The Kervaire invariant of framed manifolds and its generalization*

By WILLIAM BROWDER

In 1960, Kervaire [11] introduced an invariant for almost framed $(4k+2)$-manifolds, $(k \neq 0, 1, 3)$, and proved that it was zero for framed 10-manifolds, which was a key step in his construction of a piecewise linear 10-manifold which was not the homotopy type of a differential manifold. Haefliger [9] showed that Kervaire's invariant and the invariant of Pontrjagin [17] for 2, 6, and 14 dimensional framed manifolds, could be defined in a common fashion, and this invariant is the surgery obstruction in dimensions $4k + 2$ (see [12], [15], [6]). A central question has remained, for which dimensions can a framed manifold have a non-zero Kervaire invariant. Pontrjagin's invariant is non-zero for certain framings on $S^1 \times S^1$, $S^3 \times S^3$ and $S^7 \times S^7$, but until now all the results for the Kervaire invariant have been in the negative; Kervaire [11] showed it was zero in dimensions 10 and 18, and Brown and Peterson [8] showed it zero in dimensions $8k + 2$.

In this paper we will show that the Kervaire invariant is zero for dimensions $\neq 2^k - 2$. For dimension $2^k - 2$ we show that there is a framed manifold of Kervaire invariant 1 if and only if in the Adams spectral sequence for the stable homotopy groups of spheres the element h_k^2 in E^2 persists to E^∞, (see [1], [2]). But it is a fact due to Mahowald and Tangora, (Topology **6** (1967) 349–370, § 8) that h_4^2 in dimension 30 persists to E^∞. Hence there is a framed 30-manifold of Kervaire invariant 1. (We are informed that recently Barratt and Mahowald have shown h_5^2 persists to E^∞, so there is a framed 62-manifold of Kervaire invariant 1.)[1]

Now we list some of the geometric corollaries which follow from our result (see [12]).

COROLLARY 1. *The Pontrjagin construction from the group $F\Theta_n$ of framed homotopy spheres to the stable n-stem $\pi_{n+k}(S^k)$, k large, is onto if $n \neq 2^k - 2$. Its cokernel is Z_2 in dimensions* 2, 6, 14, *and* 30.

COROLLARY 2. *The group bP_{n+1} of homotopy n-spheres which bound parallelizable manifolds is Z_2 if $n = 4k + 1$, $n \neq 2^l - 3$. It is zero in dimen-*

* Research supported in part by an NSF grant.

[1] This implies we may add dimension 62 to the lists in Corollaries 1 and 3 below and 61 to Corollary 2.

sions 5, 13, *and* 29.

For the following corollaries see [15], [6].

COROLLARY 3. *Let X be a 1-connected finite complex satisfying Poincaré duality in dimension n, $n \geqq 5$. Suppose that $T(\xi)$ is the Thom complex of a k-plane bundle over X, $\alpha \in \pi_{n+k}(T(\xi))$ is such that $h(\alpha)$ is a generator of $H_{n+k}(T(\xi))$, where h is the Hurewicz homomorphism. If $n = 6, 14,$ or 30, then X is the homotopy of a smooth manifold with normal bundle ξ. If $n = 4k + 2$, $n \neq 2^l - 2$, and ξ is the trivial bundle, then X has the homotopy type of a π-manifold with normal invariant α if and only if the Kervaire invariant $c(\psi)$ of (X, α) is zero (see § 3 for definition of $c(\psi)$).*

This corollary covers some cases not solved in [15] or [5]. Unfortunately there are some difficulties in defining an absolute invariant, analogous to the Kervaire invariant, in the case of a general bundle, but one can do this for some bundles.

Our general approach to the problem of the Kervaire invariant is the following. Find the most general possible and simplest situation in which the Kervaire invariant can be defined and then study the place of framed manifolds in this situation.

The Kervaire invariant is the Arf invariant of a certain quadratic function defined in a $2k$-connected almost framed $(4k + 2)$-manifold. In [11] and [12] this quadratic function is obtained from the cohomology operation which detects the Whitehead product $[\iota, \iota]$, while in [7] it was shown that this could be used to define a generalization of Kervaire's invariant to Spin-manifolds of dimension $8k + 2$. In dimensions $8k + 2$ the operation is secondary, but in other dimensions the operation may have higher order.

We define here a quadratic operation in a new way, using a functional operation associated with Sq^{q+1} to define a quadratic operation ψ in a $2q$-dimensional Poincaré duality space M^{2q} with a certain extra structure. This extra structure is a lifting of the classifying map of its normal spherical fibre space [21] to a classifying space in which the Wu class $v_{q+1} = 0$. Any M^{2q} has such a lifting, but it is not unique and different lifts lead to different results. A framing induces such a lifting. We note that the definition works for all q, not just q odd. The operation ψ is defined on a subgroup of $H^q(M^{2q}; Z_2)$, and if this subgroup has certain properties the Arf invariant of ψ can be defined. It coincides with the Kervaire invariant for framed manifolds.

We show that if M^{2q} is a boundary in the cobordism theory above (using the theory where $v_{q+1} = 0$) and if the Arf invariant is defined, then it is zero. Then we study this cobordism theory and the image of framed cobordism in it to get our results.

In § 1 and 2, the theory in which $v_{q+1} = 0$ is developed, and the operation ψ defined for a Poincaré duality space M^{2q} which is oriented in this theory.

In § 3 it is shown that the Arf invariant which can be described in certain circumstances, coincides in the case of framed manifolds with Kervaire's and Pontrjagin's invariants. Then it is shown that if a framed M^{2q} is a boundary in the theory based on $v_{q+1} = 0$, then the Kervaire invariant of M^{2q} is zero. Thus if the image of framed cobordism in this new cobordism theory is zero in dimension $2q$, the Kervaire invariant is zero for framed $2q$-manifolds. This is shown in §§ 6 and 7 for $q \neq 2^l - 1$.

In § 4, we discuss some general properties of orientations (or "liftings" of the structural group of a bundle). Let $\pi: E \rightarrow B$ be a principal fibre space with fibre G, where B is the classifying space for a category $\mathcal{C}$ of fibre spaces or bundles, γ the classifying fibre space or bundle over B, $\bar{\gamma} = \pi^*(\gamma)$ over E, $\rho: \bar{\gamma} \rightarrow \gamma$ the induced fibre space or bundle map. Let ξ be in the category $\mathcal{C}$, with base space X. An E-orientation of ξ, (or lifting) is a map $b: \xi \rightarrow \bar{\gamma}$ in the category $\mathcal{C}$. We define a natural notion of equivalence of E-orientations, and show that the equivalence classes of E-orientations of ξ, are in one-to-one correspondence with cross-sections of the fibre space E' over X induced from $E \rightarrow B$ by the classifying map of ξ. This generalizes the many well known special cases in the literature, such as Spin structures (see [13]).

In § 5, we use the results of § 4 to show how to construct a structure on $S^q \times S^q$, in the theory with $v_{q+1} = 0$, which has Arf invariant 1. Then in §§ 6 and 7, it is shown that if $q = 2^l - 1$, the image of framed cobordism in our theory is exactly this element if $(h_l)^2$ in the Adams spectral sequence represents an element in the stable homotopy group of the sphere.

In § 6 it is shown in general how to calculate up to group extensions the cobordism groups in the theory with $v_{q+1} = 0$ in dimensions $\leqq 2q$, in terms of the original theory. We get an exact sequence with the original groups, the new groups, and relative groups. It happens to be easy to show that the relative groups are the homotopy groups of a space with only one non-zero k-invariant in stable dimensions $\leqq 2q + 1$.

In § 7 the k-invariant is computed and the case where $B = B_0$ is considered. In that case it is shown that the image of framed cobordism is zero if $q \neq 2^l - 1$ using Adams results [1], and that if $q = 2^l - 1$ only an element corresponding to h_l^2 can go non-trivially. In that case, it will go to the element constructed in § 5, which therefore has Kervaire invariant 1.

In a later paper we plan to further study the invariants here defined. We shall prove a product formula analogous to formulas of Brown-Peterson [8] and Sullivan (unpublished), compare our invariant in dimension $8k+2$ to Brown's, and study the invariant if q is even.

I am indebted to Ed Brown for some illuminating conversations.

1. The operation

A *cospectrum* $\mathbf{X}$ is defined to be a sequence of spaces $\mathbf{X}=\{X_0, X_1, X_2, \cdots\}$ and maps $f_n\colon X_{n+1}\to \Sigma X_n$, (where Σ is the suspension). If A is a space then define a cospectrum $\mathbf{A}$, $\mathbf{A}=\{A, \Sigma A, \Sigma^2 A, \cdots\}$, with $f_n=$ identity map. If $\mathbf{X}$ and $\mathbf{X}'$ are cospectra, a map $\boldsymbol{g}\colon \mathbf{X}\to\mathbf{X}'$ is a sequence of maps $\{g_k, g_{k+1}, \cdots\}$ for $k>l$, some l, $g_i\colon X_i\to X_i'$ such that the diagram

$$\begin{array}{ccc} X_{i+1} & \xrightarrow{f_i} & \Sigma X_i \\ {\scriptstyle g_{i+1}}\downarrow & & \downarrow{\scriptstyle \Sigma g_i} \\ X'_{i+1} & \xrightarrow{f_i'} & \Sigma X_i' \end{array}$$

is commutative.

If $g\colon A\to B$ is a map of spaces, g induces $\boldsymbol{g}\colon \mathbf{A}\to\mathbf{B}$ in the obvious way. If $\mathbf{X}=\{X_0, X_1, \cdots\}$ is a cospectrum we let $\mathbf{S}^k\mathbf{X}=\{X_k, X_{k+1}, \cdots\}$ if $k\geqslant 0$, $\mathbf{S}^k\mathbf{X}=\{*, \cdots, X_0, X_1, \cdots\}$ (X_0 in the $-k+1$ place) if $k<0$.

A cospectrum $\mathbf{X}=\{X_0, X_1, X_2, \cdots\}$ will be called a Wu $(q+1)$-cospectrum if the following conditions are satisfied:

(i) $\mathbf{X}$ is mod 2 coconnected, i.e., $H_i(\mathbf{X}; Z_2)=\lim\limits_{\overleftarrow{n}} H_{n+i}(X_n; Z_2)$ is zero for $i>0$ and $H_n(X_n; Z_2)\cong H_{n+1}(X_{n+1}; Z_2)\cong H_0(\mathbf{X}; Z_2)=Z_2$.

(ii) $\mathrm{Sq}^{q+1}\colon H^{-q-1}(\mathbf{X}; Z_2)\to H^0(\mathbf{X}; Z_2)$, is zero $\big(H^j(\mathbf{X}; Z_2)=\lim\limits_{\overrightarrow{n}} H^{n+j}(X_n; Z_2)\big)$.

We note that if $\mathbf{X}$ is a cospectrum satisfying (i) then condition (ii) is equivalent to the statement:

(ii′) $\mathrm{Sq}^{q+1}\colon H^{n-q-1}(X_n; Z_2)\to H^n(X_n; Z_2)$ is zero for every n.

If A is a space and $x\in H_j(A)$ we will denote by $\boldsymbol{x}\in H_j(\mathbf{A})\cong H_j(A)$, the corresponding element.

Let M^{2q} be a Z_2 Poincaré duality space of dimension $2q$, i.e., there is an element $\mu\in H_{2q}(M; Z_2)$ such that $\frown\mu\colon H^s(M; Z_2)\to H_{2q-s}(M; Z_2)$ is an isomorphism for all s.

Let $\mathbf{X}$ be a Wu $(q+1)$-cospectrum. An $\mathbf{X}$ orientation of M^{2q} is a map of cospectra $\boldsymbol{\eta}\colon \mathbf{X}\to \mathbf{S}^{-2q}\mathbf{M}_+$ such that $\boldsymbol{\eta}^*(\boldsymbol{\alpha})=\boldsymbol{\mu}$ where $\boldsymbol{\alpha}\in H_0(\mathbf{X}; Z_2)$ is a generator. That is, there are maps $\eta_{2q+n}\colon X_{2q+n}\to \Sigma^n M_+$ for all $n>m$, some m, such that the diagram

$$\begin{array}{ccc} X_{2q+n+1} & \xrightarrow{\eta_{2q+n+1}} & \Sigma^{n+1} M \\ {\scriptstyle f_{2q+n}}\downarrow & & \downarrow{\scriptstyle \text{identity}} \\ X_{2q+n} & \xrightarrow{\Sigma\eta_{2q+n}} & \Sigma^{n+1} M \end{array}$$

commutes and $\eta^*_{2q+n}(\alpha_{2q+n}) = \Sigma^n(\mu)$ where α_{2q+n} is a generator of $H_{2q+n}(X_{2q+n}; Z_2)$.

Suppose M^{2q} is $\mathbf{X}$ oriented where $\mathbf{X}$ is a Wu $(q+1)$-cospectrum, $\boldsymbol{\eta}$: $\mathbf{X} \longrightarrow \mathbf{S}^{-2q}\mathbf{M}_+$, let $\boldsymbol{x} \in H^{-q}(\mathbf{S}^{-2q}\mathbf{M}_+; Z_2)$ be such that $\eta^*(\boldsymbol{x}) = 0$.

This means that there is an $r > 0$ such that $\eta^*_{2q+r}(\Sigma^r(x)) = 0$ (which implies $\eta^*_{2q+s}(\Sigma^s x) = 0$, for $s \geqq r$).

We define a map ψ: $(\ker \boldsymbol{\eta})^{-q} \longrightarrow Z_2$ as follows.

Take a map f: $M_+ \longrightarrow K(Z_2, q)$ such that $f^*(\iota) = x$, where $\iota \in H^q(K(Z_2, q); Z_2)$ is the fundamental class. Let $\boldsymbol{g} = \boldsymbol{f} \circ \boldsymbol{\eta}$

$$\mathbf{X} \xrightarrow{\eta} \mathbf{S}^{-2q}\mathbf{M}_+ \xrightarrow{f} \mathbf{S}^{-2q}\mathbf{K}(Z_2, q)\ .$$

We define $\psi(\boldsymbol{x}) = \mathrm{Sq}^{q+1}{}_g(\boldsymbol{\iota})$ where

$$\mathrm{Sq}^{q+1}{}_g\colon (\ker \boldsymbol{\eta}^*)^{-q} \longrightarrow \frac{H^0(\mathbf{X}; Z_2)}{\mathrm{im}\,\mathrm{Sq}^{q+1} + \mathrm{im}\,\boldsymbol{g}^*}$$

is the functional Steenrod square (see [22]).

