ON THE EXISTENCE OF THE SELF MAP v_2^9 ON THE SMITH-TODA COMPLEX $V(1)$ AT THE PRIME 3

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Abstract. Let $V(1)$ be the Smith-Toda complex at the prime 3 whose BP homology is $BP_*/(3, v_1)$. We prove that there exists a map $v_2^9 : \Sigma^{144} V(1) \to V(1)$ that is a $K(2)$ equivalence. This map is used to construct various v_2-periodic infinite families in the 3-primary stable homotopy groups of spheres.

Let $V(1)$ be the Smith-Toda complex obtained by taking the cofiber of the self map on the Moore space $v_1 : \Sigma^4 M(3) \to M(3)$ which induces multiplication by v_1 in $K(1)$-homology. This is an example of a type 2 complex. The periodicity theorem of Hopkins and Smith [10] states that there exists a v_2-self map $v : \Sigma^N V(1) \to V(1)$ which is a $K(2)$-equivalence. The purpose of this paper is to provide a minimal such v_2-self map. The main theorem of this paper is stated below.

Theorem 0.1. There exists a self-map

$$v_2^9 : \Sigma^{144} V(1) \to V(1)$$

whose effect on $K(2)$ homology is multiplication by v_2^9.

The strategy of proving the theorem is straightforward and computational. We first prove that the element v_2^9 in the Adams spectral sequence (ASS) for computing $\pi_*(V(1))$ is a permanent cycle. We then prove that this map extends over $V(1)$. We use the ASS instead of the Adams-Novikov spectral sequence (ANSS) because v_2^9 is in Adams filtration 9, whereas it has Adams-Novikov filtration 0. Therefore there are less potential targets for a differential supported by v_2^9 in the ASS than in the ANSS. The ASS E_2-term is also easier to compute.

Our method of showing that v_2^9 is a permanent cycle in the ASS is to consider all of the elements of the ASS which could be targets of differentials supported by v_2^9 which would not be detected in the ASS for $eo_2 \wedge V(1)$. We then will show all of these potential targets either support non-trivial differentials, or are killed on earlier pages of the spectral sequence. This requires knowledge of the ASS of $eo_2 \wedge V(1)$, as well as the E_2 term of the ASS converging to $\pi_*(V(1))$.

The ASS for $eo_2 \wedge V(1)$ is well-known to many people, but has not been published. In section 1 will compute $\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*(eo_2 \wedge V(1)))$, and deduce the ASS from this computation and the computation of $EO_{2*}(V(1))$ given in [5]. All of the methods in this section derive from unpublished work of Hopkins, Mahowald, and Miller. In retrospect, the resolution we use to compute $\mathrm{Ext}(eo_2 \wedge V(1))$ should be compared to that of Ravenel [16, ch. 7].

In section 2 we compute the E_2-term of the ASS for $V(1)$ through the 144-stem (the degree of v_2^9). We rely on Tangora's computer generated tables [22] of $H^*(P_*)$ where P_* is the polynomial part of the dual Steenrod algebra. The periodic lambda

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algebra [7] allows us to compute the E_2 term $H^*(A_*//E[\tau_0, \tau_1])$ via a Bockstein spectral sequence (BSS). In some instances Christian Nassau's computer generated Ext tables were of welcome assistance.

Differentials in the ASS are computed by using the Hurewicz image in the homotopy of the spectrum eo_2. In addition, we will prove a modified 'Leibnetz rule' for differentials in the ASS for V(1). This product rule is our main tool for calculating difficult differentials in the ASS for $V(1)$. The product rule is presented in section 4. Such technology is essential because $V(1)$ is *not* a ring spectrum, so its ASS is not a spectral sequence of algebras. However, the S-module structure of $V(1)$ does make the ASS a spectral sequence of modules over the ASS for computing $\pi_*(S)$, and this is used occasionally to propagate differentials. We make heavy use of the computation of the 3-primary stable stems through the 108 stem presented in [16]. We use these computations as input for the Atiyah-Hirzebruch spectral sequence (AHSS) to make selective computations of $\pi_*(V(1))$ in certain ranges.

Section 6 is devoted to proving that v_2^9 is a permanent cycle in the ASS for $V(1)$, and therefore detects an element of $\pi_*(V(1))$.

It now remains to extend v_2^9 over $V(1)$. This is simplified by a certain splitting of the complex $D(V(1)) \wedge V(1)$, where $D(V(1))$ is the Spanier-Whitehead dual. This splitting is the subject of section 3. The splitting is also needed to prove the product rule.

Using the attaching map structure of one of the wedge summands, the obstruction to the extension of v_2^9 over $V(1)$ is identified as an element of $\pi_*(V(1))$. In showing that this obstruction is zero, it is helpful to know what power β_1^k has the property that $\beta_1^k : \Sigma^{10k}V(1) \to V(1)$ is null. In section 5, we show that the map is null precisely when $k \geq 5$.

In section 7, we proceed by considering all of the elements in the ASS that might survive to the obstruction to extending v_2^9 over $V(1)$, and show that they all either support differentials, or are the targets of differentials. Thus the map v_2^9 extends to a self map of $V(1)$, completing our proof of theorem 0.1.

In section 8 we investigate the implications on some v_2-periodic elements of the stable stems following Shimomura [18]. Theorem 0.1 allows us to deduce that certain elements of the ANSS must be permanent cycles. In particular, we have the following consequence.

Corollary 0.2. The elements β_i are permanent cycles in the ANSS for $i \equiv 0, 1, 2, 5, 6 \pmod 9$.

It should be the case that the elements β_{9t+3} exist, but we are unable to deduce this from the existence of our self map on $V(1)$. Oka [15] indicates that if the complex $M(3, v_1^3)$ has a v_2^9 self map then the elements β_{9t+3} exist. Shimomura's computations of $\pi_*(L_2V(0))$ [20] demonstrate that the elements β_{9t+3} are present in $\pi_*(L_2(S^0))$. Shimomura [19] proves that β_i cannot be a permanent cycle for $i \equiv 4, 7, 8 \pmod 9$. In [18], many relations amongst the β_i's are investigated contingent on the existence of the self-map constructed in this paper. In particular, Shimomura proves that if the elements $\beta_i\beta_1^4$ are permanent cycles for $i \equiv 1 \pmod 9$, then they are non-trivial. These elements are permanent cycles by 0.2. They should be regarded as the substitutes for the the Adams-Novikov elements β_{9t+4}, which fail to exist.

Some remarks as to how this paper came to be written are in order. The main result of this paper was the subject of the second author's dissertation completed at

Northwestern University under the direction of Mark Mahowald. The first author required the result for his dissertation work at the University of Chicago under the supervision of J. Peter May. Certain errors and gaps in the original work needed to be corrected. In the original thesis, the second author's main technique for obtaining differentials in the ASS was to lift differentials from the ANSS using a technical lemma called the 'ladder lemma'. We were unable to make the proof of this lemma rigorous, and so the product rule (Theorem 4.1) is used instead for many of the differentials.

The authors would like to thank Mark Mahowald for his constant encouragement and assistance in this project. We also thank Katsumi Shimomura, for his useful correspondence concerning the construction of the β_i from the self map. The first author would also like to thank his advisor, J. Peter May, for many useful conversations related to this project.

Conventions. Throughout this paper we shall always be working in the stable homotopy category localized at the prime 3, and all homology will be with $\mathbb{F}_3$ coefficients. We shall use the following abbreviations.

ASS: Adams spectral sequence
ANSS: Adams-Novikov spectral sequence
BSS: Bockstein spectral sequence
AHSS: Atiyah-Hirzebruch spectral sequences

The dual Steenrod algebra will be denoted by A_*. If X is a spectrum, we will often use the notation $\mathrm{Ext}(X)$ as to represent $\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*(X))$, the E_2 term of the ASS for computing $\pi_*(X)$. We will denote the E_r term of this ASS by $E_r(X)$. We shall use the notation $\doteq$ to indicate two quantities are equal up to multiplication by a unit in $\mathbb{F}_3$.

Contents

1. The Adams spectral sequence of $eo_2 \wedge V(1)$

In this section we will define $eo_2 \wedge V(1)$ and compute its ASS. The method of computing the E_2 term of the ASS is to produce a finite complex $Y(2)$ such that upon smashing it with $eo_2 \wedge V(1)$ we get a wedge of $k(2)$'s. We know the ASS of this object, and we recover the E_2 term of the ASS for $\pi_*(eo_2 \wedge V(1))$ by forming a periodic resolution of S^0 out of copies of $Y(2)$. There is nothing original in this section. Most of the material here was first discovered by Hopkins, Mahowald, and Miller, but remains unpublished.

Let E_2 be the Hopkins-Miller spectrum at $p = 3$. It is a Landweber exact cohomology theory whose coefficient ring is

$$E_{2*} = \mathbb{W}_{\mathbb{F}_9}[[u_1]][u, u^{-1}]$$

where $|u_1| = 0$ and $|u| = -2$. Here $\mathbb{W}_{\mathbb{F}_9}$ is the Witt ring with residue field $\mathbb{F}_9$. Fix a primitive 8^{th} root of unity ω in $\mathbb{F}_9$. At the risk of confusion, we will refer to it's Teichmüller lift in $\mathbb{W}_{\mathbb{F}_9}$ also as ω. The element ω satisfies the relation

$$\omega^2 + \omega + 2 = 0$$

in $\mathbb{F}_9$. The spectrum E_2 is a BP-ring spectrum, and the map $\Phi : BP \to E_2$ has the following effect on coefficient rings.

$$\begin{aligned}
\Phi(v_1) &= u^{-2}u_1 \\
\Phi(v_2) &= u^{-8} \\
\Phi(v_i) &= 0, \quad \text{for } i > 2
\end{aligned}$$

Let $\mathbb{S}_2$ be the Morava stabilizer group. It is the automorphism group of the Honda height 2 formal group law F_2 over $\mathbb{F}_9$, and is contained in the non-commutative algebra

$$\mathbb{W}_{\mathbb{F}_9}\langle S\rangle/(S^2 = p, Sa = \sigma(a)S)$$

as the multiplicative group of units. Here σ is the a lift of the Frobenius map. The Galois group $Gal = Gal(\mathbb{F}_9/\mathbb{F}_3)$ acts on $\mathbb{S}_2$ by acting on $\mathbb{W}_{\mathbb{F}_9}$. It is cyclic of order 2 generated by the Frobenius automorphism σ. One may form the semi-direct product

$$\mathbb{G}_2 = \mathbb{S}_2 \rtimes Gal.$$

The group $\mathbb{G}_2$ acts on the E_2 via E_∞ maps. The spectrum E_2 and the A_∞ action of $\mathbb{S}_2$ are presented in [17]. There is a maximal finite subgroup $G_{12} < \mathbb{S}_2$ of order 12 which is isomorphic to $C_3 \rtimes C_4$. It is generated by an element s of order 3, and an element t of order 4, given by the following formulas.

$$\begin{aligned}
s &= -\frac{1}{2}(1 + \omega S) \\
t &= \omega^2
\end{aligned}$$

These elements, as well as specific formulas for their action on E_{2*}, are given in [5]. The subgroup G_{12} is not invariant under the Galois action, so following [6] we instead investigate a maximal finite subgroup $G_{24} < \mathbb{G}_2$ which contains G_{12} and fits into the following (non-split) short exact sequence.

$$1 \to G_{12} \to G_{24} \to Gal \to 1$$

The subgroup G_{24} is generated by the elements s, t, and ψ, where we define

$$\psi = \omega\sigma \in \mathbb{G}_2.$$

The spectrum EO_2 is defined to be the homotopy fixed point spectrum $E_2^{hG_{24}}$. A complete computation of the homotopy of EO_2, and its ANSS, is given in [6]. In [5], Goerss, Henn, and Mahowald compute the ANSS for $EO_{2*}(V(1))$, but their approach must be modified since they use G_{12} instead of G_{24}. Their results may be conveniently summarized in figure 1. In this figure, dots represent additive $\mathbb{F}_3$ generators, lines of length 3 represent multiplication by α_1, lines of length 7 represent the Toda Bracket $\langle \alpha_1, \alpha_1, - \rangle$, and lines of length 10 represent multiplication by β_1.

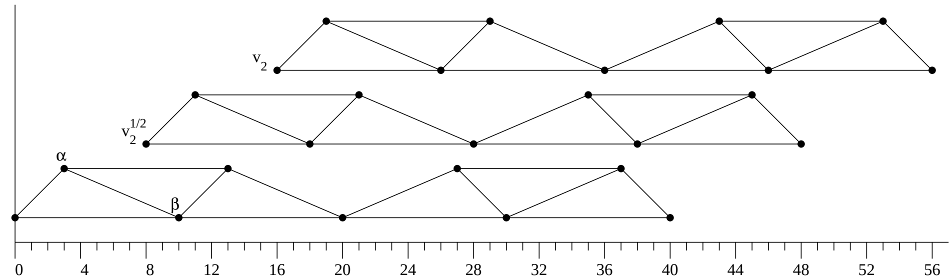

FIGURE 1. $EO_{2*}(V(1))$ is generated as a free module over $\mathbb{F}_3[v_2^{\pm 9/2}]$ on the above pattern.

These products are given by the S-module structure. The homotopy is periodic with periodicity generator $v_2^{\pm 9/2}$ of degree 72 on the displayed pattern.

We want an Adams spectral sequence, but unfortunately the v_2-periodicity in EO_2 makes its homology trivial. We therefore need to take a connective cover. There is a nice connective cover of EO_2 called eo_2 which has been constructed by Hopkins, Mahowald, and others. Since the details of this construction are quite involved, we will instead *define* $eo_2 \wedge V(1)$ to be the connective cover of $EO_2 \wedge V(1)$.

Since there is a gap in the homotopy of $EO_2 \wedge V(1)$ between the 56 stem and the 72 stem, taking the connective cover removes the periodic copies of the homotopy in negative dimensions. We remark that the reason that we cannot just define eo_2 to be the connective cover of EO_2 is that the there are infinitely many copies of $BP\langle 1\rangle$ in the homotopy supported on negative periodicity generators whose unwanted homotopy eventually appears in positive degrees. Of course, smashing with $V(1)$ kills all of this troublesome v_1-periodic homotopy.

We will now produce a finite complex $Y(2)$ which, when smashed with EO_2, splits as a wedge of Morava K-theories. Let $\gamma : S^4 \to BO$ be a generator of $\pi_4(BO) = \mathbb{Z}$. The map γ extends to a loop map $\gamma_\infty : \Omega S^5 \to BO$. Let $J_i(S^4) \hookrightarrow \Omega S^5$ be the i^{th} filtration of the James construction [11]. Then γ restricts to a map $\gamma_i : J_i(S^4) \to BO$. Let $Y(i)$ be the Thom spectrum $(J_i(S^4))^{\gamma_i}$. Then the homology of the ring spectrum $Y(\infty)$ is given by

$$H_*(Y(\infty)) = \mathbb{F}_3[b_2]$$

where b_2 has degree 4. The homology $H_*(Y(i))$ is the additive subgroup generated by x^k for $0 \le k \le i$.

There are maps $Y(i) \wedge Y(j) \to Y(i+j)$ induced from the maps $J_i(S^4) \times J_j(S^4) \to J_{i+j}(S^4)$. The complex $Y(1)$ is just the Thom spectrum $(S^4)^\gamma$, and it is a two cell complex whose top cell is attached to the bottom cell by the attaching map which is the image of γ under the J-homomorphism. Therefore, $Y(1) = S^0 \cup_{\alpha_1} e^4$. It follows that the dual Steenrod operation P^1_* acts on $H_*(Y(1))$ by the formula

$$P^1_*(b_2) = 1.$$

Using the map $Y(1) \wedge Y(1) \to Y(2)$, we obtain the following formulas for the dual action of the Steenrod algebra on $H_*(Y(2))$.

$$
\begin{aligned}
P^1_*(b_2) &= 1 \\
P^1_*(b_2^2) &= 2b_2 \\
P^2_*(b_2^2) &= 1
\end{aligned}
$$

In particular, we have the CW decomposition $Y(2) = S^0 \cup_{\alpha_1} e^4 \cup_{2\alpha_1} e^8$.

Our interest in $Y(2)$ arises from the following proposition.

