A QUICK INTRODUCTION TO OPERADS

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This document started out as notes for a talk I gave at the MIT Kan seminar in 2013 about May's Geometry of Iterated Loop Spaces, *but I've since largely rewritten it as a general introduction to A_∞ and E_∞ operads, with a mention of the recognition principle.*

1. ASSOCIATIVITY

Operads (or at least A_n and E_n operads, which is what we'll focus on) are a formalism for discussing various degrees to which associativity and commutativity can fail. Suppose I have a space X with a multiplication $\mu : X \times X \to X$ that is not associative. In particular, $\mu(-, \mu(-, -))$ and $\mu(\mu(-, -), -)$ are two different ternary operations. One approach is to consider a *space* $\mathcal{O}(3)$ of ternary operations, in which $\mu(\mu(-, -), -)$ and $\mu(-, \mu(-, -))$ are two different points. Then the geometry of this space tells us something about exactly how badly associativity fails—strict associativity corresponds to this space being one point (so in particular $\mu(-, \mu(-, -)) = \mu(\mu(-, -), -)$), and the idea is that the next best thing is for the space to be contractible. For every n, we can also consider a space $\mathcal{O}(n)$ of n-ary operations, where the operations $\mu(\mu(\mu(\dots), -), -)$ etc. are points. The best kind of associativity short of strict associativity is when all the $\mathcal{O}(n)$'s are contractible; in this case, we say the multiplication is A_∞.

All of the above is a bit of a lie: I made it sound like we're associating to every space X a collection $\{\mathcal{O}(n)\}_n$ of n-ary operations (for every n). This isn't quite true; instead, there are some stock collections $\{\mathcal{O}(n)\}$ that we care about, and you learn something about X if you can interpret $\{\mathcal{O}(n)\}$ as n-ary operations on X in a nice way.

Definition 1.1. A *non-Σ operad* is a collection $\{\mathcal{O}(n)\}_{n \geq 0}$ of spaces with some extra structure and properties inspired by thinking of $\mathcal{O}(n)$ as a space of n-ary operations:

(1) there are maps
$$\gamma : \mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) \to \mathcal{O}(k_1 + \cdots + k_n)$$
(you're supposed to think of this as taking an n-ary operation, and n other operations, and plugging those operations into the n-ary operation);

(2) there is a special element in $\mathcal{O}(1)$ that acts as the identity;

(3) $\mathcal{O}(0) = *$;

and these are subject to some compatibility conditions; see [2, Def. 1.1] for a formal definition. For a based space X, say that "$\mathcal{O}$ acts on X" (or, "X is an $\mathcal{O}$-algebra") if there is a morphism of operads[1] $\mathcal{O} \to \text{End}_X$, where $\text{End}_X(n)$ is the space of maps $X^n \to X$.

The simplest example of a non-Σ operad is the associative operad Ass, defined by setting $\text{Ass}(n) = *$ for all n. An action of Ass on a space X is a map $\mathcal{O}(n) \to \text{End}_X(n)$ for every n;

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[1]A morphism of operads $\mathcal{O} \to \mathcal{O}'$ is a collection of maps $\mathcal{O}(n) \to \mathcal{O}'(n)$ that commute with the operad structure maps that we haven't defined yet.

all this does is pick out a single operation $X^n \to X$ for every n. If μ is the chosen binary operation, then the fact that $\mathcal{O} \to \mathrm{End}_X$ preserves operad structure will ensure that the chosen ternary operation is $\mu(\mu(-,-),-) = \mu(-,\mu(-,-))$, and so on.[2] So X is an Ass-algebra iff X has a unital,[3] associative multiplication.

