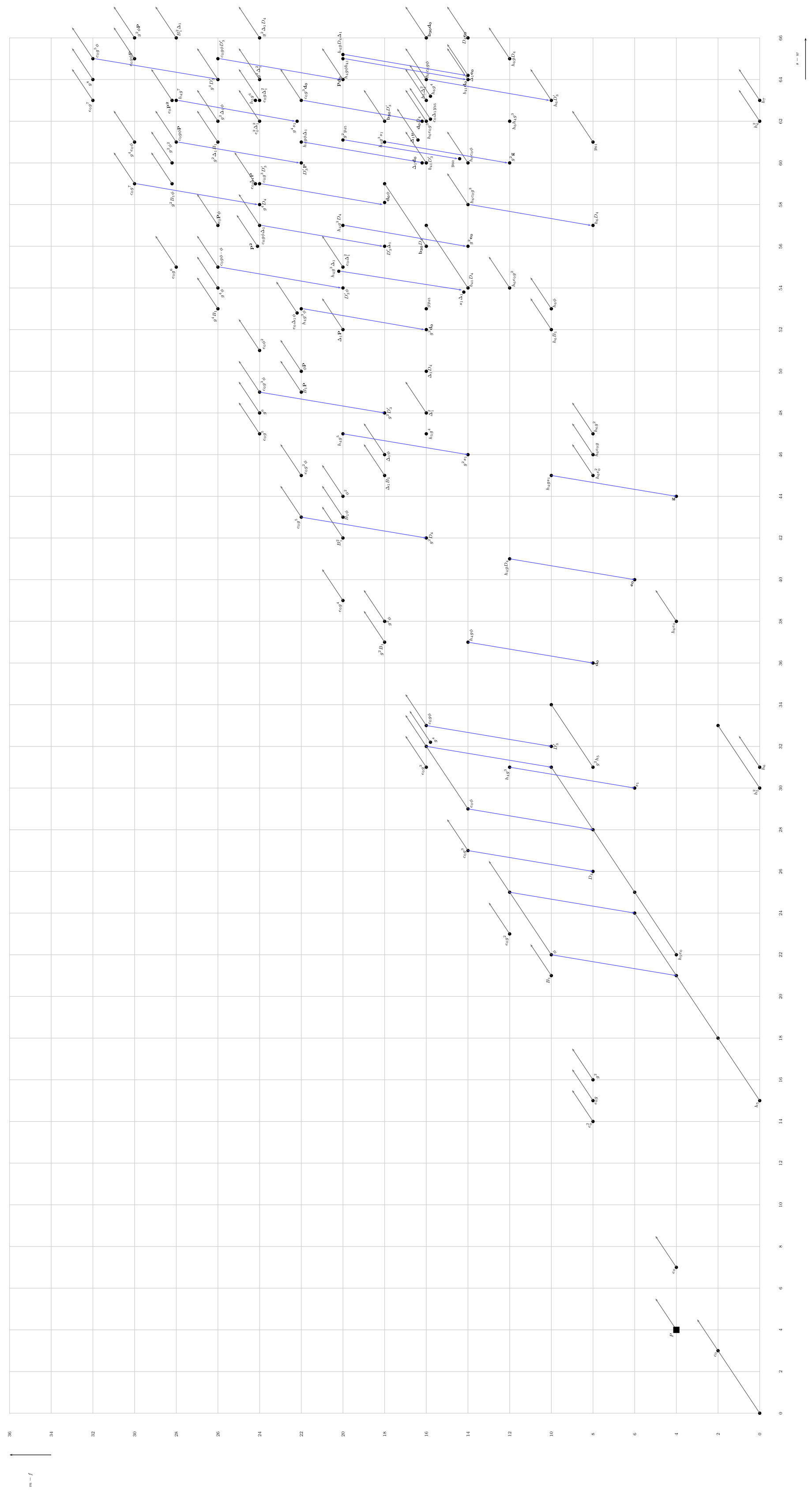
The following is a key for reading the charts.

