VANISHING LINES IN CHROMATIC HOMOTOPY THEORY

ZHIPENG DUAN, GUCHUAN LI, AND XIAOLIN DANNY SHI

ABSTRACT. We show that at the prime 2, for any height h and $G \subset \mathbb{G}_h$ a finite subgroup of the Morava stabilizer group, the $RO(G)$-graded homotopy fixed point spectral sequence for E_h has a strong horizontal vanishing line of filtration $N_{h,G}$, a specific number depending on h and G. It is a consequence of the nilpotence theorem that such homotopy fixed point spectral sequences all admit strong horizontal vanishing lines at some finite filtration. Here, we establish specific bounds for them. Our bounds are sharp for all the known computations of E_h^{hG} when G is cyclic.

Our proof is by investigating the effect of the norm functor on the Hill–Hopkins–Ravenel differentials in the Tate spectral sequence. As a consequence, we will also show that the $RO(G)$-graded slice spectral sequence of $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$ has a horizontal vanishing line at the same filtration $N_{h,G}$. As an immediate application, we establish a bound for the orientation order $\Theta(h, G)$, the smallest number such that the $\Theta(h, G)$-fold direct sum of any real vector bundle is E_h^{hG}-orientable.

CONTENTS

1. INTRODUCTION

1.1. **Motivation and main theorem.** Chromatic homotopy theory originated with Quillen's remarkable observation of the relationship between the homotopy groups of the complex cobordism spectrum and the Lazard ring [Qui69]. Later on, the work of Miller–Ravenel–Wilson on periodic phenomena in the stable homotopy groups of spheres [MRW77] and Ravenel's nilpotence conjectures led to what is now called the chromatic point of view. It is a powerful tool that studies periodic phenomena in the stable homotopy category by analyzing the algebraic geometry of smooth one-parameter formal groups. The moduli stack of formal groups has a stratification by height, which corresponds to localization with respect to the Lubin–Tate theories E_h, $h \geq 0$. This stratification organizes the search for large scale phenomena in stable homotopy theory.

More precisely, the chromatic convergence theorem of Hopkins and Ravenel [Rav92] exhibits the p-local sphere spectrum $S^0_{(p)}$ as the homotopy inverse limit of the chromatic tower

$$\cdots \longrightarrow L_{E_h}S^0 \longrightarrow \cdots \longrightarrow L_{E_1}S^0 \longrightarrow L_{E_0}S^0.$$

Each stage of the tower, $L_{E_h}S^0$, is the Bousfield localization of the sphere spectrum with respect to the height-h Lubin–Tate theory E_h. These localizations can be inductively computed via the chromatic fracture square, which is the homotopy pullback square

$$\begin{tikzcd} L_{E_h}S^0 \arrow[r] \arrow[d] & L_{K(h)}S^0 \arrow[d] \\ L_{E_{h-1}}S^0 \arrow[r] & L_{E_{h-1}}L_{K(h)}S^0. \end{tikzcd}$$

Devinatz and Hopkins [DH04] proved that $L_{K(h)}S^0 \simeq E_h^{h\mathbb{G}_h}$, where $\mathbb{G}_h$ is the Morava stabilizer group. Here, Goerss–Hopkins–Miller showed that the continuous action of $\mathbb{G}_h$ on π_*E_h can be refined to a unique $\mathbb{E}_\infty$-action of $\mathbb{G}_h$ on E_h [Rez98, GH04]. Furthermore, the $K(h)$-local E_h-based Adams spectral sequence of $L_{K(h)}S^0$ can be identified with the $\mathbb{G}_h$-homotopy fixed point spectral sequence of E_h:

$$E_2^{s,t} = H_c^s(\mathbb{G}_h, \pi_t E_h) \Longrightarrow \pi_{t-s}L_{K(h)}S^0.$$

Henn [Hen07] proposed that the $K(h)$-local sphere $L_{K(h)}S^0$ can be built up from the spectra of the form E_h^{hG}, where G is a finite subgroup of $\mathbb{G}_h$. This has been explicitly realized at heights 1 and 2 [Hen07, GHMR05, Bea15, BG18].

From this point of view, the spectra E_h^{hG} are the building blocks of the p-local stable homotopy category. In particular, their homotopy groups $\pi_*E_h^{hG}$ detects important families of elements in the stable homotopy groups of spheres [HHR16, LSWX19, BMQ20]. The computation of these homotopy groups is a central topic in chromatic homotopy theory.

In this paper, We focus on the prime $p = 2$. Hewett classified all the finite subgroups of $\mathbb{G}_h$ [Hew95] (see also [Buj12]). If $h = 2^{m-1}n$ where n is odd, then when $m \neq 2$, the maximal finite 2-subgroups of $\mathbb{G}_h$ are isomorphic to C_{2^m}, the cyclic group of order 2^m; when $m = 2$, h is of the form $4k-2$, and the maximal finite 2-subgroups are isomorphic to Q_8, the quaternion group.

Due to the difficulty of describing the explicit action of the stabilizer group $\mathbb{G}_h$ on E_h, computations at the prime 2 has been limited to heights 1 and 2 until the recent computational breakthroughs of Hill–Hopkins–Ravenel [HHR16] (norms of Real bordism and the slice spectral sequence) and Hahn–Shi [HS20] (Real orientation). This has led to the computation of $E_h^{hC_2}$ for all heights [HS20], and $E_4^{hC_4}$ [HSWX18] at height 4.

For $h > 0$ and H a finite 2-subgroup of $\mathbb{G}_h$, let $N_{h,H}$ be the positive integer defined as follows:

(1) When $(h, H) = (2^{m-1}n, C_{2^m})$, $N_{h,H} = 2^{h+m} - 2^m + 1$.
(2) When $(h, H) = (4k-2, Q_8)$, $N_{h,H} = 2^{h+3} - 7$.

Based on the classification of the finite 2-subgroups of $\mathbb{G}_h$, this accounts for all possible pairs (h, H).

The main result of this paper is the following:

Theorem. *At any height h and $G \subset \mathbb{G}_h$ a finite subgroup, let H be a Sylow 2-subgroup of G. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of E_h.*

Recall that having a strong horizontal vanishing line of filtration $N_{h,H}$ means that the spectral sequence collapses after the $E_{N_{h,H}}$-page, and there are no elements of filtration greater than or equal to $N_{h,H}$ that survives to the E_∞-page.

Our motivation for proving the main theorem is as follows: classically, it is a consequence of the nilpotence theorem that the homotopy fixed point spectral sequences of the Lubin–Tate theories all admit strong horizontal vanishing lines at *some* finite filtration (see [DH04, Section 5] and [BGH17, Section 2.3]). While theoretically useful, this fact does not help very much when doing concrete computations. Since we don't know *where* exactly the vanishing line occurs, we can't use its mere existence to prove any differentials.

The recent computations of Hill–Shi–Wang–Xu demonstrate that having a precise strong horizontal vanishing line is extremely useful for equivariant computations of Lubin–Tate theories. In [HSWX18], the authors first analyzed the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1\rangle$ (a connective model of E_2 equipped with a C_4-action). They proved that there is a horizontal vanishing line of filtration 16, and every class on or above this line must die on or before the E_{13}-page [HSWX18, Theorem 3.17]. Using this fact, they gave a much shorter proof of all the slice differentials in [HHR17].

For the next case, $BP^{(\!(C_4)\!)}\langle 2\rangle$ (a connective model of E_4 equipped with a C_4-action), a similar situation occurred: there is a horizontal vanishing line of filtration 96, and every class on or above this line must vanish on or before the E_{61}-page. This theorem is called the Vanishing Theorem [HSWX18, Theorem 9.2], and it is the key to producing all of the higher slice differentials.

The strong vanishing lines proven in our main theorem will render future computations of Lubin–Tate theories and norms of Real bordism theories much more tractable.

1.2. **Main results.** We will now give a more detailed summary of our results and describe the contents of this paper.

In Section 2, we recall some basic facts of our spectral sequences of interest. More specifically, the classical Tate diagram induces a Tate diagram of spectral sequences (see Section 2 for the definitions of these spectral sequences)

$$\begin{tikzcd}
\mathrm{HOSS(X)} \arrow[r] \arrow[d, "="] & \mathrm{SliceSS}(X) \arrow[r] \arrow[d] & \mathrm{LSliceSS}(X) \arrow[d] \\
\mathrm{HOSS}(X) \arrow[r] & \mathrm{HFPSS}(X) \arrow[r] & \mathrm{TateSS}(X).
\end{tikzcd}$$

The interaction between these spectral sequences will be crucial for proving our main theorem.

We will also recall the spectrum $BP^{(\!(G)\!)}$, its slice filtration in the case when G is a cyclic group of order 2^m, and some special classes on the E_2-page of its slice spectral sequence. We prove all the differentials in the C_2-slice spectral sequence of $i^*_{C_2}BP^{(\!(G)\!)}$ when $G = C_{2^m}$ and Q_8 (Theorem 2.3). While not stated elsewhere, this is a straightforward consequence of [HHR16, Theorem 9.9].

In Section 3, we compare the slice spectral sequence, the homotopy fixed point spectral sequence, and the Tate spectral sequence by analyzing the maps

$$\mathrm{SliceSS}(X) \longrightarrow \mathrm{HFPSS}(X) \longrightarrow \mathrm{TateSS}(X)$$

extracted from the Tate diagram of spectral sequences above. Works of Ullman [Ull13] and Böckstedt–Madsen [BM94] show that as maps between integer-graded spectral sequences, both maps induce isomorphisms in a certain range. For our purposes, we extend their isomorphism regions to the $RO(G)$-graded pages.

Theorem (Definition 3.1 and Theorem 3.3)**.** *For $V \in RO(G)$, let*

$$\tau(V) := \min_{\{e\} \subsetneq H \subset G} |H| \cdot \dim V^H.$$

The map from the slice spectral sequence to the homotopy fixed point spectral sequence induces an isomorphism on the E_2-page in the region defined by the inequality

$$\tau(V - s - 1) > |V|.$$

Furthermore, this map induces a one-to-one correspondence between the differentials in this isomorphism region.

The proof of Theorem 3.3 is by application of the main result in Hill–Yarnall [HY18] about the relationship between the slice connectivity of an equivariant spectrum and the connectivity of its geometric fixed points.

The classical analysis about the map from the homotopy fixed point spectral sequence to the Tate spectral sequence almost generalizes immediately to give an $RO(G)$-graded isomorphism region.

Theorem (Theorem 3.6)**.** *The map from the $RO(G)$-graded homotopy fixed point spectral sequence to the $RO(G)$-graded Tate spectral sequence induces an isomorphism on the E_2-page for classes in filtrations $s > 0$, and a surjection for classes in filtration $s = 0$. Furthermore, there is a one-to-one correspondence between differentials whose source is of nonnegative filtration.*

In Section 4, we give a brief summary of the norm structure in equivariant spectral sequences. The presence of the norm structure in our spectral sequences allows us to deduce the fate of certain classes in the G-equivariant spectral sequence from information in the H-equivariant spectral sequence, where $H \subset G$ is a subgroup (Proposition 4.1).

