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Secondary derived functors and the Adams spectral sequence

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Abstract

Classical homological algebra takes place in additive categories. In homotopy theory such additive categories arise as homotopy categories of "additive groupoid enriched categories", in which a secondary analog of homological algebra can be performed. We introduce secondary chain complexes and secondary resolutions leading to the concept of secondary derived functors. As a main result we show that the E_3-term of the Adams spectral sequence can be expressed as a secondary derived functor. This result can be used to compute the E_3-term explicitly by an algorithm.

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The paper introduces secondary derived functors obtained by secondary resolutions. This generalizes the concept of the classical derived functor Ext^n. It is well known that the groups Ext^n describe the E_2-term of the Adams spectral sequence. As a main application we show that the secondary Ext groups, in fact, determine the E_3-term of the Adams spectral sequence. Using the theory in [6] this yields an algorithm for the computation of the E_3-term, as described in a sequel to this paper [12]. The algorithm is achieved by taking into account the *track structure*: one considers not just homotopy classes of maps between spectra, but instead maps and homotopy classes of homotopies between maps, termed tracks. These form a *track category*, that is, a category enriched in groupoids. It then turns out that in appropriate track categories

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secondary Ext groups can be defined which are unchanged if one replaces the ambient track category with a weakly equivalent one. In fact in [6] a manageable purely algebraically described track category weakly equivalent to the track category of Eilenberg–Mac Lane spectra has been completely determined by the computation of the Hopf algebra of secondary cohomology operations. It is this algebraic model that will be used on the basis of the main result 7.3 below to compute explicitly the E_3-term of the Adams spectral sequence as a secondary Ext-group.

Since the work of Adams [1] it has been generally believed that secondary cohomology operations can be used to compute the d_2-differential and hence the E_3-term of the Adams spectral sequence. Adams gave particular examples of such computations. A global algorithm for the complete determination of d_2 and the E_3-term, however, was not achieved. In [12] it is shown precisely how such an algorithm can be realised using the algebraic model from [6] and techniques of the present paper. The corresponding algorithm in a first approach has been implemented on a computer, elaborating on the `MAPLE` package "Steenrod" by Kenneth Monks. Calculations performed so far have reached total degree 40 and confirm all the previous calculations of the E_3-term in this realm.

We are presently refining the implementation by combining our methods with a dual approach using an analog of the algebraic model from [6] for the Milnor dual of the Steenrod algebra [13]. Our goal is to compute the E_3-term as far as the E_2-term is presently known, i.e. up to degree 210 as given in the work of Nassau [22].

1. Derived functors

We first recall the notion of a resolution in an additive category from which we deduce (primary) derived functors. What follows is a version of relative homological algebra as originated in [18] and then further developed in e.g. [29] and many subsequent works. Later we introduce the secondary version of these notions in the context of an "additive track category", see Section 3.

Our initial data consist of an additive category **A** and a full additive subcategory **a** of **A**. The basic situation to have in mind is the category R-**Mod** of modules over a ring R and its subcategory R-**mod** of free (or projective) R-modules. As another motivating example, coming from topology, one considers for **A** the opposite of the stable homotopy category and for **a** its full subcategory on objects represented by finite products of Eilenberg–Mac Lane spectra over a fixed prime field $\mathbb{F}_p$; then **a** is equivalent to the category of finitely generated free modules over the mod p Steenrod algebra.

1.1. Definition. A *chain complex* (A, d) in **A** is a sequence of objects and morphisms

$$\cdots \to A_{n+1} \xrightarrow{d_n} A_n \xrightarrow{d_{n-1}} A_{n-1} \to \cdots$$

from **A**, with $d_{n-1}d_n = 0$ $(n \in \mathbb{Z})$.

A *chain map* $f : (A, d) \to (A', d')$ is a sequence of morphisms $f_n : A_n \to A'_n$ with $f_n d_n = d'_n f_{n+1}$, $n \in \mathbb{Z}$. For two maps $f, f' : (A, d) \to (A', d')$, a *chain homotopy* h from f to f' is a sequence of morphisms $h_n : A_{n-1} \to A'_n$ satisfying $f'_n = f_n + d'_n h_{n+1} + h_n d_{n-1}$, $n \in \mathbb{Z}$.

A chain complex (A, d) is called **a**-*exact* if for any object X from the subcategory **a** the (ordinary) chain complex $\mathrm{Hom}_{\mathbf{A}}(X, A_\bullet)$ of abelian groups

$$\cdots \to \mathrm{Hom}_{\mathbf{A}}(X, A_{n+1}) \xrightarrow{\mathrm{Hom}_{\mathbf{A}}(X,d_n)} \mathrm{Hom}_{\mathbf{A}}(X, A_n) \xrightarrow{\mathrm{Hom}_{\mathbf{A}}(X,d_{n-1})} \mathrm{Hom}_{\mathbf{A}}(X, A_{n-1}) \to \cdots$$

is acyclic, i.e., is an exact sequence. Explicitly, this means that for any $n \in \mathbb{Z}$, any object X from **a** and any morphism $a_n : X \to A_n$ with $d_{n-1}a_n = 0$ there exists a morphism $a_{n+1} : X \to A_{n+1}$ with $a_n = d_n a_{n+1}$.

A chain map $f : A \to A'$ is an **a**-*equivalence* if for every X in **a** the chain map $\mathrm{Hom}_{\mathbf{A}}(X, f)$ is a quasiisomorphism. Thus a chain complex (A, d) is **a**-exact if and only if the map $(0, 0) \to (A, d)$ is an **a**-equivalence.

1.2. Definition. For an object A of **A**, an A-*augmented chain complex* $A^\varepsilon_\bullet$ is a chain complex of the form

$$\cdots \to A_1 \to A_0 \to A \to 0 \to 0 \to \cdots,$$

i.e., with $A_{-1} = A$ and $A_{-n} = 0$ for $n > 1$. We will consider such an augmented chain complex as a map between chain complexes, $\varepsilon : A_\bullet \to A$, where $A_\bullet$ is the complex $\cdots \to A_1 \to A_0 \to 0 \to 0 \to \cdots$ whereas A is considered as a complex concentrated in degree 0, with $\varepsilon = d_{-1} : A_0 \to A$ called the augmentation.

An **a**-*resolution* of A is an **a**-exact A-augmented chain complex such that all A_n for $n \geqslant 0$ belong to **a**. Thus an **a**-resolution $A^\varepsilon_\bullet$ of an object A is the same as a chain complex $A_\bullet$ in **a** together with an **a**-equivalence $\varepsilon : A_\bullet \to A$.

There are obvious dual notions of an A-coaugmented complex and **a**-coresolution of A. Namely, this means a complex (resp. **a**-exact complex) with $A_1 = A$ and $A_n = 0$ for $n > 1$.

1.3. Lemma. *Suppose*

- *the coproduct of any family of objects of* **a** *exists in* **A** *and belongs to* **a** *again*;
- *there is a small subcategory* **g** *of* **a** *such that every object of* **a** *is a retract of a coproduct of a family of objects from* **g**.

Then every object of **A** *has an* **a**-*resolution*.

Proof. We begin by taking

$$A_0 = \coprod_{\substack{G \in \mathbf{g} \\ a: G \to A}} G,$$

with the obvious map $\varepsilon : A_0 \to A$ having a for the a-th component. Next, we take

$$A_1 = \coprod_{\substack{G \in \mathbf{g} \\ t_0: G \to A_0 \\ \varepsilon t_0 = 0}} G,$$

with a similar map $d_0 : A_1 \to A_0$ whose t_0th component is t_0 (so obviously $\varepsilon d_0 = 0$). One continues in this way, with

$$A_{n+1} = \coprod_{\substack{G \in \mathbf{g} \\ t_n: G \to A_n \\ d_{n-1} t_n = 0}} G,$$

$n \geqslant 1$, with $d_n : A_{n+1} \to A_n$ having t_nth component equal to t_n. Once again, $d_{n-1} d_n = 0$ is obvious.

To prove exactness, first note that if $\mathrm{Hom}_{\mathbf{A}}(X, A_\bullet)$ is exact, then for any retract A of X $\mathrm{Hom}_{\mathbf{A}}(A, A_\bullet)$ is exact as well. Similarly if $\mathrm{Hom}_{\mathbf{A}}(G_i, A_\bullet)$ are exact, so is $\mathrm{Hom}_{\mathbf{A}}(\coprod_i G_i, A_\bullet)$. Thus it suffices to show that $\mathrm{Hom}_{\mathbf{A}}(G, A_\bullet)$ is exact for any object G from $\mathbf{g}$. Thus suppose given $t_n : G \to A_n$ with $d_{n-1}t_n = 0$. Then $t_n = d_n t_{n+1}$, where $t_{n+1} : G \to A_{n+1}$ is the canonical inclusion of the t_nth component into the coproduct. □

The following is the analog, in our $\mathbf{a}$-relative setting, of the Fundamental Lemma of homological algebra on the comparison of two resolutions of an object.

1.4. Lemma. *Let $\varepsilon : A_\bullet \to A$ and $\varepsilon' : A'_\bullet \to A$ be A-augmented chain complexes. If A_n are in* $\mathbf{a}$ *for $n \geqslant 0$ and $A'_\bullet$ is* $\mathbf{a}$*-exact, then there exists a chain map $f : A_\bullet \to A'_\bullet$ over A (i.e., with f_{-1} equal to the identity of A). Moreover this map is unique up to a chain homotopy over A, i.e., for any two $f, f' : A_\bullet \to A'_\bullet$ over A there is a chain homotopy $h_\bullet$ from f to f' over A (which means $h_0 = 0$).*

Proof. Since A_0 is in $\mathbf{a}$, by $\mathbf{a}$-exactness of $A'_\bullet$ the map $\mathrm{Hom}_{\mathbf{A}}(A_0, \varepsilon')$ is surjective; in particular, there is a morphism $f_0 : A_0 \to A'_0$ with $\varepsilon' f_0 = \varepsilon$. Next, as A_1 is also in $\mathbf{a}$, and $\varepsilon' f_0 d_0 = \varepsilon d_0 = 0$, again by $\mathbf{a}$-exactness of $A'_\bullet$ there is a map $f_1 : A_1 \to A'_1$ with $f_0 d_0 = d'_0 f_1$.Continuing this way, one obtains a sequence of maps $f_n : A_n \to A'_n$ with $d'_n f_n = f_{n-1} d_n$ for all $n \geqslant 0$.

Now suppose we are given two such sequences f, f'. Take $h_0{=}0 : A \to A'_0$. Since $\varepsilon'(f_0 - f'_0){=}0$, there is a $h_1 : A_0 \to A'_1$ with $f_0 - f'_0 {=} d'_0 h_1 {=} d'_0 h_1 + h_0 d_0$. Next since $d'_0(f_1 - f'_1 - h_1 d_0) {=} (f_0 - f'_0)d_0 - d'_0 h_1 d_0 {=} 0$, there is a $h_2 : A_1 \to A'_2$ with $f_1 - f'_1 - h_1 d_0 = d'_1 h_2$. Continuing one obtains the desired chain homotopy h. □

As an immediate corollary we obtain that any two $\mathbf{a}$-resolutions $A_\bullet, A'_\bullet$ of an object are chain homotopy equivalent, i.e., there are maps $f : A'_\bullet \to A_\bullet$, $f' : A_\bullet \to A'_\bullet$ with ff' and $f'f$ chain homotopic to identity maps. We thus see that all the standard ingredients for doing homological algebra are available. So we define

1.5. Definition. The $\mathbf{a}$-*relative left derived functors* $\mathrm{L}^{\mathbf{a}}_n F$, $n \geqslant 0$, of a functor $F : \mathbf{A} \to \mathscr{A}$ from $\mathbf{A}$ to an abelian category $\mathscr{A}$ are defined by

$$(\mathrm{L}^{\mathbf{a}}_n F)A = H_n(F(A_\bullet)),$$

where $A_\bullet$ is given by any $\mathbf{a}$-resolution of A. Similarly, $\mathbf{a}$-relative right derived functors of a contravariant functor $F : \mathbf{A}^{\mathrm{op}} \to \mathscr{A}$ are given by

$$(\mathrm{R}^n_{\mathbf{a}} F)A = H^n(F(A_\bullet)).$$

By the above lemmas, $\mathrm{L}^{\mathbf{a}}_n F$ and $\mathrm{R}^n_{\mathbf{a}} F$ are indeed functors and do not depend on the choice of resolutions. Note also that these constructions are functorial in F, i.e., a natural transformation $F \to F'$ induces natural transformations between the corresponding derived functors.

In particular, we have $\mathbf{a}$-relative Ext-groups given by

$$\mathrm{Ext}^n_{\mathbf{a}}(A, X) = (\mathrm{R}^n_{\mathbf{a}}(\mathrm{Hom}_{\mathbf{A}}(\_, X)))A = H^n(\mathrm{Hom}_{\mathbf{A}}(A_\bullet, X)),$$

for objects A, X of $\mathbf{A}$ and an $\mathbf{a}$-exact $\mathbf{a}$-resolution $A_\bullet$ of A. Note that these groups can be equipped with the *Yoneda product*

$$\mathrm{Ext}^m_{\mathbf{a}}(Y, Z) \otimes \mathrm{Ext}^n_{\mathbf{a}}(X, Y) \to \mathrm{Ext}^{m+n}_{\mathbf{a}}(X, Z).$$

On representing cocycles this product can be defined as follows: given $\mathbf{a}$-exact $\mathbf{a}$-resolutions $X_\bullet$ of X and $Y_\bullet$ of Y, we can represent elements of the Ext groups in question by maps $f : Y_m \to Z$ with $fd_m = 0$ and $g : X_n \to Y$ with $gd_n = 0$. Then similarly to the proof of 1.4, we can find maps $h_0 : X_n \to Y_0, \dots, h_{m-1} : X_{n+m-1} \to Y_{m-1}, h_m : X_{n+m} \to Y_m$ giving a map of complexes, and define $[f][g] = [fh_m]$. A standard homological algebra argument then shows that this product is well-defined, bilinear and associative.

