SHEAFIFIABLE HOMOTOPY MODEL CATEGORIES

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Abstract. If a Quillen model category can be specified using a certain logical syntax (intuitively, "is algebraic/combinatorial enough"), so that it can be defined in any category of sheaves, then the satisfaction of Quillen's axioms over any site is a purely formal consequence of their being satisfied over the category of sets. Such data give rise to a functor from the category of topoi and geometric morphisms to Quillen model categories and Quillen adjunctions.

Introduction

The *homotopy model category* of the title refers to the ("closed") model categories of Quillen [36]. The intended meaning of *sheafifiable* is best illustrated by some examples:

Example 0.1. (simplicial sheaves)
Quillen's homotopy theory of simplicial sets can be extended to simplicial sheaves over a site (i.e. category $\mathcal{C}$ with a Grothendieck topology J) as follows. Choose the cofibrations to be the monomorphisms. Given a choice of local basepoint for $X \in \mathrm{Sh}(\mathcal{C}, J; SSet)$, one can construct a sheaf of homotopy groups over the basepoint. See the details in Jardine [25]. A weak equivalence is a map that induces isomorphisms on the homotopy sheaves for arbitrary local basepoints. This fixes the data for a Quillen model category, which is in fact simplicial and proper.

Example 0.2. (simplicial objects in a topos)
A topos is a category equivalent to the category of sheaves on some site; so a category of simplicial sheaves is a category of simplicial objects in a topos. In 1984, A. Joyal [29] extended Quillen's homotopy theory of simplicial sets to simplicial objects in a topos $\mathcal{E}$ as follows: for $X_\bullet \in \mathcal{E}^{\Delta^{\mathrm{op}}}$, its homotopy groups can be constructed — by purely categorical operations in $\mathcal{E}$ — as objects (with algebraic structure) over X_0. Let a weak equivalence be a morphism $X_\bullet \to Y_\bullet$ for which the induced squares

$$\begin{tikzcd} \pi_n(X_\bullet) \arrow[r] \arrow[d] & \pi_n(Y_\bullet) \arrow[d] \\ X_0 \arrow[r] & Y_0 \end{tikzcd}$$

are pullbacks. The rest is as above.

It is not hard to see that the two constructions prescribe identical homotopy model structures for the same category. For any topos $\mathcal{E}$, there exist many (in fact, a proper class) of sites whose categories of sheaves are all equivalent to $\mathcal{E}$; Joyal's result contains the additional information that to all such sites, Ex. 0.1 associates the same homotopy theory. (Historically, Ex. 0.2 preceded Ex 0.1; a detailed reworking of Joyal's proof appears in Jardine [26].)

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Example 0.3. (simplicial rings)
Quillen [36] proves that the category of simplicial rings possesses a homotopy model structure; the forgetful functor *SRing* $\to$ *SSet* detects weak equivalences and fibrations. He leaves open the question whether the same is true for *sheaves* of simplicial rings. The answer is affirmative: if $Ring(\mathcal{E})$ denotes the category of ring objects of a topos $\mathcal{E}$, then with the help of the canonical forgetful functor $Ring(\mathcal{E}) \to \mathcal{E}$ one can endow $SRing(\mathcal{E})$ with a model structure. The remark in the paragraph following Ex. 0.2 applies here as well: since $\mathcal{E}^{\Delta^{\mathrm{op}}}$ carries an intrinsic homotopy structure (i.e. independent of having chosen a site for $\mathcal{E}$ first), the same is true for $SRing(\mathcal{E})$.

Example 0.4. (small categories and categorical equivalences)
There is a "folk model structure" on *Cat*, the category of small categories: the weak equivalences are the functors that induce an equivalence of categories, and the cofibrations the functors that are injective on objects. Joyal and Tierney [30] proved that (internal) category objects of any topos carry a homotopy model structure with weak equivalences those functors that are (in the internal sense) full, faithful and essentially surjective, and cofibrations those functors that are monos on the object part. (In the topos of sets, this specializes to the case of *Cat* mentioned above. It is distinct from the homotopy theory of *Cat* introduced by Thomason [43] — which, incidentally, sheafifies as well.)

Observe that in each case

(1) One takes as granted a homotopy theory of structured sets, and attempts to build one for sheaves of such structures. The passage is not arbitrary; for example, weak equivalences in the latter are locally weak equivalences in the former.
(2) The homotopy structure is functorial in the topos, not merely in the site defining the topos. (This is the difference between Ex. 0.1 and 0.2.)
(3) If $\mathcal{E} \xrightarrow{f} \mathcal{F}$ is a topos morphism[1], then the inverse image functor f^* preserves the weak equivalences and cofibrations of the homotopy model structure associated to $\mathcal{F}$, and induces a Quillen adjoint pair between the corresponding model categories. In particular, the "stalk" at a point $Set \to \mathcal{F}$ is the set-based homotopy model category one started with; so, in this sense, one may think of the structure $\mathcal{F}$ is endowed with as a "sheaf of homotopy theories with constant stalk".
(4) The homotopy model categories in question are all cofibrantly generated.

Outline of this paper. It is not hard to find a precise sense of what was loosely referred to above as "structured sets" and "sheaves of such structures", and to guess what additional criteria the weak equivalences and cofibrations should satisfy if (3) is desired. As to (4), in the present context it rests on a theorem of Jeffrey Smith that promises to be valuable in the study of set-theoretically well-behaved Quillen model categories; see Thm. 1.7. The main result of this paper, 2.8, is a meta-theorem to the following effect: if the ingredients of a homotopy model category are given by suitable data so that they can be interpreted in any topos, and over *Set* it does satisfy Quillen's axioms, then it does so (functorially) in any topos. There are mild conditions on the syntax, and a single (annoying) set-theoretical one: cofibrations have to be generated by a set.

[1]Throughout this article, topoi and their morphisms are understood in the sense of Grothendieck. Every topos is a U-topos for a fixed universe U whose elements are called simply *sets*.

We give six instances of the main result, some known, some new. Perhaps it is worthwhile to point out that the known cases had been obtained through independent and fairly laborious methods. The goal here is to invest enough labor so that what *is* tautological becomes visibly so.

Two corollaries may deserve attention: the existence of Quillen model categories on simplicial objects in a topos with the class of cofibrations smaller than all monos (Example 2.17), and existence of the unbounded derived category of any Grothendieck abelian category (Prop. 3.13). This latter fact is certainly folklore, but the proof via presentable model categories makes localization arguments easier as well.

Motivation. Perhaps it is useful to devote a few paragraphs to sketching how a topos may be thought of as a geometric object, and why (and when) it is worthwhile to do so. This is independent of the technical content matter of the paper, and readers acquainted with Grothendieckian sheaf theory should skip to the next section.

Many types of geometric objects — manifolds, orbifolds, algebraic spaces, schemes — are locally isomorphic to a fixed collection of distinguished models, and the chief strength of sheaf theory is to make this precise and to classify the extent to which such geometric objects may be non-isomorphic globally. Grothendieck observed that in many cases, the natural notion of morphism between geometric objects leads to one and the selfsame kind of adjunction between their associated categories of sheaves. He suggested that topological (homotopical) invariants be thought of as belonging to the abstract category of sheaves (the topos), functorially in topos morphisms, rather than to the chosen representation in terms of a site associated to the geometric object. There are fruitful aspects of this point of view. An invariant constructed from a topos will have avatars in many contexts, and several global properties of the category of topoi — for example, analogues of compactness, mapping spaces and classifying spaces — indeed make them similar to "spaces". There are also drawbacks, notably the lack (to date) of any "cellular" or "skeletal" or "dimension-theoretic" approach to an abstract topos, though these are often the tools used to break up and understand a geometric object. (Instead, one may try to replace an unwieldy topos by a more combinatorial structure with isomorphic invariants; for example, a pro-simplicial set or sheaves on a simple Grothendieck topology, such as a poset.) But there is another, distinctly post-Grothendieckian approach, which sees any category of set-valued sheaves as very similar in *some* formal properties to the category of sets. Consequently the homological (homotopical) algebra of sheaves of structures — through which one hopes to capture invariants of the topos — ought to be similar in *some* formal aspects to plain homological (homotopical) algebra with no Grothendieck topology in sight. The onus is on making precise the qualifier "some". Let us believe that Quillen's axioms provide an adequate calculus for homotopical manipulations. The question I sought to answer was: what formal properties should a homotopy theory satisfy if one wishes to "lift" it into any category of sheaves as functorially as the homological algebra of R-modules can be lifted? Note that the topos realm has proven to be a fertile ground for finding cohomological functors (which ought to be thought of as just a part of homotopy theory) with prescribed properties. This note, however, is only concerned with a foundational outlook.

Machinery. The seeds of the techniques that provide an answer to the motivating question were sown (with the exception of coherent logic) by Grothendieck's school, but matured only later. The Makkai–Paré [34] theory of accessibility is both a simplification and a far-reaching extension of the notion of π-accessibility (possibly the most technical part of SGA4). See the introduction to [34] for a full discussion of the prehistory. The concept of locally presentable category was isolated by Gabriel and Ulmer by pondering localizations of abelian categories and sheaf reflections; cf. esp. Ulmer [44]. Both of these are, ultimately, set-theoretical notions that allow one (in the present context) to treat with ease arbitrary sites, even those lacking finiteness properties, points etc. As to the deployment of mathematical logic, exotic as it may seem in a paper devoted to abstract homotopy theory, it is useful for two reasons. First, it allows one to parametrize formal homotopy theories that have the right functorial behavior without lapsing into vagueness as to what constitutes a "structure" or a "correct definition". (For example, between the two ways of saying that a functor induces an equivalence of categories — namely, that it has a quasi-inverse, or that it is full, faithful and essentially surjective, cf. Ex. 0.4 — the difference is precisely that the second one proceeds within the language of limits and colimits of graphs while the first one doesn't: it mentions *there exists a functor* that is a quasi-inverse…) Secondly, one has powerful theorems to the following effect: *if* a mathematical statement, formulated within the bounds of a specific logical syntax, holds for structured sets *then* it holds for like structured objects of any topos. These theorems are essential to the main Thm. 2.8. Further comparisons with existing work are given in the text in the form of remarks which, I hope, do not distract the reader.

Omissions. Theorem 2.8 engulfs motivating examples 0.2 and 0.4, but not 0.3. The reason is that the case of the forgetful functor from simplicial algebraic theories to the underlying objects, and many other such right adjoint situations, demand a slightly more involved treatment, purely by virtue of the more complicated description of cofibrations. They are grouped together in the main theorem of [6], part of which is concerned with setting up a logical background that permits arguments similar to e.g. Cor. 2.14.

Also, this paper is concerned solely with structures (in the sense of footnote 5) whose coefficients are constant, i.e. come from the base topos *Set*. Thus, the results would apply to the category of R-modules in a topos if R is constant, but not to an arbitrary ringed topos. Many of the most interesting topos-theoretic Quillen model categories take non-constant parameters; for example, Voevodsky's I-local homotopy theory of simplicial objects, where I is an "interval object" in the topos, or G-equivariant simplicial objects, where G is a (not necessarily constant) simplicial group. The mixture of set-theoretic and logical tools used in this paper is robust enough to apply to such situations, but pursuing just those two threads is probably more important than a bid at all-encompassing generality.

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At the referee's suggestion, background material on formal homotopy theory and on categorical logic is given in the body of the text, rather than in an appendix. This makes for quite a slow start in sections 1 and 2, but my guess is that each reader will be able to skip a good half of the preliminaries — though possibly different halves.

