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Homology of the double loop space of the homogeneous space SU(n)/SO(n)

By

Younggi Choi

Abstract. We study the mod 2 homology of the double loop space of SU(n)/SO(n) using the Serre spectral sequence along with the Eilenberg–Moore spectral sequence. Then we also get the homology of the double loop space of the set of all Lagrangian subspaces of the symplectic vector space R^{2n}.

1. Introduction. Let SU(n) be the group of $n \times n$ unitary matrices of determinant 1 and SO(n) the group of $n \times n$ orthogonal matrices of determinant 1. In this paper we study the homology of the double loop space of the homogeneous space SU(n)/SO(n). For an odd prime p, we have the Harris splitting [8]

$$\mathrm{SU}(2n+1) \simeq_{(p)} (\mathrm{SU}(2n+1)/\mathrm{SO}(2n+1)) \times \mathrm{SO}(2n+1)$$

where $\simeq_{(p)}$ means homotopy equivalence localized at p. So the mod p homology of iterated loop spaces of SU(n) contains the information about that of SU(n)/SO(n). Since the mod p homology of the double loop space of SU(n) is known in [12] and [15], the computation with Z_p coefficients is not interesting. Here Z_p is the group of integers modulo p. In this paper we concentrate on Z_2 coefficients and every homology is considered as the homology with Z_2 coefficients unless mentioned otherwise. The case of SO(n) is covered in [5].

We study the cohomology of the loop space of SU(n)/SO(n) through the combined use of the Serre spectral sequence and the Eilenberg-Moore spectral sequence. In order to determine the algebra structure, we make use of the Steenrod operations in the Eilenberg-Moore spectral sequence. With this result, we compute the homology of the double loop space of SU(n)/SO(n). Since SU(n)/SO(n) is closely related with the set of all Lagrangian subspaces of the symplectic vector space R^{2n}, we also get the cohomology of the loop space and the homology of the double loop space of the set of these Lagrangian subspaces.

2. Single loop space of SU(n)/SO(n). Let $E(x)$ be the exterior algebra on x and $\Gamma(x)$ the divided power algebra on x which is free over $\gamma_i(x)$ as a Z_2–module with product $\gamma_i(x)\gamma_j(x) = \binom{i+j}{j}\gamma_{i+j}(x)$. In this paper the subscript of an element means the degree of the element, that

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is, $\deg(x_i) = i$. We recall the following fact in [4], [11].

$$H^*(\mathrm{BSO}(n)) = Z_2[w_i : 2 \leqq i \leqq n], \quad n \geqq 2$$

$$Sq^j(w_i) = \sum_{k=0}^{j} \binom{i-k-1}{j-k} w_{i+j-k} w_k, \ 0 \leqq j \leqq i.$$

For $x_i = \sigma(w_{i+1})$ in $H^i(\mathrm{SO}(n))$, we have that

$$H^*(\mathrm{SO}(n)) = V(x_i : 1 \leqq i \leqq n-1),$$

$$Sq^j(x_i) = \binom{i}{j} x_{i+j}, \ 0 \leqq j \leqq i\,.$$

Here $V(x_{i_1}, x_{i_2}, \ldots, x_{i_t})$ is the commutative associative algebra over Z_2 satisfying the following conditions,

1. $\{(x_{i_1})^{\epsilon_1}, \ldots, (x_{i_t})^{\epsilon_t}) | \epsilon_i = 0, 1\}$ is a basis.
2. $(x_{i_q})^2 = x_{2i_q}$ if $2i_q = i_s$ for some $1 \leqq s \leqq t$ and $x_{i_q}^2 = 0$ otherwise.

We also recall the following.

$$H^*(\mathrm{SU}(n)) = E(x_{2i+1} : 1 \leqq i \leqq n-1), \quad n \geqq 2.$$

We can compute $H^*(\mathrm{SU}(n)/\mathrm{SO}(n))$ from the Serre spectral sequence converging to $H^*(\mathrm{SU}(n)/\mathrm{SO}(n))$ with $E_2 = H^*(\mathrm{BSO}(n)) \otimes H^*(\mathrm{SU}(n))$. Then we have

$$H^*(\mathrm{SU}(n)/\mathrm{SO}(n)) = E(e_i : 2 \leqq i \leqq n), \quad n \geqq 2,$$

$$Sq^j(e_i) = \sum_{k=0}^{j} \binom{i-k-1}{j-k} e_{i+j-k} e_k, \ 0 \leqq j \leqq i$$

where $e_i = i^*(w_i)$ for the inclusion $i : \mathrm{SU}(n)/\mathrm{SO}(n) \to \mathrm{BSO}(n)$. We refer Theorem 6.7 of Chapter 3 in [11] for more detail explanation.

Let $\Omega^k M$ be the k–fold loop space of a space M, that is, the space of all base point preserving continuous maps from S^k to M. Now we calculate the cohomology of $\Omega\mathrm{SU}(n)/\mathrm{SO}(n)$.