Now suppose $X = \Omega Y$. There is a multiplication (concatenation of loops), but it isn't quite associative because of the parametrization: if a, b, c are loops, then $ab : [0,1]/(0 \sim 1) \to Y$ is the loop where you spend half the time on a and half the time on b, and so $a(bc) = \overset{a \quad\quad b \quad c}{\vdash\!\dashv}$ and $(ab)c = \overset{a \quad b \quad\quad c}{\vdash\!\dashv}$ are different. However, there is quite clearly a homotopy between them, in which you change the parametrization. Similarly, there are five ways to multiply four loops (without changing the order), with not only homotopies between any pair, but higher homotopies as well in a way that the operad structure will make clearer. So X is not an algebra over Ass, but I'll show it's an algebra over a non-Σ operad $\mathcal{K}$, called the Stasheff associahedron, which is essentially designed to reflect these homotopies. The first few spaces of $\mathcal{K}$ are $\mathcal{K}(1) = \mathcal{K}(2) = *$, $\mathcal{K}(3)$ is the interval labelled $\overset{(x_1x_2)x_3 \quad\quad x_1(x_2x_3)}{\bullet\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!\bullet}$, and $\mathcal{K}(4)$ is the (filled in) pentagon labelled
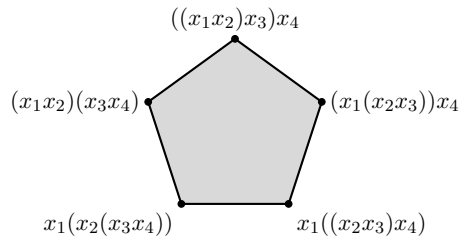
The rule is that the vertices are parenthesizations of a word on n letters, there is an edge between two parenthesizations that are related by changing one pair of parentheses, a face among vertices that are related by changing two pairs of parentheses, and so on. There's lots of interesting combinatorics involved in describing the $\mathcal{K}(n)$'s, but all we need to know is that $\mathcal{K}(n)$ is contractible. The discussion above on $X = \Omega Y$ exactly expresses the fact that there is a map $\mathcal{K}(3) \to \mathrm{End}_X(3)$ sending the vertex $(ab)c$ to $\mu(\mu(-,-),-)$, the vertex $a(bc)$ to $\mu(-,\mu(-,-))$ and the homotopy between them to the interval in $\mathcal{K}(3)$.

Note that any space with a strictly associative multiplication is also a $\mathcal{K}$-algebra, in addition to being an Ass-algebra—just precompose the collapse map $\mathcal{K}(n) \to \mathrm{Ass}(n) = *$ with the Ass-action.

We say that X is an A_∞-space if it has an action of an A_∞ [non-Σ] operad, namely an operad all of whose spaces are contractible. So $\mathcal{K}$ is an A_∞ [non-Σ] operad and it turns out that a space is A_∞ iff it has an action of $\mathcal{K}$.[4]

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[2] In particular, $\gamma(\mu; \mathbb{1}, \mu) = \mu(-, \mu(-,-))$ and $\gamma(\mu; \mu, \mathbb{1}) = \mu(\mu(-,-),-)$ are both elements of the image of $\mathrm{Ass}(3) = * \to \mathrm{Maps}(X^3, X)$, which is a point so they must be the same.

[3] Remember the spaces X have basepoints; the structure maps are forcing the basepoint to act as the unit w.r.t. the multiplication.

[4] There is a way to put a model category structure on operads, and there is a specific combinatorial way to construct cofibrant resolutions called the Boardman-Vogt W-construction (see https://ncatlab.org/nlab/

It turns out that we've already described all the *grouplike* A_∞ spaces (i.e. the ones where $\pi_0 X$ is a group):

Theorem 1.2. *[Recognition principle, $n = 1$ case] Every grouplike A_∞ space[5] has the weak homotopy type of a loop space.*

2. COMMUTATIVITY

I want to tell the same story for multiplications that fail to be commutative, but it turns out we're missing a bit of structure so far. I want to talk about a space X with a multiplication μ, and ask about a space of binary operations including $(a, b) \mapsto \mu(a, b)$ and $(a, b) \mapsto \mu(b, a)$. If the space is a point then these are the same and the multiplication is commutative; if this fails, we are less unhappy if the space is contractible.

Definition 2.1. An *operad* is a collection of spaces $\{\mathcal{O}(n)\}$ with the structure and properties of a non-Σ operad, plus an action of the symmetric group Σ_n on $\mathcal{O}(n)$ (which you're supposed to think of as permuting the variables in an n-ary operation) and additional coherences relating this to the other structure.

For example, the commutative operad Comm is defined to have $\text{Comm}(n) = *$. You can check that Comm-algebras are the same as unital, associative, commutative monoids. The idea is that $\text{Comm}(2) = *$ is forcing $(a, b) \mapsto \mu(a, b)$ and $(a, b) \mapsto \mu(b, a)$ to be the same point.

