Chapter I. Equivariant Classical Cohomology

1. G-complexes

Let G be a finite group. By a G-complex we mean a CW complex K together with a given action of G on K by cellular maps such that

(*) For each $g \in G$, $\{x \in K | g(x) = x\}$ is a subcomplex of K.

Note that for each $g \in G$, the fact that $g: K \to K$ and $g^{-1}: K \to K$ are assumed to be cellular implies that, in fact, each $g: K \to K$ in an automorphism of the given CW structure of K. Also it follows from the condition (*) that if $g \in G$ leaves any point $x \in K$ fixed then g must leave K(x) pointwise fixed. (K(A), for any subset $A \subset K$, denotes the smallest subcomplex of K containing A. It is a finite subcomplex iff A has compact closure.)

Let K be a G-complex and L a subcomplex invariant under G. Then an easy inductive argument on the skeletons of K shows that K has the equivariant homotopy extension property with respect to L. That is, if $f: K \to X$ is an equivariant map into any space X with a given G-action and if $F': L \times I \to X$ is any equivariant homotopy then there exists an equivariant homotopy $F: K \times I \to X$ extending F'.

Taking the case in which $X = L \times I \cup K \times \{0\}$ with f and F' the obvious maps we obtain the fact that $L \times I \cup K \times \{0\}$ is an equivariant retract of $K \times I$, the retraction being $F: K \times I \to L \times I \cup K \times \{0\}$. Let $B \subset X$ be the set of points x such that $F(x,1) \in L \times I$. Then B is a neighborhood of L in K and the composition

$$B\times I \xrightarrow{F} L\times I \cup K\times\{0\} \to K$$

is an equivariant strong deformation retraction of B onto L.

Now apply these facts to the G-complex $K\times I$ and the subcomplex $A = L\times I \cup K\times\{0\}$. Let U be a neighborhood of A possessing an equivariant strong deformation retraction onto A. Let $f\colon K \to I$ be a continuous function such that $f(x) = 0$ on some neighborhood of L and $f(x) = 1$ unless $x\times I \subset U$. By taking $x \to \inf\{f(g(x)) \mid g \in G\}$ we can assume that $f(g(x)) = f(x)$ for all $g \in G$. Define

$$F_t\colon K\times I \to K\times I$$

by $F_t(x,s) = (x, s(1-tf(x)))$. This forms a deformation of $K\times I$ into U which is equivariant and leaves A stationary. Following this by the deformation of U into A we see that $A = L\times I \cup K\times\{0\}$ is an equivariant strong deformation retract of $K\times I$.

Now identify $L\times\{1\}$ to a point, so that $K\times I$ becomes the mapping cylinder $M = K\times I/L\times\{1\}$ of the collapsing map $K \to K/L$. Now our deformation becomes a deformation retraction of M onto $K\times\{0\} \cup L\times I/L\times\{1\} \approx K \cup C_L$ (K with the cone C_L on L attached). On the other hand M can be deformed equivariantly into the face $K\times\{1\}/L\times\{1\} \approx K/L$. This shows that for any pair (K,L) of G-complexes, the G-complex K/L is of the same equivariant homotopy type as $K \cup C_L$.

Let us recall a construction central to the cohomology theory of CW complexes. Let K be a CW complex and pick an orientation for each cell of K. (If K is a G-complex it may be assumed that the operations of G preserve these orientations, because of (*), but this is not important.) Let $C_n(K)$ be the free abelian group generated by the n-cells of K. $C_n(K)$ is isomorphic to the singular homology group $H_n(K^n/K^{n-1};Z)$, or to $H_n(K^n,K^{n-1};Z)$.

Suppose that σ is an n-cell of K and let $f_\sigma\colon S^{n-1} \to K^{n-1}$ be a characteristic (attaching) map for σ. Collapsing K^{n-2} to a point, we obtain an induced map

$$(1.1) \qquad S^{n-1} \to K^{n-1} \to K^{n-1}/K^{n-2} = \bigvee \tau/\dot{\tau}$$

where τ ranges over the (n-1)-cells of K ($\tau/\dot{\tau}$ is an oriented (n-1)-sphere and $\bigvee$ denotes the one point union). For each τ there is a projection $\bigvee\tau/\dot{\tau} \to \tau/\dot{\tau}$ (collapsing all other spheres). Let f_σ^τ denote the composed map

$$f_\sigma^\tau\colon S^{n-1} \to \tau/\dot{\tau}$$

The map (1.1) provides a singular homology class

$$\partial\sigma \in C_{n-1}(K) = H_{n-1}(K^{n-1}/K^{n-2})$$

and we clearly have that

$$\partial\sigma = \textstyle\sum_\tau [\tau\colon \sigma]\tau$$

where $[\tau\colon \sigma] = 0$ unless τ is an (n-1)-cell and, for an (n-1)-cell τ in K,

$$[\tau\colon \sigma] = \deg f_\sigma^\tau\colon S^{n-1} \to \tau/\dot{\tau}$$

(for fixed σ this is non-zero for only a finite number of cells τ, in fact f_σ^τ is a trivial map except for a finite number of cells τ). The correspondence $\sigma \to \partial\sigma$ generates a homomorphism

$$\partial\colon C_n(K) \to C_{n-1}(K)$$

which, in fact, is just the singular homology connecting homomorphism of the triple K^n, K^{n-1}, K^{n-2}. That is, ∂ is equivalent to the composition

$$H_n(K^n,K^{n-1}) \xrightarrow{\partial_*} H_{n-1}(K^{n-1}) \xrightarrow{j_*} H_{n-1}(K^{n-1},K^{n-2}).$$

We have that $\partial^2 = 0$ since the composition

$$H_{n-1}(K^{n-1}) \xrightarrow{j_*} H_{n-1}(K^{n-1},K^{n-2}) \xrightarrow{\partial_*} H_{n-2}(K^{n-2})$$

(part of the homology sequence of the pair (K^{n-1},K^{n-2})) is zero. Note that $\partial^2 = 0$ is equivalent to the equation

$$\sum_{\tau}[\omega:\tau][\tau:\sigma] = 0 \text{ for given } \omega,\sigma.$$

2. Equivariant cohomology theories

Let G be a finite group and let $\mathcal{G}$ denote the category of G-complexes and (continuous) equivariant maps. Let $\mathcal{G}_0$ denote the category of G-complexes with base point and base point preserving equivariant maps (base points are always assumed to be left fixed by each element of G and, in the case of G-complexes, to be a vertex). Let $\mathcal{G}^2$ be the category of pairs (K,L), $L \subset K$ a subcomplex, of G-complexes.

