COMPLEX ORIENTED COHOMOLOGY THEORIES AND THE LANGUAGE OF STACKS

COURSE NOTES FOR 18.917, TAUGHT BY MIKE HOPKINS

Contents

Introduction

This text contains the notes from a course taught at MIT in the spring of 1999, whose topics revolved around the use of stacks in studying complex oriented cohomology theories. The notes were compiled by the graduate students attending the class, and it should perhaps be acknowledged (with regret) that we recorded only the mathematics and not the frequent jokes and amusing sideshows which accompanied it. Please be wary of the fact that what you have in your hands is the 'alpha-version' of the text, which is only slightly more than our direct transcription of the stream-of-consciousness lectures. Much of what is here is somewhat incoherent, and some of it is actually wrong. A 'beta-version' may perhaps appear sometime in the future, but until then the notes should probably only be circulated via the topology underground.

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In brief, the main physical goal of the course was to present proofs of the Landweber Exact Functor Theorem and the Morava/Miller-Ravenel Change-of-Rings Theorem using the language of stacks. The lecturer often prophesized that in this context those theorems would seem 'almost obvious'—the reader can decide for himself whether he ultimately buys this. There was a secondary goal of the course, however, and that was to introduce students to the general yoga of complex oriented cohomology theories. There are many lectures devoted to this background machinery, much of which appears as an aside to the main discussion. The organization is at the moment somewhat convoluted, but there also several nice vignettes to be found here. Good luck...

1. Complex Oriented Cohomology Theories

A complex oriented cohomology theory is a generalized cohomology theory E which is multiplicative and has a choice of Thom class for every complex vector bundle. The latter statement means that if $\xi \to X$ is a complex vector bundle of dimension n then we are given a class $U = U_\xi \in \tilde{E}^{2n}(X^\xi)$ with the following properties:

(a) For each $x \in X$, the image of U_ξ under the composition
$$\tilde{E}^{2n}(X^\xi) \to \tilde{E}^{2n}(*^\xi) \to \tilde{E}^{2n}(S^{2n}) \xrightarrow{\cong} E^0(*)$$
is the canonical element 1.

(b) The classes U_ξ should be natural under pullbacks: if $f : Y \to X$ then $U_{f^*\xi} = f^*(U_\xi)$.

(c) Multiplicativity: $U_{\xi\oplus\eta} = U_\xi \cdot U_\eta$.

Remark 1.1. It may appear that we had to make some choices in writing down the maps appearing in part (a)—for instance, we had to choose an identification of S^{2n} with $*^\xi$. But the fact that ξ is a complex vector bundle gives a preferred orientation of $*^\xi$, and the induced map on cohomology $\tilde{E}^{2n}(*^\xi) \to \tilde{E}^{2n}(*)$ only depends on the way the orientations match up.

Example 1.2.

(a) Both singular cohomology $H^*(-;\mathbb{Z})$ and complex K-theory K^* are complex orientable.

(b) Real K-theory KO^* is *not* complex orientable. For example, if ξ is the canonical line bundle over $\mathbb{C}P^1$ then one can show that the map
$$\mathbb{Z} \cong \widetilde{KO}^2(X^\xi) \to \widetilde{KO}^2(S^2) \cong \mathbb{Z}$$
coincides with multiplication by 2. So 1 is not in the image.

If ξ is the tautological line bundle over $\mathbb{C}P^\infty$ then the zero section $\mathbb{C}P^\infty \to (\mathbb{C}P^\infty)^\xi$ turns out to be a homotopy equivalence. The Thom class $U_\xi \in \tilde{E}^2(\mathbb{C}P^\infty)$ then pulls back to a class usually called x (or x_E) in $\tilde{E}^2(\mathbb{C}P^\infty)$.

Proposition 1.3. *Any class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ restricting to 1 under the composite*
$$\tilde{E}^2(\mathbb{C}P^\infty) \to \tilde{E}^2(\mathbb{C}P^1) = \tilde{E}^2(S^1) \cong E^0(*)$$
extends in a unique way to a complex orientation of E.

We will return to the proof later.

A complex orientation on a cohomology theory E gives rise to a Thom isomorphism
$$\cdot U_\xi : E^*(X) \xrightarrow{\cong} \tilde{E}^{*+2n}(X^\xi).$$
It also gives rise to Chern classes $c_i(\xi) \in \tilde{E}^{2i}(X)$ satisfying

(i) Naturality under pullbacks;

(ii) $c_n(\xi \oplus \eta) = \sum_{i+j=n} c_i(\xi)c_j(\eta)$;

(iii) $c_1(\mathbb{L}) = x \in \tilde{E}^2(\mathbb{C}P^\infty)$ where $\mathbb{L}$ denotes the tautological line bundle over $\mathbb{C}P^\infty$.

Question: In singular cohomology one has $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$ for line bundles L_1 and L_2 over the same base X. What can we say about $c_1(L_1 \otimes L_2)$ for an arbitrary complex oriented cohomology theory?

Answer: It turns out that $c_1(L_1 \otimes L_2)$ can be written as $F(c_1(L_1), c_1(L_2)$ for some $F(x, y) \in E^*[[x, y]]$. If we write $x +_F y$ for $F(x, y)$, then this power series will have the following properties:

(i) $x +_F y = y +_F x$ (because $L_1 \otimes L_2 \cong L_2 \otimes L_1$);
(ii) $x +_F 0 = x = 0 +_F x$ (because $L \otimes 1 \cong L$, where 1 denotes the trivial line bundle);
(iii) $(x +_F y) +_F z = x +_F (y +_F z)$ (because tensor product of line bundles is associative).

Such an F is called a 'formal group law' over the ring E^*. As far as is known, any formal group law can occur as the $F(x, y)$ for some complex oriented cohomology theory. One of the main goals of this course will be to frame the conjectural relationships between formal group laws and stable homotopy theory.

Some basic computations.
Let E be a multiplicative cohomology theory and let $x \in \tilde{E}^2(\mathbb{C}P^\infty)$ be an element restricting to 1 (as in Proposition 1.3). This gives a map $E^*[x] \to E^*(\mathbb{C}P^n)$ (for each n). One can see that x^{n+1} must map to zero: First note that $\mathbb{C}P^n$ can be covered by $n + 1$ contractible open sets U_i, and because x is a *reduced* cohomology class it must restrict to zero on each U_i. As a general rule one knows that if $a \in E^*(X, A)$ and $b \in E^*(X, B)$ then $ab \in E^*(X, A \cup B)$. But we can write $x \in E^*(\mathbb{C}P^n, U_i)$ for each i, and so x^{n+1} lies in $E^*(\mathbb{C}P^n, U_1 \cup \ldots \cup U_{n+1}) = E^*(\mathbb{C}P^n, \mathbb{C}P^n) = 0$.

We therefore get a map

$$E^*[x]/(x^{n+1}) \longrightarrow E^*(\mathbb{C}P^n).$$

Lemma 1.4. *The above map is an isomorphism.*

Proof. Use the Atiyah-Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(\mathbb{C}P^n; E^q(*)) \Rightarrow E^{p+q}(\mathbb{C}P^n)$$

(which is multiplicative). The E_2 term is isomorphic to $E^*[x]/(x^{n+1})$, and both x and the elements of E^* all have to be permanent cycles—so there can be no differentials in the spectral sequence. The rest is left to the reader. □

We want to next calculate $E^*(\mathbb{C}P^\infty)$, and we can use the Milnor sequence:

$$0 \to \lim{}^1 E^{*-1}(\mathbb{C}P^n) \to E^*(\mathbb{C}P^\infty) \to \lim_n E^*(\mathbb{C}P^n) \to 0.$$

The right-hand-term is $\lim E^*[x]/(x^{n+1}) \cong E^*[[x]]$, and the left-hand-term is zero because the maps $E^*(\mathbb{C}P^{n+1}) \to E^*(\mathbb{C}P^n)$ are all surjective. This shows that $E^*(\mathbb{C}P^\infty) \cong E^*[[x]]$.

The same argument gives that $E^*(\mathbb{C}P^\infty \times \cdots \times \mathbb{C}P^\infty) \cong E^*[[x_1, \ldots, x_n]]$ where x_i is the pullback of x along the projection to the ith factor.

Chern Classes.
A line bundle $L \to X$ is classified by a map $f : X \to \mathbb{C}P^\infty$ (which is unique up to homotopy). Define

$$c_1(L) = f^*(x).$$

Let $\mathbb{L}$ denote the universal line bundle over $\mathbb{C}P^\infty$, and consider the map

$$g : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$$

which classifies the bundle $\mathbb{L} \otimes \mathbb{L}$. This induces a map on cohomology

$$E^*[[x]] \cong E^*(\mathbb{C}P^\infty) \longrightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \cong E^*[[x, y]],$$

so the image of x will be some power series $F(x,y) \in E^*[[x,y]]$. By considering the universal example above, it is easy to check that if L_1 and L_2 are two line bundles over a space X then

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

Note that the crucial point in this argument is knowledge of $E^*(\mathbb{C}P^\infty)$.

Let $\xi \to X$ be a complex vector bundle of dimension n and let $\mathbb{P}(\xi)$ denote the projective bundle of ξ—the fibre bundle over X whose fibre over x is the projective space of ξ_x. There is a 'tautological' line bundle $\mathbb{L}_\xi$ over $\mathbb{P}(\xi)$. Let $t \in \tilde{E}^2(\mathbb{P}(\xi))$ denote $c_1(\mathbb{L}_\xi)$. An argument using the Atiyah-Hirzebruch spectral sequence, similar to the one from the last section, proves that:

Proposition 1.5. *$E^*(\mathbb{P}(\xi))$ is free over $E^*(X)$ with basis $1, t, \dots, t^{n-1}$.*

We can now mimic the Grothendieck theory of Chern classes. There exist unique elements $c_i \in E^{2i}(X)$ such that

$$t^n = c_1 t^{n-1} - c_2 t^{n-2} + \cdots + (-1)^{n-1} c_n.$$

Definition 1.6. *The ith **Chern class** of ξ is defined to be the above class c_i.*

Exercise 1.7. Verify the properties of Chern classes listed above.

Here is some motivation behind this definition (which also serves to prove that the theory of Chern classes is unique). Let $p : \mathbb{P}(\xi) \to X$ denote the projection, and form the pullback

$$\begin{tikzcd} p^*\xi \arrow[r] \arrow[d] & \xi \arrow[d] \\ \mathbb{P}(\xi) \arrow[r] & X. \end{tikzcd}$$

It is easy to see that $p^*\xi = \mathbb{L}_\xi \oplus Q$ for some new vector bundle Q.

Write $c_s(\xi) = 1 + c_1(\xi)s + c_2(\xi)s^2 + \cdots + c_n(\xi)s^n$—this is called the total Chern class. By the Cartan formula, we must have

$$p^* c_s(\xi) = c_s(p^*\xi) = c_s(\mathbb{L}_\xi) \cdot c_s(Q) = (1 + st)c_s(Q).$$

In particular, we find that $s = -\frac{1}{t}$ is a root of $p^*(c_s(\xi))$, hence of $c_s(\xi)$. This gives the relation

$$1 + c_1(\xi) \cdot \left(-\frac{1}{t}\right) + \cdots + c_n(\xi) \cdot \left(-\frac{1}{t}\right)^n = 0.$$

Multiplying through by t^n gives the relation we used to define the Chern classes.

2. Formal Group Laws

When we talk about formal group laws we really only mean those which are commutative and 1-dimensional. The reader should also be aware of the distinction between 'formal groups' and 'formal group laws'. A formal group law is essentially a formal group with a choice of coordinate—we will discuss this more in the future. For now, we'll only talk about formal group *laws*.

Definition 2.1. *A **formal group law** over a ring R is a power series $F(x,y) \in R[[x,y]]$ satisfying the properties below (where we write $x +_F y$ instead of $F(x,y)$):*

(i) $x +_F y = y +_F x$;
(ii) $x +_F 0 = x = 0 +_F x$;
(iii) $(x +_F y) +_F z = x +_F (y +_F z)$.

Remark 2.2 (Change of Base)**.** If $f : R \to S$ is a ring map and F is a formal group law over R, then we can define a new formal group law f^*F over S in the following way: if $F(x,y) = \sum a_{ij}x^i y^j$ then we let $f^*F(x,y) = \sum f(a_{ij})x^i y^j$.

Theorem 2.3. *There is a universal formal group law: that is, there is a ring R and a formal group law F_{univ} over R such that the map*

$$\mathcal{R}ing(R,S) \to \{\textit{formal group laws over } S\}$$

which sends f to f^*F_{univ} is an isomorphism. (R and F_{univ} will be unique up to unique isomorphism).

Proof. A formal group law G over a ring is a power series $G(x,y) = \sum a_{ij}x^iy^j$ with some special properties. These properties can be expressed as formulas in the a_{ij}'s:

(i) Commutativity implies $a_{ij} = a_{ji}$;
(ii) The fact that 0 is an identity implies that $a_{i0} = 1$ if $i = 1$ and $a_{i0} = 0$ otherwise;
(iii) Associativity translates into something complicated which we won't write down.

So define $R = \mathbb{Z}[a_{ij}]/(\text{above relations})$, and let $F(x,y) = \sum a_{ij}x^iy^j$. It's easy to check that this gives the desired universal formal group law. □

Remark 2.4. If we let a_{ij} have degree $2(i+j)$ and extend multiplicatively, then R becomes a graded ring (because the relations amond the a_{ij}'s are homogeneous). There's a better 'explanation' of this grading which will be discussed later.

Write $R = \bigoplus_n R_{2n}$ where R_{2n} is the homogeneous part of R in degree $2n$ and note that

(i) R is connected: $R_{2n} = 0$ for $n < 0$ and $R_0 = \mathbb{Z}$;
(ii) Each R_{2n} is a finitely generated abelian group.

Some notation: let

$$C_n(x,y) = \frac{1}{d_n}\Big[(x+y)^n - x^n - y^n\Big] \text{ where } d_n = \begin{cases} p & \text{if } n = p^e \\ 1 & \text{otherwise.} \end{cases}$$

Theorem 2.5 (Lazard)**.** *Let $L = \mathbb{Z}[x_1, x_2, \dots]$ where $\deg x_i = 2i$. Then there is a formal group law F over L with*

$$F(x,y) \equiv \sum x_n \cdot C_{n+1}(x,y) \mod (x_1, x_2, \dots)^2$$

and the map $R \to L$ classifying F is an isomorphism of graded rings. (Here R is the ring of Theorem 2.3).

Corollary 2.6. *Suppose $f : S \to T$ is a surjective map of rings and let G be a formal group law over T. Then there is a formal group law G' over S such that $f^*G' = G$.*

The proof of Lazard's theorem is a little involved. First note that since R is connected and graded, we only have to study homogeneous formal group laws over connected, graded rings. Perhaps the simplest examples of such rings are obtained by starting with an abelian group A and an $n > 0$, and defining a ring structure on $\mathbb{Z} \oplus A$ where $ab = 0$ for $a, b \in A$, with the elements of A in degree $2n$. What are the formal group laws over such a ring?

If S is a connected, graded ring then let $I = I_S = \{s \in S \mid \deg(s) > 0\}$. I/I^2 is denoted QS and called the 'module of indecomposables'. Note that graded ring homomorphisms $S \to \mathbb{Z} \oplus A_{2n}$ are in one-to-one correspondence with abelian group maps $QS_{2n} \to A$. The case $S = R$ tells us that studying formal group laws over $\mathbb{Z} \oplus A_{2n}$ will tell us about QR_{2n}.

Now a formal group law over $\mathbb{Z} \oplus A_{2n}$ must look like $x + y + f(x,y)$ for some $f(x,y) \in A \otimes \mathbb{Z}[x,y]$ which is homogeneous of degree $n-1$. This $f(x,y)$ will have the following properties:

(i) $f(x,y) = f(y,x)$ (symmetry)
(ii) $f(x,0) = 0$
(iii) $f(y,z) - f(x,y+z) + f(x+y,z) - f(x,y) = 0$ (called the '2-cocycle condition').

Part (iii) is obtained by writing out the associativity formula for $x + y + f(x,y)$ and taking the homogeneous part in degree $n-1$.

Definition 2.7. *A **symmetric 2-cocycle** with values in A is an $f(x,y) \in A \otimes \mathbb{Z}[x,y]$ satisfying the above properties.*

Proposition 2.8 (Symmetric 2-cocycle lemma)**.** *Any symmetric 2-cocycle with values in A is a linear combination of those of the form $a \otimes C_n(x, y)$.*

Proof. Postponed. □

The above result tells us that any formal group law over the ring $\mathbb{Z} \oplus A_{2n}$ has the form

$$x + y + aC_n(x, y)$$

for some $a \in A$. Now take $A = \mathbb{Z}$ and $a = 1$: there is a map $R \to \mathbb{Z} \oplus \mathbb{Z}_{2n}$ classifying the formal group law $x + y + C_n(x, y)$. The symmetric 2-cocycle lemma shows that

Corollary 2.9. *There is a canonical isomorphism $QR_{2n} \to \mathbb{Z}$ induced by the map above.*

Now let $x_n \in R$ be any element whose image in QR_{2n} maps to 1 under the above isomorphism. The x's define a map $L = \mathbb{Z}[X_1, X_2, \cdots] \to R$.

Exercise 2.10.

(a) Show that a map $S \to T$ of connected, graded rings is surjective if and only if the induced map $QS \to QT$ is surjective.
(b) Conclude that the above map $L \to R$ is surjective.

The next step is to show injectivity. We'll return to this after a brief message from our sponsor:

Maps between formal group laws.
The definition of maps between formal groups is what you would expect. If G and H are groups then a map of groups is just a map $f : G \to H$ which makes the following diagram commute

$$\begin{tikzcd}
G \times G \arrow[r, "f \times f"] \arrow[d, "\mu_G"'] & H \times H \arrow[d, "\mu_H"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

We do the same for formal group laws. We should just remember that the power series rings correspond to "rings of functions on the group" [1] and hence transform contravariantly:

$$\begin{tikzcd}
x_G +_G y_G & R[[x_G, y_G]] & R[[x_H, y_H]] \arrow[l, "f^* \otimes f^*"] & x_H +_H y_H \\
x_G \arrow[u, mapsto] & R[[x_G]] \arrow[u] & R[[x_H]] \arrow[l, "f^*"] \arrow[u] & x_H \arrow[u, mapsto]
\end{tikzcd}$$

So a map between the formal group laws F and G is a ring homomorphism f^* between the ring of functions on H and the ring of functions on G. This homomorphism is determined by where x_H is sent, i.e. by a power series in x_G. Now we just have to be careful with how we write the composition of two maps of power series rings in terms of where the generators get sent.

If $f^*(x_H) = f(x_G) \in R[[x_G]]$ and the formal group law on G sends x_G to $G(x_G, y_G)$ then the composite of the two is the map that sends $x_H \mapsto f(x_G +_G y_G)$. The upshot of all this is the following:

Definition 2.11. *Let G and H be formal group laws over a ring R. A* ***map of formal group laws*** *$f : G \to H$ is a power series $f(x) \in R[[x]]$ satisfying*

(i) $f(0) = 0$
(ii) $f(x +_G y) = f(x) +_H f(y)$.

Lemma 2.12. *f is an isomorphism if and only if $f'(0)$ is a unit in R.*

Note 2.13. A **strict** isomorphism is one where $f'(0) = 1$.

[1] Indeed, the power series ring arose as the cohomology of $\mathbb{C}P^\infty$ and hence as a ring of homotopy classes of maps.

Example 2.14 (Examples of Formal Group Laws)**.**
(1) $\mathbb{G}_a(x, y) = x + y$ (the additive formal group law).
(2) $\mathbb{G}_m(x, y) = 1 - (1 - x)(1 - y) = x + y - xy$ (the multiplicative formal group law).

Are the above two formal group laws isomorphic? For this to be true we would have to produce an $f(x)$ with $f(x + y - xy) = f(x) + f(y)$. A little playing around suggests we do something with the logarithm, and it's not hard to check that

$$f(x) = -\log(1 - x) = \sum \frac{x^n}{n}$$

is the only power series that will work. So we get an isomorphism between $\mathbb{G}_a$ and $\mathbb{G}_m$ only when we can divide by each $n \in \mathbb{Z}^+$—e.g., if our ring is a $\mathbb{Q}$-algebra.

If we start with a power series $g(x) = x + m_1x^2 + m_2x^3 + \cdots$ then we can form

$$G(x, y) = g^{-1}(g(x) + g(y))$$

and this will be a formal group law over our ring. The power series $g(x)$ gives an isomorphism $G \to \mathbb{G}_a$.

Back to our regularly scheduled program.
Remember that the only thing left to do is show that the map $L \to R$ is injective. We start by writing down the universal example of a twisted $\mathbb{G}_a$: let $U = \mathbb{Z}[m_1, m_2, \ldots]$ where $| m_n |= 2n$. Let $g(x) = x + m_1x^2 + m_2x^3 + \ldots$ and define $G(x, y) = g^{-1}(g(x) + g(y))$. This is a formal group law over U, so there is a map $R \to U$ classifying it.

To show $L \to R$ is injective, it suffices to show the composite $L \to R \to U$ is injective. And it's enough to check this on indecomposables (because U is a polynomial algebra):

$$(QL)_{2n} \xrightarrow{\cong} (QR)_{2n} \longrightarrow (QU)_{2n}.$$

So we need to write down what G looks like over the quotient $\mathbb{Z} + QU_{2n}$ of U.

Now in this quotient we have

$$g(x) \equiv x + m_nx^{n+1} \text{ and } g^{-1}(x) \equiv x - m_nx^{n+1}.$$

So

$$\begin{aligned} g^{-1}(g(x) + g(y)) &= g^{-1}(x + y + m_n(x^{n+1} + y^{n+1})) \\ &= x + y + m_n(x^{n+1} + y^{n+1}) - m_n(x + y + \cdots)^{n+1} \\ &= x + y + m_n(x^{n+1} + y^{n+1} - (x + y)^{n+1}) + \cdots \\ &\equiv x + y - d_{n+1}m_nC_{n+1}(x, y). \end{aligned}$$

This means that the map $\mathbb{Z} \cong QL_{2n} \to QU_{2n}$ sends 1 to $-d_{n+1}m_n$, hence is injective. This finishes the proof of Lazard's Theorem.

Exercise 2.15. We've seen that the functor $S \mapsto \{\text{formal group laws over } S\}$ is co-representable. Find the object representing the functor

$$S \mapsto ((\text{the category of formal group laws over } S\text{, with maps being isomorphisms})).$$

3. Proof of the Symmetric Cocycle Lemma

In this section we will give a homological proof of the symmetric 2-cocycle lemma. The reader is referred to [F] for a combinatorial proof.

Theorem 3.1 (Symmetric 2-cocycle lemma)**.** *Let A be an abelian group and let $f(x, y) \in A \otimes \mathbb{Z}[x, y]$ be homogeneous of degree n. Assume that*
(i) $f(x, y) = f(y, x)$
(ii) $f(y, z) - f(x + y, z) + f(x, y + z) - f(x, y) = 0$.
Then $f(x, y) = a \otimes C_n(x, y)$ for some $a \in A$.

Recall that

$$C_n(x,y) = \frac{1}{d_n}\Big[(x+y)^n - x^n - y^n\Big] \qquad \text{where} \qquad d_n = \begin{cases} p & \text{if } n = p^e \\ 1 & \text{otherwise.} \end{cases}$$

We begin with some simple reductions:

(a) Only finitely many elements of A are involved in the expression for f, so it suffices to prove the lemma when A is finitely-generated.
(b) The lemma is true for A and B iff it's true for $A \oplus B$.
(c) So using the structure theory for finitely-generated abelian groups, we are reduced to the case where A is either $\mathbb{Z}$ or $\mathbb{Z}/p^r$.
(d) If $B \subseteq A$ and the lemma is true for A, then it's true for B—hence we can check the case $A = \mathbb{Z}$ by checking $A = \mathbb{Q}$.
(e) The case $A = \mathbb{Z}/p^r$ follows from the case $A = \mathbb{Z}/p$:

    Proof. The argument is by induction. Let $f(x,y) \in \mathbb{Z}/p^r[x,y]$ be a symmetric 2-cocycle, where $r > 1$. If we know the theorem for $\mathbb{Z}/p^{r-1}$ then we can write

$$f(x,y) = aC_n(x,y) + p^{r-1}g(x,y)$$

    for some $g(x,y) \in A \otimes \mathbb{Z}[x,y]$. It's easy to see that $p^{r-1}g(x,y)$ will also be a symmetric 2-cocycle, and so we can think of $g(x,y)$ as a symmetric 2-cocycle over $\mathbb{Z}/p[x,y]$. But then using the theorem for $\mathbb{Z}/p$ we get that $g(x,y) = bC_n(x,y)$, and we're done. □

So we have reduced to proving the result for $A = \mathbb{Q}$ and $A = \mathbb{Z}/p$. In particular, we can assume that A is a field.

Consider the chain complex

$$A \xrightarrow{d^0} A[x] \xrightarrow{d^1} A[x,y] \xrightarrow{d^2} A[x,y,z] \tag{3.1}$$

where the maps are given by

$$\begin{aligned} d^0 a &= a \\ d^1 f(x) &= f(x+y) - f(x) - f(y) \\ d^2 g(x,y) &= g(y,z) - g(x+y,z) + g(x,y+z) - g(x,y). \end{aligned}$$

We may as well assume the modules in the complex are graded, with A in degree 0 and x, y, z in degree 2.

Note that $d^1(x^n) = d_n C_n(x,y)$, so that the 2-coboundaries (i.e., the image of d^1) are spanned by elements of the form $a \otimes d_n C_n(x,y)$. Also note that the 2-cocycles (i.e., the kernel of d^2) are precisely what we have been calling 2-cocycles all along. So the symmetric 2-cocycle lemma is saying something about 2-cocycles modulos 2-coboundaries—i.e., something homological.

Consider the co-algebra structure on $A[x]$ in which x is primitive: $x \mapsto x \otimes 1 + 1 \otimes x$. The above complex is actually the beginning of something called the cobar construction for $A[x]$. Rather than develop it in these terms, we will instead dualize and work with *algebras* instead of *coalgebras* (just because it's conceptually more familiar). Since we have graded our modules, dualization will not cause us any trouble.

Definition 3.2. *The* ***divided polynomial algebra on one variable*** $\Gamma_A[t]$ *is the quotient of the free A-algebra $A < t_1, t_2, \cdots >$ by the relations*

$$t_n \cdot t_m = \binom{n+m}{n} t_{n+m}$$

(where by convention $t_0 = 1$). We will usually write $\Gamma[t]$ instead of $\Gamma_A[t]$.

Remark 3.3. $\Gamma[t]$ is the dual of the coalgebra $A[x]$, where x is primitive. The elements t_n should be thought of as behaving 'formally' like $\frac{t^n}{n!}$ (but note that the latter expression doesn't make sense in positive characteristic). The reader may check that if A is a $\mathbb{Q}$-algebra then $\Gamma_A[t] \cong A[t]$.

The Bar Construction.

The point of this section is to show that the complex we wrote down in (3.1) computes $\mathrm{Ext}^*_{\Gamma[t]}(A, A)$, where $A = \Gamma[t]/(t_1, t_2, \dots)$ as a $\Gamma[t]$-module.

If K is a simplicial set then we can form a simplicial A-algebra $\Gamma[t] \otimes K$ whose object in degree n is

$$\Big[\Gamma[t] \otimes K\Big]_n = \bigotimes_{\sigma \in K_n} \Gamma[t].$$

Here there is one tensor factor for every n-simplex of K, and the tensor products are formed over A. There is nothing special about $\Gamma[t]$ here—we could have formed $R \otimes K$ for any A-algebra R. (It may be helpful to note that the tensor product is the coproduct in the category of A-algebras—this allows one to work out the face and degeneracy maps fairly easily).

In particular, we may form $\Gamma[t] \otimes \Delta^1$ where Δ^1 denotes the usual 1-simplex. The n-simplices of Δ^1 correspond to order-preserving maps $\{0 < 1 < \cdots < n\} \to \{0 < 1\}$: 0-simplices are $\{0, 1\}$, 1-simplices are $\{00, 01, 11\}$, 2-simplices are $\{000, 001, 011, 111\}$, etc. So the simplicial algebra $\Gamma[t] \otimes \Delta^1$ looks like

$$\Gamma[t] \leftleftarrows \Gamma[t] \otimes \Gamma[t] \;\substack{\leftarrow\\\leftarrow\\\leftarrow}\; \Gamma[t] \otimes \Gamma[t] \otimes \Gamma[t] \;\substack{\leftarrow\\\leftarrow\\\leftarrow\\\leftarrow}\; \cdots$$

Working out the face and degeneracy maps is left as an exercise for the reader. Note that what we gain by saying things in this way is that it is obvious that the simplicial relations are satisfied, no checking is necessary.

Now the inclusion of the 0th vertex $\Delta^0 \to \Delta^1$ is a simplicial homotopy equivalence, so it follows that $\Gamma[t] \otimes \Delta^0 \to \Gamma[t] \otimes \Delta^1$ is also a simplicial homotopy equivalence. The same likewise holds for

$$A \otimes_{\Gamma[t]} \otimes \Gamma[t] \otimes \Delta^0 \to A \otimes_{\Gamma[t]} \otimes \Gamma[t] \otimes \Delta^1,$$

and so the associated chain complexes are chain homotopy equivalent. But note that the chain complex associated to the left object just has homology A in degree 0 and zero everywhere else. So we conclude that the complex associated to $A \otimes_{\Gamma[t]} \Gamma[t] \otimes \Delta^1$ is a resolution of A—this complex is called the **bar construction**. Each term of the complex is free as a right $\Gamma[t]$-module.

The reader may check that the first few terms of the complex

$$\mathrm{Hom}_{\Gamma[t]}(A \otimes_{\Gamma[t]} \Gamma[t] \otimes \Delta^1, A)$$

coincide with the complex we wrote down in (3.1). In particular, the homology of that complex computes $\mathrm{Ext}^{*,*}_{\Gamma[t]}(A, A)$. We will now prove the Symmetric 2-Cocycle Lemma by computing these Ext-groups using a more efficient complex.

Computations.

Case 1: $A = \mathbb{Q}$.

In this case $\Gamma[t]$ is just isomorphic to a polynomial algebra $\mathbb{Q}[t]$, and it's easy to compute Ext over this ring by using the resolution

$$0 \longrightarrow \mathbb{Q}[t] \xrightarrow{\cdot t} \mathbb{Q}[t] \longrightarrow \mathbb{Q} \longrightarrow 0.$$

The conclusion is that

$$\mathrm{Ext}^s_{\mathbb{Q}[t]}(\mathbb{Q}, \mathbb{Q}) = \begin{cases} \mathbb{Q} & s = 0 \\ \mathbb{Q} & s = 1 \\ 0 & s > 0. \end{cases}$$

In particular, $\mathrm{Ext}^2 = 0$ and therefore every 2-cocycle (symmetric or not) is a 2-boundary. This proves the theorem in this case.

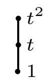
Case 2: $A = \mathbb{Z}/p$.

Let $E[t] = \mathbb{F}_p[t]/(t^p)$. It can be checked that

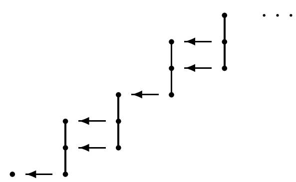
$$\Gamma[t] \cong \bigotimes_{k \geq 0} E[t_{p^k}]$$

(where the tensor product is taken over $\mathbb{F}_p$). By the Künneth formula, we then have that

$$\operatorname{Ext}_{\Gamma[t]}(A, A) \cong \bigotimes_{k \geq 0} \operatorname{Ext}_{E[t_{p^k}]}(A, A).$$

For a moment consider the case $p = 3$. $E[t]$ may be drawn pictorially as: 

The minimal resolution of $\mathbb{F}_p = E[t]/(t)$ over $E[t]$ then has the following form:



This resolution is easy to describe algebraically, and in fact this generalizes for arbitrary p: when p is odd, form the differential graded algebra $E[t] \otimes \Lambda[a] \otimes \Gamma[b]$, where the differential is determined by

$$da = t \qquad \text{and} \qquad db = t^{p-1}a$$

together with the fact that it is a derivation. It can be checked that this complex gives a resolution for $\mathbb{F}_p$ over $E[t]$. One way to see this is to recognize $E[t]$ as the group algebra of a cyclic group and notice that the resolution described above coincides with the standard resolution. One then computes that

$$\operatorname{Ext}_{E[t]}(A, A) = \Lambda[\alpha] \otimes P[\beta], \text{ where } \alpha \in Ext^1 \text{ and } \beta \in \operatorname{Ext}^2.$$

Here $\Lambda[\alpha]$ denotes an exterior algebra on the class α, and $P[\beta]$ is a polynomial algebra. So

$$\operatorname{Ext}_{\Gamma[t]}(A, A) \cong \Lambda[\alpha_k \mid k \geq 0] \otimes P[\beta_l \mid k \geq 0], \qquad \alpha_k \in \operatorname{Ext}^1 \text{ and } \beta_k \in \operatorname{Ext}^2.$$

Corollary 3.4.
(a) When p is odd, $\operatorname{Ext}^2_{\Gamma[t]}(A, A)$ has basis $\{\beta_k, \alpha_i\alpha_j\}$.
(b) When $p = 2$, $\operatorname{Ext}(A, A) = P[\alpha_i]$ and so Ext^2 has basis $\{\alpha_k^2, \alpha_i\alpha_j\}$.

Observe that one has the relations $\alpha_i\alpha_j = \alpha_j\alpha_i$.

Exercise 3.5.
(a) Show that when p is odd β_k is represented by the 2-cocycle $C_{p^k}(x, y)$ and $\alpha_i\alpha_j$ is represented by the 2-cocycle $x^{p^i}y^{p^j}$.
(b) Show that when $p = 2$, α_k^2 is represented by $C_{2^k}(x, y)$ and $\alpha_i\alpha_j$ is represented by $x^{2^i}y^{2^j}$.

