# THE KERVAIRE INVARIANT OF $8k+2$-MANIFOLDS.

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**1. Introduction.** The main results of this paper were announced in [6]. Let $\Omega_n(e)$, $\Omega_n(SU)$, and $\Omega_n(Spin)$ denote the $n$-th framed, $SU$, and $Spin$ cobordism groups respectively (see [7] and [11].). In [8] Kervaire defined a homomorphism $\Phi\colon \Omega_{4k+2}(e)\to Z_2$, $k\neq 0,1,3$, which is the obstruction to a framed $4k+2$-manifold being framed cobordant to a homotopy sphere ([9]). Kervaire showed that $\Phi=0$ for $k=2,4$. In [4] a homomorphism $\psi$: $\Omega_{8k+2}(Spin)\to Z_2$ was defined such that $\Phi=\psi\rho$ where $\rho\colon \Omega_n(e)\to\Omega_n(Spin)$ is the obvious map. The obvious map of $\Omega_n(SU)$ into $\Omega_n(Spin)$ defines a homomorphism of $\Omega_{8k+2}(SU)$ into $Z_2$ which we also denote by $\psi$. This latter map is the main object to the investigated in this paper. In an appendix we briefly discuss $\psi\colon \Omega_{8k+2}(Spin)\to Z_2$.

The main results of this paper are as follows:

Theorem 1.1. *$\Phi\colon \Omega_{8k+2}(e)\to Z_2$ is zero for $k>0$.*

The following corollaries of (1.1) are implied by the results of [9], [8] and [3].

Corollary 1.2. *$bP_{8k+2}\approx Z_2$, where $bP_{8k+2}$ is the group of homotopy spheres which bound stably parallelizable $8k+2$-manifolds* [9].

Corollary 1.3. *If $K$ is the topological manifold obtained by plumbing two copies of the tangent disc bundle of $S^{4k+1}$ together and then attaching an $8k+2$-disc, then $K$ does not admit a diffedentiable structure.*

(1.3) follows from [8] if one has the result that a $C^\infty$ manifold with underlying topological space $K$ is stably parallelizable. In Appendix 2 we give a proof of this due to John Milnor.

Corollary 1.4. *Every element of $\Omega_{8k+2}(e)$ can be represented by a homotopy sphere, $k\geqq 1$.*

Corollary 1.5. *A finite, 1-connected CW complex has the homotopy type of a stably parallelizable $8k+2$-manifold if and only if there is a stably*

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*spherical class* $m \in H_{8k+2}(X)$ *such that* $m \cap : H^q(X; Z) \approx H_{8k+2-q}(X; Z)$ *for all* $q$ *and* $\psi(X) = 0$ *(see* § 3 *for a definition of* $\psi(X)$*)* ([3], [12]).

It is known that $\Omega_2(SU) \approx Z_2$ [7]. Let $\alpha$ be the generator. Define $\psi: \Omega_2(SU) \to Z_2$ by $\psi(\alpha) = 1$.

THEOREM 1.6. *If* $\beta \in \Omega_{8k+2}(SU)$ *and* $\gamma \in \Omega_{8l}(SU)$, $k \geqq 0$, $l > 0$, *then*

$$\psi(\beta\gamma) = \psi(\beta) I(\gamma)$$

*where* $I(\gamma)$ *is the index of* $\gamma$ *reduced* mod 2. ($I(\gamma)$ = *Euler characteristic of* $\gamma$ mod 2 *also.*)

In [7] it is shown that $\Omega_{16}(SU)$ contain an element $\gamma^{16}$ with $I(\gamma^{16}) \neq 0$ mod 2.

By (1.6) we have:

COROLLARY 1.7. $\psi(\alpha\gamma^{16}) = 1$.

In § 2 we give some preliminary results about cohomology operations. In § 3 we define $\psi: \Omega_{8k+2}(SU) \to Z_2$ and in § 4 we prove Theorems (1.1) and (1.6).

**2. Some cohomology operations.** Throughout the remainder of the paper all homology and cohomology groups will have $Z_2$ coefficients unless otherwise specified. $m$ will denote an integer of the form $4k + 1$, $k > 0$. Below we show that various cohomology operations are equal. This will always mean equal modulo the largest indeterminacy involved.