(1) The horizontal axes indicate the Milnor–Witt stem, i.e., the stem minus the motivic weight.
(2) The vertical axes indicate the May filtration minus the Adams filtration.
(3) Dots indicate copies of $\mathbb{F}_2$.
(4) All of our calculations are free as a module over the polynomial ring $\mathbb{F}_2[P]$, so the charts only show the spectral sequence modulo P. The solid squares labeled P serve as a reminder of this.
(5) Lines of slope 2/3 indicate multiplications by c_0.
(6) Arrows of slope 2/3 indicate infinitely many multiplications by c_0.
(7) Horizontal blue lines indicate multiplications by $\mathbf{h_2} = h_4$.
(8) The May d_r differential takes elements one unit to the left and r units downward.

In addition to the May d_{16} differentials, the chart of the E_{16}-page also indicates the remaining higher differentials. In our range, all differentials except for d_{18} and d_{32} are trivial.

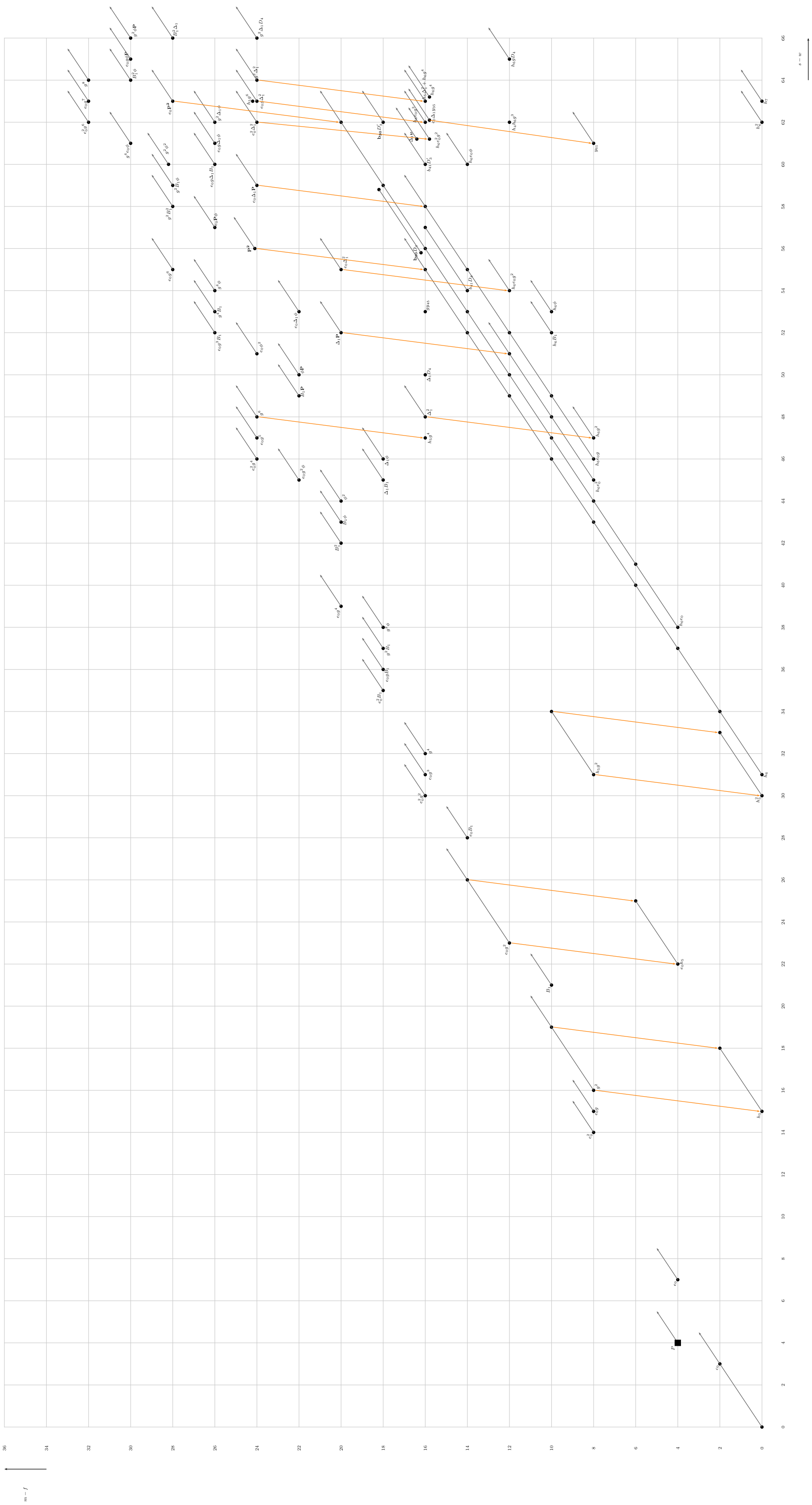


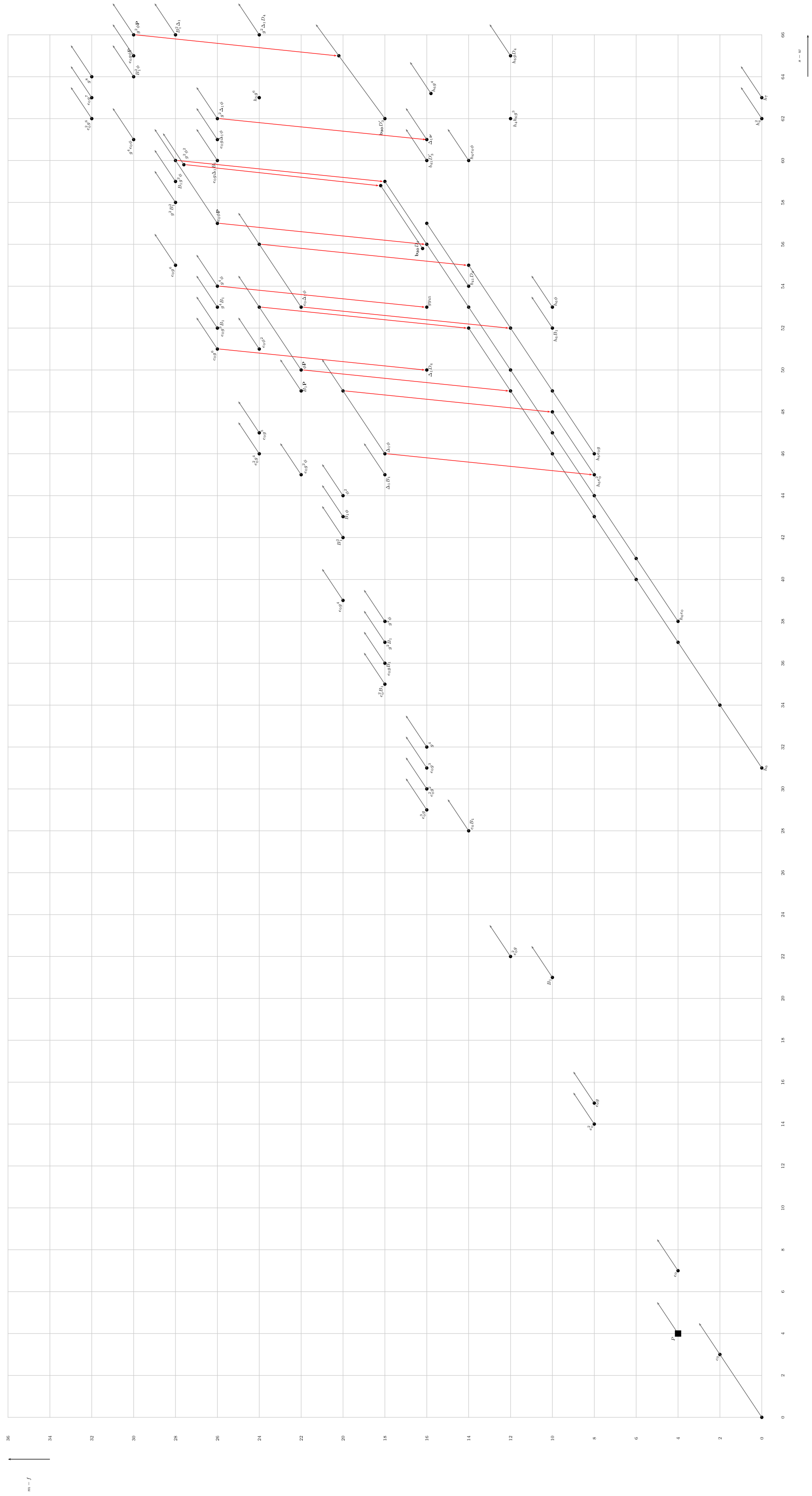
E_2-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE



E_6-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE

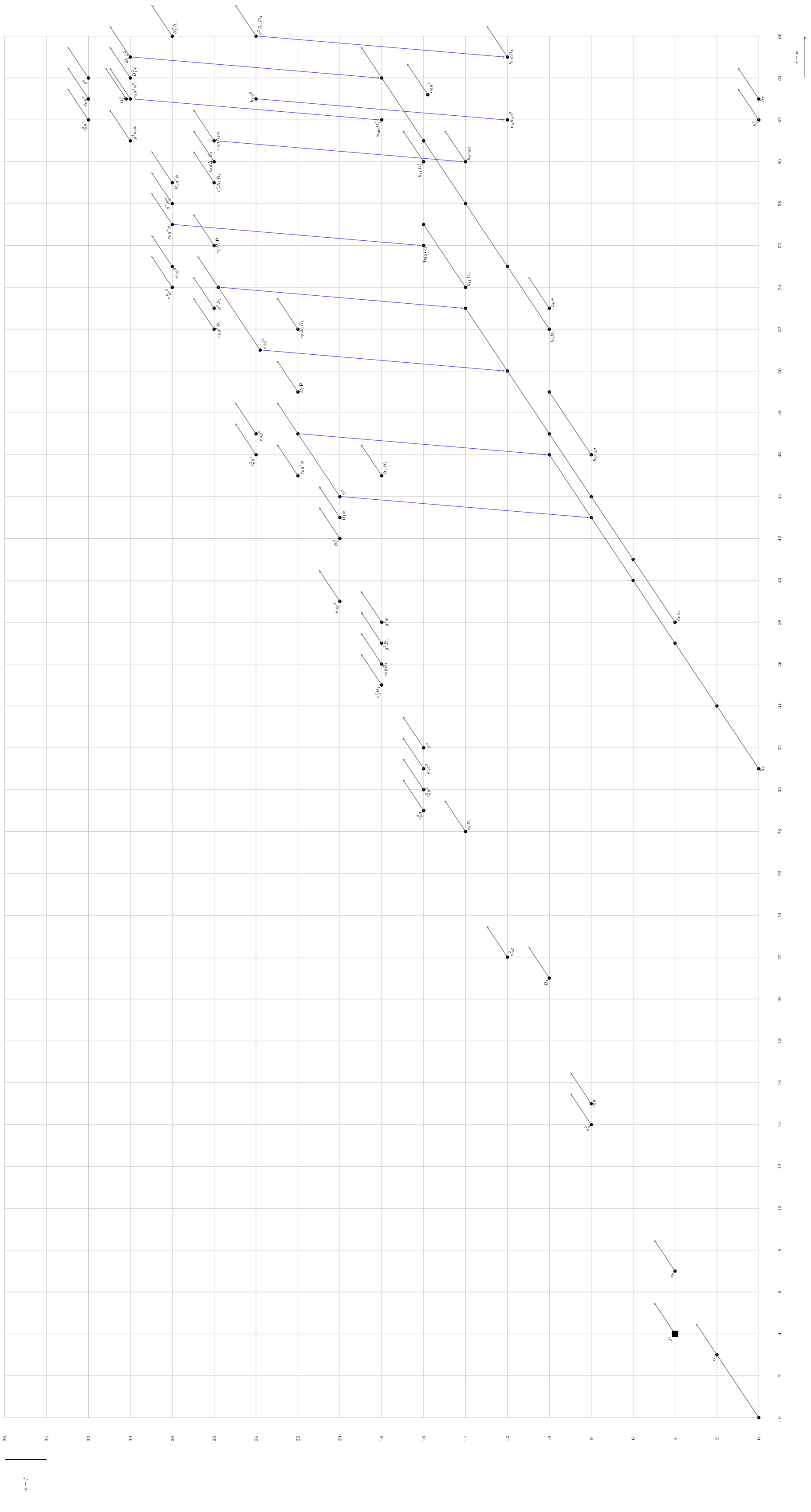
E_∞-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE





E_{10}-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE

E_{12}-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE



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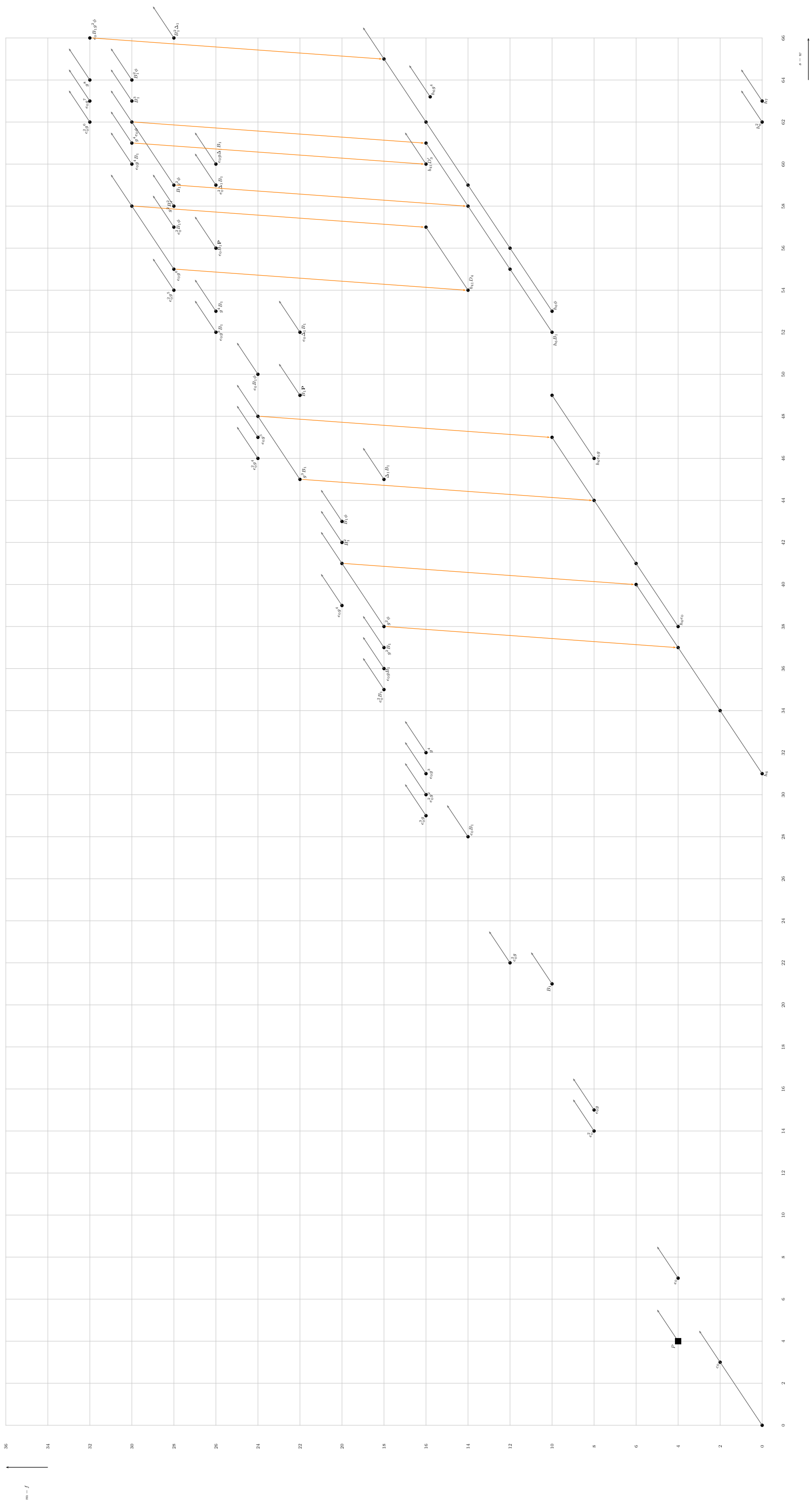

