TOWARDS $\pi_* L_{K(2)} V(0)$ AT $p = 2$

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ABSTRACT. Let $V(0)$ be the mod 2 Moore spectrum and let $\mathcal{C}$ be the super-singular elliptic curve over $\mathbb{F}_4$ defined by the Weierstrass equation $y^2 + y = x^3$. Let $F_{\mathcal{C}}$ be its formal group law and $E_{\mathcal{C}}$ be the spectrum classifying the deformations of $F_{\mathcal{C}}$. The group of automorphisms of $F_{\mathcal{C}}$, which we denote by $\mathbb{S}_{\mathcal{C}}$, acts on $E_{\mathcal{C}}$. Further, $\mathbb{S}_{\mathcal{C}}$ admits a norm whose kernel we denote by $\mathbb{S}^1_{\mathcal{C}}$. The cohomology of $\mathbb{S}^1_{\mathcal{C}}$ with coefficients in $(E_{\mathcal{C}})_*V(0)$ is the E_2-term of a spectral sequence converging to the homotopy groups of $E_{\mathcal{C}}^{h\mathbb{S}^1_{\mathcal{C}}} \wedge V(0)$, a spectrum closely related to $L_{K(2)}V(0)$. In this paper, we use the algebraic duality resolution spectral sequence to compute an associated graded for $H^*(\mathbb{S}^1_{\mathcal{C}}; (E_{\mathcal{C}})_*V(0))$. These computations rely heavily on the geometry of elliptic curves made available to us at chromatic level 2.

CONTENTS

1. Introduction

This paper can be read as a sequel to [2]. For this reason, this section builds upon the deeper discussion of [2, §2]. We give an overview of the tools that were not introduced in the prequel and state our results. The reader who wants more detail on background and motivation should refer to [2].

1.1. **Background.** Throughout this paper, I will work at the prime $p = 2$. Recall that Morava K-theory $K(2)$ is the unique ring spectrum with coefficients

$$K(2)_* = \mathbb{F}_2[v_2^{\pm 1}],$$

for v_2 in degree 6, and with formal group law the Honda formal group law F_2 of height 2. The group $\mathbb{S}_2$ is the group of automorphisms of F_2 over $\mathbb{F}_4$. The extended Morava stabilizer group $\mathbb{G}_2$ is the extension of $\mathbb{S}_2$ by the Galois group. Morava E-theory E_2 is the spectrum which classifies isomorphism classes of deformations of F_2. Its homotopy groups can be described as follows. Let ζ be a primitive third root of unity and let

$$\mathbb{W} := W(\mathbb{F}_4) \cong \mathbb{Z}_2[\zeta]$$

be the Witt vectors on $\mathbb{F}_4$. Then

$$(E_2)_* = \mathbb{W}[[u_1]][u^{\pm 1}], \tag{1.1}$$

where u_1 has degree zero and u has degree -2. The group $\mathbb{G}_2$ acts on the spectrum E_2. For any finite spectrum X, there is a weak equivalence

$$L_{K(2)}X \simeq E_2^{h\mathbb{G}_2} \wedge X.$$

Further, for closed subgroups G of $\mathbb{G}_2$ and finite spectra X, there are descent spectral sequences

$$E_2^{s,t} := H^s(G, (E_2)_t X) \Longrightarrow \pi_{t-s}(E_2^{hG} \wedge X), \tag{1.2}$$

The groups $\mathbb{S}_2$ and $\mathbb{G}_2$ both admit a norm induced by the determinant of a general linear representation of $\mathbb{S}_2$. The elements of norm one form normal subgroups denoted $\mathbb{S}_2^1$ and $\mathbb{G}_2^1$ respectively. Further,

$$\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2, \tag{1.3}$$

and

$$\mathbb{G}_2 \cong \mathbb{G}_2^1 \rtimes \mathbb{Z}_2.$$

The group $\mathbb{S}_2$ has a unique conjugacy class of maximal finite subgroups, which can be described as follows. The automorphism of F_2 given by $[-1]_{F_2}(x)$ generates a central subgroup C_2. The power series

$$\omega(x) = \zeta x$$

generates a subgroup of order three in $\mathbb{S}_2$, denoted C_3. Define

$$C_6 := C_2 \times C_3.$$

This group is contained in a subgroup

$$G_{24} := Q_8 \rtimes C_3$$

for a quaternion group Q_8. The group G_{24} is a maximal finite subgroup of $\mathbb{S}_2$.

The subgroups C_6 and G_{24} are contained in $\mathbb{S}_2^1$. However, $\mathbb{S}_2^1$ has two conjugacy classes of maximal finite subgroups. A representative for the other conjugacy class is given by

$$G'_{24} = \pi G_{24} \pi^{-1}$$

for π a topological generator of $\mathbb{Z}_2$ in (1.3).

The following result is Theorem 1.8 and Theorem 1.10 of [2].

Theorem 1.4 (Goerss, Henn, Mahowald, Rezk)**.** *There is an exact sequence of complete $\mathbb{S}_2^1$-module*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

where $\mathscr{C}_0 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]]$, $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]$ and $\mathscr{C}_0 \cong \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]]$. Further, for any finitely generated complete $\mathbb{S}_2^1$-module M, there is a first quadrant spectral sequence,

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}(\mathscr{C}_p, M) \Longrightarrow H_c^{p+q}(\mathbb{S}_2^1, M).$$

The differentials have degree

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

and

$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}; M) & \text{if } p = 0; \\ H^q(C_6; M) & \text{if } p = 1, 2; \\ H^q(G'_{24}; M) & \text{if } p = 3. \end{cases}$$

The exact sequence of Theorem 1.4 is called the *algebraic duality resolution* because it satisfies a certain duality. This is described in Theorem 1.9 of [2]. The associated spectral sequence is called the *algebraic duality spectral sequence*.

Let $V(0)$ be the mod 2 Moore spectrum. It is defined by the cofiber sequence

$$S \xrightarrow{2} S \to V(0).$$

The goal of this paper is to compute the E_∞-term of the algebraic duality spectral sequence when M is the module $(E_2)_*V(0)$. We obtain an associated graded for $H^*(\mathbb{S}_2^1; (E_2)_*V(0))$. By taking the Galois fixed points of the E_∞-term, one obtains an associated graded for the cohomology $H^*(\mathbb{G}_2^1; (E_2)_*V(0))$. Therefore, this computation gives the E_2-page of the descent spectral sequence (1.2) when $G = \mathbb{G}_2^1$ and $X = V(0)$, that is

$$H^s(\mathbb{G}_2^1; (E_2)_t V(0)) \Longrightarrow \pi_{t-s} E_2^{h\mathbb{G}_2^1} \wedge V(0).$$

Because there is a fiber sequence

$$L_{K(2)}V(0) \to E_2^{h\mathbb{G}_2^1} \wedge V(0) \to E_2^{h\mathbb{G}_2^1} \wedge V(0),$$

computing $H^*(\mathbb{S}_2^1; (E_2)_*V(0))$ is a first step for computing $\pi_* L_{K(2)}V(0)$.

The computations will be done using the fact that, at chromatic level $n = 2$, one can replace Morava K-theory $K(2)$ by a spectrum $K_{\mathcal{C}}$ whose formal group law is the formal group law of a super singular elliptic curve $\mathcal{C}$. This allows us to use the geometry of elliptic curves to get a better understanding of the action of the Morava stabilizer group $\mathbb{S}_2$ on $(E_2)_*$. Before stating the results, I will explain this point of view.

Let $\mathcal{C}$ be the unique super-singular elliptic curve over $\mathbb{F}_4$, with Weierstrass equation

$$\mathcal{C} : y^2 - y = x^3. \tag{1.5}$$

Let $F_{\mathcal{C}}$ be the formal group law of $\mathcal{C}$. It satisfies

$$[-2]_{F_{\mathcal{C}}}(x) = x^4.$$

Let $K_{\mathcal{C}}$ denote the complex oriented ring spectrum whose ring of coefficients is

$$(K_{\mathcal{C}})_* = \mathbb{F}_4[u^{\pm 1}],$$

where u is in degree -2, and whose formal group law is $F_{\mathcal{C}}$. In this paper,

$$E_{\mathcal{C}} := E(\mathbb{F}_4, F_{\mathcal{C}})$$

will denote the spectrum which represents isomorphism classes of deformations of $F_{\mathcal{C}}$. There is an isomorphism

$$(E_{\mathcal{C}})_* \cong (E_2)_*.$$

(Note that the isomorphism cannot be realized by a map of E_∞-ring spectra. Such a map would induce an $\mathbb{F}_4$-isomorphism on the formal group laws $F_{\mathcal{C}}$ and the 2-periodic extension of F_2. However, these formal group laws are not isomorphic over $\mathbb{F}_4$. They become isomorphic after passing to the algebraic closure.)

Let $\mathbb{S}_{\mathcal{C}}$ be the group of automorphisms of $F_{\mathcal{C}}$ over $\mathbb{F}_4$,

$$\mathbb{S}_{\mathcal{C}} := \mathrm{Aut}(F_{\mathcal{C}}).$$

The groups $\mathbb{S}_2$ and $\mathbb{S}_{\mathcal{C}}$ are isomorphic. An explicit isomorphism is constructed in Theorem 3.2. The group $\mathbb{S}_{\mathcal{C}}$ admits an action of the Galois group and the group $\mathbb{G}_{\mathcal{C}}$ is the extension of $\mathbb{S}_{\mathcal{C}}$ by this action. The group $\mathbb{G}_{\mathcal{C}}$ acts on the deformations. By the Goerss-Hopkins-Miller theorem [8, §7], it acts on $E_{\mathcal{C}}$ by maps of E_∞-ring spectra.

The isomorphism of Section 3 does not extend to an isomorphism of the groups $\mathbb{G}_2$ and $\mathbb{G}_{\mathcal{C}}$. In fact, these groups are not isomorphic. However, over an algebraic closure of $\mathbb{F}_2$, the formal group laws F_2 and $F_{\mathcal{C}}$ are isomorphic. Therefore, the Bousfield classes of $K(2)$ and $K_{\mathcal{C}}$ are the same. Their localization functors are weakly equivalent, so that

$$L_{K(2)}X \simeq L_{K_{\mathcal{C}}}X.$$

It follows from the work of Devinatz and Hopkins in [7] that for X a finite spectrum

$$L_{K_{\mathcal{C}}}X \cong E_{\mathcal{C}}^{h\mathbb{G}_{\mathcal{C}}} \wedge X.$$

Further, for any closed subgroup G of $\mathbb{G}_{\mathcal{C}}$ and any finite spectrum X, there is a spectral sequence analogous to (1.2). Therefore, for any finite X, there is a convergent spectral sequence

$$E_2^{s,t} := H^s(\mathbb{G}_{\mathcal{C}}, (E_{\mathcal{C}})_t X) \Longrightarrow \pi_{t-s} L_{K_{\mathcal{C}}}(X) \cong \pi_{t-s} L_{K(2)}(X),$$

where

$$(E_{\mathcal{C}})_* X := \pi_* L_{K_{\mathcal{C}}}(E_{\mathcal{C}} \wedge X).$$

The groups $\mathbb{S}_{\mathcal{C}}$ and $\mathbb{G}_{\mathcal{C}}$ also admit a norm induced by a general linear representation of $\mathbb{S}_{\mathcal{C}}$. The groups $\mathbb{S}^1_{\mathcal{C}}$ and $\mathbb{G}^1_{\mathcal{C}}$ are defined to be the norm one subgroups. Further,

$$\mathbb{S}_{\mathcal{C}} \cong \mathbb{S}^1_{\mathcal{C}} \rtimes \mathbb{Z}_2,$$

and

$$\mathbb{G}_{\mathcal{C}} \cong \mathbb{G}^1_{\mathcal{C}} \rtimes \mathbb{Z}_2.$$

Since $\mathbb{S}_{\mathcal{C}}$ is isomorphic to $\mathbb{S}_2$, the results of [2] also hold for $\mathbb{S}_{\mathcal{C}}$. In particular, the resolution of Theorem 1.4 can be constructed using $\mathbb{S}^1_{\mathcal{C}}$. Further, the algebraic duality resolution gives rise to a spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_p, (E_{\mathcal{C}})_* V(0)) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}; (E_{\mathcal{C}})_* V(0)). \tag{1.6}$$

This spectral sequence is isomorphic to the spectral sequence of Theorem 1.4. In this paper, we compute the E_∞-term of (1.6).

The main advantage of using $\mathbb{S}_{\mathcal{C}}$ is that the maximal finite subgroup of $\mathbb{S}_{\mathcal{C}}$ corresponds to those automorphisms of $F_{\mathcal{C}}$ which are induced by automorphisms of the elliptic curve $\mathcal{C}$. For the super-singular curve, Tate has shown that the natural map

$$\rho : \mathrm{End}(\mathcal{C}) \otimes \mathbb{Z}_2 \to \mathrm{End}(F_{\mathcal{C}}).$$

is an isomorphism (see [3] or [19]). Therefore, the group $\mathrm{Aut}(\mathcal{C})$ injects into $\mathbb{S}_{\mathcal{C}}$. Further,

$$\mathrm{Aut}(\mathcal{C}) \cong G_{24}, \tag{1.7}$$

so that $\mathrm{Aut}(\mathcal{C})$ is a choice of maximal finite subgroup of $\mathbb{S}_{\mathcal{C}}$. For the remainder of this paper, we let G_{24} denote $\mathrm{Aut}(\mathcal{C})$ in $\mathbb{S}_{\mathcal{C}}$.

Using level three structures, Strickland has computed the action of G_{24} on $(E_{\mathcal{C}})_*$. Strickland's results are used heavily in the computation of (1.6). They are not in print and will be described in Section 2.4.

1.2. **Statement of Results.** In order to state the results, we will describe the E_1-term of (1.6). It follows from (1.1) that

$$(E_{\mathcal{C}})_* V(0) = \mathbb{F}_4[[u_1]][u^{\pm 1}]$$

where u_1 has degree 0 and u has degree -2. Let $v_1 = u_1 u^{-1}$ in $(E_{\mathcal{C}})_* V(0)$. Let $F_{E_{\mathcal{C}}}$ be the graded formal group law of $E_{\mathcal{C}}$. Then

$$[2]_{F_{E_{\mathcal{C}}}}(x) \equiv v_1 x^2 + \dots \mod (2).$$

The element v_1 is invariant under the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_* V(0)$ that is,

$$v_1 \in H^0(\mathbb{S}^1_{\mathcal{C}}; (E_{\mathcal{C}})_* V(0)).$$

Let β be the Bockstein homomorphism associated to the exact sequence

$$0 \to (E_2)_* \xrightarrow{2} (E_2)_* \to (E_2)_* V(0) \to 0.$$

Let $h_1 = \beta(v_1)$ and $v_2 = u^{-3}$. Then

$$H^*(C_6; (E_{\mathcal{C}})_* V(0)) \cong \mathbb{F}_4[[u_1^3]][v_1, v_2^{\pm 1}, h]/(v_2^{-1} v_1^3 = u_1^3),$$

for a class $h \in H^1(C_6; (E_{\mathcal{C}})_* V(0))$ satisfying $h_1 = h v_1$. In particular,

$$H^0(C_6; (E_{\mathcal{C}})_* V(0)) = \mathbb{F}_4[[u_1^3]][v_1, v_2^{\pm 1}]/(v_2^{-1} v_1^3 = u_1^3).$$

Therefore, a set of $\mathbb{F}_4[v_1]$ generators of $H^k(C_6; (E_{\mathcal{C}})_* V(0))$ is given by

$$\{h^k v_2^n\}_{n \in \mathbb{Z}}.$$

The cohomology G_{24} is harder to describe. It is related to the cohomology of the Hopf algebroid classifying Weierstrass curves over $\mathbb{F}_4$ with their strict isomorphisms (see [1]). In particular, the G_{24} fixed points are related to modular forms modulo 2. In fact,

$$H^0(G_{24},(E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4[[j]][v_1,\Delta^{\pm 1}]/(j\Delta = v_1^{12}),$$

where v_1 as defined above is the Hasse invariant, Δ is the determinant and j is the j-invariant. The higher cohomology is described in Section 4 and is depicted in Figure 4.1. A set of $\mathbb{F}_4[v_1]$ generators for $H^0(G_{24},(E_{\mathcal{C}})_*V(0))$ is given by

$$\{\Delta^n\}_{n\in\mathbb{Z}}.$$

Theorem 1.8. *The algebraic duality resolution spectral sequence converging to $H^*(\mathbb{S}_2^1,(E_{\mathcal{C}})_*V(0))$ collapses at the E_2-term. The spectral sequence is a module over $\mathbb{F}_4[v_1,h_1]$. There exist $\mathbb{F}_4[v_1]$-generators $a_n \in E_1^{0,0}$, $b_n \in E_1^{1,0}$, $c_n \in E_1^{2,0}$ and $d_n \in E_1^{3,0}$ with*

$$\begin{aligned} b_n \equiv c_n &\equiv v_2^n \mod (v_1) \\ a_n \equiv d_n &\equiv \Delta^n \mod (v_1) \end{aligned}$$

and such that, for $k \geq 0$ and $t\in\mathbb{Z}$,

$$d_1(a_n) = \begin{cases} v_1^{6\cdot 2^k} b_{2^{k+1}(1+4t)} & n = 2^k(1+2t); \\ 0 & n=0. \end{cases}$$

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} c_{2^{k+1}(1+2t)} & n = 2^k(3+4t); \\ v_1^{3\cdot 2^{k+1}} c_{1+2^{k+1}+t2^{k+3}} & n = 1+2^{k+2}+t2^{k+3}; \\ 0 & \textit{otherwise.} \end{cases}$$

$$d_1(c_n) = \begin{cases} v_1^{3(2^{k+1}+1)} d_{2^k(1+2t)} & n = 1+2^{k+1}+2^{k+2}+t2^{k+3}; \\ 0 & \textit{otherwise.} \end{cases}$$

A differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is non-zero if and only if it is forced by h_1-linearity. All differentials $d_r : E_r^{p,q} \to E_r^{p+1,q}$ for $r\geq 2$ are zero, so that $E_2 = E_\infty$.

It is worth mentioning here that the related computation of the E_2-page of the Johnson-Wilson $E(2)$-local Adams-Novikov spectral sequence converging to L_2S was done by Shimomura and Wang in [17]. Their work is impressive, although it is hard to understand and verify. Our computations were done independently. However, historically, they depend on the work of Shimomura and Wang. Indeed, results similar to those of Theorem 1.8 can be extracted from [17], and it is using Shimomura and Wang's computation that Mahowald conjectured the existence of the duality resolution for the $K(2)$-local sphere.

1.3. **Organization of the paper.** In Section 2.1, we review the deformation theory of formal group laws. In Section 2.2, we recall how the action of the group of automorphisms of a formal group law acts on the theory classifying its deformations. In Section 2.3, we describe the universal deformation $\mathcal{C}_U$ of the super-singular elliptic curve $\mathcal{C}$. This choice of deformation is due to Strickland. This allows us to define $E_{\mathcal{C}}$. In Section 2.4, we describe the group of automorphisms of $\mathcal{C}$ and give explicit formulas for its action on $E_{\mathcal{C}}$. The author learned these results from unpublished notes of Strickland. The proofs given here are either his or constructed using his results.

Section 3 is dedicated to describing the structure of $\mathbb{S}_{\mathcal{C}}$. In Section 3.1, we give an explicit isomorphism between the group of automorphisms of the Honda formal group law $\mathbb{S}_2$ and the group $\mathbb{S}_{\mathcal{C}}$. In Section 3.2, we recall the standard filtration on $\mathbb{S}_{\mathcal{C}}$. In Section 3.3, we give the information about the action $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ that will be used in the computation of $H^*(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$. The proofs are postponed to Section 8.

The goal of Section 4 is to introduce the algebraic duality resolution spectral sequence (ADRSS) for $\mathbb{S}_{\mathcal{C}}$ and to give the information necessary to begin the computation. In Section 4.1, we recall the construction of the ADRSS that was given in [2]. The ADRSS is not multiplicative, but it has some nice properties which we describe in Section 4.2. In Section 4.3, we give a detailed description of the E_1-term. The discriminant Δ of the curve $\mathcal{C}_U$ has useful linearity properties which are given in Section 4.4.

The bulk of the paper is the computation of the E_∞-term of the ADRSS with coefficients in $(E_{\mathcal{C}})_*V(0)$. This is done in Section 5. In Sections 5.1, 5.2 and 5.3, we compute the differentials $d_1 : E_1^{p,0} \to E_1^{p+1,0}$. In Section 5.4, we compute the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$. In Section 5.5, we prove that all differentials $d_r : E_r^{p,q} \to E_r^{p,1}$ for $r \geq 2$ are zero.

This paper has three appendices. Section 6 describes some projective resolutions which are used in the above computations. These are C_3-equivariant analogues of some of the classical projective resolutions which can be found in [6]. Although we do not give references, we believe these results are folklore. Section 7 describes the v_1-Bockstein spectral sequence whose construction can be found in [6, §1]. We also use this spectral sequence to compare the cohomology of $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$ and $H^*(A_4, (E_{\mathcal{C}})_*V(0))$, where $A_4 = G_{24}/C_2$. This comparison is used in the computations of Section 5. In Section 8, we describe the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$. First, we give formulas for the minus two series of $\mathcal{C}$ and $\mathcal{C}_U$. We then use these formulas to give estimates for the action.

1.4. **Acknowledgements.** I thank Paul Goerss, Hans-Werner Henn and Peter May for their unfaltering support. I thank Neil Strickland for his unpublished notes, which were crucial to these computations. Finally, I thank Mark Mahowald whose insight continues to inspire me.

2. Morava E-theory and Elliptic Curves

In this section, I will explain how the spectrum $E_{\mathcal{C}}$ arises from deformation theory of the super singular elliptic curve $\mathcal{C}$. I will explain how this is used to compute the action of the automorphisms of $\mathcal{C}$ on the coefficients $(E_{\mathcal{C}})_*$, results which are due to Strickland.

2.1. **Deformations.** Let k be a perfect field of characteristic $p > 0$, and Γ be a formal group law of height n over k. Let $\mathcal{R}$ be the category of complete Noetherian local rings with continuous homomorphisms. Let $B \in \mathcal{R}$ with maximal ideal $\mathfrak{m}$ and projection $\pi : B \to B/\mathfrak{m}$. Then $\mathrm{Def}_\Gamma(B)$ is the groupoid whose objects are pairs (G, i) where G is a formal group law over B and i is an isomorphism

$$i : k \to B/\mathfrak{m}$$

such that

$$\pi_* G \cong i_* \Gamma.$$

A morphism between two pairs (G_1, i) and (G_2, i) with the same structure morphism i is an isomorphism $f : G_1 \to G_2$ of formal group laws such that $\pi_* f$ induces the identity on $i_*\Gamma$. These are called $\star$-isomorphisms. This defines a functor

$$\mathrm{Def}_\Gamma(-) : \mathcal{R} \to \mathcal{G}$$

where $\mathcal{G}$ denotes the category of groupoids. The Lubin-Tate theorem describes the representability of this functor.

Theorem 2.1 (Lubin, Tate)**.** *There exists a complete local ring $R(k,\Gamma)$ and a formal group law F_R over $R(k,\Gamma)$ that represents the functor $\pi_0 \mathrm{Def}_\Gamma(-)$ in the following sense. For B in $\mathcal{R}$,*

$$\mathrm{Hom}_{\mathcal{R}}(R(k,\Gamma), B) \cong \pi_0(\mathrm{Def}_\Gamma(B)).$$

Given a representative (G, i) of a $\star$-isomorphism class in $\pi_0 \mathrm{Def}_\Gamma(B)$, there is a unique ring homomorphism $\phi : R(k,\Gamma) \to B$ and a unique $\star$-isomorphism

$$f : \phi^* F_R \to G.$$

Further, if $W(k)$ denotes the Witt vectors on k, then

$$R(k,\Gamma) \cong W(k)[[u_1, \ldots, u_n]].$$

Let u be in degree -2. Then

$$F_E := uF_R(u^{-1}x, u^{-1}y)$$

defines a graded formal group law over

$$E(k,\Gamma)_* := R(k,\Gamma)[u^{\pm 1}].$$

This gives $E(k,\Gamma)_*$ the structure of a Landweber exact MU_*-module (see, for example, [14, §6]). The associated homology theory is complex oriented and two periodic. It is represented by a ring spectrum $E(k,\Gamma)$ such that

$$E(k,\Gamma)_* \simeq W(k)[[u_1, \ldots, u_n]][u^{\pm 1}].$$

By the Goerss-Hopkins-Miller theorem, $E(k,\Gamma)$ is an E_∞-ring spectrum (see [8]).

Definition 2.2. *Let $E_{\mathcal{C}} = E(\mathbb{F}_4, F_{\mathcal{C}})$, where $F_{\mathcal{C}}$ is the formal group law of the super-singular elliptic curve $\mathcal{C}$ defined in (1.5).*

2.2. **The action of** $\mathrm{Aut}(\Gamma)$**.** The group $\mathrm{Aut}(\Gamma)$ acts on $R(k,\Gamma)$ as follows. An element $\gamma \in \mathrm{Aut}(\Gamma)$ is a power series in $k[[x]]$. Let g in $R(k,\Gamma)[[x]]$ be a lift of γ. Define a new formal group law by

$$F_g(x,y) = g^{-1}F_R(g(x), g(y)).$$

Then F_g is a deformation of Γ over $R(k,\Gamma)$. By Theorem 2.1, there exists a unique ring isomorphism

$$\phi_\gamma : R(k,\Gamma) \to R(k,\Gamma) \tag{2.3}$$

and a unique $\star$-isomorphism

$$f_g : (\phi_\gamma)_* F_R \to F_g$$

which classify F_g. If h is another lift of γ, then

$$h^{-1} g f_g : (\phi_\gamma)_* F_R \to F_h$$

is a $\star$-isomorphism. Therefore, ϕ_γ is independent of the choice of lift g. This gives an action of $\mathrm{Aut}(\Gamma)$ on $R(k,\Gamma)$.

To extend this to an action on $E(k,\Gamma)_*$, let f_γ be the composite

$$(\phi_\gamma)_* F_R \xrightarrow{f_g} F_g \xrightarrow{g} F_R.$$

Define

$$\phi_\gamma(u) := f'_\gamma(0)u. \tag{2.4}$$

This is extends the action $\mathrm{Aut}(\Gamma)$ to $E(k,\Gamma)_*$ (see [14, §6]). By the Goerss-Hopkins-Miller theorem, this action can be realized through maps of E_∞-ring spectra on $E(k,\Gamma)$ (see [8]). Further, $\mathrm{Gal}(k/\mathbb{F}_p)$ acts on $W(k)$; hence, it acts on the coefficients $E(k,\Gamma)_*$. If Γ is fixed by $\mathrm{Gal}(k/\mathbb{F}_p)$, this extends the action of $\mathrm{Aut}(\Gamma)$ to an action of

$$\mathrm{Aut}(\Gamma) \rtimes \mathrm{Gal}(k/\mathbb{F}_p).$$

2.3. **The super-singular elliptic curve.** Elliptic curves over fields of characteristic $p > 0$ admit a theory of deformations which is analogous to that of formal group laws. In fact, for the super-singular elliptic curve, there is an equivalence of groupoids

$$\mathrm{Def}_{F_\mathcal{C}}(B) \simeq \mathrm{Def}_\mathcal{C}(B), \tag{2.5}$$

which can be explained as follows.

Let $\widehat{\mathcal{C}}$ be the formal group of the elliptic curve $\mathcal{C}$. The map which sends a deformation of $F_\mathcal{C}$ to its associated formal group is an equivalence of groupoids

$$\mathrm{Def}_{F_\mathcal{C}}(B) \simeq \mathrm{Def}_{\widehat{\mathcal{C}}}(B). \tag{2.6}$$

For super-singular curves over fields of characteristic $p > 0$, there is an isomorphism

$$\widehat{\mathcal{C}} \cong \mathcal{C}[p^\infty], \tag{2.7}$$

where $\mathcal{C}[p^\infty]$ denotes the p-divisible subgroup of $\mathcal{C}$. Finally, the Serre-Tate Theorem relates the deformations of $\mathcal{C}[p^\infty]$ to the deformations of the curve $\mathcal{C}$ (see [15, §2.9]).

Theorem 2.8 (Serre,Tate)**.** *Let B be in $\mathcal{R}$. There is an equivalence of groupoids*

$$\mathrm{Def}_\mathcal{C}(B) \simeq \mathrm{Def}_{\mathcal{C}[p^\infty]}(B),$$

which sends a deformation E/B of $\mathcal{C}$ to its p-divisible group $E[p^\infty]$.

Theorem 2.8 together with (2.6) and (2.7) imply (2.5). For the super-singular curve

$$\mathcal{C} : y^2 - y = x^3,$$

these facts are made concrete by the following theorem.

Theorem 2.9. *The formal group law of the elliptic curve*

$$\mathcal{C}_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $\mathbb{W}[[u_1]]$ is a universal deformation of $F_\mathcal{C}$. This specifies an isomorphism

$$(E_\mathcal{C})_* \cong \mathbb{W}[[u_1]][u^{\pm 1}],$$

and a formal group law

$$F_{E_\mathcal{C}} = uF_{\mathcal{C}_U}(u^{-1}x, u^{-1}y), \tag{2.10}$$

where $F_{\mathcal{C}_U}$ denotes the formal group law of the curve $\mathcal{C}_U$.