Theorem 2.1.

$$H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) = \bigotimes_{0 \leqq 2i \leqq n-2} \ \bigotimes_{k \geqq 0} Z_2[\gamma_{2^k}(z_{2i+1})]/(\gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)}})$$

where $\sigma(n, i)$ is the positive integer satisfying the relation

$$(2i+1)2^{\sigma(n,i)-1} \leqq n-1 < (2i+1)2^{\sigma(n,i)}\,.$$

Proof. Consider the Eilenberg–Moore spectral sequence [7], [13] converging to $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2) \\ &= \mathrm{Tor}_{E(e_2, \ldots, e_n)}(Z_2, Z_2) \\ &= \otimes_{i=2}^{n} \mathrm{Tor}_{E(e_i)}(Z_2, Z_2) \\ &= \Gamma(z_1, \ldots, z_{n-1}) \end{aligned}$$

where $z_{i-1} = \sigma(e_i)$. We claim that this spectral sequence collapses at the E_2-term. We have the map of fibrations

$$\begin{array}{ccccc} \Omega\mathrm{SU}(n) & \longrightarrow & \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) & \longrightarrow & \mathrm{SO}(n) \\ \downarrow{\scriptstyle \Omega\iota} & & \downarrow{\scriptstyle \Omega i} & & \downarrow \\ \Omega\mathrm{SU} & \longrightarrow & \Omega(\mathrm{SU}/\mathrm{SO}) & \longrightarrow & \mathrm{SO} \end{array} \tag{1}$$

where each row is a fibration. In the Eilenberg–Moore spectral sequence converging to $H^*(\Omega\mathrm{SU}(n))$ with

$$\begin{aligned} E_2 &= \mathrm{Tor}_{H^*(\mathrm{SU}(n))}(Z_2, Z_2) \\ &= \mathrm{Tor}_{E(x_3,\dots,x_{2n-1})}(Z_2, Z_2) \\ &= \Gamma(y_2, \dots, y_{2n-2}), \end{aligned}$$

the spectral sequence collapses at the E_2–term because E_2 vanishes in all odd total degrees. And so does the Eilenberg–Moore spectral sequence converging to $H^*(\Omega\mathrm{SU})$ with $E_2 = \mathrm{Tor}_{H^*(\mathrm{SU})}(Z_2, Z_2)$. Hence $(\Omega\iota)^*$ is surjective.

We consider the Serre spectral sequence for the bottom row with $E_2 = H^*(\mathrm{SO}) \otimes H^*(\Omega\mathrm{SU})$. Then we can easily check that as a graded vector space this E_2-term has the same size in every total degree as $H^*(\Omega(\mathrm{SU}/\mathrm{SO}))$ where $H^*(\Omega(\mathrm{SU}/\mathrm{SO})) = H^*(\mathrm{BO}) = Z_2[w_1, w_2, \dots]$. Note that there is a homotopy equivalence between $\Omega(\mathrm{SU}/\mathrm{SO})$ and BO [3], [4]. So the spectral sequence collapses at the E_2-term. That means that there are no nontrivial differentials. Now we consider the Serre spectral sequence for the top row in the diagram (1) with $E_2 = H^*(\mathrm{SO}(n)) \otimes H^*(\Omega\mathrm{SU}(n))$. Since the Serre spectral sequence for the bottom row collapses at the E_2–term, we have $d_r = 0$ for $r = 2, 3 \dots$. Since $(\Omega\iota)^*$ is surjective, by naturality we also have $d_r = 0$ for $r = 2, 3 \dots$ for the top row. This implies that the Serre spectral sequence for the top row also collapses at the E_2-term.

We have studied two spectral sequences going to the same destination space $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$. One is the Eilenberg–Moore spectral sequence with $E_2 = \mathrm{Tor}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)$ and the other is the Serre spectral sequence with $E_2 = H^*(\mathrm{SO}(n)) \otimes H^*(\Omega\mathrm{SU}(n))$. But as a graded vector space, the E_2-term of the Eilenberg–Moore spectral sequence has the same size in every total degree as the E_∞-term of the Serre spectral sequence. Hence the Eilenberg–Moore spectral sequence collapses at the E_2–term and $E_\infty = \Gamma(z_1, \dots, z_{n-1})$ where $z_{i-1} = \sigma(e_i)$.

Now we determine the multiplicative structure of cohomology. Since $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ is a connected, associative and commutative Hopf algebra over Z_2, by the Hopf-Borel theorem $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ is the tensor product of monogenic Hopf algebras which are of the form $Z_2[x]$ or $Z_2[x]/(x^{2^k})$ as an algebra. Since $Sq^j(e_i) = \sum_{k=0}^{j} \binom{i-k-1}{j-k} e_{i+j-k} e_k$ for $0 \leqq j \leqq i$ in $H^*(\mathrm{SU}(n)/\mathrm{SO}(n))$,

$$\begin{aligned} Sq^j z_i = Sq^j(\sigma(e_{i+1})) &= \sigma\left(\binom{i}{j} e_{i+j+1} + \binom{i-1}{j-1} e_{i+j} e_1 + \cdots\right) \\ &= \binom{i}{j} \sigma(e_{i+j+1}) = \binom{i}{j} z_{i+j} \,. \end{aligned}$$

Hence $z_i^2 = Sq^i(z_i) = z_{2i}$ for $1 \leqq i \leqq (n-1)/2$. In the bar construction, each z_i is represented by $[e_{i+1}]$ and $\gamma_{2^k}(z_i)$ is represented by $[e_{i+1}|\cdots|e_{i+1}]$ (2^k factors). Since $Sq^j e_{i+1} = 0$ for

