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Group-Like Structures in General Categories III Primitive Categories

Dedicated to Professor B. L. van der Waerden on the occasion of his sixtieth birthday

By

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1. Introduction

In [4] and [5] certain structural features of the category $\mathfrak{G}$ of groups and homomorphisms were discussed. Thus for example it was shown in [4] (see also [1]) that an $\underline{H}$-object (i.e. object with multiplication having a two-sided unit) in $\mathfrak{G}$ is just an abelian group and the $\underline{H}$-structure is just that given by the group multiplication. It follows therefore that every group-homomorphism between abelian groups is *primitive* with respect to their (unique) $\underline{H}$-structure; in other words, in the category $\mathfrak{G}$ every map between $\underline{H}$-objects preserves the $\underline{H}$-structure. It was also shown in [4] that in the canonical factorization[1]) (F') of the canonical homomorphism $\varkappa$ from the free product of n groups to their direct product, all maps except the first are isomorphisms (the first is thus equivalent to $\varkappa$ itself and hence an epimorphism).

It is the object of this paper to prove these and other results in a more general framework than that of the particular category $\mathfrak{G}$. In this way we make them available in other categories, particularly in categories of interest in topology; we are also able to avail ourselves of the formal duality principle to deduce results for comultiplicative structures. The feature of the category $\mathfrak{G}$ which we abstract in this paper is just that $\mathfrak{G}$ is a category of $\underline{M}$-objects (i.e. objects with multiplication, no further axioms being required) of another category (namely, the category $\mathfrak{S}$ of based sets and based functions), together with those maps in $\mathfrak{S}$ between the appropriate $\underline{M}$-objects which are *primitive* (or homomorphic) with respect to the given $\underline{M}$-structures. It is also of importance that $\mathfrak{G}$ is, so to say, closed with respect to direct products in $\mathfrak{S}$. Many other categories, of course, possess these properties of $\mathfrak{G}$, some being also derived from the category $\mathfrak{S}$, others from different categories which share with $\mathfrak{S}$ the property of admitting direct products.

The general notion, then, is that of an underlying category $\mathfrak{C}$ and a category $\mathfrak{D}$ of $\underline{M}$-objects of $\mathfrak{C}$ and primitive maps. Such a category $\mathfrak{D}$ we describe as a *primitive* category or, more fully, an $\underline{M}$-*primitive category over* $\mathfrak{C}$. If we restrict

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[1]) See [3], § 4, or [4].

the structures in $\mathfrak{D}$ by inserting that the appropriate axioms be satisfied (see [2]) we then get the notions of $\underline{H}$-primitive categories, $\underline{G}$-primitive categories, etc. Examples of such categories are, of course, numerous. Thus groups, semi-groups, abelian groups, nilpotent groups — and full subcategories of these — constitute examples of categories to which our theorems are applicable; in all these cases the underlying category is $\mathfrak{S}$. The topological categories $\mathfrak{T}$ and $\mathfrak{T}_h$ (see [2]) are underlying categories for the primitive categories of topological groups and H-spaces respectively. The category of Lie groups and the category of algebraic groups provide further examples of primitive categories. The dual notion may be exemplified by the category of H'-spaces and primitive classes, which is an $\overline{\underline{H}}$-primitive category over $\mathfrak{T}_h$; and by the category of free groups with preferred free generating sets, and homomorphisms between such free groups mapping preferred generator to preferred generator or identity element. This latter example is a $\overline{\underline{G}}$-primitive category over $\mathfrak{G}$; and it is pointed out in Theorem 4.16 that it is indeed category isomorphic to $\mathfrak{S}$. Thus we have a very curious "duality" between $\mathfrak{G}$ and $\mathfrak{S}$ in which $\mathfrak{G}$ is the primitive category consisting of all $\underline{G}$-objects in $\mathfrak{S}$ and $\mathfrak{S}$ is the primitive category consisting of all $\overline{\underline{G}}$-objects in $\mathfrak{G}$. Although we do not mention explicitly the duals of the theorems we prove in this paper, we do point out explicitly in sections 4 and 5 how the duality relations between $\mathfrak{S}$ and $\mathfrak{G}$ enables us to pair off certain dual features of these two categories.

Section 2 collects together certain preliminary results needed in the sequel. The proofs of the propositions enunciated are all rendered very simple by the application of the "presentation" theorem (Theorem 4.10 of [2]); one proof from first principles (that of Proposition 2.1) is given by way of illustrative contrast.

Section 3 opens with the precise definition of a primitive category, but the theorems in the section are confined to the case of commutative structures. In particular it turns out that the process of taking $\underline{G}$-structures is in a sense idempotent. Thus if $\mathfrak{D}$ is the complete[2]) $\underline{G}$-primitive category over $\mathfrak{C}$, $\mathfrak{E}$ the complete $\underline{G}$-primitive category over $\mathfrak{D}$, and $\mathfrak{F}$ the complete $\underline{G}$-primitive category over $\mathfrak{E}$, then $\mathfrak{E}$ is essentially just the complete $\underline{CG}$-primitive category over $\mathfrak{C}$ (i.e., the category of "commutative groups" in $\mathfrak{C}$); and $\mathfrak{F}$ coincides with $\mathfrak{E}$. We also describe in this section the relation of the notions of commutative category to that of additive category.

In section 4 we develop the elementary parts of the general theory of $\underline{M}$-primitive categories. In the first place we are concerned with the transport of structure from a category to a primitive category $\mathfrak{D}$ over $\mathfrak{C}$. In particular left-equalizers and intersections — more generally, inverse limits — are preserved in the passage from $\mathfrak{C}$ to $\mathfrak{D}$ provided $\mathfrak{D}$ is big enough, i.e., provided that the objects in question in $\mathfrak{C}$ lie, with their structure maps, in $\mathfrak{D}$. Thus if $\mathfrak{D}$ is complete left-equalizers and inverse limits are preserved; but, in general, right equalizers and direct limits are not. The situation is, naturally, reversed in the dual case.

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[2]) I.e., if $\mathfrak{D}$ contains all the multiplicative objects of $\mathfrak{C}$ of the sort considered.

We turn our attention next in section 4 to the question of the behaviour of epimorphisms[2a]) in a primitive category. It turns out (Theorems 4.5 and 4.6) that epimorphisms enjoy properties in primitive categories which they do not possess in arbitrary categories. We do not succeed in proving that the direct product of right-equalizers is the right-equalizer of the direct products of the maps concerned (in a primitive category) without an additional hypothesis which is verified in the cases which are familiar to us. In particular we are led to introduce the notions of a *strong* epimorphism in the primitive category $\mathfrak{D}$ over $\mathfrak{C}$; this is an epimorphism in $\mathfrak{D}$ having a right inverse in $\mathfrak{C}$. Such epimorphisms enjoy special properties and in the primitive categories over $\mathfrak{S}$ mentioned above all epimorphisms are strong. We then assume that the category $\mathfrak{D}$ is in fact an I-category (see [2]) and obtain theorems motivated by known properties of the category $\mathfrak{G}$ of groups (Theorems 4.12 and 4.15); and close the section by proving the duality theorem (Theorem 4.16) for the categories $\mathfrak{G}$ and $\mathfrak{S}$ already referred to.

Section 5 is devoted to proving Theorem 2.3 of [5] in the general context of the theory of primitive categories (in [5] it was a theorem about $\mathfrak{G}$). In particular we remark that in [5] we were content to observe that the factorization (F) of the canonical map $\varkappa : A_1 * \cdots * A_n \to A_1 \times \cdots \times A_n$ in $\mathfrak{G}$ consists of epimorphisms. In the more general context of this paper we stress that the maps of the factorization are, in fact, strong epimorphisms.

The final section is concerned with the transfer of homotopy functors to primitive categories. Here our main concern is to effect the transfer and to show that the factorization (F) is homotopy invariant in a complete primitive category $\mathfrak{D}$ over a category $\mathfrak{C}$ with homotopy.

In the course of the paper we give a few applications of the results: we emphasize that these applications are given largely to provide evidence of the relevance of the results outside the category $\mathfrak{G}$ and of the possibility of providing a common proof of assertions which are dual in the sense of [1].

2. Preliminary results

In this section we list certain useful preliminary definitions and results before we pass to primitive categories. Let $\mathfrak{C}$ be a category (with zero maps) admitting (finite) direct products; that is, in the terminology of [2], $\mathfrak{C}$ is a D-category. We should recall the canonical maps $d = d_A = \{1, 1\} : A \to A \times A$ (the diagonal map) and $\tau = \tau_{A,B} = \{p_2, p_1\} : A \times B \to B \times A$ (the reverse map).

Now let (A_1, m_1), (A_2, m_2) be two $\underline{\mathrm{M}}$-objects and let (A, m) be their direct product (see [2], Theorem 4.8 et seq.). Thus $A = A_1 \times A_2$ and $m : A_1 \times A_2 \times \times A_1 \times A_2 \to A_1 \times A_2$ is given by

$$m = \{m_1(p_1 \times p_1), m_2(p_2 \times p_2)\} .$$

[2a]) In view of terminological differences appearing in the litterature we wish to emphasize that the terms "monomorphism" and "epimorphism" are used in agreement with [9], see also [2, 3]: A map $f : X \to Y$ in the category $\mathfrak{C}$ is a monomorphism if, for all $Z \in \mathfrak{C}$ and maps $g, h : Z \to X$ in $\mathfrak{C}$, $fg = fh$ implies $g = h$; and dually for epimorphisms.

Alternatively we may describe m, using different implicit bracketing, by $m = (m_1 \times m_2)(1 \times \tau \times 1)$. This is the unique structure on $A_1 \times A_2$ which makes p_1 and p_2 primitive, cf. [2], Theorem 4.8. We will often write $m = m_1 \# m_2$, so that the direct product of (A_1, m_1) and (A_2, m_2) is the $\underline{\mathrm{M}}$-object $(A, m) = (A_1 \times A_2, m_1 \# m_2)$. We recall that the structure m satisfies any axiom of those listed in [2] provided m_1 and m_2 satisfy that axiom.

Proposition 2.1. *Let (A, m) be an $\underline{\mathrm{H}}$-object. Then $(A \times A, m \# m)$ is an $\underline{\mathrm{H}}$-object; and $m: A \times A \to A$ is primitive if and only if m is associative and commutative.*

Proof. The first assertion is already contained in Corollary 4.12 of [2]. To prove the second, we study the diagram

$$\begin{array}{ccc} A \times A \times A \times A & \xrightarrow{m \times m} & A \times A \\ \Big\downarrow {\scriptstyle m \# m} & & \Big\downarrow {\scriptstyle m} \\ A \times A & \xrightarrow{m} & A \end{array} \tag{2.2}$$

and must prove that it is commutative if and only if the $\underline{\mathrm{H}}$-structure m is associative and commutative. Now the commutativity of (2.2) expresses itself, in component form, by the equality

$$m\{m\{p_1, p_2\}, m\{p_3, p_4\}\} = m\{m\{p_1, p_3\}, m\{p_2, p_4\}\} \tag{2.3}$$

while the associativity of m asserts that

$$m\{m\{p_1, p_2\}, p_3\} = m\{p_1, m\{p_2, p_3\}\} \tag{2.4}$$

and the commutativity of m asserts that

$$m\{p_2, p_1\} = m\,. \tag{2.5}$$

Thus we must show that (2.3) holds if and only if both (2.4) and (2.5) hold. We deduce (2.4) from (2.3) by composing on the right with $\{p_1, p_2, 0, p_3\}: A \times A \times A \to A \times A \times A \times A$; and we deduce (2.5) from (2.3) by composing on the right with $\{0, p_1, p_2, 0\}: A \times A \to A \times A \times A \times A$. In both cases decisive use is made of the fact that m is an $\underline{\mathrm{H}}$-structure.