We use the abbreviation "Abel" to stand for the category of abelian groups.

An equivariant (generalized) cohomology theory on the category $\mathcal{G}$ is a sequence of contravariant functors

$$\mathcal{H}^n: \mathcal{G}^2 \to \text{Abel} \qquad (n \in Z)$$

together with natural transformations

$$\delta^n: \mathcal{H}^n(L,\emptyset) \to \mathcal{H}^{n+1}(K,L),$$

such that the following three axioms are satisfied (we put $\mathcal{H}^n(L) = \mathcal{H}^n(L,\emptyset)$):

(1) If f_0, f_1 are equivariantly homotopic maps (in $\mathcal{G}^2$) then $\mathcal{H}^n(f_0) = \mathcal{H}^n(f_1)$.

(2) The inclusion $(K, K\cap L) \subset (K\cup L, L)$ induces an isomorphism

$$\mathcal{H}^n(K\cup L, L) \xrightarrow{\approx} \mathcal{H}^n(K, K\cap L)$$

(3) If $(K,L) \in \mathcal{G}^2$ then the sequence

$$\cdots \to \mathcal{H}^n(K,L) \xrightarrow{j^*} \mathcal{H}^n(K) \xrightarrow{i^*} \mathcal{H}^n(L) \xrightarrow{\delta^*} \mathcal{H}^{n+1}(K,L) \to \cdots$$

is exact.

Remark. If G is abelian then the operations by elements of G are morphisms $\mathcal{G} \to \mathcal{G}$ (i.e. they are equivariant). Thus, in this case, each $\mathcal{H}^n(K,L)$ has a natural G-module structure.

There are functors $\mathcal{G}^2 \to \mathcal{G}_0$ and $\mathcal{G}_0 \to \mathcal{G}^2$ defined by $(K,L) \to K/L$ and $K \to (K,x_0)$ where x_0 is the base point of K, L/L is the base point of K/L (taken to be a disjoint point if $L = \emptyset$, in which case K^+ denotes $K/\emptyset$). Standard arguments can be used to translate the above axioms into an equivalent set of axioms for a "single space" theory on $\mathcal{G}_0$. (See, for example, G. W. Whitehead, Generalized homology theories, Trans. A. M. S. 102 (1962), pp. 227-283.)

In fact for $K \in \mathcal{G}_0$ let $SK = S\wedge K$ (with the obvious G action, trivial on the "circle factor" S) denote the reduced suspension of X. Then an equivariant cohomology theory on $\mathcal{G}_0$ is a sequence of contravariant functors

$$\tilde{\mathcal{H}}^n\colon \mathcal{G}_0 \to \text{Abel}$$

together with a sequence of natural transformations of functors σ^n

$$\sigma^n(K)\colon \tilde{\mathcal{H}}^n(K) \to \tilde{\mathcal{H}}^{n+1}(SK)$$

satisfying the following three axioms

(1') If f_0, f_1 are equivariantly homotopic (in $\mathcal{G}_0$) then $\tilde{\mathcal{H}}^n(f_0) = \tilde{\mathcal{H}}^n(f_1)$.

(2') $\sigma^n(K)$ is an isomorphism for each n and K.

(3') The sequence

$$\tilde{\mathcal{H}}^n(K/L) \to \tilde{\mathcal{H}}^n(K) \to \tilde{\mathcal{H}}^n(L)$$

is exact.

Most of the material of Chapter I of Eilenberg-Steenrod goes over directly to these generalized theories. Later on in these notes we shall show how to construct such theories using rather standard methods and shall consider some special cases of interest. We shall not concern ourselves with these matters at present, but shall confine ourselves to a discussion of "coefficient groups".

In non-equivariant theories the "coefficients" of the theory are defined to be $\mathcal{H}^*(pt)$ (or $\tilde{\mathcal{H}}^*(pt^+)$) and these (graded) groups are the primary distinguishing feature between different cohomology theories. In fact for (non-equivariant) "classical" theory (= cohomology theory + dimension axiom) the knowledge of the coefficient group ($\mathcal{H}^0(pt)$ in this case) allows computation of the cohomology of any finite simplicial complex. Essentially this is true because homotopy points (i.e. contractible objects such as simplexes) form the basic building blocks of all complexes.

For equivariant theory the situation is slightly more complicated, for now the "building blocks" are essentially the orbits (in an appropriate sense) of G. That is, the coset spaces G/H, where H ranges over the subgroups of G (not necessarily normal), form a representative set of building blocks.

Thus a "coefficient system" should contain all the groups $\mathcal{H}^*(G/H)$ (or $\tilde{\mathcal{H}}^*((G/H)^+)$)). But this is not enough, for we must specify how the building blocks "fit together". That is, we must consider the equivariant maps $G/H \to G/K$ and a "coefficient system" must incorporate the induced homomorphisms

$$\mathcal{H}^*(G/K) \to \mathcal{H}^*(G/H)$$

in its structure.

In the following sections we define precisely what we mean by a coefficient system.

Terminology: A cohomology theory on $\mathcal{G}$ or $\mathcal{G}_0$ will be called "classical" (= "equivariant classical cohomology" but ≠ "classical equivariant cohomology" as defined, for example, in Steenrod and Epstein, Cohomology Operations) if it satisfies the additional "dimension" axiom:

(4) $\mathcal{H}^n(G/H) = 0$ for $n \neq 0$ and all H,

or, for a single space theory,

(4') $\tilde{\mathcal{H}}^n((G/H)^+) = 0$ for $n \neq 0$ and all H.

Later on, we shall prove existence and uniqueness theorems (of the Eilenberg-Steenrod type) for such "classical" theories.

3. The category of canonical orbits.

The category of canonical orbits of G, denoted by $\mathcal{O}_G$, is defined to be the category whose objects are the left coset spaces G/H and whose morphisms are the equivariant (with respect to left translation) maps $G/H \to G/K$.