In [5] it is shown that the relation

$$Sq^2 Sq^{m-1} + Sq^1(Sq^2 Sq^{m-2}) = 0$$

on $m$-dimensional cohomology classes gives rise to a secondary operation

$$\phi: H^m(X) \cap \operatorname{Ker} Sq^{m-1} \cap \operatorname{Ker} Sq^2 Sq^{m-2} \to H^{2m}(X)/Sq^1 H^{2m-1}(X) + Sq^2 H^{2m-2}(X). \tag{2.1}$$

Furthermore, the following is proved:

2.2) If $\phi(u)$ and $\phi(v)$ are defined, $\phi(u+v)$ is defined and

$$\phi(u+v) = \phi(u) + \phi(v) + uv$$

Let $\hat{\sigma}_m \in H^m(K(Z, m); Z)$ be the generator and let $\sigma_m$ be $\hat{\sigma}_m$ reduced mod 2. Let $p: E \to K(Z, m)$ be the fibration with fibre $K(Z_2, 2m-2)$ and $k$-invariant $Sq^{m-1}\hat{\sigma}_m$. Then $\phi(p^*\sigma_m)$ is defined since, by the Adem relations,

$$Sq^2 Sq^{m-2} = Sq^m + Sq^{m-1} Sq^1 = Sq^1 Sq^{m-1} + Sq^{m-1} Sq^1$$

is zero on $p^*\sigma_m$. Choose an element $z \in \phi(p^*\sigma_m)$. $z$ defines a cohomology operation

$$\hat{\phi}\colon H^m(X,Z) \cap \operatorname{Ker} Sq^{m-1} \to H^{2m}(X)/Sq^2H^{2m-2}(X)$$

The following is immediate.

(2.3) If $\hat{u} \in H^m(X;Z)$, $Sq^{m-1}\hat{u} = 0$ and $u$ is $\hat{u}$ reduced mod 2, then

$$\hat{\phi}(\hat{u}) = \phi(u).$$

Let $\hat{u} \in H^m(X;Z)$ be viewed as a map $\hat{u}\colon X \to K(Z,m)$. Then the following is proved in [13].

(2.4) If $Sq^{m-1}\hat{u} = 0$,

$$\hat{\phi}(\hat{u}) = Sq^2{}_{\hat{u}}(Sq^{m-1}\hat{\sigma}_m)$$

Let $f\colon (X,A) \to (Y,B)$ and let $g\colon X \to Y$ be the map defined by $f$. We need a product formula for functional operations.

(2.5) If $u \in H^q(Y,B)$, $v \in H^p(Y)$ and $g^*v = Sq^2v = Sq^1u = Sq^2u = 0$, then

$$Sq^2{}_f(uv) = (f^*u)(Sq^2{}_gv)$$

(This formula is proved, in the absolute case, in [1].)

*Proof*. Let $h\colon A \to B$ be the map defined by $f$. Note that $(X,A)$ is contained in the mapping cylinders $(C_g, C_h)$ and that $A = C_h \cap X$. Hence we may assume $f$ is an inclusion map and $A = X \cap B$. Cup product with $u$ maps the exact sequence of $(Y,X)$ into the exact sequence of the triad $(Y,X,B)$ giving the following commutative diagram:

$$\begin{array}{ccccccc}
H^{p-1}(X) & \longrightarrow & H^p(Y,X) & \longrightarrow & H^p(Y) & \longrightarrow & H^p(X) \\
\downarrow{\scriptstyle \cup f^*u} & & \downarrow{\scriptstyle \cup u} & & \downarrow{\scriptstyle \cup u} & & \downarrow \\
H^{p+q-1}(X,A) & \to & H^{p+q}(Y,X\cup B) & \to & H^{p+q}(Y,B) & \to & H^{p+q}(X,A)
\end{array}$$

Since $Sq^1u = Sq^2u = 0$, $Sq^2uz = uSq^2z$ for any $z$. Hence $Sq^2$ maps the above ladder into itself giving a large commutative diagram. (2.5) now follows by chasing around this diagram.

**3. Definition of the Kervaire invariant.** Throughout this section we view classes $u \in H^q(X)$ as maps $u\colon X \to K(Z_2,q)$. Again, $m = 4k+1$, $k > 0$.

$\Omega_n(X;SU)$ will denote the $n$-th $SU$ bordism group [7]. Recall, this group is the set of equivalence classes, under an appropriately defined cobordism relation, of triples $(M, \lambda, f)$ where $M$ is a closed, compact $C^\infty$ $n$-manifold, $\lambda$ is an $SU$ reduction of the normal bundle of $M$ embedded in $R^{n+k}$ for large $k$ and $f\colon M \to X$. $\Omega_n(SU) = \Omega_n(pt; SU)$. One may easily show that if $X$ is connected, every element of $\Omega_n(X; SU)$ can be represented by $(M, \lambda, f)$ where $M$ is connected. Hereafter we assume all spaces are connected. Also we assume all manifolds have an $SU$ structure on their normal bundle, $(M, \lambda, u)$ will be denoted by $(M, u)$ and $\nu_M\colon M \to BSU_k$ will denote the map defined by this $SU$ structure.