In Section 5, we analyze the Tate spectral sequence of E_h.

Theorem (Theorem 5.1)**.** *For any height h and $G \subset \mathbb{G}_h$ a finite subgroup, let H be a Sylow 2-subgroup of G. All the classes in the $RO(G)$-graded Tate spectral sequence of E_h vanish after the $E_{N_{h,H}}$-page.*

At any prime p, Mathew and Meier have shown that the map $E_h^{hG} \to E_h$ is a faithful G-Galois extension whenever $G \subset \mathbb{G}_h$ is a finite subgroup [MM15, Example 6.2]. This implies that the Tate spectrum E_h^{tG} is contractible [Rog08, Proposition 6.3.3]. As a consequence, all the classes in the Tate spectral sequence of E_h must eventually vanish. Theorem 5.1 provides a concrete bound for the vanishing page number when $p = 2$.

The proof of Theorem 5.1 requires us to analyze the G-equivariant orientation from $BP^{(\!(G)\!)}$ to E_h, as given by [HS20]. This orientation map factors through $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$:

$$\begin{tikzcd}
BP^{(\!(G)\!)} \arrow[r] \arrow[d] & E_h \\
(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)} \arrow[ur] &
\end{tikzcd}$$

This induces a map of the corresponding Tate spectral sequences:

$$G\text{-}\operatorname{TateSS}((N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}) \longrightarrow G\text{-}\operatorname{TateSS}(E_h).$$

Equipped with the results discussed in the previous sections, we first transport the differentials from the C_2-slice spectral sequence of $i_{C_2}^*(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$ to the C_2-Tate spectral sequence of $i_{C_2}^*(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$ by using the one-to-one correspondences proven in Section 3. Then, we use the norm structure to deduce that the unit class in the $RO(G)$-graded Tate spectral sequence of $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$ must die on or before the $E_{N_{h,H}}$-page. Naturality and multiplicative structure would then imply that the unit class in the $RO(G)$-graded Tate spectral sequence of E_h must also die on or before the $E_{N_{h,H}}$-page, and hence all the other classes must vanish after this page as well.

Our proof of the Tate vanishing for E_h works for any $(N_{C_2}^G \bar{v}_h)^{-1}BP^{(\!(G)\!)}$-module to give the following vanishing theorem.

Theorem (see Remark 5.4)**.** *Let M be a $(N_{C_2}^G \bar{v}_h)^{-1}BP^{(\!(G)\!)}$-module, and H a Sylow 2-subgroup of G. All the classes in the $RO(G)$-graded Tate spectral sequence of M vanish after the $E_{N_{h,H}}$-page.*

In Section 6, we turn our attention to the homotopy fixed point spectral sequence of E_h and prove our main theorem.

Theorem (Theorem 6.1)**.** *At any height h and $G \subset \mathbb{G}_h$ a finite subgroup, let H be a Sylow 2-subgroup of G. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of E_h.*

The proof of Theorem 6.1 is by using the vanishing theorem (Theorem 5.1) in the Tate spectral sequence, combined with the comparison theorem (Theorem 3.6) between the homotopy fixed point spectral sequence and the Tate spectral sequence.

Similarly, the same strong horizontal vanishing line exists in the homotopy fixed point spectral sequence for $(N_{C_2}^G \bar{v}_h)^{-1}BP^{(\!(G)\!)}$.

Theorem (Corollary 6.3)**.** *For any $(N_{C_2}^G \bar{v}_h)^{-1}BP^{(\!(G)\!)}$-module M, there is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of M.*

Corollary (Corollary 6.4)**.** *For all $K(h)$-local finite spectra Z, the homotopy fixed point spectral sequence*

$$H^s(G, E_t Z) \Longrightarrow \pi_{t-s}(E^{hG} \wedge Z)$$

has a strong horizontal vanishing line at $N_{h,H}$.

In all the current known computations, our strong horizontal vanishing lines are sharp in the case when $H = C_{2^m}$. More precisely, When $H = C_2$, the strong horizontal vanishing line in the homotopy fixed point spectral sequence of $E_h^{hC_2}$ is at filtration exactly $2^{h+1} - 1$. When $H = C_4$, the strong horizontal vanishing lines in the homotopy fixed point spectral sequences of $E_2^{hC_4}$ and $E_4^{hC_4}$ are of filtrations exactly 13 and 61.

Conjecture. *The strong horizontal vanishing line in Theorem 6.1 is sharp when the Sylow 2-subgroup H is cyclic.*

On the other hand, Bauer's computation of tmf with the elliptic spectral sequence [Bau08] implies that in the Q_8-homotopy fixed point spectral sequence of E_2, there is a strong horizontal vanishing line of filtration 23. This is smaller than the bound given by our theorem, which is 25.

Question. *When $h = 4k - 2$, what is the sharpest bound for the strong horizontal vanishing line in the $RO(Q_8)$-homotopy fixed point spectral sequence of E_h?*

Question. *What is the sharpest bound for the strong horizontal vanishing line for the homotopy fixed point spectral sequence of $E_h^{h\mathbb{G}_h}$?*

In Section 7, we prove the existence of horizontal vanishing lines in the slice spectral sequence.

Theorem (Theorem 7.1)**.** *When $G = C_{2^m}$ or Q_8, the $RO(G)$-graded slice spectral sequence of any $(N_{C_2}^G \bar{v}_h)^{-1}BP^{(\!(G)\!)}$-module M admits a horizontal vanishing line of filtration $N_{h,G}$.*

In particular, Theorem 7.1 implies that there will be a horizontal vanishing line of filtration 121 in the C_8-slice spectral sequence of $\Omega_{\mathbb{O}}$, the detection spectrum of Hill–Hopkins–Ravenel that detects all the Kervaire invariant elements [HHR16].

It is interesting to note that when $G = Q_8$, even though there is no knowledge of the slice filtration of $BP^{(\!(Q_8)\!)}$ yet, Theorem 7.1 still applies to show that the slice spectral sequences of $(N_{C_2}^{Q_8} \bar{v}_h)^{-1}BP^{(\!(Q_8)\!)}$-modules all have strong horizontal vanishing lines of filtration N_{h,Q_8}.

Question. *What is the slice filtration of $BP^{(\!(Q_8)\!)}$?*

Nonequivariantly, Hovey and Sadofsky showed that after $L_{K(n)}$-localization, $v_n^{-1}BP$ splits as a wedge of suspensions of $v_n^{-1}BP\langle n\rangle$ [HS99, Theorem B]. Their argument generalizes to the C_2-equivariant setting to imply a splitting of $\bar{v}_n^{-1}BP_{\mathbb{R}}$ as a wedge of $(*\rho_2)$-suspensions of $\bar{v}_n^{-1}BP_{\mathbb{R}}\langle n\rangle$ [LLQ19].

When $G = C_{2^m}$, since the quotient $(N_{C_2}^G\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}\langle n\rangle$ is a $(N_{C_2}^G\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}$-module, Theorem 7.1 applies to show that there is a horizontal vanishing line of filtration $N_{h,C_{2^m}}$ in its slice spectral sequence. The fact that $(N_{C_2}^{C_{2^m}}\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}$ and $(N_{C_2}^{C_{2^m}}\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}\langle n\rangle$ have the same horizontal vanishing line suggests that a similar splitting should occur C_{2^m}-equivariantly.

Question. After $L_{K(h)}$-localization, does $(N_{C_2}^{C_{2^m}}\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}$ split as a wedge of $RO(C_{2^m})$-graded suspensions of $(N_{C_2}^{C_{2^m}}\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}\langle n\rangle$?

Without inverting $N_{C_2}^G\bar{v}_h$, there is no horizontal vanishing line in the $RO(C_{2^m})$-graded slice spectral sequence of the non-localized quotient $BP^{(\!(C_{2^m})\!)}\langle n\rangle$, as we have elements of arbitrarily high filtrations on the E_∞-page. For example, the tower a_σ^k is an infinite tower containing classes of arbitrarily high filtrations that survive to the E_∞-page. Interestingly, computations of $BP_{\mathbb{R}}\langle n\rangle$, $BP^{(\!(C_4)\!)}\langle 1\rangle$, and $BP^{(\!(C_4)\!)}\langle 2\rangle$ in [HK01, HHR17, HSWX18] suggest that horizontal vanishing lines still exist in the integer-graded slice spectral sequence of the non-localized quotients.

Conjecture. *There is a horizontal vanishing line of filtration $N_{h,C_{2^m}}$ in the integer-graded slice spectral sequence of $BP^{(\!(C_{2^m})\!)}\langle n\rangle$.*

Finally, in Section 8, we give an application of our main theorem to study E_h^{hG}-orientations of real vector bundles. For $h \geq 0$ and $G \subseteq \mathbb{G}_h$ a closed subgroup, let $\Theta(h,G)$ be the smallest number d such that the d-fold direct sum of any real vector bundle is E_h^{hG}-orientable. At the prime $p=2$ and when $G = C_2$, Kitchloo and Wilson [KW15] have studied $E_h^{hC_2}$-orientations. At all primes and when $G = C_p$, Bhattacharya and Chatham [BC21] have studied $E_{k(p-1)}^{hC_p}$-orientations.

We prove the following theorem, which uses the strong vanishing lines proven in Theorem 6.1 to give an upper bound for $\Theta(h,G)$ at the prime 2 for all heights $h \geq 0$ and $G \subset \mathbb{G}_h$ a finite group.

Theorem. *(Theorem 8.4) For any height $h \geq 0$ and $G \subset \mathbb{G}_h$ a finite subgroup, let $d = 2\cdot|G|\cdot|H|^{\frac{N_{h,H}-1}{2}}$, where H is a 2-Sylow subgroup of G. Then the d-fold direct sum of any real vector bundle is E_h^{hG}-orientable.*

1.3. **Acknowledgements.** The authors would like to thank Agnès Beaudry, Prasit Bhattacharya, Hood Chatham, Paul Goerss, Mike Hill, Tyler Lawson, Yunze Lu, Peter May, Zhouli Xu, Mincong Zeng, and Foling Zou for helpful conversations. We would like to especially thank Guozhen Wang for reading and commenting on an earlier draft of our preprint and answering our numerous questions.

2. Preliminaries

In this section, we will discuss the spectral sequences that are of interest to us. We will also collect certain facts about these spectral sequences that we will need in the later sections.