Conversely, suppose that (2.4) and (2.5) hold. Then m is associative so we may write $m^4: A \times A \times A \times A \to A$ for the unique 4-product (see Theorem 4.13 of [2]); moreover we have to prove that $m^4(1 \times \tau \times 1) = m^4$, or, equivalently, that $m^4\{p_1, p_3, p_2, p_4\} = m^4$.

Now $m^4 = m^3\{p_1, m\{p_2, p_3\}, p_4\}$; for the expression on the right is indeed a 4-product. Thus it is sufficient to show that

$$\{p_1, m\{p_2, p_3\}, p_4\}\{p_1, p_3, p_2, p_4\} = \{p_1, m\{p_2, p_3\}, p_4\}$$

But

$$\begin{aligned} \{p_1, m\{p_2, p_3\}, p_4\}\{p_1, p_3, p_2, p_4\} &= \{p_1, m\{p_3, p_2\}, p_4\} \\ &= \{p_1, m\{p_2, p_3\}, p_4\} \end{aligned}$$

by the commutativity of m, and so the proposition is completely proved.

We return now to the study of the direct product $(A_1 \times A_2, m_1 \# m_2) = (A, m)$ of two $\underline{\mathrm{M}}$-objects (A_1, m_1), (A_2, m_2). Then, by Theorem 4.8 of [2] the maps $\iota_1 = \{1, 0\} : A_1 \to A_1 \times A_2$ and $\iota_2 = \{0, 1\} : A_2 \to A_1 \times A_2$ are primitive; so too, of course, are the projections p_1 and p_2. We prove

Proposition 2.6. *If (A_1, m_1) and (A_2, m_2) are $\underline{H}$-objects, then*

$$1 = \iota_1 p_1 + \iota_2 p_2 = \iota_2 p_2 + \iota_1 p_1 : A_1 \times A_2 \to A_1 \times A_2$$

Proof. Now $p_1(\iota_1 p_1 + \iota_2 p_2) = p_1 \iota_1 p_1 + p_1 \iota_2 p_2$, by Theorem 4.7 of [2]

$$= p_1, \text{ since } p_1 \iota_1 = 1,\ p_1 \iota_2 = 0, \text{ and } m \text{ is an } \underline{H}\text{-structure.}$$

Similarly $p_2(\iota_1 p_1 + \iota_2 p_2) = p_2$ and the first equality follows from the uniqueness property of components. The second equality is proved similarly (without any assumption of commutativity of m).

Corollary 2.7. *Let $\theta, \theta' : A_1 \times A_2 \to B$ be primitive (with respect to some $\underline{M}$-structure on B) and let $\theta \iota_i = \theta' \iota_i$, $i = 1, 2$. Then $\theta = \theta'$.*

Proof. We have only to observe that, by Theorem 4.7 of [2] and Proposition 2.6,

$$\theta = \theta \iota_1 p_1 + \theta \iota_2 p_2 . \tag{2.8}$$

We bring together Proposition 2.1 and (2.8) to prove

Theorem 2.9. *Let (A, m) be an $\underline{H}$-object and let $\theta : A \times A \to A$ be an $\underline{H}$-structure on A which is primitive with respect to the $\underline{H}$-structures $m \# \dot{m}$, m on $A \times A$, A respectively. Then $\theta = m$ and m is associative and commutative.*

Proof. In the light of Proposition 2.1 it is sufficient to prove that $\theta = m$. Now

$$\begin{aligned} \theta &= \theta \iota_1 p_1 + \theta \iota_2 p_2, && \text{since } \theta \text{ is primitive [by (2.8)]} \\ &= p_1 + p_2, && \text{since } \theta \text{ is an } \underline{H}\text{-structure} \\ &= m && \text{(see Theorem 3.3 of [2]) .} \end{aligned}$$

We next introduce a notion which will be useful in the construction of primitive maps.

Definition 2.10. Let (A, m) be an $\underline{M}$-object. Then $\alpha : X \to A$, $\beta : Y \to A$ *strongly commute*[3]) if

$$m\tau(\alpha \times \beta) = m(\alpha \times \beta) : X \times Y \to A . \tag{2.11}$$

Evidently the condition (2.11) is symmetrical in α and β and equivalent to

$$m(\beta \times \alpha)\tau = m(\alpha \times \beta) : X \times Y \to A ; \tag{2.12}$$

or, alternatively, to the condition that the elements αp_1, βp_2 of the $\underline{M}$-set $H(X \times Y, A)$ commute:

$$\alpha p_1 + \beta p_2 = \beta p_2 + \alpha p_1 . \tag{2.13}$$

It is plain, for example, that α, β strongly commute if m is a commutative structure. We may prove

Proposition 2.14. (i) *If $\alpha : X \to A$, $\beta : Y \to A$ strongly commute and $\varphi : X_1 \to X$, $\psi : Y_1 \to Y$, then $\alpha\varphi$, $\beta\psi$ strongly commute.*

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[3]) We would say that $\alpha, \beta : X \to A$ commute if $\alpha + \beta = \beta + \alpha$; we make no use of this notion here. If in Definition 2.10, $X = Y$ and α, β strongly commute, then certainly α, β commute.

(ii) *If* $\alpha : X \to A$, $\beta : Y \to A$ *strongly commute and if* $\theta : (A, m) \to (B, n)$ *is primitive, then* $\theta\alpha$, $\theta\beta$ *strongly commute.*

Proof. (i) follows immediately from (2.11) since

$$\alpha\varphi \times \beta\psi = (\alpha \times \beta)(\varphi \times \psi)$$

(ii) follows from (2.13) since, if θ is primitive,

$$\theta\alpha p_1 + \theta\beta p_2 = \theta(\alpha p_1 + \beta p_2) = \theta(\beta p_2 + \alpha p_1) = \theta\beta p_2 + \theta\alpha p_1 .$$

An important example of strongly commuting maps is furnished by

Theorem 2.15. *Let* $(A_1 \times A_2, m_1 \# m_2)$ *be the direct product of the* $\underline{H}$*-objects* (A_1, m_1), (A_2, m_2). *Then* $\iota_1 : A_1 \to A_1 \times A_2$, $\iota_2 : A_2 \to A_1 \times A_2$ *strongly commute.*

Proof. It was shown in 2.6 that $\iota_1 p_1 + \iota_2 p_2 = \iota_2 p_2 + \iota_1 p_1$ $(= 1)$, which is the assertion of the theorem.

The relation of strong commutativity to primitive maps is brought out by

Theorem 2.16. *Let* $\alpha, \beta : (A, m) \to (B, n)$ *be primitive maps of* $\underline{M}$*-objects. Then if* n *is associative and* α, β *strongly commute,* $\alpha + \beta$ *is primitive.*

Proof. Set $\gamma = \alpha + \beta$. We must show that the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{\gamma \times \gamma} & B \times B \\ \Big\downarrow{\scriptstyle m} & & \Big\downarrow{\scriptstyle n} \\ A & \xrightarrow{\gamma} & B \end{array}$$

is commutative; that is, that $\gamma(p_1 + p_2) = \gamma p_1 + \gamma p_2 : A \times A \to B$. Now

$$\begin{aligned} \gamma(p_1 + p_2) &= (\alpha + \beta)(p_1 + p_2) \\ &= \alpha(p_1 + p_2) + \beta(p_1 + p_2) \\ &= (\alpha p_1 + \alpha p_2) + (\beta p_1 + \beta p_2), && \text{since } \alpha, \beta \text{ are primitive} \\ &= \alpha p_1 + (\alpha p_2 + \beta p_1) + \beta p_2, && \text{since } n \text{ is associative} \\ &= \alpha p_1 + (\beta p_1 + \alpha p_2) + \beta p_2, && \text{since } \alpha, \beta \text{ strongly commute} \\ &= (\alpha p_1 + \beta p_1) + (\alpha p_2 + \beta p_2) \\ &= (\alpha + \beta) p_1 + (\alpha + \beta) p_2 \\ &= \gamma p_1 + \gamma p_2 . \end{aligned}$$

We will also need

Theorem 2.17. *Let* (A, m) *be an associative* $\underline{M}$*-object and let* $\alpha : X \to A$; $\beta, \gamma : Y \to A$ *be maps. Then if* α *commutes strongly with* β *and* γ *it commutes strongly with* $\beta + \gamma$.

Proof. We have

$$\begin{aligned}\alpha p_1 + (\beta + \gamma)p_2 &= \alpha p_1 + (\beta p_2 + \gamma p_2)\\ &= (\alpha p_1 + \beta p_2) + \gamma p_2, \quad \text{since } m \text{ is associative}\\ &= (\beta p_2 + \alpha p_1) + \gamma p_2, \quad \text{since } \alpha, \beta \text{ commute strongly}\\ &= \beta p_2 + (\alpha p_1 + \gamma p_2),\\ &= \beta p_2 + (\gamma p_2 + \alpha p_1), \quad \text{since } \alpha, \gamma \text{ commute strongly}\\ &= (\beta p_2 + \gamma p_2) + \alpha p_1,\\ &= (\beta + \gamma)p_2 + \alpha p_1 .\end{aligned}$$

We close this section by proving two results on $\underline{G}$-objects.

Theorem 2.18. *Let (A, m) be a $\underline{G}$-object, with inverse $s: A \to A$, and let $\alpha: X \to A$, $\beta: Y \to A$ strongly commute. Then α and $s\beta$ strongly commute.*

Proof. In the group $H(B, A)$ the inverse of $f: B \to A$ is sf; thus $sf = -f$. Taking $B = X \times Y$, we have

$$\alpha p_1 + \beta p_2 = \beta p_2 + \alpha p_1 ,$$

so

$$-\beta p_2 + \alpha p_1 = \alpha p_1 - \beta p_2 ,$$

or

$$s\beta p_2 + \alpha p_1 = \alpha p_1 + s\beta p_2 ,$$

and α, $s\beta$ strongly commute.

Theorem 2.19. *Let (A, m) be a $\underline{G}$-object with inverse $s: A \to A$. Then*

$$m(s \times s) = sm\tau : A \times A \to A .$$

Proof. In the group $H(A \times A, A)$, we have $m\tau = p_2 + p_1$ and $m(s \times s) = sp_1 + sp_2$. Thus the theorem follows from the group identity $-(p_2 + p_1) = (-p_1) + (-p_2)$.