LEMMA 3.1. *If* $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$, *then* $\{M, u\} = \{M', u'\}$ *where* $M'$ *is* 1-*connected. Furthermore, there is a cobordism* $(N, v)$ *between* $(M, u)$ *and* $(M', u')$ *such that if* $i\colon M \to N$ *and* $j\colon M' \to N$ *are the inclusion maps,*

$$i^*\colon H^q(M) \approx H^q(N) \quad \textit{for } q > 2$$

$$j^*\colon H^q(M') \approx H^q(N) \quad \textit{for } q \neq 2m-1, 2m-2.$$

*Proof.* We form $M'$ by killing $\pi_1(M)$ by surgery [10]. This process yields a manifold $N$ with an $SU$ reduction such that $\partial N = M - M'$. Furthermore $N$ consists of $M \times I$ with handles $D^2 \times D^{2m-1}$ attached by maps $h\colon S^1 \times D^{2m-1} \to M \times \{0\}$. Up to homotopy type, $N$ is $M$ with 2-cells attached and $N$ is also $M'$ with $(2m-1)$-cells attached. (3.1) now follows from these properties of $N$.

Let

$$\bar{\phi}\colon \Omega_{2m}(K(Z_2, m); SU) \to Z_2$$

be defined as follows: Let $\phi$ be the cohomology operation described in (2.1). Let $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$. Since $M$ has an $SU$ reduction, its Stiefel-Whitney classes $w_1$ and $w_2$ are zero. Therefore by the Wu formulas,

$$Sq^1 H^{2m-1}(M) = w_1 H^{2m-1}(M) = 0 \text{ and } Sq^2 H^{2m-2}(M) = w_2 H^{2m-2}(M) = 0.$$

Hence, $\phi\colon H^m(M) \cap \operatorname{Ker} Sq^{m-1} \to H^{2m}(M)$. If $Sq^{m-1}u = 0$, we let

$$\bar{\phi}\{M, u\} = \phi(u)([M])$$

where $[M] \in H_{2m}(M)$ denotes the fundamental class. By (3.1) we may always choose $M$ to be 1-connected and hence so that

$$Sq^{m-1} H^m(M) \subset H^{2m-1}(M) \approx H_1(M) = 0.$$

Therefore $\bar{\phi}$ is defined on all of $\Omega_{2m}(K(Z_2, m); SU)$. We show that it is well defined. Let $\beta \in \Omega_{2m}(K(Z_2, m); SU)$. Choose an $(M_1, u_1) \in \beta$ such that

$M_1$ is 1-connected. Let $(M_2, u_2)$ be any representative of $\beta$ such that $Sq^{m-1}u_2 = 0$. We show $\phi(u_2)([M_2]) = \phi(u_1)([M_1])$. Let $(N, v)$ be a cobordism between $(M_1, u_1)$ and $(M_2, u_2)$. By surgery we make $N$ 2-connected. Let $j_i\colon M_i \to N$ be the inclusion maps. $H^{2m-1}(N, M_2) \approx H_2(N, M_1) = 0$ since $H_2(N) = H_1(M_1) = 0$. Therefore $j_2^*\colon H^{2m-1}(N) \to H^{2m-1}(M_2)$ is an injection. $j_2^* Sq^{m-1}v = Sq^{m-1}u_2 = 0$. Hence $Sq^{m-1}v = 0$ is zero. In a similar way one shows that $j_i^*\colon H^{2m}(N) \to H^{2m}(M_i)$ is an injection and hence that $Sq^2 Sq^{m-2}v = 0$. Therefore $\phi(v)$ is defined and, since $\phi(u_i) = j_i^* \phi(v)$, $\phi(u_1) = 0$ if and only if $\phi(u_2) = 0$. Thus $\bar{\phi}$ is well defined.

LEMMA. 3.2. *If $\{M, u\} \in \Omega_{2m}(K(Z_2, m); SU)$, $Sq^{m-1}u = 0$ and $u$ is the reduction* mod 2 *of an integer class $\hat{u}$, then*

$$\bar{\phi}\{M, u\} = Sq^2_{\hat{u}}\,(Sq^{m-1}\hat{\sigma}_m)([M])$$

*Proof.* By (2.3) and (2.4),

$$\phi(u) = \hat{\phi}(\hat{u}) = Sq^2_{\hat{u}}\,(Sq^{m-1}\hat{\sigma}_m).$$

Thus the only thing to check is that the indeterminacy for each of these operations is zero. The indeterminacy of $\phi$ is $Sq^2 H^{2m-2}(M) = w_2 H^{2m-2}(M) = 0$. The indeterminacy of $Sq^2{}_u$ is

$$Sq^2 H^{2m-2}(M) + \hat{u}^* H^{2m}(K(Z, m))$$

Thus we must show that $v = Sq^{i_1} Sq^{i_2} \cdots Sq^{i_l} u = 0$ if $i_1 + i_2 + \cdots + i_l = m$. By the Wu formulas $v = zu$ where $z$ is a polynomial in the Stiefel-Whitney classes of $M$. Therefore $z = \nu^*{}_M z'$ for $z' \in H^m(BSU)$. But $m$ is odd and hence $z' \in H^m(BSU) = 0$.

