The Kervaire invariant and surgery theory

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Abstract. We give an expository account of the development of the Kervaire invariant and its generalizations with emphasis on its applications to surgery and, in particular, to the existence of stably parallelizable manifolds with Kervaire invariant one.

1. Introduction

As an expository device we describe the development of this subject in chronological order beginning with Kervaire's original paper ([10]) and Kervaire-Milnor's Groups of Homotopy Spheres ([11]) followed by Frank Peterson's and my work using Spin Cobordism ([5], [7]), Browder's application of the Adams spectral sequence to the Kervaire invariant one problem ([3]), Browder-Novikov surgery ([16]) and finally an overall generalization of mine ([6]). In a final section we describe, with no detail, other work and references for these areas. We do not give any serious proofs until we get to the "overall generalization" sections where we prove the results about the generalized Kervaire invariant and Browder's Kervaire invariant one results.

2. Cobordism Preliminaries.

We make $\mathbb{R}^N \subset \mathbb{R}^{N+1}$ by identifying $x \in \mathbb{R}^N$ with $(x, 0)$. Then $BO_k = \bigcup G_{k,l}$, where $G_{k,l}$ is the space of k dimensional linear subspaces of $\mathbb{R}^{k+l}$ and the universal bundle $\zeta_k \to BO_k$ is the space of all pairs (P, v), where $v \in P \in G_{k,l}$ for some l. A vector bundle is assumed to have a metric on its fibres. Hence if ξ is a k-plane bundle over X, it has associated disc and sphere bundles, $D\xi$ and $S\xi$, a Thom space $T\xi = D\xi/S\xi$, and a Thom class $U_k \in H^k(T\xi)$ (coefficients $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ as appropriate).

Throughout this paper "m-manifold" means a smooth, compact manifold of dimension m, equipped with a smooth embedding into Euclidean space, $\mathbb{R}^{m+k}$, k large ($k > 2m + 1$). If M is such a manifold, its tangent and normal bundles are given by

$$\tau(M) = \{(x, v) \in M \times \mathbb{R}^{m+k} \mid v \text{ is tangent to } M \text{ at } x\},$$
$$\nu(M) = \{(x, v) \in M \times \mathbb{R}^{m+k} \mid v \text{ is perpendicular to } M \text{ at } x\}.$$

One associates to M a map $t : S^{m+k} = \mathbb{R}^{m+k} \cup \{\infty\} \longrightarrow T(\nu)$, the Thom construction, as follows: For $\epsilon > 0$ sufficiently small, $e : D_\epsilon(\nu(M)) \to \mathbb{R}^{m+k}$, given by $e(x, v) = x + v$, is an embedding. Let $t(u) = (x, v/\epsilon)$ if $u = e(x, v)$ and $= \{S(\nu(M))\}$ otherwise. If ξ is a k-plane bundle over X, a ξ-structure on a manifold M is a bundle map $f : \nu(M) \longrightarrow \xi$; $f_M : M \longrightarrow X$ denotes the underlying map.

We define the m^{th} ξ-cobordism group, $\Omega_m(\xi)$, to be the set of pairs (M, f), where M is a closed m-manifold with a ξ-structure f, modulo the equivalence relation generated by the following to relations. If $i : M \subset \mathbb{R}^{m+k}$ and $j : \mathbb{R}^{m+k} \subset \mathbb{R}^{m+k+1}$ are their given inclusions, then M equipped with i is equivalent to M equipped with ji. Also (M_1, f_1) and (M_2, f_2) are equivalent if they are ξ-cobordant, that is, there is a $(m + 1)$-manifold N with a ξ-structure F and an embedding $N \subset \mathbb{R}^{m+k} \times [0, 1]$ such that $\partial N = M_1 \cup M_2, N$ is perpendicular to $\mathbb{R}^{m+k} \times \{0, 1\}$, $(N, F) \cap \mathbb{R}^{m+k} \times \{i - 1\} = (M_i, f_i)$. Disjoint union of pairs (M_1, f_1) and (M_2, f_2) makes $\Omega_m(\xi)$ into an abelian group.

Theorem 2.1 (Thom [21]). *Sending* (M, f) *to* $T(f)t : S^{m+k} \longrightarrow T(\xi)$ *induces an isomorphism,* $\Psi : \Omega_m(\xi) \longrightarrow \pi_{m+k}(T(\xi))$.

Sometimes when ξ is a bundle over a particular X, we denote $T\xi$ by TX and $\Omega_m(\xi)$ by $\Omega_m(X)$, or when $X = BG_k$ by $\Omega_m(G)$.

3. Groups of Homotopy Spheres

Kervaire and Milnor ([11]) defined the group of homotopy m-spheres, θ_m, to be the set of closed, oriented m-manifolds homotopy equivalent to S^m (for $m > 4$, by Smale's Theorem, homeomorphic to S^m) modulo the relation of h-cobordism (for a cobordism N between M_1 and M_2, the inclusions of M_i into N are required to be homotopy equivalences). Addition is defined using the connected sum operation. Using Bott's computation of $\pi_*(BO)$ and results of Adams concerning the J-homomorphism, they prove:

Theorem 3.1. *If M is a homotopy m-sphere, $\nu(M)$ is trivial* (k *large*).

For the remainder of this section we assume $m > 4$. Let 0^k denote the vector bundle $\mathbb{R}^k \longrightarrow pt$. Thus an 0^k-structure on M is a framing of $\nu(M)$. If M is a homotopy m-sphere, choosing a framing of $\nu(M)$ gives an element of $\Omega_m(0^k)\approx\pi_{m+k}(S_k)$ and a simple argument shows that changing the framing adds to this element an element in the image of J. Thus we have an induced map $\Psi : \theta_m\longrightarrow\pi_{m+k}(S_k)/\operatorname{Im} J$. The Kervaire-Milnor paper is mainly devoted to computing the kernel and cokernel of this map, both of which turn out to be finite cyclic groups; the kernel is denoted by bP_{m+1} = homotopy spheres bounded by stably parallelizable $(m+1)$-manifolds.