Corollary 2.20. *If (A, m) is a $\underline{CG}$-object then $s: A \to A$ is primitive.*

The notions of this section are adequately exemplified by considering the case $\mathfrak{C} = \mathfrak{S}$, the category of based sets and based maps. The conclusions are then all familiar and elementary, and have, indeed, in many cases formed the basis of the argument in the general case. As a somewhat less familiar example, we consider the application of Corollary 2.7 to the category $\mathfrak{C} = \mathfrak{T}_h$ of based spaces and based homotopy classes of continuous maps. We then deduce that if A, B, C are H-spaces and if $f: A \times B \to C$ is a primitive map (that is, a continuous map whose homotopy class is primitive), then the homotopy class of f is uniquely determined by those of $f|A$, $f|B$. The same corollary, applied to the dual situation, tells us that if A, B, C are H′-spaces, then the homotopy class of a primitive map $f: C \to A \vee B$ is uniquely determined by those of the projections of f onto A and B.

3. Commutative categories

Let $\mathfrak{C}$ be a D-category and let $\mathfrak{D}$ be a category whose objects are $\underline{M}$-objects in $\mathfrak{C}$ and whose maps $(A, m) \to (B, n)$ are precisely the maps $A \to B$ in $\mathfrak{C}$

which are primitive with respect to the $\underline{M}$-structures m, n; it contains all identities and the zero maps between any two of its objects (cf. [2], Prop. 4.1 and the remark preceding it). We impose on $\mathfrak{D}$ the condition that it be closed with respect to direct products; by this we understand that if (A_1, m_1), (A_2, m_2) belong to $\mathfrak{D}$, then so does $(A_1 \times A_2, m_1 \# m_2)$. We remark that we can now give a precise justification for our description of $(A_1 \times A_2, m_1 \# m_2)$ as the direct product of (A_1, m_1) and (A_2, m_2), since it is indeed (together, of course, with the projections p_1, p_2) the direct product of (A_1, m_1) and (A_2, m_2) *in* $\mathfrak{D}$. Thus $\mathfrak{D}$ is itself a D-category. We call $\mathfrak{D}$ an *$\underline{M}$-primitive category over* $\mathfrak{C}$. If the objects of $\mathfrak{D}$ are all $\underline{H}$-objects, we call $\mathfrak{D}$ an $\underline{H}$-primitive category over $\mathfrak{C}$; similarly we speak of $\underline{AH}$-primitive, $\underline{G}$-primitive, $\underline{CG}$-primitive, $\underline{ACH}$-primitive categories. If $\mathfrak{D}$ contains all the $\underline{M}$-objects in $\mathfrak{C}$ it will be called the complete $\underline{M}$-primitive category over $\mathfrak{C}$; similarly we will speak of complete $\underline{H}$-primitive categories, etc.

In the next section we will discuss the general theory of primitive categories. In this section we dispose briefly of the commutative case, establishing the relationship of the theory to that of additive categories. We first prove two theorems generalizing familiar properties of the category of abelian groups.

Theorem 3.1. *If $\mathfrak{D}$ is an $\underline{ACH}$-primitive category over $\mathfrak{C}$ then $(A_1 \times A_2, m_1 \# m_2)$ is the inverse product of (A_1, m_1) and (A_2, m_2) in $\mathfrak{D}$, the injections being $\iota_1 : A_1 \to A_1 \times A_2$, $\iota_2 : A_1 \to A_1 \times A_2$.*

Proof. Let $(X, \xi) \in \mathfrak{D}$ and let $\alpha_j : A_j \to X$ be primitive maps. Then $\alpha_1 p_1$, $\alpha_2 p_2$, $A_1 \times A_2 \to X$ are primitive. Moreover $\alpha_1 p_1$, $\alpha_2 p_2$ strongly commute, since ξ is commutative, so that $\alpha_1 p_1 + \alpha_2 p_2$ is primitive, since ξ is associative (Theorem 2.16). Thus $\alpha_1 p_1 + \alpha_2 p_2 : A_1 \times A_2 \to X$ is a map in $\mathfrak{D}$ and

$$(3.2) \qquad (\alpha_1 p_1 + \alpha_2 p_2)\iota_j = \alpha_j , \qquad j = 1, 2 ,$$

since ξ is an $\underline{H}$-structure.

It remains to show that the map $\theta = \alpha_1 p_1 + \alpha_2 p_2$ is uniquely characterized by the equations (3.2). This, however, is attested by Corollary 2.7.

Theorem 3.1 asserts that, in such a category $\mathfrak{D}$, (finite) direct and inverse products coincide. In such categories therefore all objects carry unique $\underline{H}$-and $\overline{H}$-structures. Indeed if (A, m) is an object of $\mathfrak{D}$ then $m : A \times A \to A$ is primitive (Proposition 2.1) and, of course, $m\iota_1 = m\iota_2 = 1$, since m is an $\underline{H}$-structure in $\mathfrak{C}$. Moreover, $m = \langle 1, 1 \rangle$ in view of Theorem 3.1 and Proposition 4.18 of [2], so that m is the unique $\underline{H}$-structure on (A, m). Similarly $d = \{1, 1\} : A \to \to A \times A$ is the unique $\overline{H}$-structure on (A, m) in $\mathfrak{D}$. Moreover it is plain that m is an $\underline{ACH}$-structure on (A, m) and d is an $\overline{ACH}$-structure on (A, m). Indeed if m were a $\underline{CG}$-structure in $\mathfrak{C}$ then m would be a $\underline{CG}$-structure in $\mathfrak{D}$ and d would be a $\overline{CG}$-structure in $\mathfrak{D}$; the latter remark follows from Corollary 2.20.

It was observed in [4] that, in the language of this paper, the category of abelian groups is the complete $\underline{H}$-primitive category over the category of groups. We generalize this statement in Theorem 3.4 below. We first enunciate

Proposition 3.3. *Let $\mathfrak{D}$ be an $\underline{\mathrm{H}}$-primitive category over $\mathfrak{C}$ and let $\mathfrak{E}$ be an $\underline{\mathrm{H}}$-primitive category over $\mathfrak{D}$. Let $(A, m; \theta)$ be an object of $\mathfrak{E}$. Then $\theta = m$ and m is associative and commutative.*

Proof. This is just a restatement of Theorem 2.9.

It is, further, part of the statement of Proposition 2.1 that $(A, m; m)$ is an $\underline{\mathrm{H}}$-object of $\mathfrak{D}$ if m is associative and commutative; and it is a trivial observation that a map $\Phi: (A, m) \to (B, n)$ between $\underline{\mathrm{ACH}}$-objects is primitive in $\mathfrak{C}$ if and only if Φ, regarded as a map $(A, m; m) \to (B, n; n)$ is primitive in $\mathfrak{D}$. It follows that we may identify $\mathfrak{E}$, by means of the functor $(A, m; m) \to (A, m)$, with a subcategory $\mathfrak{E}_0$ of $\mathfrak{D}$ which is an $\underline{\mathrm{ACH}}$-primitive category over $\mathfrak{C}$. The category $\mathfrak{E}_0$ is, of course, a full[4]) subcategory of $\mathfrak{D}$ consisting of $\underline{\mathrm{ACH}}$-objects, and only such subcategories may be identified with an $\underline{\mathrm{H}}$-primitive category over $\mathfrak{D}$.

Theorem 3.4. *Let $\mathfrak{D}$ be an $\underline{\mathrm{H}}$-primitive category. Then the $\underline{\mathrm{H}}$-primitive categories over $\mathfrak{D}$ are essentially just the full $\underline{\mathrm{ACH}}$-primitive subcategories of $\mathfrak{D}$.*

We recall that an additive category is a category $\mathfrak{A}$ together with an abelian group structure in each $H(A, B)$, where $A, B \in \mathfrak{A}$, such that $A \times B \to H(A, B)$ is a functor from $\mathfrak{A} \times \mathfrak{A}$ to the category of abelian groups. This last requirement is equivalent to demanding two-sided distributivity of addition with respect to composition of maps.

Let us write $H_{\mathfrak{C}}(A, B)$, $A, B \in \mathfrak{C}$, for the maps of the category $\mathfrak{C}$ if it is desirable to stress the category; and let us permit ourselves to suppress the structure map from the symbol (A, m) for an object of an $\underline{\mathrm{M}}$-primitive category over $\mathfrak{C}$ if no confusion is to be feared.

Our aim is, in a certain sense, to identify $\underline{\mathrm{CG}}$-primitive categories with additive categories. We first prove

Theorem 3.5. *Let $\mathfrak{D}$ be a $\underline{\mathrm{CG}}$-primitive category over a category $\mathfrak{C}$ and let A, B be objects of $\mathfrak{D}$. Then $H_{\mathfrak{D}}(A, B)$ receives an abelian group structure from the $\underline{\mathrm{CG}}$-structure in B; and $\mathfrak{D}$ together with this abelian group structure in each $H_{\mathfrak{D}}(A, B)$ constitutes an additive category.*

Proof. It follows from Theorem 4.10 of [2] that $H_{\mathfrak{C}}(A, B)$ receives an abelian group structure from the $\underline{\mathrm{CG}}$-structure in B. That $H_{\mathfrak{D}}(A, B)$ is a subgroup follows from Theorem 2.16 and Corollary 2.20. The right distributive law is a triviality; and the left distributive law holds because we are composing with primitive maps.

We turn now to the converse. Let $\mathfrak{A}$ be an additive D-category. Then there exists, for each $B \in \mathfrak{A}$, a unique $\underline{\mathrm{CG}}$-structure $m: B \times B \to B$ inducing the given abelian group structure in $H(A, B)$, $A \in \mathfrak{A}$. Let $\mathfrak{C}$ be the underlying category of the additive category $\mathfrak{A}$ and let $\mathfrak{D}$ be the $\underline{\mathrm{CG}}$-primitive category over $\mathfrak{C}$ whose objects are the $\underline{\mathrm{CG}}$-objects (B, m). Then we have

Theorem 3.6. *The categories $\mathfrak{D}$ and $\mathfrak{A}$ coincide in the sense that*

$$H_{\mathfrak{D}}(A, B) = H_{\mathfrak{A}}(A, B)$$

as abelian groups.

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[4]) Recall that a subcategory $\mathfrak{V}$ of a category $\mathfrak{U}$ is full if $\mathfrak{V}$ contains, with two objects A, B of $\mathfrak{U}$ which it contains, all the maps $A \to B$ in $\mathfrak{U}$.