1.6. Remark. Note that, as usually in *relative* homological algebra, we do not impose any requirements to the effect that the subcategory $\mathbf{a}$ generates $\mathbf{A}$ in any sense. To put it differently, our relative derived functors are only sensitive to that part of $\mathbf{A}$ which can be "seen from $\mathbf{a}$". For example, it does not contradict anything to take $\mathbf{a}$ consisting of the zero object only—which results in trivial relative derived functors, identically zero. Thus the content of relative derived functors depend crucially on an appropriate choice of the subcategory $\mathbf{a}$.

1.7. Examples. 1. A typical situation for the above is given by a ringoid $\mathbf{g}$, with $\mathbf{A}$ being the category of $\mathbf{g}$-modules, i.e., of linear functors from $\mathbf{g}$ to abelian groups. The abelian version of the Yoneda embedding identifies $\mathbf{g}$ with the full subcategory of $\mathbf{A}$ with objects the representable functors. The natural choice for $\mathbf{a}$ is then either the category of *free* $\mathbf{g}$-*modules*, which is the closure of this full subcategory $\mathbf{g} \subset \mathbf{A}$ under arbitrary coproducts, or that of *projective* $\mathbf{g}$-*modules*—the closure under both coproducts and retracts. In particular, when $\mathbf{g}$ has only one object, we obtain the classical setup for homological algebra given by a ring R, with $\mathbf{A}$ being the category of R-modules and $\mathbf{a}$ that of free or projective R-modules.

2. When $\mathbf{A}$ has finite limits, we obtain the additive case of derived functors from [31].

1.8. Remark. There is an obvious dual version of the above which one obtains by replacing $\mathbf{A}$ with the opposite category $\mathbf{A}^{\mathrm{op}}$. Explicitly, chain complexes get replaced by cochain complexes (with differentials having degree +1 rather than −1); exactness of the complex $\mathrm{Hom}_{\mathbf{A}}(X, A_\bullet)$ becomes replaced by that of $\mathrm{Hom}_{\mathbf{A}}(A^\bullet, X)$, etc.

1.9. Example. Let $\mathbf{A}$ be the stable homotopy category of spectra and let $\mathbf{a} \subset \mathbf{A}$ be the full subcategory consisting of finite products of Eilenberg–Mac Lane spectra over a fixed prime field $\mathbb{F}_p$. Let $\mathscr{A}$ be the mod p Steenrod algebra. The mod p cohomology functor restricted to $\mathbf{a}$ yields an equivalence of categories for which the following diagram commutes

$$\begin{tikzcd}
\mathbf{A}^{\mathrm{op}} \arrow[r, "H^*"] & \mathscr{A}-\mathbf{Mod} = \mathbf{A}_{\mathscr{A}} \\
\mathbf{a}^{\mathrm{op}} \arrow[u, hook] \arrow[r, "\sim"] & \mathscr{A}-\mathbf{mod} = \mathbf{a}_{\mathscr{A}}. \arrow[u, hook]
\end{tikzcd}$$

Here $\mathbf{A}^{\mathrm{op}}$ denotes the opposite category of $\mathbf{A}$, $\mathscr{A}$-$\mathbf{Mod}$ is the category of positively graded $\mathbf{A}$-modules and $\mathscr{A}$-$\mathbf{mod}$ is its full subcategory of finitely generated free modules. Given a spectrum X, its

a-coresolution $(A^{\varepsilon}_{\bullet}, d)$

$$\cdots \leftarrow A_1 \leftarrow A_0 \leftarrow X \leftarrow 0 \leftarrow 0 \leftarrow \cdots$$

is an X-coaugmented chain complex in $\mathbf{A}$, with A_n in $\mathbf{a}$ for $n \geqslant 0$, which is $\mathbf{a}$-coexact, that is $\mathrm{Hom}_{\mathbf{A}}(A^{\varepsilon}_{\bullet}, A')$ is acyclic for all $A' \in \mathbf{a}$. Hence $(A^{\varepsilon}_{\bullet}, d)$ is an $\mathbf{a}^{\mathrm{op}}$-resolution of X in $\mathbf{A}^{\mathrm{op}}$ which is carried by the cohomology functor H^* to an $\mathbf{a}_{\mathscr{A}}$-resolution of $H^*(X)$ in $\mathbf{A}_{\mathscr{A}}$ above. For this reason we get for a spectrum Y the binatural equation

$$\mathrm{Ext}^m_{\mathbf{a}^{\mathrm{op}}}(X, Y) = \mathrm{Ext}^m_{\mathbf{a}_{\mathscr{A}}}(H^*(X), H^*(Y)).$$

Here the left hand side $\mathrm{Ext}^m_{\mathbf{a}^{\mathrm{op}}}(X, Y)$ is defined in the additive category $\mathbf{A}^{\mathrm{op}}$ which is the opposite of the stable homotopy category. Moreover the right hand side is the classical Ext group

$$\mathrm{Ext}^m_{\mathbf{a}_{\mathscr{A}}}(H^*(X), H^*(Y)) = \mathrm{Ext}^m_{\mathscr{A}}(H^*(X), H^*(Y)).$$

2. Secondary resolutions

We have seen in Section 1 how relative resolutions yield the notion of relative derived functors. We now introduce relative secondary resolutions from which we deduce relative secondary derived functors. For this we need the notion of tracks.

A *track category* is a category enriched in groupoids; in particular, for all of its objects X, Y their hom-groupoid $[\![X, Y]\!]$ is given, whose objects are maps $f : X \to Y$ and whose morphisms, denoted $\alpha : f \Rightarrow f'$, are called tracks. Detailed treatment of various aspects of this notion and its applications in homotopy theory and homological algebra can be found in [3–5,7–11,14,15,19,25,26].

Equivalently, a track category is a 2-category all of whose 2-cells are invertible. For a track $\alpha : f \Rightarrow f'$ above and maps $g : Y \to Y'$, $e : X' \to X$, the resulting composite tracks will be denoted by $g\alpha : gf \Rightarrow gf'$ and $\alpha e : fe \Rightarrow f'e$. Moreover there is a vertical composition of tracks—composition of morphisms in the groupoids $[\![X, Y]\!]$; for $\alpha : f \Rightarrow f'$ and $\beta : f' \Rightarrow f''$, it will be denoted $\beta\Box\alpha : f \Rightarrow f''$. An inverse of a track α with respect to this composition will be denoted by $\alpha^{\boxminus}$. Identity tracks will be denoted by the symbol $\equiv$.

By a *track functor* we will mean a groupoid enriched functor between track categories.

A track category $\mathbf{B}$ will be also depicted as $\mathbf{B}_1 \rightrightarrows \mathbf{B}_0$. Here $\mathbf{B}_0$ being the underlying ordinary category of $\mathbf{B}$ obtained by forgetting about the tracks, whereas $\mathbf{B}_1$ is another ordinary category with the same objects but with morphisms from X to Y being tracks $\alpha : f \Rightarrow f'$ with $f, f' : X \to Y$ in $\mathbf{B}_0$, composite of α and β in the diagram

$$\begin{tikzcd} Z & Y \arrow[l, bend right, "f"'] \arrow[l, bend left, "f'"] \arrow[l, phantom, "\Downarrow\alpha"] & X \arrow[l, bend right, "g"'] \arrow[l, bend left, "g'"] \arrow[l, phantom, "\Downarrow\beta"] \end{tikzcd}$$

being

$$\alpha\beta = \alpha g'\Box f\beta = f'\beta\Box\alpha g : fg \Rightarrow f'g'. \tag{2.1}$$

There are thus two functors $\mathbf{B}_1 \to \mathbf{B}_0$ which are identity on objects and which send a morphism $\alpha : f \Rightarrow f'$ to f, resp. f'.

A track category $\mathbf{B}$ has the *homotopy category* $\mathbf{B}_{\simeq}$—an ordinary category obtained by identifying *homotopic maps*, i.e., maps f, f' for which there exists a track $f \Rightarrow f'$. It is thus the coequalizer of $\mathbf{B}_1 \rightrightarrows \mathbf{B}_0$ in the category of categories.

We now assume given a track category $\mathbf{B}$ such that its homotopy category is an additive category like $\mathbf{A}$ from Section 1,

$$\mathbf{B}_{\simeq} = \mathbf{A},$$

and that moreover $\mathbf{B}$ has a strict zero object, that is, an object $*$ such that for every object X of $\mathbf{B}$, $[\![X, *]\!]$ and $[\![*, X]\!]$ are trivial groupoids with a single morphism. It then follows that in each $[\![X, Y]\!]$ there is a distinguished map $0_{X,Y}$ obtained by composing the unique maps $X \to *$ and $* \to Y$. The identity track of this map will be denoted just by 0. Note that $0_{X,Y}$ may also admit non-identity self-tracks; one however has

$$0_{Y,Z}\beta = 0 = \alpha 0_{X,Y} \tag{2.2}$$

for any $\alpha : f \Rightarrow f'$, $f, f' : Y \to Z$, $\beta : g \Rightarrow g'$, $g, g' : X \to Y$.

In Section 3 we introduce the notion of an "additive track category" which is the most appropriate framework for secondary derived functors and which has the properties of the track category $\mathbf{B}$.

2.3. Example. The most easily described example is the track category $\mathscr{Ch}_{\mathbf{A}}$ whose objects are chain complexes in an additive category $\mathbf{A}$, maps are chain maps, and tracks are chain homotopies.

Our basic example is the track category $\mathscr{Pair}_{\mathbf{A}}$; it is the full track subcategory of $\mathscr{Ch}$ whose objects are chain complexes concentrated in degrees 0 and 1 only. Thus objects A of $\mathscr{Pair}_{\mathbf{A}}$ are given by morphisms $\partial_A : A_1 \to A_0$ in $\mathbf{A}$, a map f from A to B is a pair of morphisms $(f_1 : A_1 \to B_1, f_0 : A_0 \to B_0)$ in $\mathbf{A}$ making the obvious square commute, and a track $f \Rightarrow f'$ for $f, f' : A \to B$ is a morphism $\phi : A_0 \to B_1$ in $\mathbf{A}$ satisfying $\phi\partial_A = f_1 - f_1'$ and $\partial_B\phi = f_0 - f_0'$.

2.4. Remark. The secondary homology $\mathscr{H}$, as defined in [6], yields a track functor

$$\mathscr{H} : \mathscr{Ch}_{\mathbf{A}} \to \mathscr{Pair}_{\mathbf{A}^{\mathbb{Z}}}.$$

Here $\mathbf{A}$ is an abelian category, $\mathbf{A}^{\mathbb{Z}}$ denotes the category of $\mathbb{Z}$-graded objects in $\mathbf{A}$, and for a chain complex (A, d) in $\mathbf{A}$ the nth component of $\mathscr{H}(A, d)$ is given by

$$\mathscr{H}_n(A, d) = (d_n : \operatorname{Coker}(d_{n+1}) \to \operatorname{Ker}(d_{n-1})).$$

2.5. Example. A further basic example we have in mind is the track category $\mathbf{B}$ which is opposite to the category of spectra, stable maps, and tracks which are stable homotopy classes of stable homotopies.

Next we describe the secondary analogues of the notions of chain complex and resolution in 1.1, 1.2.

2.6. Definition. A *secondary chain complex* (A, d, δ) in a track category $\mathbf{B}$ is a diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend right=50, "0"'] & A_{n+2} \arrow[r, "d_{n+1}"] \arrow[rr, bend left=50, "0"] \arrow[d, phantom, "\Downarrow"] & A_{n+1} \arrow[r, "d_n"] \arrow[rr, bend right=50, "0"'] \arrow[u, phantom, "\delta_n\Uparrow"] & A_n \arrow[r, "d_{n-1}"] \arrow[rr, bend left=50, "0"] \arrow[d, phantom, "\Downarrow\delta_{n-1}"] & A_{n-1} \arrow[r] \arrow[u, phantom, "\Uparrow"] & \cdots
\end{tikzcd}$$

i.e., a sequence of objects A_n, maps $d_n : A_{n+1} \to A_n$ and tracks $\delta_n : d_n d_{n+1} \Rightarrow 0$, $n \in \mathbb{Z}$, such that for each n the tracks

$$d_{n-1}d_nd_{n+1} \xRightarrow{d_{n-1}\delta_n} d_{n-1}0 \xRightarrow{\equiv} 0$$

and

$$d_{n-1}d_nd_{n+1} \xRightarrow{\delta_{n-1}d_{n+1}} 0d_{n+1} \xRightarrow{\equiv} 0$$

coincide. Equivalently, the track $\delta_{n-1}d_{n+1}\Box d_{n-1}\delta_n^{\boxminus}$ in $\hom_{[\![A_{n+2},A_{n-1}]\!]}(0,0)$ must be the identity.

It is clear that a track functor $F : \mathbf{B} \to \mathbf{B}'$ between track categories as above (which preserves the zero object) carries a secondary chain complex in $\mathbf{B}$ to a secondary chain complex in $\mathbf{B}'$.