It is equivalent to consider a single map η_{2q+k}: $X_{2q+k} \longrightarrow \Sigma^k M_+$, or suppressing subscripts η: $X \longrightarrow \Sigma^k M_+$. We may assume $\eta^*(\Sigma^k(x)) = 0$, and we may take $\psi(x) = \mathrm{Sq}^{q+1}{}_g(\Sigma^k(\iota))$, where

$$\mathrm{Sq}^{q+1}{}_g\colon (\ker g^*)^{q+k} \longrightarrow \frac{H^{2q+k}(X; Z_2)}{\mathrm{im}\,\mathrm{Sq}^{q+1} + \mathrm{im}\, g^*}\ , \qquad g = (\Sigma^k f) \circ \eta\ .$$

These two definitions of ψ can easily be seen to agree (cf. (1.2)). We will use the second for the remainder of this section.

We note that $\mathrm{Sq}^{q+1}(\iota) = 0$ since $\dim \iota = q$, so $\mathrm{Sq}^{q+1}{}_g(\Sigma^k(\iota))$ is defined. We must take $\Sigma^k K(Z_2, q)$ on the right, rather than $\Sigma^k M_+$ in order to make the indeterminacy zero, as indicated in the lemma below.

LEMMA 1.1. *The indeterminacy* $(\mathrm{im}\,\mathrm{Sq}^{q+1} + \mathrm{im}\, g^*)^{2q+k} = 0$, *so that range* $\psi = Z_2$.

PROOF. By hypothesis $(\mathrm{im}\,\mathrm{Sq}^{q+1}) \cap H^{2q+k}(X; Z_2) = 0$ so it remains to show that $(\mathrm{im}\, g^*)^{2q+k} = 0$. But $H^{2q+k}(\Sigma^k K(Z_2, q); Z_2) \cong H^{2q}(K(Z_2, q); Z_2) \cong (\mathfrak{A}_2)^q \iota$, by a theorem of Serre [1], where $(\mathfrak{A}_2)^q$ is the q-dimensional component of the Steerod algebra $\mathfrak{A}_2$. Hence $(\mathrm{im}\, g^*)^{2q+k} = (\mathfrak{A}_2)^q(g^*(\Sigma^k\iota))$, and $g^*(\Sigma^k(\iota)) = \eta^*(\Sigma^k f)^*(\Sigma^k\iota) = \eta^*\Sigma^k x = 0$, so $(\text{image}\, g^*)^{2q+k} = 0$ and the lemma follows.

LEMMA 1.2. *The value of* ψ *is independent of* k, i.e., *if one uses* $\Sigma X \xrightarrow{\Sigma\eta} \Sigma^{k+1}M$ *in place of* η *it defines the same* ψ.

LEMMA 1.3. *If* $\mathbf{Y}$ *and* $\mathbf{X}$ *are Wu* $(q+1)$*-cospectra and if* $\mathbf{Y} \longrightarrow \mathbf{X}$ *is a map of degree* 1 (i.e., *isomorphism on* H_0) *then* $\mathbf{X}$ *orientable spaces get a natural* $\mathbf{Y}$ *orientation which defines the same function* ψ *on the intersection of their domains of definition.*

THEOREM 1.4. *ψ is quadratic*, i.e., *if* $x_1, x_2 \in (\ker \eta^*)^q$,

$$\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2) + \eta^*(x_1 \smile x_2) .$$

PROOF. Let $f_i : M \to K(Z_2, q)$ such that $f_i^*(\iota) = x_i$. Then

$$M \xrightarrow{\Delta} M \times M \xrightarrow{f_1 \times f_2} K(Z_2, q) \times K(Z_2, q) \xrightarrow{\mu} K(Z_2, q)$$

represents $x_1 + x_2$, where Δ is the diagonal, and μ is the multiplication in $K(Z_2, q)$.

Let X, Y be connected complexes with base points x_0, y_0,

$i: X \to X \times Y, \quad i(x) = (x, y_0)$

$j: Y \to X \times Y, \quad j(y) = (x_0, y)$

$p: X \times Y \to X, \quad q: X \times Y \to Y,$ the projections,

$s: X \times Y \to X \wedge Y = X \times Y / X \vee Y,$ the natural identification map,

and

$h: \Sigma(X \wedge Y) \to \Sigma(X \times Y),$ the Hopf construction on the identity map.

LEMMA 1.5. $\Sigma i + \Sigma j + h: \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y) \to \Sigma(X \times Y)$

and

$$\Sigma p + \Sigma q + \Sigma s: \ \Sigma(X \times Y) \to \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$$

are natural inverse homotopy equivalences.

For a proof see [10].

Applying Lemma 1.5 to our situation we get a commutative diagram:

$$\begin{array}{ccccccc}
\Sigma M & \xrightarrow{\Sigma\Delta} & \Sigma(M \times M) & \xrightarrow{\Sigma(f_1 \times f_2)} & \Sigma(K \times K) & \xrightarrow{\Sigma\mu} & \Sigma K \\
 & & \downarrow{\scriptstyle \Sigma p + \Sigma q + \Sigma s} & & \downarrow{\scriptstyle \Sigma p + \Sigma q + \Sigma s} & & \uparrow{\scriptstyle \Sigma\mu\cdot(\Sigma i + \Sigma j + h)} \\
 & & \Sigma M \vee \Sigma M \vee \Sigma(M \wedge M) & \xrightarrow{\Sigma f_1 + \Sigma f_2 + \Sigma(f_1 \wedge f_2)} & \Sigma K \vee \Sigma K \vee \Sigma(K \wedge K) & . &
\end{array}$$

It then follows that the composite map of $\Sigma M \to \Sigma K$ which represents $x_1 + x_2$ is $(\Sigma\mu)(\Sigma i + \Sigma j + h)(\Sigma f_1 + \Sigma f_2 + \Sigma(f_1 \wedge f_2))(\Sigma p + \Sigma q + \Sigma s)(\Sigma\Delta) = \Sigma r$. Now $(\Sigma p + \Sigma q + \Sigma s)(\Sigma\Delta) = \Sigma 1_1 + \Sigma 1_2 + \Sigma\bar{\Delta}$ where $\bar{\Delta}: M \to M \times M \to M \wedge M$, $1_i: M \to M$ in i^{th} place in $M \vee M$. Now $\Sigma f_i \circ \Sigma 1_j = 0$ if $i \neq j$ and $\Sigma\mu \circ (\Sigma i + \Sigma j + h) = \Sigma\mathcal{I}_1 + \Sigma\mathcal{I}_2 + h(\mu)$ where $\mathcal{I}_i$ is identity on K in the i^{th} place in $K \vee K$. Hence $\Sigma r = \Sigma f_1 + \Sigma f_2 + h(\mu)(\Sigma(f_1 \wedge f_2))\Sigma\bar{\Delta}$.

LEMMA 1.6. *If* $\Sigma g: \Sigma A \to \Sigma B$, $\Sigma g = \Sigma a + \Sigma b$, $a^*(x) = b^*(x) = 0$ *and* $(\operatorname{im} a^*)^{2q+k} = (\operatorname{im} b^*)^{2q+k} = 0$, *then*

$$\mathrm{Sq}^k_{\Sigma g} = \mathrm{Sq}^k_{\Sigma a} + \mathrm{Sq}^k_{\Sigma b} .$$

PROOF. $\Sigma a + \Sigma b$ is defined by

$$\Sigma A \xrightarrow{\Delta} \Sigma A \vee \Sigma A \xrightarrow{\Sigma a \vee \Sigma b} \Sigma B \vee \Sigma B \xrightarrow{\Delta'} \Sigma B ,$$

where Δ' is the "folding map."

$$\begin{array}{ccc} (\Sigma B \vee \Sigma B) \smile_{(\Sigma a+\Sigma b)\Delta} C\Sigma A & \xrightarrow{1 \smile \Delta} & (\Sigma B \vee \Sigma B) \smile_{\Sigma a \vee \Sigma b} C(\Sigma A \vee \Sigma A) = (\Sigma B \smile_{\Sigma a} \Sigma A) \vee (\Sigma B \smile_{\Sigma b} C\Sigma A) \\ \Big\downarrow \Delta' \smile 1 & & \Big\downarrow \Delta' \smile 1 \\ (\Sigma B) \smile_{\Sigma g} C\Sigma A & \xrightarrow{1 \smile \Delta} & (\Sigma B) \smile_{\Delta'(\Sigma a+\Sigma b)} C(\Sigma A \vee \Sigma A) \end{array}$$

The proof is immediate from the above commutative diagram and the calculation of $\mathrm{Sq}^k(\Delta'^*(x))$ in $(\Sigma B \smile_{\Sigma a} \Sigma A) \vee (\Sigma B \smile_{\Sigma b} \Sigma A)$.

LEMMA 1.7. $\mathrm{Sq}^{q+1}{}_{h(\mu)}(\Sigma\iota) = \Sigma(\iota \wedge \iota)$.

The proof is the same as [23, (5.3)].

Then it follows from Lemmas 1.5, 1.6, and 1.7 that

$$\begin{aligned} \psi(x_1 + x_2) = \mathrm{Sq}^{q+1}{}_{\Sigma^k g} &= \mathrm{Sq}^{q+1}{}_{\Sigma^k f_1}(\Sigma^k \iota) + \mathrm{Sq}^{q+1}{}_{\Sigma^k f_2}(\Sigma^k \iota) + \eta^*(x_1 \smile x_2) \\ &= \psi(x_1) + \psi(x_2) + \eta^*(x_1 \smile x_2) \end{aligned}$$

which proves Theorem 1.4.

We shall say that an **X** oriented Poincaré duality space M^{2q} is an **X** *boundary* if there is a map $i\colon M^{2q} \to W^{2q+1}$ such that (W, M) is a relative Poincaré duality pair and the map

$$X_{2q+k} \xrightarrow{\ \eta\ } \Sigma^k M_+ \xrightarrow{\Sigma^k i} \Sigma^k W_+$$

is null-homotopic, some k.

PROPOSITION 1.8. *Image* $i^* \subset \ker \eta^*$ *and* $\psi(x) = 0$ *for* $x \in$ *Image* i^*.

The proof is obvious.

2. Wu spectra and cospectra

Let $\mathbf{Y} = \{Y_0, Y_1, Y_2, \cdots\}$ be a spectrum, i.e., there are maps $\Sigma Y_i \to Y_{i+1}$. Then **Y** will be called a Wu $(q + 1)$-spectrum if the following conditions are satisfied.

(i) $H_i(Y_i; Z_2) \cong H_{i+1}(Y_{i+1}; Z_2) \cong H_0(\mathbf{Y}; Z_2) = Z_2$ and $H_i(\mathbf{Y}; Z_2) = 0$ for $i < 0$, and

(ii) $\chi(\mathrm{Sq}^{q+1})\colon H^0(\mathbf{Y}; Z_2) \to H^{q+1}(\mathbf{Y}; Z_2)$ is zero, (where χ is the canonical anti-automorphism of the Steenrod algebra, see [23].

If **Y** is a spectrum of CW-complexes of finite type, we may define the Spanier-Whitehead dual of **Y** to be a cospectrum $\mathbf{X} = \{X_0, X_1, \cdots\}$ where $X_{3i} = S^{4i+1} - (Y_i)^{2i}$, and $X_{3i+\varepsilon} = \Sigma^\varepsilon X_{3i}$ for $\varepsilon = 1$ or 2, (where $(Y_i)^{2i}$ is the $2i$-skeleton of Y_i).

LEMMA 2.1. *If* **Y** *is a Wu* $(q + 1)$*-spectrum, the Spanier-Whitehead dual of* **Y** *is a Wu* $(q + 1)$*-cospectrum.*

PROOF. This follows immediately from the fact that Sq^{q+1} and $\chi(\mathrm{Sq}^{q+1})$ correspond under Spanier-Whitehead duality.

Now let B_n be the classifying space for an $(n-1)$-spherical fibre space theory, $n \gg 2q$. (E.g., $B_n = B_{O_n}, B_{\mathrm{PL}_n}, B_{G_n}, B_{U_{n/2}}, B_{\mathrm{SP}_{n/4}}$ etc.) Define the Wu class $V = 1 + v_1 + v_2 + \cdots$, $v_i \in H^i(B_n; Z_2)$ by the formula $V = \mathrm{Sq}^{-1} W^{-1}$, where $\mathrm{Sq}^{-1} = 1 + \chi(\mathrm{Sq}^1) + \chi(\mathrm{Sq}^2) + \cdots$ so that $\mathrm{Sq}\mathrm{Sq}^{-1} = \mathrm{Sq}^{-1}\mathrm{Sq} = 1$ (see [23, p. 36]), $W = 1 + w_1 + w_2 + \cdots$ is the total Stiefel-Whitney class defined by $W \smile U = \mathrm{Sq}\, U$, $\big(U \in H^n(T(\gamma); Z_2)$ is the Thom class$\big)$ and $W \smile W^{-1} = 1$. The following propositions give some standard properties of V.

PROPOSITION 2.2. *Let $U \in H^n\big(T(\gamma_n); Z_2\big)$ be the Thom class. Then $V \smile U = \mathrm{Sq}^{-1} U$, so $v_i \smile U = \chi(\mathrm{Sq}^i)U$.*

PROOF. $V = \mathrm{Sq}^{-1} W^{-1}$, so that

$$\begin{aligned}\mathrm{Sq}(V \smile U) &= \mathrm{Sq}\, V \smile \mathrm{Sq}\, U = (\mathrm{Sq}\,\mathrm{Sq}^{-1} W^{-1}) \smile (W \smile U) \\ &= W^{-1} \smile W \smile U = U\ .\end{aligned}$$

Hence applying Sq^{-1}, we get $V \smile U = \mathrm{Sq}^{-1} U$.

Let (X, Y) be a connected Poincaré duality pair of dimension N and let $h: X \to B_n$ be a map which induces the normal spherical fibre space ν of X from γ_n (see [21]).

PROPOSITION 2.3. *The map $\mathrm{Sq}^i: H^{N-i}(X, Y; Z_2) \to H^N(X, Y; Z_2)$ is equal to $\smile h^*(v_i)$.*

PROOF. $h^*(V) \smile U = \mathrm{Sq}^{-1} U$, where U is the Thom class in $H^n\big(T(\nu); Z_2\big)$, by naturallity. Now $H_{N+n}\big(T(\nu), T(\nu \mid Y); Z_2\big)$ is in the image of the Hurewicz homomorphism since ν is the normal spherical fibre space of X (see [21]). Hence, if $A = a_0 + a_1 + \cdots$, $a_i \in H^i\big(T(\nu), T(\nu \mid Y); Z_2\big)$, it follows that $(\mathrm{Sq}^{-1} A)^{N+n} = a_{N+n}$, (since no non-trivial cohomology operation can hit the top dimension).