Proposition 1.1. There is a splitting

$$EO_2 \wedge V(1) \wedge Y(2) \simeq K(2) \vee \Sigma^8 K(2).$$

Proof. One can easily compute $\pi_*(EO_2 \wedge V(1) \wedge Y(\infty))$ from the AHSS arising from the cellular filtration of $Y(\infty)$, but the associated graded arising from this filtration gives too much ambiguity for our purposes. We therefore will use instead the homotopy fixed point spectral sequence

$$H^*(G_{24}; \pi_*(E_2 \wedge V(1) \wedge Y(\infty))) \Rightarrow \pi_*(EO_2 \wedge V(1) \wedge Y(\infty))$$

where

$$\pi_*(E_2 \wedge V(1) \wedge Y(\infty)) = \mathbb{F}_9[u, u^{-1}, b_2].$$

In [6], the action of G_{24} on $E_{2*}(V(1))$ is given by the following formulas.

$$
\begin{aligned}
s_*(u) &= u \\
t_*(u) &= \omega^2 u \\
\psi_*(u) &= \omega u
\end{aligned}
$$

The elements $s, t \in G_{12}$ correspond to the automorphisms

$$
\begin{aligned}
s(x) &= x +_{F_2} u^{-2}\omega x^3 = x + u^{-2}\omega x^3 + \mathcal{O}(x^4) \\
t(x) &= \omega^2 x
\end{aligned}
$$

of the Honda height 2 formal group F_2 over $\mathbb{F}_9[u, u^{-1}]$, with 3-series $[3]_{F_2} = u^{-8}x^9$. Under the canonical map of Thom spectra $Y(\infty) \to MU$, b_2 maps to the element of the same name in

$$(E_2 \wedge V(1))_*(MU) = \mathbb{F}_9[u, u^{-1}][b_1, b_2, b_3, \ldots]$$

where the generators b_i coincide with those of Adams in [1, II.4.5]. These b_i correspond to the coefficient of x^{i+1} in a *strict* map of formal groups, and as such, we have

$$
\begin{aligned}
s_*(b_2) &= b_2 + u^{-2}\omega \\
t_*(b_2) &= b_2 \\
\psi_*(b_2) &= b_2.
\end{aligned}
$$

Therefore, the fixed points are given by

$$\pi_*(E_2 \wedge V(1) \wedge Y(\infty))^{G_{24}} = \mathbb{F}_3[(\omega^2 u^4)^{\pm 1}, b_2^3 - \omega^2 b_2 u^{-4}] \subset \mathbb{F}_9[u^{\pm 1}, b_2].$$

Define $a_4 = \omega^2 u^{-4}$ and $a_6 = b_2^3 - \omega^2 b_2 u^{-4}$. In [5, 1.4], the E_2 term of the homotopy fixed point spectral sequence for $EO_2 \wedge V(1)$ is computed to be

$$H^*(G_{24}; \pi_*(E_2 \wedge V(1))) = \mathbb{F}_3[a_4^{\pm 1}, \beta] \otimes E[\alpha]$$

where $\beta = \langle \alpha, \alpha, \alpha \rangle$. The cellular filtration of $Y(\infty)$ gives an Atiyah-Hirzebruch type spectral sequence that allows one to compute $H^*(G_{24}; \pi_*(E_2 \wedge V(1) \wedge Y(\infty)))$ from this. The d_4's in this spectral sequence are multiplication by α, and the d_8's are given by the application of the Massey product $\langle \alpha, \alpha, - \rangle$. Thus we conclude that

$$H^s(G_{24}; \pi_*(E_2 \wedge V(1) \wedge Y(\infty))) = \begin{cases} \pi_*(E_2 \wedge V(1) \wedge Y(\infty))^{G_{24}}, & s = 0 \\ 0, & s > 0 \end{cases}$$

and

$$\pi_*(EO_2 \wedge V(1) \wedge Y(\infty)) = \mathbb{F}_3[a_4^{\pm 1}, a_6].$$

The spectrum $EO_2 \wedge Y(\infty)$ is a ring spectrum whose homotopy is concentrated in Adams-Novikov filtration 0, and since the obstruction for $V(1)$ to be a ring spectrum lies in positive Adams-Novikov filtration (3.1), $EO_2 \wedge V(1) \wedge Y(\infty)$ is also a ring spectrum. Its homotopy is concentrated in even degrees, so it is complex-orientable [1]. The complex orientation

$$\theta : BP \to E_2 \wedge V(1) \wedge Y(\infty)$$

for which $\theta_*(v_i) = 0$ for $i \neq 2$ and $\theta_*(v_2) = u^{-8}$ lifts to a complex orientation

$$\widetilde{\theta} : BP \to EO_2 \wedge V(1) \wedge Y(\infty).$$

Here the effect on homotopy is given by $\widetilde{\theta}_*(v_i) = 0$ and $\widetilde{\theta}_*(v_2) = -a_4^2$. There are maps (for $\epsilon = 0, 1$)

$$S^{12j+8\epsilon} \wedge BP \xrightarrow{a_4^\epsilon a_6^j \wedge \phi} EO_2 \wedge V(1) \wedge Y(\infty)$$

that extend to maps

$$\psi_{12j+8\epsilon} : \Sigma^{12j+8\epsilon} K(2) \to EO_2 \wedge V(1) \wedge Y(\infty).$$

These maps give a splitting

$$EO_2 \wedge V(1) \wedge Y(\infty) = \bigvee_{j \geq 0} \Sigma^{12j} \left(K(2) \vee \Sigma^8 K(2) \right).$$

The composite

$$EO_2 \wedge V(1) \wedge Y(2) \to EO_2 \wedge V(1) \wedge Y(\infty) \to K(2) \vee \Sigma^8 K(2)$$

(the second arrow is projection onto the first two wedge summands) is an equivalence. □

Corollary 1.2. There is a splitting

$$eo_2 \wedge V(1) \wedge Y(2) \simeq k(2) \vee \Sigma^8 k(2).$$

Proof. The spectrum $k(2) \vee \Sigma^8 k(2)$ is the connective cover of $K(2) \vee \Sigma^8 K(2)$. The Atiyah-Hirzebruch spectral sequence for $(eo_2 \wedge V(1))_*(Y(2))$ is easily computed, and one finds

$$\pi_*(eo_2 \wedge V(1) \wedge Y(2)) = \mathbb{F}_3[a_4].$$

Therefore, $eo_2 \wedge V(1) \wedge Y(2)$ is the connective cover of $EO_2 \wedge V(1) \wedge Y(2)$. The previous proposition and the uniqueness of the connective cover combine to give this corollary. □

Remark 1.3. Hopkins and Miller, in [9], prove the following stronger result.

$$eo_2 \wedge Y(2) \simeq BP\langle 2 \rangle_3^\wedge \vee \Sigma^8 BP\langle 2 \rangle_3^\wedge$$

We will now construct a resolution of the sphere spectrum out of suspensions of $Y(2)$. There are cofiber sequences

$$S^0 \to Y(2) \to \Sigma^4 Y(1)$$
$$Y(1) \to Y(2) \to S^8$$

where the first maps are the evident inclusions. Splicing these together gives the following 2-periodic resolution of S^0.

$$\begin{tikzcd}
S^0 \arrow[d] & \Sigma^3 Y(1) \arrow[l] \arrow[d] & S^{10} \arrow[l] \arrow[d] & \Sigma^{13} Y(1) \arrow[l] \arrow[d] & S^{20} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
Y(2) & \Sigma^3 Y(2) & \Sigma^{10} Y(2) & \Sigma^{13} Y(2) & \Sigma^{20} Y(2) &
\end{tikzcd}$$

The homology long exact sequences associated to this resolution break up into short exact sequences as a result of the following lemma.

Lemma 1.4. The maps

$$H_*(eo_2 \wedge V(1)) \to H_*(eo_2 \wedge V(1) \wedge Y(2))$$
$$H_*(eo_2 \wedge V(1) \wedge Y(1)) \to H_*(eo_2 \wedge V(1) \wedge Y(2))$$

are injective.

Proof. The natural map of Thom spectra $Y(\infty) \to MU$ makes MU a $Y(\infty)$-ring spectrum, and therefore the Eilenberg-MacLane spectrum $H\mathbb{F}_3 = H$ is a $Y(\infty)$-ring spectrum. Thus there is a retraction

$$\begin{tikzcd}
H \arrow[r, "1\wedge\eta"] \arrow[rrr, bend right, "Id"'] & H \wedge Y(2) \arrow[r] & H \wedge Y(\infty) \arrow[r, "\mu"] & H
\end{tikzcd}$$

and we may conclude that $H_*(eo_2 \wedge V(1)) \to H_*(eo_2 \wedge V(1) \wedge Y(2))$ is an inclusion. The complex $Y(2)$ is, up to suspension, Spanier-Whitehead self-dual, and so we also have the projection map $H_*(eo_2 \wedge V(1) \wedge Y(2)) \to H_*(\Sigma^8 eo_2 \wedge V(1))$ is surjective, hence the previous map in the cofiber sequence $H_*(eo_2 \wedge V(1) \wedge Y(1)) \to H_*(eo_2 \wedge V(1) \wedge Y(2))$ must be injective. $\square$

We may therefore apply $\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*(eo_2 \wedge V(1) \wedge -))$ to this resolution, and get long exact sequences, hence a spectral sequence. Our spectral sequence takes the form (for $\epsilon = 0, 1$)

$$E_1^{2k+\epsilon,s,t} = \mathrm{Ext}^{s,t}(eo_2 \wedge V(1) \wedge \Sigma^{10k+3\epsilon} Y(2)) \Rightarrow \mathrm{Ext}^{s+2k+\epsilon, t+2k+\epsilon}(eo_2 \wedge V(1)) \tag{1.1}$$

Applying Corollary 1.2, and using the known computation $\mathrm{Ext}(k(2)) = \mathbb{F}_3[v_2]$ where $|v_2| = (1, 19)$, we may express the E_1 term of 1.1 by

$$E_1^{*,*,*} = \mathbb{F}_3[v_2, a, \beta] \otimes E[\alpha].$$

The tridegrees of these elements are $|v_2| = (0, 1, 19)$, $|a| = (0, 0, 8)$, $|\beta| = (2, 0, 10)$, and $|\alpha| = (1, 0, 3)$. The only possible differentials are

$$d_1(\alpha\beta^i v_2^j a) \doteq \beta^{i+1} v_2^j$$

but this d_1 arises from the composite $Y(2) \to S^8 \to \Sigma^8 Y(2)$. The element $v_2^j a \in \mathrm{Ext}(eo_2 \wedge V(1) \wedge Y(2))$ is born on the zero cell of $Y(2)$, and so must map to zero when

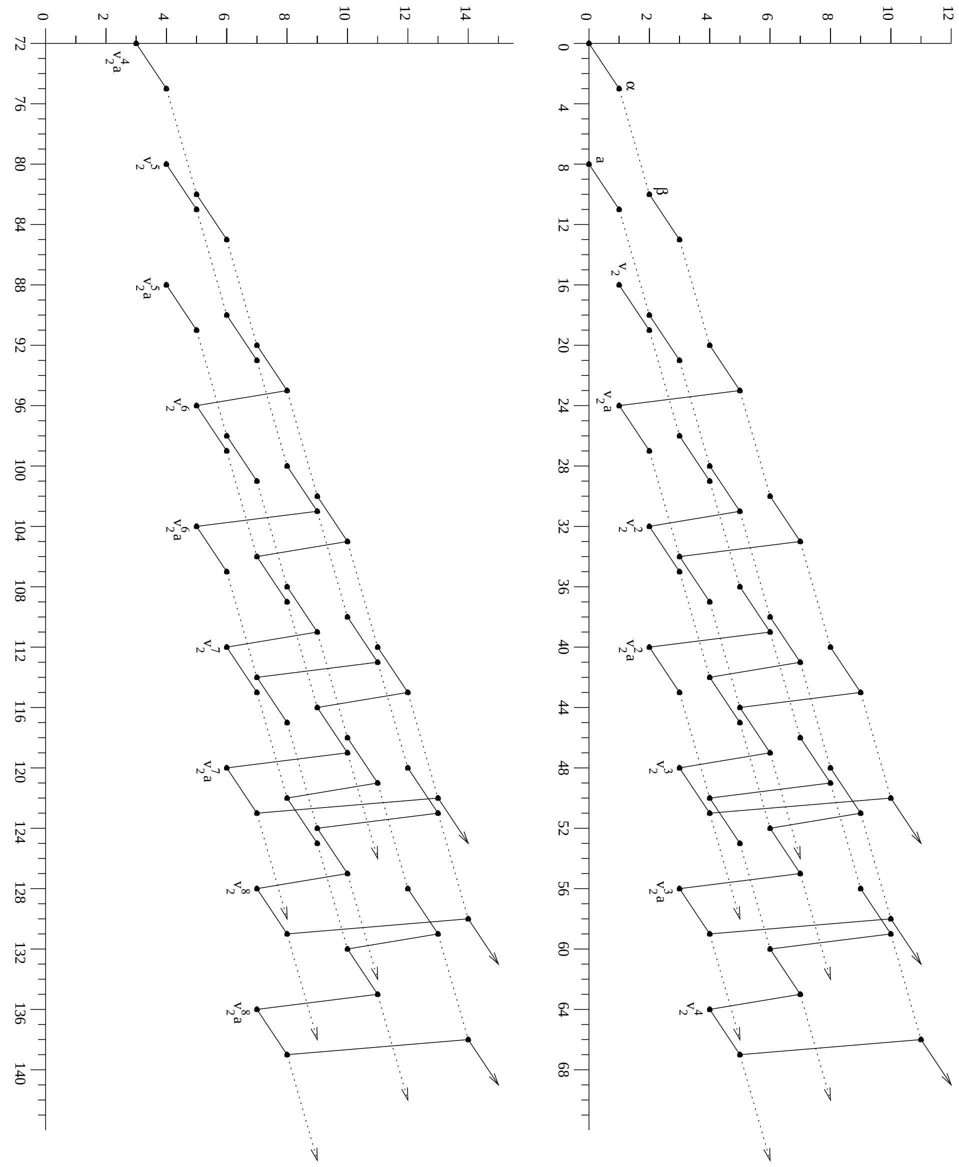

FIGURE 2. The ASS for $\pi_*(eo_2 \wedge V(1))$ is a free module over $\mathbb{F}_3[v_2^9]$ on the displayed pattern.

projected onto the 8-cell of $Y(2)$. Therefore, the spectral sequence 1.1 collapses at E_1, and we are left with a computation of the E_2-term of the ASS for $\pi_*(eo_2 \wedge V(1))$.

The differentials in the ASS are easily inferred from the differentials in the ANSS computed in [5]. Figure 2 displays the complete ASS chart. In this chart, $\mathbb{F}_3$-generators are represented by dots, α multiplication is displayed with solid lines, and the Massey product $\langle \alpha, \alpha, - \rangle$ is displayed with dotted lines. Solid lines of negative slope represent Adams differentials. The x-axis represents the $t-s$ degree, and the y-axis represents the homological degree s.

We remark that the ASS for $eo_2 \wedge V(1)$ is additively identical to the ANSS. The only difference is that v_2 has Adams filtration 1, whereas it has Adams-Novikov filtration 0.

We finish this section with a computation of the effect of the eo_2 Hurewicz homomorphism on the ASS. We will use the resolution of S^0 by the ring spectrum $Y(\infty)$. Mahowald, in [12], investigates a geometric Thom isomorphism

$$Y(\infty) \wedge Y(\infty) \simeq Y(\infty) \wedge (\Omega S^5_+)$$

under which we may make the identification

$$\pi_*(eo_2 \wedge V(1) \wedge Y(\infty) \wedge Y(\infty)) = \mathbb{F}_3[a_4, a_6, r]$$

Here r has degree 4. We may regard $\mathbb{F}_3[a_4, a_6, r]$ as being contained in $\mathbb{F}_3[u^{-1}, b_2] \otimes \mathbb{F}_3[b_2]$ where a_4 and a_6 are contained in the first factor as described earlier, and the element r corresponds to $1 \otimes b_2$. The main result of [12] states that the right unit of the associated Hopf algebroid

$$(\mathbb{F}_3[a_4, a_6], \mathbb{F}_3[a_4, a_6, r])$$

is given by the following formulas where b_2 maps to $b_2 \otimes 1 + 1 \otimes b_2$.

$$\begin{array}{ccc} \pi_*(eo_2 \wedge V(1) \wedge Y(\infty)) & \xrightarrow[\widetilde{\Delta}]{} & \pi_*(eo_2 \wedge V(1) \wedge Y(\infty) \wedge Y(\infty)) \\ \parallel & & \parallel \\ \mathbb{F}_3[a_4, a_6] & & \mathbb{F}_3[a_4, a_6, r] \end{array}$$

$$\begin{aligned} \widetilde{\Delta}(a_4) &= \Delta(\omega^2 u^{-4}) = \omega^2 u^{-4} \otimes 1 \\ &= a_4 \\ \widetilde{\Delta}(a_6) &= \Delta(b_2^3 - \omega^2 b_2 u^{-4}) = (b_2^3 - \omega^2 u^{-4} b_2) \otimes 1 - \omega^2 u^{-4} \otimes b_2 + 1 \otimes b_2^3 \\ &= a_6 - a_4 r + r^3 \end{aligned}$$

The Hurewicz image of h_0 is represented by r, and the Hurewicz image of h_1 is represented by r^3. The d_1 supported by a_6 identifies the Hurewicz image of h_1 with $h_0 a_4 = \alpha \cdot a$. This observation may be used to prove the following proposition.