This choice of universal deformation C_U is due to Strickland. In order to prove Theorem 2.9, we will use the following facts about isomorphisms of Weierstrass curves. Let $\mathcal{E}$ be an elliptic curve defined over R by a Weierstrass equation

$$\mathcal{E} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \tag{2.11}$$

If $\mathcal{E}'$ is a Weierstrass curve with coefficients a_i', an isomorphism $f : \mathcal{E} \to \mathcal{E}'$ is given by a change of coordinates of the form

$$(x, y) = (l^2x' + r, l^3y' + l^2sx' + t), \tag{2.12}$$

where (l, r, s, t) is a tuple in R and l is a unit (see [18, §III]). This forces the following relations on (l, r, s, t):

$$\begin{aligned}
la_1' &= a_1 + 2s \\
l^2a_2' &= a_2 - sa_1 + 3r - s^2 \\
l^3a_3' &= a_3 + ra_1 + 2t \\
l^4a_4' &= a_4 - sa_3 + 2ra_2 - (t + rs)a_1 + 3r^2 - 2st \\
l^6a_6' &= a_6 + ra_4 + r^2a_2 + r^3 - ta_3 - t^2 - rta_1.
\end{aligned}$$

Proof of Theorem 2.9. The equivalence (2.5) implies that there exists a ring R and an elliptic curve $\mathcal{C}_R$ whose formal group law $F_{\mathcal{C}_R}$ is a universal deformation of $F_{\mathcal{C}}$. By the Lubin-Tate theorem, there is a non-canonical isomorphism $R \cong \mathbb{W}[[u_1]]$. The curve $\mathcal{C}_U$ is a deformation of $\mathcal{C}$, hence there is a ring homomorphism

$$\phi : R \to \mathbb{W}[[u_1]]$$

such that

$$\phi^*\mathcal{C}_R \cong \mathcal{C}_U. \tag{2.13}$$

It is sufficient to show that ϕ is an isomorphism. For a complete local ring B with maximal ideal $\mathfrak{m}_B$, the tangent space of B is defined by

$$\tau_B = \mathfrak{m}_B/\mathfrak{m}_B^2.$$

A morphism of power series rings is an isomorphism if and only if it induces an isomorphism on tangent spaces. Let τ_R and τ_W denote the tangent spaces of R and $\mathbb{W}[[u_1]]$ respectively. Then τ_R and τ_W are $\mathbb{F}_4$ vector spaces of the same dimension and, therefore, the induced map

$$\tau_\phi : \tau_R \to \tau_W,$$

is an isomorphism if and only if it is surjective. The curve $\mathcal{C}_R$ is given by a Weierstrass equation

$$y^2 + a_1xy + a_3y^2 = x^3 + a_2x^2 + a_4x + a_6,$$

and the coefficients of the Weierstrass equation of $\phi^*\mathcal{C}_R$ are $\phi(a_i)$. The isomorphism (2.13) is given by a change of coordinates of the form (2.12), where (l, r, s, t) are elements of $\mathbb{W}[[u_1]]$ and l is a unit. This imposes the relation

$$\phi^*(a_1) = l^{-1}(3u_1 + 2s).$$

Hence

$$\phi^*(a_1) \equiv u_1 + 2\overline{s} \mod \mathfrak{m}_W^2,$$

where $\overline{s} \in \mathbb{F}_4$. But $\mathbb{F}_4$ is in the image of the induced map

$$\tau_\phi : \tau_R \to \tau_W,$$

which implies that τ_ϕ is surjective. □

2.4. **The automorphisms of** C**.** In unpublished notes, Strickland has computed the action of the group $\mathrm{Aut}(\mathcal{C})$ on $(E_\mathcal{C})_*$. We explain his results in this section.

The relations on the coefficients a_i for an isomorphism of Weierstrass curves $\mathcal{E}$ over R can be used to compute the automorphisms of $\mathcal{E}$. For the super-singular curve $\mathcal{C}$, this is done in [18, Appendix A]. I give the results here.

Fix a primitive third root of unity $\zeta \in \mathbb{F}_4$. For the curve $\mathcal{C}$ over $\mathbb{F}_4$, the group $\mathrm{Aut}(\mathcal{C})$ is generated by the elements

$$\omega := (\zeta^2, 0, 0, 0),$$
$$i := (1, 1, 1, \zeta),$$

The subgroup $C_3 := \langle \omega \rangle$ is cyclic of order three, so that $\omega^{-1} = \omega^2$. The element i satisfies $i^2 = -1$. Let

$$j := \omega i \omega^2,$$
$$k := \omega^2 i \omega.$$

Then $ij = k$, so that i and j generate a normal subgroup isomorphic to the quaternions Q_8 and

$$\mathrm{Aut}(\mathcal{C}) \cong Q_8 \rtimes C_3.$$

An automorphism of $\mathcal{C}$ induces an automorphism of $F_\mathcal{C}$. This gives a map

$$\rho : \mathrm{Aut}(\mathcal{C}) \to \mathrm{Aut}(F_\mathcal{C}),$$

which Tate has shown is injective (see [3]). Define

$$G_{24} := \rho(\mathrm{Aut}(\mathcal{C})).$$

Let γ be in G_{24} and ϕ_γ be as in (2.3). The curve $\phi_\gamma^* \mathcal{C}_U$ is a deformation of $\mathcal{C}$, so there exists a unique isomorphism of elliptic curves

$$f_\gamma : \phi_\gamma^* \mathcal{C}_U \to \mathcal{C}_U,$$

which covers γ. Strickland has constructed ϕ_γ and the lifts f_γ for the generators ω and i by using level three structures on the curves $\mathcal{C}$ and $\mathcal{C}_U$. First, he constructs isomorphisms ϕ_ω and ϕ_i of $(E_\mathcal{C})_0$ given by

$$\phi_\omega(\zeta) = \zeta \qquad \phi_i(\zeta) = \zeta$$
$$\phi_\omega(u_1) = \zeta u_1 \qquad \phi_i(u_1) = \frac{u_1 + 2}{u_1 - 1}.$$

Let a_i' be the coefficients of $\mathcal{C}_U$ and $a_i = \phi_\gamma(a_i')$ the coefficients of $\phi_\gamma^* \mathcal{C}_U$. The relations on the $a_i's$ determine the tuples (l, r, s, t):

$$f_\omega = (\zeta^2, 0, 0, 0)$$
$$f_i = \left(\frac{\zeta^2 - \zeta}{u_1 - 1}, 3\frac{1 - u_1^3}{(u_1 - 1)^3}, 3\frac{\zeta^2 u_1 - 1}{u_1 - 1}, 3\frac{u_1^3 - 1}{(u_1 - 1)^4}((1 - \zeta) + (1 - \zeta^2)u_1) \right).$$

Note that $\zeta^2 - \zeta$ is a square root of -3. This choice is unique up to the action of the Galois group, which preserves $\mathcal{C}$. The maps f_ω and f_i lift ω and i and the isomorphisms ϕ_ω and ϕ_i generate the action of G_{24} on $(E_2)_0$.

Finally, if $F_{\phi_\gamma^*\mathcal{C}_U}$ denotes the formal group law associated to the curve $\phi_\gamma^*\mathcal{C}_U$, then

$$F_{\phi_\gamma^*\mathcal{C}_U} = \phi_\gamma^* F_{\mathcal{C}_U}$$

and the induced isomorphism on formal group laws satisfies $f'_\gamma(0) = l$. By (2.4),

$$\phi_\omega(u) = \zeta u \qquad\qquad \phi_i(u) = u\frac{\zeta^2 - \zeta}{u_1 - 1}. \tag{2.14}$$

3. The Morava stabilizer group

The Morava stabilizer group $\mathbb{S}_2$ is the group of automorphisms of the Honda formal group law F_2, which is the p-typical formal group law over $\mathbb{F}_4$ specified by the series

$$[2]_{F_2}(x) = x^4.$$

The standard presentation for $\mathbb{S}_2$ is the non-commutative extension

$$\mathbb{S}_2 \cong \left(\mathbb{W}\langle S\rangle/(S^2 = 2, aS = Sa^\sigma)\right)^\times,$$

where S is the automorphism $S(x) = x^2$ and $a \in \mathbb{W}$ (see [13, Appendix A2] or [2] for more details.) In this section, I will specify an isomorphism $\mathbb{S}_2 \cong \mathbb{S}_{\mathcal{C}}$, whose construction I owe to Henn. I will also recall some of the key properties of the structure of the group $\mathbb{S}_2$, which transfer to properties of $\mathbb{S}_{\mathcal{C}}$ via this isomorphism.

3.1. The isomorphism of $\mathbb{S}_2$ and $\mathbb{S}_{\mathcal{C}}$.

As opposed to the Honda formal group law, it is the $[-2]$-series of the formal group law $F_{\mathcal{C}}$ which has a nice presentation. The following result is proved in Proposition 8.11 of Section 8.

Lemma 3.1. *Let $\mathcal{C}$ be the super-singular elliptic curve defined by (1.5). If $F_{\mathcal{C}}$ is the associated formal group law, then*

$$[-2]_{F_{\mathcal{C}}}(x) = x^4.$$

The curve $\mathcal{C}$ and its formal group law $F_{\mathcal{C}}$ are defined over $\mathbb{F}_2$. Therefore,

$$T(x) = x^2$$

is an endomorphism of $F_{\mathcal{C}}$. Lemma 3.1 implies that $T(T(x)) = [-2](x)$. The element ω defined in Section 2.4 induces the isomorphism

$$\omega(x) = \zeta x$$

of $F_{\mathcal{C}}$, so that $\omega T = T\omega^\sigma$. This shows that

$$\mathbb{W}\langle T\rangle/(T^2 = -2, \omega T = T\omega^\sigma) \subseteq \mathrm{End}(F_{\mathcal{C}}).$$

By Proposition 21.8.7 of [10], this must be an equality. Therefore,

$$\mathbb{S}_{\mathcal{C}} \cong \left(\mathbb{W}\langle T\rangle/(T^2 = -2, \omega T = T\omega^\sigma)\right)^\times.$$

Let σ be the Fröbenius element in $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. The action of Gal on $\mathbb{W}$ induces an action on $\mathbb{S}_{\mathcal{C}}$ defined by

$$a + bT \mapsto a^\sigma + b^\sigma T.$$

In [2], we constructed an element α in $\mathbb{W}^\times$ defined as

$$\alpha := \frac{1 - 2\omega}{\sqrt{-7}},$$

so that $\alpha = 1 + \omega T^2 + T^4 + \ldots$ and

$$\alpha\alpha^\sigma = -1.$$

Theorem 3.2. *The groups $\mathbb{S}_2$ and $\mathbb{S}_{\mathcal{C}}$ are isomorphic.*

Proof. Each element $\gamma \in \mathbb{S}_{\mathcal{C}}$ can be expressed uniquely as $a + bT$ for a and b in $\mathbb{W}$ and a a unit. On the other hand, the elements of $\mathbb{S}_2$ admit a similar representation as $a + bS$. The map $\mathbb{S}_{\mathcal{C}} \to \mathbb{S}_2$,

$$a + bT \mapsto a + b(\alpha S),$$

is an isomorphism. □

3.2. **The filtration and the norm.** Theorem 3.2 implies that all the results of [2] can be restated for the group $\mathbb{S}_{\mathcal{C}}$ instead of $\mathbb{S}_2$. Here, I briefly review those results which will be important for the computations of this paper.

As in [2], any element $\gamma \in \mathbb{S}_{\mathcal{C}}$ can be expressed as a power series

$$\gamma = \sum_{n=0}^{\infty} a_n T^n,$$

where the a_i's satisfy the equation $x^4 - x = 0$ and $a_0 \neq 0$. Let $F_{0/2}\mathbb{S}_{\mathcal{C}} := \mathbb{S}_{\mathcal{C}}$. For $n > 0$, let

$$F_{n/2}\mathbb{S}_{\mathcal{C}} := \{\gamma \in \mathbb{S}_{\mathcal{C}} \mid \gamma \equiv 1 \mod T^n \}. \tag{3.3}$$

Define

$$S_{\mathcal{C}} := F_{1/2}\mathbb{S}_{\mathcal{C}}.$$

Then $S_{\mathcal{C}}$ is the 2-Sylow subgroup of $\mathbb{S}_{\mathcal{C}}$. This filtration is compatible with the 2-adic filtration on $\mathbb{W}^\times$. Further, $\{F_{n/2}\mathbb{S}_{\mathcal{C}}\}_{n \geq 0}$ forms a system of open subgroups and $\mathbb{S}_{\mathcal{C}}$ is a profinite topological group.

Recall the following result follows from Theorem 2.29 of [2].

Proposition 3.4. *The subgroup generated by G_{24}, π and α is dense in $\mathbb{S}_{\mathcal{C}}$.*

The group $\mathbb{S}_{\mathcal{C}}$ acts on $\mathrm{End}(F_{\mathcal{C}})$ by right multiplication. This gives rise to a representation $\rho : \mathbb{S}_{\mathcal{C}} \to GL_2(\mathbb{W})$, given by

$$\rho(a + bT) = \begin{pmatrix} a & b \\ -2b^\sigma & a^\sigma \end{pmatrix}.$$

The restriction of the determinant to $\mathbb{S}_{\mathcal{C}}$ is given by

$$\det(a + bT) = aa^\sigma + 2bb^\sigma.$$

Therefore, the determinant induces a map $\det : \mathbb{S}_{\mathcal{C}} \to \mathbb{Z}_2^\times$. The *norm* is defined as the composite

$$N : \mathbb{S}_{\mathcal{C}} \xrightarrow{\det} \mathbb{Z}_2^\times \to \mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2.$$

The norm is split surjective. Indeed, let

$$\pi = 1 + 2\omega.$$

Then $\det(\pi) = 3$ projects to a topological generator of $\mathbb{Z}_2^\times/\{\pm 1\}$. The subgroup $\mathbb{S}_{\mathcal{C}}^1$ is then defined by the short exact sequence,

$$1 \to \mathbb{S}_{\mathcal{C}}^1 \to \mathbb{S}_{\mathcal{C}} \xrightarrow{N} \mathbb{Z}_2^\times/\{\pm 1\} \to 1,$$

and

$$\mathbb{S}_{\mathcal{C}} \cong \mathbb{S}^1_{\mathcal{C}} \rtimes \mathbb{Z}_2^\times/\{\pm 1\}.$$

Note that $\mathbb{Z}_2^\times/\{\pm 1\} \cong \mathbb{Z}_2$ is torsion-free; hence, G_{24} is a subgroup of $\mathbb{S}^1_{\mathcal{C}}$.

The following result was shown in Lemma 2.27 of [2], based on results of [4].

Proposition 3.5. *The group $\mathbb{S}_{\mathcal{C}}$ contains a unique conjugacy class of maximal finite subgroups isomorphic to G_{24}. Further, $\mathbb{S}^1_{\mathcal{C}}$ contains two conjugacy classes of maximal finite subgroups, represented by G_{24} and $G'_{24} = \pi G_{24} \pi^{-1}$.*

3.3. **The action of the Morava stablizer group.** In order to compute the cohomology of $\mathbb{S}_{\mathcal{C}}$, it is necessary to understand its action on $(E_{\mathcal{C}})_*$. The action of the elements of G_{24} was computed by Strickland, and his results were explained in Section 2.4. By Proposition 3.4, it thus suffices to understand the action of α and π on $(E_{\mathcal{C}})_*$ to approximate the action of any element of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$.

A concrete method for approximating the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ was developed in [11]. We describe it in Section 8. We state the key results here and prove them there.

Theorem 3.6. *For γ in $\mathbb{S}_{\mathcal{C}}$, there exists $t_0(\gamma)$ in $(E_{\mathcal{C}})_0^\times$ and $t_1(\gamma)$ in $(E_{\mathcal{C}})_0$ such that*

$$\begin{aligned} \phi_\gamma(u) &= t_0(\gamma)u, \\ \phi_\gamma(u_1) &= t_0(\gamma)u_1 + \frac{2}{3}\frac{t_1(\gamma)}{t_0(\gamma)}. \end{aligned}$$

In particular, modulo (2),

$$\begin{aligned} \phi_\gamma(u_1) &\equiv t_0(\gamma)u_1, \\ \phi_\gamma(u) &\equiv t_0(\gamma)u. \end{aligned}$$

Therefore, $v_1 = u_1u^{-1}$ is fixed by the action of $\mathbb{S}_{\mathcal{C}}$ modulo (2).

Theorem 3.7. *Let $\gamma = 1 + \sum_{i=1}^\infty a_iT^i$ be in $F_{1/2}\mathbb{S}_{\mathcal{C}}$. Then*

$$t_0(\gamma) \equiv 1 \mod (2, v_1),$$

so that $\phi_\gamma \equiv id \mod (2, v_1)$. For $\gamma = 1 + a_2T^2 + \ldots$ in $F_{2/2}\mathbb{S}_{\mathcal{C}}$, modulo $(4, 2v_1^2, v_1^{10})$,

$$t_0(\gamma) \equiv 1 + 2a_2 + 2a_3^2u_1 + (a_2 + a_2^2)u_1^3 + a_3u_1^5 + a_3u_1^8 + (a_2 + a_2^2 + a_4 + a_4^2)u_1^9.$$

and

$$t_1(\gamma) \equiv a_2^2u_1 \mod (2, v_1^3).$$

If $\gamma = 1 + a_4T^4 + \ldots$ is in $F_{4/2}\mathbb{S}_{\mathcal{C}}$, then

$$\phi_\gamma \equiv id \mod (2, v_1^9).$$

We will also use the following result. Recall that the action of $[-1]_{F_{\mathcal{C}}}(x) \in \mathbb{S}_{\mathcal{C}}$ and $\omega \in C_3$ is given by

$$\begin{aligned} \phi_{-1}(u_1) &= u_1, & \phi_\omega(u_1) &= \zeta u_1, \\ \phi_{-1}(u) &= -u, & \phi_\omega(u) &= \zeta u, \end{aligned}$$

where ζ is a primitive third root of unity. Hence,

$$(E_{\mathcal{C}})_0^{C_6} = \mathbb{W}[[u_1]]^{C_6} = \mathbb{W}[[u_1]]^{C_3} = \mathbb{W}[[u_1^3]].$$

Lemma 3.8. *Let γ in $\mathbb{S}_{\mathcal{C}}$ be an element which commutes with ω in C_3. Then*

$$t_0(\gamma) \in \mathbb{W}[[u_1^3]].$$

Proof. Since $\gamma\omega = \omega\gamma$, we have

$$\phi_\gamma \circ \phi_\omega(u) = \phi_\omega \circ \phi_\gamma(u).$$

By (2.14), $\phi_\omega(u) = \zeta u$. This forces

$$\phi_\omega(t_0(\gamma)) = t_0(\gamma).$$

Therefore, $t_0(\gamma)$ is in $\mathbb{W}[[u_1]]^{C_3}$, where the action of C_3 is the $\mathbb{W}$-linear map determined by $\phi_\omega(u_1) = \zeta u_1$. This implies that $t_0(\gamma) \in \mathbb{W}[[u_1^3]]$. □

We now apply these results to study the action of

$$\alpha \equiv 1 + \omega T^2 \mod T^4.$$

Lemma 3.9. *The unit $t_0(\alpha)$ is an element of $(E_{\mathcal{C}})_0^{C_6}$. For ϵ_0 and ϵ in $(E_{\mathcal{C}})_0 V(0)^{C_6}$,*

$$t_0(\alpha) \equiv 1 + u_1^3 + \epsilon_0 u_1^6 \mod (2),$$

and for $v_2 = u^{-3}$

$$\phi_\alpha(v_2) = v_2 + v_1^3 + v_2^{-1} v_1^6 \epsilon \mod (2).$$

Further

$$\phi_\alpha \equiv \phi_{\alpha^{-1}} \mod (2, v_1^9).$$

Proof. The element α is in $\mathbb{W}$. Therefore, it commutes with ω. Lemma 3.8 implies that $t_0(\gamma)$ is in $(E_{\mathcal{C}})_0 V(0)^{C_6} = \mathbb{W}[[u_1^3]]$. The identities for $t_0(\alpha)$ and $\phi_\alpha(v_2)$ follows from Theorem 3.7, using the fact that, for α, the coefficient $a_2 = \omega$ and $a_3 = 0$. Finally, since $\alpha^2 \in F_{4/2} S_{\mathcal{C}}$, it follows from Theorem 3.7 that ϕ_{α^2} is the identity modulo $(2, v_1^9)$. Then, the claim follows from the fact that $\phi_{\alpha^{-1}} = \phi_\alpha^{-1}$ and that $\phi_{\alpha^2} = \phi_\alpha \circ \phi_\alpha$. □

Lemma 3.10. *Let $\pi = 1 + 2\omega$. Then*

$$\phi_\pi \equiv id \mod (2, u_1^3).$$

Proof. This follows from Theorem 3.7 since $\pi \in F_{2/2}\mathbb{S}_{\mathcal{C}}$ and, for π, $a_2 = \omega$. □

4. The algebraic duality resolution spectral sequence

4.1. **Preliminaries.** The results in the following theorem were shown in [2] for the group $\mathbb{S}_2^1$. I restate them here for the group $\mathbb{S}_{\mathcal{C}}^1$. The construction of the resolution is due to Goerss, Henn, Mahowald and Rezk. The descriptions of the maps ∂_1 and ∂_2 are due to the author and Henn.

Theorem 4.1. *Let $\mathbb{Z}_2$ be the trivial $\mathbb{S}_{\mathcal{C}}^1$-module. There is an exact sequence of complete $\mathbb{S}_{\mathcal{C}}^1$-modules*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

where $\mathscr{C}_0 \cong \mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1/G_{24}]]$, $\mathscr{C}_3 \cong \mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1/G_{24}']]$ and $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1/C_6]]$. Let e be the unit in $\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1]]$ and e_p be the resulting generator of $\mathscr{C}_p$. The maps ∂_p can be chosen to satisfy:

(i) $\partial_1(e_1) = (e - \alpha)e_0$,

(ii) $\partial_2(e_2) = \Theta e_1$ *for* $\Theta \in \mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]$ *such that*

$$\Theta \equiv e + \alpha \mod (2, (IS^1_{\mathcal{C}})^2).$$

Further, there are isomorphisms of $\mathbb{S}^1_{\mathcal{C}}$*-modules* $g_p : \mathscr{C}_p \to \mathscr{C}_p$ *and an exact sequence*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial'_3} \mathscr{C}_2 \xrightarrow{\partial'_2} \mathscr{C}_1 \xrightarrow{\partial'_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$$

such that

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{C}_3 & \xrightarrow{\partial_3} & \mathscr{C}_2 & \xrightarrow{\partial_2} & \mathscr{C}_1 & \xrightarrow{\partial_1} & \mathscr{C}_0 & \xrightarrow{\varepsilon} & \mathbb{Z}_2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \| & & \\ 0 & \longrightarrow & \mathscr{C}_3 & \xrightarrow{\partial'_3} & \mathscr{C}_2 & \xrightarrow{\partial'_2} & \mathscr{C}_1 & \xrightarrow{\partial'_1} & \mathscr{C}_0 & \xrightarrow{\varepsilon} & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

is an isomorphism of complexes, and the map $\partial'_3 : \mathscr{C}_3 \to \mathscr{C}_2$ *is given by*

$$\partial'_3(e_3) = \pi(e + i + j + k)(e - \alpha^{-1})\pi^{-1}e_2.$$

The duality resolution gives rise to a spectral sequence which computes the cohomology of $\mathbb{S}^1_{\mathcal{C}}$. Indeed, let M be a finitely generated complete $\mathbb{S}^1_{\mathcal{C}}$-module. There is a first quadrant spectral sequence,

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_p, M) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, M). \tag{4.2}$$

The differentials in (4.2) have degree

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1},$$

and

$$E_1^{p,q} \cong \begin{cases} H^q(G_{24}, M) & \text{if } p = 0; \\ H^q(C_6, M) & \text{if } p = 1, 2; \\ H^q(G'_{24}, M) & \text{if } p = 3. \end{cases}$$

4.2. **Some extra structure.** In our computation, we will need to use some additional structure in the algebraic duality resolution. We record that here. For any complete $\mathbb{S}^1_{\mathcal{C}}$-modules A and B, let

$$\mathrm{Ext}(A, B) := \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(A, B).$$

If B is an $\mathbb{S}^1_{\mathcal{C}}$-module which is free over the 2-adics $\mathbb{Z}_2$, then $B/2$ is defined by

$$0 \to B \xrightarrow{2} B \to B/2 \to 0. \tag{4.3}$$

Let $\beta : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B/2)$ be the Bockstein homomorphism, that is, the reduction modulo 2 of the connecting homomorphism of the long exact sequence for (4.3). The algebraic duality resolution

$$0 \to \mathscr{C}_3 \to \mathscr{C}_2 \to \mathscr{C}_1 \to \mathscr{C}_0 \to \mathbb{Z}_2 \to 0$$

is obtained from splicing exact sequences

$$0 \to N_p \to \mathscr{C}_p \to N_{p-1} \to 0 \tag{4.4}$$

with $\mathscr{C}_3 = N_2$ and $N_{-1} = \mathbb{Z}_2$ (see [2]). The exact couple

$$\begin{array}{ccc} \mathrm{Ext}(N_*, B/2) & \overset{\delta_*}{\cdots\cdots\cdots} & \mathrm{Ext}(N_{*-1}, B/2) \\ {\scriptstyle i^*}\nwarrow & & \swarrow{\scriptstyle r_*} \\ & \mathrm{Ext}(\mathscr{C}_*, B/2) & \end{array} \tag{4.5}$$

gives rise to the algebraic duality resolution spectral sequence. Here, the dotted arrows are the connecting homomorphisms for the exact sequences (4.4), thus they increase the cohomological degree.

Lemma 4.6. *Let $x \in E_r^{p,q}$ in the algebraic duality resolution spectral sequence. Then $\beta(x) \in E_r^{p,q+1}$ and $d_r(\beta(x)) = \beta(d_r(x))$.*

Proof. The maps d_*, r_*, i_* and δ_* in the exact couple (4.5) commute with β. A diagram chase shows that $d_r(\beta(x)) = \beta(d_r(x))$. □

Lemma 4.7. *Let R be an $\mathbb{S}^1_{\mathcal{C}}$-module which is also a ring. Suppose that the action of $\mathbb{S}^1_{\mathcal{C}}$ is given by ring homomorphisms. The algebraic duality resolution with coefficients R is a module over the cohomology $H^*_c(\mathbb{S}^1_{\mathcal{C}}; R)$.*

Proof. Note that $\mathrm{Ext}(A, R)$ is a module over $\mathrm{Ext}(\mathbb{Z}_2, R)$ for any $\mathbb{S}^1_{\mathcal{C}}$-module A. Further, the maps in the algebraic duality resolution are maps of $\mathbb{Z}_2$-modules. Therefore, the maps d_*, r_*, i_* and δ_* in the exact couple giving rise to the algebraic duality resolution are morphisms of $\mathrm{Ext}(\mathbb{Z}_2, R)$-modules. This implies that the differentials in the algebraic duality resolution are linear over $\mathrm{Ext}(\mathbb{Z}_2, R)$. □

Recall that

$$(E_{\mathcal{C}})_* V(0) = (E_{\mathcal{C}})_*/(2) = \mathbb{F}_4[[u_1]][u^{\pm 1}].$$

In Theorem 2.9, it was shown that

$$F_{E_{\mathcal{C}}} = u F_{\mathcal{C}_U}(u^{-1}x, u^{-1}y),$$

where $\mathcal{C}_U$ was defined by

$$\mathcal{C}_U : y^2 + 3u_1 xy + (u_1^3 - 1)y = x^3.$$

It follows from [18, §IV.1] that

$$[2]_{F_{E_{\mathcal{C}}}}(x) \equiv u^{-1}u_1 x^2 + u^{-3}(u_1^3 + 1)x^4 + \dots \mod (2).$$

Therefore, we can define

$$v_1 := u^{-1}u_1$$

and

$$v_2 := u^{-3}.$$

Note that the element v_1 is uniquely determined modulo (2), but v_2 is only defined modulo $(2, v_1)$. The element v_1 is invariant under the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_* V(0)$, and v_1 is an element of $H^0(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_* V(0))$. However, it does not lift to an invariant in $(E_{\mathcal{C}})_*$. Therefore, v_1 has a non-zero Bockstein. Define

$$h_1 := \beta(v_1). \tag{4.8}$$

Lemma 4.9. *The algebraic duality resolution spectral sequence is a spectral sequence of modules over $\mathbb{F}_4[v_1, h_1]$.*

Proof. By Lemma 4.7, the duality resolution is a module over $H^*(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$ and $H^*(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$ is a module over $\mathbb{F}_4[v_1, h_1]$. □

4.3. **The E_1-term.** I will now describe the E_1-term of the algebraic duality spectral sequence. If $p : \mathbb{W} \to \mathbb{F}_4$ is the projection, then $p^*\mathcal{C}_U$ is defined over $(E_{\mathcal{C}})_*V(0)$ and classifies deformations of $\mathcal{C}$ to complete local $\mathbb{F}_4$-algebras. Let $\mathcal{E}$ be a Weierstrass curve with coefficients a_i (2.11) and $\omega_{\mathcal{E}}$ be a generator for the module of invariant differentials on $\mathcal{E}$. If $\mathcal{E}$ is defined over an $\mathbb{F}_4$-algebra, then $a_1\omega_{\mathcal{E}}$ is the Hasse invariant of $\mathcal{E}$. For $p^*\mathcal{C}_U$, the element $u^{-1} \in (E_{\mathcal{C}})_2V(0)$ generates the invariant differentials (see [9, §2.2]), and therefore the Hasse invariant is precisely

$$v_1 \equiv 3u_1u^{-1} \mod (2).$$

Further, the discriminant of $\mathcal{C}_U$ is $\Delta_{\mathcal{C}_U} = 27(u_1^3 - 1)^3$, so that

$$\Delta := u^{-12}27(u_1^3 - 1)^3 \equiv v_2(v_2 + v_1^3)^3 \mod (2) \tag{4.10}$$

is invariant under the action of $\mathrm{Aut}(\mathcal{C})$ (see [18, §III.1]). Finally, the element $j = v_1^{12}\Delta^{-1}$ is the j-invariant, not to be confused with the quaternion element. Theorem 4.11 follows from [9, §4.4]. Its computational content is originally due to Hopkins and Mahowald, but the best reference is Bauer, [1, §7]. We have used the notation of [1] in the statement below.