4. Surgery

The process of doing surgery on an m-manifold with respect to an embedding $g : S^n\times D^{m-n}\longrightarrow M$ (S^{q-1} and D^q are the unit sphere and disc in $\mathbb{R}^q$) consists of producing a new manifold M' and a cobordism N between M and M' as follows. Let N be a smoothed version of the identification space formed from $M\times I\cup D^{n+1}\times D^{m-n}$ by identifying $(x, y) \in S^n\times D^{m-n}$ with $(g(x, y), 1)$. The boundary of N consists of three parts, $M = M\times\{0\}, M - g(S^n\times D^{m-n})\cup D^{n+1}\times S^{m-n-1}$ and $(\partial M)\times I$. In what follows M is closed or its boundary is a homotopy $(m-1)$-sphere which we can cone off to form a topological closed manifold. Henceforth we ignore ∂M. Note M' has an embedding $D^{n+1}\times S^{m-n-1}\longrightarrow M'$, namely the inclusion, and applying surgery to it gives M. We define a ξ-surgery to be one in which M, M' and N have ξ-structures making a ξ-cobordism. For our applications to bP_{m+1} we use $\xi = 0^k$. The change in homology going from M to M' can be easily computed from the homology exact sequences of the pairs (N, M) and (N, M') and the observation that $H_q(N, M)\approx H_q(D^{n+1}, S^n)$. If $2n < m - 1$, $H_q(M)\approx H_q(M')$ for $q < n$ and $H_n(M')\approx H_n(M)/\{u\}$, where u is the homology class represented by $g(S^n, 0)$. If $2n = m - 1$ or m the outcome is more complicated. If $m = 2n$, M is oriented and if there is a $v \in H_n(M)$ such that the intersection number $u{\cdot}v = 1$, then $H_n(M')\approx H_n(M)/\{u, v\}$.

Suppose M has a ξ-structure f and u is in the kernel of

$$(f_M)_* : H_n(M)\longrightarrow H_n(X)$$

(X is the base of ξ). If X is simply connected, M can be made simply connected by a sequence of $n = 0$ and 1 ξ-surgeries. The standard procedure for killing u by ξ-surgery proceeds through the following steps:

(i) Represent u by a map $g : S^n\longrightarrow M$ such that $f_Mg : S^n\longrightarrow X$ is homotopic to zero. If M and X are simply connected and $\pi_q(X, M)$ is zero for $q \le n$, g exists. (If $X = \{pt\}$, M is $(n-1)$ -connected.)

(ii) Choose g so that it is a smooth embedding. If $2n < m$ or M is simply connected and $2n = m$, such a g exists.

(iii) Extend g to an embedding $g : S^n\times D^{m-n}\longrightarrow M$. Such an extension exists if the normal bundle of S^n in M, ν, is trivial. Since f_Mg is homotopic to zero, ν is stably trivial. Hence ν is trivial if $2n < m$, or $2n = m = 4a$ and the Euler class of ν is zero, or $2n = m = 4a+2$ and from the two possibilities for ν, trivial or $\tau(S^n)$, it is trivial. This last case is what this paper is all about.

(iv) Extend the ξ structure over the cobordism N. This follows from the hypotheses in (ii), except when $m = 2n$ and $n = 1, 3, 7$.

5. Application of surgery to the calculation of bP_m

Suppose M has an 0^k-structure and the boundary of M is null or a homotopy sphere. Starting in dimension zero, one can make it $([m/2]-1)$-connected by a sequence of 0^k-surgeries. When $m = 2n+1$, delicate arguments show that 0^k-surgery can be applied to produce an n-connected manifold and hence, by Poincaré duality, an $(m-1)$-connected manifold which is either an m-disc or a homotopy m-sphere. Hence $bP_{2n+1} = 0$ and coker $\Psi = 0$ in odd dimensions.

Suppose $m = 2n$ and M is $(n-1)$-connected. We first consider the case n even, which provides techniques and results which one tries to mimic when n is odd. As we described above, we can kill $u \in H_n(M;\mathbb{Z})$ by 0^k-surgery if there is a class $u \in H_n(M)$ such that $u{\cdot}v = 1$ and, when u is represented by an embedded n-sphere, its normal bundle, ν, is trivial; ν is trivial if and only if its Euler class is zero if and only if $u{\cdot}u = 0$. Thus one can kill $H_n(M)$ by surgery if and only if $H_n(M)$ has a basis $u_i, v_i, i = 1, 2, \ldots, r$, such that $u_i{\cdot}u_j = v_i{\cdot}v_j = 0$ and $u_i{\cdot}v_j = \delta_{i,j}$, that is, a symplectic basis. From this one can deduce that M can be made n-connected (and hence $m-1$ connected) if and only if the index of M, that is, the signature of the quadratic form on $H_n(M;\mathbb{Z})$ given by the intersection pairing, is zero ([15]). For M closed, the Hirzebruch index theorem expresses the index of M as polynomial in the Pontrjagin classes. But since M has an 0^k-structure, the Pontrjagin classes are zero. Thus the cokernel of Ψ is zero in dimensions $4a, a > 1$. Kervaire-Milnor also use the Hirzebruch index theorem to calculate bP_{4a}.

Now suppose M is as above with n odd, $n \neq 1, 3$ or 7. In the 1, 3, 7 cases there is an obstruction to extending the 0^k-structure over the cobordism N. Although our function ϕ measures this obstruction, we do not treat this case because of the difficulty of the surgery details required. Suppose M is $(n-1)$-connected. Let $\phi : H_n(M;\mathbb{Z}) \longrightarrow \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ be defined as follows: For $u \in H_n(M;\mathbb{Z})$, represent u by an embedded n-sphere and let ν_u be its normal bundle in M. Let $\phi(u) = 0$ or 1 according as ν_u is trivial or isomorphic to $\tau(S^n)$ (the only two possibilities for ν_u).