We next define the Kervaire Invariant $\psi\colon \Omega_{2m}(SU) \to Z_2$. Let $\{M\} \in \Omega_{2m}(SU)$. Choose a symplectic basis $\{u_i, v_i \mid i = 1, 2, \cdots, \nu\}$ for $H^m(M)$, that is, $u_1, \cdots, u_\nu, v_1, \cdots, v_\nu$ is a basis for $H^m(M)$, $u_i u_j = v_i v_j = 0$, and $u_i v_j = \delta_{ij}$. Since $M$ is orientable, $u^2 = 0$, $u \in H^m(M)$, and hence such a basis exists. Define

$$\psi\{M\} = \sum_{i=1}^{\nu} \bar{\phi}\{M, u_i\} \cdot \bar{\phi}\{M, v_i\}. \tag{3.3}$$

We show that $\psi$ is a homomorphism and that it is well defined. By (3.1) we may change $M$ by surgery so that it is 1-connected. Furthermore, by (3.1) $\{u_i, v_i\}$ goes over, under this process, to a symplectic basis. Hence we may assume $M$ is 1-connected. By (2.2)

$$\bar{\phi}\{M, \ \}\colon H^m(M) \to Z_2$$

is a quadratic function whose associated bilinear form, namely, cup product, is non-singular. Therefore the right side of 3.3 is independent of the choice of the symplectic basis [2]. One may easily check that the right side of 3.3 is additive with respect to the connected sum operation on manifolds. Thus $\psi$ is a homomorphism and to show that it is well defined it is sufficient to show that the right side of (3.3) is zero if $M=\partial N$. We may make $N$ 2-connected by surgery. Recall, if $j: M \to N$ is the inclusion map, $u \in H^m(N)$, $v \in H^m(N)$, then $u \cdot \delta^* v = \delta^*(j^*(u) \cdot v)$, where $\delta^*: H^m(M) \to H^{m+1}(N,M)$. From this fact and Poincare duality one may obtain classes $u_i \in H^m(N)$ and $v_i \in H^m(M)$ such that $\{j^* u_i, v_i\}$ is a symplectic basis for $H^m(M)$. $H^{2m-1}(N) \approx H_2(N,M)=0$ and $H^{2m}(N) \approx H_1(N,M)=0$. Therefore $\phi(u_i)$ is defined and equals zero. Therefore $\bar{\phi}(\{M, j^* u_i\}) = \phi(j^* u_i)([M]) = j^* \phi(u_i)([M]) = 0$. Thus $\psi$ is well defined.

*Remark* 3.4. The secondary operation $\phi$ is not uniquely determined by the relation between primary operations from which it arises, that is, it is only determined up to the addition of a stable primary operation. (3.2) shows that $\psi: \Omega_{8k+2}(SU) \to Z_2$ does not depend on the choice of $\phi$ since for any $\{M\} \in \Omega_{2m}(SU)$, one may choose $M$ 1-connected and with every element of $H^m(M)$ the reduction of an integer class (See § 4.).

Let $\alpha$ be the generator of $\Omega_2(SU) \approx Z_2$. We define $\psi: \Omega_2(SU) \to Z_2$ by $\psi(\alpha)=1$.

Finally, to complete the statement of Corollary 1.5, we define $\psi(X) \in Z_2$ when $X$ is a 1-connected, finite $CW$ complex which has a stably spherical class $m \in H_{8k+2}(X;Z)$ such that

$$\cap\, m: H^q(X;Z) \approx H_{8k+2-q}(X;Z)$$

for all $q$. Since $H_{8k+2}(X;Z)$ is generated by a stably spherical class, $Sq^i$ is zero on $H^{8k+2-i}(X)$. Also

$$Sq^{4k} H^{4k+1}(X) \subset H^{8k+1}(X) \approx H_1(X) = 0$$

Therefore $\phi$ defines a quadratic function

$$\phi: H^{4k+1}(X) \to H^{8k+2}(X)$$

Let

$$\psi(X) = \sum_{i=1}^{r} \phi(u_i)(m) \cdot \phi(v_i)(m)$$

where $\{u_i, v_i \mid i=1, \cdots, r\}$ is a symplectic basis for $H^{4k+1}(X)$.

**4. Proofs of Theorems (1.1) and (1.6).** We first prove (1.6). Let

$\beta \in \Omega_{8k+2}(SU)$, $k \geqq 0$ and let $\gamma \in \Omega_{8l}(SU)$, $l > 0$. We wish to show that $\psi(\beta\gamma) = \psi(\beta) I(\gamma)$ where $I(\gamma)$ is the index of $\gamma \bmod 2$.

Let $M \in \beta$ and $N \in \gamma$. Applying surgery to $M$ and $N$ we may choose them so that $\nu_{M*}\colon \pi_i(M) \to \pi_i(BSU)$ is an isomorphism for $i < 4k+1$ and $\nu_{N*}\colon \pi_i(N) \to \pi_i(BSU)$ is an isomorphism for $i < 4l$. Then $H^q(M) = 0$ for $q$ odd and $q \neq 4k+1$ and $H^q(N) = 0$ for $q$ odd. Furthermore the elements of $H^{4k+1}(M)$ and $H^{4l}(N)$ are reduction mod 2 of integer classes because $H^{4l+1}(N;Z) \approx H_{4l-1}(N;Z) = 0$ and $H^{4k+2}(M;Z) \approx H_{4k}(M;Z) \approx H_{4k}(BSU;Z)$ which is free abelian. Note $H^{4(k+l)+1}(M \times N) = H^{4k+1}(M) \otimes H^{4l}(N)$.