$j \geqq i+1$, by the Cartan formula,

$$\begin{aligned}(\gamma_{2^k}(z_i))^2 &= Sq^{2^k i}(\gamma_{2^k}(z_i)) = Sq^{2^k i}([e_{i+1}|\cdots|e_{i+1}])\\ &= [Sq^i e_{i+1}|\cdots|Sq^i e_{i+1}]\\ &= [e_{2i+1} + \text{ decomposables } |\cdots|e_{2i+1} + \text{ decomposables}]\\ &= [e_{2i+1}|\cdots|e_{2i+1}] + \sum_r [s_{r_1}|\cdots|s_{r_{2^k}}]\end{aligned}$$

where for each r, some s_{r_-} is a decomposable element. Here $\sum_r [s_{r_1}|\cdots|s_{r_{2^k}}]$ represents zero in $\mathrm{Tor}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)$ because of the following reason [1, p. 424–425]. The product in $\Gamma(\sigma(e_2), \cdots, \sigma(e_n))$=$\mathrm{Tor}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)$ is induced by the shuffle product in the bar construction [9, § 10.12], [10, § 7.2]. The form of the shuffle product implies that every element of $\Gamma(\sigma(e_2), \cdots, \sigma(e_n))$ have a representative $\sum_t [u_{t_1}|\cdots|u_{t_{2s}}]$ where no u_{t_-} is decomposable. And from the definition of differential of bar construction, $\sum_r [s_{r_1}|\cdots|s_{r_{2^k}}] + \sum_t [u_{t_1}|\cdots|u_{t_{2s}}]$ can not be target of the differential. Therefore $\sum_r [s_{r_1}|\cdots|s_{r_{2^k}}]$ represents zero in $\mathrm{Tor}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)$. So we have that $(\gamma_{2^k}(z_i))^2 = [e_{2i+1}|\cdots|e_{2i+1}] = \gamma_{2^k}(z_{2i})$ for $1 \leqq i \leqq \frac{n-1}{2}$ and $k \geqq 0$.

$\Gamma(z) = E(\gamma_{2^k}(z) : k \geqq 0)$ as an algebra. Let $\sigma(n,i)$ be the positive integer such that $(2i+1)2^{\sigma(n,i)-1} \leqq n-1 < (2i+1)2^{\sigma(n,i)}$. Since $z_{(2i+1)2^{m-1}}^{2^{\sigma(n,i)-m}} = z_{(2i+1)2^m}^{2^{\sigma(n,i)-m-1}}$ for $1 \leqq m \leqq \sigma(n,i)-1$, we have

$$z_{2i+1}^{2^{\sigma(n,i)-1}} = z_{(2i+1)2^{\sigma(n,i)-1}}, \quad z_{2i+1}^{2^{\sigma(n,i)}} = z^2_{(2i+1)2^{\sigma(n,i)-1}} = 0, \quad 0 \leqq 2i \leqq n-2.$$

In the same way, for $0 \leqq 2i \leqq n-2$

$$\gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)-1}} = \gamma_{2^k}(z_{(2i+1)2^{\sigma(n,i)-1}}), \quad \gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)}} = 0.$$

Hence it follows from these relations that for $0 \leqq 2i \leqq n-2$ and $k \geqq 0$,

$$E(\gamma_{2^k}(z_{2i+1})) \otimes \ldots \otimes E(\gamma_{2^k}(z_{(2i+1)2^{\sigma(n,i)-1}}))$$

in the E_∞–term produces $Z_2[\gamma_{2^k}(z_{2i+1})]/(\gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)}})$. We claim that there is no relation among $\gamma_{2^k}(z_{2i+1})$ for $0 \leqq 2i \leqq n-2$, $k \geqq 0$. Consider the Eilenberg–Moore spectral sequence [6] converging to $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ with

$$\begin{aligned}E_2 &= \mathrm{Cotor}^{H_*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)\\ &= \mathrm{Ext}_{H^*(\mathrm{SU}(n)/\mathrm{SO}(n))}(Z_2, Z_2)\\ &= \mathrm{Ext}_{E(e_2,\ldots,e_n)}(Z_2, Z_2) = Z_2[c_1, \ldots, c_{n-1}].\end{aligned}$$

Then it also collapses at the E_2–term by duality. So $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) = Z_2[c_1, \ldots, c_{n-1}]$ and

$$(\Omega i)_* : H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))) \to H_*(\Omega(\mathrm{SU}/\mathrm{SO}))$$

is injective where $H_*(\Omega(\mathrm{SU}/\mathrm{SO})) = Z_2[c_i : i \geqq 1]$. Here $H^*(\Omega(\mathrm{SU}/\mathrm{SO})) = H^*(\mathrm{BO})$ is a polynomial algebra with one generator in each degree, so that its dual $H_*(\Omega(\mathrm{SU}/\mathrm{SO}))$ has one primitives in each degree $i \geqq 1$. Since 2^k power of a primitive is also primitive, $H_*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ has primitives of degrees $(2i+1)2^k$ for $0 \leqq 2i \leqq n-2$, $k \geqq 0$. Hence from duality, $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ has generators of degrees $(2i+1)2^k$ for $0 \leqq 2i \leqq n-2$, $k \geqq 0$. So there is no relation among $\gamma_{2^k}(z_{2i+1})$ for $0 \leqq 2i \leqq n-2$, $k \geqq 0$. Hence we get the conclusion.