Proof. Since the $\underline{\mathrm{CG}}$-structure $m : B \times B \to B$ induces the abelian group structure in both $H_{\mathfrak{P}}(A, B)$ and $H_{\mathfrak{A}}(A, B)$, and $H_{\mathfrak{P}}(A, B)$ is a subset of $H_{\mathfrak{A}}(A, B)$, it remains only to demonstrate the equality of $H_{\mathfrak{P}}(A, B)$ and $H_{\mathfrak{A}}(A, B)$ as sets, that is, to show that every map $f : A \to B$ in $\mathfrak{C}$ is primitive with respect to the given structures $m : A \times A \to A$, $m : B \times B \to B$. However, the commutativity of the diagram

$$\begin{array}{ccc} A \times A & \xrightarrow{f \times f} & B \times B \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} \\ A & \xrightarrow{f} & B \end{array}$$

amounts, in terms of the abelian group structure in $H_{\mathfrak{A}}(P, Q)$, $P, Q \in \mathfrak{A}$, to the assertion

$$f(p_1 + p_2) = f p_1 + f p_2,$$

which holds by hypothesis in an additive category.

The theorems of this section apply, clearly, to categories of abelian groups or abelian monoids. Theorem 3.1, for example, generalizes the fact that, in the category of abelian groups, (finite) inverse and direct products essentially coincide; while Theorem 3.4 has the consequence that the $\underline{\mathrm{H}}$-objects in the category $\mathfrak{G}$ of the groups are just the abelian groups. We may also exemplify the theorems by reference to the category $\mathfrak{H}$ of $\underline{\mathrm{H}}$-spaces and primitive maps (that is, maps whose homotopy classes are primitive in the category $\mathfrak{T}_h$ of based spaces and based homotopy classes). Let us refer to a pair (A, m), where $A \in \mathfrak{H}$ and $m : A \times A \to A$ in $\mathfrak{H}$, as a *homotopy* $\underline{\mathrm{H}}$*-object* of $\mathfrak{H}$ if $(A, [m])$ is an $\underline{\mathrm{H}}$-object of $\mathfrak{H}_h$, where $[m]$ is the homotopy class of m and $\mathfrak{H}_h$ is the category of H-spaces and primitive homotopy classes[5]). Then we infer that the homotopy $\underline{\mathrm{H}}$-objects of $\mathfrak{H}$ are the homotopy-associative, homotopy-commutative H-spaces; and in the category of homotopy-associative, homotopy-commutative H-spaces inverse and direct products coincide (being just the cartesian product). Dually we infer that the homotopy $\overline{\mathrm{H}}$-objects of $\mathfrak{H}'$ are[6]) the homotopy-associative, homotopy-commutative H′-spaces; and in the category of homotopy-associative, homotopy-commutative H′-spaces inverse and direct products coincide (being just the disjoint union with base-points identified). We may also consider full subcategories of the $\underline{\mathrm{ACH}}$-primitive and $\overline{\mathrm{ACH}}$-primitive categories referred to above. In particular, of course, we obtain the result that inverse and direct products coincide in the category of double loop spaces (and primitive classes) and in the category of double suspensions (and primitive classes). This last category furnishes an example of a category familiar in homotopy theory, in which the objects are sets but the direct product is not the cartesian product of the sets.

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[5]) Thus $\mathfrak{H}_h$ is the complete $\underline{\mathrm{H}}$-primitive category over $\mathfrak{T}_h$.

[6]) $\mathfrak{H}'$ is the category of H′-spaces and primitive maps. In talking of H- and H′-spaces we revert to the notations and terminology of [1].

4. Primitive categories

In this section we develop certain basic notions in the theory of primitive categories. We fix once and for all an underlying category $\mathfrak{C}$ and consider primitive categories over $\mathfrak{C}$. We recall that such a category is supposed to be closed with respect to direct products; this assumption is sensible because, as remarked, we have a unique natural process for defining an $\underline{M}$-structure on the direct product *in* $\mathfrak{C}$ of two $\underline{M}$-objects, such that the direct product, so structured, becomes the direct product in $\mathfrak{D}$. Our first observation in this section is that an analogous statement holds for inverse limits (left-equalizers and intersections). We will suppose in 4.1—4.4 below that $\mathfrak{C}$ admits leftequalizers

Precisely, let (A, m_A), (B, m_B) be two objects of $\mathfrak{D}$ and let $f, g : A \to B$ be maps in $\mathfrak{D}$, i.e., primitive maps. Let $k : K \to A$ be the left-equalizer of f and g *in* $\mathfrak{C}$. Then we recall (cf. [3], prop. 1.10)

Proposition 4.1. *The object K admits a unique $\underline{M}$-structure $m_K : K \times K \to K$ with respect to which k is primitive. Moreover m_K satisfies axion N $(N = \mathrm{I}, \mathrm{II}, \mathrm{IV})$ of [2, § 4] if m_A satisfies axiom N; and m_K is a $\underline{G}$-structure if m_A and m_B are $\underline{G}$-structures.*

Let us say that $\mathfrak{D}$ is *admissible* if, for any $f, g : A \to B$ in $\mathfrak{D}$, the object (K, m_K) is in $\mathfrak{D}$ where $k : K \to A$ is the left-equalizer of f and g in $\mathfrak{C}$. Then we have as a consequence of the proposition above and Prop. 4.1 (ii) of [2].

Corollary 4.2. *If $\mathfrak{D}$ is admissible, left equalizers coincide in $\mathfrak{C}$ and $\mathfrak{D}$; more precisely, $k : (K, m_K) \to (A, m_A)$ is the left-equalizer of $f, g : A \to B$ in $\mathfrak{D}$ if $k : K \to A$ is the left-equalizer of $f, g : A \to B$ in $\mathfrak{C}$.*

As a further consequence of Proposition 4.1 we have

Corollary 4.3. *The complete $\underline{M}$-primitive ($\underline{H}$-primitive, $\underline{AH}$-primitive, $\underline{G}$-primitive, $\underline{CG}$-primitive, $\underline{ACH}$-primitive) category over $\mathfrak{C}$ is admissible.*

In the light of the method of construction of inverse limits (or directly from their definition) we may immediately infer

Theorem 4.4. *If $\mathfrak{D}$ is an admissible $\underline{M}$-primitive category over $\mathfrak{C}$ then inverse limits coincide in $\mathfrak{D}$ and $\mathfrak{C}$; that is, if $\mathscr{A}$ is an aggregate in $\mathfrak{D}$ with inverse limit $[D; d_A]$ in $\mathfrak{C}$, then D may be given a unique $\underline{M}$-structure m_D such that $(D, m_D) \in \mathfrak{D}$, each d_A, $A \in \mathscr{A}$, is primitive and $[(D, m_D); d_A]$ is the inverse limit of $\mathscr{A}$ in $\mathfrak{D}$.*

Of course, Proposition 4.1 implies, as a special case, that kernels are transferred from $\mathfrak{C}$ to $\mathfrak{D}$. It is worth remarking that cokernels are not, in general, so transferred (for example, if $\mathfrak{C} = \mathfrak{S}$, the category of based sets, and $\mathfrak{D} = \mathfrak{G}$, the category of groups, then a homomorphism $f : G \to H$ has a different cokernel according to whether it is regarded as a map in $\mathfrak{S}$ or $\mathfrak{G}$). In particular, maps of $\mathfrak{D}$ with zero kernel in $\mathfrak{D}$ have zero kernel in $\mathfrak{C}$, whereas there can be maps of $\mathfrak{D}$ with zero cokernel in $\mathfrak{D}$ and non-zero cokernel in $\mathfrak{C}$. This leads to the question whether *monomorphisms (epimorphisms) in $\mathfrak{D}$ are also monomorphisms (epimorphisms) in $\mathfrak{C}$*; it is easy to construct examples showing that this is, in general, *not* the case. It is, of course, plain that, in passing from a category $\mathfrak{C}$ to an $\underline{M}$-primitive category $\mathfrak{D}$ over $\mathfrak{C}$, any map of $\mathfrak{D}$ which is a monomorphism (epimorphism) in $\mathfrak{C}$ is also a monomorphism (epimorphism) in $\mathfrak{D}$.

In the following we concentrate attention on *epimorphisms in* $\underline{\mathrm{M}}$*-primitive (more precisely,* $\underline{\mathrm{H}}$*-primitive) categories.* Our main object is to point out that epimorphisms in $\underline{\mathrm{H}}$-primitive categories possess certain special properties not enjoyed by epimorphisms in arbitrary categories. We first recall from [3], § 5 (see also [9]) the concept of a normal epimorphism: the epimorphism ε is *normal* if the (right) annihilator of its (left) annihilator is the ideal generated by ε. Now let us consider two objects A_1, A_2 of a category $\mathfrak{C}$ and the projection $p_1 : A_1 \times A_2 \to A_1$. Then, in general, p_1 is *not normal.* For its left annihilator is the ideal generated by $\iota_2 : A_2 \to A_1 \times A_2$. No wif $\mathfrak{C} = \mathfrak{S}$, the category of based sets, then the right annihilator of ι_2 is the projection π_1 of $A_1 \times A_2$ onto the set obtained from $A_1 \times A_2$ by the identification $(o, x) = o$, $x \in A_2$, and π_1 does not factor through p_1 unless A_1 is a one-point set. Thus p_1 is not normal in $\mathfrak{S}$. On the other hand we may prove

Theorem 4.5. *Let* $\mathfrak{H}$ *be an* $\underline{\mathrm{H}}$*-primitive category. Then* $p_1 : A_1 \times A_2 \to A_1$ *is normal in* $\mathfrak{H}$*.*

Proof. Certainly $\iota_2 : A_2 \to A_1 \times A_2$ lies in $\mathfrak{H}$, so that it is indeed the kernel of p_1 in $\mathfrak{H}$. We now prove that p_1 is the cokernel of ι_2. Thus let $\theta : A_1 \times A_2 \to B$ be a map in $\mathfrak{H}$ such that $\theta\iota_2 = 0$. Then θ is primitive so that, by (2.8),

$$\theta = \theta\iota_1 p_1 + \theta\iota_2 p_2 = (\theta\iota_1) p_1 ,$$

and the theorem is proved.

We observed in [2] § 7 that a direct product of epimorphisms is not necessarily an epimorphism. Again we indicate the force of the assumption of primitivity by showing

Theorem 4.6. *Let* $\mathfrak{H}$ *be an* $\underline{\mathrm{H}}$*-primitive category, and let* $\alpha_i : A_i \to B_i$, $i = 1, 2$, *be maps in* $\mathfrak{H}$*. Then* $\alpha_1 \times \alpha_2 : A_1 \times A_2 \to B_1 \times B_2$ *is an epimorphism in* $\mathfrak{H}$ *if and only if* α_1, α_2 *are epimorphisms*[7]) *in* $\mathfrak{H}$*.*

Proof. It is true in any category that α_1, α_2 are epimorphisms if $\alpha_1 \times \alpha_2$ is an epimorphism, so that it is the converse which merits special attention. Suppose that α_1, α_2 are epimorphisms and let θ, $\varphi : B_1 \times B_2 \to C$ be maps in $\mathfrak{H}$ such that $\theta(\alpha_1 \times \alpha_2) = \varphi(\alpha_1 \times \alpha_2)$. Now $(\alpha_1 \times \alpha_2)\iota_i = \iota_i \alpha_i$, $i = 1, 2$, so that

$$\theta\iota_1\alpha_1 = \varphi\iota_1\alpha_1 , \quad \theta\iota_2\alpha_2 = \varphi\iota_2\alpha_2 .$$

But α_1, α_2 are epimorphisms so that $\theta\iota_1 = \varphi\iota_1$, $\theta\iota_2 = \varphi\iota_2$. It now follows from Corollary 2.7 that $\theta = \varphi$, whence $\alpha_1 \times \alpha_2$ is an epimorphism.