For $n \neq p^k$ we have $C_n(x, y) = d^1(x^n)$ so we have the following

Corollary 3.6. *A basis for the 2-cocycles is given by the $C_n(x, y)$ and the $x^{p^a}y^{p^b}$ when $a < b$.*

Notice that $x^{p^a}y^{p^b} + x^{p^b}y^{p^a} = C_{p^a+p^b}(x, y)$. It therefore follows that the $C_n(x, y)$ form a basis for the *symmetric* 2-cocycles. This completes the proof.

Note 3.7.

(a) The techniques used above to determine the 2-cocycles apply in more general situations. One example would be the classification of cocyles satisfying

$$f(y,z) -_G f(x +_F y, z) + f(x, y +_F z) -_G f(x,y) = 0$$

(b) One can also work out a basis for all the 2-cocycles (not symmetric) for any algebra A by using a Bockstein spectral sequence.

4. Complex Cobordism and MU

There are different approaches to complex cobordism; we will focus on the vector bundle/Thom complex perspective. The best reference for the material in this section is Chapter 2 of [A].

Consider the sequence of classifying spaces

$$BU(1) \hookrightarrow BU(2) \hookrightarrow \cdots \hookrightarrow BU.$$

Let $MU(n)$ denote the Thom space of the universal bundle ξ_n over $BU(n)$. It's easy to see that there are natural maps $\Sigma^2 MU(n) \to MU(n+1)$.

Recall that a spectrum is a sequence of pointed spaces $\{E_n\}$ together with (pointed) maps $\Sigma E_n \to E_{n+1}$. So in particular, the $MU(n)$'s assemble to give us a spectrum—this is usually called MU. Any spectrum gives rise to a generalized homology and cohomology theory defined by

$$\begin{aligned} \tilde{E}_k(X) &= \lim_{n\to\infty} \pi_{n+k}(E_n \wedge X) \\ \tilde{E}^k(X) &= \lim_{n\to\infty} [\Sigma^n X, E_{n+k}] \end{aligned}$$

The generalized cohomology theory associated to MU is called **complex cobordism**. It turns to be a ring spectrum, and in fact it's complex oriented. The latter is somewhat formal: given a vector bundle η over X, one gets a map $X \to BU(n)$ expressing η as a pullback of ξ_n. Taking Thom spaces then gives $X^\eta \to MU(n)$, which defines an element in $\widetilde{MU}^{2n}(X^\eta)$ by definition.

Now the amazing thing is that MU is actually the *universal* complex oriented cohomology theory. It can be shown that complex orientations of a spectrum E are in one-to-one correspondence with multiplicative maps $MU \to E$.

The complex orientation on MU defines a formal group law on $MU^* = \pi_*(MU)$. Therefore one gets a map of graded rings

$$\mathbb{Z}[x_1, x_2, \ldots] = L \to \pi_*(MU).$$

Theorem 4.1 (Quillen)**.** *The above map is an isomorphism.*

We've seen that complex oriented cohomology theories give rise to formal group laws. The fact that the *universal* complex oriented cohomology theory gives rise to the universal formal group law suggests a very intimate relationship between the two. We will explore this more as the course progresses. The proof of Quillen's theorem uses computations with the Adams-Novikov spectral sequence and complex oriented cohomology theories. We will begin with the former.

Adams resolutions.

We will follow the treatment by Haynes Miller [M]. The idea is the following. Let E be a spectrum which we pretend to know something about and let X be another spectrum which we want to learn about. Haynes' idea is to think of this situation as you would in homological algebra.

Definition 4.2.

*(i) A sequence of spectra $A_1 \to A_2 \to \ldots \to A_n$ is **exact** if the sequence of homotopy functors it represents is exact.*

*(ii) A map $f : A \to B$ is a **monomorphism** if $* \to A \xrightarrow{f} B$ is exact.*

*(iii) A map $f : A \to B$ is an **epimorphism** if $A \xrightarrow{f} B \to *$ is exact.*

*(iv) A sequence $A \to B \to C$ is **short exact** if $* \to A \to B \to C \to *$ is exact.*

Lemma 4.3. *A map $f : A \to B$ is a monomorphism iff there exists $g : C \to B$ such that*

$$A \vee C \xrightarrow{f \vee g} B$$

is a weak equivalence.

Proof. Consider the cofiber sequence

$$A \to B \to B/A$$

Since $A \to B$ is mono, it follows that $B \to B/A$ is epi. We can now take $C = B/A$ and g to be any map lifting the identity. □

Something similar holds for epimorphisms so the homological algebra of spectra in this naive form is kind of stupid.

Definition 4.4. *A sequence of spectra is E-**exact** if the sequence obtained from it by smashing with E is exact.*

We can now define the notions of E-monomorphism, E-epimorphism and E-short exact as above.

Definition 4.5. *A spectrum I is E-**injective** if for each E-monomorphism $f : A \to B$ and each map $g : A \to I$ there exists a map $h : B \to I$ making the following diagram commute up to homotopy*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & I \\ B \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Definition 4.6. *An E-**Adams resolution** of a spectrum X is a sequence*

$$* \to X \xrightarrow{j_0} I_0 \xrightarrow{j_1} I_1 \longrightarrow \cdots$$

such that

(i) $j_n \circ j_{n-1} \sim *$
(ii) Each I_n is E-injective.
(iii) The sequence is E-exact.

Remark 4.7. It follows from Lemma 4.3 that defining $I_{-1} = X$, we have splittings $E \wedge I_n = J_n \vee J_{n+1}$ with $J_{-1} = *, J_0 = E \wedge X, J_1 = E \wedge (I_0/X), \ldots$.

$$\begin{tikzcd} {} \arrow[r] & E \wedge X \arrow[r] \arrow[d, "\sim"] & E \wedge I_0 \arrow[r] \arrow[d, "\sim"] & E \wedge I_1 \arrow[r] \arrow[d, "\sim"] & \cdots \\ {} \arrow[r] & J_0 \arrow[r] & J_0 \vee J_1 \arrow[r] & J_1 \vee J_2 \arrow[r] & \cdots \end{tikzcd}$$

and the way one usually shows that a certain sequence is a resolution is by showing that the sequence splits in this way.

Lemma 4.8. *Let*

$$* \longrightarrow X \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

$$* \longrightarrow Y \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots$$

be E-Adams resolutions of X and Y and let $f : X \to Y$ be a map of spectra. Then there exists a map of resolutions lifting f and this is unique up to chain homotopy.

Proof. Exercise. □

Note 4.9. One can actually speak of the space of maps between two resolutions. If the spectrum E is a ring spectrum with higher order commutativity properties, the cells in the spaces parametrizing the higher order homotopies give maps of resolutions shifting dimensions by the dimension of the cell and it is in this way that all Adams differentials are derived.

Adams towers.
We will now show how Adams resolutions correspond to Adams towers.

Definition 4.10. *A **tower** is a diagram of spectra*

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& X_2 \arrow[r] \arrow[d] & \Sigma^{-2} I_3 \\
& X_1 \arrow[r] \arrow[d] & \Sigma^{-1} I_2 \\
X \arrow[r, "j_0"] \arrow[ur] \arrow[uur] & X_0 = I_0 \arrow[r] & I_1
\end{tikzcd}$$

where the sequences $X_{n+1} \to X_n \to \Sigma^{-n} I_{n+1}$ are cofibration sequences. The composites

$$\Sigma^{-n} j_{n+1} : \Sigma^{-n} I_n \to X_n \to \Sigma^{-n} I_{n+1}$$

*where the first map is the inclusion of the fiber of $X_n \to X_{n+1}$ are called the **k-invariants** of the tower.*

From a tower we get a sequence of spectra

$$X \to I_0 \xrightarrow{j_1} I_1 \xrightarrow{j_2} I_2 \longrightarrow \cdots$$

where clearly $j_n \circ j_{n-1} \sim *$.

In general this sequence carries much less information then the tower we started out with. The sequence just "remembers" the layers of the tower and the maps between them.

Definition 4.11. *A tower is an E-**Adams tower** if its corresponding sequence is an E-Adams resolution.*

It is a miracle that in the case of an E-Adams resolution we can go back to get an E-Adams tower. In general there are obstructions to constructing a corresponding tower which lie in the Toda brackets $< j_0, j_1, \dots, j_n >$. In the case of an E-Adams resolution the homological algebra somehow guarantees that they all contain 0.

Proposition 4.12. *Every Adams resolution arises from an Adams tower.*

Proof. Let $X \to I_0 \to I_1 \to \dots$ be an E-Adams resolution. Then from the definition of a tower we are forced to take $X_0 = I_0$ and $X_1 =$ fibre of $I_0 \to I_1$. We will show how to get the next stage of the tower which should make it clear how to proceed by induction.

So far we have the following diagram

$$\begin{tikzcd}
& X_1 \arrow[d] & & & \\
X \arrow[ur, dashed, "h"] \arrow[r] & X_0 = I_0 \arrow[r] & I_1 \arrow[rr] \arrow[dr] & & I_2 \\
& & & \Sigma X_1 \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

In order to proceed one more stage we need to guarantee

(i) the composite $X \xrightarrow{h} X_1 \xrightarrow{\Sigma^{-1}g} \Sigma^{-1}I_2$ is null.
(ii) the composite $X_1 \xrightarrow{g} \Sigma^{-1}I_2 \longrightarrow \Sigma^{-1}I_3$ is null.

It is now easy to check that these hold because the two maps are null after smashing with E and hence must actually be null as the I_n's are E-injective. $\square$

5. The Adams spectral sequence

The E**-Adams spectral sequence** is the homotopy spectral sequence associated to an E-Adams tower. It turns out that this spectral sequence is useful even when we don't know anything about the spectrum E! We'll see examples of this later. There are two obvious questions one can ask:

Question 1: What is the E_2-term of this spectral sequence?
Question 2: To what does this spectral sequence converge?
Answer to Question 1: The E_1 term is the complex

$$\pi_* I_0 \longrightarrow \pi_* I_1 \longrightarrow \dots$$

so the E_2-term is the cohomology of this complex. By Lemma 4.8 this is independent of the choice of resolution. In general, this is all you can say. But in good cases we can give a homological description of the E_2 term. We will discuss this in more detail in the next section.

Answer to Question 2: The spectral sequence converges conditionally to $\pi_* \lim_{\leftarrow} X_n$. That is, we can tell what $\lim^1_{\leftarrow} \pi_* X_n$ and $\lim_{\leftarrow} \pi_* X_n$ are from the spectral sequence and there is the Milnor sequence relating these to $\pi_* \lim_{\leftarrow} X_n$. I think this is what conditional convergence means.

There is also a map

$$X \to \lim_{\leftarrow} X_n$$

The space on the right hand side is called the E**-nilpotent completion** of X, which in good cases coincides with $L_E X$, the Bousfield localization of X. Rather than worry about to which extent this map is an equivalence, the point of view we will take is that it is the completion of X we are interested in.

In order to try to give a description of E_2 we will need to assume that E is a *ring spectrum.* Pretty much nothing is known if this is not the case and this might be an interesting question to consider since we *are* losing generality by assuming that E is a ring spectrum. Ignoring for a moment the difference between localization and completion, note that the abutment of the spectral sequence depends upon the Bousfield class of E and it is not true that every Bousfield class contains a ring spectrum. So all that we will say next does not apply in any way to the study of localizations with respect to Bousfield classes not containing a ring spectrum.

So assume E is a **ring spectrum**, i.e. that we have a diagram

$$\begin{tikzcd}
S^0 \wedge E \arrow[r] \arrow[dr, "1\wedge id"'] & E \wedge E \arrow[d, "\mu"] & E \wedge S^0 \arrow[l] \arrow[dl, "id\wedge 1"] \\
 & E &
\end{tikzcd}$$

Lemma 5.1. *A spectrum I is E-injective iff $I \to E \wedge I$ is the inclusion of a retract.*

Proof. Exercise. $\square$

Lemma 5.2. *For any spectrum X, the map $X \to E \wedge X$ is an E-monomorphism.*

Proof. The diagram

$$\begin{tikzcd} E \wedge X \arrow[r] \arrow[dr, "1"'] & E \wedge E \wedge X \arrow[d, "\mu\wedge id"] \\ & E \wedge X \end{tikzcd}$$

shows that $E \wedge X \to E \wedge E \wedge X$ is the inclusion of a retract. □

Corollary 5.3. *Any spectrum X has an E-Adams resolution.*

Proof. Let $\overline{E}$ denote the cofiber of the unit map $S^0 \to E$. Then it is easy to check that the top row of the following diagram is an E-Adams resolution for X

$$\begin{tikzcd} X \arrow[r] & E \wedge X \arrow[rr] \arrow[dr] & & E \wedge \overline{E} \wedge X \arrow[rr] \arrow[dr] & & E \wedge \overline{E} \wedge \overline{E} \wedge X \arrow[r] & \cdots \\ & & \overline{E} \wedge X \arrow[ur] & & \overline{E} \wedge \overline{E} \wedge X \arrow[ur] & & \end{tikzcd}$$

□

The resolution described in the previous proof is called the **normalized E-Adams resolution**. This is not ideal for some purposes. For instance, if X is also a ring spectrum, one would like to have a resolution made of ring spectra and maps of ring spectra and this is not the case with the normalized resolution.

The standard resolution.
Let $I_n = E \wedge \ldots \wedge E \wedge X$ (n+1 copies of E indexed from 0 to n) and define maps δ^i for $i = 0, \ldots, n+1$

$$\delta^i : I_n = E \wedge \ldots \wedge E \wedge X \xrightarrow{\sim} E \wedge \ldots \wedge S^0 \wedge \ldots \wedge E \wedge X \longrightarrow I_{n+1}$$

with the second map given by inclusion of the unit on the i-th factor.

Definition 5.4. *The **standard resolution** of X is the resolution*

$$X \xrightarrow{\delta} E \wedge X \xrightarrow{\delta} E \wedge E \wedge X \xrightarrow{\delta} \ldots$$

where $\delta = \sum_{i=0}^{n} (-1)^i \delta^i$.

The standard resolution is also called the *bar construction.*

Remark 5.5.
(i) The functor which associates

$$[n] \mapsto I_n = E \wedge \ldots \wedge E \wedge X$$

extends to a cosimplicial spectrum. If X is a ring spectrum then this is a cosimplicial ring spectrum which we will denote by $E^\bullet X$.
(ii) The tot tower of this cosimplicial spectrum is an Adams tower associated to this Adams resolution.

The E_2 term.
We will now talk about the nice cases in which the E_2 term of the spectral sequence gets a homological name. Recall that the E_2 term is the cohomology of the cochain complex

$$\pi_* E \wedge X \rightrightarrows \pi_* E \wedge E \wedge X \;\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}\; \cdots \tag{5.1}$$

There is a convenient assumption that we will make. Since E is a ring spectrum, $\pi_* E = E_*$ and $\pi_* E \wedge E = E_* E$ is also a ring. There are two maps given by smahing with the unit on the left and right respectively

$$E_* \underset{\eta_R}{\overset{\eta_L}{\rightrightarrows}} E_* E$$

Note that the two maps differ by the flip automorphism of $E \wedge E$.

Assumption: The map $\eta_L : E_* \to E_*E$ is *flat*. When this assumption is satisfied we say that E is flat. Note that η_L is flat iff η_R is.

Remark 5.6. This assumption is not too restrictive although there are several interesting cases in which it is not satisfied.

Recall that the smash product gives natural homomorphisms

$$\pi_*A \otimes \pi_*B \longrightarrow \pi_*A \wedge B$$

which together with the definition of the tensor product as a coequalizer give the natural map

$$\begin{tikzcd} E_*E \otimes E_* \otimes E_*X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E_*E \otimes E_*X \arrow[r] \arrow[d] & E_*E \otimes_{E_*} E_*X \arrow[d, dashed, "\exists!"] \\ \pi_*E \wedge E \wedge E \wedge E \wedge X \arrow[r, shift left, "1\wedge\mu\wedge 1\wedge 1"] \arrow[r, shift right, "1\wedge 1\wedge\mu\wedge 1"'] & \pi_*E \wedge E \wedge E \wedge X \arrow[r, "1\wedge\mu\wedge 1"] & \pi_*E \wedge E \wedge X \end{tikzcd}$$

where the action of E_* on E_*E in the top right hand corner of the diagram is through η_R.

Proposition 5.7. *If E is flat then the natural map*

$$E_*E \otimes_{E_*} E_*X \longrightarrow \pi_*E \wedge E \wedge X$$

is an isomorphism.

Proof.

(i) The result clearly holds when X is a sphere.
(ii) Since E is flat, both the domain and range of the map are homology theories in X (i.e. take cofiber sequences to long exact sequences of abelian groups)
(iii) Both sides take infinite wedges to infinite direct sums.

We conclude that the map is an isomorphism for all X since a map of homology theories which is an isomorphism on coefficients is an isomorphism. □

Notation: We will write

$$A = E_* \qquad \Gamma = E_*E \qquad M = E_*X.$$

Proposition 5.7 lets us write the complex (5.1) in a purely algebraic fashion. It is the complex

$$M \rightrightarrows \Gamma \otimes_A M \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \Gamma \otimes_A \Gamma \otimes_A M \cdots$$

and we now have to unravel what the natural maps in this complex are. Let's start first with the case $X = S^0$ or equivalently $M = A$. In this case all the natural maps can be derived from the following

(i) The maps induced in homotopy by η_L and η_R which we denote by the same name

$$\begin{tikzcd} A \arrow[r, shift left, "\eta_L"] \arrow[r, shift right, "\eta_R"'] & \Gamma \end{tikzcd}$$

(ii) The maps induced in homotopy by the three maps

$$E \wedge E \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} E \wedge E \wedge E$$

Recall that $\pi_*E \wedge E = \Gamma$ and $\pi_*E \wedge E \wedge E = \Gamma \otimes_A \Gamma$. It is easy to check that under these identifications

$$S^0 \wedge E \wedge E \longrightarrow E \wedge E \wedge E$$

induces

$$x \in \Gamma \mapsto 1 \otimes x \in \Gamma \otimes_A \Gamma$$

and similarly

$$E \wedge E \wedge S^0 \longrightarrow E \wedge E \wedge E$$

induces

$$x \in \Gamma \mapsto x \otimes 1 \in \Gamma \otimes_A \Gamma$$

but the remaining map

$$E \wedge S^0 \wedge E \longrightarrow E \wedge E \wedge E$$

induces a map that we have to name. We will denote it by $\Psi : \Gamma \to \Gamma \otimes_A \Gamma$.

Exercise 5.8.
(a) Check that $\Psi(\eta_L(a)x) = \eta_L(a) \otimes 1.\Psi(x)$ and $\Psi(x.\eta_R(a)) = \Psi(x).1 \otimes \eta_R(a)$.
(b) Check that all other maps in the bar construction can be derived from η_R, η_L and Ψ.

There are a couple of other maps which are useful although not really necessary to describe the bar construction. The multiplication $E \wedge E \xrightarrow{\mu} E$ induces the *augmentation* $\epsilon : \Gamma \to A$ and the flip automorphism $c : E \wedge E \longrightarrow E \wedge E$ induces a map $\Gamma \to \Gamma$ which we still denote by c. It is easy to check that we have the following identities

$$\epsilon \circ \eta_L = \epsilon \circ \eta_R = 1$$

Hopf algebroids.
A ring A is determined by the functor it corepresents

$$\mathcal{R}ing(A, -) : \text{ Rings } \to \text{ Sets}$$

This is often a better point of view than thinking of rings in terms of elements. In this language, a map of rings is a natural transformation of functors. The category of functors from rings to sets is actually a lot like the category of sets and it is often convenient to think of objects in this category as sets.

We have described the E_2 term of the Adams spectral sequence in terms of some complicated algebraic data $(A, \Gamma, \eta_R, \eta_L, \epsilon, c, \Psi)$. In our new language, thinking of the elements in the functor category as sets, this data defines a *groupoid*. Writing

$$X_0 = \mathcal{R}ing(A, -) \quad \text{and} \quad X_1 = \mathcal{R}ing(\Gamma, -)$$

we have
(i) X_1 represents the set of morphisms.
(ii) X_0 represents the set of objects.
(iii) η_R represents range.
(iv) η_L represents domain.
(v) ϵ represents the identity morphism.
(vi) c represents the inverse map.
(vii) Ψ represents composition.

For the last item note that the set of composable maps is the fiber product $X_1 \times_{X_0} X_1$ and so composition $X_1 \times_{X_0} X_1 \longrightarrow X_1$ is represented by $\Psi : \Gamma \longrightarrow \Gamma \otimes_A \Gamma$. Also note that the above data gives the first three terms of a simplicial object

$$\begin{tikzcd} X_0 \arrow[r, "\epsilon"] & X_1 \arrow[l, bend right, "\eta_L"'] \arrow[l, bend left, "\eta_R"] & X_1 \times_{X_0} X_1. \arrow[l, "\Psi"'] \arrow[l, bend right, "\pi_1"'] \arrow[l, bend left, "\pi_2"] \end{tikzcd}$$

(where we haven't drawn the degeneracies from X_1 to $X_1 \times_{X_0} X_1$ for typographical reasons).

Definition 5.9. *A* ***Hopf algebroid*** *is a pair of rings* (A, Γ) *together with maps* $(\eta_R, \eta_L, \Psi, c, \epsilon)$ *as above such that*

1. η_L *is flat*
2. $X_0 = \mathcal{R}ing(A, -)$, $X_1 = \mathcal{R}ing(\Gamma, -)$ *and* $(\eta_R, \eta_L, \Psi, c, \epsilon)$ *define a functor from Rings to Groupoids.*

Exercise 5.10. Check that the bar construction on (A, Γ) regarded as a functor from rings to simplicial sets assigns to each ring the nerve of the groupoid associated to that ring by (A, Γ).

Exercise 5.11. Consider the functor $Q : \text{ Rings } \longrightarrow \text{ Groupoids }$ with objects

$$\mathrm{ob}(Q) = R \times R = \{x^2 + bx + c | b, c \in R\}$$

and maps

$$\mathrm{Map}((b_1, c_1), (b_2, c_2)) = \{r \in R | (x + r)^2 + b_1(x + r) + c_1 = x^2 + b_2 x + c_2\}$$

This is the groupoid of quadratic expressions and changes of variables. Find explicitly a Hopf algebroid (A, Γ) representing this functor.

We also need to consider the case when X is not a sphere. In this case $M = E_* X$ is an A-comodule over (A, Γ):

Definition 5.12. *A* ***comodule*** M *over the Hopf algebroid* (A, Γ) *is a left* A*-module* M *together with a coaction map*

$$\eta : M \longrightarrow \Gamma \otimes_A M$$

of left A*-modules satisfying*

(a) The composite $M \longrightarrow \Gamma \otimes_A M \xrightarrow{\epsilon \otimes 1} M$ *is the identity. (Counital property)*
(b) $(\Psi \otimes id) \circ \eta = (id_\Gamma \otimes \eta) \circ \eta$ *(Coassociativity)*

Facts:

1. Co-modules over (A, Γ) form an abelian category with enough injectives (this requires flatness of η_L).
2. The E_2 term of the Adams spectral sequence is
$$\mathrm{Ext}^*_{(A,\Gamma)}(A, M)$$
This is also written $\mathrm{Ext}^*_\Gamma(A, M)$ and $\mathrm{Ext}_{E_*E}(E_*, E_*X)$.

Note that different Hopf algebroids can have the same cohomology. This is true if the groupoids they represent are equivalent (or even locally equivalent). This suggests that the Hopf algebroid is still not the natural object we should be considering when analyzing the Adams spectral sequence.

6. The Hopf Algebroid (MU_*, MU_*MU) and formal groups

We have already seen Quillen's theorem that the ring MU_* is isomorphic to the Lazard ring which corepresents the functor

$$S \mapsto \{\text{formal group laws over } S\}$$

from rings to sets. We will now examine the Hopf algebroid (MU_*, MU_*MU) and the functor from rings to groupoids which it corepresents.

Computation of $E_*\mathbb{C}P^\infty, E_*BU, E_*MU$. Rather than specifically computing MU_*MU, we'll compute the ring E_*MU where E is any complex oriented cohomology theory. The computation in this general case is no different from the special case $E = MU$.

We have seen that $E^*(\mathbb{C}P^\infty) \cong E^*[[x_E]]$, where $x_E \in E^2(\mathbb{C}P^\infty)$ is the complex orientation of E, and that $E^*(\mathbb{C}P^n) \cong E^*[x]/(x^{n+1})$, where x denotes the restriction of the class x_E. To compute $E_*(\mathbb{C}P^n)$ and $E_*(\mathbb{C}P^\infty)$, we make use of the pairing of Atiyah-Hirzebruch Spectral Sequences

$$\begin{array}{ccc} H^*(\mathbb{C}P^n; E_*) & \Longrightarrow & E^*(\mathbb{C}P^n) \\ \times & & \times \\ H_*(\mathbb{C}P^n; E_*) & \Longrightarrow & E^*(\mathbb{C}P^n) \\ \downarrow & & \downarrow \\ E_* & & E_* \end{array} .$$

The nonsingularity of the pairing in the E_2-term, together with the vanishing of all differentials in the spectral sequence for cohomology, shows that all differentials in the spectral sequence for homology likewise vanish. Hence $E_*(\mathbb{C}P^n)$ is a free E_*-module on classes $b_0, \dots, b_n$ with b_i dual to x^i. Passing to the colimit, we find that $E_*(\mathbb{C}P^\infty)$ is a free E_*-module on generators $b_0, b_1, b_2, \dots$ dual to $1, x, x^2, \dots$.

Remark 6.1. Note that the natural map $E^*(\mathbb{C}P^\infty) \to \mathrm{Hom}_{E_*}(E_*\mathbb{C}P^\infty, E_*)$ is an isomorphism. For a power series $\sum a_n x_E^n \in E^*(\mathbb{C}P^\infty)$, the corresonding map $E_*(\mathbb{C}P^\infty) \to E_*$ is determined by $b_n \mapsto a_n$.

Before describing the structure of E_*MU it will help to discuss Thom complexes of vector bundles and virtual bundles. Suppose V is a vector bundle over a space X. Let X^V denote the Thom spectrum of this bundle, i.e. the suspension spectrum of the Thom complex. If $\mathbf{n}$ is a trivial bundle of real dimension n, then

$$X^{V+\mathbf{n}} = \Sigma^n X^V.$$

So take this to be the definition of $X^{V+\mathbf{n}}$ even when n is a negative integer. If X is compact, this allows us to define X^V for any virtual bundle $V \in KO(X)$, using the fact that any such V may be written as $W - \mathbf{n}$ for some vector bundle W. For more general spaces X (i.e. for paracompact X), we define X^V by passage to the colimit along compact subspaces. (This definition of Thom spectra is a bit troublesome because we made so many choices along the way. Later, when we define spectra more carefully, we'll see that Thom spectra may be defined in a way which is obviously functorial and independent of the choices.)

Define $\beta_i \in E_*(\mathbb{C}P^\infty)^{\mathbb{L}-1}$ to be the class corresponding to b_i under the Thom isomorphism, and observe that the correspondence

$$(\mathbb{C}P^\infty)^{\mathbb{L}-1} = \Sigma^{-2}(\mathbb{C}P^\infty)^{\mathbb{L}} \xleftarrow{\sim} \Sigma^{-2}\mathbb{C}P^\infty$$

maps β_n to $\Sigma^{-2} b_{n+1}$.

Proposition 6.2.

$$E_*(BU) = \mathrm{Sym}_{E_*}[E_*\mathbb{C}P^\infty]/(b_0 - 1) = E_*[b_1, b_2, \dots]$$

$$E_*(MU) = \mathrm{Sym}_{E_*}[E_*(\mathbb{C}P^\infty)^{\mathbb{L}-1}]/(\beta_0 - 1) = E_*[\beta_1, \beta_2, \dots]$$

Proof. Both statements follow from the known case $E = H\mathbb{Z}$, using the Atiyah-Hirzebruch Spectral Sequence. The multiplicative structure of E_*MU is as indicated because the multiplication map $MU \wedge MU \to MU$ is derived from the Whitney sum map $BU \times BU \to BU$ upon passing to Thom complexes. □

E_*MU **and formal groups.** Now, we want to understand the ring E_*MU, and the functor it corepresents, in terms of formal groups, but again it helps to work in greater generality. Let E, M be complex oriented cohomlogy theories with complex orientations x_E, x_M, respectively. The spectrum $E \wedge M$ has two complex orientations, which we shall also call x_E, x_M, coming from the two maps

$$\Sigma^{-2}\mathbb{C}P^\infty \overset{x_E}{\to} E \to E \wedge M, \qquad \Sigma^{-2}\mathbb{C}P^\infty \overset{x_M}{\to} M \to E \wedge M$$

Likewise, if F, G are the associated formal groups over E_*, M_*, then their images under the natural maps $E_* \to (E \wedge M)_*$, $M_* \to (E \wedge M)_*$ define two formal groups over $(E \wedge M)_*$ which we will also call F, G. We know that $(E \wedge M)^*(\mathbb{C}P^\infty) = (E \wedge M)_*[[x_E]]$. In particular, the element x_M may be expressed as a power series

$$x_M = t_0 x_E + t_1 (x_E)^2 + \dots$$

In fact, $t_0 = 1$ because complex orientations must restrict to a preferred class in $(E \wedge M)^2(\mathbb{C}P^1)$. Let $g(x)$ be the power series $x + t_1 x^2 + \dots$. Looking at

$$\begin{array}{rcl} (E \wedge M)^*(\mathbb{C}P^\infty) & \to & (E \wedge M)^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \\ x_E & \mapsto & x_E +_F y_E \\ g(x_E) & \mapsto & g(x_E +_F y_E) \\ g(x_E) = x_M & \mapsto & x_M +_G y_M = g(x_E) +_G g(y_E) \end{array}$$

we see that $g(x +_F y) = g(x) +_G g(y)$. Hence g is an isomorphism from F to G. In fact g satisfies the additional property that $g'(0) = 1$; such an isomorphism is called a **strict isomorphism**.

Definition 6.3. *Let R, S be rings, and F, G formal group laws over R, S, respectively. The functor* $\mathrm{StrictIso}(F, G) : \mathit{Rings} \to \mathit{Sets}$ *is defined by*

$$T \mapsto \{f : R \to T, g : S \to T, \phi : f^*F \to g^*G \textit{ a strict isomorphism}\}.$$

The previous discussion shows that a pair of complex oriented theories E, M with formal group laws F, G determines a natural transformation $\tau_{E,M} : \mathcal{R}ing(\pi_*(E \wedge M), -) \to \mathrm{StrictIso}(F, G)$ of functors from rings to sets.

Lemma 6.4. *If E, M, N are three complex oriented theories with formal group laws F, G, H, then the following diagram of natural transformations commutes:*

$$\begin{tikzcd} & \mathcal{R}ing(\pi_* E \wedge M \wedge N, -) \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathcal{R}ing(\pi_* E \wedge M, -) \times \arrow[d, "\tau_{E,M}"] & \mathcal{R}ing(\pi_* M \wedge N, -) \arrow[d, "\tau_{M,N}"] & \mathcal{R}ing(\pi_* E \wedge N, -) \arrow[d, "\tau_{E,N}"] \\ \mathrm{StrictIso}(F, G) \times & \mathrm{StrictIso}(G, H) \arrow[r] & \mathrm{StrictIso}(F, H) \end{tikzcd}$$

Here the map $\mathrm{StrictIso}(F, G) \times \mathrm{StrictIso}(G, H) \to \mathrm{StrictIso}(F, H)$ *is defined by composition of strict isomorphisms. The map* $\mathcal{R}ing(\pi_*(E \wedge M \wedge N), -) \to \mathcal{R}ing(\pi_*(E \wedge M), -)$ *is induced by the natural map* $\pi_*(E \wedge M) \to \pi_*(E \wedge M \wedge N)$*, and likewise for $M \wedge N$ and $E \wedge N$.*

Proof. Comparing the two complex orientations on $E \wedge M$, and on $M \wedge N$, and on $E \wedge N$, gives us three power series:

- g_1, an isomorphism from F to G over $(E \wedge M)_*$
- g_2, an isomorphism from G to H over $(M \wedge N)_*$
- g_3, an isomorphism from F to H over $(E \wedge N)_*$.

The lemma asserts that $g_3 = g_2 \circ g_1$ if we consider all three power series as being defined over the ring $(E \wedge M \wedge N)_*$. But in the ring $(E \wedge M \wedge N)^*(\mathbb{C}P^\infty)$, we have

$$g_3(x_E) = x_N = g_2(x_M) = g_2(g_1(x_E))$$

which verifies that $g_3 = g_2 \circ g_1$. ☐

Proposition 6.5.

1. *The natural transformation $\tau_{E,MU}$ is an isomorphism. In other words, E_*MU corepresents the functor which assigns to each ring R the set of triples (f, G, g), where $f : E_* \to R$ is a ring homomorphism, G is a formal group over R, and g is a strict isomorphism between f^*F and G over R.*
2. *The Hopf algebroid (MU_*, MU_*MU) corepresents the functor Rings $\to$ Groupoids which associates to each ring R the groupoid of formal group laws over R and strict isomorphisms between them.*

Proof. First observe that the information in a triple (f, G, g) as in part (1) is redundant, because the formal group G is uniquely determined by f and g. Furthermore, since $E_*MU \cong E_*[\beta_1, \beta_2, \dots]$, a map $\phi : E_*MU \to R$ is clearly determined by the map $f : E_* \to R$ and the power series $\sum \phi(\beta_n)x^{n+1}$. It remains to show that this power series is equal to $g(x)$. For this, see the lemma below.