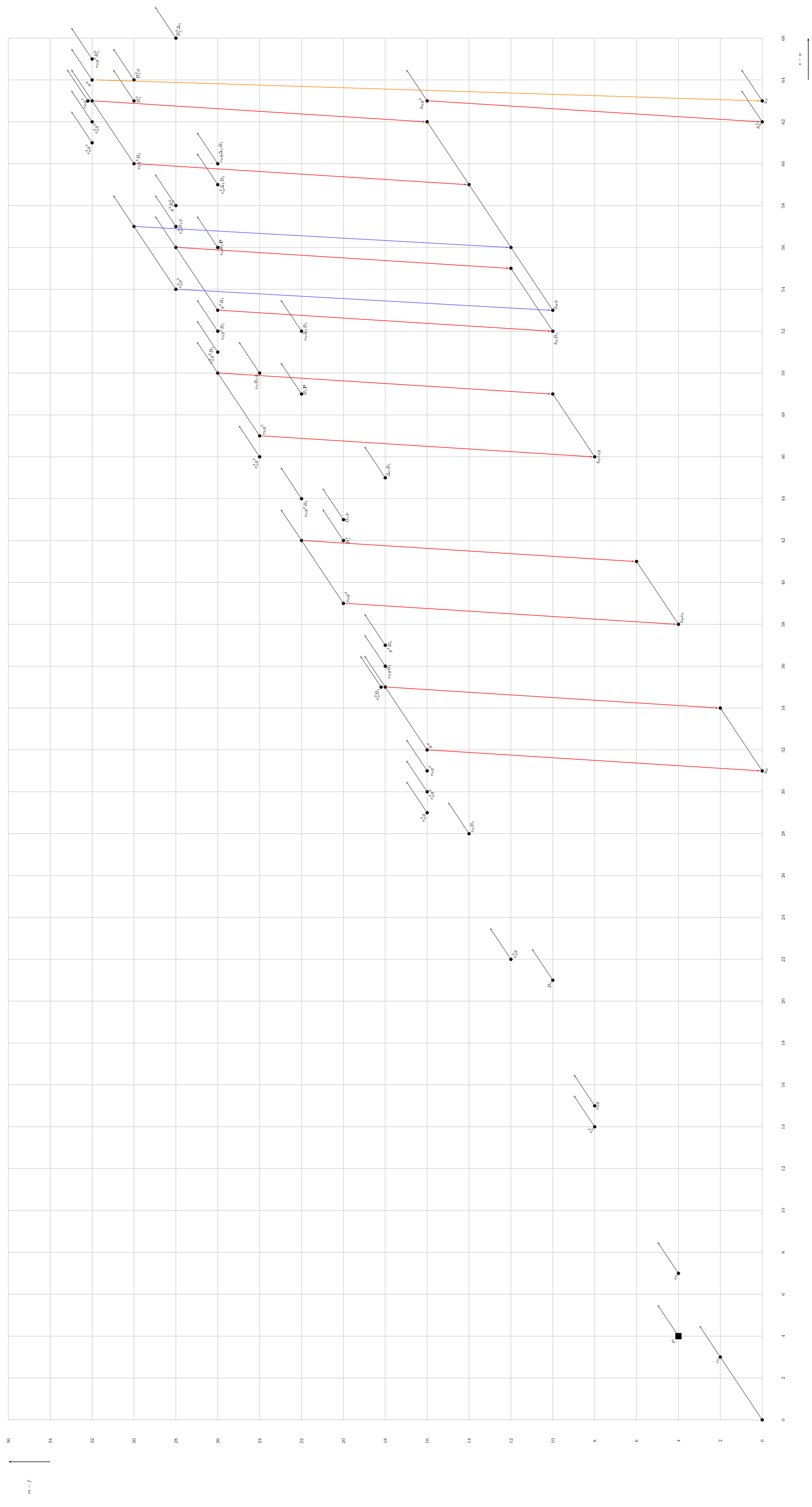

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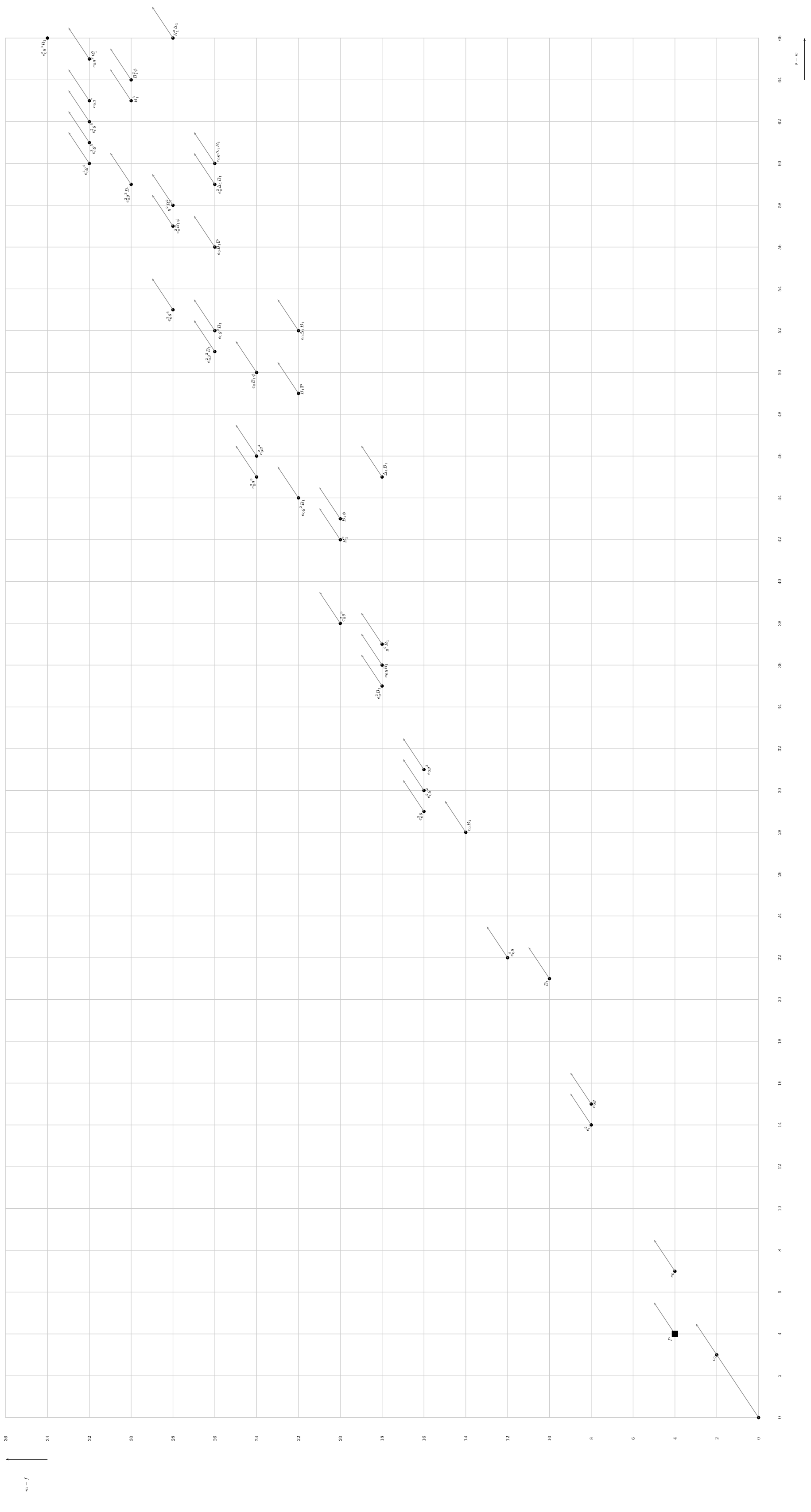

E_∞-PAGE OF THE h_1-LOCALIZED MOTIVIC MAY SPECTRAL SEQUENCE

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