Of particular interest to us among the epimorphisms are the right-equalizers. In any category the direct product of left-equalizers is the left-equalizer of the product maps (cf. [3], Prop. 1.9). On the other hand there certainly are categories in which the direct product of right-equalizers is not the right-equalizer of the product maps; such a category, for example, is the category $\mathfrak{G}^*$ dual to the category of groups, for in $\mathfrak{G}$ the free product of kernels is not, in general, the kernel of the free product of the maps.

[7]) It is true in any category that $\alpha_1 \times \alpha_2$ is a monomorphism (isomorphism) if and only if α_1 and α_2 are monomorphisms (isomorphisms).

We have not succeeded in proving that in a primitive category the direct product of right-equalizers is the right-equalizer of the product maps, except under an additional hypothesis. In fact we prove

Theorem 4.7. *Let $\mathfrak{D}$ be an $\underline{\mathrm{H}}$-primitive category over $\mathfrak{C}$ and let $f_i, g_i : A_i \to B_i$ be maps in $\mathfrak{D}$ with right-equalizers $c_i : B_i \to C_i$, $i = 1, 2$. Then $c_1 \times c_2$ is the right-equalizer of $f_1 \times f_2$ and $g_1 \times g_2$ provided either (a) $c_1 \times c_2 \times c_1 \times c_2$ is an epimorphism in $\mathfrak{C}$ or (b) $c_1 \times c_2$ is normal in $\mathfrak{D}$.*

Proof. Certainly $c_1 \times c_2$ is an epimorphism in $\mathfrak{D}$ (Theorem 4.6) and $(c_1 \times c_2)(f_1 \times f_2) = (c_1 \times c_2)(g_1 \times g_2)$. It remains then to show that if $\theta : B_1 \times B_2 \to D$ is a map in $\mathfrak{D}$ such that $\theta(f_1 \times f_2) = \theta(g_1 \times g_2)$, then $\theta = \varphi(c_1 \times c_2)$ for some $\varphi : C_1 \times C_2 \to D$ in $\mathfrak{D}$. We proceed to show, without invoking hypotheses (a) or (b) that $\theta = \varphi'(c_1 \times c_2)$ for some $\varphi' : C_1 \times C_2 \to D$ in $\mathfrak{C}$.

Now since $\theta(f_1 \times f_2) = \theta(g_1 \times g_2)$ it follows that $\theta\iota_i f_i = \theta\iota_i g_i$, $i = 1, 2$. Thus, by hypothesis, $\theta\iota_i = \varphi_i c_i$ for some map $\varphi_i : C_i \to D$ in $\mathfrak{D}$. Then

$$\begin{aligned}
\theta &= \theta\iota_1 p_1 + \theta\iota_2 p_2 \\
&= \varphi_1 c_1 p_1 + \varphi_2 c_2 p_2 \\
&= m\{\varphi_1 c_1 p_1, \varphi_2 c_2 p_2\} \\
&= m(\varphi_1 c_1 \times \varphi_2 c_2) \\
&= m(\varphi_1 \times \varphi_2)(c_1 \times c_2) \\
&= \varphi'(c_1 \times c_2)
\end{aligned}$$

where $\varphi' = m(\varphi_1 \times \varphi_2)$. Now φ' is a map in $\mathfrak{C}$; it is not obviously a map in $\mathfrak{D}$ unless $\mathfrak{D}$ is an $\underline{\mathrm{ACH}}$-primitive category[8]). To complete the proof we invoke either hypothesis (a) or hypothesis (b). Indeed hypothesis (a) implies that φ' is already itself primitive; for we may invoke the elementary

Proposition 4.8. *Let (A, m_A), (B, m_B), (C, m_C) be $\underline{\mathrm{M}}$-objects in $\mathfrak{C}$ and let $f : A \to B$, $g : B \to C$ be maps in $\mathfrak{C}$ such that* (i) *f and gf are primitive, and* (ii) *$f \times f$ is an epimorphism. Then g is primitive.*

The conclusion of Theorem 4.7 follows from hypothesis (b) in the light of

Proposition 4.9. *If ε is a normal epimorphism in $\mathfrak{D}$ and α' is a map in $\mathfrak{C}$ such that $\alpha'\varepsilon$ is a map in $\mathfrak{D}$, then $\alpha'\varepsilon = \alpha\varepsilon$ where α is a map in $\mathfrak{D}$.*

For plainly $\alpha'\varepsilon\xi = 0$ whenever $\varepsilon\xi = 0$ so that $\alpha'\varepsilon$ is a left multiple of ε *in* $\mathfrak{D}$.

Thus the proof of Theorem 4.7 is complete. Hypothesis (a) while appearing artificial and very restrictive, is, in fact, verified in the particular categories considered in this series of papers; it holds, of course, whenever epimorphisms in $\mathfrak{D}$ are also epimorphisms in $\mathfrak{C}$. Hypothesis (b), on the other hand, is quite unrestrictive in the special case of cokernels, since it would be implied by the conclusion of the theorem.

Hypothesis (a) may be rendered more acceptable if the notion of *strong epimorphism* in $\mathfrak{D}$ is introduced.

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[8]) Thus if $\mathfrak{D}$ is an $\underline{\mathrm{ACH}}$-primitive category, the conclusion of Theorem 4.7 follows without any special assumption (a) or (b).

A map $f: A \to B$ in $\mathfrak{D}$ is a *strong epimorphism* in an $\underline{\mathrm{M}}$-primitive category $\mathfrak{D}$ if there exists a map $g: B \to A$ in $\mathfrak{C}$ with $fg = 1$. Plainly a strong epimorphism in $\mathfrak{D}$ is an epimorphism in both $\mathfrak{C}$ and $\mathfrak{D}$; and direct products of strong epimorphisms are strong epimorphisms. Thus hypothesis (a) is certainly satisfied if c_1 and c_2 are strong epimorphisms. Now since every epimorphism in $\mathfrak{S}$ has a right inverse and since every epimorphism in $\mathfrak{G}$ is indeed an epimorphism in $\mathfrak{S}$, all epimorphisms in $\mathfrak{G}$ are strong. An important property of the category $\mathfrak{G}$ is thus generalized in the following theorem.

Theorem 4.10. *Let $\mathfrak{C}$ be a category with kernels and let $\mathfrak{D}$ be an admissible*[9]) *$\underline{\mathrm{G}}$-primitive category over $\mathfrak{C}$. Then in $\mathfrak{D}$ all strong epimorphisms are normal.*

Proof. Let $f: A \to B$ be a strong epimorphism in $\mathfrak{D}$ and let $g: B \to A$ be a map in $\mathfrak{C}$ such that $fg = 1$. Then $f = fgf$, whence, f being a primitive map of $\underline{\mathrm{G}}$-objects, $f(1 - gf) = 0$. Thus if $k: K \to A$ is the kernel of f in $\mathfrak{C}$, $1 - gf = kq$, or

$$(4.11) \qquad 1 = kq + gf$$

for some map $q: A \to K$ in $\mathfrak{C}$.

Now k is also the kernel of f in $\mathfrak{D}$ (Proposition 4.1) so that we prove the normality of f by showing that if $l: A \to C$ is a map in $\mathfrak{D}$ such that $lk = 0$ then $l = mf$ with m in $\mathfrak{D}$. By (4.11) and the primitivity of l we have

$$l = lkq + lgf = lgf = mf,$$

where $m = lg$. Finally we invoke Proposition 4.8 to deduce the primitivity of m; for $f \times f$ certainly is an epimorphism in $\mathfrak{C}$ since f is a strong epimorphism in $\mathfrak{D}$.

Thus, in fact, hypotheses (a) and (b) of Theorem 4.7 are both verified if c_1 and c_2 are strong epimorphisms. We return to strong epimorphisms in the next section; meanwhile (and also in preparation for the next section) we discuss the situation in which a primitive category admits inverse products. At this point we merely recall notations, prove one simple fact taken from group theory, and a second fact also taken from group theory and given a purely group theoretic proof in [4].

Suppose that $\mathfrak{D}$ is a $\underline{\mathrm{H}}$-primitive over $\mathfrak{C}$ and that $\mathfrak{D}$ is also an I-category (thus, in fact, $\mathfrak{D}$ is a DI-category). We use the notations of [2],

$$q_i: A_i \to A_1 * \cdots * A_n, \qquad i = 1, 2, \ldots, n$$
$$\pi_i: A_1 * \cdots * A_n \to A_i, \qquad i = 1, 2, \ldots, n$$

for the injections and projections associated with the inverse product of n objects $A_k \in \mathfrak{D}$; we also adopt the notation

$$\varkappa: A_1 * \cdots * A_n \to A_1 \times \cdots \times A_n$$

for the canonical map from the inverse to direct product ((3.34) of [2]). Thus $\varkappa$ is given by

$$\varkappa q_i = \iota_i, \quad \text{all } i,$$

or by

$$p_i \varkappa = \pi_i, \quad \text{all } i,$$

[9]) Here it is sufficient that the kernels of maps of $\mathfrak{D}$ belong to $\mathfrak{D}$.

or, symmetrically, by

$$p_i \varkappa q_j = \delta^i_j ,$$

where $\delta^i_j : A_j \to A_i$ is the Kronecker symbol. We prove

Theorem 4.12. *Let $\mathfrak{H}$ be an $\underline{\mathrm{H}}$-primitive category. Then*

$$\varkappa = \iota_1\pi_1 + \iota_2\pi_2 + \cdots + \iota_n\pi_n : A_1 * \cdots * A_n \to A_1 \times \cdots \times A_n$$

where the bracketing and the order of the summation of the $\iota_i\pi_i$ is arbitrary. Moreover $\varkappa$ is a strong epimorphism.

Proof. The first assertion is an immediate consequence of the following generalization of Proposition 2.6.

Proposition 4.13. *Let $\mathfrak{H}$ be an $\underline{\mathrm{H}}$-primitive category. Then*

$$1 = \iota_1 p_1 + \iota_2 p_2 + \cdots + \iota_n p_n : A_1 \times \cdots \times A_n \to A_1 \times \cdots \times A_n$$

where the bracketing and the order of the summation of the $\iota_i p_i$ is arbitrary.

