THREE MODELS FOR THE HOMOTOPY THEORY OF HOMOTOPY THEORIES

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ABSTRACT. Given any model category, or more generally any category with weak equivalences, its simplicial localization is a simplicial category which can rightfully be called the "homotopy theory" of the model category. There is a model category structure on the category of simplicial categories, so taking its simplicial localization yields a "homotopy theory of homotopy theories." In this paper we show that there are two different categories of diagrams of simplicial sets, each equipped with an appropriate definition of weak equivalence, such that the resulting homotopy theories are each equivalent to the homotopy theory arising from the model category structure on simplicial categories. Thus, any of these three categories with the respective weak equivalences could be considered a model for the homotopy theory of homotopy theories. One of them in particular, Rezk's complete Segal space model category structure on the category of simplicial spaces, is much more convenient from the perspective of making calculations and therefore obtaining information about a given homotopy theory.

1. INTRODUCTION

Classical homotopy theory considers topological spaces, up to weak homotopy equivalence. Eventually, the structure of the category of topological spaces making it possible to talk about its "homotopy theory" was axiomatized; it is known as a model category structure. In particular, given a model category structure on an arbitrary category, we can talk about its homotopy category. More generally, we can think about the "homotopy theory" given by that category with its particular class of weak equivalences, where the homotopy theory encompasses the homotopy category as well as higher-order information. One might ask what specifically is meant by a homotopy theory.

One answer to this question uses simplicial categories, which in this paper we will always take to mean categories enriched over simplicial sets. Given a model category $\mathcal{M}$, taking its simplicial localization with respect to its subcategory of weak equivalences yields a simplicial category $\mathcal{L}\mathcal{M}$ [9, 4.1]. The simplicial localization encodes the known homotopy-theoretic information of the model category, so one point of view is that this simplicial category is the homotopy theory associated to the model category structure. Set-theoretic issues aside, we can also construct the simplicial localization for any category with a subcategory of weak equivalences, so

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therefore we can speak of an associated homotopy theory even in this more general situation.

Given two homotopy theories, one can ask whether they are equivalent to one another in some natural sense. There is a notion of weak equivalence between two simplicial categories which is a simplicial analogue of an equivalence between categories. These weak equivalences are known as *DK-equivalences*, where the "DK" refers to the fact that they were first defined by Dwyer and Kan in [8]. In fact, there is a model category structure $\mathcal{SC}$ on the category of all (small) simplicial categories in which the weak equivalences are these DK-equivalences [3, 1.1]. The associated homotopy theory of simplicial categories is what we will refer to as the homotopy theory of homotopy theories.

In [17], Rezk takes steps toward finding a model other than that of simplicial categories for the homotopy theory of homotopy theories. He defines complete Segal spaces, which are simplicial spaces satisfying some nice properties (Definitions 3.4 and 3.6 below) and constructs a functor which assigns a complete Segal space to any simplicial category. He considers a model category structure $\mathcal{CSS}$ on the category of all simplicial spaces in which the weak equivalences are levelwise weak equivalences of simplicial sets and then localizes it in such a way that the local objects are the complete Segal spaces (Theorem 3.8).

However, Rezk does not construct a functor from the category of complete Segal spaces to the category of simplicial categories, nor does he discuss the model category $\mathcal{SC}$. In this paper, we complete his work by showing that $\mathcal{SC}$ and $\mathcal{CSS}$ have equivalent homotopy theories. This result is helpful in that the weak equivalences between complete Segal spaces are easy to identify (see Proposition 3.11 below), unlike the weak equivalences between simplicial categories, and therefore making any kind of calculations would be much easier in $\mathcal{CSS}$. Using terminology of Dugger [7], this model category $\mathcal{CSS}$ is a *presentation* for the homotopy theory of homotopy theories, since it is a localization of a category of diagrams of spaces.

In order to prove this result, we make use of an intermediate category. Consider the full subcategory $\mathcal{SeCat}$ of the category of simplicial spaces whose objects are simplicial spaces with a discrete simplicial set in degree zero. We will prove the existence of two model category structures on $\mathcal{SeCat}$, each with the same class of weak equivalences. The first of these structures, which we denote $\mathcal{SeCat}_c$, has as cofibrations the maps which are levelwise cofibrations of simplicial sets. (An alternate proof of the existence of this model category structure is given by Hirschowitz and Simpson [13, 2.3]. They actually prove the existence of such a model category structure for Segal n-categories, whereas we consider only the case where $n = 1$.) The second model category structure, which we denote $\mathcal{SeCat}_f$, has as fibrations maps which can be thought of as localizations of levelwise fibrations of simplicial sets, although strictly speaking they cannot be obtained in this way. We use these model category structures to produce a chain of Quillen equivalences

$$\mathcal{SC} \leftrightarrows \mathcal{SeCat}_f \rightleftarrows \mathcal{SeCat}_c \rightleftarrows \mathcal{CSS}.$$

(In each case, the topmost arrow is the left adjoint of the adjoint pair.) Notice that we can obtain a single Quillen equivalence $\mathcal{SeCat}_f \rightleftarrows \mathcal{CSS}$ via composition. Since Quillen equivalent model categories have DK-equivalent simplicial localizations (Proposition 2.8), all three of these categories with their respective weak equivalences give models for the homotopy theory of homotopy theories.

1.1. **Organization of the Paper.** We begin in section 2 by recalling standard information about model category structures and simplicial objects. In section 3, we state the definitions of simplicial categories, complete Segal spaces, and Segal categories, and we give some basic results about each. In section 4, we set up some constructions on Segal precategories that we will need in order to prove our model category structures. In section 5, we prove the existence of a model category structure $\mathcal{S}e\mathcal{C}at_c$ on the category of Segal precategories which we then in section 6 show is Quillen equivalent to Rezk's complete Segal space model category structure $\mathcal{CSS}$. In section 7, we prove the existence of the model category structure $\mathcal{S}e\mathcal{C}at_f$ on the category of Segal precategories and prove that it is Quillen equivalent to $\mathcal{S}e\mathcal{C}at_c$. We then show in section 8 that $\mathcal{S}e\mathcal{C}at_f$ is Quillen equivalent to the model category structure $\mathcal{SC}$ on simplicial categories. Section 9 contains the proofs of some technical lemmas.

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2. Background on Model Categories and Simplicial Objects

2.1. **Model Categories.** Recall that a model category structure on a category $\mathcal{C}$ is a choice of three distinguished classes of morphisms: fibrations ($\twoheadrightarrow$), cofibrations ($\hookrightarrow$), and weak equivalences ($\xrightarrow{\sim}$). A (co)fibration which is also a weak equivalence is an *acyclic (co)fibration*. With this choice of three classes of morphisms, $\mathcal{C}$ is required to satisfy five axioms MC1-MC5 which can be found in [10, 3.3].

In all the model categories we use, the factorizations given by axiom MC5 can be chosen to be functorial [14, 1.1.1]. An object X in a model category is *fibrant* if the unique map $X \to *$ to the terminal object is a fibration. Dually, X is *cofibrant* if the unique map from the initial object $\phi \to X$ is a cofibration. Given any object X, the functorial factorization of the map $X \to *$ as the composite of an acyclic cofibration followed by a fibration

$$X \overset{\sim}{\hookrightarrow} X^f \twoheadrightarrow *$$

gives us the object X^f, the *fibrant replacement* of X. Dually, we can define its *cofibrant replacement* X^c using the functorial factorization

$$\phi \hookrightarrow X^c \overset{\sim}{\twoheadrightarrow} X.$$

All the model category structures that we work with are cofibrantly generated. In a cofibrantly generated model category, there are two sets of specified morphisms, the generating cofibrations and the generating acyclic cofibrations, such that a map is an acyclic fibration if and only if it has the right lifting property with respect to the generating cofibrations, and a map is a fibration if and only if it has the right lifting property with respect to the generating acyclic cofibrations [12, 11.1.2]. To prove that a particular category with a choice of weak equivalences has a cofibrantly generated model category structure, we need the following definition.

Definition 2.2. [12, 10.5.2] Let $\mathcal{C}$ be a category and I a set of maps in $\mathcal{C}$. Then an *I-injective* is a map which was the right lifting property with respect to every map in I. An *I-cofibration* is a map with the left lifting property with respect to every *I*-injective.

We are now able to state the theorem that we use in this paper to prove the existence of specific model category structures.

Theorem 2.3. [12, 11.3.1] *Let $\mathcal{M}$ be a category with a specified class of weak equivalences which satisfies model category axioms MC1 and MC2. Suppose further that the class of weak equivalences is closed under retracts. Let I and J be sets of maps in $\mathcal{M}$ which satisfy the following conditions:*

(1) *Both I and J permit the small object argument [12, 10.5.15].*
(2) *Every J-cofibration is an I-cofibration and a weak equivalence.*
(3) *Every I-injective is a J-injective and a weak equivalence.*
(4) *One of the following conditions holds:*
    (i) *A map that is an I-cofibration and a weak equivalence is a J-cofibration, or*
    (ii) *A map that is both a J-injective and a weak equivalence is an I-injective.*

Then there is a cofibrantly generated model category structure on $\mathcal{M}$ in which I is a set of generating cofibrations and J is a set of generating acyclic cofibrations.

We now define our notion of "equivalence" between two model categories. Recall that for categories $\mathcal{C}$ and $\mathcal{D}$ a pair of functors

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is an *adjoint pair* if for each object X of $\mathcal{C}$ and object Y of $\mathcal{D}$ there is an isomorphism $\varphi : \mathrm{Hom}_{\mathcal{D}}(FX, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, RY)$ which is natural in X and Y [15, IV.1].

Definition 2.4. [14, 1.3.1] If $\mathcal{C}$ and $\mathcal{D}$ are model categories, then the adjoint pair

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is a *Quillen pair* if one of the following equivalent statements is true:

(1) F preserves cofibrations and acyclic cofibrations.
(2) R preserves fibrations and acyclic fibrations.

Definition 2.5. [14, 1.3.12] A Quillen pair is a *Quillen equivalence* if for all cofibrant X in $\mathcal{C}$ and fibrant Y in $\mathcal{D}$, a map $f : FX \to Y$ is a weak equivalence in $\mathcal{D}$ if and only if the map $\varphi f : X \to RY$ is a weak equivalence in $\mathcal{C}$.

We will use the following proposition to prove that a Quillen pair is a Quillen equivalence. Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ *reflects* a property if, for any morphism f of $\mathcal{C}$, whenever Ff has the property, then so does f.

Proposition 2.6. [14, 1.3.16] *Suppose that*

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : R$$

is a Quillen pair. Then the following statements are equivalent:

(1) *This Quillen pair is a Quillen equivalence.*
(2) *F reflects weak equivalences between cofibrant objects and, for every fibrant Y in $\mathcal{D}$, the map $F((RY)^c) \to Y$ is a weak equivalence.*
(3) *R reflects weak equivalences between fibrant objects and, for every cofibrant X in $\mathcal{C}$, the map $X \to R((FX)^f)$ is a weak equivalence.*

The existence of a Quillen equivalence between two model categories is actually a stronger condition than we need, but it is a convenient way to show that two homotopy theories are the same. Here, we take the viewpoint that simplicial categories are models for homotopy theories. A *simplicial category* is a category $\mathcal{C}$ enriched over simplicial sets, or a category such that, for objects x and y of $\mathcal{C}$, there is a simplicial set of morphisms $\text{Hom}_{\mathcal{C}}(x, y)$ between them. We will use the following notion of equivalence of simplicial categories.

Definition 2.7. [8, 2.4] A functor $f \colon \mathcal{C} \to \mathcal{D}$ between two simplicial categories is a *DK-equivalence* if it satisfies the following two conditions:

(1) for any objects x and y of $\mathcal{C}$, the induced map $\text{Hom}_{\mathcal{C}}(x, y) \to \text{Hom}_{\mathcal{D}}(fx, fy)$ is a weak equivalence of simplicial sets, and

(2) the induced map of categories of components $\pi_0 f \colon \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ is an equivalence of categories.

Recall that the *category of components* $\pi_0 \mathcal{C}$ of a simplicial category $\mathcal{C}$ is the category with the same objects as $\mathcal{C}$ and such that

$$\text{Hom}_{\pi_0 \mathcal{C}}(x, y) = \pi_0 \text{Hom}_{\mathcal{C}}(x, y).$$

Now, the following result tells us that model categories which are Quillen equivalent actually have equivalent homotopy theories.

Proposition 2.8. [8, 5.4] *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent model categories. Then the simplicial localizations $\mathcal{L}\mathcal{C}$ and $\mathcal{L}\mathcal{D}$ are DK-equivalent.*

2.9. **Simplicial Objects.** Recall that a *simplicial set* is a functor $\mathbf{\Delta}^{op} \to \mathcal{S}ets$, where the *cosimplicial category* $\mathbf{\Delta}$ has as objects the finite ordered sets $[n] = \{0, \ldots, n\}$ and as morphisms the order-preserving maps, and $\mathbf{\Delta}^{op}$ is its opposite category. In particular, for $n \geq 0$, we have $\Delta[n]$, the n-simplex, $\dot{\Delta}[n]$, the boundary of $\Delta[n]$, and, for $n > 0$ and $0 \leq k \leq n$, $V[n, k]$, which is $\dot{\Delta}[n]$ with the kth face removed [11, I.1]. For any simplicial set X, we denote by X_n the image of $[n]$. There are face maps $d_i \colon X_n \to X_{n-1}$ for $0 \leq i \leq n$ and degeneracy maps $s_i \colon X_n \to X_{n+1}$ for $0 \leq i \leq n$, satisfying certain compatibility conditions [11, I.1]. We denote by $|X|$ the topological space given by geometric realization of the simplicial set X [11, I.2].

There is a model category structure on simplicial sets in which the weak equivalences are the maps which become weak homotopy equivalences of topological spaces after geometric realization [11, I.11.3]. We denote this model category structure by $\mathcal{S}\mathcal{S}ets$. Note in particular that it is cofibrantly generated. The generating cofibrations are the maps $\dot{\Delta}[m] \to \Delta[m]$ for all $m \geq 0$, and the generating acyclic cofibrations are the maps $V[m, k] \to \Delta[m]$ for all $m \geq 1$ and $0 \leq k \leq m$ [14, 3.2.1]. This model category structure is Quillen equivalent to the standard model category structure on topological spaces [14, 3.6.7]. In light of this fact, we will sometimes refer to simplicial sets as "spaces."

More generally, a simplicial object in a category $\mathcal{C}$ is a functor $\mathbf{\Delta}^{op} \to \mathcal{C}$ [14, 3.1]. In particular, a *simplicial space* (or *bisimplicial set*) is a functor $\mathbf{\Delta}^{op} \to \mathcal{S}\mathcal{S}ets$ [11, IV.1]. Given a simplicial set X, we also use X to denote the constant simplicial space with the simplicial set X in each degree. By X^t we denote the simplicial space such that $(X^t)_n$ is the constant simplicial set X_n, or the simplicial set which has the set X_n in each degree.

Notice, however, that our definition of "simplicial category" in this paper is inconsistent with this terminology. There is a more general notion of simplicial category by which is meant a simplicial object in the category of small categories. Such a simplicial category is a functor $\mathbf{\Delta}^{op} \to \mathcal{C}at$ where $\mathcal{C}at$ is the category with objects the small categories and morphisms the functors between them. Our definition of simplicial category coincides with this one when the extra condition is imposed that the face and degeneracy maps be the identity map on objects [8, 2.1].

We also require the following additional structure on some of our model category structures. A *simplicial model category* is a model category which is also a simplicial category satisfying two additional axioms [12, 9.1.6]. (Again, the terminology is potentially confusing because a simplicial model category is not a simplicial object in the category of model categories.) The important part of this structure that we use is the fact that, given objects X and Y of a simplicial model category, it makes sense to talk about the *function complex*, or simplicial set $\mathrm{Map}(X, Y)$.