Proposition 1.5. The Hurewicz homomorphism

$$h : \mathrm{Ext}(V(1)) \to \mathrm{Ext}(eo_2 \wedge V(1))$$

is described by

$$\begin{array}{lll} h(h_0) = \alpha & h(b_0) = \beta & h(v_2) = v_2 \\ h(h_1) = \alpha_1 a & h(g_0) = \beta a & \end{array}$$

2. Calculation of the Adams E_2 term $\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*(V(1)))$

The ASS for computing $\pi_*(V(1))$ has as its E_2 term

$$\mathrm{Ext}_{A_*}(\mathbb{F}_3, H_*(V(1)))$$

As a comodule over the dual Steenrod algebra, we have

$$H_*(V(1)) = E[\tau_0, \tau_1].$$

This is a subalgebra of the Steenrod algebra, but $V(1)$ is not a ring spectrum, so this algebra structure is not the consequence of a geometric multiplication. For the

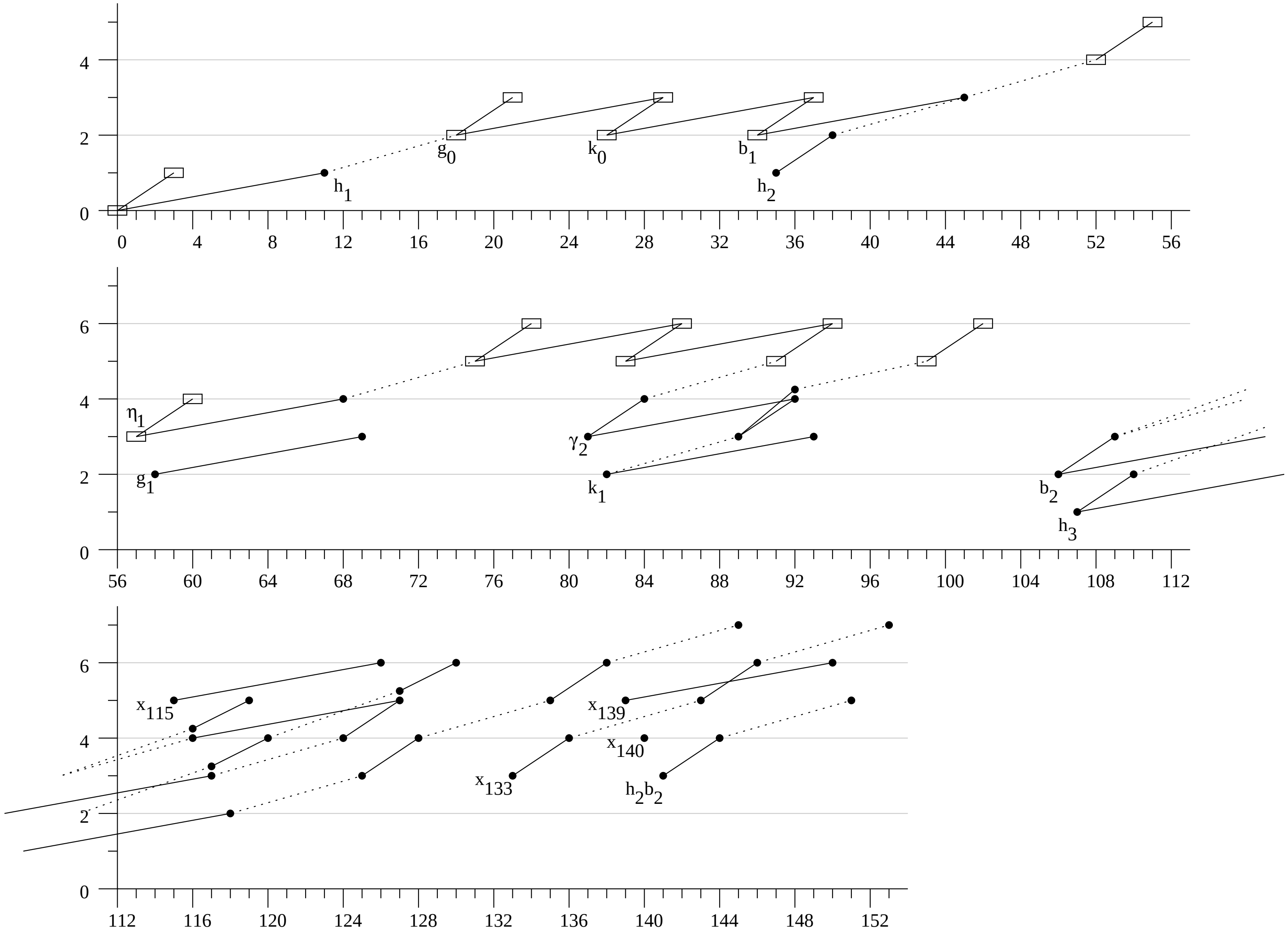

FIGURE 3. $\mathrm{Ext}_{P_*}(\mathbb{F}_3, \mathbb{F}_3)$, from Tangora's tables.

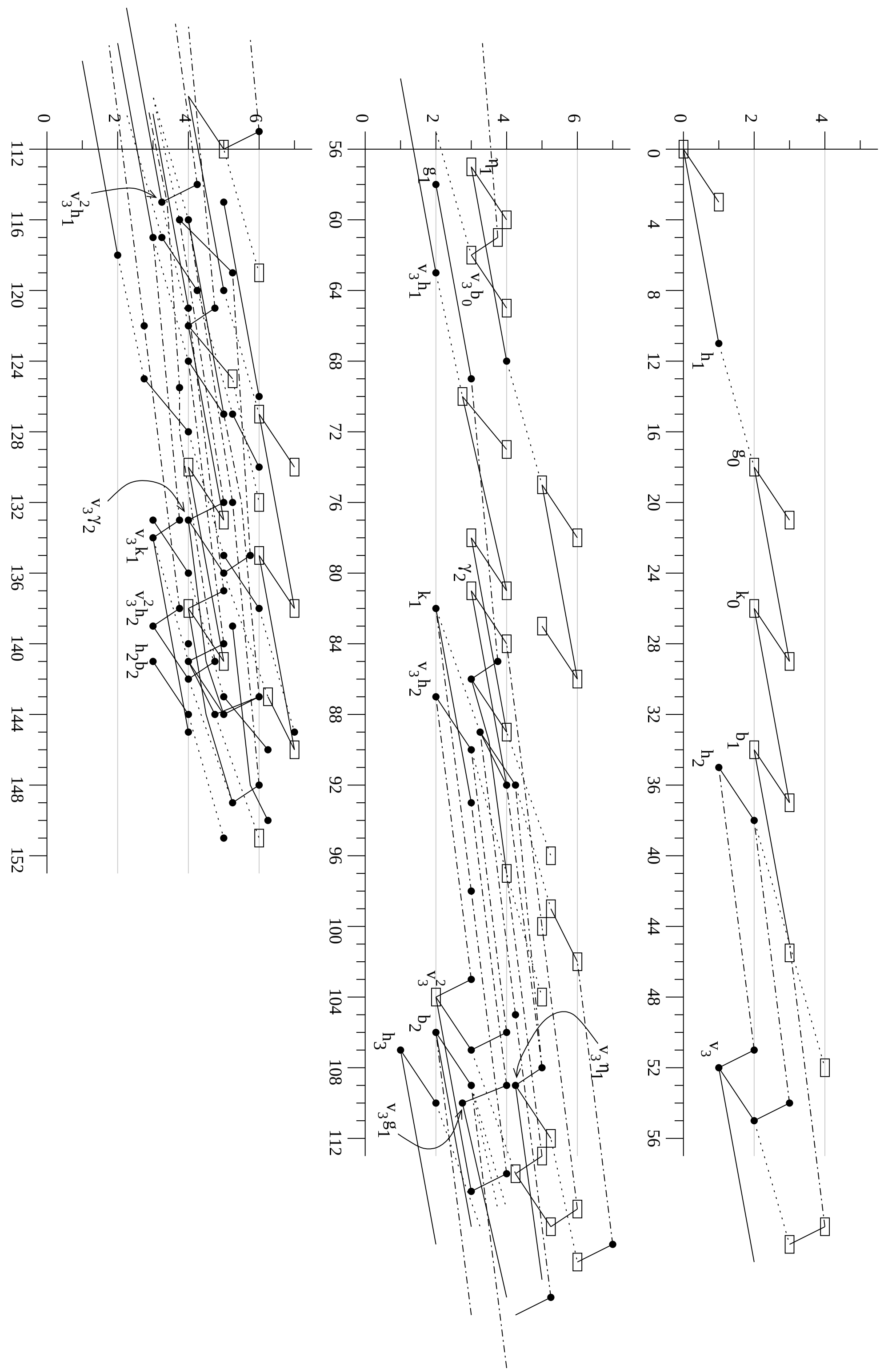

FIGURE 4. The BSS $\mathrm{Ext}_{A_*//E[\tau_0,\tau_1,\tau_2]}(\mathbb{F}_3,\mathbb{F}_3)\otimes P[v_2] \Rightarrow \mathrm{Ext}_{A_*//E[\tau_0,\tau_1]}(\mathbb{F}_3,\mathbb{F}_3)$.

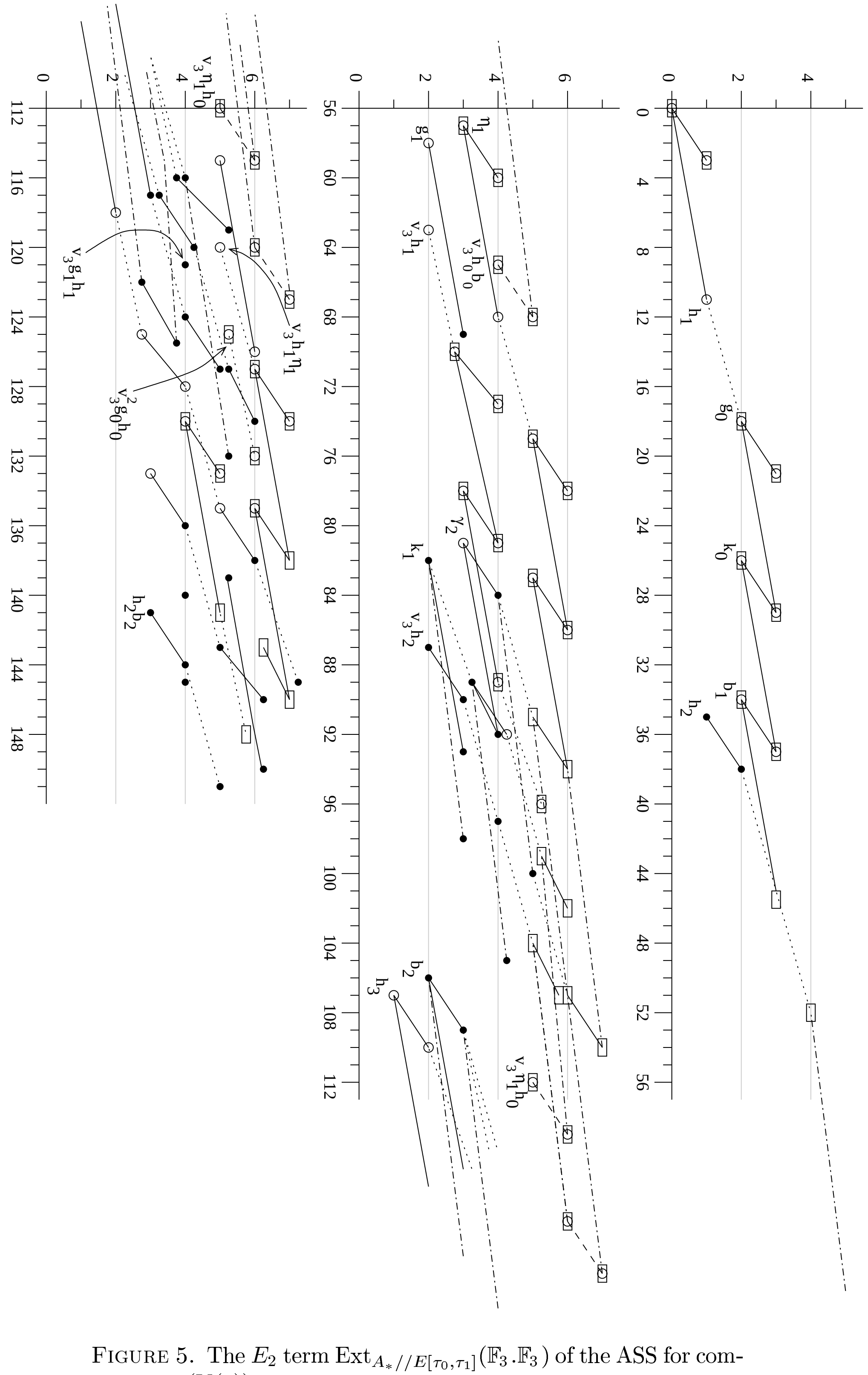

FIGURE 5. The E_2 term $\mathrm{Ext}_{A_*//E[\tau_0,\tau_1]}(\mathbb{F}_3.\mathbb{F}_3)$ of the ASS for computing $\pi_*(V(1))$.

purposes of computing Ext, though, we may use the algebra structure. A change of rings isomorphism identifies the E_2 term of the ASS as the cohomology of a Hopf algebra.

$$\mathrm{Ext}_{A_*}(\mathbb{F}_3, E[\tau_0,\tau_1]) \cong \mathrm{Ext}_{A_*//E[\tau_0,\tau_1]}(\mathbb{F}_3,\mathbb{F}_3) = H^*(A_*//E[\tau_0,\tau_1]).$$

Of course, we may identify

$$A_*//E[\tau_0,\tau_1] = P[\xi_1,\xi_2,\ldots] \otimes E[\tau_2,\tau_3,\ldots].$$

The cohomology of this Hopf algebra is the cohomology of the subalgebra of the periodic lambda algebra (see [7] for a description of the periodic lambda algebra) given by

$$\overline{\Lambda}_{(1)} = \langle \lambda_i, v_j \;:\; i \geq 0, j \geq 2\rangle \subset \overline{\Lambda}.$$

We only need the E_2 term of the Adams spectral sequence through the dimension of v_2^9, which is 144. Since the dimension of v_4 is 160, the cohomology of $\overline{\Lambda}_{(2)}$ coincides with $H^*(P_*) \otimes P[v_3]$ in the range we are interested in.

Figure 3 displays $H^*(P_*)$. It was produced from Tangora's Curtis tables in [22]. The y axis is the homological degree s, and the x-axis is $t-s$, where t is the internal degree. Solid lines of different slopes indicate multiplication by h_0 and h_1. Dotted lines indicate the Massey product $\langle -, h_0, h_0\rangle$. The element $b_0 = \langle h_0, h_0, h_0\rangle$, so b_0 multiplication can be read off as composites of h_0 multiplication and application of the above Massey product. Rectangles indicate that a generator supports a polynomial algebra on b_0, that is, all multiples of b_0 on the generator are non-zero in the calculated range, but they have not been written down in an effort to make the chart less cluttered. Certain generators have been given names. The Massey product representatives are the ones produced by using the full tags produced by the Curtis algorithm.

Figure 4 is a chart of the BSS

$$\mathrm{Ext}_{A_*//E[\tau_0,\tau_1,\tau_2]}(\mathbb{F}_3,\mathbb{F}_3) \otimes P[v_2] \Rightarrow \mathrm{Ext}_{A_*//E[\tau_0,\tau_1]}(\mathbb{F}_3,\mathbb{F}_3)$$

It is straightforward to calculate the differentials of this spectral sequence completely by explicitly finding the differentials in the periodic lambda algebra $\overline{\Lambda}_{(1)}$ and then finding the representatives in the Curtis table.

In figure 4, the E_1-term consists of $H^*(A_*//E[\tau_0,\tau_1,\tau_2]) \otimes P[v_2]$ which is isomorphic to $H^*(P_*) \otimes P[v_2, v_3]$ in our range of computation. It is implicit in the chart that every generator supports a $P[v_2]$, but all v_2 multiples are omitted unless they are targets of differentials, or otherwise contribute to hidden extensions. When v_2 multiples are displayed, they are represented by dash-dot lines. Hidden extensions are represented by dashed lines, and differentials are represented by negatively sloped solid lines.

We now have computed the E_2 term of the ASS. It is displayed in figure 5. unlike in figure 4, in this chart v_2 is not implicit unless specifically indicated. Small solid dots on the chart represent, like all of the previous charts, $\mathbb{F}_3$-basis elements. Small circles represent polynomial algebras on v_2. Otherwise v_2 multiplication is represented explicitly by dash-doted lines. It should be noted that if an element is represented by a circle, it does not mean that that element supports infinitely many non-trivial multiplications by v_2. It just means that throughout the indicated range, all multiplications by v_2 are non-trivial.

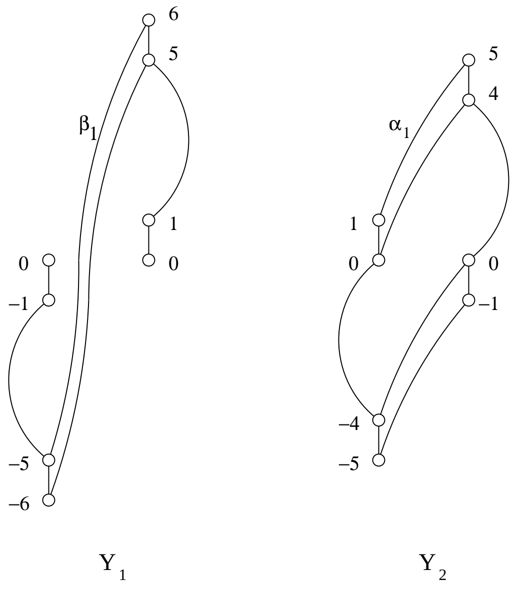

FIGURE 6. The irreducible subcomplexes of $D(V(1)) \wedge V(1)$

3. THE SPLITTING OF $D(V(1)) \wedge V(1)$

The complex $V(1)$ may be visualized with the following cell diagram.