For part (2), we have already shown that MU_*MU corepresents the functor $\mathrm{StrictIso}(F_{univ}, F_{univ})$ which assigns to R the set of morphisms in the groupoid of formal group laws over R. It remains to show that $\eta_L, \eta_R, \epsilon, \Psi$ identify sources, targets, identity maps, and composition in this groupoid. All of these assertions are easy; the fact that Ψ induces the composition law for strict isomorphisms is a consequence of Lemma 6.4. ☐

Lemma 6.6. *The two complex orientations x_E, x_{MU} of $E \wedge MU$ are related by the equation $x_{MU} = \Sigma\beta_n x_E^{n+1}$.*

Proof. By the remark following the computation of $E_*\mathbb{C}P^\infty$, the power series coefficients of a class in $E^*(\mathbb{C}P^\infty) = E_*[[x_E]]$ may be extracted by considering the induced map $E_*(\mathbb{C}P^\infty) \to E_*$. We are working with $E \wedge MU$, and the coefficient of x_E^{n+1} in the power series expansion of x_{MU} is the image of b_{n+1} under this induced map, i.e. it is the element in $\pi_{2n}(E \wedge MU)$ represented by the class

$$S^{2n} \xrightarrow{\Sigma^{-2} b_{n+1}} E \wedge (\mathbb{C}P^\infty)^{\mathbb{L}-1} \xrightarrow{1 \wedge x_M U} E \wedge MU \quad ,$$

which by definition is β_n. ☐

On the proof of Quillen's theorem. Until now, we have been using Quillen's theorem without having seen a proof of it. A complete proof appears in [A]; here we will only provide a short sketch of the proof.

In the above discussion, consider what happens if we take $E = H$, the integral Eilenberg-MacLane spectrum. The formal group associated to H is $\mathbb{G}_a$, the additive formal group. Hence the ring $H_*MU = \mathbb{Z}[\beta_1, \beta_2, \dots]$ corepresents the functor $\mathrm{StrictIso}(\mathbb{G}_a, F_{univ})$. In other words, there is an isomorphism between $\mathbb{G}_a$ and F_{univ} over H_*MU, and it is the universal ring over which such an isomorphism exists. We considered this ring during the proof of Lazard's Theorem, and we showed that the map $L \to \mathbb{Z}[\beta_1, \beta_2, \dots]$ was a monomorphism and that the induced map on indecomposables,

$$\begin{tikzcd} (QL)_{2n} \arrow[r] \arrow[d, "\sim"] & (Q\mathbb{Z}[\beta_1, \beta_2, \dots])_{2n} \arrow[d, "\sim"] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \text{ (generated by } \beta_n) \end{tikzcd}$$

is multiplication by d_n. Milnor, using the Adams Spectral Sequence, showed that $\pi_*MU \to H_*MU$ is a monomorphism and induces the same map on indecomposables. This implies Quillen's theorem.

7. More on isomorphisms, strict isomorphisms, and $\pi_* E \wedge E$.

If R, S are rings and F, G are formal groups over R, S, respectively, we defined in the last section a functor $\mathrm{StrictIso}(F, G)$ from rings to sets by

$$T \mapsto \{f : R \to T, g : S \to T, \phi : f^*F \to g^*G \text{ a strict isomorphism}\}$$

Similarly, we can define functors $\mathrm{Iso}(F, G), \mathrm{Hom}(F, G)$ by considering isomorphisms or homomorphisms from f^*F to g^*G in place of strict isomorphisms.

Proposition 7.1. *The functors* $\mathrm{Iso}(F, G), \mathrm{StrictIso}(F, G), \mathrm{Hom}(F, G)$ *are corepresentable.*

Proof. The functor $T \mapsto \{f : R \to T, g : S \to T\}$ is corepresented by $R \otimes S$. Pushing forward F and G via the natural maps $R \to R \otimes S, S \to R \otimes S$, we may interpret F, G as being defined over $R \otimes S$. A homomorphism between them is a power series $\phi(x) = a_0 x + a_1 x^2 + \dots$ satisfying

$$\phi(x +_F y) = \phi(x) +_G \phi(y)$$

Expanding the two sides of this equation as power series in x and y and equating the coefficients of the monomials $x^i y^j$ gives a set of relations among the a_i's. Let $I \subset R \otimes S[a_0, a_1, \dots]$ be the ideal generated by these relations. Then it is easy to see that:

- $\mathrm{Hom}(F, G)$ is corepresented by $R \otimes S[a_0, a_1, \dots]/I$
- $\mathrm{Iso}(F, G)$ is corepresented by $R \otimes S[a_0^{\pm 1}, a_1, \dots]/I$
- $\mathrm{StrictIso}(F, G)$ is corepresented by $R \otimes S[a_0^{\pm 1}, a_1, \dots]/(I + (a_0 - 1))$

$\square$

Recall from the last section that MU_*MU corepresents the functor $\mathrm{StrictIso}(F_{univ}, G_{univ})$. This reliance on strict isomorphisms is a weird facet of the grading. Eventually we wll restrict our attention to complex oriented theories E for which $\pi_2 E$ contains a unit, e.g. K-theory. In such cases the formal group law is defined over $\pi_0 E$, and the "model" in such cases is that $\pi_0(E_1 \wedge E_2)$ corepresents $\mathrm{Iso}(F_1, F_2)$. In other words, there is always a natural transformation $\mathcal{R}ing(\pi_0(E_1 \wedge E_2), -) \to \mathrm{Iso}(F_1, F_2)$, and in good cases this natural transformation is an isomorphism. For example, after formulating an appropriate definition of "flat formal group law" we will prove:

Theorem 7.2. *The natural transformation* $\mathcal{R}ing(\pi_0(E_1 \wedge E_2), -) \to \mathrm{Iso}(F_1, F_2)$ *is an isomorphism if one of* F_1, F_2 *is flat.*

The Hopf algebroid $(H\mathbb{Z}/2_*, H\mathbb{Z}/2_* H\mathbb{Z}/2)$**.** The original impetus for thinking about formal groups in the context of cohomology theories came from an observation of Atiyah and Hirzebruch concerning the dual Steenrod algebra [AH]. Consider imitating the analysis of complex oriented cohomology theories but with "real orientations" instead of complex orientations. In other words, we let $\mathbb{R}P^\infty$ play the role of $\mathbb{C}P^\infty$, define a real orientation of a cohomology theory E to be a class in $E^1(\mathbb{R}P^\infty)$ whose restriction to $\mathbb{R}P^1$ is the suspension of the class $1 \in E^0(S^0)$, and associate a formal group to a real oriented cohomology theory using the map $\mathbb{R}P^\infty \times \mathbb{R}P^\infty \to \mathbb{R}P^\infty$ which classifies the Whitney sum of real vector bundles.

The formal group associated to the mod-2 Eilenberg-MacLane spectrum $H\mathbb{Z}/2$ is the additive formal group law over the field $\mathbb{F}_2$. Over $H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$ we have two "orientations"

$$\begin{tikzcd}
 & & (\Sigma H\mathbb{Z}/2) \wedge H\mathbb{Z}/2 \\
\mathbb{R}P^\infty \arrow[r, "x"] \arrow[urr, bend left, "x_L"] \arrow[drr, bend right, "x_R"'] & \Sigma H\mathbb{Z}/2 \arrow[ur] \arrow[dr] & \\
 & & H\mathbb{Z}/2 \wedge (\Sigma H\mathbb{Z}/2)
\end{tikzcd} .$$

We know x_R can be written as a power series $f(x_L)$, where $f(t) = t +$ higher order terms. Taking into account the formal group law, f has to satisfy

$$f(x_L + y_L) = x_R + y_R = f(x_L) + f(y_L)$$

i.e. $f(s+t) = f(s) + f(t)$. Since we are working in characteristic 2, this implies

$$f(t) = \sum_{n\geq 0} \zeta_n t^{2^n},$$

where $\zeta_0 = 1$. In fact, the functor $\mathrm{StrictIso}(\mathbb{G}_a, \mathbb{G}_a)$ on $\mathbb{F}_2$-algebras is corepresented by the ring $\mathbb{F}_2[\zeta_1, \zeta_2, \dots]$ with $|\zeta_n| = 2^n - 1$. The above formula for $f(t)$ defines a map

$$\mathbb{F}_2[\zeta_1, \zeta_2, \dots] \to (H\mathbb{Z}/2)_* H\mathbb{Z}/2,$$

and Milnor's computation of the dual Steenrod algebra $A_* = (H\mathbb{Z}/2)_* H\mathbb{Z}/2$ shows that this map is an isomorphism.

Consider the three maps $H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \rightrightarrows H\mathbb{Z}/2 \wedge H\mathbb{Z}/2 \wedge H\mathbb{Z}/2$ obtained by smashing the unit map $\iota : S^0 \to H\mathbb{Z}/2$ on the first, second, or third factor with the identity map on the remaining two factors. Applying π_*, we get three maps $A_* \rightrightarrows A_* \otimes A_*$, namely $a \mapsto 1 \otimes a, a \mapsto \Psi a$, and $a \mapsto a \otimes 1$, respectively. We may use the composition law for strict isomorphisms of $\mathbb{G}_a$ to work out the formula for Ψ. If our two strict isomorphisms are

$$\begin{aligned} f(t) &= \sum (\zeta_n \otimes 1) t^{2^n} \\ g(t) &= \sum (1 \otimes \zeta_n) t^{2^n} \end{aligned}$$

then their composition $g \circ f$ is

$$\begin{aligned} g \circ f(t) &= \sum_{m,n} (1 \otimes \zeta_n)((\zeta_m \otimes 1) t^{2^m})^{2^n} \\ &= \sum_{m,n} (\zeta_m^{2^n} \otimes \zeta_n) t^{2^{m+n}} \end{aligned}$$

Hence

$$\Psi(\zeta_n) = \sum_{i+j=n} \zeta_i^{2^j} \otimes \zeta_j$$

which is the familiar formula for the coproduct in the dual Steenrod algebra.

To extend this analysis to odd primes p, in place of $H^*(\mathbb{R}P^\infty; \mathbb{Z}/2)$ we would consider $H^*(B\mathbb{Z}/p; \mathbb{Z}/p) = E[a] \otimes P[b]$, where a, b are classes in H^1, H^2, respectively. The comultiplication is given by $a \mapsto a \otimes 1 + 1 \otimes a, \quad b \mapsto b \otimes 1 + 1 \otimes b$, which is like an additive formal group but with two variables instead of one. To deal with this, we introduce the category of "super-rings", i.e. $\mathbb{Z}/2$-graded rings with multiplication satisfying the graded commutativity relation $ab = (-1)^{|a||b|} ba$. In the commutative case, we have affine n-space $\mathbb{A}^n$ whose ring of functions is $k[x_1, \dots, x_n]$, so that $H^*(\mathbb{R}P^\infty)$ can be identified with the ring of functions on $\mathbb{A}^1$. The additive group structure on $\mathbb{A}^1$ gives rise to a group $\mathbb{G}_a$ whose formal completion is the formal group law over H_*. In the "super" case, we may think of $P[x_1, \dots, x_k] \otimes E[y_1, \dots, y_\ell]$ as the ring of functions on affine (k, ℓ)-space $\mathbb{A}^{k,\ell}$. This identifies $H^*(B\mathbb{Z}/p)$ with the ring of functions on $\mathbb{A}^{1,1}$, and the additive group structure on $\mathbb{A}^{1,1}$ gives rise to a "supergroup" $\mathbb{G}_a^{1,1}$ whose formal completion is the formal group law over H_*.

Exercise 7.3. Show that $\mathrm{StrictIso}(\mathbb{G}_a^{1,1}, \mathbb{G}_a^{1,1})$ is corepresented in the category of $\mathbb{F}_p$-superalgebras by the mod-p dual Steenrod algebra.

Remark 7.4. We have indicated how *isomorphisms* of the formal group associated to a complex oriented cohomology theory are related to stable operations in that cohomology theory. It turns out that *endomorphisms* of the formal group are related to unstable operations. For example, in K-theory, upon completing at a prime p, one has unstable Adams operations ψ^k for any $k \in \mathbb{Z}_p$, corresponding to the "multiplication by k" endomorphism of the formal group. These operations extend to stable operations only in the case where k is a p-adic unit, i.e. only if the endomorphism is an automorphism of the formal group.

The formal group associated to K-theory. Let K be the spectrum representing K-theory. Bott periodicity tells us that $\pi_*(K) = \mathbb{Z}[v, v^{-1}]$, where $|v| = 2$. The isomorphism $\pi_2(K) \xrightarrow{\sim} \tilde{K}^0(S^2)$ maps v to $1 - L$, where L is the restriction to $\mathbb{C}P^1 = S^2$ of the tautological line bundle $\mathbb{L}$ over $\mathbb{C}P^\infty$. Therefore the class $x = v^{-1}(1 - \mathbb{L}) \in \tilde{K}^2(\mathbb{C}P^\infty)$ is a complex orientation for K-theory.

Remark 7.5. We could also take $x' = 1 - \mathbb{L}$ as our complex orientation, leading to a formal group law over $\pi_0 K$.

To determine the formal group law associated to K-theory, we begin by observing that the map $K^2(\mathbb{C}P^\infty) \to K^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$ sends $v^{-1}(1 - \mathbb{L})$ to $v^{-1}(1 - \mathbb{L} \otimes \mathbb{L})$. In $K^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$, let $x = v^{-1}(1 - \mathbb{L} \otimes 1),\ y = v^{-1}(1 - 1 \otimes \mathbb{L})$. Then

$$\begin{aligned} \mathbb{L} \otimes \mathbb{L} &= (1 - vx)(1 - vy) \\ &= 1 - v(x + y - vxy) \\ v^{-1}(1 - \mathbb{L} \otimes \mathbb{L}) &= x + y - vxy \end{aligned}$$

hence the formal group law is $x \mapsto x + y - vxy$. Let's call this formal group law $\mathbb{G}_m^v$. If we set $v = 1$ we get the formal group law

$$x \mapsto x + y - xy,$$

which is called the multiplicative formal group law and is denoted by $\mathbb{G}_m$. We will see later that the formal group law for K-theory is flat, hence $\pi_0(K \wedge K)$ corepresents $\mathrm{Iso}(\mathbb{G}_m, \mathbb{G}_m)$.

To calculate $\mathrm{Iso}(\mathbb{G}_m, \mathbb{G}_m)$ we consider power series of the form

$$1 - (1 - x)^b = bx - \binom{b}{2} x^2 + \binom{b}{3} x^3 - \dots$$

These determine endomorphsms of $\mathbb{G}_m$ which if b is invertible will be isomorphisms. Let

$$\mathbb{Z}[b_0^{\pm 1}, b_1, \dots]/rels \hookrightarrow \mathbb{Q}[b^{\pm 1}]$$

$$b_{n-1} \mapsto \binom{b}{n},$$

where the above relations are imposed by the requirment that the homomorphism defined above be a monomorphism. Since $\mathbb{Z}[b_0^{\pm 1}, b_1, \dots]/rels$ is constructed to represent some automorphisms of $\mathbb{G}_m$ there is a naural transformation of functors

$$\operatorname{Spec} \mathbb{Z}[b_0^{\pm 1}, b_1, \dots]/rels \to \mathrm{Iso}(\mathbb{G}_m, \mathbb{G}_m).$$

Proposition 7.6. *This natural transformations is an isomorphism.*

$$\operatorname{Spec} \mathbb{Z}[b_0^{\pm 1}, b_1, \dots]/rels \cong \mathrm{Iso}(\mathbb{G}_m, \mathbb{G}_m).$$

Note 7.7. Clark and Adams prove this from a different perspective.

Logarithm of a FGL. Let G be a formal group law over a ring R and $l(x)$ a power series which defines a strict isomorphism (i.e. $l'(0) = 1$) of G with the additive formal group law $\mathbb{G}_a$:

$$l(x +_G y) = l(x) + l(y).$$

Taking the derivative of the above equation with respect to y and then evaluating at $y = 0$ we calculate

$$\frac{d}{dy}|_{y=0} : l'(x +_G y) G_y(x, 0) = 1.$$

$$l'(x) G_y(x, 0) = 1$$

$$l'(x) = \frac{1}{G_y(x, 0)}$$

$$l(x) = \int \frac{1}{G_y(x,0)} dx.$$

For all G we call the power series $\int \frac{1}{G_y(x,0)} dx$ the logarithm of G. This series though is not necessarily defined over R, but rather over $R \otimes \mathbb{Q}$, because integrating the coefficient of x^n will require dividing by n. By the above calculation the logarithm defines an isomorphism of G with $\mathbb{G}_a$ if and only if this power series is defined over R. The differential $\frac{1}{G_y(x,0)} dx$, which is always defined over R, is called the **invariant differential** of G.

Example 7.8. $\mathbb{G}_m(x,y) = 1 - (1-x)(1-y)$.

$$G_y(x,0) = 1 - x$$

$$log_{\mathbb{G}_m}(x) = \int \frac{1}{1-x} dx = \sum \frac{x^n}{n}$$

This calculation proves that over any $\mathbb{F}_p$-algebra R the formal group laws $\mathbb{G}_m$ and $\mathbb{G}_a$ are not isomorphic.

Exercise 7.9. Work out the relationship between

- $\pi_* H\mathbb{Q} \wedge E$ and $\mathrm{Iso}(\mathbb{G}_a, G_E)$.
- $\pi_0 H\mathbb{Q}[u^{\pm 1}] \wedge E$ and $\mathrm{Iso}(\mathbb{G}_a, G_E)$.

8. STACKS

We start with an example to give intuition. Let X be a topological space, we think of X as a category whose objects are the open sets of X and morphisms inclusions, denoted $\mathcal{C}_X$. For a topological group G consider the assignment to each open $U \hookrightarrow X$ the groupoid of principle G-bundles and isomorphisms. To each inclusion $V \hookrightarrow U$ in X there corresponds a pullback functor from principle G-bundles over U to those over V. This assignment is (almost) a functor from $\mathcal{C}_X$ to groupoids.

Definition 8.1. *A **sheaf** on X is a contravariant functor $F : \mathcal{C}_X \to Sets$ satisfying the "sheaf condition" :*

If $\{U_i\}$ is a covering of U then

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j)$$

is an equalizer sequence.

The simplest example of a sheaf on X assigns to each open U the set of continuous real valued functions on U.

Example 8.2. (**Continued**) The assignment to the open set U the groupoid of principle G-bundles consists of both the assignment of the set of objects and the set of morphisms in the groupoid. Each of these assignments is a sheaf (in as much as it is an actual functor to sets).

(We will mention the following technical difficulties but will ignore them for now:

- The collection of G-bundles on U is a class and not a set.
- Pullback of G-bundles is not functorial. Given $E \to U$ a principle G-bundle over U and

$$W \overset{i}{\hookrightarrow} V \overset{j}{\hookrightarrow} U$$

it is true that $(j \circ i)^* E \cong j^* \circ i^* E$ but $(j \circ i)^* E \neq j^* \circ i^* E$ as G-bundles over W.)

Even though both the object and morphism assignment are sheaves there is an even stronger "lifting" property that holds for this groupoid valued functor. The stronger lifing property roughly says that given a cover and objects on each member of the cover which are isomorphic (not necessarily equal) on the intersections then there is a lifting. More precisely, given a cover of U by $\{U_i\}$, G-bundles $E_i \to U_i$, and isomorphisms we call "gluing data" $\alpha_{ij} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$ satisfying the "cocycle condiditon" $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$, there is a principle G-bundle $E \to U$ and isomorphisms $f_i : E|_{U_i} \to E_i$, compatible with the gluing data:

$$\begin{tikzcd} E_i|_{U_i \cap U_j} \arrow[dr, "\alpha_{ij}"'] & E|_{U_i \cap U_j} \arrow[l, "f_i"'] \arrow[d, "f_j"] \\ & E_j|_{U_i \cap U_j}. \end{tikzcd}$$

We would like to generalize this stronger sheaf property. If (X_0, X_1) is a sheaf of groupoids and $\{U_i\}$ a covering of U, let $Desc_{\{U_i\}}$ be the "category of descent datum" for the covering $\{U_i\}$.

- **Objects:** collections of objects $E_i \in X_0(U_i)$ together with isomorphisms $\alpha_{ij} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$ in $X_1(U_i \cap U_j)$ satisfying the "cocycle condiditon" $\alpha_{jk} \circ \alpha_{ij} = \alpha_{ik}$.
- **Morphisms:** $Desc_{\{U_i\}}((E_i, \alpha_{ij}), (E_i', \alpha_{ij}'))$ consists of "morphisms" $f_i : E_i \to E_i'$ in $X_1(U_i)$ which are compatible with α_{ij} in the sense that:

$$\begin{tikzcd} E_i|_{U_i \cap U_j} \arrow[r, "f_i"] \arrow[d, "\alpha_{ij}"] & E_i'|_{U_i \cap U_j} \arrow[d, "\alpha_{ij}'"] \\ E_j|_{U_i \cap U_j} \arrow[r, "f_j"] & E_j'|_{U_i \cap U_j} \end{tikzcd}$$

  commutes.
- **Gluing Property - Descent Condition:** The functor

$$(X_0, X_1)(U) \to Desc_{\{U_i\}}$$

  is an equivalence of groupoids.

Definition 8.3. *A **stack** on X is a sheaf of groupoids (X_0, X_1) satisfying the descent condition for each open cover $\{U_i \to U\}$.*

Remark 8.4. A Hopf algebroid will have a stack associated to it. Also, a stack turns out to be a like a space - it has coverings and cohomological invariants.

Reference 8.5. The reader is referred to Mumford's article [Mu] for a nice discussion of stacks (which never actually uses the word 'stack', however).

Example 8.6. How do we make principle G-bundles into a stack?

- Let $\mathcal{E}$ be the category whose objects are pairs (U, E_U) where E_U is a principle G-bundle over U. Morphisms in $\mathcal{E}((V, E_V), (U, E_U)$ are pullback squares:

$$\begin{tikzcd} E_V \arrow[r] \arrow[d] & E_U \arrow[d] \\ V \arrow[r] & U \end{tikzcd}.$$

- Consider the projection funtor $\mathcal{E} \to \mathcal{C}_X$. Let $\mathcal{C}_{X/U}$ denote the "over category of U".
- The category of sections of

$$\begin{tikzcd} & \mathcal{E} \arrow[d] \\ \mathcal{C}_{X/U} \arrow[r] \arrow[ur, dashed] & \mathcal{C}_X \end{tikzcd}$$

  is a groupoid. The assignment which sends U to this category of sections is a sheaf of groupoids (and a stack).

If $\mathcal{M}$ and $\mathcal{N}$ are stacks on X then $Stacks(\mathcal{M}, \mathcal{N})$ is the set of functors between them (as members of $Groupoids^{\mathcal{C}_X}$). Considering in addition the natural transformations we see that Stacks over X is a 2-category.

Stacks on a Grothendieck Topology. More generally we can define stacks over any category with a Grothendieck topology.

Definition 8.7. *A **Grothendieck topology** on a category $\mathcal{C}$ (with finite limits) is a "notion of covering" J. Here J is a collection of sets $\{U_i \to U\}$ called "coverings" which satisfy:*

1. *Isomorphisms are coverings.*
2. *Transistivity: $\{U_i \to U\}$ and $\{V_{ij} \to U_i\}$ are coverings implies $\{V_{ij} \to U\}$ is also a covering.*
3. *$\{U_i \to U\}$ covering and $V \to U$ any map implies $\{V \times_U U_i \to V\}$ is also a covering.*

Example 8.8.

1. X is a space, $\mathcal{C}_X$ is equiped with the usual notion of cover.
2. $\mathcal{C}$ is the category of all spaces with the usual notion of open covering.
3. $(Spaces)_{et}$ is the category of spaces where coverings are collections of maps $\{U_i \to U\}$, whose image is an open cover of U, and in which each $U_i \to U$ is a covering space of its image. Note that the "open subsets" in this example have nontrivial automorphisms, unlike the above cases.

Definition 8.9. *Let $(\mathcal{C}, J)$ be a category with a Grothendieck topology. A **sheaf** on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to Sets$ with the property that for any cover $\{U_i \to U\}$ the sequence*

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)$$

is an equalizer.

Example 8.10 (Affine schemes in the flat topology)**.**
Let $\mathcal{C} = (Rings)^{op}$. Coverings are the opposite of collections $\{R \to R_i\}$ where

1. Each $R \to R_i$ is flat.
2. (Faithful) If M is an R-module such that $M \otimes_R R_i = 0$ for all i, then $M = 0$.

Verify that this is a Grothendieck topology.

Example 8.11. $\mathcal{C}$ is Spaces, usual open covers, G is a topological group. $\mathcal{B}G$ is the stack of principle G-bundles. $\mathcal{B}G$ is equivalent to the fundamental groupoid of BG^X (if X is a reasonable space).

Exercise 8.12. Check this. Is it true in general or just for G discrete? (Difficulty may arise in $X \mapsto$(morphisms in fundamental groupoid of BG^X) being a sheaf?)

Groupoid Objects in $\mathcal{C}$. A category in $\mathcal{C}$ consists of a pair of objects $X_0, X_1 \in Ob(\mathcal{C})$ corresponding to the sets of objects and morphisms in a category, and morphisms corresponding to the usual functions between these sets....

$$\begin{tikzcd} X_1 \arrow[r, bend left, "range"] \arrow[r, bend right, "domain"'] & X_0 \arrow[l, "id"'] \end{tikzcd}.$$

$$X_1 \times_{X_0} X_1 \xrightarrow{composition} X_1$$

Given a category we can construct a simplicial set called the nerve of the category by setting

$$(N\mathcal{C})_n = \{x_0 \to \ldots \to x_n \in \mathcal{C}\}$$

(with face and degeneracy maps given by composition and the insertion of identity morphisms). Similarly given a category in $\mathcal{C}$ the same nerve construction assigns to it a simplicial object in $\mathcal{C}$. A simplical object is the nerve of a category if and only if the map $X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$ is an isomorphism. So equivalently we can think of a category object in $\mathcal{C}$ as a simplicial object satisfying the above condition (assuming $\mathcal{C}$ has fiber products).

A groupoid in $\mathcal{C}$ is a category in $\mathcal{C}$ (X_0, X_1) together with a map $i : X_1 \to X_1$ corresponding to the inverse of a morphism and satisfying the obvious properties.

A simplicial set is the nerve of a groupoid if and only if any one of the following hold:

1. It is the nerve of a category and it is a Kan complex.
2. The map $X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ (where the product is over d_1 and on the ith component the map $X_n \to X_1$ is the one induced by $(0 \mapsto 0, 1 \mapsto i)$ check this !!)
3. The nerve is a cyclic set in the sense of Connes.

The representable functors associated to groupiods in $\mathcal{C}$ are sheaves in groupoids (or groupoid valued sheaves) which is our starting point for the discussion of stacks.

Associated Stacks. From here onward we assume that the Grothendieck topology $(\mathcal{C}, J)$ is subcanonical (this means that the representable functors are sheaves).

The *associated stack* to a sheaf of groupoids (X_0, X_1) is the stack $\mathcal{M}_{(X_0,X_1)}$ nearest to (X_0, X_1) from the right. Morally:

$$ShGroupoids((X_0, X_1), \mathcal{N}) = Stacks(\mathcal{M}_{(X_0,X_1)}, \mathcal{N}).$$

(Really this is a 2-category equivalence....will explain latter.)

Definition 8.13. $\mathcal{M}_{(X_0,X_1)}(U) = \operatorname{colim} Desc_{\{U_i \to U\}}(X_0, X_1)$.

Example 8.14. $X_0 = pt.$ and $X_1 = G$. Given $\{U_i \to X\}$, $Desc_{\{U_i\}}(pt, G)$ is the category of priciple G-bundles over X together with a trivialization over $\{U_i\}$. So we have $\mathcal{M}_{(pt,G)} = \mathcal{B}G$.

Example 8.15. Suppose S is a space and G is a group acting on S. $(S, S \times G)$ is a groupoid in the category of spaces. $\mathcal{M}_{(S,S\times G)}$ is called the "orbifold" of this group action.

Example 8.16. $\mathcal{C}$ is the category of smooth manifolds (no limits, strictly speaking can't have Grothendieck topology). G is a Lie group, $\mathfrak{g}$ the associated Lie algebra, and Ω^1 the sheaf of 1-forms. The sheaf G acts on $\Omega^1 \otimes \mathfrak{g}$ by $(a, \omega) \mapsto a\omega a^{-1} + da \cdot a^{-1}$. $(\Omega^1 \otimes \mathfrak{g}, G \times \Omega^1 \otimes \mathfrak{g})$ gives rise to a sheaf of groupoids. The associated stack $\mathcal{M}_{(\Omega^1\otimes\mathfrak{g}, G\times\Omega^1\otimes\mathfrak{g})}$ is called the stack of G-bundles with connections.

Morphisms of Stacks.

Definition 8.17. *A **2-category** consists of:*

1. *A collection $\mathcal{C}$ of objects.*
2. *For $x, y \in \mathcal{C}$ a category of morphisms denoted $\mathcal{C}(x, y)$.*
3. *A "composition law" functor $\mathcal{C}(x, y) \times \mathcal{C}(y, z) \to \mathcal{C}(x, z)$ which is associative.*

Example 8.18. The category CAT of (small) categories.

Definition 8.19. *The objects of $\mathcal{C}(x, y)$ are called* 1*-morphisms. The morphisms of $\mathcal{C}(x, y)$ are called* 2*-morphisms.*

We have embedding functors

$$\mathcal{C} \hookrightarrow Shv(\mathcal{C}, J) \hookrightarrow Stacks(\mathcal{C}, J)$$

where the last inclusion assigns to a sheaf of sets the sheaf of groupoids whose objcts are the memeber of the set with only identity morphisms (check this is a stack). So we can think of $\mathcal{C}$ as a subcategory of $Stacks(\mathcal{C}, J)$, or $Stacks(\mathcal{C}, J)$ as an enlargement of $\mathcal{C}$ and we will try to extend the notions and constructions from $\mathcal{C}$. The Yondeda lemma implies that for $X \in \mathcal{C}, \mathcal{M} \in Stacks(\mathcal{C}, J)$ we have

$$Stacks(X, \mathcal{M}) = \mathcal{M}(X).$$

The moral is we have inflated $\mathcal{C}$ so as to incorporate new "classifying spaces". A *representable stack* is an object of $\mathcal{C}$ regarded as a stack. (**MY Note** or equivalent to one??). We will extend this now to the notion of a *representable morphism.*

Definition 8.20. *Let*

$$\begin{tikzcd} & \mathcal{M}_2 \arrow[d, "j"] \\ \mathcal{M}_1 \arrow[r, "i"] & \mathcal{N} \end{tikzcd}$$

be a diagram of stacks. The **2-category fiber product** $\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2$ *is given by:*

1. $Ob(\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2)(X) = \{a \in \mathcal{M}_1, b \in \mathcal{M}_2, \phi : i(a) \cong j(b)\}$.
2. $Mor(\mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2)(X) =$ *what you would guess.*

The diagram

$$\begin{tikzcd} \mathcal{M}_1 \times_{\mathcal{N}} \mathcal{M}_2 \arrow[r] \arrow[d] & \mathcal{M}_2 \arrow[d, "j"] \\ \mathcal{M}_1 \arrow[r, "i"] & \mathcal{N} \end{tikzcd}$$

satisfies a certain univeral property, but does not commute (we will talk about this more latter).

Example 8.21.

$$\begin{tikzcd} & pt \arrow[d, "j"] \\ pt \arrow[r, "i"] & \mathcal{B}G \end{tikzcd}$$

$pt \times_{\mathcal{B}G} pt = G$—here by G we mean the stack assigning to each open the groupoid whose objects are the members of G with only identity morphisms.

Check: an object of $pt \times_{\mathcal{B}G} pt(X)$ is an automorphism of the trivial principle G-bundle over X - this is just an element of G.

Example 8.22.

$$\begin{tikzcd} & \mathcal{B}H \arrow[d, "j"] \\ X \arrow[r, "i"] & \mathcal{B}G \end{tikzcd}$$

$H \hookrightarrow G$, X a space. The map $X \to \mathcal{B}G$ classifies a priciple G bundle $E \to X$. $X \times_{\mathcal{B}G} \mathcal{B}H = E \times_G G/H = E|_H$. Check this as an exercise.

Definition 8.23. *A* 1*-morphism* $\mathcal{M} \to \mathcal{N}$ *is called* ***representable*** *if for each* $X \in \mathcal{C}$ *and each map* $X \to \mathcal{N}$ *the* 2 *category fiber product* $X \times_{\mathcal{N}} \mathcal{M}$ *is equivalent to a representable stack.*

Example 8.24.

1. If H is a subgroup of G, then $\mathcal{B}H \to \mathcal{B}G$ is representable.
2. $pt \to \mathcal{B}G$ is representable.

The principle here is to extend properties of maps in $\mathcal{C}$ to properties of representable 1-morphisms in stacks.

Example 8.25. A collection of representable 1-morphisms $\{\mathcal{N}_i \to \mathcal{N}\}$ is a *cover* if for each $X \in \mathcal{C}$ and each 1-morphism $X \to \mathcal{N}$ the collection $\{X \times_{\mathcal{N}} \mathcal{N}_i \to X\}$ is a cover.

9. Stacks and Associated Stacks

In this section we are going to do some general theorems about stacks. We want to say something about the relationship between a groupoind (X_0, X_1) on a category $\mathcal{C}$ with a Grothendieck topology $\mathcal{J}$ that is subcanonical and its associated stack $\mathcal{M}_{(X_0,X_1)}$ as defined in the last section. We denote by *Stacks* the 2-category of stacks on $(\mathcal{C}, \mathcal{J})$.

Now let $\mathcal{M}$ be a stack. Define the category $Stacks/\mathcal{M}$ of stacks over the stack $\mathcal{M}$ to be the category with

- Objects: 1-morphisms $\mathcal{N} \to \mathcal{M}$ of stacks, and
- Morphisms: $Stacks/\mathcal{M}((\mathcal{N}_1 \to \mathcal{M}), (\mathcal{N}_2 \to \mathcal{M})) = \{\text{sections } \mathcal{N}_1 \to \mathcal{N}_1 \times_{\mathcal{M}} \mathcal{N}_2\}$.