Let X be a G-spectrum, and let $P^\bullet X$ be the slice tower of X. The Tate diagram

$$\begin{array}{ccccc}
EG_+\wedge X & \longrightarrow & X & \longrightarrow & \tilde{E}G\wedge X \\
\downarrow\simeq & & \downarrow & & \downarrow \\
EG_+\wedge F(EG_+,X) & \longrightarrow & F(EG_+,X) & \longrightarrow & \tilde{E}G\wedge F(EG_+,X)
\end{array}$$

induces a diagram of towers:

$$\begin{tikzcd}
EG_+ \wedge P^\bullet X \arrow[r] \arrow[d, "\simeq"] & P^\bullet X \arrow[r] \arrow[d] & \tilde{E}G \wedge P^\bullet X \arrow[d] \\
EG_+ \wedge F(EG_+, P^\bullet X) \arrow[r] & F(EG_+, P^\bullet X) \arrow[r] & \tilde{E}G \wedge F(EG_+, P^\bullet X).
\end{tikzcd}$$

This diagram of towers further induces a Tate diagram of spectral sequences

$$\begin{tikzcd}
\mathrm{HOSS(X)} \arrow[r] \arrow[d, "="] & \mathrm{SliceSS}(X) \arrow[r] \arrow[d, "①"] & \mathrm{LSliceSS}(X) \arrow[d] \\
\mathrm{HOSS}(X) \arrow[r] & \mathrm{HFPSS}(X) \arrow[r, "②"] & \mathrm{TateSS}(X).
\end{tikzcd} \tag{2.1}$$

All the spectral sequences in (2.1) are $RO(G)$-graded spectral sequences. We pause to briefly discuss notations:

(1) The spectral sequence associated with the tower $EG_+ \wedge P^\bullet X$ is the *homotopy orbit spectral sequence* (HOSS) of X. It is a third and fourth quadrant spectral sequence, and it converges to $\underline{\pi}_\star EG_+ \wedge X$. In the integer-graded page at the G/G-level, the spectral sequence converges to $\pi_*^G EG_+ \wedge X = \pi_* X_{hG}$.
(2) The spectral sequence associated with the tower $P^\bullet X$ is the *slice spectral sequence* (SliceSS) of X. It is a first and third quadrant spectral sequence, and it converges to $\underline{\pi}_\star X$. In the integer graded page at the G/G-level, the spectral sequence converges to $\pi_*^G X = \pi_* X^G$.
(3) Following the treatment of a forthcoming paper by Meier–Shi–Zeng, the spectral sequence associated with the tower $\tilde{E}G \wedge P^\bullet X$ is called the *localized slice spectral sequence of* X and is denoted by $\mathrm{LSliceSS}(X)$. It converges to $\underline{\pi}_\star \tilde{E}G \wedge X$.
(4) The spectral sequence associated with the tower $F(EG_+, P^\bullet X)$ is the *homotopy fixed point spectral sequence* (HFPSS) of X. It is a first and second quadrant spectral sequence, and it converges to $\underline{\pi}_\star F(EG_+, X)$. In the integer-graded page at the G/G-level, the spectral sequence converges to $\pi_*^G F(EG_+, X) = \pi_* X^{hG}$.
(5) The spectral sequence associated with the tower $\tilde{E}G \wedge F(EG_+, P^\bullet X)$ is the *Tate spectral sequence* (TateSS) of X. It has classes in all four quadrants, and it converges to $\underline{\pi}_\star \tilde{E}G \wedge F(EG_+, X)$. In the integer-graded page at the G/G-level, the spectral sequence converges to $\pi_*^G \tilde{E}G \wedge F(EG_+, X) = \pi_* X^{tG}$.

Let ρ_2 denote the regular C_2-representation. In [BHSZ21], it is shown that there are generators

$$\bar{t}_i \in \pi^{C_2}_{(2^i-1)\rho_2} BP^{(\!(C_{2^m})\!)}$$

such that

$$\pi^{C_2}_{*\rho_2} BP^{(\!(C_{2^m})\!)} \cong \mathbb{Z}_{(2)}[C_m \cdot \bar{t}_1, C_m \cdot \bar{t}_2, \ldots,].$$

For a precise definitions of these generators, see formula (1.3) in [BHSZ21] (also see [HHR16, Section 5] for analogous generators in $\pi^{C_2}_{*\rho_2} MU^{(\!(C_{2^m})\!)}$). In particular, for $BP_{\mathbb{R}}$, the generators $\bar{t}_i$ can be taken to be the $\bar{v}_i$'s, which are the C_2-equivariant lifts of the classical v_i-generators in $\pi_{2*} BP$.

Similar to the treatment of $MU^{(\!(C_{2^m})\!)}$ in [HHR16], we can build an equivariant refinement

$$S^0[C_{2^m} \cdot \bar{t}_1, C_{2^m} \cdot \bar{t}_2, \ldots] \longrightarrow BP^{(\!(C_{2^m})\!)}$$

from which we can apply the Slice Theorem [HHR16, Theorem 6.1] to show that the slice associated graded of $BP^{(\!(C_{2^m})\!)}$ is the graded spectrum

$$H\underline{\mathbb{Z}}[C_{2^m} \cdot \bar{t}_1, C_{2^m} \cdot \bar{t}_2, \ldots].$$

Here, the degree of a summand corresponding to a monomial in the $\bar{t}_i$-generators and their conjugates is the underlying degree.

As a consequence, the slice spectral sequence for the $RO(C_{2^m})$-graded homotopy groups of $BP^{(\!(C_{2^m})\!)}$ has E_2-term the $RO(C_{2^m})$-graded homotopy of $H\underline{\mathbb{Z}}[C_{2^m} \cdot \bar{t}_1, C_{2^m} \cdot \bar{t}_2, \ldots]$. To compute this, note that $S^0[C_{2^m} \cdot \bar{t}_1, C_{2^m} \cdot \bar{t}_2, \ldots]$ can be decomposed into a wedge sum of slice cells of the form

$$C_{2^m+} \wedge_{H_p} S^{\frac{|p|}{|H_p|}\rho_{H_p}},$$

where p ranges over a set of representatives for the orbits of monomials in the $\gamma^j \bar{t}_i$-generators, and $H_p \subset C_{2^m}$ is the stabilizer of p (mod 2). Therefore, it suffices to compute the equivariant homology groups of the representations spheres $S^{\frac{|p|}{|H_p|}\rho_{H_p}}$ with coefficients in the constant Mackey functor $\underline{\mathbb{Z}}$.

We recall some distinguished elements in the $RO(G)$-graded homotopy groups that we will need in order to name the relevant classes on the E_2-page of the slice spectral sequence (see [HHR16, Section 3.4] and [HSWX18, Section 2.2]).

Definition 2.1. Let V be a G-representation. We will use $a_V : S^0 \to S^V$ to denote its *Euler class*. This is an element in $\pi^G_{-V} S^0$. We will also denote its Hurewciz image in $\pi^G_{-V} H\underline{\mathbb{Z}}$ by a_V.

If the representation V has nontrivial fixed points (i.e. $V^G \neq \{0\}$), then $a_V = 0$. Moreover, for any two G-representations V and W, we have the relation $a_{V \oplus W} = a_V a_W$ in $\pi^G_{-V-W}(S^0)$.

Definition 2.2. Let V be an orientable G-representation. Then a choice of orientation for V gives an isomorphism $H^G_{|V|}(S^V; \underline{\mathbb{Z}}) \cong \mathbb{Z}$. In particular, the restriction map

$$H^G_{|V|}(S^V, \underline{\mathbb{Z}}) \longrightarrow H_{|V|}(S^{|V|}, \mathbb{Z})$$

is an isomorphism. Let $u_V \in H^G_{|V|}(S^V; \underline{\mathbb{Z}})$ be the generator that maps to 1 under this restriction isomorphism. The class u_V is called the *orientation class* of V.

The orientation class u_V is stable in V. More precisely, if 1 is the trivial representation, then $u_{V \oplus 1} = u_V$. Moreover, if V and W are two orientable G-representations, then $V \oplus W$ is also orientable, and $u_{V \oplus W} = u_V u_W$.

The Euler class a_V and the orientation class u_V behave well with respect to the Hill–Hopkins–Ravenel norm functor. More precisely, for $H \subset G$ a subgroup and V a H-representation, we have the equalities

$$N^G_H(a_V) = a_{\operatorname{Ind} V} \tag{2.2}$$

$$u_{\operatorname{Ind}|V|} N^G_H(u_V) = u_{\operatorname{Ind} V} \tag{2.3}$$

where $\operatorname{Ind} V = \operatorname{Ind}^G_H V$ is the induced representation.

When $G = C_{2^m}$, let λ_i, $1 \leq i \leq m$ denote the 2-dimensional real C_{2^m}-representation corresponding to rotation by $\left(\frac{2\pi}{2^i}\right)$. In particular, when $i = 1$, the representation λ_1 corresponds to rotation by π and thus equals to 2σ, where σ is the real sign representation of C_{2^m}. When localized at 2, the representations that will be relevant to us are 1, σ, λ_2, λ_3, …, λ_m.

When $G = Q_8$, $RO(Q_8) = \mathbb{Z}\{1, \sigma_i, \sigma_j, \sigma_k, \mathbb{H}\}$. The representations σ_i, σ_j, and σ_k are one-dimensional representations whose kernels are $\langle i \rangle$, $\langle j \rangle$, and $\langle k \rangle$, respectively. The representation $\mathbb{H}$ is a four-dimensional irreducible representation, obtained by the action of Q_8 on the quaternion algebra $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ by left multiplication.

For $h \geq 1$, let $\bar{v}_h \in \pi^{C_2}_{(2^h-1)\rho_2} BP^{(\!(G)\!)}$ denote the images of $\bar{v}_h$-generators under the map

$$BP_{\mathbb{R}} \longrightarrow i^*_{C_2} BP^{(\!(G)\!)},$$

which is inclusion into the first factor. The following theorem describes all the differentials in the slice spectral sequence of $i^*_{C_2} BP^{(\!(G)\!)}$.

Theorem 2.3. *Let $G = C_{2^m}$ or Q_8. In the C_2-slice spectral sequence of $i^*_{C_2} BP^{(\!(G)\!)}$, the differentials are generated under multiplicative structures by the differentials*

$$d_{2^{h+1}-1}(u^{2^{h-1}}_{2\sigma_2}) = \bar{v}_h a^{2^{h+1}-1}_{\sigma_2}, \;\; h \geq 1.$$

Proof. When $G = C_2$, the claim is immediate from the Slice Differential Theorem of Hill–Hopkins–Ravenel [HHR16, Theorem 9.9]. When G is larger than C_2, the unit map $BP_{\mathbb{R}} \to i^*_{C_2} BP^{(\!(G)\!)}$ induces a map

$$\text{SliceSS}(BP_{\mathbb{R}}) \longrightarrow \text{SliceSS}(i^*_{C_2} BP^{(\!(G)\!)}) \tag{2.4}$$

of slice spectral sequences. For degree reasons, the only possible differentials in $\text{SliceSS}(i^*_{C_2} BP^{(\!(G)\!)})$ are of lengths $2^{h+1}-1$, $h \geq 1$.