2.7. Examples. 1. In the example $\mathscr{P}\!\mathit{air}_{\mathbf{A}}$, a secondary chain complex looks like

$$\begin{tikzcd}
\cdots \arrow[r] & A_{1,n+2} \arrow[r, "d_{1,n+1}"] \arrow[d, "\partial_{n+2}"] & A_{1,n+1} \arrow[r, "d_{1,n}"] \arrow[d, "\partial_{n+1}"] & A_{1,n} \arrow[r, "d_{1,n-1}"] \arrow[d, "\partial_n"] & A_{1,n-1} \arrow[r] \arrow[d, "\partial_{n-1}"] & \cdots \\
\cdots \arrow[r] \arrow[ur, "\delta_{n+1}"] & A_{0,n+2} \arrow[r, "d_{0,n+1}"'] \arrow[urr, "\delta_n"] & A_{0,n+1} \arrow[r, "d_{0,n}"'] \arrow[urr, "\delta_{n-1}"] & A_{0,n} \arrow[r, "d_{0,n-1}"'] \arrow[urr] & A_{0,n-1} \arrow[r] & \cdots
\end{tikzcd}$$

with the equations $\partial_n d_{1,n} = d_{0,n}\partial_{n+1}$, $d_{1,n-1}d_{1,n} = \delta_{n-1}\partial_{n+1}$, $d_{0,n-1}d_{0,n} = \partial_{n-1}\delta_{n-1}$ and $d_{1,n-1}\delta_n = \delta_{n-1}d_{0,n+1}$ satisfied for all n.

More generally for $\mathscr{C}\!\mathit{h}_{\mathbf{A}}$ what one obtains is a bigraded group $A_{m,n}$ with differentials $\partial_{m,n} : A_{m+1,n} \to A_{m,n}$, $\partial_{m,n}\partial_{m+1,n} = 0$, and maps $d_{m,n} : A_{m,n+1} \to A_{m,n}$, $\delta_{m,n} : A_{m-1,n+2} \to A_{m,n}$ satisfying analogous equalities for all m and n.

One thus obtains a structure strongly related to what is called multicomplex or twisted complex in the literature; cf. [17,20,24].

2. In [30], the notion of complex of categories with abelian group structure is investigated. One can show that a slightly strictified version of their notion coincides with that of the secondary chain complex in an appropriate track category. On the other hand we could relax the requirement of existence of the

strict zero object to that of a weak zero object; then the construction of [30] would be subsumed in full generality.

2.8. Definition. A *secondary chain map* (f, ϕ) between secondary chain complexes (A, d, δ) and (A', d', δ') is a sequence of maps and tracks as indicated

$$\begin{tikzcd}
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_n"] \arrow[rr, bend left=50, "0"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\Downarrow\phi_n"] & A_n \arrow[r, "d_{n-1}"] \arrow[d, "f_n"] \arrow[u, phantom, "\delta_{n-1}\Uparrow"] \arrow[dr, phantom, "\Downarrow\phi_{n-1}"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_{n+1} \arrow[r, "d'_n"'] \arrow[rr, bend right=50, "0"'] & A'_n \arrow[r, "d'_{n-1}"'] \arrow[d, phantom, "\Downarrow\delta'_{n-1}"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

such that pasting of tracks in this diagram yields the identity track $\equiv: 0 \Rightarrow 0$, that is, the resulting track diagrams

$$\begin{tikzcd}
 & d'_{n-1}f_nd_n \arrow[r, Rightarrow, "d'_{n-1}\phi_n"] & d'_{n-1}d'_nf_{n+1} \arrow[drr, Rightarrow, "\delta'_{n-1}f_{n+1}"] & & \\
f_{n-1}d_{n-1}d_n \arrow[ur, Rightarrow, "\phi_{n-1}d_n"] \arrow[r, Rightarrow, "f_{n-1}\delta_{n-1}"] & f_{n-1}0 \arrow[rr, Rightarrow, "\equiv"] & & 0 \arrow[r, Rightarrow, "\equiv"] & 0f_{n+1}
\end{tikzcd} \tag{2.9}$$

commute for all $n \in \mathbb{Z}$.

For secondary chain maps $(f, \phi) : (A, d, \delta) \to (A', d', \delta')$ and $(f', \phi') : (A', d', \delta') \to (A'', d'', \delta'')$, their *composite* is given by $(f'_nf_n, \phi'_nf_{n+1}\Box f'_n\phi_n)$, $n \in \mathbb{Z}$. It is straightforward to check that this indeed defines a secondary chain map, and that the resulting composition operation is associative. Thus these operations determine the category of secondary chain complexes.

As in Section 1 we now fix a full track subcategory $\mathbf{b}$ of $\mathbf{B}$, with $\mathbf{a} = \mathbf{b}_{\simeq}$.

2.10. Definition. For a secondary complex (A, d, δ) in $\mathbf{B}$ and an integer n, a $\mathbf{b}$-*chain of degree n* of (A, d, δ) is a map $X \to A_n$ for some object X of $\mathbf{b}$. A $\mathbf{b}$-*cycle* is a pair (c, γ) consisting of a $\mathbf{b}$-chain $c : X \to A_n$ and a track $\gamma : d_{n-1}c \Rightarrow 0$ such that the track $d_{n-2}\gamma : d_{n-2}d_{n-1}c \Rightarrow d_{n-2}0 \equiv 0$ is equal to $\delta_{n-2}c : d_{n-2}d_{n-1}c \Rightarrow 0c \equiv 0$. A $\mathbf{b}$-cycle (b, β) of degree n is a $\mathbf{b}$-*boundary* if there exists a $\mathbf{b}$-chain a of degree $n+1$ and a track $\alpha : b \Rightarrow d_na$ such that the following diagram of tracks commutes:

$$\begin{tikzcd}
 & d_{n-1}d_na \arrow[r, Rightarrow, "\delta_{n-1}a"] & 0a \arrow[dr, Rightarrow, "\equiv"] & \\
d_{n-1}b \arrow[ur, Rightarrow, "d_{n-1}\alpha"] \arrow[rrr, Rightarrow, "\beta"] & & & 0.
\end{tikzcd}$$

A secondary complex (A, d, δ) is called **b**-*exact* if all of its **b**-cycles are **b**-boundaries. In other words, every diagram consisting of solid arrows below

$$\begin{tikzcd}
 & & X \arrow[dl, dashed, "a"'] \arrow[d, "c"] \arrow[dr, "0"] & & & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "d_n"] \arrow[rr, bend right=50, "0"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & A_n \arrow[r, "d_{n-1}"] \arrow[rr, bend right=50, "0"'] \arrow[d, phantom, "\delta_{n-1}\Downarrow"] \arrow[ur, phantom, "\overset{\gamma}{\Longrightarrow}"] & A_{n-1} \arrow[r, "d_{n-2}"] \arrow[d, phantom, "\Downarrow\delta_{n-2}"] & A_{n-2} \arrow[r] & \cdots \\
 & & {} & {} & &
\end{tikzcd}$$

in which the pasted track from $d_{n-2}0_{X,A_{n-1}}$ to $0_{A_n,A_{n-2}}c$ is the identity track can be completed by the dashed arrows in such a way that the resulting pasted track from $0_{A_{n+1},A_{n-1}}a$ to $0_{X,A_{n-1}}$ is the identity track.

2.11. Example. Consider the track category $\mathscr{P}air_{\mathbf{A}}$ from 2.3, with **A** the category of modules over a ring R, and choose for **b** the full track subcategory on the objects $0 \to R^n$, $n \geqslant 0$. Then for a secondary chain complex as in 2.7, a secondary cycle of degree n is a pair $(c, \gamma) \in A_{0,n} \times A_{1,n-1}$ satisfying $d_{0,n-1}c = \partial_{n-1}\gamma$ and $\delta_{n-2}c = d_{1,n-2}\gamma$. Such a cycle is a boundary if there exist elements $a \in A_{0,n+1}$ and $\alpha \in A_{1,n}$ with $c = d_{0,n}a + \partial_n\alpha$ and $\gamma = \delta_{n-1}a + d_{1,n-1}\alpha$.

Note that we can arrange for a *total complex*

$$\cdots \leftarrow A_{0,n-1} \oplus A_{1,n-2} \xleftarrow{\begin{pmatrix} d_{0,n-1} & -\partial_{n-1} \\ \delta_{n-2} & -d_{1,n-2} \end{pmatrix}} A_{0,n} \oplus A_{1,n-1} \xleftarrow{\begin{pmatrix} d_{0,n} & -\partial_{n} \\ \delta_{n-1} & -d_{1,n-1} \end{pmatrix}} A_{0,n+1} \oplus A_{1,n} \leftarrow \cdots$$

in such a way that secondary cycles and boundaries will become usual cycles and boundaries in this total complex. In particular then, secondary exactness of the secondary chain complex of type 2.7 is equivalent to the exactness in the ordinary sense of the above total complex.

We now turn to the secondary analog of the notion of resolution from 1.2.

2.12. Definition. For an object B in **B**, a *B-augmented secondary chain complex* is a secondary chain complex (B, d, δ) with $B_{-1} = B$, $B_{-n} = 0$ for $n > 1$, and δ_{-n} equal to identity track for $n > 1$. For a full track subcategory **b** of **B**, a B-augmented secondary chain complex is called a **b**-*resolution* of B if it is **b**-exact as a secondary chain complex and moreover all B_n for $n \geqslant 0$ belong to **b**.

As in the primary case, denoting $\varepsilon = d_{-1}$, $\hat{\varepsilon} = \delta_{-1}$, a B-augmented secondary chain complex can be considered as a secondary chain map $(\varepsilon, \hat{\varepsilon}) : B_\bullet \to B$ from the secondary chain complex $B_\bullet$ given by $\cdots \to B_1 \to B_0 \to 0 \to 0 \to \cdots$ with δ_{-n} identities for all $n > 0$, to the secondary chain complex B concentrated in degree 0, with trivial differentials:

$$\begin{tikzcd}
\cdots \arrow[r] & B_2 \arrow[r, "d_1"] \arrow[rr, bend left=50, "0"] \arrow[d] \arrow[dr, phantom, "\equiv"] & B_1 \arrow[r, "d_0"] \arrow[d] \arrow[u, phantom, "\delta_0\Uparrow"] \arrow[dr, phantom, "\Downarrow\hat{\varepsilon}"] & B_0 \arrow[r] \arrow[d, "\epsilon"] \arrow[dr, phantom, "\equiv"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & B \arrow[r] & 0 \arrow[r] & \cdots.
\end{tikzcd}$$

Accordingly such an augmented secondary chain complex will be denoted $B_\bullet^{\hat{\varepsilon},\varepsilon}$, and the pair $(\varepsilon, \hat{\varepsilon})$ will be called its augmentation.

Dually, we have the notion of a *B-coaugmented secondary chain complex*—the one satisfying $B_1 = B$, $B_n = 0$ for $n > 1$, and δ_n equal to the identity track for $n > 1$. Accordingly, there is a notion of a **b**-coresolution of B.

To have the analog of 1.3 we need an appropriate notion of coproduct; we might in principle use groupoid enriched, or *strong* coproducts, but for further applications more suitable is the less restrictive notion of *weak coproduct*, which we now recall.

2.13. Definition. A family of maps $(i_k : A_k \to A)_{k\in K}$ in a track category is a *weak* (respectively, *strong*) *coproduct diagram* for the family of objects $(A_k)_{k\in K}$ if for every object X the induced functor

$$[\![A, X]\!] \to \prod_{k\in K} [\![A_k, X]\!]$$

is an equivalence (resp., isomorphism) of groupoids.

Thus being a weak coproduct diagram means two things:

(1) for any object X and any maps $x_k : A_k \to X$, $k \in K$, there is a map $x : A \to X$ and a family of tracks $\chi_k : x_k \Rightarrow xi_k$, $k \in K$;
(2) for any $x, x' : A \to X$ and any family of tracks $(\chi_k : xi_k \Rightarrow x'i_k)_{k\in K}$ there is a unique track $\chi : x \to x'$ satisfying $\chi_k = \chi i_k$ for all $k \in K$,

whereas for a strong coproduct one must have

(1′) for any object X and any maps $x_k : A_k \to X$, $k \in K$, there is a unique map $x : A \to X$ satisfying $x_k = xi_k$ for all $k \in K$

and (2).

We will use notation $A = \widetilde{\coprod}_k A_k$ (resp., $A = \coprod_k A_k$) to indicate that A occurs in a weak (resp. strong) coproduct diagram for the family $(A_k)_k$ as above; we will say then, that A is a weak (resp. strong) coproduct of the A_k.

Note that any weak (*a fortiori* strong) coproduct diagram in a track category $\mathbf{B}$ becomes a coproduct diagram in its homotopy category $\mathbf{B}_\simeq$. In particular, if $(i_k : A_k \to A)_k$ and $(i'_k : A_k \to A')_k$ are weak coproduct diagrams with the same family $(A_k)_k$ then the objects A and A' are canonically isomorphic in $\mathbf{B}_\simeq$, i.e., homotopy equivalent in $\mathbf{B}$.

We can also weaken the notion of retract in 1.3: call an object X a *weak retract* of an object Y if there exist maps $j : X \to Y$, $p : Y \to X$ and a track $1_X \Rightarrow pj$.

2.14. Lemma. *Suppose*

- *the weak coproduct of any family of objects of* **b** *exists in* **B** *and belongs to* **b** *again*;
- *there is a small track subcategory* **g** *of* **b** *such that every object of* **b** *is a weak retract of a weak coproduct of a family of objects from* **g**.

Then every object of **B** *has a* **b**-*resolution*.

Proof. The first step is exactly as in the primary case: for an object B we take

$$B_0 = \widetilde{\coprod_{\substack{G\in\mathbf{g}\\ b:G\to B}}} G,$$

i.e., we choose any object B_0 occurring in a weak coproduct diagram for the indicated family. Thus in particular there is a map $d_{-1} : B_0 \to B$ and a family of tracks $\iota_b : b \Rightarrow d_{-1}i_b$ for each $b : G \to B$.