Given a model category $\mathcal{M}$, or more generally a category with weak equivalences, a *homotopy function complex* $\mathrm{Map}^h(X, Y)$ is a simplicial set which is the morphism space between X and Y in the simplicial localization $\mathcal{L}\mathcal{M}$ [8, §4]. If $\mathcal{M}$ is a simplicial model category, X is cofibrant in $\mathcal{M}$, and Y is fibrant in $\mathcal{M}$, then $\mathrm{Map}^h(X, Y)$ is weakly equivalent to $\mathrm{Map}(X, Y)$.

2.10. **Localized Model Category Structures.** Several of the model category structures that we use are obtained by localizing a given model category structure with respect to a map or a set of maps. Suppose that $S = \{f \colon A \to B\}$ is a set of maps with respect to which we would like to localize a model category (or category with weak equivalences) $\mathcal{M}$. We define an *S-local* object W to be an object of $\mathcal{M}$ such that for any $f \colon A \to B$ in S, the induced map on homotopy function complexes

$$f^* \colon \mathrm{Map}^h(B, W) \to \mathrm{Map}^h(A, W)$$

is a weak equivalence of simplicial sets. (If $\mathcal{M}$ is a model category, a local object is usually required to be fibrant.) A map $g \colon X \to Y$ in $\mathcal{M}$ is then defined to be an *S-local equivalence* if for every local object W, the induced map on homotopy function complexes

$$g^* \colon \mathrm{Map}^h(Y, W) \to \mathrm{Map}^h(X, W)$$

is a weak equivalence of simplicial sets.

The following theorem holds for model categories $\mathcal{M}$ which are left proper and cellular. We will not define these conditions here, but refer the reader to [12, 13.1.1, 12.1.1] for more details. We do note, in particular, that a cellular model category is cofibrantly generated. All the model categories that we localize in this paper can be shown to satisfy both these conditions.

Theorem 2.11. [12, 4.1.1] *Let $\mathcal{M}$ be a left proper cellular model category. There is a model category structure $\mathcal{L}_S\mathcal{M}$ on the underlying category of $\mathcal{M}$ such that:*

(1) *The weak equivalences are the S-local equivalences.*
(2) *The cofibrations are precisely the cofibrations of $\mathcal{M}$.*
(3) *The fibrations are the maps which have the right lifting property with respect to the maps which are both cofibrations and S-local equivalences.*
(4) *The fibrant objects are the S-local objects which are fibrant in $\mathcal{M}$.*
(5) *If $\mathcal{M}$ is a simplicial model category, then its simplicial structure induces a simplicial structure on $\mathcal{L}_S\mathcal{M}$.*

In particular, given an object X of $\mathcal{M}$, we can talk about its functorial fibrant replacement LX in $\mathcal{L}_S\mathcal{M}$. The object LX is an S-local object which is fibrant in $\mathcal{M}$, and we will refer to it as the *localization* of X in $\mathcal{L}_S\mathcal{M}$.

2.13. **Model Category Structures for Diagrams of Spaces.** Suppose that $\mathcal{D}$ is a small category and consider the category of functors $\mathcal{D} \to \mathcal{SS}ets$, denoted $\mathcal{SS}ets^{\mathcal{D}}$. This category is also called the category of $\mathcal{D}$-diagrams of spaces. We would like to consider model category structures on $\mathcal{SS}ets^{\mathcal{D}}$.

A natural choice for the weak equivalences in $\mathcal{SS}ets^{\mathcal{D}}$ is the class of levelwise weak equivalences of simplicial sets. Namely, given two $\mathcal{D}$-diagrams X and Y, we define a map $f\colon X \to Y$ to be a weak equivalence if and only if for each object d of $\mathcal{D}$, the map $X(d) \to Y(d)$ is a weak equivalence of simplicial sets.

There is a model category structure $\mathcal{SS}ets^{\mathcal{D}}_f$ on the category of $\mathcal{D}$-diagrams with these weak equivalences and in which the fibrations are given by levelwise fibrations of simplicial sets. The cofibrations in $\mathcal{SS}ets^{\mathcal{D}}_f$ are then the maps of simplicial spaces which have the left lifting property with respect to the maps which are levelwise acyclic fibrations. This model structure is often called the *projective* model category structure on $\mathcal{D}$-diagrams of spaces [11, IX, 1.4]. Dually, there is a model category structure $\mathcal{SS}ets^{\mathcal{D}}_c$ in which the cofibrations are given by levelwise cofibrations of simplicial sets, and this model structure is often called the *injective* model category structure [11, VIII, 2.4]. The small category $\mathcal{D}$ which we use in this paper is $\mathbf{\Delta}^{op}$, so that the diagram category $\mathcal{SS}ets^{\mathbf{\Delta}^{op}}$ is just the category of simplicial spaces.

Consider the Reedy model category structure on simplicial spaces [16]. In this structure, the weak equivalences are again the levelwise weak equivalences of simplicial sets. The Reedy model category structure is cofibrantly generated, where the generating cofibrations are the maps

$$\dot{\Delta}[m] \times \Delta[n]^t \cup \Delta[m] \times \dot{\Delta}[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $n, m \geq 0$. The generating acyclic cofibrations are the maps

$$V[m,k] \times \Delta[n]^t \cup \Delta[m] \times \dot{\Delta}[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $n \geq 0$, $m \geq 1$, and $0 \leq k \leq m$ [17, 2.4].

It turns out that the Reedy model category structure on simplicial spaces is exactly the same as the injective model category structure on this same category, as given by the following result.

Proposition 2.14. [12, 15.8.7, 15.8.8] *A map $f\colon X \to Y$ of simplicial spaces is a cofibration in the Reedy model category structure if and only if it is a monomorphism. In particular, every simplicial space is Reedy cofibrant.*

In light of this result, we denote the Reedy model structure on simplicial spaces by $\mathcal{SS}ets^{\mathbf{\Delta}^{op}}_c$. Both $\mathcal{SS}ets^{\mathbf{\Delta}^{op}}_c$ and $\mathcal{SS}ets^{\mathbf{\Delta}^{op}}_f$ are simplicial model categories. In each case, given two simplicial spaces X and Y, we can define $\mathrm{Map}(X,Y)$ by

$$\mathrm{Map}(X,Y)_n = \mathrm{Hom}(X \times \Delta[n], Y)$$

where the set on the right-hand side consists of maps of simplicial spaces.

To establish some notation we need later in the paper, we recall the definition of fibration in the Reedy model category structure. If X is a simplicial space, let $\mathrm{sk}_n X$ denote its n-skeleton, generated by the spaces in degrees less than or equal to n, and let $\mathrm{cosk}_n X$ denote the n-coskeleton of X [16, §1]. A map $X \to Y$ is a fibration in $\mathcal{SS}ets^{\mathbf{\Delta}^{op}}_c$ if

- $X_0 \to Y_0$ is a fibration of simplicial sets, and
- for all $n \geq 1$, the map $X_n \to P_n$ is a fibration, where P_n is defined to be the pullback in the following diagram:

$$
\begin{array}{ccc}
P_n & \longrightarrow & Y_n \\
\downarrow & & \downarrow \\
(\mathrm{cosk}_{n-1}X)_n & \longrightarrow & (\mathrm{cosk}_{n-1}Y)_n
\end{array}
$$

Notice in particular that this pullback diagram is actually a homotopy pullback diagram, as follows. If $f \colon X \to Y$ is a Reedy fibration, then it has the right lifting property with respect to all Reedy acyclic cofibrations. In particular, there is a dotted arrow lift in the following diagram, where $m \geq 1$, $0 \leq k \leq m$, and $n \geq 0$:

$$
\begin{array}{ccc}
V[m,k] \times \dot{\Delta}[n]^t & \longrightarrow & X \\
\downarrow & \nearrow & \downarrow \\
\Delta[m] \times \dot{\Delta}[n]^t & \longrightarrow & Y.
\end{array}
$$

Since the functors sk_n and cosk_n are adjoint [16, §1], we have that

$$
(\mathrm{cosk}_{n-1}X)_n \simeq \mathrm{Map}(\Delta[n], \mathrm{cosk}_n X) \simeq \mathrm{Map}(\mathrm{sk}_n\Delta[n], X) \simeq \mathrm{Map}(\dot{\Delta}[n], X).
$$

Therefore, we have a dotted arrow lift in each diagram

$$
\begin{array}{ccc}
V[m,k] & \longrightarrow & (\mathrm{cosk}_{n-1}X)_n \\
\downarrow & \nearrow & \downarrow \\
\Delta[m] & \longrightarrow & (\mathrm{cosk}_{n-1}Y)_n.
\end{array}
$$

In particular, the right-hand vertical arrow is a fibration of simplicial sets. Thus, the simplicial set P_n is a homotopy pullback and therefore homotopy invariant.

We also make use of the projective model category structure $\mathcal{SS}ets_f^{\Delta^{op}}$ on simplicial spaces. This model category is also cofibrantly generated; the generating cofibrations are the maps

$$
\dot{\Delta}[m] \times \Delta[n]^t \to \Delta[m] \times \Delta[n]^t
$$

for all $m, n \geq 0$ [11, IV.3.1].

In the next section, we localize the Reedy (or injective) and projective model category structures on simplicial spaces with respect to a map to obtain model category structures in which the fibrant objects are Segal spaces (Definition 3.4). We will further localize them to obtain model category structures in which the fibrant objects are complete Segal spaces (Definition 3.6).

3. Some Definitions and Model Category Structures

In this section, we define and discuss in turn the three main structures that we will use in the course of this paper: simplicial categories, complete Segal spaces, and Segal categories.

3.1. **Simplicial Categories.** Simplicial categories, most simply stated, are categories enriched over simplicial sets, or categories with a simplicial set of morphisms between any two objects. So, given any objects x and y in a simplicial category $\mathcal{C}$, there is a simplicial set $\text{Hom}_{\mathcal{C}}(x, y)$.

Fix an object set $\mathcal{O}$ and consider the category of simplicial categories with object set $\mathcal{O}$ such that all morphisms are the identity on the objects. Dwyer and Kan define a model category structure $\mathcal{SC}_{\mathcal{O}}$ in which the weak equivalences are the functors $f \colon \mathcal{C} \to \mathcal{D}$ of simplicial categories such that given any objects x and y of $\mathcal{C}$, the induced map

$$\text{Hom}_{\mathcal{C}}(x, y) \to \text{Hom}_{\mathcal{D}}(x, y)$$

is a weak equivalence of simplicial sets [9, §7]. The fibrations are the functors $f \colon \mathcal{C} \to \mathcal{D}$ for which these same induced maps are fibrations, and the cofibrations are the functors which have the left lifting property with respect to the acyclic fibrations.

It is more useful, however, to consider the category of all small simplicial categories with no restriction on the objects. Before describing the model category structure on this category, we need a few definitions. Recall from Definition 2.7 above that if $\mathcal{C}$ is a simplicial category, then we denote by $\pi_0 \mathcal{C}$ the category of components of $\mathcal{C}$.

If $\mathcal{C}$ is a simplicial category and x and y are objects of $\mathcal{C}$, a morphism $e \in \text{Hom}_{\mathcal{C}}(x, y)_0$ is a *homotopy equivalence* if the image of e in $\pi_0 \mathcal{C}$ is an isomorphism.

Theorem 3.2. [3, 1.1] *There is a model category structure on the category $\mathcal{SC}$ of small simplicial categories defined by the following three classes of morphisms:*

(1) *The weak equivalences are the maps $f \colon \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:*
- *(W1) For any objects x and y in $\mathcal{C}$, the map*

$$Hom_{\mathcal{C}}(x, y) \to Hom_{\mathcal{D}}(fx, fy)$$

  is a weak equivalence of simplicial sets.
- *(W2) The induced functor $\pi_0 f \colon \pi_0 \mathcal{C} \to \pi_0 \mathcal{D}$ on the categories of components is an equivalence of categories.*

(2) *The fibrations are the maps $f \colon \mathcal{C} \to \mathcal{D}$ satisfying the following two conditions:*
- *(F1) For any objects x and y in $\mathcal{C}$, the map*

$$Hom_{\mathcal{C}}(x, y) \to Hom_{\mathcal{D}}(fx, fy)$$

  is a fibration of simplicial sets.
- *(F2) For any object x_1 in $\mathcal{C}$, y in $\mathcal{D}$, and homotopy equivalence $e \colon fx_1 \to y$ in $\mathcal{D}$, there is an object x_2 in $\mathcal{C}$ and homotopy equivalence $d \colon x_1 \to x_2$ in $\mathcal{C}$ such that $fd = e$.*

(3) *The cofibrations are the maps which have the left lifting property with respect to the maps which are fibrations and weak equivalences.*

Notice that the weak equivalences are precisely the DK-equivalences that we defined above (Definition 2.7).

The proof of this theorem actually shows that this model category structure is cofibrantly generated. Define the functor $U \colon \mathcal{SSets} \to \mathcal{SC}$ such that for any simplicial set K, the simplicial category UK has two objects, x and y, and only

nonidentity morphisms the simplicial set $K = \mathrm{Hom}(x, y)$. Using this functor, we define the generating cofibrations to be the maps of simplicial categories

- (C1) $U\dot{\Delta}[n] \to U\Delta[n]$ for $n \geq 0$, and
- (C2) $\phi \to \{x\}$, where ϕ is the simplicial category with no objects and $\{x\}$ denotes the simplicial category with one object x and no nonidentity morphisms.

The generating acyclic cofibrations are defined similarly [3, §1].

3.3. **Segal Spaces and Complete Segal Spaces.** Complete Segal spaces, defined by Rezk in [17], are more difficult to describe, but ultimately they are actually easier to work with than simplicial categories. The name "Segal" refers to the similarity between Segal spaces and Segal's Γ-spaces [18].

We begin by defining Segal spaces. In [17, 4.1], Rezk defines for each $0 \leq i \leq k-1$ a map $\alpha^i \colon [1] \to [k]$ in $\mathbf{\Delta}$ such that $0 \mapsto i$ and $1 \mapsto i+1$. Then for each k he defines the simplicial space

$$ G(k)^t = \bigcup_{i=0}^{k-1} \alpha^i \Delta[1]^t \subset \Delta[k]^t. $$

He shows that, for any simplicial space X, there is a weak equivalence of simplicial sets

$$ \mathrm{Map}^h_{\mathcal{SS}ets^{\mathbf{\Delta}^{op}}}(G(k)^t, X) \to \underbrace{X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{k}, $$

where the right hand side is the homotopy limit of the diagram

$$ X_1 \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 \xrightarrow{d_0} \cdots \xrightarrow{d_0} X_0 \xleftarrow{d_1} X_1 $$

with k copies of X_1.

Now, given any k, define the map $\varphi^k \colon G(k)^t \to \Delta[k]^t$ to be the inclusion map. Then for any simplicial space W there is a map

$$ \varphi_k = \mathrm{Map}^h_{\mathcal{SS}ets^{\mathbf{\Delta}^{op}}}(\varphi^k, W) \colon \mathrm{Map}^h_{\mathcal{SS}ets^{\mathbf{\Delta}^{op}}}(\Delta[k]^t, W) \to \mathrm{Map}^h_{\mathcal{SS}ets^{\mathbf{\Delta}^{op}}}(G(k)^t, W). $$

More simply written, this map is

$$ \varphi_k \colon W_k \to \underbrace{W_1 \times^h_{W_0} \cdots \times^h_{W_0} W_1}_{k} $$

and is often called a *Segal map*.

Definition 3.4. [17, 4.1] A Reedy fibrant simplicial space W is a *Segal space* if for each $k \geq 2$ the map φ_k is a weak equivalence of simplicial sets. In other words, the Segal maps

$$ \varphi_k \colon W_k \to \underbrace{W_1 \times^h_{W_0} \cdots \times^h_{W_0} W_1}_{k} $$

are weak equivalences for all $k \geq 2$.

Notice that if W is a Segal space, or more generally if W is Reedy fibrant, we can use ordinary function complexes and a limit in the definition of the Segal maps [17, §4].