For future reference we shall classify the equivariant maps $G/H \to G/K$. Suppose f is any map

$$f: G/H \to G/K$$

and put

$$f(H) = aK \quad \text{where } a \in G.$$

Then f is equivariant iff $f(gH) = gaK$ for all $g \in G$. Conversely, the formula $f(gH) = gaK$ defines a map (which must be equivariant) provided that

$$f(ghH) = f(gH)$$

for all $h \in H$. That is, we must have $ghaK = gaK$ for all $h \in H$. This is equivalent to $haK = aK$ and hence to

(3.1) $$a^{-1}Ha \subset K.$$

Thus we have the following result: Let $a \in G$ be such that $a^{-1}Ha \subset K$. Define

$$\hat{a}: G/H \to G/K$$

by

$$\hat{a}(gH) = gaK.$$

Then $\hat{a}$ is equivariant, that is, $\hat{a} \in \hom(G/H, G/K)$ and every equivariant map has this form. Also, clearly, $\hat{a} = \hat{b}$ iff $aK = bK$, that is, iff $a^{-1}b \in K$.

Suppose that (3.1) is satisfied. Then the inclusion $a^{-1}Ha \subset K$ induces a natural projection $G/a^{-1}Ha \to G/K$ (equivariant) and, similarly, the inclusion $H \subset aKa^{-1}$ induces $G/H \to G/aKa^{-1}$. Now <u>right</u> translation by a induces an equivariant map $R_a: G/H \to G/a^{-1}Ha$ (given by $gH \to gHa = ga(a^{-1}Ha)$) and also $R_a: G/aKa^{-1} \to G/K$. Clearly the diagram

(3.2)
$$\begin{array}{ccc} G/H & \longrightarrow & G/aKa^{-1} \\ {\scriptstyle R_a}\downarrow & \searrow^{\hat{a}} & \downarrow{\scriptstyle R_a} \\ G/a^{-1}Ha & \longrightarrow & G/K \end{array}$$

commutes. Thus equivariant maps are precisely those maps induced by inclusions of subgroups and by right translations.

In particular hom(G/H,G/H) consists of the right translations by elements of the normalizer N(H) of H (i.e. $a \in N(H)$ yields $gH \to gHa = gaH$). Since $R_a R_b = R_{ba}$, and generally $\hat{a}\hat{b} = \widehat{ba}$, the correspondence $a \to R_a^{-1}$ yields an isomorphism

(3.3) $$N(H)/H \approx \hom(G/H,G/H).$$

For example, let $G = Z_p$, where p is prime. Then $\mathcal{O}_G$ consists of the objects G/G and G/{e} (that is essentially of a point P and of G) together with the following morphisms

$$P \to P$$

$$G \to P$$

$$\hat{a}: G \to G \quad \text{for each } a \in G$$

(where here $\hat{a} = R_a$ takes g into ga).

4. <u>Generic coefficient systems</u>

(4.1) <u>Definition</u>. <u>A (generic) coefficient system</u> (<u>for</u> G) <u>is defined to be a contravariant functor</u> $\mathcal{O}_G \to$ Abel.

If M,N: $\mathcal{O}_G \to$ Abel are coefficient systems, a <u>morphism</u> T: M → N is a natural transformation of functors. With this definition, the (generic) coefficient systems for G form an abelian category $\mathcal{C}_G = \text{Dgram}(\mathcal{O}_G^*, \text{Abel})$. ($\mathcal{O}_G^*$ denotes the dual category to $\mathcal{O}_G$ and the fact that $\mathcal{C}_G$ is an abelian category is a special case of a result of Grothendieck; see Maclane, Homology, IX, 3.1, p. 258.)

Examples:

(1) Let $\mathcal{H}$ be an equivariant cohomology theory and let q be an integer. Define

$$h^q: \mathcal{O}_G \to \text{Abel}$$

by $h^q(G/H) = \mathcal{H}^q(G/H)$ and if $f: G/H \to G/K$ is equivariant, let $h^q(f) = \mathcal{H}^q(f): \mathcal{H}^q(G/K) \to \mathcal{H}^q(G/H)$.

(2) Let A be a G-module. Define

$$M: \mathcal{O}_G \to \text{Abel}$$

as follows: Let $M(G/H) = A^H$ (the set of stationary points of H in A). For $g \in G$ with $H \subset gKg^{-1}$ note that the operation by $g: A \to A$ takes A^K into A^H, (for $a \in A^K$ implies that $Hga \subset gKg^{-1}ga = gKa = ga$). Denote this map $A^K \to A^H$ by $g_{H,K}$. If $\hat{g} = \widehat{g'}$ so that $g^{-1}g' \in K$, then clearly $g_{H,K} = g'_{H,K}$. Thus, for $\hat{g}: G/H \to G/K$ we let

$$M(\hat{g}) = g_{H,K}: A^K \to A^H.$$

(3) Let Y be a G-space with a base point y_0. Define $\tilde{\omega}_q(Y) \in \mathcal{C}_G$, that is $\tilde{\omega}_q(Y): \mathcal{O}_G \to \text{Abel}$, as follows:

$$\tilde{\omega}_q(Y)(G/H) = \pi_q(Y^H, y_0)$$

$$\tilde{\omega}_q(Y)(\hat{g}) = g_\#: \pi_q(Y^K, y_0) \to \pi_q(Y^H, y_0)$$

where $g \in G$ satisfies $H \subset gKg^{-1}$, so that g maps $Y^K \to Y^H$ (see example 2). (In this example we assume each $\pi_1(Y^H, y_0)$ to be abelian when $q = 1$.)

<u>Remark</u>. Since hom(G/H,G/H) $\approx$ N(H)/H we have that, for any coefficient system $M \in \mathcal{C}_G$, M(G/H) possess a natural N(H)/H-module structure.

Let $M \in \mathcal{C}_G$. Since $\mathcal{O}_G$ contains, in particular, the objects G = G/{e} and P = G/G with the morphisms

$$1: P \to P$$

$$r: G \to P$$

$$\hat{a}: G \to G$$

we have that M "contains" the abelian groups M(P) and M(G) with the homomorphisms M(1) = 1 and

$$\varepsilon = M(r): M(P) \to M(G)$$

$$a_* = M(\hat{a}): M(G) \to M(G)$$

which satisfy $M(\widehat{ab}) = M(\hat{b}\hat{a}) = M(\hat{a})M(\hat{b})$ and $M(\hat{a})M(r) = M(r\hat{a}) = M(r)$; that is,

$$\begin{cases} (ab)_* = a_* b_* \\ a_*\varepsilon = \varepsilon . \end{cases}$$

Thus we may consider M(G) to have a G-module structure defined by $(a,m) \to a_*(m)$ and M(P) to have a <u>trivial</u> G-module structure and $\varepsilon : M(P) \to M(G)$ to be an equivariant homomorphism (i.e. $\varepsilon: M(P) \to M(G)^G$).