If your space fails to have a commutative multiplication, the next best thing to ask for is an action of an operad, all of whose spaces are contractible.

Definition 2.2. An E_∞ operad is an operad $\mathcal{E}$ such that $\mathcal{E}(n)$ is contractible for all n, and the Σ_n action is free.[6]

(The free action is a technical condition.) For example, recall that $E\Sigma_n$ is a contractible space with a free Σ_n-action; it is the total space for the classifying space $B\Sigma_n$. Then I can define an operad with $\mathcal{E}(n) = E\Sigma_n$ and structure maps coming from maps of Σ_i's; this is called the Barratt-Eccles operad.

Given a non-Σ operad $\mathcal{C}$, you can always turn it into an operad $\mathcal{C}^\Sigma$ (non-standard notation) by defining $\mathcal{C}^\Sigma(n) = \mathcal{C}(n) \times \Sigma_n$, where Σ_n acts freely on the right. All the ideas in the previous section are unchanged by this modification, and so we almost always talk about operads, not non-Σ-operads. For example, if $\mathcal{K}^\Sigma$ is the non-(non-Σ) version of $\mathcal{K}$, then $\mathcal{K}^\Sigma(2)$ consists of

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`show/Boardman-Vogt+resolution`). It turns out that $\mathcal{K}$ is the W-construction of the associative operad Ass. Some technicalities related to this are hinted at here: https://mathoverflow.net/questions/265308/stasheffs-operad-as-a-relative-w-construction.

[5]Technically, "space" has to mean "compactly generated weakly Hausdorff space with non-degenerate base point" (i.e. $(X, *)$ is an NDR-pair).

[6]More abstractly, an E_∞ operad is any cofibrant resolution of Comm; see https://ncatlab.org/nlab/show/E-infinity+operad.

two points labelled $x_1 x_2$ and $x_2 x_1$, and $\mathcal{K}^\Sigma(3)$ consists of six copies of the interval:

$(x_1 x_2)x_3$ $x_1(x_2 x_3)$ $(x_1 x_3)x_2$ $x_1(x_3 x_2)$ $(x_3 x_1)x_2$ $x_3(x_1 x_2)$

$(x_2 x_1)x_3$ $x_2(x_1 x_3)$ $(x_3 x_2)x_1$ $x_3(x_2 x_1)$ $(x_2 x_3)x_1$ $x_3(x_3 x_1)$

A structure-preserving collection of maps $\mathcal{K}^\Sigma(n) \to \mathrm{End}_X(n)$ (where $\mathrm{End}_X(n)$ has the obvious Σ_n action) is enforcing the associativity conditions as above, but is not making any new demands about commutativity.

3. A_n AND E_n

Another candidate for "not quite as good as an associative multiplication" is a homotopy associative multiplication – that is, there is a homotopy between the maps $\mu(\mu(-,-),-)$ and $\mu(-,\mu(-,-))$. How does this fit in to the above discussion? This looks like the condition for $\mathcal{K}(3)$; more precisely, being homotopy associative is the same thing as saying there is an action of just $\mathcal{K}(i)$ for $i \leq 3$. Alternatively, you can truncate $\mathcal{K}$ to form another operad $\mathcal{K}_3$ (taking care to make sure it's closed under the structure maps); then a space with a homotopy associative multiplication is the same as a $\mathcal{K}_n$-space. Similarly, if X is an H-space (a space with a unital multiplication), it might not have an action of all of $\mathcal{K} = \{\mathcal{K}(n)\}$, but you can write down coherent maps from $\{\mathcal{K}(i)\}_{i \leq 2}$, since a map from $\mathcal{K}(2)$ is just picking out a multiplication, and the structure maps make it unital w.r.t. the basepoint. We say that X is an A_n-*space* if it has an "action" of $\mathcal{K}(i)$ for $i \leq n$ (or equivalently, is an algebra over a properly truncated operad $\mathcal{K}_n$).

So there is a hierarchy:

X is an H-space $\Longleftarrow$ X has htpy. assoc. mult $\Longleftarrow$ ... $\Longleftarrow$ X is A_n $\Longleftarrow$ ... $\Longleftarrow$ X is A_∞ $\Longleftarrow$ X has a strictly assoc. mult.