Rezk defines the coproduct of all these inclusion maps

$$ \varphi = \coprod_{k \geq 0} (\varphi^k \colon G(k)^t \to \Delta[k]^t). $$

Using this map φ, we have the following result.

Theorem 3.5. [17, 7.1] *There is a model category structure on simplicial spaces which can be obtained by localizing the Reedy model category structure with respect to the map φ. This model category structure has the following properties :*

    (1) *The weak equivalences are the maps f for which $\mathrm{Map}^h_{\mathcal{SS}ets^{\Delta^{op}}}(f, W)$ is a weak equivalence of simplicial sets for any Segal space W.*

    (2) *The cofibrations are the monomorphisms.*

    (3) *The fibrant objects are the Reedy fibrant φ-local objects, which are precisely the Segal spaces.*

We will refer to this model category structure on simplicial spaces as the *Segal space model category structure* and denote it $\mathcal{SeSp}_c$.

The properties of Segal spaces enable us to speak of them much in the same way that we speak of categories. Heuristically, a simple example of a Segal space is the nerve of a category $\mathcal{C}$, regarded as a simplicial space $\mathrm{nerve}(\mathcal{C})^t$. (We need to take a Reedy fibrant replacement of this nerve to be an actual Segal space.) In particular, we can define "objects" and "maps" of a Segal space. We summarize the particular details here that we need; a full description is given by Rezk [17, §5].

Given a Segal space W, define its set of *objects*, denoted $\mathrm{ob}(W)$, to be the set of 0-simplices of the space W_0, namely, the set $W_{0,0}$. Given any two objects x, y in $\mathrm{ob}(W)$, define the *mapping space* $\mathrm{map}_W(x, y)$ to be the homotopy fiber of the map $(d_1, d_0)\colon W_1 \to W_0 \times W_0$ over (x, y). (Note that since W is Reedy fibrant, this map is a fibration, and therefore in this case we can just take the fiber.) Given a 0-simplex x of W_0, we denote by id_x the image of the degeneracy map $s_0\colon W_0 \to W_1$. We say that two 0-simplices of $\mathrm{map}_W(x, y)$, say f and g, are *homotopic*, denoted $f \sim g$, if they lie in the same component of the simplicial set $\mathrm{map}_W(x, y)$.

Given $f \in \mathrm{map}_W(x, y)_0$ and $g \in \mathrm{map}_W(y, z)_0$, there is a composite $g \circ f \in \mathrm{map}_W(x, z)_0$, and this notion of composition is associative up to homotopy. We define the *homotopy category* $\mathrm{Ho}(W)$ of W to have as objects the set $\mathrm{ob}(W)$ and as morphisms between any two objects x and y, the set $\mathrm{map}_{\mathrm{Ho}(W)}(x, y) = \pi_0 \mathrm{map}_W(x, y)$.

A map g in $\mathrm{map}_W(x, y)_0$ is a *homotopy equivalence* if there exist maps $f, h \in \mathrm{map}_W(y, x)_0$ such that $g \circ f \sim \mathrm{id}_y$ and $h \circ g \sim \mathrm{id}_x$. Any map in the same component as a homotopy equivalence is itself a homotopy equivalence [17, 5.8]. Therefore we can define the space W_{hoequiv} to be the subspace of W_1 given by the components whose zero-simplices are homotopy equivalences.

We then note that the degeneracy map $s_0\colon W_0 \to W_1$ factors through W_{hoequiv} since for any object x the map $s_0(x) = \mathrm{id}_x$ is a homotopy equivalence. Therefore, we have the following definition:

Definition 3.6. [17, §6] A *complete Segal space* is a Segal space W for which the map $s_0\colon W_0 \to W_{\mathrm{hoequiv}}$ is a weak equivalence of simplicial sets.

We now consider an alternate way of defining a complete Segal space which is less intuitive but will enable us to localize the Segal space model category structure further in such a way that the complete Segal spaces are the new fibrant objects. Consider the category $I[1]$ which consists of two objects x and y and exactly two non-identity maps which are inverse to one another, $x \to y$ and $y \to x$. Denote by E the nerve of this category, and by E^t the corresponding simplicial space. There are two maps $\Delta[0]^t \to E^t$ given by the inclusions of $\Delta[0]^t$ to the objects x and y,

respectively. Let $\psi\colon \Delta[0]^t \to E^t$ be the map which takes $\Delta[0]^t$ to the object x. (It does not actually matter which one of the two maps we have chosen, as long as it is fixed.) This map then induces, for any Segal space W, a map on homotopy function complexes

$$\psi^*\colon \mathrm{Map}^h_{\mathcal{SS}ets^{\Delta^{op}}}(E^t, W) \to \mathrm{Map}^h_{\mathcal{SS}ets^{\Delta^{op}}}(\Delta[0]^t, W) = W_0.$$

Proposition 3.7. [17, 6.4] *For any Segal space W, the map ψ^* of homotopy function complexes is a weak equivalence of simplicial sets if and only if W is a complete Segal space.*

Given this proposition, we can further localize the category of simplicial spaces with respect to this map.

Theorem 3.8. [17, 7.2] *Taking the localization of the Reedy model category structure on simplicial spaces with respect to the maps φ and ψ above results in a model category structure which satisfies the following properties:*

(1) *The weak equivalences are the maps f such that $\mathrm{Map}^h_{\mathcal{SS}ets^{\Delta^{op}}}(f, W)$ is a weak equivalence of simplicial sets for any complete Segal space W.*

(2) *The cofibrations are the monomorphisms.*

(3) *The fibrant objects are the complete Segal spaces.*

We refer to this model category structure on simplicial spaces as the *complete Segal space model category structure*, denoted $\mathcal{CSS}$. It turns out that when the objects involved are Segal spaces, the weak equivalences in this model category structure can be described more explicitly.

Definition 3.9. A map $f\colon U \to V$ of Segal spaces is a *DK-equivalence* if

(1) for any pair of objects $x, y \in U_0$, the induced map $\mathrm{map}_U(x, y) \to \mathrm{map}_V(fx, fy)$ is a weak equivalence of simplicial sets, and

(2) the induced map $\mathrm{Ho}(f)\colon \mathrm{Ho}(U) \to \mathrm{Ho}(V)$ is an equivalence of categories.

We then have the following result by Rezk:

Theorem 3.10. [17, 7.7] *Let $f\colon U \to V$ be a map of Segal spaces. Then f is a DK-equivalence if and only if it becomes a weak equivalence in $\mathcal{CSS}$.*

Note that these weak equivalences have been given the same name as the ones in $\mathcal{SC}$. While this may at first seem strange, the two definitions are very similar, in fact rely on the same generalization of the idea of equivalence of categories to a simplicial setting.

However, what is especially nice about the complete Segal space model category structure is the simple characterization of the weak equivalences between the fibrant objects.

Proposition 3.11. [17, 7.6] *A map $f\colon U \to V$ between complete Segal spaces is a DK-equivalence if and only if it is a levelwise weak equivalence.*

This proposition is actually a special case of a more general result. In any localized model category structure, a map is a local equivalence between fibrant objects if and only if it is a weak equivalence in the original model category structure [12, 3.2.18].

It is also possible to localize the projective model category structure $\mathcal{SS}ets^{\Delta^{op}}_f$ on the category of simplicial spaces to obtain analogous model category structures.

We will denote the localization of the projective model category structure by with respect to the map φ by $\mathcal{S}e\mathcal{S}p_f$. There is also a localization of the projective model category structure with respect to the maps φ and ψ analogous to the model category structure $\mathcal{C}\mathcal{S}\mathcal{S}$, but we do not need this structure here.

3.12. **Segal Categories.** Lastly, we consider the Segal categories. We begin by defining the preliminary notion of a Segal precategory.

Definition 3.13. [13, §2] A *Segal precategory* is a simplicial space X such that the simplicial set X_0 in degree zero is discrete, i.e., a constant simplicial set.

In the case of Segal precategories, it again makes sense to talk about the Segal maps

$$\varphi_k \colon X_k \to \underbrace{X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1}_{k}$$

for each $k \geq 2$. Since X_0 is discrete, we can actually take the limit

$$\underbrace{X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{k}$$

on the right-hand side.

Definition 3.14. [13, §2] A *Segal category* X is a Segal precategory such that each Segal map φ_k is a weak equivalence of simplicial sets for $k \geq 2$.

Note that the definition of a Segal category is similar to that of a Segal space, with the additional requirement that the degree zero space be discrete. However, Segal categories are not required to be Reedy fibrant, so they are not necessarily Segal spaces.

Given a fixed set $\mathcal{O}$, we can consider the category $\mathcal{S}\mathcal{S}ets_{\mathcal{O}}^{\mathbf{\Delta}^{op}}$ whose objects are the Segal precategories with $\mathcal{O}$ in degree zero and whose morphisms are the identity on this set. There is a model category structure $\mathcal{S}\mathcal{S}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$ on this category in which the weak equivalences are levelwise [5, 3.7]. In other words, $f \colon X \to Y$ is a weak equivalence if for each $n \geq 0$, the map $f_n \colon X_n \to Y_n$ is a weak equivalence of simplicial sets. Furthermore, the fibrations are also levelwise. This model structure can then be localized with respect to a map similar to the map which we used to obtain the Segal space model category structure.

We first need to determine what this map should be. We begin by considering the maps of simplicial spaces $\varphi^k \colon G(k)^t \to \Delta[k]^t$ and adapting them to the case at hand.

The first problem is that $\Delta[k]^t$ is not going to be in $\mathcal{S}\mathcal{S}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$ for all values of k. Instead, we need to define a separate k-simplex for any k-tuple $x_0, \ldots, x_k$ of objects in $\mathcal{O}$, denoted $\Delta[k]_{x_0,\ldots,x_k}^t$, so that the objects are preserved. Note that this object $\Delta[k]_{x_0,\ldots,x_k}^t$ also needs to have all elements of $\mathcal{O}$ as 0-simplices, so we add any of these elements that have not already been included in the x_i's, plus their degeneracies in higher degrees.

Then we can define

$$G(k)_{x_0,\ldots,x_k}^t = \bigcup_{i=0}^{k-1} \alpha^i \Delta[1]_{x_i,x_{i+1}}^t.$$

Now, we need to take coproducts not only over all values of k, but also over all k-tuples of vertices. Hence, the resulting map $\varphi_{\mathcal{O}}$ looks like

$$\varphi_{\mathcal{O}} = \coprod_{k \geq 0} (\coprod_{(x_0,\ldots,x_k) \in \mathcal{O}^{k+1}} (G(k)^t_{x_0,\ldots,x_k} \to \Delta[k]^t_{x_0,\ldots,x_k})).$$

Setting $\underline{x} = (x_0, \ldots, x_k)$, we can write the component maps as $G(k)^t_{\underline{x}} \to \Delta[k]^t_{\underline{x}}$. We can then localize $\mathcal{SS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},f}$ with respect to the map $\varphi_{\mathcal{O}}$ to obtain a model category which we denote $\mathcal{LSS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},f}$.

There are also analogous model category structures $\mathcal{SS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},c}$ and $\mathcal{LSS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},c}$ on the category of Segal precategories with a fixed set $\mathcal{O}$ in degree zero with the same weak equivalences but where the cofibrations, rather than the fibrations, are defined levelwise, and then we can localize with respect to the same map [5, 3.9], [19, A.1.1].

However, we would like a model category structure on the category of all Segal precategories, not just on these more restrictive subcategories. In the course of this paper, we prove the existence of two model category structures on Segal precategories. Unlike in the fixed object set case, we cannot actually obtain the model category structure via localization of a model category structure with levelwise weak equivalences since it is not possible to put a model structure on the category of Segal precategories in which the weak equivalences are levelwise and in which the cofibrations are monomorphisms.

To see that there is no such model structure, suppose that one did exist and consider the map $f \colon \Delta[0]^t \amalg \Delta[0]^t \to \Delta[0]^t$. By model category axiom MC5, f could be factored as the composite of a cofibration $\Delta[0]^t \amalg \Delta[0]^t \to X$ followed by an acyclic fibration $X \to \Delta[0]^t$. However, since the weak equivalences would be levelwise weak equivalences, X_0 would have to consist of one point. However, the only map $(\Delta[0]^t \amalg \Delta[0]^t)_0 \to X_0$ is not a monomorphism. Thus, there is no such factorization of the map f, and therefore there can be no model category structure satisfying the two given properties.

3.15. **Relationship Between Simplicial Categories and Segal Categories in Fixed Object Set Cases.** Recall from above that there is a model category structure $\mathcal{SC}_{\mathcal{O}}$ on the category whose objects are the simplicial categories with a fixed set $\mathcal{O}$ of objects and whose morphisms are the functors which are the identity on the objects and that there is a model category structure $\mathcal{LSS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},f}$ on the category whose objects are the Segal precategories with the set $\mathcal{O}$ in degree zero and whose morphisms are the identity on degree zero.

Theorem 3.16. [5, 5.5] *There is an adjoint pair*

$$F_{\mathcal{O}} : \mathcal{LSS}ets^{\boldsymbol{\Delta}^{op}}_{\mathcal{O},f} \rightleftarrows \mathcal{SC}_{\mathcal{O}} : R_{\mathcal{O}}$$

which is a Quillen equivalence.

The proof of this theorem uses a generalization of a result by Badzioch [1, 6.5] which relates strict and homotopy algebras over an algebraic theory. This generalization uses the notion of multi-sorted algebraic theory [4].

A key step in this proof is a explicit description of the localization of the objects $\Delta[n]^t_{\underline{x}}$. Up to homotopy, this localization is the same as the localization of the

objects $G(n)_{\underline{x}}^t$ and is obtained by taking the colimit of stages of a filtration

$$G(n)_{\underline{x}}^t = \Psi_1 G(n)_{\underline{x}}^t \subseteq \Psi_2 G(n)_{\underline{x}}^t \subseteq \cdots$$

Let e_i denote the nondegenerate 1-simplex $x_{i-1} \to x_i$ in $G(n)_{\underline{x}}^t$ and let w_j denote a word in the e_i's which can be obtained via "composition" of these 1-simplices. The k-th stage of the filtration is given by

$$(\Psi_k(G(n)_{\underline{x}}^t))_m = \{(w_1 \mid \cdots \mid w_m) \mid \ell(w_1 \cdots w_m) \leq k\}$$

where $\ell(w_1 \cdots w_n)$ denotes the length of the word $w_1 \cdots w_n$. The colimit of this filtration is weakly equivalent to $L_c G(n)_{\underline{x}}^t$ in $\mathcal{LSSets}_{\mathcal{O},f}^{\Delta^{op}}$.

We show in the proof of [5, 4.2] that for each $i \geq 1$ the map

$$\Psi_i G(n)_{\underline{x}}^t \to \Psi_{i+1} G(n)_{\underline{x}}^t$$

is a DK-equivalence, and that the unique map from $G(n)_{\underline{x}}^t$ to the colimit of this directed system is also a DK-equivalence.

In the current paper, we use some of the ideas of the proof from the fixed object set case, but we no longer use multi-sorted theories as we pass from $\mathcal{SC}_{\mathcal{O}}$ to $\mathcal{SC}$ and $\mathcal{SSets}_{\mathcal{O}}^{\Delta^{op}}$ to $\mathcal{SeCat}$.

4. Methods of Obtaining Segal Precategories from Simplicial Spaces

In the course of proving the existence of these two model category structures $\mathcal{SeCat}_c$ and $\mathcal{SeCat}_f$, we need sets of generating cofibrations and generating acyclic cofibrations which are similar to those of the Reedy and projective model category structures on simplicial spaces. However, we need to modify these maps so that they are actually maps between Segal precategories. The purpose of this section is to define two methods of modifying the generating cofibrations and generating acyclic cofibrations so that they are actually maps between Segal precategories, and to prove a result which we need to prove the existence of the model structures $\mathcal{SeCat}_c$ and $\mathcal{SeCat}_f$.