LEMMA 4.1. *If* $u \in H^{4k+1}(M)$ *and* $v \in H^{4l}(N)$,

$$\begin{aligned} \phi(u \otimes v) &= \phi(u) \otimes v^2 \quad \text{if } k > 0 \\ &= 0 \qquad\qquad\quad \text{if } k = 0 \text{ and } v^2 = 0. \end{aligned}$$

*Proof.* Let $\hat{u} \in H^{4k+1}(M;Z)$ and $\hat{v} \in H^{4l}(N;Z)$ be classes which give $u$ and $v$ when reduced mod 2. Below we denote $Sq^2{}_f u$ by $Sq^2(f,u)$.

$$\phi(u \otimes v) = \hat{\phi}(\hat{u} \otimes \hat{v}) \tag{4.2}$$

$$= Sq^2(\hat{u} \otimes \hat{v}, Sq^{4(k+l)} \hat{\sigma}_{4(k+l)+1}) \tag{4.3}$$

$$= Sq^2((\hat{u} \times id)(\hat{\sigma}_{4k+1} \otimes \hat{v}), Sq^{4(k+l)} \hat{\sigma}_{4(k+l)+1})$$

$$= Sq^2(\hat{u} \times id, Sq^{4(k+l)}(\hat{\sigma}_{4k+1} \otimes v)) \tag{4.4}$$

$$= Sq^2(u \times id, Sq^{4k} \hat{\sigma}_{4k+1} \otimes v^2) \tag{4.5}$$

$$= Sq^2(u, Sq^{4k} \hat{\sigma}_{4k+1}) \otimes v^2 \tag{4.6}$$

$$= \phi(u) \otimes v^2 \tag{4.7}$$

(4.2) follows from (2.3), (4.2) and (4.7) from (3.2), (4.4) from the naturality of $Sq^2{}_f$, (4.5) from the Cartan formula, and (4.6) from (2.5). In the case $k = 0$, one needs $v^2 = 0$, in order that $\phi(u \otimes v)$ be defined and (4.5) yields $\phi(u \otimes v) = 0$.

We continue with the proof of (1.6). Let $v_{4l}(N) \in H^{4l}(N)$ be the class such that $z^2 = z v_{4l}(N)$ for all $z \in H^{4l}(N)$. Recall, $v_{4l}(N)^2([N]) = \text{index}\, N \bmod 2$ (the proof of this is contained in the argument below). Let $u \in H^{4l}(N)$ be a class such that $u = 0$ if $v_{4l}(N) = 0$, $u = v_{4l}(N)$ if $v_{4l}(N)^2 \neq 0$ and $u v_{4l}(N) \neq 0$ if $v_{4l}(N) \neq 0$ and $v_{4l}(N)^2 = 0$. Let $V \subset H^{4l}(N)$ be the subspace spanned by $u$ and $v_{4l}(N)$ and let $U \subset H^{4l}(N)$ be its orthogonal complement, that is, $U = \{z \in H^{4l}(N) \mid zu = z v_{4l}(N) = 0\}$. $H^{4k+1}(M) \otimes U$ is the orthogonal complement of $H^{4k+1}(M) \otimes V$ in $H^{4(k+l)+1}(M \times N)$. Hence a symplectic basis for each of these subspaces will provide a symplectic basis for $H^{4(k+l)+1}(M\times N)$. By (4.1) $H^{4k+1}(M)\otimes U$ makes no contribution to $\psi\{M\times N\}$ as $z^2=zv_{4l}(N)=0$ if $z\in U$. Let $\{x_i,y_i\}$ be a symplectic basis for $H^{4k+1}(M)$. We now consider four cases.

*Case* I. $v_{4l}(N)=0$. Then $V=0$ and $\psi\{M\times N\}=0=\psi\{M\}v_{4l}^2(N)$.

*Case* II. $v_{4l}^2(N)=0$, $v_{4l}(N)\neq 0$. A symplectic basis for $H^{4k+1}(M)\otimes V$ is given by $\{x_i\otimes v_{4l}(N), y_i\otimes v_{4l}(N)\}$ as the first group of terms and $\{x_i\otimes(v_{4l}(N)+u), y_i\otimes u\}$ as the second group. By (4.1) $\phi(x_i\otimes v_{4l}(N)) =\phi(y_i\otimes v_{4l}(N))=0$. Hence $\psi\{M\times N\}=0$.