This proposition is proved by taking any such sum λ of the $\iota_i p_i$ and showing that $p_i \lambda = p_i$, $i = 1, 2, \ldots, n$. The details may be omitted. Finally, to prove that $\varkappa$ is a strong epimorphism, we observe that

$$\begin{aligned}\varkappa(q_1 p_1 + \cdots + q_n p_n) &= \varkappa q_1 p_1 + \cdots + \varkappa q_n p_n \\ &= \iota_1 p_1 + \cdots + \iota_n p_n = 1 ,\end{aligned}$$

by proposition 4.13, where again the bracketing and the order of summation of the $q_i p_i$ is arbitrary; of course we do not here claim that the value of the sum $q_1 p_1 + \cdots + q_n p_n$ is independent of the bracketing and order of summation. This completes the proof of the theorem.

Now we assume that $\mathfrak{H}$ is a DI-category admitting right-equalizers (or, what is the same thing, admitting direct limits). We may then factorize $\varkappa : A_1 * A_2 * A_3 \to A_1 \times A_2 \times A_3$ as

$$A_1 * A_2 * A_3 \xrightarrow{\varkappa''} T(A_1, A_2, A_3) \xrightarrow{\varkappa'} A_1 \times A_2 \times A_3 ; \tag{4.14}$$

(see [6] or [3], where notations are different). We repeat here the definition of $T(A_1, A_2, A_3) = T$ and of the factorization (4.14) in an arbitrary DI-category with unions. Let $\iota^j_i = \{1, 0\}$ be the "embedding" of A_i in $A_i \times A_j$; we will identify $A_i \times A_j$ with $A_j \times A_i$ so that the notation ι^j_i is preferable to one involving components. Then T is characterized up to canonical equivalence by the conditions (a) there exist maps $\beta_{ij} : A_i \times A_j \to T$ with $\beta_{12}\iota^2_1 = \beta_{13}\iota^3_1$, $\beta_{12}\iota^1_2 = \beta_{23}\iota^3_2$, $\beta_{13}\iota^1_3 = \beta_{23}\iota^2_3$; (b) given maps $\gamma_{ij} : A_i \times A_j \to U$ with $\gamma_{12}\iota^2_1 = \gamma_{13}\iota^3_1$; $\gamma_{12}\iota^1_2 = \gamma_{23}\iota^3_2$, $\gamma_{13}\iota^1_3 = \gamma_{23}\iota^2_3$, there exists a unique map $\gamma : T \to U$ with $\gamma\beta_{ij} = \gamma_{ij}$.

By taking $U = A_1 \times A_2 \times A_3$ and $\gamma_{ij} = \iota_{ij}$, the "embedding" of $A_i \times A_j$ in $A_1 \times A_2 \times A_3$, we obtain from (b) a unique map $\varkappa' : T \to A_1 \times A_2 \times A_3$ with $\varkappa' \beta_{ij} = \iota_{ij}$. We define $\varkappa'' : A_1 * A_2 * A_3 \to T$ by $\varkappa'' = \langle \beta_1, \beta_2, \beta_3 \rangle$, where $\beta_1 = \beta_{12}\iota^2_1$, $\beta_2 = \beta_{12}\iota^1_2$, $\beta_3 = \beta_{13}\iota^1_3$. Then

$$\varkappa'\varkappa'' = \langle \varkappa'\beta_1, \varkappa'\beta_2, \varkappa'\beta_3 \rangle = \langle \iota_{12}\iota^2_1, \iota_{12}\iota^1_2, \iota_{13}\iota^1_3 \rangle = \langle \iota_1, \iota_2, \iota_3 \rangle = \varkappa .$$

We now prove

Theorem 4.15. *If $\mathfrak{D}$ is an $\underline{\mathrm{AH}}$-primitive* DI*-category with right equalizers, then*

$$\varkappa' : T(A_1, A_2, A_3) \to A_1 \times A_2 \times A_3$$

is an equivalence[10]).

Proof. Since $\varkappa$ is an epimorphism (Theorem 4.12) so is $\varkappa'$. Thus it remains only to construct a map $\eta : A_1 \times A_2 \times A_3 \to T$ in $\mathfrak{D}$ such that $\eta\varkappa' = 1$. We define

$$\eta = \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3 : A_1 \times A_2 \times A_3 \to T;$$

since the structures in $\mathfrak{D}$ are associative the map η is well-defined.

We now prove η primitive. By Theorem 2.15 ι_1^2 and ι_2^1 strongly commute, so that (Proposition 2.14) $\iota_1^2 p_1$ and $\iota_2^1 p_2$ strongly commute and, applying β_{12} on the left, $\beta_1 p_1$ and $\beta_2 p_2$ strongly commute. Similarly $\beta_1 p_1$ and $\beta_3 p_3$, $\beta_2 p_2$ and $\beta_3 p_3$ strongly commute. We now apply Theorems 2.16 and 2.17 to deduce that $\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3$ is primitive; note that $\mathfrak{D}$ has been assumed $\underline{\mathrm{AH}}$-primitive.

It remains to show that $\eta\varkappa' = 1$. Now $\eta\iota_j = (\beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3)\iota_j = \beta_j$ so that $\eta\varkappa' \beta_{12}\iota_1^2 = \eta\iota_{12}\iota_1^2 = \eta\iota_1 = \beta_1 = \beta_{12}\iota_1^2$, and, similarly, $\eta\varkappa'\beta_{12}\iota_2^1 = \beta_{12}\iota_2^1$. Thus, by corollary 2.7 $\eta\varkappa'\beta_{12} = \beta_{12}$. Similarly $\eta\varkappa'\beta_{13} = \beta_{13}$, $\eta\varkappa'\beta_{23} = \beta_{23}$, so that, by condition (b) above, $\eta\varkappa' = 1$ and the theorem is proved.

A known case of this theorem is that of the category $\mathfrak{G}$ (see [4]). It is also known that the category $\mathfrak{S}^*$ dual to the category of based sets is one in which $\varkappa'$ is an equivalence. (In other words in the dual factorization in the category $\mathfrak{S}$ the map corresponding to $\varkappa'$ is an equivalence.) In fact this second case is indeed an example of Theorem 4.15 in the light of the following theorem which expresses a curious "duality" between the categories $\mathfrak{S}$ and $\mathfrak{G}$. Of course $\mathfrak{G}$ is the complete $\underline{\mathrm{G}}$-primitive category over $\mathfrak{S}$; however we also have

Theorem 4.16. *The category $\mathfrak{S}$ is equivalent to the complete $\overline{\mathrm{G}}$-primitive category over $\mathfrak{G}$.*

Proof. Kan's theorem ([8], Theorem 3.10 or [4], Theorem 1.6) asserts that an $\overline{\mathrm{AH}}$-object in $\mathfrak{G}$ is a free group G together with a homomorphism $\mu : G \to G * G$ given on a free set of generators $S = \{x\}$ by

$$\mu x = q_1 x \cdot q_2 x; \tag{4.17}$$

moreover the set $\bar{S} = S \cup (e)$, where e is the identity of G, is characterized as the set of solutions in G of (4.17); or, as we may say, as the set of *primitive elements* of G with respect to μ. It is plain, further, that (G, μ) is in fact a $\overline{\mathrm{G}}$-object with inverse[11]) $\varrho : G \to G$ given by $\varrho x = x^{-1}$, $x \in S$. Thus there exists a one-to-one correspondence F between objects $\bar{S}$ of $\mathfrak{S}$ and $\overline{\mathrm{G}}$-objects $(F(\bar{S}), \mu)$ of $\mathfrak{G}$, $F(\bar{S})$ being the free group freely generated by $S = \bar{S} - (e)$, where e is the base point of $\bar{S}$ and is to be identified with the identity of G; the $\overline{\mathrm{G}}$-structure on $F(S)$ is that given by (4.17).

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[10]) Theorem 1.1 of [6] enables us to generalize this immediately to more than 3 factors.

[11]) Recall that ϱ is the inverse with respect to the comultiplication μ.

Theorem 1.9 of [4] may now be paraphrased as saying that, given two $\overline{\text{G}}$-objects $(F(\bar{S}_1), \mu_1)$ and $(F(\bar{S}_2), \mu_2)$ a primitive map from the first to the second is just the unique homomorphism $F(\bar{S}_1) \to F(\bar{S}_2)$ extending a given based function $\bar{S}_1 \to \bar{S}_2$. If we extend the domain of F by declaring the image, $F(\Phi)$, of such a based function $\Phi : \bar{S}_1 \to \bar{S}_2$ to be the unique homomorphism $F(\bar{S}_1) \to F(\bar{S}_2)$ extending Φ it is then plain that F is a functor from $\mathfrak{S}$ onto the complete $\overline{\text{G}}$-primitive category over $\mathfrak{G}$ which is one-to-one on objects and maps and thus establishes the equivalence of the two categories.

This theorem has the effect of explaining "dual" phenomena in the categories $\mathfrak{S}$ and $\mathfrak{G}$ as twin manifestations of a general theorem on primitive categories. We have already mentioned that Theorem 4.15 applies both to $\mathfrak{S}$ and to $\mathfrak{G}$; it yields, in the language of [3], § 5, the fact that, for any group G, $\bar{l}(G) \leqq 2$, and, for any set S, $l(S) \leqq 2$. A simpler example of this duality is provided by the assertion in Theorem 4.12 that $\varkappa$ is a strong epimorphism in an $\underline{\text{H}}$-primitive DI-category. As a consequence of this assertion the two statements "$\varkappa$ is a strong epimorphism in $\mathfrak{G}$", "$\varkappa$ is a strong monomorphism in $\mathfrak{S}$" may both be deduced. Again Theorem 4.6 enables us to deduce simultaneously that direct products of epimorphisms in $\mathfrak{G}$ are epimorphisms and inverse products (= "unions with base point identified") of monomorphisms in $\mathfrak{S}$ are monomorphisms. A less trivial example will be found in the next section.

5. The canonical factorization in $\underline{\text{G}}$-primitive categories

Our main purpose in this section is to generalize Theorem 2.3 of [5]. Let

$$(5.1)\qquad \begin{aligned} A_1 * \cdots * A_n = A^n &\xrightarrow{\varkappa^{n-1}} A^{n-1} \longrightarrow \ldots A^{p+1} \xrightarrow{\varkappa^p} A^p \ldots \xrightarrow{\varkappa^1} A^1 \\ &= A_1 \times \cdots \times A_n \end{aligned}$$

be the factorization (F) of [3], Theorem 4.4, where $A_1, \ldots, A_n$ are objects of a DI-category $\mathfrak{U}$ with left equalizers. Then it was shown in [5] that if $\mathfrak{U} = \mathfrak{G}$ then each $\varkappa^p$ is an epimorphism. Our generalization is as follows. We suppose $\mathfrak{C}$ to be a D-category with left-equalizers and take $\mathfrak{D}$ to be an admissible $\underline{\text{G}}$-primitive DI-category over $\mathfrak{C}$. We may thus construct the factorization (F) (5.1) in $\mathfrak{D}$ and we prove

Theorem 5.2. *In the factorization* (F) *in* $\mathfrak{D}$ *each* $\varkappa^p$ *is a strong epimorphism.*

Proof. We adopt the notation of [3, 4, 5]. Given strings $I, J \subset N = (1, 2, \ldots, n)$ with $J \subset I$, let A_I, A_J be the associated inverse products and let $\pi^I_J : A_I \to A_J$, $q^J_I : A_J \to A_I$ be the projections and injections. Then A^p is the inverse limit of the $\mathfrak{D}$-aggregate $\mathscr{A}^{p,p} = (A_I, \pi^I_J \text{ with } |I| \leqq p)$, and there are canonical maps

$$\xi^p_I : A^p \to A_I$$

with $\pi^I_J \xi^p_I = \xi^p_J$. Moreover a unique map

$$\lambda^p : A^n \to A^p$$

is defined by

$$(5.3)\qquad \xi^p_I \lambda^p = \pi_I,$$

where π_I is the projection $A^n \to A_I$. Since

$$\lambda^p = \varkappa^p \circ \varkappa^{p+1} \circ \cdots \circ \varkappa^{n-1} \tag{5.4}$$

it is plainly sufficient to prove that λ^p is a strong epimorphism. We will need the relation

$$\pi^I_K q^I_J = q^K_L \pi^J_L, \quad \text{where} \quad L = J \cap K; \tag{5.5}$$

the proof of this relation will be omitted.

