# The Homotopy Theory of Cyclic Sets

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# THE HOMOTOPY THEORY OF CYCLIC SETS[1]

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ABSTRACT. The aim of this note is to show that the homotopy theory of the *cyclic* sets of Connes [3] is equivalent to that of SO(2)-*spaces* (i.e. spaces with a circle action) and hence to that of *spaces over* $K(Z, 2)$.

## 1. Introduction.

1.1 *Summary*. The aim of this note is to study the homotopy theory of the cyclic sets of Connes [3]. These are simplicial sets with some additional structure, or more precisely, diagrams of sets indexed by a category $\Lambda^{\mathrm{op}}$ of cyclic operators which contains the category $\Delta^{\mathrm{op}}$ of simplicial operators as a subcategory.

After a few preliminaries (§2), we construct a Quillen *closed model category structure* on the category of cyclic sets (§3) and show that the resulting homotopy theory is equivalent to (§4) that of SO(2)-*spaces* (i.e. spaces with a circle action) as well as to (§5) that of *simplicial sets over the nerve of* $\Lambda^{\mathrm{op}}$, which has the homotopy type of $K(Z, 2)$.

The first and the last of these results hold more generally, as their proofs depend only on certain properties (2.2 and 2.6) of the inclusion functor $j\colon \Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$. Some additional exampls are (§6) a kind of covering $\mathbf{K}^{\mathrm{op}}$ of $\Lambda^{\mathrm{op}}$ and, for every simplicial group $G$, its flattening $\mathbf{b}G$. (The homotopy theory of $\mathbf{b}G$-sets (i.e. functors $\mathbf{b}G \to \mathbf{sets}$) is equivalent to the homotopy theory of simplicial sets over the classifying complex of $G$.)

1.2 *Notation, terminology, etc.* (i) *The category* $\Delta^{\mathrm{op}}$ *of simplicial operators.* This is the category with objects $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$ and generating maps

$$d_i\colon \mathbf{n} \to \mathbf{n-1}, \qquad 0 \leqslant i \leqslant n, n > 0,$$
$$s_i\colon \mathbf{n} \to \mathbf{n+1}, \qquad 0 \leqslant i \leqslant n,$$

subject to the relations

$$(*)\qquad \begin{aligned} d_i d_j &= d_{j-1} d_i, && 0 < j - i,\\ s_j s_i &= s_i s_{j-1}, && 0 < j - i,\\ d_i s_j &= s_{j-1} d_i\colon \mathbf{n} \to \mathbf{n}, && 0 < j - i \leqslant n,\\ &= \mathrm{id}, && -1 \leqslant j - i \leqslant 0,\\ &= s_j d_{i-1}, && j - i < -1. \end{aligned}$$

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(ii) *The category* $\Lambda^{\mathrm{op}}$ *of cyclic operators*. This is the category with objects **0**, **1**, **2**,... and generating maps

$$d_i: n \to \mathbf{n-1}, \qquad 0 \leqslant i \leqslant n, n > 0,$$
$$s_i: n \to \mathbf{n+1}, \qquad 0 \leqslant i \leqslant n+1,$$

subject to the above relations (*) as well as the *cyclic relations*

$$(d_0 s_{n+1})^{n+1} = \mathrm{id}: \mathbf{n} \to \mathbf{n}, \qquad n \geqslant 0.$$

Note that, except for these cyclic relations, there is *no* relation involving the first face operator and last degeneracy operator, i.e. $d_0 s_{n+1} \neq s_n d_0: \mathbf{n} \to \mathbf{n}$.

(iii) *The categories* **S** *of simplicial sets* (*i.e.* $\Delta^{\mathrm{op}}$-*sets*) *and* $\mathbf{S}^c$ *of cyclic sets* (*i.e.* $\Lambda^{\mathrm{op}}$-*sets*). These are the categories which have as objects the functions $\Delta^{\mathrm{op}} \to$ **sets** and $\Lambda^{\mathrm{op}} \to$ **sets**, respectively, and as maps the natural transformations between them. A cyclic set $X$ is thus a simplicial set together with an *extra degeneracy map* $s_{n+1}: X_n \to X_{n+1}$ in each dimension $n \geqslant 0$, with the cyclic property that $(d_0 s_{n+1})^{n+1} = \mathrm{id}: X_n \to X_n$. However, in general, $d_0 s_{n+1} \neq s_n d_0: X_n \to X_n$.