1. Presentable homotopy model categories

Homotopy model categories. In this article, the terms *Quillen model category* and *homotopy model category* will both refer to what Quillen [36] calls *closed model category.* (The reason for dropping the qualifier "closed" is that the interference with *closed category* is unfortunate, and the weaker "non-closed" axioms [36] will not be used.) Good introductions to the subject include Dwyer–Spaliński [16], Hovey [21] and Goerss–Jardine [18]. Quillen's axioms [37] are:

M1: $\mathcal{C}$ has finite limits and colimits.

M2: If f and g are composable morphisms in $\mathcal{C}$, and if two of f, g and fg are weak equivalences, then so is the third.

M3: A retract (in the category of morphisms of $\mathcal{C}$) of a fibration, cofibration or weak equivalence is respectively a fibration, cofibration or weak equivalence.

M4: Given the commuting solid arrow diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dashed, "l"] & Y \end{tikzcd}$$

with i a cofibration and p a fibration, if (i) p or (ii) i is a weak equivalence then a lifting l exists making both triangles commute. (One also says, "i has the left lifting property with respect to p" or "p has the right lifting property with respect to i" when an l exists in every commutative square of this type.)

M5: Every morphism can be factored as (i) an acyclic cofibration followed by a fibration, and also as (ii) a cofibration followed by an acyclic fibration.

(Co)fibrations that are also weak equivalences will be called *acyclic* (rather than *trivial* or *aspherical*).

Quillen adjoint pairs. Let $\mathcal{M}$, $\mathcal{N}$ be homotopy model categories and $\mathcal{M} \underset{R}{\overset{L}{\leftrightarrows}} \mathcal{N}$ an adjunction. The following three conditions are equivalent: (i) L preserves cofibrations and R preserves fibrations (ii) L preserves cofibrations and acyclic cofibrations (iii) R preserves fibrations and acyclic fibrations. $\mathsf{HOMODEL}$ is formed by the "class" of homotopy model categories with these adjunctions as morphisms.[2] By convention, the right adjoint gives the direction of the arrow. (2-categorical aspects are ignored; note, in particular, that equivalence of homotopy model categories in Quillen's sense does *not* coincide with equivalence in $\mathsf{HOMODEL}$.)

Transfinite composition. Let $\mathcal{C}$ be a category and I a class of morphisms of $\mathcal{C}$. A morphism $F(0) \to \operatorname{colim} F$ is said to be a *transfinite composition* of arrows in I if

- it is the part of a colimit cocone on $F : \alpha \to \mathcal{C}$ from $F(0)$ to the colimit. Here α is an ordinal (thought of as an ordered set, hence diagram) and $0 \in \alpha$ its smallest element.
- F takes all successor arrows $\beta \prec \beta^+$ in α into a morphism of I,
- and F is *continuous*: for every limit ordinal $\beta \prec \alpha$, F restricted to the diagram $\{\gamma \preccurlyeq \beta\}$ is a colimiting cocone in $\mathcal{C}$ on F restricted to $\{\gamma \prec \beta\}$.

Definition 1.1. Let $\mathcal{C}$ be a cocomplete category, I any class of morphisms of $\mathcal{C}$.

- Close the class of all pushouts of I under transfinite composition. This defines the class $\mathsf{cell}(I)$ of *relative I-cellular maps.*

[2] Since a typical Quillen model category is not accessible, an extra Grothendieck universe is needed to form $\mathsf{HOMODEL}$. If one restricts to presentable homotopy model categories as below, then $\mathsf{HOMODEL}$ lives in the same universe as TOPOI; both are subcategories of ACC, the Makkai–Paré [34] category of accessible categories and functors.

- The class $\mathsf{cof}(I)$ of I-cofibrations is defined as follows: $X \xrightarrow{c} Y \in \mathsf{cof}(I)$ iff c is a retract of an $X \xrightarrow{r} Z \in \mathsf{cell}(I)$ in the category $X/\mathcal{C}$ of objects under X.
- I-fibrations, or I-injectives, denoted $\mathsf{inj}(I)$, are the morphisms with the right lifting property w.r.t. I; that is, such that in any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r] \arrow[ur, dotted] & \bullet \end{tikzcd}$$

with $i \in I$, $p \in \mathsf{inj}(I)$, a dotted lift making both triangles commute exists.

Remark 1.2. Morally, a Quillen model category $\mathcal{C}$ is *cofibrantly generated* if (it is cocomplete and) the class of its cofibrations is of the form $\mathsf{cof}(I)$ for some *set* I, and its acyclic cofibrations are $\mathsf{cof}(J)$ for some set J. The fibrations (resp. acyclic fibrations) must then be $\mathsf{inj}(J)$ ($\mathsf{inj}(I)$, resp). The converse implication holds if the small object argument applies; and that is the definition of cofibrant generation chosen by Dwyer–Hirschhorn–Kan [14], Hovey [21]. If a Quillen model category is locally presentable, as will always be in this article, it is cocomplete and the small object argument (and more) applies to any set of maps; hence, for the purposes of this paper, cofibrant generation is the "moral" referred to above.

The transfinite small object argument. The inventor's account (with a mild inaccuracy related to not requiring a certain cardinal to be regular) is Bousfield [10]. The proof will be quite abbreviated since so many versions are already in the literature; see e.g. Hovey [21] (which builds on Hirschhorn [20] and Dwyer–Hirschhorn–Kan [14]) for full details. For an introduction to the lore of locally presentable and accessible categories, see Adámek–Rosický [1] or Borceux [8] vol.II.

Proposition 1.3. *Let $\mathcal{C}$ be a locally presentable category and I a* set *of morphisms of $\mathcal{C}$.*

– *Every $m \in \operatorname{mor}\mathcal{C}$ can be factored as fc with $c \in \mathsf{cell}(I)$, $f \in \mathsf{inj}(I)$. The factorization is not unique, but can be performed functorially.*
– $\mathsf{cof}(I)$ *is exactly the class of morphisms having the left lifting property w.r.t. all elements of* $\mathsf{inj}(I)$*, and* $\mathsf{inj}(I)$ *is exactly the class of morphisms with the right lifting property w.r.t. all elements of* $\mathsf{cof}(I)$*.*

Proof. Define a functor $\mathrm{Mor}(\mathcal{C}) \xrightarrow{F} \mathcal{C}$ and natural transformations $\mathsf{dom} \xrightarrow{\Theta} F$, $F \xrightarrow{\Psi} \mathsf{codom}$ (these being the domain resp. codomain functors) factoring the identity $\mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{C})$ as follows. For a map $X \xrightarrow{m} Y$ of $\mathcal{C}$, let S be the set of all commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "m"] \\ B \arrow[r] & Y \end{tikzcd}$$

such that $i \in I$, and $A_S \to B_S$ the coproduct of all these arrows i; A_S comes with a natural map to X. The value of F at m is the pushout of

$$\begin{tikzcd} A_S \arrow[r] \arrow[d] & X \\ B_S & \end{tikzcd}$$

with its natural induced map $\Theta(m)$ (resp. $\Psi(m)$) from X (resp. to Y). Beginning with $F_0 := F$, $\Theta_0 := \Theta$, $\Psi_0 := \Psi$, build by transfinite induction a diagram of functors $\mathrm{Mor}(\mathcal{C}) \xrightarrow{F_\alpha} \mathcal{C}$ and compatible natural transformations $\mathsf{dom} \xrightarrow{\Theta_\alpha} F_\alpha$, $F_\alpha \xrightarrow{\Psi_\alpha} \mathsf{codom}$. For a successor ordinal $\alpha + 1$, $F_{\alpha+1}(-) := F(\Psi_\alpha(-))$, $\Psi_{\alpha+1}(-) := \Psi(\Psi_\alpha(-))$, $\Theta_{\alpha+1} := \Theta(\Psi_\alpha(-)) \circ \Theta_\alpha$. For a limit ordinal α, F_α resp. Ψ_α, Θ_α are colimits along the chain $\beta \prec \alpha$. Let κ be a regular cardinal greater than the rank of presentability of the domain of any arrow in I. The requisite factorization of m is $X \xrightarrow{\Theta_\kappa(m)} F_\kappa(m) \xrightarrow{\Psi_\kappa(m)} Y$. ($\Theta_\kappa(m)$ is a transfinite composition of coproducts of pushouts of arrows from I, but a coproduct of arrows is a transfinite composition of pushouts, and a transfinite composition of transfinite compositions is one such again.) The second part of the claim follows by factoring any $m \in \mathsf{cof}(I)$ as fc with $f \in \mathsf{inj}(I)$ and $c \in \mathsf{cell}(I)$, and noting that m has the left lifting property w.r.t. $\mathsf{inj}(I)$, in particular, w.r.t. f, which entails that it is a retract of c of the type claimed. □

Remark 1.4. For the proof to work, it is enough for $\mathcal{C}$ to be cocomplete and for the domains X of the maps in I to be such that $\hom_{\mathcal{C}}(X, -)$ commutes with transfinite compositions of morphisms from $\mathsf{cell}(I)$ provided they are "long enough". The assumption that $\mathcal{C}$ is locally presentable is much stronger; for example, it implies the existence of a set of dense generators. There exist cocomplete categories other than locally presentable ones where every object X has a *rank*, i.e. $\hom(X, -)$ commutes with all κ-filtered colimits for some κ depending on X (take e.g. free cocompletions of certain *large* categories). Finally, even such categories as topological spaces or various topological spectra, where not every object has a rank, may possess *certain* sets of morphisms I which "permit the small object argument", in the terminology of Dwyer–Hirschhorn–Kan [14], Hovey [21]. In our context of algebraic models for homotopy types, however, local presentability is a convenient ground assumption.

Definition 1.5. (the solution set conditions)
Let $\mathcal{C}$ be a category, W a class of morphisms, $m \in \mathrm{mor}\,\mathcal{C}$. Say that W *satisfies the solution set condition at* m if there exists a subset W_m of W such that any commutative square

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "m"'] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with $w \in \mathsf{W}$ allows to be factorized by a commutative diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \star \arrow[r] \arrow[d, "w_m"] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \star \arrow[r] & \bullet \end{tikzcd}$$

with $w_m \in \mathsf{W}_m$. Let I be a class of morphisms; W satisfies the solution set condition at I if it satisfies it at each $m \in I$. If the solution set condition is satisfied for every $m \in \mathrm{mor}\,\mathcal{C}$, say simply that W satisfies the solution set condition.

Remark 1.6. Let $\mathrm{Mor}(\mathcal{C})$ be the category of morphisms of $\mathcal{C}$ (maps in $\mathrm{Mor}(\mathcal{C})$ are commutative squares in $\mathcal{C}$), and $\mathrm{Mor}(\mathsf{W})$ the full subcategory of $\mathrm{Mor}(\mathcal{C})$ whose objects belong to W. Def. 1.5 is Freyd's solution set condition for the inclusion functor $\mathrm{Mor}(\mathsf{W}) \hookrightarrow \mathrm{Mor}(\mathcal{C})$. (Even if W is a subcategory of $\mathcal{C}$, the inclusion $\mathsf{W} \hookrightarrow \mathcal{C}$ will not be considered in this context, so the terminology of Def. 1.5 should result in no confusion.)

The following theorem was announced by Jeffrey Smith at the 1998 Barcelona conference in Algebraic Topology. It greatly amplifies and simplifies results of Goerss and Jardine [17] and the author [7]. I am indebted to him for explaining his proof, and for his permission to reproduce it here.