Lemma 5.1. *ϕ is well defined and satisfies:*

$$\phi(u+v) = \phi(u) + \phi(v) + u{\cdot}v.$$

Since n is odd, $u{\cdot}u = 0$, and therefore $\phi(2u) = 0$. Hence we do not lose anything by taking u in $H_n(M;\mathbb{Z}_2)$. For the remainder of this paper $H_n(M) = H_n(M;\mathbb{Z}_2)$. As above, we may make M $(m-1)$-connected by 0^k-surgery if $H_n(M)$ has a symplectic basis u_i, v_i such that $\phi(u_i) = 0$ for all i. Arf associated to quadratic functions such as ϕ a $\mathbb{Z}_2$ invariant given by

$$A(\phi) = \sum \phi(u_i)\phi(v_i).$$

He also proved that given the pairing, $A(\phi)$ classifies such quadratic functions and $H_n(M)$ has a symplectic basis u_i, v_i such that $\phi(u_i) = 0$, for all i, if and only if $A(\phi) = 0$. Thus, if $A(\phi) = 0$, M can be made $(2n-1)$-connected. Starting with a closed $2n$-manifold, n odd $\neq 1, 3, 7$ and 0^k-structure f on M, by a sequence of surgeries one can produce an $(n-1)$ connected (M', f') and then a $\phi_{M'}$. Then $A(\phi_{M'})$ is the Kervaire invariant of M and the following was proved in [10]:

Theorem 5.2. *Sending (M, f) to $A(\phi_{M'})$ induces a homomorphism*

$$\alpha : \Omega_{2n}(0^k) \longrightarrow \mathbb{Z}_2\ .$$

An element $z \in \Omega_{2n}(0^k)$ can be represented by a homotopy sphere, if and only if $\alpha(z) = 0$.

Corollary 5.3. *In dimensions $4a+2$, the kernel and cokernel of Ψ are 0 or $\mathbb{Z}_2$; ker $\Psi = 0$ if and only if $\alpha = 0$ and coker $\Psi = 0$ if and only if $\alpha \neq 0$.*

The present state of knowledge on α is:

Theorem 5.4. *$\alpha \neq 0$ for $2n = 2, 6, 14$ ([11]), 30 ([24]), 62 ([2]) and $\alpha = 0$ for $2n = 10$ ([10]), $8a+2$ ([7]), $\neq 2^j - 2$ ([3]). $\alpha \neq 0$ if and only if h_j^2 lives to E_∞ in the Adams spectral sequence for stable homotopy groups of spheres.* ([3]) (*Should such an element exist it is called θ_j.*)

We prove Browder's results in section 8.

An $(n-1)$-connected $2n$-manifold M with 0^k-structure, boundary a homotopy sphere and $A(\phi_M) = 1$ may be constructed as follows: One plumbs together two copies of $D\tau(S^n)$ as follows: Let $h : D^n \longrightarrow S^n$ be a homeomorphism onto the upper hemisphere of S^n given by $h(x) = (x, \sqrt{1-|x|^2})$ and $r : D^n \times D^n \longrightarrow \tau(S^n)$ be a bundle map covering h. Let M be a smoothed version of $D\tau(S^n) \times \{0\} \cup D\tau(S^n) \times \{1\}$ with $(h(x,y), 0)$ identified with $(h(y,x), 1)$ for all x, y. An easy cell decomposition of M shows that its boundary is two n-discs glued together along their boundaries and hence the boundary of M is a homotopy sphere Σ. Let N be M with Σ coned off.

Theorem 5.5. *If Σ is diffeomorphic to S^{2n-1}, N is smoothable and has Kervaire invariant one; otherwise N is a topological manifold which does not admit a differentiable structure* ([10]) *and Σ generates $bP_{2n} \approx \mathbb{Z}_2$.*

The proofs of the results cited in theorem 5.4 follow a very orderly path which we now outline. We switch from homology to cohomology, $H^n(M) = H^n(M;\mathbb{Z}_2)$. Composing with Poincaré duality, ϕ becomes $\phi : H^n(M) \longrightarrow \mathbb{Z}_2$. One wants to associate to a closed $2n$-manifold M, n odd, ϕ satisfying:

(5.6) $\phi(u+v) = \phi(u) + \phi(v) + uv([M])$ where uv denotes cup product.

(5.7) If the Poincaré dual of u can be represented by an embedded n-sphere with stably trivial normal bundle ν and $n \neq 1, 3, 7$, then $\phi(u) = 0$ if and only if ν is trivial.

In [10] Kervaire defines ϕ for an $(n-1)$-connected M as follows. Recall the loop space $\Omega = \Omega(S^{n+1})$ has cohomology generators e_i in dimensions ni and under the multiplication $\Omega\times\Omega \longrightarrow \Omega$, e_1 goes to $e_1 \otimes 1 + 1 \otimes e_1$ and e_2 goes to $e_2 \otimes 1 + 1 \otimes e_2 + e_1 \otimes e_1$. For $u \in H^n(M)$ there is a map $f_u : M \longrightarrow \Omega$ such that $f_u^*(e_1) = u$; ϕ is then defined by $\phi(u) = f_u^*(e_2)$ and satisfies 5.6 and 5.7. Then it is shown that $\alpha : \pi_{2n+k}(S^k) = \Omega_{2n} \longrightarrow \mathbb{Z}_2$, as above, is a well defined homomorphism. Kervaire proves that α is zero for $n = 5$ using knowledge of $\pi_{10+k}(S^k)$, namely, this group has a unique element a of order two and $a = bc, b \in \pi_{k+1}(S^k)$ which represents a manifold $S^1 \times M'$ which then can be surgered to a homotopy sphere. We remark that an equivalent way of defining ϕ would be to represent u by a map $F : SM \longrightarrow S_{n+1}$ (S= suspension) and define $\phi(u)$ to be the functional squaring operation $Sq_F^{n+1}(s_{n+1}), s_{n+1}$ the cohomology generator.

In [5] $\alpha : \Omega_{8a+2}(\mathrm{Spin}) \longrightarrow \mathbb{Z}_2$ was defined as follows: $\Omega_m(\mathrm{Spin}) = \Omega_m(\xi)$, where $\xi \longrightarrow B\mathrm{Spin}_k$ is the universal Spin_k vector bundle. For $n = 4a+1$, the Adem relation,

$$Sq^{n+1} = Sq^2 Sq^{n-1} + Sq^1 Sq^2 Sq^{n-2}$$

gives a relation on $H^n(M)$,

$$Sq^2 Sq^{n-1} + Sq^1 Sq^2 Sq^{n-2} = 0$$

which in turn gives a secondary cohomology operation on M with a Spin structure,

$$\phi' : H^n(M) \longrightarrow H^{2n}(M) \ .$$

Define $\phi(u) = \phi'(u)(M)$. The Spin structure is used to ensure that ϕ' is defined on all of $H^n(M)$ with zero indeterminacy. For example, $Sq^1 : H^{2n-1}(M) \longrightarrow H^{2n}(M)$ is given by $Sq^1(v) = w_1 v$. Then ϕ satisfies 5.6

and 5.7 and defines α. In [7] we showed that α was zero on the image of $\Omega_{8k+2}(0^k)$ (framed cobordism) in $\Omega_{8a+2}(\text{Spin})$ using the result of Conner and Floyd and Lashof and Rothenberg that if $A \in \Omega_{8k+2}(SU)$ goes to zero in $\Omega_{8k+2}(U)$, then $A = B^2C, B \in \Omega_{8k}(SU)$.