Example 9.1. Let $\mathcal{C} = \mathcal{T}op$ be the category of topological spaces, and let $\mathcal{J}$ be the usual Grothendieck topology on $\mathcal{T}op$. Let BG be the stack of principal G-bundles. Since we always identify an object with the stack it represents, we have $\mathcal{T}op/BG \subset Stacks/BG$. Explicity, $\mathcal{T}op/BG$ is the category with

- Objects: all pairs (X, E) where X is a space and E is a principal G-bundle over X, and
- Morphisms: $f : X \to Y$ together with an isomorphism $E_X \to f^*E_Y$ of principal G-bundles.

Proposition 9.2. *The category $Stacks/\mathcal{M}$ together with the notion of covering as defined in the previous section is a Grothendieck topology.*

Proof. Postponed. □

Remark 9.3. This proposition allows us to think about stacks as somewhat like spaces. In particular, we can talk about sheaves, cohomology, etc. in $Stacks/\mathcal{M}$.

Stacks and associated stacks. Fix a category $\mathcal{C}$ with a Grothendieck topology $\mathcal{J}$ that is subcanonical. Suppose that (X_0, X_1) is a representable groupoid in $\mathcal{C}$. Recall that $\mathcal{M}_{(X_0,X_1)}$ is the associated stack of (X_0, X_1).

Lemma 9.4. *The map*

$$X_1 \longrightarrow X_0 \times_{\mathcal{M}} X_0$$

induced by the domain and range maps $X_1 \to X_0$ is an isomorphism of stacks.

The proof will be given after the following remarks.

Remark 9.5. The above lemma says that

$$\begin{tikzcd} X_1 \arrow[r, "range"] \arrow[d, "dom"] & X_0 \arrow[d] \\ X_0 \arrow[r] & \mathcal{M}_{(X_0,X_1)} \end{tikzcd}$$

is a pullback square of stacks.

Remark 9.6 (Descent datum for morphisms)**.** Suppose $\{U_i \to U\}$ is a cover, and suppose (E_i, g_{ij}^E) and (F_i, g_{ij}^F) are objects in the descent category $\mathrm{Desc}_{\{U_i\}}$. What is a morphism between them? The answer is: It's what you think it is. So a morphism $h : (E_i, g_{ij}^E) \to (F_i, g_{ij}^F)$ is a collection of maps $h_i : E_i \to F_i$ compatible with the g's on intersections, i.e. on $U_{ij} = U_i \times_U U_j$ the diagram

$$\begin{tikzcd} E_i \arrow[r, "g_{ij}^E"] \arrow[d, "h_i"'] & E_j \arrow[d, "h_j"] \\ F_i \arrow[r, "g_{ij}^F"] & F_j \end{tikzcd}$$

commutes. In particular, if both g_{ij}^E and g_{ij}^F are identities for all i and j, then $h_i = h_j$ on U_{ij}.

Question: What does this tell us?
Answer: You never need to refine a cover to make a map. In down-to-earth language, we know that the h_i's agree on intersections, $U_{ij} = U_i \times_U U_j$. So the sheaf axiom tells us that $\{h_i\}$ actually comes from a global section.

Now we can prove the lemma.

Proof. Let S be an object of $\mathcal{C}$. We as usual identify it with the stack it represents. A 1-morphism $S \to X_0 \times_{\mathcal{M}_{(X_0,X_1)}} X_0$ consists of two elements $a, b \in Stacks(S, X_0) = X_0(S)$ (by the Yoneda lemma) together with an isomorphism $a \to b$ in $\mathcal{M}_{(X_0,X_1)}(S)$. By the construction of the associated stack, giving such an isomorphism is equivalent to giving a cover $\{U_i \to S\}$ of S and an isomorphism in the descent category $\mathrm{Desc}_{\{U_i\}}$. The above remark implies that $h_i = h_j$ on U_{ij}, so we can paste them together to get an element of $X_1(S)$, or equivalently, a 1-morphism $S \to X_1$. ☐

Local presentability.

Definition 9.7. *A stack $\mathcal{M}$ for which the diagonal map $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is a representable 1-morphism is called **locally presentable**.*

We have alternative characterizations of local presentability.

Proposition 9.8. *The following conditions on a stack $\mathcal{M}$ are equivalent.*

(1) $\mathcal{M}$ is locally presentable.
(2) Given representable stacks $A, B \in \mathcal{C}$ and 1-morphisms $A \to \mathcal{M}$, $B \to \mathcal{M}$ of stacks, the 2-category fiber product $A \times_{\mathcal{M}} B$ is representable.
(3) Every 1-morphism $B \to \mathcal{M}$ from a representable stack $B \in \mathcal{C}$ to $\mathcal{M}$ is representable.

Proof. Conditions (2) and (3) are clearly equivalent by definition.

(2) ⇒ (1). Let $A \in \mathcal{C}$ be a representable stack, and let $(E_1, E_2) : A \to \mathcal{M} \times \mathcal{M}$ be a 1-morphism of stacks. Let $\mathcal{N}$ be the 2-category fiber product $A \times_{\mathcal{M}\times\mathcal{M}} \mathcal{M}$.

$$\begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta"'] \\ A \arrow[r, "(E_1,E_2)"'] & \mathcal{M} \times \mathcal{M} \end{tikzcd}$$

We will show that $\mathcal{N}$ is equivalent to the stack $A \times_{\mathcal{M}} A$, which by assumption is representable. If X is any stack, then giving a 1-morphism $F : X \to \mathcal{N}$ is equivalent to giving a 1-morphism $f : X \to A$ and isomorphisms

$$f^*E_1 \xrightarrow{c_1} F \xleftarrow{c_2} f^*E_2.$$

But the groupoid of $(f, F, \text{the above sequence})$ is equivalent to the groupoid of $(f, f^*E_1 \xrightarrow{d} f^*E_2)$ via the maps

$$(c_1, c_2) \mapsto c_2^{-1} \circ c_1$$
$$d \mapsto (F = f^*E_1, c_1 = d, c_2 = id).$$

So $\mathcal{N}$ is equivalent to the stack $A \times_{\mathcal{M}} A$.

(1) ⇒ (2). Suppose given representable stacks A, B and 1-morphisms $A \xrightarrow{f} \mathcal{M}$, $B \xrightarrow{g} \mathcal{M}$ of stacks. Then $A \times B$ is also a representable stack. Now $A \times_{\mathcal{M}} B$ is representable because $\Delta : \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is by assumption locally presentable, and there is a pullback square

$$\begin{tikzcd} A \times_{\mathcal{M}} B \arrow[r] \arrow[d] & \mathcal{M} \arrow[d, "\Delta"'] \\ A \times B \arrow[r, "f\times g"] & \mathcal{M} \times \mathcal{M}. \end{tikzcd}$$

☐

Proposition 9.9. *Assume the objects of $(\mathcal{C}, \mathcal{J})$ descend (defined below). If (X_0, X_1) is a representable groupoid on $\mathcal{C}$, then the associated stack $\mathcal{M}_{(X_0,X_1)}$ is locally presentable. Conversely, if $\mathcal{M}$ is a locally presentable stack, then there exists a groupoid (X_0, X_1) on $\mathcal{C}$ with mboxdom $: X_1 \to X_0$ a cover, for which $\mathcal{M}$ is equivalent to the associated stack $\mathcal{M}_{(X_0,X_1)}$.* Is this correct?

Proof. Postponed. ☐

Remark 9.10. dom : $X_1 \to X_0$ is a cover if and only if range : $X_1 \to X_0$ is a cover. Indeed, there is an isomorphism $c : X_1 \to X_1$ (the "flip" map) of stacks that takes $f \in X_1$ to $f^{-1} \in X_1$, and there is a commutative triangle

$$\begin{tikzcd} X_1 \arrow[rr, "c"] \arrow[dr, "dom"'] & & X_1 \arrow[dl, "range"] \\ & X_0 & \end{tikzcd}.$$

Definition 9.11. *We say that in* $(\mathcal{C}, \mathcal{J})$ ***objects descend*** *if for every representable sheaf* $X \in \mathcal{C}$, $\mathcal{F}$ *a sheaf on* $\mathcal{C}$ *with a map* $\mathcal{F} \to X$, *and* $\{U_\alpha \to X\}$ *a cover of* X *in* $\mathcal{C}$, *for which each pullback* $\mathcal{F}_\alpha := U_\alpha \times_X \mathcal{F}$ *is representable, then* $\mathcal{F}$ *is also representable.*

See the following pullback diagram:

$$\begin{tikzcd} \mathcal{F}_\alpha \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ U_\alpha \arrow[r] & X. \end{tikzcd}$$

This condition says that one can test the representability of a sheaf locally. If $X \in \mathcal{C}$, let $\mathcal{C}/X$ be the groupoid with

- Objects: morphisms $Y \to X$ in $\mathcal{C}$,
- Morphisms: commutative triangles

$$\begin{tikzcd} Y_1 \arrow[rr, "\sim"] \arrow[dr] & & Y_2 \arrow[dl] \\ & X & \end{tikzcd}$$

Then the objects of $(\mathcal{C}, \mathcal{J})$ descend if and only if the functor $\mathcal{C}/- : X \mapsto \mathcal{C}/X$ is a stack on $\mathcal{C}$. Note that this functor is not contravariant, but one can rigidify it.

Remark 9.12. If $\mathcal{C}$ is the category of schemes, then objects do *not* descend.

10. More on Stacks and associated stacks

We continue our journey on stacks and associated stacks. We will prove a proposition that was stated in the previous section.

Proposition 10.1. *If* $\mathcal{C}/-$ *is a stack (i.e. the objects of* $(\mathcal{C}, \mathcal{J})$ *descend), then for any representable groupoid* (X_0, X_1), *the associated stack* $\mathcal{M} = \mathcal{M}_{(X_0,X_1)}$ *is locally presentable.*

Proof. By a proposition in the previous section, we need to show that given a diagram

$$\begin{tikzcd} & B \arrow[d, "j"] \\ A \arrow[r, "i"'] & \mathcal{M} \end{tikzcd}$$

with $A, B \in \mathcal{C}$, the pullback $A \times_{\mathcal{M}} B$ is equivalent to an object in $\mathcal{C}$ (i.e. is representable). We do this in several steps.

Case 1. $A = B = X_0$ with i, j the canonical maps. We showed that $X_0 \times_{\mathcal{M}} X_0$ is isomorphic to X_1 in the previous section, so it is representable.

Case 2. Suppose $A \to \mathcal{M}$ factors through a map $A' \to \mathcal{M}$ such that $A' \times_{\mathcal{M}} B$ is representable. Then we have the diagram

$$\begin{tikzcd} & & B \arrow[d] \\ A \arrow[r] & A' \arrow[r] & \mathcal{M}. \end{tikzcd}$$

But $A \times_{\mathcal{M}} B = A \times_{A'} (A' \times_{\mathcal{M}} B)$. Since $A' \times_{\mathcal{M}} B$ is representable, so is $A \times_{\mathcal{M}} B$. In particular, it holds if $A \to \mathcal{M}$ factors through X_0.

Case 3. Suppose there is a cover $\{U_i \to A\}$ of A in $\mathcal{C}$ such that each $U_i \times_{\mathcal{M}} B$ is representable. Then $A \times_{\mathcal{M}} B$ is also representable. Indeed, since objects of $\mathcal{C}$ descend, $A \times_{\mathcal{M}} B$ is obtained from the $U_i \times_{\mathcal{M}} B$ by descent, so it is representable.

Case 4. $B = X_0$ and j is the canonical map. By definition of $\mathcal{M} = \mathcal{M}_{(X_0,X_1)}$, a map $A \to \mathcal{M}$ is represented by a cover $\{U_i \to A\}$ of A together with maps $U_i \to X_0$, plus gluing data. So we have a diagram

$$\begin{tikzcd} \coprod U_i \arrow[r] \arrow[d] & X_0 \arrow[d] \\ A \arrow[r] & \mathcal{M} \end{tikzcd}$$

that is commutative on the nose. Now this case follows from Case 3 and the following diagram

$$\begin{tikzcd} & & X_0 \arrow[d] \\ U_i \arrow[r] & A \arrow[r] & \mathcal{M}. \end{tikzcd}$$

In more detail: There is a diagram

$$\begin{tikzcd} & & X_0 \arrow[d] \\ U_i \arrow[r] & X_0 \arrow[r] & \mathcal{M} \end{tikzcd}$$

for each i. By Case 1, $X_0 \times_{\mathcal{M}} X_0$ is representable, so Case 2 implies that $U_i \times_{\mathcal{M}} X_0$ is also representable. Now Case 3 says that $A \times_{\mathcal{M}} X_0$ is representable.

Case 5 (General case). As in Case 4, to give a map $B \to \mathcal{M}$ is to give a cover $\{V_j \to B\}$ of B together with maps $V_j \to X_0$, plus gluing data. See the following diagram.

$$\begin{tikzcd} & & A \arrow[d] \\ V_j \arrow[r] & X_0 \arrow[r] & \mathcal{M} \end{tikzcd}$$

Now $A \times_{\mathcal{M}} X_0$ is representable by Case 4. So Case 2 implies that $V_j \times_{\mathcal{M}} A$ is representable for all j. By Case 3, since we know $V_j \times_{\mathcal{M}} A$ is representable for all j, so is $A \times_{\mathcal{M}} B$. $\square$

Remark 10.2. Something is not right about this proof. Stacks are supposed to encode all the descent datum. So you don't need to talk about covers when using a stack. The proof above does not exploit the stack properties of $\mathcal{C}/-$. The diligent reader might want to find a proof that does use the stack properties of $\mathcal{C}/-$ (as a good exercise).

Question: What is special about the following map?

$$\begin{tikzcd} X_0 \arrow[d] \\ \mathcal{M} = \mathcal{M}_{(X_0,X_1)} \end{tikzcd}$$

We know that we can recover X_1 by taking the fiber product $X_0 \times_{\mathcal{M}} X_0$. Suppose $\mathcal{M}$ is locally presentable. Consider a 1-morphism $E : A_0 \to \mathcal{M}$. Define A_1 by the pullback diagram

$$\begin{tikzcd} A_1 \arrow[r, "range"] \arrow[d, "dom"'] & A_0 \arrow[d, "E"] \\ A_0 \arrow[r, "E"'] & \mathcal{M}. \end{tikzcd}$$

We claim that (A_0, A_1) forms a groupoid. Indeed,

$$\mathcal{C}(-, A_0) \qquad \text{and} \qquad \mathcal{C}(-, A_1)$$

represent a functor $\mathcal{C}^{op} \to \mathit{Groupoids}$ with

- Objects: maps $X \xrightarrow{p} A_0$,
- Morphisms: p_1 to p_2 are isomorphisms $p_1^* E \xrightarrow{\sim} p_2^* E$ in $\mathcal{M}(X)$.

Now we can form the associated stack $\mathcal{M}_{(A_0,A_1)}$ and get a 1-morphism $\mathcal{M}_{(A_0,A_1)} \to \mathcal{M}$. The above question is equivalent to the following
Question: When is this 1-morphism an equivalence of stacks?
Answer 1: When $E : A_0 \to \mathcal{M}$ is a cover.
Answer 2: When there is a cover $X \to \mathcal{M}$ for which the map $A \times_{\mathcal{M}} X \to X$ admits a section.

Proof. Postponed, as usual. □

Example 10.3. Let $\mathcal{C} = \mathit{Rings}^{op}$, the opposite category of the category of commutative rings. The Grothendieck topology $\mathcal{J}$ here is the **flat topology**. For a commutative ring R, we use the notation $\operatorname{Spec} R$ to denote the object R in $\mathcal{C}$. Then the flat topology is defined as follows: $\{\operatorname{Spec} U_i \to \operatorname{Spec} R\}$ is a cover if

- each $R \to U_i$ is flat, and
- if $M \otimes_R U_i = 0$ for all i, then $M = 0$.

Equivalently,

- each $R \to U_i$ is flat, and
- given a prime ideal p in R, there exists a prime ideal q in some U_i such that $p = q \cap R$.

<u>Example</u>. $\{\operatorname{Spec} \mathbb{Z}[\frac{1}{2}] \to \operatorname{Spec} \mathbb{Z}, \operatorname{Spec} \mathbb{Z}[\frac{1}{3}] \to \operatorname{Spec} \mathbb{Z}\}$ is a cover.

<u>Example</u>. $\{\operatorname{Spec} R[x]/f(x) \to \operatorname{Spec} R\}$ is a cover, where $f(x) = a_0 + \cdots + a_{n-1}x^{n-1} + x^n$.

Proof. The set $\{1, x, \ldots, x^{n-1}\}$ is a basis of $R[x]/f(x)$ over R, so it is free over R of rank n. Hence it is flat, and it is now clear that it is a cover. □

Claim 10.4. *For $\mathcal{C} = \mathit{Rings}^{op}$, $\mathcal{C}/-$ is a stack, i.e. objects descend.*

Proof. It is a dumb exercise in commutative algebra, so we postpone it as usual. The reader can read about faithfully flat descent in, e.g. Milne's book [Mi]. □

Hopf algebroids and stacks. A Hopf algebroid (A, Γ) gives a groupoid $(\operatorname{Spec} A, \operatorname{Spec} \Gamma)$ on $\mathcal{C}$. The domain map is a cover because η_L is flat. $\mathcal{M}_{(\operatorname{Spec} A, \operatorname{Spec} \Gamma)}$ will be denoted by $\mathcal{M}_{(A,\Gamma)}$.

Example 10.5. Take $A = \mathbb{Z}[b, c]$, $\Gamma = A[r]$. Then $(A, \Gamma)(R)$ is the groupoid of $b, c \in R$ together with maps $b \mapsto b + 2r$, $c \mapsto r^2 + br + c$. Let $\mathcal{M}$ be the associated stack $\mathcal{M}_{(A,\Gamma)}$.
Question: What is a map $\operatorname{Spec} R \to \mathcal{M}$?
Answer: To give a map $\operatorname{Spec} R \to \mathcal{M}$ is to give a faithfully flat map $R \to E$, elements $b, c \in E$, and 'gluing data': an element $r \in E \otimes_R E$ such that

(i) $b \otimes 1 = 1 \otimes b + 2r$ and $c \otimes 1 = 1 \otimes c + (1 \otimes b)r + 2r$, and
(ii) $d^0 r - d^1 r + d^2 r = 0$, where the d^i are the three maps

$$E \otimes_R E \rightrightarrows\!\!\!\to E \otimes_R E \otimes_R E$$

each of which takes the value 1 in the i^{th} position.

Question: Is there another way to define $\mathcal{M}$?
Answer: Yes, we'll actually get a much smaller presentation of it.
So let $B = \mathbb{Z}[b] = A/(c)$. Consider the map $\operatorname{Spec} B \to \mathcal{M}$, which actually factors through $\operatorname{Spec} A$.

Claim 10.6. *The map* $\operatorname{Spec} B \to \mathcal{M}$ *is a cover.*

Before proving this claim, note that it will imply (by Answer 1) that there is an equivalence $\mathcal{M} \cong \mathcal{M}_{(B,\Gamma_B)}$, the associated stack of (B, Γ_B).

Proof. We need to show that given any representable stack $\operatorname{Spec} R$ and 1-morphism $j : \operatorname{Spec} R \to \mathcal{M}$, the dotted arrow in the diagram

$$\begin{tikzcd} \operatorname{Spec} R \times_{\mathcal{M}} \operatorname{Spec} B \arrow[r] \arrow[d, dashed] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} R \arrow[r, "j"] & \mathcal{M} \end{tikzcd}$$

is a cover. Note that it is sufficient to show that there is a cover $\{U_i \to \operatorname{Spec} R\}$ of $\operatorname{Spec} R$ such that $\{U_i \times_{\mathcal{M}} \operatorname{Spec} B \to U_i\}$ is a cover for each i. See the following diagram:

$$\begin{tikzcd} & & \operatorname{Spec} B \arrow[d] \\ U_i \arrow[r] & \operatorname{Spec} R \arrow[r, "j"] & \mathcal{M} \end{tikzcd}$$

We can find a cover $\operatorname{Spec} E \to \operatorname{Spec} R$ so that

$$\begin{tikzcd} \operatorname{Spec} E \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d] \\ \operatorname{Spec} R \arrow[r] & \mathcal{M} \end{tikzcd}$$

commutes. So we may assume that $\operatorname{Spec} R \to \mathcal{M}$ factors through $\operatorname{Spec} A$. Let $\operatorname{Spec} W$ be the pullback

$$\begin{tikzcd} \operatorname{Spec} W \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d, "c=0"] \\ \operatorname{Spec} R \arrow[r, "(\beta,\gamma)"'] & \mathcal{M}. \end{tikzcd}$$

It turns out that W has to be $R[r]/(r^2 + \beta r + \gamma = 0)$, and $B \to W$ has to be the map $b \mapsto \beta + 2r$. Now it is clear that $R \to W$ is flat and gives a cover $\operatorname{Spec} W \to \operatorname{Spec} R$. $\square$

Setting $R = B$, it follows that $\Gamma_B = B[r]/(r^2 + br)$, (B, Γ_B) is a Hopf algebroid, and the map $\mathcal{M}_{(B,\Gamma_B)} \to \mathcal{M}$ is an equivalence by Answer 1.

11. Sheaves on stacks

don't think this the ef is very clear in c 8 however e def in section 9 uld be more clear

Recall that $(\mathcal{C}, \mathcal{J})$ denotes a category $\mathcal{C}$ with a Grothendieck topology $\mathcal{J}$. Also $\mathcal{S}tacks := \mathcal{S}tacks(\mathcal{C}, \mathcal{J})$ denotes the 2-category of stacks on $(\mathcal{C}, \mathcal{J})$ as defined in Section ?? . Let $\mathcal{M}$ denote a fixed object in $\mathcal{S}tacks$ and let $\mathcal{S}tacks/\mathcal{M}$ denote the over-category of the 2-category $\mathcal{S}tacks$ over $\mathcal{M}$ (see Section ??) .

Definition 11.1. *A sheaf on $\mathcal{M}$ is defined to be a sheaf on $\mathcal{S}tacks/\mathcal{M}$, that is a functor $\mathcal{S}tacks/\mathcal{M} \to \mathcal{S}et$ which satisfies the sheaf condition.*

The category of sheaves on $\mathcal{M}$ is defined to be the category with objects such sheaves and morphisms maps of sheaves.

Consider the following two functors:

$$\mathrm{Sh} : \mathcal{C}^{op} \to Groupoids; X \mapsto \text{category of sheaves on } \mathcal{C}/X \text{ and isos}$$

$$\text{Sh-map} : \mathcal{C}^{op} \to Groupoids; X \mapsto \text{cat. with obj. maps of sheaves and morphisms isos:}$$

$$\begin{tikzcd}
\mathcal{F}_0 \arrow[r] \arrow[d, "\simeq"] & \mathcal{G}_0 \arrow[d, "\simeq"] \\
\mathcal{F}_1 \arrow[r] & \mathcal{G}_1
\end{tikzcd}$$

These two functors are easily seen to satisfy the descent condition i.e. they are in fact stacks (Def 8.3??). Now the pair (Sh, Sh-map), together with the domain and range maps Sh-map $\to$ Sh form a category object in $\mathcal{S}tacks$.

Theorem 11.2. *The category of sheaves on $\mathcal{M}$ is equivalent to the category of morphisms of stacks*

$$\mathcal{M} \to (\mathrm{Sh}, \text{Sh-map})$$

That is the objects are the 1-morphisms, and the morphisms are the 2-morphisms.

h yeah?

Proof. We will prove this next time. ☐

We now proceed into what is hopefully an oasis of truth:

Definition 11.3. *A sheaf on a groupoid (X_0, X_1) is defined to be the following data:*

1. *A sheaf $\mathcal{F}$ on X_0*
2. *An isomorphism $\mathrm{Domain}^* \mathcal{F} \to \mathrm{Range}^* \mathcal{F}$ satisfying the cocycle condition for sheaves on (X_0, X_1) (see below).*

The category of sheaves on a groupoid (X_0, X_1) is denoted $Shv(X_0, X_1)$.

Definition 11.4 (The cocycle condition for sheaves on (X_0, X_1))**.** *Let $p_1, p_2 \in X_1$ be elements such that $\mathrm{Range}(p_1) = \mathrm{Domain}(p_2)$. Composition $\mu : X_1 \times_{X_0} X_1 \to X_1$ gives an element $\mu(p_1, p_2) \in X_1$ with domain $\mathrm{Domain}(p_1)$ and range $\mathrm{Range}(p_2)$. The cocycle condition demands that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Domain}(p_1)^*\mathcal{F} \arrow[r] \arrow[drr, "\mu(p_1,p_2)"'] & \mathrm{Range}(p_1)^*\mathcal{F} \arrow[r, "\simeq"] & \mathrm{Domain}(p_2)^*\mathcal{F} \arrow[d] \\
& & \mathrm{Range}(p_2)^*\mathcal{F}
\end{tikzcd}$$

Proposition 11.5. *The functor $Shv(\mathcal{M}_{(X_0,X_1)}) \to Shv(X_0, X_1)$ is an isomorphism of categories.*

Proof. For any stack $\mathcal{N}$, we have, by definition of stackification as left adjoint to the forgetful functor, that $\mathcal{S}tacks(\mathcal{M}_{(X_0,X_1)}, \mathcal{N}) = Groupoid((X_0, X_1), \mathcal{N})$. Now let $\mathcal{N}$ be the stack (Sh, Sh-map) from before. The left-hand side gets identified with sheaves on $\mathcal{M}_{(X_0,X_1)}$ by Theorem 11.2 and writing out what the right-hand side means, gives the requirements for a sheaf on (X_0, X_1). ☐

Quasi-coherent sheaves on stacks on $\mathcal{A}ff$. Let $(\mathcal{C}, \mathcal{J}) = (\mathcal{A}ff, flat)$ be the Grothendieck category of affine schemes with the flat topology. A *quasi-coherent sheaf* $\mathcal{F}$ on $\mathcal{A}ff/\operatorname{Spec}(R)$ is a sheaf on $\mathcal{A}ff/\operatorname{Spec}(R)$ such that for all (flat?) ring homomorphisms $R \to S$ we have functorial maps and isomorphisms:

$$\begin{tikzcd}
S \otimes \mathcal{F}(\operatorname{Spec}(S)) \arrow[r] & \mathcal{F}(\operatorname{Spec}(S)) \\
S \otimes (S \otimes_R \mathcal{F}(\operatorname{Spec}(R))) \arrow[r] \arrow[u, squiggly, "\simeq"'] & S \otimes_R \mathcal{F}(\operatorname{Spec}(R)) \arrow[u, squiggly, "\simeq"']
\end{tikzcd}$$

So, the category of quasi-coherent sheaves on $\mathcal{A}ff/\operatorname{Spec}(R)$ is equal to the opposite category of R-modules.

Using this definition of quasi-coherent sheaves on $\mathcal{A}ff/\operatorname{Spec}(R)$ we now get an obvious definition of the pair $(\mathrm{Sh}_{q.-c.}, \text{Sh-map}_{q.-c.})$ by requiring $(\mathrm{Sh}, \text{Sh-map})$ to take values in quasi-coherent sheaves. We now define a *quasi-coherent sheaf on a stack $\mathcal{M}$ on $\mathcal{A}ff$* to be a 1-morphism $\mathcal{M} \to (\mathrm{Sh}_{q.-c.}, \text{Sh-map}_{q.-c.})$.

Let (A, Γ) be a Hopf algebroid, and let $\mathcal{M}_{(A,\Gamma)}$ be the corresponding stack on $\mathcal{A}ff$. The following proposition is establishes the fundamental link between stacks and algebraic topology.

Proposition 11.6. *We have an equivalence of categories*

$$\{\text{ quasi-coherent sheaves on } \mathcal{M}_{(A,\Gamma)}\} \leftrightarrow (A, \Gamma) - \text{ comodules}$$

Proof. This is really easy, but I'll leave it to you. ∎

I think we could in this one, if wanted...

Let us at least explain this equivalence: Let $p : \operatorname{Spec} A \to \mathcal{M}_{(A,\Gamma)}$ be the map induced by the counit $\operatorname{Spec} A \to \operatorname{Spec} \Gamma$ composed with the stackification. We have a pull-back square

$$\begin{tikzcd}
\operatorname{Spec} \Gamma \arrow[r] \arrow[d] & \operatorname{Spec} A \arrow[d, "p"] \\
\operatorname{Spec} A \arrow[r, "p"] & \mathcal{M}
\end{tikzcd}$$

so A corresponds to $\mathcal{O}_{\mathcal{M}}$ and Γ corresponds to $p_*\mathcal{O}_{\operatorname{Spec}(A)}$ since $\operatorname{Spec}(\Gamma)$ is the pushforward of the pullback.

I'm afraid this planation is not ev quasi-coherent

12. A calculation and the link to topology

A calculation. Let $A = \mathbb{Z}[b, c]$, $\Gamma_A = A[r]$, and let (A, Γ) be the associated Hopf algebroid from Example 10.5??. That is (A, Γ) is the functor $\mathcal{R}ing \to$ Groupoid which to each ring R associates the groupoid with objects quadratic expressions $x^2 + bx + c$,$b, c \in R$ and morphisms $x^2 + bx + c \mapsto x^2 + b'x + c'$ induced by a translation coordinate changes $x \mapsto x + r$,$r \in R$ (i.e. $b \mapsto b + 2r$, $c \mapsto r^2 + br + c$). Likewise we want to consider (B, Γ_B), where $B = \mathbb{Z}[b]$ and $\Gamma_B = B[r]/(r^2 + br)$. In Section 10?? we saw the following theorem:

Theorem 12.1. *The stacks $\mathcal{M}_{(A,\Gamma_A)}$ and $\mathcal{M}_{(B,\Gamma_B)}$ are equivalent. Especially the category of quasi-coherent sheaves on $\mathcal{M}_{(A,\Gamma_A)}$ and $\mathcal{M}_{(B,\Gamma_B)}$ coincide.*

Using our equivalence of categories from last time we get the following corollary.

Corollary 12.2. *The category of (A, Γ_A) comodules is equivalent to the category of (B, Γ_B)-comodules. Especially*

$$\operatorname{Ext}^*_{(A,\Gamma_A)}(A, A) = \operatorname{Ext}^*_{(B,\Gamma_B)}(B, B).$$

In the first part of this lecture we will be concerned with calculating $\mathrm{Ext}^*_{(B,\Gamma_B)}(B,B)$. From now on we set $\Gamma := \Gamma_B$. We want to find a resolution

$$0 \to B \to I_0 \to I_1 \to \cdots$$

where the I_n have the property that $\mathrm{Ext}^s_{(B,\Gamma)}(B, I_n) = 0$ for all $s > 0$. By a spectral sequence argument we see that such a resolution can be used just as an injective resolution to calculate Ext, i.e. that

$$\mathrm{Ext}^s_{(B,\Gamma)}(B,B) = H^s(\mathrm{Hom}_{(B,\Gamma)}(B, I_*))$$

We want to see that we can construct a resolution with the above properties by setting $I_n = \Gamma$.

Lemma 12.3. *Let $B \to \Gamma$ be the inclusion $b \mapsto b + 0r$ (equal to η_L) and let $p : \Gamma \to B$ be the projection $b + b'r \mapsto b'$. Then*

$$0 \to B \xrightarrow{\eta_L} \Gamma \xrightarrow{p} B \to 0$$

is a short exact sequence of (B,Γ)-comodules.

Proof. Since exactness in the category of (B,Γ)-comodules is the same as exactness in abelian groups, the only thing which needs to be justified is that η_L and p are maps of (B,Γ)-comodules.

To prove that η_L is a map of (B,Γ)-comodules we to check that the following diagram commutes:

$$\begin{tikzcd} B \arrow[r, "\eta_L"] \arrow[d, "\eta_L"] & \Gamma \arrow[d, "\psi"] \\ \Gamma = \Gamma \otimes_B B \arrow[r, "1\otimes\eta_L"] & \Gamma \otimes_B \Gamma \end{tikzcd}$$

Taking the high route in the diagram associates to a pair of composable maps the domain of the composition. Taking the low route associates to a pair of composable maps the domain of the first map. But these are same same, so the diagram commutes.

To prove that p is a map of (B,Γ)-comodules we need to check that the following diagram commutes:

$$\begin{tikzcd} \Gamma \arrow[r, "p"] \arrow[d, "\psi"] & B \arrow[d, "\eta_L"] \\ \Gamma \otimes_B \Gamma \arrow[r, "\Gamma\otimes_B p"] & \Gamma \otimes_B B \end{tikzcd}$$

Taking the high route we get $b + b'r \mapsto b' \mapsto \eta_L(b') = b' \otimes 1$. Taking the low route we get $b + b'r \mapsto (b \otimes 1 + (r \otimes 1 + 1 \otimes r)(b' \otimes 1)) = (b \otimes 1 + b'r \otimes 1 + b' \otimes r) \mapsto b' \otimes 1$. $\square$

Lemma 12.4 (Shapiro's lemma)**.** *The functor $\Gamma \otimes_B -$: B-modules $\to$ (B,Γ)-comodules is right adjoint to the forgetful functor, and sends injectives to injectives.*

Especially $\mathrm{Ext}^*_{(B,\Gamma)}(M, \Gamma \otimes_B N) = \mathrm{Ext}^*_B(M,N)$

Proof. The adjointness follows by checking the definition. That $\Gamma \otimes_B -$ sends injective objects to injective objects now follows from the definition of an injective object, using the adjunction together with the fact that the forgetful functor preserves injections. The statement about Ext now follows from the definition of Ext via injective resolutions. $\square$

Remark 12.5. Note that $\Gamma \otimes_B -$ is a *right* adjoint, not a left adjoint as usual, since we are working with comodules.