Of course, if $G = Z_p$ where p is prime, then this is all of the structure of an $M \in \mathcal{C}_G$. That is, in this case, a coefficient system consists of an abelian group M_0, an abelian group M_1 with a G-module structure and an homomorphism $\varepsilon : M_0 \to M_1^G$. Moreover, a morphism between two such systems M and M' is a commutative diagram of G-module homomorphisms:

$$\begin{array}{ccc} M_0 & \xrightarrow{\varepsilon} & M_1 \\ \downarrow & & \downarrow \\ M_0' & \xrightarrow{\varepsilon'} & M_1' \end{array}$$

For example, when $G = Z_p$ and Y is a G-space with base point, $\tilde{\omega}_q(Y)$ consists of the group $\pi_q(Y^G)$, the group $\pi_q(Y)$ on which G acts by the induced homomorphisms $g_{\#}: \pi_q(Y) \to \pi_q(Y)$, and the homomorphism $\varepsilon : \pi_q(Y^G) \to \pi_q(Y)^G \subset \pi_q(Y)$ induced by inclusion $Y^G \subset Y$.

5. Coefficient systems on a G-complex.

Let K be a G-complex. From K we form a category $\mathcal{K}$ whose objects are the finite subcomplexes of K and whose morphisms are as follows: If L and L' are finite subcomplexes of K, then hom(L,L') consists of all maps $g: L \to gL \subset L'$ for $g \in G$ (hom(L,L') may be empty). Note that we do not distinguish between maps induced by different elements of G if they are the same map.

Clearly the morphisms of $\mathcal{K}$ are just the inclusion maps $L \subset L'$, the maps $a: L \to aL$ induced by operations by elements of G, and the compositions of these.

We should note that for most purposes only the objects $K(\sigma)$ of $\mathcal{K}$ for cells σ of K are of importance, but for some constructions one needs the more general subcomplexes.

We define a canonical contravariant functor

$$\theta: \mathcal{K} \to \mathcal{O}_G$$

as follows: For $L \subset K$ a finite subcomplex, let $G_L = \{g \in G | g \text{ leaves } L \text{ pointwise fixed}\}$. We put

$$\theta(L) = G/G_L .$$

If $gL \subset L'$ and f denotes the map $L \to L'$ induced by operation by $g \in G$, then we see that

$$g^{-1}G_{L'}g \subset G_L$$

and we put $\theta(f) = \hat{g}: \theta(L') \to \theta(L)$, that is $\theta(f)$ is $\hat{g}: G/G_{L'} \to G/G_L$ which takes $g'G_{L'}$ into $g'gG_L$.

In other words, if $L \subset L'$ then $G_{L'} \subset G_L$ and θ(inclusion) is the natural map $G/G_L \to G/G_{L'}$, while if $g: L \to gL$ then $G_{gL} = gG_Lg^{-1}$ and $\theta(g: L \to gL): \theta(gL) = G/gG_Lg^{-1} \to G/G_L = \theta(L)$ is right multiplication by g.

Now if $M \in \mathcal{C}_G$ is a generic coefficient system, that is, if $M: \mathcal{O}_G \to$ Abel is a contravariant functor, then

$$M\theta: \mathcal{K} \to \text{Abel}$$

is a covariant functor and is called a (simple) coefficient system on K. We generalize this as follows:

A local coefficient system on K is a covariant functor

$$\mathcal{L} : \mathcal{K} \to \text{Abel} .$$

Again by Grothendieck's result, the local coefficient systems on K form an abelian category $\mathcal{L}\mathcal{C}_K = \text{Dgram}(\mathcal{K}, \text{Abel})$.

The coefficient systems $M\theta: \mathcal{K} \to$ Abel, for $M \in \mathcal{C}_G$, clearly form a subcategory $\mathcal{C}_K$ of $\mathcal{L}\mathcal{C}_K$.

Notation. If $\mathcal{L} \in \mathcal{L}\mathcal{C}_K$ and σ is a cell we let $\mathcal{L}(\sigma) = \mathcal{L}(K(\sigma))$ and for $K(\tau) \subset K(\sigma)$ we let $\mathcal{L}(\tau \to \sigma)$ denote $\mathcal{L}$(inclusion: $K(\tau) \to K(\sigma)$). Note that if $[\tau : \sigma] \neq 0$ then $K(\tau) \subset K(\sigma)$ so that $\tau \to \sigma$ is "in" $\mathcal{K}$.

6. Cohomology

Let $\mathcal{L}: \mathcal{K} \to$ Abel be in $\mathcal{L}\mathcal{C}_K$. Orient the cells of K in such a way that G preserves the orientations and define

$$C^q(K;\mathcal{L})$$

to be the group of all functions f on the q-cells of K with

$$f(\sigma) \in \mathcal{L}(\sigma).$$

Define $\delta : C^q(K;\mathcal{L}) \to C^{q+1}(K;\mathcal{L})$ by

$$(6.1) \qquad (\delta f)(\sigma) = \sum_{\tau} [\tau : \sigma] \mathcal{L}(\tau \to \sigma) f(\tau)$$

(which makes sense since $K(\tau) \subset K(\sigma)$ whenever $[\tau : \sigma] \neq 0$). In other words $(\delta f)(\sigma)$ is defined by "pushing" all coefficients to $\mathcal{L}(\sigma)$ and then taking the usual coboundary. This remark shows that $\delta\delta = 0$ since to compute $(\delta\delta f)(\omega)$ we push coefficients to $\mathcal{L}(\omega)$ and then compute (classical) coboundaries twice which necessarily gives zero. Of course, $\delta\delta = 0$ also follows by direct computation.

Now we define an operation of G on $C^q(K;\mathcal{L})$ as follows: If $g \in G$ and $f \in C^q(K;\mathcal{L})$ we put

$$(6.2) \qquad g(f)(\sigma) = \mathcal{L}(g)(f(g^{-1}\sigma)).$$

Here $\mathcal{L}(g)$ refers to $\mathcal{L}(g: K(g^{-1}\sigma) \to K(\sigma))$. Let us abbreviate

$$\mathcal{L}(g) = g_*.$$

Replacing σ by $g(\sigma)$ in (6.2) we obtain

$$(6.3) \qquad g(f)(g\sigma) = g_*(f(\sigma))$$

It is clear that the automorphism $f \to g(f)$ of $C^*(K;\mathcal{L})$ defines an <u>action</u> of G on $C^*(K;\mathcal{L})$ by chain mappings. Thus the fixed point set

$$C^q(K;\mathcal{L})^G = \{f \in C^q \mid g(f) = f \text{ for all } g \in G\}$$

is a subcomplex. It is also denoted by $C^q_G(K;\mathcal{L})$. By (6.3) $C^*(K;\mathcal{L})^G$ consists precisely of the <u>equivariant</u> cochains f (i.e. such that $f(g\sigma) = g_*(f(\sigma))$).