Suppose now we are given a (-1)-dimensional $\mathbf{b}$-cycle (b, β) in the $\mathbf{b}$-resolution. This means just a map $b : X \to B$ for an object X of $\mathbf{b}$, since $\beta : d_{-2}b \Rightarrow 0$ is then necessarily the trivial track. By hypothesis we then can find some weak coproduct $G = \widetilde{\coprod}_{k\in K} G_k$ of objects from $\mathbf{g}$, maps $j : X \to G$ and $p : G \to X$, and a track $\theta : 1_X \Rightarrow pj$. Then by the weak coproduct property, for the maps $i_{bpi_k} : G_k \to B_0$, where $i_k : G_k \to G$ are the weak coproduct structure maps, there exists a map $f_0 : G \to B_0$ and a family of tracks $\iota_k : i_{bpi_k} \Rightarrow f_0 i_k$, $k \in K$. This then gives composite tracks

$$d_{-1} f_0 i_k \xLeftarrow{d_{-1}\iota_k} d_{-1} i_{bpi_k} \xLeftarrow{\iota_{bpi_k}} bpi_k.$$

Again by the defining property of weak coproducts there is then a track $\phi : bp \Rightarrow d_{-1} f_0$ with $d_{-1}\iota_k \square \iota_{bpi_k} = \phi i_k$ for all $k \in K$. Denoting $f_0 j$ by a, one then obtains a track $\alpha : b \Rightarrow d_{-1}a$, namely the composite

$$b \xRightarrow{b\theta} bpj \xRightarrow{\phi j} d_{-1} f_0 j,$$

which means that (b, β) is a boundary, since both β and $\delta_{-2}a \square d_{-2}\alpha$ are zero for trivial reasons.

We next take

$$B_1 = \widetilde{\coprod_{\substack{G\in\mathbf{g}\\ t_0:G\to B_0\\ \tau:d_{-1}t_0\Rightarrow 0}}} G.$$

Then by the weak coproduct property, for the family $(t_0 : G \to B_0)_{\tau:d_{-1}t_0\Rightarrow 0}$ there exists $d_0 : B_1 \to B_0$ and tracks $\iota_\tau : t_0 \Rightarrow d_0 i_\tau$, where the $i_\tau : G \to B_1$ are the structure maps of the weak coproduct. Moreover for the family

$$\left(d_{-1}d_0 i_\tau \xRightarrow{d_{-1}\iota_\tau^{\boxminus}} d_{-1}t_0 \xRightarrow{\tau} 0 = 0_{G,B} = 0_{B_1,B}\, i_\tau \right)_{\tau:d_{-1}t_0\Rightarrow 0}$$

there exists $\delta_{-1} : d_{-1}d_0 \Rightarrow 0$ with

$$\delta_{-1} i_\tau = \tau \square d_{-1}\iota_\tau^{\boxminus} \tag{$\dagger_0$}$$

for all $\tau : d_{-1}t_0 \Rightarrow 0$. Since δ_{-2} by definition must be the identity track of the zero map, whereas d_{-2} is the unique map to the zero object, the condition $d_{n-1}\delta_n = \delta_{n-1}d_{n+1}$ from 2.6 is trivially satisfied at $n = -1$.

To prove $\mathbf{b}$-exactness at B_0, let $b_0 : X \to B_0$ and $\beta : d_{-1}b_0 \Rightarrow 0$ be given, for some object X of $\mathbf{b}$. By hypothesis, there is a weak retraction $j : X \to G$, $p : G \to X$, $\theta : 1_X \Rightarrow pj$ for some weak coproduct $G = \widetilde{\coprod}_{k\in K} G_k$ of objects G_k from $\mathbf{g}$. Then for the family $(i_{\beta p i_k} : G_k \to B_1)_{k\in K}$, where $i_k : G_k \to G$ are the structure maps of the weak coproduct, there exists a map $f_1 : G \to B_1$ and tracks $\iota_k : i_{\beta p i_k} \Rightarrow f_1 i_k$, $k \in K$. One thus obtains the composite tracks

$$d_0 f_1 i_k \xLeftarrow{d_0 \iota_k} d_0 i_{\beta p i_k} \xLeftarrow{\iota_{\beta p i_k}} b_0 p i_k.$$

Then again by the second property of weak coproducts there is a track $\phi_0 : b_0 p \Rightarrow d_0 f_1$ with $\phi_0 i_k = d_0 \iota_k \square \iota_{\beta \pi_k}$, $k \in K$. One then gets $a_1 = f_1 j$ and $\alpha = \phi_0 j \square b_0 \theta : b_0 \Rightarrow d_0 a_1$. To prove that (a_1, α) exhibits (b_0, β) as a boundary, it remains to show $\beta = \delta_{-1} a_1 \square d_{-1} \alpha$, that is, $\beta = \delta_{-1} f_1 j \square d_{-1} \phi_0 j \square d_{-1} b_0 \theta$. Now we have

$$\begin{aligned}\delta_{-1} f_1 i_k \square d_{-1} \phi_0 i_k &= \delta_{-1} f_1 i_k \square d_{-1} d_0 \iota_k \square d_{-1} \iota_{\beta p i_k}\\ &= \delta_{-1} i_{\beta p i_k} \square d_{-1} \iota_{\beta p i_k}.\end{aligned}$$

On the other hand by $(\dagger_0)$ one has $\delta_{-1} i_{\beta p i_k} = \beta p i_k \square d_{-1} \iota^{\boxminus}_{\beta p i_k}$, so one obtains

$$\delta_{-1} f_1 i_k \square d_{-1} \phi_0 i_k = \beta p i_k$$

for all k; by the weak coproduct property this then implies $\delta_{-1} f_1 \square d_{-1} \phi_0 = \beta p$, hence

$$\begin{aligned}\delta_{-1} f_1 j \square d_{-1} \phi_0 j \square d_{-1} b_0 \theta &= \beta p j \square d_{-1} b_0 \theta\\ &= 0\theta \square \beta\\ &= \beta.\end{aligned}$$

Now take some $n \geqslant 1$ and suppose all the B_i, d_{i-1} and δ_{i-2} have been already constructed for $i \leqslant n$ in such a way that the conditions of 2.6 and **b**-exactness are satisfied up to dimension $n-1$. Moreover we can assume by induction that exactness is *constructively established for* **b**-*cycles originating at* **g**, that is, for each $(n-1)$-cycle $(t_{n-1} : G \to B_{n-1}, \tau_{n-2} : d_{n-2} t_{n-1} \Rightarrow 0)$, $G \in \mathbf{g}$, with $d_{n-3} \tau_{n-2} = \delta_{n-3} t_{n-1}$, we are given explicit maps $i_{\tau_{n-2}} : G \to B_{n-1}$ and tracks $\iota_{\tau_{n-2}} : t_{n-1} \Rightarrow d_{n-1} i_{\tau_{n-2}}$ satisfying $\tau_{n-2} = \delta_{n-2} i_{\tau_{n-2}} \square d_{n-2} \iota_{\tau_{n-2}}$. At least this induction hypothesis is certainly satisfied for $n = 1$, by $(\dagger_0)$ above.

We then define

$$B_{n+1} = \widetilde{\coprod_{\substack{G \in \mathbf{g}\\ t_n : G \to B_n\\ \tau_{n-1} : d_{n-1} t_n \Rightarrow 0\\ d_{n-2} \tau_{n-1} = \delta_{n-2} t_n}}} G.$$

Then for the family $(t_n : G \to B_n)_{\{\tau_{n-1} : d_{n-1} t_n \Rightarrow 0 \mid d_{n-2} \tau_{n-1} = \delta_{n-2} t_n\}}$ there exists $d_n : B_{n+1} \to B_n$ and tracks $\iota_{\tau_{n-1}} : t_n \Rightarrow d_n i_{\tau_{n-1}}$, where the $i_{\tau_{n-1}} : G \to B_{n+1}$ are the coproduct structure maps. Moreover for the family

$$\left(d_{n-1} d_n i_{\tau_{n-1}} \xRightarrow{d_{n-1} \iota^{\boxminus}_{\tau_{n-1}}} d_{n-1} t_n \xRightarrow{\tau_{n-1}} 0 = 0 i_{\tau_{n-1}} \right)_{\{\tau_{n-1} : d_{n-1} t_n \Rightarrow 0 \mid d_{n-2} \tau_{n-1} = \delta_{n-2} t_n\}}$$

there exists $\delta_{n-1} : d_{n-1} d_n \Rightarrow 0$ with

$$\delta_{n-1} i_{\tau_{n-1}} = \tau_{n-1} \square d_{n-1} \iota^{\boxminus}_{\tau_{n-1}} \qquad (\dagger_n)$$

for all $\tau_{n-1}: d_{n-1}t_n \Rightarrow 0$ with $d_{n-2}\tau_{n-1} = \delta_{n-2}t_n$. To prove the condition $d_{n-2}\delta_{n-1} = \delta_{n-2}d_n$ from 2.6, it suffices by the weak coproduct property to prove

$$d_{n-2}\tau_{n-1}i_{\tau_{n-1}}\Box d_{n-2}d_{n-1}\imath_{\tau_{n-1}} = \delta_{n-2}d_n i_{\tau_{n-1}}\Box d_{n-2}d_{n-1}\imath_{\tau_{n-1}}$$

for each $\tau_{n-1}: d_{n-1}t_n \Rightarrow 0$, $t_n: G \to B_n$, with $d_{n-2}\tau_{n-1} = \delta_{n-2}t_n$. Now by $(\dagger_n)$ we have

$$d_{n-2}\delta_{n-1}i_{\tau_{n-1}}\Box d_{n-2}d_{n-1}\imath_{\tau_{n-1}} = d_{n-2}\tau_{n-1},$$

whereas by naturality we have

$$\delta_{n-2}d_n i_{\tau_{n-1}}\Box d_{n-2}d_{n-1}\imath_{\tau_{n-1}} = 0\imath_{\tau_{n-1}}\Box\delta_{n-2}t_n = \delta_{n-2}t_n.$$

Next note that the maps $i_{\tau_{n-1}}$ and tracks $\imath_{\tau_{n-1}}$ fulfil the induction hypothesis, i.e., explicitly exhibit cycles with domains from $\mathbf{g}$ as boundaries. Finally to prove exactness at B_n, consider any X, any weak coproduct $G = \widetilde{\coprod}_{k\in K} G_k$ of objects from $\mathbf{g}$, any weak retraction $j: X \to G$, $p: G \to X$, $\theta: 1_X \Rightarrow pj$, and any $b_n: X \to B_n$, $\beta_{n-1}: d_{n-1}b_n \Rightarrow 0$ with $d_{n-2}\beta_{n-1} = \delta_{n-2}b_n$. Then for each coproduct inclusion $i_k: G_k \to G$ one has cycles given by $b_n p i_k: G_k \to B_n$, $\beta_{n-1}p i_k: d_{n-1}b_n p i_k \Rightarrow 0$, hence for the family $(i_{\beta_{n-1}pi_k}: G_k \to B_{n+1})_{k\in K}$ there exists a map $f_{n+1}: G \to B_{n+1}$ and tracks $\imath_k: i_{\beta_{n-1}pi_k} \Rightarrow f_{n+1}i_k$. We then can consider the composite tracks

$$d_n f_{n+1} i_k \overset{d_n\imath_k}{\Longleftarrow} d_n i_{\beta_{n-1}pi_k} \overset{\imath_{\beta_{n-1}pi_k}}{\Longleftarrow} b_n p i_k$$

and by the weak coproduct property of G find for them a track $\phi_n: b_n p \Rightarrow d_n f_{n+1}$ with $d_n\imath_k\Box\imath_{\beta_{n-1}pi_k} = \phi_n i_k$ for all $k \in K$. This gives us an $(n+1)$-chain $a_{n+1} = f_{n+1}j$ and a track $\alpha = \phi_n j\Box b_n\theta: b_n \Rightarrow d_n a_{n+1}$. To show that these exhibit (b_n, β_{n-1}) as a boundary, one has to prove $\beta_{n-1} = \delta_{n-1}a_{n+1}\Box d_{n-1}\alpha$. The proof goes exactly as for the case $n = 0$ above. $\Box$

Now to the analog of 1.4.

2.15. Lemma. *Let $B_\bullet$ and $B'_\bullet$ be B-augmented secondary chain complexes. If all B_n belong to* **b** *and $B'_\bullet$ is* **b**-*exact, then there exists a secondary chain map $(f, \phi): B_\bullet \to B'_\bullet$ over B (i.e., with f_{-1} equal to the identity of B).*

Proof. The pair $d_{-1}: B_0 \to B$, identity$_0: d_{-2}d_{-1} \Rightarrow 0$ can be considered as a (-1)-cycle in $B'_\bullet$, so by **b**-exactness of $B'_\bullet$ there exist $f_0: B_0 \to B'_0$ and $\phi_{-1}: d_{-1} \Rightarrow d'_{-1}f_0$. Next f_0d_0, $\delta_{-1}\Box\phi^{\boxminus}_{-1}d_0: d'_{-1}f_0d_0 \Rightarrow d_{-1}d_0 \Rightarrow 0$ is a 0-cycle in $B'_\bullet$, so again by exactness of $B'_\bullet$ there are $f_1: B_1 \to B'_1$ and $\phi_0: f_0d_0 \Rightarrow d'_0f_1$ with

$$\delta_{-1}\Box\phi^{\boxminus}_{-1}d_0 = \delta'_{-1}f_1\Box d'_{-1}\phi_0, \tag{$*$}$$

which ensures the condition of 2.8 for $n = 0$. Then f_1d_1, $f_0\delta_0\Box\phi^{\boxminus}_0 d_1: d'_0f_1d_1 \Rightarrow f_0d_0d_1 \Rightarrow f_00 = 0$ is a 1-cycle in $B'_\bullet$. Indeed $(*)$ above implies $\delta'_{-1}f_1d_1 = \delta_{-1}d_1\Box\phi^{\boxminus}_{-1}d_0d_1\Box d'_{-1}\phi^{\boxminus}_0 d_1$; on the other hand $\delta_{-1}d_1\Box\phi^{\boxminus}_{-1}d_0d_1 = d_{-1}\delta_0\Box\phi^{\boxminus}_{-1}d_0d_1 = \phi^{\boxminus}_{-1}0\Box d'_{-1}f_0\delta_0 = d'_{-1}f_0\delta_0$, so $\delta'_{-1}f_1d_1 = d'_{-1}f_0\delta_0\Box d'_{-1}\phi^{\boxminus}_0 d_1$,which precisely means that the cycle condition is fulfilled. One thus obtains $f_2: B_2 \to B'_2$ and $\phi_1: f_1d_1 \Rightarrow d'_1f_2$ such that $f_0\delta_0\Box\phi^{\boxminus}_0 d_1 = \delta'_0f_2\Box d'_0\phi_1$, so the condition of 2.8 at $n = 1$ is also satisfied.