We will proceed by using induction on h. For the base case, when $h = 1$, we have the d_3-differential

$$d_3(u_{2\sigma_2}) = \bar{v}_1 a^3_{\sigma_2}$$

in $\text{SliceSS}(BP_{\mathbb{R}})$. Under the map (2.4), the source is mapped to $u_{2\sigma_2}$ and the target is mapped to $\bar{v}_1 a^3_{\sigma_2}$. By naturality, $\bar{v}_1 a^3_{\sigma_2}$ must be killed by a differential of length at most 3. Since the lowest possible differential length is 3 by degree reasons, the d_3-differential

$$d_3(u_{2\sigma_2}) = \bar{v}_1 a^3_{\sigma_2}$$

must occur in $\text{SliceSS}(i^*_{C_2} BP^{(\!(G)\!)})$. Multiplying this differential by permanent cycles determines the rest of the d_3-differentials. These are all the d_3-differentials because after these, there are no more room for other d_3-differentials.

Suppose now that the induction hypothesis holds for all $1 \leq k \leq h-1$. For degree reasons, after the d_{2^h-1}-differentials, the next possible differential is of length $d_{2^{h+1}-1}$. In $\text{SliceSS}(BP_{\mathbb{R}})$, consider the differential

$$d_{2^{h+1}-1}(u^{2^{h-1}}_{2\sigma_2}) = \bar{v}_h a^{2^{h+1}-1}_{\sigma_2}.$$

The map (2.4) sends both the source and the target of this differential to nonzero classes of the same name in $\text{SliceSS}(i^*_{C_2} BP^{(\!(G)\!)})$. By naturality, the image of the target, $\bar{v}_h a^{2^{h+1}-1}_{\sigma_2}$, must be killed by a differential of length at most $2^{h+1}-1$. By degree reasons, it is impossible for this class to be killed by a differential of length smaller than $2^{h+1}-1$. It follows that the differential

$$d_{2^{h+1}-1}(u^{2^{h-1}}_{2\sigma_2}) = \bar{v}_h a^{2^{h+1}-1}_{\sigma_2}$$

exists in $\text{SliceSS}(i^*_{C_2} BP^{(\!(G)\!)})$. The rest of the $d_{2^{h+1}-1}$-differentials are determined by multiplying this differential with permanent cycles. After these differentials, there is no room for other $d_{2^{h+1}-1}$-differentials by degree reasons. This completes the induction step. □

3. Comparison of spectral sequences

In [Ull13] and [BM94], it is shown that the maps ① and ② in (2.1) induce isomorphisms in a certain range in the integer-graded page. For our purposes, we will extend their integral-graded isomorphism ranges to $RO(G)$-graded isomorphism ranges.

Definition 3.1. For $V \in RO(G)$, let

$$\tau(V) := \min_{\{e\} \subsetneq H \subset G} |H| \cdot \dim V^H.$$

Lemma 3.2. *For $V \in RO(G)$, the spectrum $S^V \wedge \tilde{E}G$ is of slice $\geq \tau(V)$.*

Proof. By [HY18, Theorem 2.5], $S^V \wedge \tilde{E}G$ is of slice $\geq n$ if and only if the geometric fixed points $\Phi^H(S^V \wedge \tilde{E}G) \in \tau^{Post}_{\geq n/|H|}$ for all $H \subset G$. For $\tilde{E}G$, its underlying space is contractible and its H-fixed points is S^0 whenever H is a nontrivial subgroup of G. Since $\Phi^H S^V = S^{V^H} \in \tau^{Post}_{\geq \dim V^H}$, $S^V \wedge \tilde{E}G$ is of slice $\geq \tau(V)$. □

Theorem 3.3. *The map from the $RO(G)$-graded slice spectral sequence to the $RO(G)$-graded homotopy fixed point spectral sequence*

$$\begin{array}{ccc} \pi^G_{V-s}P^{|V|}_{|V|}X & \longrightarrow & \pi^G_{V-s}F(EG_+, P^{|V|}_{|V|}X) \\ \Downarrow & & \Downarrow \\ \pi^G_{V-s}X & \longrightarrow & \pi^G_{V-s}F(EG_+, X) \end{array}$$

induces an isomorphism on the E_2-page in the region defined by the inequality

$$\tau(V-s-1) > |V|.$$

Furthermore, the map induces a one-to-one correspondence between the differentials in this isomorphism region.

Proof. Applying the functor $F(-, P^{|V|}_{|V|}X)$ to the cofiber sequence

$$EG_+ \longrightarrow S^0 \longrightarrow \tilde{E}G$$

produces the cofiber sequence

$$F(\tilde{E}G, P^{|V|}_{|V|}X) \longrightarrow P^{|V|}_{|V|}X \longrightarrow F(EG_+, P^{|V|}_{|V|}X).$$

The long exact sequence in homotopy groups imply that the map

$$\pi^G_{V-s}P^{|V|}_{|V|}X \longrightarrow \pi^G_{V-s}F(EG_+, P^{|V|}_{|V|}X)$$

is an isomorphism when both $\pi^G_{V-s}F(\tilde{E}G, P^{|V|}_{|V|}X)$ and $\pi^G_{V-s-1}F(\tilde{E}G, P^{|V|}_{|V|}X)$ are trivial. Since $\pi^G_\star F(\tilde{E}G, P^{|V|}_{|V|}X) = F(S^\star \wedge \tilde{E}G, P^{|V|}_{|V|}X)$ and $P^{|V|}_{|V|}X$ is a $|V|$-slice, it suffices to find pairs (V, s) such that $S^{V-s-1} \wedge \tilde{E}G$ is of slice greater than $|V|$. By Lemma 3.2, this is equivalent to (V, s) satisfying the inequality $\tau(V-s-1) > |V|$.

We will now use induction on r to show that the map of spectral sequences induces a one-to-one correspondence between all the d_r-differentials whose source and target are both in the isomorphism region. The base case of the induction, when $r = 1$, is trivial.

For the induction step, suppose that the map induces a one-to-one correspondence between all the $d_{r'}$-differentials in the isomorphism region for all $r' < r$. Let $d_r(x) = y$ be a d_r-differential in SliceSS(X) such that both x and y are in the isomorphism region. By naturality, y' (the image of y) must be killed by a differential of length at most r in HFPSS(X). If the length of this differential is r, then the source must be x' (the image of x) and we are done. If the length of this differential is smaller than r, then the induction hypothesis implies that the same differential must appear in SliceSS(X). This would mean that y is killed by a differential of length smaller than r, which is a contradiction. Therefore all the d_r-differentials in SliceSS(X) that are in the isomorphism region appear in HFPSS(X).

On the other hand, let $d_r(x') = y'$ be a d_r-differential in HFPSS(X) such that both x' and y' are in the isomorphism region. Let x be the pre-image of x'. By naturality, x must support a differential of length at most r. If this differential is of length exactly r, then naturality implies that the target must be y, the unique preimage of y'. If the length is smaller than r, then by the induction hypothesis, x' must support a differential of length smaller than r as well. This is a contradiction. Therefore all the d_r-differentials in HFPSS(X) that are in the isomorphism region appear in SliceSS(X). This completes the induction step. □

Remark 3.4. In the integer-graded page, let $V = t \in \mathbb{Z}$. Let $m(G)$ be the order of the smallest nontrivial subgroup of G. When $t - s \geq 1$, $\tau(t-s-1) = m(G)(t-s-1)$, and the isomorphism region in Theorem 3.3 is defined by the inequality

$$m(G)(t-s-1) > t.$$

This recovers Theorem I.9.4 in [Ull13].

Example 3.5. When $G = C_{2^m}$, all the nontrivial C_{2^m}-representations are rotations, and they have no H-fixed points when $H \subset C_{2^m}$ is a nontrivial subgroup. Therefore, if $V' \in RO(G)$ is an element consisting only of nontrivial representations, then $\tau(t + V' - s - 1) = 2(t-s-1)$ when $t - s > 1$.

In the $(* + V')$-graded page, the isomorphism region in Theorem 3.3 contains pairs $(t + V', s)$ that satisfies the inequality

$$2(t-s-1) > t + |V'|,$$

or equivalently

$$s < (t-s) - 2 - |V'|.$$

In particular, the inequality shows that in any of the $RO(C_{2^m})$-graded page, the isomorphism region is always bounded above by a line of slope 1.

Theorem 3.6. *The map from the $RO(G)$-graded homotopy fixed point spectral sequence to the $RO(G)$-graded Tate spectral sequence induces an isomorphism on the E_2-page for classes in filtrations $s > 0$, and a surjection for classes in filtration $s = 0$. Furthermore, there is a one-to-one correspondence between differentials whose source is of nonnegative filtration.*

Proof. The E_2-page of the Tate spectral sequence of X is

$$E_2^{s,V} = \hat{H}^s(G, \pi_0(S^{-V} \wedge X)) \Longrightarrow \pi_V^G \, \tilde{E}G \wedge F(EG_+, X),$$

and the E_2-page of the homotopy fixed point spectral sequence is

$$E_2^{s,V} = H^s(G, \pi_0(S^{-V} \wedge X)) \Longrightarrow \pi_V^G \, F(EG_+, X).$$

By the definition of Tate cohomology, $\hat{H}^s = H^s$ when $s > 0$. Furthermore, the map $H^0 \to \hat{H}^0$ is a surjection whose kernel is the image of the norm map. This proves the claim about the E_2-page. The proof for the one-to-one correspondence of differentials is exactly the same as the proof in Theorem 3.3. □

We end this section by discussing the invertibility of certain Euler classes in the Tate spectral sequence. Recall that if V is a G-representation such that the fixed point set V^H is empty whenever $H \subset G$ is nontrivial, then $S(\infty V)$ is a geometric model for EG, and $S^{\infty V}$ is a geometric model for $\tilde{E}G$. Therefore, for any G-spectrum X,

$$\tilde{E}G \wedge X \simeq S^{\infty V} \wedge X = a_V^{-1} X.$$

Specialized to the case when $G = C_{2^m}$ and Q_8, we see that $\tilde{E}C_{2^m} \simeq S^{\infty \lambda_m}$ and $\tilde{E}Q_8 \simeq S^{\infty \mathbb{H}}$. Moreover, if X is a G-spectrum, then the Tate spectral sequence of X is the spectral sequence associated to the tower $\tilde{E}G \wedge F(EG_+, P^\bullet X)$. This implies that the class a_{λ_m} is invertible in all the C_{2^m}-Tate spectral sequences, and the class $a_{\mathbb{H}}$ is invertible in all the Q_8-Tate spectral sequences.

4. The norm structure

In this section, we give a brief summary of results for the norm structure in equivariant spectral sequences. For more detailed discussions, see [Ull13, Chapter I.5], [HHR17, Section 4], and [MSZ20, Section 3.4].