The categories **S** and $\mathbf{S}^c$ are connected by the rather useful *forgetful functor* $j^*$: $\mathbf{S}^c \to \mathbf{S}$, induced by the *inclusion functor* $j$: $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$.

(iv) *The category* $\mathbf{T}^c$ *of* SO(2)-*spaces and the forgetful functor* $u$: $\mathbf{T}^c \to \mathbf{T}$. We denote by **T** the category of topological spaces, by $\mathbf{T}^c$ the category of SO(2)-spaces, i.e. topological spaces with a continuous SO(2)-action, and by $u$: $\mathbf{T}^c \to \mathbf{T}$ the functor which sends each SO(2)-space to its underlying topological space.

**2. The standard cyclic sets** $\Lambda[n]$. This is a brief discussion of the standard cyclic sets $\Lambda[n]$ (the cyclic analogs of the standard simplicial sets $\Delta[n]$), much of which is explicit or implicit in [**8**].

2.1 *The standard cyclic sets in* $\Lambda[n]$. For every integer $n \geqslant 0$, the *standard cyclic set* $\Lambda[n]$ is given by

$$\Lambda[n] = \hom^{\Lambda^{\mathrm{op}}}(\mathbf{n}, -): \Lambda^{\mathrm{op}} \to \mathbf{sets}.$$

It has the *universal property* that, for every cyclic set $X$ and every $n$-simplex $x \in X$, there is a unique map (1.2(iii)) $c_x$: $\Lambda[n] \to X \in \mathbf{S}^c$ such that $c_x i_n = x$ (where $i_n \in \Lambda[n]$ denotes the generating $n$-simplex, i.e. the identity map of **n**). This gives rise to a natural isomorphism

$$\hom^{\mathbf{S}^c}(\Lambda[n], X) \approx \hom^{\mathbf{S}}(\Delta[n], j^*X) \approx X_n.$$

The $\Lambda[n]$ and the maps between them form a $\Lambda$-diagram of cyclic sets, i.e. a functor $\Lambda[-]$: $\Lambda \to \mathbf{S}^c$, with the following property:

2.2 PROPOSITION. *In the induced* (1.2(iii)) *diagram of simplicial sets* $j^*\Lambda[-]$: $\Lambda \to \mathbf{S}$, *all maps are weak* (*homotopy*) *equivalences.*

2.3 *The cyclic sets* $\Lambda[n, k]$. As in the simplicial theory [**2**, Chapter VIII, 3.3] we need the cyclic subsets $\Lambda[n, k] \subset \Lambda[n]$ $(0 \leqslant k \leqslant n)$ which are spanned by the $(n-1)$-simplices $d_i i_n$ $(0 \leqslant i \leqslant n, i \neq k)$, which clearly give rise to *natural isomorphisms*

$$\hom^{\mathbf{S}^c}(\Lambda[n, k], X) \approx \hom^{\mathbf{S}}(\Delta[n, k], j^*X), \qquad 0 \leqslant k \leqslant n.$$

Moreover, it is not difficult to prove (using 2.2)

2.4 **Proposition.** *The inclusions* $\Lambda[n, k] \to \Lambda[n] \in \mathbf{S}^c$, $(0 \leqslant k \leqslant n)$ *induce weak equivalences* $j^*\Lambda[n, k] \to j^*\Lambda[n] \in \mathbf{S}$.

2.5 *The cyclic sets* $\dot{\Lambda}[n]$. Also useful are the cyclic subsets $\dot{\Lambda}[n] \subset \Lambda[n]$ $(n > 0)$ spanned by the $(n-1)$-simplices $d_i i_n$ $(0 \leqslant i \leqslant n)$. They give rise to *natural isomorphisms*

$$\hom^{\mathbf{S}^c}(\dot{\Lambda}[n], X) \approx \hom^{\mathbf{S}}(\dot{\Delta}[n], j^*X), \qquad n > 0.$$

Moreover, each $\dot{\Lambda}[n]$ is closely related to the direct limit $\partial\Lambda[n]$ of the diagram in $\mathbf{S}^c$ which consists of

(i) for every integer $i$ with $0 \leqslant i \leqslant n$, a copy $\Lambda[n]_i$ of $\Lambda[n-1]$, and

(ii) for every pair of integers $(i, j)$ with $0 \leqslant i < j \leqslant n$, a copy $\Lambda[n]_{i,j}$ of $\Lambda[n-2]$ together with a pair of maps