The first method we call reduction, and we use it to define the generating cofibrations in $\mathcal{SeCat}_c$. Consider the forgetful functor from the category of Segal precategories to the category of simplicial spaces. This map has a left adjoint, which we call the reduction map. Given a simplicial space X, we denote its reduction by $(X)_r$. The degree n space of $(X)_r$ is obtained from X_n by collapsing the subspace $s_0^n X_0$ of X_n to the discrete space $\pi_0(s_0^n X_0)$, where s_0^n is the iterated degeneracy map.

Recall that the cofibrations in the Reedy model category structure on simplicial spaces are monomorphisms (Proposition 2.14) and that the Reedy generating cofibrations are of the form

$$\dot{\Delta}[m] \times \Delta[n]^t \cup \Delta[m] \times \dot{\Delta}[n]^t \to \Delta[m] \times \Delta[n]^t$$

for all $n, m \geq 0$. In general, these maps are not in $\mathcal{SeCat}$ because the objects involved are not Segal precategories. Therefore, we apply this reduction functor to these maps.

Thus, we consider the maps

$$(\dot{\Delta}[m] \times \Delta[n]^t \cup \Delta[m] \times \dot{\Delta}[n]^t)_r \to (\Delta[m] \times \Delta[n]^t)_r.$$

However, we still need to make some modifications to assure that all these maps are actually monomorphisms. In particular, we need to check the case where $n = 0$.

If $n = m = 0$, and if ϕ denotes the empty simplicial space, we obtain the map $\phi \to \Delta[0]^t$, which is a monomorphism. However, when $n = 0$ and $m = 1$, we get the map $\Delta[0]^t \amalg \Delta[0]^t \to \Delta[0]^t$, which is not a monomorphism. When $n = 0$ and $m \geq 2$, we obtain the map $\Delta[0]^t \to \Delta[0]^t$. This map is an isomorphism, and thus there is no reason to include it in the generating set. Therefore, we define the set

$$I_c = \{(\dot\Delta[m] \times \Delta[n]^t \cup \Delta[m] \times \dot\Delta[n]^t)_r \to (\Delta[m] \times \Delta[n]^t)_r\}$$

for all $m \geq 0$ when $n \geq 1$ and for $n = m = 0$. This set I_c will be a set of generating cofibrations of $\mathcal{S}e\mathcal{C}at_c$.

This reduction process works well in almost all situations, but we have problems when we try to reduce some of the generating cofibrations in $\mathcal{S}\mathcal{S}ets_f^{\boldsymbol{\Delta}^{op}}$, namely the maps

$$\dot\Delta[1] \times \Delta[n]^t \to \Delta[1] \times \Delta[n]^t$$

for any $n \geq 0$. The object $\Delta[1] \times \Delta[n]^t$ reduces to a Segal precategory with $n + 1$ points in degree zero, but the object $\dot\Delta[1] \times \Delta[n]^t$ reduces to a Segal precategory with $2(n + 1)$ points in degree zero. In other words, the reduced map in this case is no longer a monomorphism.

Consider the set $\Delta[n]_0$ and denote by $\Delta[n]_0^t$ the doubly constant simplicial space defined by it. For $m \geq 1$ and $n \geq 0$, define $P_{m,n}$ to be the pushout of the diagram

$$
\begin{array}{ccc}
\dot\Delta[m] \times \Delta[n]_0^t & \longrightarrow & \dot\Delta[m] \times \Delta[n]^t \\
\downarrow & & \downarrow \\
\Delta[n]_0^t & \longrightarrow & P_{m,n}.
\end{array}
$$

If $m = 0$, then we define $P_{m,0}$ to be the empty simplicial space. For all $m \geq 0$ and $n \geq 1$, define $Q_{m,n}$ to be the pushout of the diagram

$$
\begin{array}{ccc}
\Delta[m] \times \Delta[n]_0^t & \longrightarrow & \Delta[m] \times \Delta[n]^t \\
\downarrow & & \downarrow \\
\Delta[n]_0^t & \longrightarrow & Q_{m,n}.
\end{array}
$$

For each m and n, the map $\dot\Delta[m] \times \Delta[n]^t$ induces a map $i_{m,n} \colon P_{m,n} \to Q_{m,n}$. We then define the set $I_f = \{i_{m,n} \colon P_{m,n} \to Q_{m,n} \mid m, n \geq 0\}$. Note that when $m \geq 2$ this construction gives exactly the same objects as those given by reduction, namely that $P_{m,n}$ is precisely $(\dot\Delta[m] \times \Delta[n]^t)_r$ and likewise $Q_{m,n}$ is precisely $(\Delta[m] \times \Delta[n]^t)_r$.

Given a Segal precategory X, we denote by $X_n(v_0, \ldots v_n)$ the fiber of the map $X_n \to X_0^{n+1}$ over $(v_0, \ldots, v_n) \in X_0^{n+1}$, where this map is given by iterated face maps of X. More specifically, $X_0^{n+1} = (\mathrm{cosk}_0 X)_n$ and the map $X_n \to X_0^{n+1}$ is given by the map $X \to \mathrm{cosk}_0 X$.

If Hom denotes morphism set and X is an arbitrary simplicial space, notice that we can use the pushout diagrams defining the objects $P_{m,n}$ and $Q_{m,n}$ to see that

$$\mathrm{Hom}(P_{m,n}, X) \cong \coprod_{v_0,\ldots,v_n} \mathrm{Hom}(\dot\Delta[m], X_n(v_0, \ldots, v_n))$$

and

$$\mathrm{Hom}(Q_{m,n}, X) \cong \coprod_{v_0,\ldots,v_n} \mathrm{Hom}(\Delta[m], X_n(v_0, \ldots, v_n)).$$

We now state and prove a lemma using the maps in I_f.

Lemma 4.1. *Suppose a map $f \colon X \to Y$ has the right lifting property with respect to the maps in I_f. Then the map $X_0 \to Y_0$ is surjective and each map*

$$X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is an acyclic fibration of simplicial sets for each $n \geq 1$ and $(v_0, \ldots, v_n) \in X_0^{n+1}$.

Proof. The surjectivity of $X_0 \to Y_0$ follows from the fact that f has the right lifting property with respect to the map $P_{0,0} \to Q_{0,0}$.

In order to prove the remaining statement, it suffices to show that there is a dotted arrow lift in any diagram of the form

(4.2)
$$\begin{array}{ccc} \dot{\Delta}[m] & \longrightarrow & X_n(v_0, \ldots, v_n) \\ \downarrow & \nearrow & \downarrow \\ \Delta[m] & \longrightarrow & Y_n(fv_0, \ldots, fv_n) \end{array}$$

for $m, n \geq 0$.

By our hypothesis, there is a dotted arrow lift in diagrams of the form

(4.3)
$$\begin{array}{ccc} P_{m,n} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ Q_{m,n} & \longrightarrow & Y \end{array}$$

for all $m, n \geq 0$. The existence of the lift in diagram 4.3 is equivalent to the surjectivity of the map $\operatorname{Hom}(Q_{m,n}, X) \to P$ in the following diagram, where P denotes the pullback and Hom denotes morphism set:

$$\begin{array}{ccc} \operatorname{Hom}(Q_{m,n}, X) \longrightarrow P \longrightarrow \operatorname{Hom}(P_{m,n}, X) \\ \downarrow \qquad\qquad\qquad\qquad \downarrow \\ \operatorname{Hom}(Q_{m,n}, Y) \longrightarrow \operatorname{Hom}(P_{m,n}, Y). \end{array}$$

Now, as noted above we have that

$$\operatorname{Hom}(Q_{m,n}, X) \cong \coprod_{v_0, \ldots, v_n} \operatorname{Hom}(\Delta[m], X_n(v_0, \ldots, v_n))$$

and analogous weak equivalences for the other objects of the diagram.

Using these weak equivalences and being particularly careful in the cases where $m = 1$ and $m = 0$, one can show that for each $m, n \geq 0$ the dotted-arrow lift in diagram 4.2 exists and therefore that each map

$$X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is an acyclic fibration of simplicial sets for each $n \geq 1$. $\qquad\square$

5. A Segal Category Model Category Structure on Segal Precategories

In this section, we prove the existence of the model category structure $\mathcal{S}e\mathcal{C}at_c$.

We would like to define a functorial "localization" functor L_c on $\mathcal{S}e\mathcal{C}at$ such that, given any Segal precategory X, its localization L_cX is a Segal space which is a Segal category weakly equivalent to X in $\mathcal{S}e\mathcal{S}p_c$. We begin by considering a functorial localization functor in $\mathcal{S}e\mathcal{S}p_c$ and then modifying it so that it takes values in $\mathcal{S}e\mathcal{C}at$. In the case of $\mathcal{S}e\mathcal{S}p_c$, this localization functor is precisely the functorial fibrant replacement functor.

A choice of generating acyclic cofibrations for $\mathcal{S}e\mathcal{S}p_c$ is the set of maps

$$V[m,k] \times \Delta[n]^t \cup \Delta[m] \times G(n)^t \to \Delta[m] \times \Delta[n]^t$$

for $n \geq 0$, $m \geq 1$, and $0 \leq k \leq m$ [12, §4.2]. Therefore, one can use the small object argument to construct a functorial localization functor by taking a colimit of pushouts, each of which is along the coproduct of all these maps [12, §4.3].

If we apply this functor to a Segal precategory, the maps with $n = 0$ are problematic because taking pushouts along them will not result in a space which is discrete in degree zero. We claim that we can obtain a functorial localization functor L_c on the category $\mathcal{S}e\mathcal{C}at$ by taking a colimit of iterated pushouts along the maps

$$V[m,k] \times \Delta[n]^t \cup \Delta[m] \times G(n)^t \to \Delta[m] \times \Delta[n]^t$$

for $n, m \geq 1$ and $0 \leq k \leq m$.

To see that this restricted set of maps is sufficient, consider a Segal precategory X and the Segal category L_cX we obtain from taking such a colimit. Then for any $0 \leq k \leq m$, consider the diagram

$$
\begin{array}{ccc}
V[m,k] & \longrightarrow & \mathrm{Map}^h(G(0)^t, L_cX) \\
\downarrow & \nearrow & \downarrow \\
\Delta[m] & \longrightarrow & \mathrm{Map}^h(\Delta[0]^t, L_cX).
\end{array}
$$

Since $\Delta[0]^t$ is isomorphic to $G(0)^t$, and since L_cX is discrete in degree zero, the right-hand vertical map is an isomorphism of discrete simplicial sets. Therefore, a dotted arrow lift exists in this diagram. It follows that the map $L_cX \to \Delta[0]^t$ has the right lifting property with respect to the maps

$$V[m,k] \times \Delta[n]^t \cup \Delta[m] \times G(n)^t \to \Delta[m] \times \Delta[n]^t$$

for all $n \geq 0$, $m \geq 1$, and $0 \leq k \leq m$. Therefore, L_cX is fibrant in $\mathcal{S}e\mathcal{S}p_c$, namely, a Segal space.

Since L_cX is a Segal space, it makes sense to talk about the mapping space $\mathrm{map}_{L_cX}(x,y)$ and the homotopy category $\mathrm{Ho}(L_cX)$. Given these facts, we show that there exists a model category structure $\mathcal{S}e\mathcal{C}at_c$ on Segal precategories with the following three distinguished classes of morphisms:

- Weak equivalences are the maps $f : X \to Y$ such that the induced map $L_cX \to L_cY$ is a DK-equivalence of Segal spaces. (Again, we will call such maps *DK-equivalences*.)
- Cofibrations are the monomorphisms. (In particular, every Segal precategory is cofibrant.)

- Fibrations are the maps with the right lifting property with respect to the maps which are both cofibrations and weak equivalences.

Theorem 5.1. *There is a cofibrantly generated model category structure $\mathcal{S}e\mathcal{C}at_c$ on the category of Segal precategories with the above weak equivalences, fibrations, and cofibrations.*

We first need to define sets I_c and J_c as our candidates for generating cofibrations and generating acyclic cofibrations, respectively.

We take as generating cofibrations the set

$$I_c = \{(\dot\Delta[m] \times \Delta[n]^t \cup \Delta[m] \times \dot\Delta[n]^t)_r \to (\Delta[m] \times \Delta[n]^t)_r\}$$

for all $m \geq 0$ when $n \geq 1$ and for $n = m = 0$. Notice that since taking a pushout along such a map amounts to attaching an m-simplex to the space in degree n, any cofibration can be written as a directed colimit of pushouts along the maps of I_c.

We then define the set $J_c = \{i\colon A \to B\}$ to be a set of representatives of isomorphism classes of maps in $\mathcal{S}e\mathcal{C}at$ satisfying two conditions:

(1) For all $n \geq 0$, the spaces A_n and B_n have countably many simplices.
(2) The map $i\colon A \to B$ is a monomorphism and a weak equivalence.

Given these proposed generating acyclic cofibrations, we need to show that any acyclic cofibration in $\mathcal{S}e\mathcal{C}at_c$ is a directed colimit of pushouts along such maps. To prove this result, we require several lemmas. The proofs of the first three we omit here; proofs can be found in the author's thesis [6].

Lemma 5.2. *Let $A \to B$ be a CW-inclusion. The following statements are equivalent:*

(1) $A \to B$ is a weak equivalence of topological spaces.
(2) For all $n \geq 1$, any map of pairs $(D^n, S^{n-1}) \to (B, A)$ extends over the map of cones (CD^n, CS^{n-1}).
(3) For all $n \geq 1$, any map $(D^n, S^{n-1}) \to (B, A)$ is homotopic to a constant map.

Lemma 5.3. *Let $f\colon X \to Y$ be a an inclusion of simplicial sets which is a weak equivalence, and let W and Z be simplicial sets such that we have a diagram of inclusions*

$$
\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

Let $u\colon (D^n, S^{n-1}) \to (|Z|, |W|)$ be a relative map of CW-pairs. Then the inclusion $i\colon (|Z|, |W|) \to (|Y|, |X|)$ can be factored as a composite

$$(|Z|, |W|) \to (|K|, |L|) \to (|Y|, |X|)$$

where K is a subspace of Y obtained from Z by attaching a finite number of non-degenerate simplices, L is a subspace of X, and the composite map of relative CW-complexes

$$(D^n, S^{n-1}) \to (|Z|, |W|) \to (|K|, |L|)$$

is homotopic rel S^{n-1} to a map $D^n \to |L|$.

Lemma 5.4. *Let (Y, X) be a CW-pair such that X and Y have only countably many cells. Then for a fixed $n \geq 0$, there are only countably many homotopy classes of maps $(D^n, S^{n-1}) \to (Y, X)$.*

If $A \to B$ is a monomorphism of Segal precategories, then taking the localization via the small object argument gives us that $L_c A \to L_c B$ is a monomorphism of Segal categories. In particular, if $A \subseteq B$ is an inclusion, then we can regard $L_c A \subseteq L_c B$ as an inclusion as well.

Lemma 5.5. *Let A and B be Segal precategories such that $A \subseteq B$. Let σ be a simplex in $L_c B$ which is not in $L_c A$. Then there exists a Segal precategory A' such that A' is obtained from A by attaching a finite number of nondegenerate simplices and σ is in LA'.*

Proof. By our description of our localization functor at the beginning of the section, $L_c B$ is obtained from B by taking a colimit of pushouts, each of which is along the map

$$\coprod_{m,k,n} V[m,k] \times \Delta[n]^t \cup \Delta[m] \times G(n)^t \to \coprod_{m,k,n} \Delta[m] \times \Delta[n]^t$$

for $n, m \geq 1$ and $0 \leq k \leq m$. The Segal category $L_c B$ is the colimit of a filtration

$$B \subseteq \Psi^1 B \subseteq \Psi^2 B \subseteq \cdots$$

where each Ψ^i is given by a colimit of iterated pushouts along this map. Since σ is a single simplex, it is small and therefore σ is in $\Psi^n B$ for some n.