Here the uncurved lines represent the Bockstein β (attaching map $\cdot 3$) and the curved line represents the Steenrod operation P^1 (attaching map α_1). The top $V(0)$ is attached to the bottom $V(0)$ by v_1, but this is not explicitly indicated in the cell diagram. Let $D(V(1)) \simeq \Sigma^{-6}V(1)$ be the Spanier-Whitehead dual of $V(1)$. In this section we will decompose $D(V(1)) \wedge V(1)$ into irreducible subcomplexes. Of course, since $V(1)$ is self dual, we have also provided a splitting of $V(1) \wedge V(1)$.

Define finite complexes Y_1 and Y_2 as follows.

$$Y_1 = \text{cofiber}\left(\Sigma^{-1}V(1) \xrightarrow{\nu} \Sigma^4 V(0) \xrightarrow{\beta_1} \Sigma^{-6}V(1)\right)$$

$$Y_2 = \text{cofiber}\left(\Sigma^{-2}V(1) \xrightarrow{\alpha_1} \Sigma^{-5}V(1)\right)$$

Here ν is projection onto the top $V(0)$. Figure 6 displays cell diagrams of these complexes.

We will prove the following

Proposition 3.1. There is a splitting

$$D(V(1)) \wedge V(1) \simeq Y_1 \vee Y_2.$$

Proof. Since $D(V(1)) \wedge V(1) \simeq \Sigma^{-6}V(1) \wedge V(1)$, it suffices to split the latter. Consider the twist map

$$\tau : V(1) \wedge V(1) \to V(1) \wedge V(1)$$

The map τ decomposes $V(1) \wedge V(1)$ into $+1$ and -1 eigenspaces T_+ and T_-. We claim that these are $\Sigma^6 Y_1$ and $\Sigma^6 Y_2$, respectively. More precisely, the self-maps $(1+\tau)/2$ and $(1-\tau)/2$ are idempotents on $V(1) \wedge V(1)$, and thus give a splitting

$$V(1) \wedge V(1) \simeq T_+ \vee T_-.$$

The structure of T_- is straightforward from the action of the Steenrod algebra. To prove the existence of the β_1 attaching map in T_+, we will use the secondary cohomology operation ϕ corresponding to the Adem relation $P^2 P^1 = 0$. This secondary operation detects β_1. In [23], Thomas proves the Cartan formula

$$\phi(xy) = \phi(x)y + x\phi(y) + (P^1\beta(x))(\beta P^1 \beta(y)) + (\beta P^1 \beta(x))(P^1 \beta(y)).$$

Let e_i denote the generator of $H^*(V(1))$ in dimension i. Evaluating ϕ on $e_0 \wedge e_0$, we get

$$\phi(e_0 \wedge e_0) = 0 \wedge e_0 + e_0 \wedge 0 + e_5 \wedge e_6 + e_6 \wedge e_5 = e_5 \wedge e_6 + e_6 \wedge e_5.$$

Similarly, we see $\phi(e_1 \wedge e_0 + e_0 \wedge e_1) = -e_6 \wedge e_6$. □

4. The product rule

The statement of the product rule will require some notation related to Adams resolutions which we will give presently. Let $\overline{H}$ be the fiber of the unit $\eta : S^0 \to H$, where H is the Eilenberg-MacLane spectrum $H\mathbb{F}_3$. The standard Adams resolution is defined by letting $W_s = \overline{H}^{(s)}$ and then defining $W_{s,r}$ to be the cofiber $\overline{H}^{(s)}/\overline{H}^{(s+r)}$. In particular, $W_{s,\infty} = W_s = \overline{H}^{(s)}$ and $W_{s,1} = H \wedge \overline{H}^{(s)}$. For a spectrum X the resolution may be written as

$$\begin{tikzcd}
X \arrow[d] & W_1 \wedge X \arrow[l] \arrow[d] & W_2 \wedge X \arrow[l] \arrow[d] & \cdots \arrow[l] \\
W_{0,1} \wedge X & W_{1,1} \wedge X & W_{2,1} \wedge X &
\end{tikzcd}$$

The ASS for $\pi_*(W_{s,r})$ is a truncated version of the ASS for $\pi_*(X)$; it is the spectral sequence obtained by only including the $E_1^{i,j}$ terms for $s \le i < s+r$, and omitting differentials supported on $E_r^{i,j}$ for $i < s$. There are several important maps relating the spectra $W_{s,r}$.

$$\begin{aligned}
f_{s,r,k} &: W_{s+r,k} \to W_{s,r+k} \\
g_{s,r,k} &: W_{s,r+k} \to W_{s,r} \\
\partial_{s,r,k} &: W_{s,r} \to \Sigma W_{s+r,k} \\
\mu_{s_1,s_2,r} &: W_{s_1,r} \wedge W_{s_2,r} \to W_{s_1+s_2,r}
\end{aligned}$$

μ is induced by the product on E. The remaining maps are compatible in all of the ways one might expect them to be, and the sequence

$$W_{s+r,k} \xrightarrow{f} W_{s,r+k} \xrightarrow{g} W_{s,r} \xrightarrow{\partial} \Sigma W_{s+r,k}$$

is a cofiber sequence.

It is easy to see that an element $x \in \pi_*(W_{s,1})$ persists to the E_r term of the ASS if and only if it lifts to an element $\tilde{x} \in \pi_*(W_{s,r})$. In fact, we have

$$E_r^{s,t} = \frac{\mathrm{Im}\{\pi_{t-s}(W_{s,r}) \to \pi_{t-s}(W_{s,1})\}}{\mathrm{Im}\{\pi_{t-s+1}(W_{s-r,r}) \xrightarrow{\partial} \pi_{t-s}(W_{s,r}) \to \pi_{t-s}(W_{s,1})\}}$$

and $d_r(x)$ is computed as $\partial(\tilde{x}) \in \pi_*(W_{s+r,1})$.

Define, for x_i in $\pi_*(V(1))$,

$$F(x_1,x_2) = \overline{x_1} \cdot \overline{\overline{x_2}} + (-1)^{|x_1|}\overline{\overline{x_1}} \cdot \overline{x_2} \in \pi_*(V(0))$$

where $\overline{x_i}$ is the image of x_i in $\pi_*(V(0))$ under the projection $V(1) \to \Sigma^5 V(0)$, $\overline{\overline{x_i}}$ is the image of $\overline{x_i}$ in $\pi_*(S)$ under the projection $V(0) \to S^1$. If $x_i \in \pi_*(W_{s_i,r} \wedge V(1))$, then $F(x_1,x_2)$ will be regarded as an element of $\pi_*(W_{s_1+s_2,r} \wedge V(0))$.

Theorem 4.1 (Product Rule). Suppose $x_i \in E_1(V(1))$ persist to the E_r-term of the ASS and $\beta_1 \cdot F(x_1,x_2) = 0$, thought of an element of $\pi_*(W_{s+2,r-2} \wedge V(1))$ (the product is induced from the $V(0)$-module structure of $V(1)$). Then it follows that $\beta_1 \cdot F(x_1,x_2) \in \pi_*(W_{s+2,r-1} \wedge V(1))$ lifts to an element $G(x_1,x_2)$ in $\pi_*(W_{s+r,1} \wedge V(1))$ and we have the following formula for $d_r(x_1x_2)$.

$$d_r(x_1x_2) = (d_rx_1)\cdot x_2 + (-1)^{|x_1|}x_1 \cdot (d_rx_2) - G(x_1,x_2)$$

Example 4.2. We will use the product rule to compute $d_3(v_2^2)$. The element v_2 is a permanent cycle for dimensional reasons. We have $\overline{v_2} = \widetilde{\beta_1} = h_1$ and $\overline{\overline{v_2}} = \beta_1 = b_0$. Here we have given the ASS names of these elements. Therefore,

$$G(v_2,v_2) = (h_1b_0 + b_0h_1)b_0 = -h_1b_0^2$$

so the product rule says that

$$d_3(v_2^2) = d_3(v_2)v_2 + v_2d_3(v_2) - G(v_2,v_2) = h_1b_0^2.$$

Of course, one can also use the Hurewicz homomorphism $V(1) \to eo_2 \wedge V(1)$ to get this formula.

Proof of product rule. Let X be the cofiber of $\beta_1 : \Sigma^{10}V(0) \to V(1)$. The closest substitute for a product on $V(1)$ is the map

$$\mu : V(1) \wedge V(1) \to X$$

formed by projecting onto the wedge summand Y_1 (Proposition 3.1) and collapsing out the bottom two cells of the top $V(1)$. If y_i are elements of $\pi_*(V(1))$, then the image of $y_1 \wedge y_2 \in \pi_*(V(1) \wedge V(1))$ under the composition

$$V(1) \wedge V(1) \to X \to \Sigma^{11}V(0)$$

is $F(y_1,y_2)$.

We shall need various filtered forms of X. Define $X_{s,r} = W_{s,r} \wedge X$, and define $\widetilde{X}_{s,r}$ and $\widetilde{\widetilde{X}}_{s,r}$ to be the following cofibers.

$$\Sigma^{10}W_{s,r} \wedge V(0) \xrightarrow{\beta_1} W_{s+2,r-2} \wedge V(1) \to \widetilde{X}_{r,s}$$
$$\Sigma^{10}W_{s,r} \wedge V(0) \xrightarrow{\beta_1} W_{s+2,r-1} \wedge V(1) \to \widetilde{\widetilde{X}}_{r,s}$$

Then we have the following cofiber sequences (by Verdier's axiom).

$$\widetilde{X}_{s,r} \to X_{s,r} \to W_{s,2} \wedge V(1)$$
$$\widetilde{\widetilde{X}}_{s,r} \to \widetilde{X}_{s,r} \to \Sigma W_{s+r,1} \wedge V(1)$$
$$\widetilde{X}_{s,r+1} \to \widetilde{X}_{s,r} \to \Sigma X_{s+r,1}$$
$$\widetilde{X}_{s,r+1} \to \widetilde{\widetilde{X}}_{s,r} \to \Sigma^{12}W_{s+r,1} \wedge V(0)$$

Since the Adams filtration of β_1 is greater than 0, $H \wedge X \simeq H \wedge V(1) \vee H \wedge \Sigma^{11} V(0)$, thus $X_{s,1}$ splits in a similar manner. We need a splitting map that behaves well with respect to the other maps floating about. Consider the splitting j induced on the cofibers below (the rows are cofiber sequences).

$$\begin{tikzcd}
\widetilde{X}_{s,r+1} \arrow[r] \arrow[d, equal] & \widetilde{\widetilde{X}}_{s,r} \arrow[r] \arrow[d] & \Sigma^{12} W_{s+r,1} \wedge V(0) \arrow[d, dotted, "j"] \arrow[dd, bend left=60, "Id"] \\
\widetilde{X}_{s,r+1} \arrow[r] & \widetilde{X}_{s,r} \arrow[r] \arrow[d] & \Sigma X_{s+r,1} \arrow[d] \\
 & \Sigma^{12} W_{s+r,1} \wedge V(0) \arrow[r, equal] & \Sigma^{12} W_{s+r,1} \wedge V(0)
\end{tikzcd}$$

We consider j to be a nice splitting, because the projection ν it induces onto the other wedge summand fills in the following triad of cofiber sequences.

$$\begin{tikzcd}
\widetilde{X}_{s,r+1} \arrow[r] \arrow[d, equal] & \widetilde{\widetilde{X}}_{s,r} \arrow[r] \arrow[d] & \Sigma^{12} W_{s+r,1} \wedge V(0) \arrow[d, dotted, "j"] \\
\widetilde{X}_{s,r+1} \arrow[r] \arrow[d] & \widetilde{X}_{s,r} \arrow[r, "\partial_1"'] \arrow[d, "\partial_2"'] & \Sigma X_{s+r,1} \arrow[d, dotted, "\nu"] \\
* \arrow[r] & \Sigma W_{s+r,1} \wedge V(1) \arrow[r, equal] & \Sigma W_{s+r,1} \wedge V(1)
\end{tikzcd} \tag{4.1}$$

Let $\widetilde{x}_i \in \pi_*(W_{s_i,r})$ be lifts of $x_i \in \pi_*(W_{s_i,1})$. Let $s = s_1 + s_2$. It is useful to keep in mind the following diagram.

$$\begin{tikzcd}
W_{s,1} \wedge V(1) \arrow[d, "\iota"] & W_{s,r} \wedge V(1) \arrow[l, "g"'] \arrow[r, "\partial"'] \arrow[d, "\iota"'] & \Sigma W_{s+r,1} \wedge V(1) \arrow[d, "\iota"'] \\
X_{s,1} \arrow[u, bend left, "\nu"] & X_{s,r} \arrow[l, "g"'] \arrow[r, "\partial"'] & \Sigma X_{s+r,1} \arrow[u, bend right, "\nu"'] \\
W_{s_1,1} \wedge W_{s_2,1} \wedge V(1)^{(2)} \arrow[u, "\mu"'] & W_{s_1,r} \wedge W_{s_2,r} \wedge V(1)^{(2)} \arrow[l, "g"'] \arrow[r, "\partial"'] \arrow[u, "\mu"'] & \begin{matrix} \Sigma W_{s_1+r,1} \wedge W_{s_2,r} \wedge V(1)^{(2)} \\ \vee \Sigma W_{s_1,r} \wedge W_{s_2+r,1} \wedge V(1)^{(2)} \end{matrix} \arrow[u, "\mu"']
\end{tikzcd}$$

The element $x_1 \wedge x_2 \in \pi_*(W_{s_1,1} \wedge W_{s_2,1} \wedge V(1)^{(2)})$ lifts to $\widetilde{x_1} \wedge \widetilde{x_2} \in \pi_*(W_{s_1,r} \wedge W_{s_2,r} \wedge V(1)^{(2)})$. We then have

$$\partial(\widetilde{x_1} \wedge \widetilde{x_2}) = d_r(x_1) \wedge x_2 + (-1)^{|x_1|} x_1 \wedge d_r(x_2).$$

The element $x_1 \cdot x_2$ is equal to $\nu \circ \mu(x_1 \wedge x_2) \in \pi_*(W_{s,1} \wedge V(1))$. We want to compute $d_r(x_1 \cdot x_2)$, which means we first need to lift $x_1 \cdot x_2$ to $\pi_*(W_{s,r} \wedge V(1))$. Now $\mu(\widetilde{x_1} \wedge \widetilde{x_2})$ is a lift of $\mu(x_1 \wedge x_2)$, but this element will not lift to a lift of $x_1 \cdot x_2$ without a little modification. The following sequence is exact.

$$\pi_*(\widetilde{X}_{s,r}) \to \pi_{*-11}(W_{s,r} \wedge V(0)) \to \pi_{*-1}(W_{s+2,r-2} \wedge V(1))$$

Our assumption that $\beta_1 \cdot F(\widetilde{x_1}, \widetilde{x_2}) \in \pi_*(W_{s+2,r-2} \wedge V(1))$ is trivial implies that $F(\widetilde{x_1}, \widetilde{x_2})$ lifts to an element $\widetilde{F} \in \pi_*(\widetilde{X}_{s,r})$. Let y be the image of $\widetilde{F}$ in $X_{s,r}$, and define

$$z = \mu(\widetilde{x_1} \wedge \widetilde{x_2}) - y \in \pi_*(X_{s,r}).$$

We claim that (1) z lifts to $\widetilde{z} \in \pi_*(W_{s,r} \wedge V(1))$, and (2) $\widetilde{z}$ is a lift of $x_1 \cdot x_2 \in \pi_*(W_{s,1} \wedge V(1))$.

With regard to claim (1), we need only check that the image of z in $\pi_*(\Sigma^{11} W_{s,r} \wedge V(0))$ is zero. The image of both $\mu(\widetilde{x_1} \wedge \widetilde{x_2})$ and y in $\pi_*(\Sigma^{11} W_{s,r} \wedge V(0))$ is $F(\widetilde{x_1}, \widetilde{x_2})$, therefore the image of z, their difference, is zero. Claim (2) is established by noting that the sequence

$$\pi_*(\widetilde{X}_{s,r}) \to \pi_*(X_{s,r}) \to \pi_*(W_{s,2} \wedge V(1))$$

is exact. Therefore the image of y in $\pi_*(W_{s,2} \wedge V(1))$ is zero, so its image $\nu \circ g(y) \in \pi_*(W_{s,1} \wedge V(1))$ is zero. So, we have

$$g(\widetilde{z}) = \nu \circ \iota \circ g(\widetilde{z}) = \nu \circ g(z) = \nu \circ \mu(x_1 \wedge x_2) = x_1 \cdot x_2$$

and claim (2) is established.