We define the equivariant cohomology group

(6.4) $$H^q_G(K;\mathcal{L}) = H^q(C^*(K;\mathcal{L})^G).$$

If $M \in \mathcal{C}_G$ (so that $M\theta \in \mathcal{C}_K \subset \mathcal{L}\mathcal{C}_K$) we use the abbreviation

(6.5) $$H^q_G(K;M) = H^q_G(K;M\theta).$$

If L is a subcomplex of K, invariant under G, then there is a restriction map $C^*(K;\mathcal{L}) \to C^*(L;\mathcal{L})$ whose kernel is the relative cochain group $C^*(K,L;\mathcal{L})$. There is a splitting homomorphism $C^*(L;\mathcal{L}) \to C^*(K;\mathcal{L})$ defined by extension of a cochain by zero (not a chain map). This clearly commutes with operations by G so that the sequence

$$0 \to C^*(K,L;\mathcal{L})^G \to C^*(K;\mathcal{L})^G \to C^*(L;\mathcal{L})^G \to 0$$

is exact. With the obvious definitions we obtain an induced cohomology exact sequence

$$\cdots \to H^n_G(K,L;\mathcal{L}) \to H^n_G(K;\mathcal{L}) \to H^n_G(L;\mathcal{L}) \to H^{n+1}_G(K,L;\mathcal{L}) \to \cdots$$

7*. <u>Equivariant maps</u>.

This section is not necessary to our main line of thought and it is included merely for the sake of completeness.

Let G and G' be finite groups and let $\varphi : G \to G'$ be a homomorphism. Let K be a G-complex, K' a G'-complex and let $\psi : K \to K'$ be a <u>cellular</u> map which is equivariant (i.e.

$\psi(g(x)) = \varphi(g)(\psi(x)))$. The map ψ (together with φ) induces a functor

$$\Psi : \mathcal{K} \to \mathcal{K}'$$

(between the categories associated with K and K' respectively) as follows: If $L \subset K$, let $\Psi(L) = K'(\psi(L))$ and if f is the composition $L \xrightarrow{g} gL \subset L_1$ then $\Psi(f)$ is the obvious composition

$$K'(\psi(L)) \to \varphi(g)K'(\psi(L)) = K'(\varphi(g)\psi(L)) = K'(\psi(gL)) \subset K'(\psi(L_1)).$$

(By abuse of notation we might define Ψ on morphisms by writing $\Psi(g) = \varphi(g)$.)

Let $\mathcal{L}'$: $\mathcal{K}' \to$ Abel be a local coefficient system on K'. Then $\mathcal{L}'\Psi$: $\mathcal{K} \to$ Abel is a local coefficient system on K. Suppose that $\mathcal{L}$: $\mathcal{K} \to$ Abel is any local coefficient system on K. Then we define a Ψ-<u>morphism</u> λ from $\mathcal{L}'$ to $\mathcal{L}$ to be a natural transformation

$$\lambda : \mathcal{L}'\Psi \to \mathcal{L}$$

of functors on $\mathcal{K}$. Now there is an obvious chain map $C^*(K;\mathcal{L}'\Psi) \to C^*(K;\mathcal{L})$ induced by λ and this is clearly equivariant with respect to the actions by G. Thus λ induces a homomorphism

(7.1) $$\lambda^* : H^*_G(K;\mathcal{L}'\Psi) \to H^*_G(K;\mathcal{L}).$$

We shall define a canonical homomorphism

(7.2) $$\Psi^* : H^*_{G'}(K';\mathcal{L}') \to H^*_G(K;\mathcal{L}'\Psi)$$

so that together with (7.1) we will obtain a homomorphism

$$\lambda^*\Psi^* : H^*_{G'}(K';\mathcal{L}') \to H^*_G(K;\mathcal{L})$$

(also denoted merely by λ^*).

In fact note that the cellularity of ψ implies that ψ induces a map $K^n/K^{n-1} \to K'^n/K'^{n-1}$ and hence induces a chain map

$$\psi_*: C_n(K) \to C_n(K').$$

Define

(7.3) $\Psi^*: C^*(K';\mathcal{L}') \to C^*(K;\mathcal{L}'\Psi)$

by

$$\Psi^*(f)(\sigma) = f(\psi_*(\sigma))$$

where the right hand side is shorthand for

$$\sum_\alpha n_\alpha \mathcal{L}'\big(K'(\tau_\alpha) \to K'(\psi(\sigma))\big) f(\tau_\alpha) \in \mathcal{L}'(K'(\psi(\sigma))) = \mathcal{L}'\Psi(\sigma)$$

where $\psi_*(\sigma) = \Sigma n_\alpha \tau_\alpha \in C_n(K')$.

Now we compute

$$\Psi^*(\varphi(g)(f))(\sigma) = (\varphi(g)(f))(\psi_*(\sigma)) = \mathcal{L}'(\varphi(g))(f(\varphi(g)^{-1}\psi_*(\sigma)))$$
$$= (\mathcal{L}'\Psi)(g)(f(\psi_*(g^{-1}\sigma))) = (\mathcal{L}'\Psi)(g)(\Psi^*(f)(g^{-1}\sigma))$$
$$= g(\Psi^*(f))(\sigma).$$

Thus, if $\varphi(g)(f) = f$ for all $g \in G$, then

$$g(\Psi^*(f)) = \Psi^*(\varphi(g)(f)) = \Psi^*(f).$$

Therefore (7.3) takes $C^*(K';\mathcal{L}')^{\varphi(G)}$ into $C^*(K;\mathcal{L}'\Psi)^G$. Since $C^*(K';\mathcal{L}')^{G'} \subset C^*(K';\mathcal{L}')^{\varphi(G)}$ we obtain a chain map $C^*(K';\mathcal{L}')^{G'} \to C^*(K;\mathcal{L}'\Psi)^G$ which induces our promised map (7.2) upon passage to homology.