Therefore, σ is obtained by attaching $\Delta[m] \times \Delta[n]^t$ along a finite number of nondegenerate simplices of $\Psi^{n-1} B$. We can then apply the preceding argument to each of these simplices and inductively obtain a finite number of nondegenerate simplices of B which form a sub-Segal precategory which we will call C. We then define $A' = A \cup C$. $\square$

We then state one more lemma, which is a generalization of a lemma given by Hirschhorn [12, 2.3.6].

Lemma 5.6. *Let the map $g \colon A \to B$ be an inclusion of Segal precategories, each of which has countably many simplices. If X is a Segal precategory with countably many simplices, then its localization LX with respect to the map g has only countably many simplices.*

We are now able to state and prove our result about generating cofibrations.

Proposition 5.7. *Any acyclic cofibration $j \colon C \to D$ in $\mathcal{S}e\mathcal{C}at_c$ can be written as a directed colimit of pushouts along the maps in J_c.*

Proof. Note that by definition $j \colon C \to D$ is a monomorphism of Segal precategories. We assume that it is an inclusion. Let U be a subsimplicial space of D such that U has countably many simplices in each degree. Apply the localization functor L_c to obtain a diagram of Segal categories

$$
\begin{array}{ccc}
L_c(U \cap C) & \longrightarrow & L_c U \\
\downarrow & & \downarrow \\
L_c C & \longrightarrow & L_c D.
\end{array}
$$

Since U has only countably many simplices, this localization process adds at most a countable number of simplices to the original simplicial space by Lemma 5.6.

We would like to find a Segal precategory W such that $U \subseteq W \subseteq D$ and such that the map $W \cap C \to W$ is in the set J_c.

First consider the map

$$\text{Ho}(L_c(U \cap C)) \to \text{Ho}(L_c U)$$

which we want to be an equivalence of categories. If it is not an equivalence, then there exists $z \in (L_c U)_0$ which is not equivalent to some $z' \in (L_c(U \cap C))_0$. However, there is such a z' when we consider z as an element of $(L_c D)_0$, since $j : C \to D$ is a DK-equivalence. If this z' is not in $(U \cap C)_0$, then we add it. Repeat this process for all such z.

Now for each such z, consider the four mapping spaces in $L_c U$ involving the objects z and z': $\text{map}_{L_c U}(z, z)$, $\text{map}_{L_c U}(z, z')$, $\text{map}_{L_c U}(z', z)$, and $\text{map}_{L_c U}(z', z')$. We want the sets of components of these four spaces to be isomorphic to one another in $\text{Ho}(L_c U)$. We can attach a countable number of simplices via an analogous argument to the one in the proof of Lemma 5.5 such that these sets of components are isomorphic. We then repeat the same argument to assure that $\pi_0 \text{map}_{L_c U}(x, z)$ is isomorphic to $\pi_0 \text{map}_{L_c U}(x, z')$ for each $x \in U_0$ and analogously for the sets of components of the mapping spaces out of each such x.

By repeating this process for each such z, we obtain a Segal precategory Y with a countable number of simplices such that $\text{Ho}(L_c(Y \cap C)) \to \text{Ho}(L_c Y)$ is an equivalence of categories. However, we do not necessarily have that for each $x, y \in L_c(Y \cap C)$,

$$\text{map}_{L_c(Y \cap C)}(x, y) \to \text{map}_{L_c Y}(x, y)$$

is a weak equivalence of simplicial sets. Therefore we consider all maps

$$(D^n, S^{n-1}) \to (|\text{map}_{L_c Y}(x, y)|, |\text{map}_{L_c(Y \cap C)}(x, y)|) \to (|\text{map}_{L_c D}(x, y)|, |\text{map}_{L_c C}(x, y)|)$$

for each $x, y \in (Y \cap C)_0$ and $n \geq 0$. Identify all x, y, and n such that the map

$$(D^n, S^{n-1}) \to (|\text{map}_{L_c Y}(x, y)|, |\text{map}_{L_c(Y \cap C)}(x, y)|)$$

is not homotopic to a constant map.

However each composite map

$$(D^n, S^{n-1}) \to (|\text{map}_{L_c Y}(x, y)|, |\text{map}_{L_c(Y \cap C)}(x, y)|) \to (|\text{map}_{L_c D}(x, y)|, |\text{map}_{L_c C}(x, y)|)$$

is homotopic to a constant map by Lemma 5.2 since

$$|\text{map}_{L_c C}(x, y)| \to |\text{map}_{L_c D}(x, y)|$$

is a weak equivalence.

For each such x, y, and n, it follows from Lemma 5.3 that there exists some pair of simplicial sets

$$(\text{map}_{L_c Y}(x, y), \text{map}_{L_c(Y \cap C)}(x, y)) \subseteq (K, L) \subseteq (\text{map}_{L_c D}(x, y), \text{map}_{L_c C}(x, y))$$

such that the composite map

$$(D^n, S^{n-1}) \to (|\text{map}_{L_c Y}(x, y)|, |\text{map}_{L_c(Y \cap C)}(x, y)|) \to (|K|, |L|)$$

is homotopic to a constant map, and the pair (K, L) is obtained from the pair $(\text{map}_{L_c Y}(x, y), \text{map}_{L_c(Y \cap C)}(x, y))$ by attaching a finite number of nondegenerate

simplices. We apply Lemma 5.5 to each of these new simplices obtained by considering each nontrivial homotopy class to obtain some Segal precategory Y' with a countable number of number of simplices such that each composite map

$$(D^n, S^{n-1}) \to (|\mathrm{map}_{L_cY}(x,y)|, |\mathrm{map}_{L_c(Y\cap C)}(x,y)|) \to (|\mathrm{map}_{L_cY'}(x,y)|, |\mathrm{map}_{L_c(Y'\cap C)}(x,y)|)$$

is homotopic to a constant map.

However, the process of adding simplices may have created more maps

$$(D^n, S^{n-1}) \to (|\mathrm{map}_{L_cY'}(x,y)|, |\mathrm{map}_{L_c(Y'\cap C)}(x,y)|)$$

that are not homotopic to a constant map. Therefore we repeat this argument, perhaps countably many times, until, taking a colimit over all of them, we obtain a Segal precategory W such that each map

$$(D^n, S^{n-1}) \to (|\mathrm{map}_{L_cW}(x,y)|, |\mathrm{map}_{L_c(W\cap C)}(x,y)|)$$

is homotopic to a constant map. Since each of these steps added only countably many simplices to the original Segal precategory U, and since by Lemma 5.2

$$\mathrm{map}_{L_c(W\cap C)}(x,y) \to \mathrm{map}_{L_cW}(x,y)$$

is a weak equivalence for all $x, y \in (L_c(W \cap C))_0$, the map $W \cap C \to W$ is in the set J_c.

Now, take some $\widetilde{U}$ obtained from W by adding a countable number of simplices, consider the inclusion map $\widetilde{U} \cap C \to \widetilde{U}$, and repeat the entire process. To show that we can repeat this argument, taking a (possibly transfinite) colimit, and eventually obtain the map $j : C \to D$, it suffices to show that the localization functor L_c commutes with arbitrary directed colimits of inclusions. However, this fact follows from [12, 2.2.18]. $\qquad\square$

Now, we have two definitions of acyclic fibration that we need to show coincide: the fibrations which are weak equivalences, and the maps with the right lifting property with respect to the maps in I_c.

Proposition 5.8. *The maps with the right lifting property with respect to the maps in I_c are fibrations and weak equivalences.*

Before giving a proof of this proposition, we begin by looking at the maps in I_c and determining what an I_c-injective looks like. Recall the definition of the coskeleton of a simplicial space from the paragraph following Proposition 2.14. If $f : X \to Y$ has the right lifting property with respect to the maps in I_c, then for each $n \geq 1$, the map $X_n \to P_n$ is an acyclic fibration of simplicial sets, where P_n is the pullback in the diagram

$$
\begin{array}{ccc}
P_n & \longrightarrow & Y_n \\
\downarrow & & \downarrow \\
(\mathrm{cosk}_{n-1}X)_n & \longrightarrow & (\mathrm{cosk}_{n-1}Y)_n.
\end{array}
$$

In the case that $n = 0$, the restrictions on m and n give us that the map $X_0 \to Y_0$ is a surjection rather than the isomorphism we get in the Reedy case. Notice that by the same argument given for the Reedy model category structure (in the section following Proposition 2.14 above), the simplicial sets P_n can be characterized up to weak equivalence as homotopy pullbacks and are therefore homotopy invariant.

This characterization of the maps with the right lifting property with respect to I_c will enable us to prove Proposition 5.8. Before proceeding to the proof, however, we state a lemma, whose proof we defer to section 9.

Lemma 5.9. *Suppose that $f : X \to Y$ is a map of Segal precategories which is an I_c-injective. Then f is a DK-equivalence.*

Proof of Proposition 5.8. Suppose that $f \colon X \to Y$ is an I_c-injective, or a map which has the right lifting property with respect to the maps in I_c. Note that f then has the right lifting property with respect to all cofibrations. Since, in particular, it has the right lifting property with respect to the acyclic cofibrations, it is a fibration by definition. It remains to show that f is a weak equivalence.

However, this fact follows from Lemma 5.9, proving the proposition.   □

We now state the converse, which we prove in section 9.

Proposition 5.10. *The maps in $\mathcal{S}e\mathcal{C}at_c$ which are both fibrations and weak equivalences are I_c-injectives.*

Now we prove a lemma which we need to check the last condition for our model category structure.

Lemma 5.11. *A pushout along a map of J_c is also an acyclic cofibration in $\mathcal{S}e\mathcal{C}at_c$.*

Proof. Let $j \colon A \to B$ be a map in J_c. Notice that j is an acyclic cofibration in the model category $\mathcal{C}\mathcal{S}\mathcal{S}$. Since $\mathcal{C}\mathcal{S}\mathcal{S}$ is a model category, we know that a pushout along an acyclic cofibration is again an acyclic cofibration [10, 3.14(ii)]. If all the objects involved are Segal precategories, then the pushout will again be a Segal precategory and therefore the pushout map will be an acyclic cofibration in $\mathcal{S}e\mathcal{C}at_c$.   □

Proposition 5.12. *If a map of Segal precategories is a J_c-cofibration, then it is an I_c-cofibration and a weak equivalence.*

Proof. By definition and Proposition 5.7, a J_c-cofibration is a map with the left lifting property with respect to the maps with the right lifting property with respect to the acyclic cofibrations. However, by the definition of fibration, these maps are the ones with the left lifting property with respect to the fibrations.

Similarly, using Propositions 5.8 and 5.10, an I_c-cofibration is a map with the left lifting property with respect to the acyclic fibrations. Thus, we need to show that a map with the left lifting property with respect to the fibrations has the left lifting property with respect to the acyclic fibrations and is a weak equivalence. Since the acyclic fibrations are fibrations, it remains to show that the maps with the left lifting property with respect to the fibrations are weak equivalences.

Let $f \colon A \to B$ be a map with the left lifting property with respect to all fibrations. By Lemma 5.11 above, we know that a pushout along maps of J_c is an acyclic cofibration. Therefore, we can use the small object argument [12, 10.5.15] to factor the map $f \colon A \to B$ as the composite of an acyclic cofibration $A \to A'$ and a fibration $A' \to B$. Then there exists a dotted arrow lift in the diagram

$$
\begin{array}{ccc}
A & \overset{\simeq}{\longrightarrow} & A' \\
\downarrow & \nearrow & \downarrow \\
B & \underset{\mathrm{id}}{\longrightarrow} & B
\end{array}
$$

showing that the map $A \to B$ is a retract of the map $A \to A'$ and therefore a weak equivalence. □

Proof of Theorem 5.1. Axiom MC1 follows since limits and colimits of Segal precategories (computed as simplicial spaces) still have discrete zero space and are therefore Segal precategories. MC2 and MC3 (for weak equivalences) work as usual, for example see [10, 8.10].

It remains to show that the four conditions of Theorem 2.3 are satisfied. The set I_c permits the small object argument because the generating cofibrations in the Reedy model category structure do. We can show that the objects A which appear as the sources of the maps in J_c are small using an analogous argument to the one for simplicial sets [12, 10.4.4], so the set J_c permits the small object argument. Thus, condition 1 is satisfied.

Condition 2 is precisely the statement of Proposition 5.12. Condition 3 and condition 4(ii) are precisely the statements of Propositions 5.8 and 5.10. □

Note that the reduced Reedy acyclic cofibrations

$$(V[m,k] \times \Delta[n]^t \cup \Delta[m] \times \dot{\Delta}[n]^t)_r \to (\Delta[m] \times \Delta[n]^t)_r$$

are acyclic cofibrations in $\mathcal{S}e\mathcal{C}at_c$ for $m \geq 0$ when $n \geq 1$ and for $n = m = 0$.

Corollary 5.13. *The fibrant objects in $\mathcal{S}e\mathcal{C}at_c$ are Reedy fibrant Segal categories.*

Proof. Suppose that X is fibrant in $\mathcal{S}e\mathcal{C}at_c$. Then, since the reduced Reedy cofibrations are acyclic cofibrations in $\mathcal{S}e\mathcal{C}at_c$ and since X has discrete zero space, it follows that X is Reedy fibrant.

Then, since the maps

$$(\Delta[m] \times G(n)^t)_r \to (\Delta[m] \times \Delta[n]^t)_r$$

for all $m, n \geq 0$ are acyclic cofibrations in $\mathcal{S}e\mathcal{C}at_c$, it follows that X is a Segal category. □

The converse statement, that the Reedy fibrant Segal categories are fibrant in $\mathcal{S}e\mathcal{C}at_c$, also holds [2].

6. A Quillen Equivalence Between $\mathcal{S}e\mathcal{C}at_c$ and $\mathcal{C}\mathcal{S}\mathcal{S}$

In this section, we will show that there is a Quillen equivalence between the model category structure $\mathcal{S}e\mathcal{C}at_c$ on Segal precategories and the complete Segal space model category structure $\mathcal{C}\mathcal{S}\mathcal{S}$ on simplicial spaces. We first need to show that we have an adjoint pair of maps between the two categories.

Let $I \colon \mathcal{S}e\mathcal{C}at_c \to \mathcal{C}\mathcal{S}\mathcal{S}$ be the inclusion functor of Segal precategories into the category of all simplicial spaces. We will show that there is a right adjoint functor $R \colon \mathcal{C}\mathcal{S}\mathcal{S} \to \mathcal{S}e\mathcal{C}at_c$ which "discretizes" the degree zero space.

Let W be a simplicial space. Define simplicial spaces $U = \text{cosk}_0(W_0)$ and $V = \text{cosk}_0(W_{0,0})$. There exist maps $W \to U \leftarrow V$. Then we take the pullback RW in the diagram

$$
\begin{array}{ccc}
RW & \longrightarrow & V \\
\downarrow & & \downarrow \\
W & \longrightarrow & U.
\end{array}
$$

Note that RW is a Segal precategory. If W is a complete Segal space, then so are U and V, and in this case RW is a Segal category, which we can see as follows. The pullback at degree 1 gives

$$
\begin{array}{ccc}
(RW)_1 & \longrightarrow & W_{0,0} \times W_{0,0} \\
\downarrow & & \downarrow \\
W_1 & \longrightarrow & W_0 \times W_0
\end{array}
$$

and at degree 2 we get

$$
\begin{array}{ccc}
(RW)_1 \times_{(RW)_0} (RW)_1 & \longrightarrow & (W_{0,0})^3 \\
\downarrow & & \downarrow \\
W_2 \simeq W_1 \times_{W_0} W_1 & \longrightarrow & W_0 \times W_0 \times W_0.
\end{array}
$$

Looking at these pullbacks, and the analogous ones for higher n, we notice that RW is in fact a Segal category.

We define the functor $R\colon \mathcal{CSS} \to \mathcal{SeCat}_c$ which takes a simplicial space W to the Segal precategory RW given by the description above.