We are left with identifying $\partial(\widetilde{z})$. We have

$$\partial \widetilde{z} = \nu \circ \partial(\mu(\widetilde{x_1} \wedge \widetilde{x_2}) - y) = d_r(x_1) \cdot x_2 + (-1)^{|x_1|} x_1 \cdot d_r(x_2) - \nu \circ \partial(y)$$

We must evaluate $\nu \circ \partial(y)$. In diagram 4.1 the boundary maps ∂_1 and ∂_2 are displayed. There is a map of cofiber sequences relating ∂_1 to ∂ in the commutative diagram displayed below.

$$\begin{tikzcd}
\widetilde{X}_{s,r} \arrow[rr] \arrow[dr, "\partial_1"'] & & X_{s,r} \arrow[dl, "\partial"] \\
& \Sigma X_{s+r,1} &
\end{tikzcd}$$

Therefore, $\partial y = \partial_1 \widetilde{F}$. Furthermore, diagram 4.1 reveals the relationship between ∂_1 and ∂_2. Thus we have $\nu \circ \partial_1(\widetilde{F}) = \partial_2(\widetilde{F})$, and we just need an explicit description of the latter. The map of cofiber sequences

$$\begin{tikzcd}
W_{s+r,1} \wedge V(1) \arrow[r] \arrow[d, "f"'] & \widetilde{\widetilde{X}}_{s,r} \arrow[r] \arrow[d, equal] & \widetilde{X}_{s,r} \arrow[r, "\partial_2"'] \arrow[d] & \Sigma W_{s+r,1} \wedge V(1) \arrow[d, "f"'] \\
W_{s+2,r-1} \wedge V(1) \arrow[r] & \widetilde{\widetilde{X}}_{s,r} \arrow[r] & \Sigma^{11} W_{s,r} \wedge V(0) \arrow[r, "\beta_1"'] & \Sigma W_{s+2,r-1} \wedge V(1)
\end{tikzcd}$$

tells us that $\partial_2(\widetilde{F})$ is a lift of $\beta_1 \cdot F(\widetilde{x_1}, \widetilde{x_2})$ to $\pi_*(W_{s+r,1} \wedge V(1))$ and as such, deserves to be called $G(x_1, x_2)$. This completes our verification of the formula. $\square$

Remark 4.3. The theorem holds under a weaker assumption. The proof of the theorem does not require x_1 and x_2 to survive to E_r, but only that $\partial(x_1) \cdot x_2 + (-1)^{|x_1|} x_1 \cdot \partial(x_2)$ have Adams filtration greater than or equal to $s+r$. We will need this technical generalization for some of our applications of the product rule.

5. The order of the β_1 action on $V(1)$

In this section we will prove the following proposition.

Proposition 5.1. The map

$$\beta_1^5 : \Sigma^{50} V(1) \to V(1)$$

induced from smashing the map $\beta_1^5 : S^{50} \to S^0$ with $V(1)$ is null.

Corollary 5.2. Regarding $\pi_*(V(1))$ as a module over $\pi_*(S)$, we have the relation

$$\beta_1^5 \cdot x = 0$$

for all $x \in \pi_*(V(1))$.

We remark that in $\pi_*(S)$ we have the relation $\beta_1^6 = 0$, and β_1^5 is non-zero. The power of β_1^5 in Proposition 5.1 is minimal, since in $\pi_*(V(1))$ the image of the element β_1^4 under the inclusion of the bottom cell is non-trivial.

Corollary 5.2 follows from Proposition 5.1 since the element $\beta_1^5 \cdot x$ may be expressed as the following composite.

$$S^{50+k} \xrightarrow{x} \Sigma^{50} V(1) \xrightarrow{\beta_1^5} V(1)$$

We will first prove the following lemma.

Lemma 5.3. The element β_1^5 in $\pi_{50}(V(1))$ is trivial.

Proof. We must show that the element b_0^5 in the ASS for $\pi_*(V(1))$ is the target of a differential. In the ASS for $\pi_*(eo_2 \wedge V(1))$ there is a differential

$$d_6(v_2^3 h_0) \doteq b_0^5.$$

Using the results of Proposition 1.5, we may conclude that if $v_2^3 h_0$ supports no shorter differentials in the ASS for $V(1)$, then it must kill b_0^5. Upon investigating the E_2 term of the ASS for $V(1)$, we see that there is no element in smaller Adams filtration that could be the target of a shorter differential. $\square$

Proof of Proposition 5.1. We will demonstrate that the Spanier-Whitehead adjoint of β_1^5

$$\beta_1^5 : S^{50} \to D(V(1)) \wedge V(1)$$

is null. Let X be the fiber of the composite

$$V(1) \to \Sigma^5 V(0) \xrightarrow{\beta_1} \Sigma^{-5} V(0)$$

where the first arrow is projection onto the top $V(0)$. By Proposition 3.1, X may be regarded as a subcomplex of Y_1, which may in turn be regarded as a subcomplex of $D(V(1)) \wedge V(1)$. We wish to show that the composite

$$S^{50} \xrightarrow{\beta_1^5} S^0 \hookrightarrow X \hookrightarrow Y_1 \hookrightarrow D(V(1)) \wedge V(1)$$

is null. We will show that the shorter composite $S^{50} \to Y_1$ is null.

Consider the following diagram, whose two bottom rows are cofiber sequences.

$$\begin{tikzcd}
& S^{50} \arrow[d, "\beta_1^5"'] \arrow[ddl, dotted, "f"'] \arrow[ddr, "*"] & \\
& S^0 \arrow[d] & \\
\Sigma^{-6}V(0) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & V(1) \arrow[d, equal] \\
\Sigma^{-6}V(1) \arrow[r] & Y_1 \arrow[r] & V(1)
\end{tikzcd}$$

In this diagram, the map $S^{50} \to V(1)$ is null by Lemma 5.3. Therefore the lift f exists making the diagram commute. We will complete the proof of the proposition once we establish the following

Claim. The image of the map $\pi_{56}(V(0)) \to \pi_{56}(V(1))$ is trivial.

The claim follows easily from the AHSS for $\pi_*(V(0))$ and $\pi_*(V(1))$. A portion of the AHSS for $\pi_*(V(0))$ is displayed below (elements are given their E_1 name in the form $\gamma[n]$ where $\gamma \in \pi_*(S)$ and n is the cell supporting it).

Stem 55	Stem 56	Stem 57
$\alpha_{14}[0]$	$\alpha_{14}[1]$	
$\beta_2^2\alpha_1[0]$	$\beta_2^2\alpha_1[1]$	

There are no differentials, and $\pi_{56}(V(0))$ is of rank 2. We now consider the image in $V(1)$. The same portion of the AHSS for $V(1)$ is displayed below, with differentials.

Stem 55	Stem 56	Stem 57
$\alpha_{14}[0]$	$\alpha_{14}[1]$	$\beta_2^2[5]$
$\beta_2^2\alpha_1[0]$	$\beta_2^2\alpha_1[1]$	$\alpha_{13}[6]$
$\beta_1^5[5]$	$\alpha_{13}[5]$	
$\beta_2\beta_1^2\alpha_1[6]$	$\beta_1^5[6]$	

The generators of $\pi_{56}(V(0))$ have been killed by differentials, and the claim follows. □

6. Proof that v_2^9 is a permanent cycle

In this section we will prove that the element v_2^9 is a permanent cycle in the ASS for $\pi_*(V(1))$. We will let $h : V(1) \to eo_2 \wedge V(1)$ be the Hurewicz homomorphism. We will also use the notation $E_r(X)$ for the E_r-term of the ASS for $\pi_*(X)$.

We will first use the product rule (4.1) to determine $d_2(v_2^i)$.

Lemma 6.1. There are the following differentials on v_2^i in the ASS.

$$\begin{array}{lll} d_*(v_2) = 0 & d_3(v_2^2) = h_1 b_0^2 & d_2(v_2^3) = -b_0 k_0 h_1 \\ d_2(v_2^4) = -b_0 k_0 h_1 v_2 & d_2(v_2^5) = -b_0 k_0 h_1 v_2^2 & d_2(v_2^6) = b_0 k_0 h_1 v_2^3 \\ d_2(v_2^7) = b_0 k_0 h_1 v_2^4 & d_2(v_2^8) = b_0 k_0 h_1 v_2^5 & d_2(v_2^9) = 0 \end{array}$$

Proof. These formulas are just obtained by iterated application of the product rule. The differential $d_3(v_2^2)$ is computed in this manner in Example 4.2. One then uses the following formulas in the ANSS, which are derived in [16, 5.1.20]

$$\overline{v_2^k} = \widetilde{\beta_k} \equiv k h_1 v_2^{k-1} \pmod{v_1}$$

$$\overline{\overline{v_2^k}} = \beta_k \equiv \binom{k}{2} v_2^{k-2} k_0 + k v_2^{k-1} b_0 \pmod{3, v_1}$$

to inductively determine $d_*(v_2^{k+1})$ from $d_*(v_2^k)$. We should point out that this formula differs by a sign from the formula in [16] because the elements we are referring to as b_0 and k_0 are normalized to their lambda algebra names, which make them the negatives of the elements b_0 and k_0 in [16]. □

Remark 6.2. In [15], Oka demonstrates that v_2^5 is a permanent cycle in the ANSS for $\pi_*(V(1))$. Above, we have shown that it supports a d_2 in the ASS for $\pi_*(V(1))$. In fact, there is a differential

$$d_2(v_3 b_0 g_0) \doteq b_0 k_0 h_1 v_2^2$$

and $v_2^5 \pm v_3 b_0 g_0$ is a permanent cycle in the ASS. This differential is established in Lemma 7.3.

We must eliminate the possibility that v_2^9 supports a d_r for $r > 2$. We will make a list of all elements in the ASS in the 143-stem of Adams filtration greater than 11. It is given in the table below, with references to the lemma that takes care of it, as well as the Adams filtration (AF).

AF	Element	Lemma
29	$h_0 b_0^{14}$	6.3
23	$g_0 h_0 v_2^2 b_0^9$	6.3
22	$b_1 h_0 v_2 b_0^9$	6.4
18	$h_0 b_0^6 v_2^5$	6.3
	$\eta_1 v_2 b_0^7$	6.5
	$v_3 h_1 b_0^8$	6.6
17	$k_0 h_0 v_2^4 b_0^5$	6.9
	$\eta_1 k_0 b_0^6$	6.10
13	$v_3 h_0 b_0^4 v_2^3$	6.13
12	$g_0 h_0 b_0 v_2^7$	6.3
	$\eta_1 g_0 v_2^3 b_0^2$	6.15
	$v_3 k_0 h_0 v_2^2 b_0^3$	6.14

Lemma 6.3. If $x \in E_2(V(1))$ is an element of the ASS for $V(1)$, and its Hurewicz image $h(x) \in E_2(eo_2 \wedge V(1))$ is non-zero, then x cannot be the target of a differential supported by v_2^9.

Proof. We have $h(d_r(v_2^9)) = d_r h(v_2^9)$, but $h(v_2^9) \in E_2(eo_2 \wedge V(1))$ is a permanent cycle, so $d_r(v_2^9)$ must be in the Hurewicz kernel. $\square$

Lemma 6.4. Suppose that $y \in E_6(V(1))$. Then $h_0 b_0^3 y = 0$ in $E_6(V(1))$. Similarly, if $z \in E_7(V(1))$, then $b_0^6 z = 0$ in $E_7(V(1))$. In particular, if x is an element of $E_2(V(1))$ and $x = h_0 b_0^3 y$ or $x = b_0^6 z$ for y or z as before, than x is not the target of a non-trivial d_r for $r \geq 6$ or $r \geq 7$, respectively.

Proof. The element $h_0 b_0^3$ dies in $E_6(S^0)$ (this is just the Toda differential $d_5 b_1 \doteq h_0 b_0^3$). Similarly, there is a differential $d_6(h_0 b_0 v_2^3) \doteq b_0^6$ giving the relation $b_0^6 = 0$ in $E_7(S^0)$. Then use the S^0-module structure of $V(1)$. $\square$

The following lemmas take care of the other possible targets. We work from highest to lowest Adams filtration to eliminate the possibility of intervening differentials as we go along.

Lemma 6.5. In $E_5(V(1))$ there is a non-trivial differential

$$d_5(\eta_1 v_2 b_0^7) \doteq k_0 v_2 b_0^{10}.$$

Proof. In the $E_5(S^0)$, we have $d_5 \eta_1 \doteq k_0 b_0^3$. $v_2 b_0^7$ is a permanent cycle in the ASS for $V(1)$, so the differential follows from the S-module structure of $V(1)$. $\square$

Lemma 6.6. In $E_4(V(1))$ there is a non-trivial differential

$$d_4(v_3 h_1 b_0^8) \doteq b_0^9 k_0^2.$$

Proof. We will first prove $\pi_{72}(V(1)) = 0$. Consider the AHSS for computing $\pi_*(V(1))$ from $\pi_*(S^0)$. The computation of $\pi_{72}(V(1))$ is summarized in the chart below. The notation $x[k]$ for $x \in \pi_*(S^0)$ indicates the element of the E_1-term born on the k-cell of $V(1)$.

$$\begin{tikzcd}
\underline{\text{Stem 71}} & \underline{\text{Stem 72}} & \underline{\text{Stem 73}} \\
\alpha_{18/3}[0] & \beta_2^2\beta_1^2[0] & \beta_2^2\beta_1^2[1] \\
 & \alpha_{18/3}[1] \arrow[ul, "\cdot 3"'] & x_{68}[5] \arrow[ul] \\
 & \alpha_{17}[5] \arrow[uul] & \alpha_{17}[6] \arrow[ul]
\end{tikzcd}$$

The d_1's above are given by multiplication by 3, and the d_5's by the Toda bracket $\langle 3, \alpha_1, - \rangle$. The only differential which isn't clear is $d_5(x_{68}[5])$. We compute

$$\alpha_1 \langle 3, \alpha_1, x_{68} \rangle \doteq 3 \langle \alpha_1, \alpha_1, x_{68} \rangle \neq 0.$$

This is a hidden extension in the ANSS for $\pi_*(S^0)$ in the computations in [16]. The indeterminacy of $\langle 3, \alpha_1, x_{68} \rangle$ is trivial, and the indeterminacy of $\langle \alpha_1, \alpha_1, x_{68} \rangle$ is contained in $\alpha_1 \cdot \pi_{72}(S^0) = 3 \cdot \pi_{75}(S^0)$, so it doesn't enter into the above computation. We conclude that $\langle 3, \alpha_1, x_{68} \rangle \neq 0$, so it has no choice but to be a non-zero multiple of $\beta_2^2 \beta_1^2$.

Therefore, the ASS for $\pi_*(V(1))$ should have no non-trivial permanent cycles in the 72-stem. The E_2 term contains $k_0^2 b_0^2$, $b_0^3 k_0 v_2$, and $b_0^4 v_2^2$. The element $b_0^4 v_2^2$ supports a non-trivial d_3, and $d_5 \eta_1 v_2 \doteq b_0^3 k_0 v_2$. The only possibility for eliminating $k_0^2 b_0^2$ is for $d_4(v_3 h_1 b_0) \doteq k_0^2 b_0^2$. Therefore $d_4(b_0^8 v_3 h_1) \doteq k_0^2 b_0^9$. $\square$

For Lemma 6.9 we need to know the differentials supported by $v_2^i g_0$ for small i. These are given below.

Lemma 6.7. We have the following Adams differentials on $g_0v_2^i$ in $E_r(V(1))$.

$$\begin{array}{lll} d_4(g_0v_2) = b_0^3h_0 & d_3(g_0v_2^2) = b_0^2k_0h_0 & d_2(g_0v_2^3) = -g_0b_0k_0h_1 \\ d_3(g_0v_2^4) = 0 & & d_3(g_0v_2^6) = v_2^4b_0^2k_0h_0 \end{array}$$

Remark 6.8. In the proof of Lemma 6.12, the element $g_0v_2^4$ is shown to be a permanent cycle, and this should be regarded as anomalous. It is related to the anomalous relation $\alpha_1\beta_5 = 0$ in $\pi_*(S)$.

Proof. We will first explain how the term $G(g_0v_2^i, v_2)$ and the term $G(g_0v_2^i, \overline{v_2^2})$ in the product rule is computed. The Adams-Novikov element which detects $\overline{g_0v_2^i}$ is given by

$$v_2^ib_0h_0 - iv_2^{i-1}k_0h_0 \pmod{v_1}.$$

(We will just work modulo (v_1) since we will be mapping everything into $V(1)$ for the product rule anyways.) The Adams-Novikov element which detects $\overline{\overline{g_0v_2^i}}$ is given by

$$\binom{i}{2}v_2^{i-2}k_0g_0 + iv_2^{i-1}b_0g_0 \pmod{v_1}.$$

We recall from Lemma 6.1 the following formulas.

$$\begin{array}{ll} \overline{v_2} = h_1 & \overline{v_2^2} = -h_1v_2 \\ \overline{\overline{v_2}} = b_0 & \overline{\overline{v_2^2}} = k_0 - b_0v_2 \end{array}$$

We remind the reader in what follows that there are relations

$$\begin{array}{ll} h_1b_0 = h_0g_0 & h_1g_0 = h_0k_0 \\ h_1k_0 = h_0b_1 & v_2 \cdot b_1h_1 = 0 \end{array}$$

Applying the product rule (4.1) iteratively and applying these formulas now gives the requisite differentials. Specifically, first apply the product rule to $g_0 \cdot v_2$, to get $d_4(v_2g_0)$, then apply the product rule to $(g_0v_2) \cdot v_2$ to get $d_3(g_0v_2^2)$. In $E_2(S^0)$, g_0 supports a d_2, and v_2^3 supports a d_2 in $E_2(V(1))$, thus $d_2(v_2^3g_0)$ may be deduced from the S-module pairing of Adams spectral sequences. The problem is, we can no longer apply the product rule and v_2 multiplication to get $d_3v_2^4g_0$. However, we may instead apply the product rule to the product $v_2^2 \cdot v_2^2g_0$ to get the formula for $d_3(v_2^4g_0)$ and similarly to $v_2^2 \cdot v_2^4g_0$ to get the formula for $d_3(v_2^6g_0)$. $\square$

In particular, we have the following lemma, which follows immediately.