Note that $\smile U: H^k(X, Y; Z_2) \to H^{n+k}\big(T(\nu), T(\nu \mid Y); Z_2\big)$ is an isomorphism. Then if $x \in H^{N-i}(X, Y; Z_2)$, $(\mathrm{Sq}\, x \smile U)^{N+n} = (\mathrm{Sq}^i\, x) \smile U$. But

$$\big(\mathrm{Sq}^{-1}(\mathrm{Sq}\, x \smile U)\big)^{N+n} = (x \smile \mathrm{Sq}^{-1} U)^{N+n} = (x \smile h^* V \smile U)^{N+n} = x \smile h^* v_i \smile U\ ,$$

but

$$\big(\mathrm{Sq}^{-1}(\mathrm{Sq}\, x \smile U)\big)^{N+n} = \big((\mathrm{Sq}\, x) \smile U\big)^{N+n} = (\mathrm{Sq}^i\, x) \smile U\ .$$

Hence $x \smile h^* v_i \smile U = (\mathrm{Sq}^i\, x) \smile U$, and since $\smile U$ is an isomorphism, $x \smile h^* v_i = \mathrm{Sq}^i\, x$.

Definition. $B_n\langle v_{q+1}\rangle$ is the total space of the fibre space $\pi: B_n\langle v_{q+1}\rangle \to B_n$ with $K(Z_2, q)$ (the Eilenberg-MacLane space) as fibre induced from the space of paths of $K(Z_2, q+1)$ by a map $f: B_n \to K(Z_2, q+1)$ such that $f^*(\iota) = v_{q+1}$.

The space $B_n\langle v_{q+1}\rangle$ and its associated spaces and theories will play a central role in this work.

We let $\bar{\gamma}_n = \pi^*(\gamma_n)$ be the canonical fibre space pulled back to $B_n\langle v_{q+1}\rangle$, and we note that since $\pi^*(v_{q+1}) = 0$, it follows that $\chi(\mathrm{Sq}^{q+1})\bar{U} = 0$ where $\bar{U} \in H^n(T(\bar{\gamma}_n); Z_2)$ is the Thom class. Therefore we have proved

PROPOSITION 2.4. *$T(\bar{\gamma}_n)$ defines a* Wu $(q + 1)$*-spectrum, so that the dual* $\mathbf{X}$ *of $T(\bar{\gamma}_n)$ defines a* Wu $(q + 1)$*-cospectrum.*

Since the index n plays no role if it is large enough, we will suppress it from now on, i.e., $B = B_n$, $\gamma = \gamma_n$, etc.

If M^{2q} is a Poincaré duality space (or smooth manifold, PL manifold, etc.), a $B\langle v_{q+1}\rangle$-structure on M^{2q} will be a map of fibre spaces (or vector bundles, or PL bundles) of ν into $\bar{\gamma}$, where ν is the normal spherical fibre space (or normal bundle) of M^{2q} in S^{2q+n}. Then this gives a map of $T(\nu) \to T(\bar{\gamma})$ and hence the S-dual is a map of the Wu $(q + 1)$-cospectrum $\mathbf{X} \to \mathbf{S}^{-2q}\mathbf{M}_+$ which gives M an $\mathbf{X}$ orientation.

M^{2q} is a $B\langle v_{q+1}\rangle$-boundary if there is an $i: M \to W$ such that (W, M) is a relative Poincaré duality space and the map $\nu_M \to \bar{\gamma}$ extends to a map of $\nu_W \to \bar{\gamma}$, where ν_W is the normal spherical fibre space of W^{2q+1} in D^{2q+1+n}, $(M^{2q} \subset S^{2q+n} = \partial D^{2q+1+n})$.

PROPOSITION 2.5. *If M^{2q} is a $B\langle v_{q+1}\rangle$-boundary, then M is an* $\mathbf{X}$*-boundary in the sense of* § 1, (*where* $\mathbf{X}$ *is dual to* $T(\bar{\gamma})$).

PROOF. Considering the map of ν_W as going into $\bar{\gamma} \times I$ over $B\langle v_{q+1}\rangle \times I$ we get a map of pairs $(\nu_W, \nu_M) \to (\bar{\gamma} \times I, \bar{\gamma} \times 0)$, inducing a map

$$T(\nu_W)/T(\nu_M) \longrightarrow T(\bar{\gamma}) \times I/* \times I \cup T(\bar{\gamma}) \times 0$$

such that the diagram commutes:

$$\begin{array}{ccc} T(\nu_W)/T(\nu_M) & \longrightarrow & T(\bar{\gamma}) \times I/* \times I \cup T(\bar{\gamma}) \times 0 \\ \downarrow & & \downarrow \\ \Sigma T(\nu_M) & \xrightarrow{\Sigma g} & \Sigma T(\bar{\gamma}) \end{array}$$

where $g: T(\nu_M) \to T(\bar{\gamma})$ is the map induced by the $B\langle v_{q+1}\rangle$-orientation. Taking S-duals and using the Milnor-Spanier-Atiyah theorem [4] that the S-dual of $T(\nu_W)/T(\nu_M)$ is $\Sigma^{k+1}W_+$, we get

$$\begin{array}{ccc} \Sigma X & \longrightarrow & \Sigma^{k+1}M_+ \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Sigma^{k+1}W_+ \end{array}$$

where Y is the dual of $T(\bar{\gamma}) \times I/* \times I \cup T(\bar{\gamma}) \times 0$. But the latter space is clearly contractible, so Y is also contractible and the result follows.

3. The Kervaire invariant

Using the results of § 1 and 2 we may now define an Arf invariant in certain circumstances.

Let M^{2q} be a $B\langle v_{q+1}\rangle$-oriented Poincaré duality space and suppose that the bilinear form given by the $\smile$ product is symplectic and non-singular on a subspace $A \subset (\ker \eta^*)^q$. Then the Arf invariant [3] of ψ on A may be defined as follows.

Let $x_i, y_i \in A$ be a symplectic basis, i.e., $x_i x_j = y_i y_j = 0$, all i, j, $x_i y_j = \delta_{ij}$. Then $c(\psi, A) = \sum_i \psi(x_i)\psi(y_i)$ is the Arf invariant of ψ on A and is independent of the choice of basis, (see [3]). Note that it is defined this way for any q.

In general $c(\psi, A)$ will depend on the choice of A, as well as the $B\langle v_{q+1}\rangle$-orientation of M^{2q}.

LEMMA 3.1. *If M^{2q} is $B\langle v_{q+1}\rangle$-boundary of W and if* $(\text{im } i^*)^q \cap A$ *is a submodule of* rank $= 1/2(\text{rank } A)$, *then* $c(\psi, A) = 0$ *($i: M \to W$ is the inclusion).*

PROOF. $(\text{Image } i^*)^q$ is a self-annihilating subspace, so that a basis of $(\text{Image } i^*)^q \cap A$ is 1/2 a symplectic basis for A. By Proposition 1.8, $\psi = 0$ on image i^* so that $c(\psi, A) = 0$.

An important case is when the $\smile$ product is symplectic and non-singular on all of $(\ker \eta^*)^q$. In that case we set $c(\psi, (\ker \eta^*)^q) = c(\psi)$. (We emphasize again that the definition of ψ and hence $c(\psi)$ depends strongly on the $B\langle v_{q+1}\rangle$-orientation, as will be demonstrated in § 5.) This will be the case for example if $(\ker \eta^*)^q = H^q(M; Z_2)$, which will be the case if M is a π-manifold.

Now if M^{2q} is framed, then the framing is a reduction of ν_n to the trivial bundle, i.e., the fibre over a point of γ_n. Thus a framing induces a $B\langle v_{q+1}\rangle$-structure of M^{2q}. (In fact a trivialization over the q-skeleton of ν_M induces a $B\langle v_{q+1}\rangle$-structure on M^{2q}.) Now we relate the Kervaire invariant (see [11], [9], [12]) to $c(\psi)$.

THEOREM 3.2. *If (M^{2q}, F) is a framed manifold, q odd, and ψ is defined using the induced $B\langle v_{q+1}\rangle$-structure, then $c(\psi) =$ Kervaire invariant of (M, F).*

(Actually this holds with similar proof if F is a framing on the q-skeleton of M.)

PROOF. To define the Kervaire invariant, we do framed surgery on M^{2q} to make it $(q-1)$-connected. By Lemma 3.1, this will not affect the invariant $c(\psi)$. Thus we may assume M^{2q} is $(q-1)$-connected, and in fact using more surgery we may assume $H^q(M; Z_2) = Z_2 + Z_2$, with generators x, y, such that $x \smile x = y \smile y = 0$, $x \smile y = 1$, (see [12]).

Now $q \neq 1, 3, 7$ the Kervaire invariant is defined as $\varepsilon(x)\varepsilon(y)$ where $\varepsilon(x) = 0$ if the normal bundle of an embedded S^q dual to x is trivial, and $\varepsilon(x) = 1$ if the normal bundle is non-trivial. Hence to prove the theorem when $q \neq 1, 3, 7$, it suffices to show that $\psi(x) = \varepsilon(x)$ in this case.

Let $S^q \subset M^{2q}$ be an embedded sphere, ξ^q its normal bundle and let $\alpha: M^{2q} \to T(\xi^q)$ be the natural collapsing map. Let $U \in H^q(T(\xi); Z_2)$ be the Thom class, so that $\alpha^* U = x$, the cohomology class dual to S^q. Then the map $f: M \to K(Z_2, q)$ such that $f^*\iota = x$ may be factored

$$M \xrightarrow{\alpha} T(\xi) \xrightarrow{f'} K(Z_2, q)$$

where $f'^*(\iota) = U$.

Now the map $T(\nu_M) \to T(\overline{\gamma}_n)$ factors through $T(\nu_M) \to S^n \to T(\overline{\gamma}_n)$ since it comes from a framing, so that the dual map η factors

$$X \longrightarrow S^{2q+k} \xrightarrow{\beta} \Sigma^k M_+ .$$

Hence by Lemma 1.3, ψ may be defined using β instead of η.

Now suppose ξ is trivial, so that $T(\xi) = S^q \vee S^{2q}$. Then we have

$$(*) \qquad S^{2q+k} \xrightarrow{\beta} \Sigma^k M_+ \xrightarrow{\Sigma^k\alpha} \Sigma^k T(\xi) \xrightarrow{\Sigma^k f'} \Sigma^k K(Z_2, q)$$

so that we may look at the map

$$\begin{array}{ccc} S^{2q+k} & \xrightarrow{\Sigma^k\alpha\cdot\beta} & \Sigma^k T(\xi) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \Sigma^k p} \\ \Sigma^k K(Z_2, q) & \xleftarrow{\Sigma^k f''} & \Sigma^k S^q \end{array}$$

where $p: T(\xi) \to S^q$ comes from the trivialization of ξ and $f'': S^q \to K(Z_2, q)$ is such that $f''^*(\iota) = \omega$, generator of $H^q(S^q; Z_2)$. It follows easily that

$$\mathrm{Sq}^{q+1}{}_g(\Sigma^k(\iota)) = \mathrm{Sq}^{q+1}{}_{\bar g}(\Sigma^k(\omega))$$

where

$$g = (\Sigma^k f'')\cdot(\Sigma^k p)\cdot(\Sigma^k\alpha)\cdot\beta$$

and

$$\bar g = (\Sigma^k p)\cdot(\Sigma^k\alpha)\cdot\beta: S^{2q+k} \longrightarrow S^q .$$

But if $\mathrm{Sq}^{q+1}{}_{\bar g}(\Sigma^k(\omega)) \neq 0$, then $\bar g$ is a map of Hopf invariant 1, so that is impossible by the Theorem of Adams [1], if $q \neq 1, 3, 7$. Hence if $\varepsilon(x) = 0$, then $\psi(x) = 0$.

Now suppose $\varepsilon(x) = 1$ so that ξ^q is non-trivial, and since ξ^q is stably trivial, it follows that $\xi^q = \tau_{S^q}$. So we must consider a map δ,

$$S^{2q+k} \xrightarrow{\delta} \Sigma^k T(\xi) \xrightarrow{\Sigma^k f'} \Sigma^k K(Z_2, q) .$$

Set $t = (\Sigma^k f')\cdot\delta$.

If $\delta = (\Sigma^k\alpha)\cdot\beta$, then $\psi(x) = \mathrm{Sq}^{q+1}{}_t(\Sigma^k(\iota))$. Then the result follows from the following.

LEMMA 3.3. *If $\xi^q = \tau_{S^q}$, $q \neq 1, 3, 7$, and $\delta\colon S^{2q+k} \to T(\xi)$ is any map such that*

$$\delta^*\colon H^{2q+k}(\Sigma^k T(\xi); Z_2) \longrightarrow H^{2q+k}(S^{2q+k}; Z_2)$$

is an isomorphism, then $\mathrm{Sq}^{q+1}{}_t(\Sigma^k(\iota)) \neq 0$.

PROOF. First let us note that τ_{S^q} = normal bundle of the diagonal in $S^q \times S^q$. Hence we have the natural map $S^q \times S^q \xrightarrow{\alpha} T(\xi)$ such that $\alpha^*(U) = \omega \otimes 1 + 1 \otimes \omega \in H^q(S^q \times S^q; Z_2)$. Hence, if $f''\colon T(\xi) \to K(Z_2, q)$ is such that $f''^*(\iota) = U$, then $(f'' \circ \alpha)^*(\iota) = \omega \otimes 1 + 1 \otimes \omega$. Embedding $S^q \times S^q$ in S^{2q+1} with trivial normal line bundle, we get

$$S^{2q+1} \xrightarrow{\bar\delta} \Sigma(S^q \times S^q) \xrightarrow{\Sigma \bar f} \Sigma K(Z_2, q)$$

where $\bar f = f''\cdot\alpha$.

Now $\bar\delta$ is the Hopf construction on the identity $S^q \times S^q \to S^q \times S^q$, and $\bar f = \mu(a \times a)$, where $a\colon S^q \to K(Z_2, q)$ such that $a^*(\iota) = \omega$. Hence $\Sigma \bar f \cdot \delta = h(\bar f)$, the Hopf construction on $\bar f$. But $h(\bar f) = h(\mu)\cdot s$, where $s\colon S^{2q+1} \to \Sigma(K(Z_2, q) \wedge K(Z_2, q))$ is a map such that s^* is an isomorphism on H^{2q+1}, so that $\mathrm{Sq}^{q+1}{}_{h(\bar f)}(\Sigma(\iota)) = s^* \mathrm{Sq}^{q+1}{}_{h(\mu)}(\Sigma(\iota))$. But $\mathrm{Sq}^{q+1}{}_{h(\mu)}(\Sigma(\iota)) \neq 0$ by Lemma 1.7. Hence the lemma is proved for $\delta = (\Sigma\alpha)\cdot\bar\delta$.