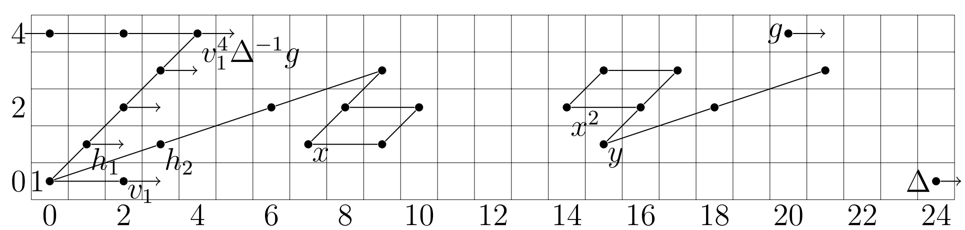

FIGURE 4.1. The cohomology $H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$, drawn in the Adams grading $(t-s, s)$. It is periodic of period $t = 24$ with respect to the element Δ. It is periodic of period 4 with respect to the element g. A $\bullet$ denotes a copy of $\mathbb{F}_4$. Lines of slope 1 denote multiplication by h_1 and lines of slope 1/3 denote multiplication by h_2. Horizontal lines denote multiplication by v_1. Classes attached to horizontal arrows are free over $\mathbb{F}_4[v_1]$.

Theorem 4.11. *There is an isomorphism*

$$H^*(G_{24}, (E_{\mathcal{C}})_*V(0)) \cong \mathbb{F}_4[[j]][v_1, \Delta^{\pm 1}, h_1, h_2, x, y, g]/(\sim),$$

where the degrees (s, t) (for s the cohomological grading, and t the internal grading) are given by

$$|v_1| = (0,2), \quad |\Delta| = (0,24), \quad |h_1| = (1,2), \quad |h_2| = (1,4)$$
$$|x| = (1,8), \quad |y| = (1,16), \quad |g| = (4,24), \quad |j| = (0,0),$$

and "$\sim$" denotes the following relations:

$$v_1h_2 = 0, \quad v_1y = 0, \quad v_1^2x = 0 \quad h_1h_2 = 0, \quad h_2^3 = h_1^2x, \quad h_1v_1x = h_2x$$
$$h_1^2x^2 = 0, \quad h_1y = v_1x^2, \quad h_2^3y = 0, \quad x^3 = 0 \quad v_1^4g = h_1^4\Delta, \quad \Delta^{-1}v_1^{12} = j.$$

Proof. Let $\mathcal{M}_{Weier}$ denote the stack of Weierstrass curves and ω be the canonical quasi-coherent sheaf of invariant 1-forms on $\mathcal{M}_{Weier}$. Let (A, Γ) be the Hopf algebroid classifying Weierstrass curves and their strict isomorphisms. Then

$$H^*(\mathcal{M}_{Weier}, \omega^{\otimes *}) \cong H^*(A, \Gamma).$$

The smooth locus of $\mathcal{M}_{sm} \subseteq \mathcal{M}_{Weier}$ is given by the points where the determinant Δ is invertible, hence

$$H^*(\mathcal{M}_{sm}, \omega^{\otimes *}) \cong H^*(A, \Gamma)[\Delta^{-1}].$$

Let $\widehat{\mathcal{M}}_{ss}$ be the formal neighborhood of the super-singular locus in $\mathcal{M}_{sm} \otimes \mathbb{F}_2$. Then

$$H^*(\widehat{\mathcal{M}}_{ss}, \omega^{\otimes *}) \cong (H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2)[\Delta^{-1}])^{\wedge}_{(j)}.$$

Let $G_{48} = G_{24} \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. By Lubin-Tate theory,

$$H^*(\widehat{\mathcal{M}}_{ss}, \omega^{\otimes *}) \cong H^*(G_{48}, (E_{\mathcal{C}})_*V(0)).$$

The groups G_{48} and G_{24} differ by the action of the Galois group on the coefficients $\mathbb{F}_4$, so that

$$H^*(G_{24}; (E_{\mathcal{C}})_*V(0)) \cong \left(H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2)[\Delta^{-1}]\right)^{\wedge}_{(j)} \otimes_{\mathbb{F}_2} \mathbb{F}_4,$$

where the completion is done degree-wise. Finally, in [1, §7], Bauer computes $H^*(A \otimes \mathbb{F}_2, \Gamma \otimes \mathbb{F}_2)$. The result then follows from his computation. □

The cohomology of C_6 can be computed using the formulas of Section 2.4.

Lemma 4.12. *The cohomology of C_6 with coefficients in $(E_{\mathcal{C}})_*V(0)$ is given by*

$$H^*(C_6; (E_{\mathcal{C}})_*V(0)) = \mathbb{F}_4[[u_1^3]][v_1, v_2^{\pm 1}, h]/(v_1^3 = v_2 u_1^3),$$

where $|h| = (1,0)$, $|v_2| = (0,6)$, $|v_1| = (0,2)$ and $|u_1^3| = (0,0)$. Further, the action of h_1 is determined by

$$h_1 \cdot 1 = v_1 h. \tag{4.13}$$

Proof. Recall that $C_2 = \{\pm 1\}$ denotes the center of G_{24} and that $C_6 = C_2 \times C_3$. Because C_2 acts trivially on $(E_{\mathcal{C}})_*V(0)$,

$$H^*(C_2, (E_{\mathcal{C}})_*V(0)) = ((E_{\mathcal{C}})_*V(0))[h],$$

where h is in (s,t) degree $(1,0)$. The order of C_3 is coprime to 2, so that

$$\begin{aligned} H^*(C_6; (E_{\mathcal{C}})_*V(0)) &\cong H^*(C_2; (E_{\mathcal{C}})_*V(0))^{C_3} \\ &= (E_{\mathcal{C}})_*V(0)^{C_3}[h] \\ &= \mathbb{F}_4[[u_1^3]][v_1, v_2^{\pm 1}, h]/(v_1^3 = v_2 u_1^3). \end{aligned}$$

To prove (4.13), first note that C_2 acts on $(E_{\mathcal{C}})_* = \mathbb{W}[[u_1]][u^{\pm 1}]$ by

$$\begin{aligned} \phi_{-1}(u) &= -u \\ \phi_{-1}(u_1) &= u_1. \end{aligned}$$

This follows from (2.4) and the fact that -1 fixes the curve $\mathcal{C}_U$. Using this and the standard resolution of the trivial C_2-module $\mathbb{Z}_2$, one can compute that the Bockstein

$$\beta : (E_{\mathcal{C}})_*V(0) \to (E_{\mathcal{C}})_*V(0)$$

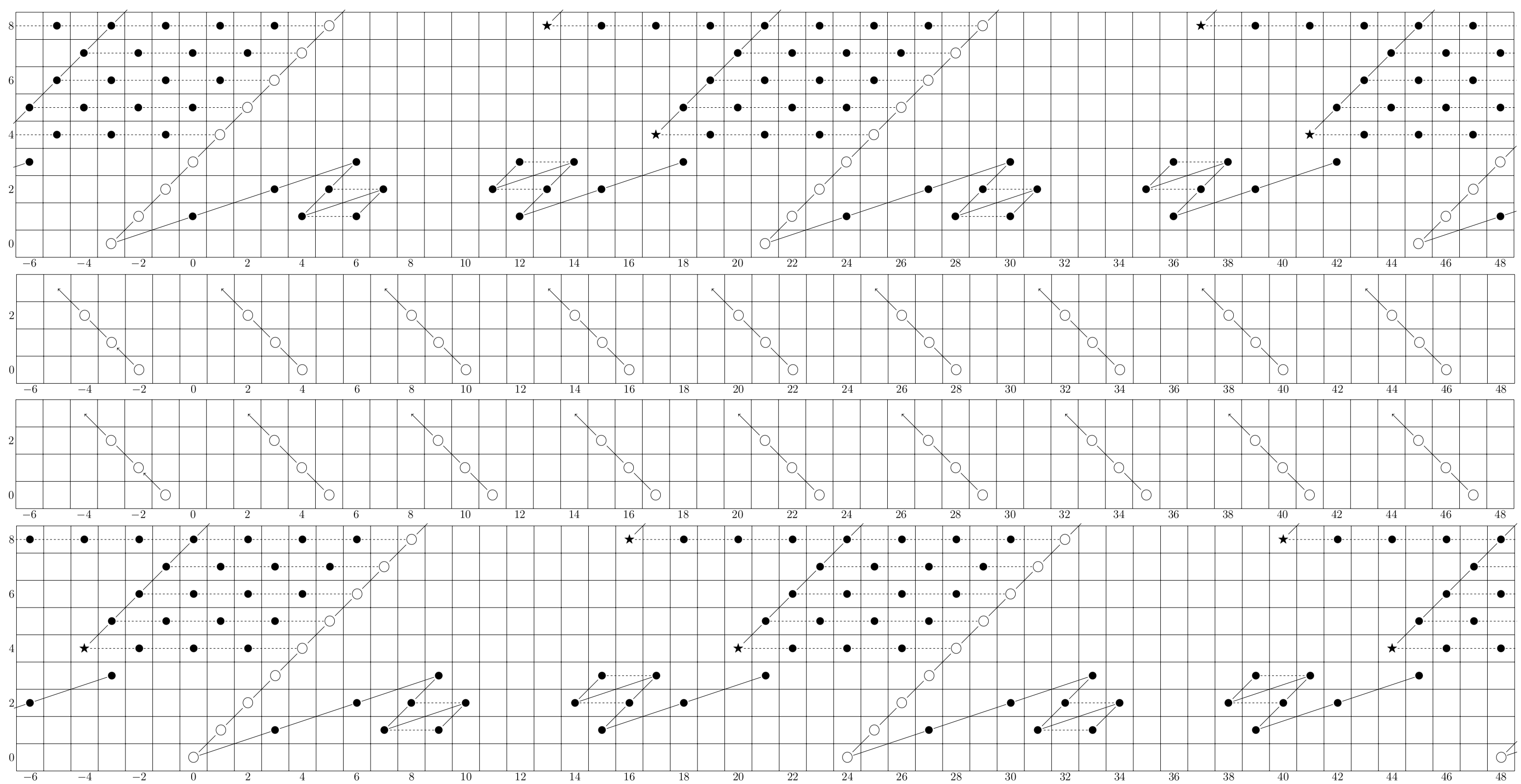

FIGURE 4.2. The E_1-term for the ADRSS with coefficients $(E_{\mathcal{C}})_*V(0)$. The rows represent $E_1^{p,*}$, where the top row corresponds to $p = 3$. The grading is given by $(t - q - p, q)$, where t is the internal grading, so that $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ decreases the horizontal grading by 1. A $\bullet$ denotes a copy of $\mathbb{F}_4$. Dashed horizontal lines denote multiplication by v_1, and a $\bigcirc$ denotes a copy of $\mathbb{F}_4[v_1]$. A $\bigstar$ is a copy of Figure 4.3.

induced by

$$0 \to (E_{\mathcal{C}})_* \xrightarrow{2} (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_* V(0) \to 0,$$

satisfies

$$\beta(v_1) = v_1 h.$$

The identity (4.13) follows by naturality and the fact that $\beta(v_1) = h_1$ (see (4.8)). $\square$

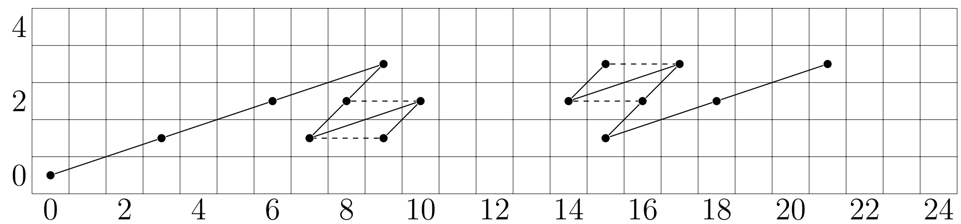

FIGURE 4.3. The pattern ★ in Figure 4.2.

Lemma 4.14. *Let* $\pi = 1 + 2\omega$ *in* $\mathbb{S}_{\mathcal{C}}$. *Let* $G'_{24} = \pi G_{24} \pi^{-1}$. *Let* $\phi_\pi : (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*$ *give the action of* π *on* $(E_{\mathcal{C}})_*$. *Then* ϕ_π *induces an* $\mathbb{F}_4[v_1, h_1]$*-linear isomorphism*

$$H^*(G_{24}, (E_{\mathcal{C}})_* V(0)) \cong H^*(G'_{24}, (E_{\mathcal{C}})_* V(0)).$$

Proof. For M an $\mathbb{S}_{\mathcal{C}}$-module, define a map $F_\pi : M \to M$ by $m \mapsto \pi \cdot m$. Although this is not a morphism of $\mathbb{S}_{\mathcal{C}}$-modules, it induces a natural isomorphism

$$F_\pi : (-)^{G_{24}} \to (-)^{G'_{24}}.$$

Indeed, for another $\mathbb{S}_{\mathcal{C}}$-module N and a morphism of $\mathbb{S}_{\mathcal{C}}$-modules $f : M \to N$,

$$\begin{array}{ccc} (M)^{G_{24}} & \xrightarrow{F_\pi} & (M)^{G'_{24}} \\ \Big\downarrow{\scriptstyle (f)^{G_{24}}} & & \Big\downarrow{\scriptstyle (f)^{G'_{24}}} \\ (N)^{G_{24}} & \xrightarrow{F_\pi} & (N)^{G'_{24}} \end{array}$$

is commutative. Therefore, F_π induces an isomorphism on the right derived functors, i.e., on group cohomology. For $(E_{\mathcal{C}})_* V(0)$, the map F_π is induced by ϕ_π. The linearity follows from the fact that v_1 is invariant under the action of π, and $h_1 = \beta(v_1)$. $\square$

To avoid ambiguities, define $\Delta' := \phi_\pi(\Delta)$ and $j' := \phi_\pi(j)$.

4.4. **Approximate Δ-linearity.** In this section, I explain some additional properties of the action of $\mathbb{S}_{\mathcal{C}}$. These will be used in the computations of the differentials $d_1 : E_1^{p,0} \to E_1^{p+1,0}$. Recall from (4.10) that there is an element $\Delta \in (E_{\mathcal{C}})_*^{G_{24}}$ such that

$$\Delta = 27 v_2 (v_1^3 + v_2)^3 \equiv v_2 (v_2 + v_1^3)^3 \mod (2).$$

As in Theorem 4.11, we abuse notation by denoting

$$\Delta = v_2 (v_2 + v_1^3)^3 \qquad (4.15)$$

in $(E_{\mathcal{C}})_* V(0)^{G_{24}}$. The key observation in the computation of Section 5 is that the action of $(IS^1_{\mathcal{C}})^2$ is *approximately* Δ*-linear*. The following theorem makes this precise.

Theorem 4.16. *Let x be in $(E_{\mathcal{C}})_*V(0)$. Let $\sum a_{g,h}(e-g)(e-h)$ be an element of $(IS^1_{\mathcal{C}})^2$, where $a_{g,h}$ is in $\mathbb{Z}_2[[\mathbb{S}^1_2]]$. Then, modulo $(2, v_1^{1+3\cdot 2^{k+1}})$,*

(4.17)

$$\sum a_{g,h}(id-\phi_g)(id-\phi_h)\left(x\Delta^{2^k(1+2t)}\right) \equiv \sum a_{g,h}(id-\phi_g)(id-\phi_h)(x)\Delta^{2^k(1+2t)}.$$

Further,

$$\sum a_{g,h}(id-\phi_g)(id-\phi_h)(\Delta) \equiv 0 \mod (2, v_1^8). \tag{4.18}$$

The next results are needed to prove Theorem 4.16.

Lemma 4.19. *The action of α on Δ is given by*

$$\phi_\alpha(\Delta) \equiv \Delta(1+v_2^{-2}v_1^6) \mod (2, v_1^9). \tag{4.20}$$

Proof. By (3.9)

$$\begin{aligned}\phi_\alpha(\Delta) &\equiv (v_2+v_1^3+v_1^6\epsilon)(v_2+v_1^6\epsilon)^3\\ &\equiv v_2^3(v_2+v_1^3) \mod (2, v_1^9),\end{aligned}$$

so that

$$\begin{aligned}\phi_\alpha(\Delta)\Delta^{-1} &\equiv v_2^2(v_2+v_1^3)^{-2}\\ &\equiv 1+v_2^{-2}v_1^6 \mod (2, v_1^9).\end{aligned}$$

□

Lemma 4.21. *The group G_{24} acts as the identity on $\phi_\alpha(\Delta)$ modulo $(2, v_1^8)$.*

Proof. First note that Δ itself is fixed by G_{24}. The group G_{24} is generated by ω and i. As α and ω commute,

$$\begin{aligned}\phi_\omega(\phi_\alpha(\Delta)) &= \phi_\alpha(\phi_\omega(\Delta))\\ &= \phi_\alpha(\Delta).\end{aligned}$$

Further, it follows from Strickland's computations, which were described in Section 2.4, that, for $t_0(i)$ as in Theorem 3.6,

$$t_0(i) \equiv (1+u_1)^{-1} \mod (2).$$

Therefore,

$$\phi_i(v_2) = (1+u_1)^3 v_2.$$

It follows that

$$\begin{aligned}\phi_i(\phi_\alpha(\Delta)) &= \Delta(1+v_2^{-2}(1+u_1)^{-6}v_1^6)\\ &= \Delta(1+v_2^{-2}(1+u^2v_1^2)^{-3}v_1^6)\\ &\equiv \Delta(1+v_2^{-2}v_1^6)\\ &\equiv \phi_\alpha(\Delta) \mod (2, v_1^8).\end{aligned}$$

□

Lemma 4.22. *Let γ be in $F_{3/2}\mathbb{S}_{\mathcal{C}}$. Then γ acts trivially on Δ and $\phi_\alpha(\Delta)$ modulo $(2, v_1^8)$.*

Proof. First suppose that γ is in $F_{4/2}\mathbb{S}_{\mathcal{C}}$. By Theorem 3.7, the action of γ is trivial modulo $(2, u_1^6)$. This implies that

$$\phi_\gamma(v_2) \equiv v_2 + v_2^{-1}v_1^6\epsilon_1 \mod (2, v_1^8).$$

Hence, modulo $(2, v_1^8)$,

$$\begin{aligned}
\phi_\gamma(\Delta) &\equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)(v_2 + v_1^3 + v_2^{-1}v_1^6\epsilon_1)^3 \\
&\equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)((v_2 + v_1^3)^3 + (v_2 + v_1^3)^2 v_2^{-1}v_1^6\epsilon_1) \\
&\equiv (v_2 + v_2^{-1}v_1^6\epsilon_1)((v_2 + v_1^3)^3 + v_2 v_1^6\epsilon_1) \\
&\equiv v_2(v_2 + v_1^3)^3 + v_2^{-1}v_1^6\epsilon_1(v_2 + v_1^3)^3 + v_2^2 v_1^6\epsilon_1 \\
&\equiv v_2(v_2 + v_1^3)^3 + 2v_2^2 v_1^6\epsilon_1 \\
&\equiv \Delta \mod (2, v_1^8).
\end{aligned}$$

Now suppose that $\gamma \in F_{3/2}\mathbb{S}_{\mathcal{C}}$. It was shown in [2] that $F_{3/2}S_{\mathcal{C}}/F_{4/2}S_{\mathcal{C}}$ is generated by α_i and α_j, where

$$\alpha_\tau = \tau\alpha\tau^{-1}\alpha^{-1}.$$

Thus, $\gamma = \alpha_i\gamma_0$, $\gamma = \alpha_j\gamma_0$ or $\gamma = \alpha_i\alpha_j\gamma_0$, where $\gamma_0 \in F_{4/2}\mathbb{S}_{\mathcal{C}}^1$. But $\alpha_j = \omega\alpha_i\omega^2$ and both ω and γ_0 fix Δ modulo $(2, v_1^8)$, so it is enough to verify the case when $\gamma = \alpha_i$. Using Lemma 3.9 and Lemma 4.21,

$$\begin{aligned}
\phi_{\alpha_i}(\Delta) &\equiv \phi_i \circ \phi_\alpha \circ \phi_{i^{-1}} \circ \phi_{\alpha^{-1}}(\Delta) \\
&\equiv \phi_i \circ \phi_\alpha \circ \phi_{i^{-1}}(\phi_\alpha(\Delta)) \\
&\equiv \phi_i \circ \phi_{\alpha^2}(\Delta) \\
&\equiv \Delta \mod (2, v_1^8),
\end{aligned}$$

To prove that $\phi_\alpha(\Delta)$ is fixed, note that $\gamma\alpha = \alpha\gamma'$, where $\gamma' = \alpha^{-1}\gamma\alpha$ is in $F_{3/2}\mathbb{S}_{\mathcal{C}}^1$. Hence

$$\begin{aligned}
\phi_\gamma(\phi_\alpha(\Delta)) &= \phi_\alpha(\phi_{\gamma'}(\Delta)) \\
&\equiv \phi_\alpha(\Delta) \mod (2, v_1^8).
\end{aligned}$$

□

Lemma 4.23. *Let γ be in $\mathbb{S}_{\mathcal{C}}^1$. Modulo $(2, v_1^8)$, the action of γ either permutes Δ and $\phi_\alpha(\Delta)$, or it fixes both.*

Proof. Write $\gamma = \gamma_0\tau$ where $\gamma_0 \in K^1$ and $\tau \in G_{24}$. That such a representation is possible follows from the fact that $\mathbb{S}_{\mathcal{C}}^1 \cong K^1 \rtimes G_{24}$. It is sufficient to show that, modulo $(2, v_1^8)$,

$$\phi_\gamma(\Delta) \equiv \begin{cases} \phi_\alpha(\Delta) & \text{if } \gamma_0 \notin F_{3/2}\mathbb{S}_{\mathcal{C}}^1; \\ \Delta & \text{if } \gamma_0 \in F_{3/2}\mathbb{S}_{\mathcal{C}}^1. \end{cases}$$

Because τ acts trivially on Δ,

$$\phi_\gamma(\Delta) \equiv \phi_{\gamma_0}(\Delta) \mod (2, v_1^8).$$

If γ_0 is not in $F_{3/2}\mathbb{S}_{\mathcal{C}}^1$, then $\gamma_0 = \alpha\gamma_1$ for $\gamma_1 \in F_{3/2}\mathbb{S}_{\mathcal{C}}^1$. Modulo $(2, v_1^8)$, the element γ_1 acts trivially on Δ, so that $\phi_{\gamma_0}(\Delta) \equiv \phi_\alpha(\Delta)$. The same proof works for $\phi_\alpha(\Delta)$. □

Proof of Theorem 4.16. Elements of $(IS^1_{\mathcal{C}})^2$ are possibly infinite linear combinations of elements of the form $(e-g)(e-h)$ for g and h in $S^1_{\mathcal{C}}$. It suffices to show that, for these generators,

$$(id+\phi_g)(id+\phi_h)\left(x\Delta^{2^k(1+2t)}\right) \equiv (id+\phi_g+\phi_h+\phi_{gh})(x)\cdot\Delta^{2^k(1+2t)} \tag{4.24}$$

modulo $(2, v_1^{1+3\cdot 2^{k+1}})$. By Lemma 4.23, the elements g and h either fix Δ and $\phi_\alpha(\Delta)$ or permute them modulo $(2, v_1^8)$. If both permute Δ and $\phi_\alpha(\Delta)$, then gh fixes them. Therefore, up to relabeling, one can assume that h fixes Δ and $\phi_\alpha(\Delta)$. There are two cases depending on the action of g.

If g and h fix Δ modulo $(2, v_1^8)$, then they fix $\Delta^{2^k(1+2t)}$ modulo $(2, v_1^{2^{k+3}})$. Hence, modulo $(2, v_1^{2^{k+3}})$,

$$(id+\phi_h)(id+\phi_g)(x\Delta^{2^k(1+2t)}) \equiv (x+\phi_h(x)+\phi_g(x)+\phi_{gh}(x))\Delta^{2^k(1+2t)}. \tag{4.25}$$

This implies (4.18) for the case of $x=1$ and $k,t=0$. Further, since

$$2^{k+3} > 1+3\cdot 2^{k+1},$$

(4.25) trivially implies (4.17).

If g permutes Δ and $\phi_\alpha(\Delta)$ modulo $(2, v_1^8)$, then it permutes $\Delta^{2^k(1+2t)}$ and $\phi_\alpha(\Delta)^{2^k(1+2t)}$ modulo $(2, v_1^{2^{k+3}})$. Therefore,

$$\begin{aligned}(id+\phi_h)(id+\phi_g)(x\Delta^{2^{s-2}+2^{s-1}t}) &\equiv (x+\phi_h(x))\Delta^{2^k(1+2t)}\\ &\quad + (\phi_g(x)+\phi_{gh}(x))\phi_\alpha(\Delta)^{2^k(1+2t)}.\end{aligned}$$

But

$$\begin{aligned}(\phi_g(x)+\phi_{gh}(x))\phi_\alpha(\Delta)^{2^k(1+2t)} &\equiv (\phi_g(x)+\phi_{gh}(x))\Delta^{2^k(1+2t)}(1+v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}})\\ &\equiv \Big((\phi_g(x)+\phi_{gh}(x))\\ &\qquad + \phi_g(x+\phi_h(x))v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}}\Big)\Delta^{2^k(1+2t)}.\end{aligned}$$

When $x=1$ and $k,t=0$, this implies (4.18). Because h is in $S^1_{\mathcal{C}}$, Theorem 3.7 implies that

$$x+\phi_h(x)\equiv 0 \mod (2, v_1),$$

so that

$$(\phi_g(x)+\phi_{gh}(x))\phi_\alpha(\Delta)^{2^k(1+2t)} \equiv (\phi_g(x)+\phi_{gh}(x))\Delta^{2^k(1+2t)} \mod (2, v_1^{1+3\cdot 2^{k+1}}).$$

Therefore, modulo $(2, v_1^{1+3\cdot 2^{k+1}})$,

$$(id+\phi_h)(id+\phi_g)(x\Delta^{2^k(1+2t)}) \equiv (x+\phi_g(x)+\phi_h(x)+\phi_{gh}(x))\Delta^{2^k(1+2t)}.$$

□

5. Computation of the E_∞-Term

Now we turn to the computation of the algebraic duality resolution spectral sequence

$$E_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_p, (E_{\mathcal{C}})_*V(0)) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0)), \tag{5.1}$$

whose construction was described in Section 4. Recall also that the spectral sequence comes from a resolution

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_2} \mathscr{C}_1 \xrightarrow{\partial_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0,$$

The differentials

$$d_1 : E_1^{p,q} \to E_1^{p+1,q}$$

are thus induced by

$$d_1 = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\partial_{p+1}, (E_{\mathcal{C}})_* V(0)).$$

The morphisms ∂_{p+1} were described in Theorem 4.1. We will use these descriptions together with our partial knowledge of the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$ to compute the d_1 differentials.

Recall that $E_1^{0,0} \cong (E_{\mathcal{C}})_*^{G_{24}}$ and $E_1^{p,0} \cong (E_{\mathcal{C}})_*^{C_6}$ for $p = 1$ and $p = 2$. Since there is an inclusion

$$(E_{\mathcal{C}})_*^{G_{24}} \to (E_{\mathcal{C}})_*^{C_6},$$

there is an action of $(E_{\mathcal{C}})_*^{G_{24}}$ on $E_1^{p,0}$ for $0 \leq p \leq 2$. Therefore, it will make sense to talk about the image of Δ defined in Theorem 4.11 in $E_1^{p,0}$. To avoid ambiguity, we will use the convention

$$\Delta^k[p] = \Delta^k \cdot 1 \in E_1^{p,0}$$

in the statement of the results. However, in the proofs, we will assume that the context is sufficient to determine which elements are meant. Similarly, $v_2 \in (E_{\mathcal{C}})_*^{C_6}$. To distinguish between $E_1^{1,0}$ and $E_1^{2,0}$, we let

$$v_2^k[p] = v_2^k \cdot 1 \in E_1^{p,0}.$$

Finally, recall that the differentials are v_1-linear. This will be used without mention.

5.1. **The differential** $d_1 : E_1^{0,0} \to E_1^{1,0}$**.** The differential $d_1 : E_1^{0,0} \to E_1^{1,0}$ is induced by the map

$$\partial_1 : \mathscr{C}_1 \to \mathscr{C}_0,$$

given by $\partial_1(\gamma e_1) = \gamma(e - \alpha)e_0$. Here, e_i is the canonical generator of $\mathscr{C}_i$. Therefore,

$$d_1 = id + \phi_\alpha : E_1^{0,0} \to E_1^{1,0}.$$

Recall from Theorem 4.11 that the powers of the element

$$\Delta = v_2(v_2 + v_1^3)^3$$

generate $H^0(G_{24}, (E_{\mathcal{C}})_* V(0)) \cong (E_{\mathcal{C}})_*^{G_{24}}$ as an $\mathbb{F}_4[v_1]$-module. So it is sufficient to compute d_1 on $\Delta^n[0]$ for $n \in \mathbb{Z}$.