Lemma 6.9. In $E_3(V(1))$ there is a differential

$$d_3(v_2^6b_0^3g_0) = k_0h_0v_2^4b_0^5.$$

Lemma 6.10. In $E_5(V(1))$ there is a differential

$$d_5(v_2^4h_0b_0^2\eta_1) \doteq \eta_1k_0b_0^6.$$

Proof. We need to compute the differential supported by $v_2^4h_0$. Observe that the differential $d_6(v_2^3h_0) \doteq b_0^5$ in the ASS for $eo_2 \wedge V(1)$ lifts to a differential in $E_6(V(1))$. The element $v_2^2h_0$ must be a permanent cycle since there is nothing for it to kill. Hence, by the product rule,

$$b_0^5 \doteq d_6(v_2^3h_0) = -G(v_2^2h_0, v_2) = -b_0(\overline{h_0v_2^2}b_0 - \overline{\overline{h_0v_2^2}}h_1).$$

We conclude that the image of $\overline{h_0v_2^2}$ in $\pi_*(V(1))$ is $\pm\beta_1^3$. The image of $\overline{\overline{h_0v_2^2}}$ in $\pi_*(V(1))$ has to be in Adams filtration greater than or equal to that of the image of $\overline{h_0v_2^2}$, and so we may conclude that the image of $\overline{\overline{h_0v_2^2}}$ is actually zero. Whereas $d_3(v_2^2)$ is non-zero, the differential $(d_3(v_2^2))v_2^2h_0$ is zero, and there are no permanent cycles in higher Adams filtration, so $(\partial(v_2^2))\cdot v_2^2h_0 = 0$, and we are in a position to use the version of the product rule explained in Remark 4.3. Since $\overline{\overline{v_2^2}}$ is detected by k_0, we have

$$d_5(v_2^4h_0) = -G(v_2^2, v_2^2h_0) = -b_0(\overline{v_2^2}\cdot\overline{\overline{v_2^2h_0}} + \overline{\overline{v_2^2}}\cdot\overline{v_2^2h_0}) \doteq -b_0(k_0\cdot b_0^3) = -k_0b_0^4.$$

We then use the S-module structure of $V(1)$, and the differential $d_5\eta_1 \doteq k_0b_0^3$ to obtain

$$d_5(\eta_1\cdot v_2^4h_0b_0^2) = \pm k_0h_0b_0^5v_2^4 \pm \eta_1k_0b_0^6.$$

Of course, by Lemma 6.9, $k_0h_0b_0^5v_2^4$ is the target of a d_3. □

We will need a couple of lemmas to prove Lemmas 6.14 and 6.13.

Lemma 6.11. In $E_4(V(1))$ there is a differential

$$d_4(v_3h_0b_0) \doteq b_1b_0^3.$$

Proof. We will compute $\pi_{64}(V(1))$ and $\pi_{65}(V(1))$ using the AHSS. The differentials are given in the chart below.

$$\begin{tikzcd}
\underline{\text{Stem 63}} & \underline{\text{Stem 64}} & \underline{\text{Stem 65}} & \underline{\text{Stem 66}} \\
\alpha_{16}[0] & \alpha_{16}[1] & \beta_2^2\beta_1\alpha_1[0] & \beta_2^2\beta_1\alpha_1[1] \\
\beta_2^2\beta_1[1] & \alpha_{15/2}[5] \arrow[ul] & \alpha_{15/2}[6] \arrow[ul] \arrow[l, "\cdot 3"'] &
\end{tikzcd}$$

In the Adams spectral sequence for $\pi_*(V(1))$ in these stems, we have the following elements and differentials.

$$\begin{tikzcd}
\underline{\text{Stem 63}} & \underline{\text{Stem 64}} & \underline{\text{Stem 65}} & \underline{\text{Stem 66}} \\
b_0^6h_0 & b_0^3g_0v_2 \arrow[l] & b_0^3h_0v_2^2 & b_0^5v_2 \\
b_0^2v_2^2h_1 & b_1b_0^3 & b_0^2h_0k_0v_2 & b_0^4k_0 \\
b_1b_0h_0v_2 & v_2^4 \arrow[l] & b_0^2b_1h_1 & v_2^3g_0 \arrow[l] \\
v_3h_1 & & v_3h_0b_0 \arrow[uul, dotted, "(2)"] & b_1v_2^2 \arrow[uuul, dotted, "(1)"'] \arrow[uul, dotted]
\end{tikzcd}$$

Here we know for dimensional reasons that b_1, k_0, v_2, $h_0v_2^2$, and $h_0k_0v_2$ are permanent cycles, and so all b_0 multiples of them are permanent cycles. We have the solid differentials from Lemmas 6.1 and 6.7. The only means by which the correct homotopy can be achieved is for there to be one of the dotted differentials (1) to exist and the dotted differential (2) to exist. The differential (2) is the desired differential. □

Lemma 6.12. The element $v_2^3b_1$ is a permanent cycle.

Proof. We first compute $\pi_{81}(V(1))$ and $\pi_{82}(V(1))$ using the AHSS. The chart is displayed below.

$$
\begin{tikzcd}[row sep=small]
\underline{\text{Stem } 80} & \underline{\text{Stem } 81} & \underline{\text{Stem } 82} & \underline{\text{Stem } 83} \\
\alpha_{20}[1] & \langle \beta_5, \alpha_1, \alpha_1 \rangle[0] & \beta_{6/3}[0] & \alpha_{21/2}[0] \\
\alpha_{19}[5] & \gamma_2[0] & \langle \beta_5, \alpha_1, \alpha_1 \rangle[1] & \beta_{6/3}[1] \\
\frac{\beta_2^2\beta_1^2\alpha_1}{3}[5] & \alpha_{19}[6] \arrow[uul] & \gamma_2[1] & \beta_2^3[5] \\
\beta_2^2\beta_1^2\alpha_1[5] & \frac{\beta_2^2\beta_1^2\alpha_1}{3}[6] \arrow[l] & & \\
\beta_5[6] & \beta_2^2\beta_1^2\alpha_1[6] & &
\end{tikzcd}
$$

All of these differentials are completely straightforward, except that we must justify there is not a differential $d(\beta_2^3[5]) \doteq \beta_{6/3}[0]$. If there were such a differential we would have the following Toda bracket in $\pi_*(S)$.

$$\beta_{6/3} \in \pm\langle \beta_2^3, \alpha_1, 3 \rangle$$

But then we would have

$$\alpha_1\beta_{6/3} \in \pm\alpha_1\langle \beta_2^3, \alpha_1, 3 \rangle \doteq 3\langle \beta_2^3, \alpha_1, \alpha_1 \rangle = 0$$

which provides a contradiction, since $\alpha_1\beta_{6/3} \neq 0$.

We will now infer differentials in the ASS for $\pi_*(V(1))$. The result is displayed below.

$$
\begin{tikzcd}[row sep=small]
\underline{\text{Stem } 80} & \underline{\text{Stem } 81} & \underline{\text{Stem } 82} & \underline{\text{Stem } 83} \\
b_0^8 & b_0^7h_1 & b_0^5v_2^2 \arrow[l] & h_0b_0^8 \\
b_0^3g_0v_2^2 & b_0^3v_2^3h_0 \arrow[ul] & b_0^4k_0v_2 & b_0^4h_1v_2^2 \\
b_0^3b_1v_2 & b_0^2h_0k_0v_2^2 & b_0^3k_0^2 & b_0^3h_0b_1v_2 \\
b_0^2h_0\eta_1 & v_2v_3h_0b_0 \arrow[l] & g_0v_2^4 & \eta_1v_2b_0 \arrow[uul] \\
v_3g_0b_0 & v_3k_0h_0 & v_2^3b_1 & b_0^2v_3h_1 \arrow[uul] \\
v_2^5 & \gamma_2 & k_1 & h_0v_2^5 \arrow[uuul, dotted]
\end{tikzcd}
$$

The differentials shown above mostly follow from our previous work: $d(v_2^3h_0)$ in the proof of Lemma 6.10, $d(v_2^2)$ in Lemma 6.1, $d(v_3h_1b_0)$ in the proof of Lemma 6.6, and $d(\eta_1v_2)$ follows from $d_5(\eta_1) \doteq b_0^3k_0$ in $E_5(S^0)$, and the S-module action on $V(1)$. One cannot preclude the possibility that $h_0v_2^5$ supports the indicated dotted differential. In fact, it must support this differential, because $v_2^5 \pm v_3b_0g_0$ is a permanent cycle (see Remark 6.2 and Lemma 7.3), so the element

$$h_0 \cdot (v_2^5 \pm v_3b_0g_0) = v_2^5h_0 \pm v_3b_0^2h_1$$

must be a permanent cycle as well. The differential $d_3(v_2v_3h_0b_0) = b_0^2h_0\eta_1$ is derived using the product rule (4.1). One has the following formulas.

$$\overline{v_3h_0b_0} = -h_0\eta_1$$

$$\overline{\overline{v_3h_0b_0}} = \text{an element in Adams filtration} > 5$$

The first of these formulas may be derived from explicit computer-assisted lambda algebra manipulation. For the second formula, we know $filt(\overline{\overline{v_3 h_0 b_0}}) > 4$ since $h_0 \eta_1 \in E_*(V(0))$ is in the image of the inclusion of the bottom cell. The only element in Adams filtration 5 is $b_1 v_2 b_0$, yet in $E_4(V(1))$ we have $\overline{\overline{b_1 b_0 v_2}} = b_1 b_0^2 \neq 0$, thus $b_1 b_0 v_2$ in $E_4(V(1))$ cannot detect $\overline{\overline{v_3 h_0 b_0}}$, because if it did, its image under projection onto the top cell of $V(1)$ would be zero. The product rule now completes the computation of this differential.

$$d_3(v_3 h_0 b_0 \cdot v_2) = -b_0 G(v_3 h_0 b_0, v_2) = b_0^2 h_0 \eta_1$$

Our AHSS calculation shows us that $\pi_{81}(V(1))$ and $\pi_{82}(V(1))$ both must have rank 3, so there can be no more differentials. In particular $v_2^3 b_1$ must be a permanent cycle and corresponds to the element $\beta_{6/3}[0]$. As an aside related to the remark following Lemma 6.7, we point out that $g_0 v_2^4$ must also be a permanent cycle. It corresponds to the element $\langle \beta_5, \alpha_1, \alpha_1 \rangle [1]$ in the AHSS. □

Lemma 6.13. In $E_2(V(1))$ there is a non-trivial differential

$$d_2(v_3 h_0 b_0^4 v_2^3) \doteq v_2 h_0 \eta_1 k_0 b_0^4.$$

Proof. In Lemma 6.11, we showed that $d_2(v_3 h_0 b_0) = 0$. In Lemma 6.1, we showed that

$$d_2(v_2^3) = -b_0 k_0 h_1 = -b_0 b_1 h_0.$$

We also know $\overline{v_2^3}$, and hence $\overline{\overline{v_2^3}}$, are trivial in E_2 (they are in higher Adams filtration). We will now use the product rule (Theorem 4.1) to compute the differential on $(v_3 h_0 b_0) \cdot v_2^3$. The term $G(v_3 h_0 b_0, v_2^3)$ is trivial in E_2 by the previous considerations. The following manipulations are made possible with the hidden extension $h_0 \cdot (v_3 h_0 b_0) \doteq b_1 g_0 v_2$.

$$\begin{aligned} d_2((v_3 h_0 b_0) \cdot v_2^3) &= (v_3 h_0 b_0) \cdot (b_0 b_1 h_0) \\ &\doteq b_1^2 g_0 b_0 v_2 \\ &\doteq v_2 h_0 \eta_1 k_0 b_0 \end{aligned}$$

Now multiply by b_0^3. □

Lemma 6.14. In $E_5(V(1))$ there is a non-trivial differential

$$d_5(v_3 k_0 h_0 v_2^2 b_0^3) \doteq b_1 b_0^6 v_2^3.$$

Proof. In $E_2(V(1))$, there is a Massey product

$$v_2^3 b_1 b_0^3 \in \langle b_1 b_0^3, v_1, v_1^2 h_2 \rangle.$$

We are regarding the elements in the Massey product as detecting the following maps in homotopy.

$$\begin{aligned} b_1 b_0^3 &\leftrightarrow \beta_2 \beta_1^3 \alpha_1 [5] : \Sigma^{64} V(0) \to V(1) \\ v_1 &\leftrightarrow v_1 : \Sigma^4 V(0) \to V(0) \\ v_1^2 h_2 &\leftrightarrow \widetilde{\beta_3} : \Sigma^{43} V(0) \to V(0) \end{aligned}$$

This Massey product therefore detects the Toda bracket

$$\langle \beta_2 \beta_1^3 \alpha_1 [5], v_1, \widetilde{\beta_3} \rangle. \tag{6.1}$$

We saw in Lemma 6.11 that $b_0^3 b_1 = 0$ in $E_5(V(1))$. Therefore, the Toda bracket is detected in a higher Adams filtration modulo an indeterminacy contained in the subgroup

$$[\Sigma^{69} V(0), V(1)] \circ \widetilde{\beta_3}.$$

We may compute $\pi_{69}(V(1))$ and $\pi_{70}(V(1))$ as usual using the AHSS, whose chart is given below.

$$\begin{tikzcd}[row sep=small]
\underline{\text{Stem } 68} & \underline{\text{Stem } 69} & \underline{\text{Stem } 70} & \underline{\text{Stem } 71} \\
x_{68}[0] & x_{68}[1] & \beta_2^2\beta_1\alpha_1[5] & \alpha_{18/3}[0] \\
\alpha_{17}[1] & \alpha_{16}[6] \arrow[l] & & \\
\alpha_{16}[5] & & & \\
\beta_2^2\beta_1[6] & & &
\end{tikzcd}$$

So we have $[\Sigma^{69} V(0), V(1)]$ is of rank 2, generated by elements $\beta_2^2\beta_1\alpha_1[5]$ and $\widetilde{x_{68}[1]}$. We will conclude that the indeterminacy is zero using the $V(0)$-module structure of $V(1)$.

$$\beta_2^2\beta_1\alpha_1[5] \circ \widetilde{\beta_3} = \beta_3 \cdot \beta_2^2\beta_1\alpha_1[5] = 0 \tag{6.2}$$

$$\widetilde{x_{68}[1]} \circ \widetilde{\beta_3} = \beta_3[1] \cdot x_{68}[1] = 0 \tag{6.3}$$

The equation 6.2 follows from the relation $\beta_3\beta_1 = 0$ in $\pi_*(S)$.

In equation 6.3, we mean the product of $\beta_3[1] \in \pi_{43}(V(0))$ and $x_{68}[1] \in \pi_{69}(V(1))$ under the module map $\mu : V(0) \wedge V(1) \to V(1)$. The following diagram commutes.

$$\begin{tikzcd}
\pi_{43}(V(0)) \otimes \pi_{69}(V(0)) \arrow[r, "\mu"'] \arrow[d] & \pi_{112}(V(0)) \arrow[d] \\
\pi_{43}(V(0)) \otimes \pi_{69}(V(1)) \arrow[r, "\mu"'] & \pi_{112}(V(1))
\end{tikzcd}$$

Since $x_{68}[1] \in \pi_{69}(V(1))$ is born on the 1-cell, we may lift it to an element $x_{68}[1] \in \pi_{69}(V(0))$. Thus in order to show there is no indeterminacy it suffices to show that $\beta_3[1] \cdot x_{68}[1] = 0 \in \pi_{112}(V(0))$. At this point we remind the reader that by our descriptions of the elements $\beta_3[1]$ and $x_{68}[1]$ as elements in $\pi_*(V(1))$, we mean that their images under the projection on to the top cell is β_3 and x_{68}, respectively. These elements are not necessarily uniquely determined in $\pi_*(V(0))$, but any two representatives will differ by the image of an element of $\pi_*(S)$ under the inclusion of the bottom cell of $V(0)$. Now $\pi_{69}(S) = 0$, so $x_{68}[1]$ is uniquely determined. However, $\pi_{43}(S)$ is generated by α_{11}, so any two elements of $\pi_{43}(V(0))$ which project to β_3 on the top cell must differ by $\pm\alpha_{11}[0] \in \pi_{43}(V(0))$. Under the product map $V(0) \wedge V(0) \to V(0)$, all homotopy carried by the smash product of the 1-cells is annihilated. We will necessarily have $\beta_3[1] \cdot x_{68}[1] = 0 \in \pi_{112}(V(0))$ for all possible representatives of $\beta_3[1]$ if we can show

$$\alpha_{11}[0] \cdot x_{68}[1] = 0 \in \pi_{112}(V(0)).$$

This is straightforward: in $\pi_{43}(V(0))$ we have $\alpha_{11}[0] = v_1^{10}\alpha_1[0]$, and $\alpha_1 \cdot x_{68} = 0 \in \pi_*(S)$. Thus the Toda bracket that the Massey product 6.1 detects has no indeterminacy.