It is clear that continuing in this way one indeed obtains a secondary chain map. $\Box$

3. Additive track categories

The secondary analogue of an additive category is an additive track category considered in this section. For related conditions, see [16].

3.1. Definition. A track category $\mathbf{B}$ is called *additive* if it has a strict zero object $*$, the homotopy category $\mathbf{A} = \mathbf{B}_{\simeq}$ is additive and moreover $\mathbf{B}$ is a linear track extension

$$D \to \mathbf{B}_1 \rightrightarrows \mathbf{B}_0 \to \mathbf{A}$$

of $\mathbf{A}$ by a biadditive bifunctor

$$D : \mathbf{A}^{\mathrm{op}} \times \mathbf{A} \to \mathscr{Ab}.$$

Explicitly, this means the following: a biadditive bifunctor D as above is given together with a system of isomorphisms

$$\sigma_f : D(X, Y) \to \mathrm{Aut}_{[\![X,Y]\!]}(f) \tag{3.2}$$

for each 1-arrow $f : X \to Y$ in $\mathbf{B}$, such that for any $f : X \to Y, g : Y \to Z, a \in D(X, Y), b \in D(Y, Z)$, $\alpha : f \Rightarrow f'$ one has

$$\sigma_{gf}(ga) = g\sigma_f(a);$$
$$\sigma_{gf}(bf) = \sigma_g(b)f;$$
$$\alpha\square\sigma_f(a) = \sigma_{f'}(a)\square\alpha.$$

3.3. Remark. Using 3.2 we can identify the bifunctor D via the natural equation

$$D(X, Y) = \mathrm{Aut}(0_{X,Y}),$$

where $0 = 0_{X,Y} : X \to * \to Y$ is the unique morphism factoring through the zero object.

A *strict equivalence* between additive track categories $\mathbf{B}$, $\mathbf{B}'$ is a track functor $\mathbf{B} \to \mathbf{B}'$ which induces the identity on $\mathbf{A}$ and is compatible with the actions 3.2 above. Thus for fixed $\mathbf{A}$ and D as above, one obtains a category whose objects are additive track categories which are linear track extensions of $\mathbf{A}$ by D and morphisms are strict equivalences. This category will be denoted by Trext(A;D). For an additive category $\mathbf{A}$ and a biadditive bifunctor D on it, there is a bijection

$$\pi_0(\mathrm{Trext}(\mathbf{A}; D)) \approx H^3(\mathbf{A}; D),$$

where $\pi_0(\mathbb{C})$ denotes the set of connected components of a small category $\mathbb{C}$. Two additive track categories are called *equivalent* if they are in the same connected component of Trext($\mathbf{A}$;D). Thus in particular (as shown in [7,25]) each additive track category $\mathbf{B}$ as above determines a class $\langle\mathbf{B}\rangle \in H^3(\mathbf{A}; D)$.

As shown in [27], when $\mathbf{A}$ is the category of finitely generated free modules over a ring R and D is given by $D(X, Y) = \mathrm{Hom}_R(X, B\otimes_R Y)$ for some R-R-bimodule B, there are isomorphisms

$$H^3(\mathbf{A}; D) \cong HML^3(R; B) \cong THH^3(HR; HB),$$

where HML^* denotes Mac Lane cohomology, THH is topological Hochschild cohomology, and HR and HB are the Eilenberg–Mac Lane spectra corresponding to R and B.

3.4. Definition. An additive track category $\mathbf{B}$ is *Σ-additive* if an additive endofunctor $\Sigma : \mathbf{A} \to \mathbf{A}$ is given which left represents the bifunctor D, i.e., $\mathbf{B}$ is a linear track extension of $\mathbf{A}$ by the bifunctor

$$D(X, Y) = \mathrm{Hom}_{\mathbf{A}}(\Sigma X, Y).$$

Dually, $\mathbf{B}$ is *Ω-additive* if an additive endofunctor $\Omega : \mathbf{A} \to \mathbf{A}$ is given such that $\mathbf{B}$ is a linear track extension of $\mathbf{A}$ by the bifunctor

$$D(X, Y) = \mathrm{Hom}_{\mathbf{A}}(X, \Omega Y).$$

For objects X, Y in a Σ-, resp. Ω-additive track category $\mathbf{B}$ we will denote the group $\mathrm{Hom}_{\mathbf{A}}(\Sigma^m X, Y)$, resp. $\mathrm{Hom}_{\mathbf{A}}(X, \Omega^m Y)$ by $[X, Y]^m$.

In examples from topology the functor Σ is the suspension and the functor Ω is the loop space, compare also [8].

3.5. Example. As in 1.9 let $\mathbf{A}$ be the stable homotopy category of spectra. Since the category of spectra has a Quillen model structure we know that $\mathbf{A}$ is the homotopy category of all spectra which are fibrant and cofibrant. Using the cylinder of such spectra we obtain the additive track category $\mathbf{B}$. That is, $\mathbf{B}$ consists of spectra which are fibrant and cofibrant, of maps between such spectra, and tracks between such maps. Then $\mathbf{B}$ is both Σ-additive and Ω-additive, if one takes for Σ the suspension and for Ω the loop functor.

Σ- or Ω-additivity of a track category enables one to relate secondary exactness of a secondary chain complex to exactness of the corresponding chain complex in the homotopy category.

3.6. Lemma. *Let $\mathbf{B}$ be a track category with the additive homotopy category $\mathbf{A} = \mathbf{B}_{\simeq}$, let $\mathbf{b}$ be a full track subcategory of $\mathbf{B}$ and denote $\mathbf{a} = \mathbf{b}_{\simeq}$. Suppose that one of the following conditions is satisfied*:

(a) *$\mathbf{B}$ is Σ-additive and $\mathbf{a}$ is closed under suspensions* (*i.e., for each $X \in \mathbf{a}$ one has $\Sigma X \in \mathbf{a}$*); *or*
(b) *$\mathbf{B}$ is Ω-additive and the functor Ω is $\mathbf{a}$-exact* (*i.e., for an $\mathbf{a}$-exact complex $A_\bullet$ in $\mathbf{A}$, $\Omega A_\bullet$ is also $\mathbf{a}$-exact*).

Then for any secondary chain complex (A, d, δ) in $\mathbf{B}$, $\mathbf{a}$-exactness of its image $(A, [d])$ in $\mathbf{A}$ implies $\mathbf{b}$-exactness of (A, d, δ).

If moreover (A, d, δ) is bounded below, then conversely its $\mathbf{b}$-exactness implies $\mathbf{a}$-exactness of $(A, [d])$.

Proof. Unraveling definitions, we have that for any $a_n : X \to A_n$ with X in $\mathbf{b}$ and for any track $\alpha_{n-1} : d_{n-1}a_n \Rightarrow 0$ there exists $a_{n+1} : X \to A_{n+1}$ and a track $\alpha_n : a_n \Rightarrow d_n a_{n+1}$. From this, we have then to deduce that for (a_n, α_{n-1}) as above with the additional property $d_{n-2}\alpha_{n-1} = \delta_{n-2}a_n$ one can actually find $(\tilde{a}_{n+1}, \tilde{\alpha}_n)$ as above with the additional property $\alpha_{n-1} = \delta_{n-1}\tilde{a}_{n+1} \square d_{n-1}\tilde{\alpha}_n$.

Indeed for any $a_n : X \to A_n$, $\alpha_{n-1} : d_{n-1}a_n \Rightarrow 0$ with $d_{n-2}\alpha_{n-1} = \delta_{n-2}a_n$ and any $a_{n+1} : X \to A_{n+1}$, $\alpha_n : a_n \Rightarrow d_n a_{n+1}$ consider the element $\omega_{n-1} \in \mathrm{Aut}(d_{n-1}a_n)$ given by the composite

$$d_{n-1}a_n \xRightarrow{d_{n-1}\alpha_n} d_{n-1}d_n a_{n+1} \xRightarrow{\delta_{n-1}a_{n+1}} 0a_{n+1} \equiv 0 \xRightarrow{\alpha_{n-1}^{\boxminus}} d_{n-1}a_n.$$

For this element one has $d_{n-2}\omega_{n-1} = 0$. Indeed, this equality is equivalent to the equality

$$d_{n-2}\delta_{n-1}a_{n+1} \square d_{n-2}d_{n-1}\alpha_n = d_{n-2}\alpha_{n-1}$$

of tracks $\mathrm{Aut}(0_{X,A_{n-2}})$. But $d_{n-2}\alpha_{n-1} = \delta_{n-2}a_n$. Moreover by naturality there is a commutative diagram

$$\begin{tikzcd}
d_{n-2}d_{n-1}a_n \arrow[rr, Rightarrow, "d_{n-2}d_{n-1}\alpha_{n+1}"] \arrow[d, Rightarrow, "\delta_{n-2}a_n"'] & & d_{n-2}d_{n-1}d_na_{n+1} \arrow[d, Rightarrow, "\delta_{n-2}d_na_{n+1}"] \\
0a_n \arrow[rr, Rightarrow, "0\alpha_{n+1}"] \arrow[dr, Rightarrow, "\equiv"'] & & 0d_na_{n+1} \arrow[dl, Rightarrow, "\equiv"] \\
& 0 &
\end{tikzcd}$$

showing that $\delta_{n-2}a_n = \delta_{n-2}d_na_{n+1}\square d_{n-2}d_{n-1}\alpha_{n+1}$. It thus follows that $d_{n-2}\omega_{n-1} = 0$ if and only if one has

$$d_{n-2}\delta_{n-1}a_{n+1}\square d_{n-2}d_{n-1}\alpha_{n+1} = \delta_{n-2}d_na_{n+1}\square d_{n-2}d_{n-1}\alpha_{n+1},$$

which is clear since (A, d, δ) is a secondary complex.

Now if (a) is satisfied, then there is a commutative diagram

$$\begin{tikzcd}
{[\Sigma X, A_n]} \arrow[r, "{[d_{n-1}]_-}"] \arrow[d, "\cong"'] & {[\Sigma X, A_{n-1}]} \arrow[r, "{[d_{n-2}]_-}"] \arrow[d, "\cong"'] & {[\Sigma X, A_{n-2}]} \arrow[d, "\cong"'] \\
\mathrm{Aut}(a_n) \arrow[r, "d_{n-1}-"] & \mathrm{Aut}(d_{n-1}a_n) \arrow[r, "d_{n-2}-"] & \mathrm{Aut}(d_{n-2}d_{n-1}a_n)
\end{tikzcd}$$

Similarly if (b) holds, then one has the diagram

$$\begin{tikzcd}
{[X, \Omega A_n]} \arrow[r, "{\Omega[d_{n-1}]_-}"] \arrow[d, "\cong"'] & {[X, \Omega A_{n-1}]} \arrow[r, "{\Omega[d_{n-2}]_-}"] \arrow[d, "\cong"'] & {[X, \Omega A_{n-2}]} \arrow[d, "\cong"'] \\
\mathrm{Aut}(a_n) \arrow[r, "d_{n-1}-"] & \mathrm{Aut}(d_{n-1}a_n) \arrow[r, "d_{n-2}-"] & \mathrm{Aut}(d_{n-2}d_{n-1}a_n).
\end{tikzcd}$$

In both cases, it follows that there exists $\omega_n \in \mathrm{Aut}(a_n)$ such that $\omega_{n-1} = d_{n-1}\omega_n$.

Let us then choose

$$\tilde{a}_{n+1} = a_{n+1},$$
$$\tilde{\alpha}_n = \alpha_n\square\omega_n^{\boxminus}.$$

Then $\alpha_{n-1}^{\boxminus}\square\delta_{n-1}a_{n+1}\square d_{n-1}\tilde{\alpha}_n = \omega_{n-1}\square d_{n-1}\omega_n^{\boxminus}$ is the identity track of $d_{n-1}a_n$, that is,

$$\alpha_{n-1} = \delta_{n-1}a_{n+1}\square d_{n-1}\tilde{\alpha}_n,$$

as desired.

For the converse, by boundedness we can assume by induction that $(A, [d])$ is exact in all degrees $< n$. Let us then consider any **a**-cycle $[c] \in [X, A_n]$ in $(A, [d])$, choose a representative map $c : X \to A_n$ and a track $\gamma : 0 \Rightarrow d_{n-1}c$ and consider the composite track $\omega = \delta_{n-2}c\square d_{n-2}\gamma$ in $\mathrm{Aut}(0_{X,A_{n-2}})$.

