NOTES ON DELTA-GENERATED SPACES

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These notes concern my limited understanding of some ideas of Jeff Smith. Any mistakes are my fault, not his.

1.1. **General setup.** Let $\mathcal{T}op$ denote the category of all topological spaces, and continuous maps. Fix a nonempty full subcategory $\mathcal{I}$ of $\mathcal{T}op$. It is convenient for one's intuition to assume $\mathcal{I}$ contains the one-point space; however, this is not necessary for any of the claims below. Much (if not all) of the general machinery I describe in this section is developed in the nice paper [V]. When I omit proofs of Lemmas and Propositions it is because they are elementary exercises.

Definition 1.2. *A space X is called **$\mathcal{I}$-generated** if it has the property that a subset $S \subseteq X$ is open if and only if $f^{-1}(S)$ is open for every continuous map $f\colon Z \to X$ with $Z \in \mathcal{I}$. Let $\mathcal{T}op_{\mathcal{I}}$ be the full subcategory of $\mathcal{I}$-generated spaces.*

Proposition 1.3. *Any object of $\mathcal{I}$ is $\mathcal{I}$-generated. Any colimit of $\mathcal{I}$-generated spaces is again $\mathcal{I}$-generated.*

Let $(\mathcal{I} \downarrow X)$ be the overcategory, and let $(\mathcal{I} \downarrow X) \to \mathcal{T}op$ be the canonical diagram sending $[f\colon Z \to X]$ to Z. The colimit of this diagram will be denoted $\operatorname{colim}_{Z \to X} Z$, or by $k_{\mathcal{I}}(X)$. Note that this is an $\mathcal{I}$-generated space by the above lemma, and there is a canonical map $k_{\mathcal{I}}(X) \to X$.

Remark 1.4. If you are worried about $(\mathcal{I} \downarrow X)$ being a 'big' indexing category for the colimit, just go ahead and assume $\mathcal{I}$ is small. This rules out $\mathcal{I}$ being the subcategory of compact Hausdorff spaces, but it's not the end of the world.

Proposition 1.5. *(a) $k_{\mathcal{I}}(X) \to X$ is a set-theoretic bijection.*
(b) X is $\mathcal{I}$-generated if and only if $k_{\mathcal{I}}(X) \to X$ is a homeomorphism.
(c) A space is $\mathcal{I}$-generated if and only if it is a colimit of some diagram whose objects belong to $\mathcal{I}$.
(d) The two obvious maps $k_I(k_I(X)) \to k_I(X)$ are identical.
(e) The functors $i\colon \mathcal{T}op_{\mathcal{I}} \rightleftarrows \mathcal{T}op\colon k_{\mathcal{I}}$ are an adjoint pair (with i the left adjoint), where i is the inclusion.

Note that $\mathcal{T}op_{\mathcal{I}}$ is a co-complete category, and the colimits are the same as those in $\mathcal{T}op$. The adjoint functors show that $\mathcal{T}op_{\mathcal{I}}$ is also complete; limits are computed by first taking the limit in $\mathcal{T}op$ and then applying $k_I(-)$.

Example 1.6. Let Δ be the full subcategory of $\mathcal{T}op$ consisting of the topological simplices Δ^n.

(a) By Proposition 1.3, $\mathcal{T}op_{\Delta}$ contains the geometric realization of every simplicial set.

(b) Convince yourself that the closed topologist's sine curve does not belong to $\mathcal{T}op_\Delta$.

Example 1.7. Prove that if X is $\mathcal{I}$-generated and $\sim$ is an equivalence relation on X, then the quotient space $X/\sim$ is also $\mathcal{I}$-generated (the idea of the proof is more obvious if you assume $\mathcal{I}$ contains the one-point space). Prove that the Sierpinski space and the Hawaiian earings are both quotient spaces of Δ^1, hence Δ-generated.

Example 1.8. In $\mathcal{T}op$ the Sierpinski space is not small with respect to any ordinal λ—an argument is given in [Ho]. We will show that Hovey's argument does not work in $\mathcal{T}op_\Delta$, because the spaces involved will not be Δ-generated for large λ.