Theorem 5.2. *Let* $n = 2^k(2t+1)$*, then*

$$d_1(\Delta^n[0]) = v_1^{6\cdot 2^k} v_2^{2^{k+1}(4t+1)}[1] \mod (2, v_1^{9\cdot 2^k}).$$

Proof. Recall that d_1 is induced by $id + \phi_\alpha$. Using Lemma 4.19, one computes

$$\begin{aligned}
\Delta^n + \phi_\alpha(\Delta^n) &= (v_2(v_2 + v_1^3)^3)^n + (v_2(v_2 + v_1^3)^3(1 + v_2^{-2} v_1^6 + v_1^9 \epsilon))^n \\
&= (v_2(v_2 + v_1^3)^3)^n (1 + (1 + v_2^{-2^{k+1}} v_1^{6\cdot 2^k} + v_1^{9\cdot 2^k} \epsilon^{2^k})^{2t+1}) \\
&\equiv v_2^{2^{k+1}(4t+1)} v_1^{6\cdot 2^k} \mod (2, v_1^{9\cdot 2^k}).
\end{aligned}$$

□

5.2. **The differential** $d_1 : E_1^{1,0} \to E_1^{2,0}$**.** The differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ is induced by the map

$$\partial_2 : \mathscr{C}_2 \to \mathscr{C}_1.$$

Recall from Theorem 4.1 that

$$\partial_2(\gamma e_2) = \gamma \Theta e_1$$

for $\Theta \in \mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]$ such that

$$\Theta \equiv e + \alpha \mod (2, (IS^1_2)^2).$$

Let

$$\Theta = e + \alpha + \mathscr{E}, \tag{5.3}$$

where $\mathscr{E} = \sum a_{g,h}(e-g)(e-h)$ is in $(IS^1_{\mathcal{C}})^2$ and is to be thought of as the *error*. Further, let

$$\phi_{\mathscr{E}} = \sum a_{g,h}(id - \phi_g)(id - \phi_h). \tag{5.4}$$

The goal of this section is to prove the following theorem:

Theorem 5.5. *Let $n = 2^k(1+2t)$ where $t \in \mathbb{Z}$ and $k \geq 0$. There exist homogenous elements b_n, such that*

$$b_n \equiv v_2^n[1] \mod (2, v_1).$$

The elements b_n satisfy

$$d_1(\Delta^n[0]) = \begin{cases} v_1^{6\cdot 2^k} b_{2^{k+1}(1+4t)} & n = 2^k(1+2t) \\ 0 & n = 0, \end{cases}$$

and

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} v_2^{2^{k+1}(1+2t)}[2] \mod (2, v_1^{3\cdot 2^k+3}) & n = 2^k(3+4t) \\ v_1^{3\cdot 2^{k+1}} v_2^{m-2^{k+1}}[2] \mod (2, v_1^{3\cdot 2^{k+1}+3}) & n = 1+2^{k+2}+2^{k+3}t \\ 0 & n = 0, 1 \text{ and } 2^{k+1}(1+4t). \end{cases}$$

The idea for the next theorem comes from Mahowald and Rezk's computations of the homotopy of TMF, specifically, Corollary 6.2 in [12]. The idea is to consider the spectral sequence

$$\widetilde{E}_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_p, (E_{\mathcal{C}})_*) \Longrightarrow H^{p+q}(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*). \tag{5.6}$$

Let

$$f : \widetilde{E}_1^{p,q} \to E_1^{p,q} \tag{5.7}$$

be the map of spectral sequences induced by the map $(E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*V(0)$ on the coefficients. We will show that there is a permanent cycle $B_1 \in \widetilde{E}_1^{1,0}$ such that

$$f(B_1) \equiv v_1 v_2 \mod (2, v_1).$$

This will allow us to define a permanent cycle $b_1 \in E_1^{1,0}$ by

$$b_1 = v_1^{-1} f(B_1).$$

Theorem 5.8. *There is an element $b_1 \in E_1^{1,0}$ such that*

$$b_1 \equiv v_2[1] \mod (2, v_1^3),$$

and

$$d_1(b_1) = 0.$$

Proof. There is a modular form c_4 in $(E_{\mathcal{C}})_*^{G_{24}}$ given by

$$c_4 = 9(v_1^4 + 8v_1v_2) = 9u^{-4}u_1(u_1^3 + 8)$$

(see, for example, [18, §III.1]). I claim that there is an element $B_1 \in (E_{\mathcal{C}})_*^{C_6}$ such that

$$B_1 \equiv v_1v_2 \mod (2, v_1^2)$$

and

$$d_1(c_4) = c_4 - \phi_\alpha(c_4) = 16B_1,$$

where d_1 here denotes the differential in the spectral sequence $\widetilde{E}_r^{p,q}$ defined by (5.6).

The first step is to show that

$$d_1(c_4) \equiv 0 \mod (16). \tag{5.9}$$

Let $t_0 = t_0(\alpha)$ and $t_1 = t_1(\alpha)$ as defined in Theorem 3.6. A direct computation using Theorem 3.6 implies that

$$d_1(c_4) \equiv 8u^{-4}\left(u_1 + \frac{3u_1}{t_0^3} + \frac{t_1^2u_1^2}{t_0^4} + \frac{t_1u_1^3}{t_0^2} + \frac{2t_1}{t_0^5} + \frac{2t_1^4}{t_0^8}\right) \mod (32).$$

Let $A = d_1(c_4)/(8u^{-4})$. Then

$$A \equiv u_1t_0^{-4}\left(t_0 + t_0^4 + u_1t_1^2 + u_1^2t_1t_0^2\right) \mod (2).$$

It follows from Proposition 8.21 of Section 8 that

$$t_0 \equiv t_0^4 + u_1t_1^2 + u_1^2t_1t_0^2 \mod 2.$$

This proves that $d_1(c_4) \equiv 0 \mod 16$.

The next step is to show that

$$A \equiv 2u_1 \bmod (4, u_1^2).$$

Theorem 3.7 applied to α gives

$$t_0 \equiv 1 + 2\omega \bmod (4, u_1^2),$$
$$t_1 \equiv u_1\omega^2 \bmod (2, u_1^2).$$

This implies that, modulo $(4, u_1^2)$,

$$\begin{aligned} A &\equiv u_1 + \frac{3u_1}{t_0^3} + \frac{2t_1}{t_0^5} + \frac{2t_1^4}{t_0^8} \\ &\equiv u_1 + \frac{3u_1}{(1+2\omega)^3} + \frac{2\omega^2u_1}{(1+2\omega)^5} \\ &\equiv 2u_1. \end{aligned}$$

Define

$$B_1 = \frac{d_1(c_4)}{16}.$$

Because $v_1v_2 = u_1u^{-4}$, this implies that

$$B_1 \equiv v_1v_2 \mod (2, v_1^2)$$

so that

$$d_1(c_4) \equiv 16(v_1v_2 + \ldots) \mod (32).$$

Let $f : \widetilde{E}_1^{p,q} \to E_1^{p,q}$ be the map defined in (5.7). Since $f(B_1)$ is divisible by v_1, we can define an element $b_1 \in E_1^{1,0}$ by

$$b_1 := v_1^{-1}f(B_1).$$

Then $b_1 \equiv v_2$ modulo $(2, v_1)$. But b_1 is an element of

$$(E_{\mathcal{C}})_6 V(0)^{C_6} = \mathbb{F}_4[[u_1^3]]\{v_2\}.$$

This forces the congruence

$$b_1 \equiv v_2 \mod (2, v_1^3).$$

It remains to show that $d_1(b_1) = 0$. In the spectral sequence $\widetilde{E}_r^{p,q}$, we have

$$d_1^2(c_4) = d_1(16B_1) = 16 d_1(B_1).$$

Since $d_1^2 = 0$, and there is no torsion in $(E_{\mathcal{C}})^{C_6}$, this implies that

$$d_1(B_1) = 0.$$

in $\widetilde{E}_1^{2,0}$. Therefore,

$$d_1(f(B_1)) = 0.$$

Since $B_1 = v_1 b_1$, this implies that

$$d_1(v_1 b_1) = 0$$

in $E_1^{2,0}$. But the differential $d_1 : E_1^{1,0} \to E_1^{2,0}$ is v_1-linear, so that

$$d_1(v_1 b_1) = v_1 d_1(b_1) = 0.$$

Because there is no v_1-torsion in $E_1^{2,0}$, we must have $d_1(b_1) = 0$. This finishes the proof. □

Lemma 5.10. *For* $d_1 : E_1^{1,0} \to E_1^{2,0}$

$$d_1(\Delta[1]) \equiv v_1^6 v_2^2[2] \mod (2, v_1^8).$$

Proof. Using Theorem 4.16, with $\phi_{\mathscr{E}}$ as defined by (5.4),

$$\begin{aligned} d_1(\Delta) &= \Delta + \phi_\alpha(\Delta) + \phi_{\mathscr{E}}(\Delta) \\ &\equiv \Delta + \Delta(1 + v_2^{-2} v_1^6) \\ &\equiv v_2^2 v_1^6 \mod (2, v_1^8). \end{aligned}$$

□

Lemma 5.11. *For* $d_1 : E_1^{1,0} \to E_1^{2,0}$

$$d_1(v_2^3[1]) \equiv v_1^3 v_2^2[2] \mod (2, v_1^5).$$

Proof. Because $b_1 \equiv v_2$ modulo $(2, v_1^3)$,

$$\Delta \equiv b_1^4 + v_2^3 v_1^3 + b_1^2 v_1^6 \mod (2, v_1^8).$$

Hence,

$$\begin{aligned} d_1(\Delta) &\equiv d_1(b_1^4 + v_2^3 v_1^3 + b_1^2 v_1^6) \\ &\equiv d_1(b_1)^4 + v_1^3 d_1(v_2^3) + v_1^6 d_1(b_1)^2 \\ &\equiv v_1^3 d_1(v_2^3) \mod (2, v_1^8). \end{aligned}$$

It thus follows from Lemma 5.10 that

$$v_1^3 d_1(v_2^3) \equiv v_1^6 v_2^2 \mod (2, v_1^8).$$

As there is no v_1-torsion in $E_1^{2,0}$, this proves the claim. □

Lemma 5.12. *For $\sum a_g g$ in $\mathbb{Z}_2[[S^1_{\mathcal{C}}]]$, where $a_g \in \mathbb{Z}_2$,*

$$\sum a_g \phi_g(v_2^{3+4t}) \equiv \sum a_g \phi_g(v_2^3) v_2^{4t} \mod (2, v_1^4).$$

Proof. As $g \in S^1_{\mathcal{C}}$, it follows from Theorem 3.7 that $t_0(g) \equiv 1$ modulo $(2, v_1)$. Hence, $t_0(g)^{4t} \equiv 1$ modulo $(2, v_1^4)$, and

$$\begin{aligned} \sum a_g \phi_g(v_2^{3+4t}) &\equiv \sum a_g \phi_g(v_2^3) \phi_g(v_2^{4t}) \\ &\equiv \sum a_g \phi_g(v_2^3) t_0(g)^{4t} v_2^{4t} \\ &\equiv \sum a_g \phi_g(v_2^3) v_2^{4t} \mod (2, v_1^4). \end{aligned}$$

□

Proof of Theorem 5.5. Let $t \in \mathbb{Z}$ and $k \geq 0$

$$b_n := \begin{cases} b_1^n & n = 0, 1; \\ v_2^n & n = 2^k(3+4t); \\ b_1 \Delta^{2^k + 2^{k+1} t} & n = 1 + 2^{k+2} + t 2^{k+3}; \\ v_1^{-6 \cdot 2^k} d_1 \left(\Delta^{2^k(2t+1)} \right) & n = 2^{k+1}(4t+1). \end{cases}$$

The element b_n is in degree $6n$ and

$$b_n \equiv v_2^n \mod (2, v_1^3).$$

By Theorem 5.5, for $n = 0, 1$,

$$d_1(b_n) = 0.$$

Let $n = 2^{k+1}(1+4t)$. Then

$$d_1(v_1^{6 \cdot 2^k} b_n) = d_1^2 \left(\Delta^{2^k(2t+1)} \right) = 0.$$

The map d_1 is v_1-linear and there is no v_1-torsion in $E_1^{2,0}$; hence, $d_1(b_n) = 0$.

Next, let $n = 2^k(3+4t)$, so that $b_n = (v_2^{3+4t})^{2^k}$. For any b in $E_r^{p,q}$,

$$d_r(b^2) = d_r(b)^2 \mod (2),$$

so it is sufficient to prove the claim when $k = 0$. Let $\phi_{\mathscr{E}}$ be as in (5.4). It follows from Lemma 5.11 and Lemma 5.12 that, modulo $(2, v_1^4)$,

$$\begin{aligned} d_1(v_2^{3+4t}) &= v_2^{3+4t} + \phi_\alpha(v_2^{3+4t}) + \phi_{\mathscr{E}}(v_2^{3+4t}) \\ &\equiv v_2^{3+4t} + \phi_\alpha(v_2^3) v_2^{4t} + \phi_{\mathscr{E}}(v_2^3) v_2^{4t} \\ &\equiv \left(v_2^3 + \phi_\alpha(v_2^3) + \phi_{\mathscr{E}}(v_2^3) \right) v_2^{4t} \\ &\equiv d_1(v_2^3) v_2^{4t} \\ &\equiv v_1^3 v_2^{2+4t} \mod (2, v_1^4). \end{aligned}$$

Finally, let $n = 1 + 2^{k+2} + 2^{k+3} t$, so that $b_n = b_1 \Delta^{2^k(1+2t)}$. By Theorem 4.16,

$$\phi_{\mathscr{E}}(b_n) \equiv \phi_{\mathscr{E}}(b_1) \Delta^{2^k(1+2t)} \mod v_1^{1+3 \cdot 2^{k+1}}.$$

Therefore,

$$\begin{aligned} d_1(b_n) &\equiv b_n + \phi_\alpha(b_n) + \phi_{\mathscr{E}}(b_n) \\ &\equiv b_1\Delta^{2^k(1+2t)} + \phi_\alpha(b_1)\Delta^{2^k(1+2t)}(1+v_2^{-2}v_1^6)^{2^k(1+2t)} + \phi_{\mathscr{E}}(b_1)\Delta^{2^k(1+2t)} \\ &\equiv b_1\Delta^{2^k(1+2t)} + \phi_\alpha(b_1)\Delta^{2^k(1+2t)}(1+v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}}) + \phi_{\mathscr{E}}(b_1)\Delta^{2^k(1+2t)} \\ &\equiv (b_1 + \phi_\alpha(b_1) + \phi_{\mathscr{E}}(b_1))\,\Delta^{2^k(1+2t)} + \phi_\alpha(b_1)v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}}\Delta^{2^k(1+2t)} \\ &\equiv d_1(b_1)\Delta^{2^k(1+2t)} + \phi_\alpha(b_1)v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}}\Delta^{2^k(1+2t)} \mod (2, v_1^{1+3\cdot 2^{k+1}}). \end{aligned}$$

But $d_1(b_1) = 0$ and

$$\phi_\alpha(b_1) \equiv v_2 \mod (2, v_1^3).$$

Furthermore, $\Delta^{2^k(1+2t)} \equiv v_2^{2^{k+2}+2^{k+3}t}$, so that

$$\begin{aligned} d_1(b_n) &\equiv v_1^{3\cdot 2^{k+1}} v_2^{1-2^{k+1}+2^{k+2}+2^{k+3}t} \\ &\equiv v_1^{3\cdot 2^{k+1}} v_2^{1+2^{k+1}+2^{k+3}t} \mod (2, v_1^{1+3\cdot 2^{k+1}}). \end{aligned}$$

This complete the proof of Theorem 5.5. □

5.3. **The differential** $d_1 : E_1^{2,0} \to E_1^{3,0}$**.** Recall that

$$E_1^{3,0} \cong H^q(G'_{24}, (E_{\mathcal{C}})_*V(0)) = \mathbb{F}_4[[j']][v_1, \Delta']/(j' = v_1^{12}\Delta'^{-1}).$$

We let

$$\Delta'[3] = \Delta' \cdot 1 \in E_1^{3,0}.$$

The next goal will be to prove:

Theorem 5.13. *Let $n = 2^k(1+2t)$ where $t \in \mathbb{Z}$ and $k \geq 0$. There exist homogenous elements c_n such that*

$$c_n \equiv v_2^n[2] \mod (2, v_1) \tag{5.14}$$

and

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} c_{2^{k+1}(1+2t)} & n = 2^k(3+4t) \\ v_1^{3\cdot 2^{k+1}} c_{1+2^{k+1}+t2^{k+3}} & n = 1+2^{k+2}+t2^{k+3} \\ 0 & \textit{otherwise.} \end{cases}$$

Further,

$$d_1(c_n) = v_1^{3(1+2^{k+1})}\Delta'^{2^k(1+2t)}[3] \mod (2, v_1^{3(1+2^{k+1})+12})$$

if $n = 1 + 2^{k+1}(3+4t)$ and is zero otherwise.

Let $(F_r^{s,t}, d'_r)$ be the duality resolution spectral sequence associated to the resolution

$$0 \to \mathscr{C}_3 \xrightarrow{\partial'_3} \mathscr{C}_2 \xrightarrow{\partial'_2} \mathscr{C}_1 \xrightarrow{\partial'_1} \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0, \tag{5.15}$$

described in Theorem 4.1. Recall also that there are isomorphisms

$$g_p : \mathscr{C}_p \to \mathscr{C}_p,$$

which induce an isomorphism of resolutions

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathscr{C}_3 & \xrightarrow{\partial_3} & \mathscr{C}_2 & \xrightarrow{\partial_2} & \mathscr{C}_1 & \xrightarrow{\partial_1} & \mathscr{C}_0 & \xrightarrow{\varepsilon} & \mathbb{Z}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \| & & \\
0 & \longrightarrow & \mathscr{C}_3 & \xrightarrow{\partial'_3} & \mathscr{C}_2 & \xrightarrow{\partial'_2} & \mathscr{C}_1 & \xrightarrow{\partial'_1} & \mathscr{C}_0 & \xrightarrow{\varepsilon} & \mathbb{Z}_2 & \longrightarrow & 0,
\end{array}$$

As g_p is an isomorphism, the map $\mathbb{F}_2 \otimes_{\mathbb{Z}_2[[S^1_{\mathcal{C}}]]} g_p : \mathbb{F}_2 \to \mathbb{F}_2$ is non-zero, so that

$$g_p(e_p) = (e + IS^1_{\mathcal{C}})e_p. \tag{5.16}$$

Let g_p^* be the map induced by g_p

$$g_p^* := \mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(g_p, (E_{\mathcal{C}})_* V(0))).$$

Theorem 4.1 implies that there is an isomorphism of complexes,

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & F_1^{0,0} & \xrightarrow{d'_1} & F_1^{1,0} & \xrightarrow{d'_1} & F_1^{2,0} & \xrightarrow{d'_1} & F_1^{3,0} & \longrightarrow & 0\ , \\
 & & \downarrow{\scriptstyle g_0^*} & & \downarrow{\scriptstyle g_1^*} & & \downarrow{\scriptstyle g_2^*} & & \downarrow{\scriptstyle g_3^*} & & \\
0 & \longrightarrow & E_1^{0,0} & \xrightarrow{d_1} & E_1^{1,0} & \xrightarrow{d_1} & E_1^{2,0} & \xrightarrow{d_1} & E_1^{3,0} & \longrightarrow & 0
\end{array} \tag{5.17}$$

and Theorem 3.7 together with (5.16) implies that

$$g_p^* \equiv id \mod (2, v_1). \tag{5.18}$$

Proof of Theorem 5.13. I construct the c_n inductively. For $n = 2^k(3 + 4t)$ and $n = 1 + 2^{k+2} + 2^{k+3}t$, define c_n by the identities

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} c_{2^{k+1}(1+2t)} & n = 2^k(3+4t); \\ v_1^{3\cdot 2^{k+1}} c_{1+2^{k+1}+2^{k+3}t} & n = 1 + 2^{k+2} + 2^{k+3}t. \end{cases}$$

Then c_n satisfies equation (5.14) and

$$d_1(c_n) = 0.$$

Now note that the morphism ϕ_π for $\pi = 1 + 2\omega$ restricts to an isomorphism of $F_1^{2,0} \cong ((E_{\mathcal{C}})_* V(0))^{C_6}$. The isomorphism of complexes (5.17) implies that, for

$$m := 1 + 2^{k+1} + 2^{k+3}t,$$

there exist c'_m in $F_1^{2,0}$ and x_m in $E_1^{1,0}$ such that

$$g_2^*(\phi_\pi(c'_m)) = c_m + d_1(x_m).$$

But, for $x_m \in E_1^{1,0}$,

$$d_1(x_m) \equiv 0 \mod (2, v_1^3),$$

so that

$$c'_m \equiv v_2^{1+2^{k+1}+2^{k+3}t} \mod (2, v_1). \tag{5.19}$$

For $n = 1 + 2^{k+1} + 2^{k+2} + t2^{k+3}$, let

$$c'_n := c'_m \Delta^{2^k}$$

and define

$$c_n := g_2^*(\phi_\pi(c'_n)).$$

Because $\Delta^{2^k} \equiv v_2^{2^{k+1}}$ modulo $(2, v_1)$, the elements c_n satisfy (5.14). Further, using the fact that $d_1 g_2^* = g_3^* d_1'$,

$$d_1(c_n) = g_3^*(d_1'(\phi_\pi(c_n'))).$$

By Theorem 4.1, the map $d_1' : F_1^{2,0} \to F_1^{3,0}$ is given by

$$\phi_\pi(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})\phi_\pi^{-1},$$

so that

$$d_1'(\phi_\pi(c_n')) = \phi_\pi(id + \phi_i + \phi_j + \phi_k)(id + \phi_\alpha^{-1})(c_n').$$

I will compute this in three steps.

By Lemma 3.9,

$$\phi_{\alpha^{-1}}(\Delta) = \Delta(1 + v_2^{-2}v_1^6 + v_1^9\epsilon).$$

Hence,

$$\begin{aligned}(id + \phi_{\alpha^{-1}})(c_n') &= c_n' + \phi_{\alpha^{-1}}(c_n') \\ &= c_m'\Delta^{2^k} + \phi_{\alpha^{-1}}(c_m')\Delta^{2^k}(1 + v_2^{-2}v_1^6 + v_1^9\epsilon)^{2^k} \\ &= c_m'\Delta^{2^k} + \phi_{\alpha^{-1}}(c_m')\Delta^{2^k}(1 + v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}} + v_1^{9\cdot 2^k}\epsilon^{2^k}) \\ &= (id + \phi_{\alpha^{-1}})(c_m') \cdot \Delta^{2^k} + \phi_{\alpha^{-1}}(c_m')(v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}} + v_1^{9\cdot 2^k}\epsilon^{2^k})\Delta^{2^k}.\end{aligned}$$

Now note that i, j and k fix Δ, so that

$$\begin{aligned}\phi_\pi\left((id + \phi_i + \phi_j + \phi_k)\left((id + \phi_{\alpha^{-1}})(c_m')\Delta^{2^k}\right)\right) &= d_1'(\phi_\pi(c_m'))\phi_\pi(\Delta^{2^k}) \\ &= 0.\end{aligned}$$

The second equality follows from the fact that

$$g_3^* d_1'(\phi_\pi(c_m')) = d_1(c_m) = 0$$

and g_3^* is an isomorphism, so is injective. Therefore,

$$d_1'(\phi_\pi(c_n')) = \phi_\pi\left((id + \phi_i + \phi_j + \phi_k)\left(\phi_{\alpha^{-1}}(c_m')(v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}} + v_1^{9\cdot 2^k}\epsilon^{2^k})\Delta^{2^k}\right)\right)$$

The morphism $\phi_{\alpha^{-1}}(c_m') \equiv c_m'$ modulo $(2, v_1^3)$, so that (5.19) implies that

$$\phi_{\alpha^{-1}}(c_m') = v_2^m + v_1^3 v_2^{2^{k+1}+2^{k+3}t}\epsilon_0,$$

and,

$$\begin{aligned}d_1'(\phi_\pi(c_n')) = \phi_\pi\Big(&(\mathrm{id} + \phi_i + \phi_j + \phi_k)\big((v_2^m + v_1^3 v_2^{2^{k+1}+2^{k+3}t}\epsilon_0) \\ &\cdot (v_2^{-2^{k+1}}v_1^{3\cdot 2^{k+1}} + v_1^{9\cdot 2^k}\epsilon^{2^k})\Delta^{2^k}\big)\Big) \\ \equiv \phi_\pi\Big(&(id + \phi_i + \phi_j + \phi_k)\left(v_2^{1+2^{k+3}t}v_1^{3\cdot 2^{k+1}} + v_1^{3(1+2^{k+1})}v_2^{2^{k+3}t}\epsilon_1\right)\Delta^{2^k}\Big).\end{aligned}$$

Here, I have used the fact that

$$3(2^{k+1} + 1) \leq 9 \cdot 2^k < 9 \cdot 2^k + 3.$$

Because i, j and k are in $S_\mathcal{C}^1$, they act as the identify modulo $(2, v_1)$. Further,

$$\omega(e + i + j + k)\omega^{-1} = e + i + j + k.$$

Hence, for $x \in (E_{\mathcal{C}})_* V(0)^{C_6}$,

$$\phi_\omega((id + \phi_i + \phi_j + \phi_k)(x)) = (id + \phi_i + \phi_j + \phi_k)(x)$$

so that

$$(id + \phi_i + \phi_j + \phi_k)(x) \in (E_{\mathcal{C}})_* V(0)^{C_6}.$$

Therefore,

$$(id + \phi_i + \phi_j + \phi_k)(v_2^{2^{k+3}t}\epsilon_1) \equiv 0 \mod (2, v_1^3).$$

Hence, modulo $(2, v_1^{3(1+2^{k+1})+3})$,

$$d_1'(\phi_\pi(c_n')) \equiv \phi_\pi \left((id + \phi_i + \phi_j + \phi_k)(v_2^{1+2^{k+3}t}) v_1^{3\cdot 2^{k+1}} \Delta^{2^k} \right).$$

Finally, using Strickland's formulas from Section 2.4,

$$\begin{aligned} t_0(i)^{-1} &= 1 + u_1 \\ t_0(j)^{-1} &= 1 + \zeta u_1 \\ t_0(k)^{-1} &= 1 + \zeta^2 u_1, \end{aligned}$$

and

$$d_1'(\phi_\pi(c_n')) \equiv \left(1 + t_i^{-3(1+2^{k+3}t)} + t_j^{-3(1+2^{k+3}t)} + t_k^{-3(1+2^{k+3}t)}\right) v_2^{1+2^{k+3}t} v_1^{3\cdot 2^{k+3}} \Delta^{2^k}.$$

Note that

$$3(1 + 2^{k+3}t) = 1 + 2 + 2^{k+3}t + 2^{k+4}t \equiv 1 + 2 \mod 8.$$

Modulo 2, the binomial coefficients satisfy,

$$\binom{a_0 + 2a_1 + 2^2 a_2 + \ldots + 2^n a_n}{b_0 + 2b_1 + 2^2 b_2 + \ldots + 2^n b_n} \equiv \binom{a_0}{b_0} \cdot \binom{a_1}{b_1} \cdots \binom{a_n}{b_n}.$$

This implies that the binomial coefficients

$$\binom{3(1 + 2^{k+3}t)}{i} \equiv \begin{cases} 1 & \text{if } 0 \le i \le 3; \\ 0 & \text{if } 3 < i \le 6. \end{cases}$$

Hence, modulo $v_1^{3(1+2^{k+1})+3}$,

$$\begin{aligned} d_1'(\phi_\pi(c_n')) &\equiv \phi_\pi \left(\left(1 + \sum_{s=0}^{2} (1 + \zeta^s u_1 + \zeta^{2s} u_1^2 + u_1^3)\right) v_2^{1+2^{k+3}t} v_1^{3\cdot 2^{k+1}} \Delta^{2^k} \right) \\ &\equiv \phi_\pi \left(u_1^3 v_2^{1+2^{s+1}t} v_1^{3\cdot 2^{k+1}} \Delta^{2^k} \right) \\ &\equiv v_1^{3(1+2^{k+1})} v_2^{2^{k+3}t} \phi_\pi(\Delta^{2^k}). \end{aligned}$$

The last equivalence uses the fact that $\phi_\pi \equiv id$ modulo $(2, v_1^3)$, which was shown in Lemma 3.10.

Now, recall from Lemma 4.14 that the powers of

$$\Delta' := \phi_\pi(\Delta)$$

form a set of $\mathbb{F}_4[v_1]$-generators of $H^0(G_{24}, (E_{\mathcal{C}})_* V(0))$. Because, $d_1'(\phi_\pi(c_n'))$ is G_{24}'-invariant, it must be a linear combination of powers of $\Delta'^{\pm 1}$ and powers of v_1. This implies that

$$d_1'(\phi_\pi(c_n')) \equiv v_1^{3(1+2^{k+1})} \Delta'^{2^k + 2^{k+1}t} \mod (2, v_1^{3(1+2^{k+1})+3}).$$

Finally,

$$d_1(c_n) = g_3^*(d_1'(\phi_\pi(c_n')))$$

must satisfy the same congruence by (5.18).