Theorem 1.7. (J. Smith)
Let $\mathcal{C}$ be a locally presentable category, W a subcategory, and I a set of morphisms of $\mathcal{C}$. Suppose they satisfy the criteria:

c0 W *is closed under retracts and has the 2-of-3 property (Quillen's axiom **M2**).*
c1 $\mathsf{inj}(I) \subseteq \mathsf{W}$.
c2 *The class* $\mathsf{cof}(I) \cap \mathsf{W}$ *is closed under transfinite composition and under pushout.*
c3 W *satisfies the solution set condition at I.*

Then setting weak equivalences:=W*, cofibrations:=*$\mathsf{cof}(I)$ *and fibrations:=*$\mathsf{inj}(\mathsf{cof}(I)\cap\mathsf{W})$*, one obtains a cofibrantly generated Quillen model structure on $\mathcal{C}$.*

Proof. (J. Smith) The strategy is to exhibit a set J of morphisms such that $\mathsf{cof}(J) = \mathsf{cof}(I) \cap \mathsf{W}$. From there, Quillen's axioms follow in a well-known way. Two small object arguments yield the factorization axiom **M5**; the part of **M4** that is not the definition follows from **M5**, **M2** and the retract argument; **M3** holds by the definition of (co)fibrations and **c0**; finally, a locally presentable category is complete and cocomplete, so **M1** is satisfied.

J itself will be constructed in two steps. Lemma 1.8 shows that if a collection J of morphisms is "dense" between I and W, then $\mathsf{cof}(J) = \mathsf{cof}(I) \cap \mathsf{W}$. Lemma 1.9, using **c3**, constructs such a J that is only a set.

Lemma 1.8. *Let $J \subseteq \mathsf{cof}(I) \cap \mathsf{W}$ be a collection (set or possibly proper class) of maps in $\mathcal{C}$ such that for any commutative square*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

with $i \in I$, $w \in \mathsf{W}$ there exists $j \in J$ that factors it:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \star \arrow[r] \arrow[d, "j"] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \star \arrow[r] & \bullet \end{tikzcd}$$

Then any $f \in \mathsf{W}$ can be factored as hg with $g \in \mathsf{cell}(J)$, $h \in \mathsf{inj}(I)$.

Corollary. Under the assumptions of the previous lemma, $\mathsf{cof}(J) = \mathsf{cof}(I) \cap \mathsf{W}$.

Proof of the corollary. $\mathsf{cof}(J)$ is the saturation of J under pushout, transfinite composition and retracts, $J \subseteq \mathsf{cof}(I)\cap\mathsf{W}$ and $\mathsf{cof}(I)\cap\mathsf{W}$ is supposed to be closed under these operations, so $\mathsf{cof}(J) \subseteq \mathsf{cof}(I) \cap \mathsf{W}$. Conversely, consider any $f \in \mathsf{cof}(I) \cap \mathsf{W}$ and write it $f = hg$ as above. Since $f \in \mathsf{cof}(I)$ and $h \in \mathsf{inj}(I)$, f is a retract of g (in the category of objects under the domain of f). So $f \in \mathsf{cof}(J)$.

Proof of Lemma 1.8. This is rather like the ordinary small object argument, save that one glues on the "interpolating" maps J instead of the I. More precisely, we wish to build by transfinite induction on λ certain factorizations

$$X =: P_0 \to P_1 \to \ldots \to P_\alpha \to P_{\alpha+1} \to \ldots \to P_\lambda \xrightarrow{h_\lambda} Y$$

of f such that (this is the induction hypothesis) the diagram $P_0 \to \ldots \to P_\lambda$ is a continuous composition of maps belonging to $\mathsf{cell}(J)$. Thence the composite itself will belong to $\mathsf{cell}(J)$. Since $\mathsf{cell}(J) \subseteq \mathsf{W}$ and $f \in \mathsf{W}$, the 2-of-3 property of W implies that $h_\lambda \in \mathsf{W}$.
Set $P_0 := X$, $h_0 := f$. At a successor stage, let S_λ be the set of all commutative squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & P_\lambda \arrow[d, "h_\lambda"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $i \in I$. The density assumption on J means the existence of a factorization

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & A_s \arrow[r, "t_s"] \arrow[d, "j_s"] & P_\lambda \arrow[d, "h_\lambda"] \\ \bullet \arrow[r] & B_s \arrow[r] & Y \end{tikzcd}$$

with $j_s \in J$, for each square $s \in S_\lambda$. Let $P_{\lambda+1}$ be the pushout

$$\begin{tikzcd} \coprod A_s \arrow[r] \arrow[d, "\coprod j_s"'] & P_\lambda \arrow[d] \\ \coprod B_s \arrow[r] & P_{\lambda+1} \end{tikzcd}$$

along the canonical $\coprod_{s \in S_\lambda} A_s \xrightarrow{\{t_s \mid s \in S_\lambda\}} P_\lambda$. Let $h_{\lambda+1}$ be the canonical pushout corner map from $P_{\lambda+1}$ to Y. The connecting map $P_\lambda \to P_{\lambda+1}$ is a pushout of coproducts of morphisms from J. But any coproduct of maps is a transfinite composition (starting from the coproduct of the domains), so the connecting map belongs to $\mathsf{cell}(J)$.
At a limit ordinal λ, $X \to P_\lambda$ is the colimit of the diagram $X \to \ldots \to P_\alpha$, $\alpha \prec \lambda$.

Let now κ be a regular cardinal exceeding the rank of presentability of all the objects that occur as domains of maps in I. The required factorization of f is $X \xrightarrow{g} P_\kappa \xrightarrow{h} Y$.

Indeed, consider any lifting problem

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & P_\kappa \arrow[d, "h"] \\ B \arrow[r] & Y \end{tikzcd}$$

with $i \in I$. Since κ is regular, the diagram $X \to \ldots \to P_\kappa$ is κ-filtered, and since $\hom(A, -)$ commutes with κ-filtered colimits by assumption, a factors through a prior stage $A \to P_\lambda \to P_\kappa$. If the lifting problem

$$\begin{tikzcd} A \arrow[r] \arrow[dd, "i"'] & P_\lambda \arrow[d] \\ & P_\kappa \arrow[d, "h"] \\ B \arrow[r] & Y \end{tikzcd}$$

is indexed by $s \in S_\lambda$, the solution to the original one is the bottom composite

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & \star \arrow[r] \arrow[d, "j_s"] & \bigstar \arrow[r] \arrow[d, "\coprod j_s"] & P_\lambda \arrow[d] \\ B \arrow[r] & \star \arrow[r] & \bigstar \arrow[r] & P_{\lambda+1} \arrow[r] & P_\kappa . \end{tikzcd}$$

Lemma 1.9. *There exists a set J with the property required in Lemma 1.8.*

Indeed, consider the set of all morphisms (in the category of arrows) from $i \in I$ to the solution set W_i:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[d, "w_i \in \mathsf{W}_i"] \\ \bullet \arrow[r] & Y \end{tikzcd}$$

form the pushout P and the canonical corner map c

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[d, "i'"'] \arrow[ddr, "w_i"] & \\ \bullet \arrow[r] \arrow[drr] & P \arrow[dr, "c"] & \\ & & Y \end{tikzcd}$$

and factor c as $P \xrightarrow{p} Q \xrightarrow{q} Y$ with $p \in \mathsf{cell}(I)$, $q \in \mathsf{inj}(I)$. Set $j := pi'$. J is the set of such j (one for each morphism from $i \in I$ to W_i). Indeed, $i' \in \mathsf{cell}(I)$ and $p \in \mathsf{cell}(I)$, so $j \in \mathsf{cof}(I)$. $q \in \mathsf{inj}(I) \subseteq \mathsf{W}$ by **c1**, so $w_0 = qpi' \in \mathsf{W}$ and 2-of-3 imply $j \in \mathsf{W}$. Finally, any morphism from I to W

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

allows to be factored, using **c3**, as

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "i"'] & X \arrow[r, equal] \arrow[d, "j"'] & X \arrow[r] \arrow[d, "w_i"] & \bullet \arrow[d, "w"] \\ \bullet \arrow[r] & Q \arrow[r, "q"] & Y \arrow[r] & \bullet . \end{tikzcd}$$

This completes the proof of lemma 1.9 and of Jeff Smith's theorem. □

Remark 1.10. There are numerous variations on the above themes, notably Dwyer-Hirschhorn-Kan [14], Hirschhorn [20] and Stanley [40], but J. Smith's theorem seems to be the first to identify Freyd's solution set condition as the "culprit" for there being a set of generating acyclic cofibrations, provided a set of generating cofibrations is known to exist.

Remark 1.11. By virtue of the way the set J is found in lemma 1.9, lemma 1.8 would go through by assuming merely that the domains of I are small w.r.t. long enough transfinite $\mathsf{cell}(I)$-compositions. To prove **M5** however, one also needs to do a small object argument on J. Since the solution set is non-canonical, it is best to assume that for every object X,

$\hom(X, -)$ commutes with κ-filtered sequential colimits for some κ; and this is guaranteed by the assumption that $\mathcal{C}$ is locally presentable.

There is a notable type of Quillen model categories whose cofibrations are identifiable explicitly as the monomorphisms. In all such cases I am aware of, the following proposition applies to show that they are generated by a set. The proof really goes back to Grothendieck; see also Barr [2].

Proposition 1.12. *Let $\mathcal{C}$ be a category; write* mono *for its class of monomorphisms. Suppose*

(i) *$\mathcal{C}$ is locally presentable.*
(ii) *Subobjects have effective unions in $\mathcal{C}$. That is,*

$$\begin{tikzcd} A \cap B \arrow[rr, tail] \arrow[dd, tail] & & A \arrow[dl, "a"'] \arrow[dd, tail] \\ & A \cup B \arrow[dr, tail, "m"] & \\ B \arrow[ur, "b"] \arrow[rr, tail] & & X \end{tikzcd}$$

given any two subobjects A, B of an object X, form their intersection $A \cap B = A \times_X B$ and their pushout $A \cup B$ over their intersection; the induced maps a, b, m are to be monomorphisms (whence $A \cup B$ really is the supremum of A and B in the subobject lattice of X, i.e. their union).

(iii) mono *is closed under transfinite composition.*

Then $\mathsf{mono} = \mathsf{cell}(I) = \mathsf{cof}(I)$ *for some set* $I \subset \mathsf{mono}$.

Remark 1.13. Any $AB5$ abelian category and any elementary topos satisfies (ii)-(iii). (i) restricts them to Grothendieck abelian categories (see Prop. 3.10) resp. Grothendieck topoi. If a category satisfies (i)-(ii)-(iii), so do all its over-, under-, and diagram categories, and left exact localizations.

Proof. Let $\mathcal{G}$ be a set of strong generators for $\mathcal{C}$, so that the functors $\hom(G, -)$, $G \in \mathcal{G}$, collectively reflect isomorphisms. (Any locally presentable category has such $\mathcal{G}$.) Let $\mathcal{Q}$ be the set of (isomorphism types of) regular quotients of these generators; finally, let I be the set of all (isomorphism types of) subobjects of members of $\mathcal{Q}$. Then $\mathsf{mono} = \mathsf{cell}(I)$ (a fortiori $\mathsf{mono} = \mathsf{cof}(I)$, since monos are closed under retract).

Argue by contradiction. Suppose $X \xrightarrow{m} Y$ is a mono but $m \notin \mathsf{cell}(I)$. By transfinite induction, we will build a chain $X := P_0 \to P_1 \to P_2 \to \ldots \to P_\lambda \to \ldots \to Y$ of subobjects of Y that is (i) properly increasing (ii) satisfies $X \to P_\lambda \in \mathsf{cell}(I)$ for $0 \prec \lambda$. This contradicts that $\mathcal{C}$ is well-powered.