The track $d_{n-3}\omega$ is the identity track 0 of $0_{X,A_{n-3}}$. Indeed $d_{n-3}\delta_{n-2} = \delta_{n-3}d_{n-1}$ by definition of a secondary chain complex, so $d_{n-3}\omega = \delta_{n-3}d_{n-1}c\square d_{n-3}d_{n-2}\gamma$. Then by (2.1) for $\delta_{n-3} : d_{n-3}d_{n-2} \Rightarrow 0_{A_{n-1},A_{n-3}}$ and $\gamma : 0_{X,A_{n-1}} \Rightarrow d_{n-1}c$ one has $\delta_{n-3}d_{n-1}c\square d_{n-3}d_{n-2}\gamma = 0_{A_{n-1},A_{n-3}}\gamma\square\delta_{n-3}0_{X,A_{n-1}}$ and by (2.2) both of the constituents in the last composition are identity tracks.

Now by induction hypothesis $(A,[d])$ is **a**-exact in degree $n-2$, hence if (a), resp. (b) holds, then the diagram

$$\begin{tikzcd}
{[\Sigma X, A_{n-1}]} \arrow[r, "{[d_{n-2}]\text{-}}"] \arrow[d, "\cong"'] & {[\Sigma X, A_{n-2}]} \arrow[r, "{[d_{n-3}]\text{-}}"] \arrow[d, "\cong"'] & {[\Sigma X, A_{n-3}]} \arrow[d, "\cong"'] \\
\mathrm{Aut}(0_{X,A_{n-1}}) \arrow[r, "d_{n-2}\text{-}"] & \mathrm{Aut}(0_{X,A_{n-2}}) \arrow[r, "d_{n-3}\text{-}"] & \mathrm{Aut}(0_{X,A_{n-3}}),
\end{tikzcd}$$

resp.

$$\begin{tikzcd}
{[X, \Omega A_{n-1}]} \arrow[r, "{\Omega[d_{n-2}]\text{-}}"] \arrow[d, "\cong"'] & {[X, \Omega A_{n-2}]} \arrow[r, "{\Omega[d_{n-3}]\text{-}}"] \arrow[d, "\cong"'] & {[X, \Omega A_{n-3}]} \arrow[d, "\cong"'] \\
\mathrm{Aut}(0_{X,A_{n-1}}) \arrow[r, "d_{n-2}\text{-}"] & \mathrm{Aut}(0_{X,A_{n-2}}) \arrow[r, "d_{n-3}\text{-}"] & \mathrm{Aut}(0_{X,A_{n-3}}).
\end{tikzcd}$$

shows that there exists $\alpha \in \mathrm{Aut}(0_{X,A_{n-1}})$ such that $\omega = d_{n-2}\alpha$. Then for $\tilde{\gamma} = \gamma\Box\alpha^{\boxminus}$ one has $\delta_{n-2}c\Box d_{n-2}\tilde{\gamma} = \delta_{n-2}c\Box d_{n-2}\gamma\Box d_{n-2}\alpha = \omega\Box\omega^{\boxminus} = 0$, so that $(c,\tilde{\gamma})$ is a secondary cycle. Then by secondary **b**-exactness of (A,d,δ) there is a $b : X \to A_{n+1}$ and $\beta : c \Rightarrow d_n b$, so $[c]$ is the boundary of $[b]$ in $[X,(A,[d])]$. Thus $(A,[d])$ is exact in degree n and we are done. $\Box$

3.7. Remark. Note that the additive track category from our 3.5 above satisfies both hypotheses of 3.6.

4. Secondary Ext

In this section we deduce from a secondary resolution a differential defined on "primary" derived functors as studied in Section 1. This differential is the analogue of the d_2-differential in a spectral sequence. We use the secondary differential to define certain "secondary" derived functors.

Let **B** be an additive track category with the additive homotopy category $\mathbf{A} = \mathbf{B}_{\simeq}$. Let us furthermore fix a full additive subcategory **a** in **A**; it determines the full track subcategory **b** of **B** on the same objects. It is clear that if **b** satisfies the conditions of 2.14, then **a** will satisfy those of 1.3. We can then consider the **a**-derived functors in **A**. In particular, the Ext groups $\mathrm{Ext}^n_{\mathbf{a}}(X,Y)$ are defined for any objects X, Y in **B**. Moreover if **B** is Σ-, resp. Ω-additive, then derived functors of the functor $D(X,Y) = \mathrm{Aut}(0_{X,Y})$ are given by

$$D^n_{\mathbf{a}}(X,Y) \cong \mathrm{Ext}^n_{\mathbf{a}}(\Sigma X, Y),$$

resp.

$$D^n_{\mathbf{a}}(X,Y) \cong \mathrm{Ext}^n_{\mathbf{a}}(X, \Omega Y).$$

We will use these isomorphisms to introduce the graded Ext groups $\mathrm{Ext}^n_{\mathbf{a}}(X,Y)^m = \mathrm{Ext}^n_{\mathbf{a}}(\Sigma^m X, Y)$, resp. $\mathrm{Ext}^n_{\mathbf{a}}(X,Y)^m = \mathrm{Ext}^n_{\mathbf{a}}(X, \Omega^m Y)$. Evidently if **B** is both Σ- and Ω-additive, these groups coincide.

We will from now on assume in what follows that for the pair $(\mathbf{B}, \mathbf{a})$ one of the conditions in 3.6 is satisfied, i.e., either **B** is Σ-additive and **a** is closed under Σ or **B** is oΩ-additive and Ω preserves **a**-exactness of chain complexes in $\mathbf{B}_{\simeq}$; moreover in the latter case we also assume that **a** is closed under Ω.

We are going to define the *secondary differential*

$$d_{(2)} = d^{n,m}_{(2)} : \mathrm{Ext}^n_{\mathbf{a}}(X,Y)^m \to \mathrm{Ext}^{n+2}_{\mathbf{a}}(X,Y)^{m+1}.$$

Replacing, if needed, X by $\Sigma^m X$ (resp. Y by $\Omega^m Y$) we might clearly assume $m = 0$ here. Moreover by 2.14 we may suppose that a **b**-exact **b**-resolution $(X_\bullet, d_\bullet, \delta_\bullet)$ of X is given. Then by 3.6 it determines an **a**-exact **a**-resolution $(X_\bullet, [d_\bullet])$ of X in **A**. Hence an element of $\mathrm{Ext}^n_{\mathbf{a}}(X, Y)$ gets represented by an n-dimensional cocycle in that resolution, i.e., by a homotopy class $[c] : X_n \to Y$ with $[c][d_n] = 0$. Thus we may choose a map $c \in [c]$ and a track $\gamma : 0 \Rightarrow cd_n$ in **B**, as in the diagram below:

$$\begin{tikzcd}
X_{n+3} \arrow[r, "d_{n+2}"] \arrow[rr, bend right=40, "0"'] \arrow[rr, phantom, bend right=20, "\Downarrow\delta_{n+1}"] & X_{n+2} \arrow[r, "d_{n+1}"] \arrow[rr, bend left=40, "0"] \arrow[rr, phantom, bend left=20, "\delta_n\Uparrow"] & X_{n+1} \arrow[r, "d_n"] \arrow[dr, "0"'] \arrow[dr, phantom, shift left=4, "\gamma\Nearrow"] & X_n \arrow[d, "c"] \\
 & & & Y
\end{tikzcd} \tag{4.1}$$

Then the composite track $c\delta_n \square \gamma d_{n+1} \in \mathrm{Aut}(0_{X_{n+2},Y})$ determines an element $\Gamma = \Gamma_{c,\gamma}$ in the group $\mathrm{Aut}(0_{X_{n+2},Y})$. One then has $\Gamma d_{n+2} = 0$. Indeed

$$\begin{aligned}
\Gamma d_{n+2} &= (c\delta_n \square \gamma d_{n+1})d_{n+2} \\
&= c\delta_n d_{n+2} \square \gamma d_{n+1} d_{n+2} \\
&= c d_n \delta_{n+1} \square \gamma d_{n+1} d_{n+2} \\
&= \gamma 0 \square 0 \delta_{n+1} \\
&= 0.
\end{aligned}$$

Thus Γ determines an $(n+2)$-cocycle in $\mathrm{Aut}(0_{(X_\bullet,[d_\bullet]),Y}) \cong [(X_\bullet, [d_\bullet]), Y]^1$. We then have

4.2. Theorem. *The above construction does not depend on the choice of* c, γ *and the resolution, up to coboundaries in* $[(X_\bullet, [d_\bullet]), Y]^1$; *hence the assignment* $[c] \mapsto [\Gamma_{c,\gamma}]$ *gives a well-defined homomorphism*

$$d_{(2)}^{n,m} : \mathrm{Ext}^n_{\mathbf{a}}(X, Y)^m \to \mathrm{Ext}^{n+2}_{\mathbf{a}}(X, Y)^{m+1}.$$

4.3. Remark. Of course the above homomorphism depends on the additive track category **B** in which we define the secondary resolution. In fact, $d_{(2)}$ depends only on the track subcategory $\mathbf{b}\{X, Y\} \subset \mathbf{B}$ obtained by adding to **b** the objects X and Y and all morphisms and tracks from $[\![Z, X]\!]$, $[\![Z, Y]\!]$ for all objects Z from **b**. We shall see in Section 5 below that additive track categories $\mathbf{B}$, $\mathbf{B}'$ with subcategories $\mathbf{b}$, $\mathbf{b}'$ such that the track categories $\mathbf{b}\{X, Y\}$ and $\mathbf{b}'\{X, Y\}$ are track equivalent yield the same differential $d_{(2)}$.

If the composites $d_{(2)}^{n,m} d_{(2)}^{n-2,m-1}$ are all zero (as this is the case for examples derived from spectral sequences), we define the *secondary* Ext *groups*

$$\mathrm{Ext}^n_{\mathbf{b}}(X, Y)^m := \ker(d_{(2)}^{n,m}) / \mathrm{im}(d_{(2)}^{n-2,m-1}). \tag{4.4}$$

This then will be, in examples, the E_3-term of a spectral sequence. We point out that the secondary Ext-groups are well defined and do not depend on the choice of the secondary resolution. We shall use the secondary Ext-groups for the computation of the E_3-term in the Adams spectral sequence, see [12].

Proof. We will first show that the cocycles corresponding to (c,γ) and (c,γ') for $\gamma,\gamma' : 0 \Rightarrow cd_n$ are cohomologous. Indeed the first one is $c\delta_n\square\gamma d_{n+1}$ and the second is

$$\begin{aligned} c\delta_n\square\gamma' d_{n+1} &= c\delta_n\square\gamma d_{n+1}\square\gamma^{\boxminus} d_{n+1}\square\gamma' d_{n+1} \\ &= c\delta_n\square\gamma d_{n+1}\square(\gamma^{\boxminus}\square\gamma')d_{n+1}, \end{aligned}$$

so these cocycles indeed differ by the coboundary of $\gamma^{\boxminus}\square\gamma'$. Thus we obtain a map $d_{(2)}$ from the group of n-cocycles of $\mathrm{Hom}_{\mathbf{A}}((X_\bullet,[d_\bullet]),Y)$ to $H^{n+2}(\mathrm{Aut}(0_{(X_\bullet,[d_\bullet]),Y}))\cong \mathrm{Ext}^{n+2}_{\mathbf{a}}(X,Y)^1$.

Next let us show that the map we just constructed is actually a homomorphism.

To see this, let us choose maps $p_1,\nabla,p_2 : Y\oplus Y\to Y$ in the homotopy classes $([1_Y],0)$, $([1_Y],[1_Y])$, $(0,[1_Y])\in[Y\oplus Y,Y]$ respectively. Thus for any two maps $c_1,c_2 : X\to Y$ there is a map $c_{1,2} : X\to Y\oplus Y$ such that there exist tracks $\pi_i : p_ic_{1,2}\Rightarrow c_i$, $i=1,2$, and moreover $[c_1]+[c_2]=[\nabla c_{1,2}]$. Now suppose c_i represent cocycles, then $[c_{1,2}][d_n]=([c_1][d_n],[c_2][d_n])=(0,0)\in[X_{n+1},Y]\times[X_{n+1},Y]\approx[X_{n+1},Y\oplus Y]$, so there is a track $\gamma : 0\Rightarrow c_{1,2}d_n$. Consequently, the cohomology class $d_{(2)}([c_1]+[c_2])=d_{(2)}([\nabla c_{1,2}])$ can be represented by the cocycle

$$\nabla c_{1,2}\delta_n\square\nabla\gamma d_{n+1} = \nabla(c_{1,2}\delta_n\square\gamma d_{n+1}).$$

On the other hand $d_{(2)}([f_i])$, $i=1,2$, can be represented by

$$\begin{aligned} c_i\delta_n\square\pi_i d_nd_{n+1}\square p_i\gamma d_{n+1} &= \pi_i 0\square p_ic_{1,2}\delta_n\square p_i\gamma d_{n+1} \\ &= p_ic_{1,2}\delta_n\square p_i\gamma d_{n+1} \\ &= p_i(c_{1,2}\delta_n\square\gamma d_{n+1}) \end{aligned}$$

(see the diagram below).

$$\begin{tikzcd}
X_{n+2} \arrow[r, "d_{n+1}"'] \arrow[rr, bend left, "0"] \arrow[rr, phantom, bend left=20, "\Uparrow\delta_n"] & X_{n+1} \arrow[r, "d_n"] \arrow[dr, "0"] \arrow[ddr, "0"'] \arrow[dr, phantom, bend left=30, "\gamma"] & X_n \arrow[d, "c_{1,2}"] \arrow[dr, "c_i"] \arrow[dr, phantom, bend right=20, "\pi_i"] \\
& & Y\oplus Y \arrow[r, "p_i"'] \arrow[d, "\nabla"] & Y \\
& & Y
\end{tikzcd}$$

But by assumption Aut(0) is biadditive, which in particular means that the map

$$(p_{1\_},p_{2\_}) : \mathrm{Aut}(0_{X,Y\oplus Y})\to\mathrm{Aut}(0_{X,Y})\times\mathrm{Aut}(0_{X,Y})$$

is an isomorphism, and moreover addition in $\mathrm{Aut}(0_{X,Y})$ is given by the composite of the left action $\nabla\_$ with the inverse of that isomorphism. This obviously means $d_{(2)}([c_1]+[c_2])=d_{(2)}([c_1])+d_{(2)}([c_2])$.