Proposition 12.6. *For any Hopf algebroid (B,Γ) we have that $\mathrm{Ext}^s_{(B,\Gamma)}(B,\Gamma) = 0$ for $s > 0$, and the map $B \to \mathrm{Hom}_{(B,\Gamma)}(B,\Gamma)$, $b \mapsto \eta_L(x)\eta_R(b)$ is an isomorphism.*

Proof. From the lemma we get $\mathrm{Ext}^s_{(B,\Gamma)}(B,\Gamma) = \mathrm{Ext}^s_B(B,B) = \left\{ \begin{array}{ll} B & \text{if } s = 0 \\ 0 & \text{if } s > 0 \end{array} \right.$.

To finish the proof one just has to navigate through the isomorphism $\mathrm{Hom}_{(B,\Gamma)}(B,\Gamma) = B$ to see that it is the one claimed. $\square$

Theorem 12.7.

$$\mathrm{Ext}^*_{(B,\Gamma)}(B,B) = \mathbb{Z}[b^2][\eta]/(2\eta)$$

where $\eta \in \mathrm{Ext}^1$.

Proof. The additive part of the statement is that $\mathrm{Ext}^0 = \mathbb{Z}[b^2]$ while $\mathrm{Ext}^s = \mathbb{F}_2[b^2]$ for $s > 0$, and we will restrict ourself to this. (The easiest way to see the multiplicative structure is to use the periodic resolution together with composition of Yoneda extensions.)

By splicing our exact sequences $0 \to B \to \Gamma \to B \to 0$ we get a resolution

$$0 \to \Gamma \to \Gamma \to \cdots$$

of the (B,Γ)-comodule B, where the boundary map is given by $b + b'r \mapsto b'$. By the previous proposition we can use this resolution to calculate Ext. Applying $\mathrm{Hom}_{(B,\Gamma)}(B,-)$ to the resolution, yealds the sequence:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{(B,\Gamma)}(B,\Gamma) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{(B,\Gamma)}(B,\Gamma) \arrow[r] \arrow[d, equal] & \cdots \\
0 \arrow[r] & B \arrow[r, "d"] & B \arrow[r, "d"] & \cdots
\end{tikzcd}$$

From the previous lemma we get that under the isomorphism $B \simeq \mathrm{Hom}_{(B,\Gamma)}(B,\Gamma)$, $b \mapsto (1 \mapsto \eta_R(b) = b+2r)$, $2 \mapsto (1 \mapsto 2)$, so $d(b) = 2$. We also need to find $d(b^n)$. First note that $\eta_R(b^2) = (b+2r)^2 = b^2+4rb+4r^2 = b^2$, since $r^2+rb = 0$. Hence, more generally, $\eta_R(b^{2n}) = b^{2n}$ and $\eta_R(b^{2n+1}) = b^{2n}(b+2r)$. But this shows that $d(b^{2n}) = 0$ while $d(b^{2n+1}) = 2b^{2n}$. We have now described the differential d completely, and taking homology we see that we get the claimed result. □

Exercise 12.8. Compute $\mathrm{Ext}^*_{(A,\Gamma)}(A,A)$ where $A = \mathbb{Z}[a_1,\ldots,a_p]$, $\Gamma = A[r]$. That is, analogous to before, we look at degree p polynomial expressions $x^p + a_1x^{p-1} + \cdots + a_p$ and get monoidal structure by looking at the effect on the coefficients of the coordinate change $x \mapsto x+r$.

This is easiest done modulo Ext^0, and in fact unknown in general. You should get $E(y_1) \otimes P(x_2)$ (i.e. exterior in degree 1 tensor polynomial in degree 2).

A fundamental example and link to topology. Let (A,Γ) be the Hopf algebra of formal group laws and isomorphisms. That is, A is the Lazard ring $A = L = \mathbb{Z}[x_1,x_2,\ldots]$ and Γ the ring of universal isomorphisms $\Gamma = A[b_0^{-1}, b_0, b_1, b_2, \ldots]$.

With a bad notation, let $MU_+(X) = MU_*(X) \otimes \mathbb{Z}[u,u^{-1}]$ where $|u| = 2$. Let MU_+ be the Thom spectrum of $\mathbb{Z} \times MU \to \mathbb{Z} \times BO$. (This will have a lot of structure, e.g. an E_∞ structure, if you know what that means (it actually has the structure, even if you don't)). We get

1. $\pi_0(MU_+) = L$
2. $\pi_0(MU_+ \wedge MU_+) = \Gamma$

Note that by inverting u we get isomorphisms instead of strict isomorphisms, and hence a good connection to algebraic geometry.

Until now we have been talking about formal group laws. We are now ready to define what we mean by a formal group. (We will make this definition more explicit in a later lecture.)

Definition 12.9. *Let* $\mathcal{M}_{FG} = \mathcal{M}_{(A,\Gamma)}$. *The category of formal groups over* R *and isomorphisms is the category of maps* $\mathcal{S}tacks(\mathrm{Spec}\,R, \mathcal{M}_{FG})$ *(remember, stacks is a 2-category).*

We now have a functor which to a spectrum X associates the (A,Γ)-comodule $(MU_+)_n(X)$. (Note that we get one for each n, but all even ones are isomorphic, since we've inverted u.) This last object corresponds, via our equivalence of categories to a quasi-coherent sheaf $\mathcal{M}_n(X)$ on $\mathcal{M}_{FG}$. If $\mathrm{Spec}\,L \xrightarrow{p} \mathcal{M}_{FG}$ denotes the counit composed with stackification, then $p^*\mathcal{M}_n(X)$ corresponds to the L-module $\pi_n((MU_+) \wedge X)$.

If $X \to Y \to Z$ is a cofiber sequence, then we get a long exact sequence of sheaves

$$\cdots \to \mathcal{M}_n(X) \to \mathcal{M}_n(Y) \to \mathcal{M}_n(Z) \to \mathcal{M}_{n-1}(X) \to \cdots,$$

either by appealing to the equivalence of category or arguing directly, using that $\operatorname{Spec} L \to \mathcal{M}_{FG}$ is a cover (i.e. flat). Also the wedge axiom is satisfied $\mathcal{M}_n(\vee_\alpha X_\alpha) = \oplus_\alpha \mathcal{M}_n(X_\alpha)$. So we get a cohomology theory. We generalize this:

Definition 12.10. *A formal group over a ring R is* flat *if the classifying map $\operatorname{Spec} R \xrightarrow{p} \mathcal{M}_{FG}$ is flat.*

If $\operatorname{Spec} R \xrightarrow{p} \mathcal{M}_{FG}$ is flat, then $X \mapsto p^*\mathcal{M}_n(X) \in R$-modules still defines a cohomology theory. This amounts to unraveling the definition and using that locally we are just tensoring up over a flat extension, which preserves exactness. Note that if we suppose that p comes from a formal group, then we would have $\operatorname{Spec} R \to \operatorname{Spec} L \to \mathcal{M}_{FG}$, and hence $p^*\mathcal{M}_n(X) = \pi_n((MU_+) \wedge X) \otimes_{\pi_0(MU_+)} R$. However p might be flat without the the map $\operatorname{Spec} R \to \operatorname{Spec} L$ being flat. We want to find out how to recognize flat formal groups.

Another, slightly tangential question which comes up is the following: In the category of comodules the forgetful functor is not a right adjoint as usual so the inverse limit functor has higher derived functors. One should try to find a way to compute the derived functors of inverse limits of comodules. Any takers? pand or leave out?

13. Formal groups in prime characteristic

In this section, we will be interested in maps from $\operatorname{Spec} K$ to a stack $\mathcal{M}$. Observe the following analogy with schemes: given a scheme X, we can consider the set of all maps $\operatorname{Spec} K \to X$ where K is an (algebraically closed) field. If we regard two such maps $\operatorname{Spec} K_1 \to X$, $\operatorname{Spec} K_2 \to X$ as equivalent if $\operatorname{Spec} K_1$ and $\operatorname{Spec} K_2$ are isomorphic over X then, in the case where $X = \operatorname{Spec} R$ is affine, the set of all such maps corresponds bijectively to the set of prime ideals of R. Given a map $R \to K$, the kernel is a prime ideal; conversely, given any prime ideal $\mathfrak{p} \triangleleft R$, $R \to R/\mathfrak{p} \to (R/\mathfrak{p})_{(0)}$ is a map into a field with kernel $\mathfrak{p}$.

Now, suppose that $\mathcal{M} = \mathcal{M}_{(A,\Gamma)}$ is the stack associated to a Hopf algebroid.

Definition 13.1. *An* ***invariant prime ideal*** *(for (A,Γ)) is a prime ideal $\mathfrak{p} \triangleleft A$ such that $(\eta_R(\mathfrak{p})) = (\eta_L(\mathfrak{p}))$ where (X) denotes the ideal generated by the set X.*

Next, we will study maps from fields into the stack $\mathcal{M}_{FG}$ of formal groups. Note that fields are local rings.

Lemma 13.2. *Let R be a local ring. Then every 1-morphism $\operatorname{Spec} R \to \mathcal{M}_{FG}$ factors (up to an isomorphism) through $\operatorname{Spec} L \to \mathcal{M}_{FG}$, where L is the Lazard ring. In other words, every formal group over a local ring comes from a formal group law.*

Proof: postponed.

Corollary 13.3. *If R is a local ring then the map of groupoids*

$$\{\textit{formal group laws over } R \textit{ with isomorphisms}\} \longleftrightarrow \mathcal{S}tacks(\operatorname{Spec} R, \mathcal{M}_{FG})$$

is an equivalence.

This result shows that the classical theory of formal groups over local rings carries over to the stack setting.

Until now, we have not seen any method for showing that two given formal groups are not isomorphic. We will now look for an invariant that can distinguish them.

Let G be a formal group over R and p be a prime. Define the ***p*-series** of G to be $[p](x) = x +_G \cdots +_G x$. Since G is commutative and associative, $[p]$ is an endomorphism of G.

A formal group can be considered as a group object in the opposite category of the category of linearly topologised complete local rings or something like that. Then the ***p*-torsion** of G, ${}_pG$, is defined to be the kernel of the map of groups $(\operatorname{Spec} R[\![x]\!], G) \xrightarrow{[p]} (\operatorname{Spec} R[\![x]\!], G)$ in this category. It fits into a diagram

$$\begin{tikzcd} {}_pG \arrow[r] \arrow[d] & G \arrow[d, "{[p]}"] \\ 1 \arrow[r] & G \end{tikzcd}$$

Taking rings of functions, this translates to

$$\begin{tikzcd} \mathcal{O}_{{}_pG} & R[\![x]\!] \arrow[l] & {[p](x)} \\ R \arrow[u] & R[\![x]\!] \arrow[l] \arrow[u] & x \arrow[u, mapsto] \\ 0 & x \arrow[l, mapsto] & \end{tikzcd}$$

We measure the size of ${}_pG$ by $R[\![x]\!]/[p](x)$.

- Case 1: $R = K =$ field of characteristic 0. Then $[p](x) = px + \dots$. Since p is a unit in K, $K[\![x]\!]/[p](x) \cong K$ (by inversion of the power series $[p](x)$).
- Case 2: $R = K =$ field of characteristic p.

Lemma 13.4. *Let R be an $\mathbf{F}_p$-algebra, F, G formal group laws over R, $\phi : F \to G$ a homomorphism. Then there exists an integer n and a power series $g \in R[\![x]\!]$ such that*

- $g'(0) \neq 0$ *in R*
- $\phi(x) = g(x^{p^n})$

Proof. If $f \in R[\![x]\!]$ with $f'(x) = 0$ then $f(x) = g(x^p)$ for some g. To see this, let $f(x) = \sum a_n x^n$. $f'(x) = 0$ implies that $na_n = 0$ for $(n,p) = 1$.

If $\phi \neq 0$ is a homomorphism with $\phi'(0) = 0$, then we automatically have $\phi'(x) = 0$. To show this, we take the equality $G(\phi(x), \phi(y)) = \phi(F(x,y))$,apply $\frac{\partial}{\partial y}$ at $y = 0$ and get:

$$G_2(\phi(x), 0)\phi(0) = \phi'(F(x,0))F_2(x,0) = \phi'(x)F_2(x,0).$$

Since $F_2(x,0)$ is a unit and the left hand side is zero, we get $\phi'(x) = 0$.

To return to the assertion of the Lemma, we proceed inductively. If $\phi'(0) \neq 0$, we are done by taking $n = 0$, $\phi = g$.

Now assume $\phi'(0) = 0$. This means we can write $\phi(x) = h(x^p)$.

Claim: There is a unique formal group law F' such that the map $x \mapsto x^p$ induces a homomorphism of formal group laws $F \to F'$.

Let us write $F(x,y) = \sum a_{ij} x^i y^j$. We have to show that $(x +_F y)^p = x^p +_{F'} y^p$ for a unique F'. Since we are in characteristic p, we have

$$\left(\sum a_{ij} x^i y^j\right)^p = \sum a_{ij}^p x^{ip} y^{jp},$$

and hence we must define

$$F'(x,y) = \sum a_{ij}^p x^i y^j.$$

F' is certainly a formal group since it is the image of F under the ring homomorphism $R \to R$, $x \mapsto x^p$.

Hence we get

$$\begin{tikzcd} F \arrow[rr] \arrow[dr, "\phi"'] & & F' \arrow[dl, "h"] \\ & G. & \end{tikzcd}$$

We check that h is a homomorphism of formal group laws:

$$h(x^p +_{F'} y^p) = h(x +_F y)^p = \phi(F(x, y)) = \phi(x) +_G \phi(y) = h(x^p) +_G h(y_p).$$

Hence we have $h(x +_{F'} y) = h(x) +_G h(y)$ and the result follows by induction in applying it to h. $\square$

Definition 13.5. *The **height** of a formal group law F over a field of characteristic p is the unique integer $n = \operatorname{ht} F$ such that $[p](x) = g(x^{p^n})$ with $g'(0) \neq 0$. If $[p](x) = 0$ we say $\operatorname{ht} F = \infty$. In characteristic 0, the height of every formal group law is defined to be 0.*

- The height measures the p-rank of ${}_pF$. $\operatorname{ht} F = \log_p \dim R[\![x]\!]/[p](x)$.

Lemma 13.6. *The height is independent of the coordinate of the formal group, i.e. if F and G are isomorphic then their heights are equal.*

Example 13.7. $R = \mathbf{F}_p$. Then $\operatorname{ht} \mathbf{G}_a = \infty$, but since $1 - (1 - x)^p = x^p$, $\operatorname{ht} \mathbf{G}_m = 1$. Hence $\mathbf{G}_a \ncong \mathbf{G}_m$.

A good reference for the following is Serre's article in [CF], pp. 148ff. The original proof can be found in [LT1].

Theorem 13.8 (Dieudonné, Lubin–Tate)**.**
(a) If $\operatorname{char} K = p > 0$ then there exists a formal group law of height n for every $n \in \mathbf{N} \cup \{\infty\}$.
(b) If additionally $K = \overline{K}$ then any two formal group laws of the same height are isomorphic.

We first turn to the construction of formal group laws with certain given endomorphisms over complete local rings.

Let A be a complete local domain with maximal ideal $\mathfrak{m} = (\pi)$ and residue field $k = A/\mathfrak{m} = \mathbf{F}_{p^n} = \mathbf{F}_q$, such that the associated graded ring satisfies

$$\bigoplus \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} = k[\pi].$$

The examples to have in mind are:

- $A = \mathbf{Z}_p$ (which is not the topologist's $\mathbf{Z}/(p)$ but the p-adics) with $\pi = p$, $n = 1$;
- $A = \mathbf{Z}_{p^n} := \mathbf{Z}_p[\zeta_{p^n-1}]$ where ζ_{p^n-1} is a primitive $(p^n - 1)$st root of unity, $\pi = p$. This is the ring of p-typical Witt vectors on $\mathbf{F}_{p^n}$, as described below in Section 16;
- $A = \mathbf{Z}_p[\pi]/(\pi^k - p)$;
- $A = k[\![\pi]\!]$.

Let

$$\mathcal{F}_\pi = \left\{ f \in A[\![x]\!] \mid f(x) = \pi x + O(x^2);\ f(x) \equiv x^q\ (\pi) \right\}$$

be the set of "Eisenstein polynomials" with respect to the uniformiser π.

Lemma 13.9. *Let $f, g \in \mathcal{F}_\pi$ and $\phi_1(x_1, \ldots, x_n)$ a linear form. Then there exists a unique $\phi(x_1, \ldots, x_n) \in A[\![x_1, \ldots, x_n]\!]$ satisfying*
(a) $\phi \equiv \phi_1$ mod deg 2 and
(b) $f \circ \phi(x_1, \ldots, x_n) = \phi\left(g(x_1), \ldots, g(x_n)\right) =: \phi(g(x))$.

Proof. Suppose by induction $\phi(g(x)) \equiv f(\phi(x))$ mod deg $n + 1$ for a (mod deg $n + 1$) unique ϕ satisfying also a. This is true for $n = 1$ by the definition of $\mathcal{F}_\pi$. Then we can write

$$f(\phi(x)) - \phi(g(x)) \equiv E_{n+1} \quad \text{mod deg } n + 2$$

for a unique term E_{n+1} homogeneous of degree $n + 1$. Now we try to correct ϕ by adding ϕ_{n+1} of degree $n + 1$. We have:

$$\left.\begin{aligned} f(\phi + \phi_{n+1}) &\equiv f(\phi) + \pi\phi_{n+1} \\ \phi(g(x)) + \phi_{n+1}(g(x)) &\equiv \phi(g(x)) + \pi^{n+1}\phi_{n+1} \end{aligned}\right\} \text{mod deg } n + 2.$$

Subtracting yields $E_{n+1} = (\pi - \pi^{n+1})\phi_{n+1}$. Since A is a domain, we must take $\phi_{n+1} = \frac{1}{\pi(1-\pi^n)} E_{n+1}$ if this exists, i.e. if $\pi | E_{n+1}$ (note that the factor $1 - \pi^n$ is a unit in A).

Computing mod π, we find that

$$\phi(g(x)) \equiv \phi(x^q);\ f(\phi(x)) \equiv \phi(x)^q;$$

so $E_{n+1} \equiv \phi(x)^q - \phi(x^q) \equiv 0$ since $A/(\pi) = \mathbf{F}_q$. □

Application 13.10. Take $f = g$, $\phi_1(x, y) = x + y$. We get a $\phi(x, y) = F(x, y)$ which turns out to be a formal group law. This follows from other applications of Lemma 13.9. For example, $F(x, y) = F(y, x)$ by uniqueness; $(x +_F y) +_F z = x +_F (y +_F z)$: both sides commute with F and equal $x + y + z$ mod deg 2, so again by uniqueness, they coincide.

Take two $f, g \in \mathcal{F}_\pi$ and denote the formal group laws obtained as above by F^f, F^g. Then F^f and F^g are canonically isomorphic. Once again, this follows from Lemma 13.9 by taking $\phi_1(x) = x$. We get a unique ϕ with $f(\phi(x)) = \phi(g(x))$. We show that $\phi : F^g \xrightarrow{\cong} F^f$. Set

$$\begin{aligned} h(x, y) &= F^f\left(\phi(x), \phi(y)\right), \\ h'(x, y) &= \phi\left(F^g(x, y)\right). \end{aligned}$$

Both series are congruent to $x + y$ mod deg 2, and the calculations

$$f(h(x, y)) = f\left(F^f\left(\phi(x), \phi(y)\right)\right) = F^f\left(f(\phi(x), f(\phi(y))\right) = F^f\left(\phi(g(x)), \phi(g(y))\right) = h(g(x), g(y));$$
$$f(h'(x, y)) = f\left(\phi\left(F^g(x, y)\right)\right) = \phi\left(g(F^g(x, y))\right) = \phi\left(F^g(g(x), g(y))\right) = h'(g(x), g(y))$$

show that h and h' must agree, by Lemma 13.9.

F^f (or its reduction mod π) is called the **Lubin–Tate formal group**.

We will finally be interested in the endomorphism ring of a formal group law F. Lubin–Tate formal group laws come with a useful ring homomorphism $A \to \mathrm{End}(F)$ that determines a lot of endomorphisms. One says that F is a formal A-module or, in analogy with elliptic curves, that F has complex multiplication. We will now construct this homomorphism.

Pick any $f \in \mathcal{F}_\pi$, $F = F^f$. For $a \in A$, define $\phi_1(x) = ax$. Setting $f = g$, Lemma 13.9 produces a power series $\phi_a(x) =: [a](x)$. To check that it is an endomorphism, we have to see that $\phi_a(x +_F y) = \phi_a(x) +_F \phi_a(y)$. This is true for the linear part since $a(x+y) = ax+ay$. Furthermore, both sides commute with f: $\phi_a\left(f(x) +_F f(y)\right) = \phi_a\left(f(x +_F y)\right) = f(\phi_a(x +_F y))$, similarly for $\phi_a(x) +_F \phi_a(y)$. So, by uniqueness again, they agree.

Lemma 13.11. *For the Lubin–Tate formal group $F = F^f$ over the ring A, $[-]$ is a homomorphism of rings*

$$A \xrightarrow{[-](x)} \mathrm{End}(F)$$

satisfying $[\pi](x) = f(x)$.

Proof. It only remains to show that $[a + b](x) = [a](x) +_F [b](x)$ and $[ab](x) = [a]\left([b](x)\right)$. Both sides agree mod deg 2, so they are equal if all four terms commute with f. But this is trivial since f commutes by construction both with F and $[a]$ for all $a \in A$.

The second assertion follows since $f(x) \equiv [\pi](x) \equiv \pi x$ mod deg 2 and both sides commute with f. □

Example 13.12.

1. $A = \mathbf{Z}_{p^n} = \mathbf{W}\mathbf{F}_{p^n} = \mathbf{Z}_p[\zeta_{p^n-1}]$, $\pi = p$, $f(x) = px + x^q$. We get a formal group law F with $[p]_F(x) = f(x) \equiv x^q$ mod p. Its reduction mod p therefore has height n. F has coefficients in $\mathbf{Z}_p$ since $f(x)$ has (this follows by the invariance under the choice of the primitive $p^n - 1st$ root of unity).

2. There is a variation of this, due to Ravenel. The equation $f(x) = px +_F x^q$ has a unique solution for f, which can be seen as follows (due to Bobby):

   We start with $f_0(x) = px + x^q$ and get a formal group law F_0 with p-series $f_0(x)$. Inductively, we define $f_{i+1}(x) = px +_{F_i} x^q$ and F_{i+1} as its Lubin–Tate formal group law. We have to show that the F_i converge. However, in general, if $f(x) \equiv g(x) \bmod x^n$ then $F^f(x,y) \equiv F^g(x,y) \bmod (x,y)^n$. The easiest way to see this is to regard f and g as equal elements of $\mathcal{F}_\pi$ for the complete local ring $R/\mathfrak{m}^n$, hence F^f and F^g agree mod $(x,y)^n$.

   We claim that $f_i \equiv f_{i-1} \bmod x^{iq+2-i}$. Suppose it is true for i. Then $f_{i+1}(x) = F_i(px, x^q) \equiv F_{i-1}(px, x^q) = f_i(x) \bmod (px, x^q)^{iq+2-i}$. The lowest term where they could differ is $x^{iq+1-i}x^q = x^{(i+1)q+1-i}$, so they are congruent mod $x^{(i+1)q+2-(i+1)}$.

   This group law is superior because it has a managable logarithm: We have $\log_F(px) + \log_F(x^q) = \log_F(px +_F x^q) = \log_F([p](x)) = p\log_F(x)$.

   don't think this is ue. When I do it, ere is a (unit but ot 1) factor in each oefficient.

   F therefore satisfies the identity
$$\log_F(x) = \sum_n \frac{x^{q^n}}{p^n}.$$
3. $A = \mathbf{Z}_p[\pi]/(\pi^n - p)$. Then $A/(\pi)\mathbf{F}_p$. The Lubin–Tate formal group for A satisfies $[p](x) = [\pi^n](x) = x^{p^n} \bmod \pi$, so it has height n.

Now let $F \in \mathbf{F}_{p^n}[\![x,y]\!]$ be the mod p reduction of the formal group law F as in the first example. We have already constructed a map
$$\mathbf{Z}_{p^n} \xrightarrow{[-]} \mathrm{End}(F),$$
but this map is not surjective. Indeed, the equality $F(x,y)^p = F(x^p, y^p)$ shows that the map V which sends x to x^p is another element in $\mathrm{End}(F)$. Note that V need not commute with $[a]$, $a \in \mathbf{Z}_{p^n}$; hence we get a map $\mathbf{Z}_{p^n}\langle\!\langle V \rangle\!\rangle \longrightarrow \mathrm{End}\,F$, where $R\langle\!\langle x \rangle\!\rangle$ denotes the following noncommutative power series ring in x:
$$\mathbf{Z}_{p^n}\langle\!\langle V \rangle\!\rangle = \left\{ \sum a_m V^m \,\middle|\, a_m \in \mathbf{Z}_{p^n},\ Va = a^\sigma V \right\},$$
where $a \mapsto a^\sigma$ is the Frobenius automorphism which sends ζ to ζ^p.

To check the identity $Va = a^\sigma V$, it is enough to consider $a = p$, where it is obvious, and $a = \zeta^k$, which is easy.

One also checks that $\underbrace{V \circ \cdots \circ V}_{n}(x) = x^{p^n} = [p](x)$.

Proposition 13.13. *The maps*
$$\mathbf{Z}\langle\!\langle V \rangle\!\rangle/(V^n - p) \longrightarrow \mathrm{End}(F)$$
and
$$\mathbf{S}_n := (\mathbf{Z}\langle\!\langle V \rangle\!\rangle)^\times \longrightarrow \mathrm{Aut}(F)$$
are isomorphisms. The latter group is called the ***nth Morava stabiliser group****.*

The proof will be given later.

We have shown the first part of Theorem 13.8, we still have to show uniqueness over algebraically closed fields. Thus, let $K = \overline{K}$ be an algebraically closed field of characteristic $p > 0$, F any formal group law over K of height n, and F_{LT} be the Lubin–Tate formal group law coming from $\mathbf{Z}_{p^n}$, passing through $\mathbf{F}_{p^n}$ and arriving at K (with respect to $f(x) = px + x^q$). For this, we fix an inclusion $\mathbf{F}_{p^n} \hookrightarrow K$ such that $\mathbf{F}_{p^n} = \{t \in K \mid t^{p^n} = t\}$.

Theorem 13.8 will be proven once we have shown the following.

Proposition 13.14. *F and F_{LT} are isomorphic over K.*

Proof. We proceed in two steps:

Step I: F is isomorphic to a formal group law G which satisfies:

1. $G(x,y) = \sum a_{ij} x^i y^j$ with all $a_{ij} \in \mathbf{F}_{p^n}$;
2. $[p]_G(x) = x^{p^n}$.

Step II: Choose any lift of G to $\tilde{G}$ over $\mathbf{Z}_{p^n}$. Then $[p]_{\tilde{G}}(x) \in \mathcal{F}_{(p)}$, hence $\tilde{G}$ comes from $[p]_{\tilde{G}}$ by the Lubin–Tate construction and is therefore isomorphic to any other Lubin–Tate formal group law.

Step I: Since F has height n, we can write $[p]_F(x) = g(x^{p^n})$ with $g'(0) \neq 0$. We use 2) as an Ansatz and try to solve for ϕ in the commutative diagram:

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d, "g(x^{p^n})"'] & G \arrow[d, "x^{p^n}"] \\ F \arrow[r, "\phi"] & G, \end{tikzcd}$$

i.e. $\phi(x)^{p^n} = \phi(g(x)^{p^n})$. Write

$$\phi(x) = \sum_i b_i x^i \quad \text{and} \quad g(x) = \sum_k a_k x^k.$$

Then

$$\phi(x)^{p^n} = \sum_i b_i^{p^n} \left(x^{p^n}\right)^i =: \phi^\sigma(x^{p^n}).$$

We need to solve $\phi^{-1}\phi^\sigma = g$.

First look at the coefficient of x:

$$b_1^{p^n-1} = a_1$$

This is solvable since $K = \overline{K}$. Now assume inductively that we have replaced F by a formal group law for which

$$g(x) = x + a_k x^k + O(x^{k+1}).$$

We look for a ϕ satisfying $\phi(x) = x + b_k x^k$. We have

$$\phi^{-1} \circ \phi^\sigma = x + \left(b^{p^n} - b_k\right) + O(x^{k+1}),$$

so we need b_k to satisfy $b_k^{p^n} - b_k = a_k$; this is again possible since K is algebraically closed.

In Step I, 2) actually implies 1): We have $[p]_G(x) +_G [p]_G(y) = [p](x +_G y)$, hence by 2), $G(x^{p^n}, y^{p^n}) = G(x,y)^{p^n}$. Hence

$$\sum a_{ij} \left(x^{p^n}\right)^i \left(y^{p^n}\right)^j = \sum a_{ij}^{p^n} \left(x^{p^n}\right)^i \left(y^{p^n}\right)^j,$$

meaning that all $a_{ij} = a_{ij}^{p^n}$ and therefore $a_{ij} \in \mathbf{F}_{p^n}$. $\square$

Recapitulation of what was done above. Let K be an algebraically close field of characteristic $p > 0$. Let F be a formal group law over K of height n, and choose an inclusion $\mathbb{F}_{p^n} \hookrightarrow K$.

Goal: F is isomorphic to the Lubin-Tate formal group law coming from $\mathbb{Z}_{p^n}$ and $f(x) = px + x^{p^n}$.

The proof proceeds in two steps:

Claim: F is isomorphic to a formal group law G satisfying

(i) $G(x,y) = \sum a_{ij} x^i y^j$ with $a_{ij} \in \mathbb{F}_{p^n}$;
(ii) $[p]_G(x) = x^{p^n}$.

Conclusion: Choose any lift of G to $\tilde{G}$ over $\mathbb{Z}_{p^n}$. Then $[p]_{\tilde{G}}(x) \in \mathfrak{F}_\pi$, so $\tilde{G}$ is isomorphic to the Lubin-Tate formal group law for $[p]_{\tilde{G}}(x)$ (and hence isomorphic to any other Lubin-Tate group).

Proof of the claim. We prove (ii) first. Write $[p]_F(x) = g(x^{p^n})$ for some $g(x)$ with $g'(0) \neq 0$. We're looking for a formal group law G and an isomorphism $\varphi : F \to G$ such that the square

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "g(x^{p^n})"'] & G \arrow[d, "x^{p^n}"] \\ F \arrow[r, "\varphi"'] & G \end{tikzcd}$$

commutes. So we need $\varphi(x)^{p^n} = \varphi(g(x^{p^n}))$. Recall that if $f(x) = \sum b_i x^i$ then f^σ is defined to be the power series $f^\sigma(x) = \sum b_i^{p^n} x^i$. So our equation becomes

$$\varphi^\sigma(x^{p^n}) = (\varphi \circ g)(x^{p^n}).$$

In other words, we need to solve

$$\varphi^{-1} \circ \varphi^\sigma = g.$$

Write $g(x) = \sum a_i x^i$ and $\varphi(x) = \sum b_i x^i$. We have to solve for the b_i's. Looking at the coefficient of x in the above equation, one finds that $b_1^{p^n - 1} = a_1$. Since K is algebraically closed, we can find a b_1 such that this holds. By twisting F by $x \mapsto b_1 x$, we see that we can replace F by an isomorphic formal group law for which $a_1 = 1$.

Now suppose by induction that we can replace F by a formal group law for which $g(x) = x + a_k x^k + \mathcal{O}(x^{k+1})$. The goal is to show that we can also get rid of a_k, and to do this we look for a φ of the form $\varphi(x) = x + b_k x^k$.

The equation $\varphi^{-1} \circ \varphi^\sigma = g$ becomes $g(x) = x + (b_k^{p^n} - b_k)x^k + \mathcal{O}(x^{k+1})$, and so we need to solve

$$(b_k)^{p^n} - b_k = a_k.$$

Again, since K is algebraically closed we can do this, and so we're done by induction.

Now we must go back and prove (i). We've found a power series $\varphi(x)$, and we let G be the twist of F by this series. By construction of $\varphi(x)$ we have $[p]_G(x) = x^{p^n}$. Now write $G(x, y) = \sum a_{ij} x^i y^j$. G must commute with its own p-series, so one finds that

$$G(x^{p^n}, y^{p^n}) = [p]_G(x) +_G [p]_G(y) = [p](x +_G y) = G(x, y)^{p^n},$$

or that

$$\sum a_{ij} (x^{p^n})^i (y^{p^n})^j = \left(\sum a_{ij} x^i y^j\right)^{p^n} = \sum a_{ij}^{p^n} (x^{p^n})^i (y^{p^n})^j.$$

So $a_{ij} = a_{ij}^{p^n}$, which means $a_{ij} \in \mathbb{F}_{p^n}$. ∎

14. The automorphism group of the Lubin-Tate formal group laws

Consider the Lubin-Tate formal group law with p-series $[p](x) = px +_F x^{p^n}$, and let Γ denote its reduction to $\mathbb{F}_{p^n}$. (Note that the coefficients of Γ actually lie in $\mathbb{F}_p$, as we have seen before). The goal of this section is to calculate $\operatorname{Aut}\Gamma$, and also $\operatorname{End}\Gamma$. There are essentially two parts to this:

(1) Can we explicitly write down all the automorphism and endomorphisms?
(2) The functors

$$\mathbb{F}_{p^n} - \mathcal{A}lg \longrightarrow \mathcal{G}roup \qquad (\mathbb{F}_{p^n} \xrightarrow{i} R) \mapsto \operatorname{Aut}(i^*\Gamma)$$

and

$$\mathbb{F}_{p^n} - \mathcal{A}lg \longrightarrow \mathcal{R}ing \qquad (\mathbb{F}_{p^n} \xrightarrow{i} R) \mapsto \operatorname{End}(i^*\Gamma)$$

are both co-representable. Can we determine the representing objects?