In fact, when n goes to infinity, each $\gamma_{2^k}(z_{2i+1})$ gets to have infinite height, so that each $\gamma_{2^k}(z_{2i+1})$ becomes a generator in $H^*(\Omega(\mathrm{SU}/\mathrm{SO})) = H^*(\mathrm{BO})$. Note that numbers $(2i+1)2^k$ for $i, k \geqq 0$ cover whole natural numbers. □

Consider the fibration $\mathrm{SU}(n)/\mathrm{SO}(n) \to U(n)/O(n) \xrightarrow{f} K(Z,1)$ where $[f]$ is a generator of $H^1(U(n)/O(n); Z) = Z$. By looping one more time, we get

$$\Omega(U(n)/O(n)) \cong \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \times Z\,.$$

So we get $\Omega_0(U(n)/O(n)) \cong \Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$ where $\Omega_0(U(n)/O(n))$ is the zero component of $\Omega(U(n)/O(n))$. On the other hand, it is well-known [14] that $U(n)/O(n)$ is diffeomorphic to $\mathscr{L}(R^{2n})$, the set of all Lagrangian subspaces of the symplectic vector space R^{2n} with the symplectic form $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$. So we obtain the cohomology of $\Omega_0\mathscr{L}(R^{2n})$.

Corollary 2.2. *The space $\Omega_0\mathscr{L}(R^{2n})$ has the same cohomology algebra as $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$, which is explicitly given in Theorem* 2.1.

Since $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ has the same size as $\Gamma(z_1, \ldots, z_{n-1})$ in every total degree as a graded vector space, the Poincaré series of the space $\Omega_0\mathscr{L}(R^{2n})$ is given by

$$\begin{aligned} P_t(\Omega_0\mathscr{L}(R^{2n})) &= \sum_{i \geqq 0} \dim_{Z_2} H^i(\Omega_0\mathscr{L}(R^{2n}))t^i \\ &= (1 + t^1 + t^2 + \cdots)\cdots(1 + t^{n-1} + t^{2n-2} + \cdots) \\ &= \prod_{i=1}^{n-1} \frac{1}{(1-t^i)}\,. \end{aligned}$$

For $n \geqq 3$, the i–th Betti number grows at least linearly as i increases. So does the i–th Betti number with coefficients in Z.

3. Homology of the double loop space of SU(n)/SO(n). We will compute $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ using the Serre spectral sequence for the fibration

$$\Omega^2\mathrm{SU}(n) \longrightarrow \Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)) \longrightarrow \Omega_0\mathrm{SO}(n)\,.$$

Hence we need to know $H_*(\Omega^2\mathrm{SU}(n))$ and $H_*(\Omega_0\mathrm{SO}(n))$. Recall the following fact in [12].

Theorem 3.1.

$$\begin{aligned} H_*(\Omega^2\mathrm{SU}(n)) = E&\left(u_{2^{k+1}l-1} : 0 < l \leqq \frac{n-1}{2}, k \geqq 0, l\ odd\right) \\ &\otimes Z_2\left[v_{2^{k+2}l-2} : 0 < l \leqq \frac{n-1}{2}, 2^{k+1}l \geqq n-1, l\ odd\right] \\ &\otimes Z_2\left[u_{2^{k+1}l-1} : \frac{n-1}{2} \leqq l \leqq n-1, k \geqq 0, l\ odd\right]. \end{aligned}$$

Since Spin(n) is a double covering space of SO(n), $\Omega_0\mathrm{SO}(n)$ is homeomorphic to $\Omega\mathrm{Spin}(n)$. It is known [11, Chapter 4, Theorem 2.19] that

$$H^*(\mathrm{Spin}(n)) = V\big(x_i | 3 \leqq i \leqq n-1 \text{ and } i \neq 2^j\big) \otimes E(z_{2^{s+1}-1})$$
$$Sq^r(x_i) = \binom{i}{r} x_{i+r},\ r \leqq i, \quad i+r \leqq n-1$$

where s is the number satisfying $2^s < n \leqq 2^{s+1}$. The following lemma comes from the computation of the Eilenberg–Moore spectral sequence converging to $H_*(\Omega_0\mathrm{SO}(n)) = H_*(\Omega\mathrm{Spin}(n))$ with

$$E^2 = \mathrm{Cotor}^{H_*(\mathrm{Spin}(n))}(Z_2, Z_2) = \mathrm{Ext}_{H^*(\mathrm{Spin}(n))}(Z_2, Z_2)\,.$$

Lemma 3.2.

$$\begin{aligned} H_*(\Omega_0\mathrm{SO}(n)) = E&\left(a_{2i} : 1 \leqq i \leqq \frac{n-3}{4}\right)\\ &\otimes Z_2\left[a_{2i} : \frac{n-2}{4} \leqq i \leqq \frac{n-2}{2}\right]\\ &\otimes Z_2\left[b_{4i+2} : \frac{n-4}{4} \leqq i \leqq \frac{n-3}{2}\right]. \end{aligned}$$