The situation with simple coefficient systems is slightly more complicated, and we shall now discuss this case. We define a functor

$$\Phi : \mathcal{O}_G \to \mathcal{O}_{G'}$$

by putting $\Phi(G/H) = G'/\varphi(H)$ and, if $a^{-1}Ha \subset K$ as in (3.1), so that $\varphi(a)^{-1}\varphi(H)\varphi(a) \subset \varphi(K)$ we put $\Phi(\hat{a}: G/H \to G/K) = \widehat{\varphi(a)}: G'/\varphi(H) \to G'/\varphi(K)$.

The diagram

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\Psi} & \mathcal{K}' \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\theta'} \\ \mathcal{O}_G & \xrightarrow{\Phi} & \mathcal{O}_{G'} \end{array}$$

does not generally commute since

$$\theta'\Psi(L) = \theta'(K'(\psi(L))) = G'/G'_{\psi(L)}$$

while

$$\Phi\theta(L) = \Phi(G/G_L) = G'/\varphi(G_L)$$

and $\varphi(G_L) \subset G'_{\psi(L)}$ are not generally equal. However the projection

$$G'/\varphi(G_L) \to G'/G'_{\psi(L)}$$

is clearly functorial and provides a natural transformation

(7.4) $$\Phi\theta \to \theta'\Psi$$

of functors. Let $M' \in \mathcal{C}_{G'}$ be a generic coefficient system for G'. Since M' is a contravariant functor $\mathcal{O}_{G'} \to$ Abel, the transformation (7.4) induces a natural transformation

(7.5) $$M'\theta'\Psi \to M'\Phi\theta$$

of functors $\mathcal{K} \to$ Abel. In other words, (7.5) is a Ψ-morphism

(7.6) $$M'\theta' \to M'\Phi\theta.$$

Thus we have an induced homomorphism

(7.7) $$H^*_{G'}(K';M') \to H^*_G(K;M'\Phi)$$

(where the θ and θ' have been dropped in accordance with our notation conventions).

If $M \in \mathcal{C}_G$ and $M' \in \mathcal{C}_{G'}$, we define a φ-morphism $M' \to M$ to be a natural transformation

$$M'\Phi \to M$$

of functors $\mathcal{O}_G \to \text{Abel}$. Clearly, in combination with (7.7), every φ-morphism $M' \to M$ induces a homomorphism

$$H^*_{G'}(K';M') \to H^*_G(K;M). \tag{7.8}$$

8.* Products

Suppose that K is a G-complex and K' is a G'-complex. Then $K\times K'$ with the product cell-structure and the weak topology is a $G\times G'$-complex in the obvious way. If $\mathcal{L}$ and $\mathcal{L}'$ are local coefficient systems on K and K' respectively then define

$$\mathcal{L} \hat{\otimes} \mathcal{L}' \in \mathcal{L}\mathcal{C}_{K\times K'}$$

by $(\mathcal{L} \hat{\otimes} \mathcal{L}')(W) = \mathcal{L}(\pi_1 W) \otimes \mathcal{L}'(\pi_2 W)$ where $\pi_1 \colon K\times K' \to K$ and $\pi_2 \colon K\times K' \to K'$ are the projections. The definition of $\mathcal{L} \hat{\otimes} \mathcal{L}'$ on morphisms is obvious.

Suppose that $f \in C^p(K;\mathcal{L})$ and $f' \in C^q(K';\mathcal{L}')$. Define

$$f\times f' \in C^{p+q}(K\times K'; \mathcal{L} \hat{\otimes} \mathcal{L}')$$

by

$$(f\times f')(\sigma\times\tau) = f(\sigma) \otimes f'(\tau)$$

where σ and τ are (oriented) p and q-cells respectively ($f\times f'$ vanishes elsewhere). $(f,f') \to f\times f'$ is obviously bilinear.

If $g \in G$ and $g' \in G'$ then clearly

$$(g\times g')(f\times f') = g(f) \times g'(f').$$

It is also clear that $\delta(f\times f') = (\delta f)\times f' + (-1)^p f\times\delta f'$. Thus $\times$ induces a chain map

$$C^p_G(K;\mathcal{L}) \otimes C^q_{G'}(K';\mathcal{L}') \to C^{p+q}_{G\times G'}(K\times K';\mathcal{L}\hat{\otimes}\mathcal{L}')$$

and consequently, a "cross-product":

$$H^p_G(K;\mathcal{L}) \otimes H^q_{G'}(K';\mathcal{L}') \to H^{p+q}_{G\times G'}(K\times K';\mathcal{L}\hat{\otimes}\mathcal{L}').$$

If $\mathcal{L}$ and $\mathcal{L}'$ are simple then so is $\mathcal{L}\hat{\otimes}\mathcal{L}'$ as the reader can check.

An internal product, the "cup-product" can be derived from the cross-product by means of equivariant diagonal approximations. However, we have not given the necessary background for this since the definition of the cup product is more easily obtained as a consequence of general facts which we shall develop later in these notes.

9*. Another description of cochains.

We define an element

$$\underline{C}_n(K;Z) \in \mathcal{C}_G$$

by $\underline{C}_n(K;Z)(G/H) = C_n(K^H;Z)$ together with the obvious values on morphisms of $\mathcal{O}_G$. These objects, for $n = 0,1,2,\ldots$, form a chain complex in the abelian category $\mathcal{C}_G$. We can form the homology $\underline{H}_n(K;Z) = H_n(\underline{C}_*(K;Z)) \in \mathcal{C}_G$ of this chain complex. Clearly, this is just $\underline{H}_n(K;Z)(G/H) = H_n(K^H;Z)$ together, again, with the obvious values on morphisms. Similar considerations apply to the relative case.