Now if δ' is another map $\delta'\colon S^{2q+k} \to \Sigma^k T(\xi)$ such that δ'^* is an isomorphism on H^{2q+k}, then $\delta' = \delta + i_*\zeta$, where $\zeta \in \pi_{2q+k}(S^{q+k})$ and $i\colon S^{q+k} \to \Sigma^k T(\xi)$ is the suspension of $S^q \to T(\xi)$, which comes from the inclusion of the fibre. Let $a = \Sigma^k \bar f\cdot\delta$, $l = \Sigma^k \bar f\cdot(i\cdot\zeta)$ so that $a + l = (\Sigma^k \bar f)\cdot\delta'$. Then by Lemma 1.6, $\mathrm{Sq}^{q+1}{}_{a+l} = \mathrm{Sq}^{q+1}{}_a + \mathrm{Sq}^{q+1}{}_l$. But, if $\mathrm{Sq}^{q+1}{}_l(\Sigma^k(\iota)) \neq 0$, it follows easily that ζ has Hopf invariant 1, which is impossible by Adams [1]. Hence $\mathrm{Sq}^{q+1}{}_l = 0$ and the lemma follows.

This completes the proof of Theorem 3.2 if $q \neq 1, 3, 7$.

In case $q = 1, 3$ or 7, one may give a similar proof, using the different definition of the Kervaire invariant (i.e., the Pontrjagin invariant) in these dimensions, which depends on a framing. Here the normal bundle ξ^q to an embedding $S^q \subset M$ is always trivial, but the map $\delta\colon S^{2q+k} \to T(\xi)$ (as in (3.3)) plays an important role. We omit the details. (See [6, Ch. III, § 4].)

COROLLARY 3.4. *If (M^{2q}, F) is a framed manifold and as a $B\langle v_{q+1}\rangle$-manifold (M, F) is cobordant to zero, then the Kervaire invariant of*

$(M, F) = 0$.

This follows immediately from Theorem 3.2 and Lemma 3.1, using Poincaré duality.

Hence if the image of $\Omega_{2q}^{\text{framed}}$ in $B\langle v_{q+1}\rangle$-cobordism of smooth manifolds in dimension $2q$ is zero, then the Kervaire invariant of any framed $2q$-manifold is zero. We will show in § 7 that this is the case if $q \neq 2^n - 1$.

4. Orientations

In this section we discuss some generalities about "orientations" or "liftings of the structural group" of a bundle, which will be convenient to have. Special cases of these results are well known such as in the case of Spin structures [13]. We then apply these generalities to the case of $B\langle v_{q+1}\rangle$-orientations.

Let γ_n be the canonical bundle over B, the classifying space for "bundles" in some category, (such as linear or PL bundles, or even spherical fibre spaces), and let $\bar{\gamma}_n = \pi^*(\gamma_n)$, the induced bundle over E, where $\pi\colon E \to B$ is a fibre map. Let $\rho\colon \bar{\gamma} \to \gamma$ be the natural bundle map, and let ξ be a bundle in the category over a space X. An *E-orientation* of ξ is a bundle map $b\colon \xi \to \bar{\gamma}$. Two E-orientations $b_0, b_1\colon \xi \to \bar{\gamma}$ are *equivalent* if there is an orientation $\bar{b}\colon \xi \times I \to \bar{\gamma}$ where $\xi \times I$ is the induced bundle over $X \times I$ and $\bar{b}(x, i) = b_i(x)$, $i = 0$ or 1, $x \in \xi$.

Let $c\colon \xi \to \gamma$ be a bundle map, and call an E-orientation $b\colon \xi \to \bar{\gamma}$ *canonical* (with respect to c) if $\rho b = c$. Two canonical orientations b_0, b_1 will be called *canonically equivalent* if there is an equivalence $\bar{b}\colon \xi \times I \to \bar{\gamma}$ which is canonical for each $t \in I$.

Let us denote by $\mathcal{O}(\xi)$ the set of equivalence classes of E-orientations of ξ, and by $\mathcal{O}(\xi, c)$ the set of canonical equivalence classes of canonical E-orientations of ξ. There is a natural map $\eta\colon \mathcal{O}(\xi, c) \to \mathcal{O}(\xi)$.

LEMMA 4.1. *η is a one-to-one correspondence.*

PROOF. First we show η is onto. Let $b\colon \xi \to \bar{\gamma}$ be an E-orientation of ξ. Since γ is universal, the maps c and ρb, on $\xi \times 0$ and $\xi \times 1$ respectively, extend to a bundle map $e\colon \xi \times I \to \gamma$. If $h\colon X \times I \to B$ is the map of base spaces covered by e, then by the covering homotopy theorem for $\pi\colon E \to B$, h is covered by $\bar{h}\colon X \times I \to E$. Then the pair of maps $(e, \bar{h})$ defines a map of $\xi \times I$ into the induced bundle $\pi^*(\gamma) = \bar{\gamma}$ which defines an equivalence of b with a canonical orientation.

Now we show η is one-to-one. Let b_0, b_1 be canonical E-orientations and let $\bar{b}\colon \xi \times I \to \bar{\gamma}$ be an (ordinary) equivalence between them. We must find a canonical equivalence.

Now we define a bundle map $a: \xi \times I^2 \mid X \times \partial I^2 \to \gamma$ as follows. On $\xi \times I \times 0 \cup \xi \times 0 \times I \cup \xi \times 1 \times I$ we take $a(x, t, s) = c(x)$, $x \in \xi$, $s = 0$ or $t = 0$ or 1. On $\xi \times 1 \times I$ define $a(x, t, s) = \bar{b}(x, t)$. Since γ is universal, a extends to $\bar{a}: \xi \times I^2 \to \gamma$. Let h be the map, $h: X \times I \times I \to B$ covered by $\bar{a}$, and note that if $Y = X \times I \times 0 \cup X \times I \times 1 \cup X \times 1 \times I$, $h \mid Y$ is covered by a map $H: Y \to E$. Since Y is a deformation retract of $X \times I^2$, the covering homotopy theorem implies that H extends to $\bar{H}: X \times I^2 \to E$. Then $(\bar{a}, \bar{H})$ together define a map $\xi \times I^2 \to \bar{\gamma}$, and on $\xi \times I \times 0$ it yields a canonical equivalence between b_0 and b_1.

This proves (4.1).

THEOREM 4.2. *The set $\mathcal{O}(\xi)$ of E-orientations of ξ is in one-to-one correspondence with the set of homotopy classes of cross-sections of $f^*(E)$, where $f: X \to B$ is a classifying map for ξ.*

PROOF. By Lemma 4.1 it suffices to prove this for $\mathcal{O}(\xi, c)$, where c covers f. Now homotopy classes of cross-sections of $f^*(E)$ are in one-to-one correspondence with homotopy classes of maps $g: X \to E$ such that $\pi g = f$, (where the projections of the homotopies are also constantly f). Now a map $g: X \to E$ such that $\pi g = f$, together with $c: \xi \to \gamma$, induces a bundle map $b: \xi \to \bar{\gamma} = \pi^*(\gamma)$ such that $\rho b = c$, i.e., a canonical E-orientation. Similarly a homotopy lying over f induces a canonical equivalence so that a map is defined from classes of cross-sections to $\mathcal{O}(\xi, c)$. On the other hand, if $b \in \mathcal{O}(\xi, c)$, b covers a map $g: X \to E$ such that $\pi g = f$ (since $\rho b = c$). Similarly, canonical equivalences yield homotopies lying over f, so that a map is defined from $\mathcal{O}(\xi, c)$ to cross-sections. It is easy to verify these two maps are inverses, which proves Theorem 4.2.

COROLLARY 4.3. *If $\pi: E \to B$ is a principal fibre space with group G, and if ξ has an E-orientation, then $\mathcal{O}(\xi)$ is in one-to-one correspondence with $[X, G]$(= the homotopy classes of maps $X \to G$).*

For $[X, G]$ is in one-to-one correspondence with sections of a trivial G-bundle, and since $f^*(E)$ has a section, it is trivial.

COROLLARY 4.4. *If a bundle ξ over X has a $B\langle v_{q+1}\rangle$-structure, then the equivalence classes of $B\langle v_{q+1}\rangle$-structures are in one-to-one correspondence with elements of $H^q(X; Z_2)$.*

For $B\langle v_{q+1}\rangle \to B$ is a principal fibre space with group $K(Z_2, q)$.

We note that if $\beta: X \to G$ and $g: X \to E$ such that $\pi g = f$, the new map corresponding to β is given by $g'(x) = \mu(\beta(x), g(x))$, where $x \in X$ and $\mu: G \times E \to E$ is the action of the group on E.

5. Examples of manifolds with $c(\psi) = 1$

LEMMA 5.1. *With any $B\langle v_{q+1}\rangle$-orientation on a π-manifold M^{2q}, $(\ker \eta^*)^q = H^q(M; Z_2)$.*

PROOF. Since $\pi: B\langle v_{q+1}\rangle \to B$ is a fibre space with $K(Z_2, q)$ as fibre, and non-zero k-invariant v_{q+1},

$$\pi_*: H_q(B\langle v_{q+1}\rangle; Z_2) \longrightarrow H_q(B; Z_2)$$

is mono. Since M is a π-manifold, the classifying map $c: M \to B$ is null-homotopic and if $a: M \to B\langle v_{q+1}\rangle$ is an orientation, $\pi a = c$, so $\pi_* a_* = c_*$. Hence $a_*: H_q(M; Z_2) \to H_q(B\langle v_{q+1}\rangle; Z_2)$ is zero, and by the Thom isomorphism,

$$T(b)_*: H_{q+n}(T(\nu_M); Z_2) \longrightarrow H_{q+n}(T(\bar{\gamma}_n); Z_2)$$

is zero. Since for the map of Spanier-Whitehead duals,

$$\eta: X \longrightarrow \Sigma^k M_+ ,$$
$$\eta^* = (T(b)_*)^*, \ \eta^*: H^{q+k}(M; Z_2) \longrightarrow H^{q+k}(X; Z_2)$$

is zero.

Let us consider $S^q \times S^q$ framed in S^{2q+1}, so that it is naturally framed cobordant to zero. Then, of course, $c(\psi) = 0$, for the ψ coming from this framing. We will construct a new $B\langle v_{q+1}\rangle$-orientation on $S^q \times S^q$ so that $c(\psi) = 1$, for each q. (Here B may be taken to be B_{O_n}, B_{PL_n}, B_{G_n}, etc.)

Let $q: S^q \times S^q \to B\langle v_{q+1}\rangle$ be the map coming from the framing above, which we may take to be the constant map, and let $d: S^q \times S^q \to K(Z_2, q)$ be such that $d^*(\iota) = g \otimes 1 + 1 \otimes g$, $g =$ generator of $H^q(S^q; Z_2)$. Then let us consider the $B\langle v_{q+1}\rangle$-orientation corresponding to $\bar{q}(x) = \mu(d(x), q(x))$ (see § 4).

THEOREM 5.2. *With the $B\langle v_{q+1}\rangle$-orientation on $S^q \times S^q$ corresponding to $\bar{q}$, $c(\psi) = 1$.*

PROOF. By (5.1), ψ is defined on all of $H^q(S^q \times S^q; Z_2)$. We must show that $\psi(g \otimes 1) = \psi(1 \otimes g) = 1$, for this ψ. Now the map $f: S^q \times S^q \to K(Z_2, q)$ such that $f^*(\iota) = g \otimes 1$, factors through the projection $p: S^q \times S^q \to S^q$, $p(x, y) = x$, i.e., $f = rp$ where $r: S^q \to K(Z_2, q)$ is such that $r^*(\iota) = g$. Hence we have $X \xrightarrow{\eta} \Sigma^k(S^q \times S^q)_+ \xrightarrow{\Sigma^k p} \Sigma^k S^q \xrightarrow{\Sigma^k r} \Sigma^k K(Z_2, q)$ and if $t = (\Sigma^k p)\cdot\eta$, and $t' = (\Sigma^k r)\cdot t$, then clearly $\mathrm{Sq}^{q+1}{}_t((\Sigma^k r)^*(\Sigma^k \iota)) = \mathrm{Sq}^{q+1}{}_{t'}(\Sigma^k(\iota))$, and then $\psi(g \otimes 1) = \mathrm{Sq}^{q+1}{}_t(\Sigma^k(g))$.

Now considering the natural embedding of $S^q \times S^q$ in S^{2q+1}, one can deduce from the Milnor-Spanier theorem that $(S^q \times S^q)_+$ is stably $(2q+1)$-dual to itself (see [4]) and that the inclusion in the second factor $i: S^q \to S^q \times S^q$ is S-dual to p. Hence $\Sigma^n i: \Sigma^n S^q \to \Sigma^n(S^q \times S^q)_+$ is S-dual to p, and the composite $l: \Sigma^n S^q_+ \to \Sigma^n(S^q \times S^q)_+ \to T(\bar{\gamma}_n)$ is S-dual to t. It follows from

[16] (see also [23]) that $\mathrm{Sq}^{q+1}{}_t\colon H^{q+k}(\Sigma^k S^q; Z_2) \to H^{2q+k}(X; Z_2)$ is dual to $\chi(\mathrm{Sq}^{q+1})_l\colon H^n(T(\bar{\gamma}_n); Z_2) \to H^{n+q}(\Sigma^n S^q_+; Z_2)$. We will show $\chi(\mathrm{Sq}^{q+1})_l$ is non-zero so that $\mathrm{Sq}^{q+1}{}_t$ is onto, and hence $\mathrm{Sq}^{q+1}{}_t(\Sigma^k(g)) \neq 0$. (We use here the fact that if $f\colon X \to Y$, $\mathrm{Sq}^{q+1}{}_f$ is calculated from Sq^{q+1} in the space $Y \smile_f cX$, and $(Y \smile_f cX)^* = X^* \smile_{f^*} cY^*$, where $*$ denotes S-dual.)

Now the composite map $s = \bar{q}i$, $S^q \xrightarrow{i} S^q \times S^q \xrightarrow{\bar{q}} B\langle v_{q+1}\rangle$ generates the (kernel $\pi_*)_q$, $\pi_*\colon \pi_q(B\langle v_{q+1}\rangle) \to \pi_q(B)$. Hence, $\bar{\pi}\colon B\langle v_{q+1}\rangle \smile_s e^{q+1} \to B$ which extends π, is an isomorphism on π_i, $i \leqq q$.

If $q = 1, 3$, or 7, then the inclusion $j\colon K(Z_2, q) \to B\langle v_{q+1}\rangle$, is zero on $\pi_q(K(Z_2, q)) = Z_2$, while if $q \neq 1, 3, 7$,

$$j_*\colon \pi_q(K(Z_2, q)) \longrightarrow \pi_q(B\langle v_{q+1}\rangle)$$

is mono. This comes from the fact that there is a spherical fibre space over S^{q+1} with $v_{q+1} \neq 0$ if and only if $q + 1 = 2, 4$ or 8, (cf. proof of (5.3)).