Set $P_0 := X \xrightarrow{m} Y$. At a successor stage, find $G \xrightarrow{g} Y$, $G \in \mathcal{G}$, that does not factor through $P_\lambda \to Y$. Factor g as a regular epi followed by a mono: $G \twoheadrightarrow Q \rightarrowtail Y$. Form an effective subobject union diagram as above:

$$\begin{tikzcd}
& Q \cap P_\lambda \arrow[rr, tail] \arrow[dd, tail] & & P_\lambda \arrow[dl, "a"] \arrow[dd, tail] \\
& & Q \cup P_\lambda \arrow[dr, tail] & \\
G \arrow[r, two heads] & Q \arrow[ur] \arrow[rr, tail] & & Y
\end{tikzcd}$$

and define $P_{\lambda+1} := Q \cup P_\lambda$. Note $a \in \mathsf{cell}(I)$. $P_{\lambda+1}$ is bigger than P_λ since g factors through it, so the induction hypotheses are satisfied.

At a limit ordinal λ, set $P_\lambda := \operatorname{colim}_{\alpha \prec \lambda} P_\alpha$ and use assumption (iii). □

Solution sets vs. accessibility. We recall results on the second named set-theoretic constraint. In the context of classes of morphisms, it is stronger than 1.5 but much better behaved under (2-)categorical operations. All in all, it may be easier to check for accessibility than to solve for a solution set. The interaction with homotopy theory (or, rather, Quillen model categories whose underlying category is locally presentable) is solely through the solution set condition for weak equivalences, and their closure under retracts (see Prop. 1.19). The reader may wish to skip to the next section and refer back only as necessary.

Definition 1.14. (accessibility of a class of maps)
Let $\mathcal{C}$ be a locally presentable category, W a class of morphisms of $\mathcal{C}$. Let $\mathrm{Mor}(\mathsf{W})$ and $\mathrm{Mor}(\mathcal{C})$ be as in remark 1.6. Say that W *is an accessible class of maps*[3] if $\mathrm{Mor}(\mathsf{W})$ is an accessible category, and it is closed in $\mathrm{Mor}(\mathcal{C})$ under κ-filtered colimits for some κ.

In the terminology of Adámek–Rosický [1], $\mathrm{Mor}(\mathsf{W})$ is an accessibly embedded, accessible subcategory of $\mathrm{Mor}(\mathcal{C})$. We refer to Adámek–Rosický [1], Borceux [8] vol.II or Makkai–Paré [34] for further background. Note that accessibility of a class of maps lacks meaning unless the ground category is locally presentable (or accessible, at least). In what follows, fix a locally presentable $\mathcal{C}$ and class of maps W.

Proposition 1.15. *If* W *is accessible, it satisfies the solution set condition.*

Proof. More generally, any accessible functor satisfies the solution set condition (at every object); see Adámek–Rosický [1] Corollary 2.45. □

Remark 1.16. The distinction between being an accessible class and satisfying the solution set condition is subtle. (These notions have the obvious meaning for any class of objects in a locally presentable category $\mathcal{K}$, and the statements about to be quoted apply to this more general case.) A theorem due to H. Hu and M. Makkai [22] asserts that a class of objects closed in $\mathcal{K}$ under κ-filtered colimits (for some κ) is accessible iff it satisfies the solution set condition. J. Rosický and W. Tholen [38] prove that the set-theoretical statement known as Vopěnka's Principle (a so-called *large cardinal axiom*) implies that any class of objects satisfying the solution set condition will be accessible. This diminishes the chances of finding, without the aid of axioms external to ZFC, any class of objects that is not accessible but satisfies the solution set condition. Such a counterexample would disprove Vopěnka's Principle from ZFC, and current set theoretical intuition is that this is unlikely.

[3] The last part of remark 1.6 applies here as well.

Remark 1.17. One reason to cherish ACC, the (very large) category of accessible categories and functors is the Limit theorem of Makkai and Paré [34]: ACC is closed in the (very large) category of (large) categories under weighted pseudo-limits. As a corollary, ACC is stable under forming under-, over-, comma-, diagram categories…; the intersection of a set of accessible subcategories is accessible… and more. A category is locally presentable iff it is accessible and cocomplete; as further corollaries, the category of algebras (over an accessible monad), the category of coalgebras (over an accessible comonad), the category of [commutative] monoids (w.r.t. an accessible [symmetric] monoidal structure) is locally presentable if the ground category is so. Here we single out one corollary that ties in quite directly with the subject matter.

Proposition 1.18. *Let* $\mathrm{Mor}(\mathcal{A}) \xrightarrow{F} \mathrm{Mor}(\mathcal{B})$ *be an accessible functor, and* W *an accessible class in* $\mathcal{B}$*.* $F^{-1}(\mathsf{W})$ *(i.e. the class of morphisms in* $\mathcal{A}$ *taken into* W *by* F*) is an accessible class.*

Proof. The full inverse image of a full, accessible subcategory by an accessible functor is again accessible; see Adámek–Rosický [1] Remark 2.50. □

In contrast to the previous ones, the next fact is elementary.

Proposition 1.19. *Any accessible class* W *of maps is closed under retracts (in the category of morphisms).*

Proof. In fact, any full subcategory of cocomplete category closed under κ-filtered colimits for some κ must be closed under retracts. This is since a retract is a colimit on an "idempotent loop" diagram

$$\begin{tikzcd} \bullet \arrow[loop above] \end{tikzcd}$$

which is ∞-filtered, ie. κ-filtered for every κ. □

Here is a way to make homotopical machinery accessible. By virtue of Kan's combinatorial description of weak equivalences, the class of weak equivalences in *SSet* is accessible.[4] Let now $\mathcal{C}$ be a locally presentable category, to be made into a Quillen model category. Suppose there exists a detection functor: $\mathcal{C} \xrightarrow{F} SSet$ such that $w \in \mathrm{mor}\,\mathcal{C}$ is a weak equivalence iff $F(w)$ is one. If F is accessible, that is to say, it preserves κ-filtered colimits for some κ, then weak equivalences form an accessible class in $\mathcal{C}$, and **c0** and **c3** of Theorem 1.7 are automatic. J. Smith conjectures that every cofibrantly generated model category whose underlying category is locally presentable, arises this way.

2. Defining and "sheafifying" homotopy model theories

What is common to simplicial objects, simplicial rings and categories is that they are structures definable in terms of finite limits. On the intuitive level, this notion has been

[4] See Example 3.1. The claim does not appear to be a formal consequence of the definition / theorem that a map in *SSet* is a weak equivalence iff its geometric realization is. There is no convenient category of topological spaces known that is also locally presentable.

around since the beginning of category theory[5] and has been given several equivalent formal definitions since then.

Logics. A *language* for (first-order, many-sorted) logic consists of a set of *types* (also called *sorts*), function symbols, relations symbols, logical connectives $\wedge, \vee, \implies, \neg$ and quantifiers $\forall, \exists$. For convenience, we also add the symbols $\top$, $\perp$ and $\exists!$ for the logical constants "true", "false" and the quantifier "there exists precisely one".[6] There's to be a set of (dummy) *variables*, each assigned one of the types (and thought of as running over the set of things of that type). Similarly, each function symbol is supposed to come with its *arity* $\langle t_1, t_2, \dots, t_n\rangle \to t$, meaning it takes an n-tuple of things, the i^{th} of which is type t_i, and turns them into a thing of type t. Analogously for relation symbols. *Constants* may be thought of as 0-ary functions. Refer to any textbook for the precise formation rules of *well-formed formulas*. A formula containing no free variables (i.e. such that any variable falls under the scope of a quantifier) is called a *sentence*. An axiom system is a logical language together with a set of sentences (axioms).

Example 2.1. In the most natural way of formalizing the notion of *vector space*, there exist two types (scalars and vectors), functions of "scalar multiplication" (of arity $\langle$scalar,vector$\rangle \to$ vector), vector addition..., symbols for three constants: the zero vector and the scalars 0 and 1, numerous axioms, and no relation symbols other than equality. (Since only things of the same type can be equal, and every relation symbol is to have its arity, there's supposed to be a separate $=$ for comparing two scalars, and one for comparing two vectors.)

An *interpretation* of a language $\mathcal{L}$ (in the category *Set*) is the assignment of a set T_i to each type t_i of $\mathcal{L}$, an actual function $T_1 \times T_2 \times \cdots \times T_n \to T$ for each function symbol of the respective arity, and a subset R of $T_1 \times T_2 \times \cdots \times T_n$ for each relation symbol r of arity $\langle t_1, t_2, \dots, t_n\rangle$. It is a *model* if the axioms are satisfied.

The category of models. Models for a fixed axiom system form a category, in fact a full subcategory of the category of interpretations of the language. Let M, N be models, with underlying sets for types T_i^M, T_i^N, resp. A morphism from M to N is given by a function $T_i^M \xrightarrow{f_i} T_i^N$ for each type such that they commute with the interpretations of the function symbols:

$$\begin{tikzcd} T_1^M \times T_2^M \times \cdots \times T_n^M \arrow[r] \arrow[d, "\times\{f_i\}"] & T^M \arrow[d, "f"] \\ T_1^N \times T_2^N \times \cdots \times T_n^N \arrow[r] & T^N \end{tikzcd}$$

and for each relation symbol r, R^M is to be taken into a subset of R^N by $T_1^M \times T_2^M \times \cdots \times T_n^M \xrightarrow{\times\{f_i\}} T_1^N \times T_2^N \times \cdots \times T_n^N$.

[5]SGA4, Tome 1, Expose 1, 2.9 speaks thus: "Soit γ une espèce de structure algébrique 'définie par limites projectives finies'. (Le lecteur est prié de donner un sens mathématique à la phrase précédente. Notons seulement que les structures de groupes, groupes abéliens, anneux, modules, etc... sont de telles structures.)"

[6]One could, in several ways, start with a smaller set of logical symbols as basic and express the rest in terms of them, but there's no reason for such parsimony for us.

Interpreting logic in a topos. Let now $\mathcal{E}$ be a topos. An interpretation of a language in $\mathcal{E}$ is the assignment of an object T_i to each type t_i there is, a morphism $T_1 \times T_2 \times \cdots \times T_n \to T$ for each function symbol, and a subobject R of $T_1 \times T_2 \times \cdots \times T_n$ for each relation symbol of the respective arity. Using a calculus of subobjects — intersections, unions, pseudo-complements, direct and inverse images of subobjects — and via induction on the complexity of formulas, every sentence of $\mathcal{L}$ gets assigned a truth value of "true" or "false". The rules reduce to the usual ones when $\mathcal{E} = \mathit{Set}$; however, it is not the case that two formulas that are logically equivalent in *Set* (i.e. always evaluate to the same truth value) will always be logically equivalent in any topos. The reader is referred to the textbooks of MacLane–Moerdijk [33] or Borceux [8] vol.III for details. The category of models in $\mathcal{E}$ of an axiom system can now be defined by analogy with the case of *Set*.

In a *fragment of logic* only expressions of some specific form are permitted. In this paper, the emphasis is on the following two fragments.

Cartesian logic. A language for cartesian logic[7] is a (many-sorted, first-order) language containing *no relation symbols other than equality* and making no use of the logical symbols $\vee, \neg, \exists, \perp$. Moreover, in a cartesian axiom system, only sentences of the following kind are allowed:

$$\forall \vec{x}\big(\Phi \implies \exists ! \vec{y}(\Psi)\big)$$

where $\vec{x}$, $\vec{y}$ are shorthand for a string of variables, and Φ and Ψ must be finite conjunctions of expressions of the form $\mathbf{t}_1 = \mathbf{t}_2$ where $\mathbf{t}_1$, $\mathbf{t}_2$ are *terms*, that is, meaningful combinations of constants, dummy variables and function symbols.