Proposition 6.1. *The functor $R\colon \mathcal{CSS} \to \mathcal{SeCat}_c$ is right adjoint to the inclusion map $I\colon \mathcal{SeCat}_c \to \mathcal{CSS}$.*

Proof. We need to show that there is an isomorphism

$$\mathrm{Hom}_{\mathcal{SeCat}_c}(Y, RW) \cong \mathrm{Hom}_{\mathcal{CSS}}(IY, W)$$

for any Segal precategory Y and simplicial space W.

Suppose that we have a map $Y = IY \to W$. Since Y is a Segal precategory, Y_0 is equal to $Y_{0,0}$ viewed as a constant simplicial set. Therefore, we can restrict this map to a unique map $Y \to V$, where V is the Segal precategory defined above. Then, given the universal property of pullbacks, there is a unique map $Y \to RW$. Hence, we obtain a map

$$\varphi\colon \mathrm{Hom}_{\mathcal{CSS}}(IY, W) \to \mathrm{Hom}_{\mathcal{SeCat}_c}(Y, RW).$$

This map is surjective because given any map $Y \to RW$ we can compose it with the map $RW \to W$ to obtain a map $Y \to W$.

Now for any Segal precategory Y, consider the diagram

$$
\begin{array}{ccc}
Y & & \\
& RW \longrightarrow V & \\
& W \longrightarrow U &
\end{array}
$$

Because this diagram must commute and the image of the map $Y_0 \to W_0$ is contained in $W_{0,0}$ since Y is a Segal precategory, this map uniquely determines what the map $Y \to V$ has to be. Therefore, given a map $Y \to RW$, it could only have come from one map $Y \to W$. Thus, φ is injective. $\qquad\square$

Now, we need to show that this adjoint pair respects the model category structures that we have.

Proposition 6.2. *The adjoint pair of functors*

$$I : \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{C}\mathcal{S}\mathcal{S} : R$$

is a Quillen pair.

Proof. It suffices to show that the inclusion map I preserves cofibrations and acyclic cofibrations. I preserves cofibrations because they are defined to be monomorphisms in each category. Also in each of the two categories, a map is a weak equivalence if it is a DK-equivalence after localizing to obtain a Segal space, as given in Theorem 3.10. In each case an acyclic cofibration is an inclusion satisfying this property. Therefore, the map I preserves acyclic cofibrations. $\square$

Theorem 6.3. *The Quillen pair*

$$I : \mathcal{S}e\mathcal{C}at_c \rightleftarrows \mathcal{C}\mathcal{S}\mathcal{S} : R$$

is a Quillen equivalence.

Proof. We need to show that I reflects weak equivalences between cofibrant objects and that for any fibrant object W (i.e., complete Segal space) in $\mathcal{C}\mathcal{S}\mathcal{S}$, the map $I((RW)^c) = IRW \to W$ is a weak equivalence in $\mathcal{S}e\mathcal{C}at_c$.

The fact that I reflects weak equivalences between cofibrant objects follows from the same argument as the one in the proof of the Quillen pair. To prove the second part, it remains to show that the map $j \colon RW \to W$ in the pullback diagram

$$
\begin{array}{ccc}
RW & \longrightarrow & V \\
\downarrow{\scriptstyle j} & & \downarrow \\
W & \longrightarrow & U
\end{array}
$$

is a DK-equivalence. It suffices to show that the map of objects $\mathrm{ob}(RW) \to \mathrm{ob}(W)$ is surjective and that the map $\mathrm{map}_{RW}(x,y) \to \mathrm{map}_W(jx, jy)$ is a weak equivalence, where the object set of a Segal space is defined as in section 3.3. However, notice by the definition of RW that $\mathrm{ob}(RW) = \mathrm{ob}(W)$. In particular, $jx = x$ and $jy = y$. Then notice, using the pullback that defines $(RW)_1$, that $\mathrm{map}_{RW}(x,y) \simeq \mathrm{map}_W(x,y)$. Therefore, the map $RW \to W$ is a DK-equivalence. $\square$

7. Another Segal Category Model Category Structure on Segal Precategories

The model category structure $\mathcal{S}e\mathcal{C}at_c$ that we defined above is helpful for the Quillen equivalence with the complete Segal space model category structure, but there does not appear to be a Quillen equivalence between it and the model category structure $\mathcal{S}\mathcal{C}$ on simplicial categories. Therefore, we need another model category structure $\mathcal{S}e\mathcal{C}at_f$ to obtain such a Quillen equivalence.

In the model category structure $\mathcal{S}e\mathcal{C}at_c$, we started with the generating cofibrations in the Reedy model category structure and adapted them to be generating cofibrations of Segal precategories. In this second model category structure, we use modified generating cofibrations from the projective model category structure on simplicial spaces so that the objects involved are Segal precategories.

We make the following definitions for a model category structure $Se\mathcal{C}at_f$ on the category of Segal precategories.

- The weak equivalences are the same as those of $Se\mathcal{C}at_c$.
- The cofibrations are the maps which can be formed by taking iterated pushouts along the maps of the set I_f defined in section 4.
- The fibrations are the maps with the right lifting property with respect to the maps which are cofibrations and weak equivalences.

Notice that to define the weak equivalences in this case we want to use a functorial localization in $Se\mathcal{S}p_f$ rather than $Se\mathcal{S}p_c$. We define a localization functor L_f in the same way that we defined L_c at the beginning of section 5 but making necessary changes in light of the fact that we are starting from the model structure $Se\mathcal{S}p_f$. So, in a sense, the weak equivalences are not defined identically in the two categories, since they make use of the same localization of different model category structures on the category of simplicial spaces. However, in each case the weak equivalences are the same in the unlocalized model category, so we can define homotopy function complexes using only the underlying category and the weak equivalences. Recall by the definition of local objects that a map $X \to Y$ is a local equivalence if and only if the induced map of homotopy function complexes

$$\mathrm{Map}^h(Y, Z) \to \mathrm{Map}^h(X, Z)$$

is a weak equivalence of simplicial sets for any local Z. In particular, the weak equivalences of the localized category depend only on the weak equivalences of the unlocalized category. Therefore the weak equivalences in $Se\mathcal{C}at_c$ and $Se\mathcal{C}at_f$ are actually the same.

Theorem 7.1. *There is a cofibrantly generated model category structure $Se\mathcal{C}at_f$ on the category of Segal precategories in which the weak equivalences, fibrations, and cofibrations are defined as above.*

We define the set J_f to be a set of isomorphism classes of maps $\{i\colon A \to B\}$ such that

(1) for all $n \geq 0$, the spaces A_n and B_n have countably many simplices, and
(2) $i\colon A \to B$ is an acyclic cofibration.

We would like to show that I_f (defined in section 4) is a set of generating cofibrations and that J_f is a set of generating acyclic cofibrations for $Se\mathcal{C}at_f$.

We begin with the following lemma.

Lemma 7.2. *Any acyclic cofibration $j\colon C \to D$ in $Se\mathcal{C}at_f$ can be written as a directed colimit of pushouts along the maps in J_f.*

Proof. The argument that we used to prove Proposition 5.7 still holds, applying the functor L_f rather than L_c. $\square$

Proposition 7.3. *A map $f\colon X \to Y$ is an acyclic fibration in $Se\mathcal{C}at_f$ if and only if it is an I_f-injective.*

Proof. First suppose that f has the right lifting property with respect to the maps in I_f. Then we claim that for each $n \geq 0$ and $(v_0, \ldots, v_n) \in X_0^{n+1}$, the map $X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$ is an acyclic fibration of simplicial sets. This fact, however, follows from Lemma 4.1. In particular, it is a weak equivalence, and therefore we can apply the proof of Lemma 5.9 to show that the map $X \to Y$

is a DK-equivalence, completing the proof of the first direction. (The proof does not follow precisely in this case, in particular because not all monomorphisms are cofibrations. However, we can use the fact that weak equivalences are the same in $\mathcal{S}e\mathcal{C}at_c$ and $\mathcal{S}e\mathcal{C}at_f$ to see that the argument still holds.)

Then, to prove the converse, assume that f is a fibration and a weak equivalence. Then we can apply the proof of Proposition 5.10, making the factorizations in the projective model category structure rather than in the Reedy model category structure. The argument follows analogously. $\qquad\square$

Proposition 7.4. *A map in $\mathcal{S}e\mathcal{C}at_f$ is a J_f-cofibration if and only if it is an I_f-cofibration and a weak equivalence.*

Proof. This proof follows just as the proof of Proposition 5.12, again using the projective structure rather than the Reedy structure. $\qquad\square$

Proof of Theorem 7.1. As before, we must check the conditions of Theorem 2.3. Condition 1 follows just as in the proof of Theorem 5.1. Condition 2 is precisely the statement of Proposition 7.4. Condition 3 and condition 4(ii) follow from Proposition 7.3 after applying Lemma 7.2. $\qquad\square$

We now prove that both our model category structures on the category of Segal precategories are Quillen equivalent.

Theorem 7.5. *The identity functor induces a Quillen equivalence*

$$I : \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}e\mathcal{C}at_c : J.$$

Proof. Since both maps are the identity functor, they form an adjoint pair. We then show that this adjoint pair is a Quillen pair.

We first make some observations between the two categories. Notice that the cofibrations of $\mathcal{S}e\mathcal{C}at_f$ form a subclass of the cofibrations of $\mathcal{S}e\mathcal{C}at_c$ since they are monomorphisms. Similarly, the acyclic cofibrations of $\mathcal{S}e\mathcal{C}at_f$ form a subclass of the acyclic cofibrations of $\mathcal{S}e\mathcal{C}at_c$.

In particular, these observations imply that the left adjoint $I\colon \mathcal{S}e\mathcal{C}at_f \to \mathcal{S}e\mathcal{C}at_c$ preserves cofibrations and acyclic cofibrations. Hence, we have a Quillen pair.

It remains to show that this Quillen pair is a Quillen equivalence. To do so, we must show that given any cofibrant X in $\mathcal{S}e\mathcal{C}at_f$ and fibrant Y in $\mathcal{S}e\mathcal{C}at_c$, a map $f\colon IX \to Y$ is a weak equivalence in $\mathcal{S}e\mathcal{C}at_f$ if and only if $\varphi f\colon X \to JY$ is a weak equivalence in $\mathcal{S}e\mathcal{C}at_c$. However, this follows from the fact that the weak equivalences are the same in each category. $\qquad\square$

Note. One might ask at this point why we could not just use the $\mathcal{S}e\mathcal{C}at_f$ model category structure and show a Quillen equivalence between it and the model category structure $\mathcal{C}\mathcal{S}\mathcal{S}_f$ where we localize the projective model category structure (rather than the Reedy) with respect to the maps φ and ψ. The existence of such a Quillen equivalence would certainly simplify this paper!

However, if we work with "complete Segal spaces" which are fibrant in the projective model structure rather than in the Reedy structure, then for a fibrant object W the map $W \to U$ used in defining the right adjoint $\mathcal{C}\mathcal{S}\mathcal{S} \to \mathcal{S}e\mathcal{C}at_c$ is no longer necessarily a fibration. Therefore, the pullback RW is no longer a homotopy pullback and in particular not homotopy invariant. If RW is not homotopy invariant,

then there is no guarantee that the map $RW \to W$ is a DK-equivalence, and the argument for a Quillen equivalence fails. Thus, the $\mathcal{S}e\mathcal{C}at_c$ and $\mathcal{C}\mathcal{S}\mathcal{S}$ model structures are necessary.

8. A QUILLEN EQUIVALENCE BETWEEN $\mathcal{S}\mathcal{C}$ AND $\mathcal{S}e\mathcal{C}at_f$

We begin, as above, by defining an adjoint pair of functors between the two categories $\mathcal{S}\mathcal{C}$ and $\mathcal{S}e\mathcal{C}at_f$. We have the nerve functor $R\colon \mathcal{S}\mathcal{C} \to \mathcal{S}e\mathcal{C}at_f$. In order to define a left adjoint to this map, we need some terminology.

Definition 8.1. Let $\mathcal{D}$ be a small category and $\mathcal{S}\mathcal{S}ets^{\mathcal{D}}$ the category of functors $\mathcal{D} \to \mathcal{S}\mathcal{S}ets$. Let S be a set of morphisms in $\mathcal{S}\mathcal{S}ets^{\mathcal{D}}$. An object Y of $\mathcal{S}\mathcal{S}ets^{\mathcal{D}}$ is *strictly S-local* if for every morphism $f\colon A \to B$ in S, the induced map on function complexes

$$f^*\colon \mathrm{Map}(B, Y) \to \mathrm{Map}(A, Y)$$

is an isomorphism of simplicial sets. A map $g\colon C \to D$ in $\mathcal{S}\mathcal{S}ets^{\mathcal{D}}$ is a *strict S-local equivalence* if for every strictly S-local object Y in $\mathcal{S}\mathcal{S}ets^{\mathcal{D}}$, the induced map

$$g^*\colon \mathrm{Map}(D, Y) \to \mathrm{Map}(C, Y)$$

is an isomorphism of simplicial sets.

Now, we can view Segal precategories as functors $\mathbf{\Delta}^{op} \to \mathcal{S}\mathcal{S}ets$. Because we require the image of $[0]$ to be a discrete simplicial set, the category of Segal precategories is a subcategory of the category of all such functors. In this section, we are going to regard simplicial categories as the strictly local objects in $\mathcal{S}e\mathcal{C}at_f$ with respect to the map φ described in section 3.3.

Although we are actually working in a subcategory, we can still use the following lemma to obtain a left adjoint functor F to our inclusion map R, since the construction will always produce a simplicial space with discrete 0-space when applied to such a simplicial space.

Lemma 8.2. [4, 5.6] *Consider two categories, the category of all diagrams $X\colon \mathcal{D} \to \mathcal{S}\mathcal{S}ets$ and the category of strictly local diagrams with respect to the set of maps $S = \{f\colon A \to B\}$. The forgetful functor from the category of strictly local diagrams to the category of all diagrams has a left adjoint.*

We define the functor $F\colon \mathcal{S}e\mathcal{C}at_f \to \mathcal{S}\mathcal{C}$ to be this left adjoint to the inclusion functor of strictly local diagrams into all diagrams $R\colon \mathcal{S}\mathcal{C} \to \mathcal{S}e\mathcal{C}at_f$.

Proposition 8.3. *The adjoint pair*

$$F : \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}\mathcal{C} : R$$

is a Quillen pair.

Proof. We prove that this adjoint pair is a Quillen pair by showing that the left adjoint F preserves cofibrations and acyclic cofibrations. We begin by considering cofibrations.

Since F is a left adjoint functor, it preserves colimits. Therefore, it suffices to show that F preserves the set I_f of generating cofibrations in $\mathcal{S}e\mathcal{C}at_f$. Recall that the elements of this set are the maps $P_{m,n} \to Q_{m,n}$ as defined in section 4. We begin by considering the maps $P_{n,1} \to Q_{n,1}$ for any $n \geq 0$. The strict localization of such a map is precisely the map of simplicial categories $U\dot{\Delta}[n] \to U\Delta[n]$ (section 3.1) which is a generating cofibration in $\mathcal{S}\mathcal{C}$. We can also see that the strict localization

of any $P_{m,n} \to Q_{m,n}$ can be obtained as the colimit of iterated pushouts along the generating cofibrations of $\mathcal{SC}$. Therefore, F preserves cofibrations.

We now need to show that F preserves acyclic cofibrations. To do so, first consider the model category structure $\mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$ (defined in section 3.15) on Segal precategories with a fixed set $\mathcal{O}$ in degree zero and the model category structure $\mathcal{SC}_{\mathcal{O}}$ of simplicial categories with a fixed object set $\mathcal{O}$. Recall from section 3.15 that there is a Quillen equivalence

$$F_{\mathcal{O}} : \mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}} \rightleftarrows \mathcal{SC}_{\mathcal{O}} : R_{\mathcal{O}}.$$

In particular, if X is a cofibrant object of $\mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$, then there is a weak equivalence $X \to R_{\mathcal{O}}((F_{\mathcal{O}}X)^f)$. Notice that $F_{\mathcal{O}}$ agrees with F on Segal precategories with the set $\mathcal{O}$ in degree zero, and similarly for $R_{\mathcal{O}}$ and R.