Consider a tower

$$\cdots \longrightarrow P^{i+1} \longrightarrow P^i \longrightarrow P^{i-1} \longrightarrow \cdots$$

of G-spectra and let $E_*^{*,\star}$ be the associated spectral sequence. Set $P_n^m = \mathrm{fib}(P^m \to P^{n-1})$ and $P_n = P_n^\infty$. The towers that will be of relevant to us in this paper are the towers for the slice spectral sequence, the homotopy fixed point spectral sequence, and the Tate spectral sequence.

Let $H \subset G$ be a subgroup. Suppose we have maps $N_H^G P_n \to P_{|G/H|n}$ and $N_H^G P_n^n \to P_{|G/H|n}^{|G/H|n}$ that are (up to homotopy) compatible with the maps $P_n \to P_{n-1}$ and $P_n \to P_n^n$. This is called a *norm structure*. It induces norm maps

$$N_H^G : E_2^{s,V+s} \longrightarrow E_2^{|G/H|s, \mathrm{Ind}_H^G V + |G/H|s}.$$

If X is a commutative G-spectrum, then its slice spectral sequence, homotopy fixed point spectral sequence, and Tate spectral sequence all have the norm structure (for the Tate spectral sequence, there is a norm structure as long as $H \neq e$, as discussed in [MSZ20, Example 3.9]).

The following proposition ([MSZ20, Proposition 3.7]) is a restatement of [Ull13, Proposition I.5.17] and [HHR17, Theorem 4.7]. It describes the behaviour of differentials under the norm structure.

Proposition 4.1. *Let $x \in E_2(G/H)$ be an element representing zero in $E_{r+1}(G/H)$. Then $N_H^G(x)$ represents zero in $E_{|G/H|(r-1)+2}(G/G)$.*

In other words, Proposition 4.1 states that if $x \in E_2^{s,V+s}(G/H)$ is killed by a d_r-differential, then $N_H^G(x) \in E_2^{|G/H|s, \mathrm{Ind}_H^G V + |G/H|s}(G/G)$ must be killed by a differential of length at most $|G/H|(r-1)+1$.

Let σ_2 be the sign representation of C_2. As an immediate consequence of Equations (2.2) and (2.3), we have the following proposition.

Proposition 4.2. *The following equalities hold:*

$$\begin{aligned}
N_{C_2}^{C_{2^m}}(a_{\sigma_2}) &= a_{\lambda_m}^{2^{m-2}} \\
N_{C_2}^{C_{2^m}}(u_{2\sigma_2}) &= \frac{u_{\lambda_m}^{2^{m-1}}}{u_{2\sigma}\prod_{i=2}^{m-1} u_{\lambda_i}^{2^{i-1}}} \\
N_{C_2}^{Q_8}(a_{\sigma_2}) &= a_{\mathbb{H}} \\
N_{C_2}^{Q_8}(u_{2\sigma_2}) &= \frac{u_{\mathbb{H}}^2}{u_{2\sigma_i}u_{2\sigma_j}u_{2\sigma_k}}.
\end{aligned}$$

Proof. The equalities follow from (2.2), (2.3), and the following facts about induced representations:

$$\begin{aligned}
\mathrm{Ind}_{C_2}^{C_{2^m}}(1) &= 1 + \sigma + \sum_{i=2}^{m-1} 2^{i-2}\lambda_i \\
\mathrm{Ind}_{C_2}^{C_{2^m}}(\sigma_2) &= 2^{m-2}\lambda_m \\
\mathrm{Ind}_{C_2}^{Q_8}(1) &= 1 + \sigma_i + \sigma_j + \sigma_k \\
\mathrm{Ind}_{C_2}^{Q_8}(\sigma_2) &= \mathbb{H}.
\end{aligned}$$

$\square$

Theorem 4.3.

(1) *The class* $N_{C_2}^{C_{2^m}}(\bar{v}_h)a_{\lambda_m}^{2^{m-2}(2^{h+1}-1)}$ *in the* C_{2^m}*-slice spectral sequence of* $BP^{(\!(C_{2^m})\!)}$ *dies on or before the* $E_{2^{h+m}-2^m+1}$*-page.*

(2) *The class* $N_{C_2}^{Q_8}(\bar{v}_h)a_{\mathbb{H}}^{2^{h+1}-1}$ *in the* Q_8*-slice spectral sequence of* $BP^{(\!(Q_8)\!)}$ *dies on or before the* $E_{2^{h+3}-7}$*-page.*

Proof. By Theorem 2.3, we have the differential

$$d_{2^{h+1}-1}(u_{2\sigma_2}^{2^{h-1}}) = \bar{v}_h a_{\sigma_2}^{2^{h+1}-1}$$

in the C_2-slice spectral sequence of $i_{C_2}^* BP^{(\!(C_{2^m})\!)}$ and $i_{C_2}^* BP^{(\!(Q_8)\!)}$. Our claims follow by applying Proposition 4.1 and the equations in Proposition 4.2 to $(H, G, x, r) = (C_2, C_{2^m}, \bar{v}_h a_{\sigma_2}^{2^{h+1}-1}, 2^{h+1}-1)$ and $(C_2, Q_8, \bar{v}_h a_{\sigma_2}^{2^{h+1}-1}, 2^{h+1}-1)$. □

5. Vanishing in the Tate spectral sequence

By the work of Hahn and Shi [HS20], the Lubin–Tate theory E_h admits an equivariant orientation. More specifically, for $G \subset \mathbb{G}_h$ a finite subgroup, there is a G-equivariant map from $BP^{(\!(G)\!)}$ to E_h. Furthermore, this G-equivariant map factors through $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$:

$$\begin{tikzcd} BP^{(\!(G)\!)} \arrow[r] \arrow[d] & E_h \\ (N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)} \arrow[ur] & \end{tikzcd}$$

This equivariant orientation induces the following diagram of spectral sequences:

$$\begin{tikzcd} \mathrm{SliceSS}(BP^{(\!(G)\!)}) \arrow[r] & \mathrm{HFPSS}(BP^{(\!(G)\!)}) \arrow[r] \arrow[d] & \mathrm{TateSS}(BP^{(\!(G)\!)}) \arrow[d] \\ & \mathrm{HFPSS}(E_h) \arrow[r] & \mathrm{TateSS}(E_h). \end{tikzcd}$$

For $h > 0$ and H a finite 2-subgroup of $\mathbb{G}_h$, let $N_{h,H}$ be a positive integer defined as follows:

(1) When $(h, H) = (2^{m-1}n, C_{2^m})$, $N_{h,H} = 2^{h+m} - 2^m + 1$.

(2) When $(h, H) = (4k - 2, Q_8)$, $N_{h,H} = 2^{h+3} - 7$.

Based on the classification of 2-subgroups of $\mathbb{G}_h$, this accounts for all possible pairs (h, H).

Theorem 5.1. *For any height* h *and* $G \subset \mathbb{G}_h$ *a finite subgroup, let* H *be a Sylow 2-subgroup of* G*. All the classes in the* $RO(G)$*-graded Tate spectral sequence of* E_h *vanish after the* $E_{N_{h,H}}$*-page.*

In order to prove Theorem 5.1, we will first prove the following lemmas:

Lemma 5.2. *Let* G *be a finite group and* $H \subset G$ *a* p*-Sylow subgroup. For a* p*-local* G*-spectrum* X*, if the* $RO(H)$*-graded Tate spectral sequence of* X *vanish after the* E_r*-page, then the* $RO(G)$*-graded homotopy fixed point spectral sequence of* X *will also vanish after the* E_r*-page.*

Proof. The restriction and transfer maps induce the following maps of spectral sequences:

$$G\text{-}\mathrm{TateSS}(X) \xrightarrow{\mathrm{res}} H\text{-}\mathrm{TateSS}(X) \xrightarrow{\mathrm{tr}} G\text{-}\mathrm{TateSS}(X).$$

The composition map $\mathrm{tr} \circ \mathrm{res}$ is the degree-$|G/H|$ map. Since $|G/H|$ is coprime to p and X is p-local, the composition $\mathrm{tr} \circ \mathrm{res}$ is an isomorphism. This exhibits the $RO(G)$-grated Tate spectral sequence as a retract of the $RO(H)$-graded Tate spectral sequence.

Consider a nonzero class x on the E_2-page of the $RO(G)$-graded Tate spectral sequence of X. Its image, $y = \mathrm{res}(x)$, must be nonzero. This is because if $y = 0$, then $0 = \mathrm{tr}(y) = \mathrm{tr} \circ \mathrm{res}(x) = x$, which contradicts our assumption. Since y vanishes after the E_r-page, it either supports a differential or gets killed by a differential of length at most r. If y supports a $d_{r'}$-differential, where $r' \leq r$, then by naturality of the restriction map, x must also support a differential of length at most r'. On the other hand, if y is killed by a $d_{r'}$-differential, then by by naturality of the transfer map, x must also be killed by a differential of length at most r'. It follows that x must also vanish after the E_r-page. $\square$

Lemma 5.3.

(1) *At height $h = 2^{m-1}n$, the unit class in the $RO(C_{2^m})$-graded Tate spectral sequence of $(N_{C_2}^{C_{2^m}}\bar{v}_h)^{-1}BP^{(\!(C_{2^m})\!)}$ must be killed on or before the $(2^{h+m} - 2^m + 1)$-page.*

(2) *At height $h = 4k - 2$, the unit class in the $RO(Q_8)$-graded Tate spectral sequence of $(N_{C_2}^{Q_8}\bar{v}_h)^{-1}BP^{(\!(Q_8)\!)}$ must be killed on or before the $(2^{h+3} - 7)$-page.*

Proof. For $G = C_{2^m}$ and Q_8, consider the map from the C_2-slice spectral sequence of $i^*_{C_2}BP^{(\!(G)\!)}$ to the C_2-Tate spectral sequence of $i^*_{C_2}BP^{(\!(G)\!)}$. Theorem 2.3, combined with the isomorphisms in Theorem 3.3 and Theorem 3.6, shows that we have the differential

$$d_{2^{h+1}-1}(u_{2\sigma_2}^{2^{h-1}}) = \bar{v}_h a_{\sigma_2}^{2^{h+1}-1}$$

in the C_2-Tate spectral sequence of $i^*_{C_2}BP^{(\!(G)\!)}$. Since a_{σ_2} is invertible by our discussion in Section 3, after inverting $\bar{v}_h$, we have the differential

$$d_{2^{h+1}-1}(\bar{v}_h^{-1}u_{2\sigma_2}^{2^{h-1}}a_{\sigma_2}^{1-2^{h+1}}) = 1$$

in the C_2-Tate spectral sequence of $i^*_{C_2}(N_{C_2}^G v_h)^{-1}BP^{(\!(G)\!)}$. Our claims now follow by applying Proposition 4.1 to $(H, G, x, r) = (C_2, C_{2^m}, 1, 2^{h+1} - 1)$ and $(C_2, Q_8, 1, 2^{h+1} - 1)$. $\square$

Proof of Theorem 5.1. By Lemma 5.2, it suffices to prove the claim for $G = C_{2^m}$ and Q_8. The G-equivariant map

$$(N_{C_2}^G\bar{v}_h)^{-1}BP^{(\!(G)\!)} \longrightarrow E_h$$

induces a map of the corresponding Tate spectral sequences. By naturality and Lemma 5.3, the unit class in the $RO(C_{2^m})$-graded Tate spectral sequence is killed on or before the $(2^{h+m}-2^m+1)$-page, and the unit class in the Q_8-Tate spectral sequence is killed on or before the $(2^{h+3}-7)$-page. Our claim now follows from the fact that the Tate spectral sequence is multiplicative. $\square$

Remark 5.4. If M is a $(N_{C_2}^G\bar{v}_h)^{-1}BP^{(\!(G)\!)}$-module, its Tate spectral sequence will also be a module over the Tate spectral sequence of $(N_{C_2}^G\bar{v}_h)^{-1}BP^{(\!(G)\!)}$. The same proof as the one used in Theorem 5.1 will apply to show the same vanishing results in the Tate spectral sequence of M.