The only element c_n which has not been constructed is c_1. Its existence follows from Lemma 5.20 below. □

Lemma 5.20. *There exists a sequence of elements* $\{c_{1,n}\}$ *such that*
(1) $c_{1,n} \equiv v_2$ *modulo* $(2, v_1^6)$,
(2) $d_1(c_{1,n}) \equiv 0$ *modulo* $(2, v_1^{3(1+4n)})$,
(3) $c_{1,n+1} - c_{1,n} \equiv 0$ *modulo* $(2, v_1^{6n})$.
If $(E_{\mathcal{C}})_6V(0)$ *is given the topology induced by the maximal ideal*

$$\mathfrak{m} = (v_1),$$

then the limit

$$c_1 := \lim_{n\to\infty} c_{1,n}$$

exists. The element c_1 *satisfies equation (5.14) and*

$$d_1(c_1) = 0.$$

Proof. The construction of $\{c_{1,n}\}$ is by induction on n. First, define

$$c_{1,1} := v_2$$

and note that

$$c_{1,1} + \phi_{\alpha^{-1}}(c_{1,1}) \equiv v_1^3 + v_1^6\epsilon.$$

The $\mathbb{F}_4$-vector space with basis

$$\{v_1^3, v_1^{3\cdot 5}\Delta'^{-1}, v_1^{3\cdot 9}\Delta'^{-2}, \ldots, v_1^{3(1+4s)}\Delta'^{-s}, \ldots\}$$

is dense in $((E_{\mathcal{C}})_6V(0))^{G_{24}'}$. Hence,

$$d_1(c_{1,1}) \equiv 0 \mod (2, v_1^6).$$

Now suppose that $c_{1,n}$ has been defined. If $d_1(c_{1,n}) = 0$, then let $c_{1,N} := c_{1,n}$ for all $N \geq n$. Otherwise,

$$d_1(c_{1,n}) = v_1^{3+12s_n}\Delta'^{-s_n} + \ldots \tag{5.21}$$

for $s_n \geq n$. Let $s_n = 2^{k_n}(1 + 2t_n)$ and let $m_n = 3 \cdot 2^{k_n+1}(1 + 4t_n)$. Then

$$m_n \geq 6n.$$

For

$$r_n = 1 + 2^{k_n+1} + 2^{k_n+2} + 2^{k_n+3}(-t_n - 1),$$

(5.21) together with the fact that

$$d_1(c_{r_n}) = v_1^{3(1+2^{k_n+1})}\Delta'^{2^k_n(1+2(-t_n-1)} + \ldots,$$

implies that

$$d_1(c_{1,n}) = v_1^{m_n} d_1(c_{r_n}) + \ldots$$

Define

$$c_{1,n+1} := c_{1,n} + v_1^{m_n} c_{r_n}.$$

Then $c_{1,n+1}$ satisfies properties (1), (2) and (3).

Now consider the sequence $\{c_{1,n}\}$. Since $m_{n+k} \geq 6n$ for $k \geq 0$,

$$c_{1,n+k} - c_{1,n} = v_1^{m_{n+1}} c_{r_{n+1}} + \ldots + v_1^{m_{n+k}} c_{r_{n+k}} \in (v_1)^{6n}.$$

The sequence $\{c_{1,n}\}$ is Cauchy in the topology generated by $\mathfrak{m}$. Since $((E_{\mathcal{C}})_6 V(0))^{C_6}$ is complete with respect to $\mathfrak{m}$, the limit

$$c_1 := \lim_{n\to\infty} c_{1,n}$$

exists. The map d_1 is continuous, so that,

$$d_1(c_1) = \lim_{n\to\infty} d_1(c_{1,n}).$$

But $d_1(c_{1,n}) \in \mathfrak{m}^{3(1+4N)}$ for all $n \geq N$, which implies that

$$d_1(c_1) \in \bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0.$$

□

We can now combine the results of this section to prove the first part of Theorem 1.8. We restate it here for convenience.

Theorem 5.22. *The algebraic duality resolution spectral sequence converging to* $H^*(\mathbb{S}_2^1, (E_{\mathcal{C}})_* V(0))$ *collapses at the* E_2*-term. The spectral sequence is a module over* $\mathbb{F}_4[v_1, h_1]$*. There exist* $\mathbb{F}_4[v_1]$*-generators* $a_n \in E_1^{0,0}$, $b_n \in E_1^{1,0}$, $c_n \in E_1^{2,0}$ *and* $d_n \in E_1^{3,0}$ *with*

$$\begin{aligned} a_n &\equiv \Delta^n[0] \mod (v_1) \\ b_n &\equiv v_2^n[1] \mod (v_1) \\ c_n &\equiv v_2^n[2] \mod (v_1) \\ d_n &\equiv \Delta'^n[3] \mod (v_1) \end{aligned}$$

and such that, for $k \geq 0$ *and* $t \in \mathbb{Z}$,

$$d_1(a_n) = \begin{cases} v_1^{6\cdot 2^k} b_{2^{k+1}(1+4t)} & n = 2^k(1+2t) \\ 0 & n = 0. \end{cases}$$

$$d_1(b_n) = \begin{cases} v_1^{3\cdot 2^k} c_{2^{k+1}(1+2t)} & n = 2^k(3+4t) \\ v_1^{3\cdot 2^{k+1}} c_{1+2^{k+1}+t2^{k+3}} & n = 1 + 2^{k+2} + t2^{k+3} \\ 0 & \textit{otherwise.} \end{cases}$$

$$d_1(c_n) = \begin{cases} v_1^{3(2^{k+1}+1)} d_{2^k(1+2t)} & n = 1 + 2^{k+1} + 2^{k+2} + t2^{k+3} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Define

$$a_n := \begin{cases} \Delta^n[0] & n = 2^k(1+2t) \\ 1 \cdot [0] & n = 0, \end{cases}$$

$$d_n := \begin{cases} v_1^{-3(1+2^{k+1})} d_1(c_{1+2^{k+1}+2^{k+2}+2^{k+3}t}) & n = 2^k(1+2t) \\ 1 \cdot [3] & n = 0. \end{cases}$$

Then Theorem 5.2, Theorem 5.5 and Theorem 5.13 together prove the theorem. □

5.4. **The differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$.** The goal of this section is to compute the remaining d_1 differentials and obtain the E_2-term.

Although $V(0)$ is not a ring spectrum,

$$(E_{\mathcal{C}})_* V(0) \cong (E_{\mathcal{C}})_*/2,$$

and a canonical generator is given by the image of the unit in $(E_{\mathcal{C}})_0$ in the long exact sequence

$$\ldots \to (E_{\mathcal{C}})_* \xrightarrow{2} (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_* V(0) \to \ldots$$

Thus, we can give $(E_{\mathcal{C}})_* V(0)$ the ring structure induced by that of $(E_{\mathcal{C}})_*$. Then, Lemma 4.7 implies that the algebraic duality resolution is a module over the cohomology $H^*(\mathbb{S}_{\mathcal{C}}, (E_{\mathcal{C}})_* V(0))$. The canonical inclusion

$$\mathbb{F}_4 \to (E_{\mathcal{C}})_* V(0)$$

induces a map

$$H^*(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4) \to H^*(\mathbb{S}_{\mathcal{C}}^1, (E_{\mathcal{C}})_* V(0)).$$

Therefore, the algebraic duality resolution spectral sequence for $(E_{\mathcal{C}})_* V(0)$ is also a module over $H^*(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4)$.

Let

$$F_1^{p,q} = \mathrm{Ext}^q_{\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1]]}(\mathscr{C}_p, \mathbb{F}_4) \Longrightarrow H^{p+q}(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4). \tag{5.23}$$

Let $g_0 \in F_1^{0,4}$ be the periodicity generator for the cohomology of G_{24},

$$g_0 \in H^4(G_{24}, \mathbb{F}_4) = H^4(Q_8, F_4)^{C_3}.$$

The extension

$$1 \to K^1 \to \mathbb{S}_{\mathcal{C}}^1 \to G_{24} \to 1$$

is split. Therefore, the map

$$H^*(\mathbb{S}_{\mathcal{C}}^1, \mathbb{F}_4) \to H^*(G_{24}, \mathbb{F}_4)$$

induced by the inclusion of G_{24} in $\mathbb{S}_{\mathcal{C}}^1$ is split surjective. This implies that the image of g_0 is a permanent cycle in $F_1^{0,4}$. Therefore, it represents a class

$$g_0 \in H^4(\mathbb{S}_{\mathcal{C}}; \mathbb{F}_4),$$

and the differentials in the algebraic duality spectral sequence commute with the action of g_0. To make sense of this, we must compute the action of g_0 on $E_1^{p,q}$. First, note that g_0 acts by multiplication by $\Delta^{-1}g$ in $E_1^{0,q}$ and by $\Delta'^{-1}g'$ in $E_1^{3,q}$. Further, the map

$$H^*(\mathbb{S}_{\mathcal{C}}^1; \mathbb{F}_4) \to H^*(C_6; (E_{\mathcal{C}})_* V(0))$$

factors through the map

$$H^*(G_{24}; (E_{\mathcal{C}})_* V(0)) \to H^*(C_6; (E_{\mathcal{C}})_* V(0))$$

induced by the inclusion $C_6 \to G_{24}$. Therefore, g_0 acts by multiplication by h^4 on $E_1^{p,q}$ for $p = 1$ and $p = 2$.

We collect these remarks in the following lemma.

Lemma 5.24. *The differentials in the algebraic duality resolution are g_0-linear, where the action of g_0 is given by multiplication by $\Delta^{-1}g$ on $E_r^{0,*}$, by multiplication by $\Delta'^{-1}g$ on $E_r^{3,*}$, and by multiplication by h^4 on $E_r^{p,*}$ for $p = 1, 2$.*

This lemma will allow us to compute some of the differentials $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ for $q > 0$ based on our results for $q = 0$.

Lemma 5.25. *Let $x \in E_1^{0,q}$. The differential $d_1 : E_1^{0,q} \to E_1^{1,q}$ is zero unless $x = h_1^t\Delta^s$ or $x = g^t\Delta^s$, in which case it is given by*

$$d_1(h_1^t\Delta^s) = h_1^t d_1(\Delta^s)$$

and

$$d_1(g^t\Delta^s) = h^{4t}d_1(\Delta^{s+t}).$$

Proof. There is no v_1-torsion in $E_1^{1,q}$, and d_1 is v_1-linear. Therefore, if x is v_1-torsion, we must have $d_1(x) = 0$. The only classes in $E_1^{0,q}$ which are not v_1 torsion are of the form $x = h_1^t\Delta^s$ or $x = g^t\Delta^s$. The statement for $x = h_1^t\Delta^s$ follows from h_1-linearity of the differentials. For $x = g^t\Delta^s$, rewrite x as $g^t\Delta^{-1}\Delta^{s+t}$. The statement then follows from Lemma 5.24. □

Lemma 5.26. *Let $x \in E_1^{1,q}$. The differential $d_1 : E_1^{1,q} \to E_1^{2,q}$ satisfies*

$$h^k d_1(x) = d_1(h^k x).$$

Proof. This follows from the fact that the differentials are $h_1 = hv_1$ and v_1-linear. Indeed, since $h_1 = v_1 h$, we have the following equalities

$$v_1^k h^k d_1(x) = h_1^k d_1(x) = d_1(h_1^k x) = d_1(v_1^k h^k x) = v_1^k d_1(h^k x).$$

Since there is no v_1-torsion and no h-torsion in $E_1^{1,q}$ and $E_1^{2,q}$, $h^k d_1(x) = d_1(h^k x)$. □

Understanding the differential $d_1 : E_1^{2,q} \to E_1^{3,q}$ is more subtle as there is v_1-torsion in $E_1^{3,q}$ for $q > 0$. We will use the following result. Its proof is postponed until the end of the section.

Lemma 5.27. *Let $x \in E_1^{2,q}$. There exists $y \in E_1^{3,q}$ such that $d_1(x) = v_1^3 y$.*

Lemma 5.28. *Let $x \in E_1^{2,0}$. Consider $d_1 : E_1^{2,q} \to E_1^{3,q}$. Then*

$$d_1(h^i x) = v_1^{-i} h_1^i d_1(x),$$

where we make sense of division by v_1 as follows: if $d_1(x) = v_1^k y$, then

$$v_1^{-i} h_1^i d_1(x) := v_1^{k-i} h_1^i y.$$

This includes the case when $y = 0$, in which case the formula reads as $d_1(h^i x) = 0$.

Proof. Since the differentials are g_0-linear and the action of g_0 on x is given by multiplication by h^4, it suffices to consider the cases $d_1(h^i x)$ for $1 \le i \le 3$.

Now, suppose instead that $x \in E_1^{2,0}$. Note that since d_1 is $\mathbb{F}_4[v_1, h_1]$-linear and $h_1 = v_1 h$, we have

$$v_1^i d_1(h^i x) = h_1^i d_1(x). \tag{5.29}$$

We will treat the cases $d_1(x) = 0$ and $d_1(x) \neq 0$ seperately.

First, suppose that $d_1(x) \neq 0$. By Lemma 5.27, there is some $y \in E_1^{3,0}$ such that

$$d_1(x) = v_1^k y$$

for $k \ge 3$. Hence,

$$v_1^i d_1(h^i x) = h_1^i d_1(x) = v_1^k h_1^i y.$$

Since $E_1^{3,0}$ is free over $\mathbb{F}_4[v_1, h_1]$, $d_1(x)$ is not annihilated by v_1 or h_1. Hence, neither is y, so that

$$d_1(h^i x) = v_1^{k-i} h_1^i y.$$

Suppose that $d_1(x) = 0$. We must show that $d_1(h^i x) = 0$ for $i = 1, 2, 3$. Since d_1 is h_1-linear,

$$v_1^i d_1(h^i x) = d_1(h_1^i x) = 0.$$

Hence, in this case, $d_1(h^i x)$ must be v_1-torsion. By Lemma 5.27, if $z \in E_1^{2,q}$, then $d_1(z) = v_1^3 y$ for some $y \in E_1^{3,q}$. Letting $z = h^i x$, this implies that $d_1(h^i z)$ is divisible by v_1^3. However, there are no non-zero elements in $E_1^{3,q}$ which are both v_1-torsion and divisible by v_1^3. Indeed, all the v_1-torsion is annihilated by v_1^2. This finishes the proof. ☐

To prove Lemma 5.27, we will use the following fact. To state it, recall that

$$x_0 = i + j + k, \qquad x_1 = i + \zeta j + \zeta^2 k, \qquad x_2 = i + \zeta^2 j + \zeta k.$$

Further, let $B_* \to \mathbb{W}$ be the projective resolution of the trivial C_6-module $\mathbb{W}$ which was constructed in Lemma 6.2. Let $C_* \to \mathbb{W}$ be the projective resolution of the trivial G_{24}-module $\mathbb{W}$ which was constructed in Lemma 6.1.

Lemma 5.30. *Let e_0 denote the canonical generator of $\mathbb{W}[[\mathbb{S}_{\mathcal{C}}^1/G_{24}]]$ and e_1 denote the canonical generator of $\mathbb{W}[[\mathbb{S}_{\mathcal{C}}^1/C_6]]$. The map*

$$F : \mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(\mathbb{W}) \to \mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(\mathbb{W})$$

given by

$$F(\gamma e_0) = \gamma(e + i + j + k)(e - \alpha^{-1})e_1$$

has a lift F_*

$$\begin{array}{ccc} \mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(C_*) & \xrightarrow{F_*} & \mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(B_*) \\ \downarrow & & \downarrow \\ \mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(\mathbb{W}) & \xrightarrow{F} & \mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(\mathbb{W}). \end{array}$$

such that, for γ, γ_1 and γ_2 in $\mathbb{W}[[\mathbb{S}_{\mathcal{C}}^1]]$,

$$\begin{aligned} F_0(\gamma c_{0,0}) &= \gamma(e + x_0)(e - \alpha^{-1})b_0 \\ F_1(\gamma_1 c_{1,1}, \gamma_2 c_{1,2}) &= -(\gamma_1 \zeta^2 x_1 + \gamma_2 \zeta x_2)(e - \alpha^{-1})b_1 \\ F_2(\gamma_1 c_{2,1}, \gamma_2 c_{2,2}) &= (\gamma_1(\zeta^2 - \zeta)x_2 + \gamma_2(\zeta - \zeta^2)x_1)(e - \alpha^{-1})b_2 \\ F_3(\gamma c_{3,0}) &= -3\gamma x_0(e - \alpha^{-1})b_3 \\ F_4(\gamma c_{4,0}) &= -3\gamma x_0(e + x_0)i^2(e - \alpha^{-1})b_4. \end{aligned}$$

Further, for $0 \le k < 4$, if we define $F_{4t+k} \otimes_{\mathbb{W}} \mathbb{F}_4 = F_k \otimes_{\mathbb{W}} \mathbb{F}_4$, then

$$F_* \otimes_{\mathbb{W}} \mathbb{F}_4 : \mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(C_*) \otimes_{\mathbb{W}} \mathbb{F}_4 \to \mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(B_*) \otimes_{\mathbb{W}} \mathbb{F}_4$$

is a periodic lift of $F \otimes_{\mathbb{W}} \mathbb{F}_4$.

Proof. The chain complexes $\mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(C_*)$ and $\mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(B_*)$ are projective resolutions of $\mathbb{S}_{\mathcal{C}}^1$-module of $\mathrm{Ind}_{G_{24}}^{\mathbb{S}_{\mathcal{C}}^1}(C_*) \cong \mathbb{W}[[\mathbb{S}_{\mathcal{C}}^1/G_{24}]]$ and $\mathrm{Ind}_{C_6}^{\mathbb{S}_{\mathcal{C}}^1}(B_*) \cong \mathbb{W}[[\mathbb{S}_{\mathcal{C}}^1/C_6]]$ respectively. A direct computation shows that for $0 \le k \le 4$, $F_{k-1}d_k = d_k F_k$. Since these are complexes of projective $\mathbb{S}_{\mathcal{C}}^1$-modules, there exists F_k, $k > 4$ such that F_* lifts F.

Finally, note that

$$-3x_0(e + x_0)i^2 \equiv (x_0 + x_0^2)i^2 \equiv (e + x_0) \mod (2).$$

Therefore,

$$-3x_0(e+x_0)i^2(e-\alpha^{-1}) \equiv (e+x_0)(e-\alpha^{-1}) \mod (2)$$

and we can choose F_* so that $F_* \otimes_W \mathbb{F}_4$ is periodic. □

Proof of Lemma 5.27. Let $M = (E_{\mathcal{C}})_* V(0)$. Recall from Theorem 4.1 that there is a commutative diagram

$$\begin{array}{ccc} \mathscr{C}_3 & \xrightarrow{\partial_3} & \mathscr{C}_2 \\ \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_2} \\ \mathscr{C}_3 & \xrightarrow{\partial_3'} & \mathscr{C}_2, \end{array}$$

where

$$\partial_3'(\gamma e_3') = \gamma\pi(e+i+j+k)(e-\alpha^{-1})\pi^{-1}(e_2').$$

Let $g_\pi : \mathscr{C}_3 \to \mathscr{C}_0$ be the map of $\mathbb{S}^1_{\mathcal{C}}$-modules such that

$$g_\pi(e_3) = \pi e_0.$$

This is well-defined. Indeed, if $\tau' \in G'_{24}$, then $\tau' = \pi\tau\pi^{-1}$ for $\tau \in G_{24}$. Hence,

$$\begin{aligned} g_\pi(\tau' e_3) &= \tau' g_\pi(e_3) \\ &= \pi\tau\pi^{-1}\pi e_0 \\ &= \pi e_0. \end{aligned}$$

Similarly, the map $g_{\pi^{-1}} : \mathscr{C}_1 \to \mathscr{C}_2$ of $\mathbb{S}^1_{\mathcal{C}}$-modules given by

$$g_\pi(e_1) = \pi^{-1} e_2$$

is well-defined because π commutes with the elements of C_6. Let

$$f : \mathscr{C}_0 \to \mathscr{C}_1$$

be the map of $\mathbb{S}^1_{\mathcal{C}}$-modules given by

$$f(\gamma e_0) = \gamma(e+i+j+k)(e-\alpha^{-1})e_1.$$

Then, $\partial_3 = g_2^{-1} g_{\pi^{-1}} f g_\pi g_3$, that is, it is the composite

$$\mathscr{C}_3 \xrightarrow{g_3} \mathscr{C}_3 \xrightarrow{g_\pi} \mathscr{C}_0 \xrightarrow{f} \mathscr{C}_1 \xrightarrow{g_{\pi^{-1}}} \mathscr{C}_2 \xrightarrow{g_2^{-1}} \mathscr{C}_2.$$

Since $g_\pi g_3$ and $g_2^{-1} g_{\pi^{-1}}$ are isomorphisms and the map they induce on M are v_1-linear, it is sufficient to prove that the map

$$f^* : \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_1, M) \to \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathscr{C}_0, M)$$

induced by f has image divisible by v_1^3. Further, note that

$$\mathrm{Ext}_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}/H]], M) \cong \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{Z}_2[[\mathbb{S}^1_{\mathcal{C}}/H]], M).$$

Therefore, it is sufficient to prove that the map

$$F^* : \mathrm{Ext}_{\mathbb{W}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{W}_2[[\mathbb{S}^1_{\mathcal{C}}/C_6]], M) \to \mathrm{Ext}_{\mathbb{W}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{W}_2[[\mathbb{S}^1_{\mathcal{C}}/G_{24}]], M)$$

induced by

$$F = \mathbb{W} \otimes_{\mathbb{Z}_2} f : \mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}/G_{24}]] \to \mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}/C_6]]$$

has image divisible by v_1^3.

To compute F^*, we use the lift described in Lemma 5.30. Let

$$F_k^* = \mathrm{Hom}_{\mathbb{W}_2[[\mathbb{S}^1_{\mathcal{C}}]]}(F_k, M).$$

Let $\mathcal{E}_\lambda$ be the λ-eigenspace with respect to the action of ω. Note that $\mathcal{E}_\lambda$ is an $\mathbb{F}_4[v_1]$-module. Define

$$\phi_{x_0} = \phi_i + \phi_j + \phi_k, \qquad \phi_{x_1} = \phi_i + \zeta\phi_j + \zeta^2\phi_k, \qquad \phi_{x_2} = \phi_i + \zeta^2\phi_j + \zeta\phi_k.$$

Let

$$\begin{aligned} G_0^* &= (\phi_e + \phi_{x_0}) : \mathcal{E}_1 \to \mathcal{E}_1, \\ G_1^* &= -(\zeta^2\phi_{x_1}, \zeta\phi_{x_2}) : \mathcal{E}_1 \to \mathcal{E}_{\zeta^2} \oplus \mathcal{E}_\zeta, \\ G_2^* &= ((\zeta^2 - \zeta)\phi_{x_2}, (\zeta - \zeta^2)\phi_{x_1}) : \mathcal{E}_1 \to \mathcal{E}_\zeta \oplus \mathcal{E}_{\zeta^2}, \\ G_3^* &= -\phi_3\phi_{x_0} : \mathcal{E}_1 \to \mathcal{E}_1, \\ G_4^* &= -\phi_3\phi_{x_0}(\phi_e + \phi_{x_0})\phi_{i^2} : \mathcal{E}_1 \to \mathcal{E}_1 \end{aligned}$$

so that

$$F_k^* = G_k^*(\phi_e - \phi_\alpha^{-1}),$$

where $(\phi_e - \phi_\alpha^{-1}) : \mathcal{E}_1 \to \mathcal{E}_1$. Let x be an element of $\mathrm{Ext}^k_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}/C_6]], M)$. Choose a representative $\widetilde{x} \in \mathrm{Hom}_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathrm{Ind}_{C_6}^{\mathbb{S}^1_{\mathcal{C}}}(B_k), M)$. Since $(\phi_e - \phi_\alpha^{-1})(\widetilde{x}) = v_1^3\widetilde{x}'$, and G_k^* is a v_1-linear map, we have

$$F_k^*(\widetilde{x}) = v_1^3 G_k^*(\widetilde{x}')$$

Let $\widetilde{y} = G_k^*(\widetilde{x}')$. Since $\widetilde{x}$ is a cocycle, so is $v_1^3\widetilde{y}$. Since, the differential d of $\mathrm{Hom}_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathrm{Ind}_{G_{24}}^{\mathbb{S}^1_{\mathcal{C}}}(C_*), M)$ is v_1-linear, and $\mathrm{Hom}_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathrm{Ind}_{G_{24}}^{\mathbb{S}^1_{\mathcal{C}}}(C_*), M)$ has no v_1-torsion, $\widetilde{y}$ is also a cocycle. Let

$$y \in \mathrm{Ext}^k_{\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{W}[[\mathbb{S}^1_{\mathcal{C}}/G_{24}]], M)$$

be the class detected by $\widetilde{y}$. Then $F^*(x) = v_1^3 y$. □

This completes the computation of the E_2-page. A small sample is shown in Figure 5.1.

5.5. **Higher Differentials.** In this section, we prove that all differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ for $r \geq 2$ are zero. Because of the sparsity of the spectral sequence, the only differentials d_r for $r \geq 2$ which do not have a zero target are

$$\begin{aligned} d_2 &: E_2^{0,q} \to E_2^{2,q-1}, \ q \geq 2 \\ d_2 &: E_2^{1,q} \to E_2^{3,q-1}, \ q \geq 2 \\ d_3 &: E_3^{0,q} \to E_3^{3,q-2}, \ q \geq 3. \end{aligned}$$

The proof of the following result is a direct computation. A similar computation is done in [12, §4], and our notation corresponds to theirs.

Lemma 5.31. *Let v_1 have degree $(s,t) = (2,0)$, v_2 have degree $(6,0)$, and h have degree $(0,1)$. Let $x = v_2^3 h$. Then*

$$H^*(C_6; (E_{\mathcal{C}})_*) \cong \mathbb{W}[[u_1^3]][v_1^2, v_1v_2, v_2^{\pm 1}, x]/(2x).$$

Lemma 5.32. *All differentials $d_2 : E_2^{1,q} \to E_2^{3,q-1}$ are zero.*

Proof. Let b_n be as in Theorem 5.5. The set

$$B = \{h^k b_n \mid n = 0, 1, 2^{s+1}(1+4t),\ 0 \leq k \leq 3,\ 0 \leq s\}$$

generates $E_2^{1,*}$ as an $\mathbb{F}_4[v_1, g_0]$-module, for g_0 as in Lemma 5.24. Because the differentials are $\mathbb{F}_4[v_1, g_0]$-linear, it suffices to show that the d_2 differentials on the

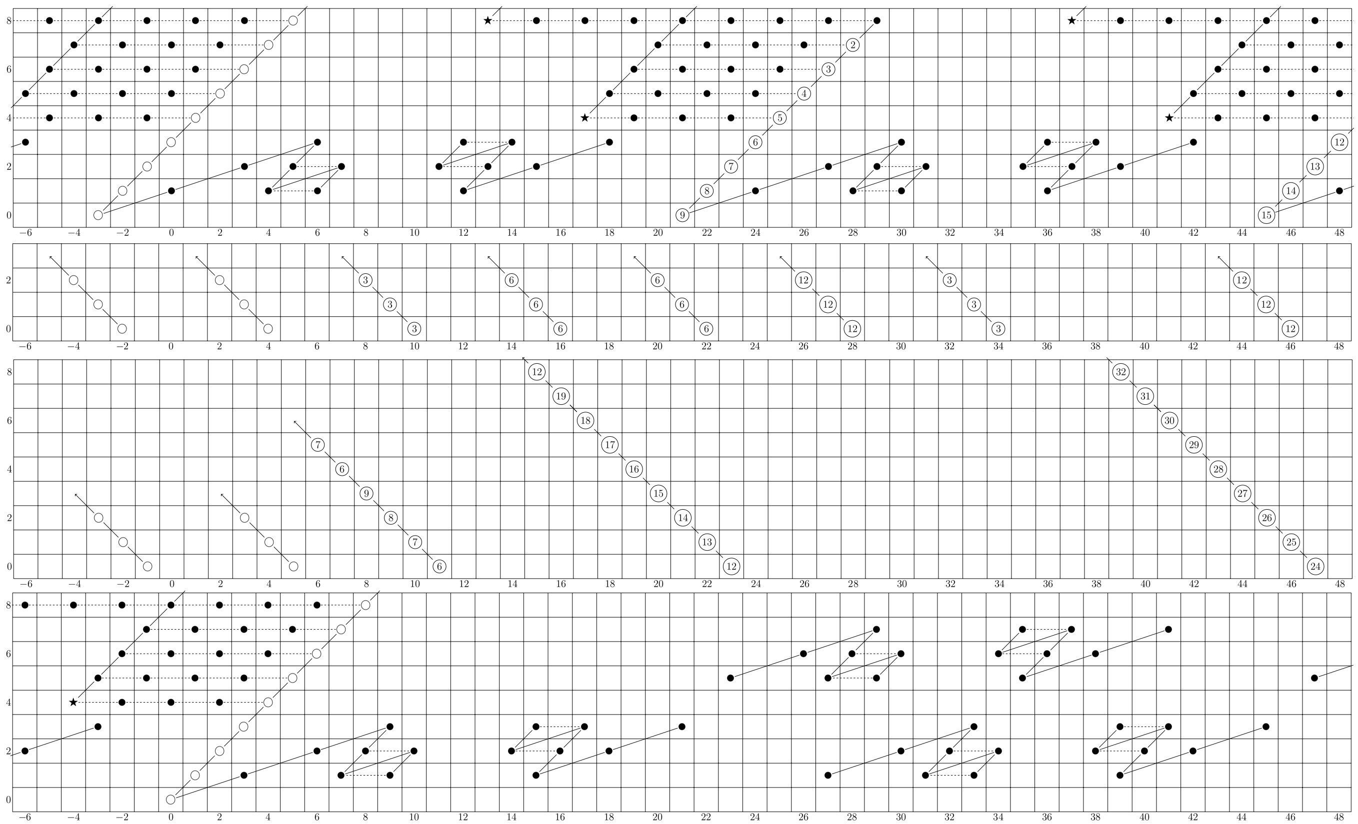

FIGURE 5.1. The E_2-term of the ADRSS with coefficients $(E_{\mathcal{C}})_* V(0)$. The notation and grading is as in Figure 5.1. In addition, a ⓒ is a copy of $\mathbb{F}_4[v_1]/(v_1^c)$. In Section 5.5, we prove that $E_2 \cong E_\infty$. Therefore, this is also the E_∞-term of the ADRSS.

elements of B are zero. First, note that $d_2(b_n) = 0$ for all n, since the targets of these differentials are zero. Hence, it suffices to show that $d_2(h^k b_n) = 0$ for $1 \leq k \leq 3$.