Let λ be an ordinal which is of greater cardinality than any lesser ordinal (that is to say, λ is a cardinal). Assume that λ is actually a regular cardinal (cf. [H, 10.1.11]). Let $\bar{\lambda} = \lambda \cup \{\lambda\}$, and make this into a topological space by giving it the order topology. To avoid confusion, let's actually denote the new point of $\bar{\lambda}$ by $[\lambda]$.

(a) Show that if $\{U_a\}$ is a set of neighborhoods of $[\lambda]$, indexed by a set of cardinality less than λ, then there is a neighborhood $[\lambda] \in V$ which is contained in all the U_a's.
(b) If $\lambda > 2^{\aleph_0}$ and $f\colon \Delta^n \to \bar{\lambda}$ is a continuous map, use (a) to show that $f^{-1}([\lambda])$ must be open. Observe that $f^{-1}([\lambda])$ obviously must be closed, and so it is either empty or all of Δ^n.
(c) Conclude that when $\lambda > 2^{\aleph_0}$, the space $\bar{\lambda}$ is not Δ-generated.

1.9. **Model structures.** Assume that the simplices Δ^n are all $\mathcal{I}$-generated (which will follow if $\Delta \subseteq \mathcal{I}$, for instance). Note that it then follows that the spaces $\partial\Delta^n$ and $\Lambda^{n,k}$ are also $\mathcal{I}$-generated. Also, the map $k_I(X) \to X$ has the right-lifting-property with repsect to the maps $\partial\Delta^n \to \Delta^n$.

The same proofs as in [Ho] for the cases of $\mathcal{T}op$ and k-spaces show that $\mathcal{T}op_\mathcal{I}$ has a model structure in which the weak equivalences and (Serre) fibrations have the usual definitions. The model structure is cofibrantly-generated, with the usual sets of generating cofibrations and acyclic cofibrations.

If $\mathcal{I} \subseteq \mathcal{J}$ are full subcategories of $\mathcal{T}op$, then $\mathcal{T}op_\mathcal{I} \subseteq \mathcal{T}op_\mathcal{J}$ (a colimit of things from $\mathcal{I}$ is *a fortiori* a colimit of things from $\mathcal{J}$). The functor $k_\mathcal{I}$ is a right adjoint to this inclusion. One checks easily that $k_\mathcal{I}$ preserves fibrations and trivial fibrations, so these are Quillen functors. The fact that $k_I(X) \to X$ is a trivial fibration (having the right-lifting-property with respect to the maps $\partial\Delta^n \to \Delta^n$) shows that they're in fact a Quillen equivalence—this is the same as the argument in [Ho] showing that all topological spaces and k-spaces give Quillen equivalent model categories.

1.10. **The locally presentable condition.** For background on locally presentable categories we refer to [AR]. Recall that a small, full subcategory $\mathcal{A} \subseteq \mathcal{K}$ is called **dense** if every object of $\mathcal{K}$ is its canonical colimit with respect to $\mathcal{A}$. By combining [AR, Remark following 1.23] and [AR, Th. 1.20] one finds that a co-complete category is locally presentable iff it has a small, dense subcategory in which every object is λ-presentable for some λ.

We have already seen that $\mathcal{T}op_\mathcal{I}$ is co-complete, and that $\mathcal{I}$ is dense in $\mathcal{T}op_\mathcal{I}$. If $\mathcal{I}$ is small, then to show $\mathcal{T}op_\mathcal{I}$ is locally presentable we must only show that every element of $\mathcal{I}$ is λ-presentable for a suitable λ. This is not so obvious, even in a concrete case like $\mathcal{I} = \Delta$.

Vopenka's Principle (cf. [AR, Section 6, Appendix A]) is essentially a set-theoretic axiom related to the existence of huge cardinals. If one accepts Vopenka's

Principle then everything we want follows for free: [AR, 6.14] immediately shows that $\mathcal{T}op_{\mathcal{I}}$ is locally presentable when $\mathcal{I}$ is small. Depending on one's taste, Vopenka's Principle may or may not be acceptable. The set-theorists seem assured that no harm can come from assuming it. On the other hand, it seems ridiculous to assume such an elaborate axiom in order to prove something fairly concrete about topological spaces. Jeff Smith has found a proof that $\mathcal{T}op_{\Delta}$ is locally presentable which doesn't use Vopenka's Principle. I don't know what this proof is. However, Vopenka's Principle at least tells me that such a proof should exist, and so I believe it.