P r o o f. We consider n modulo 4. Once we find $H_*(\Omega_0\mathrm{SO}(4n))$, the other cases can be obtained by a similar way. Here we compute $H_*(\Omega\mathrm{Spin}(4n))$ instead.

$$H^*(\mathrm{Spin}(4n)) = V(x_i | 3 \leqq i \leqq 4n-1 \text{ and } i \neq 2^j) \otimes E(z_{2^{s+1}-1})$$
$$Sq^r(x_i) = \binom{i}{r} x_{i+r},\ r \leqq i, \quad i+r \leqq 4n-1$$

where $2^s < 4n \leqq 2^{s+1}$. As in Theorem 2.1, $\sigma(n,i)$ is the number satisfying the relation, $(2i+1)2^{\sigma(4n,i)-1} \leqq 4n-1 < (2i+1)2^{\sigma(4n,i)}$. From the Steenrod actions on x_i, we have $x_{2i} = Sq^i(x_i) = x_i^2$ for $1 \leqq i \leqq 2n-1$. By applying the method in the proof of Theorem 2.1, we get $x_{2i+1}^{2^{\sigma(4n,i)-1}} = x_{(2i+1)2^{\sigma(4n,i)-1}}$ and $x_{2i+1}^{2^{\sigma(4n,i)}} = (x_{(2i+1)2^{\sigma(4n,i)-1}})^2 = 0$ for $1 \leqq i \leqq 2n-1$. So we get

$$\begin{aligned} H^*(\mathrm{Spin}(4n)) = &\Big\{\bigotimes_{1\leqq i\leqq n-1} Z_2[x_{2i+1}]/((x_{2i+1})^{2^{\sigma(4n,i)}})\Big\}\\ &\otimes\Big\{\bigotimes_{n\leqq i\leqq 2n-1} E(x_{2i+1})\Big\} \otimes E(z_{2^{s+1}-1})\,. \end{aligned}$$

Consider the Eilenberg–Moore spectral sequence converging to $H_*(\Omega\mathrm{Spin}(4n))$ with

$$\begin{aligned} E^2 &= \mathrm{Ext}_{H^*(\mathrm{Spin}(4n))}(Z_2, Z_2)\\ &= E(a_{2i} : 1 \leqq i \leqq n-1) \otimes Z_2[b_{(2i+1)2^{\sigma(4n,i)}-2} : 1 \leqq i \leqq n-1]\\ &\quad \otimes Z_2[a_{2i} : n \leqq i \leqq 2n-1] \otimes Z_2[b_{2^{s+1}-2}]\,. \end{aligned}$$

Since $\{(2i+1)2^{\sigma(4n,i)-1} : 1 \leqq i \leqq n-1\} \cup \{2^s\} = \{2i : 2n \leqq 2i \leqq 4n-1\}$, $\{(2i+1)2^{\sigma(4n,i)} - 2 : 1 \leqq i \leqq n-1\} \cup \{2^{s+1}-2\} = \{4i-2 : n \leqq i \leqq 2n-1\}$. Hence we get

$$\begin{aligned} E^2 &= E(a_{2i} : 1 \leqq i \leqq n-1) \otimes Z_2[a_{2i} : n \leqq i \leqq 2n-1]\\ &\quad \otimes Z_2[b_{4i+2} : n-1 \leqq i \leqq 2n-2]\,. \end{aligned}$$

Since E^2 vanishes in all odd degrees, the spectral sequence collapses at the E^2-term. Hence $E^2 = E^\infty$.

We claim that there are no multiplicative extension problems here. First we recall the following in [2]. The generating variety for the homology of $\Omega\mathrm{Spin}(4n)$ is $V_{4n-2} = \mathrm{SO}(4n)/(\mathrm{SO}(2) \times \mathrm{SO}(4n-2))$ and $\lim\limits_{n\to\infty} V_{4n-2} = CP^{\infty}$ is the generating variety for the homology of $\Omega\mathrm{Spin}$, that is, we have a map from V_{4n-2} to $\Omega\mathrm{Spin}(4n)$ such that the image of $H_*(V_{4n-2})$ under the induced map generates $H_*(\Omega\mathrm{Spin}(4n))$ as an algebra. Here $H^*(V_{4n-2}) = Z_2[u_2]/(u_2^{2n}) \otimes E(v_{4n-2})$ and $H_*(V_{4n-2})$ is free on $\{1, \alpha_2, \alpha_4, \cdots, \alpha_{4n-2}\} \cup \{\beta_{4n-2}, \beta_{4n}, \cdots, \beta_{8n-4}\}$. By the Hopf-Borel theorem, $H^*(\Omega\mathrm{Spin}(4n))$ is the tensor product of monogenic Hopf algebras which are of the form $Z_2[x]$ or $Z_2[x]/(x^{2^k})$ as an algebra. If there were any extension, the only possible extension would occur in $E(a_{2i} : 1 \leqq i \leqq n-1)$ to make the square of a_{2i} equal a_{4i} by the degree reason.