*Case* III. $v_{4l}^2(N)\neq 0$, $k>0$. $\{x_i\otimes v_{4l}(N), y_i\otimes v_{4l}(N)\}$ is a symplectic basis for $H^{4k+1}(M)\otimes V$. Therefore by (4.1),

$$\begin{aligned}\psi\{M\times N\}&=\sum\phi(x_i\otimes v_{4l}(N))([M\times N])\phi(y_i\otimes v_{4l}(N))([M\times N])\\&=\sum\phi(x_i)([M])\phi(y_i)([M])v_{4l}^2(N)\\&=\psi(\{M\})I(\{N\}).\end{aligned}$$

*Case* IV. $v_{4l}^2(N)\neq 0$, $k=0$. The generator of $\Omega_2(SU)$ is represented by $M=S^1\times S^1$ with the non-trivial $SU$ reduction of its normal bundle. Let $x\in H^1(S^1)$ be the generator. $\{x\otimes 1\otimes v_{4l}(N), 1\otimes x\otimes v_{4l}(N)\}$ is a symplectic basis for $H^1(M)\otimes V$. Hence

$$\psi\{M\times N\}=\bar\phi\{M\times N, x\otimes 1\otimes v_{4l}(N)\}\bar\phi\{M\times N, 1\otimes x\otimes v_{4l}(N)\}.$$

By a symmetry argument this equals $\bar\phi\{M\times N, x\otimes 1\otimes v_{4l}(N)\}$. By Wu formulas $v_{4l}(N)$ is a polynomial in the Stiefel-Whitney classes of N. Therefore $v_{4l}(N)=\nu_N{}^*(z_{4l})$ where $z_{4l}\in H^{4l}(BSU)$. $z_{4l}$ is the reduction mod 2 of an integer class $\hat z_{4l}$. Hence if $\partial P=S^1\times N-N'$

$$\partial(P,\nu_p{}^*z_{4l})=(S^1\times N, 1\otimes v_{4l}(N))-(N',\nu_{N'}{}^*z_{4l})$$

By (3.1) we may choose $N'$ to be 1-connected. Let $y=\nu_{N'}{}^*z_{4l}$. $y^2\in H^{8l}(N')\approx H_1(N')=0$. Therefore $\phi(x\otimes y)$ is defined. Let $\hat y=\nu_{N'}{}^*\hat z_{4l}$ and let $\hat x\in H^1(S^1;Z)$ be the generator. By (2.3) and (2.4),

$$\psi\{M\times N\}=\hat\phi(\hat x\otimes\hat y)([S^1\times N'])$$

$$\begin{aligned}\hat\phi(\hat x\otimes\hat y)&=Sq^2{}_{\hat x\otimes\hat y}(Sq^{4l}\hat\sigma_{4l+1})\\&=Sq^2{}_{\mathrm{id}\times\hat y}(Sq^{4l}(\hat x\otimes\hat\sigma_{4l}))\\&=Sq^2{}_{\mathrm{id}\times\hat y}(x\otimes\sigma_{4l}{}^2)\\&=x\otimes Sq^2{}_{\hat y}(\sigma_{4l}{}^2).\end{aligned}$$

Hence we must show that $Sq^2{}_{\hat y}(\sigma_{4l}{}^2)\neq 0$.

Let $\nu$ be the normal bundle of $N'$ embedded in $R^{8l+2r}$ for large $r$. Let $\zeta_r$ be the canonical, real $2r$ bundle over $BSU_r$. let $\lambda\colon \nu \to \zeta_r$ be the $SU$ reduction of $\nu$, $T(\nu)$ and $T(\zeta_r)$ the Thom spaces, $U \in H^{2r}(T(\zeta_r);Z)$ the Thom class, and let $f\colon S^{8l+2r} \to T(\nu)$ be the map obtained by the Thom construction. Note $T(\lambda)f$ is homotopic to $g\eta$ where $\eta$ is suspension of the Hopf map and $g\colon S^{8l+2r-1} \to T(\zeta_r)$ is the map obtained from $N$ by the Thom construction. This is because $N'$ and $S^1 \times N$ are $SU$ cobordant.

$$Sq^2{}_{\hat{y}}(\sigma_{4l}{}^2) = Sq^2{}_{\nu_{N'}}(z_{4l}{}^2) \text{ as } \hat{y} = \hat{z}_{4l} \circ \eta_{N'}.$$

Note $Sq^1U = w_1U = 0$ and $Sq^2U = w_2U = 0$.

$$(4.8) \qquad \begin{aligned} f^*(Sq^2{}_{\nu_{N'}}(z_{4l}{}^2) \cdot T(\lambda)^*U) &= f^*Sq^2{}_{T(\lambda)}(z_{4l}{}^2U) \\ &= Sq^2{}_{T(\lambda)f}(z_{4l}{}^2U) \\ &= Sq^2{}_{g\eta}(z_{4l}{}^2U) \\ &= Sq^2{}_{\eta}(g^*z_{4l}{}^2U). \end{aligned}$$

But $Sq^2{}_\eta$ is an isomorphism and $g^*(z_{4l}{}^2U)([S^{8k+2r-1}]) = v_{4l}{}^2([N]) \neq 0$. This completes the proof of (1.6).