We conclude the Toda bracket 6.1 must be zero in $\pi_*(V(1))$ modulo higher Adams filtration. There are only three elements in the correct range to kill this Massey product, and these elements are $v_2^3v_3h_0b_0$, $\gamma_2v_2^2$, and $v_3k_0h_0v_2^2$. In Lemma 6.13, we proved that $v_2^3v_3h_0b_0$ supported a non-trivial d_2. The element $\gamma_2v_2^2$ must support a non-trivial d_3; this follows from the S-module pairing and the d_3 supported by v_2^2 that was proved in Lemma 6.1. Thus we must have

$$d_5(v_3k_0h_0v_2^2) \doteq v_2^3b_1b_0^3$$

and the lemma is proven after multiplying both sides by b_0^3. $\square$

Lemma 6.15. In $E_2(V(1))$ there is a differential

$$d_2(\eta_1g_0v_2^3b_0^2) = -\eta_1g_0k_0b_0^3h_1 \doteq k_1b_0^6.$$

Proof. This lemma follows immediately from the S-module pairing of Adams spectral sequences. We know $b_0^2\eta_1$ survives to $E_5(S)$ and we have computed $d_2(g_0v_2^3)$ in $E_2(V(1))$ as part of Lemma 6.7. $\square$

We have established that every possible target of a differential supported by v_2^9 is either the target of a shorter differential or the source of a differential. We may conclude that v_2^9 is a permanent cycle.

7. Proof that v_2^9 extends over $V(1)$

In this section we will prove that if $v_2^9 : S^{144} \to V(1)$ is a map detected by the element $v_2^9 \in E_2(V(1))$, then it extends over $V(1)$ to a self-map

$$\widetilde{v_2^9} : \Sigma^{144}V(1) \to V(1).$$

Applying Spanier-Whitehead duality, this is equivalent to finding an extension corresponding to the dotted arrow in the diagram below.

$$\begin{tikzcd} & V(1)\wedge DV(1) \arrow[d] \\ S^{144} \arrow[ur, dotted] \arrow[r, "v_2^9"'] & V(1) \end{tikzcd}$$

It suffices to extend v_2^9 over the complex Y_1, which is a wedge summand of $V(1) \wedge DV(1)$, as displayed below.

$$\begin{tikzcd} & Y_1 \arrow[d] \\ S^{144} \arrow[ur, dotted] \arrow[r, "v_2^9"'] & V(1) \arrow[d, "\delta"] \\ & \Sigma^{-5}V(1) \end{tikzcd}$$

The vertical column forms a cofiber sequence where δ is given as the composite

$$V(1) \xrightarrow{\nu} \Sigma^5V(0) \xrightarrow{\beta_1} \Sigma^{-5}V(0) \xrightarrow{\iota} \Sigma^{-5}V(1).$$

Here, ν is projection onto the top $V(0)$ and ι is inclusion of the bottom $V(0)$. Thus there is a solution to the extension problem if and only if the composite $\delta \circ v_2^9 = 0$. The map δ is also given by the composite

$$\begin{tikzcd} V(1) \arrow[r, "\nu"] \arrow[rr, bend right, "\delta'"'] & \Sigma^5 V(0) \arrow[r, "\iota"] & \Sigma^5 V(1) \arrow[r, "\beta_1"] & \Sigma^{-5} V(1) \end{tikzcd}$$

Here δ' is the geometric v_1-Bockstein. In this section we will prove

$$\beta_1 \cdot \delta'(v_2^9) = 0 \tag{7.1}$$

from which it follows that $\delta \circ v_2^9 = 0$, and thus v_2^9 extends over $V(1)$.

Since v_2^9 has Adams filtration 9, $\delta'(v_2^9)$ has Adams filtration ≥ 9. We may calculate δ' in Ext using the periodic lambda algebra. We have

$$d(v_2^9) = v_1^9 \lambda_{27}$$

thus we have

$$\begin{array}{ccccc} \overline{\Lambda}_{(1)} & \xrightarrow[\nu_*]{} & \overline{\Lambda}_{(0)} & \xrightarrow[\iota_*]{} & \overline{\Lambda}_{(1)} \\ v_2^9 & \longmapsto & v_1^8 \lambda_{27} & \longmapsto & 0 \end{array}$$

and so $\delta'_*(v_2^9) = 0$ viewed as an element of $E_2(V(1))$. We may conclude that the Adams filtration of $\delta'(v_2^9)$ is ≥ 10. Our strategy to prove that $\beta_1 \cdot \delta'(v_2^9) = 0$ is to make a list of all of the elements in $E_2(V(1))$ in Adams filtration greater than or equal to 10 in the 139-stem. We then will prove that each of these elements is either not a permanent cycle, or is killed by a differential or at least has the property that composition with β_1 is zero. A list of the elements, as well as the lemmas that deal with it, is given below.

AF	Element	Lemma
26	$h_0 v_2 b_0^{12}$	6.4
25	$k_0 h_0 b_0^{11}$	6.4
20	$g_0 h_0 v_2^3 b_0^7$	7.1
19	$b_1 h_0 v_2^2 b_0^7$	7.2
15	$v_2^6 h_0 b_0^4$	7.5, 7.6
	$\eta_1 v_2^2 b_0^5$	7.5, 7.7
	$v_3 h_1 v_2 b_0^6$	7.5, 7.7
14	$\eta_1 k_0 v_2 b_0^4$	7.5, 7.8
	$v_2^5 k_0 h_0 b_0^3$	7.5, 7.9
	$v_3 k_0 h_1 b_0^5$	7.5, 7.8, 7.9
13	$\langle k_1, h_0, h_0 \rangle b_0^5$	7.10
10	$v_3 h_0 b_0^2 v_2^4$	7.11

Lemma 7.1. If x is an element of $E_2(V(1))$ whose Hurewicz image $h(x) \in E_2(eo_2 \wedge V(1))$ supports a non-trivial differential, then x cannot be a permanent cycle.

Proof. This is obvious; h is a map of spectral sequences. $\square$

Lemma 7.2. The element $b_1 h_0 v_2^2 b_0^7 \in E_6(V(1))$ must be zero.

Proof. We will show that $v_2^2 b_1 h_0 \in E_r(V(1))$ is a permanent cycle. The result then follows from the fact that in $E_7(S)$ there is a relation $b_0^6 = 0$. Consider the portion of the AHSS chart for $\pi_*(V(1))$ given below.

Stem 56	Stem 57	Stem 58
$\beta_2^2\alpha_1[1]$	$\beta_2^2[5]$	$\beta_2^2[6]$
$\alpha_{14}[1]$	$\alpha_{13}[6]$	
$\alpha_{13}[5]$		

We use this to deduce the following portion of the ASS chart for $\pi_*(V(1))$.

Stem 56	Stem 57	Stem 58	Stem 59
$b_0^4 v_2$	$b_0^3 h_1 v_2$	$b_0^4 g_0$	$v_2 b_0^4 h_0$
$b_0^3 k_0$	$b_1 b_0^2 h_0$	$b_0 v_2^3$	$k_0 b_0^3 h_0$
	η_1	$v_2^2 k_0$	$h_1 v_2^3$
		g_1	

The differential on η_1 follows from the differential in $E_5(S)$. The differential on $b_0 v_2^3$ is a consequence of Lemma 6.1. Since the AHSS calculation told us that $\pi_{57}(V(1)) = 0$, we may conclude that one of the dotted differentials (1) exists. Also, we have shown that $\pi_{58}(V(1))$ has rank 1, hence something must kill $b_0^4 g_0$. Both $v_2 b_0^4 h_0$ and $k_0 b_0^3 h_0$ are permanent cycles, so the only candidate to support the dotted differential (2) is $h_1 v_2^3$. Note that this differential is present in the ASS for $eo_2 \wedge V(1)$. Armed with this differential, we may propagate it with b_0 multiplication and investigate the stems in the vicinity of $h_0 b_1 v_2^2$. The AHSS in these stems is shown below.

Stem 66	Stem 67	Stem 68	Stem 69
$\beta_2^2\beta_1\alpha_1[1]$	$\alpha_{17}[0]$	$x_{68}[0]$	$x_{68}[1]$
	$\beta_2^2\beta_1[5]$	$\alpha_{17}[1]$	$\alpha_{16}[6]$
		$\alpha_{16}[5]$	
		$\beta_2^2\beta_1[6]$	

We may infer the differentials in these stems for the ASS for $\pi_*(V(1))$. They are displayed below.

Stem 67	Stem 68	Stem 69
$v_2 b_0^4 h_1$	$b_0^5 g_0$	$b_0^5 v_2 h_0$
$b_1 b_0^3 h_0$	$b_0^2 v_2^3$	$b_0^4 h_0 k_0$
$\eta_1 b_0$	$b_0 v_2^2 k_0$	$b_0 h_1 v_2^3$
$v_2^4 h_0$	$v_2 k_0^2$	$h_0 b_1 v_2^2$
	$\eta_1 h_1$	$g_1 h_1$

The two solid differentials are propagated from our previous calculations. Our AHSS calculations tell us that $\pi_{67}(V(1)) = 0$. Now $v_2 b_0^4 h_1$, $b_0^5 g_0$, and $v_2 k_0^2$ must be permanent cycles since $v_2 h_1$, g_0, and $v_2 k_0$ are permanent cycles for dimensional reasons. Since $v_2 b_0^4 h_1$ must vanish, one of the dotted differentials must occur. We will find later (see the proof of Lemma 7.5) that in fact $\eta_1 h_1$ must be a permanent cycle, and thus the dotted differential supported on $v_2^2 k_0 b_0$ must be the non-trivial one. We have computed $\pi_{68}(V(1))$ to be of rank 2, so there can be no more differentials originating from the 69 stem. Thus we conclude that $h_0 b_1 v_2^2$ is a permanent cycle, and the lemma follows. $\square$

We are now in a position to prove the differential promised in Remark 6.2. We will need this differential later in this paper.

Lemma 7.3. In $E_2(V(1))$, there is a non-trivial differential

$$d_2(v_3 g_0) \doteq k_0 h_1 v_2^2 = b_1 h_0 v_2^2.$$

Proof. Consider the portion of the AHSS for $\pi_*(V(1))$ displayed below.

$$\begin{tikzcd}
\underline{\text{Stem } 68} & \underline{\text{Stem } 69} & \underline{\text{Stem } 70} \\
x_{68}[0] & x_{68}[1] & \beta_2^2\beta_1\alpha_1[5] \\
\alpha_{17}[1] & \alpha_{16}[6] \arrow[l] & \\
\alpha_{16}[5] & & \\
\beta_2^2\beta_1[6] & &
\end{tikzcd}$$

We have shown that $\pi_{69}(V(1))$ has rank 1. In the proof of Lemma 7.2, we computed all of the Adams differentials in $E_*(V(1))$ supported in the 69 stem. That data is used to compute the following portion of the ASS for $\pi_*(V(1))$.

$$\begin{tikzcd}
\underline{\text{Stem } 68} & \underline{\text{Stem } 69} & \underline{\text{Stem } 70} \\
b_0^5 g_0 & b_0^5 v_2 h_0 & b_0^7 \\
b_0^2 v_2^3 & b_0^4 h_0 k_0 & b_0^2 g_0 v_2^2 \arrow[l] \\
b_0 v_2^2 k_0 & b_0 h_1 v_2^3 \arrow[uul] & b_0^2 v_2 b_1 \arrow[uul] \\
v_2 k_0^2 & h_0 b_1 v_2^2 & h_0 b_0 \eta_1 \\
\eta_1 h_1 & g_1 h_1 & v_3 g_0 \arrow[ul, dotted]
\end{tikzcd}$$

The differential supported by $b_0^2 g_0 v_2^2$ is a consequence of Lemma 6.7. The differential supported by $b_0^2 v_2 b_1$ is a consequence of the Toda differential on b_1 in $E_5(S)$ and the S-module structure of $V(1)$. One more element in the 69 stem must be the target of an Adams differential, and the only possibility is the dotted differential. This is the differential we wanted. $\square$

There is an elaborate pattern of activity in Adams filtration 13-15 in the 139-stem. We would like to understand which elements in this range of filtrations are permanent cycles, and which aren't, and this is accomplished in Lemma 7.5. As a consequence, certain linear combinations of the possible obstructions to the extension of v_2^9 will be eliminated. First we need the following differential.

Lemma 7.4. In $E_3(V(1))$, there is a non-trivial differential

$$d_3(v_3h_1v_2) = -b_0\eta_1h_1.$$

Proof. In Lemma 6.6, we showed that v_3h_1 supports a non-trivial d_4 and thus is a d_3 cycle. We now apply the product formula (4.1) to deduce the differential on $v_3h_1 \cdot v_2$. Computer assisted lambda algebra calculations yield the formula

$$\overline{v_3h_1} = -g_1 \in E_*(V(0)).$$

We wish to compute $\overline{\overline{v_3h_1}}$, which is obtained by computing the Bockstein on g_1. Now g_1 is d_1-cycle when it is considered as an element of $E_1(S^0)$, so we need to compute the Adams differential of g_1 in $E_*(S^0)$ to get this Bockstein. We use the main theorem of Bruner ([3, VI.1.1]) to understand the relationship between Steenrod operations in Ext and differentials in the ASS. Using Bruner's formula, we may compute

$$d_2(g_1) = d_2(P^0(g_0)) = v_0 \cdot \beta P^0(g_0) = -v_0\eta_1.$$

Here we should point out that our indexing in the Steenrod operations is different of that of Bruner's, but conforms to perhaps the more common indexing as given in [13]. The element v_0 detects the degree p map on spheres. The Steenrod operation $\beta P^0(g_0) = -\eta_1$ is computed using an explicit combinatorial formula for βP^0 on the lambda algebra. We conclude that $\overline{\overline{v_3h_1}} = \eta_1$. We then apply the product formula, keeping in mind that $\overline{v_2} = h_1$ and $\overline{\overline{v_2}} = b_0$, and get

$$d_3(v_3h_1 \cdot v_2) = -b_0(\eta_1h_1 - g_1b_0) = -b_0\eta_1h_1.$$

□

Lemma 7.5. We have the following differentials in the 139 stem of the ASS for $V(1)$ in Adams filtrations 13–15.

$$\begin{aligned}
d_3(\eta_1v_2^2b_0^5) &\doteq b_0^7h_1\eta_1 \\
d_3(v_3h_1b_0^6v_2) &\doteq b_0^7h_1\eta_1 \\
d_3(v_2^6h_0b_0^4) &\doteq b_0^7h_1\eta_1 \quad \text{or } v_2^6h_0 \text{ is a permanent cycle} \\
d_5(v_2^5k_0h_0b_0^3) &\doteq b_0^7k_0^2v_2 \\
d_5(\eta_1k_0v_2b_0^4) &\doteq b_0^7k_0^2v_2 \\
d_5(v_3k_0b_0^5h_1) &\doteq b_0^7k_0^2v_2
\end{aligned}$$

An $\mathbb{F}_3$-basis of permanent cycles in this range of Adams filtration is listed below.

$$\begin{gathered}
v_2^6h_0b_0^4 + a_1(\eta_1v_2^2b_0^5) + a_2(v_3h_1b_0^6v_2) \\
\eta_1v_2^2b_0^5 \pm v_3h_1b_0^6v_2 \\
v_2^5k_0h_0b_0^3 \pm v_3k_0b_0^5h_1 \\
\eta_1k_0v_2b_0^4 \pm v_3k_0b_0^5h_1 \\
\langle k_1, h_0, h_0\rangle b_0^5
\end{gathered}$$

Here a_1 and a_2 are elements of $\mathbb{F}_3$. We are unable to determine the $\pm$ signs or the coefficients a_i. A diagram of this portion of the ASS chart is given below for the reader's convenience.

AF	Stem 138	Stem 139
19	$k_0^2 v_2 b_0^7$	
18	$b_0^7 h_1 \eta_1$	
15		$v_2^6 h_0 b_0^4$
		$\eta_1 v_2^2 b_0^5$
		$v_3 h_1 b_0^6 v_2$
14		$v_2^5 k_0 h_0 b_0^3$
		$\eta_1 k_0 v_2 b_0^4$
		$v_3 k_0 b_0^5 h_1$
13		$\langle k_1, h_0, h_0 \rangle b_0^5$

Proof. The method of proof is to divide these elements by a maximal power of b_0, and then multiply by b_0 successively until all of the elements in question are present in the E_2 term. We begin with the vicinity of the 79 stem. The AHSS for $\pi_*(V(1))$ is displayed below.

Stem 77	Stem 78	Stem 79	Stem 80
$\beta_2^2\beta_1^2[5]$	$\beta_2^3[0]$	$\alpha_{20}[0]$	$\alpha_{20}[1]$
$\alpha_{18/3}[6]$	$\beta_2^2\beta_1^2[6]$ (1)	$\beta_2^3[1]$	$\alpha_{19}[5]$
		$\beta_5[5]$	$\beta_2^2\beta_1^2\alpha_1[5]$
			$\frac{\beta_2^2\beta_1^2\alpha_1}{3}[5]$
			$\beta_5[6]$

At this point we do not know if the dotted differential (1) exists because we are unsure of whether or not $\beta_2^3 \in \pm\langle \beta_5, \alpha_1, 3\rangle$. We conclude that either $\pi_{78}(V(1))$ has rank 2 and $\pi_{79}(V(1))$ has rank 1, or $\pi_{78}(V(1))$ has rank 1 and $\pi_{79}(V(1))$ is trivial. We will see shortly however that the latter is the case, i.e. that the dotted differential (1) must exist. We display below the ASS in the same range.