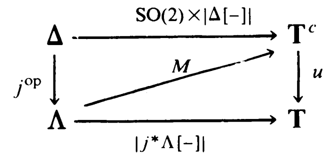
$$\Lambda[n]_i \overset{c_{d_{j-1}i_{n-1}}}{\longleftarrow} \Lambda[n]_{i,j} \overset{c_{d_i i_{n-1}}}{\longrightarrow} \Lambda[n]_j$$

and which has the following nontrivial property:

2.6 **Proposition.** *For each integer* $n > 0$, *the obvious map* $\partial\Lambda[n] \to \dot{\Lambda}[n] \in \mathbf{S}^c$ *is an isomorphism.*

Propositions 2.2 and 2.6 follow readily from

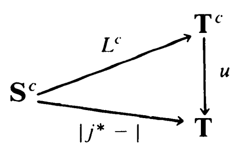
2.7 **Proposition.** *Let* $\mathrm{SO}(2) \times |\Delta[-1]|\colon \mathbf{\Delta} \to \mathbf{T}^c$ (1.2(iv)) *denote the obvious functor which sends* $\mathbf{n}$ $(n \geqslant 0)$ *to the product of* SO(2) *and the geometric realization of* $\Delta[n]$, *with* SO(2) *acting on the left on itself and trivially on* $|\Delta[n]|$. *Then there exists a functor* $M\colon \mathbf{\Lambda} \to \mathbf{T}^c$ *such that the following diagram commutes up to natural equivalences*:




*In particular*, $|j^*\Lambda[n]|$ *is homeomorphic to* $\mathrm{SO}(2) \times |\Delta[n]|$ $(n \geqslant 0)$.

As, for $X \in \mathbf{S}^c$, the geometric realization $|j^*X|$ can be expressed as a quotient of the disjoint union $\coprod_n X_n \times |j^*\Lambda[n]|$ by the obvious identifications, Proposition 2.7 readily implies

2.8 **Proposition.** *There is a functor* $L^c\colon S^c \to T^c$ *such that the following diagram commutes up to a natural equivalence*:



It thus remains to give a

2.9 *Proof of Proposition* 2.7. Consider the "twisted" product $\Delta[1] \times_t \Delta[n] \in \mathbf{S}$ which has as $k$-simplices ($k \geqslant 0$) the $(k+1)$-tuples of pairs of integers

$$((0, i_1), \ldots, (0, i_a), (1, j_1), \ldots, (1, j_b))$$

such that $0 \leqslant j_1 \leqslant \cdots \leqslant j_b \leqslant i_1 \leqslant \cdots \leqslant i_a \leqslant n$ and $a, b \geqslant 0$, with the obvious faces and degeneracies. Then there is a homeomorphism between the geometric realizations

$$|\Delta[1] \times_t \Delta[n]| \approx |\Delta[1] \times \Delta[n]|$$

which is natural in $n$. As $j^*\Lambda[n]$ is exactly the simplicial set obtained from $\Delta[1] \times_t \Delta[n]$ by identifying the simplices

$$((0, i_1), \ldots, (0, i_a)) \quad \text{and} \quad ((1, i_1), \ldots, (1, i_a))$$

for every sequence of integers $(i_1, \ldots, i_a)$ with $0 \leqslant i_1 \leqslant \cdots \leqslant i_a \leqslant n$, the above homeomorphism induces a homeomorphism $|j^*\Lambda[n]| \approx SO(2) \times |\Delta[n]|$ and the desired result now follows readily.

**3. A homotopy theory for cyclic sets.** In Theorem 3.1 below we turn the category $\mathbf{S}^c$ of cyclic sets (see subsection 1.2) into a *closed model category* in the sense of Quillen, i.e. [**2**, p. 241] we define notions of *weak equivalence*, *fibration* and *cofibration* such that the following five axioms are satisfied:

CM1. *The category is closed under finite and direct and inverse limits.*

CM2. *If $f$ and $g$ are maps such that $gf$ is defined and if two of $f$, $g$ and $gf$ are weak equivalences, then so is the third.*

CM3. *If $f$ is a retract of $g$ (i.e. if there are, in the category of maps, maps $a\colon f \to g$ and $b\colon g \to f$ such that $ba = \mathrm{id}_f$) and $g$ is a weak equivalence, a fibration or a cofibration, then so is $f$.*

CM4. *Given a commutative solid arrow diagram*

$$\begin{array}{ccc} U & \rightarrow & X \\ i\downarrow & \nearrow & \downarrow p \\ V & \rightarrow & Y \end{array}$$

*in which $i$ is a cofibration, $p$ is a fibration, and either $p$ or $i$ is a weak equivalence, then the dotted arrow exists* (and one says that $i$ has the left lifting property with respect to $p$, or equivalently, that $p$ has the right lifting property with respect to $i$).