Lemma 14.1. *Let F be the Lubin-Tate formal group law defined over $\mathbb{Z}_p$ with p-series $[p](x) = px +_F x^{p^n}$. Then over $\mathbb{Z}_{p^n}[t]/(t^{p^n} - t)$ the map $\varphi : x \mapsto tx$ is an endomorphism of F.*

Proof. What must be shown is that $F(tx, ty) = tF(x, y)$. Since our ring is torsion-free, it suffices to do this after tensoring with $\mathbb{Q}$. But rationally we have an isomorphism

$$\log : F \longrightarrow \mathbb{G}_a$$

and so it suffices to show that

$$\log(tx +_F ty) = \log(t(x +_F y)).$$

Now F is defined so that its log has the form

$$\log_F(x) = \sum \frac{x^{p^{nk}}}{p^k}$$

and it follows directly that $\log(t\lambda) = t \cdot \log(\lambda)$ (using that $t^{p^n} = t$). So now one just computes that

$$\begin{aligned} \log(tx +_F ty) = \log(tx) + \log(ty) = t\log(x) + t\log(y) &= t(\log(x) + \log(y)) \\ &= t(\log(x +_F y)) \\ &= \log(t(x +_F y)). \end{aligned}$$

□

Let R be an $\mathbb{F}_{p^n}$-algebra via a map $i : \mathbb{F}_{p^n} \to R$.

Corollary 14.2. *If $t \in R$ satisfies $t^{p^n} = t$ then $x \mapsto tx$ is in* $\mathrm{End}(i^*\Gamma)$.

Proof. We can define a map $\mathbb{Z}_p[t]/(t^{p^n} - t) \to R$ in the obvious way, and then we use the fact that $x \mapsto tx$ was an endomorphism of Γ over the first ring. □

So we have now constructed a simple class of endomorphisms of $i^*\Gamma$. Since Γ is actually defined over $\mathbb{F}_p$, there is also the 'Frobenius' endomorphism v defined by $v(x) = x^p$.

Lemma 14.3. *Suppose $\varphi(x) = \sum a_k x^k \in \mathrm{End}(\Gamma)$. Then $a_k^{p^n} = a_k$.*

Proof. Since φ is an endomorphism, we must have that

$$\varphi([p]_\Gamma(x)) = [p]_\Gamma(\varphi(x)).$$

But $[p]_\Gamma(x) = x^{p^n}$, and the lemma follows by comparing the coefficients on either side of the equation.

□

Corollary 14.4. *Any endomorphism of $i^*\Gamma$ is of the form $\sum^\Gamma t_n x^{p^n}$ for some $t_n \in R$ such that $t^{p^n} = t$.*

Proof. Let $\varphi \in \mathrm{End}(i^*\Gamma)$, and set $t_0 = \varphi'(0)$. By the lemma above we know $t_0^{p^n} = t^0$. Now suppose by induction that we've found $t_0, t_1, \ldots, t_{k-1}$ and we want to find t_k. Consider the power series

$$g(x) = \varphi(x) -_\Gamma \left(\sum\nolimits^\Gamma t_i x^{p^i}\right).$$

This is an endomorphism of $i^*\Gamma$ (since the maps $x \mapsto x^{p^i}$ and $x \mapsto tx$ are both endomorphisms). The first nonvanishing term of $g(x)$ is of the form $u_m x^{p^m}$ for some $m \geq k$, and so we define $t_m = u_m$ and $t_l = 0$ for $k \leq l < m$. Now repeat ad infinitum. □

Corollary 14.5.
(a) The functor $R \mapsto \mathrm{End}(i^*\Gamma)$ is co-represented by $\mathbb{F}_{p^n}[t_0, t_1, \ldots]/(t_k^{p^n} - t_k)$.
(b) The functor $R \mapsto \mathrm{Aut}(i^*\Gamma)$ is co-represented by $\mathbb{F}_{p^n}[t_0^{\pm 1}, t_1, \ldots]/(t_k^{p^n} - t_k)$.

Corollary 14.6. *If R is an $\mathbb{F}_p$-algebra which is a domain, then the obvious map*

$$\mathbb{Z}_{p^n}\langle\langle v \rangle\rangle/(v^n - p) \xrightarrow{\alpha} \mathrm{End}(i^*\Gamma)$$

is an isomorphism.

Proof. We first observe that any element $a \in \mathbb{Z}_{p^n}\langle\langle v\rangle\rangle/(v^n - p)$ can be uniquely written in the form $a = \sum a_k v^k$ where each a_k is either zero or a (p^n-1)st root of unity (in other words, $a_k^{p^n} = a_k$). The map α sends such an element to $\sum^F a_k x^{p^k}$, and so the observation shows that α is a bijection. $\square$

Presentations of End(Γ)**.** Let Γ be the Lubin-Tate formal group law of height n over $\mathbb{F}_{p^n}$. Recall that this is defined over $\mathbb{F}_p$ even though it is constructed over $\mathbb{F}_{p^n}$. We have seen that over $\mathbb{F}_{p^n}$,

$$\mathrm{End}(\Gamma) = \mathbb{Z}_{p^n}\langle\langle V\rangle\rangle/(V^n - p)$$

where $Va = a^\sigma V$ for $a \in \mathbb{Z}_{p^n}$. This is a very important algebra in homotopy theory. There are two useful ways of thinking about it.

(i) Write $x \in \mathrm{End}(\Gamma)$ as

$$x = \sum_{k=0}^{\infty} t_k V^k$$

where $t_k^{p^n} = t_k$. The t_k's are continuous functions on End(Γ) if End(Γ) is given the topology of the filtration by powers of the ideal generated by V (note that $V^n = p$).

(ii) The elements $\{1, V, \ldots, V^{n-1}\}$ form a basis for End(Γ) as a right $\mathbb{Z}_{p^n}$-module. End(Γ) acts on itself on the left by right $\mathbb{Z}_{p^n}$-module maps which gives us a matrix representation of End(Γ).

Example 14.7. Take $n = 2$. Then one easily checks that $a + bV$ is represented by the matrix

$$\begin{bmatrix} a & pb \\ b^\sigma & a^\sigma \end{bmatrix}$$

In general, we note that if $a_0 + a_1 V + \ldots + a_{n-1}V^{n-1}$ corresponds to the matrix A then $a_0^\sigma + a_1^\sigma V + \ldots + a_{n-1}^\sigma V^{n-1}$ will correspond to the matrix A^σ obtained from A by applying σ to all its entries.

The element V clearly corresponds to the matrix (also denoted by V)

$$\begin{bmatrix} 0 & \cdots & \cdots & p \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & \cdots & 1 & 0 \end{bmatrix}$$

It is not hard to check that

$$\mathrm{End}(\Gamma) = \{A \in M_n(\mathbb{Z}_{p^n}) : VA = A^\sigma V\}$$

This second point of view is very useful. For example, it gives us a non-trivial homomorphism $\det : \mathrm{End}(\Gamma)^\times \longrightarrow \mathbb{Z}_{p^n}$ which would have no easy description from the point of view of Hopf algebroids and formal groups.

15. Formal Groups

The purpose of this brief section is to finally write down a definition of formal groups, as opposed to formal group *laws*.

Faithfully flat descent. Suppose $R \to S$ is a map of rings. We define a category Desc(S/R) called the category of 'descent datum' for S/R.

(i) An object is an S-module N together with an isomorphism $\varphi : S \otimes_R N \to N \otimes_R S$ of $S \otimes_R S$-modules, which is required to satisfy the 'cocycle condition' saying that the following diagram commutes:

$$\begin{tikzcd} S \otimes_R S \otimes_R N \arrow[r, "1\otimes\varphi"] \arrow[dr] & S \otimes_R N \otimes_R S \arrow[d, "\varphi\otimes 1"] \\ & N \otimes_R S \otimes_R S. \end{tikzcd}$$

(ii) The morphisms are the obvious candidate.

Remark 15.1. If $R \to S$ is flat and we set $A = S$, $\Gamma = S \otimes_R S$, then (A, Γ) is a Hopf algebroid. One can check that the category $\mathrm{Desc}(S/R)$ is equivalent (or isomorphic?) to the category of (A,Γ)-comodules.

There is a canonical map

$$\mathcal{M}od_R \to \mathrm{Desc}(S/R) \qquad M \mapsto S \otimes_R M.$$

This is left-adjoint to the functor

$$\mathrm{Desc}(S/R) \to \mathcal{M}od_R \qquad N \mapsto \{n \in N \mid \varphi(1 \otimes n) = n \otimes 1\}.$$

Recall that a map $R \to S$ is said to be **faithfully flat** if

(i) The functor $S \otimes_R (-)$ is exact (flatness), and
(ii) A map of R-modules $M \to N$ is an isomorphism if and only if $S \otimes_R M \to S \otimes_R N$ is an isomorphism of S-modules (faithfulness).

Remark 15.2. If $R \to S$ is flat, then the condition that it be faithful is equivalent to requiring that $S \otimes_R M = 0$ iff $M = 0$.

Proposition 15.3. *If $R \to S$ is faithfully flat, then the functors $\mathcal{M}od_R \rightleftarrows \mathrm{Desc}(S/R)$ are an equivalence of categories.*

Proof. We check that the unit and co-unit of the adjunction are isomorphisms.

Step 1: This is easy if the map $R \to S$ admits a retraction.
Step 2: By applying $S \otimes_R (-)$—which replaces R with $S \otimes_R R$ and S with $S \otimes_R S$—one reduces to the above case.

$\square$

One can repeat this discussion with R-modules replaced by R-*algebras*, and one obtains the following

Corollary 15.4. *If $R \to S$ is faithfully flat then the functor $R - \mathcal{A}lg \longrightarrow \mathrm{AlgDesc}(S/R)$ is an equivalence.*

Definition 15.5. *Let R be a ring. A **formal group** over R is*

(1) An augmented R-algebra $A \xrightarrow{\epsilon} R$ with augmentation ideal m, with the properties that
   (i) A is complete with respect to m—i.e., the map $A \to \lim A/m^n$ is an isomorphism;
   (ii) m/m^2 is locally free of rank one over R;
   (iii) $Gr_m A \cong \mathrm{Sym}_R(m/m^2)$.

(2) A map $A \xrightarrow{F} \lim A/m^n \otimes A/m^n =: A\hat{\otimes}A$ making the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "F"] \arrow[dr, "F"'] & A\hat{\otimes}A \arrow[d, "t"] \\ & A\hat{\otimes}A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "F"] \arrow[dr, "id"'] & A\hat{\otimes}A \arrow[d, "1\otimes\epsilon"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "F"] \arrow[d, "F"'] & A\hat{\otimes}A \arrow[d, "1\otimes F"] \\ A\hat{\otimes}A \arrow[r, "F\otimes 1"'] & A\hat{\otimes}A\hat{\otimes}A. \end{tikzcd}$$

Remark 15.6.

(a) A formal group over the ring $R[[x]]$ is exactly what we've been calling a formal group *law*.

(b) If we have a formal group for which m/m^2 is *free* over R and we pick a generator, then there is an isomorphism $R[[x]] \xrightarrow{\cong} A$ and one gets an actual formal group law.
(c) If R is a field—or more generally, a local ring—then every locally free module is actually free (more-or-less by definition). So every formal group over such a ring actually comes from a formal group law.

Remark 15.7. A formal group should be something which is locally a formal group law in the flat topology. The previous definition gave something which is locally a formal group law in the Zariski topology. These turn out to be equivalent. The reason boils down to the fact that if a module over a local ring R becomes free of rank 1 under a faithfully flat base change $R \to S$, then it was free of rank 1 to start with.

16. Witt Vectors

The p-adic integers $\mathbb{Z}_p$ are in some sense constructed by starting with $\mathbb{Z}/p$. A similar construction can be done starting with $\mathbb{F}_{p^n}$, and leads to the ring of Witt vectors.

One of the first things you learn in math is to add numbers. The digits represent a number in the associated graded of $\mathbb{Z}$ given by $\mathbb{Z} \supset 10\mathbb{Z} \supset 100\mathbb{Z} \supset \cdots$. So in some sense we've built the integers out of $\mathbb{Z}/10$. The usual algorithm for addition is not so good, because it is not algebraic: it requires the mysterious process of 'carrying'. Witt's idea is that if we choose different coset representatives, then we can actually make the addition and multiplication laws completely algebraic. There is a presentation of this with a lot of good motivation in the appendix to [Mu1].

Let p be a prime. Good coset representatives for $p\mathbb{Z}_p \subset \mathbb{Z}_p$ are 0 and roots of unity. Write $a \in \mathbb{Z}_p$ as $a = \sum_{k\geq 0} a_k p^k$ where the a_k's are 0 or roots of unity. Define

$$w_n = a_0^{p^n} + a_1^{p^{n-1}} p + \cdots + a_n p^n = \sum_{i+j=n} a_i^{p^j} p^i \quad .$$

Note that in $\mathbb{Z}/p$, $a_i^p = a_i$ so that $w_n = a_0 + a_1 p + \cdots + a_n p^n$.

For $\mathbf{a} = (a_0, a_1, \cdots)$ and $\mathbf{b} = (b_0, b_1, \cdots) \in R^\infty$, we'd like to define the Witt addition and multiplication in the way that $w_n(\mathbf{a} +_w \mathbf{b}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ and $w_n(\mathbf{a} \cdot_w \mathbf{b}) = w_n(\mathbf{a}) \cdot w_n(\mathbf{b})$. Since we want to do this for any ring R, we may as well work in the universal case.

Example 16.1. Observe that if $\mathbf{c} = (c_0, c_1, \cdots) = \mathbf{a} +_w \mathbf{b}$, then one is forced to have $c_0 = a_0 + b_0$. c_1 has to have the property that

$$\begin{aligned} c_0^p + pc_1 &= (a_0^p + pa_1) + (b_0^p + pb_1) \\ (a_0 + b_0)^p + pc_1 &= (a_0^p + pa_1) + (b_0^p + pb_1) \quad . \end{aligned}$$

So that

$$c_1 = (a_1 + b_1) - \frac{1}{p} \sum_{i=1}^{p-1} \binom{p}{i} a_0^i b_0^{p-i} \quad .$$

The formulae get complicated in general.

Let $A = \mathbb{Z}[a_0, a_1, \cdots; b_0, b_1, \cdots]$, the universal ring. Consider the map

$$A^\infty \xrightarrow{(w_0, w_1, w_2, \cdots)} A^\infty$$
$$(x_0, x_1, x_2, \cdots) \mapsto (x_0, x_0^p + px_1, x_0^{p^2} + px_1^p + p^2 x_2, \cdots) \quad .$$

We'll show that there are unique $+_w$ and $\cdot_w$ on the domain which map to componentwise addition and multiplication.

Remark 16.2. If our ring is a $\mathbb{Q}$-algebra, then we can solve $x_0^{p^n} + \cdots + p^n x_n = w_n$ for the x's in terms of w's. So there is nothing to check in this case. This also implies that if the ground ring is torsion free and the operations $+_w$ and $\cdot_w$ exist, then they are unique.

Lemma 16.3 (Dwork). *Suppose R is a torsion-free ring with a ring homomorphism $\sigma : R \to R$ satisfying $\sigma(r) \equiv r^p \mod p$.*

Then a sequence $(w_0, w_1, \cdots) \in R^\infty$ is of the form $(w_0(\mathbf{a}), w_1(\mathbf{a}), \cdots)$ for some $\mathbf{a} \in R^\infty$ if and only if

$$w_n - w_{n-1}^\sigma \equiv 0 \mod p^n \ .$$

Moreover, the sequence $\mathbf{a}$ is unique.

The idea of using this lemma is thinking of σ as raising to the p-th power so that

$$\begin{aligned} w_n &= a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p + p^n a_n \\ w_{n-1} &= a_0^{p^{n-1}} + \cdots + p^{n-1}a_{n-1} \\ w_{n-1}^\sigma &= a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p \\ w_n - w_{n-1}^\sigma &\equiv 0 \mod p^n \ . \end{aligned}$$

Proof. If p is a prime and $x \equiv y \mod p$, then $x^{p^{k-1}} \equiv y^{p^{k-1}} \mod p^k$ since $x \equiv y \mod p^{k-1}$ implies

$$x^p = (y + p^{k-1}r)^p = y^p + p(p^{k-1}r)y^{p-1} + \binom{p}{2}(p^{k-1}r)^2 y^{p-2} + \cdots \equiv y^p \mod p^k \ .$$

We know $a_0 = w_0$. Suppose we've found $a_0, \cdots, a_{n-1}$ with $w_k = a_0^{p^k} + \cdots + p^k a_k$ for $k < n$. From $w_n = a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p + p^n a_n$, we need to see that

$$w_n - (a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p) \equiv 0 \mod p^n \ .$$

The previous discussion shows

$$(a_k^\sigma)^{p^{n-1-k}} \equiv (a_k^p)^{p^{n-1-k}} \equiv a_k^{p^{n-k}} \mod p^{n-k} \ .$$

Hence

$$w_{n-1}^\sigma = (a_0^\sigma)^{p^{n-1}} + \cdots + p^{n-1}a_{n-1}^\sigma \equiv a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p \mod p^n$$

and

$$w_n - (a_0^{p^n} + \cdots + p^{n-1}a_{n-1}^p) \equiv w_n - w_{n-1}^\sigma \equiv 0 \mod p^n \ .$$

□

Go back to $A = \mathbb{Z}[a_0, a_1, \cdots; b_0, b_1, \cdots]$. Define $\sigma : A \to A$ by $a_i^\sigma = a_i^p$ and $b_i^\sigma = b_i^p$. By the Dwork lemma, the existence of $\mathbf{c} \in A^\infty$ with $w_n(\mathbf{c}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ is implied by

$$w_n(\mathbf{a}) + w_n(\mathbf{b}) - (w_{n-1}(\mathbf{a}) + w_{n-1}(\mathbf{b}))^\sigma \equiv 0 \mod p^n$$

which is fairly clear.

So there is a unique $\mathbf{c} \in A^\infty$ such that $w_n(\mathbf{c}) = w_n(\mathbf{a}) + w_n(\mathbf{b})$ for $n \geq 0$, and similarly there is a unique $\mathbf{d} \in A^\infty$ such that $w_n(\mathbf{d}) = w_n(\mathbf{a}) \cdot w_n(\mathbf{b})$.

Definition 16.4. *For any ring R, the **ring of Witt vectors** of R, $\mathbb{W}(R)$, is the set R^∞ with the above addition and multiplication.*

Sometimes these are called the **p-typical Witt vectors** and denoted by $\mathbb{W}_{p^\infty}(R)$.

Exercise 16.5. Think through these definitions to show that $\mathbb{W}(\mathbb{Z}/p) = \mathbb{Z}_p$.

Big Witt vectors.
Define $W_n(\mathbf{a}) = \sum_{d|n} da_d^{n/d}$ for $\mathbf{a} = (a_1, a_2, \cdots)$. If $n = p^k$, this is an expression similar to that used before. There is a similar game which produces the **big Witt vectors**.

Lemma 16.6 (Big Dwork lemma). *Let R be a torsion-free ring. Suppose that for each prime p there is an endomorphism $\sigma_p : R \to R$ such that $\sigma_p(x) \equiv x^p \mod p$. For $\mathbf{a} = (a_1, a_2, \cdots) \in R^\infty$, let*

$$W_n(\mathbf{a}) = \sum_{d|n} da_d^{n/d}.$$

Then the set of equations $W_n = \sum_{d|n} da_d^{n/d}$ can be solved uniquely for the a's if and only if

$$W_{p^n m} \equiv W_{p^{n-1}m}^{\sigma_p} \mod p^n$$

for each prime p and each m with $(m, p) = 1$.

Proof. Left to reader. You might need to assume the σ_p's commute, but probably not. □

The additive group of big Witt vectors comes up in a different context:

Proposition 16.7. *Given a ring R, let $(1 + xR[[x]])^\times = (\{f \in R[[x]] | f(0) = 1\}, \text{multiplication})$. This group is isomorphic to the additive group of big Witt vectors.*

Proof. Any element of $(1 + xR[[x]])^\times$ can be uniquely written in the form

$$\prod_{n=1}^{\infty}(1 - a_n x^n) \ .$$

So we need to check

$$\prod(1 - a_n x^n) \cdot \prod(1 - b_n x^n) = \prod(1 - c_n x^n)$$

where $\mathbf{c} = \mathbf{a} +_W \mathbf{b}$ in the big Witt vectors. Again, it suffices to check this in the universal case $A = \mathbb{Z}[a_1, \cdots ; b_1, \cdots]$. In fact, we can check over $\mathbb{Q}[a_1, \cdots ; b_1, \cdots]$. Taking the log of both sides shows it:

$$-\log\Big[\prod_n (1 - a_n x^n)\Big] = \sum_{n,j} na_n^j \frac{x^{nj}}{nj} = \sum_k W_k(\mathbf{a}) \frac{x^k}{k}$$

$$\begin{aligned} -\log\Big[\prod(1 - a_n x^n) \cdot \prod(1 - b_n x^n)\Big] &= -\sum \Big[\log(1 - a_n x^n) + \log(1 - b_n x^n)\Big] \\ &= \sum [W_n(\mathbf{a}) + W_n(\mathbf{b})] \frac{x^n}{n} \\ &= \sum W_n(\mathbf{c}) \frac{x^n}{n} = -\log\Big[\prod(1 - c_n x^n)\Big] \end{aligned}$$

□

Now, we are going to relate big Witt vectors to p-typical ones.

Proposition 16.8. *If R is a $\mathbb{Z}_{(p)}$-algebra, then*

$$\text{bigWitt}(R) \cong \prod_{(m,p)=1} \mathbb{W}(R)$$

as rings.

Proof. Let us define the map in the universal case $A = \mathbb{Z}_{(p)}[a_1, a_2, \cdots]$. Take $\sigma_p(a_i) = a_i^p$ and $\sigma_q(a_i) = 0$ for primes q other than p. Notice that $(W_1(\mathbf{a}), W_2(\mathbf{a}), \cdots)$ splits to $W^m(\mathbf{a}) = (W_{p^0 m}(\mathbf{a}), W_{p^1 m}(\mathbf{a}), W_{p^2 m}(\mathbf{a}), \cdots)$ for $(m, p) = 1$. Each sequence $W^m(\mathbf{a})$ has the property of the Dwork lemma, and therefore has the form $w(\mathbf{b}^m)$ for some unique $\mathbf{b}^m \in A^\infty$. This defines the map

$$\text{bigWitt}(R) \longrightarrow \prod_{(m,p)=1} \mathbb{W}(R)$$

$$\mathbf{a} \mapsto \{\mathbf{b}^m\} \ .$$

It is easy to see this is an isomorphism. $\square$

Remark 16.9. We have an explicit isomorphism

$$(1 + x\mathbb{Z}/p[[x]])^\times \cong \prod_{(m,p)=1} \mathbb{Z}_p \ .$$

We can do something similar for

$$\big(1 + x\mathbb{Z}/p[[x]]/(x^n)\big)^\times \cong \text{(product of cyclic groups of determined orders)} \ .$$

The role of the log (actually exp) for the p-Witt vectors.

Suppose R is a torsion-free $\mathbb{Z}_{(p)}$-algebra with $\sigma : R \to R$ such that $\sigma(x) \equiv x^p \mod p$. Then $\exp\left(\sum v_n \frac{t^{p^n}}{p^n}\right)$ has coefficients in R provided $v_n \equiv v_{n-1}^\sigma \mod p^n$.

Example 16.10. In the case where $R = \mathbb{Z}_{(p)}$ and $\sigma(x) = x$, we would find $\exp\big(\sum \frac{t^{p^n}}{p^n}\big) \in \mathbb{Z}_{(p)}[[t]]$ which is called the Artin-Hasse exponential. This comes up in writing the map back from

$$\prod_{(m,p)=1} \mathbb{W}(R) \longrightarrow \text{bigWitt}(R).$$

17. Classifying Lifts — The Lubin-Tate Space

Up until now we have only looked very carefully at the points of this stack. Now we will try to expand our vision to the local picture, a small neighborhood of these points. This involves analysis of formal groups over complete local rings whose residue field is our point. We will begin by discussing the contents of the second important paper of Lubin and Tate [LT2] on formal groups, which deals with deformation theory—i.e. the theory of how to lift a structure from a residue feild to a complete local ring sitting over it.

Let Γ be the Lubin-Tate formal group law of height n over $\mathbb{F}_p$. Let B be a local ring with nilpotent maximal ideal $\mathfrak{m}$.

Definition 17.1. *A **deformation** of Γ to B consists of a triple (G, i, t) where*
(a) G is a FGL over B
(b) $i : \mathbb{F}_p \longrightarrow B/\mathfrak{m}$ is an inclusion
(c) $t : \overline{G} \longrightarrow i^*(\Gamma)$ is an isomorphism, where $\overline{G}$ denotes the image of G in $B/\mathfrak{m}$.

Remark 17.2. The map i is superfluous since it is unique. We have included it in the definition of deformation because, following Drinfeld, this is the thing to do if we consider fields other than $\mathbb{F}_p$.

We will denote a deformation (G, i, t) by a diagram

$$G \longrightarrow i^*\Gamma$$

Definition 17.3. *An **isomorphism** $(G_1, i, t_1) \xrightarrow{f,g} (G_2, j, t_2)$ consists of an isomorphism $G_1 \xrightarrow{f} G_2$ such that $t_2 \circ f = (j \circ i^{-1})^* \circ t_1$, i.e.*

$$\begin{tikzcd} G_1 \arrow[rr, "f"] \arrow[d, "t_1"'] & & G_2 \arrow[d, "t_2"] \\ i^*\Gamma \arrow[r, "(j\circ i^{-1})^*"] & j^*\Gamma \arrow[r, "g"] & j^*\Gamma \end{tikzcd}$$

*An isomorphism is called a $*$-**isomorphism** if we can take $g = id$ in the above definition.*

The space of deformations modulo $*$-isomorphism is called the **Lubin-Tate space**. The category of deformations modulo $*$-isomorphism is the **Lublin-Tate category**, where the objects are $*$-isomorphism classes and the morphisms are isomorphisms modulo $*$-isomorphisms.

Note that $g \in \mathrm{Aut}(i^*\Gamma)$ acts on the space of deformations by $t \mapsto g \circ t$

$$\begin{tikzcd} G \arrow[d, "t"'] & \\ i^*\Gamma \arrow[r, "g"] & i^*\Gamma. \end{tikzcd}$$

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Here (G, i, t) and $g(G, i, t) = (G, i, g \circ t)$ are isomorphic via a non-$*$-isomorphism which we call $\tilde{g}$ given by

$$\begin{tikzcd} G \arrow[r, "="] \arrow[d, "t"'] & G \arrow[d, "g\circ t"] \\ i^*\Gamma \arrow[r, "g"] & i^*\Gamma \end{tikzcd}$$

It is easy to see that this action preserves $*$-isomorphism classes and as such descends to a faithful action on the Lubin-Tate space. Writing a typical morphism in the category

$$\begin{tikzcd} G_1 \arrow[rr, "f"] \arrow[d, "t_1"'] & & G_2 \arrow[d, "t_2"] \\ i^*\Gamma \arrow[r, "(j\circ i^{-1})^*"] & j^*\Gamma \arrow[r, "g"] & j^*\Gamma \end{tikzcd}$$

we see that it factors as $(G_1, i, t_1) \xrightarrow{*iso} (G_1, j, (j\circ i^{-1})^* \circ t_1) \xrightarrow{\tilde{g}} (G_1, j, g\circ(j\circ i^{-1})^* \circ t_1) \xrightarrow{f*iso} (G_2, j, t_2)$, so morphisms in the Lublin-Tate category are given by pairs consisting of an object and an element of $\mathrm{Aut}(i^*\Gamma)$.

The Lublin-Tate category coresponds to the fiber category of the projection $FGL(B) \to FGL(\mathbb{F}_q)$ over Γ. Geometrically, Γ is a point of the moduli stack of formal groups and the Lublin-Tate category is a infinitesimal neighborhood of this point given by ways of completing the diagram:

$$\begin{tikzcd} \operatorname{Spec} \mathbb{F}_q \arrow[d, hook] \arrow[dr, "\Gamma"] & \\ \operatorname{Spec} B \arrow[r, dashed] & \mathcal{M}_{FG} \end{tikzcd}$$

To best understand what these neighborhoods "looks like" is to find a representing pair for the Lublin-Tate category. This is equivalent to finding a universal lift of Γ and universal isomorphism. Already we have reduced this problem to representing the objects of this category (since the previous section showed how to represent $\mathrm{Aut}(i^*\Gamma)$ from the previous section).

The Lubin-Tate space is a basic object in homotopy theory. Many unsolved problems in homotopy theory have to do with the fact that this action is very hard to describe.

Theorem 17.4 (Lubin-Tate)**.**

(a) *There exists a formal group law F over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]$ for which*
 (i) $F(x+y) = x + y + u_1 C_p(x,y) + \ldots + u_{n-1} C_{p^{n-1}}(x,y) + C_{p^n}(x,y) \bmod \mathfrak{n}^2$ *where* $\mathfrak{n} = (p, u_1, \dots, u_{n-1})$
 (ii) $F \equiv \Gamma \bmod \mathfrak{n}$

(b) *Given such an F, the functor*

$$\begin{array}{rcl} \mathcal{R}ing(\mathbb{Z}_p[[u_1, \dots, u_{n-1}]], B) \cong \mathfrak{m}^{\times(n-1)} & \longrightarrow & \{\mathit{Deformations}\}/(* - \mathit{iso}) \\ f & \longmapsto & f^*F \end{array}$$

is (pro)corepresented by $(F, 1, 1)$ over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]$, and this map is in isomorphism.

(c) *More precisely, in the category of complete local rings with residue field an $\mathbb{F}_p$-algebra the functor of Deformations is corepresented by $(F, 1, 1)$ over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]$.*

We will first give several lemmas which prove the case $B = \mathbb{F}_p \oplus \mathfrak{m}, \mathfrak{m}^2 = 0$. Then we will deduce the general case.

NONE OF THIS MAKES ANY SENSE TO ME

For (a): Here it suffices to construct F' over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]/(u_1, \dots, u_{n-1})^2$. Let $l(x)$ be the logarythm of the Lublin-Tate formal group law over $\mathbb{Z}_p$. wrong log?

$$l(x) = \sum \frac{x^{p^{nk}}}{p^k}$$

We can show that how??

$$l(x) + \frac{u}{p} l(x^p) + \cdots + \frac{u_{n-1}}{p} l(x^{p^{n-1}})$$

is the log of a formal group law over $\mathbb{Z}_p[[u_1, \dots, u_{n-1}]]/(u_1, \dots, u_{n-1})^2$. (We just have to check that $l(x +_F y)$ has integer coefficients, and this is easy since it is true for l and $v_i^2 = v_i v_j = 0$). Given this we will prove (b).

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Lemma 17.5. *Let $B = \mathbb{F}_p \oplus \mathfrak{m}, \mathfrak{m}^2 = 0$. Then*

$$\{\mathrm{Def}\}/ * iso \cong H^2(\Gamma; \mathfrak{m}) \cong \text{ symmetric 2-cocycles/coboundaries.}$$

- *A symmetric 2-cocycle is an $\varepsilon(x, y) \in \mathfrak{m}[[x, y]]$ satisfying*
$$\varepsilon(y, z) - \varepsilon(x +_\Gamma y, z) + \varepsilon(x, y +_\Gamma z) - \varepsilon(x, y) = 0.$$
- *Coboudaries are of the form*
$$\delta\alpha(x, y) = \alpha(x +_\Gamma y) - \alpha(x) - \alpha(y).$$
for $\alpha(x) \in \mathfrak{m}[[x]]$.

The correspondence is given by $G(x, y) = \Gamma(x, y) +_\Gamma \varepsilon(x, y)$.

Proof of Lemma. We have Γ and a deformation G. By using an appropriate $*$-isomorphism we can suppose that $t = id$ and $G \equiv \Gamma \bmod \mathfrak{m}$. Now write $G(x, y) = x +_\Gamma y +_\Gamma \varepsilon(x, y)$, where $\varepsilon(x, y) \in \mathfrak{m}[[x, y]]$. Now

$$x +_\Gamma y +_\Gamma \varepsilon(x, y) = x +_\Gamma y + \Gamma_2(x +_\Gamma y, 0)\varepsilon(x, y)$$

since the coeficients are in $\mathfrak{m}$ and square to 0 (here Γ_2 denotes the derivative in the second variable of Γ).

$\varepsilon(x, y)$ satisfies:

- $\varepsilon(x, y) = \varepsilon(y, x)$
- $\varepsilon(0, y) = y$
- $\varepsilon(y, z) +_\Gamma \varepsilon(x, y +_\Gamma z) = \varepsilon(x, y) +_\Gamma \varepsilon(x +_\Gamma y, z)$.

The latter statement follows from associativity of G:

$$\begin{aligned}
(x +_G y) +_G z &= (x +_G y) +_\Gamma z +_\Gamma \varepsilon(x +_G y, z) \\
&= (x +_\Gamma y +_\Gamma \varepsilon(x, y)) +_\Gamma z +_\Gamma \varepsilon(x +_\Gamma y, z) x +_G (y +_G z) \\
&= x +_\Gamma (y +_G z) +_\Gamma \varepsilon(x, y +_G z) \\
&= x +_\Gamma (y +_\Gamma z +_\Gamma \varepsilon(y, z) +_\Gamma \varepsilon(x, y +_\Gamma z).
\end{aligned}$$

So there is a correspondence between the set of such G and $\varepsilon(x,y)$ satisfying the above properties—these are symmetric 2-cocycles on Γ with values in $\mathbb{G}_a$.