Now we consider the following diagram

$$\begin{array}{ccc} V_{4n-2} & \longrightarrow & \Omega\mathrm{Spin}(4n) \\ {\scriptstyle i}\downarrow & & {\scriptstyle \Omega\iota}\downarrow \\ CP^{\infty} & \longrightarrow & \Omega\mathrm{Spin} \end{array}$$

where i and ι are inclusion maps. Then for $1 \leqq i \leqq 2n-1$, the image of each α_{2i} under the generating map is a_{2i} modulo decomposables in $H_*(\Omega\mathrm{Spin}(4n))$. From the diagram we can find that each a_{2i} under $(\Omega\iota)_*$ corresponds a generator in $H_*(\Omega\mathrm{Spin})$ which is the exterior algebra on generators of every even degree [4], [11]. In fact, there are choices of generators such that $H_*(\Omega\mathrm{Spin})$ is $E(a_2, a_4, a_6, \dots)$. Note that $\Omega\iota$ is an H-map. So if $a_{2i}^2 = a_{4i}$,

$$0 \neq (\Omega\iota)_*(a_{4i}) = (\Omega\iota)_*\big(a_{2i}^2\big) = (\Omega\iota)_*(a_{2i})(\Omega\iota)_*(a_{2i}) = 0\,.$$

This is a contradiction. So there is no extension. In fact, we can also derive the same result from [2]. Now we get

$$\begin{aligned} H_*(\Omega\mathrm{Spin}(4n)) = E(a_{2i} : 1 \leqq i \leqq n-1) \otimes Z_2[a_{2i} : n \leqq i \leqq 2n-1] \\ \otimes Z_2[b_{4i+2} : n-1 \leqq i \leqq 2n-2]\,. \end{aligned}$$

In a similar way, we also get

$$\begin{aligned} H_*(\Omega\mathrm{Spin}(4n+1)) &= E(a_{2i} : 1 \leqq i \leqq n-1) \otimes Z_2[a_{2i} : n \leqq i \leqq 2n-1] \\ &\quad \otimes Z_2[b_{4i+2} : n \leqq i \leqq 2n-1], \\ H_*(\Omega\mathrm{Spin}(4n+2)) &= E(a_{2i} : 1 \leqq i \leqq n-1) \otimes Z_2[a_{2i} : n \leqq i \leqq 2n] \\ &\quad \otimes Z_2[b_{4i+2} : n \leqq i \leqq 2n-1], \\ H_*(\Omega\mathrm{Spin}(4n+3)) &= E(a_{2i} : 1 \leqq i \leqq n) \otimes Z_2[a_{2i} : n+1 \leqq i \leqq 2n] \\ &\quad \otimes Z_2[b_{4i+2} : n \leqq i \leqq 2n]\,. \end{aligned}$$

We can rewrite the above as follows.

$$\begin{aligned} H_*(\Omega\mathrm{Spin}(n)) = E&\left(a_{2i} : 1 \leqq i \leqq \frac{n-3}{4}\right) \\ &\otimes Z_2\left[a_{2i} : \frac{n-2}{4} \leqq i \leqq \frac{n-2}{2}\right] \\ &\otimes Z_2\left[b_{4i+2} : \frac{n-4}{4} \leqq i \leqq \frac{n-3}{2}\right]. \quad \Box \end{aligned}$$

From the homotopy exact sequence

$$\cdots \to \pi_2(\mathrm{SU}(n)) \to \pi_2(\mathrm{SU}(n)/\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SU}(n)) \to \cdots,$$

we get $\pi_2(\mathrm{SU}(2)/\mathrm{SO}(2)) = Z$ and $\pi_2(\mathrm{SU}(n)/\mathrm{SO}(n)) = Z_2$ for $n \geqq 3$. Since there is an Hopf fibration for $n = 2$, $\mathrm{SO}(2) \to \mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{SO}(2) \cong S^2$, we obtain that $\Omega^2(\mathrm{SU}(2)/\mathrm{SO}(2)) \cong \Omega^2\mathrm{SU}(2) \times Z$. Hence by Theorem 3.1,

$$H_*\big(\Omega_0^2(\mathrm{SU}(2)/\mathrm{SO}(2))\big) = H_*(\Omega^2\mathrm{SU}(2)) = Z_2[u_{2^{k+1}-1} : k \geqq 0].$$

Theorem 3.3. *For* $n \geqq 3$, $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ *is*

$$\begin{aligned}
&E\left(x_{(2i+1)2^k-1} : 1 \leqq i \leqq \frac{n-3}{4}, k \geqq 0\right)\\
&\otimes Z_2\left[y_{(2i+1)2^k 2^{\sigma(n,i)}-2} : 1 \leqq i \leqq \frac{n-3}{4}, k \geqq 0\right]\\
&\otimes Z_2\left[x_{(2i+1)2^k-1} : \frac{n-3}{4} < i \leqq \frac{n-2}{2}, k \geqq 0\right] \otimes E(x_{2^{k+1}-1} : k \geqq 0)\\
&\otimes Z_2[y_{2^{k+1}2^{\sigma(n,0)}-2} : k \geqq 0] \otimes Z_2[y_{2^{\sigma(n,0)}-2}]
\end{aligned}$$

where $\sigma(n,i)$ *is the integer defined in Theorem* 2.1.