Example 2.2. Here's how to say *groupoid* in this language. There are two sorts of things, objects A_0 and arrows A_1. There are the usual functions of "source" $\mathbf{s}$, "target" $\mathbf{t}$, "identity" $\mathbf{i}$, and "inverse"; their commutation relations (to conform to the standard that implication has to be used precisely once…) can be expressed in the form

$$\forall a\big(\top \implies \mathbf{s}(\mathbf{i}(a)) = a\big)$$

and so on (here $\top$ is the True, and a is a variable of the type *object*). To express composition and its associativity, one has to introduce the auxiliary type P for the "set of" composable pairs of arrows. There are projections $P \xrightarrow{\mathbf{p}_1} A_1$, $P \xrightarrow{\mathbf{p}_2} A_1$; composition is a function $P \to A_1$. Another typical axiom is

$$\forall a_1, a_2\big(\mathbf{t}(a_1) = \mathbf{s}(a_2) \implies \exists ! p(\mathbf{p}_1(p) = a_1 \wedge \mathbf{p}_2(p) = a_2)\big)$$

where p is a variable of type P.

In fact, groupoids are (qualitatively) the most complicated type of example that can occur. Every axiom $\forall \vec{x}\big(\Phi \implies \exists ! \vec{y}(\Psi)\big)$ can be seen as expressing the existence of a (vector-valued) function with domain specified by Φ and codomain specified by Ψ, and asserting some equational conditions about components of that vector. So every cartesian structure is an equational theory of partial algebras, and actually of a special class of partial algebras: the domain of the partial operations has to be specified by a conjunction of equational conditions between total operations (morally, "by pullbacks").

Cartesian logic provides a solution to the exercise of SGA4 recalled in footnote 5: "define *definable in terms of finite limits*". M. Coste's *lim-logic* (see Coste [13]), Lawvere's *cartesian*

[7]See Johnstone [28] p.180 and p.221 for the origin of this term.

theories (see Barwise [4] A.8, Barr–Wells [3]), Ehresmann's *finite limit sketches* (see Adámek–Rosický [1], Barr–Wells [3], Borceux [8] vol.III) and P. Freyd's *essentially algebraic theories* (see Adámek–Rosický [1]) have the same expressive strength in a topos as cartesian logic does; in fact, any two of them are functorially interdefinable.

The following proposition compresses all that will be needed of finite limit=cartesian structures below.

Proposition 2.3. *Given a structure S defined in terms of finite limits, one can associate to a topos $\mathcal{E}$ the category of S-structures (or "models of the axiom system S") in $\mathcal{E}$, to be denoted* $\mathrm{Mod}_S(\mathcal{E})$.[8]

(i) $\mathrm{Mod}_S(\mathcal{E})$ *is a locally presentable category.*
(ii) *A topos morphism $\mathcal{E} \xrightarrow{f} \mathcal{F}$ (i.e. adjoint pair $f^* \dashv f_*$ such that f^* preserves finite limits) induces an adjunction between* $\mathrm{Mod}_S(\mathcal{E})$ *and* $\mathrm{Mod}_S(\mathcal{F})$.
(iii) *For any small category $\mathcal{D}$,* $\mathrm{Mod}_S(\mathcal{E})^{\mathcal{D}}$ *is canonically isomorphic to* $\mathrm{Mod}_S(\mathcal{E}^{\mathcal{D}})$.
(iv) *A morphism of S-structures is itself definable in terms of finite limits; more precisely, there exists a finite limit structure* $\mathrm{Mor}(S)$ *such that* $\mathrm{Mor}(\mathrm{Mod}_S(\mathcal{E}))$ *is canonically equivalent to* $\mathrm{Mod}_{\mathrm{Mor}(S)}(\mathcal{E})$, *for any* $\mathcal{E}$.

Example 2.4. The following *Set*-based structures are finite limit definable: simplicial set, more generally $\mathcal{D}$-set, for any diagram $\mathcal{D}$; monoid, group, ring… more generally, any species of finitary equational universal algebras (possibly many-sorted); (small) category, groupoid, 2-category, double category…, diagram of a fixed shape within the category of any of the aforementioned structures. The corresponding notion of morphism of structures is always (strict) homomorphism, preserving all operations on the nose.

Remark 2.5. As pointed out by the referee, both 2.3(iii) and (iv) are special cases of the classical fact that given cartesian structures S_1 and S_2, there exists a "tensor product" cartesian theory $S_1 \otimes S_2$ such that models of S_1 in the category of S_2-structures, as well as models of S_2 in the category of S_1-structures, are canonically isomorphic to $S_1 \otimes S_2$-structures.

The prototype of (iv) is the observation that a natural transformation between two diagrams, say of shape $\mathcal{D}$, can itself be thought of as a diagram, indexed by $\{\bullet \to \star\} \times \mathcal{D}$. It reduces the task of understanding how to specify well-behaved classes of morphisms between finite limit structures — notably, the weak equivalences and (co)fibrations — to specifying subcategories of models (of a different theory). Now recall property (3) of the motivating examples, that weak equivalences are preserved by inverse images, and the solution set condition **c3** of Theorem 1.7. These are characteristic of notions that can be defined in terms of finite limits and arbitrary colimits, using a set's worth of data. Since the early 70's, categorical logicians have provided several equivalent formal analyses of such notions, a convenient one being

Coherent logic. This fragment is richer than cartesian logic. The language can contain (besides typed variables and function symbols, as usual) relation symbols as well as the logical connectives $\wedge, \vee, \implies, \forall, \exists, \top, \bot$. However, only sentences of the following form are allowed:

$$\forall \vec{x}\big(\Phi \implies \Psi\big)$$

[8] Let $(\mathcal{C}, J)$ be a site; $\mathrm{Mod}_S(\mathrm{Sh}(\mathcal{C}, J))$ is equivalent to the category of $\mathrm{Mod}_S(\mathit{Set})$-valued sheaves on $(\mathcal{C}, J)$. E.g. a sheaf of abelian groups is an abelian group object in sheaves. Finite limit definable structures form the largest class of first-order structures to which this extends.

where $\vec{x}$ stands for a string of variables, and Φ and Ψ must be built up via the use of $\exists$, finite conjunctions and arbitrary (i.e. possibly infinite) disjunctions from "atomic formulas", which are meaningful combinations of constants, variables and function and relation symbols.

Example 2.6. Within each category of type $\mathrm{Mod}_S(Set)$ below, S a finite limit structure, the notions listed are coherently definable in the language of S:

- *SSet*: fibrant simplicial sets (i.e. those satisfying the Kan extension condition), acyclic simplicial sets
- Mor(*SSet*): monomorphisms, Kan fibrations, "topological" weak equivalences
- Groups: divisible groups, torsion groups
- Rings: local rings, fields, fields of characteristic $p > 0$, separably closed fields, algebraically closed fields
- Mor(Rings): flat homomorphisms
- $\mathcal{D}_+$-diagrams in Mod_S, where $\mathcal{D}_+$ is the cocone on a diagram $\mathcal{D}$ (i.e. is the result of formally adding a disjoint terminal object to the small category $\mathcal{D}$): being an initial cocone, i.e. a colimit on a functor $\mathcal{D} \to \mathrm{Mod}_S(Set)$.
- $\mathcal{D}_-$-diagrams in Mod_S, where $\mathcal{D}_-$ is the cone on a finite diagram $\mathcal{D}$ (i.e. is the result of formally adding a disjoint initial object to a category $\mathcal{D}$ with finitely many arrows): being a terminal cone, i.e. a limit on a functor $\mathcal{D} \to \mathrm{Mod}_S(Set)$.

The reader wishing to see coherent logic in action (in addition to an introduction to it) may enjoy reading Wraith [46] that constructs the étale topos of a scheme without the intervening use of a topology on the category of schemes, or any subcategory thereof. (The general étale topos is glued together from the affines, and the étale topos of a ring is constructed as a classifying topos.) Johnstone's related [27] shows how to obtain various spectral (i.e. sheaf) representations of rings directly by looking at the axioms and also treats Coste's logic.

Much as cartesian logic matches finite limit sketches, the language of coherent logic is interdefinable with colimit-and-finite limit sketches; see Makkai–Paré [34], Borceux [8] vol.III.

CAVEAT. The terms *geometric* and *coherent* logic are sometimes used interchangeably. When not, they mean almost the same thing: one of them refers to a logic permitting infinitary disjunctions (arbitrary colimits) while the other to the same logic with only finitary disjunctions (resp. finite colimits). In this article, the term "geometric logic" is not used and "coherent logic" allows arbitrary infinite disjunctions.

The next proposition is a "reader's digest" of facts of coherent logic, tailored for the needs of the present paper.

Proposition 2.7. *Let S be a structure defined in terms of finite limits, $\mathcal{E}$ a topos, $\mathcal{A}$ a set of coherent sentences in the language of S. If $X \in \mathrm{Mod}_S(\mathcal{E})$ satisfies these sentences ("axioms"), one writes $X \models \mathcal{A}$. Let $\mathrm{Mod}_{S,\mathcal{A}}(\mathcal{E})$ be the full subcategory of $\mathrm{Mod}_S(\mathcal{E})$ with objects those X that satisfy $\mathcal{A}$.*

(i) *Let $\mathcal{E} \xrightarrow{f} \mathcal{F}$ be a topos morphism, $X \in \mathrm{Mod}_S(\mathcal{F})$. If $X \models \mathcal{A}$ then $f^*(X) \models \mathcal{A}$.*
(ii) *$\mathrm{Mod}_{S,\mathcal{A}}(\mathcal{E})$ is closed in $\mathrm{Mod}_S(\mathcal{E})$ under filtered colimits.*
(iii) *$\mathrm{Mod}_{S,\mathcal{A}}(\mathcal{E})$ is an accessibly embedded, accessible full subcategory of $\mathrm{Mod}_S(\mathcal{E})$.*
(iv) *Let $\mathcal{D}$ be a small category. For any $d \in \mathrm{ob}\,\mathcal{D}$, let ev_d denote the "evaluation at d" functor $\mathcal{E}^{\mathcal{D}} \to \mathcal{E}$, which is the inverse image of a topos morphism. For $X \in \mathrm{Mod}_S(\mathcal{E}^{\mathcal{D}})$, $X \models \mathcal{A}$ iff $\mathsf{ev}_d(X) \models \mathcal{A}$ for all objects d of $\mathcal{D}$.*

Example 2.6 may suggest that one can transplant a Quillen model category from *Set* to an arbitrary topos by borrowing the "formulaic" definition of weak equivalences, cofibrations and fibrations. This is too much to ask, as pointed out by Jardine [25]: any Eilenberg–MacLane sheaf is locally fibrant (being a sheaf of simplicial abelian groups); if it were globally fibrant as well, sheaf cohomology would be trivial. Experience has shown that the "local" (i.e. stalkwise, i.e. "formulaic" in the sense of the internal logic) weak equivalences are the right ones. Any two of the three classes defining a Quillen model category determine the third. This suggests that one borrows the logical description of weak equivalences and one of the classes of cofibrations and fibrations. The second possibility, in general, leads to a local homotopy theory, or "homotopy theory of fibrant objects", in the sense of K. Brown [12]; see also Jardine [24]. Note that Quillen's axioms are self-dual; the asymmetry is inherent in the set-theoretic aspect of sheaf theory (cf. cofibrant generation etc.).

The first choice leads to a Quillen model category and is the subject of this paper. Note that Jeff Smith's theorem prefigures the non-uniqueness of the choice of cofibrations: provided one deals with an accessible class of weak equivalences, which therefore satisfy the solution set condition everywhere, one may replace the I of 1.7 by any set $I' \subset \mathsf{cof}(I)$ as long as condition **c1** is not violated. For this reason, the main theorem below is split into two parts; the first one produces cofibrations less uniformly in TOPOI, but under less stringent conditions.