CM5. *Any map $f$ can be factored in two ways:*

(i) *$f = pi$, where $i$ is a cofibration and $p$ is a trivial fibration* (i.e. a fibration as well as a weak equivalence), *and*

(ii) *$f = pi$, where $p$ is a fibration and $i$ is a trivial cofibration* (i.e. a cofibration as well as a weak equivalence).

3.1 THEOREM. *The category $\mathbf{S}^c$ of cyclic sets admits a closed model category structure in which a map $X \to Y \in \mathbf{S}^c$ is a weak equivalence or a fibration whenever the induced* (1.2) *map $j^*X \to j^*Y \in \mathbf{S}$ is so, and in which the cofibrations are* (*see also* 3.4) *the retracts of the* (*possibly transfinite*) *compositions of cobase extensions of the inclusions $\Lambda[n] \to \dot{\Lambda}[n]$ $(n \geqslant 0)$.*

To prove this we first note that 2.3 and 2.5 imply

3.2 PROPOSITION. *A map in* $\mathbf{S}^c$ *is a fibration iff it has the right lifting property with respect to the inclusions* $\Lambda[n, k] \to \Lambda[n]$ $(0 \leqslant k \leqslant n)$.

3.3 PROPOSITION. *A map in* $\mathbf{S}^c$ *is a trivial fibration iff it has the right lifting property with respect to the inclusions* $\dot{\Lambda}[n] \to \Lambda[n]$ $(n \geqslant 0)$.

PROOF OF THEOREM 3.1. Verification of CM1, CM2, CM3 and the first part of CM4 is easy. To prove CM5 one combines the small object argument of [**9**, Chapter II, §3] with 3.3 or 2.4 and 3.2. This proof of CM5 implies that a trivial cofibration $U \to V$ admits a factorization $U \to V' \to V$ in which (by construction) the map $U \to V'$ is a trivial cofibration which has the left lifting property with respect to the fibrations and $V' \to V$ is a (necessarily trivial) fibration. The remaining part of CM4 now follows from the fact that the map $U \to V$ is (easily seen to be) a retract of the map $U \to V'$.

We end with some alternate descriptions of the cofibrations and the weak equivalences.

3.4 *Cofibrations in* $\mathbf{S}^c$. Call a map $X \to Y \in \mathbf{S}^c$ *free* if it is 1-1 and if, for every integer $n \geqslant 0$, the group of automorphisms of $\mathbf{n}$ (which is cyclic of order $n + 1$) acts freely on the $n$-simplices of $Y$ which are not in the image of $X$. Clearly *the inclusions* $\Lambda[n, k] \to \Lambda[n]$ *and* $\dot{\Lambda}[n] \to \Lambda[n]$ *are free and so are their cobase extensions.* In fact one readily verifies

3.5 PROPOSITION. *The free maps in* $\mathbf{S}^c$ *are exactly the* (*possibly transfinite*) *compositions of cobase extensions of the inclusions* $\dot{\Lambda}[n] \to \Lambda[n]$ $(n \geqslant 0)$. *Hence a map in* $\mathbf{S}^c$ *is a cofibration iff it is a free map.*

3.6 *Weak equivalences in* $\mathbf{S}^c$. Given a cyclic set $X$, i.e. a functor $X\colon \Lambda^{\mathrm{op}} \to$ **sets**, one can compose it with the inclusion functor $i$: **sets** $\to \mathbf{S}$ and take the homotopy direct limit [**2**, Chapter XIII] $\underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}} iX \in \mathbf{S}$. Then one has

3.7 PROPOSITION *A map* $X \to Y \in \mathbf{S}^c$ *is a weak equivalence iff the induced map* $\underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}} iX \to \underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}} iY \in \mathbf{S}$ *is so.*

PROOF. Let $\mathbf{X}$ be the category which has as objects the simplices of $X$ and as maps the cyclic operators between them. Then the nerve $N\mathbf{X}$ is clearly naturally isomorphic to $\underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}} iX$. Similarly, if $j^*\mathbf{X} \subset \mathbf{X}$ is the subcategory consisting of the simplicial operators (the objects remain the same), then the nerve $Nj^*\mathbf{X}$ is naturally isomorphic to $\underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}} ij^*\mathbf{X}$. Moreover, by [**2**, Chapter XII, 4.3] the latter simplicial set is naturally weakly equivalent to $j^*X$. The under categories of the inclusion functor $j^*\mathbf{X} \to \mathbf{X}$ are readily verified to be isomorphic to the under categories $\mathbf{n}\downarrow j$ of the inclusion functor $j\colon \Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ and, by the above arguments, the nerves of the latter are naturally isomorphic to $\underrightarrow{\mathrm{holim}}^{\Delta^{\mathrm{op}}} ij^*\Lambda[n]$ and hence naturally weakly equivalent to $j^*\Lambda[n]$. The desired result now follows immediately from 2.2 and Quillen's theorem $B$ [**10**, p. 97].