Suppose, then, that X is an object of $\mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$, Y is an object of $\mathcal{LSS}ets_{\mathcal{O}',f}^{\mathbf{\Delta}^{op}}$, and $X \to Y$ is an acyclic cofibration in $\mathcal{S}e\mathcal{C}at_f$. We have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\simeq} & L_f X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\simeq} & L_f Y
\end{array}
$$

where the upper and lower horizontal maps are weak equivalences not only in $\mathcal{S}e\mathcal{C}at_f$, but in $\mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$ and $\mathcal{LSS}ets_{\mathcal{O}',f}^{\mathbf{\Delta}^{op}}$, respectively. However, using the fixed-object case Quillen equivalence, the functors $F_{\mathcal{O}}$ and $F_{\mathcal{O}'}$ (and hence F) will preserve these weak equivalences, giving us a diagram

$$
\begin{array}{ccc}
FX & \xrightarrow{\simeq} & FL_f X \\
\downarrow & & \downarrow \\
FY & \xrightarrow{\simeq} & FL_f X.
\end{array}
$$

Using these weak equivalences and our assumption that $L_f X \to L_f Y$ is a DK-equivalence, we obtain a diagram

$$
\begin{array}{ccc}
L_f X & \xrightarrow{\simeq} & RFL_f X \\
\downarrow{\scriptstyle \simeq} & & \downarrow \\
L_f Y & \xrightarrow{\simeq} & RFL_f Y
\end{array}
$$

in which the upper horizontal arrow is a weak equivalence in $\mathcal{LSS}ets_{\mathcal{O},f}^{\mathbf{\Delta}^{op}}$ and the lower horizontal arrow is a weak equivalence in $\mathcal{LSS}ets_{\mathcal{O}',f}^{\mathbf{\Delta}^{op}}$. The commutativity of this diagram implies that the map $RFL_f X \to RFL_f Y$ is a DK-equivalence also. Thus, we have shown that F preserves acyclic cofibrations between cofibrant objects.

It remains to show that F preserves all acyclic cofibrations. Suppose that $f \colon X \to Y$ is an acyclic cofibration in $\mathcal{S}e\mathcal{C}at_f$. Apply the cofibrant replacement functor to the map $X \to Y$ to obtain an acyclic cofibration $X' \to Y'$, and notice

that in the resulting commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

the vertical arrows are levelwise weak equivalences.

Now consider the following diagram, where the top square is a pushout diagram:

$$\begin{array}{ccc} X' & \xrightarrow{\simeq} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{\simeq} & Y'' \\ =\downarrow & & \downarrow \\ X & \xrightarrow{\simeq} & Y. \end{array}$$

Notice that all three of the horizontal arrows are acyclic cofibrations in $Se\mathcal{C}at_f$, the upper and lower by assumption and the middle one because pushouts preserve acyclic cofibrations [10, 3.14]. Now we apply the functor F to this diagram to obtain a diagram

(8.4)
$$\begin{array}{ccc} FX' & \xrightarrow{\simeq} & FY' \\ \downarrow & & \downarrow \\ FX & \xrightarrow{\simeq} & FY'' \\ \downarrow & & \downarrow \\ FX & \longrightarrow & FY. \end{array}$$

The top horizontal arrow is an acyclic cofibration since F preserves acyclic cofibrations between cofibrant objects. Furthermore, since F is a left adjoint and hence preserves colimits, the middle horizontal arrow is also an acyclic cofibration because the top square is a pushout square.

Now, recall that, given an object X in a model category $\mathcal{C}$, the *category of objects under X* has as objects the morphisms $X \to Y$ in $\mathcal{C}$ for any object Y, and as morphisms the maps $Y \to Y'$ in $\mathcal{C}$ making the appropriate triangular diagram commute [12, 7.6.1]. There is a model category structure on this under category in which a morphism is a weak equivalence, fibration, or cofibration if it is in $\mathcal{C}$ [12, 7.6.5]. In particular, a object $X \to Y$ is cofibrant in the under category if it is a cofibration in $\mathcal{C}$.

With this definition in mind, to show that the bottom horizontal arrow of diagram 8.4 is an acyclic cofibration, consider the following diagram in the category of cofibrant objects under X:

$$\begin{array}{ccc} X & \longrightarrow & Y'' \\ & \searrow & \downarrow \\ & & Y. \end{array}$$

Now, let $\mathcal{O}''$ denote the set in degree zero of Y'' (and also of Y) which is not in the image of the map from X. Now we have the diagram in the category of cofibrant

objects under $X \amalg \mathcal{O}''$ with the same set in degree zero

$$X \amalg \mathcal{O}'' \longrightarrow Y''$$
$$\searrow \qquad \downarrow$$
$$Y.$$

However, since we are now working in a fixed object set situation, we know by Theorem 3.16 that $F_{\mathcal{O}''}$ is the left adjoint of a Quillen pair, and therefore the map $F_{\mathcal{O}''}Y'' \to F_{\mathcal{O}''}Y$ is a weak equivalence in $\mathcal{SC}_{\mathcal{O}''}$, and in particular a DK-equivalence when regarded as a map in $\mathcal{SC}$. It follows that the map $FX \to FY$ is a weak equivalence, and F preserves acyclic cofibrations. $\square$

Recall that we are regarding a Segal category as a local diagram and a simplicial category as a strictly local diagram in $\mathcal{SeCat}_f$.

Lemma 8.5. *The map $X \to FX$ is a DK-equivalence for every cofibrant object X in $\mathcal{SeCat}_f$.*

Proof. First consider a free diagram in $\mathcal{SeCat}_f$, namely some $\amalg_i Q_{m_i,n_i}$, where each Q_{m_i,n_i} is defined as in section 4. If Y is a fibrant object in $\mathcal{SeCat}_f$, then we have

$$\mathrm{Map}_{\mathcal{SeCat}_f}(\coprod_i Q_{m_i,n_i}, Y) \simeq \prod_i \mathrm{Map}_{\mathcal{SeCat}_f}(Q_{m_i,n_i}, Y)$$

$$\simeq \prod_i \coprod_{v_0,\ldots,v_n} \mathrm{Map}_{\mathcal{SSets}}(\Delta[m_i], Y_{n_i}(v_0,\ldots,v_n))$$

$$\simeq \prod_i, \coprod_{v_0,\ldots,v_n} \mathrm{Map}_{\mathcal{SSets}}(\Delta[0], Y_{n_i}(v_0,\ldots,v_n))$$

$$\simeq \mathrm{Map}_{\mathcal{SeCat}_f}(\coprod_i Q_{0,n_i}, Y)$$

$$\simeq \mathrm{Map}_{\mathcal{SeCat}_f}(\coprod_i \Delta[n_i]^t, Y)$$

Therefore, it suffices to consider free diagrams $\amalg_i \Delta[n_i]^t$. Such a diagram is a Segal category. It is also the nerve of a category and thus a strictly local diagram. It follows that the map

$$\coprod_i \Delta[n_i]^t \to F(\coprod_i \Delta[n_i]^t)$$

is a DK-equivalence.

Now suppose that X is any cofibrant object in $\mathcal{SeCat}_f$. Then X can be written as a directed colimit $X \simeq \mathrm{colim}_{\mathbf{\Delta}^{op}} X_j$, where each X_j can be written as $\amalg_i \Delta[n_i]^t$. As before we regard FX as a strictly local object in $\mathcal{SeCat}_f$. If Y is a fibrant object in $\mathcal{SeCat}_f$ which is strictly local, we have

$$\mathrm{Map}_{\mathcal{SeCat}_f}(\mathrm{colim}_{\mathbf{\Delta}^{op}} X_j, Y) \simeq \varprojlim_{\mathbf{\Delta}} \mathrm{Map}_{\mathcal{SeCat}_f}(X_j, Y)$$

$$\simeq \varprojlim_{\mathbf{\Delta}} \mathrm{Map}_{\mathcal{SeCat}_f}(FX_j, Y)$$

$$\simeq \mathrm{Map}_{\mathcal{SeCat}_f}(\mathrm{colim}_{\mathbf{\Delta}^{op}} FX_j, Y)$$

$$\simeq \mathrm{Map}_{\mathcal{SeCat}_f}(F(\mathrm{colim}_{\mathbf{\Delta}^{op}}(FX_j)), Y)$$

We can now apply the result that

$$F(\mathrm{colim}(FX_j)) \simeq F(\mathrm{colim}X_j).$$

(This fact is proved in [5, 4.1] for ordinary localization, but it holds for strict localization in this case since each X_j is cofibrant and F preserves cofibrant objects.) Therefore we have

$$\mathrm{Map}_{\mathcal{S}e\mathcal{C}at_f}(F(\mathrm{colim}_{\mathbf{\Delta}^{op}}(FX_j)), Y) \simeq \mathrm{Map}_{\mathcal{S}e\mathcal{C}at_f}(FX, Y).$$

It follows that the map $X \to FX$ is a DK-equivalence.   □

We are now able to prove the main result of this section.

Theorem 8.6. *The Quillen pair*

$$F : \mathcal{S}e\mathcal{C}at_f \rightleftarrows \mathcal{S}\mathcal{C} : R$$

is a Quillen equivalence.

Proof. We first show that F reflects weak equivalences between cofibrant objects. Let $f : X \to Y$ be a map of cofibrant Segal precategories such that $Ff : FX \to FY$ is a weak equivalence of simplicial categories. (Since F preserves cofibrations, both FX and FY are again cofibrant.) Then consider the following diagram:

$$
\begin{array}{ccccc}
FX & \longrightarrow & L_f FX & \longleftarrow & L_f X \\
\simeq \downarrow & & \downarrow & & \downarrow \\
FY & \longrightarrow & L_f FY & \longleftarrow & L_f Y.
\end{array}
$$

By assumption, the leftmost vertical arrow is a DK-equivalence. The horizontal arrows of the left-hand square are also DK-equivalences by definition. Since X and Y are cofibrant, Lemma 8.5 shows that the horizontal arrows of the right-hand square are DK-equivalences. The commutativity of the whole diagram shows that the map $L_f FX \to L_f FY$ is a DK-equivalence and then that the map $L_f X \to L_f Y$ is also. Therefore, F reflects weak equivalences between cofibrant objects.

Now, we will show that given any fibrant simplicial category Y, the map

$$F((RY)^c) \to Y$$

is a DK-equivalence. Consider a fibrant simplicial category Y and apply the functor R to obtain a Segal category which is levelwise fibrant and therefore fibrant in $\mathcal{S}e\mathcal{C}at_f$. Its cofibrant replacement will be DK-equivalent to it in $\mathcal{S}e\mathcal{C}at_f$. Then, by the above argument, strictly localizing this object will again yield a DK-equivalent simplicial category.   □

9. Proofs of Lemma 5.9 and Proposition 5.10

In this section, we give a proof of two results stated in section 5. We begin with a lemma which we will use in the proof of Lemma 5.9.

Lemma 9.1. *Suppose that $f : X \to Y$ is a map of Segal precategories with the right lifting property with respect to the maps in I_c. Then*

(1) *The map $f_0 : X_0 \to Y_0$ is surjective, and*
(2) *The map $X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$ is a weak equivalence of simplicial sets for all $n \geq 1$ and $(v_0, \ldots, v_n) \in X_0^{n+1}$.*

Proof. Since $f : X \to Y$ has the right lifting property with respect to the maps in I_c, it has the right lifting property with respect to all cofibrations. In particular, it has the right lifting property with respect to the maps in the set I_f. Therefore we can apply Lemma 4.1 and the result follows.   □

Proof of Lemma 5.9. To prove Lemma 5.9, we consider a given map $f\colon X \to Y$ with the right lifting property with respect to the maps in I_c. It follows from Lemma 9.1 that the map $X_0 \to Y_0$ is surjective and such that for all $n \geq 1$ and $(v_0, \ldots v_n) \in X_0^{n+1}$ the map

$$X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is a weak equivalence of simplicial sets.

We must prove that $\mathrm{map}_{L_c X}(x, y) \to \mathrm{map}_{L_c Y}(fx, fy)$ is a weak equivalence of simplicial sets. Given that fact, combining it with the surjectivity of the map $X_0 \to Y_0$ implies that $\mathrm{Ho}(L_c X) \to \mathrm{Ho}(L_c Y)$ is an equivalence of categories.

We construct a factorization $X \to \Phi Y \to Y$ such that $(\Phi Y)_0 = X_0$ and the map

$$(\Phi Y)_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is an isomorphism of simplicial sets for all $(v_0, \ldots, v_n) \in (\Phi Y)_0^{n+1}$. We begin by defining the object ΦY as the pullback of the diagram

$$
\begin{array}{ccc}
\Phi Y & \longrightarrow & Y \\
\downarrow & & \downarrow \\
\mathrm{cosk}_0(X_0) & \longrightarrow & \mathrm{cosk}_0(Y_0).
\end{array}
$$

Note in particular that $(\Phi Y)_0 = X_0$.

Now, notice that for each $n \geq 1$ and $(v_0, \ldots, v_n) \in (\Phi Y)_0^{n+1}$ the map

$$(\Phi Y)_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is an isomorphism of simplicial sets. Since each

$$X_n(v_0, \ldots, v_n) \to Y_n(fv_0, \ldots, fv_n)$$

is a weak equivalence, we can apply model category axiom MC2 to simplicial sets to see that the map

$$X_n(v_0, \ldots, v_n) \to (\Phi Y)_n(v_0, \ldots, v_n)$$

is a weak equivalence for each $n \geq 1$ and $(v_0, \ldots v_n)$ also.

Thus we have shown that if $X \to Y$ has the right lifting property with respect to the maps in I_c, then each map $X_n(v_0, \ldots, v_n) \to (\Phi Y)_n(v_0, \ldots, v_n)$ is a weak equivalence of simplicial sets for $n \geq 1$ and $(v_0, \ldots, v_n) \in X_0^{n+1}$. Since $X_0 = (\Phi Y)_0$, the map $X \to \Phi Y$ is actually a Reedy weak equivalence and therefore also a DK-equivalence. To prove Lemma 5.9, it remains to show that the map $\Phi Y \to Y$ is a DK-equivalence, implying that the map $X \to Y$ is also. We will prove this fact by induction on the skeleta of Y.

We will denote by $\mathrm{sk}_n Y$ the n-skeleton of Y, as defined above in the paragraph below Proposition 2.14. We seek to prove that the map

$$\Phi(\mathrm{sk}_n Y) \to \mathrm{sk}_n Y$$

is a DK-equivalence for all $n \geq 0$.

We first consider the case where $n = 0$. In this case, $\mathrm{sk}_0(\Phi Y)$ and $\mathrm{sk}_0 Y$ are already Segal categories. They can be observed to be DK-equivalent as follows. In the case of $\mathrm{sk}_0 Y$, given any pair of elements $(x, y) \in (\mathrm{sk}_0 Y)_0 \times (\mathrm{sk}_0 Y)_0$, the mapping space $\mathrm{map}_{\mathrm{sk}_0 Y}(x, y)$ is the homotopy fiber of the map

$$(\mathrm{sk}_0 Y)_1 = (\mathrm{sk}_0 Y)_0 \times (\mathrm{sk}_0 Y)_0 \to (\mathrm{sk}_0 Y)_0 \times (\mathrm{sk}_0 Y)_0$$

over (x, y). If $x = y$, this fiber is just the point (x, y), since in this case this map is the identity. If $x \neq y$, then the fiber is empty. For $(a, b) \in (\mathrm{sk}_0 \Phi Y)_0 \times (\mathrm{sk}_0 \Phi Y)_0$, the fiber of the analogous map over (a, b) is equivalent to (a, b) if a and b map to the same point x in Y_0. Otherwise the fiber is empty. The definition of ΦY and the map $\Phi Y \to Y$ then show that the two are DK-equivalent.