6. Horizontal vanishing lines in the homotopy fixed point spectral sequence

The vanishing of the Tate spectral sequence (Theorem 5.1) leads to the existence of strong horizontal vanishing lines in the homotopy fixed point spectral sequences of Lubin–Tate theories.

Theorem 6.1. *At any height h and $G \subset \mathbb{G}_h$ a finite subgroup, let H be a Sylow 2-subgroup of G. There is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of E_h.*

We first prove the following lemma, which reduces the general case to the case when G is a finite 2-subgroup of $\mathbb{G}_h$.

Lemma 6.2. *Let G be a finite group and $H \subset G$ a p-Sylow subgroup. For a p-local G-spectrum X, if the $RO(H)$-graded homotopy fixed point spectral sequence of X has a vanishing line L_H, then the $RO(G)$-graded homotopy fixed point spectral sequence of X will also have L_H as a vanishing line.*

Proof. The restriction and transfer maps induce the following maps of spectral sequences:

$$G\text{-}\,\mathrm{HFPSS}(X) \xrightarrow{\text{res}} H\text{-}\,\mathrm{HFPSS}(X) \xrightarrow{\text{tr}} G\text{-}\,\mathrm{HFPSS}(X).$$

The composition map $\mathrm{tr} \circ \mathrm{res}$ is the degree-$|G/H|$ map. Since $|G/H|$ is coprime to p and X is p-local, the composition $\mathrm{tr} \circ \mathrm{res}$ is an isomorphism. The $RO(G)$-graded homotopy fixed point spectral sequence is a retract of the $RO(H)$-graded homotopy fixed point spectral sequence. It follows that the vanishing line in the middle spectral sequence, H-HFPSS(X), will force the same vanishing line in G-HFPSS(X). $\square$

Proof of Theorem 6.1. By Lemma 6.2, we just have to prove the claim for all (h, H). Consider the map

$$H\text{-}\,\mathrm{HFPSS}(E_h) \longrightarrow H\text{-}\,\mathrm{TateSS}(E_h).$$

By Theorem 3.6, this map induces an isomorphism of classes above filtration 0 and a one-to-one correspondence of differentials whose sources are in non-negative filtrations.

By Theorem 5.1, all the classes in the Tate spectral sequence vanish after the $N_{h,H}$-page. In particular, this implies that the longest differential is of length at most $N_{h,H}$, and any class of filtration at least $N_{h,H}$ must die from a differential whose source and target both have nonnegative filtrations. Combined with the isomorphism in Theorem 3.6, this implies that the homotopy fixed point spectral sequence collapses after the $N_{h,H}$-page, and there is a strong horizontal vanishing line of filtration $N_{h,H}$. $\square$

Corollary 6.3. *For any $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$-module M, there is a strong horizontal vanishing line of filtration $N_{h,H}$ in the $RO(G)$-graded homotopy fixed point spectral sequence of M.*

Proof. By Remark 5.4, the proof is the same as the proof of Theorem 6.1. $\square$

Corollary 6.4. *For all $K(h)$-local finite spectra Z, the homotopy fixed point spectral sequence*

$$H^s(G, E_t Z) \Longrightarrow \pi_{t-s}(E^{hG} \wedge Z)$$

has a strong horizontal vanishing line of filtration $N_{h,H}$.

Remark 6.5. The existence of concrete strong horizontal vanishing lines (as given by Theorem 6.1) is very useful for equivariant computations. In [HSWX18], a crucial observation is that there is a horizontal vanishing line of filtration 16 in the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1 \rangle$ (hence in HFPSS(E_2)) and there is a horizontal vanishing line of filtration 96 in the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2 \rangle$ (hence in HFPSS(E_4)). The authors of that paper called this the Vanishing Theorem (see [HSWX18, Theorem 3.17, Theorem 9.2]). Using the Vanishing Theorem, Hill–Shi–Wang–Xu gave a much easier recomputation of the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 1 \rangle$ (first computed in [HHR17]) and completely computed all the differentials in the slice spectral sequence of $BP^{(\!(C_4)\!)}\langle 2 \rangle$.

In a forthcoming paper, we will apply Theorem 6.1 to give an alternative computation of $E_2^{hG_{24}}$ (the 2-primary $K(2)$-local TMF) by using the equivariant structures present in the homotopy fixed point spectral sequence. Furthermore, we will also use Theorem 6.1 to prove new $RO(G)$-graded periodicities in the homotopy fixed point spectral sequences of E_h (for example, E_4 has a periodicity of $36 - 16\lambda - 4\sigma$).

Example 6.6. When $G = C_2$ and at all heights h, there is a $d_{2^{h+1}-1}$-differential in the C_2-homotopy fixed point spectral sequence of E_h, and there is a nonzero class $\bar{v}_h^2 a_\sigma^{2^{h+1}-2}$ in bidegree $(2^{h+1}-2, 2^{h+1}-2)$. Therefore the vanishing line in Theorem 6.1 is sharp for $E_h^{hC_2}$.

Example 6.7. The computations in [HHR17] implies that in the $RO(C_4)$-homotopy fixed point spectral sequence of E_2, there exists a d_{13}-differential

$$d_{13}(N_2^4(\bar{t}_1)^5 u_{4\lambda} u_{4\sigma} a_\lambda a_\sigma) = N_2^4(\bar{t}_1)^8 u_{8\sigma} a_{8\lambda}$$

(where we let $\lambda = \lambda_2$ and $N_2^4(-) = N_{C_2}^{C_4}(-)$ for convenience). Moreover, the class $N_2^4(\bar{t}_1)^{10} u_{4\lambda} u_{10\sigma} a_{6\lambda}$ in bidegree $(28, 12)$ (representing κ^2) that survives to the E_∞-page. Therefore, our vanishing line is sharp for $E_2^{hC_4}$.

Example 6.8. The computations in [HSWX18] implies that in the $RO(C_4)$-homotopy fixed point spectral sequence of E_4, there is a d_{61}-differential

$$d_{61}(N_2^4(\bar{t}_2)^{11} u_{16\lambda} u_{32\sigma} a_{17\lambda} a_\sigma) = N_2^4(\bar{t}_2)^{16} u_{48\sigma} a_{48\lambda}$$

Moreover, the class $N_2^4(\bar{t}_2)^{24} N_2^4(\bar{t}_1) u_{44\lambda} u_{74\sigma} a_{30\lambda}$ in bidegree $(236, 60)$ survives to the E_∞-page. Therefore, our vanishing line is sharp for $E_4^{hC_4}$.

Conjecture 6.9. *The vanishing line given by Theorem 6.1 is sharp when the Sylow 2-subgroup H is cyclic.*

Example 6.10. Consider the $RO(Q_8)$-homotopy fixed point spectral sequence of E_2. Theorem 6.1 implies that there is a strong horizontal vanishing line of filtration 25. However, the actual vanishing line is of filtration 23. More specifically, by Bauer's computation [Bau08], there is a d_{23}-differential

$$d_{23}(\eta\Delta^5) = \Delta^6 k^6$$

(the notations are adopted from [Bea17] where Δ and k are generators on E_2-page). This implies that in the Tate spectral sequence, there is a d_{23}-differential

$$d_{23}(\eta\Delta^{-1}k^{-6}) = 1.$$

By the same argument as the one given in the proof of Theorem 6.1 and Bauer's computation, the sharpest vanishing line in the homotopy fixed point spectral sequence is of filtration 23.

Question 6.11. *When $h = 4k-2$, what is the sharpest bound for the strong horizontal vanishing line in the $RO(Q_8)$-homotopy fixed point spectral sequence of E_h?*

7. Horizontal vanishing lines in the slice spectral sequence

We will now prove explicit horizontal vanishing lines for the slice spectral sequences of $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$-modules. Recall from Section 5 that when $(h, G) = (2^{m-1}n, C_{2^m})$, $N_{h,C_{2^m}} = 2^{h+m} - 2^m + 1$, and when $(h, G) = (4k-2, Q_8)$, $N_{h,Q_8} = 2^{h+3} - 7$.

Theorem 7.1. *When $G = C_{2^m}$ or Q_8, the $RO(G)$-graded slice spectral sequence of any $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$-module M admits a horizontal vanishing line of filtration $N_{h,G}$.*

Lemma 7.2. *When $G = C_{2^m}$ or Q_8, any $(N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$-module is cofree.*

Proof. By [HHR16, Corollary 10.6], we need to show that $\Phi^H (N_{C_2}^G \bar{v}_h)^{-1} BP^{(\!(G)\!)}$ is contractible for all non-trivial $H \subset G$. To do so, it suffices to check that $\Phi^H(N_{C_2}^G \bar{v}_h) = 0$ for all nontrivial $H \subset G$. Recall that $\bar{v}_h \in \pi_{(2^h-1)\rho_2}^{C_2} BP^{(\!(G)\!)}$ is defined to be the composition

$$S^{(2^h-1)\rho_2} \xrightarrow{\bar{v}_h} BP_{\mathbb{R}} \longrightarrow i_{C_2}^* BP^{(\!(G)\!)}.$$

The claim now follows from the fact that for the class $\bar{v}_h \in \pi_{(2^h-1)\rho_2}^{C_2} BP_{\mathbb{R}}$, $\Phi^{C_2}(\bar{v}_h) = 0$ and therefore

$$\Phi^H(N_{C_2}^H \bar{v}_h) = \Phi^{C_2}(\bar{v}_h) = 0$$

for all nontrivial $H \subset G$. $\square$

Proof of Theorem 7.1. Since the spectrum M is cofree by Lemma 7.2, both the slice spectral sequence and the homotopy fixed point spectral sequence converge to the same homotopy groups:

$$\begin{array}{ccc} \mathrm{SliceSS}(M) & \longrightarrow & \mathrm{HFPSS}(M) \\ \Downarrow & & \Downarrow \\ \pi_\star^G M & \xrightarrow{=} & \pi_\star^G F(EG_+, M). \end{array}$$

Consider a class x on the E_2-page of the slice spectral sequence. We claim that if the filtration of x is at least $N_{h,G}$, then x cannot survive to the E_∞-page. This is because if x survives to represent an element in $\pi_\star^G M$, then there must be a class y on the E_2-page of the homotopy fixed point spectral sequence that also survives to represent the same element in

$$\pi_\star^G F(EG_+, M) = \pi_\star^G M.$$

Moreover, the filtration of y must be at least the filtration of x, which is $\geq N_{h,G}$. This is a contradiction because by Corollary 6.3, there is a strong horizontal vanishing line of filtration $N_{h,G}$ in the homotopy fixed point spectral sequence. $\square$

8. E_h^{hG}-orientation of real vector bundles

For $h \geq 0$ and $G \subseteq \mathbb{G}_h$ a closed subgroup, let $\Theta(h, G)$ be the smallest number d such that the d-fold direct sum of any real vector bundle is E_h^{hG}-orientable (defined in Definition 8.1). At the prime $p = 2$ and when $G = C_2$, Kitchloo and Wilson [KW15] have studied $E_h^{hC_2}$-orientations. At all primes and when $G = C_p$, Bhattacharya and Chatham [BC21] have studied $E_{k(p-1)}^{hC_p}$-orientations.