There is a similar hierarchy for commutativity, with the intermediate stages called E_n. We could try to truncate e.g. the Barratt-Eccles operad, but the most important theory surrounding this has to do with truncations of another E_∞ operad, the little ∞-cubes operad. We'll focus on the nth truncation, called the little n-cubes operad.

Let $I = [0,1] \subset \mathbb{R}$; specifying a linear map $I \to I$ is equivalent to just specifying the range $[a,b] \subset [0,1]$. Specifying n of these maps is the same as specifying a "little cube" $[a_1, b_1] \times \cdots \times [a_n, b_n] \subset I^n$; this is a (particularly constrained) linear map $c_n : I^n \to I^n$.

Definition 3.1 (Little n-cubes operad)**.** The little n-cubes operad $\mathcal{C}_n$ is defined so an element of $\mathcal{C}_n(j)$ is a map $\underbrace{I^n \sqcup \cdots \sqcup I^n}_{j} \to I^n$ that specifies j little cubes in I^n that do not overlap.
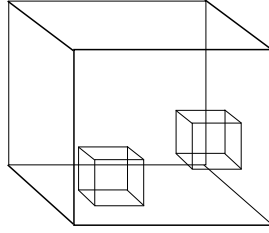
FIGURE 1. An element of $\mathcal{C}_3(2)$

One reason to care about $\mathcal{C}_n$ is that it acts on $\Omega^n X$. This is the space of maps $(S^n, *) \to (X, *)$, but we can think of it as the space of maps $(I^n, \partial I^n) \to (X, *)$. Given j little n-cubes $\langle c_1, \cdots, c_j \rangle$ and j maps $f_i : (I^n/\partial I^n) \to (X^*) \in \Omega^n X$, we can create another map $f : (I^n, \partial I^n) \to (X, *)$ as follows:

$$f : \mathbf{t} \mapsto \begin{cases} f_i(c_i^{-1}(\mathbf{t})) & \text{if } \mathbf{t} \in \operatorname{im} c_i \\ 0 & \text{otherwise.} \end{cases}$$
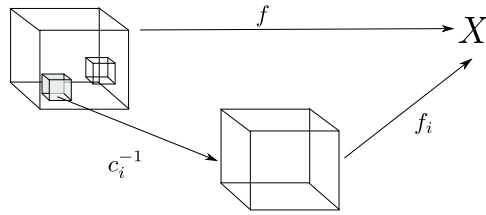

FIGURE 2. The map $(f : (I^n, \partial I^n) \to (X, *)) \in \Omega^n X$

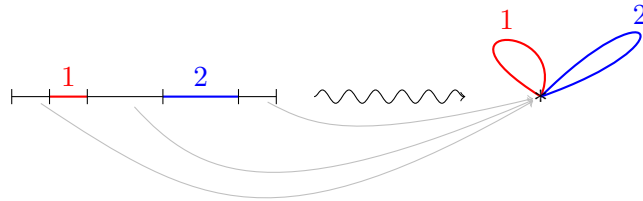If $n = 1$, this is just the ordinary composition of j loops.

FIGURE 3. The map $(f : I^1 \to X) \in \Omega X$ associated to an element of $\mathcal{C}_1(2)$

Moreover, you can show that the little intervals operad $\mathcal{C}_1$ has the right homotopy type to be an A_∞ operad. That is, E_1 spaces are A_∞ spaces (and the converse is true as well). Thus we have a hierarchy:

$$A_\infty = E_1 \impliedby E_2 \impliedby \ldots \impliedby E_n \impliedby \ldots \impliedby E_\infty \impliedby \text{strictly commutative.}$$

One can also identify E_2 in a manner analogous to A_3: an A_∞ space is E_2 iff it is homotopy commutative.

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One of the useful things about operads (and one of the reasons we use them instead of their historical predecessor, PROPs) is that every operad gives rise to a monad.