The first remark is that, if $d_2(h^k b_n) = 0$, then

$$v_1 d_2(h^{k+1} b_n) = d_2(v_1 h^{k+1} b_n) = d_2(h_1 h^k b_n) = h_1 d_2(h^k b_n) = 0.$$

Hence, if $d_2(h^k b_n) = 0$, then $v_1 d_2(h^{k+1} b_n) = 0$. Further,

$$v_1^k d_2(h^k b_n) = d_2(h_1^k b_n) = h_1^k d_2(b_n).$$

Since $d_2(b_n) = 0$, we must have that $v_1^k d_2(h^k b_n) = 0$ for all $k \geq 0$.

Let $1 \leq k \leq 3$. Then $d_2(h^k b_0)$ is an element of internal degree $t = 0$ in $E_2^{3,k-1}$. Since $d_2(b_0) = 0$, $v_1 d_2(h b_0) = 0$. However, there is no v_1-torsion in $(E_2^{3,0})_0$, hence $d_2(h b_0) = 0$. Further, $(E_2^{3,1})_0$ and $(E_2^{3,2})_0$ are zero and $d_2(h^k b_0) = 0$ for $k = 2, 3$ for degree reasons.

Next, consider the elements of the form $h^k b_1$ for $1 \leq k \leq 3$. Since $d_2(h^k b_1)$ is in $E_2^{3,k-1}$ and there is no v_1-torsion in $E_2^{3,k-1}$ for $1 \leq k \leq 3$, these differentials must be zero.

The classes $h^k b_{2^{s+1}(1+4t)}$ have internal degree $3 \cdot 2^{s+2}(1+4t)$. Hence, their degree is congruent to zero modulo 3.

First, consider the case when $k = 1$. The possible targets for the d_2 differentials on these classes are in $E_2^{3,0}$ and must be annihilated by v_1. Therefore, they must be of the form

$$v_1^{3(1+2^{s'+1})-1} d_{2^{s'}(1+2t')}.$$

However, such classes have internal degree congruent to 1 modulo 3, since the degree of $d_{2^{s'}(1+2t')}$ is $24 \cdot 2^s(1+2t')$ and the degree of v_1 is 2. Hence, there is no appropriate target for these differentials. Further, this implies that $d_2(h^2 b_n)$ is annihilated by v_1.

The classes which are annihilated by v_1 in $E_2^{3,1}$ are of one of the forms

$$v_1^{3(1+2^{s'+1})-2} h_1 d_{2^{s'}(1+2t')},$$

$h_2 d_{2^{s'}(1+2t')}$, $v_1 x d_{2^{s'}(1+2t')}$, or $y d_{2^{s'}(1+2t')}$. Here, h_2 has internal degree 4, x has internal degree 8 and y has internal degree 16. Again, such classes have internal degree congruent to 1 modulo 3, so there is no possible target for the differentials. This, in turn, implies that $d_2(h^3 b_n)$ is annihilated by v_1.

The classes in $E_2^{3,2}$ which are annihilated by v_1 are of one of the forms

$$v_1^{3(1+2^{s'+1})-3} h_1^2 d_{2^{s'}(1+2t')},$$

$h_2^2 d_{2^{s'}(1+2t')}$, $v_1 h_1 x d_{2^{s'}(1+2t')}$, $h_1 y d_{2^{s'}(1+2t')}$ or $h_2 y d_{2^{s'}(1+2t')}$. Of these classes, $v_1 h_1 x d_{2^{s'}(1+2t')}$ and $h_1 y d_{2^{s'}(1+2t')}$ have internal degree congruent to 0 modulo 3, so we must make a more careful analysis.

Note that $3 \cdot 2^{s+2}(1+4t) \equiv 0 \mod 24$ if $s \geq 1$, and $3 \cdot 2^2(1+4t) \equiv 12 \mod 24$. Since the internal degree of $h_1 y d_{2^{s'}(1+2t')}$ is congruent to 18 modulo 24, it cannot be hit by a differential. Further, the internal degree of $v_1 h_1 x d_{2^{s'}(1+2t')}$ is 12 modulo 24. Therefore, the only possible differentials are $d_2(h^3 b_{2(1+4t)})$, with target $v_1 h_1 x d_{2t}$.

Let β be the Bockstein homomorphism described in Lemma 4.6. To finish the proof, we show that

$$\beta(d_2(h^3 b_{2(1+4t)})) = 0$$

and that

$$\beta(v_1 h_1 x d_{2t}) \neq 0.$$

Since $b_{2(1+4t)} = v_2^{2(1+4t)} f$ for some power series $f \in \mathbb{F}_4[[u_1^3]]$, there is a class in $H^0(C_6, (E_{\mathcal{C}})_*)$ which reduces to $b_{2(1+4t)}$. Hence,

$$\beta(b_{2(1+4t)}) = 0.$$

Since

$$\beta(h^{2t+1}) = h^{2t+2},$$

it follows that

$$\beta(h^3 b_{2(1+4t)}) = \beta(h^3) b_{2(1+4t)} + h^3 \beta(b_{2(1+4t)}) = h^4 b_{2(1+4t)}.$$

By Lemma 4.6,

$$\beta(d_2(h^3 b_{2(1+4t)})) = d_2(\beta(h^3 b_{2(1+4t)})) = d_2(h^4 b_{2(1+4t)}) = g_0 d_2(b_{2(1+4t)}) = 0.$$

The cohomology $H^*(G_{24}, (E_{\mathcal{C}})_*)$ can be obtained from [1] (see Figure 5.2). The classes $v_1 h_1 x d_{2t}$ are not integral cohomology class. Further, $\beta(v_1 h_1 x d_{2t}) = h_2^3 d_{2t}$. In particular, $\beta(v_1 h_1 x d_{2t}) \neq 0$. Therefore,

$$d_2(h^3 b_{2(1+4t)}) \neq v_1 h_1 x d_{2t},$$

and we must have $d_2(h^3 b_{2(1+4t)}) = 0$. □

The next few results will be necessary to prove that all remaining higher differentials are zero. First, note that the algebraic duality resolution is a resolution of $P\mathbb{S}^1_{\mathcal{C}}$-modules, where $\mathscr{C}_0 \cong \mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/A_4]]$, $\mathscr{C}_1 \cong \mathscr{C}_2 \cong \mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/C_3]]$ and $\mathscr{C}_3 \cong \mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/A_4']]$. There is a corresponding algebraic duality resolution spectral sequence. Let $F_r^{p,q}$ be this spectral sequence for coefficients in $(E_{\mathcal{C}})_* V(0)$. This spectral sequence converges to $H^*(P\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_* V(0))$. There is a map of spectral sequences $F_r^{p,q} \to E_r^{p,q}$ induced by the projection $\mathbb{S}^1_{\mathcal{C}} \to P\mathbb{S}^1_{\mathcal{C}}$. The induced map $F_1^{0,q} \to E_1^{0,q}$ is the map

$$\varphi : H^q(A_4, (E_{\mathcal{C}})_* V(0)) \to H^q(G_{24}, (E_{\mathcal{C}})_* V(0)).$$

Theorem 5.33. *The map*

$$\varphi : H^*(A_4, (E_{\mathcal{C}})_* V(0)) \to H^*(G_{24}, (E_{\mathcal{C}})_* V(0))/(h_1)$$

induced by the projection $G_{24} \to G_{24}/C_2 \cong A_4$ is surjective in degrees $* \leq 3$. In particular,

$$H^0(A_4, (E_{\mathcal{C}})_* V(0)) \cong H^0(G_{24}, (E_{\mathcal{C}})_* V(0)),$$

and the classes h_2, h_2^2, h_2^3, x, $v_1 x$, x^2, $v_1 x^2$, y, $h_2 y$, $h_2^2 y$ (defined in Theorem 4.11) and their translations by powers of Δ are in the image of φ.

Proof. The 2-Sylow subgroups of G_{24} and A_4 are Q_8 and $V = Q_8/C_2$ respectively. The E_∞-terms of the v_1-Bockstein spectral sequences for Q_8 and V are computed in Proposition 7.1 and Proposition 7.6. Since C_2 acts trivially on $(E_{\mathcal{C}})_* V(0)$ and on Q_8, the projection $p : Q_8 \to V$ induces a morphism of v_1-Bockstein spectral sequences $E_r^{s,t,w}(V) \to E_r^{s,t,w}(Q_8)$. Further, this morphism is induced by the map constructed in Lemma 6.4. Using this, one can compute the image of the projection p on the associated graded. Taking C_3-fixed points finishes the proof. □

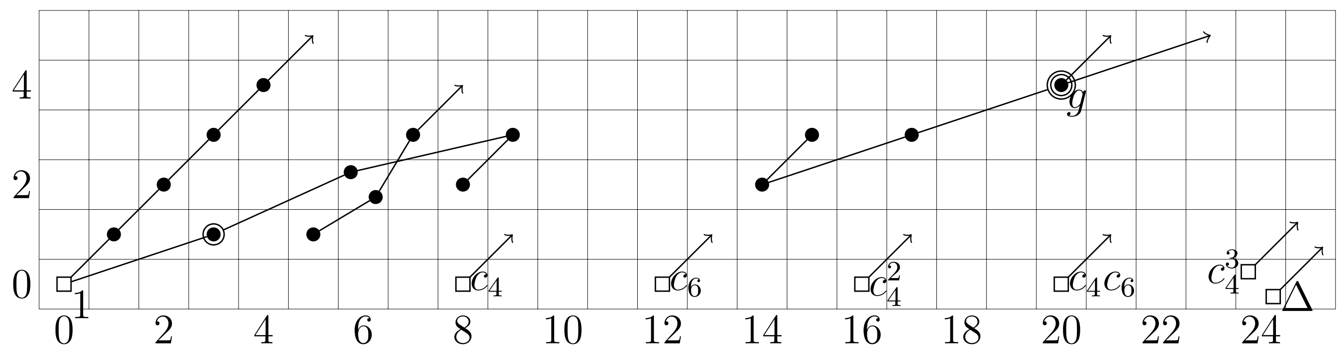

FIGURE 5.2. The cohomology $H^q(G_{24}, (E_{\mathcal{C}})_*)$ for $q \leq 5$. It is periodic of period 24 on Δ. A $\square$ denotes a copy of $\mathbb{W}[[j]]$. A $\bullet$ denotes a copy of $\mathbb{F}_4$. A $\odot$ denotes an extension by multiplication by 2.

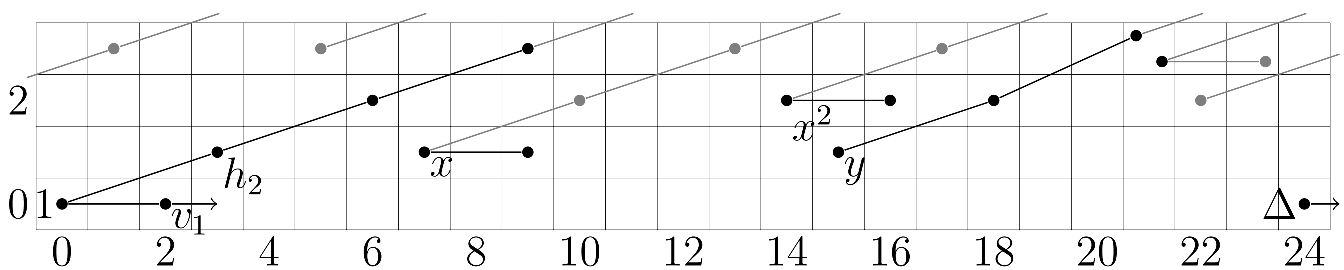

FIGURE 5.3. The cohomology $H^*(A_4, (E_{\mathcal{C}})_*V(0))$ in degrees $* \leq 3$. Gray classes go to zero under the map $\varphi : H^*(A_4, (E_{\mathcal{C}})_*V(0)) \to H^*(G_{24}, (E_{\mathcal{C}})_*V(0))$.

Theorem 5.33 is depicted in Figure 5.3. It implies that, modulo the image of multiplication by h_1, the map φ is surjective in degrees $q \leq 3$. All classes of degree $q \geq 4$ in $E_r^{0,q}$ are multiples of g_0, so their differentials will be determined by differentials on classes of degree $q \leq 3$. Further, by h_1-linearity it suffices to show that the differentials on the classes in the image of φ are zero. It is therefore sufficent to compute of some of differentials $d_r : F_r^{0,q} \to F_r^{r,q-r+1}$ for $q \leq 3$. The advantage of this method is that the spectral sequence $F_r^{p,q}$ is sparser than $E_r^{p,q}$. Indeed, $\mathscr{C}_1$ and $\mathscr{C}_2$ are projective $P\mathbb{S}^1_{\mathcal{C}}$-modules. Hence, for $p = 1$ or $p = 2$,

$$F_1^{p,q} \cong \mathrm{Ext}^q_{\mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}]]}(\mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/C_3]], (E_{\mathcal{C}})_* V(0))$$

is zero when $q > 0$. Hence, $F_r^{p,q} = 0$ when $q \geq 0$ for $p = 1$ or and $p = 2$. Further, $E_1^{p,0} \cong F_1^{p,0}$ for all p, so the computation of $F_2^{p,q}$ follows immediately from that of $E_2^{p,q}$,

$$F_2^{p,q} \cong \begin{cases} E_2^{p,q} & q = 0 \\ F_1^{p,q} & q > 0. \end{cases}$$

The following results are generalizations of results that can be found in [11, §6]. The first result we state is Lemma 6.1 of [11].

Lemma 5.34 (Henn-Karamanov-Mahowald)**.** *Let R be a $\mathbb{Z}_2$-algebra and M be an R-module. Let*

$$0 \to \mathscr{C}_3 \xrightarrow{\partial_3} \mathscr{C}_2 \xrightarrow{\partial_3} \mathscr{C}_1 \xrightarrow{\partial_2} \mathscr{C}_0 \xrightarrow{\partial_1} \mathbb{Z}_2 \to 0$$

be an exact sequence of R-modules such that $\mathscr{C}_1$ and $\mathscr{C}_2$ are projective. Define N_i recursively by $0 \to N_i \to \mathscr{C}_i \xrightarrow{\partial_i} N_{i-1} \to 0$, and let $E_r^{s,t}$ be the first quadrant spectral sequence of the exact couple

$$\begin{array}{ccc} \mathrm{Ext}_R(N_i, M) & \dashrightarrow & \mathrm{Ext}_R(N_{i-1}, M) \\ & \nwarrow \quad \swarrow & \\ & \mathrm{Ext}_R(\mathscr{C}_i, M) & \end{array}$$

Then $E_1^{p,q} = 0$ for $0 < p < 3$ and $q > 0$. Further, there are isomorphisms

$$\mathrm{Ext}^q_R(N_0, M) \cong \begin{cases} \ker(E_1^{1,0} \to E_1^{2,0}) & q = 0 \\ E_2^{q+1,0} \cong E_3^{q+1,0} & q = 1, 2 \\ E_3^{q-2,0} & q \geq 3. \end{cases}$$

Let $j : N_0 \to \mathscr{C}_0$ be the inclusion. The only possible non-zero higher differentials are of the form $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$, and they can be identified with the map $\mathrm{Ext}^q_R(\mathscr{C}_0, M) \to \mathrm{Ext}^q_R(N_0, M)$ induced by j.

Let $P_* = \mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}]] \otimes_{\mathbb{Z}_2[A_4]} D_*$ for D_* as defined in Lemma 6.3 below, so that P_* is a projective resolution of $\mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/A_4]]$. Let P'_* be the analogous projective resolution of $\mathbb{Z}_2[[P\mathbb{S}^1_{\mathcal{C}}/A'_4]]$. Let N_0 be defined by the exact sequence $0 \to N_0 \to \mathscr{C}_0 \xrightarrow{\varepsilon} \mathbb{Z}_2 \to 0$. One can splice P'_* with the algebraic duality resolution to obtain a $P\mathbb{S}^1_{\mathcal{C}}$-projective resolution Q_* of N_0.

Lemma 5.35. *There is a map $\phi : Q_* \to P_*$ such that*

$$\phi_0 : Q_0 \to P_0$$

covers the map $j : N_0 \to \mathscr{C}_0$ which sends $e_1 \to (e - \alpha)e_0$.

Proof. Note that $Q_0 \cong P_0 \cong \mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}/C_3]]$. So the map which sends the generator of $e \otimes 1 \in Q_0$ to $(e-\alpha)\otimes 1 \in P_0$ is well defined and covers j. By the theory of acyclic models, this extends to a chain map ϕ. □

The following is an observation in [11, §6]. It follows from Lemma 5.34.

Lemma 5.36. *Let $T_{*,*}$ be the double complex satisfying $T_{*,0} = P_*$ and $T_{*,1} = Q_*$ with vertical differentials δ_P and δ_Q and horizontal differentials $\phi_s : Q_s \to P_s$. Up to reindexing, the filtration of the spectral sequence of this double complex agrees with that of the algebraic duality resolution spectral sequence.*

The following result is an adaptation of part of Lemma 6.5 of [11].

Lemma 5.37. *Let $s > 0$. Let $z \in H^s(A_4, (E_{\mathcal{C}})_*V(0))$ be an element of internal degree $2t$ such that*

$$v_1^k z = 0.$$

Let $c \in \mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(P_s, M)$ be a cocyle which represents x. Choose an element h in $\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(P_{s-1}, M)$ such that

$$\delta_P(h) = v_1^k c.$$

Let

$$\phi^* : \mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(P_*, M) \to \mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(Q_*, M)$$

be induced by ϕ. Then there are elements d and d' in $\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(Q_{s-1}, M)$ and an element d'' in $\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(Q_s, M)$ such that

$$\phi^*_{s-1}(h) = d' + v_1^k d \tag{5.38}$$

and

$$\delta_Q(d') = v_1^k d''.$$

For d'' as above,

$$j^*(z) = [d''] \in \mathrm{Ext}^s_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}((N_0, M).$$

Proof. Let $M = (E_{\mathcal{C}})_*V(0)$. Recall that $\mathcal{E}_\lambda$ denotes the λ-eigenspace with respect to the action of ω in C_3. Consider $\phi^*_{s-1}(h)$ in $\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(Q_{s-1}, M)$. Identify

$$\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}^1_{\mathcal{C}}]]}(Q_{s-1}, M) \cong \begin{cases} \mathcal{E}_1 & s = 1, 2 \\ \bigoplus_{n+m=s-3} \mathcal{E}_{\zeta^{2n+m}} & s \geq 3. \end{cases}$$

Since x has degree $2t$, so do h and $\phi^*_{s-1}(h)$. Hence,

$$\phi^*_{s-1}(h) \cong \begin{cases} (\mathcal{E}_1)_{2t} & s = 1, 2 \\ \bigoplus_{n+m=s-3} (\mathcal{E}_{\zeta^{2n+m}})_{2t} & s \geq 3. \end{cases}$$

Therefore, it is the sum of terms of the form

$$u^{-t} \sum_{i=0}^{\infty} a_i u_1^i = u^{-t} \sum_{i=0}^{k-1} a_i u_1^k + v_1^k \left(u^{-t+k} \sum_{i=k}^{\infty} a_i u_1^{i-k} \right),$$

and we can write

$$\phi^*_{s-1}(h) = d' + v_1^k d.$$

To prove (5.38), note that

$$\begin{aligned}\delta_Q(d') + v_1^k\delta_Q(d) &= \delta_Q(d' + v_1^k d)\\ &= \delta_Q(\phi_{s-1}^*(h))\\ &= \phi_s^*(\delta_P(h))\\ &= \phi_s^*(v_1^k c)\\ &= v_1^k\phi_s^*(c).\end{aligned}$$

Hence, $\delta_Q(d') \equiv 0$ modulo (v_1^k), that is,

$$\delta_Q(d') = v_1^k d''$$

for some d''. So the first claim holds.

Now, note that

$$\begin{aligned}v_1^k j^*(c) &= v_1^k\phi_s^*(c)\\ &= \phi_s^*(v_1^k c)\\ &= \phi_s^*(\delta_P(h))\\ &= \delta_Q(\phi_{s-1}^*(h))\\ &= \delta_Q(d' + v_1^k d)\\ &= v_1^k d'' + v_1^k\delta_Q(d).\end{aligned}$$

Since there is no v_1-torsion in the double complex $\mathrm{Hom}_{\mathbb{F}_4[[P\mathbb{S}_\mathcal{C}^1]]}(T_{*,*}, M)$, we must have

$$j^*(c) = d'' + \delta_Q(d).$$

This reduces to

$$j^*(z) = [d''] \in \mathrm{Ext}^s_{\mathbb{F}_4[[P\mathbb{S}_\mathcal{C}^1]]}((N_0, M).$$

□

Lemma 5.39. *Let z be in $F_2^{0,q}$. Then $d_2(z) = 0$.*

Proof. If $q > 1$, then $d_2(z) = 0$ since the target of the differential is zero. Suppose that $q = 1$. Then z is v_1-torsion. Let k be the smallest integer such that $v_1^k z = 0$. The computations in Section 7 show that $k = 1$ or $k = 2$ (see Figure 7.4). Choose h as in Lemma 5.37 and write

$$\phi_0(h) = (e - \phi_\alpha)(h) = d' + v_1^k d.$$

However,

$$\phi_\alpha \equiv \mathrm{id} \mod v_1^3.$$

So we must have

$$d' = 0.$$

By Lemma 5.34 and Lemma 5.37, this implies that $d_2(z) = 0$ in the algebraic duality spectral sequence for $P\mathbb{S}_\mathcal{C}^1$. □

Corollary 5.40. *All differentials $d_2 : E_2^{0,q} \to E_2^{2,q-1}$ are zero.*

Lemma 5.41. *All differentials $d_3 : E_3^{0,q} \to E_3^{3,q-1}$ are zero.*

Proof. Differentials $d_3 : E_3^{0,q} \to E_3^{3,q-2}$ are zero for degree reasons if $0 \leq q < 2$. By Corollary 5.40, the classes $h_2\Delta^k$ survive to the E_3-term, and hence they must be permanent cycles. Thus, they represent cohomology classes in $H^*(\mathbb{S}^1_{\mathcal{C}}, (E_{\mathcal{C}})_*V(0))$. By Lemma 4.7, the differentials are $h_2\Delta^k$-linear for all $k \in \mathbb{Z}$. Using this fact and linearity with respect to h_1 and v_1, the problem reduces to verifying the claim for $x^2\Delta^k$. However, by the same argument, x is a permanent cycle and

$$d_3(x^2\Delta^k) = xd_3(x\Delta^k) = 0.$$

□

Lemma 5.42. *All differentials $d_r : E_r^{0,q} \to E_r^{r,q-r+1}$ are zero.*

Proof. By Lemma 5.39 and Lemma 5.41, $E_2^{*,*} \cong E_4^{*,*}$. The spectral sequence collapses at the E_4-term since the targets of all higher differentials are zero. □

6. Appendix I: Some projective resolutions

The group G_{24} contains the central subgroup C_2. Further,

$$G_{24}/C_2 \cong A_4,$$

where A_4 is the alternating group on four letters. It will be necessary for the computation to know which classes are in the image of the map induced by the projection $G_{24} \to A_4$. To do this computation, one can use explicit projective resolutions for these groups, and a morphism of resolutions in order to compare their cohomology. I construct these resolutions and this map in this section. The resolutions described are C_3-equivariant versions of the classical projective resolutions for the finite groups involved. Having these resolutions, their cohomology with coefficients in $(E_{\mathcal{C}})_*V(0)$ can be computed using v_1-Bockstein spectral sequences as described in [6]. This is carried out in Section 7.

Choose generators i and j for Q_8 such that $\omega i\omega^{-1} = j$, and $\omega^2 i\omega^{-2} = ij$. Let e be the identity. For any subgroup H of G_{24} containing $i^2 = -1$, I will call this element i^2 even if the group H does not contain i. I do this in order to avoid confusion with the coefficients $-1 \in \mathbb{Z}_2$.

Let χ_{ζ^s} be the representation of C_3 whose underlying module is $\mathbb{W}$ and such that $\omega \in C_3$ acts by multiplication by ζ^s. For a representation χ of C_3 and a group G which contains C_3, we can form the induced module

$$\mathrm{Ind}_{C_3}^{G}(\chi) := \mathbb{W}[G] \otimes_{\mathbb{W}[C_3]} \chi.$$

The modules $\mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta^s})$ are projective Q_8-modules. Define the following elements of $\mathbb{W}[G_{24}]$:

$$x_0 = i + j + ij,$$
$$x_1 = i + \zeta j + \zeta^2 ij,$$
$$x_2 = i + \zeta^2 j + \zeta ij.$$

These are eigenvectors for the action of ω with eigenvalues 1, ζ^2 and ζ respectively.

The following proposition gives a periodic projective resolution of the trivial G_{24}-module $\mathbb{W}$. In essence, this is a C_3-equivariant version of the Cartan-Eilenberg resolution for Q_8 described in [5, XII§7]. For any ring R, let $R\{x\}$ denote the free R-module on the generator x.

Lemma 6.1. *There is a periodic projective resolution of the trivial G_{24}-module $\mathbb{W}$ given by*

$$C_k = \begin{cases} \mathbb{W}[G_{24}/C_3]\{c_{k,0}\} & k \equiv 0,3 \mod 4 \\ \mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta^2})\{c_{k,1}\} \oplus \mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta})\{c_{k,2}\} & k \equiv 1 \mod 4, \\ \mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta})\{c_{k,1}\} \oplus \mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta^2})\{c_{k,2}\} & k \equiv 2 \mod 4. \end{cases}$$

where the differentials

$$d_{4k+i} : C_{4k+i} \to C_{4k+i-1}$$

are given by

$$\begin{aligned} d_{4k+1}(c_{4k+1,0}) &= x_1 c_{4k,0} + x_2 c_{4k,2} \\ d_{4k+2}(c_{4k+2,1}) &= -x_1 c_{4k+1,1} + (e+i^2)c_{4k+1,2} \\ d_{4k+2}(c_{4k+2,2}) &= (e+i^2)c_{4k+1,1} - x_2 c_{4k+1,2} \\ d_{4k+3}(c_{4k+3,1}) &= x_1 c_{4k+2,0} \\ d_{4k+3}(c_{4k+3,2}) &= x_2 c_{4k+2,0} \\ d_{4k+4}(c_{4k+4,0}) &= (e+x_0)(e+i^2)c_{4k+3,0}. \end{aligned}$$

That is, the differentials are given by the right action of the following matrices:

$$d_{4k+1} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad d_{4k+2} = \begin{pmatrix} -x_1 & e+i^2 \\ e+i^2 & -x_2 \end{pmatrix}$$

$$d_{4k+3} = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \qquad d_{4k+4} = (e+x_0)(e+i^2)$$

Lemma 6.2. *There is a periodic projective resolution of the trivial C_6-module $\mathbb{W}$ given by*

$$B_k = \mathbb{W}[C_6/C_3]\{b_k\},$$

and whose differentials are $d_{k+1} : B_{k+1} \to B_k$ given by

$$d_{k+1}(b_{k+1}) = (e + (-1)^k i^2)b_k.$$

Applying $\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1]]\otimes_{\mathbb{Z}_2[G_{24}]}$ to $C_\bullet$ and $\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1]]\otimes_{\mathbb{Z}_2[C_6]}$ to $B_\bullet$ gives C_3-equivariant projective resolutions of the $\mathbb{S}_{\mathcal{C}}^1$-modules of $\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1/G_{24}]]$ and $\mathbb{Z}_2[[\mathbb{S}_{\mathcal{C}}^1/C_6]]$.

The following is a C_3-equivariant analogue of the Cartan-Eilenberg resolution of the trivial $\mathbb{F}_4[C_2 \times C_2]$-module $\mathbb{F}_4$.

Lemma 6.3. *Let $A_4 \cong G_{24}/C_2$. Let $\overline{\chi} = \chi \otimes_{\mathbb{W}} \mathbb{F}_4$. Denote also by x_1 and x_2 the image of their projections under the natural map $\mathbb{W}[G_{24}] \to \mathbb{F}_4[A_4]$. There is a projective resolution of the trivial $\mathbb{F}_4[A_4]$-module $\mathbb{F}_4$ which, in degree k, is given by*

$$D_k = \bigoplus_{s+t=k} \mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta^{2s+t}})\{d_{s,t}\}.$$

The differential on D_* is given by the unique A_4-linear maps determined by

$$d(d_{s,t}) = x_1 d_{s-1,t} + x_2 d_{s,t-1}.$$

Proof. The modules $\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta^{2s+t}})$ are projective $\mathbb{F}_4[A_4]$-modules. Using the fact that $x_i^2 = 0$, a direct computation shows that $d^2 = 0$. Further, the differential commutes with the left action of C_3. It remains to show that (D_*, d) is an exact chain complex.