In [3] Browder developed $\alpha : \Omega_{2n}(\xi) \longrightarrow \mathbb{Z}_2$, n even or odd, $\xi \longrightarrow X$, as follows: One may assume that X is a smooth closed N-manifold, N large, with normal bundle in $\mathbb{R}^{N+k}$ equal to ξ. Suppose M is a smooth closed $2n$-manifold with ξ-structure f. One may assume f_M is an embedding. Then the normal bundle of M in X is trivial and trivializing and using the Thom construction one obtains a map $F : X \longrightarrow S^kM$ (S^kM, the k-fold suspension of M). Then ϕ is defined by

$$\phi(u) = Sq_F^{n+1}(S^k u).$$

In order for $\phi(u)$ to be defined, $F^*(S^k u)$ must equal zero, and for there to be no indeterminacy, $Sq^{n+1} : H^{N-n-1}(X) \longrightarrow H^N(X)$ must be zero. $Sq^{n+1}(w) = v_{n+1}w$ for $w \in H^{N-n-1}(X)$, where v_{n+1} is the Wu class of ξ. One restricts the choice of ξ to bundles with $v_{n+1} = 0$. Then $\phi : P \longrightarrow \mathbb{Z}_2$ where $P = \{u \in H^n(M) | F^*(S^k u) = 0\}$. Where defined ϕ satisfies 5.6 and 5.7. One restricts α to those (M, f) such that ϕ is zero on all $u \in P$ such that $uv(M) = 0$ for all $v \in P$. The Arf invariant algebra then works to give an integer mod 2. Then α is related to the Adams spectral sequence by computing a Postnikov system, up to the relevant dimension of $MO[v_{n+1}]$. We give the details in section 8.

6. Generalized Groups of Homotopy Spheres

Several people, most notably Novikov, discovered that the Groups of Homotopy Spheres paper ([11]) could be generalized by the following two step process. Replace 0^k, the bundle $\mathbb{R}^k \longrightarrow pt$, by O^k_m the bundle $S^m \times \mathbb{R}^k \longrightarrow S^m$. Then the coker Ψ question asks: "Which elements $\{M, f\}$, where f_M has degree one, can be represented by $\{M, f\}$, where f_M is a homotopy equivalence?" (For there to be an (M, f) with f_M of degree one, the top homology class of $T(\xi)$ must be spherical.) The bP_{m+1} question asks, "If $\{M, f\} = \{S^m, id\}$, can the cobordism between them be chosen to be an h-cobordism?" Now replace all occurrences of O^k_m with $\xi \longrightarrow X$, where X is a simply connected CW complex of finite type satisfying Poincaré duality in dimension m, that is there a class $x \in H_m(X; \mathbb{Z})$ such that cap product with x gives an isomorphism, $H^n(X) \longrightarrow H_{m-n}(X)$ for all n. Then everything in Kervaire-Milnor goes through. Suppose $\{M, f\} \in \Omega_m(\xi)$ and f_M has degree one. The trick is to consider the commutative diagram:

$$\begin{array}{ccc} H^n(M) & \xleftarrow[f_M{}^*]{} & H^n(X) \\ \Big\downarrow{\scriptstyle [M]\cap} & & \Big\downarrow{\scriptstyle x\cap} \\ H_{m-n}(M) & \xrightarrow{f_{M*}} & H_{m-n}(X) \end{array}$$

By Poincaré duality f_{M*} is an epimorphism and $f_M{}^*$ is a monomorphism. For $2n < m$ one can kill elements in the kernel of f_{M*} just as in [11]. This material is thoroughly described in Browder's book "Surgery on Simply Connected Manifolds" ([4]). This material, when X is not simply connected, is the subject of Wall's book "Surgery on Compact Manifolds" ([22]), where the general pattern of the above is followed but surgery in the middle dimensions is much more complicated and leads to Wall's L-groups, in which the obstructions to doing the middle dimension surgery lie. Ranicki develops in "Exact Sequences in the Algebraic Theory of Surgery" ([19]) a completely algebraic approach to these surgery obstructions, replacing manifolds by their chain complex analogs. Michael Weiss refines this algebraic approach to surgery obstructions and makes it more calculable ([23], [24]).

7. A Further Generalization of ϕ

We state the main theorems of this section and then prove them.

In the remainder of this section most spaces have base points, M^+ is M with a disjoint base point, $[X,Y]$ is the homotopy classes of maps from X to Y, $\{X,Y\} = \lim[S^kX, S^kY]$ and $\eta : [X,Y] \longrightarrow \{X,Y\}$ sends $[f]$ to $\{f\}$. Let $s : S^n \longrightarrow K_n$ be the π_* generator.

Lemma 7.1. *The Hopf construction $h(\lambda) : S^{2n+1} \longrightarrow SK_n$ on*

$$\lambda : S^n \times S^n \xrightarrow{s\times s} K_n \times K_n \xrightarrow{\mu} K_n$$

with μ the multiplication, gives a generator of $\{S^{2n}, K_n\}\approx\mathbb{Z}_2$.

Suppose M is a closed $2n$-manifold (n even or odd). We form an abelian group $G(M) = H^n(M) \times H^{2n}(M)$ with addition

$$(u,v) + (u',v') = (u+u', v+v'+uu').$$

Let $j : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ be the homomorphism sending 1 to 2. Then functions $\phi : H^n(M) \longrightarrow \mathbb{Z}_4$ satisfying $\phi(u+v) = \phi(u) + \phi(v) + j(uv([M]))$ are in one to one correspondence with homomorphisms $h : G(M) \longrightarrow \mathbb{Z}_4$ such that $h(0,v) = j(v([M]))$. We will see that such functions occur in nature.