We next prove (1.1). Suppose $\gamma = \{M\} \in \Omega_{8k+2}(SU)$ where $M$ is stably parallelizable. We must show that $\psi(\gamma) = 0$. Conner and Floyd [7] and Lashof and Rothenberg (unpublished) show that if $\mu \in \Omega_{2n}(SU)$ has all its Chern numbers zero, then $\mu = \alpha\beta$ where $\alpha \in \Omega_2(SU)$ is the generator and $\beta \in \Omega_{2n-2}(SU)$. Hence $\gamma = \alpha\beta$. By (1.6) $\psi(\gamma) = \psi(\alpha)I(\beta) = I(\beta) = v_{4l}(N)^2([N])$ where $v_{4l}(N)$ and $N$ are as above. Suppose $v_{4l}{}^2(N) \neq 0$. Then by (4.3) in Case IV above, $Sq^2{}_{g\eta}(z_{4l}{}^2U) \neq 0$.

Let $\theta$ be the secondary cohomology operation associated with the relation $Sq^2Sq^2 = 0$ on integer classes.

$$\theta\colon H^q(X;Z) \cap \operatorname{Ker} Sq^2 \to H^{q+3}(X)/Sq^2H^{q+1}(X).$$

Let $\theta_f$ denote the associated functional operation. In [14] it is shown that

$$(4.9) \qquad \theta_{\eta\eta}x = Sq^2{}_\eta Sq^2{}_\eta x.$$

$Sq^2(z_{4l}{}^2U) = 0$, hence $\theta(z_{4l}{}^2U) \in H^{8l+3+2r}(T(\zeta_r)) = 0$ is defined and is zero.

$$\begin{aligned} Sq^2{}_\eta Sq^2{}_\eta(z_{4l}{}^2U) &= Sq^2{}_\eta Sq^2{}_\eta(g^*(z_{4l}{}^2U)) \\ &= \theta_{\eta\eta}(g^*(z_{4l}{}^2U)) \\ &= \theta_{g\eta\eta}(z_{4l}{}^2U). \end{aligned}$$

We show that $\theta_{g\eta\eta}(z_{4l}{}^2U)$ is zero and has zero indeterminacy. Since $Sq^2{}_\eta$ is an isomorphism, this shows that $Sq^2{}_{g\eta}(z_{4l}{}^2U) = 0$, which is the contradiction we seek. Note $g\eta\eta\colon S^{8l+2+2r} \to T(\zeta_r)$ is the map corresponding to $S^1 \times S^1 \times N$

under the Thom construction. By hypothesis, $S^1 \times S^1 \times N$ is $SU$ cobordant to a stably parallelizable manifold. Hence $g\eta\eta$ is homotopic to $ih$ where $i\colon S^{2r} \to T(\zeta_r)$ is the inclusion of a fibre and $h\colon S^{8l+2+2r} \to S^{2r}$. Therefore,

$$\theta_{g\eta\eta}(z_{4l}{}^2U) = \theta_h(i^* z_{4l}{}^2U) = 0.$$

The indeterminacy of $\theta_{g\eta\eta}(z_{4l}{}^2U)$ is

$$(g\eta\eta)^*(H^{8l+2r}(T(\zeta_r))) + \theta(H^{8l+2r-3}(S^{8l+2r})) \\ + Sq^2(H^{8l+2r-2}(S^{8l+2r})) + Sq^2{}_{g\eta\eta}(H^{8l+2r-1}(T(\zeta_r))) = 0.$$

The above argument shows that if $\beta \in \Omega_{8l}(SU)$ and $f\colon S^{8l+2+2r} \to T(\zeta_r)$ is the map associated with $\alpha\beta$, then

$$\psi(\alpha\beta) = \theta_f(z_{4l}{}^2U)([S^{8l+2+2r}])$$

By examining how the operation $\theta_f$ is related to the Thom isomorphism one may prove:

THEOREM 4.10. *If $\alpha \in \Omega_2(SU)$ is the generator and $\beta \in \Omega_{8l}(SU)$, then*

$$\psi(\alpha\beta) = \theta_{\nu_M}(z_{4l}{}^2)([M])$$

*where $M$ is a 3-connected manifold representing $\alpha\beta$.*

## Appendix 1.

We state here, without proof, those parts of our results which go through for $\psi\colon \Omega_{8k+2}(Spin) \to Z_2$.

The first difficulty in generalizing our results to the $Spin$ case is that $\bar{\phi}\colon \Omega_{2m}(K(Z_2, m); Spin) \to Z_2$ depends on the choice of the operation $\phi$ associated to the given relation among primary operations. One choice would be to choose $\phi$ so that the third suspension of $\phi$ is zero on all classes of dimension $m - 3$. This is possible and gives rise to two choices for $\phi$, each of which give the same $\bar{\phi}$.