Proposition 1.11 (Smith)**.** *The category $\mathcal{T}op_{\Delta}$ is locally presentable.*

Proof. ????? See [Sm]. □

1.12. **Function complexes and adjointness.** For $X, Y \in \mathcal{T}op_{\mathcal{I}}$ I will write $X \otimes Y$ for the product in $\mathcal{T}op_{\mathcal{I}}$, and $X \times Y$ for the usual Cartesian product in $\mathcal{T}op$. Recall that $X \otimes Y = k_I(X \times Y)$.

For $X, Y \in \mathcal{T}op$ let $F(X, Y)$ denote the function space with the compact-open topology. Let $F_{\mathcal{I}}(X, Y) = k_I(F(X, Y))$.

Recall that if Y is locally compact then the functors $(-) \times Y$ and $F(Y, -)$ are an adjoint pair $\mathcal{T}op \rightleftarrows \mathcal{T}op$. In particular, $(-) \times Y$ preserves colimits.

Consider the following hypothesis:

Axiom 1.13. Every object of $\mathcal{I}$ is locally compact, and the Cartesian product of any two elements of $\mathcal{I}$ is $\mathcal{I}$-generated.

Note that the axiom applies to the subcategory Δ.

Proposition 1.14. *Assume Axiom 1.13. If $X, Y \in \mathcal{T}op_{\mathcal{I}}$ and X is locally compact, then $X \times Y$ is $\mathcal{I}$-generated; so the natural map $X \otimes Y \to X \times Y$ is a homeomorphism.*

Proposition 1.15. *If Axiom 1.13 holds, then the functors $(-) \otimes Y$ and $F_{\mathcal{I}}(Y, -)$ are an adjoint pair $\mathcal{T}op_{\mathcal{I}} \rightleftarrows \mathcal{T}op_{\mathcal{I}}$, for every $Y \in \mathcal{T}op_{\mathcal{I}}$.*

Proof. See [V, Section 3]. □

1.16. **Miscellaneous other properties.**

Lemma 1.17. *Suppose that for every $Z \in \mathcal{I}$, every open subset of Z is $\mathcal{I}$-generated. Then every open subset of an $\mathcal{I}$-generated space is still $\mathcal{I}$-generated.*

Proposition 1.18. *(a) Every open subset of Δ^n is Δ-generated.*
(b) An open subset of a Δ-generated space is still Δ-generated.

Proof. It is enough to prove (a), so let U be an open set in Δ^n. By looking at very small neighborhoods of every point in U, one sees that U is the colimit of a diagram whose objects are all homeomorphic either to $\mathbb{R}^n$ or the half-space $\mathbb{H} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$. Both these spaces can obviously be constructed as colimits of Δ^n's, and so U is Δ-generated. □

Remark 1.19. A closed subspace of a Δ-generated space need not be Δ-generated. The closed topologist's sine curve is a closed subspace of Δ^2, for instance.

Assume $\mathcal{I}$ is small. Consider the 'singular functor' $S\colon \mathcal{T}op_{\mathcal{I}} \to Set^{\mathcal{I}^{op}}$—for every $X \in \mathcal{T}op_{\mathcal{I}}$, $S(X)$ is the functor $Z \mapsto \mathcal{T}op(Z, X)$.

Proposition 1.20. *S is a full and faithful embedding, with a left adjoint. It identifies $\mathcal{T}op_{\mathcal{J}}$ with a full, reflective subcategory of $\mathcal{S}et^{\mathcal{J}^{op}}$.*

Proof. This is just a translation of the facts that $\mathcal{J}$ is dense in $\mathcal{T}op_{\mathcal{J}}$, and $\mathcal{T}op_{\mathcal{J}}$ is co-complete. See [AR, 1.26,1.29]. □

References

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