Proof. We have the morphism of fibrations

$$\begin{array}{ccccc}
\Omega^2\mathrm{SU}(n) & \longrightarrow & \Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)) & \longrightarrow & \Omega_0\mathrm{SO}(n)\\
{\scriptstyle \Omega^2 i}\big\downarrow & & {\scriptstyle \Omega^2 j}\big\downarrow & & {\scriptstyle \Omega^2 \iota}\big\downarrow\\
\Omega^2\mathrm{SU} & \longrightarrow & \Omega_0^2(\mathrm{SU}/\mathrm{SO}) & \longrightarrow & \Omega_0\mathrm{SO}
\end{array} \tag{2}$$

where i, j, and ι are inclusion maps. By Bott periodicity, we have the following [3], [4], [11].

$$\begin{aligned}
H_*(\Omega^2\mathrm{SU}) &= E(u_{2i+1} : i \geqq 0)\\
H_*\big(\Omega_0^2(\mathrm{SU}/\mathrm{SO})\big) &= E(x_i : i \geqq 1)\\
H_*(\Omega_0\mathrm{SO}) &= E(a_{2i} : i \geqq 1).
\end{aligned}$$

So the Serre spectral sequence converging to $H_*(\Omega_0^2(\mathrm{SU}/\mathrm{SO}))$ for the bottom row collapses at the E^2–term because as a graded vector space the size of the E^2–term is the same as that of the total space. We know from [12, Theorem 1.11] that for all u_{2i+1} in $H_*(\Omega^2\mathrm{SU}(n))$,

$$(\Omega^2 i)_*(u_{2i+1}) = u_{2i+1}.$$

Now consider the Serre spectral sequence converging to $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ for the top row. We claim that it collapses at the E^2–term. Since it is a spectral sequence of an Hopf algebra, the first nontrivial differential which is from an indecomposable element to a primitive element acts in a transgressive manner. Assume that it does not collapse at the E^2–term. Then there exists a first nontrivial differential from a_{2i} or b_{4i+2} for some i. Since u_{2i-1} and u_{4i+1} are the only primitives of possible degree, we have

$$d_{2i}(a_{2i}) = u_{2i-1} \quad \text{or} \quad d_{4i+2}(b_{4i+2}) = u_{4i+1}.$$

Since the spectral sequence for the bottom row of the diagram (2) collapses at E_2–term, by naturality

$$(\Omega^2 i)_*(d_{2i}(a_{2i})) = d_{2i}((\Omega\iota)_*(a_{2i})) = 0\,.$$

If $d_{2i}(a_{2i}) = u_{2i-1}$, $(\Omega^2 i)_*(d_{2i}(a_{2i})) = (\Omega^2 i)_*(u_{2i-1}) = u_{2i-1}$ which is a contradiction. So $d_{2i}(a_{2i}) = 0$. Similarly we can show that $d_{4i+2}(b_{4i+2}) = 0$. Hence the spectral sequence collapses at the E^2–term and the E^∞–term is

$$\begin{aligned}
&E\left(u_{2^{k+1}l-1} : 0 < l \leqq \frac{n-1}{2}, k \geqq 0, l \text{ odd}\right)\\
&\otimes Z_2\left[v_{2^{k+2}l-2} : 0 < l \leqq \frac{n-1}{2}, 2^{k+1}l \geqq n-1, l \text{ odd}\right]\\
&\otimes Z_2\left[u_{2^{k+1}l-1} : \frac{n-1}{2} \leqq l \leqq n-1, k \geqq 0, l \text{ odd}\right] \otimes E\left(a_{2i} : 1 \leqq i \leqq \frac{n-3}{4}\right)\\
&\otimes Z_2\left[a_{2i} : \frac{n-2}{4} \leqq i \leqq \frac{n-2}{2}\right] \otimes Z_2\left[b_{4i+2} : \frac{n-4}{4} \leqq i \leqq \frac{n-3}{2}\right].
\end{aligned} \tag{3}$$

Since all u_i, v_i, a_i and b_i in E^2 survive to $E^\infty = E^0(H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n))))$, we consider them as elements in $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$. From the diagram (2), we know that $(\Omega^2 j)_*(u_i) = x_i$ and $(\Omega^2 j)_*(a_{2i}) = x_{2i}$ for all u_i, a_{2i} in $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$. We will show that there are no multiplicative extensions. Here $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ is the tensor product of monogenic Hopf algebras which are of the form $Z_2[x]$ or $Z_2[x]/(x^{2^k})$ as an algebra. If there were extensions, there would be two possibilities. One is $u_i^2 = a_{2i}$. Then these elements u_i, a_{2i} are mapping to x_i, x_{2i} by $(\Omega^2 j)_*$, respectively in $H_*(\Omega_0^2(\mathrm{SU}/\mathrm{SO}))$. Since $\Omega^2 j$ is a H-map, in $H_*(\Omega_0^2(\mathrm{SU}/\mathrm{SO}))$ we have

$$x_{2i} = (\Omega^2 j)_*(a_{2i}) = (\Omega^2 j)_*(u_i^2) = (\Omega^2 j)_*(u_i)(\Omega^2 j)_*(u_i) = x_i x_i = 0\,.$$