The most general $*$-isomorphism is

$$g(x) = x +_\Gamma \alpha(x), \alpha(x) \in \mathfrak{m}[[x]].$$

One readily checks that $g^{-1}(x) = x -_\Gamma \alpha(x)$, since $\mathfrak{m}^2 = 0$. Then

$$\begin{aligned}
g^{-1}G(g(x), g(y)) &= G(g(x), g(y)) -_\Gamma \alpha(G(g(x), g(y))) \\
&= g(x) +_\Gamma g(y) +_\Gamma \varepsilon(g(x), g(y)) -_\Gamma \alpha(g(x) +_\Gamma g(y) +_\Gamma \varepsilon(g(x), g(y))) \\
&= x +_\Gamma \alpha(x) +_\Gamma y +_\Gamma \alpha(y) +_\Gamma \varepsilon(x,y) -_\Gamma \alpha(x +_\Gamma y) \\
&= x +_\Gamma y +_\Gamma \tilde{\varepsilon}(x,y)
\end{aligned}$$

where $\tilde{\varepsilon}(x,y) = \alpha(x) +_\Gamma \alpha(y) +_\Gamma \varepsilon(x,y) -_\Gamma \alpha(x +_\Gamma y)$ is a coboundary by definition. So the $*$-isomorphism sends $\varepsilon(x,y) \to \tilde{\varepsilon}(x,y)$. Thus

$$\{\text{Def}\}/*\,iso = H^2(\Gamma; \mathfrak{m}) = \text{ symmetric 2-cocycles/coboundaries.}$$

We have calculated this before using homological algebra on a 2-stage chain complex. This chain complex is the tangent space to our stack at the point Γ. □

Lemma 17.6. *$H^2(\Gamma; \mathfrak{m})$ is free of rank $n-1$ on generators which start out looking like*

$$C_{p^k}(x,y) + \textit{ monomials of degrees higher than } p^k$$

for $k = 1, \ldots, n-1$.

Proof. We look at the complex which computes H^2:

$$\mathbb{Z}/p \Longrightarrow \mathbb{Z}/p[[x]] \Longrightarrow \mathbb{Z}/p[[x,y]] \Longrightarrow \mathbb{Z}/p[[x,y,z]] \ldots$$

where

$$\alpha(x) \mapsto \alpha(y) - \alpha(x +_\Gamma y) + \alpha(x),$$
$$f(x,y) \mapsto f(y,z) - f(x +_\Gamma y, z) + f(x, y +_\Gamma z) - f(x,y).$$

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(We can think of this as the cohomology of $\mathbb{Z}/p$ with respect to the triple on $\otimes\mathbb{Z}_p[[x]]$ with transformations given by $x \mapsto 0$ and Γ.)

Recall the analgous complex with $+ = +_{\mathbb{G}_a}$, instead of $+_\Gamma$, whose cohomology we already computed.

$$H^1(\mathbb{G}_a; \mathfrak{m}) \text{ has basis } \{\alpha_k := x^{p^k} | k = 0, 1, \ldots\}$$

$$H^2(\mathbb{G}_a; \mathfrak{m}) \text{ has basis } \{x^{p^i} x^{p^j} - x^{p^j} x^{p^i}, i < j; \beta_k := C_{p^k} = \frac{(x+y)^{p^k} + x^{p^k} + y^{p^k}}{p}\}$$

$$H^*(\mathbb{G}_a; \mathfrak{m}) = E[\alpha_i] \otimes P[\beta_{??}].$$

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ith C_p and β_j??

Now filter the Γ-complex by powers of the maximal ideal. The associated graded complex for this filtration is the $+$-complex. So we have a spectral sequence with $E_2 = H^*(\mathbb{G}_a; \mathfrak{m})$. We need now to compute the remaining differentials.

Let $\tilde{\Gamma}$ be the Lublin-Tate Formal group over $\mathbb{Z}_p$ and Γ be the Lublin-Tate Formal group over $\mathbb{Z}/p$. Recall:

$$\log_{\tilde{\Gamma}}(x) = x + \frac{x^{p^n}}{p} + \ldots \quad \text{and} \quad \exp_{\tilde{\Gamma}}(x) = x - \frac{x^{p^n}}{p} + \ldots$$

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e right log

Then

$$\begin{aligned}\Gamma(x,y) = \exp(\log(x)+\log(y)) &= x + y - \frac{(x+y)^{p^n} + x^{p^n} + y^{p^n}}{p} + o(x^{p^n}) \\ &= x + y + C_{p^n}(x,y) + \dots\end{aligned}$$

In the Γ-complex we have

$$\delta(\alpha_0) = \alpha_0(x) + \alpha_0(y) - \alpha_0(x +_\Gamma y) = C_{p^n}(x,y) + \dots.$$

We can conclude from this that in the spectral sequence α_0 hits β_n. Similarly,

$$\begin{aligned}\delta(\alpha_k) = \alpha_k(x) + \alpha_k(y) - \alpha_k(x +_\Gamma y) &= x^{p^k} + y^{p^k} - (x +_\Gamma y)^{p^k} \\ &= x^{p^k} + y^{p^k} - (x + y - C_{p^n}(x,y))^{p^k} \\ &= C_{p^{n+k}}(x,y) + \text{ higher terms.}\end{aligned}$$

The last equality follows from computing

- $(x+y)^{p^{k-1}} \equiv x^{p^{k-1}} + y^{p^{k-1}} \mod (p)$.
- $(x+y)^{p^k} \equiv (x^{p^{k-1}} + y^{p^{k-1}})^p \equiv x^{p^k} + y^{p^k} + pC_p(x^{p^{k-1}}, y^{p^{k-1}}) \mod (p^2)$.
- $\Rightarrow C_{p^k}(x,y) \equiv C_p(x^{p^{k-1}}, y^{p^{k-1}}) \mod (p)$
- Combining this with the fact that $C_{p^n}(x,y)^p \equiv C_{p^{n+1}}(x,y) \mod (p)$, we discover finally that $C_{p^n}(x,y)^p \equiv C_p(x^{p^{n+1}}, y^{p^{n+1}}) \mod (p)$.

So α_k hits β_{n+k}. It now follows that the E_∞-term of the spectral sequence is $\mathbb{F}_p[[\beta_1, \dots \beta_{n-1}]]$ so $H^2(\Gamma; \mathbb{Z}/p)$ is free of rank $n-1$ on classes $C_{p^i}(x,y)+$ higher terms.

□

Corollary 17.7. *Let $b_k(x,y)$ be a symmetric 2-cocycle lifting the cohomology class β_k. Then the map*

$$\begin{aligned}\mathfrak{m}^{n-1} &\to H^2(\Gamma; \mathfrak{m}) \\ (x_1, \dots, x_{n-1}) &\mapsto \sum x_i b_i\end{aligned}$$

is an isomorphism.

Proof of Theorem 17.4 for general B.
Part (a): Let $\Gamma^1(x,y) = \Gamma(x,y) +_\Gamma v_1 b_1(x,y) +_\Gamma \cdots +_\Gamma v_{n-1} b_{n-1}(x,y)$. This looks right over $\mathbb{F}_p[v_1, \dots, v_{n-1}]/(v_1, \dots, v_{n-1})^2$ and it is a formal group law. We can let F be any lift to $\mathbb{Z}_p[[v_1, \dots, v_{n-1}]]$.

Part (b): The proof is by induction on k. Suppose we have proved the result for $B/\mathfrak{m}^{k-1}$ (i.e. where the maximal ideal to the $(k-1)$-st power is 0). In other words, assume that we've shown

$$\mathfrak{m}/\mathfrak{m}^{k-1} \times \cdots \times \mathfrak{m}/\mathfrak{m}^{k-1} \to \mathrm{Def}(B/\mathfrak{m}^{k-1})/*\!-\!\mathrm{iso}$$

is an isomorphism.

Now we want to show that if we have two deformations over $B/\mathfrak{m}^k$ which agree mod $(\mathfrak{m}^{k-1})$ then they differ by an element of ??????.

Consider the following diagram whose columns are exact:

$$\begin{tikzcd}
\mathfrak{m}^{k-1}/\mathfrak{m}^k \times \cdots \times \mathfrak{m}^{k-1}/\mathfrak{m}^k \arrow[r] \arrow[d] & ker \arrow[d] \\
\mathfrak{m}/\mathfrak{m}^k \times \cdots \times \mathfrak{m}/\mathfrak{m}^k \arrow[r] \arrow[d] & \mathrm{Def}(B/\mathfrak{m}^k)/*\!-\!\mathrm{iso} \arrow[d] \\
\mathfrak{m}/\mathfrak{m}^{k-1} \times \cdots \times \mathfrak{m}/\mathfrak{m}^{k-1} \arrow[r, "\cong"] & \mathrm{Def}(B/\mathfrak{m}^{k-1})/*\!-\!\mathrm{iso}
\end{tikzcd}$$

We want to show that the middle row is also an isomorphism, and for this it suffices to check that the top row is an isomorphism. Suppose G,G^1 are formal group laws over $B/\mathfrak{m}^k$, with

$$G \equiv G^1 \equiv \Gamma \mod (\mathfrak{m})$$

and $G \equiv G^1 \bmod (\mathfrak{m}^{k-1})$. The upper right hand corner consists of the set of such G^1 modulo isomorphisms which reduce to the identity mod$\mathfrak{m}^{k-1}$. We have then $G^1(x,y) = G(x,y) +_G \varepsilon(x,y), \varepsilon(x,y) \in \mathfrak{m}^{k-1}/\mathfrak{m}^k[[x,y]] \subset B/\mathfrak{m}^k[[x,y]]$. Thus the set of such G^1 modulo isomorphisms which reduce to the identity mod$\mathfrak{m}^{k-1}$ is in bijective correspondence with these symmetric 2-cocycles ε modulo coboundaries which is again $H^2(\Gamma; \mathfrak{m}^{k-1}/\mathfrak{m}^k)$. This is just the case where $\mathfrak{m}^2 = 0$, so $H^2(\Gamma; \mathfrak{m}^{k-1}/\mathfrak{m}^k) \cong \mathfrak{m}^{k-1}/\mathfrak{m}^k \times \cdots \times \mathfrak{m}^{k-1}/\mathfrak{m}^k$.

□

The chromatic spectral sequence.

We will discuss the chromatic spectral sequence, which provides much of the motivation for the formal group point of view in homotopy theory. The cornerstone on which it rests is the Miller-Ravenel (or sometimes Morava) change of rings theorem. Moreover, the spectral sequence can be realized geometrically via the Landweber exact functor theorem, which we will also have to discuss. The idea behind both of these is to find a different presentation for the stack of height-n formal groups.

The Adams-Novikov spectral sequence is derived from the cosimplicial spectrum

$$MU \Longrightarrow MU \wedge MU \Rrightarrow MU \wedge MU \wedge MU \Rrightarrow \cdots$$

We have already seen that applying $\pi_*(-)$ to the above complex gives the nerve of the category of formal groups and strict isomorphisms. The E_2-term, the cohomology of the above complex, is $\mathrm{Ext}^*_{MU_*MU}(MU_*, MU_*)$. Note that this is also the cohomology of the structure sheaf over the moduli stack of formal groups $\mathcal{M}_{FG}$. From now on we'll denote $\mathrm{Ext}^*_{MU_*MU}(MU_*, N)$ by $\mathrm{Ext}(N)$ for short.

To help us compute more of $\mathrm{Ext}(MU_*)$ we look at short exact sequences like

$$0 \to MU_* \to MU_* \otimes \mathbb{Q} \to MU_* \otimes \mathbb{Q}/\mathbb{Z} \to 0.$$

This leads to a long exact sequence of Ext groups. It's easy to compute the middle terms coming from $\mathrm{Ext}(MU_* \otimes \mathbb{Q})$, since formal groups over a $\mathbb{Q}$-algebra are all isomorphic to $\mathbb{G}_a$ with isomorphisms classified by the log of the formal group law. This is equivalent to the trivial groupoid whose one object is $\mathbb{G}_a$ (since the automorphism group of $\mathbb{G}_a$ is trivial). It suffices to compute the cohomology over this equivalent groupoid represented by the Hopf algebroid $(\mathbb{Q},\mathbb{Q})$-in $\mathbb{Q}$-algebras. The category of such comodules is equivalent to $\mathbb{Q}$-vector spaces so there are no higher derived functors and we have:

$$\mathrm{Ext}^{0,0}(MU_* \otimes \mathbb{Q}) = \mathbb{Q}$$
$$\mathrm{Ext}^{s,t}(MU_* \otimes \mathbb{Q}) = 0, (s,t) \neq (0,0).$$

Getting back to our computation of $\mathrm{Ext}^*(MU_*)$ we need still to figure out $\mathrm{Ext}^*(MU_* \otimes \mathbb{Q}/\mathbb{Z})$. We will be working one prime at a time so we want to calculate $\mathrm{Ext}^*(MU_* \otimes \mathbb{Q}/\mathbb{Z}_{(p)})$. We can write $\mathbb{Q}/\mathbb{Z}_{(p)} = \lim \mathbb{Z}/p^n\mathbb{Z}$ and we have short exact sequences

$$0 \to \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n+1}\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

So if we found $\mathrm{Ext}(MU_*/p)$ then we could work inductively to find each $\mathrm{Ext}(MU_*/p^n)$, and thus deduce $\mathrm{Ext}(MU_* \otimes \mathbb{Q}/\mathbb{Z}_{(p)})$.

Let G be a formal group law over a $\mathbb{Z}/p$-algebra. Then

$$[p]_G(x) = v_1 x^p + \dots$$

for some v_1 (possibly zero). This gives us an element $v_1 \in \pi_{2p-2}MU/p$ which is invariant, in the sense that it is equalized by the two arrows $\pi_*(MU/p) \rightrightarrows \pi_*(MU/p \wedge MU/p)$. Therefore we get an element $v_1 \in \mathrm{Ext}^{0,2p-2}(MU_*/p)$. In fact the natural map $\mathbb{F}_p[v_1] \to \mathrm{Ext}^{0,*}(MU_*/p)$ is an isomorphism. Now to find the higher Ext groups MU_*/p we can play a similar trick with v_1 as we did before with p and look at the sequence

$$0 \to MU_*/p \to v_1^{-1}MU_*/p \to MU_*/(p, v_1^\infty) \to 0.$$

Here too the middle terms $\mathrm{Ext}(v_1^{-1}MU_*/p)$ are easy to compute and so $\mathrm{Ext}^0(MU_*/(p, v_1^\infty))$ will give us $\mathrm{Ext}^1(MU_*/p)$ (and eventually $\mathrm{Ext}^2(MU_*)$).

To find $\mathrm{Ext}(v_1^{-1}MU_*/p)$ we consider formal groups over $\mathbb{F}_p$-algebras where v_1 is a unit—these are the formal group laws of height 1. The groupoid of such formal groups is equivalent to the groupoid whose sole object is the Lubin-Tate formal group Γ_1 of height 1, and whose morphisms are all its automorphisms.

Theorem 17.8 (Morava, Miller-Ravenel Change of Rings Theorem)**.**

$$\mathbb{F}_p[v_1^\pm] \otimes H^*(\mathrm{Aut}(\Gamma_1)) \cong \mathrm{Ext}(v_1^{-1}MU_*/p).$$

Note that $\mathrm{Aut}(\Gamma_1) \cong \mathbb{Z}_p^*$.

Returning to $\mathrm{Ext}^*(MU_*/(p, v_1^\infty))$ we write $MU_*/(p, v_1^\infty) = \lim MU_*/(p, v_1^n)$ where to start the induction again we look at $\mathrm{Ext}^*(MU_*/(p, v_1))$. Over $MU_*/(p, v_1)$ the universal formal group law has $[p]_F(x) = v_2 x^{p^2} +$ higher terms. This gives a new element v_2 in $\pi_{2p^2-2}(MU/(p, v_1)$, and in fact yields a corresponding $v_2 \in \mathrm{Ext}^{0,2(p^2-1)}(MU_*/(p, v_1))$.

Theorem 17.9. $\mathbb{F}_p[v_2] \cong \mathrm{Ext}^{0,*}(MU_*/(p, v_1))$

Following the same pattern we can try to get information from the sequence

$$0 \to MU_*/(p, v_1) \to v_2^{-1}MU_*/(p, v_1) \to MU_*/(p, v_1, v_2^\infty) \to 0.$$

Again we have a change of rings theorem which says

$$H^*_{cont}(\mathrm{Aut}'(\Gamma_2); \mathbb{F}_p[v_2^\pm]) \otimes \mathbb{F}_{p^2} \cong \mathrm{Ext}(v_2^{-1}MU_*/(p, v_1)) \otimes \mathbb{F}_{p^2}.$$

Here the $H^2(-)$ is to be interpreted as continuous cohomology, and $\mathrm{Aut}'(\Gamma_2)$ is defined in the following way: Any element $x \in \mathrm{Aut}(\Gamma_2)$ has the form $t_0 x +_\Gamma \ldots +_\Gamma t_m x^{p^m} +_\Gamma \ldots$ for some unique t_i's, and the assigment $x \mapsto t_0$ gives a homomorphism

there are some m details about t ch.rng.thm h which i don't get all

$$\mathrm{Aut}(\Gamma_2) \to \mathbb{F}_{p^2}^*.$$

The group $\mathrm{Aut}'(\Gamma_2)$ is the kernel of this map.

The pattern of the above analysis continues, and allows one to express the Adams-Novikov E_2-term in terms of the continuous cohomology of certain profinite groups.

Note: It would be really nice if we could get this last section cleaned up at some point. —dd

18. Cohomology of stacks, with applications

Let $\mathcal{M}$ be a stack of the form $\mathcal{M}_{(A,\Gamma)}$, where (A, Γ) is some Hopf algebroid. In some sense all that we will really need about $\mathcal{M}$ is that

- The diagonal is representable, and
- $\mathcal{M}$ admits a covering by representables.

(In fact even these criteria can be relaxed somewhat).

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}$. The **cohomology of $\mathcal{M}$ with coefficients in $\mathcal{F}$** are the groups

$$H^*(\mathcal{M}; \mathcal{F}) := \mathrm{Ext}^*(\mathcal{O}_\mathcal{M}, \mathcal{F}).$$

We will often abbreviate these to $H^*(\mathcal{F})$.

There is a standard recipe for computing cohomology in this case: cover $\mathcal{M}$ by the map $\operatorname{Spec} A \to \mathcal{M}$ and use the Čech nerve of this covering.

$$\begin{tikzcd}
\cdots \operatorname{Spec} A \times_{\mathcal{M}} \operatorname{Spec} A \times_{\mathcal{M}} \operatorname{Spec} A \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, equal] & \operatorname{Spec} A \times_{\mathcal{M}} \operatorname{Spec} A \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \operatorname{Spec} A \arrow[d, equal] \\
\cdots \operatorname{Spec} \Gamma \otimes_A \Gamma \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \operatorname{Spec} \Gamma \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} A.
\end{tikzcd}$$

Note that the simpicial object on the bottom is $\operatorname{Spec}(-)$ applied to the cobar construction for (A, Γ).

Write $M = \mathcal{F}(\operatorname{Spec} A)$. Then there is a canonical isomorphism $\mathcal{F}(\operatorname{Spec} \Gamma) \cong \Gamma \otimes_A M$, and M becomes an (A, Γ)-comodule in a canonical way. Applying $\mathcal{F}$ to the above simplicial stack, one gets

$$\begin{tikzcd}
\cdots \mathcal{F}(\operatorname{Spec} \Gamma \otimes_A \Gamma) \arrow[d, equal] & \mathcal{F}(\operatorname{Spec} \Gamma) \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] \arrow[d, equal] & \mathcal{F}(\operatorname{Spec} A) \arrow[l, shift left] \arrow[l, shift right] \arrow[d, equal] \\
\cdots \Gamma \otimes_A \Gamma \otimes_A M & \Gamma \otimes_A M \arrow[l, shift left=2] \arrow[l] \arrow[l, shift right=2] & M, \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}$$

and this is of course the cobar construction for the comodule M.

The spectral sequence associated to the Čech nerve is now seen to be

$$H^p(\operatorname{Spec} A \times \cdots \times \operatorname{Spec} A, \mathcal{F}) \Rightarrow H^{p+q}(\mathcal{M}; \mathcal{F}),$$

where there are q copies of $\operatorname{Spec} A$ in the product. Since $\mathcal{F}$ is quasi-coherent it has no higher cohomology on affines, and therefore the spectral sequence collapses to the $p = 0$ line. We find that $H^*(\mathcal{M}; \mathcal{F})$ in this case is just the cohomology of the cobar complex for M as an (A, Γ)-comodule.

Now assume that $\operatorname{Spec} R \to \mathcal{M}$ is an arbitrary map. We can form the Čech nerve just as before, and we'd like to ask the following:

Question: If $\operatorname{Spec} R^n$ denotes $\operatorname{Spec} R \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} \operatorname{Spec} R$ (with n copies), when does the complex $H^0(\operatorname{Spec} R^n; \mathcal{F})$ compute $H^*(\mathcal{M}; \mathcal{F})$?

If $\operatorname{Spec} R \to \mathcal{M}$ is a *cover* then one can repeat the argument from above, and the claim is almost automatic. But in fact one can get by with even less: all we need is that $\operatorname{Spec} R \to \mathcal{M}$ is 'surjective in the flat topology'.

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on.

Definition 18.1. *A map* $\operatorname{Spec} R \to \mathcal{M}$ *is said to be* ***surjective*** *if for any given map* $\operatorname{Spec} B \to \mathcal{M}$ *there exists some faithfully flat cover* $\operatorname{Spec} \tilde{B} \to \operatorname{Spec} B$ *and a map* $\operatorname{Spec} \tilde{B} \to \operatorname{Spec} R$ *with the property that the diagram*

$$\begin{tikzcd}
 & & \operatorname{Spec} R \arrow[d] \\
\operatorname{Spec} \tilde{B} \arrow[r] \arrow[urr, dotted] & \operatorname{Spec} B \arrow[r] & \mathcal{M}
\end{tikzcd}$$

commutes up to a 2-morphism

$$(\operatorname{Spec} \tilde{B} \to \operatorname{Spec} R \to \mathcal{M}) \Rightarrow (\operatorname{Spec} \tilde{B} \to \operatorname{Spec} B \to \mathcal{M}).$$

Proposition 18.2. *If* $\operatorname{Spec} R \to \mathcal{M}$ *is surjective, then the Čech complex computes the cohomology* $H^*(\mathcal{M}; \mathcal{F})$.

Proof. Consider the two maps

$$\begin{tikzcd}
 & \operatorname{Spec} R \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \mathcal{M}.
\end{tikzcd}$$

By surjectivity, we can find a faithfully flat extension $A \to \tilde{A}$ and a map $\operatorname{Spec} \tilde{A} \to \operatorname{Spec} R$ giving us the diagram

$$\begin{tikzcd}
 & & \operatorname{Spec} R \arrow[d] \\
\operatorname{Spec} \tilde{A} \arrow[r] \arrow[urr, dotted] & \operatorname{Spec} A \arrow[r] & \mathcal{M}.
\end{tikzcd}$$

We can replace A by $\tilde{A}$ and Γ by $\Gamma \otimes_A \tilde{A}$ without effecting anything, and so we reduce to the case where the map $\operatorname{Spec} A \to \mathcal{M}$ factors through $\operatorname{Spec} R$ (up to a 2-morphism).

Now let $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} A$. Form the double complex

$$\begin{tikzcd}
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U \times_{\mathcal{M}} U \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & \mathcal{M} \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U \times_{\mathcal{M}} U \times_{\mathcal{M}} V \arrow[r, shift left] \arrow[r, shift right] \arrow[u] & U \times_{\mathcal{M}} V \arrow[r] \arrow[u] & V \arrow[u] \\
\cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & U \times_{\mathcal{M}} U \times_{\mathcal{M}} V^2 \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left] \arrow[u, shift right] & U \times_{\mathcal{M}} V^2 \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & V^2 \arrow[u, shift left] \arrow[u, shift right] \\
\vdots \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & & \vdots \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & \vdots \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2]
\end{tikzcd}$$

Here V^n is short for $V \times_{\mathcal{M}} \cdots \times_{\mathcal{M}} V$. The assumption that $V \to \mathcal{M}$ factors through U can be seen to imply that the horizontal simplicial sets (except for the top one) admit a contracting homotopy.

Now apply $\mathcal{F}(-)$ to the double complex $U^n \times_{\mathcal{M}} V^m$ above, and take the associated chain complexes. One gets:

$$\begin{tikzcd}
 & \vdots & \vdots \\
\cdots & \mathcal{F}(U^2 \times V^3) \arrow[l] \arrow[u] & \mathcal{F}(U \times V^3) \arrow[l] \arrow[u] \\
\cdots & \mathcal{F}(U^2 \times V^2) \arrow[l] \arrow[u] & \mathcal{F}(U \times V^2) \arrow[l] \arrow[u] \\
\cdots & \mathcal{F}(U^2 \times V) \arrow[l] \arrow[u] & \mathcal{F}(U \times V) \arrow[l] \arrow[u]
\end{tikzcd}$$

The horizontal rows are seen to be acyclic, with H^0 the complex

$$F(V) \to F(V^2) \to F(V^3) \to \cdots$$

So the homology of the above bi-complex is just the homology of this ordinary complex, which is $H^*(\mathcal{M}; \mathcal{F})$ (as we have seen already).

But now let's think about the columns in the double complex. These came as the result of applying $\mathcal{F}$ to the Čech nerve of the map $U^k \times_{\mathcal{M}} V \to U^k$. But this map is a cover (because $U \to \mathcal{M}$ was a cover, and covers pull back). So the cohomology of the columns are computing $H^*(U^k; \mathcal{F})$. Since each U^k is affine the higher cohomology vanishes, and so the columns are acyclic and the H^0 is the complex

$$\cdots \leftarrow F(U^3) \leftarrow F(U^2) \leftarrow F(U).$$

Hence, we find that the cohomology of the double complex is the cohomology of $\mathcal{F}(-)$ applied to the Čech nerve of $U = \operatorname{Spec} R \to \mathcal{M}$. The conclusion is then that this latter complex computes $H^*(\mathcal{M}; \mathcal{F})$, which is what we wanted. □

For the remainder of the section we fix a prime p.

Definition 18.3. *$\mathcal{M}_{FG}^{(n)}$ is the moduli stack over $\mathbb{F}_p$-algebras consisting of formal groups G with the property that*

$$[p]_G(x) = v_n x^{p^n} + \cdots$$

where v_n is a unit. (One can check that this property doesn't depend on the choice of a coordinate).

It's easy to see that $\mathcal{M}_{FG}^{(n)} = \mathcal{M}_{(A,\Gamma)}$ where $A = v_n^{-1}L/(p, v_1, \dots, v_{n-1})$ and $\Gamma = \Gamma_{FG} \otimes_L A = A \otimes_L \Gamma_{FG}$.

Proposition 18.4. *The 1-morphism $\operatorname{Spec} \mathbb{F}_p \to \mathcal{M}_{FG}^{(n)}$ classifying the Lubin-Tate formal group law is surjective.*

Remark 18.5. The map is not just surjective, but is actually a cover—proving this requires some extra work, however, and being surjective is good enough for our purposes.

Proof. The proof is very similar to something we've seen already.

Given a map $\operatorname{Spec} B \to \mathcal{M}_{FG}^{(n)}$ classifying some formal group G, we need to show that there is a faithfully flat extension $f : B \to \tilde{B}$ such that f^*G is isomorphic to the Lubin-Tate group Γ_{LT}.

First note that we can find a flat extension $f : B \to B_0$ for which f^*G is a formal group *law*, and for which the p-series of f^*G has the form

$$[p](x) = v_n x^{p^n} + \cdots$$

for some $v_n \in B^\times$ (essentially by the definition of $\mathcal{M}_{FG}^{(n)}$). Let's now write G for f^*G, and write the p-series as $[p](x) = g(x^{p^n})$ for some $g(x) = v_n x + \cdots$.

Next we try to find an isomorphism $\varphi(x) : G \to \Gamma_{LT}$. The isomorphism must preserve the p-series, in the sense that

$$\varphi(x)^{p^n} = \varphi([p]_G(x)).$$

If we write $\varphi(x) = \sum a_k x^k$ and $\varphi^\sigma(x) = \sum a_k^{p^n} x^k$, then the above equation translates into

$$\varphi^\sigma(x^{p^n}) = \varphi(g(x^{p^n})),$$

or

$$\varphi^{-1} \circ \varphi^\sigma = g.$$

This is secretly a collection of equations in the a_k, and what we must show is that we can solve them after some faithfully flat extension of B_0.

Since $\varphi(x) = a_1 x + \mathcal{O}(x^2)$ and $g(x) = v_n x + \mathcal{O}(x^2)$, it's easy to see that we must have $a^{p^n - 1} = v_n$ for the above equation to be satisfied. So we set $B_1 = B_0[a]/(a^{p^n-1} - v_n)$ and make the faithfully flat extension $f_1 : B_0 \to B_1$ (it is faithfully flat because the target is free as a B_0-module).

We again follow the practice of just writing G for f_1^*G. By twisting G via the isomorphism $\varphi(x) = ax$, we have arranged for the resulting formal group law to have p-series $g(x^{p^n})$ with $g(x) = x + \mathcal{O}(x^2)$.

Now assume by induction that $B \to B_{k-1}$ is a faithfully flat extension such that G is isomorphic over B_{k-1} with a formal group law having p-series $g(x^{p^n})$ where $g(x)$ has the form

$$g(x) = x + cx^k + \mathcal{O}(x^{k+1}).$$

We will show that we can eliminate c by extending B_{k-1} even further.

As we did a few lectures back, we look for a $\varphi(x)$ of the form $\varphi(x) = x + a_k x^k + \cdots$. Then

$$\varphi^{-1} \circ \varphi^{\sigma}(x) = x + (a_k^{p^n} - a_k)x^k + \mathcal{O}(x^{k+1}),$$

and we want this to equal $g(x) = x + cx^k + \mathcal{O}(x^{k+1})$. So form the extension $B_k \to B_k[a_k]/(a_k^{p^n} - a_k - c)$, which is again faithfully flat by inspection. This does it.

We have now shown that after a faithfully flat base extension the original G is isomorphic to a formal group law whose p-series if $[p](x) = x^{p^n}$ (and so we may as well assume that this *is* the p-series of G). We want to conclude that G is isomorphic to Γ_{LT}. There are two basic approaches:

(1) Since G must commute with its own p-series, we know that $G(x^{p^n}, y^{p^n}) = G(x,y)^{p^n}$. So if $G(x,y) = \sum a_{ij}x^i y^j$ then it must be that $a_{ij}^{p^n} = a_{ij}$. One must show that G is isomorphic to a formal group law coming from $\mathbb{F}_{p^n}$ with the same p-series (perhaps after a faithfully flat base change). This will be left to the reader as an exercise.

(2) Make use of the theory of p-typical formal group laws (to be discussed in the next section): Choose a p-typical coordinate on G, and apply the Quillen idempotent.

□

19. p-typical Formal Group Laws.

In this section we explain how specializing to $\mathbb{Z}_{(p)}$-algebras greatly simplifies the Hopf algebroid of formal group laws and isomorphisms.

The group of curves. In the study of Lie groups, it is useful to associate to each Lie group its Lie algebra. This has the effect of producing a functor from Lie groups to Lie algebras, a category which is much more manageable from an algebraic point of view. We will try doing something of the same sort for *formal* groups. The algebraic object which plays the role of the Lie algebra will be something called the Dieudonné module of the formal group.

Definition 19.1. *The* group of curves *on G is the set $CG = \{f(x) = a_1x + a_2x^2 + \ldots : a_i \in R\}$ with addition law given by $(f+g)(x) := f(x) +_G g(x)$.*

Geometrically, CG is the group of functions $\mathbb{A}^1 \longrightarrow G$. The analog of the Lie algebra is the quotient of CG by the subgroup of curves whose derivative vanishes at 0. Usually one recovers the group from the Lie algebra using the Campbell-Baker-Hausdorff formula. However, for us, this is not an option since the formula requires introducing denominators which may not exist if R is not a $\mathbb{Q}$-algebra. Instead we can consider the full group of curves on G together with certain natural operations on CG:

1. *Homothety:* Given $r \in R$ let $(\langle r\rangle g)(x) := g(rx)$. Geometrically this is precomposition with multiplication by r:
$$\mathbb{A}^1 \xrightarrow{r} \mathbb{A}^1 \xrightarrow{g} G$$
2. *n-th Verschiebung:* Given $n \in \mathbb{N}$, define $V_n g(x) := g(x^n)$. Geometrically this is precomposition with the n-th power map:
$$\mathbb{A}^1 \xrightarrow{x^n} \mathbb{A}^1 \xrightarrow{g} G$$
3. *n-th Frobenius:* Given $n \in N$ and letting ζ denote a primitive n-th root of unity, we define
$$(F_n g)(x) = \sum_{i=1,\ldots,n}^{G} g(\zeta^i x^{1/n}) = \sum_{y_i\ n\text{-th root of x}}^{G} g(y_i)$$
This formula requires a little explanation: the expression on the right is a power series on $x^{1/n}$ with coefficients in $R[\zeta]$. Since the formal group law is commutative and associative the coefficients can be expressed in terms of the elementary symmetric functions on the ζ^i. But

these all vanish except for $\sigma_n(1, \zeta, \dots, \zeta^{n-1}) = (-1)^n$ so the expression is really a power series in x with coefficients in R.

Another way to think of the Frobenius is as the Verschiebung on the Pontryagin dual (or character group) of G.

The algebra D generated by the operations $\langle r\rangle$, V_n and F_n modulo certain universal relations is called the **Dieudonné algebra**. CG with its D-module structure is called the **Dieudonné module** of G and it is the sought after analog of the Lie algebra: one can show that the functor which associates to each formal group law its Dieudonné module is an equivalence of categories.

To get a better understanding of the Dieudonné algebra we can consider the case when R has no additive torsion. When $R \hookrightarrow R \otimes \mathbb{Q}$ we can understand the operators $\langle r\rangle$, V_n and F_n in terms of the logarithm.