We now assume that the map $\Phi(\mathrm{sk}_{n-1} Y) \to \mathrm{sk}_{n-1} Y$ is a DK-equivalence and seek to show that the map

$$\Phi(\mathrm{sk}_n Y) \to \mathrm{sk}_n Y$$

is also for $n \geq 2$. Notice that $\mathrm{sk}_n Y$ is obtained from $\mathrm{sk}_{n-1} Y$ via iterations of pushouts of diagrams of the form

$$(9.2) \qquad Q_{m,n} \xleftarrow{\quad} P_{m,n} \xrightarrow{\quad} \mathrm{sk}_{n-1} Y$$

For simplicity, we will assume that $m = 0$ and we require only one such pushout to obtain $\mathrm{sk}_n Y$. Notice that $(\mathrm{sk}_{n-1} Y)_0 = (\mathrm{sk}_n Y)_0 = Y_0$ and that the map

$$\mathrm{sk}_{n-1} Y \to \mathrm{sk}_n Y$$

is the identity on the discrete space in degree zero. Therefore we use the distinct n-simplex $\Delta[n]^t_{y_0, \dots, y_n}$ for each $(y_0, \dots, y_n) \in Y_0^{n+1}$ as defined above in section 3.12. Setting $\underline{y} = (y_0, \dots y_n)$, we write this n-simplex as $\Delta[n]^t_{\underline{y}}$.

We can then apply the map Φ to diagram 9.2 (and its pushout) to obtain the diagram

$$(9.3)$$

$$
\begin{array}{ccc}
\Phi \dot{\Delta}[n]^t_{\underline{y}} & \longrightarrow & \Phi \mathrm{sk}_{n-1} Y \\
\downarrow & & \downarrow \\
\Phi \Delta[n]^t_{\underline{y}} & \longrightarrow & \Phi \mathrm{sk}_n Y.
\end{array}
$$

We would like to know that we still have a pushout diagram. In other words, we want to know that the functor Φ preserves pushouts. To see that it does, consider the levelwise pullback diagram defining $(\Phi Y)_n$:

$$
\begin{array}{ccc}
(\Phi Y)_n & \longrightarrow & Y_n \\
\downarrow & & \downarrow \\
X_0^{n+1} & \longrightarrow & Y_0^{n+1}.
\end{array}
$$

We can regard the map $f \colon X \to Y$ as inducing a pullback functor f^* from the category of simplicial sets over Y_0^{n+1} to the category of simplicial sets over X_0^{n+1}. (Recall that the category of objects *over* a simplicial set Z has as objects maps of simplicial sets $W \to Z$ and as morphisms the maps of simplicial sets making the appropriate triangle commute.) However, this functor between over categories can be shown to have a right adjoint. Therefore it is a left adjoint and hence preserves pushouts.

We know that the maps

$$\Phi \dot{\Delta}[n]^t_{\underline{y}} \to \dot{\Delta}[n]^t_{\underline{y}}$$

and

$$\Phi(\mathrm{sk}_{n-1} Y) \to \mathrm{sk}_{n-1} Y$$

are DK-equivalences by our inductive hypothesis, since the nondegenerate simplices in each case are concentrated in degrees less than n. Since the left-hand vertical

maps of diagrams 9.2 and 9.3 above are cofibrations, the right-hand vertical map in diagram 9.3 is also a cofibration, and therefore it remains only to show that the map $\Phi\Delta[n]_{\underline{y}}^t \to \Delta[n]_{\underline{y}}^t$ is a DK-equivalence in order to show that the pushouts of the two diagrams are weakly equivalent.

If $n = 0$, then $\Phi\Delta[0]_{\underline{y}}^t \to \Delta[0]_{\underline{y}}^t$ is a DK-equivalence since everything is already local and $\Phi\Delta[0]_{\underline{y}}^t$ is just the nerve of some contractible category. In fact, given any $n \geq 0$ and $\underline{y} = (y_0, \ldots, y_n)$, if $y_i \neq y_j$ for each $0 \leq i, j \leq n$, the map $\Phi\Delta[n]_{\underline{y}}^t \to \Delta[n]_{\underline{y}}^t$ is a DK-equivalence, since $\Delta[n]_{\underline{y}}^t$ is already local.

Now suppose that $n = 1$ and $\underline{y} = (y_0, y_0)$. Consider $g : \Phi\Delta[1]_{\underline{y}}^t \to \Delta[1]_{\underline{y}}^t$ and let k be the number of 0-simplices of $g^{-1}(y_0)$. If C_k denotes the category with k objects and a single isomorphism between any two objects, then we have that

$$\Phi\Delta[1]_{\underline{y}}^t \simeq \Delta[1]_{\underline{y}}^t \times \mathrm{nerve}(C_k).$$

Thus, it suffices to show that

$$L_c\Phi\Delta[1]_{\underline{y}}^t \simeq L_c\Delta[1]_{\underline{y}}^t \times L_c\mathrm{nerve}(C_k).$$

To prove this fact, first note that the fibrant objects in $\mathcal{S}e\mathcal{S}p_c$ are closed under internal hom, namely that given a Segal space W and any simplicial space Y, there is a Segal space W^Y given by $(W^Y)_k = \mathrm{Map}^h(Y \times \Delta[k]^t, W)$ [17, 7.1]. Therefore, given any Segal precategories X and Y and any Segal space W, we can work in the category $\mathcal{S}e\mathcal{S}p_c$ and make the following calculation.

$$\begin{aligned}
\mathrm{Map}^h(L_cX \times L_cY, W) &\simeq \mathrm{Map}^h(L_cX, W^{L_cY}) \\
&\simeq \mathrm{Map}^h(X, W^Y) \\
&\simeq \mathrm{Map}^h(X \times Y, W) \\
&\simeq \mathrm{Map}^h(L_c(X \times Y), W)
\end{aligned}$$

In other words, the map

$$L_c(X \times Y) \to L_cX \times L_cY$$

is a DK-equivalence, and in particular the statement above for $L_c\Phi\Delta[n]_{\underline{y}}^t$ holds.

Now consider the case where $n = 2$. Then if $\underline{y} = (y_0, y_1, y_2)$, we have that $G(2)_{\underline{y}}^t$ can be written as a pushout

(9.4)
$$\begin{array}{ccc}
G(0)_{y_1}^t & \longrightarrow & G(1)_{y_0, y_1}^t \\
\downarrow & & \downarrow \\
G(1)_{y_1, y_2}^t & \longrightarrow & G(2)_{\underline{y}}^t.
\end{array}$$

Now consider the map $g : \Phi G(2)_{\underline{y}}^t \to G(2)_{\underline{y}}^t$. We have that $g^{-1}(G(0)_{y_1}^t)$ is the nerve of some contractible category. Similarly, the map $g^{-1}(G(1)_{y_0, y_1}^t) \to G(1)_{y_0, y_1}^t$ is a DK-equivalence, as is the map $g^{-1}(G(1)_{y_1, y_2}^t) \to G(1)_{y_1, y_2}^t$. Since we have a

pushout diagram

$$\begin{array}{ccc} g^{-1}(G(0)^t_{y_1}) & \longrightarrow & g^{-1}(G(1)^t_{y_0,y_1}) \\ \downarrow & & \downarrow \\ g^{-1}(G(1)^t_{y_1,y_2}) & \longrightarrow & \Phi G(2)^t_{\underline{y}} \end{array}$$

and the left-hand vertical maps of this diagram and of diagram 9.4 are cofibrations, it follows that the map $\Phi G(2)^t_{\underline{y}} \to G(2)^t_{\underline{y}}$ is a DK-equivalence. In fact, for any $n \geq 2$, $G(n)^t_{\underline{y}}$ can be obtained by iterating such pushouts. Therefore, we have shown that the map $\Phi G(n)^t_{\underline{y}} \to G(n)^t_{\underline{y}}$ is a DK-equivalence.

To see that $\Phi \Delta[n]^t_{\underline{y}} \to \Delta[n]^t_{\underline{y}}$ is a DK-equivalence for any choice of $\underline{y}$, we need a variation on this argument. Again using a pushout construction, we will use the fact that this map is a DK-equivalence when each y_i is distinct to show that it is also a DK-equivalence even if $y_i = y_j$ for some $i \neq j$. We will describe this construction for a specific example, but it works in general. Specifically, we show that $\Phi \Delta[2]^t_{y_0,y_1,y_0} \to \Delta[2]^t_{y_0,y_1,y_0}$ is a DK-equivalence.

Define the Segal precategory $\widetilde{Y} = Y \amalg \{\widetilde{y}\}$, where $\widetilde{y}$ is a 0-simplex not in Y_0, and we regard $\{\widetilde{y}\}$ as a doubly constant simplicial space. Then, using the map $g \colon \Phi Y \to Y$ and some vertex y_0 of Y, we let Z be a Segal precategory isomorphic to $(g^{-1}y_0)$ and define $\widetilde{X} = X \amalg Z$. There is a map $\widetilde{X} \to \widetilde{Y}$ such that Z maps to $\widetilde{y}$. We define a functor $\widetilde{\Phi}$ and factorization

$$\widetilde{X} \longrightarrow \widetilde{\Phi}\widetilde{Y} \xrightarrow{g} \widetilde{Y}$$

just as we defined ΦY above. More generally, we apply $\widetilde{\Phi}$ to any Segal precategory with 0-simplices those of $\widetilde{Y}$ to obtain a Segal precategory with 0-simplices those of $\widetilde{X}$, just as we have been doing with Φ.

Now consider the objects $G(2)^t_{y_0,y_1,\widetilde{y}}$ and $\Delta[2]^t_{y_0,y_1,\widetilde{y}}$, each with 0-simplices those of $\widetilde{Y}$. There is a natural map

$$G(2)^t_{y_0,y_1,\widetilde{y}} \to G(2)^t_{y_0,y_1,y_0}$$

where $\widetilde{y} \mapsto y_0$, and an analogous map

$$\Delta[2]^t_{y_0,y_1,\widetilde{y}} \to \Delta[2]^t_{y_0,y_1,y_0}.$$

We have a pushout diagram

$$\begin{array}{ccc} G(2)^t_{y_0,y_1,\widetilde{y}} & \longrightarrow & G(2)^t_{y_0,y_1,y_0} \\ \downarrow & & \downarrow \\ \Delta[2]^t_{y_0,y_1,\widetilde{y}} & \longrightarrow & \Delta[2]^t_{y_0,y_1,y_0}. \end{array}$$

Since the left-hand vertical map is a cofibration, this map is actually a homotopy pushout diagram.

Now, from above we know that the maps

$$\widetilde{\Phi} G(2)^t_{y_0,y_1,\widetilde{y}} \to G(2)^t_{y_0,y_1,\widetilde{y}}$$

and

$$\Phi G(2)^t_{y_0,y_1,y_0} \to G(2)^t_{y_0,y_1,y_0}$$

are DK-equivalences. We also know that the map

$$\widetilde{\Phi}\Delta[2]^t_{y_0,y_1,\widetilde{y}} \to \Delta[2]^t_{y_0,y_1,\widetilde{y}}$$

is a DK-equivalence since the 0-simplices $y_0, y_1, \widetilde{y}$ are distinct. We can consider the pushout diagram

$$
\begin{array}{ccc}
\widetilde{g}^{-1}G(2)^t_{y_0,y_1,\widetilde{y}} & \longrightarrow & g^{-1}G(2)^t_{y_0,y_1,y_0} \\
\downarrow & & \downarrow \\
\widetilde{g}^{-1}\Delta[2]^t_{y_0,y_1,\widetilde{y}} & \longrightarrow & \Phi\Delta[2]^t_{y_0,y_1,y_0}.
\end{array}
$$

which is again a homotopy pushout diagram. It follows that the map

$$\Phi\Delta[2]^t_{y_0,y_1,y_0} \to \Delta[2]^t_{y_0,y_1,y_0}$$

is a DK-equivalence, completing the proof. $\square$

We now proceed with the other remaining proof from section 5.

Proof of Proposition 5.10. Suppose that $f\colon X \to Y$ is a fibration and a weak equivalence. First, consider the case where $f_0\colon X_0 \to Y_0$ is an isomorphism. Without loss of generality, assume that $X_0 = Y_0$ and factor the map $f\colon X \to Y$ functorially in $\mathcal{SS}ets_c^{\Delta^{op}}$ as the composite of a cofibration and an acyclic fibration in such a way that the Y_0' remains a discrete space:

$$X \hookrightarrow Y' \overset{\simeq}{\twoheadrightarrow} Y.$$

(We can obtain a Y' with discrete zero space by taking a factorization in $\mathcal{SS}ets_c^{\Delta^{op}}$ analogous to the one we defined for $\mathcal{SeSp}_c$ at the beginning of section 5.) Since the map $X \to Y$ is a DK-equivalence and the map $Y' \to Y$ is a Reedy weak equivalence and therefore a DK-equivalence, it follows that the map $X \to Y'$ is a DK-equivalence. In particular, $X \to Y'$ is an acyclic cofibration and therefore by the definition of fibration in $\mathcal{SeCat}_f$ the dotted arrow lift exists in the following solid-arrow diagram:

$$
\begin{array}{ccc}
X & \overset{\mathrm{id}}{\longrightarrow} & X \\
\downarrow & \nearrow & \downarrow \\
Y' & \longrightarrow & Y.
\end{array}
$$

Thus, $f\colon X \to Y$ is a retract of $Y' \to Y$ and therefore a Reedy acyclic fibration. In particular, f has the right lifting property with respect to the maps in I_c, since they are monomorphisms and therefore Reedy cofibrations.

Now consider the general case, where $X_0 \to Y_0$ is surjective but not necessarily an isomorphism. Then, as in the proof of Lemma 5.9, define the object ΦY and consider the composite map $X \to \Phi Y \to Y$. since by the first case $X \to \Phi Y$ has the right lifting property with respect to the maps in I_c, it remains to show that $\Phi Y \to Y$ has the right lifting property with respect to the maps in I_c.

Let $A \to B$ be an acyclic cofibration. Then there is a dotted arrow lift in any solid-arrow diagram of the form

(9.5)

$$
\begin{array}{ccccc}
A & \longrightarrow & X & \stackrel{=}{\longrightarrow} & X \\
{\scriptstyle \simeq}\downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & \Phi Y & \longrightarrow & Y \\
& & \downarrow & & \downarrow \\
& & \mathrm{cosk}_0 X_0 & \longrightarrow & \mathrm{cosk}_0 Y_0
\end{array}
$$

We would like to know that this lift $B \to X$ also makes the upper left-hand square commute.

Suppose that $A_0 = B_0 = X_0$. In this case, a map $B \to Y$ together with a lifting

$$
\begin{array}{ccc}
& & X_0 \\
& \nearrow & \downarrow \\
B_0 & \longrightarrow & Y_0
\end{array}
$$

completely determines a map $B \to \Phi Y$. Therefore, in this fixed object set case, there is only one possible lifting $B \to X$ in diagram 9.5, and one which makes the upper left-hand square commute.

The map $X \to \Phi Y$ is a fibration in the fixed object model category structure $\mathcal{LSS}ets_{\mathcal{O},f}^{\Delta^{op}}$ where $\mathcal{O} = X_0$. However, since the cofibrations in $\mathcal{LSS}ets_{\mathcal{O},f}^{\Delta^{op}}$ are precisely the monomorphisms, the acyclic fibrations are Reedy acyclic fibrations. Therefore, the map $X \to \Phi Y$ is a Reedy acyclic fibration and thus has the right lifting property with respect to all monomorphisms of simplicial spaces. In particular, it has the right lifting property with respect to the maps in I_c.

Using the construction of ΦY and the fact that $X \to Y$ is a fibration and a weak equivalence, we can see that $X_0 \to Y_0$ is surjective. In particular, the map $\mathrm{cosk}_0 X_0 \to \mathrm{cosk}_0 Y_0$ has the right lifting property with respect to the maps in I_c. Using the universal property of pullbacks, we can see that the map $\Phi Y \to Y$ also has the right lifting property with respect to the maps in I_c. $\square$

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