Let $f \in C^n_G(K;M)$ where $M \in \mathcal{C}_G$. Then for an n-cell σ, $f(\sigma) \in M(G/G_\sigma)$. Suppose that $\sigma \in K^H$. Then $H \subset G_\sigma$ so that we have an element

$$M(G/H \to G/G_\sigma)f(\sigma) \in M(G/H).$$

Denote this element by $\hat{f}(G/H)(\sigma)$. This map clearly extends to a homomorphism

(9.1) $$\hat{f}(G/H) \colon C_n(K^H;Z) \to M(G/H).$$

It is easily checked that (9.1) is natural with respect to the morphisms of $\mathcal{O}_G$, so that $\hat{f}\colon \underline{C}_n(K;Z) \to M$ is a natural transformation of functors. That is,

(9.2) $$\hat{f} \in \mathrm{Hom}(\underline{C}_n(K;Z),M)$$

where Hom refers to the morphisms of the abelian category $\mathcal{C}_G$. Conversely, suppose we are given an element $\hat{f} \in \mathrm{Hom}(\underline{C}_n(K;Z),M)$. Let σ be an n-cell of K and regard σ as an element of $C_n(K^{G_\sigma};Z)$. Define

$$f(\sigma) = \hat{f}(G/G_\sigma)(\sigma) \in M(G/G_\sigma)$$

so that $f \in C^n(K;M)$. Let us check that f is equivariant. Applying the fact that $\hat{f}$ is natural to the morphism $\hat{g}\colon G/G_{g\sigma} = G/gG_\sigma g^{-1} \to G/G_\sigma$ of $\mathcal{O}_G$, we see that the diagram

$$\begin{array}{ccc}
C_n(K^{G_\sigma};Z) & \xrightarrow{\hat{f}(G/G_\sigma)} & M(G/G_\sigma) \\
\Big\downarrow g_* & & \Big\downarrow g_* = M(\hat{g}) \\
C_n(K^{G_{g\sigma}};Z) & \xrightarrow{\hat{f}(G/G_{g\sigma})} & M(G/G_{g\sigma})
\end{array}$$

commutes. Thus $f(g\sigma) = \hat{f}(G/G_{g\sigma})(g\sigma) = g_*(\hat{f}(G/G_\sigma)(\sigma)) = g_*(f(\sigma))$ as claimed.

We have demonstrated an isomorphism

(9.3) $$C^n_G(K;M) \approx \mathrm{Hom}(\underline{C}_n(K;Z),M)$$

given by $f \to \hat{f}$. It is clear that this isomorphism preserves the coboundary operators. Thus we may pass to homology and obtain the isomorphism

(9.4) $\quad H^n_G(K;M) \approx H^n(\mathrm{Hom}(\underline{C}_*(K;Z),M))$.

Since Hom is left exact on $\mathcal{C}_G$ we obtain a canonical homomorphism

(9.5) $\quad H^n_G(K;M) \to \mathrm{Hom}(\underline{H}_n(K;Z),M)$.

It is also easy to check that <u>if</u> K <u>has no</u> (n-1)-<u>cells</u>, so that $\underline{C}_{n-1}(K;Z) = 0$, then (9.5) is an <u>isomorphism</u> (triviality of $\underline{H}_{n-1}(K;Z)$, or even of $\underline{H}_q(K;Z)$ for $0 < q < n$, is not sufficient for this).

<u>Remark</u>. If A is a G-module and $M \in \mathcal{C}_G$ is the corresponding coefficient system as defined in §4, example 2, then an equivariant homomorphism $C_n(K;Z) \to A$ must take $C_n(K^H;Z) \subset C_n(K;Z)^H$ into $A^H = M(G/H)$. Thus it is clear that we have an isomorphism

$$\mathrm{Hom}_{Z(G)}(C_n(K;Z),A) \approx \mathrm{Hom}(\underline{C}_n(K;Z),M) \approx C^n_G(K;M).$$

The left hand side is, by definition, the classical equivariant cochain group with coefficients in the G-module A.

10*. <u>A spectral sequence</u>.

We shall show that the abelian category $\mathcal{C}_G$ contains sufficiently many projectives and injectives. However, projective resolutions of length one (or even of finite length) do not generally exist, in contrast to the category Abel. Thus instead of a universal coefficient sequence linking homology and cohomology we obtain a <u>spectral</u> sequence.

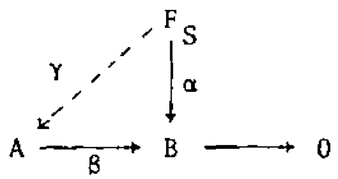
For a set S let F(S) denote the free abelian group based on S. Suppose that S is a G-set. Define an element

$$F_S \in \mathcal{C}_G \quad \text{by} \quad F_S(G/H) = F(S^H)$$

together with the obvious values on morphisms of $\mathcal{O}_G$ (see §4, example 2). For example, if S is the set of n-cells of a G-complex K which are not in the G-subcomplex L, then $F_S \approx \underline{C}_n(K,L;Z)$.

(10.1) <u>Proposition</u>. F_S <u>is projective</u>.

<u>Proof</u>. Let



be a diagram in $\mathcal{C}_G$ with exact row and with γ to be constructed. Let $S' \subset S$ be a subset containing exactly one element from each orbit of G on S. Given $s \in S'$, consider s as an element of $F(S^{G_s}) = F_S(G/G_s)$. Then $\alpha(s) \in B(G/G_s)$. Define $\gamma(s) \in A(G/G_s)$ to be any element with $\beta(\gamma(s)) = \alpha(s)$. For $g \in G$ we let $\gamma(gs) = g_*\gamma(s) \in A(G/G_{gs})$ (where $g_* = A(\hat{g}\colon G/G_{gs} \to G/G_s)$). For $H \subset G_s$ let j denote the projection $G/H \to G/G_s$. The element s represents an element of $F(S^H) = F_S(G/H)$, namely $F_S(j)(s)$. We define $\gamma(F_S(j)(s)) = A(j)\gamma(s)$. Now γ has been defined on a set of free generators of $F_S(G/H)$ for every $H \subset G$. Thus there is a unique extension to $F_S(G/H)$ for all H. This extension is clearly a morphism $F_S \to A$ with $\beta\gamma = \alpha$, as claimed.

(10.2) <u>Corollary</u>. $C^n_G(K,L;M) \approx \mathrm{Hom}(\underline{C}_n(K,L;Z),M)$.

<u>Proof</u>. The exact sequence

$$0 \to \underline{C}_n(L;Z) \to \underline{C}_n(K;Z) \to \underline{C}_n(K,L;Z) \to 0$$

of projective objects in $\mathcal{C}_G$ induces an exact sequence via the functor Hom(· ,M) and the result follows.

(10.3) Corollary. $C^n_G(K,L;M)$ is an exact functor of M.

Proof. This is immediate from (10.2).

It follows from (10.3) that an exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{C}_G$ induces a long exact cohomology sequence of (K,L).

At the end of this section we shall show that $\mathcal{C}_G$ contains sufficiently many projectives. In fact if S is the disjoint union of all of the G-sets G/H for $H \subset G$ then F_S is a (projective) generator of the category $\mathcal{C}_G$. Since $\mathcal{C}_G$ obviously satisfies Grothendieck's axiom AB5 (arbitrary direct sums and exactness of the direct limit functor) it follows by a result of Grothendieck that $\mathcal{C}_G$ possesses sufficiently many injectives (see Mitchell: Theory of Categories).