We next consider the maps ξ^p_I, $|I| \leqq t$; since $\mathfrak{C}$ admits left-equalizers and, hence, intersections, and since $\mathfrak{D}$ is admissible, we may construct the intersection of the kernels of these particular maps to obtain an object $F_t = F_t(A^p)$ and monomorphism $\mu_t : F_t \to A^p$. The map μ_t is a map in $\mathfrak{D}$ and is, as we have stressed, the intersection of the kernels of the ξ^p_I, $|I| \leqq t$, both as a map in $\mathfrak{C}$ and as a map in $\mathfrak{D}$. Note that $F_0 = A^p$, $F_p = 0$; and, of course, $\mu_0 = 1$, $\mu_p = 0$; and that μ_t is the intersection of the kernels of the ξ^p_I, $|I| = t$.

We will use the symbol m for the $\underline{G}$-structure in any object of $\mathfrak{D}$; indeed, for simplicity, we will still use the symbol m even if a k-product is involved[12]) for $k < 2$. We will use the symbol s for the inverse throughout $\mathfrak{D}$; and we will use a symbol like $\{f_j\}$ for the map, into a direct product, with components f_j and may indicate the range of j in a subscript.

With these conventions we proceed to construct a right-inverse of λ^p in $\mathfrak{C}$. As in [5] we order the r-strings for each r and we define the map (in $\mathfrak{C}$)

$$\gamma_{t,I} : A^p \to A_I,\ |I| \geqq t+1$$

by

$$\gamma_{t,I} = m\left\{\xi^p_I,\ s m \{q^I_J \xi^p_J\}_{|J|=t+1,\, J \subset I}\right\}. \tag{5.6}$$

The notation $\{q^I_J \xi^p_J\}_{|J|=t+1,\, J \subset I}$ indicates the map whose components are $q^I_J \xi^p_J$ where J proceeds over the given range in the order laid down for the strings.

We prove next that, if $|J| = t+1$,

$$\begin{aligned} \pi^I_K q^I_J \xi^p_J \mu_t &= 0, \quad \text{if} \quad J \not\subset K \\ &= q^K_J \xi^p_J \mu_t, \quad \text{if} \quad J \subset K \end{aligned} \tag{5.7}$$

For, if $L = J \cap K$,

$$\begin{aligned} \pi^I_K q^I_J \xi^p_J \mu_t &= q^K_L \pi^J_L \xi^p_J \mu_t, \\ &= q^K_L \xi^p_L \mu_t. \end{aligned}$$

by (5.5)

Now if $J \not\subset K$ then $|L| \leqq t$ so that $\xi^p_L \mu_t = 0$; if $J \subset K$ then $L = J$, so (5.7) is proved.

Thus, since π^I_K is primitive,

$$\begin{aligned} \pi^I_K \gamma_{t,I} \mu_t &= m\left\{\pi^I_K \xi^p_I \mu_t,\ s m \{\pi^I_K q^I_J \xi^p_J \mu_t\}_{|J|=t+1,\, J\subset I}\right\} \\ &= m\left\{\xi^p_K \mu_t,\ s m \{q^K_J \xi^p_J \mu_t\}_{|J|=t+1,\, J \subset K}\right\} \\ &= \gamma_{t,K} \mu_t. \end{aligned}$$

[12]) This is legitimate since our structures are associative, see [2].

It follows that the maps $\gamma_{t,I}\mu_t$ together determine a map

(5.8) $$\gamma_t : F_t \to A^p \text{ (in } \mathfrak{C})$$

such that

(5.9) $$\xi_I^p \gamma_t = \gamma_{t,I}\mu_t .$$

(Here we are, of course, invoking the fact that A^p is the inverse limit of the maps π_K^I in $\mathfrak{C}$ as in $\mathfrak{D}$.)

Let $|I| = t + 1$; then

$$\begin{aligned}\xi_I^p \gamma_t = \gamma_{t,I}\mu_t &= m\left\{\xi_I^p \mu_t, s\,m\{q_J^I \xi_J^p \mu_t\}_{|J|=t+1, J\subset I}\right\}\\ &= m\{\xi_I^p \mu_t, s\,\xi_I^p \mu_t\}\\ &= 0 .\end{aligned}$$

Thus γ_t induces

(5.10) $$\alpha_t : F_t \to F_{t+1}$$

with

(5.11) $$\mu_{t+1}\alpha_t = \gamma_t .$$

We define $\delta_t : A^p \to A^n$ by

$$\delta_t = m\{q_J \xi_J^p\}_{|J|=t+1}$$

where q_J embeds A_J in A^n and then define

(5.12) $$\beta_t = \delta_t \mu_t : F_t \to A^n .$$

We next prove the decisive equality

(5.13) $$\mu_t = m\{\gamma_t, \lambda^p \beta_t\} .$$

We have to show that each side of (5.13) composes with ξ_I^p to give the same map. Now

$$\begin{aligned}\xi_I^p m\{\gamma_t, \lambda^p \beta_t\} &= m\{\xi_I^p \gamma_t, \xi_I^p \lambda^p \beta_t\} &&\text{since } \xi_I^p \text{ is primitive}\\ &= m\{\gamma_{t,I}\mu_t, \pi_I \beta_t\}, &&\text{by (5.9) and (5.3)}\\ &= m\{\gamma_{t,I}\mu_t, \pi_I \delta_t \mu_t\} &&\text{by (5.12)}\\ &= m\left\{\xi_I^p \mu_t, s\,m\{q_J^I \xi_J^p \mu_t\}_{|J|=t+1, J\subset I}, m\{\pi_I q_J \xi_J^p \mu_t\}_{|J|=t+1,}\right\}\\ &= m\left\{\xi_I^p \mu_t, s\,m\{q_J^I \xi_J^p \mu_t\}_{|J|=t+1, J\subset I}, m\{q_J^I \xi_J^p \mu_t\}_{|J|=t+1, J\subset I}\right\} &&\text{by (5.7)} .\end{aligned}$$

This expression is equal to $\xi_I^p \mu_t$, and thus (5.13) is established. We now define maps $\theta_t : A^p \to F_t$, $0 \leqq t \leqq p$, by

(5.14) $$\theta_0 = 1, \theta_t = \alpha_{t-1}\theta_{t-1}, \qquad t \geqq 1 .$$

We prove, by induction on t, that

(5.15) $$1 = m\{\gamma_t \theta_t, \lambda^p \beta_t \theta_t, \lambda^p \beta_{t-1}\theta_{t-1}, \ldots, \lambda^p \beta_0 \theta_0\} .$$

Now $m\{\gamma_0, \lambda^p \beta_0\} = 1$ by (5.13) so (5.15) holds with $t = 0$. We suppose (5.15) established with $t = \tau - 1$. Then

$$\begin{aligned} m\{\gamma_\tau\theta_\tau, \lambda^p \beta_\tau\theta_\tau, \lambda^p \beta_{\tau-1}\theta_{\tau-1}, \ldots, \lambda^p \beta_0\theta_0\} \\ = m\big\{m\{\gamma_\tau, \lambda^p \beta_\tau\}\theta_\tau, \lambda^p \beta_{\tau-1}\theta_{\tau-1}, \ldots, \lambda^p \beta_0\theta_0\big\} \\ = m\{\mu_\tau\theta_\tau, \lambda^p \beta_{\tau-1}\theta_{\tau-1}, \ldots, \lambda^p \beta_0\theta_0\} \quad \text{by (5.13)} \\ = m\{\gamma_{\tau-1}\theta_{\tau-1}, \lambda^p \beta_{\tau-1}\theta_{\tau-1}, \ldots, \lambda^p \beta_0\theta_0\} \quad \text{by (5.11)} \\ = 1, \quad \text{by the inductive hypothesis.} \end{aligned}$$

This establishes (5.15) which we apply with $t = p - 1$. Then

$$\begin{aligned} 1 = m\{\gamma_{p-1}\theta_{p-1}, \lambda^p \beta_{p-1}\theta_{p-1}, \ldots, \lambda^p \beta_0\theta_0\} \\ = m\{\mu_p\theta_p, \lambda^p \beta_{p-1}\theta_{p-1}, \ldots, \lambda^p \beta_0\theta_0\} \\ = m\{0, \lambda^p \beta_{p-1}\theta_{p-1}, \ldots, \lambda^p \beta_0\theta_0\} \quad \text{since} \quad \mu_p = 0 \\ = \lambda^p m\{\beta_{p-1}\theta_{p-1}, \ldots, \beta_0\theta_0\} \quad \text{since } \lambda^p \text{ is primitive.} \end{aligned}$$

We have thus constructed a right inverse of λ^p in $\mathfrak{C}$, namely $m\{\beta_{p-1}\theta_{p-1}, \ldots, \beta_0\theta_0\}$, and the theorem is proved.

This theorem has already been established in $\mathfrak{G}$ (see [5] Theorem 2.3), the present proof being, of course modeled on that in $\mathfrak{G}$. However Theorem 5.2 does enable us, by virtue of Theorem 4.16, to recognize the statement that, in the factorization (F') in $\mathfrak{S}$, the maps ${}^p\varkappa$ are strong monomorphisms, as a manifestation of the same phenomenon as that described by Theorem 2.3 of [5].

We recall that in [3], § 5, we defined not only the concept of length but also the concept of weak length. Thus in a category $\mathfrak{U}$ we have $\underline{l}(A) < n$ if there exists $g : \underline{T}(A) \to A$ with[13] $g\varkappa^{n-1} = \bar{d} = \langle 1, 1, \ldots, 1\rangle : A^n \to A$ (see (5.1)); while $\underline{wl}(A) < n$ if $\bar{d}k = 0$ where $k : K \to A^n$ is the kernel of $\varkappa^{n-1}$. However Theorems 4.10 and 5.2 enable us to infer (cf. [3], Theorem 5.11)

Corollary 5.16. *In the admissible $\underline{G}$-primitive* I*-category* $\mathfrak{D}$ *length and weak length coincide.*

This was merely noted in passing in [5] for the category $\mathfrak{G}$.