Hence if $q \neq 1, 3, 7$, then $\bar{\pi}_*$ is onto $\pi_{q+1}(B)$, for any choice of $\bar{\pi}$ extending π, while if $q = 1, 3$ or 7, then s is homotopic to a constant so that

$$B\langle v_{q+1}\rangle \smile_s e^{q+1} \cong B\langle v_{q+1}\rangle \vee S^{q+1}\ ,$$

and if we choose $\bar{\pi}$ on S^{q+1} to go onto a generator of $\pi_{q+1}(B)$, $\bar{\pi}$ will map onto $\pi_{q+1}(B)$. Therefore we may assume $\bar{\pi}_*\colon \pi_i(B\langle v_{q+1}\rangle \smile_s e^{q+1}) \to \pi_i(B)$ is an isomorphism for $i \leqq q$ and onto for $i = q + 1$, and hence $\pi_i(\bar{\pi}) = 0$ for $i \leqq q + 1$. By the relative Hurewicz theorem, $H_i(\bar{\pi}) = 0$ for $i \leqq q + 1$ so that $H_i(\bar{\pi}; Z_2) = 0$ and $H^i(\bar{\pi}; Z_2) = 0$ for $i \leqq q + 1$. Hence $\bar{\pi}^*\colon H^i(B; Z_2) \to H^i(B\langle v_{q+1}\rangle \smile_s e^{q+1}; Z_2)$ is mono for $i \leqq q + 1$, and hence $\bar{\pi}^*(v_{q+1}) \neq 0$.

Now $S^q \to S^q \times S^q \to B\langle v_{q+1}\rangle$ is covered by a bundle map inducing l,

$$\Sigma^n S^q_+ \longrightarrow \Sigma^n(S^q \times S^q)_+ \longrightarrow T(\bar{\gamma}_n)$$

and hence $T(\bar{\gamma}_n) \smile_l e^{n+q+1}$ is the Thom complex of the induced bundle $\bar{\pi}^*(\gamma_n)$ over $B\langle v_{q+1}\rangle \smile_s e^{q+1}$. Since $\bar{\pi}^*(v_{q+1}) \neq 0$, it follows from (2.2) that $\chi(\mathrm{Sq}^{q+1})U \neq 0$ in $T(\bar{\pi}^*(\gamma_n))$, and hence $\chi(\mathrm{Sq}^{q+1})_l \neq 0$ on $H^n(T(\bar{\gamma}_n); Z_2)$, and $\mathrm{Sq}^{q+1}{}_t(\Sigma^k g) \neq 0$, which completes the proof of Theorem 5.2.

We note the following

PROPOSITION 5.3. *The structure $\bar{q}$ on $S^q \times S^q$ comes from a framing if and only if $q = 1, 3$, or 7.*

PROOF. If $\bar{q}$ comes from a framing, then $\bar{q}$ is homotopic to a constant map into $B\langle v_{q+1}\rangle$, so that if $j\colon K(Z_2, q) \to B\langle v_{q+1}\rangle$, $j_*(\pi_q(K(Z_2, q))) = 0$ and $\partial\colon \pi_{q+1}(B) \to \pi_q(K(Z_2, q))$ is onto, where ∂ is the boundary in the exact sequence of the fibre space. By the natural relation between ∂ and the transgression in cohomology, since $\iota \in H^q(K(Z_2, q); Z_2)$ transgresses to v_{q+1}, it follows that if

$\partial\alpha$ generates $\pi_q(K(Z_2, q))$, $\alpha: S^{q+1} \to B$, then $\alpha^*(v_{q+1}) \neq 0$ and α induces a spherical fibre space over S^{q+1} with $v_{q+1} \neq 0$. But then the Thom complex has two cells and $\chi(\mathrm{Sq}^{q+1}) \neq 0$ (by (2.2)) so that $q = 1, 3$, or 7 by Adams [1].

If $q = 1, 3$, or 7, then there are linear bundles over S^{q+1} with $v_{q+1} \neq 0$, and by an argument similar to the above, we may deduce from this that $\bar{q}$ is homotopic to a constant.

The structure $\bar{q}$ on $S^q \times S^q$ determines an element of $B\langle v_{q+1}\rangle$-cobordism for any classifying space B, any q, which is non-zero since it has $c(\psi) = 1$.

THEOREM 5.4. *The structure $\bar{q}$ on $S^q \times S^q$ has Thom invariant (in $\pi_{n+2q}(T(\bar{\gamma}))$) which is equal to $\bar{j}_*(\omega)$, where $\bar{j}: \Sigma^n K(Z_2, q) \to T(\bar{\gamma})$ coming from the inclusion of $K(Z_2, q) \to B\langle v_{q+1}\rangle$, and where $\omega \in \pi_{n+2q}(\Sigma^n K(Z_2, q)) = Z_2$ is a generator.*

The proof is routine from the definitions, using the fact that the Hopf construction on $\mu: K(Z_2, q) \times K(Z_2, q) \to K(Z_2, q)$ sends the generator of $\pi_{2q+1}(\Sigma K(Z_2, q) \wedge K(Z_2, q))$ into the generator of $\pi_{2q+1}(\Sigma K(Z_2, q))$.

6. $B\langle x\rangle$-cobordism

Let B be a classifying space as usual and let $\pi: E \to B$ be a fibre space with $K(Z_2, q)$ as fibre, induced by $f: B \to K(Z_2, q+1)$ such that $f^*(\iota) = x \in H^{q+1}(B; Z_2)$. Let $\bar{\gamma} = \pi^*(\gamma)$. In this section we shall compute $\pi_i(T(\bar{\gamma}))$ for $i \leqq n + 2q$ up to a group extension in terms of $\pi_*(T(\gamma))$ and $H_*(T(\gamma); Z_2)$. If $x = v_{q+1}$ and B is a classifying space for which transversality holds (such as $B_{O_n}, B_{SO_n}, B_{PL_n}$, etc.), this gives us a computation of $B\langle v_{q+1}\rangle$-cobordism in dimensions $\leqq 2q$ in terms of B-cobordism and $H_*(B; Z_2)$. We will use these results in § 7 to prove that the Kervaire invariant of a framed manifold M^{2q} is zero unless $q = 2^m - 1$, and in that case that the Kervaire invariant is related to the element h_m^2 in the Adams spectral sequence.

Below we use concepts of "disk" bundle associated with a fibre space with some fibre F, etc., notions which are explained in detail in an appendix. Intuitively one may think of the "disk" bundle as the mapping cylinder of the projection and make analogous constructions as in the case of linear bundles.

Let $\pi: E \to B$ be a fibre space with fibre $K(Z_2, q)$, let B be connected, and let γ be an $(n-1)$-spherical fibre space over B, n very large, $\bar{\gamma} = \pi^*\gamma$. Let $\bar{\pi}: \bar{E} \to B$ be the disk bundle associated with π, with contractible fibre $cK(Z_2, q)$, (see appendix) and let $\hat{\pi}: \hat{E} \to B$ be the sum of π with a trivial S^0 bundle, so that the fibre of $\hat{\pi}$ is $\Sigma K(Z_2, q)$. Let $\bar{\pi}^*(\gamma)$ also be denoted by γ, and let $\hat{\gamma} = \hat{\pi}^*(\gamma)$. The fibre space $\hat{E} = \bar{E}_+ \cup \bar{E}_-$, where $\bar{E}_+, \bar{E}_-$ are two copies of $\bar{E}$, $\bar{E}_+ \cap \bar{E}_- = E$, and let γ_+, γ_- be the restrictions of $\hat{\gamma}$ to E_+, E_-, (so $\gamma_\pm$

corresponds to γ under the identification of $\bar{E}_{\pm}$ with $\bar{E}$).

Now we have the exact sequence of the pair $(T(\gamma), T(\bar{\gamma}))$,

$$\cdots \longrightarrow \pi_{n+i+1}(T(\gamma), T(\bar{\gamma})) \longrightarrow \pi_{n+i}(T(\bar{\gamma})) \longrightarrow \pi_{n+i}(T(\gamma)) \longrightarrow \pi_{n+i}(T(\gamma), T(\bar{\gamma})) \longrightarrow \cdots .$$

THEOREM 6.1. *There is a map $h: T(\gamma)/T(\bar{\gamma}) \to K(Z_2, q+1) \wedge T(\gamma)$ which is an isomorphism on $H^{n+i}(\ ; Z_2)$ for $i \leqq 2q+1$, and with kernel h^* in dimension $n+2q+2$ generated by a single element X.*

THEOREM 6.2. *$\pi_{n+i}(T(\gamma), T(\bar{\gamma})) \cong H_{n+i-q-1}(T(\gamma); Z_2)$, for $i \leqq 2q$ and is detected by the Hurewicz homomorphism* mod 2, *and for $i = 2q+1$, the sequence*

$$0 \longrightarrow Z_2 \longrightarrow \pi_{n+2q+1}(T(\gamma), T(\bar{\gamma})) \longrightarrow H_{n+q}(T(\gamma); Z_2) \longrightarrow 0$$

is exact, and the Z_2 on the left is the kernel of the Hurewicz homomorphism and generated by the image of

$$\pi_{n+2q+1}(c\Sigma^n K(Z_2, q), \Sigma^n K(Z_2, q)) = Z_2 \ in\ \pi_{n+2q+1}(T(\gamma), T(\bar{\gamma})) .$$

The first part of (6.2) is equivalent to

THEOREM 6.2′. *$K(Z_2, q+1) \wedge T(\gamma)$ is a product of $K(Z_2, t_i)$ in dimensions $\leqq n+2q+2$, with $\pi_*(K(Z_2, q+1) \wedge T(\gamma))$ in these dimensions isomorphic to $H_{q+1}(K(Z_2, q+1); Z_2) \otimes \bar{H}_*(T(\gamma); Z_2)$.*

PROOF OF THEOREM 6.1. If we include the pair $(\bar{E}_+, E) \to (\hat{E}, \bar{E}_-)$, this is an excision map so that it induces an isomorphism on $H_*(\ ; Z_2)$. Hence the map of Thom complexes above this inclusion $(T(\gamma_+), T(\bar{\gamma})) \to (T(\hat{\gamma}), T(\gamma_-))$ also induces an isomorphism in $H_*(\ ; Z_2)$. We are interested in stable homotopy groups and since n is assumed very large, we have that $(T(\gamma_+), T(\bar{\gamma})) \to (T(\gamma_+)/T(\bar{\gamma}), *)$ and $(T(\hat{\gamma}), T(\gamma_-)) \to (T(\hat{\gamma})/T(\gamma_-), *)$ induce isomorphisms on stable homotopy groups. Hence we shall deal with $T(\hat{\gamma})/T(\gamma_-)$. But if $\alpha: B \to \hat{E}$ is the canonical cross-section $\alpha(B) \subset \bar{E}_-$, then $(\hat{E}, \bar{E}_-)$ is homotopy equivalent to $(\hat{E}, \alpha(B))$, and $T(\hat{\gamma})/T(\gamma_-)$ is homotopy equivalent to $T(\hat{\gamma})/T(\gamma)$, where γ is identified with $\hat{\gamma} \mid \alpha(B)$.

Now $\hat{\pi}: \hat{E} \to B$ is a fibre space with $\Sigma K(Z_2, q)$ as fibre, and with a cross-section $\alpha: B \to \hat{E}$ (as above). Hence, it follows that there exists a $g' \in H^{q+1}(\hat{E}; Z_2)$ such that $j^*(g') = \Sigma(\iota)$ the generator of $H^{q+1}(\Sigma K(Z_2, q); Z_2)$, where $j: \Sigma K(Z_2, q) \to \hat{E}$ is the inclusion. If $g = g' - \hat{\pi}^*\alpha^*(g')$, then $j^*(g) = \Sigma(\iota)$ also, for $j^*\hat{\pi}^* = (\hat{\pi}j)^* = 0$ as $\hat{\pi}j$ is the constant map. Further

$$\alpha^*(g) = \alpha^*(g') - \alpha^*\hat{\pi}^*\alpha^*(g') = \alpha^*g' - \alpha^*g' = 0 ,$$

since $\alpha^*\hat{\pi}^* = (\hat{\pi}\alpha)^* = (\text{identity})^*$, ($\alpha$ being a section of $\hat{\pi}$). Let $\beta: \hat{E} \to$

$K(Z_2, q+1)$ be such that $\beta^*(\iota) = g$ and define

$$f\colon \hat{E} \longrightarrow K(Z_2, q+1) \times B \quad \text{by} \quad f(x) = (\beta(x), \hat{\pi}(x)) .$$

Then the diagram

$$\begin{array}{ccc} \hat{E} & \xrightarrow{\ f\ } & K(Z_2, q+1) \times B \\ {\scriptstyle \hat{\pi}}\downarrow & & \downarrow{\scriptstyle \text{projection}} \\ B & \xrightarrow{\text{identity}} & B \end{array}$$

clearly commutes, so that f is a map of fibre spaces. Now

$$f' = f \mid \Sigma K(Z_2, q)\colon \Sigma K(Z_2, q) \longrightarrow K(Z_2, q+1)$$

is the usual suspension map, so that f'^* is an isomorphism on $H^i(\ ; Z_2)$ for $i \leqq 2q+1$, and $(\text{kernel} f'^*)^{2q+2}$ is generated by $\mathrm{Sq}^{q+1}(\iota_{q+1}) \in H^{2q+2}(K(Z_2, q+1); Z_2)$, (see [19]).