Let $M \in \mathcal{C}_G$ and let M^* be an injective resolution of M. Consider the double complex

$$\mathrm{Hom}(\underline{C}_*(K,L;Z),M^*).$$

Standard homological algebra applied to this double complex yields a spectral sequence with

$$(10.4) \qquad E_2^{p,q} = \mathrm{Ext}^p(\underline{H}_q(K,L;Z),M) \Longrightarrow H_G^{p+q}(K,L;M).$$

(This notation means that $E_r^{p,q}$ converges to $E_\infty^{p,q}$ which is the graded group associated with a filtration of $H_G^{p+q}(K,L;M)$. Also Ext^p refers to the p^{th} right derived functor of Hom in the category $\mathcal{C}_G$.)

By way of illustration we shall compute $\mathrm{Ext}^p(A,M)$ in two rather elementary cases.

<u>Example 1</u>. Let $A \in \mathcal{C}_G$ be defined by $A(G) = Z$, with trivial G-operators, and $A(G/H) = 0$ for $H \neq \{e\}$. Let F_* be a $Z(G)$-free resolution of Z. Then $\underline{F}_*$, defined by $\underline{F}_*(G) = F_*$ and $\underline{F}_*(G/H) = 0$ for $H \neq \{e\}$, is a projective resolution of A in $\mathcal{C}_G$. Clearly $\mathrm{Hom}(\underline{F}_*;M) \approx \mathrm{Hom}_{Z(G)}(F_*;M(G))$ so that

$$\mathrm{Ext}^p(A,M) \approx H^p(G;M(G)),$$

where the right hand side is the classical cohomology of G with coefficients in the G-module M(G). If K is a connected G-complex on which G acts <u>freely</u> and such that

$$H_q(K;Z) = 0 \quad \text{for} \quad 0 < q < N$$

then in (10.4) we have $E_2^{p,q} \approx \delta_0^q \mathrm{Ext}^p(A,M) \approx \delta_0^q H^p(G;M(G))$ for $q < N$. Consequently, we have an isomorphism

$$H_G^n(K;M) \approx H^n(G;M(G)) \text{ for } n < N.$$

<u>Example 2</u>. Let B be an abelian group and let $\underline{B} \in \mathcal{C}_G$ be defined by $\underline{B}(G/H) = B$ and $\underline{B}(j) = 1$ for all morphisms j in $\mathcal{O}_G$. Then, if M^* is an injective resolution of M, we have

$$\mathrm{Hom}(\underline{B},M^*) \approx \mathrm{Hom}(\underline{B}(P),M^*(P)) = \mathrm{Hom}(B,M^*(P))$$

where P is the point G/G. $M^*(P)$ is clearly an injective resolution of M(P) in Abel. Hence

$$\mathrm{Ext}^p(\underline{B},M) \approx \mathrm{Ext}^p(B,M(P))$$

where the right hand side is Ext in Abel. That is

$$\begin{cases} \mathrm{Ext}^0(\underline{B},M) = \mathrm{Hom}(\underline{B},M) \approx \mathrm{Hom}(B,M(P)) \\ \mathrm{Ext}^1(\underline{B},M) \approx \mathrm{Ext}(B,M(P)) \\ \mathrm{Ext}^p(\underline{B},M) = 0 \quad \text{for} \quad p > 1. \end{cases}$$

In particular, if B is free abelian then $\text{Ext}^p(\underline{B},M) = 0$ for $p > 0$, that is, $\underline{B}$ is projective in $\mathcal{C}_G$ if B is projective in Abel. (Of course, this also follows directly from (10.1) in the case in which G acts trivially on S.)

Let us return to the general discussion. There is an edge homomorphism

$$H^n_G(K,L;M) \to \text{Hom}\,(\underline{H}_n(K,L;Z),M)$$

of (10.4) (coinciding with (9.5) when $L = \emptyset$). Clearly this is an <u>isomorphism</u> if each $\underline{H}_q(K,L;Z)$ is projective for $q < n$.

For example suppose that $n > 1$, that K possesses stationary points (e.g. k_0) and that

$$\tilde{\underline{\omega}}_q(K,k_0) = 0 \text{ for } q < n.$$

The Hurewicz theorem, applied to each K^H, shows that the (obvious) Hurewicz homomorphism (in $\mathcal{C}_G$)

$$\tilde{\underline{\omega}}_q(K,k_0) \to \underline{H}_q(K;Z)$$

is an isomorphism for $0 < q \leq n$. Thus

$$(10.5) \qquad H^n_G(K;M) \approx \text{Hom}(\tilde{\underline{\omega}}_n(K,k_0),M)$$

in this case.

We shall now justify our earlier contention that there are enough projectives in $\mathcal{C}_G$. For any G-sets S and T let $E(S,T)$ denote the set of equivariant maps $S \to T$. For $K \subset G$, the assignment $f \to f(K)$ clearly yields a one-one correspondence

$$E(G/K,S) \xrightarrow{\approx} S^K.$$

(It is of interest to reconsider the material of §3 and the examples of §4 in this light.) Thus

$$F_{G/H}(G/K) = F((G/H)^K) = F(E(G/K,G/H)).$$

Now if $\alpha \in M(G/H)$ the map $f \to M(f)(\alpha)$ of

$$E(G/K,G/H) \to M(G/K)$$

induces a homomorphism $F(E(G/K,G/H)) \to M(G/K)$. This is clearly natural in G/K and hence is a morphism

$$\varphi_\alpha: F_{G/H} \to M$$

in $\mathcal{C}_G$. It is also clear that the generator $H/H \in F_{G/H}(G/H)$ corresponds to $1 \in E(G/H,G/H)$ and hence that φ_α maps it into $\alpha \in M(G/H)$.

We shall now explicitly exhibit a projective which maps onto a given $M \in \mathcal{C}_G$. For $\alpha \in M(G/H)$ let S_α be a copy of the G-set G/H and let $S(M) = \bigcup_\alpha S_\alpha$ be the disjoint union of these for all $\alpha \in M(G/H)$ and all $H \subset G$. Then $F_{S(M)} = \sum_\alpha F_{S_\alpha}$. The homomorphisms $\varphi_\alpha: F_{S_\alpha} \to M$ yield a homomorphism

$$\varphi = \sum \varphi_\alpha : F_{S(M)} \to M \tag{10.6}$$

which is clearly surjective.