In this section, we will use the strong vanishing lines proven in Theorem 6.1 to give an upper bound for $\Theta(h, G)$ at the prime 2 for all heights $h \geq 0$ and $G \subset \mathbb{G}_h$ a finite group.

Definition 8.1. Let E be a multiplicative cohomology theory with multiplication $\mu_E : E \wedge E \to E$, and ξ a virtual k-dimensional real vector bundle over a space X. Denote the Thom spectrum of ξ by $M\xi$. An *E-orientation* for ξ is a Thom class $u : M\xi \to \Sigma^k E$ such that for any map $f : Y \to X$, the pull-back $u_{f^*(\xi)} : Mf^*(\xi) \to M\xi \to \Sigma^k E$ induces an equivalence

$$F(\Sigma^k Y_+, E) \xrightarrow{\simeq} F(Mf^*(\xi), E), \tag{8.1}$$

where (8.1) is defined by sending a map $g : \Sigma^k Y_+ \to E$ to the composition

$$\begin{aligned} Mf^*(\xi) = S^0 \wedge Mf^*(\xi) &\xrightarrow{\iota_E \wedge \mathrm{id}} E \wedge Mf^*(\xi) \xrightarrow{\mathrm{id}\wedge\Delta} E \wedge Mf^*(\xi) \wedge Y_+ \xrightarrow{\mathrm{id}\wedge u_{f^*(\xi)}\wedge \mathrm{id}} E \wedge \Sigma^k E \wedge Y_+ \\ &\xrightarrow{\mu_E \wedge \mathrm{id}} E \wedge \Sigma^k Y_+ \xrightarrow{\mathrm{id}\wedge g} E \wedge E \xrightarrow{\mu_E} E. \end{aligned}$$

Here, $\Delta : Mf^*(\xi) \to Mf^*(\xi) \wedge Y_+$ is the Thom diagonal map.

Remark 8.2. If ξ is E-oriented, then the equivalence (8.1) induces a Thom isomorphism

$$E^{*-k}(Y_+) \xrightarrow{\cong} E^*(Mf^*(\xi))$$

for any map f. In particular, when f is the identity map, there is a Thom isomorphism

$$E^{*-k}(X_+) \xrightarrow{\cong} E^*(M\xi)\ .$$

Note that it follows immediately from Definition 8.1 that for any E-oriented bundle ξ, its pull back bundle $f^*(\xi)$ is also E-oriented. Our definition also recovers the classical definition of orientations. More precisely, if we take Y to be a point, then the Thom space of the pull back is S^k, and the restriction of the Thom class u under the map

$$E^k(Th(\xi)) \longrightarrow E^k(S^k)$$

is a unit.

For X a non-equivariant spectrum, we can treat it as a G-spectrum equipped with the trivial G-action. We have the equivalence

$$F(EG_+, F(X, E_h))^G \simeq F(X, F(EG_+, E_h))^G \simeq F(X, F(EG_+, E_h)^G) = F(X, E_h^{hG}).$$

This equivalence allows us to use the homotopy fixed point spectral sequence to compute $(E_h^{hG})^*(X)$. The E_2-page of this homotopy fixed point spectral sequence is

$$E_2^{s,t} = H^s(G; E_h^t X) \Longrightarrow (E_h^{hG})^{t+s}(X).$$

Let γ be the universal bundle on BO (it is of virtual dimension zero). Following Kitchloo–Wilson [KW15], we will denote the Thom spectrum of $n\gamma$ by $MO[n]$. Let $MO[0] = S^0$ and define $\Pi MO := \bigvee_{k\geq 0} MO[2k]$.

Lemma 8.3. *The homotopy fixed point spectral sequence for $(E_h^{hG})^*(\Pi MO)$ is a multiplicative spectral sequence whose multiplication on the E_2-page is commutative.*

Proof. By [HR20], in order to ensure that the homotopy fixed point spectral sequence has a multiplicative structure, it suffices to construct a G-equivariant map

$$F(\Pi MO, E_h) \wedge F(\Pi MO, E_h) \to F(\Pi MO, E_h).$$

Consider the following map

$$BO \xrightarrow{\Delta} BO \times BO \xrightarrow{[2i,2j]} BO.$$

It induces a map of the corresponding Thom spectra

$$MO[2i+2j] \longrightarrow MO[2i] \wedge MO[2j].$$

If we fix n and combine all such maps for the pairs (i, j) such that $2i + 2j = 2n$, we get a map

$$MO[2n] \longrightarrow \bigvee_{2i+2j=2n} MO[2i] \wedge MO[2j].$$

Taking the wedge sum of all such maps for all $n \geq 0$ gives the map

$$\Delta\colon \Pi MO = \bigvee_{n\geq 0} MO[2n] \longrightarrow \bigvee_{n\geq 0} \left(\bigvee_{2i+2j=2n} MO[2i] \wedge MO[2j] \right) = \Pi MO \wedge \Pi MO.$$

We can then define the G-equivariant multiplication on $F(\Pi MO, E_h)$ as the following composition:

$$\begin{aligned} F(\Pi MO, E_h) \wedge F(\Pi MO, E_h) \longrightarrow F(\Pi MO \wedge \Pi MO, E_h \wedge E_h) &\xrightarrow{(\mu_{E_h})_*} F(\Pi MO \wedge \Pi MO, E_h) \\ &\xrightarrow{\Delta^*} F(\Pi MO, E_h) \end{aligned}$$

To show that the multiplication on the E_2-page is commutative, note that since $k\gamma \otimes \mathbb{C} = 2k\gamma$ as real vector bundles, $2k\gamma$ is E_h-oriented. For each $k \geq 0$, we have a Thom isomorphism

$$E_h^*(MO[2k]) \cong E_h^*(BO_+) \cdot u_{2k}.$$

We will show that the E_2-page is equal to $E_h^*(BO_+)[u_2]$, equipped with the obvious multiplicative structure (which is commutative).

Consider the composition map

$$MO[2k] \longrightarrow \overbrace{MO[2] \wedge \cdots \wedge MO[2]}^{k} \xrightarrow{u_2 \wedge \cdots \wedge u_2} \overbrace{E_h \wedge \cdots \wedge E_h}^{k} \xrightarrow{\mu} E_h.$$

This composition is u_2^k by the definition of our multiplicative structure on $E_h^*(\Pi MO)$. We claim that u_2^k is a Thom class for $MO[2k]$.

Note that by iteratively applying adjunction and the Thom isomorphism, we have

$$\begin{aligned} F(MO[2] \wedge \cdots \wedge MO[2], E_h) &= F(MO[2] \wedge \cdots \wedge MO[2], F(MO[2], E_h)) \\ &= F(MO[2] \wedge \cdots \wedge MO[2], F(BO_+, E_h)) \\ &= F(MO[2] \wedge \cdots \wedge MO[2] \wedge BO_+, E_h) \\ &= \cdots \\ &= F(BO_+ \wedge \cdots \wedge BO_+, E_h), \end{aligned}$$

and this is given by the Thom class $u_2 \wedge \cdots \wedge u_2$. Pulling back this Thom class via the diagonal map $BO_+ \to BO_+ \wedge \cdots \wedge BO_+$ gives u_2^k, and it induces the Thom isomorphism

$$E_h^*(MO[2k]) \cong E_h^*(BO_+) \cdot u_2^k.$$

It follows that we can express the induced multiplication on E_2-page $H^*(G, E_h^*(\Pi MO))$ by using the multiplication on $\bigoplus_{k \geq 0} E^*(BO_+) \cdot u_2^k$ via the Thom isomorphisms. □

Theorem 8.4. *For any height $h \geq 0$ and $G \subset \mathbb{G}_h$ a finite subgroup, let $d = 2 \cdot |G| \cdot |H|^{\frac{N_{h,H}-1}{2}}$, where H is a 2-Sylow subgroup of G. Then the d-fold direct sum of any real vector bundle is E_h^{hG}-orientable.*

