Cyclotomic Redshift

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- R a ring spectrum, $K(R) := K(\mathrm{Mod}_R^{\omega})$.

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$$\begin{tikzcd}
(2,\infty) & & (3,\infty) & & (5,\infty) & & (7,\infty) \\
\vdots & & \vdots & & \vdots & & \vdots \\
(2,2) \arrow[u] & & (3,2) \arrow[u] & & (5,2) \arrow[u] & & (7,2) \arrow[u] \\
(2,1) \arrow[u] & & (3,1) \arrow[u] & & (5,1) \arrow[u] & & (7,1) \arrow[u] \\
& & & 0 \arrow[ulll] \arrow[ul] \arrow[ur] \arrow[urrr] & & &
\end{tikzcd}$$

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$$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}], \quad |v_n| = 2(p^n - 1).$$

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- Completion "at the prime ideal": $\mathbb{S}_{T(n)}$.
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Corresponding categories of "complete modules": $\mathrm{Sp}_{T(n)}$ and $\mathrm{Sp}_{K(n)}$.

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For $n = 0$:

Theorem (Mitchell)

If R is a (discrete) ring then $L_{T(n)}K(R) = 0$ for $n > 1$.

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Theorem (Purity, Clausen-Land-Mathew-Meier-Neumann-Noel-Tamme)

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Theorem (Burklund-Schlank-Yuan)

For a commutative ring spectrum R in $\mathrm{Sp}_{T(n)}$:

$$R \neq 0 \implies L_{T(n+1)}K(R) \neq 0.$$

K-Theory at the Boundary of Redshift

Heuristic Idea: $L_{T(n+1)}K(R)$ is an "étale-topological invariant" of $\mathrm{Spec}(R)$ for R a $T(n)$-local ring spectrum.

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Theorem (Thomason)

$L_{K(1)}K$ *satisfies étale descent on discrete commuative rings.*

This has been partially generalized to higher heights:

Theorem (CMNN)

The functor

$$L_{T(n+1)}K \colon \{L_n^f\text{-local categories}\} \to \{T(n+1)\text{-local spectra}\}$$

preserves fixed points for finite p-group actions.

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*Same for colimits instead of limits (Ambidexterity!).

Rough Proof Sketch

- Induction on the truncatedness of A, *starting from CMNN.*

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- (Mathew): Both sides preserve (suitable) geometric realizations $\implies$ use the Bar construction $A \simeq \varinjlim_{\Delta^{\mathrm{op}}} \Omega A^k$ and the inductive hypothesis.

Chromatic Cyclotomics

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There is a decomposition

$$\mathbb{S}_{T(n)}[B^nC_{p^k}] \simeq \mathbb{S}_{T(n)}[B^nC_{p^{k-1}}] \times \mathbb{S}_{T(n)}[\omega_{p^k}^{(n)}].$$

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Definition

$\mathbb{S}_{T(n)}[\omega_{p^k}^{(n)}]$ is the p^k-th *chromatic cyclotomic extension*.

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Theorem (BCSY)

For a commutative $T(n)$-local ring spectrum R:

$$L_{T(n+1)}K(R[\omega_{p^k}^{(n)}]) \simeq L_{T(n+1)}K(R)[\omega_{p^k}^{(n+1)}]$$

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The case $n = 0$ is a result of Bhatt-Clausen-Mathew:

Theorem (BCM)

For a commutative ring R:

$$L_{K(1)}K(R[\zeta_{p^\infty}]) \simeq L_{K(1)}(K(R) \otimes \mathrm{KU}).$$

Fourier Transform

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Theorem (Barthel-C.-Schlank-Yanovski)

There is a natural Fourier isomorphism

$$\mathfrak{F}_\omega\colon R[M] \xrightarrow{\sim} R^{\Omega^\infty \Sigma^n M^*}$$

Fourier Redshift

By Cyclotomic Redshift: A map $\omega \colon \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}] \to R$ gives a map

$$\tilde{\omega} \colon \mathbb{S}_{T(n+1)}[\omega^{(n+1)}_{p^k}] \to L_{T(n+1)} K(R)$$

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Theorem (B*CSY)

There is a natural commutative square

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L_{T(n+1)}K(R)[\Sigma M] \arrow[r, "\mathfrak{F}_{\tilde{\omega}}"] & L_{T(n+1)}K(R)^{\Omega^\infty \Sigma^{n+1} (\Sigma M)^*}
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**Ben-Moshe.*

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A map $f\colon \Sigma^n M^* \to \mathrm{Pic}(R)$ "categorifies" to a map

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Cyclotomic Hyperdescent

- The cyclotomic tower:

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- Devinatz-Hopkins: $\mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}]$ is a Galois extension:

$$\mathbb{S}_{K(n)}[\omega_{p^\infty}^{(n)}]^{h\mathbb{Z}} \simeq \mathbb{S}_{K(n)}.$$

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Theorem (Cyclotomic Hyperdescent, BCSY)

For a $T(n)$-local ring spectrum R:

$$L_{K(n+1)}K(R[\omega_{p^\infty}^{(n)}])^{h\mathbb{Z}} \simeq L_{K(n+1)}K(R).$$

The Telescope Conjecture

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- Fact (Burklund-Hahn-Levy-Schlank): There exists a $\mathbb{Z}$-action on $\mathrm{BP}\langle n\rangle$ such that the map

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- By Cyclotomic Hyperdescent:

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The Telescope Conjecture II

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Corollary

$\mathrm{Sp}_{K(n)} \neq \mathrm{Sp}_{T(n)}$ *for* $n \geq 2$.

Thank You!