**4. Comparison with** SO(2)**-spaces.** In order to show that the homotopy theory of cyclic sets of §3 is equivalent, in the strong sense of [**5**, §5], to the homotopy theory of SO(2)-spaces (resulting from its usual model category structure (see Theorem 4.1)), we verify in Theorem 4.2 the existence of a pair of adjoint functors

$$L^c\colon (\textbf{cyclic sets}) \leftrightarrow (\mathrm{SO}(2)\textbf{-spaces})\colon R^c$$

satisfying the *equivalence conditions* of Quillen [**9**, Chapter I, Theorem 3]:

EQ1. *The left adjoint $L^c$ sends cofibrations into cofibrations and weak equivalences between cofibrant objects into weak equivalences.*

EQ2. *The right adjoint $R^c$ sends fibrations into fibrations and weak equivalences between the fibrant objects into weak equivalences.*

EQ3. *For every cofibrant cyclic set $X$ and every fibrant* SO(2)*-space $Y$, a map $X \to R^cY$ is a weak equivalence iff its adjoint $L^cX \to Y$ is so.*

First we recall from [**6**, 1.2 and 2.2]

4.1 THEOREM. *The category $\mathbf{T}^c$ of* SO(2)*-spaces* (*see subsection* 1.2) *admits a closed model category structure in which a map $X \to Y \in \mathbf{T}^c$ is a weak equivalence or a fibration whenever the underlying map of topological spaces $uX \to uY \in \mathbf{T}$ is a weak homotopy equivalence or a Serre fibration, and in which the cofibrations are the retracts of the* (*possibly transfinite*) *compositions of cobase extensions of the inclusions* $\mathrm{SO}(2) \times |\dot{\Delta}[n]| \to \mathrm{SO}(2) \times |\Delta[n]|$ ($n \geqslant 0$).

In view of Proposition 2.8 one can then formulate

4.2 THEOREM. *The functor $L^c\colon \mathbf{S}^c \to \mathbf{T}^c$ has as right adjoint the functor* $R^c = \hom(L^c\Lambda[-], -)\colon \mathbf{T}^c \to \mathbf{S}^c$. *Moreover, this pair of adjoint functors satisfies the* (*above*) *equivalence conditions of Quillen.*

4.3 COROLLARY. *The categories $\mathbf{S}^c$ and $\mathbf{T}^c$ have equivalent homotopy theories in the strong sense that* [5] *their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of* [**5**, §2].

PROOF. One verifies successively and without much difficulty the following properties:

(i) the adjointness;

(ii) $L^c$ preserves weak equivalences;

(iii) for every object $Y \in \mathbf{T}^c$, the simplicial set $j^*R^cY$ is just the singular complex of the underlying topological space $uY$;

(iv) $R^c$ preserves weak equivalences and fibrations;

(v) $L^c$ preserves cofibrations (use (iv) and adjointness);

(vi) for every object $Y \in \mathbf{T}^c$, the underlying map $uL^cR^cY \to uY$ of the adjunction map is the usual adjunction map from the geometric realization of the singular complex of $uY$ back to $uY$;

(vii) EQ3 is satisfied.

**5. Comparison with simplicial sets over** $K(Z, 2)$. As the homotopy theory of SO(2)-spaces is (well known to be [**4**]) equivalent to that of simplicial sets over $K(Z, 2)$, Theorem 4.2 implies that the same holds for the homotopy theory of cyclic sets of §3. We will now give a direct proof of this result (Theorem 5.1) which only uses the fact that the functor $j$: $\Delta^{\mathrm{op}} \to \Lambda^{\mathrm{op}}$ has the properties described in Propositions 2.2 and 2.6.