It follows that in order to show that $d_{(2)}$ factors through a homomorphism from the group $\mathrm{Ext}^n_{\mathbf{a}}(X,Y) = H^n([(X_\bullet,[d_\bullet]),Y])$ it suffices to show that $d_{(2)}$ vanishes on coboundaries, i.e., on cocycles of the form $[c]=[ad_{n-1}]$, for some map $a : X_{n-1}\to Y$. But for such a cocycle we may choose the track

$\gamma : 0 \Rightarrow ad_{n-1}d_n$ to be $a\delta^{\boxminus}_{n-1}$, and then the value of $d_{(2)}$ on it will be represented by the cocycle $ad_{n-1}\delta_n \square a\delta_{n-1}d_n = 0$—see the diagram.

$$
\begin{tikzcd}
X_{n+2} \arrow[r, "d_{n+1}"'] \arrow[rr, bend left, "0"] & X_{n+1} \arrow[r, "d_n"] \arrow[u, phantom, "\delta_n\Uparrow"] \arrow[rr, bend right, "0"'] \arrow[drr, "0"'] & X_n \arrow[r, "d_{n-1}"] \arrow[d, phantom, "\Uparrow\ \delta^{\boxminus}_{n-1}"] & X_{n-1} \arrow[dl, "a"] \\
& & Y \arrow[u, phantom, "\equiv"] &
\end{tikzcd}
$$

Finally we must show that $d_{(2)}$ does not depend on the choice of the secondary resolution. Indeed consider any two **b**-exact **b**-resolutions $(X_\bullet, d_\bullet, \delta_\bullet)$ and $(X'_\bullet, d'_\bullet, \delta'_\bullet)$ of X. By 2.15 there is a secondary chain map (f, ϕ) between them over X. Obviously then $[f]$ determines a chain map between $(X_\bullet, [d_\bullet])$ and $(X'_\bullet, [d'_\bullet])$ inducing isomorphisms f^* on cohomology of the cochain complexes obtained by applying $[\_, Y]$ and $\mathrm{Aut}(0_{\_,Y})$. We must then show that the diagrams

$$
\begin{tikzcd}
H^n([(X_\bullet, [d_\bullet]), Y]) \arrow[r, "d_{(2)}"] \arrow[d, "f^*"'] & H^n(\mathrm{Aut}(0_{(X_\bullet,[d_\bullet]),Y})) \arrow[d, "f^*"] \\
H^n([(X'_\bullet, [d'_\bullet]), Y]) \arrow[r, "d_{(2)}"] & H^n(\mathrm{Aut}(0_{(X'_\bullet,[d'_\bullet]),Y}))
\end{tikzcd}
$$

commute. This can be seen from the diagram

$$
\begin{tikzcd}
X'_{n+2} \arrow[r, "d'_{n+1}"] \arrow[rr, bend left, "0"] \arrow[d, "f_{n+2}"'] \arrow[dr, phantom, "\Downarrow\phi_{n+1}"] & X'_{n+1} \arrow[r, "d'_n"] \arrow[u, phantom, "\delta'_n\Uparrow"] \arrow[d, "f_{n+1}"] \arrow[dr, phantom, "\Downarrow\phi_n"] & X'_n \arrow[d, "f_n"] \\
X_{n+2} \arrow[r, "d_{n+1}"'] \arrow[rr, bend right, "0"'] & X_{n+1} \arrow[r, "d_n"'] \arrow[d, phantom, "\Downarrow\delta_n"] \arrow[dr, "0"'] & X_n \arrow[d, "c"] \arrow[dl, phantom, "\overset{\gamma}{\Rightarrow}"] \\
& & Y;
\end{tikzcd}
$$

in more detail, one considers the track diagram

$$
\begin{tikzcd}
& cf_n0 \arrow[dl, Rightarrow, "\equiv"'] \arrow[dd, Rightarrow, "\equiv"] & cf_nd'_nd'_{n+1} \arrow[l, Rightarrow, "cf_n\delta'_n"'] & cd_nf_{n+1}d'_{n+1} \arrow[l, Rightarrow, "c\phi^{\boxminus}_n d'_{n+1}"'] \arrow[dd, Rightarrow, "cd_n\phi_{n+1}"'] & 0f_{n+1}d'_{n+1} \arrow[l, Rightarrow, "\gamma f_{n+1}d'_{n+1}"'] \arrow[dd, Rightarrow, "0\phi_{n+1}"'] \arrow[dr, Rightarrow, "\equiv"] & \\
0 \arrow[dr, Rightarrow, "\equiv"'] & & & & & 0 \arrow[dl, Rightarrow, "\equiv"] \\
& c0f_{n+2} & & cd_nd_{n+1}f_{n+2} \arrow[ll, Rightarrow, "c\delta_nf_{n+2}"'] & 0d_{n+1}f_{n+2} \arrow[l, Rightarrow, "\gamma d_{n+1}f_{n+2}"'] &
\end{tikzcd}
$$

whose left part commutes by 2.9 and the right part by naturality. Now the lower composition of this diagram is

$$(c\delta_n \square \gamma d_{n+1}) f_{n+2} = f^* d_{(2)}([c]),$$

whereas the upper one is

$$(cf_n)\delta'_n \square (c\phi_n^{\boxminus} \square \gamma f_{n+1}) d'_{n+1},$$

which represents $d_{(2)}(f^*([c]))$, since we might choose for $\gamma' : 0 \Rightarrow f^*(c)d'_n$ the track $c\phi_n^{\boxminus} \square \gamma f_{n+1} : 0 \Rightarrow cf_n d'_n \equiv f^*(c)d'_n$. $\square$

5. Invariance of the secondary differential in the equivalence class of the track extension

In this section we will prove that the secondary differential

$$d_{(2)}^{n,m} : \mathrm{Ext}_{\mathbf{a}}^n(X, Y)^m \to \mathrm{Ext}_{\mathbf{a}}^{n+2}(X, Y)^{m+1}$$

constructed from an additive track category $\mathbf{B}$ depends only on the class

$$\langle \mathbf{B} \rangle \in H^3(\mathbf{A}; D).$$

More precisely one has

5.1. Theorem. *For any additive track categories* $\mathbf{B}$ *and* $\mathbf{B}'$ *with* $\mathbf{B}_{\simeq} = \mathbf{A} = \mathbf{B}'_{\simeq}$ *and any additive subcategory* $\mathbf{a} \subset \mathbf{A}$, *the secondary differentials* $d_{(2)}^{n,m}$ *constructed from* $\mathbf{B}$ *and* $\mathbf{B}'$ *coincide provided there is a strict equivalence of track subcategories* $\mathbf{b}\{X, Y\} \xrightarrow{\sim} \mathbf{b}'\{X, Y\}$.

Proof. Recall the construction of

$$d_{(2)}^{n,m} : \mathrm{Ext}_{\mathbf{a}}^n(X, Y)^m \to \mathrm{Ext}_{\mathbf{a}}^{n+2}(X, Y)^{m+1}.$$

Let $\mathbf{b} \subset \mathbf{B}$, $\mathbf{b}' \subset \mathbf{B}'$ be the full track subcategories in $\mathbf{B}$, resp. $\mathbf{B}'$, on objects from $\mathbf{a}$. Then $F(\mathbf{b}) \subset \mathbf{b}'$. One starts from a $\mathbf{b}$-exact $\mathbf{b}$-resolution $(X_\bullet, d_\bullet, \delta_\bullet)$ of X in $\mathbf{B}$; according to 4.2, the resulting $d_{(2)}$ does not depend on the choice of such a resolution. Suppose now we are given an element in $\mathrm{Ext}_{\mathbf{a}}^n(X, Y)$ represented by a Y-valued n-cocycle $[c] \in [X_n, Y]$ in the $\mathbf{a}$-exact $\mathbf{a}$-resolution $(X_\bullet, [d_\bullet])$ of X in $\mathbf{A}$. By our construction, the value on this element of the $d_{(2)}$ corresponding to $\mathbf{B}$ is obtained by choosing a representative $[c] \ni c : X_n \to Y$ and a track $\gamma : 0 \Rightarrow cd_n$ in $\mathbf{B}$, as in 4.1. One then has

$$d_{(2)}([c]) = [c\delta_n \square \gamma d_{n+1}].$$

But it is clear that $F(X_\bullet, d_\bullet, \delta_\bullet)$ is a $\mathbf{b}'$-exact $\mathbf{b}'$-resolution of X in $\mathbf{B}'$. We then might choose $F(c)$ and $F(\gamma)$ for the corresponding data in $\mathbf{B}'$, which would give us the element of $\mathrm{Aut}_{\mathbf{B}'}(0_{X_{n+2},Y})$ equal to $F(c)F(\delta_n) \square F(\gamma)F(d_{n-1}) = F(c\delta_n \square \gamma d_{n_1})$. Since by assumption F induces the identity on $\mathrm{Aut}(0)$, the theorem follows. $\square$

6. Resolutions of the Adams type

Let **B** be a track category with a strict zero object and homotopy category $\mathbf{B}_{\simeq} = \mathbf{A}$.

6.1. Definition. For an object X of **A**, an *X-coaugmented sequence* $\mathscr{R}$ is a diagram in **A** of the form

$$\mathscr{R}: \quad \cdots \longleftarrow Y_{n+1} \xleftarrow{\bar{p}_n} A_n \xleftarrow{\bar{i}_n} Y_n \longleftarrow \cdots \longleftarrow Y_2 \xleftarrow{\bar{p}_1} A_1 \xleftarrow{\bar{i}_1} Y_1 \xleftarrow{\bar{p}_0} A_0 \xleftarrow{\bar{i}_0} Y_0 = X$$

satisfying

$$\bar{p}_n \bar{i}_n = 0$$

in **A** for all $n = 0, 1, 2, \ldots$. The *associated X-coaugmented cochain complex* of such a sequence is then defined to be

$$C_{\mathbf{A}}(\mathscr{R}): \quad \cdots \xleftarrow{\bar{i}_{n+1}\bar{p}_n} A_n \xleftarrow{\bar{i}_n\bar{p}_{n-1}} \cdots \xleftarrow{\bar{i}_2\bar{p}_1} A_1 \xleftarrow{\bar{i}_1\bar{p}_0} A_0 \xleftarrow{\bar{i}_0} X.$$

For an additive subcategory $\mathbf{a} \subset \mathbf{A} = \mathbf{B}_{\simeq}$, an X-coaugmented sequence $\mathscr{R}$ as above will be called an **a**-*sequence* if A_n belongs to **a** for all n. Moreover it will be called **a**-*exact* if for any object A from **a**, the induced sequence

$$\mathrm{Hom}_{\mathbf{A}}(Y_{n+1}, A) \to \mathrm{Hom}_{\mathbf{A}}(A_n, A) \to \mathrm{Hom}_{\mathbf{A}}(Y_n, A)$$

is a short exact sequence of abelian groups for all $n \geqslant 0$. Thus in this case, the chain complex $C_{\mathbf{A}}(\mathscr{R})$ is **a**-exact in the sense of 1.1. In fact, for any object A in **a** the differential $\bar{d}_n : \mathrm{Hom}_{\mathbf{A}}(A_{n+1}, A) \to \mathrm{Hom}_{\mathbf{A}}(A_n, A)$ in $\mathrm{Hom}_{\mathbf{A}}(C_{\mathbf{A}}(\mathscr{R}), A)$ is then $\mathrm{Hom}_{\mathbf{A}}(\bar{i}_{n+1}\bar{p}_n, A)$, and one has

$$\ker(\bar{d}_{n-1}) = \mathrm{im}(\bar{d}_n) = \mathrm{Hom}_{\mathbf{A}}(Y_{n+1}, A)$$

for all n and all $A \in \mathbf{a}$.