Theorem 7.2. *Let $F : G(M) \longrightarrow \{M^+, K_n\}$, given by $F(u,v) = \eta(u) + h(\lambda)g_v$, where $g_v \longrightarrow S^{2n}$ has degree one. Then F is an isomorphism.*

Let ν be the normal bundle of M in $\mathbb{R}^{2n+k}$ and $\Delta : T\nu \longrightarrow T\nu \wedge M^+$ be the diagonal map sending v to $(v, p(v)), p : \nu \longrightarrow M$. Then $S^{2n+k} \xrightarrow{t} T\nu \xrightarrow{\Delta} T\nu \wedge M^+$ is an S (Spanier-Whitehead) duality map ([20]). Then $\{M^+, K_n\} \approx \{S^2n, T\nu \wedge K_n\}$ under the map sending $S^lM \longrightarrow S^lK_n$ to

$$S^{2n+k+l} \longrightarrow S^lT\nu \longrightarrow S^l(T\nu \wedge M^+) = T\nu \wedge S^lM^+ \longrightarrow T\nu \wedge S^lK_n\ .$$

Let $q(\lambda) \in \{S^{2n}, T\nu \wedge K_n\}$ be the image of $h(\lambda)$ under this map.

Lemma 7.3. *If $f : \nu \longrightarrow \xi$, the image of $f_*(q(\lambda))$ is non-zero if and only if $v_{n+1}(\xi) = 0$ and it is at most divisible by 2.*

We call ξ a Wu-n spectrum if $v_{n+1}(\xi) = 0$ in which case we can choose a homomorphism $\omega : \{S^{2n}, T\xi \wedge K_n\} \longrightarrow \mathbb{Z}_4$ such that $\omega((f_*(q(\lambda))) = 2$. Let $\phi = \phi(M, f, \omega)$ be the composition $H^n(M) = [M^+, K_n] \longrightarrow \{M^+, K_n\} \longrightarrow \{S^{2n}, T\nu \wedge K_n\} \longrightarrow \{S^{2n}, T\xi \wedge K_n\} \longrightarrow \mathbb{Z}_4$. Hence,

Theorem 7.4. *$\phi = \phi(M, f, \omega)$ satisfies*

$$\phi(u+v) = \phi(u) + \phi(v) + j(uv([M])).$$

Let $BO_k[v_{n+1}] \longrightarrow BO_k$ be the fibration with fibre K_n and k-invariant v_{n+1} and let $\xi[v_{n+1}]$ be the pull back of the universal bundle over BO_k. Suppose M has a $\xi[v_{n+1}]$-structure, $S^n \subset M$ has normal bundle μ and $\nu|S^n$ is trivial. Then $S^n \longrightarrow M \longrightarrow BO_k[v_{n+1}]$ factors through K_n. Let $\epsilon_1 = 0$ or $1 =$ degree of this map. Let $\epsilon_2 = 0$ or 1 according to whether μ is trivial or $\tau(S^n)$. Let u be the Poincaré dual of the homology class represented by $S^n \longrightarrow M$.

Lemma 7.5. *If n is odd and $\neq 1, 3, 7$, then $\phi(u) = 2(\epsilon_1 + \epsilon_2)$.*

Remark. *If n is even, $\epsilon_2 =$ Euler number of μ mod 4. If $n = 1, 3,$ or 7 and $\epsilon_2 = 0$, the element in $\pi_{2n+k}(T\xi \wedge K_n)$ involves a map $g : S^{2n+k} \longrightarrow S^k$ such that $\epsilon_1 =$Hopf invariant of g.*

Let $S^n \times S^n \longrightarrow BO_k[v_{n+1}]$ be the composition of $K_n \longrightarrow BO_k[v_{n+1}]$ and $s \otimes 1 + 1 \otimes s$. This lifts to a $\xi[v_{n+1}]$-structure q. Then 7.5 gives

Lemma 7.6. *The ϕ associated to $(S^n \times S^n, q)$. satisfies $\phi(s \otimes 1) = \phi(1 \otimes s) = 2$.*

The following gives an analog of the Arf invariant for these quadratic functions.

Definition 7.7. *Let V be a finite dimensional vector space over $\mathbb{Z}_2$. A function $\phi : V \longrightarrow \mathbb{Z}_4$ is a (nonsingular) quadratic if it satisfies $\phi(u+v) = \phi(u) + \phi(v) + jt(u,v)$ where $j : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ sends 1 to 2 and t is a nonsingular bilinear pairing. If $\phi_i : V_i \longrightarrow \mathbb{Z}_4, i = 1, 2$, are such functions $\phi_1 \approx \phi_2$ if there is an isomorphism $T : V_1 \approx V_2$ such that $\phi_1 = \phi_2 T$. $(\phi_1 + \phi_2) : V_1 + V_2 \longrightarrow \mathbb{Z}_4, -\phi$ and $(\phi_1\phi_2) : V_1 \otimes V_2 \longrightarrow \mathbb{Z}_4$ are defined by $(\phi_1 + \phi_2)(u,v) = \phi_1(u) + \phi_2(v), (-\phi)(u) = -\phi(u)$ and $(\phi_1\phi_2)(u \otimes v) = \phi_1(u)\phi_2(v)$.*

A proof of the following appears in [6] and is straightforward. The first part of the theorem is proved by showing that the Grothendieck group of these functions is cyclic of order eight.