To formulate Theorem 5.1, let $N\Lambda^{\mathrm{op}}$ be the nerve of $\Lambda^{\mathrm{op}}$ (which [**3**] is weakly equivalent to $K(Z,2)$), let the category $\mathbf{S}/N\Lambda^{\mathrm{op}}$ of simplicial sets over $N\Lambda^{\mathrm{op}}$ have the closed model category structure induced by the usual one on $\mathbf{S}$ [**9**, Chapter II], let $*$: $\Lambda^{\mathrm{op}} \to \mathbf{sets} \in \mathbf{S}^c$ be the terminal object and let $L^{\Lambda^{\mathrm{op}}}$: $\mathbf{S}^c \to \mathbf{S}/N\Lambda^{\mathrm{op}}$ be the functor which sends an object $X \in \mathbf{S}^c$ to the map (see 3.6)

$$\underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}}\, iX \to \underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}}\, i* = N\Lambda^{\mathrm{op}}.$$

Then one has

5.1 THEOREM. *The functor* $L^{\Lambda^{\mathrm{op}}}$: $\mathbf{S}^c \to \mathbf{S}/N\Lambda^{\mathrm{op}}$ *has as right adjoint the functor*

$$R^{\Lambda^{\mathrm{op}}} = \hom\left(L^{\Lambda^{\mathrm{op}}}\Lambda[-],-\right): \mathbf{S}/N\Lambda^{\mathrm{op}} \to \mathbf{S}^c.$$

*Moreover, this pair of adjoint functors satisfies the* (*see* §4) *equivalence conditions of Quillen.*

5.2 COROLLARY. *The categories* $\mathbf{S}$ *and* $\mathbf{S}/N\Lambda^{\mathrm{op}}$ *have equivalent homotopy theories in the strong sense that* [**5**] *their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of* [**5**, §2].

PROOF. The adjointness is obvious and Proposition 3.7 implies that a map $X \to Y \in \mathbf{S}^c$ is a weak equivalence iff the induced map $L^{\Lambda^{\mathrm{op}}}X \to L^{\Lambda^{\mathrm{op}}}Y \in \mathbf{S}/N\Lambda^{\mathrm{op}}$ is so. Moreover, $L^{\Lambda^{\mathrm{op}}}$ preserves cofibrations and hence it follows from the adjointness that $R^{\Lambda^{\mathrm{op}}}$ preserves fibrations and trivial fibrations. By [**1**, 1.2 and 1.3], this implies EQ2. Finally, to prove EQ3, it suffices to show that, for every fibrant object $Y \in \mathbf{S}/N\Lambda^{\mathrm{op}}$, the adjunction map $L^{\Lambda^{\mathrm{op}}}R^{\Lambda^{\mathrm{op}}}Y \to Y \in \mathbf{S}/N\Lambda^{\mathrm{op}}$ is a weak equivalence.

To do this let $Y' = (Nj)^*Y \in \mathbf{S}/N\Delta^{\mathrm{op}}$ be the pull back over $Nj$: $N\Delta^{\mathrm{op}} \to N\Lambda^{\mathrm{op}}$ and let

$$L^{\Delta^{\mathrm{op}}}: \mathbf{S} \leftrightarrow \mathbf{S}/N\Delta^{\mathrm{op}}: R^{\Delta^{\mathrm{op}}}$$

be the pair of adjoint functors analogous to $L^{\Lambda^{\mathrm{op}}}$ and $R^{\Lambda^{\mathrm{op}}}$. Then it is not difficult to verify that the adjunction map $L^{\Delta^{\mathrm{op}}}R^{\Delta^{\mathrm{op}}}Y' \to Y' \in \mathbf{S}/N\Delta^{\mathrm{op}}$ is a weak equivalence and that this map admits an obvious factorization $L^{\Delta^{\mathrm{op}}}R^{\Delta^{\mathrm{op}}}Y' \to L^{\Delta^{\mathrm{op}}}j^*R^{\Lambda^{\mathrm{op}}}Y \to Y'$. Using the argument of Proposition 3.7 and the fact that $Y \in \mathbf{S}/N\Lambda^{\mathrm{op}}$ is fibrant, one now proves that the adjunction map $L^{\Lambda^{\mathrm{op}}}R^{\Lambda^{\mathrm{op}}}Y \to Y \in \mathbf{S}/N\Lambda^{\mathrm{op}}$ is a weak equivalence iff the map $L^{\Delta^{\mathrm{op}}}j^*R^{\Lambda^{\mathrm{op}}}Y \to Y' \in \mathbf{S}/N\Delta^{\mathrm{op}}$ is so. It thus remains to prove that the map $L^{\Delta^{\mathrm{op}}}R^{\Delta^{\mathrm{op}}}Y' \to L^{\Delta^{\mathrm{op}}}j^*R^{\Lambda^{\mathrm{op}}}Y \in \mathbf{S}/N\Delta^{\mathrm{op}}$ is a weak equivalence and to do this one observes [**4**, §5] that the obvious maps $\underrightarrow{\mathrm{holim}}^{\Delta^{\mathrm{op}}}\, i\Delta[n] \to \underrightarrow{\mathrm{holim}}^{\Lambda^{\mathrm{op}}}\, i\Lambda[n] \in \mathbf{S}$ ($n \geqslant 0$) are weak equivalences and that the $\Delta$-diagram $L^{\Lambda^{\mathrm{op}}}\Lambda[-] \circ j^{\mathrm{op}}$: $\Delta \to \mathbf{S}/N\Lambda^{\mathrm{op}}$ is a cosimplicial resolution [**5**, 4.3]. To prove this last statement one notes that Proposition 2.6 implies that the $\Delta$-diagram $\Lambda[-] \circ j^{\mathrm{op}}$: $\Delta \to \mathbf{S}^c$ is a cosimplicial resolution and that the functor $L^{\Lambda^{\mathrm{op}}}$ satisfies EQ1 and commutes with the direct limits in question.