Now from the commutative diagram

$$\text{(S)} \qquad \begin{array}{ccccc} \Sigma K(Z_2, q) & \xrightarrow{\ j\ } & \hat{E} & \longrightarrow & B \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ K(Z_2, q+1) & \xrightarrow{\ j'\ } & K(Z_2, q+1) \times B & \longrightarrow & B \end{array}$$

we get a commutative diagram of Thom complexes

$$\text{(T)} \qquad \begin{array}{ccccc} \Sigma^n(\Sigma K(Z_2, q)_+) & \xrightarrow{\ \bar{j}\ } & T(\hat{\gamma}) & \longrightarrow & T(\gamma) \\ {\scriptstyle \bar{f}'}\downarrow & & {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle 1} \\ \Sigma^n(K(Z_2, q+1)_+) & \xrightarrow{\ \bar{j}'\ } & K(Z_2, q+1)_+ \wedge T(\gamma) & \longrightarrow & T(\gamma) . \end{array}$$

Now the section $\alpha\colon B \to \hat{E}$ has the property that $p_1 f \alpha$ is null-homotopic where $p_1\colon K(Z_2, q+1) \times B \to K(Z_2, q+1)$ is the projection. Hence $f\alpha\colon B \to K(Z_2, q+1) \times B$ is homotopic to the inclusion of the second factor $B \to x_0 \times B$, $x_0 \in K(Z_2, q+1)$. Then we get

$$\text{(R)} \qquad \begin{array}{ccc} T(\hat{\gamma}) & \longrightarrow & T(\hat{\gamma})/T(\gamma_-) \\ {\scriptstyle \bar{f}}\downarrow & & \downarrow{\scriptstyle \bar{f}_0} \\ K(Z_2, q+1)_+ \wedge T(\gamma) & \longrightarrow & K(Z_2, q+1)_+ \wedge T(\gamma)/T(\gamma) \end{array}$$

where $T(\gamma) = x_0 \times T(\gamma) \subset K(Z_2, q+1)_+ \wedge T(\gamma)$. Hence

$$K(Z_2, q+1)_+ \wedge T(\gamma)/T(\gamma) = K(Z_2, q+1) \wedge T(\gamma) .$$

Using diagram (S), the fact that f'^* is an isomorphism on $H^i(\ ; Z_2)$ for $i \leqq 2q+1$, and kernel $f'^* \subset H^{2q+2}(K(Z_2, q+1); Z_2)$ is generated by one element $\mathrm{Sq}^{q+1}(\iota_{q+1})$, we get using the spectral sequences of the two fibre maps that

$$f^*: H^i(K(Z_2, q+1) \times B; Z_2) \longrightarrow H^i(\hat{E}; Z_2)$$

is an isomorphism for $i \leqq 2q+1$, and j'^* maps $(\text{kernel } f^*)^{2q+2}$ isomorphically onto $(\text{kernel } f'^*)^{2q+2}$. Hence $(\text{kernel } f^*)^{2q+2}$ is generated by a single element Y, and $j'^* Y = \text{Sq}^{q+1}(\iota_{q+1})$. Similarly, it follows that $(\text{cokernel } f^*)^{2q+2}$ is generated by a single element Z and $j^*(Z) = \Sigma(\iota \smile \text{Sq}^1 \iota)$.

Then using the Thom isomorphism for the spherical fibre spaces γ, $\hat{\gamma}$, etc., we find that in diagram (T),

$$\bar{f}^*: H^{i+n}(K(Z_2, q+1)_+ \wedge T(\gamma); Z_2) \longrightarrow H^{i+n}(T(\hat{\gamma}); Z_2)$$

is an isomorphism for $i \leqq 2q+1$, and $(\text{kernel } \bar{f}^*)^{2q+2}$ is generated by a single element $X = \Phi(Y)$, (Φ is Thom isomorphism) and $\bar{j}'^*(X) = \Sigma^n(\text{Sq}^{q+1} \iota_{q+1}) = \text{Sq}^{q+1}(\Sigma^n \iota_{q+1})$. This proves Theorem 6.1.

To prove Theorem 6.2 we have to compute $H^*(K(Z_2, q+1) \wedge T(\gamma); Z_2)$ and show that it is a free module over the Steenrod algebra $\mathfrak{A}_2$ in dimensions $\leqq n + 2q + 2$, with basis in dimension l isomorphic to $H^{l-q-1}(T(\gamma); Z_2)$, $l \leqq 2q + 2 + n$.

Now we have the following well known lemma. Let $\mathfrak{A}$ be a connected, locally finitely generated (l.f.g.) Hopf algebra over a field, associative as an algebra, (see [14]).

LEMMA 6.3. *If F is a free* l.f.g. *module over $\mathfrak{A}$ in dimensions $\leqq r$, $F_i = 0$ for $i < 0$, and M is any* l.f.g. *module over $\mathfrak{A}$, with $M_i = 0$ for $i < n$, then $F \hat{\otimes} M$ is free over $\mathfrak{A}$ in dimensions $\leqq r + n$, with basis, $V \otimes M$, where $F = \mathfrak{A} \otimes V$ (where $\hat{\otimes}$ is as modules over the Hopf algebra $\mathfrak{A}$, see* [23]).

PROOF. $F \hat{\otimes} M$ is additively isomorphic to $\mathfrak{A} \otimes V \otimes M$ in dimensions $\leqq r + n$, and has the structure of an $\mathfrak{A}$-module using the diagonal map, i.e., if $\psi a = \sum_i a_i \otimes a'_i$,

$$a(\bar{a} \otimes v \otimes m) = \sum_i a_i \bar{a} \otimes v \otimes a'_i m$$

where $\psi: \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ is the diagonal of $\mathfrak{A}$. We will show that $F \hat{\otimes} M$ is isomorphic as an $\mathfrak{A}$-module to $\mathfrak{A} \otimes (V \otimes M)$ where $a(\bar{a} \otimes (v \otimes m)) = a\bar{a} \otimes (v \otimes m)$.

Since $\mathfrak{A} \otimes (V \otimes M)$ is free over $\mathfrak{A}$, the inclusion $V \otimes M \to F \hat{\otimes} M$ extends to a map of $\mathfrak{A}$-modules, $f: \mathfrak{A} \otimes (V \otimes M) \to F \hat{\otimes} M$.

Now we show that f is onto in dimensions $\leqq r + n$. By induction suppose f is onto $\sum_{i<m} \mathfrak{A} \otimes V_i \otimes M \subset F \hat{\otimes} M$ in dimensions $\leqq r + n$, and let $x \in \mathfrak{A} \otimes V_m \otimes M$, dimension $x = s \leqq r + n$, (where if $m < 0$, this is trivial). Assume by a second induction that f is onto $\mathfrak{A}_k \otimes V_m \otimes M$, $k < l$, (where for $k = 0$ this is obvious). Let $a \otimes v \otimes m \in \mathfrak{A}_l \otimes V_m \otimes M$. Since $\mathfrak{A}$ is connected, $\psi a = a \otimes 1 + \sum_i a_i \otimes a'_i$, $\dim a_i < l$. Then

$$f(a \otimes (v \otimes m)) = a(1 \otimes v \otimes m) = \sum_i a_i \otimes v \otimes a_i' m + a \otimes v \otimes m \; .$$

Hence $a \otimes v \otimes m - f(a \otimes (v \otimes m)) \in \sum_{k<l} \mathcal{A}_k \otimes V_m \otimes M$, so that $a \otimes v \otimes m - f(a \otimes (v \otimes m)) \in (\text{image } f)$ by induction, and hence f is onto $\mathcal{A}_l \otimes V_m \otimes M$. Hence by induction $\mathcal{A} \otimes V_m \otimes M \subset \text{image } f$, so by induction f is onto.

But $F \hat{\otimes} M$ and $\mathcal{A} \otimes (V \otimes M)$ are isomorphic as graded vector spaces, and finite dimensional in each dimension (i.e., l.f.g.). Hence f is an isomorphism in dimensions $\leqq r + n$.

We apply Lemma 6.3 to $\mathcal{A} = \mathcal{A}_2$, the Steenrod algebra, $F = \bar{H}^*(K(Z_2, q+1); Z_2)$, $r = 2q + 2$, $M = \bar{H}^*(T(\gamma_n); Z_2)$, $n = n$. Hence $H^*(K(Z_2, q+1) \wedge T(\gamma); Z_2)$ is free over $\mathcal{A}_2$ in dimensions $\leqq 2q + 2 + n$, so that $K(Z_2, q+1) \wedge T(\gamma)$ is homotopy equivalent to a product of $K(Z_2, k_i)$'s in these dimensions where the homotopy groups are isomorphic to $\bar{H}^{q+1}(K(Z_2, q+1); Z_2) \otimes \bar{H}^*(T(\gamma); Z_2)$ in these dimensions, by Lemma 6.3.

Let $t\colon T(\hat{\gamma})/T(\gamma) \to K(Z_2, n + 2q + 2)$ such that $t^*(\iota) = \Phi(Z)$. Then

$$\bar{f} \times t\colon T(\hat{\gamma})/T(\gamma) \longrightarrow (K(Z_2, q+1) \wedge T(\gamma)) \times K(Z_2, n + 2q + 2)$$

is a map into a product of $K(Z_2, k_i)$'s, $(\bar{f} \times t)^*$ is an isomorphism on $H^{n+i}(\;; Z_2)$, $i \leqq 2q + 1$, $(\bar{f} \times t)^*$ is onto for $i = 2q + 2$, and kernel $(\bar{f} \times t)^{2q+2}$ is generated by X, where

$$\bar{j}'^*(X) = \mathrm{Sq}^{q+1}(\Sigma^n \iota_{q+1}) \in H^{n+2q+2}(\Sigma^n K(Z_2, q+1); Z_2) \; .$$

Then in dimensions $\leqq n + 2q + 2$, $\bar{f} \times t$ is the projection into the first stage of a generalized Postnikov system for $T(\hat{\gamma})/T(\gamma)$ (generalized meaning we use products of Eilenberg-MacLane spaces). Using diagram (T), since $\mathrm{Sq}^{q+1}(\Sigma^n \iota_{q+1})$ generates $(\text{kernel } \bar{f}')^{n+2q+2}$, it follows that, for a generator $\omega \in \pi_{n+2q+1}(\Sigma^{n+1} K(Z_2, q))$, $\bar{j}_*(\omega)$ generates the Z_2 subgroup of $\pi_{n+2q+1}(T(\hat{\gamma})/T(\gamma))$ detected by the element X, considered as a cohomology operation.

This completes the proof of Theorem 6.2.

7. The Kervaire invariant of framed manifolds

In this section we show that the image of framed cobordism $\Omega_{2q}^{\text{framed}}$ in $B_0\langle v_{q+1}\rangle$-cobordism is zero if $q \neq 2^l - 1$, so that by Corollary 3.5 the Kervaire invariant is zero if $q \neq 2^l - 1$.

If $q = 2^l - 1$, we shall show that image $\Omega_{2q}^{\text{framed}}$ in $B_0\langle v_{q+1}\rangle$-cobordism is the subgroup generated by $(S^q \times S^q, \bar{q})$ (see § 5) if the element h_l^2 in the Adams spectral sequence (see [2]) represents an element in the stable homotopy groups of spheres, and the image $\Omega_{2q}^{\text{framed}}$ is 0 otherwise. But for $(S^q \times S^q, \bar{q})$, $c(\psi) = 1$, (see (5.2)) so that we get

THEOREM 7.1. *There exists a framed manifold M^{2q} of Kervaire invariant* 1 *if and only if $q = 2^l - 1$ and the element h_l^2 in the Adams spectral sequence represents an element of the stable homotopy groups of spheres.*

COROLLARY. *There is a framed manifold of dimension* 30 *with Kervaire invariant* 1. *Equivalently, the Kervaire manifold of dimension* 30 *is smoothable.*

It has been proved by Mahowald and Tangora (Topology **6** (1967), 349–370 § 8) that h_4^2 persists to E_∞ in the Adams spectral sequence. Hence the corollary follows from (7.1).

Coupling this example with the framed $S^1 \times S^1$, $S^3 \times S^3$, and $S^7 \times S^7$ of Pontrjagin, we get

There are framed manifolds of Kervaire invariant 1 *in dimensions* 2, 6, 14, *and* 30.

Now the inclusion of the fibre of $\bar{\gamma}$ induces the map $S^n \to T(\bar{\gamma})$ and the question about the image of $\Omega_{2q}^{\text{framed}}$ in $B_0\langle v_{q+1}\rangle$-cobordism is, by transversality, equivalent to the question: "What is the image of $\pi_{n+2q}(S^n)$ in $\pi_{n+2q}(T(\bar{\gamma}))$?"

Consider the exact sequence of the pair $(T(\gamma), T(\bar{\gamma}))$:

$$\cdots \longrightarrow \pi_{2q+1+n}(T(\gamma), T(\bar{\gamma})) \xrightarrow{\partial} \pi_{2q+n}(T(\bar{\gamma})) \xrightarrow{\bar{p}^*} \pi_{2q+n}(T(\gamma)) \longrightarrow \cdots .$$

Since by the theorem of Thom, $T(\gamma) = \prod K(Z_2, t_i)$, so that the Hurewicz homomorphism is mono in $T(\gamma)$, it follows that

$$\text{image}\,(\pi_{n+2q}(S^n)) \subset \text{image}\,\partial \subset \pi_{n+2q}(T(\bar{\gamma})) .$$

We first prove

PROPOSITION 7.2. image $(\pi_{n+2q}(S^n)) \subset \partial(\bar{j}_*(\pi_{n+2q+1}(c\Sigma^n K(Z_2, q), \Sigma^n K(Z_2, q))))$ *so that the only possible non-zero element in* image $\pi_{n+2q}(S^n)$ *is the Thom invariant of* $(S^q \times S^q, \bar{q})$.

Here, $\bar{j}: (c\Sigma^n K(Z_2, q), \Sigma^n K(Z_2, q)) \to (T(\gamma), T(\bar{\gamma}))$ is the map of Thom complexes induced by $j: (cK(Z_2, q), K(Z_2, q)) \to (\bar{E}, E)$, (see § 6).

PROOF OF PROPOSITION 7.2. We consider the map of $i: S^n \to T(\bar{\gamma})$, and suppose $\alpha: S^{n+2q} \to S^n$ represents some element of $\pi_{n+2q}(S^n)$. Then $\bar{p}i\alpha$ is homotopic to zero, so we may factor $\Sigma(i\alpha)$ through the map $\eta: T(\gamma)/T(\bar{\gamma}) \to \Sigma T(\gamma)$. Since we are considering a stable homotopy group, it suffices to study $\Sigma(i\alpha) = (\Sigma i)(\Sigma\alpha)$.

If we set $Y = S^n \cup_\alpha e^{n+2q+1}$, we get a commutative diagram (see [18]),

$$
\text{(S)}\qquad
\begin{array}{ccccccccc}
S^{n+2q} & \xrightarrow{\alpha} & S^n & \xrightarrow{\beta} & Y & \xrightarrow{\delta} & S^{n+2q+1} & \xrightarrow{\Sigma\alpha} & S^{n+1} \longrightarrow \cdots \\
 & & \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle i_c} & & \big\downarrow{\scriptstyle i_s} & & \big\downarrow{\scriptstyle \Sigma i} \\
 & & T(\bar\gamma) & \longrightarrow & T(\gamma) & \xrightarrow{\delta'} & T(\gamma)/T(\bar\gamma) & \xrightarrow{\eta} & \Sigma T(\bar\gamma)
\end{array}
$$

which yields a map of exact sequences (coefficients Z_2 throughout)

$$
\text{(H)}\qquad
\begin{array}{ccccccccc}
H^k(S^{n+2q}) & \longleftarrow & H^k(S^n) & \xleftarrow{\beta^*} & H^k(Y) & \xleftarrow{\delta^*} & H^k(S^{n+2q+1}) & \xleftarrow{\Sigma\alpha^*} & H^k(S^{n+1}) \\
 & & \big\uparrow{\scriptstyle i^*} & & \big\uparrow{\scriptstyle i_c^*} & & \big\uparrow{\scriptstyle i_s^*} & & \big\uparrow{\scriptstyle (\Sigma i)^*} \\
 & & H^k(T(\bar\gamma)) & \longleftarrow & H^k(T(\gamma)) & \xleftarrow{\delta'^*} & H^k(T(\gamma)/T(\bar\gamma)) & \xleftarrow{\eta^*} & H^k(\Sigma T(\bar\gamma))
\end{array}
$$

where the horizontal sequences are the usual cohomology exact sequences.