Construction 4.1 (Monad associated to $\mathcal{C}$). Given an operad $\mathcal{C}$ we can define a monad (C, μ, η) where C is an endofunctor $\mathrm{Top}_* \to \mathrm{Top}_*$:

$$CX = \bigsqcup_{j \geq 0} \mathcal{C}(j)_+ \wedge_{\Sigma_j} X^{\wedge j}.$$

Very explicitly, this is $CX = \bigsqcup_{j \geq 0} \mathcal{C}(j) \times X^j / \sim$, where in the case $\mathcal{C} = \mathrm{End}_X$ (and other operads are analogous) $\sim$ is described as:

- "including a basepoint works well": $\left(\begin{smallmatrix} f:X^n \to X, \\ (a_1, \cdots, \underset{j}{\underbrace{*}}, \cdots, a_{n-1}) \end{smallmatrix} \right) \sim \left(\begin{smallmatrix} \widetilde{f}:X^{n-1} \overset{x_j := *}{\to} X^n \overset{f}{\to} X, \\ (a_1, \cdots, a_{n-1}) \end{smallmatrix} \right)$;

- permuting the variables in f is the same as permuting the order of the copies of X in X^j.

Remark 4.2. This is the monad arising from the free-forgetful adjunction $\mathrm{Top}_* \rightleftarrows \mathcal{C}$-algebras, and the category of $\mathcal{C}$-algebras is equivalent to the category of C-algebras.

Example 4.3 (James reduced product (free monoid on X)). Let's do this to the associative operad Ass^Σ (this is the non-symmetric operad Ass made into an actual operad as discussed in Section 2, i.e. by setting $\mathrm{Ass}^\Sigma(j) = \underbrace{\mathrm{Ass}(j)}_{*} \times \Sigma_j = \Sigma_j$); call the associated monad A. By Remark 4.2, we know that AX is supposed to be the free monoid on X, i.e. the James construction, but let's watch this come out of the definition. We have $AX = \bigsqcup_j \mathrm{Ass}^\Sigma(j) \times_{\Sigma_j} X^j / \sim$ and every element $(\sigma, \mathbf{x})$ has a unique representative under $\sim$ of the form $(\mathbb{1}, \mathbf{y})$. So we can think of elements of AX as ordered strings $x_1 \cdots x_n$, and the basepoint condition just guarantees that this plays nicely with inclusion of the basepoint (i.e. the monoid identity) anywhere in the string. That is,

$$AX \cong \bigvee_j X^{\wedge j}.$$

Now let's go back to the little cubes operad $\mathcal{C}_n$; recall we had an action of $\mathcal{C}_n$ on $\Omega^n X$ for any space X. If (C_n, μ, η_n) is the monad corresponding to $\mathcal{C}_n$, this induces a map $\theta_n : C_n \Omega^n X \to \Omega^n X$.

Construction 4.4 (Morphism of monads $C_n \to \Omega^n \Sigma^n$). There is an adjunction $\Sigma^n : \mathrm{Top}_* \to \mathrm{Top}_* : \Omega^n$ (where Σ^n is the suspension functor) so $\Omega^n \Sigma^n$ is a monad. Then there is a map

$$\alpha_n : C_n X \overset{C_n \eta_n}{\Longrightarrow} C_n \Omega^n \Sigma^n X \overset{\theta_n}{\longrightarrow} \Omega^n \Sigma^n X.$$

Theorem 4.5 ("Approximation Theorem"). α_n *is a weak equivalence.*

Proof. Read [2, Ch. 6-7]. □

The idea is that you should think of C_n and $\Omega^n\Sigma^n$ as the same monad. Indeed:

Theorem 4.6 (Recognition principle)*. For $1 \le n \le \infty$, grouplike E_n-spaces (i.e. C_n-algebras) are the same as n-fold loop spaces.*

Since $A_\infty = E_1$, we see that Theorem 1.2 is the $n = 1$ case of this.

This clears up what (grouplike) E_n-spaces are. But we could have done this entire story over a different category – that is, the operad pieces $\mathcal{O}(n)$ and the $\mathcal{O}$-algebras live in a category other than spaces, such as spectra or chain complexes. Then there is no analogue of the recognition principle to explain everything, and figuring out which things are E_∞ (or E_n) is a major area of current research.

References

[1] Adams, J. F. *Infinite Loop Spaces.* Princeton University Press, 1978.
[2] May, Peter. *The Geometry of Iterated Loop Spaces.* Springer-Verlag, Lecture Notes in Mathematics, 1972.
[3] Stasheff, Jim. "What is an operad?" http://www.ams.org/notices/200406/what-is.pdf.