Recall that

$$A_4 \cong (C_2 \times C_2) \rtimes C_3.$$

To prove that the complex D_* is exact, we will prove that it is exact as a complex of $\mathbb{F}_4[C_2 \times C_2]$-modules. As left $\mathbb{F}_4[C_2 \times C_2]$-modules,

$$\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta^{2s+t}}) \cong \mathbb{F}_4[C_2 \times C_2].$$

Let $E(x_1, x_2)$ be the exterior sub-algebra of $\mathbb{F}_4[C_2 \times C_2]$ generated by x_1 and x_2. The natural inclusion

$$\iota : E(x_1, x_2) \to \mathbb{F}_4[C_2 \times C_2]$$

induces an isomorphism of left $\mathbb{F}_4[C_2 \times C_2]$-modules. Let $\Gamma[\gamma]$ denote the divided power algebra generated by γ. Consider the projective resolution of the trivial $E(x_1, x_2)$-modules $\mathbb{F}_4$ given by

$$X = E(x_1, x_2) \otimes \Gamma[\gamma_1, \gamma_2]$$

and differential $d(\gamma_1^s \gamma_2^t) = x_1 \gamma_1^{s-1} \gamma_2^t + x_2 \gamma_1^s \gamma_2^{t-1}$. Then X is a a projective resolution of the trivial $E(x_1, x_2)$-module $\mathbb{F}_4$. Further, the map of chain complexes

$$\varphi : X \to D_*$$

determined by

$$\varphi(a(\gamma_1^s \gamma_2^t)) = \iota(a) d_{s,t}$$

is an isomorphism of chain complexes. □

Lemma 6.4. *The complex D_* is a complex of G_{24}-modules via restriction along the natural map $G_{24} \to A_4$. The map $\phi : C_* \to D_*$ determined by*

$$\phi(c_{k,i}) = \begin{cases} d_{0,0} & k = i = 0 \\ d_{1,0} & k = i = 1, \\ d_{0,1} & k = 1, i = 2, \\ d_{2,0} & k = 2, i = 1, \\ d_{0,2} & k = i = 2, \\ d_{3,0} + d_{0,3} & k = 3, i = 0 \\ 0 & \text{otherwise} \end{cases}$$

lifts the canonical map $\mathbb{W} \to \mathbb{F}_4$.

To compute with these resolutions, it is necessary to understand the eigenspaces of $(E_{\mathcal{C}})_* V(0) = \mathbb{F}_4[[u_1]][u^{\pm 1}]$ with respect to the C_3 action.

Lemma 6.5. *Let $\mathcal{E}_\lambda$ be the λ-eigenspace of $(E_{\mathcal{C}})_* V(0)$ with respect to the action of the generator $\omega \in C_3$. For $k \in \mathbb{Z}$,*

$$\mathcal{E}_\lambda = \begin{cases} \mathbb{F}_4[v_1]\{u^{3k}\}_{k \in \mathbb{Z}} & \lambda = 1 \\ \mathbb{F}_4[v_1]\{u^{3k+1}\}_{k \in \mathbb{Z}} & \lambda = \zeta \\ \mathbb{F}_4[v_1]\{u^{3k+2}\}_{k \in \mathbb{Z}} & \lambda = \zeta^2 \end{cases}$$

Further,

$$(E_{\mathcal{C}})_* V(0) \cong \mathcal{E}_1 \oplus \mathcal{E}_\zeta \oplus \mathcal{E}_{\zeta^2}$$

as C_3-modules.

Proof. This holds since $\phi_\omega(u_1) = \zeta u_1$ and $\phi_\omega(u) = \zeta u$. □

7. Appendix II: The v_1-Bockstein spectral sequence

Our next goal is to introduce the v_1-Bockstein spectral sequence described in [6, §1]. Recall that $S_{\mathcal{C}}$ is the 2-Sylow subgroup of $\mathbb{S}_{\mathcal{C}}$. Fix a closed subgroup G of either $S_{\mathcal{C}}$ or $PS_{\mathcal{C}}$, where

$$PS_{\mathcal{C}} = S_{\mathcal{C}}/C_2.$$

Consider the exact couple

$$\begin{array}{ccc} H^*(G;\mathbb{F}_4[[u_1]][u^{\pm 1}]/(u_1^n)) & \longleftarrow & H^*(G;\mathbb{F}_4[[u_1]][u^{\pm 1}]/(u_1^{n+1})) \\ & \searrow & \uparrow v_1^n \\ & & H^*(G;\mathbb{F}_4[[u_1]][u^{\pm 1}]/(u_1)) \end{array}$$

Let s denote the cohomological degree, t denote the internal degree and w denote the filtration degree. The above exact couple gives rise to a strongly convergent tri-graded spectral sequence

$$\bigoplus_{w\geq 0} H^s(G;\mathbb{F}_4[u^{\pm 1}]_t) \Longrightarrow H^*(G;\mathbb{F}_4[[u_1]][u^{\pm 1}]).$$

We will show in Theorem 3.7 that the action of $S_{\mathcal{C}}$ on $(E_2)_*$ is trivial modulo $(2, v_1)$. Therefore, for any subgroup G of $S_{\mathcal{C}}$ or $PS_{\mathcal{C}}$

$$E_1^{s,t,w} = H^s(G;\mathbb{F}_4[u^{\pm 1}]_t) \cong H^s(G;\mathbb{F}_4) \otimes \mathbb{F}_4[u^{\pm 1}]_t.$$

The differentials are given by

$$d_r : E_r^{s,t,w} \to E_r^{s+1,t-2r,w+2r}.$$

They can be computed using any $\mathbb{F}_4[G]$-projective resolution P_* of $\mathbb{F}_4$ as follows.

Let $\partial_s : P_s \to P_{s-1}$ denote the differentials of P_*. Let x be in $H^s(G;\mathbb{F}_4[u^{\pm 1}])$. Choose a representative $\widetilde{x}$ in the complex $\mathrm{Hom}_G(P_s, \mathbb{F}_4[u^{\pm 1}])$. Let $\widetilde{x}'$ be a lift of $\widetilde{x}$ in the complex $\mathrm{Hom}_G(P_s, \mathbb{F}_4[[u_1]][u^{\pm 1}])$. Let

$$\partial^{s+1} : \mathrm{Hom}_G(P_s, \mathbb{F}_4[[u_1]][u^{\pm 1}]) \to \mathrm{Hom}_G(P_{s+1}, \mathbb{F}_4[[u_1]][u^{\pm 1}])$$

be the morphism induced by ∂_{s+1}. Then

$$\partial^{s+1}(\widetilde{x}') = v_1^r \widetilde{y}'$$

for some $\widetilde{y}'$ in $\mathrm{Hom}_G(P_s, \mathbb{F}_4[[u_1]][u^{\pm 1}])$. Let y be the class in $H^{s+1}(G;\mathbb{F}_4[u^{\pm 1}])$ detected by the image of $\widetilde{y}'$ in $\mathrm{Hom}_G(P_{s+1}, \mathbb{F}_4[u^{\pm 1}])$. The differential d_r is defined by

$$d_r(x) = y.$$

The differentials are v_1-linear and this is a spectral sequence of modules over $H^*(G;\mathbb{F}_4)$.

Proposition 7.1. *Let $E_r^{s,t,w}(Q_8)$ be the v_1-Bockstein spectral sequence computing $H^*(Q_8;(E_{\mathcal{C}})_*V(0))$. Then*

$$\begin{aligned} E_\infty^{*,*,*} = \mathbb{F}_4[g_0, u^{-4}] \otimes \Big(& \mathbb{F}_4[v_1]\{1, h_{1,0}u^{-1}, h_{1,0}u^{-2}, h_{1,0}u^{-3}\} \\ & \oplus \mathbb{F}_4[v_1]/(v_1^2)\{h_{1,0}, h_{1,0}^2, h_{1,0}^2 u^{-1}, h_{1,0}^3 u^{-1}\} \\ & \oplus \mathbb{F}_4\{h_{1,1}, h_{1,1}^2, h_{1,1}^3, h_{1,1}u^{-2}, h_{1,1}^2 u^{-2}, h_{1,1}^3 u^{-2}\}. \Big) \end{aligned}$$

A class is named by the name of the class which detects it, and

$$H^*(Q_8, \mathbb{F}_4) \cong \mathbb{F}_4[g_0, h_{1,0}, h_{1,1}]/(h_{1,0}h_{1,1}, h_{1,0}^3 + h_{1,1}^3) \tag{7.2}$$

where $h_{1,i}$ *has cohomological degree* 1 *and* g_0 *has cohomological degree* 4*. Further, the action of* C_3 *is determined by*

$$\phi_\omega(u) = \omega u$$
$$\phi_\omega(h_{1,i}) = \zeta^{2^i} h_{1,i}.$$

Proof. Let $\widetilde{h}_{1,0}$ be the canonical generator of $\mathrm{Hom}_{Q_8}(\mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta^2}), \mathbb{F}_4)$ where

$$\mathrm{Hom}_{Q_8}(\mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta^2}), \mathbb{F}_4) \subseteq \mathrm{Hom}_{Q_8}(C_1, \mathbb{F}_4)$$

and let $\widetilde{h}_{1,1}$ be the canonical generator of $\mathrm{Hom}_{Q_8}(\mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta}), \mathbb{F}_4)$ where

$$\mathrm{Hom}_{Q_8}(\mathrm{Ind}_{C_3}^{G_{24}}(\chi_{\zeta}), \mathbb{F}_4) \subseteq \mathrm{Hom}_{Q_8}(C_1, \mathbb{F}_4).$$

Let $h_{1,i}$ be the corresponding cohomology classes in $H^1(Q_8, \mathbb{F}_4)$. Then $H^*(Q_8, \mathbb{F}_4)$ is given by (7.2). Note that

$$\phi_\omega(h_{1,i}) = \zeta^{2^i} h_{1,i}.$$

The cohomology $H^*(Q_8, (E_{\mathcal{C}})_*V(0))$ is computed by the following periodic resolution:

$$\mathcal{E}_1 \xrightarrow{\begin{pmatrix}\phi_{x_1}\\ \phi_{x_2}\end{pmatrix}} \mathcal{E}_{\zeta^2} \oplus \mathcal{E}_\zeta \xrightarrow{\begin{pmatrix}\phi_{x_1} & 0\\ 0 & \phi_{x_2}\end{pmatrix}} \mathcal{E}_\zeta \oplus \mathcal{E}_{\zeta^2} \xrightarrow{(\phi_{x_1}, \phi_{x_2})} \mathcal{E}_1 \xrightarrow{0} \mathcal{E}_1.$$

Therefore, the cohomology of $H^*(Q_8, (E_{\mathcal{C}})_*V(0))$ is periodic of period 4 with respect to the cohomological grading s. The periodicity generator is given by the image of g_0 in $H^4(Q_8, (E_{\mathcal{C}})_*V(0))$. It is also periodic of period 8 with respect to its internal grading. Indeed, define the Q_8-invariant

$$\delta = (u \cdot \phi_i(u) \cdot \phi_j(u) \cdot \phi_k(u))^{-1} \equiv u^{-4}(1 + u_1^3) \mod (2). \tag{7.3}$$

Note that $\delta^3 = \Delta$. Since δ is invertible, it induces an isomorphism

$$H^*(Q_8, (E_{\mathcal{C}})_t) \to H^*(Q_8, (E_{\mathcal{C}})_{t+8}).$$

The map induced by δ on the associated graded

$$\bigoplus E_1^{s,t,w} \to \bigoplus E_1^{s,t+8,w}$$

is given by multiplication by u^{-4}. Therefore, all differentials in the v_1-Bockstein spectral sequence are u^{-4}-linear.

This reduces the computation to a few simple verifications. It is sufficient to compute the differentials for u^{-r} when $r = 0, 1, 2, 3$ and the differential on $h_{1,1}u^{-3}$. Let ϕ_{x_i} denote the action of x_i on $(E_2)_*V(0)$. When $r = 0$, all differentials are zero. If $r = 1$, we have

$$\phi_{x_1}(u^{-r}) \equiv \begin{cases} 0 & r = 0 \\ 0 & r = 1 \\ v_1^2 & r = 2 \\ u^{-1}v_1^2 & r = 3. \end{cases} \tag{7.4}$$

and

$$\phi_{x_2}(u^{-r}) \equiv \begin{cases} 0 & r = 0 \\ v_1 & r = 1 \\ 0 & r = 2 \\ u^{-2}v_1 & r = 3. \end{cases} \tag{7.5}$$

This gives the following key differentials

$$d_1(u^{-1}) = v_1 h_{1,1},\ d_1(u^{-3}) = u^{-2}v_1 h_{1,1}$$

and

$$d_2(u^{-2}) = v_1^2 h_{1,0},\ d_2(u^{-3}h_{1,0}) = u^{-1}v_1^2 h_{1,0}^2.$$

All other differentials are determined by v_1 and u^{-4} linearity. □

Figure 7.1 illustrates the E_∞-term. Classes have been named according to the element which detects them. Taking C_3-fixed points, we obtain an associated graded for $H^*(G_{24}, (E_{\mathcal{C}})_* V(0))$, which is depicted in Figure 7.3.

Proposition 7.6. *Let $V = Q_8/C_2$. Let $E_r^{s,t,w}(V)$ be the v_1-Bockstein spectral sequence computing $H^*(V; (E_{\mathcal{C}})_* V(0))$. Then,*

$$\begin{aligned} E_\infty^{s\le 3,*,*} = \mathbb{F}_4[u^{-4}] \otimes \Big(& \mathbb{F}_4[v_1]\{1\} \oplus \mathbb{F}_4[v_1]/(v_1^2)\{h_{1,0}, h_{1,0}^2, h_{1,0}^3\} \\ & \oplus \mathbb{F}_4\{h_{1,1}, h_{1,1}^2, h_{1,0}h_{1,1}, h_{1,0}^2 h_{1,1}, h_{1,0}h_{1,1}^2, h_{1,1}^3, \\ & h_{1,1}u^{-2}, h_{1,1}^2 u^{-2}, h_{1,0}h_{1,1}u^{-2}, h_{1,0}^2 h_{1,1}u^{-2}, h_{1,1}^3 u^{-2}, h_{1,0}h_{1,1}^2 u^{-2}\}.\Big) \end{aligned}$$

A class is named by the name of the class which detects it, where

$$H^*(V, \mathbb{F}_4) \cong \mathbb{F}_4[h_{1,0}, h_{1,1}] \tag{7.7}$$

with $h_{1,i}$ of cohomological degree 1. Further, the action of C_3 is determined by

$$\phi_\omega(u) = \omega u$$
$$\phi_\omega(h_{1,i}) = \zeta^{2^i} h_{1,i}.$$

Proof. Let $\widetilde{h}_{1,0}$ be the canonical generator of $\mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta^2}), \mathbb{F}_4)$ where

$$\mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta^2}), \mathbb{F}_4) \subseteq \mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(D_1, \mathbb{F}_4).$$

Let $\widetilde{h}_{1,1}$ the canonical generator of $\mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta}), \mathbb{F}_4)$ where

$$\mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(\overline{\chi}_{\zeta}), \mathbb{F}_4) \subseteq \mathrm{Hom}_V(\mathrm{Ind}_{C_3}^{A_4}(D_1, \mathbb{F}_4).$$

Let $h_{1,i}$ be the corresponding cohomology classes in $H^1(V, \mathbb{F}_4)$. Note that under the morphism of chain complexes induced by the map $C_* \to D_*$ described in Lemma 6.4, the element $\widetilde{h}_{1,i}$ in $\mathrm{Hom}_V(D_1, \mathbb{F}_4)$ is sent to the element of the same name in $\mathrm{Hom}_{Q_8}(C_1, \mathbb{F}_4)$. This justifies our choice of notation. Then $H^*(V, \mathbb{F}_4)$ is given by (7.7) and $\phi_\omega(h_{1,i}) = \zeta^{2^i} h_{1,i}$. The cohomology $H^*(V, (E_{\mathcal{C}})_* V(0))$ is computed using the following resolution:

$$\mathcal{E}_1 \xrightarrow{\begin{pmatrix} \phi_{x_1} \\ \phi_{x_2} \end{pmatrix}} \mathcal{E}_{\zeta^2} \oplus \mathcal{E}_\zeta \xrightarrow{\begin{pmatrix} \phi_{x_1} & 0 \\ \phi_{x_2} & \phi_{x_1} \\ 0 & \phi_{x_2} \end{pmatrix}} \mathcal{E}_\zeta \oplus \mathcal{E}_1 \oplus \mathcal{E}_{\zeta^2} \longrightarrow \dots$$

The element δ defined by (7.3) is again invariant and invertible. Therefore, the cohomology $H^*(V,(E_{\mathcal{C}})_*V(0))$ is also periodic of period 8 with respect to its internal grading. Multiplication by δ induces an isomorphism

$$\bigoplus E_1^{s,t,w} \to \bigoplus E_1^{s,t+8,w}$$

in the associated v_1-Bockstein spectral sequence given by multiplication by u^{-4}. The formulas given by (7.4) and (7.5) give the two key differentials:

$$d_1(u^{-1}) = v_1 h_{1,1},\ d_2(u^{-2}) = v_1^2 h_{1,0}.$$

All other differentials follow from linearity under multiplication by v_1, u^{-4} and elements of $H^*(V,\mathbb{F}_4)$. □

The E_∞-term is drawn in Figure 7.2, where classes are named according to the class that detects them in the E_1-term of the spectral sequence. Taking C_3-fixed points gives the associated graded for $H^*(A_4,(E_{\mathcal{C}})_*V(0))$, which is depicted in Figure 7.4.

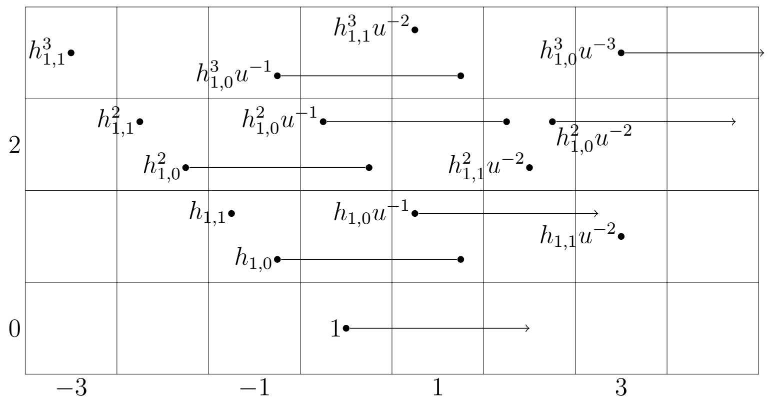

FIGURE 7.1. The E_∞-term for the v_1-Bockstein spectral sequence computing $H^*(Q_8, (E_{\mathcal{C}})_* V(0))$. Horizontal lines denote multiplication by v_1. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The E_∞-term is periodic of period 8 with respect to the internal grading t on a class $\delta = u^{-4}(1 + u_1^3)$, which is detected by u^{-4} in degree $(t - s, s) = (8, 0)$. It is periodic of period 4 with respect to the cohomological grading s on a class detected by g_0 (see (7.2)) in degree $(t - s, s) = (-4, 4)$.

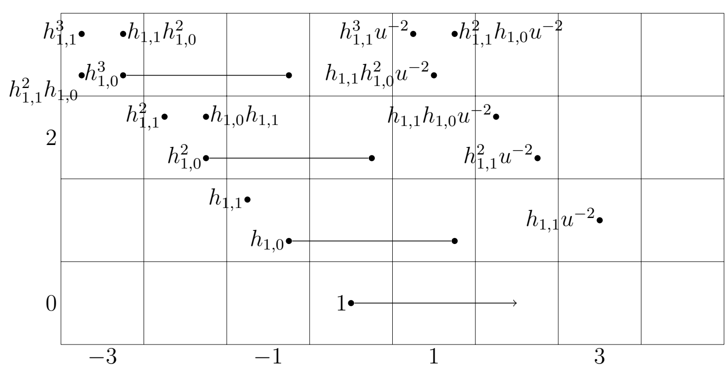

FIGURE 7.2. The E_∞-term for the v_1-Bockstein spectral sequence computing $H^*(V, (E_{\mathcal{C}})_* V(0))$. Horizontal lines denote multiplication by v_1. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The E_∞-term is periodic of period 8 with respect to the internal grading t on a class $\delta = u^{-4}(1 + u_1^3)$, which is detected by u^{-4} in degree $(t - s, s) = (8, 0)$.

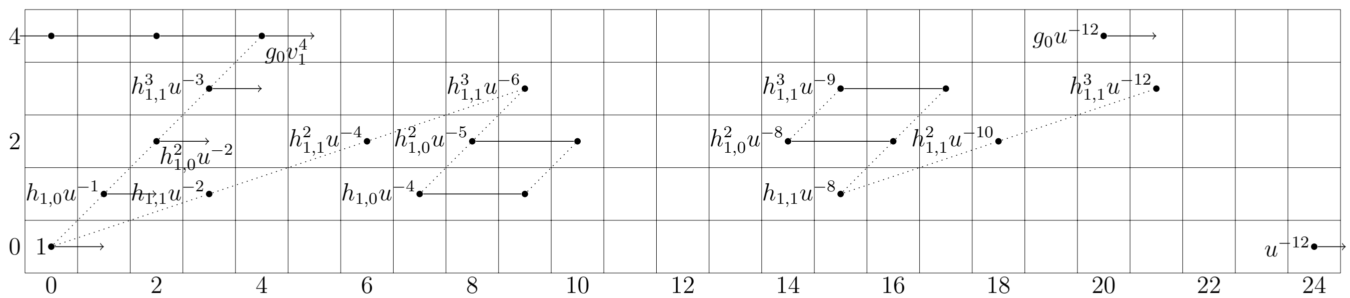

FIGURE 7.3. The associated graded for $H^*(G_{24}, (E_\mathcal{C})_* V(0))$. Horizontal lines denote multiplication by v_1. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The associated graded is periodic of period 24 with respect to the internal grading t on the class $\Delta = u^{-12}(1 + u_1^3)^3$, which is detected by u^{-12} in degree $(t - s, s) = (24, 0)$. It is periodic of period 4 with respect to the cohomological degree s on a class detected by g_0 in degree $(t-s, s) = (-4, 4)$. Dotted lines denote extensions which are known from [1, §7]. The element $h_{1,0}u^{-1}$ detects h_1.

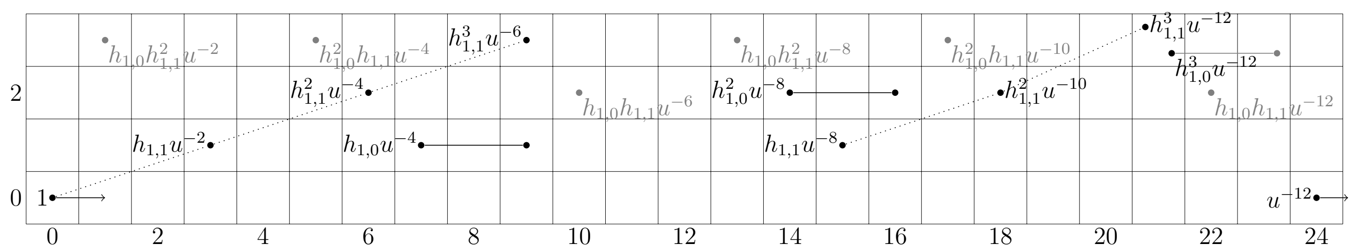

FIGURE 7.4. The associated graded for $H^*(A_4, (E_\mathcal{C})_* V(0))$. Horizontal lines denote multiplication by v_1. Horizontal arrows denote classes which are free over $\mathbb{F}_4[v_1]$. The associated graded is periodic of period 24 with respect to the internal grading t on the class $\Delta = u^{-12}(1 + u_1^3)^3$, which is detected by u^{-12} in degree $(t - s, s) = (24, 0)$. Gray classes map to zero under the map $H^*(A_4, (E_\mathcal{C})_* V(0)) \to H^*(G_{24}, (E_\mathcal{C})_* V(0))$.

8. Appendix III: The action of the Morava stabilizer group

The goal of this appendix is to approximate the action of elements of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$. Some of our results are stronger than needed for the computations of this paper, but the more precise estimates are necessary for future computations. Note that the results of this section prove Theorems 3.6 and 3.7.

8.1. **The formal group laws.** Let $\mathcal{E}$ be an elliptic curve with Weierstrass equation

$$\mathcal{E} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Let $F_{\mathcal{E}}(z_1, z_2)$ be the formal group law of $\mathcal{E}$, where the coordinates (z, w) at the origin are chosen so that

$$w(z) = z^3 + a_1zw(z) + a_2z^2w(z) + a_3w(z)^2 + a_4zw(z)^2 + a_6w(z)^3. \tag{8.1}$$

That the group $\mathbb{S}_{\mathcal{C}}$ acts on $(E_{\mathcal{C}})_*$ is a consequence of the fact that the formal group law $F_{E_{\mathcal{C}}}$ of $E_{\mathcal{C}}$ is a universal deformation of the formal group law $F_{\mathcal{C}}$ of the elliptic curve

$$\mathcal{C} : y^2 + y = x^3$$

defined over any field extension of $\mathbb{F}_2$. Further, $F_{E_{\mathcal{C}}}$ is the formal group law of an elliptic curve, namely

$$\mathcal{C}_U : y^2 + 3u_1xy + (u_1^3 - 1)y = x^3$$

defined over $(E_{\mathcal{C}})_0$. That is,

$$F_{E_{\mathcal{C}}} = F_{\mathcal{C}_U}.$$

The goal of this section is to gather information about $F_{\mathcal{C}_U}$. These results will be used in the next section to compute the action of $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$. We will also compute information about the formal group law of the curve

$$\mathcal{C}_{\mathbb{Z}} : y^2 - y = x^3$$

defined over $\mathbb{Z}$. The curve $\mathcal{C}_{\mathbb{Z}}$ is a lift of $\mathcal{C}$ to $\mathbb{Z}$, and $\mathcal{C}_U$ reduces to $\mathcal{C}_{\mathbb{Z}}$ modulo u_1. Therefore, we will derive information about $F_{\mathcal{C}_U}$ from information about $F_{\mathcal{C}_{\mathbb{Z}}}$.

The following results are proved using the methods described in [18, §4]. We recall the key tools here. We restrict to elliptic curves $\mathcal{E}$ with homogenous Weierstrass equation of the form

$$\mathcal{E} : y^2z + a_1xyz + a_3yz^2 = x^3.$$

Let $z = -\frac{x}{y}$ and $w = -\frac{z}{y}$, so that $(z, w(z))$ is a coordinate chart of $\mathcal{E}$ at the origin, with

$$w(z) = z^3 + a_1zw(z) + a_3w(z)^2.$$

This can be used to write $w(z)$ as a power series in z. Letting

$$\lambda(z_1, z_2) = \frac{w(z_2) - w(z_1)}{z_2 - z_1},$$

the line through the points $(z_1, w(z_1))$ and $(z_2, w(z_2))$ has equation

$$w(z) = \lambda(z_1, z_2)z + w(z_1) - \lambda(z_1, z_2)z_1. \tag{8.2}$$

(Note that there is a sign mistake in [18, §4.1] in the equation of the connecting line. This was pointed out to the author by Hans-Werner Henn.) Substituting (8.2) in (8.1), we obtain a monic cubic polynomial whose roots are z_1, z_2 and

$[-1]_{F_\mathcal{E}}(F(z_1,z_2))$. The coefficient of z^2 is $a_1\lambda(z_1,z_2)+a_3\lambda(z_1,z_2)^2$. This implies that

$$[-1]_{F_\mathcal{E}}(F(z_1,z_2)) = -z_1 - z_2 - a_1\lambda(z_1,z_2) - a_3\lambda(z_1,z_2)^2. \tag{8.3}$$

Noting that

$$\lambda(z,z) = \lim_{s\to z}\frac{w(s)-w(z)}{s-z} = w'(z),$$

it follows that

$$[-2]_{F_\mathcal{E}}(z) = -2z - a_1 w'(z) - a_3 w'(z)^2.$$

Finally, the series $[-1]_{F_\mathcal{E}}(z)$, which is $[-1]_{F_\mathcal{E}}(F(z,0))$, is given by

$$[-1]_{F_\mathcal{E}}(z) = -z - a_1\frac{w(z)}{z} - a_3\frac{w^2(z)}{z^2}. \tag{8.4}$$

so that $F_\mathcal{E}$ can be computed by composing (8.4) with (8.3).

The following two results give formulas for the $[-2]$-series of the curve $\mathcal{C}_U$, both integrally and modulo 2. Corollary 8.7 was conjectured by the author using the algorithms described of [18, §4]. The proofs are due to Hans-Werner Henn.