Since $\bar{j}_*\pi_{n+2q+1}(c\Sigma^n K(Z_2, q), \Sigma^n K(Z_2, q))$ is the kernel of the Hurewicz homomorphism mod 2 in $(T(\gamma), T(\bar\gamma))$ (by Theorem 6.2), it follows that Proposition 7.2 is equivalent to the statement that $i_s^* = 0$ in diagram (H). So we consider, in (H), $k = n + 2q + 1$, and we note that $i_s^*\eta^* = 0$ since $(\Sigma i)^* = 0$ in this dimension, and $i_s^*\eta^* = (\Sigma\alpha)^*(\Sigma i)^*$. Hence, if $x \in H^k(T(\gamma)/T(\bar\gamma))$, and $i_s^*(x) \neq 0$, then $\delta'^*x \neq 0$. Now $T(\gamma)$ is (in the stable range) a product of $K(Z_2, t_i)$, so that δ'^*x corresponds to a 1st order cohomology operation mod 2. Now $\delta^*: H^k(S^{n+2q+1}) \to H^k(Y)$ is an isomorphism for $k = n + 2q + 1$, so that $i_c^*(\delta'^*)(x) = \delta^* i_s^*(x) \neq 0$, and hence the operation corresponding to δ'^*x is non-zero in Y. But by the result of Adams [1], the only 1st order cohomology operations mod 2 which can be non-zero in a space Y with only two cells are Sq^1, Sq^2, Sq^4 and Sq^8, and since δ'^*x changes dimension by an odd integer $2q + 1$ and $q > 0$, we get a contradiction.

Hence it follows that $i_s^* \equiv 0$ and the Proposition 7.2 is proved.

Thus to prove Theorem 7.1 it remains to determine under what conditions the element in the kernel of the Hurewicz homomorphism in $\pi_{n+2q+1}(T(\gamma)/T(\bar\gamma))$ may be in the image of i_{s*}, (where i_s is as in diagram (S) above). We will do this in a similar fashion to the proof of Proposition 7.2, but in order to do this we must identify the k-invariant X of $T(\hat\gamma)/T(\gamma)$, and find out which elements in $\pi_{n+2q}(S^n)$ can be detected by the second order cohomology operation which X will determine.

Let $\iota \in H^{q+1}(K(Z_2, q + 1); Z_2)$ be the generator, $U \in H^n(T(\gamma); Z_2)$ the Thom class, and $w_i \in H^i(B_0; Z_2)$ the Stiefel-Whitney classes of γ, $v_i \in H^i(B_0; Z_2)$ the Wu classes, so that $w_i \smile U = \mathrm{Sq}^i U$, $v_i \smile U = \chi(\mathrm{Sq}^i)U$, where $\chi: \mathfrak{A}_2 \to \mathfrak{A}_2$ is the canonical anti-automorphism of the Steenrod algebra.

PROPOSITION 7.3. *The first non-zero k-invariant of $T(\gamma)/T(\bar\gamma)$ is given by*

$$X = \mathrm{Sq}^{q+1}(\iota \smile U) + \iota \smile v_{q+1} \smile U + \sum_{\substack{0 \leqq i \leqq q \\ i+j=q+1}} (\mathrm{Sq}^i \iota) \smile (w_j \smile U)$$
$$\in H^{n+2q+2}\big(K(Z_2, q+1) \wedge T(\gamma); Z_2\big) , \qquad \big(\textit{see } (6.1) \textit{ and } (6.2')\big) .$$

To prove this proposition we need the following elementary lemmas, which generalize the standard property of the Euler class.

LEMMA 7.4. *If* $j: X \rightarrow (X, Y)$ *is the inclusion,* $x \in H^n(X, Y; Z_2)$, *then* $\mathrm{Sq}^n(x) = x \smile j^*x$.

PROOF. $\mathrm{Sq}^n x = x^2 = x \smile j^* x$, by the standard property of relative cup product.

LEMMA 7.5. *If* $g \in H^{q+1}(\bar{E}, E; Z_2)$ *is the generator, where* E *is the total space of the fibre space* $\pi: B_0\langle v_{q+1}\rangle \rightarrow B_0$, $\bar{E}$ *is the associated "disk bundle", then* $\mathrm{Sq}^{q+1}(g) = g \smile v_{q+1}$.

PROOF. By (7.4) $\mathrm{Sq}^{q+1} g = g \smile (j^* g)$ where $j: \bar{E} \rightarrow (\bar{E}, E)$. But j^*g is the lowest dimensional element in kernel π^*,

$$\pi^*: H^{q+1}(B_0; Z_2) \longrightarrow H^{q+1}\big(B_0\langle v_{q+1}\rangle; Z_2\big)$$

so $j^*g =$ "k-invariant" of $\pi = v_{q+1}$.

PROOF OF (7.3). By (6.1), (6.2′) X is the unique non-zero element in $(\text{kernel } h^*)^{n+2q+2}$, $h: T(\gamma)/T(\bar{\gamma}) \rightarrow K(Z_2, q+1) \wedge T(\gamma)$. Now $h^*(\iota \smile U) = g \smile U$ (where $H^*\big(T(\gamma)/T(\bar{\gamma}); Z_2\big)$ is considered as a left $H^*(\bar{E}, E; Z_2)$ module). Hence

$$h^*\big(\mathrm{Sq}^{q+1}(\iota \smile U)\big) = \mathrm{Sq}^{q+1}(g \smile U) = \sum_{i+j=q+1} (\mathrm{Sq}^i g) \smile (\mathrm{Sq}^j U)$$
$$= g \smile v_{q+1} \smile U + \sum_{\substack{i+j=q+1 \\ i<q+1}} (\mathrm{Sq}^i g) \smile (w_j \smile U) .$$

But $h^*(\iota \smile v_{q+1} \smile U) = g \smile v_{q+1} \smile U$, and $h^*\big(\mathrm{Sq}^i \iota \smile (w_j \smile U)\big) = \mathrm{Sq}^i g \smile (w_j \smile U)$ (since h^* is a map of $\mathcal{A}_2$-modules, and also of $H^*(B_0; Z_2)$ modules). Hence, it follows that $h^*(X) = 0$. But $X \neq 0$, since $\bar{j}'^*(X) = \Sigma^n(\mathrm{Sq}^{q+1}\iota) \neq 0$, where $\bar{j}': \Sigma^n K(Z_2, q+1) \rightarrow K(Z_2, q+1) \wedge T(\gamma)$ (since $\bar{j}'^*(I) = 0$ where I is the ideal generated by $\bar{H}^*(B_0; Z_2)$). Hence X is the first non-zero k-invariant of $T(\gamma)/T(\bar{\gamma})$, and Proposition 7.3 is proved.

Now if $i_s: S^{n+2q+1} \rightarrow T(\gamma)/T(\bar{\gamma})$ (from diagram (S)) represents a non-zero element in $\pi_{n+2q+1}\big(T(\gamma)/T(\bar{\gamma})\big)$ then it is detected by the functional 1st order cohomology operation defined by X. Then we shall show that the composite map

$$(\Sigma i)(\Sigma \alpha): S^{n+2q+1} \longrightarrow \Sigma T(\bar{\gamma})$$

will be detected by a functional 2nd order operation which X defines in a natural way, and we shall use the theory of Adams [1] of second order cohomology operations to study it.

To define a second order cohomology operation, we consider the composite map

$$T(\gamma) \xrightarrow{\delta'} T(\gamma)/T(\bar{\gamma}) \xrightarrow{h} K(Z_2, q+1) \wedge T(\gamma) .$$

Now $T(\gamma)$ is a product of $K(Z_2, t_i)$'s in the stable range, and by (6.2′), $K(Z_2, q+1) \wedge T(\gamma)$ is a product of $K(Z_2, t_i)$'s in these dimensions, so that the composite $d = h\delta'$ is a map of "generalized Eilenberg-MacLane spaces", and using the methods of Adams [1], is a universal model for certain 2^{nd} order cohomology operations. On the other hand $(h\delta')^*(X) = 0$ since $h^*(X) = 0$ so that, in the notation of [1], the pair (d, X) defines a stable 2^{nd} order cohomology operation Φ. Since $K(Z_2, q+1) \wedge T(\gamma)$ is a product of many $K(Z_2, t_i)$'s, Φ is an operation of many variables.

PROPOSITION 7.6. *The functional operation associated with Φ detects the element $\partial\bar{j}_*(\omega) \in \pi_{n+2q}(T(\bar{\gamma}))$, where ω generates $\pi_{n+2q+1}(\Sigma^{n+1}K(Z_2, q))$.*

PROOF. First we recall the construction of Φ from (d, X) following Adams [1].

The map d gives us a map of generalized Eilenberg-MacLane spaces which is realized geometrically in our context as

$$T(\gamma) \xrightarrow{\delta'} T(\gamma)/T(\bar{\gamma}) \xrightarrow{h} K(Z_2, q+1) \wedge T(\gamma) .$$

Then we let $p: U \to T(\gamma)$ be the induced fibre space from the path fibration over $K(Z_2, q+1) \wedge T(\gamma)$, so that the fibre $F = \Omega(K(Z_2, q+1) \wedge T(\gamma))$ is a generalized Eilenberg-MacLane space. Given a space Y, and a map $\alpha: Y \to T(\gamma)$ (which is essentially a collection of elements in $H^*(Y, Z_2)$) such that $(h\delta')\alpha \sim 0$ (which means that the 1^{st} order operation d is zero on this collection), we lift the map α to $\alpha_1: Y \to U$, such that $p\alpha_1 = \alpha$. The choice of maps α_1 depends on an element of $[Y, F]$.

Let $X' \in H^{n+2q+1}(U; Z_2)$ be such that $k^*(X') = \sigma^*(X)$, $k: F \to U$ is inclusion,

$$\sigma^*: H^{n+2q+2}(K(Z_2, q+1) \wedge T(\gamma); Z_2) \longrightarrow H^{n+2q+1}(F; Z_2)$$

the cohomology suspension. Then Φ is defined as $\alpha_1^*(X')$ and the ambiguity of Φ depends on the choice of α_1 and X' with the above properties.

We show first that for $Y = T(\bar{\gamma})$, $\alpha: Y \to T(\gamma)$ the natural map $\alpha = \bar{\rho}$ induced from the bundle map ρ, we can choose α_1, X' so that $\Phi = 0$.

Since we are dealing with stable operations we consider the suspended situation (where we have made obvious identifications which are valid in the dimension range of our interest).

$$\begin{array}{ccccccc}
\Sigma^{n+1}K(Z_2, q) & & & & & & \\
\downarrow{\scriptstyle \bar{j}} & & & & & & \\
T(\gamma)/T(\bar{\gamma}) & \xrightarrow{\eta} & \Sigma T(\bar{\gamma}) & \xrightarrow{\bar{\rho}'} & \Sigma T(\gamma) & \xrightarrow{\Sigma\delta'} & \Sigma\big(T(\gamma)/T(\bar{\gamma})\big) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle h'} \\
K(Z_2, q+1) \wedge T(\gamma) & \xrightarrow{k'} & U' & \xrightarrow{\rho'} & \Sigma T(\gamma) & \xrightarrow{\Sigma d} & K(Z_2, q+1) \wedge \Sigma T(\gamma)
\end{array}$$

Then α_2 is defined as the map of "fibre spaces" induced by the map h', so that h is the map on fibres. Such interpretations are valid in the stable range; here $U = \Omega U'$, $\alpha_1 = \Omega\alpha_2$. Now $k'^*(X') = X$, and $h^*(X) = 0$ so by exactness, $\alpha_2^*(X') \in \text{image } \bar{\rho}'^*$. Hence $X' \in H^*(U'; Z_2)$ may be changed by an element in image ρ'^*, $X'' = X' + \rho'^*(a)$ so that $\alpha_2^* X'' = 0$.

On the other hand, since the functional operation associated with X detects $\bar{j}_*(\omega)$, it follows from naturality that the functional operation associated with X'' detects $\eta_*\bar{j}_*(\omega) = \Sigma\big(\partial\bar{j}_*(\omega)\big)$, which completes the proof of (7.6).

Let $i: S^n \to T(\bar{\gamma})$ be the inclusion of a fibre. Then from Proposition (7.2), and (7.6), it follows that $i_*\pi_{n+2q}(S^n) \neq 0$ if and only if the functional operation Φ (using the map $S^n \xrightarrow{i} T(\bar{\gamma}) \xrightarrow{\bar{\rho}} T(\gamma)$ to define it) detects an element in $\pi_{n+2q}(S^n)$. Such a question can be settled in many cases using the techniques of Adams [1] and]2].

Now to detect an element of $\pi_{n+2q}(S^n)$ the operation Φ reduces to an operation of one variable Ψ of dimension n. Adams [1] has described such 2nd order operations in terms of $\text{Ext}^2_{\mathfrak{A}_2}(Z_2, Z_2)$, which is the E_2 term in the Adams spectral sequence. It follows from Adams work that Ψ corresponds to some relation in the Steenrod algebra $\mathfrak{A}_2$, and we will try to identify it.

Now the element $X \in H^{n+2q+2}\big(K(Z_2, q+1) \wedge T(\gamma); Z_2\big)$ can be written in the form $\sum a_i c_i$ where $a_i \in \mathfrak{A}_2$ and c_i are part of an $\mathfrak{A}_2$-base for the free $\mathfrak{A}_2$-module $H^*\big(K(Z_2, q+1) \wedge T(\gamma); Z_2\big)$. We consider the map

$$d = h\delta' : T(\gamma) \longrightarrow K(Z_2, q+1) \wedge T(\gamma)$$

and a map

$$\xi : K(Z_2, n) \longrightarrow T(\gamma)$$

where $\xi^*(U) = \iota \in H^n\big(K(Z_2, n); Z_2\big)$ is the generator, and Ψ is defined on

$$g = h\delta'\xi : K(Z_2, n) \longrightarrow K(Z_2, q+1) \wedge T(\gamma) .$$

Then $g^*(c_i) = b_i(\iota)$, $b_i \in \mathfrak{A}_2$, and the relation defining Ψ is $\sum a_i b_i = 0$.