Proposition 19.2. *Suppose R is a $\mathbb{Q}$-algebra and let* $\log$ *denote the logarithm of G. Then if $g \in CG$ and* $\log(g(x)) = \sum_{n\geq 1} a_n x^n$
(a) $\log(\langle r\rangle g(x)) = \sum r^n a_n x^n$
(b) $\log(V_k g(x)) = \sum a_n x^{nk}$
(c) $\log(F_k g(x)) = \sum k a_{nk} x^n$

Proof. Only (c) requires proof. Writing ζ for a primitive k-th root of unity we have

$$\begin{aligned}\log(F_k g(x)) &= \log\left(\sum_{i=1,\dots,k}^{G} g(\zeta^i x^{1/k})\right)\\ &= \sum_{i=1}^{k} \log g(\zeta^i x^{1/k})\\ &= \sum_{i=1}^{k}\sum a_n \left(\zeta^i x^{1/k}\right)^n\end{aligned}$$

Since

$$\sum_{i=1}^{k}(\zeta^{in}) = \begin{cases} k & \text{if } k|n\\ 0 & \text{otherwise}\end{cases}$$

we see that

$$\log(F_k g(x)) = \sum k a_{nk} x^n$$

as required. □

Using the formulas of the previous proposition it is easy to deduce the relations that the operators $\langle r\rangle$, V_n and F_n satisfy.

p-typical curves. If we consider only formal group laws over $\mathbb{Z}_{(p)}$-algebras the theory is simplified. From now on we assume that R is a $\mathbb{Z}_{(p)}$-algebra. We begin with a definition due to Cartier.

Definition 19.3. *Let p be a prime. A curve $g \in CG$ is **p-typical** if for every n such that $(n, p) = 1$ we have $F_n g = 0$. The subgroup of CG consisting of all p-typical curves is denoted by $C_{p^\infty}(G)$.*

Remark 19.4. If $R \hookrightarrow R \otimes \mathbb{Q}$, Proposition 19.2 implies that a curve g is p-typical if and only if

$$\log(g(x)) = \sum_{n\geq 0} m_n x^{p^n}$$

for some $m_n \in R$.

The operators V_p, F_p and $\langle r\rangle$ act on $C_{p^\infty}(G)$. The algebra generated by these operators modulo certain universal relations is called the **Cartier algebra**. One can show that a formal group law G over a $\mathbb{Z}_{(p)}$-algebra is determined by the Cartier module structure on $C_{p^\infty}(G)$.

Definition 19.5. *A curve $g(x)$ on G is a **coordinate** if $g'(0)$ is a unit in R.*

Note that a coordinate $y \mapsto g(x)$ determines a formal group law on $R[[y]]$ in the usual way: $F(x,y) = g^{-1}(G(g(x), g(y))$.

Definition 19.6. *A formal group law is **p-typical** if the coordinate x is a p-typical curve, i.e. if $F_n(x) = 0$ for every n such that $(n,p) = 1$.*

The following proposition shows that every formal group law over a $\mathbb{Z}_{(p)}$-algebra is isomorphic in a natural way to a p-typical one.

Proposition 19.7 (Cartier)**.** *If R is a $\mathbb{Z}_{(p)}$-algebra then any formal group law over R has a p-typical coordinate.*

Proof. Suppose first that R is torsion free. We need to find a coordinate g so that $\log(g(x)) = \sum m_n x^{p^n}$. We can use the Dieudonné module structure on CG to change any coordinate g on G to one which has this form as follows.

Let $\log(g(x)) = \sum a_n x^n$ then

$$\log(V_l F_l g(x) = \sum l a_{nl} x^{nl}$$

If $(l,p) = 1$ then the l-series $[l](x)$ is invertible in $R[[x]]$. Let $[1/l](x)$ denote the inverse. Then

$$\log([1/l]V_l F_l g(x)) = \sum a_{nl} x^{nl}$$

and

$$\log(g(x) -_G [1/l]V_l F_l g(x)) = \sum_{(n,l)=1} a_n x^n$$

Iterating this process for all l such that $(l,p) = 1$ clearly gives us a p-typical coordinate.

In order to deal with the general case, however, it is useful to have an expression for the p-typical curve thus obtained. Let μ denote the Möbius function defined by

$$\mu(m) = \begin{cases} 0 & \text{if } l^2 | m \text{ for some prime } l \\ (-1)^k & \text{if } m = p_1 \dots p_k \text{ with } p_i \text{ distinct primes} \end{cases}$$

Then defining the operator

$$\epsilon = \sum_{(m,p)=1} \left[\frac{\mu(m)}{m}\right] V_m F_m \tag{19.1}$$

one easily checks that for any $g \in CG$, ϵg is p-typical. □

Remark 19.8. The operator ϵ defined by (19.1) is easily seen to be an idempotent. It defines a projection

$$CG \xrightarrow{\epsilon} C_{p^\infty}(G)$$

In topology it corresponds to the Quillen idempotent.

Parameterizations of p-typical formal group laws. It is helpful to have different parameterizations of the ring classifying p-typical formal group laws. The next proposition is useful for this purpose.

Lemma 19.9. *Suppose $R \hookrightarrow R \otimes \mathbb{Q}$ and G is a formal group law over R. Then G is p-typical if and only if there exist $v_i \in R$ such that*

$$[p]_G(x) = px +_G v_1 x^p +_G \ldots +_G v_n x^{p^n} +_G + \ldots \tag{19.2}$$

Proof. The expression (19.2) is equivalent to

$$p \log(x) = \sum_{n \geq 0} \log(v_n x^{p^n}) \tag{19.3}$$

with $v_0 = p$ or equivalently since R is torsion free,

$$\log(x) = \sum \frac{1}{p} \log(v_n x^{p^n})$$

If this is the case then clearly $\log(x) = \sum m_n x^{p^n}$ and hence the formal group law is p-typical by Remark 19.4. Conversely, if G is p-typical $\log(x)$ is of this form and so one clearly can inductively choose $v_n \in R$ such that (19.3) holds. Moreover the choice is unique. □

Proposition 19.10 (Cartier)**.** *Let G_1 and G_2 be p-typical formal group laws over a $\mathbb{Z}_{(p)}$-algebra. If $[p]_{G_1}(x) = [p]_{G_2}(x)$ then $G_1 = G_2$.*

Proof. Let $L = \mathbb{Z}[x_1, x_2, \ldots]$ be the Lazard ring. Consider the universal formal group law $\tilde{F}$ over $L \otimes \mathbb{Z}_{(p)}$ with coordinate z and let $x = \epsilon(z)$ be the p-typicalization of z. Let G denote the corresponding p-typical formal group law and let φ be the ring endomorphism

$$L \otimes \mathbb{Z}_{(p)} \xrightarrow{\varphi} L \otimes \mathbb{Z}_{(p)}$$

classifying the formal group law G. Then φ is idempotent (because ϵ is). We write $L_{p^\infty} = \varphi(L \otimes \mathbb{Z}_{(p)})$. Then clearly, G defined over L_{p^∞} is the universal p-typical formal group law.

By the Lemma 19.9, there are elements $v_n \in L_{p^\infty}$ such that

$$[p]_G(x) = {\sum}^G v_n x^{p^n}$$

Applying $\log_G$ to this equation we see that in the module of indecomposables of $L \otimes \mathbb{Q}$, up to a unit in $\mathbb{Z}_{(p)}$, we have

$$v_n = p m_n$$

The same argument used to prove Lazard's theorem now implies that

$$L_{p^\infty} = \mathbb{Z}_{(p)}[v_1, \ldots, v_n]$$

which concludes the proof. □

The previous proposition describes one of the two usual schemes for parameterizing p-typical formal group laws - the **Kudo-Araki parameterization**.

The other scheme is the **Hazewinkel parameterization**. The parameters are traditionally also called v_n although they *do not coincide with the v_n's above.* Let σ be the endomorphism of L_{p^∞} defined by

$$\sigma(v_k) = v_k^p$$

and write f^σ for the power series obtained from f by applying σ to the coefficients. Then the Hazewinkel parameters v_i are defined by the equation

$$\log_G(x) = x + \frac{v_1}{p} \log_G^\sigma(x^p) + \ldots + \frac{v_n}{p} \log_G^{\sigma^n}(x^{p^n}) + \ldots \tag{19.4}$$

If we write

$$\log_G(x) = \sum m_n x^{p^n}$$

then in the Kudo-Araki case we have,

$$pm_n = v_n + m_1 v_{n-1}^p + \ldots + m_{n-1} v_1^{p^{n-1}} + m_n p^{p^n}$$

and in the Hazewinkel case,

$$pm_n = v_n + v_{n-1} m_1^{\sigma^{n-1}} + \ldots + v_1 m_{n-1}^{\sigma}$$

Note that the Hazewinkel parameterization gives easy closed formulas for the m_n's in terms of the v_n's, unlike the Kudo-Araki parameterization.

Example 19.11. *The Lubin-Tate formal group law.* The p-series is given by

$$[p](x) = px +_F x^{p^n}$$

so the Kudo-Araki parameters are simple. The expression for the log is mildly complicated.

The isomorphic formal group law with Hazewinkel parameters $v_i = 0$ for $i \neq n$ and $v_n = 1$ has a log satisfying the functional equation

$$\log(x) = x + \frac{1}{p}\log(x^{p^n})$$

This gives an easy expression for the log:

$$\log(x) = \sum_{k \geq 0} \frac{x^{p^n k}}{p^k}$$

but a slightly complicated p-series.

Remark 19.12. Over $L_{p^\infty} \otimes \mathbb{Z}/p$, the Kudo-Araki and Hazewinkel parameters coincide.

20. Stacks: what's up with that?

In this section we will discuss what there is to gain by using the point of view of stacks in homotopy theory and also talk a little about prospects for the future.

The basic miracle about complex oriented (not just orientable) cohomology theories is their close relationship with the algebra of formal group laws. The language of stacks allows us to go much deeper in this correspondence between algebra and topology.

There are two main theorems coming from the correspondence between complex oriented cohomology theories and formal group laws

1. Landweber's exact functor theorem.
2. The Morava, Miller-Ravenel change of rings theorem.

Both of these have nice proofs in the language of stacks. But there is much more to be gained from this language.

Spectra associated to sheaves over $\mathcal{M}_{FG}$. Let E be a multiplicative spectrum satisfying

(a) E is complex orientable
(b) $\pi_2 E$ contains a unit
(c) $\pi_1 E = 0$

Remark 20.1. Properties (b) and (c) imply property (a).

We have seen that this data gives us a map

$$\operatorname{Spec}\pi_0 E \longrightarrow \mathcal{M}_{FG}$$

so we have a way of assigning to certain spectra affine stacks over $\mathcal{M}_{FG}$. It is natural to ask the following naive

Question: Can we go back?

This turns out to be a surprisingly good question. The answer is that in certain cases (more than you would think) we can and there is a nice criterion for this.

Let $\mathcal{M}U$ be the spectrum determined by

$$\mathcal{M}U^*(X) = MU^*(X) \otimes \mathbb{Z}[u^{\pm 1}]$$

Alternatively, $\mathcal{M}U$ is the Thom spectrum associated to the identity map

$$\mathbb{Z} \times BU \longrightarrow \mathbb{Z} \times BU$$

just as MU is the Thom spectrum associated to $\mathbb{Z} \times BU \longrightarrow BU$. Then $\pi_0 \mathcal{M}U = L$ is the Lazard ring and under the correspondence above, $\mathcal{M}U$ corresponds to a very special element in the category $Aff/\mathcal{M}_{FG}$, namely the canonical map

$$\operatorname{Spec} L \xrightarrow{l} \mathcal{M}_{FG}$$

Let $\mathcal{F}$ be a flat quasi-coherent sheaf over $\mathcal{M}_{FG}$. In some cases we can associate to $\mathcal{F}$ a spectrum which we will call $E(\mathcal{F})$. It will be defined by descent.

The motivating fact for the construction is the following. Suppose that E is a spectrum satisfying our assumptions. Then you should recognize that we have proved that

$$\begin{tikzcd}
\operatorname{Spec}\pi_0 \mathcal{M}U \wedge E \arrow[r] \arrow[d, "k_1"] & \operatorname{Spec}\pi_0 E \arrow[d, "k_2"] \\
\operatorname{Spec}\pi_0 \mathcal{M}U \arrow[r, "l"] & \mathcal{M}_{FG}
\end{tikzcd}$$

is a pull back square of 2-categories.

This diagram tells us that the smash product of (certain) spectra in topology corresponds to the fibre product over $\mathcal{M}_{FG}$ in algebra. Moreover, since l is a cover, the statement should be true in general as k_1 should be determined by k_2 and descent data.

If we have a flat quasi-coherent sheaf $\mathcal{F}$ over $\mathcal{M}_{FG}$, we get a flat L-module $l^*\mathcal{F}$ and by flatness a homology theory

$$X \mapsto l^*\mathcal{F} \otimes_{\pi_0 \mathcal{M}U} \mathcal{M}U_*(X)$$

which, in view of the remarks above, should be represented by the spectrum $\mathcal{M}U \wedge E(\mathcal{F})$ (even though we haven't defined $E(\mathcal{F})$ yet).

Similarly, considering the maps

$$\operatorname{Spec}\Gamma \rightrightarrows^{\pi_1}_{\pi_2} \operatorname{Spec} L \xrightarrow{l} \mathcal{M}_{FG}$$

by flatness we get a diagram of cohomology theories

$$l^*\mathcal{F} \otimes_{\pi_0 \mathcal{M}U} \mathcal{M}U_*(-) \rightrightarrows (l \circ \pi_1)^*\mathcal{F} \otimes_{\Gamma} (\mathcal{M}U \wedge \mathcal{M}U)_*(-) \tag{20.1}$$

We would like to define $E(\mathcal{F})_*(-)$ to be the equalizer of (20.1) but we can't guarantee that the equalizer will be exact. This will be the case however, if $\mathcal{F}$ comes from a flat map

$$\operatorname{Spec} R \longrightarrow \mathcal{M}_{FG}$$

which factors through $\operatorname{Spec} A$, i.e. a formal group which comes from a formal group law. In this case the equalizer above will split and we will get a homology theory, namely

$$X \mapsto R \otimes_L MU_*(X)$$

Remark 20.2. If we are more fancy we can actually see that this works for any representable sheaf, without any flatness assumption.

Harder question: What about more general maps? Given a flat map

$$\mathcal{N} \longrightarrow \mathcal{M}_{FG}$$

can we associate to $\mathcal{N}$ a cohomology theory $E(\mathcal{N})$?

Every time we can do this we obtain an interesting cohomology theory. The answer is yes in many cases but there is no general theorem giving a functor from such maps to cohomology theories and this is an important problem.

We can try to define the cohomology theory $E(\mathcal{N})$ using the following approach: the cover $\operatorname{Spec} L \longrightarrow \mathcal{M}_{FG}$ gives us a simplicial stack over $\mathcal{M}_{FG}$ and taking pullbacks under the boundary maps d_0

$$\begin{tikzcd} \mathcal{N}_2 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{N}_1 \arrow[r, shift left] \arrow[r, shift right] & \mathcal{N}_0 \arrow[r] & \mathcal{N} \arrow[d] \\ \operatorname{Spec} \Gamma \otimes_L \Gamma \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \operatorname{Spec} \Gamma \arrow[r, shift left] \arrow[r, shift right] & \operatorname{Spec} L \arrow[r] & \mathcal{M}_{FG} \end{tikzcd}$$

we obtain from $\mathcal{N}$ a simplicial stack over $\mathcal{M}_{FG}$

$$\mathcal{N}_\bullet \longrightarrow \mathcal{M}_{FG}$$

which we would like to think of as the nerve of a covering of $\mathcal{N}$.

As the map $\mathcal{N} \longrightarrow \mathcal{M}_{FG}$ is flat, it is in particular representable so we have

$$\mathcal{N}_\bullet = \operatorname{Spec} R_\bullet$$

for some cosimplicial ring $R_\bullet$, with R_k flat over $\Gamma \otimes_L \cdots \otimes_L \Gamma$. Hence we have a cosimplicial homology theory

$$X \mapsto R_k \otimes_{\Gamma \otimes_L \cdots \otimes_L \Gamma} \mathcal{M}U \wedge \ldots \wedge \mathcal{M}U_*(X)$$

This is where we run into trouble. *If* we could refine this to a cosimplicial spectrum $E(\operatorname{Spec} R_\bullet)$ we could define

$$E(\mathcal{N}) := \operatorname{Tot}(E(\operatorname{Spec} R_\bullet))$$

but if we pick spectra representing the various homology theories, the cosimplicial identities will only hold up to homotopy modulo phantom maps.

It is definitely not the case that we can always rigidify such a cosimplicial diagram of cohomology theories, but by a miracle this seems to happen in important examples.

Properties of the construction. Assuming however that we did have this fantasy correspondence between flat stacks over $\mathcal{M}_{FG}$ and spectra it would have the following properties:

1. $E(\mathcal{N}_1 \times_{\mathcal{M}_{FG}} \mathcal{N}_2) = E(\mathcal{N}_1) \wedge E(\mathcal{N}_2)$ up to natural weak equivalence.
2. There is a spectral sequence converging to $\pi_* E(\mathcal{N})$

Property 1 is very useful because the smash product in spectra is so much easier to understand than the fiber product in stacks. It seems to be a new part of this miraculous correspondence between the algebra of formal group laws and algebraic topology.

The spectral sequence in 2 is just the spectral sequence of a cosimplicial spectrum but there is a nice homological name for the E_2 term which we now describe.

There is a map

$$\mathcal{M}_{FG} \xrightarrow{\omega} B\mathbb{G}_m$$

classifying a line bundle. In the language of formal groups, ω is the sheaf of invariant differentials on the universal formal group. It is determined by the map of groupoids corepresented by

$$\begin{tikzcd} \Gamma & \mathbb{Z}[\lambda^{\pm 1}] \arrow[l, "\varphi"'] \\ L \arrow[u, shift left] \arrow[u, shift right] & \mathbb{Z} \arrow[l] \arrow[u, shift left] \arrow[u, shift right] \end{tikzcd}$$

where φ corepresents the map assigning to an isomorphism between formal group laws its derivative at 0.

As fairly immediate consequences of the setup and denoting by the same symbol the pullback of ω to a stack over $\mathcal{M}_{FG}$, one finds that

$$\begin{aligned} \pi_{2m}E(\operatorname{Spec} R) &= H^0(\operatorname{Spec} R; \omega^m) \\ \pi_{2m+1}E(\operatorname{Spec} R) &= 0 \end{aligned}$$

From this it is easy to identify the E_2 term of the spectral sequence. We get

$$E_2^{s,t} = H^s(\mathcal{N}, \omega^t) \Rightarrow \pi_{2t-s}E(\mathcal{N})$$

Remark 20.3. There are cohomological obstructions to rigidifying cosimplicial diagrams of cohomology theories. Stacks allow us to give these a natural cohomological name. This is what got me started thinking about stacks in the first place. I then realized that this could be used to push the analogy between FGLs and cohomology theories further.

The sphere spectrum. We will start with the worst possible example. We take $\mathcal{N} = \mathcal{M}_{FG}$ and the identity map

$$\mathcal{N} \longrightarrow \mathcal{M}_{FG}$$

In this case, the cosimplicial homology theory we get is represented by

$$\mathcal{M}U \rightrightarrows \mathcal{M}U \wedge \mathcal{M}U \Rrightarrow \cdots$$

which is the Adams-Novikov resolution of the sphere. Thus

$$E(\mathcal{N}) = S^0$$

and the spectral sequence that we described above is just the Novikov spectral sequence. Note how we have brought the sphere into this picture of formal groups in a systematic way.

Elliptic cohomology. If C is a plane curve meeting each line in 3 points, we can define a group structure on C by decreeing that colinear points add up to 0 and choosing a point of inflection as the unit. Such a curve C is called an elliptic curve. We want to describe the moduli stack of elliptic curves $\mathcal{M}_{Ell}$.

If we choose ∞ to be a point of inflection and pick the line at infinity to be the tangent to this point, it is an easy exercise to check that C will be defined by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_4x + a_6$$

The isomorphisms between such curves will be the automorphisms of P^2 which preserve the line at infinity. It is easy to check that a general isomorphism is given by

$$\begin{aligned} x &\mapsto \lambda^{-2}x + r \\ y &\mapsto \lambda^{-3}y + sx + t \end{aligned}$$

Thus we can define a Hopf algebroid (A, Γ) with

$$\begin{aligned} A &= \mathbb{Z}[a_1, a_2, a_3, a_4, a_6] \\ \Gamma &= A[r, s, t, \lambda^{\pm 1}] \end{aligned}$$

and we define the moduli stack of elliptic curves to be the associated stack

$$\mathcal{M}_{Ell} = \mathcal{M}_{(A,\Gamma)}$$

Remark 20.4. Locally every elliptic curve embeds in the plane so this is really the moduli stack of elliptic curves.

There is a map

$$\mathcal{M}_{Ell} \longrightarrow \mathcal{M}_{FG}$$

obtained by expanding the group law of C in the parameter x/y, which is a local coordinate near ∞. By the procedure described above this yields a spectrum

$$eo_2 := E(\mathcal{M}_{Ell})$$

and there is a spectral sequence

$$H^s(\mathcal{M}_{Ell}; \omega^n) \Rightarrow \pi_* eo_2 \tag{20.2}$$

Given that $H^0(\mathcal{M}_{Ell}; \omega^n)$ is the group of modular forms of weight n over $\mathbb{Z}$, (20.2) gives us a spectral sequence relating modular forms and homotopy groups.

The spectrum eo_2 seems to play an important role in homotopy theory. For example, it is very easy to compute the homotopy groups of spheres through dimension 60 using eo_2.

In the $60's$ using K-theory it was possible to describe in a geometric way the first layer of the homotopy groups of spheres (the image of J). Similarly, elliptic cohomology promises to describe the second layer of the homotopy groups of spheres in a way which is close to geometry.

21. The Landweber exact functor theorem

In this section we will see how using the language of stacks we can get both a simpler proof and a stronger statement of the Landweber exact functor theorem.

Let R be a ring and G a formal group law over R. Write

$$[p]_G(x) = \sum_i a_i x^i$$

Then for each prime p we have a sequence of elements of R

$$v_i := a_{p^i}$$

for $i \geq 0$ where $v_0 = p$. This is not an intrinsic definition of the v_i's as it depends on the choice of the coordinate x. However it turns out that the choice of v_n is more or less unique modulo the ideal $(p, v_1, \dots, v_{n-1})$ in the following sense: if we change coordinate by

$$x \mapsto g(x) = \lambda x + \dots$$

then modulo $(p, v_1, \dots, v_{n-1})$,

$$[p](x) = v_n x^{p^n} + \dots$$

is sent to

$$g \circ [p](g^{-1}(x)) = \lambda^{1-p^n} v_n x^{p^n} + \dots$$

In the language of stacks this says that v_n is a section of the line bundle ω^{p^n-1} over $\mathcal{M}_{FG}$, which is canonically defined over the stack $\mathcal{M}_{FG}^{[n]}$ defined by

$$\mathcal{M}_{FG}^{[n]} := \mathcal{M}_{(A,\Gamma)}$$

with $A = L/(p, v_1, \dots, v_{n-1})$ and $\Gamma = \Gamma_{FG} \otimes_L A$.

Now let $A = L$ be the Lazard ring. We have elements $v_i \in A$ corresponding to a universal formal group law over A. Let $\Gamma = L[t_0^{\pm 1}, t_1, \dots]$ be the ring corepresenting isomorphisms of formal group laws. Then we can state the following theorem of Landweber

Theorem 21.1. *Let M be an (A, Γ)-comodule such that for each prime p and each $n \geq 1$ the map*

$$v_n : M/(p, v_1, \dots, v_{n-1}) \hookrightarrow M/(p, v_1, \dots, v_{n-1})$$

is an injection. Then for X a spectrum the assignment

$$X \mapsto MU_*(X) \otimes_A M$$

is a homology theory on finite *spectra.*

Note that the (A, Γ)-comodule $MU_*(X)$ with X a finite spectrum is a very special kind of comodule.

Remark 21.2. Note that it doesn't matter whether we take M to be just an A-module or an (A, Γ)-comodule because we can always make an A-module an (A, Γ)-comodule and come down by faithfully flat descent.

We will now formulate a variation of Theorem 21.1 in the language of stacks and give a proof of this stronger version.

In the language of stacks, M corresponds to a quasi-coherent sheaf $\mathcal{F}(M)$ over $\mathcal{M}_{FG}$ and $\mathcal{MU}_*(X)$ corresponds to a graded quasi-coherent sheaf $\mathcal{F}(X)$.

Moreover we have that under the map

$$\mathrm{Spec}(L) \xrightarrow{f} \mathcal{M}_{FG}$$

$$f^*\mathcal{F}(X) \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} \mathcal{F}(M) = \mathcal{MU}_*(X) \otimes_L M$$

Since f is flat, $M \otimes_L \mathcal{MU}_*(X)$ is exact in the X variable if and only if $\mathcal{F}(M) \otimes_{\mathcal{O}_{\mathcal{M}_{FG}}} \mathcal{F}(X)$ is exact in the X variable. So the question we have to answer is: when is $\mathcal{F}(M)$, or more generally a quasi-coherent sheaf over $\mathcal{M}_{FG}$, flat?

Exercise 21.3. Check that the Landweber conditions given in the statement of Theorem 21.1 are necessary for flatness.

Landweber's conditions are also close to being sufficient. They are not quite because we are only testing exactness tensoring with special kinds of comodules. We can actually get sufficient conditions for flatness. It is usually the case that when we apply Theorem 21.1 these stronger conditions are satisfied.

Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{M}_{FG}$. As we have pointed out above, we have sections v_n of the line bundle ω^{p^n-1} over $\mathcal{M}_{FG}$. We define inductively the sheaves $cF/(p, \dots, v_n)$ by

$$\mathcal{F}/(p) := \mathrm{coker}(\mathcal{F} \xrightarrow{p} \mathcal{F})$$

$$\mathcal{F}/(p, \dots, v_n) := \mathrm{coker}\left(\omega^{1-p^n} \otimes \mathcal{F}/(p, \dots, v_{n-1}) \xrightarrow{v_n} \mathcal{F}/(p, \dots, v_{n-1})\right)$$

Theorem 21.4. *If $\mathcal{F}$ satisfies*

(i) $v_i : \omega^{1-p^i} \otimes \mathcal{F}/(p, v_1, \dots, v_{i-1}) \hookrightarrow \mathcal{F}/(p, v_1, \dots, v_{i-1})$ *is an inclusion for each $i \geq 1$.*
(ii) $\mathcal{F}/(p, v_1, \dots, v_{n-1}) = 0$ *for $n \gg 0$.*

then $\mathcal{F}$ is flat.

Proof. Let $\mathcal{N}$ be a quasi-coherent sheaf over $\mathcal{M}_{FG}$. We need to show that

$$Tor_1(\mathcal{F}, \mathcal{N}) = 0$$

It suffices to do this after tensoring $\mathcal{F}$ with $\mathbb{Z}_{(p)}$ for each prime p. The short exact sequence

$$\mathcal{F} \xrightarrow{p} \mathcal{F} \longrightarrow \mathcal{F}/(p)$$

gives

$$Tor_2(\mathcal{F}/(p), \mathcal{N}) \longrightarrow Tor_1(\mathcal{F}, \mathcal{N}) \xrightarrow{p} Tor_1(\mathcal{F}, \mathcal{N})$$

so it suffices to show that

(i) $Tor_2(\mathcal{F}/(p), \mathcal{N}) = 0$
(ii) $Tor_1(p^{-1}\mathcal{F}, \mathcal{N}) = 0$

Now consider the short exact sequence

$$\omega^{1-p} \otimes \mathcal{F}/(p) \xrightarrow{v_1} \mathcal{F}/(p) \longrightarrow \mathcal{F}/(p, v_1)$$

yields the long exact sequence

$$Tor_3(\mathcal{F}/(p, v_1), \mathcal{N}) \longrightarrow Tor_2(\omega^{1-p} \otimes \mathcal{F}/(p), \mathcal{N}) \xrightarrow{v_1} Tor_1(\mathcal{F}/(p), \mathcal{N})$$

and defining

$$v_1^{-1}\mathcal{F}/(p) := \operatorname{colim}\left(\mathcal{F}/(p) \xrightarrow{v_1} \omega^{p-1} \otimes \mathcal{F}/(p) \xrightarrow{v_1} \omega^{2(p-1)} \otimes \mathcal{F}/(p) \xrightarrow{v_1} \cdots\right)$$

we see that is suffices to show

(i) $Tor_1(p^{-1}\mathcal{F}, \mathcal{N}) = 0$
(ii) $Tor_2(v_1^{-1}\mathcal{F}/(p), \mathcal{N}) = 0$
(iii) $Tor_3(\mathcal{F}/(p, v_1), \mathcal{N}) = 0$

Since by hypothesis $\mathcal{F}/(p, \dots, v_{n-1}) = 0$ for large n, continuing in this way we see that the theorem is reduced to proving

$$Tor_{k+1}(v_k^{-1}\mathcal{F}/(p, \dots, v_{k-1}), \mathcal{N}) = 0 \tag{21.1}$$

for all k.

To see why this is true, recall that there is a natural map of stacks

$$\mathcal{M}_{FG}^{(n)} \xrightarrow{f^{(n)}} \mathcal{M}_{FG}$$

which gives a pair of adjoint functors

$$f^{(n)*} : q-coh/\mathcal{M}_{FG}^{(n)} \rightleftarrows q-coh/\mathcal{M}_{FG} : f_*^{(n)}$$

($f_*^{(n)}$ is the right adjoint). Since $\mathcal{M}_{FG}^{(n)} = \mathcal{M}_{(A_n, \Gamma_n)}$ with

$$\begin{aligned} A_n &= v_n^{-1}L/(p, \dots, v_{n-1}) \\ \Gamma_n &= \Gamma \otimes A_n \end{aligned}$$

in terms of comodules we have

$$\begin{aligned} f^{(n)*}M &= v_n^{-1}M/(p, \dots, v_{n-1}) \\ f_*^{(n)}N &= N \text{ regarded as an } (L, \Gamma) \text{ comodule} \end{aligned}$$

Moreover $f_*^{(n)}$ embeds $q-coh/\mathcal{M}_{FG}^{(n)}$ as the full subcategory of comodules where $p, \dots, v_{n-1}$ act as 0 and v_n acts as a unit (this is independent of the choice of v_n). Thus we can rephrase (21.1) as saying that

$$Tor_{k+1}(f_*^{(k)} f^{(k)*}\mathcal{F}, \mathcal{N}) = 0 \tag{21.2}$$

This result now follows from the following proposition. □

Proposition 21.5. *If $\mathcal{G}$ is a quasi-coherent sheaf on $\mathcal{M}_{FG}^{(n)}$ and $\mathcal{N}$ is a quasi-coherent sheaf on $\mathcal{M}_{FG}$ then $Tor_m(f_*^{(n)}\mathcal{G}, \mathcal{N}) = 0$ for $m > n$*

Proof. We must show that the m-th left derived funtors of the functor

$$\mathcal{N} \mapsto f_*^{(n)}\mathcal{G} \otimes \mathcal{N}$$

vanish for $m > n$. But, as is easily checked on the level of comodules, we have

$$f_*^{(n)}\mathcal{G} \otimes \mathcal{N} = f_*^{(n)}(\mathcal{G} \otimes f^{(n)*}\mathcal{N})$$

and so, since $f_*^{(n)}$ is an exact functor, it is enough to check that the m-th left derived functors of the composite functor

$$q - coh/\mathcal{M}_{FG} \xrightarrow{f^{(n)*}} q - coh/\mathcal{M}_{FG}^{(n)} \xrightarrow{\mathcal{G}\otimes -} q - coh/\mathcal{M}_{FG}^{(n)}$$

$$\mathcal{N} \longrightarrow f^{(n)*}\mathcal{N} \longrightarrow \mathcal{G} \otimes f^{(n)*}\mathcal{N}$$

vanish for $m > n$. This will follow from the composite functor spectral sequence and the following two results

(a) $\mathbf{L}_t f^{(n)*} = 0$ for $t > n$

(b) If $\mathcal{A}, \mathcal{B}$ are quasi-coherent sheaves over $\mathcal{M}_{FG}^{(n)}$ then $Tor_s(\mathcal{A}, \mathcal{B}) = 0$ for $s > 0$

Proof of (a): There is a pullback diagram

$$\begin{tikzcd} \operatorname{Spec} A_n \arrow[r] \arrow[d, "i"] & \operatorname{Spec} L \arrow[d, "j"] \\ \mathcal{M}_{FG}^{(n)} \arrow[r, "f^{(n)}"] & \mathcal{M}_{FG} \end{tikzcd}$$

Since j is faithfully flat, so is i. Pulling back under i takes nonzero objects to nonzero objects so it is enough to check that if N is an L-module then

$$Tor_t(N, A_n) = 0$$

for $t > n$. This is true because there is a Koszul resolution for A_n given by

$$A_n \otimes E[b_0, \dots, b_{n-1}]$$

with $d(b_k) = v_k$, which is a flat resolution of length n.

Proof of (b): The map

$$\operatorname{Spec} \mathbb{F}_p \xrightarrow{\Gamma_n} \mathcal{M}_{FG}^{(n)}$$

classifying the mod$p$ reduction of the Lubin-Tate formal group law is faithfully flat (we have only proved that it was a cover but looking at the argument more closely reveals that it is faithfully flat).

Then since all $\mathbb{F}_p$-modules are flat,

$$Tor_s^{\mathbb{F}_p}(\Gamma_n^* A, \Gamma_n^* B) \implies Tor_s^{\mathcal{M}_{FG}^{(n)}}(\mathcal{A}, \mathcal{B}) = 0$$

for $s > 0$

This completes the proof. $\square$

Remark 21.6. The proof just given is not really simpler than Landweber's since it uses the language of stacks. The good thing about it is that it uses the same idea as the proof of the change of rings theorem and so fits in a general framework.

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