5.3 REMARK. The last step in the above proof is the only place where we used Proposition 2.6.

**6. A generalization to other categories under $\Delta^{op}$.** As mentioned in 1.1, Theorems 3.1 and 5.1 hold more generally, as their proofs only use the fact that the functor $j$: $\Delta^{op} \to \Lambda^{op}$ has the properties described in Propositions 2.2 and 2.6. In order to formulate these generalizations, we start with a brief discussion of

6.1 *Categories under* $\Delta^{op}$. Let $\Sigma$ be a small category and let $j$: $\Delta^{op} \to \Sigma^{op}$ be a functor. For every integer $n \geqslant 0$, one can then define the *standard* $\Sigma^{op}$-set $\Sigma[n]$ by

$$\Sigma[n] = \hom^{\Sigma^{op}}(j\mathbf{n}, -) \colon \Sigma^{op} \to \mathbf{sets}.$$

Clearly, these $\Sigma^{op}$-sets give rise to a $\Sigma$-diagram $\Sigma[-]$: $\Sigma \to \Sigma^{op}$-**sets** of $\Sigma^{op}$-sets as well as an induced $\Sigma$-diagram $j^*\Sigma[-]$: $\Sigma \to \mathbf{S}$ of simplicial sets. One can also consider the $\Sigma^{op}$-subset $\dot{\Sigma}[n] \subset \Sigma[n]$ spanned by the "faces" of the generating element, i.e. the identity map of $j\mathbf{n}$, and the $\Sigma^{op}$-set $\partial\Sigma[n]$, defined by means of a direct limit as in 2.5.

We will assume that the functor $j$: $\Delta^{op} \to \Sigma^{op}$ satisfies one or both of the following conditions:

(i) *In the diagram of simplicial sets* $j^*\Sigma[-]$: $\Sigma \to \mathbf{S}$, *all maps are weak equivalences.*

(ii) *For every integer* $n > 0$, *the obvious map* $\partial\Sigma[n] \to \dot{\Sigma}[n] \in \Sigma^{op}$-**sets** *is an isomorphism.*

The arguments of the proof of Theorem 3.1 then yield

6.2 THEOREM. *Let* $\Sigma$ *be a small category and let*: $\Delta^{op} \to \Sigma^{op}$ *be a functor satisfying* 6.1(i). *Then the category* $\Sigma^{op}$-**sets** (*of functors* $\Sigma^{op} \to$ **sets** *and natural transformations between them*) *admits a closed model category structure in which a map* $X \to Y \in \Sigma^{op}$-**sets** *is a weak equivalence or a fibration whenever the induced map* $j^*X \to j^*Y \in \mathbf{S}$ *is so and in which the cofibrations are the retracts of the* (*possibly transfinite*) *compositions of cobase extensions of the inclusions* $\dot{\Sigma}[n] \to \Sigma[n]$ ($n \geqslant 0$).