6. Homotopy systems in primitive categories[14])

Let $\mathfrak{C}$, $\mathfrak{C}'$ be D-categories and let $H : \mathfrak{C} \to \mathfrak{C}'$ be a D-functor. Then (see [2]) if (A, m) is an $\underline{M}$-object of $\mathfrak{C}$, (HA, Hm) is an $\underline{M}$-object of $\mathfrak{C}'$ satisfying all the axioms satisfied by (A, m). We may prove

Proposition 6.1. *Let $H, K : \mathfrak{C} \to \mathfrak{C}'$ be two* D*-functors and let $t : H \to K$ be a natural transformation. Then for any $\underline{M}$-object (A, m) of $\mathfrak{C}$*

$$t_A : HA \to KA$$

is primitive.

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[13]) The terminology is here as in [3], § 5: A^n is the inverse product of n copies of A and $\underline{T}(A)$ the second object in the factorization $(F) : A^n \xrightarrow{\varkappa^{n-1}} \underline{T}(A)$.

[14]) See [3], § 6 for the concept of (left or right) homotopy and homotopy system in a general category.

Proof. We recall that, for any objects X_1, X_2 of $\mathfrak{C}$ and any D-functor $H: \mathfrak{C} \to \mathfrak{C}'$ there is an equivalence $\theta = \theta(H) = \{Hp_1, Hp_2\}: H(X_1 \times X_2) \to HX_1 \times \times HX_2$ in $\mathfrak{C}'$ such that $p_j \theta = Hp_j$, $j = 1, 2$. Then the diagram

$$\text{(6.2)} \qquad \begin{array}{ccc} H(X_1 \times X_2) & \xrightarrow{t_{X_1 \times X_2}} & K(X_1 \times X_2) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} \\ HX_1 \times HX_2 & \xrightarrow{t_{X_1} \times t_{X_2}} & KX_1 \times KX_2 \end{array}$$

commutes. For $p_j(t_{X_1} \times t_{X_2})\theta = t_{X_j} p_j \theta = t_{X_j} H p_j = K p_j t_{X_1 \times X_2}$, since t is a natural transformation; thus

$$p_j(t_{X_1} \times t_{X_2})\theta = p_j \theta t_{X_1 \times X_2}.$$

In the light of (6.2), the primitivity of t_A is just the assertion that the diagram

$$\begin{array}{ccc} H(A \times A) & \xrightarrow{t_{A \times A}} & K(A \times A) \\ \downarrow{\scriptstyle Hm} & & \downarrow{\scriptstyle Km} \\ HA & \xrightarrow{t_A} & KA \end{array}$$

commutes; and this is an immediate consequence of the naturality of t.

Let (A, m_A), (B, m_B) be $\underline{\mathrm{M}}$-objects of $\mathfrak{C}$ and let $Q: A \to PB$ be a homotopy between primitive maps $f, g: A \to B$, where P is the path functor belonging to a right homotopy system on $\mathfrak{C}$. Then by definition P is a D-functor so that it is meaningful to demand of Q that it be a primitive map. If it is we call Q a *primitive homotopy* between the primitive maps f, g. This terminology is justified by the observation that a right homotopy system in $\mathfrak{C}$ is a functor $P: \mathfrak{C} \to \mathfrak{C}$ together with some natural transformations $P \to I$, $I \to P$, where I is the identity functor; and that, by proposition 6.1, the resulting maps $PA \to A$, $A \to PA$ are primitive for all $\underline{\mathrm{M}}$-objects (A, m). Let us express this in the

Corollary 6.3. *Let $\mathfrak{D}$ be a primitive category over $\mathfrak{C}$ and let $S(P; t, b, p)$ be a right homotopy system in $\mathfrak{C}$ such that $(PA, Pm) \in \mathfrak{D}$ whenever $(A, m) \in \mathfrak{D}$. Then S induces a primitive right homotopy system $S_{\mathfrak{D}}$ on $\mathfrak{D}$ by the rule $P_{\mathfrak{D}}(A, m) = (PA, Pm)$, the natural transformations of $S_{\mathfrak{D}}$ being just those of S, transferred*[15]) *to $\mathfrak{D}$.*

We now recall that a right homotopy system is *faithful* if P preserves left-equalizers, and deduce from corollary 4.2

Proposition 6.4. *Let S be a faithful right homotopy system in $\mathfrak{C}$ and let $\mathfrak{D}$ be an admissible primitive category over $\mathfrak{C}$ such that $(PA, Pm) \in \mathfrak{D}$ if $(A, m) \in \mathfrak{D}$, where P is the path functor belonging to S. Then the induced primitive right homotopy system $S_{\mathfrak{D}}$ on $\mathfrak{D}$ is also faithful.*

[15]) Any further axioms which we wished to impose on our right homotopy system (cf. [3], § 6) would equally well transfer from $\mathfrak{C}$ to $\mathfrak{D}$.

We draw attention to the implication of this proposition. The system $S_{\mathfrak{D}}$ may well fail to have a left adjoint in $\mathfrak{D}$; but, if S has a left adjoint in $\mathfrak{C}$, then $S_{\mathfrak{D}}$ will still be faithful.

Now it was shown in [3], § 6 that the factorization (F) in a category is invariant under faithful right homotopy. To recall the details, let P be a path functor $\mathfrak{U} \to \mathfrak{U}$ belonging to some faithful right homotopy system S. For any objects $B_1, B_2, \ldots, B_n$ of $\mathfrak{U}$ let

$$F(B): \qquad B^n \longrightarrow B^{n-1} \longrightarrow \cdots \longrightarrow B^{q+1} \xrightarrow{\varkappa^q} B^q \longrightarrow \cdots \longrightarrow B^1$$

$$F(PB): \qquad (PB)^n \longrightarrow (PB)^{n-1} \longrightarrow \cdots \longrightarrow (PB)^{q+1} \xrightarrow{\varkappa^q} (PB)^q \longrightarrow \cdots \longrightarrow (PB)^1$$

be the F-factorization associated with $B_1, B_2, \ldots, B_n$; $PB_1, PB_2, \ldots, PB_n$ respectively. Then there is a natural map $\varphi : F(PB) \to PF(B)$; that is, a sequence of natural maps $\varphi^q : (PB)^q \to P(B^q)$, $q = 1, 2, \ldots, n$, such that the diagram

$$\begin{array}{ccc} (PB)^{q+1} & \xrightarrow{\varkappa^q} & (PB)^q \\ \Big\downarrow{\scriptstyle \varphi^{q+1}} & & \Big\downarrow{\scriptstyle \varphi^q} \\ P(B^{q+1}) & \xrightarrow{P(\varkappa^q)} & P(B^q) \end{array}$$

is commutative for each q. Moreover the map φ has the property that if $Q_i : A_i \to PB_i$ is a homotopy between f_i and $g_i : A_i \to B_i$, $i = 1, 2, \ldots, n$, where $A_1, A_2, \ldots, A_n$ are objects of $\mathfrak{U}$, and f_i, g_i, Q_i induce

$$f^q : A^q \to B^q, \quad g^q : A^q \to B^q, \quad Q^q : A^q \to (PB)^q,$$

then $\varphi^q Q^q$ is a homotopy between f^q and g^q. From this it is an immediate consequence that if the maps $f_i : A_i \to B_i$ are homotopy equivalences (with respect to the faithful system S) then so are the maps $f^q : A^q \to B^q$ and thus the factorization (F) is, in a precise sense, homotopy invariant.

Now let S be a faithful right homotopy system on a category $\mathfrak{C}$ and let $\mathfrak{D}$ be an admissible primitive category over $\mathfrak{C}$ which is closed with respect to the path functor of S, as in proposition 6.4; suppose further that $\mathfrak{D}$ is an I-category. Then the discussion above is applicable to the induced right homotopy system $S_{\mathfrak{D}}$ on $\mathfrak{D}$ and we have proved

Theorem 6.5. *Let $\mathfrak{D}$ be an admissible primitive* DI-*category over a category $\mathfrak{C}$ and let $S_{\mathfrak{D}}$ be the primitive right homotopy system on $\mathfrak{D}$ induced by a faithful system S on $\mathfrak{C}$. Then the factorization (F) in $\mathfrak{D}$ is homotopy invariant with respect to $S_{\mathfrak{D}}$.*

We may exemplify this theorem by considering the factorization (F) in the category of group complexes (or c.s.s. groups (see [5])). Then Kan's notion of "homotopy through homomorphisms" or loop homotopy (see [7]) coincides precisely with the equivalence relation on maps generated by the homotopy system $S_{\mathfrak{D}}$, where the path functor P in S is just the functor $X \to X^I$. Certainly S is faithful (since P has a left adjoint), so we may conclude that the factorization (F) is homotopy invariant, in the sense of Kan, in the category of group complexes.

We note that the introduction of the homotopy system $S_{\mathfrak{D}}$ enables us to define, in $\mathfrak{D}$, not only the length and weak length, but also the homotopy-length and weak homotopy-length of objects of $\mathfrak{D}$. The fact that the homotopy-length and weak homotopy-length are homotopy invariants, with respect to $S_{\mathfrak{D}}$, where S is faithful, is an immediate consequence of the homotopy invariance of the factorization (F); however, as remarked in [3] § 6, this fact follows quite simply from the fact that homotopy classification is compatible with composition of maps and thus does not depend on the faithfulness of S.

Finally we point out that although length and weak length coincide in an admissible $\underline{G}$-primitive category (Corollary 5.16), there is no reason to suppose that homotopy-length and weak homotopy-length coincide in such a category furnished with a primitive right homotopy system.

References

[1] Eckmann, B., and P. J. Hilton: Groupes d'homotopie et dualité 1. Groupes absolus. Compt. rend. **246**, 2444 (1958); 2. Suites exactes. Compt. rend. **246**, 2555 (1958); 3. Coefficients. Compt. rend. **246**, 2993 (1958).
[2] — — Group-like structures in general categories I. Multiplications and comultiplications. Math. Ann. **145**, 227 (1962).
[3] — — Group-like structures in general categories II. Equalizers. Limits. Lengths. Math. Ann. (to appear).
[4] — — Structure maps in group theory. Fundamenta Math. **50**, 207 (1961).
[5] Hilton, P. J.: Remark on the free product of groups. Trans. Am. Math. Soc. **96**, 478 (1960).
[6] — Note on free and direct products. Bull. Soc. Math. Belgique **13**, 38 (1961).
[7] Kan, D. M.: On homotopy theory and c.s.s. groups. Ann. Math. **68**, 38 (1958).
[8] — Monoids and their dual. Bol. Soc. Math. Mex. **3**, 52 (1958).
[9] Kurosh, A. C., A. Kh. Livshits and E. G. Shul'geifer: Foundations of the theory of categories. Uspekhi Matem. Nauk XV, **6**, 1 (1960).

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