Stem 77	Stem 78	Stem 79	Stem 80
$v_2 b_0^5 h_1$	$b_0^6 g_0$	$v_2 b_0^6 h_0$	b_0^8
$b_1 b_0^4 h_0$	$v_2^3 b_0^3$	$k_0 b_0^5 h_0$	$b_0^3 g_0 v_2^2$
$b_0^2 \eta_1$	$v_2^2 k_0 b_0^2$	$v_2^3 b_0^2 h_1$	$b_0^3 v_2 b_1$
$v_2^4 h_0 b_0$	$b_0 v_2 k_0^2$	$b_1 v_2^2 b_0 h_0$	$\eta_1 b_0^2 h_0$
$v_2^3 k_0 h_0$	$\eta_1 h_1 b_0$	$v_3 h_1 v_2$	$b_0 v_3 g_0$
(2)	$v_3 k_0$		v_2^5

By comparing with the ASS chart in the vicinity of the 69-stem in the proof of Lemma 7.2, we see that the elements $b_0 v_2 k_0^2$, $v_2 b_0^6 h_0$, $k_0 b_0^5 h_0$, and $b_1 v_2^2 b_0 h_0$ are permanent cycles, and the elements $v_2^3 b_0^3$ and $v_2^3 b_0^2 h_1$ support the indicated differentials. The differential on $v_3 h_1 v_2$ was the subject of Lemma 7.4. In Lemma 7.2, we were unable to determine whether $v_2 h_1 b_0^5$ was killed by $v_2^2 k_0 b_0^2$ or $\eta_1 h_1 b_0$. Since $\eta_1 h_1 b_0$ is the target of a differential, this ambiguity is now resolved: $d_4(v_2^2 k_0 b_0^2) \doteq v_2 b_0^5 h_1$. The differential supported by $b_0^3 g_0 v_2^2$ was established in Lemma 6.7. The differential supported by $b_0^3 v_2 b_1$ follows from the Toda differential on b_1 in $E_*(S^0)$. The differentials supported by $b_0 v_3 g_0$ and v_2^5 were proven in Lemmas 7.3 and 6.1, respectively. There is nothing remaining in stem 79, so we conclude that $\pi_{79}(V(1))$ is trivial. Therefore the dotted differential (1) exists in the previous AHSS chart. The AHSS chart now tells us that $\pi_{78}(V(1))$ is of rank 1, and the only way for the ASS to produce the same answer is for there to exist the dotted differential (2) since neither $b_0^2 \eta_1$ nor $v_2^4 h_0 b_0$ are permanent cycles.

We now multiply everything by b_0 and move into the vicinity of the 89 stem. The AHSS for $\pi_*(V(1))$ is displayed below.

Stem 87	Stem 88	Stem 89	Stem 90
$\alpha_{22}[0]$	$\alpha_{22}[1]$	$\gamma_2\alpha_1[5]$	$\beta_6[0]$
$\beta_{6/2}[1]$	$\alpha_{21/2}[5]$	$\beta_5\beta_1[5]$	$\beta_{6/3}\alpha_1[5]$
$\beta_{6/3}[5]$	$\beta_{6/3}[6]$	$\alpha_{21/2}[6]$	$\langle\beta_2^3,\alpha_1,\alpha_1\rangle[5]$
$\langle\beta_5,\alpha_1,\alpha_1\rangle[6]$			$\gamma_2\alpha_1[6]$
$\gamma_2[6]$			$\beta_5\beta_1[6]$

The corresponding ASS chart is displayed below.

Stem 87	Stem 88	Stem 89	Stem 90
$b_0^6 h_1 v_2$	$b_0^7 g_0$	$v_2 b_0^7 h_0$	b_0^9
$b_1 b_0^5 h_0$	$v_2^3 b_0^4$	$k_0 b_0^6 h_0$	$v_2^2 b_0^4 g_0$
$\eta_1 b_0^3$	$v_2^2 k_0 b_0^3$	$v_2^3 h_1 b_0^3$	$b_1 v_2 b_0^4$
$v_2^4 b_0^2 h_0$	$b_0^2 v_2 k_0^2$	$b_1 v_2^2 b_0^2 h_0$	$\eta_1 b_0^3 h_0$
$b_0 h_0 v_2^3 k_0$	$b_0^2 h_1 \eta_1$	$v_2^2 \eta_1$	$b_0^2 v_3 g_0$
$v_3 h_2$	$v_3 k_0 b_0$	$b_0 v_2 v_3 h_1$	$v_2^5 b_0$
		$h_1 v_3 k_0$	$v_2^4 k_0$
		$\langle k_1, h_0, h_0\rangle$	$v_2^2 g_1$
			$v_3 h_2 h_0$

All of the indicated differentials follow from our computations near the 79 stem except for the one supported by $h_1 v_3 k_0$. For that we consider the image under projection onto the top cell of $V(1)$. We saw in the proof of 7.4 that $\overline{\overline{v_3 h_1}} = \eta_1$.

We have

$$\overline{\overline{d(v_3h_1k_0)}} = d(\overline{\overline{v_3h_1k_0}}) = d(\eta_1k_0) \doteq k_0^2b_0^3 = \overline{\overline{b_0^2v_2k_0^2}}.$$

This can only happen if $d_5(v_3h_1k_0) = k_0^2b_0^2v_2$. Our AHSS calculation tells us that $\pi_{89}(V(1))$ is of rank 2, so there can be no more differentials originating from the 90 stem.

We now multiply once more by b_0 and arrive in the crucial region around the 99 stem. First we work out the AHSS in this region.

Stem 97	Stem 98	Stem 99	Stem 100
$x_{92}[5]$	$\beta_6\alpha_1[5]$	$\alpha_{25}[0]$	$\beta_2\beta_5[0]$
$\beta_{6/3}\beta_1[5]$	$\frac{\beta_6\alpha_1}{3}[5]$	$\langle x_{92}, \alpha_1, \alpha_1\rangle[0]$	$\alpha_{25}[1]$
$\alpha_{23}[6]$	$x_{92}[6]$	$\gamma_2\beta_1\alpha_1[5]$	$\langle x_{92}, \alpha_1, \alpha_1\rangle[1]$
$\langle\beta_5\beta_1, \alpha_1, \alpha_1\rangle[6]$	$\beta_{6/3}\beta_1[6]$	$\beta_5\beta_1^2[5]$	$\alpha_{24/2}[5]$
$\gamma_2\beta_1[6]$		$\beta_6\alpha_1[6]$	$\beta_{6/3}\beta_1\alpha_1[5]$
		$\frac{\beta_6\alpha_1}{3}[6]$	$\gamma_2\beta_1\alpha_1[6]$
			$\beta_5\beta_1^2[6]$

Therefore the rank of $\pi_{98}(V(1))$ is 3 and the rank of $\pi_{99}(V(1))$ is 4. We turn now to the ASS.

Stem 97	Stem 98	Stem 99	Stem 100
$b_0^7v_2h_1$	$b_0^8g_0$	$v_2b_0^8h_0$	b_0^{10}
$b_1b_0^6h_0$	$v_2^3b_0^5$	$k_0b_0^7h_0$	$b_0^5g_0v_2^2$
$b_0^4\eta_1$	$b_0^4v_2^2k_0$	$v_2^3b_0^4h_1$	$b_0^5v_2b_1$
$v_2^4b_0^3h_0$	$b_0^3k_0^2v_2$	$v_2^2b_1b_0^3h_0$	$b_0^4h_0\eta_1$
$v_2^3b_0^2h_0k_0$	$b_0^3h_1\eta_1$	$v_2v_3b_0^2h_1$	$b_0^3v_3g_0$
$v_2^2v_3h_0b_0$	$b_0^2v_3k_0$	$h_0v_2^6$	$v_2^5b_0^2$
$v_2\gamma_2$	$g_0v_2^5$	$b_0v_2^2\eta_1$	$v_2^4k_0b_0$
$b_0v_3h_2$	$v_2^4b_1$	$v_2\eta_1k_0$	$v_2^3k_0^2$
	v_2k_1	$b_0h_1v_3k_0$	$v_2^2h_1\eta_1$
		$\langle x_{92}, h_0, h_0\rangle$	$h_0v_2\gamma_2$

(3) (4)

Aside from the differentials supported by (possibly) $g_0v_2^5$, $v_2^6h_0$, and $v_2\eta_1k_0$, all of the differentials displayed follow from our calculations near the 89 stem. If we only had the differentials arising from b_0 multiplication on elements in the vicinity of the 89 stem, we would have created groups of the correct rank in the 98 and 99 stem as predicted from the AHSS. Therefore any additional differentials must preserve the rank of the E_∞ term. We easily see that $d_5(\eta_1 v_2 k_0) = b_0^3 v_2 k_0^2$ from the differential on η_1 and the S-module structure of $V(1)$.

The problem is that $h_0 v_2^6$ could support a d_4 killing $b_0^3 k_0^2 v_2$, and this would make both $v_2 \eta_1 k_0$ and $b_0 h_1 v_3 k_0$ into permanent cycles. We claim that this cannot happen. For suppose that $d_4(v_2^6 h_0) \doteq b_0^3 k_0^2 v_2$. Then there is no linear combination of elements containing $v_2^6 h_0$ which is a permanent cycle. Our AHSS calculation indicates there is some element in $\pi_{99}(V(1))$ such that its image in $\pi_{93}(S)$ under projection onto the top cell of $V(1)$ is $\beta_6 \alpha_1$. The only element which can account for this is $v_2^6 h_0$. Therefore, if $v_2^6 h_0$ is not a permanent cycle, it must support a d_3 killing $b_0^3 h_1 \eta_1$. This possibility is indicated by the dashed differential (4).

Also, we cannot eliminate the possibility that $d_3(g_0 v_2^5) \doteq v_2^3 b_0^2 h_0 k_0$, since $g_0 v_2^5$ is in the same Adams filtration as $b_0^2 v_3 k_0$. This possibility is indicated with the dashed differential (3). Upon taking the pattern of differentials supported by the 99 stem and multiplying by b_0^4, we get the promised pattern of differentials supported in the 139 stem. □

Lemma 7.6. Choose the correct coefficients $a_i \in \mathbb{F}_3$ so that $v_2^6 h_0 b_0^4 + a_1(\eta_1 v_2^2 b_0^5) + a_2(v_3 h_1 b_0^6 v_2)$ is a permanent cycle. The composition with any element that this permanent cycle detects with β_1 must be null.

Proof. The element $v_2^6 h_0 + a_1(\eta_1 v_2^2 b_0) + a_2(v_3 h_1 b_0^2 v_2)$ is a permanent cycle by Lemma 7.5. Now apply Corollary 5.2. Comparing with the elements present in the 149 stem of $E_2(V(1))$ we see that there is no possibility of a hidden β_1 extension. □

Lemma 7.7. Choose the correct sign so that $\eta_1 v_2^2 b_0^5 \pm v_3 h_1 b_0^6 v_2$ is a permanent cycle. This element must be the target of a differential.

Proof. This follows from the fact that $\eta_1 v_2^2 \pm v_3 h_1 b_0 v_2$ is a permanent cycle (Lemma 7.5), and Corollary 5.2. □

Lemma 7.8. Choose the correct sign so that $\eta_1 k_0 v_2 b_0^4 \pm v_3 k_0 b_0^5 h_1$ is a permanent cycle. The composition with any element that this permanent cycle detects with β_1 must be null.

Proof. Again, apply Corollary 5.2. Comparing with the elements present in the 149 stem of $E_2(V(1))$ we see that there is no possibility of a hidden β_1 extension. □

Lemma 7.9. Choose the correct sign so that $v_2^5 k_0 h_0 b_0^3 \pm v_3 k_0 b_0^5 h_1$ is a permanent cycle. This element must be the target of a differential.

Proof. In Lemma 7.3, it was established that $v_2^5 \pm v_3 g_0 b_0$ was a permanent cycle, therefore the element

$$h_0 \cdot (v_2^5 \pm v_3 g_0 b_0) = h_0 v_2^5 \pm v_3 h_1 b_0^2$$

is a permanent cycle. Now $d_5(\eta_1) \doteq k_0 b_0^3$ in the ASS for $\pi_*(S)$. Using the S-module structure of $V(1)$, we deduce that there is a differential

$$d_5(\eta_1 \cdot (h_0 v_2^5 \pm v_3 h_1 b_0^2)) \doteq v_2^5 k_0 h_0 b_0^3 \pm v_3 k_0 b_0^5 h_1.$$

Note that $v_2^5 k_0 h_0 b_0^3 \pm v_3 k_0 b_0^5 h_1$ might be the target of a shorter differential, in which case the conclusion of the lemma is still satisfied. □

Lemma 7.10. The element $\langle k_1, h_0, h_0 \rangle b_0^5$ must be the target of a differential.

Proof. This follows immediately from Corollary 5.2 and the fact that $\langle k_1, h_0, h_0\rangle$ is a permanent cycle (Lemma 7.5). □

Lemma 7.11. In $E_2(V(1))$ there is a non-trivial differential

$$d_2(v_3 h_0 b_0^2 v_2^4) \doteq v_2^2 h_0 \eta_1 k_0 b_0^2.$$

Proof. In Lemma 6.11, we showed that $d_2(v_3 h_0 b_0) = 0$. In Lemma 6.1, we showed that

$$d_2(v_2^4) = -b_0 k_0 h_1 v_2 = -b_0 b_1 h_0 v_2.$$

Computer assisted lambda algebra computations reveal that in E_2, we have

$$\begin{aligned} \overline{v_3 h_0 b_0} &= -h_0 \eta_1 \\ \overline{\overline{v_3 h_0 b_0}} &= 0 \end{aligned}$$

We also have $\overline{v_2^4} = h_1 v_2^3$ and $\overline{\overline{v_2^4}} = 0$ in E_2. We will now use the product rule (Theorem 4.1) to compute the differential on $(v_3 h_0 b_0)\cdot v_2^4$. The term $G(v_3 h_0 b_0, v_2^4)$ is trivial in E_2 by the previous considerations. The following manipulations are made possible with the hidden extension $h_0 \cdot (v_3 h_0 b_0) \doteq b_1 g_0 v_2$.

$$\begin{aligned} d_2((v_3 h_0 b_0)\cdot v_2^4) &= (v_3 h_0 b_0)\cdot(b_0 b_1 h_0 v_2) \\ &\doteq b_1^2 g_0 b_0 v_2^2 \\ &\doteq v_2^2 h_0 \eta_1 k_0 b_0 \end{aligned}$$

Now multiply by b_0. □

8. Some consequences in v_2-periodic homotopy

In this section we will indicate the construction of some of the β_i elements in $\pi_*(S)$ using the self-map

$$v_2^9 : \Sigma^{144} V(1) \to V(1).$$

These constructions were communicated to the authors by Katsumi Shimomura.

Corollary (0.2). The elements β_i are permanent cycles in the ANSS for $i \equiv 0, 1, 2, 5, 6 \pmod 9$.

Proof. For dimensional reasons, v_2 is a permanent cycle of the ANSS which detects a map

$$v_2 : S^{16} \to V(1).$$

By [15], or Remark 6.2, the element v_2^5 in the ANSS is a permanent cycle which detects a map

$$v_2^5 : S^{80} \to V(1).$$

We denote the Spanier-Whitehead dual of v_2 by

$$v_2^* : \Sigma^{16} DV(1) = \Sigma^{10} V(1) \to S^0.$$

Let $\nu : V(1) \to S^6$ be projection onto the top cell. The elements β_i in Corollary 0.2 are constructed by the following compositions.

$$\beta_{9t} : S^{144t-6} \hookrightarrow \Sigma^{144t-6}V(1) \xrightarrow{v_2^{9t}} \Sigma^{-6}V(1) \xrightarrow{\nu} S^0$$
$$\beta_{9t+1} : S^{144t+10} \xrightarrow{v_2} \Sigma^{144t-6}V(1) \xrightarrow{v_2^{9t}} \Sigma^{-6}V(1) \xrightarrow{\nu} S^0$$
$$\beta_{9t+2} : S^{144t+26} \xrightarrow{v_2} \Sigma^{144t+10}V(1) \xrightarrow{v_2^{9t}} \Sigma^{10}V(1) \xrightarrow{v_2^*} S^0$$
$$\beta_{9t+5} : S^{144t+74} \xrightarrow{v_2^5} \Sigma^{144t-6}V(1) \xrightarrow{v_2^{9t}} \Sigma^{-6}V(1) \xrightarrow{\nu} S^0$$
$$\beta_{9t+6} : S^{144t+90} \xrightarrow{v_2^5} \Sigma^{144t+10}V(1) \xrightarrow{v_2^{9t}} \Sigma^{10}V(1) \xrightarrow{v_2^*} S^0$$

□

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