For the rest of this section, let S stand for the definition of a structure in terms of finite limits. For brevity, write $\mathsf{S}(\mathcal{E})$ for $\mathrm{Mod}_S(\mathcal{E})$, $\mathcal{E}$ a topos. Let W resp. C be two sets of axioms of coherent logic in the language of morphisms of models of S. Write $\mathsf{W}(\mathcal{E}) := \{f \in \mathrm{mor}\,\mathsf{S}(\mathcal{E}) \mid f \models W\}$ and $\mathsf{C}(\mathcal{E}) := \{f \in \mathrm{mor}\,\mathsf{S}(\mathcal{E}) \mid f \models C\}$.

Theorem 2.8. *Consider the hypotheses*

(i) $\mathsf{S}(Set)$ *with weak equivalences* $\mathsf{W}(Set)$ *and cofibrations* $\mathsf{C}(Set)$ *is a Quillen model category; and* $\mathsf{C}(Set)$ *is of the form* $\mathsf{cof}(I)$, I *a set.*

(ii) *For every topos* $\mathcal{E}$, $\mathsf{C}(\mathcal{E}) = \mathsf{cof}(I_{\mathcal{E}})$ *for some set* $I_{\mathcal{E}}$ *of maps.*

(iii) *One of the following:* (†) W *and* C *belong to a fragment of coherent logic that has enough models in Set (for example, axioms of finite length; the countable fragment of coherent logic; essentially algebraic theories) or* (‡) $\mathsf{S}(\mathcal{E})$ *with weak equivalences* $\mathsf{W}(\mathcal{E})$ *and cofibrations* $\mathsf{C}(\mathcal{E})$ *is a Quillen model category for every topos* $\mathcal{E}$ *of the form* $\mathrm{Sh}(\mathcal{B})$, $\mathcal{B}$ *being a complete Boolean algebra with its canonical topology.*

(1) *(i) implies that for every topos* $\mathcal{E}$ *with enough points,* $\mathsf{S}(\mathcal{E})$ *with weak equivalences* $\mathsf{W}(\mathcal{E})$ *and cofibrations a certain subclass of* $\mathsf{C}(\mathcal{E})$ *is a cofibrantly generated Quillen model category. In the presence of (iii), the conclusion extends to every topos.*

(2) *(ii) implies that for every topos* $\mathcal{E}$ *with enough points, the cofibrations can be chosen to be* $\mathsf{C}(\mathcal{E})$*, the rest being as in* **(1)***. In the presence of (iii), the conclusion extends to every topos.*

Remark 2.9. It is immediate that in case **(2)**, a topos morphism $\mathcal{E} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{F}$ induces a Quillen pair $\mathsf{S}(\mathcal{E}) \leftrightarrows \mathsf{S}(\mathcal{F})$. (2.3(ii) and 2.7(i) show that the left adjoint preserves *all* weak equivalences and cofibrations.) One thus has a "coherently definable" or "sheafifiable" homotopy theory and a functor $\mathsf{TOPOI} \to \mathsf{HOMODEL}$. See remark 2.18 as to functoriality under **(1)**.

Remark 2.10. The choice of cofibrations is not unique, even if all assumptions are satisfied. There may be more than one functorial cofibration class as well.

Remark 2.11. In all examples I am aware of, if (i) holds, so does (ii). In fact, (ii) is likely to be a formal consequence of its being true for the case $Set = \mathcal{E}$. Alas, as things stand now, (ii) has to be checked by hand, unless 1.12 applies.

Remark 2.12. I know a single example where one doesn't seem to get by with countably many defining axioms of countable length, or with some set of axioms of finite length, for W and C — cases when (iii)(†) applies — and that is homological localization of simplicial sets (simplicial objects in a topos). That example is much better seen as a direct consequence of Jeff Smith's theorem, together with 1.12 and a bit of accessibility.

The statement of 2.8 begs its proof: observe that the hypotheses of Jeff Smith's theorem hold for $\mathsf{S}(Set)$; transfer them to $\mathsf{S}(\mathcal{E})$ via logical methods; forward applications of Thm. 1.7 yield the conclusions. To begin with, one has unconditionally

Lemma 2.13. *For any topos $\mathcal{E}$, $\mathsf{S}(\mathcal{E})$ is a locally presentable category. $\mathsf{W}(\mathcal{E})$ is closed under retracts and satisfies the solution set condition in* $\mathrm{Mor}(\mathsf{S}(\mathcal{E}))$.

Proof. By 2.3(i), 2.7(iii), 1.19 and 1.15. □

The next two lemmas state facts in tandem, preceded by the hypotheses they need.

Lemma 2.14.
(i) *For every topos $\mathcal{E}$ with enough points: $\mathsf{W}(\mathcal{E})$ has the 2-of-3 property; $\mathsf{C}(\mathcal{E})$ is closed under composition; $\mathsf{C}(\mathcal{E}) \cap \mathsf{W}(\mathcal{E})$ is closed under pushout.*
(i,iii) *Same conclusions for* every *topos $\mathcal{E}$.*

Proof. Each of these properties has the following form: a set of coherent sentences (in the language of the appropriate diagram of S-structures) implies certain coherent sentences. Topoi with enough points inherit the truth of such statements from *Set*. As to (†), the definition of having enough models in *Set* is that such conclusions extend to an arbitrary topos. It is a theorem of Makkai–Reyes that countable coherent logic has enough models in *Set*, of Deligne-Joyal that finitary coherent sentences do. The case of universal Horn logic / essentially algebraic theories is classical (see Makkai–Reyes [35]). The Boolean case (‡) is Barr's theorem. □

Lemma 2.15.
(i) *For every topos $\mathcal{E}$ with enough points: $\mathsf{W}(\mathcal{E})$ and $\mathsf{C}(\mathcal{E})$ are closed under transfinite composition.*
(i,iii) *Same conclusions for* every *topos $\mathcal{E}$.*

Proof. By Prop. 2.7, $\mathsf{W}(\mathcal{E})$ and $\mathsf{C}(\mathcal{E})$ are closed under filtered colimits in the category of morphisms of $\mathsf{S}(\mathcal{E})$. If they are closed under composition, they must be closed under transfinite composition. (Use transfinite induction.) □

Lemma 2.16. *(i) implies: for any topos $\mathcal{E}$, $\mathsf{inj}(\mathsf{C}(\mathcal{E})) \subseteq \mathsf{W}(\mathcal{E})$.*

Proof. This holds for $\mathcal{E} = Set$ by assumption, since acyclic fibrations are weak equivalences. Extend the conclusion from *Set* to presheaf topoi $\mathrm{Pre}(\mathcal{D})$. Let d be any object of the small category $\mathcal{D}$, and consider the adjunction

$$\mathsf{S}(Set)^{\mathcal{D}^{\mathrm{op}}} \underset{\mathsf{ev}_d}{\overset{L}{\leftrightarrows}} \mathsf{S}(Set)$$

where ev_d is the "evaluation at d" functor and L its left adjoint (the left Kan extension). Let $f \in \mathsf{C}(\mathit{Set})$. $L(f)$ is a $\mathcal{D}^{\mathrm{op}}$-diagram in $\mathsf{S}(\mathit{Set})$ that is, at every object of $\mathcal{D}$, a copower of f. Recall that $\mathsf{S}(\mathit{Set})^{\mathcal{D}^{\mathrm{op}}}$ is the "same" as $\mathsf{S}(\mathrm{Pre}(\mathcal{D}))$. Since any copower of f is a cofibration in $\mathsf{S}(\mathit{Set})$, and since coherent axioms are evaluated "objectwise" in functor categories (cf. 2.7(iv)), $L(f)$ will belong to the class $\mathsf{C}(\mathrm{Pre}(\mathcal{D}))$. Suppose $g \in \mathsf{inj}(\mathsf{C}(\mathrm{Pre}(\mathcal{D}))$. A fortiori $g \in \mathsf{inj}(L(\mathsf{C}(\mathit{Set})))$. By adjunction, $\mathsf{ev}_d(g) \in \mathsf{inj}(\mathsf{C}(\mathit{Set}))$, so $\mathsf{ev}_d(g) \in \mathsf{W}(\mathit{Set})$. But since the class $\mathsf{W}(\mathrm{Pre}(\mathcal{D}))$ is defined again by coherent axioms, $\mathsf{ev}_d(g) \in \mathsf{W}(\mathit{Set})$, for every object d of $\mathcal{D}$, implies $g \in \mathsf{W}(\mathrm{Pre}(\mathcal{D}))$.

Consider now an arbitrary topos $\mathcal{E}$. Choose a site $(\mathcal{D}, J)$ of definition for $\mathcal{E}$, and consider the inclusion $\mathcal{E} \underset{i}{\overset{\ell}{\leftrightarrows}} \mathrm{Pre}(\mathcal{D})$, where ℓ is sheafification. It induces an adjunction $\mathsf{S}(\mathcal{E}) \leftrightarrows \mathsf{S}(\mathrm{Pre}(\mathcal{D}))$. (Out of laziness, retain the same letters to denote these adjoints.) Take any $f \in \mathsf{inj}(\mathsf{C}(\mathcal{E}))$. Since sheafification (being an inverse image part of a topos morphism) preserves the coherently defined class of cofibrations, by adjointness one has $i(f) \in \mathsf{inj}(\mathsf{C}(\mathrm{Pre}(\mathcal{D})))$ whence $i(f) \in \mathsf{W}(\mathrm{Pre}(\mathcal{D}))$. Since sheafification must preserve weak equivalences too, $f \cong \ell i(f) \in \mathsf{W}(\mathcal{E})$. □

Part **(2)** of 2.8 is now simply Jeff Smith's theorem 1.7 applied to the data $\mathsf{S}(\mathcal{E})$, $\mathsf{W}(\mathcal{E})$, and the $I_{\mathcal{E}}$ of assumption (ii), using Prop. 2.3(i) and lemmas 2.13 through 2.16.

Part **(1)** follows by a closer look at the argument in 2.16. Assume (i). By Jeff Smith's theorem, $\mathsf{S}(\mathit{Set})$ is a cofibrantly generated Quillen model category. By a theorem of Dwyer–Hirschhorn–Kan [20], for any cofibrantly generated model category $\mathcal{M}$ and small category $\mathcal{D}$, $\mathcal{M}^{\mathcal{D}}$ possesses a model structure with the weak equivalences and fibrations being $\mathcal{D}$-objectwise. Cofibrations in $\mathcal{M}^{\mathcal{D}}$ can be described as follows. Let $\mathcal{D}_\delta$ be the diagram $\mathcal{D}$ "made discrete", that is, $\mathcal{D}_\delta$ is the set of objects of $\mathcal{D}$ and their identities. There is an adjunction $\mathcal{M}^{\mathcal{D}} \leftrightarrows \mathcal{M}^{\mathcal{D}_\delta}$ where the right adjoint is the forgetful functor (i.e. precomposition with $\mathcal{D}_\delta \to \mathcal{D}$) and the left adjoint its left Kan extension L_K. $M^{\mathcal{D}_\delta}$ is (tautologously) a cofibrantly generated model category whose set of generating cofibrations, $I_{\mathcal{D}_\delta}$, is the set of tuples that are coordinatewise generating cofibrations in $\mathcal{M}$. With the above choice of weak equivalences and fibrations, cofibrations in $\mathcal{M}^{\mathcal{D}}$ are $\mathsf{cof}(L_K(I_{\mathcal{D}_\delta}))$.