This is a contradiction. The other possibility is $u_i^2 = b_{2i}$. We show that it cannot occur. Consider the map $h : \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \to K(Z_2, 1)$ such that $h^*(\omega_1) = z_1$ in $H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n)))$ where $H^*(K(Z_2, 1)) = H^*(RP^\infty) = Z_2[\omega_1]$. Then we have the fibration

$$\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))\langle 1\rangle \longrightarrow \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \xrightarrow{\ h\ } K(Z_2, 1)$$

where $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))\langle 1\rangle$ is a 1–connected cover of $\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))$. Since the lowest generator z_1 has the height $2^{\sigma(n,0)}$ by Theorem 2.1, $w_1^{2^{\sigma(n,0)}}$ in $H^*(RP^\infty)$ should be the target of a differential in the Serre spectral sequence for above fibration. Since this is a spectral sequence of a Hopf algebra, there should be some generator of degree $\sigma(n, 0) - 1$, let $c_{2^{\sigma(n,0)}-1}$, such that

$$\begin{aligned}
H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))\langle 1\rangle) = \Big\{ \bigotimes_{2\leqq 2i\leqq n-2} \Big\{ \bigotimes_{k\geqq 0} Z_2[\gamma_{2^k}(z_{2i+1})]/(\gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)}})\Big\}\Big\}\\
\otimes \Big\{ \bigotimes_{k\geqq 0} Z_2[\gamma_{2^k}(z_2)]/(\gamma_{2^k}(z_2)^{2^{\sigma(n,0)}})\Big\} \otimes E(c_{2^{\sigma(n,0)}-1})\,.
\end{aligned}$$

Note that $Z_2[\gamma_{2^k}(z_{2i+1}) : \frac{n-3}{4} < i \leqq \frac{n-2}{2}, k \geqq 0]/(\gamma_{2^k}(z_{2i+1})^{2^{\sigma(n,i)}})$ is $E(\gamma_{2^k}(z_{2i+1}) : \frac{n-3}{4} < i \leqq \frac{n-2}{2}, k \geqq 0)$. Consider the Eilenberg–Moore spectral sequence converging to

$H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ where E^2–term is

$$\begin{aligned}
&\mathrm{Ext}_{H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))\langle 1\rangle)}(Z_2, Z_2)\\
&= \left\{ \bigotimes_{k\geqq 0} \left(E\left(x_{(2i+1)2^k-1} : 1 \leqq i \leqq \frac{n-3}{4} \right) \right.\right.\\
&\qquad \left.\left. \otimes Z_2\left[y_{(2i+1)2^k 2^{\sigma(n,i)}-2} : 1 \leqq i \leqq \frac{n-3}{4} \right]\right)\right\}\\
&\otimes \left\{ \bigotimes_{k\geqq 0} Z_2\left[x_{(2i+1)2^k-1} : \frac{n-3}{4} < i \leqq \frac{n-2}{2} \right]\right\}\\
&\otimes \left\{ \bigotimes_{k\geqq 0} (E(x_{2^{k+1}-1}) \otimes Z_2[y_{2^{k+1}2^{\sigma(n,0)}-2}])\right\} \otimes Z_2[y_{2^{\sigma(n,0)}-2}]\,.
\end{aligned}$$

Then by simple calculation, this E^2-term has the same size in every total degree as the E^∞-term (3) as a graded vector space. In fact, inspecting on degree of each generator, we can find that both are exactly same as an algebra. Hence the above Eilenberg–Moore spectral sequence collapses at the E^2–term and $E^2 = E^\infty$.

Now we go back to the extension problem. If there were such an extension, $u_i^2 = b_{2i}$ in (3), then it would force an extension $x_i^2 = y_{2i} +$ some decomposables in above E^∞–term. This implies that y_{2i} becomes a decomposable element in $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$. Consider the Eilenberg–Moore spectral sequence converging to $H^*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$ with

$$E_2 = \mathrm{Tor}_{H^*(\Omega(\mathrm{SU}(n)/\mathrm{SO}(n))\langle 1\rangle)}(Z_2, Z_2)\,.$$

By duality, this Eilenberg–Moore spectral sequence also collapses at the E_2–term. Then it follows that y_{2i} is dual to a transpotence element which becomes a primitive element in $H^*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$. Hence y_{2i} becomes an indecomposable element, which gives a contradiction. So there is no extension and we get $H_*(\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n)))$. □

Since $\Omega(U(n)/O(n)) \cong \Omega(\mathrm{SU}(n)/\mathrm{SO}(n)) \times Z$, by looping one more time, we get $\Omega^2(U(n)/O(n)) \cong \Omega^2(\mathrm{SU}(n)/\mathrm{SO}(n))$. With the same argument for Corollary 2.2, we also get the homology of the double loop space of $\mathscr{L}(R^{2n})$.

Corollary 3.4. *The space $\Omega_0^2\mathscr{L}(R^{2n})$ has the same homology algebra as $\Omega_0^2(\mathrm{SU}(n)/\mathrm{SO}(n))$, which is $\bigotimes_{k\geqq 0} Z_2[u_{2^{k+1}-1}]$ for $n = 2$ and is explicitly given in Theorem 3.3 for $n \geqq 3$.*

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Anschrift des Autors:

Younggi Choi
Department of Mathematics Education
Seoul National University
Seoul 151-742
Korea
yochoi@plaza.snu.ac.kr

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