Similarly, if the category $\mathbf{S}/N\Sigma^{op}$ of simplicial sets over $N\Sigma^{op}$ has the closed model category structure induced by the usual one on $\mathbf{S}$ [**9**, Chapter II], if $* \in \Sigma^{op}$-**sets** is the terminal object and if $L^{\Sigma^{op}}$-**sets** $\to \mathbf{S}/N\Sigma^{op}$ denotes the functor which sends an object $X \in \Sigma^{op}$-**sets** to the map (see 3.6)

$$\underrightarrow{\text{holim}}^{\Sigma^{op}}\, iX \to \underrightarrow{\text{holim}}^{\Sigma^{op}}\, i* = N\Sigma^{op},$$

then the arguments of the proof of Theorem 5.1 yield

6.3 THEOREM. *Let* $\Sigma$ *be a small category with connected nerve and let* $j$: $\Delta^{op} \to \Sigma^{op}$ *be a functor satisfying* 6.1(i) *and* (ii). *Then the functor* $L^{\Sigma^{op}}$ *has as right adjoint the functor*

$$R^{\Sigma^{op}} = \hom\left(L^{\Sigma^{op}}\Sigma[-], -\right) \colon \mathbf{S}/N\Sigma^{op} \to \Sigma^{op}\text{-}\mathbf{sets}$$

*and this pair of adjoint functors satisfies the* (*see* §4) *equivalence conditions of Quillen.*

6.4 COROLLARY. *The categories* $\Sigma^{op}$-**sets** *and* $\mathbf{S}/N\Sigma^{op}$ *have equivalent homotopy theories in the strong sense that* [**5**] *their simplicial localizations with respect to the weak equivalences are weakly equivalent in the sense of* [**5**, §2].

6.5 EXAMPLE. If in the definition of $\Lambda^{\mathrm{op}}$ (see 1.2(ii)) one omits the cyclic relations $(d_0 s_{n+1})^{n+1} = \mathrm{id}\colon \mathbf{n} \to \mathbf{n}$ $(n \geqslant 0)$, one gets a category $\mathbf{K}^{\mathrm{op}}$ which is a kind of covering of $\Lambda^{\mathrm{op}}$. There is an obvious functor $j\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{K}^{\mathrm{op}}$ which satisfies 6.1(i) and (ii). Moreover, the nerve $N\mathbf{K}^{\mathrm{op}}$ is contractible and *the homotopy theory of* $\mathbf{K}^{\mathrm{op}}$*-sets thus is equivalent to the usual homotopy theory of simplicial sets.*

6.6 EXAMPLE. Given a simplicial group $G$, one can form its *flattening* $\mathbf{b}G$, i.e. [**7**, §7] the category which has as objects the $\mathbf{n}$ $(n \geqslant 0)$ and as maps $\mathbf{k} \to \mathbf{n}$ the pairs $(e, g)$, where $e$ is a map $e\colon \mathbf{k} \to \mathbf{n} \in \mathbf{\Delta}^{\mathrm{op}}$ and $g$ is an $n$-simplex of $G$. There is an obvious functor $j\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathbf{b}G$ given by $je = (e, 1)$ for all $e \in \mathbf{\Delta}^{\mathrm{op}}$. Moreover, one readily verifies that this functor satisfies 6.1(i) and (ii) and that $N\mathbf{b}G$ is weakly equivalent to the classifying complex of $G$. Hence, *the resulting homotopy theory of* $\mathbf{b}G$*-sets is equivalent to the homotopy theory of simplicial sets over the classifying complex of* $G$. This is, however, not surprising as $\mathbf{b}G$*-sets* are essentially the same as *simplicial sets with a G-action.*

We end with observing that in all three cases (i.e. $\mathbf{\Delta}^{\mathrm{op}}$, $\mathbf{K}^{\mathrm{op}}$ and $\mathbf{b}G$) *the functor j was onto on objects*. Actually this is not surprising in view of

6.7 PROPOSITION. *Let* $\mathbf{\Sigma}$ *be a small category with connected nerve and let* $j$: $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Sigma}^{\mathrm{op}}$ *be a functor satisfying* 6.1(i) *and* (ii). *If* $\mathbf{\Sigma}_0^{\mathrm{op}} \subset \mathbf{\Sigma}^{\mathrm{op}}$ *denotes the full subcategory spanned by the image of* $j$, *then the resulting functor* $j_0$: $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Sigma}_0^{\mathrm{op}}$ *also satisfies* 6.1(i) *and* (ii) *and the inclusion* $\mathbf{\Sigma}_0^{\mathrm{op}} \to \mathbf{\Sigma}^{\mathrm{op}}$ *induces an equivalence between the homotopy theory of* $\mathbf{\Sigma}^{\mathrm{op}}$*-sets and that of* $\mathbf{\Sigma}_0^{\mathrm{op}}$*-sets.*

PROOF. This follows immediately from Theorem 6.3 and Quillen's Theorem B [**10**, p. 97].

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