Since $K(Z_2, q+1) \wedge T(\gamma)$ is $(q+n)$-connected, it follows that dimension $c_i \geqq q + 1 + n$ for all i. We recall some theorems of Adams (see [1, § 2.1, § 3.8 and Th. 2.5.1]).

THEOREM 7.7 (Adams). *A basis over $\mathcal{A}_2$ for stable 2^{nd} order cohomology operations of one variable is in natural one-to-one correspondence with a basis (over Z_2) of* $\text{Ext}^{2,*}_{\mathcal{A}_2}(Z_2, Z_2)$. *Further there exist elements* $h_i \in \text{Ext}^{1,2^i}_{\mathcal{A}_2}(Z_2, Z_2) =$ *Primitive elements of $\mathcal{A}_2^*$ in dimension 2^i, so that these $\{h_i\}$ are a basis for* $\text{Ext}^{1,*}_{\mathcal{A}_2}(Z_2, Z_2)$ *and so that the products $h_j h_i$, $j \geqq i \geqq 0$, $j \neq i + 1$, form a basis for* $\text{Ext}^{2,*}_{\mathcal{A}_2}(Z_2, Z_2)$.

The following theorem is also contained in the work of Adams (see [1, in particular Lemma 2.2] and [2]).

THEOREM 7.8. *Let Ψ be a stable 2^{nd} order cohomology operation based on the relation in the Steenrod algebra $\sum_i a_i b_i = 0$, $a_i, b_i \in \mathcal{A}_2$. Then Ψ detects an element in $\pi_*(S^n)$ if and only if $(h_j h_k)(\sum a_i b_i) = \sum_i h_j(a_i) h_k(b_i) \neq 0$ for some j,k and $(h_j h_k)$ is a permanent cycle in the Adams spectral sequence.*

Here $h_i \in \text{Ext}^{1,2^i}_{\mathcal{A}_2}(Z_2, Z_2)$ is considered as an element of $\text{Hom}_{Z_2}(\mathcal{A}^{2^i}, Z_2)$, which annihilates decomposable elements, and $h_i(\text{Sq}^{2^i}) \neq 0$.

Now our operation Ψ is based on the relation $\sum a_i b_i = 0$, where dimension $b_i \geqq q + 1$, dimension $a_i b_i = 2q + 2$. Hence for dimension reasons,

$$(h_j h_k)\left(\sum a_i b_i\right) = \sum h_j(a_i) h_k(b_i) = 0 \qquad \text{if } j > k .$$

Hence the only element in $\text{Ext}^{2,*}_{\mathcal{A}_2}(Z_2, Z_2)$ which may have a non-zero value on $\sum a_i b_i$ is h_k^2, so that Ψ cannot detect an element in $\pi_{n+2q}(S^n)$ if $q + 1 \neq 2^k$. Hence there is no framed manifold M^{2q} of Kervaire invariant 1 if $q \neq 2^k - 1$, which proves part of Theorem 7.1.

Now suppose $q = 2^k - 1$. Consider

$$X = \text{Sq}^{q+1}(\iota \smile U) + \sum_{\substack{i+j=q+1 \\ 0 \leqq i \leqq q}} (\text{Sq}^i \iota) \smile (w_j U) + \iota \smile v_{q+1} \smile U ,$$

the k-invariant of $T(\gamma)/T(\bar{\gamma})$. Since $q + 1 = 2^k$, Sq^{q+1} is indecomposable, and it follows that $\chi(\text{Sq}^{q+1}) = \text{Sq}^{q+1} + D$, where $D \in$ subalgebra of $\mathcal{A}$ generated by Sq^{2^i}, $i < k$. Hence $v_{q+1} = w_{q+1} + e$, where $e \in$ subalgebra of $H^*(B_0; Z_2)$ generated by w_i, $i < q + 1$, (see [25]). Then $X = \text{Sq}^{q+1}(\iota \smile U) + \sum_{0 \leqq i \leqq q} \zeta_i \smile x_i \smile U$ where $\zeta_i \in H^{i+q+1}(K(Z_2, q + 1); Z_2)$, and $x_i \in H^{q+1-i}(B_0; Z_2)$.

Now $\iota \smile U$ goes to $\chi(\text{Sq}^{q+1})U$ in $H^*(T(\gamma); Z_2)$, so that $b_1 = \chi(\text{Sq}^{q+1})$ and since $q = 2^k - 1$, $h_k(\chi(\text{Sq}^{q+1})) = 1$. Also $a_1 = \text{Sq}^{q+1}$, so $(h_k^2)(a_1 b_1) = 1$.

Let us write $\sum \zeta_j \smile x_j \smile U = \sum_{i \geqq 2} a_i c_i$, $a_i \in \mathcal{A}$, c_i are generators of $H^*(K(Z_2, q + 1) \wedge T(\gamma); Z_2)$ over $\mathcal{A}$, so that dimension $c_i \geqq q + 1$. It follows that if dimension $c_2 = q + 1$ and dimension $c_i > q + 1$ for $i > 2$, then $(h_k)^2(\sum a_i c_i) = h_k(a_2)$. Now $a_2 \in \mathcal{A}^{q+1}$, so $(h_k^2)(\sum a_i c_i) \neq 0$ if and only if $a_2 = \text{Sq}^{q+1} + D$ where D is a decomposable element in $\mathcal{A}_2$. But $c_2 = \iota \smile U$, so that would mean that $a_2 c_2 = \text{Sq}^{q+1}(\iota \smile U) + D(\iota \smile U)$ and hence $\bar{j}'^*(a_2 c_2) = \text{Sq}^{q+1}(\iota_{n+q+1}) + D(\iota_{n+q+1})$ where

$$\bar{j}'\colon \Sigma^n(K(Z_2, q+1)) \longrightarrow K(Z_2, q+1) \wedge T(\gamma)\,.$$

Now since dimension $c_i > q+1$, for $i > 2$, it follows that $\bar{j}'^*(c_i) = e_i \Sigma^n \iota_{q+1}$ where $e_i \in \mathcal{A}$, so that $\bar{j}'^*(a_i c_i) = a_i e_i \Sigma^n \iota_{q+1}$. Hence $\bar{j}'^*(a_i c_i) = S_i \Sigma^n \iota_{q+1}$ where S_i is a decomposable element of $\mathcal{A}$ when $i > 2$, while $\bar{j}'^*(a_2 c_2) = \mathrm{Sq}^{q+1}\Sigma^n \iota_{q+1} + D\Sigma^n \iota_{q+1}$ and Sq^{q+1} is indecomposable since $q+1 = 2^k$. Hence $\bar{j}'^*(\sum_{i\geqq 2} a_i c_i) \neq 0$. But $\sum_{i\geqq 2} a_i c_i = \sum \zeta_j \smile x_j \smile U$ where $\zeta_j \in H^*(K(Z_2, q+1); Z_2)$, $x_j \in \bar{H}^*(B_0; Z_2)$. Now $\bar{j}'^*(\zeta \smile x \smile U) = 0$ if dimension $x > 0$, $x \in H^*(B_0; Z_2)$, since $\bar{j}'$ is induced by the inclusion map $K(Z_2, q+1) \to K(Z_2, q+1) \times B_0$, and $\zeta \smile x \smile U = \zeta \smile p^*(x) \smile U$ where $p\colon K(Z_2, q+1) \times B_0 \to B_0$ is the projection on the second factor. Hence $\bar{j}'^*(\sum \zeta_i \smile x_i \smile U) = 0$, and hence $h_k^2(\sum \zeta_i \smile x_i \smile U) = 0$. Therefore, $h_k^2(X) = h_k^2(\mathrm{Sq}^{q+1}(\iota \smile U)) \neq 0$, and it follows that the image of $\Omega_{2q}^{\mathrm{framed}}$ in $B_0\langle v_{q+1}\rangle$-cobordism is non-zero if and only if $q = 2^k - 1$ and h_k^2 persists to E^∞ in the Adams spectral sequence. Then the remainder of Theorem 7.1 follows from Proposition 7.2 and Theorem 5.2.

Appendix

"Disk" bundles, etc. In this appendix we describe some constructions of fibre spaces which we used in §§ 6 and 7. One could use different constructions, analogous to those of linear bundle theory, but it is harder to verify the necessary facts with this approach. We refer to [20] for the constructions used here.

Let $f\colon A \to B$ be a map. Using the mapping cylinder construction we may replace f by an equivalent map $g\colon A \to C$ so that g is an inclusion with the homotopy extension property.

Consider the function space C^I, and the two projections $r, l\colon C^I \to C$, $r(\alpha) = \alpha(1)$, $l(\alpha) = \alpha(0)$, for $\alpha \in C^I$. Both r and l are fibre maps, where the homotopy lifting is functorial, using the path defined by the homotopy in C to define the homotopy in the path space. The fibres of r and l are just the path space P of C beginning at a base point, so are contractible.

Let $r'\colon C' \to A$ be the induced fibre space over A from $r\colon C^I \to C$, so $C' = \{(x, \alpha) \in A \times C^I$ such that $x = r(\alpha) = \alpha(1)\}$. Then $i\colon A \to C'$ is defined by $i(x) = (x, \alpha_x)$ where $\alpha_x(t) = x$ all t. Then r' and i are inverse homotopy equivalences. We have a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\ \rho\ } & C^I \\ \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} \\ A & \xrightarrow{\ g\ } & C \end{array}$$

where $\rho(x, \alpha) = \alpha$.

Define $g'\colon C' \to C$ by $g' = l\rho$. Clearly l is homotopic to r so g' is homotopic to gr', and further g' is a fibre map, using the functorial lifting of homotopies.

If $f\colon A \to B$ is a fibre map it is routine to show that there is a homotopy equivalence of fibre spaces

$$\begin{array}{ccc} A & \longrightarrow & C' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g'} \\ B & \longrightarrow & C \end{array}$$

so that the new situation is equivalent to the original, for homotopy theory. So we replace f by g'.

Now $C' \subset C^I$, and the fibre of $l\colon C^I \to C$ is contractible, being the based path space of C. Hence we shall call $l\colon C^I \to C$ the "disk bundle" or "cone bundle" of g' (or f) and its fibre is a contractible space containing the fibre of g'. Further, the pair (C^I, C') satisfies the homotopy extension property, since the pair (C, A) does, and (C^I, C') is a fibre pair (under r) over (C, A).

Then we may define the "Whitney sum of $f\colon A \to B$ with a trivial S^0-bundle" as follows. Take two copies of C^I, call them C^I_+ and C^I_- and identify C'_+ and C'_- to get $\hat{C}$. Since the covering homotopies for C^I_+, C^I_- and C' are all given functorially by the same formula, the covering homotopy theorem works for $\hat{C} = C^I_+ \cup C^I_-$. Clearly the fibre of $g\colon \hat{C} \to C$ is the union of the fibres, so it is $P_+ \cup P_-$ with $P_+ \cap P_- =$ fibre of g'. Since P_+ and P_- are contractible, this is homotopy equivalent to the suspension of the fibre of g'.

We will use the following (abuse of) notation.

If $\pi\colon E \to B$ with fibre F, $\bar{\pi}\colon \bar{E} \to B$ will be the "disk bundle" of π with fibre cF, $\hat{\pi}\colon \hat{E} \to B$ will be the sum of π with a trivial S^0-bundle, with fibre ΣF and $\bar{E}/E = \hat{E}/E_-$ is the Thom complex of π.

PRINCETON UNIVERSITY

BIBLIOGRAPHY

[1] J. F. ADAMS, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. **72** (1960), 20-104.
[2] ———, *On the structure and applications of the Steenrod algebra*, Comment. Math. Helv. **32** (1958), 180-214.
[3] C. ARF, *Untersuchungen über quadratische Formen in Körpern der Charakteristik* 2, J. Reine Angew. Math. **183** (1941), 148-167.
[4] M. ATIYAH, *Thom complexes*, Proc. L.M.S. **11** (1961), 291-310.
[5] W. BROWDER, "Homotopy type of differentiable manifolds," in Colloquium of Algebraic Topology, Aarhus, 1962, p. 42-46.
[6] ———, Surgery on simply connected manifolds.
[7] E. H. BROWN, *Note on an invariant of Kervaire*, Michigan Math. J. **12** (1965), 23-24.

[8] ——— and F. P. PETERSON, *The Kervaire invariant of* $(8k+2)$*-manifolds*, Bull. Amer. Math. Soc. **71** (1965), 190-193.

[9] A. HAEFLIGER, *Knotted* $(4k-1)$*-spheres in* $6k$*-space*, Ann. of Math. **75** (1962), 452-466.

[10] P. HILTON, Homotopy Theory and Duality, New York, 1965.

[11] M. KERVAIRE, *A manifold which does not admit any differentiable structure*, Comment. Math. Helv. **34** (1960), 257-270.

[12] ——— and J. MILNOR, *Groups of homotopy spheres*: I, Ann. of Math. **77** (1963), 504-537.

[13] J. MILNOR, *Spin structures on manifolds*, L'Enseignement Math. **9** (1963), 198-203.

[14] ——— and J. C. MOORE, *On the structure of Hopf algebras*, Ann. of Math. **81** (1965), 211-264.

[15] S. P. NOVIKOV, *Homotopy equivalent smooth manifolds*: I, (in Russian), Izv. Akad. Nauk SSSR, Ser. Mat. **28** (1964), 365-474.

[16] F. P. PETERSON, *Some non-embedding problems*, Bol. Soc. Math. Mexicana, (1957), 9-15.

[17] L. PONTRJAGIN, Smooth manifolds and their applications in homotopy theory, Amer. Math. Soc. Translations, Ser 2, II (1959), 1-114.

[18] D. PUPPE, *Homotopiemengen und Ihre Induzierten Abbildungen*: I, Math. Z. **69** (1958), 299-344.

[19] J-P. SERRE, *Cohomologie modulo* 2 *des complexes d'Eilenberg-MacLane*, Comment. Math. Helv. **27** (1953), 198-232.

[20] ———, *Groupes d'homotopie et classes de groupes abeliens*, Ann. of Math. **58** (1953), 258-294.

[21] M. SPIVAK, *Spaces satisfying Poincaré duality*, Topology **6** (1967), 77-102.

[22] N. STEENROD, *Cohomology invariants of mappings*, Ann. of Math. **50** (1949), 954-988.

[23] ——— and D. B. A. EPSTEIN, Cohomology operations, Annals of Math. Study No. 50, Princeton, N.J., 1962.

[24] H. TODA, Composition methods in homotopy groups of spheres, Annals of Math. Study No. 49, Princeton, N.J., 1962.

[25] W. T. WU, *Les i-carrés dans une variété grassmannienne*, C.R. Acad. Sci. Paris, **230** (1950), 918-920.

(Received August 18, 1967)
(Revised January 30, 1969)