Theorem 7.8. *There is a unique function σ from quadratic functions as in 7.1 to $\mathbb{Z}_8$ satisfying:*

(i) *If $\phi_1 \approx \phi_2$, then $\sigma(\phi_1) = \sigma(\phi_2)$*
(ii) $\sigma(\phi_1 + \phi_2) = \sigma(\phi_1) + \sigma(\phi_2)$
(iii) $\sigma(-\phi) = -\sigma(\phi)$
(iv) $\sigma(\gamma) = 1$, *where $\gamma : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4$ by $\gamma(0) = 0$ and $\gamma(1) = 1$.*

Furthermore σ satisfies:

(v) *If $\phi = j\phi', \sigma(\phi) = 4\,\mathrm{Arf}(\phi')$.*
(vi) *If $\phi : V \longrightarrow \mathbb{Z}_4, \sigma(\phi) = \dim V$ mod 2.*
(vii) $\sigma(\phi_1\phi_2) = \sigma(\phi_1)\sigma(\phi_2)$.
(viii) *If U is a finitely generated abelian group, $\tau : U \otimes U \longrightarrow \mathbb{Z}$ is a symmetric unimodular form, $\Psi(u) = \tau(u,u)$ and $\phi : U/2U \longrightarrow \mathbb{Z}_4$ is defined by $\phi(u) = \Psi(u) \mod 4$, then ϕ is quadratic and $\sigma(\phi) = (\mathrm{signature}\,\Psi) \mod 8$.*
(ix) *Suppose t is the bilinear form of $\phi : V \longrightarrow \mathbb{Z}_4$, $V_1 \xrightarrow{\nu} \nu V \xrightarrow{\delta} V_2$ is an exact sequence and $t' : V_1 \otimes V_2 \longrightarrow \mathbb{Z}_2$ is a nonsingular bilinear form such that $t'(u, \delta v) = t(\nu u, v)$. If $\phi\nu = 0$, then $\sigma(\phi) = 0$.*
(x) *With $i = \sqrt{-1}$,*

$$\sigma(\phi) = (4i/\pi)\ln(2^{(\dim V)/2}/(\sum_{u \in V} i^{\phi(u)})).$$

Theorem 7.9. *Sending (M,f) to $\sigma(\phi(M,f,\omega))$ induces a homomorphism $\sigma_\omega : \Omega_{2n}(\xi[v_{n+1}]) \longrightarrow \mathbb{Z}_8$ such that σ_ω composed with*

$$\Omega_{2n}(0^k) \longrightarrow \Omega_{2n}(\xi[v_{n+1}])$$

gives the Kervaire invariant.

We can apply σ_ω to the general, simply connected surgery problem as follows.

Definition 7.10. *For n odd, a $2n$-Poincaré quadruple (X, ξ, β, ω) is a connected finite CW complex X, a k-plane bundle ξ over X, ω a homomorphism as above and $\beta \in \pi_{m+k}(T\xi)$ such that*

$$S^{m+k} \xrightarrow{\beta} T\xi \xrightarrow{\Delta} T\xi \wedge X_+$$

is an S duality (which makes ξ a Wu bundle).

Theorem 7.10. *Then by the Thom Theorem, β gives a $2n$-manifold M with a ξ-structure f such that f_M has degree one and the surgery obstruction to making f_M a homotopy equivalence is $\sigma(\phi(X, id_X, \omega)) - \sigma(\phi(M, f, \omega))$.*

Proof of 7.1 and 7.2. Let ι be the generator of $H^n(K_n)$. Let $E \longrightarrow K_{n+1}$ be the fibration with fibre K_{2n+1} and k-invariant ι_{n+1}^2. Then $S\iota_n \longrightarrow K_{n+1}$ lifts $g : S\iota_n \longrightarrow E$ and on homotopy groups π_i $\pi_i(g)$ is an isomorphism for $i \leq 2n+1$. Then since M is $2n$ dimensional $\{M, K_n\} = [SM, SK_n] \approx [SM, E] \approx [M, \Omega E] = [M, K_n \times K_{2n}] = H^n(M) \times H^{2n}(M)$. The additive structure comes from the fact that under multiplication $\Omega E \times \Omega E \longrightarrow \Omega E$, ι_{2n} goes to $\iota_{2n} \otimes 1 + 1 \otimes \iota_{2n} + \iota_n \otimes \iota_n$. Applying this to $M = S^n \times S^n$ gives 7.1.

Proof of 7.3. Let $V(X) = \{S^{2n+k+l}, X \wedge S^l(K_n)\}$. We want to know the image of $V(S^k)$ in $V(T\xi)$ and how divisible it is. We can assume $T\xi$ is a finite $(k-1)$-connected CW complex and $S^l(K_n) \longrightarrow K_{n+l}$ is a fibration with fiber K_{2n+l} and k-invariant $Sq^{n+1}\iota_{n+l}$. Applying $\{S^{2n+k+l}, T\xi \wedge (\ )\}$, this gives an exact sequence

$$\longrightarrow H_{k+n+1}(T\xi) \xrightarrow{Sq^{n+1}} H_k(T\xi) \longrightarrow V(T\xi) \longrightarrow H_{k+n}(T\xi) \longrightarrow$$

and the same with $T\xi$ replaced by S^k. For $x \in H_{k+n+1}(T\xi)$ and U the Thom class,

$$U(Sq^{n+1}(x)) = \chi(Sq^{n+1})(U)(x) = v_{n+1}U(x)\ .$$

The two exact sequences make a ladder from which the desired result can be read off.

Proof of 7.5. Note, applying the Thom construction and the Thom class give maps $S^n \times S^n \longrightarrow T\tau(S^n) \longrightarrow K_n$ and the Hopf construction gives $S^{2n+1} \longrightarrow ST\tau(S^n) \longrightarrow SK_n$. Also the element $a \in \pi_{2n+k}(T\xi \wedge K_n)$ such that $\phi(u)$ comes from it is given by $S^{2n+k} \longrightarrow T\nu_{S^n} \longrightarrow S^k(S^{n+}) \wedge T\mu \longrightarrow S^k(K_n) \wedge K_n = S^kK_n \vee S^kK_n \wedge K_n \longrightarrow T\xi \wedge K_n$. Combining these two gives the desired result.

Proof of 7.9. To show that $\sigma(\omega)$ is well defined suppose $(M, f) = \partial(N, F)$. By virtue of 7.8(ix) and the exact sequence

$$H^n(N) \xrightarrow{j^*} H^n(M) \longrightarrow H^{n+1}(N, M),$$

it is sufficient to show that $\phi(j^*(u) = 0$. The element $a \in \pi_{2n+k}(T\xi \wedge K_n)$ such that $\phi(j^*(u) = \omega(a)$ is

$$\begin{aligned} S^{2n+k+1} \longrightarrow & T\nu_N / T\nu_M \longrightarrow T\nu_N \wedge (N/M) \longrightarrow \\ & T\nu_N \wedge SM^+ \longrightarrow T\nu_N \wedge SN^+ \longrightarrow T\xi \wedge SK_n \end{aligned}$$

But this is zero because $N/M \longrightarrow SM^+ \longrightarrow SN^+$ is zero.