Proof. It suffices to show that for the universal bundle γ on BO, its d-fold direct sum $d\gamma$ is E_h^{hG}-orientable. Let $u_2 : MO[2] \to E_h$ be a Thom class for 2γ, as considered in Lemma 8.3. For any element $g \in G$, let $gu_2 : MO[2] \to E_h$ denote the composition

$$MO[2] \xrightarrow{u_2} E_h \xrightarrow{g} E_h .$$

Define the map $u_G : MO[2 \cdot |G|] \to E_h$ to be the composition

$$u_G : MO[2 \cdot |G|] \xrightarrow{\Delta} \overbrace{MO[2] \wedge \cdots \wedge MO[2]}^{|G|} \xrightarrow{g_1 u_2 \wedge \cdots \wedge g_{|G|} u_2} \overbrace{E_h \wedge \cdots \wedge E_h}^{|G|} \xrightarrow{\mu_{E_h}} E_h,$$

where g_1, g_2, …, $g_{|G|}$ are all the elements of the group G. It is clear from the definition that u_G is an element in $H^0(G, E_h^0(MO[2 \cdot |G|]))$. By a similar argument as the one used in the proof of Lemma 8.3, the class $u_G^k \in H^0(G, E_h^0(MO[2 \cdot |G| \cdot k]))$ is a Thom class for $MO[2 \cdot |G| \cdot k]$ in the sense that there is a Thom isomorphism

$$E_h^*(BO_+) \xrightarrow{\cdot u_G^k} E_h^*(MO[2 \cdot |G| \cdot k]) .$$

If the class u_G^k is a permanent cycle in the homotopy fixed point spectral sequence for $(E_h^{hG})^*(MO[2 \cdot |G| \cdot k])$, then by naturality, the map of spectral sequences

$$\begin{tikzcd} H^*(G, E_h^*(BO_+)) \arrow[r, "\cdot u_G^k"] \arrow[d, Rightarrow] & H^*(G, E_h^*(MO[2 \cdot |G| \cdot k])) \arrow[d, Rightarrow] \\ (E_h^{hG})^*(BO_+) \arrow[r] & (E_h^{hG})^*(MO[2 \cdot |G| \cdot k]) \end{tikzcd}$$

induces an isomorphism

$$(E_h^{hG})^*(BO_+)\cdot u_G^k \cong (E_h^{hG})^*(MO[2\cdot|G|\cdot k])$$

on the E_∞-page. Moreover, for any map $f : Y \to BO$, the pull back of the class u_G^k, $f^*(u_G^k) \in H^0(G, E_h^0(Mf^*(2\cdot|G|\cdot k\gamma)))$, is also going to be a permanent cycle by naturality:

$$\begin{array}{ccc} H^*(G, E_h^*(MO[2\cdot|G|\cdot k])) & \longrightarrow & H^*(G, E_h^*(Mf^*(2\cdot|G|\cdot k\gamma))) \\ \Downarrow & & \Downarrow \\ (E_h^{hG})^*(MO[2\cdot|G|\cdot k]) & \longrightarrow & (E_h^{hG})^*(Mf^*(2\cdot|G|\cdot k\gamma)). \end{array}$$

Therefore, it will induce a Thom isomorphism on the E_∞-page of the homotopy fixed point spectral sequence for $(E_h^{hG})^*(Mf^*(2\cdot|G|\cdot k\gamma))$ as well.

It remains to find a k such that u_G^k is a permanent cycle. Since there is a splitting map

$$E_h^*(MO[2\cdot|G|\cdot k]) \longrightarrow E_h^*(\Pi MO) \longrightarrow E_h^*(MO[2\cdot|G|\cdot k]),$$

the homotopy fixed point spectral sequence for $(E_h^{hG})^*(MO[2\cdot|G|\cdot k])$ is a retract of the homotopy fixed point spectral sequence for $(E_h^{hG})^*(\Pi MO)$. In particular, the class u_G^k is a permanent cycle in the homotopy fixed point spectral sequence for $(E_h^{hG})^*(MO[2\cdot|G|\cdot k])$ if and only if it is a permanent cycle in the homotopy fixed point spectral sequence for $(E_h^{hG})^*(\Pi MO)$.

By Lemma 8.3, the homotopy fixed point spectral sequence for $(E_h^{hG})^*(\Pi MO)$ has a multiplicative structure that is commutative on the E_2-page. Furthermore, all the classes on the E_2-page are $|H|$-torsion, and there can only be differentials of odd length by degree reasons. Since this spectral sequence is a module over the spectral sequence for $(E_h^{hG})^*(S^0)$, it has a strong horizontal vanishing line of filtration $N_{h,H}$. It follows that when $k = |H|^{\frac{N_{h,H}-1}{2}}$, the class u_G^k must be a permanent cycle. This finishes the proof of the theorem. $\square$

Remark 8.5. Theorem 8.4 implies that $\Theta(h, G) \le 2\cdot|G|\cdot|H|^{\frac{N_{h,H}-1}{2}}$. We would like to remark that our bound is by no means optimal, as we haven't done any explicit computations with the homotopy fixed point spectral sequence. In [KW15, Theroem 1.4], Kitchloo and Wilson proved via explicit computation of $(E_h^{hC_2})^*(BO(q))$ that the 2^{h+1}-fold direct sum of any real vector bundle is $E_h^{hC_2}$-orientable. Our bound in this case is $\Theta(h, C_2) \le 2^{2^h+1}$.

The only facts we used to obtain our bounds are that there is a strong horizontal vanishing line of filtration $N_{h,H}$, and that the E_2-page is H-torsion. With more computational knowledge of the homotopy fixed point spectral sequence of $(E_h^{hG})^*(\Pi MO)$, there will be more data to obtain a much better upper bound for $\Theta(h, G)$.

References

[Bau08] Tilman Bauer. Computation of the homotopy of the spectrum `tmf`. In *Groups, homotopy and configuration spaces*, volume 13 of *Geom. Topol. Monogr.*, pages 11–40. Geom. Topol. Publ., Coventry, 2008.

[BC21] Prasit Bhattacharya and Hood Chatham. On the EO-orientability of vector bundles. *arXiv: 2003.03795*, 2021.

[Bea15] Agnès Beaudry. The algebraic duality resolution at $p = 2$. *Algebr. Geom. Topol.*, 15(6):3653–3705, 2015.

[Bea17] Agnès Beaudry. Towards the homotopy of the $K(2)$-local Moore spectrum at $p = 2$. *Adv. Math.*, 306:722–788, 2017.

[BG18] Irina Bobkova and Paul G. Goerss. Topological resolutions in $K(2)$-local homotopy theory at the prime 2. *J. Topol.*, 11(4):918–957, 2018.

[BGH17] Agnès Beaudry, Paul G. Goerss, and Hans-Werner Henn. Chromatic splitting for the $K(2)$-local sphere at $p = 2$. *arXiv: 1712.08182*, 2017.

[BHSZ21] Agnès Beaudry, Michael A. Hill, XiaoLin Danny Shi, and Mingcong Zeng. Models of Lubin-Tate spectra via real bordism theory. *Adv. Math.*, 392:Paper No. 108020, 58, 2021.

[BM94] M. Bökstedt and I. Madsen. Topological cyclic homology of the integers. *Astérisque*, (226):7–8, 57–143, 1994. K-theory (Strasbourg, 1992).

[BMQ20] Mark Behrens, Mark Mahowald, and J.D. Quigley. The 2-primary Hurewicz image of tmf. *arXiv: 2011.08956*, 2020.

[Buj12] Cedric Bujard. Finite subgroups of extended morava stabilizer groups. *arXiv: 1206.1951*, 2012.

[DH04] Ethan S. Devinatz and Michael J. Hopkins. Homotopy fixed point spectra for closed subgroups of the Morava stabilizer groups. *Topology*, 43(1):1–47, 2004.

[GH04] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[GHMR05] P. Goerss, H.-W. Henn, M. Mahowald, and C. Rezk. A resolution of the $K(2)$-local sphere at the prime 3. *Ann. of Math. (2)*, 162(2):777–822, 2005.

[Hen07] Hans-Werner Henn. On finite resolutions of $K(n)$-local spheres. In *Elliptic cohomology*, volume 342 of *London Math. Soc. Lecture Note Ser.*, pages 122–169. Cambridge Univ. Press, Cambridge, 2007.

[Hew95] Thomas Hewett. Finite subgroups of division algebras over local fields. *J. Algebra*, 173(3):518–548, 1995.

[HHR16] M. A. Hill, M. J. Hopkins, and D. C. Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.

[HHR17] Michael A. Hill, Michael J. Hopkins, and Douglas C. Ravenel. The slice spectral sequence for the C_4 analog of real K-theory. *Forum Math.*, 29(2):383–447, 2017.

[HK01] Po Hu and Igor Kriz. Real-oriented homotopy theory and an analogue of the Adams-Novikov spectral sequence. *Topology*, 40(2):317 – 399, 2001.

[HR20] Alice Hedenlund and John Rognes. A multiplicative tate spectral sequence for compact lie group actions. *arXiv: 2008.09095*, 2020.

[HS99] Mark Hovey and Hal Sadofsky. Invertible spectra in the $E(n)$-local stable homotopy category. *J. London Math. Soc. (2)*, 60(1):284–302, 1999.

[HS20] Jeremy Hahn and XiaoLin Danny Shi. Real orientations of Lubin-Tate spectra. *Invent. Math.*, 221(3):731–776, 2020.

[HSWX18] Michael A. Hill, XiaoLin Danny Shi, Guozhen Wang, and Zhouli Xu. The slice spectral sequence of a C_4-equivariant height-4 lubin-tate theory. *arXiv: 1811.07960*, 2018.

[HY18] Michael A. Hill and Carolyn Yarnall. A new formulation of the equivariant slice filtration with applications to C_p-slices. *Proc. Amer. Math. Soc.*, 146(8):3605–3614, 2018.

[KW15] Nitu Kitchloo and W. Stephen Wilson. The $ER(n)$-cohomology of $BO(q)$ and real Johnson-Wilson orientations for vector bundles. *Bull. Lond. Math. Soc.*, 47(5):835–847, 2015.

[LLQ19] Guchuan Li, Vitaly Lorman, and JD Quigley. Tate blueshift and vanishing for real oriented cohomology. *arXiv: 1910.06191*, 2019.

[LSWX19] Guchuan Li, XiaoLin Danny Shi, Guozhen Wang, and Zhouli Xu. Hurewicz images of real bordism theory and real Johnson-Wilson theories. *Adv. Math.*, 342:67–115, 2019.

[MM15] Akhil Mathew and Lennart Meier. Affineness and chromatic homotopy theory. *J. Topol.*, 8(2):476–528, 2015.

[MRW77] Haynes R. Miller, Douglas C. Ravenel, and W. Stephen Wilson. Periodic phenomena in the Adams-Novikov spectral sequence. *Ann. of Math. (2)*, 106(3):469–516, 1977.

[MSZ20] Lennart Meier, XiaoLin Danny Shi, and Mingcong Zeng. The localized slice spectral sequence, norms of Real bordism, and the Segal conjecture. *arXiv e-prints*, page arXiv:2008.04963, August 2020.

[Qui69] Daniel Quillen. On the formal group laws of unoriented and complex cobordism theory. *Bull. Amer. Math. Soc.*, 75:1293–1298, 1969.

[Rav92] Douglas C. Ravenel. *Nilpotence and periodicity in stable homotopy theory*, volume 128 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[Rez98] Charles Rezk. Notes on the Hopkins-Miller theorem. In *Homotopy theory via algebraic geometry and group representations (Evanston, IL, 1997)*, volume 220 of *Contemp. Math.*, pages 313–366. Amer. Math. Soc., Providence, RI, 1998.

[Rog08] John Rognes. Galois extensions of structured ring spectra. Stably dualizable groups. *Mem. Amer. Math. Soc.*, 192(898):viii+137, 2008.

[Ull13] John Richard Ullman. *On the Regular Slice Spectral Sequence*. ProQuest LLC, Ann Arbor, MI, 2013. Thesis (Ph.D.)–Massachusetts Institute of Technology.

School of mathematical sciences, Nanjing Normal University
Email address: zhipeng@njnu.edu.cn

Department of mathematics, University of Michigan
Email address: guchuan@umich.edu

Department of Mathematics, University of Chicago
Email address: dannyshixl@gmail.com