By virtue of 2.3(iii) and 2.7(iv), the above prescription gives a cofibrantly generated Quillen model structure on $\mathsf{S}(\mathrm{Pre}(\mathcal{D}))$ with weak equivalences the $\mathsf{W}(\mathrm{Pre}(\mathcal{D}))$ and cofibrations a — for non-trivial $\mathcal{D}$, proper — subclass of $\mathsf{C}(\mathrm{Pre}(\mathcal{D}))$. (This follows also by a direct application of Jeff Smith's theorem.) Now for an arbitrary topos $\mathcal{E}$, choose a site, i.e. topos inclusion $\mathcal{E} \underset{i}{\overset{\ell}{\leftrightarrows}} \mathrm{Pre}(\mathcal{D})$. Write, for brevity, K for the set of generating cofibrations in $\mathsf{S}(\mathrm{Pre}(\mathcal{D}))$. Apply Jeff Smith's theorem to the data $\mathsf{S}(\mathcal{E})$, $\mathsf{W}(\mathcal{E})$, $\ell(K)$. Since $\mathsf{S}(\mathcal{E})$ is a full, reflective subcategory of $\mathsf{S}(\mathrm{Pre}(\mathcal{D}))$, $\mathsf{cof}(\ell(K)) = \ell(\mathsf{cof}(K))$. That is, cofibrations in $\mathsf{S}(\mathcal{E})$ are sheafifications of the cofibrations in $\mathsf{S}(\mathrm{Pre}(\mathcal{D}))$. So $\mathsf{cof}(\ell(K)) \subseteq \mathsf{C}(\mathcal{E})$, which lets one deduce that a pushout of an acyclic cofibration is acyclic from lemma 2.14. Use 2.13 and 2.15 for **c0**, **c3** and the other part of **c2**. The adjunction argument used in the second part of the proof of 2.16 establishes **c1**. This completes the proof of 2.8. □

Example 2.17. Here's how to construct a model category on simplicial objects in a topos with Joyal's weak equivalences, whose class of fibrations is intermediate between the local fibrations (or fibrations in the internal sense; when the topos has enough points, these are the maps that are stalkwise fibrations) and the global ones, i.e. maps with the right lifting property w.r.t. every acyclic mono. Say $\mathcal{E} = \mathrm{Sh}(\mathcal{D}, J)$. Consider the Quillen model structure on $\mathrm{Pre}(\mathcal{D})^{\Delta^{\mathrm{op}}} = SSet^{\mathcal{D}^{\mathrm{op}}}$ created by the forgetful functor to $SSet^{\mathcal{D}_\delta}$. (It is the Bousfield–Kan model structure, see [11] XI.8.) Its class of cofibrations bk is a proper subclass of the monomorphisms in general. Letting $\mathcal{E} \overset{\ell}{\underset{i}{\leftrightarrows}} \mathrm{Pre}(\mathcal{D})$ be the canonical adjunction, define cofibrations in $\mathcal{E}^{\Delta^{\mathrm{op}}}$ to be the class $\ell(\mathsf{bk})$, weak equivalences W as in 0.2 and fibrations to be $\mathsf{inj}(\ell(\mathsf{bk}) \cap \mathsf{W})$. Proof: choose a set I such that $\mathsf{cof}(I) = \mathsf{bk}$ in $\mathrm{Pre}(\mathcal{D})^{\Delta^{\mathrm{op}}}$. Then $\ell(\mathsf{bk}) = \mathsf{cof}(\ell(I))$ in $\mathcal{E}^{\Delta^{\mathrm{op}}}$. Apply Jeff Smith's theorem.

If the topology J is "too strong", $\ell(\mathsf{bk})$ may include all monomorphisms in $\mathrm{Sh}(\mathcal{D}, J)$. (For example, should it happen that $\mathrm{Sh}(\mathcal{D}, J)$ is equivalent to Set, note that $\mathsf{cof}(I)$ is all injections as soon as I contains a single non-trivial injection.) There seems to be no a priori reason for this to happen for all non-trivial topologies, however.

Remark 2.18. This formal argument extends to the upcoming examples, and even to the ones in [6]. Despite the unsettling vagueness of the "correct" cofibration class, any two choices have a common super-class, thus giving rise to Quillen-equivalent model categories [5]. In fact, the Quillen equivalence type of any presentable model category is fixed by the category itself and its subcategory of weak equivalences. This suggests that the proper target of $\mathsf{TOPOI} \to \mathsf{HOMODEL}$ may be the category of Quillen model categories and Quillen pairs *modulo* Quillen equivalence. Note that existing work of Dwyer–Kan and Rezk also suggest that "a homotopy theory" (as such) is determined by the category of models and its subcategory of weak equivalences.

3. Examples

As soon as there is one example of a sheafifiable homotopy model category — in the sense of part **(2)** of 2.8 — there are infinitely many, by the following cheap observation: since $\mathsf{S}(\mathcal{E})^{\mathcal{D}}$ is equivalent to $\mathsf{S}(\mathcal{E}^{\mathcal{D}})$, and $\mathcal{E}^{\mathcal{D}}$ is a topos if $\mathcal{E}$ is, there exists a sheafifiable homotopy theory of $\mathcal{D}$-diagrams of S-structures with the weak equivalences and *cofibrations* being $\mathcal{D}$-objectwise. This is reminiscent of (and contains as a special case) the model structure on simplicial diagrams that Heller [19] denotes *right*: let the cofibrations be all the monos. It is in general distinct from (but Quillen-equivalent to) the one existing on diagrams over any cofibrantly generated model category (reducing to the Bousfield–Kan structure on simplicial diagrams that Heller named *left*). We list six non-trivial cases of the main theorem here. Each satisfies all three hypotheses of 2.8; (ii) holds either because Prop. 1.12 applies directly, or because the structure arises via an adjunction from one where it applies. This is examined in more detail in [6]. We also digress into the unbounded derived category.

Example 3.1. (simplicial sets)
The observation that weak equivalences of simplicial sets are definable in terms of finite limits and countable colimits goes back (at least) to Illusie [23]. It is not necessary to construct the homotopy group objects internally, however. To begin with, consider a map $X_\bullet \xrightarrow{f} Y_\bullet$ between Kan complexes in $SSet$. f is a weak equivalence iff it induces a bijection between embedded "singular spheres", that is, simplices all of whose lower-dimensional faces are one and the same 0-simplex and its degeneracies. That the map is onto amounts to: *for every 0-simplex y_0 in $Y_\bullet$ and n-simplex y_n all of whose faces are y_0 (and its degeneracies), there exists an n-simplex x_n all of whose faces are some x_0 (and its degeneracies) such that $f(x_n)$ is in the same based homotopy class of singular simplices as y_n*, where for two n-simplices to be in the same homotopy class amounts to the existence of an $n+1$-simplex with suitable

face matching conditions. The sentence in italics translates verbatim into a coherent axiom in the language of morphisms of simplicial objects. That f induces an injection is similar. To deal with an arbitrary simplicial map, one has to use some fibrant replacement functor $SSet \to SSet$ definable in coherent logic. Kan's [31] Ex^∞ is such. The functor Ex — together with the natural transformation $\mathsf{Id} \to \mathsf{Ex}$ — is defined in terms of finite limits, and Ex^∞ is the colimit along the (countable) chain of iterations of $\mathsf{Id} \to \mathsf{Ex}$.

Example 3.2. (homological localizations of simplicial sets)
Let h_* be a homology theory on $SSet$. We need to choose a representation of h_* in the sense of G. Whitehead [45],

$$h_n(X) = \operatorname{colim} \pi_n(X_+ \wedge E_i) \tag{3.1}$$

where E_i is a "naive spectrum", ie. sequence of pointed simplicial sets and connecting maps from the suspension of E_i to E_{i+1}. One may describe by coherent axioms (or, colimit and finite limit constructions of simplicial sets) the following, in turn: adding a disjoint basepoint to X; smashing two pointed simplicial sets; suspending a simplicial set (one ought to use Kan's model here); computing $\pi_n(-)$ of a simplicial set (after fibrant replacement); the colim_i of homotopy groups (here one has to use that the connecting maps are definable too); finally, that for a simplicial map $X \xrightarrow{f} Y$, $h_n(f)$ is an isomorphism.

Theorem 1.7 and Prop. 1.12 can now be applied; this is one of the (rare!) cases when **c2** can be checked directly. Note that by using the machinery of accessible functors and classes, Bousfield's original proof in [9] can also be made to work in any topos [7]. By working directly in the site, Goerss and Jardine [17] establish homological localization for simplicial sheaves as well.

Remark 3.3. The calculus of accessibility highlights one aspect of the homology–cohomology dichotomy. A homology theory is a functor $SSet \xrightarrow{h_*} \mathcal{A}$ where $\mathcal{A}$ is, typically, a locally presentable category. ($\mathcal{A}$ will be so as soon as it is a cocomplete $AB5$ abelian category with generator — such as, graded abelian groups. See Prop. 3.10.) By Milnor's axiom, h_* preserves filtered colimits. In fact, as soon as h_* preserves κ-filtered colimits for some κ, it is an accessible functor, and $h_*^{-1}(\mathsf{iso})$ will be accessible in $\mathrm{Mor}(SSet)$, yielding the solution set condition. By contrast, a cohomology theory is a functor $SSet \xrightarrow{h^*} \mathcal{A}^{\mathrm{op}}$. If the opposite category of a locally presentable category is accessible, they must be (equivalent to) small posets; see Adámek–Rosický [1] 1.64. So $\mathcal{A}^{\mathrm{op}}$ is not accessible; the accessibility of $h_*^{-1}(\mathsf{iso})$ in $SSet$ is then an open question.

From the set-theoretical point of view, all that was used of $SSet$ is that it is a locally presentable category. Indeed, J. Smith's theorem implies that any presentable model category allows localizations w.r.t. any accessible homology theory.

Remark 3.4. This is a good place to discuss (in)dependence of the choices made. After all, the class of weak equivalences in $SSet$ admits a canonical definition: the maps whose topological realizations are weak homotopy equivalences. In Ex. 3.1, *some* combinatorial characterization of this had to be found. Similarly, in Ex. 3.1, *a* representing spectrum had to be chosen. Dependence on these choices is not a frivolous issue; there are examples (albeit very artificial, from the point of view of homotopy theory) of coherent axioms that cannot be satisfied in Set (so, from the point of view of sets, parametrize the empty collection) but do have models in other topoi. On the positive side, one has

Proposition 3.5. *Let A_1, A_2 be two coherent axiom systems in the same (first-order, many-sorted) language $\mathcal{L}$. Suppose that in the category of sets, the same structures satisfy them.*

(i) *Then the same is true in any topos with enough points.*

(ii) *Suppose that there is a "reason" for this coincidence, more precisely, a formal deduction via coherent logic of A_1 from A_2, and vice versa. Then the same class of structures satisfies them in any topos.*

(iii) *Suppose that both A_1 and A_2 belong to a fragment of coherent logic that has enough models in Set. (For example, A_i, $i = 1, 2$, may be finitary, or employ countable disjunctions but have only countably many axioms, or be an essentially algebraic theory.) Then again, the same structures will satisfy them in any topos.*

Proof. (The reader looking for relevance to abstract homotopy theory may wish to skip to the examples following the proof.) (i) is trivial, given that points of a topos preserve and any conservative collection of points reflect the truth of coherent axioms. (ii) is the mere fact that coherent deductions remain valid in any topos. (iii) is a "degenerate case" of the Conceptual Completeness theorem of Joyal–Makkai–Reyes [35]. A_1 and A_2, by assumption, have enough models and have the same models in *Set*. Hence they have the same class of coherent consequences. This means that their associated syntactic sites $\mathrm{Def}(A_1)$, $\mathrm{Def}(A_2)$ (see Makkai–Reyes [35] or MacLane–Moerdijk [33]) are equivalent. (In fact, $\mathrm{Def}(A_1)$ and $\mathrm{Def}(A_2)$ can be constructed to be the *same* — not only isomorphic, but identical — since A_1 and A_2 were assumed to share the same language $\mathcal{L}$.) But this implies they have the same models in any topos. □

(i) already covers all but a few types of topoi that tend to arise in algebraic geometry and topology. (ii) applies, for example, to the choice of representing spectra. Let E_1 and E_2 be weakly homotopy equivalent spectra, and let A_1 say "it (i.e. a variable spectrum) has the same weak homotopy type as E_1"; analogously for A_2. Should these two classes coincide, there is a witness for that, namely a *map of spectra* $E_1 \xrightarrow{f} E_2$ *inducing isomorphisms on the stable homotopy groups.* The statement in italics is equivalent to a set of sentences of coherent logic. This allows the formal demonstration of the equivalence of "the spectrum X has the same weak homotopy type as E_1" with "the spectrum X has the same weak homotopy type as E_2" valid in any topos. (iii) implies, for example, that any "purely combinatorial" definition of a simplicial map being a weak equivalence can be chosen as long as it has the intended meaning for simplicial *sets.* The technical sense of "purely combinatorial" is: statements in the countable fragment of coherent logic, with signature Δ (and its extension by coproducts and coequalizers of equivalence relations). Intuitively, the definition must operate directly with simplices and the face and degeneracy maps (no mapping spaces, fundamental groupoids etc. can be used unless these had been so defined beforehand). The axioms must be "if–then" type, demanding the existence of simplices, or sequences of simplices, satisfying finitary face-matching conditions whenever other conditions of this type are met; but one cannot employ uncountably many conditions nor uncountable strings of simplices.