8. Proof of Theorem 5.4

Let $B = BO_k$, $K_n = K(\mathbb{Z}_2, n)$, $v_{n+1} : B \longrightarrow K_{n+1}$ represent the $(n+1)$-th Wu class, $C = \{(b, a) \in B \times K_{n+1}^I \mid a(1) = v_{n+1}(b)\}$, $D = BO_k[v_{n+1}] = \{(b, a) \in C \mid a(0) = *\}$, $i : B \longrightarrow C$ by $i(b) = (b, a_b)$ where $a_b(t) = v_{n+1}(b)$ and $\pi : (C, D) \longrightarrow (B \times K_{n+1}, B \times \{*\})$ and $\pi' : C \longrightarrow B \times K_{n+1}$ by $(b, a) \to (b, a(0))$. The following is easily verified:

Lemma 8.1. *The map i is a homotopy equivalence. π is a fibre map with fibre K_n and in the Serre cohomology spectral sequence for π, $E_2^{p,q} = 0$ for $q > 0$ and $p+q \leq 2n+2$ except $E_2^{n+1,n} \approx \mathbb{Z}_2$. Hence $\pi^* : H^p(B \times K_{n+1}, B \times \{*\}) \approx H^p(C, D)$ for $p \leq 2n+2$ except it may have a kernel isomorphic to $\mathbb{Z}_2$ for $p = 2n+2$.*

Lemma 8.2. *The kernel of*

$$\pi^* : H^{2n+2}(B \times K_{n+1}, B \times \{*\}) \longrightarrow H^{2n+2}(C, D)$$

is generated by $v_{n+1} \otimes \iota_{n+1} + 1 \otimes \iota_{n+1}^2$.

Proof. Recall that via the inclusion map $j : X \longrightarrow (X, A)$, $H^*(X)$ acts on $H^*(X, A)$, and for $u \in H^*(X, A)$, $u^2 = j^*(u)u$. Note $i\pi'$ sends b to $(b, v_{n+1}(b))$ and hence $(i\pi')^*(1 \otimes \iota_{n+1}) = v_{n+1}$. Thinking of $1 \otimes \iota_{n+1} \in H^{n+1}(B \times K_{n+1}, B \times \{*\})$,

$$\begin{aligned} \pi^*(1 \otimes \iota_{n+1}^2) &= \pi'^* j^*(1 \otimes \iota_{n+1})(1 \otimes \iota_{n+1}) \\ &= \pi^*(v_{n+1} \otimes 1)(1 \otimes \iota_{n+1}) \\ &= \pi^*(v_{n+1} \otimes \iota_{n+1}). \end{aligned}$$

Under the map $C \longrightarrow B$ sending (b, a) to b, the universal bundle over B pulls back to bundles over C and D whose Thom spaces we denote by TD, TC, and TB and whose Thom classes we denote by U. The map π induces a map $T\pi : TC \wedge TD \longrightarrow TB \wedge K_{n+1}$. And,

Lemma 8.3. *The kernel of $H^q(T\pi)$ for $q \leq 2n+2+k$ is generated by*

$$\begin{aligned} v_{n+1}U \otimes \iota_{n+1} + U \otimes \iota_{n+1}^2 &= \chi(Sq^{n+1})U \otimes \iota_{n+1} + 1 \otimes \iota_{n+1}^2 \\ &= \sum_{i>0} \chi(Sq^i)(U \otimes Sq^{n+1-i}\iota_{n+1}). \end{aligned}$$

The same Steenrod square manipulations yield:

Lemma 8.4. *For $j < n+1$, $TB \longrightarrow TC \longrightarrow TC/TB \longrightarrow TC \wedge K_{n+1}$ sends $(U \otimes Sq^j \iota_{n+1})$ to*

$$\sum_{k>0} (Sq^k(v_{n+1}v_{j-k}U) + \chi(Sq^k)((Sq^{j+1-k}v_j)U)).$$

We very briefly describe the portion of Adams' work on cohomology operations ([1]) relevant to this proof. All the spaces we will deal with will be approximately k-connected, and the results we state will be correct in a range of dimensions up to about $2k$. Suppose L and K are spaces and $F : L \longrightarrow K$ is a map, E is the space of paths in L starting at a base point and $P : E \longrightarrow L$ sends a path to its end point. Note E is contractible and P is a fibre map with fibre the loops on $K, \Omega(K)$. Let $p : E_F = F^*E \longrightarrow L$ be the induced fibration. Suppose $G_1 : L_1 \longleftarrow L_2$ is an inclusion and $F : L_2 \longrightarrow L_2/L_1$ is the quotient map. Then $L_1 = E_F$ and $p = G$. Suppose $G_2 : L_2 \longrightarrow L_3$ and G_2G_1 is homotopic to the constant map. Then G_1 lifts to $G_1' : L_1 \longrightarrow E_{G_2}$ and we form $E_{G_1'}$. There is a map $G_2' : E_{G_1} \longrightarrow \Omega(L_2)$ such that $E_{G_2'} = E_{G_1'}$. Call this space $E(G_1, G_2)$. We apply this to our situation by taking $L_1 = TB$, $L_2 = TB \wedge K_{n+1}, L_3 = K_{2n+2+k}, G_1 : TB \longrightarrow TC/TD \longrightarrow TB \wedge K_{n+1}$ and $G_2 : TB \wedge K_{n+1} \longrightarrow K_{2n+2+k}$ representing the cohomology element in Lemma 8.3. The map $TD = T(BO_k[v_{n+1}]) \longrightarrow T(BO_k) = TB$ lifts to $h : TD \longrightarrow E(G_1, G_2)$, and

Lemma 8.5. *$\pi_q(h)$ is an isomorphism for $q \leq 2n+k$.*

Proof. $TC/TD \longrightarrow TB \wedge K_{n+1}$ lifts to $r : TC/TD \longrightarrow E_{G_2}$. $H^q(r)$ is an isomorphism for $q \leq 2n+2+k$, hence $\pi_q(r)$ is an isomorphism for $q \leq 2n+1+k$ and hence $\pi_q(h)$ is as an isomorphism for $q \leq 2n+k$.