The only part of 1.6 that we can prove is the case $k = 0$, namely:

THEOREM A1.1. *If $\alpha \in \Omega_2(Spin)$ is the generator and $\beta \in \Omega_{8k}(Spin)$,*

$$\psi(\alpha\beta) = I(\beta)$$

*where $I(\beta)$ is the index of $\beta$* mod 2.

This theorem follows from the arguments used to prove (1.6), except that one must use slightly more complicated cohomology operations. One also sees that $\psi(\alpha\beta)$ is independent of the choice of $\phi$.

Recall, quaternionic projective space $PQ_n$ admits a *Spin* structure and has Euler characteristic 1 if $n$ is even. Hence,

COROLLARY A1.2. *If $\alpha \in \Omega_2(Spin)$ is the generator,*

$$\psi(\alpha\{PQ_{2k}\}) = 1$$

The proof of (1.6) breaks down in the *Spin* case for $k > 0$ because $H^{4(k+l)+1}(M \times N)$ is very complicated even if $M$ and $N$ are simplified by surgery. Also, it is not clear whether an analogous theorem to 4.5 goes through in the spin case.

## Appendix 2.

In this appendix we give a proof, due to John Milnor, that a differentiable manifold of the same homotopy type as a Kervaire manifold ([8]) is stably parallelizable.

Let $n$ be odd and $n \neq 1, 3, 7$, let $p: T \to S^n$ be the tangent disc bundle of $S^n$, let $D^n$ be the closed $n$-disc, $h: D^n \to S^n$ a homeomorphism into, $k: D^n \times D^n \to T$ a bundle map covering $h$, $\bar{T}$ a copy of $T$ and let $L = T \cup \bar{T}$ with $h(x,y)$ identified to $h(y,x)$ for each $(x,y) \in D^n \times D^n$. $L$ is a manifold with boundary and $\partial L$ is homeomorphic to $S^{2n-1}$. Let $f: S^{2n-1} \to \partial L$ be a homeomorphism and let $K^{2n} = L \bigcup_f D^{2n}$. $K^{2n}$ is the manifold constructed by Kervaire.

THEOREM A2.1. *If $M$ is a differentiable manifold with the same homotopy type as $K^{2n}$, then $M$ is stably parallelizable.*

*Proof.* Let $S^n$, $\bar{S}^n$ and $S_1{}^n \subset K^{2n}$ be the zero cross-section of $T$, the zero cross-section of $\bar{T}$ and a cross-section of the associated sphere bundle of $T$, respectively. Clearly $K^{2n}$ has a cell structure $S^n \vee \bar{S}^n \cup e^{2n}$ and hence, since $S^n$ and $S_1{}^n$ are isotopic,

$$K^{2n} = S_1{}^n \vee \bar{S}^n \cup e^{2n}. \qquad K^{2n}/S_1{}^n = \bar{S}^n \cup e^{2n}. \qquad K^{2n}/K^{2n} - \mathrm{Int}\, T$$

is the Thom space of $\tau(S^n)$ which is $S^n \bigcup_{[\iota,\iota]} e^{2n}$. Consider the quotient map

$$u: K^{2n}/S_1{}^n \to K^{2n}/K^{2n} - \mathrm{Int}\, T$$

$u$ is of degree one on the $n$ and $2n$ cells and hence is a homotopy equivalence. Therefore $K^{2n}/S^n = K^{2n}/S_1{}^n = S^n \bigcup_{[\iota,\iota]} e^{2n}$.

Let $g: K^{2n} \to M$ be a homotopy equivalence. We first show that $M$ is almost parallelizable by showing that $g^*\tau(M)$ is trivial on $S^n \vee \bar{S}^n$. Choose $g$ so that $g \mid S^n$ is a smooth embedding and let $\nu$ be the normal bundle of $g(S^n)$. Let $T(\nu)$ be the Thom space of $\nu$, $t: M \to T(\nu)$ the usual map and

$\bar{t}\colon K^{2n} \to K^{2n}/S^n$ the quotient map. $tg \mid S^n$ is homotopically trivial. Hence, up to homotopy, $tg = \bar{g}\bar{t}$ for some $\bar{g}$. One may easily check that $\bar{g}$ is a homotopy equivalence. Hence $T(\nu) = S^n \bigcup_\alpha e^{2n}$ where $\alpha = [\iota, \iota]$. Recall, $\alpha = J(\beta)$, $\beta \in \pi_{n-1}(0_n)$ where $\beta$ is the characteristic class of $\nu$. The stable $J$ homomorphism on $\pi_{n-1}(0)$ is a monomorphism and hence $\beta$ is stably trivial. Hence $\nu$ is stably trivial and therefore $g^*\tau(M) \mid S^n$ is trivial. The same argument shows that $g^*\tau(M) \mid \bar{S}^n$ is trivial.

Recall, the obstruction to an almost parallelizable $m$-manifold being stably parallelizable is in the kernel of $J$ on $\pi_{m-1}(0)$. $J$ is a monomorphism on $\pi_{2n-1}(0)$. Hence $M$ is stably parallelizable.

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