Proposition 8.5. *The formal group law $F_{\mathcal{C}_U}$ has $[-2]$-series*

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 9z\frac{zu_1 - 2z^2u_1^2 + z^3(u_1^3-1)}{1-6zu_1+9z^2u_1^2-4z^3(u_1^3-1)},$$

so that

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 9u_1z^2 - 36u_1^2z^3 + 9z^4 - 144u_1^3z^4 + O(z^5). \tag{8.6}$$

Proof. For the curve $\mathcal{C}_U$, we have

$$w'(z) = \frac{3(z^2+u_1w(z))}{1-3u_1z-2(u_1^3-1)w(z)}.$$

Using the fact that

$$[-2]_{F_{\mathcal{C}_U}}(z) = -2z - 3u_1w'(z) - (u_1^3-1)w'(z)^2$$

and the fact that

$$(u_1^3-1)w(z)^2 = w(z) - z^3 - 3u_1zw(z),$$

one expands and obtains the formula for $[-2]_{F_{\mathcal{C}_U}}(z)$. Computing the Taylor expansion gives the estimate (8.6). □

Corollary 8.7.

$$[-2]_{F_{\mathcal{C}_U}}(x) \equiv u_1x^2 + \sum_{k\geq 0}u_1^{2k}x^{2k+4} \mod (2).$$

Proof. It follows from Proposition 8.5 that modulo 2,

$$[-2]_{F_{\mathcal{C}_\mathcal{U}}}(z) \equiv \frac{u_1z^2+u_1^3z^4+z^4}{1+u_1^2z^2}.$$

Therefore, modulo 2,

$$\begin{aligned}[-2]_{F_{\mathcal{C}_\mathcal{U}}}(z) &\equiv \left(u_1z^2+u_1^3z^4+z^4\right)\sum_{k\geq 0}u_1^{2k}z^{2k}\\ &\equiv u_1z^2 + \sum_{k\geq 0}u_1^{2k}z^{2k+4}.\end{aligned}$$

□

Some of the key ingredients for the proof of the next result were brought to the author's attention by Inna Zakharevich. Let

$$C_k = \frac{(2k)!}{k!(k+1)!}.$$

be the k'th Catalan number. Let

$$C(y) = \sum_{k \geq 0} C_k y^k \tag{8.8}$$

be their generating series. Let $D(y) = yC(y)$. It is a standard fact that

$$D(y) = \frac{1 - \sqrt{1 - 4y}}{2}. \tag{8.9}$$

Proposition 8.10. *Let $\mathcal{C}_{\mathbb{Z}}$ be the elliptic curve defined over $\mathbb{Z}$ by the Weierstrass equation*

$$\mathcal{C}_{\mathbb{Z}} : y^2 - y = x^3.$$

Then

$$[-2]_{\mathcal{C}_{\mathbb{Z}}}(z) = -2z + 9z^4 \sum_{n \geq 0} (-1)^n 4^n z^{3n}.$$

For $(z, w(z))$ a coordinate chart at the origin with $w(z) = z^3 - w(z)^2$,

$$w(z) = -D((-z)^3) = \sum_{n \geq 0} (-1)^n C_n z^{3(n+1)} = \frac{\sqrt{1 + 4z^3} - 1}{2}.$$

Further,

$$[-1]_{\mathcal{C}_{\mathbb{Z}}} F_{\mathcal{C}_{\mathbb{Z}}}(z_1, z_2) = -z_1 - z_2 + \frac{(z_1^3 + z_2^3) + D(-(z_1^3 + z_2^3 + 4z_1^3 z_2^3))}{(z_2 - z_1)^2}.$$

Proof. It follows from Proposition 8.5 that, modulo u_1,

$$[-2]_{\mathcal{C}_{\mathbb{Z}}}(z) = -2z + 9z^4 \frac{1}{1 + 4z^3}.$$

This proves the first claim. The second claim is equivalent to showing that

$$w(z) = z^3 C((-z)^3)$$

It is a standard result that

$$C(z) = 1 + zC(z)^2.$$

Therefore,

$$C((-z)^3) = 1 + (-z)^3 C((-z)^3)^2,$$

so that

$$z^3 C((-z)^3) = z^3 - (z^3 C((-z)^3))^2.$$

Since $w(z)$ and $z^3 C((-z)^3)$ satisfy the same functional equation, they must be equal. Further, this implies that

$$w(z) = \frac{\sqrt{1 + 4z^3} - 1}{2}.$$

Finally, note that

$$\begin{aligned}\lambda(z_1,z_2) &= \frac{1}{z_2-z_1}\left(\frac{\sqrt{1+4z_2^3}-1}{2}-\frac{\sqrt{1+4z_1^3}-1}{2}\right)\\ &= \frac{\sqrt{1+4z_2^3}-\sqrt{1+4z_1^3}}{2(z_2-z_1)}.\end{aligned}$$

Therefore,

$$\begin{aligned}[-1]_{F_{\mathcal{C}_{\mathbb{Z}}}}(F_{\mathcal{C}_{\mathbb{Z}}}(z_1,z_2)) &= -z_1-z_2+\lambda(z_1,z_2)^2\\ &= -z_1-z_2+\left(\frac{\sqrt{1+4z_2^3}-\sqrt{1+4z_1^3}}{2(z_2-z_1)}\right)^2\\ &= -z_1-z_2+\frac{2(z_1^3+z_2^3)+1-\sqrt{1+4(z_1^3+z_2^3+4z_1^3z_2^3)}}{2(z_2-z_1)^2}\\ &= -z_1-z_2+\frac{(z_1^3+z_2^3)+D(-(z_1^3+z_2^3+4z_1^3z_2^3))}{(z_2-z_1)^2}.\end{aligned}$$

□

Proposition 8.11. *Let $\mathcal{C}$ be the elliptic curve defined over $\mathbb{F}_4$ by the Weierstrass equation*

$$\mathcal{C}: y^2+y=x^3.$$

The local uniformizer at the origin $w(z)=z^3+w(z)^2$, satisfies

$$w(z)=\sum_{k\geq 0} z^{3\cdot 2^k}.$$

Further,

$$[-2]_{F_{\mathcal{C}}}(z)=z^4,$$

and

$$[-1]_{F_{\mathcal{C}}}(F_{\mathcal{C}}(z_1,z_2)) = z_1+z_2+\sum_{k\geq 1}\sum_{n=0}^{3\cdot 2^{k-1}-1}(z_1^{2(3\cdot 2^{k-1}-1-n)}z_2^{2n}). \tag{8.12}$$

Further,

$$[-1]_{F_{\mathcal{C}}}(z)=\sum_{k\geq 0} z^{3\cdot 2^k-2},$$

so that

$$\begin{aligned}F_{\mathcal{C}}(z_1,z_2) &= z_1+z_2+z_1^2z_2^2+z_1^6z_2^4+z_1^4z_2^6+z_1^8z_2^8+z_1^{12}z_2^4+z_1^4z_2^{12}+z_1^{12}z_2^{10}+z_1^{10}z_2^{12}\\ &\quad +z_1^{14}z_2^8+z_1^8z_2^{14}+z_1^{16}z_2^{12}+z_1^{12}z_2^{16}+z_1^{24}z_2^4+z_1^4z_2^{24}+\dots\end{aligned}$$

where the next term has order 34.

Proof. One can compute directly that $w(z)=\sum_{k\geq 0} z^{3\cdot 2^k}$. This implies that $C_n\neq 0$ modulo 2 if and only if $n+1=2^k$. Therefore, we have the following identity of power series

$$D(y)=\sum_{n\geq 0} C_n y^{n+1}=\sum_{k\geq 0} y^{2^k}.$$

Hence,

$$
\begin{aligned}
[-1]_{F_{\mathcal{C}}}\left(F_{\mathcal{C}}(z_1, z_2)\right) &= \frac{z_1 z_2 (z_1 + z_2) + D(z_1^3 + z_2^3)}{(z_1 + z_2)^2} \\
&= \frac{1}{(z_1 + z_2)^2}\left(z_1 z_2 (z_1 + z_2) + \sum_{k \geq 0} (z_1^{3 \cdot 2^k} + z_2^{3 \cdot 2^k}) \right) \\
&= \frac{1}{(z_1 + z_2)}\left(z_1 z_2 + \sum_{k \geq 0} \sum_{r=0}^{3 \cdot 2^k - 1} (z_1^{3 \cdot 2^k - 1 - r} z_2^r) \right).
\end{aligned}
$$

The key observation is that if n is odd, then

$$
\begin{aligned}
\sum_{r=0}^{n} (z_1^{n-r} z_2^r) &= \sum_{r=0}^{n-1} (z_1 z_2)^r (z_1^{n-2r} + z_2^{n-2r}) \\
&= (z_1 + z_2) \sum_{r=0}^{n} \sum_{s=0}^{n-1-2r} (z_1^{n-1-r-s} z_2^{r+s})
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
[-1]_{F_{\mathcal{C}}}\left(F_{\mathcal{C}}(z_1, z_2)\right) &= \frac{1}{(z_1 + z_2)}\left(z_1^2 + 2 z_1 z_2 + z_2^2 + \sum_{k \geq 1} \sum_{r=0}^{3 \cdot 2^k - 1} (z_1^{3 \cdot 2^k - 1 - r} z_2^r) \right) \\
&= z_1 + z_2 + \sum_{k \geq 1} \sum_{r=0}^{3 \cdot 2^{k-1} - 1} \sum_{s=0}^{2(3 \cdot 2^{k-1} - 1 - r)} (z_1^{3 \cdot 2^k - 2 - (r+s)} z_2^{r+s}).
\end{aligned}
$$

To simplify this expression, we must count the ways in which an integer n such that $0 \leq n \leq 2(3 \cdot 2^{k-1} - 1)$ can be written as a linear combination $r + s$ where

$$0 \leq r \leq 3 \cdot 2^{k-1} - 1$$

and

$$0 \leq s \leq 2(3 \cdot 2^{k-1} - 1 - r)$$

If $n \leq 3 \cdot 2^{k-1} - 1$, there are $n + 1$ combinations. If $3 \cdot 2^{k-1} - 1 < n \leq 2(3 \cdot 2^{k-1} - 1)$, there are $2(3 \cdot 2^{k-1} - 1) - n + 1$ combinations. Therefore, the coefficient of $z_1^{3 \cdot 2^k - 2 - n} z_2^n$ is zero if and only if n is even. This implies that

$$[-1]_{F_{\mathcal{C}}}\left(F_{\mathcal{C}}(z_1, z_2)\right) = z_1 + z_2 + \sum_{k \geq 1} \sum_{n=0}^{3 \cdot 2^{k-1} - 1} (z_1^{2(3 \cdot 2^{k-1} - 1 - n)} z_2^{2n}).$$

To prove the given estimate of $F_{\mathcal{C}}(z_1, z_2)$, note that since $w(z) = z^3 + w(z)^2$, we have that

$$
\begin{aligned}
[-1]_{F_{\mathcal{C}}}(z) &= z + \frac{w(z)^2}{z^2} \\
&= \frac{z^3 + w(z)^2}{z^2} \\
&= \frac{w(z)}{z^2} \\
&= \sum_{k \geq 0} z^{3 \cdot 2^k - 2}.
\end{aligned}
$$

A direct computation using (8.12) proves the claim. □

8.2. **The technique for computing the action of $\mathbb{S}_{\mathcal{C}}$.** The method presented here is an adaptation of the techniques used in [11], which we describe here. Let γ be in $\mathbb{S}_{\mathcal{C}}$. Then $\gamma \in \mathbb{F}_4[[x]]$ is a power series which satisfies

$$\gamma F_{\mathcal{C}}(x, y) = F_{\mathcal{C}}(\gamma(x), \gamma(y)).$$

Recall from Section 2.4 that γ gives rise to an isomorphism $\phi_\gamma : (E_{\mathcal{C}})_* \to (E_{\mathcal{C}})_*$ and a lift of γ,

$$f_\gamma : \phi_\gamma^* F_{E_{\mathcal{C}}} \to F_{E_{\mathcal{C}}},$$

where

$$f_\gamma \in (E_{\mathcal{C}})_0[[x]].$$

The action of γ on $(E_{\mathcal{C}})_*$ is given precisely by ϕ_γ.

The isomorphism ϕ_γ is linear over $\mathbb{W}$; hence it is sufficient to specify $\phi_\gamma(u)$ and $\phi_\gamma(u_1)$. The morphism f_γ is given by a power series

$$f_\gamma(x) = t_0(\gamma)x + t_1(\gamma)x^2 + t_2(\gamma)x^3 + \dots$$

where

$$t_i \in (E_{\mathcal{C}})_0 = \mathbb{W}[[u_1]].$$

By (2.4)

$$\phi_\gamma(u) = f'_\gamma(0)u = t_0(\gamma)u, \tag{8.13}$$

which gives the action of γ on u.

The morphism f_γ must satisfy

$$f_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x)) = [-2]_{F_{E_{\mathcal{C}}}}(f_\gamma(x)). \tag{8.14}$$

This imposes a set of relations on the parameters $t_i(\gamma)$ and $\phi_\gamma(u_1)$. Further, f_γ is a lift of γ, so that

$$f_\gamma \equiv \gamma \mod (2, u_1). \tag{8.15}$$

This specifies the parameters $t_i(\gamma)$ modulo $(2, u_1)$. With this information, the relations imposed by (8.14) are sufficient to approximate ϕ_γ. Before executing this program, we prove a preliminary result.

Proposition 8.16. *If $\gamma \in \mathbb{Z}_2^\times \cap \mathbb{S}_{\mathcal{C}}$, so that $\gamma = \sum_{i \geq 0} a_i T^{2i}$, for $a_i \in \{0, 1\}$. Let*

$$\ell = \sum_{i \geq 0} a_i 2^i$$

in $\mathbb{Z}_2^\times \subseteq (E_{\mathcal{C}})_0$. Then $\phi_\gamma(u_1) = u_1$ and $\phi_\gamma(u) = \ell u$.

Proof. The element γ is given by

$$\gamma(x) = a_0 x +_{F_{\mathcal{C}}} a_1[-2]_{F_{\mathcal{C}}}(x) +_{F_{\mathcal{C}}} a_2[4]_{F_{\mathcal{C}}}(x) +_{F_{\mathcal{C}}} \dots$$

Let g be the lift for γ given by

$$g(x) = a_0 x +_{F_{E_{\mathcal{C}}}} a_1[-2]_{F_{E_{\mathcal{C}}}}(x) +_{F_{E_{\mathcal{C}}}} a_2[4]_{F_{E_{\mathcal{C}}}}(x) +_{F_{E_{\mathcal{C}}}} \dots$$

Then g is an automorphism of $F_{\mathcal{C}_U}$, hence $\phi_g : (E_{\mathcal{C}})_0 \to (E_{\mathcal{C}})_0$ is the identity. Therefore, the automorphism f_g described in Section 2.2 is the identity, and hence

$$f_\gamma(x) = g(x).$$

Since

$$g(x) = \ell x + \dots,$$

we have $g'(0)=\ell$. Hence, $\phi_\gamma(u)=\ell u$. □

Theorem 8.17. *Let $\gamma\in\mathbb{S}_{\mathcal{C}}$ and $t_i=t_i(\gamma)$. Then*

$$\phi_\gamma(u_1)=u_1t_0+\frac{2}{3}\frac{t_1}{t_0}. \tag{8.18}$$

Proof. Recall from (8.14) that

$$f_\gamma([-2]_{\phi_\gamma^*F_{E_{\mathcal{C}}}}(x))=[-2]_{F_{E_{\mathcal{C}}}}(f_\gamma(x)).$$

Using (8.6), one obtains the following relation on the coefficients of x^2,

$$-9\phi_\gamma(u_1)t_0+4t_1=-9u_1t_0^2-2t_1.$$

Because ϕ_γ is an isomorphism, t_0 is invertible. Isolating $\phi_\gamma(u_1)$ and dividing both sides by $-9t_0$ proves the claim. □

Therefore, to approximate the action of an element γ in $\mathbb{S}_{\mathcal{C}}$ on $(E_{\mathcal{C}})_*$, it suffices to approximate the parameters $t_0(\gamma)$ and $t_1(\gamma)$.

8.3. **Approximations for the parameters** $t_i(\gamma)$**.** In this section, we use the technique described in Section 8.2 to approximate the parameters $t_i(\gamma)$. Our goal is to give an approximation of the action of γ modulo $(2,u_1^3)^3$. By Theorem 8.17, in order to do this, we must approximate $t_0(\gamma)$ modulo $(2,u_1^3)^3$ and $t_1(\gamma)$ modulo $(2,u_1^3)^2$ for γ in $S_{\mathcal{C}}$, the 2-Sylow subgroup of $\mathbb{S}_{\mathcal{C}}$. The goal of this section is to obtain these estimates.

Corollary 8.19. *Modulo $(2,u_1^6)$,*

$$\begin{aligned}t_s &\equiv t_s^4+u_1t_{2s+1}^2+\binom{s+2}{2}t_0^2t_{s+1}u_1^2+\sum_{i=0}^{s-1}u_1^2t_i^4t_{2s-1-2i}^2\\ &+\left(\binom{s}{1}t_0^4t_{s-1}+\binom{s}{2}t_0^4t_{s-1}+\binom{s+3}{4}t_0^4t_{s+2}+\binom{s+2}{1}t_{\frac{s-1}{2}}^8\right)u_1^4.\end{aligned}$$

Proof. Let $f_\gamma(x)=\sum_{i=0}^\infty t_ix^{i+1}$. Using Corollary 8.7, we obtain

$$\begin{aligned}f_\gamma([-2]_{\phi_\gamma^*F_{E_{\mathcal{C}}}}(x)) &= \sum_{i=0}^\infty t_i\left(t_0u_1x^2+x^4+\sum_{i=1}^\infty (t_0u_1)^{2i}x^{4+2i}\right)^{i+1}\\ &\equiv \sum_{i=0}^\infty t_i\left(t_0u_1x^2+x^4+t_0^2u_1^2x^6+t_0^4u_1^4x^8\right)^{i+1}\\ &\equiv \sum_{i=0}^\infty t_i\left(x^{4(i+1)}+\binom{i+1}{1}(t_0u_1x^{4i+2}+t_0^2u_1^2x^{4i+6}+t_0^4u_1^4x^{4i+8})\right.\\ &+\binom{i+1}{2}(t_0^2u_1^2x^{4i}+t_0^4u_1^4x^{4i+8})\\ &+\binom{i+1}{3}(t_0^3u_1^3x^{4i-2}+t_0^4u_1^4x^{4i+2}+t_0^5u_1^5x^{4i+6})\\ &\left.+\binom{i+1}{4}t_0^4u_1^4x^{4i-4}+\binom{i+1}{5}t_0^5u_1^5x^{4i-6}\right).\end{aligned}$$

Further,

$$[-2]_{F_{E_{\mathcal{C}}}}(f_\gamma(x)) = u_1\left(\sum_{i=0}^{\infty} t_i x^{i+1}\right)^2 + \left(\sum_{i=0}^{\infty} t_i x^{i+1}\right)^4 + \sum_{k=1}^{\infty} u_1^{2k}\left(\sum_{i=0}^{\infty} t_i x^{i+1}\right)^{2k+4}$$
$$\equiv \sum_{i=0}^{\infty}\left(u_1 t_i^2 x^{2(i+1)} + t_i^4 x^{4(i+1)} + u_1^4 t_i^8 x^{8(i+1)}\right)$$
$$+ u_1^2\left(\sum_{i=0}^{\infty} t_i^2 x^{2(i+1)}\right)^3$$

Next, note that

$$\left(\sum_{i\geq 0} a_i x^i\right)^3 \equiv \sum_{k\geq 0}\sum_{2i+j=k} a_i^2 a_j x^k.$$

Therefore,

$$u_1^2\left(\sum_{i=0}^{\infty} t_i^2 x^{2(i+1)}\right)^3 \equiv \sum_{k\geq 0}\sum_{2i+j=k} u_1^2 t_i^4 t_j^2 x^{2k+6}$$

Now, using (8.14), the coefficient of $x^{4(s+1)}$ gives the relation

$$t_s \equiv t_s^4 + u_1 t_{2s+1}^2 + \binom{s+2}{2} t_0^2 t_{s+1} u_1^2 + \sum_{2i+j=2s-1} u_1^2 t_i^4 t_j^2$$
$$+ \left(\binom{s}{1} + \binom{s}{2}\right) t_0^4 t_{s-1} u_1^4 + \binom{s+3}{4} t_0^4 t_{s+2} u_1^4$$
$$+ \binom{s+2}{1} t_{\frac{s-1}{2}}^8 u_1^4$$

(Note that the the coefficient of the last term is chosen to be zero when s is even, so that when $t_{\frac{s-1}{2}}$ has a non-zero coefficient, $(s-1)/2$ is an integer.) □

Proposition 8.20. *For $t_i = t_i(\gamma)$ where $\gamma \in \mathbb{S}_{\mathcal{C}}$, then*

$$t_i \equiv t_i^4 + u_1 t_{2i+1}^2 + 2t_{4i+3} + 2\sum_{\substack{r+s=2i\\0\leq r<s}} t_r^2 t_s^2 \mod (2, u_1)^2$$

Proof. Modulo $(4, u_1)$, we have

$$[-2]_{F_{\mathcal{C}_U}}(x) \equiv 2x + x^4.$$

This gives

$$f_\gamma([-2]_{\phi_\gamma^* F_{E_{\mathcal{C}}}}(x)) \equiv \sum_{i=0}^{\infty} t_i\left(x^{4(i+1)} + 2\binom{i+1}{1} x^{4i+1}\right)$$

and

$$[-2]_{F_{E_{\mathcal{C}}}}(f_\gamma(x)) \equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \left(\sum_{i=0} t_i x^{i+1}\right)^4$$
$$\equiv \sum_{i=0}^{\infty} 2t_i x^{i+1} + \sum_{i=0} t_i^4 x^{4(i+1)} + \sum_{i=1}^{\infty} x^{4+2i} 2\sum_{r+s=i} t_r^2 t_s^2.$$

Using (8.14), the coefficient of $x^{4(i+1)}$ gives the relation

$$t_i = 2t_{4i+3} + t_i^4 + 2 \sum_{\substack{r+s=2i \\ 0 \le r < s}} t_r^2 t_s^2 \mod (4, u_1).$$

The claim then follows from Corollary 8.19. □

Proposition 8.21. *Modulo* (4)

$$t_0 \equiv t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1 + 3t_0^2 t_1 u_1^2. \tag{8.22}$$

Modulo (2),

$$t_1 \equiv t_1^4 + t_3^2 u_1 + t_0^4 t_1^2 u_1^2 + t_0^2 t_2 u_1^2 + t_0^5 u_1^4 + t_0^8 u_1^4 + t_0^4 t_3 u_1^4.$$

Proof. Modulo (8), the coefficient of x^4 in $f_\gamma([-2]_{\phi_\gamma^* F_{E_\mathcal{C}}}(x))$ is given by

$$t_0 + \phi_\gamma(u_1)^2 t_1$$

and the coefficient of $[-2]_{F_{E_\mathcal{C}}}(f_\gamma(x))$ is given by

$$t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1$$

Recall from Theorem 8.17 that $\phi_\gamma(u_1) = u_1 t_0 + \frac{2}{3}\frac{t_1}{t_0}$. This and (8.14) imply that

$$t_0 + t_0^2 t_1 u_1^2 \equiv t_0^4 + 2t_3 + 3t_1^2 u_1 + 2t_0 t_2 u_1.$$

Isolating t_0 proves the first claim.

Similarly, the coefficients of x^8 give the desired relation for t_1. □

Recall that $\gamma \in \mathbb{S}_\mathcal{C}$ has an expansion of the form

$$\gamma = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + \dots$$

Here the a_i are solutions to the equation $x^4 - x = 0$. Recall from Section 3 that if $\omega^s \in \mathrm{End}(F_\mathcal{C})$ is a solution to the equation $x^4 - x = 0$, then it corresponds the automorphism

$$\omega^s(x) = \zeta^s x,$$

where $\zeta \in \mathbb{F}_4 = (E_\mathcal{C})_*/(2, v_1)$. Recall that there is a canonical copy of $\mathbb{F}_4$ in $\mathrm{End}(F_\mathcal{C})$ given by the ring generated by the automorphism $\omega(x)$. Further, $(E_\mathcal{C})_*/(2, v_1)$ is isomorphic to $\mathbb{F}_4$, with canonical generator the image of ζ. Define a map

$$f : \mathbb{F}_4 \subseteq \mathrm{End}(F_\mathcal{C}) \to (E_\mathcal{C})_*/(2, v_1) \cong \mathbb{F}_4$$

by

$$f(\omega^s(x)) = \zeta^s.$$

If γ is as above, using the fact that $T(x) = x^2$,

$$\gamma(x) = f(a_0)x +_{F_\mathcal{C}} f(a_1)x^2 +_{F_\mathcal{C}} f(a_2)x^4 +_{F_\mathcal{C}} f(a_3)x^8 + \dots$$

For simplicity, we will identify a_i with $f(a_i)$ and write

$$\gamma(x) = a_0 x +_{F_\mathcal{C}} a_1 x^2 +_{F_\mathcal{C}} a_2 x^4 +_{F_\mathcal{C}} a_3 x^8 + \dots \tag{8.23}$$

Proposition 8.24. *For* $\gamma \in S_\mathcal{C}$,

$$\gamma(x) = x + a_1 x^2 + a_2 x^4 + a_1^2 x^6 + a_3 x^8 + a_2^2 x^{10} + a_1^2 a_2^2 x^{12} + a_1 x^{14} + (a_1^3 + a_4)x^{16}$$

modulo (x^{18}).

Proof. This is a direct computation using (8.23) and the formal group law of Proposition 8.11, noting that for $\gamma \in S_\mathcal{C}$, $a_0 = 1$. □

For the remainder of this section, we assume that $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$.

Corollary 8.25. *Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1)$,*

$$t_0 \equiv 1 \qquad t_3 \equiv a_2, \qquad t_7 \equiv a_3, \qquad t_9 \equiv a_2^2, \qquad t_{15} \equiv a_4,$$

and t_1, t_5, t_{11}, t_{13} and t_{2i} for $0 < i < 8$ are zero modulo $(2, u_1)$.

Proof. Since $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$, $a_1 = 0$. The claim follows from Proposition 8.24, noting that t_i is congruent to the coefficient of x^{i+1} modulo $(2, u_1)$. □

Proposition 8.26. *Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1^2)$*

$$t_0 \equiv 1 \qquad t_2 \equiv 0 \qquad t_4 \equiv a_2u_1 \qquad t_6 \equiv 0$$
$$t_1 \equiv a_2^2u_1 \qquad t_3 \equiv a_2 + a_3^2u_1 \qquad t_5 \equiv 0 \qquad t_7 \equiv a_3 + a_4^2u_1.$$

Proof. This follows from Proposition 8.20 and Corollary 8.25. □

Proposition 8.27. *Let $t_i = t_i(\gamma)$ where $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1^4)$*

$$t_0 \equiv 1 + (a_2 + a_2^2)u_1^3,$$
$$t_3 \equiv a_2 + a_3^2u_1 + a_4u_1^3.$$

Modulo $(2, u_1^3)$

$$t_1 \equiv a_2^2u_1,$$
$$t_5 \equiv (a_2 + a_2^3)u_1^2,$$

Modulo $(2, u_1^6)$

$$t_2 \equiv a_2^2u_1^2 + a_3u_1^4 + (a_2 + a_2^3)u_1^5.$$

Proof. It follows from Corollary 8.19 that, modulo $(2, u_1^4)$,

$$t_3 \equiv t_3^4 + t_7^2u_1 + t_1^2t_2^4u_1^2 + t_1^4t_3^2u_1^2 + t_0^4t_5^2u_1^2$$
$$t_5 \equiv t_5^4 + t_{11}^2u_1 + t_3^6u_1^2 + t_1^2t_4^4u_1^2 + t_2^4t_5^2u_1^2 + t_0^2t_6u_1^2 + t_1^4t_7^2u_1^2 + t_0^4t_9^2u_1^2.$$

The results for t_3 and t_5 then follow from Corollary 8.25 and Proposition 8.26. It also follows from Corollary 8.19 that, modulo $(2, u_1^6)$,

$$t_2 \equiv t_2^4 + t_5^2u_1 + t_1^6u_1^2 + t_0^4t_3^2u_1^2 + t_0^4t_1u_1^4 + t_0^4t_4u_1^4.$$

The identity for t_2 then follows from the Corollary 8.25 and Proposition 8.26, using the identity for t_5 modulo $(2, u_1^3)$. □

Proposition 8.28. *Let $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(2, u_1^8)$,*

$$t_1(\gamma) \equiv a_2^2u_1 + a_3u_1^3 + a_3^2u_1^5 + a_3u_1^6 + (a_2^2 + a_2^3 + a_4 + a_4^2)u_1^7.$$

Modulo $(2, u_1^{10})$,

$$t_0(\gamma) \equiv 1 + (a_2 + a_2^2)u_1^3 + a_3u_1^5 + a_3u_1^8 + (a_2 + a_2^2 + a_4 + a_4^2)u_1^9.$$

Proof. The estimate for t_1 follows from Propositions 8.21 and 8.27. The estimate for t_0 follow from Proposition 8.21 using the result for t_1. □

Proposition 8.29. *Let $\gamma \in F_{2/2}\mathbb{S}_{\mathcal{C}}$. Modulo $(4, 2u_1^2, u_1^{10})$,*

$$t_0(\gamma) \equiv 1 + 2a_2 + 2a_3^2u_1 + (a_2 + a_2^2)u_1^3 + a_3u_1^5 + a_3u_1^8 + (a_2 + a_2^2 + a_4 + a_4^2)u_1^9.$$

Proof. This follows from Propositions 8.21, 8.26 and 8.28. □

Proof of Theorem 3.6. The claim follows from Theorem 8.17, noting that

$$\phi(u_1) \equiv t_0 u_1 \mod (2)$$

□

Proof of Theorem 3.7. The claim for $t_0(\gamma)$ is Proposition 8.29. The claim for $t_1(\gamma)$ follows from Proposition 8.28 by reducing the identity for t_1 modulo $(2, u_1^3)$. If γ is in $F_{4/2}\mathbb{S}_{\mathcal{C}}$, that the action of γ is trivial modulo $(2, v_1^9)$ follows from the fact that $t_0(\gamma) \equiv 1$ modulo $(2, u_1^9)$. □

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