The cohomology of both $TC = TB$ and $TB \wedge K_{n+1}$ are free modules over the mod two Steenrod algebra A. If $\{u_i\}$ is a basis for $H^*(TB)$ over A and $\{v_i\}$ is a basis for $H^*(K_{n+1}), \{u_i \otimes v_j\}$ is a free A-basis for $H^*(TB \wedge K_{n+1})$. Up to homotopy type $TB = \prod K_{|u_i|}$ and similarly for $TB \wedge K_{n+1}$.

Let $s : S^k \longrightarrow TD$ represent the generator of $\pi_k(TD)$. Then the map of $2n$-framed cobordism to $2n_BO_k[v_{n+1}]$ cobordism corresponds to s_* :

$\pi_{2n+k}(S^k) \longrightarrow \pi_{2n+k}(TD) \approx \pi_{2n+k}(E(G_1,G_2))$. Let $V : K_k \longrightarrow TB$ be the map such that U, the Thom class pulls back to ι_k. Then $S^k \longrightarrow TD \longrightarrow TB$ factors through $V : K_k \longrightarrow TB$ and hence $S^k \longrightarrow TD \longrightarrow E(G_1,G_2)$ factors through $E(G_1V,G_2)$ giving a map $t : S^k \longrightarrow E(G_1V,G_2)$. In [1] Adams (with some refinements) that one may of viewing $E(G_1V,G_2)$ is as the beginning of a tower building S^k. This tower gives a spectral sequence with $E_2 = Ext_A(\mathbb{Z}_2,\mathbb{Z}_2)$ and a map $S^{2n+k} \longrightarrow E(G_1V,G_2)$ representing an element of E_2 is said to live to E_∞ if it lifts all the way up the tower giving a map $S^{2n+k} \longrightarrow S^k$ (more or less). At this two stage level, the relevant elements of E_2 have names "h_ih_j", $i \leq j$. Adams proves that if a map $S^{2n+k} \longrightarrow S^k \longrightarrow E(G_1V,G_2)$ is non zero, then G_1V and G_2 satisfy the following condition: The algebra A is generated by elements Sq^{2^i}. Let $h_i : A \longrightarrow \mathbb{Z}_2$ be the linear map which is zero on decomposables and $h_i(Sq^{2^j} = \delta_{i,j}$. Let $x, \{y_i\}$ and z be A generators of $H^*(K_k), H^*(TB \wedge K_{n+1})$ and $H^*(K_{k+2n+2})$. Then $G_2^*(z) = \sum a_iy_i$, and $G_1^*(y_i) = b_ix$. (Since G_2G_1V is homotopic to zero, $\sum a_ib_i = 0$.)

Theorem 8.6 (Adams). *If a map $S^{2n+k} \longrightarrow S^k \longrightarrow E(G_1V,G_2)$ is non-zero, then for some s and $t \leq s$, $\sum h_s(a_i)h_t(b_i) = 0$.*

Using the fact that $\chi(Sq^{2^i}) = Sq^{2^i}+$ decomposables, and inspecting the elements in 8.2 and 8.4, one sees that the condition in 8.6 is satisfied exactly when n is of the form $2^i - 1$. If a framed $2n$-manifold has Kervaire invariant one, it will be non-zero in $\Omega_{2n}(BO[v_{n+1}])$. Conversely, if h_i^2 lives to E_∞, there is a non-zero map $s : S^{2^i-2+k} \longrightarrow S^k \longrightarrow E(G_1,G_2)$. By an easy Hopf invariant one type argument, it goes to zero in E_{G_1} and hence must be the π_* generator of the fiber of $E(G_1,G_2) \longrightarrow E_{G_1}, K_{2^i-2+k}$. But this generator corresponds to $(S^{2^i-1} \times S^{2^i-1}, q)$ since by 7.6 this manifold has Kervaire invariant one and the underlying map of q factors through K_n. Hence,

Corollary 8.7 (Browder). *There is a framed $2n$-manifold with Kervaire invariant one, if and only if $n = 2^i - 1$ and h_i^2 lives to E_∞ in the Adams spectral sequence for $\pi_*(S^0)$.*

9. Other Work

An amusing low dimensional application of the generalized Kervaire invariant is afforded by immersions of surfaces = closed, compact smooth 2-manifolds in $\mathbb{R}^3$. If $f : S \longrightarrow \mathbb{R}^3$ is such an immersion, associate to it $\phi : H_1(S) = H_1(S;\mathbb{Z}_2) \longrightarrow \mathbb{Z}_4$ as follows. Represent $u \in H_1(S)$ by an embedded circle and let T be a tubular neighborhood in S of this circle. Define $\phi(u)$ to be the number of half twists (by 180°) of the twisted strip

$f(T)$. This makes sense mod 4 and ϕ has the quadratic property with respect to the intersection pairing on $H_1(S)$. Then the quadratic functions associated to the intersection pairing are in one to one correspondence with the regular homotopy classes of immersion of S in $\mathbb{R}^3$ and the Kervaire invariant gives an isomorphism of the cobordism group of such immersions onto $\mathbb{Z}_8$ ([6]).

Ochanine has generalized the above to surfaces immersed in 3- manifolds and to a $(8a+2)$-manifold V immersed in a $(8a+4)$-manifold and dual to $w_2(M)$. He also related KO characteristic classes for Spin $(8a+2)$-manifolds to these issues ([18]).

A variant of the above is to take S with boundary S^1 and $f : S \longrightarrow \mathbb{R}^3$ an embedding. Then $f(\partial S)$ is a knot. In this connection the Kervaire invariant appears in a number of knot and link theory papers. For example, Levine expresses the Kervaire invariant of a knot in terms of its Alexander polynomial ([12]).

There are a number of papers in homotopy theory studying the existence of framed manifolds having Kervaire invariant one, for example [13]. The existence of such manifolds in dimensions 30 and 62 was first proved by homotopy groups of spheres calculations ([2], [14]). In [8] Jones constructed a stably framed 30-manifold with Kervaire invariant one and also proved that a similar construction does not work in dimension 62. In [9] Browder's results for $2n \neq 2^j - 2$ are deduced from the Kahn-Priddy theorem.

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