Example 3.6. (small categories and categorical equivalences)
Motivating example 0.4 is a case of Theorem 2.8 where not all the monomorphisms are cofibrations. I am indebted to Charles Rezk for pointing out that this model structure arises from the Reedy model structure on simplicial spaces (i.e. simplicial objects in *SSet*) via a right adjoint; this gives a simple proof of condition (ii) of 2.8.

Example 3.7. (groupoids and categorical equivalences)
Restrict the previous example to groupoids, leaving the definitions of cofibrations and weak equivalences the same. (Fibrations are functors with the right lifting property w.r.t. the inclusion $\{\bullet\} \hookrightarrow \{\bullet \to \star\}$.) This model structure too was sheafified "by hand" in Joyal–Tierney [30]. Via the nerve functor, the existence of this model category can also be deduced from that on simplicial objects; this proves 2.8(ii).

Example 3.8. (cosimplicial simplicial objects, Reedy model structure)
For indexing categories $\mathcal{D}$ satisfying a certain combinatorial property, work of D. Kan (see [14] or Hovey [21]) shows the existence of a Quillen model structure on $\mathcal{M}^{\mathcal{D}}$ with weak equivalences objectwise but neither cofibrations nor fibrations so. For the case $\mathcal{D} = \Delta$ and $\mathcal{M} = SSet$, this model structure made its appearance in Bousfield–Kan [11] X.4. It has as cofibrations those monomorphisms in $SSet^{\Delta}$ that induce isomorphisms on the "maximal augmentation". Here the maximal augmentation of $X^{\bullet} \in \operatorname{ob} SSet^{\Delta}$ is a subobject of X^0, the equalizer of the coface maps d^0, d^1. This is a coherently definable condition. That 2.8(ii) is satisfied follows by Kan's methods, or one can use the following mild extension of Prop. 1.12: keep hypotheses (i), (ii) and let C be an accessible subclass of the monomorphisms closed under pushout and transfinite composition and satisfying the following cancellation condition: if $gf \in \mathsf{C}$, $f \in \mathsf{C}$ and $g \in \mathsf{mono}$, then $g \in \mathsf{C}$. Then $\mathsf{C} = \mathsf{cof}(I)$ for some set I. (Accessibility lets one find a set of generators; after that, the argument is the same. See [7].)

Example 3.9. (cyclic sets, "weak" model structure)
Let Λ^{op} be Connes' indexing diagram for *cyclic objects*, containing the simplicial indexing diagram Δ^{op} as a subcategory. Dwyer–Hopkins–Kan [15] show the existence of a Quillen model structure on $\mathrm{Pre}(\Lambda)$, or "cyclic sets", created by the forgetful functor $\mathrm{Pre}(\Lambda) \to SSet$. That is to say, $f \in \operatorname{mor} \mathrm{Pre}(\Lambda)$ belongs to W iff it is a weak equivalence considered just as a map of simplicial sets. Thanks to Ex. 3.1, this makes the class W coherently definable. Dwyer–Hopkins–Kan define a cyclic map to be a fibration iff it is a Kan fibration when considered just as a map of simplicial sets, and give the following combinatorial description of cofibrations: $X \xrightarrow{f} Y \in \mathrm{Pre}(\Lambda)$ is a cofibration iff it is an injection such that for each object $[\mathbf{n}]$ of Λ^{op}, its group of automorphisms (which is a cyclic group of order $n+1$) acts freely on the elements of $Y([\mathbf{n}])$ not in the image of $X([\mathbf{n}])$. Taking the contrapositive, this becomes a coherent condition.

Spaliński's [39] strong model category structure on cyclic sets is dealt with in [6].

Another application of Theorem 1.7 and Prop. 1.12 concerns the unbounded derived category. It is valid for Grothendieck abelian categories, which are automatically locally presentable. Though this fact is a simple conjunction of theorems of Gabriel–Popescu and Gabriel–Ulmer, we write it out in longhand, since the identification is not commonly made in the literature.

Proposition 3.10. *Let $\mathcal{A}$ be an abelian category that satisfies Grothendieck's axiom $AB5$ ("directed colimits are exact", i.e. filtered colimits commute with finite limits). Then the following are equivalent:*

(i) *$\mathcal{A}$ is cocomplete and has a generator, i.e. object G such that* $\hom(G, -)$ *is faithful* [9]
(ii) *$\mathcal{A}$ is locally presentable.*

[9] MacLane [32] suggests the name *separator* for such an object, which is certainly reasonable, given the plenitude of senses of "generator" in different categorical contexts.

Historically, an $AB5$ category satisfying (i) has been called a *Grothendieck (abelian) category.*

Proof. (ii) $\Longrightarrow$ (i) is trivial: if $\mathcal{A}$ is locally presentable, then it is (by definition) cocomplete and has a set of dense generators, a fortiori a set of generators. Their coproduct will do as a single generator (in the sense of the proposition), since $\mathcal{A}$ has a zero object.

(i) $\Longrightarrow$ (ii): by the Gabriel-Popescu theorem there exists a ring R and an adjoint pair $\mathcal{A} \underset{i}{\overset{L}{\leftrightarrows}} \mathrm{Mod}_R$ where i is full and faithful, and the left adjoint L is exact. That is, A is equivalent to a localization of Mod_R. (Here a *localization* of a category is a full, reflective, isomorphism-closed subcategory with the reflector preserving finite limits.) Such localizations biject with Gabriel topologies on R, i.e. collections $\mathcal{F}$ of right ideals of R with certain closure properties, via associating with $\mathcal{F}$ the full subcategory of Mod_R of $\mathcal{F}$-closed modules. (See e.g. Stenström [41].) A module M is $\mathcal{F}$-closed iff it is orthogonal to the maps $I \hookrightarrow R$, $I \in \mathcal{F}$, i.e. $\hom_R(R, M) \to \hom_R(I, M)$ is an isomorphism for all $I \in \mathcal{F}$. Mod_R is locally presentable; by the theorem of orthogonal reflection in locally presentable categories (see e.g. Adámek–Rosický [1] Cor. 1.40), so is its full subcategory of $\mathcal{F}$-closed modules. □

Remark 3.11. The second implication is nontrivial. $\mathcal{A}$ being locally presentable entails, for example, that every object of $\mathcal{A}$, in particular, the generator, has a *rank*, that is, $\hom(G, -)$ commutes with κ-filtered colimits for some κ. This is not even implicit in the definition of a generator (as "separator").

Remark 3.12. Prop. 3.10 is the additive analogue of the case of a topos. Giraud's theorem says that if a category $\mathcal{E}$ is cocomplete, has well-behaved coproducts and quotients of equivalence relations, and a set of generators, then it is a topos. It is then equivalent to a localization of a functor category $\mathrm{Pre}(\mathcal{C})$, where $\mathcal{C}$ is small. Localizations of $\mathrm{Pre}(\mathcal{C})$ biject with Grothendieck topologies on $\mathcal{C}$, via passing to sheaves on a topology; and the sheaf condition can be phrased as orthogonality w.r.t. a certain set of maps in $\mathrm{Pre}(\mathcal{C})$. The conclusion that $\mathcal{E}$ is a locally presentable category now follows as before.

Proposition 3.13. *Let $\mathcal{A}$ be a Grothendieck abelian category. There is a Quillen model structure on $\mathrm{Ch}_{\mathbb{Z}}(\mathcal{A})$ (i.e. unbounded chain complexes in $\mathcal{A}$) where cofibrations are the monomorphisms, and weak equivalences the quasi-isomorphisms.*

Proof. Apply Thm. 1.7. If $\mathcal{A}$ is Grothendieck abelian, so is $\mathrm{Ch}_{\mathbb{Z}}(\mathcal{A})$. The homology functor $\mathrm{Ch}_{\mathbb{Z}}(\mathcal{A}) \to \mathcal{A}^{\mathbb{Z}}$ (here $\mathbb{Z}$ stands for just the discrete set) commutes with filtered colimits and $\mathcal{A}^{\mathbb{Z}}$ is locally presentable. The class of isomorphisms is accessible in any locally presentable category. Prop. 1.18 gives **c3**, and Prop. 1.12 grants the generating cofibrations. □

Corollary 3.14. *The unbounded derived category of a Grothendieck abelian category exists (i.e. has small hom-sets).*

Remark 3.15. Even when a localization problem $\mathcal{C}[W^{-1}]$ allows a calculus of fractions, a genuine set-theoretical difficulty remains in establishing $\mathcal{C}[W^{-1}]$ has small hom-sets (one has to exhibit cofinal small subcategories of the large, filtered index categories that arise). This may be the reason that, to the best of this author's knowledge, a proof of 3.14 in this generality has not appeared in print yet. It is certainly a folk theorem, however; see e.g. Joyal [29]. A recent preprint of Tarrío–López–Salorio [42] gives a triangulated proof.

It is quite ironic to observe that while the simple (inductive) injective replacement arguments break down for unbounded complexes, the proof strategy of 3.13 can only apply to a category of complexes that is cocomplete. The fibrant replacement functor it produces in abstracto is quite horrible; it seems that if one wishes to work with explicit resolutions of unbounded complexes, one should find a judicious sheaf representation of the category first.

Remark 3.16. In his Tohôku classic, Grothendieck was able to demonstrate remarkable (ultimately, set-theoretical) features of what are now called Grothendieck abelian categories, notably the existence of enough injectives (via showing, essentially, that the class of monomorphisms is generated by a set). With hindsight, when working with Grothendieck abelian categories, one encounters a class of examples with much stronger set-theoretic bounds than is apparent from the concept of generator ("separator"). When limits and colimits are not as well-behaved as for abelian categories and topoi, it seems necessary to posit the extra set theoretic control explicitly. Conversely, perhaps it is not frivolous to say that presentable Quillen model categories are "convenient categories to do homotopical algebra in", and to view them as non-abelian counterparts of Grothendieck abelian categories.

As a closing remark, note that [6] lists about a dozen Quillen model categories whose weak equivalences are coherently definable and whose cofibrations are of the form $\mathsf{cof}(I)$, where I is the image under a suitable left adjoint of a coherently defined class of maps. (They thus generalize examples 3.6 and 3.7.) Their qualitative features — sheafifiability, existence of non-canonical as well as functorial choices for cofibrations, and the essential uniqueness of cofibrations — are so similar to those above that it did not seem worthwhile to burden this paper with the extra logical machinery needed to "automate" their construction.

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