6.2. Proposition. *For each X-coaugmented sequence $\mathscr{R}$ in* **A**, *any choice of representatives $i_n \in \bar{i}_n$, $p_n \in \bar{p}_n$ in $\mathbf{B}_0$ and of tracks $\alpha_n : p_n i_n \Rightarrow 0_{Y_n, Y_{n+1}}$ determines an X-coaugmented secondary chain complex in* **B** *of the form*

$$C_{\mathbf{B}}(\mathscr{R}): \quad \begin{tikzcd}[column sep=large]
\cdots & A_3 \arrow[l, "i_4p_3"'] & A_2 \arrow[l, "i_3p_2"'] \arrow[ll, bend left=50, "0"] \arrow[ll, phantom, bend left=25, "\Downarrow i_4\alpha_3p_2"] & A_1 \arrow[l, "i_2p_1"'] \arrow[ll, bend right=50, "0"'] \arrow[ll, phantom, bend right=25, "\Uparrow i_3\alpha_2p_1"] & A_0 \arrow[l, "i_1p_0"'] \arrow[ll, bend left=50, "0"] \arrow[ll, phantom, bend left=25, "\Downarrow i_2\alpha_1p_0"] & X. \arrow[l, "i_0"'] \arrow[ll, bend right=50, "0"'] \arrow[ll, phantom, bend right=25, "\Uparrow i_1\alpha_0"]
\end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd}
\cdots & Y_5 \arrow[l, "i_5"'] & A_4 \arrow[l, "p_4"'] & Y_4 \arrow[l, "i_4"'] \arrow[ll, bend right, "0"'] \arrow[ll, phantom, bend right=20, "\alpha_4\Uparrow"] & A_3 \arrow[l, "p_3"'] \arrow[llll, bend right=50, "0"'] & Y_3 \arrow[l, "i_3"'] \arrow[ll, bend left, "0"] \arrow[ll, phantom, bend left=20, "\Downarrow\alpha_3"] & A_2 \arrow[l, "p_2"'] \arrow[llll, bend left=50, "0"] & Y_2 \arrow[l, "i_2"'] \arrow[ll, bend right, "0"'] \arrow[ll, phantom, bend right=20, "\alpha_2\Uparrow"] & A_1 \arrow[l, "p_1"'] \arrow[llll, bend right=50, "0"'] & Y_1 \arrow[l, "i_1"'] \arrow[ll, bend left, "0"] \arrow[ll, phantom, bend left=20, "\Downarrow\alpha_1"] & A_0 \arrow[l, "p_0"'] \arrow[llll, bend left=50, "0"] & X. \arrow[l, "i_0"'] \arrow[ll, bend right, "0"'] \arrow[ll, phantom, bend right=20, "\alpha_0\Uparrow"] \arrow[lll, bend right=50, "0"']
\end{tikzcd}$$

That this diagram yields on $C_{\mathbf{B}}(\mathscr{R})$ above the structure of a secondary chain complex, is equivalent to the identities

$$i_{n+1}p_n i_n \alpha_{n-1} p_{n-2} = i_{n+1}\alpha_n p_{n-1} i_{n-1} p_{n-2}.$$

These are satisfied since one actually has

$$p_n i_n \alpha_{n-1} = \alpha_n p_{n-1} i_{n-1},$$

as the next lemma shows. □

6.3. Lemma. *For any maps* $f : X \to Y$, $f' : Y \to Z$ *and tracks* $\alpha : f \Rightarrow 0_{X,Y}$, $\alpha' : f' \Rightarrow 0_{Y,Z}$ *one has*

$$f'\alpha = \alpha' f.$$

Proof. This is a particular case of 2.1. □

6.4. Remark. Strictly speaking, $C_{\mathbf{B}}(\mathscr{R})$ depends on the choice of the i_n, p_n and α_n; however it will be harmless in what follows to suppress these from the notation.

6.5. Example. Let $\mathbf{B}$ be a track category and suppose that $\mathbf{A} = \mathbf{B}_{\simeq}$ is equipped with the structure of a triangulated category. Thus there is a self-equivalence $\Omega : \mathbf{A} \to \mathbf{A}$, with an inverse equivalence Ω^{-1} which we will call *delooping* in the following, and one has a distinguished class of diagrams of the form

$$A \leftarrow B \leftarrow C \leftarrow \Omega A,$$

called exact triangles, which satisfy certain axioms (see e.g. [23]). A *fiber tower* $\mathscr{T}$ over an object X is a diagram in $\mathbf{A}$

$$\begin{tikzcd}
X \arrow[r, equal] & X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
 & A_0 & A_1 & A_2 & \cdots
\end{tikzcd} \tag{6.6}$$

such that each $A_n \leftarrow X_n \leftarrow X_{n+1} \leftarrow \Omega A_n$ is an exact triangle in $\mathbf{A}$. In particular, the composites $\Omega^{-1} X_{n+1} \leftarrow A_n \leftarrow X_n$ are zero maps in $\mathbf{A}$. For an additive subcategory $\mathbf{a}$ in $\mathbf{A}$, call a fiber tower $\mathscr{T}$

a-*exact* if $A_i \in \mathbf{a}$ for all i and moreover each of its exact triangles induces a short exact sequence

$$0 \to \mathrm{Hom}(\Omega^{-1}X_{n+1}, A) \to \mathrm{Hom}(A_n, A) \to \mathrm{Hom}(X_n, A) \to 0$$

for all $A \in \mathbf{a}$.

A fiber tower yields a system of coaugmented sequences in **A** of the form

$$\begin{tikzcd}
 & & & & A_0 & X \arrow[l] \\
 & & A_1 & X_1 \arrow[l] & \Omega A_0 \arrow[l] & \Omega X \arrow[l] \\
A_2 & X_2 \arrow[l] & \Omega A_1 \arrow[l] & \Omega X_1 \arrow[l] & \Omega^2 A_0 \arrow[l] & \Omega^2 X \arrow[l] \\
 & & & \vdots & &
\end{tikzcd} \tag{6.7}$$

which via delooping in **A** yields the X-coaugmented sequence

$$\mathscr{R}(\mathscr{T}): \quad \cdots \leftarrow \Omega^{-2}A_2 \leftarrow \Omega^{-2}X_2 \leftarrow \Omega^{-1}A_1 \leftarrow \Omega^{-1}X_1 \leftarrow A_0 \leftarrow X.$$

Thus by 6.2 each fiber tower over X gives rise to an X-coaugmented secondary chain complex $C_{\mathbf{B}}(\mathscr{R}(\mathscr{T}))$.

6.8. Remark. Before the authors obtained the construction from 6.2, a direct topological proof that Adams resolutions give rise to a secondary complex has been kindly provided to them by Birgit Richter [28].

One then has

6.9. Theorem. *Assume either* **B** *is Σ-additive and $A \in \mathbf{a}$ implies $\Sigma A \in \mathbf{a}$, or* **B** *is Ω-additive and Ω preserves* **a**-*exactness of complexes in* **A** (*cf.* 3.6). *Then for any* **a**-*exact fiber tower over an object X, any X-coaugmented secondary chain complex associated to it* (*as in* 6.5 *and* 6.2) *is a* **b**-*coresolution of X. Hence for any object Y there is a secondary differential*

$$d_{(2)}: \mathrm{Ext}^n_{\mathbf{a}^{\mathrm{op}}}(X, Y)^m \to \mathrm{Ext}^{n+2}_{\mathbf{a}^{\mathrm{op}}}(X, Y)^{m+1},$$

where $\mathrm{Ext}^*_{\mathbf{a}^{\mathrm{op}}}(X, Y)^m$ *denotes either* $\mathrm{Ext}^*_{\mathbf{a}^{\mathrm{op}}}(\Sigma^m X, Y)$ *or* $\mathrm{Ext}^*_{\mathbf{a}^{\mathrm{op}}}(X, \Omega^m Y)$ *in* $\mathbf{A}^{\mathrm{op}}$. *The differential $d_{(2)}$ is well-defined by the cohomology class* $\langle \mathbf{B} \rangle \in H^3(\mathbf{A}; D)$ *with D in* 3.4.

Proof. This follows directly from 3.6. $\square$

7. The E_3 term of the Adams spectral sequence

Let us return to our main Examples 1.9 and 3.5. Thus let **A** be the stable homotopy category of spectra, let $\mathbf{a} \subset \mathbf{A}$ be the full subcategory of finite products of Eilenberg–Mac Lane spectra over a fixed prime field $\mathbb{F}_p$, and let **B** and **b** be the corresponding additive track category and its subcategory as in 3.5. Thus **B** is both Σ- and Ω-additive, with Σ and Ω having their usual meaning (i.e. suspension and loop functors).

Let X be a spectrum of finite type, that is, one for which the cohomology groups $H^i(X;\mathbb{F}_p)$ are finite dimensional $\mathbb{F}_p$-vector spaces for all i. Then the *Adams fiber tower* of X is given by

$$\begin{tikzcd} X \arrow[r, equal] & X_0 \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\ & H\wedge X_0 & H\wedge X_1 & H\wedge X_2 & \cdots. \end{tikzcd} \tag{7.1}$$

Here $H = H\mathbb{F}_p$ is the Eilenberg–Mac Lane spectrum, the map $X_i \to H\wedge X_i$ is given by smashing $S^0 \to H$ with X_i, and X_{i+1} is the fiber of this map. Since X is of finite type all spectra $H\wedge X_i$ can be considered to be objects of $\mathbf{a}$. By construction the Adams fiber tower is $\mathbf{a}$-exact.

As noted in 3.7, conditions of 3.6 are satisfied, so that we can apply Theorem 6.9 to the Adams fiber tower. Hence we get for a spectrum Y the following diagram whose top row is defined by any secondary $\mathbf{b}$-coresolution of X and the bottom row is the differential $d_{(2)}$ in the Adams spectral sequence.

$$\begin{tikzcd} \mathrm{Ext}^n_{\mathbf{a}^{\mathrm{op}}}(X,Y)^m \arrow[r, "d_{(2)}"] \arrow[d, "\cong"] & \mathrm{Ext}^{n+2}_{\mathbf{a}^{\mathrm{op}}}(X,Y)^{m+1} \arrow[d, "\cong"] \\ \mathrm{Ext}^n_{\mathscr{A}}(H^*X,H^*Y)^m \arrow[r, "d_{(2)}"] & \mathrm{Ext}^{n+2}_{\mathscr{A}}(H^*X,H^*Y)^{m+1}. \end{tikzcd} \tag{7.2}$$

The vertical isomorphisms in this diagram are defined in Example 1.9.

7.3. Theorem. *The diagram* 7.2 *commutes.*

This shows that $d_{(2)}d_{(2)}=0$ so that the secondary Ext in Section 4 coincides with the E_3-term of the Adams spectral sequence. In the book [6] a pair algebra $\mathscr{B}$ is computed which can be used to describe algebraic models for secondary $\mathbf{b}$-coresolutions. This, in fact, yields an algorithm computing the $d_{(2)}$ differential in the Adams spectral sequence since we can use Theorem 7.3.

Proof. In our terms the second differential of the Adams spectral sequence can be understood in the following way: one is given a fiber tower $\mathscr{T}$ like 6.6 or 7.1 over an object X in the stable homotopy category, with the associated X-coaugmented sequence $\mathscr{R}(\mathscr{T})$ as in 6.5. To it corresponds by 6.1 the associated X-coaugmented cochain complex

$$C_{\mathbf{A}}(\mathscr{R}(\mathscr{T})):\quad \cdots \longleftarrow \Omega^{-n-1}A_{n+1} \xleftarrow{\Omega^{-n}d^n} \Omega^{-n}A_n \longleftarrow \cdots \xleftarrow{\Omega^{-1}d^1} \Omega^{-1}A_1 \xleftarrow{d^0} A_0 \longleftarrow X$$

where $d^n: A_n \to \Omega^{-1}A_{n+1}$ are the composites $A_n \to \Omega^{-1}X_{n+1} \to \Omega^{-1}A_{n+1}$ of maps in the exact triangles $X_{n+1}\to X_n\to A_n\to\Omega^{-1}X_{n+1}$ and $A_{n+1}\to\Omega^{-1}X_{n+2}\to\Omega^{-1}X_{n+1}\to\Omega^{-1}A_{n+1}$. Here all A_n are $\mathbb{F}_p$-module spectra, i.e., Eilenberg–MacLane spectra of $\mathbb{F}_p$-vector spaces, and moreover the sequences $X_n\to A_n\to\Omega^{-1}X_{n+1}$ are $\mathbb{F}_p$-exact, i.e., applying $H^*(\_;\mathbb{F}_p)$ to them yields short exact sequences. In particular, $H^*(C_{\mathbf{A}}(\mathscr{R}(\mathscr{T}));\mathbb{F}_p)$ is an $\mathscr{A}$-projective resolution of $H^*(X;\mathbb{F}_p)$.

Now choose new spectra B_n fitting in exact triangles

$$B_n \xrightarrow{-i^n} A_n \xrightarrow{-d^n} \Omega^{-1}A_{n+1} \longrightarrow \Omega^{-1}B_n$$

and observe that by the octahedron axiom there is a commutative diagram of (co)fibre sequences of the form

$$
\begin{tikzcd}[column sep=small]
X_{n+2} \arrow[dr] \arrow[rr, bend left] & & X_n \arrow[dr] \arrow[rr, bend left] & & A_n \arrow[dr] \arrow[rr, bend left] & & \Omega^{-1}A_{n+1} \arrow[dr] \arrow[rr, bend left] & & \cdots \\
& X_{n+1} \arrow[ur] \arrow[dr] & & B_n \arrow[ur] \arrow[dr] & & \Omega^{-1}X_{n+1} \arrow[ur] \arrow[dr] & & \Omega^{-1}B_n \arrow[ur] \arrow[dr] & \cdots \\
& & A_{n+1} \arrow[ur] \arrow[rr, bend right] & & \Omega^{-1}X_{n+2} \arrow[ur] \arrow[rr, bend right] & & \Omega^{-1}X_n \arrow[ur] \arrow[rr, bend right] & & \cdots
\end{tikzcd}
\tag{7.4}
$$

so that in particular the original fiber tower $\mathscr{T}$ "doubles" to give two new fiber towers $\mathscr{T}^{(2)}$ starting at X_0, resp. X_1, of the form

$$
\begin{tikzcd}
\cdots & X_n \arrow[l] \arrow[d] & X_{n+2} \arrow[l] \arrow[d] & X_{n+4} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & B_n & B_{n+2} & B_{n+4} & \cdots .
\end{tikzcd}
$$

The associated sequences $\mathscr{R}(\mathscr{T}^{(2)})$ and the cochain complexes

$$C_{\mathbf{A}}(\mathscr{R}(\mathscr{T}^{(2)})): \quad \cdots \longleftarrow \Omega^{-n-2}B_{n+4} \longleftarrow \Omega^{-n-1}B_{n+2} \xleftarrow{\Omega^{-n}d(2)^n} \Omega^{-n}B_n \longleftarrow \cdots,$$

where $d(2)^n : B_n \to \Omega^{-1}B_{n+2}$ is the composite $B_n \to \Omega^{-1}X_{n+2} \to \Omega^{-1}B_{n+2}$, are then obtained as in 6.5.

Let us now take any spectrum Y and apply the stable homotopy classes functor $\{Y, \_\}$ to the whole business. Because of the exact triangles $B_n \to A_n \to \Omega^{-1}A_{n+1} \to \Omega^{-1}B_n$, there are isomorphisms

$$\operatorname{im}(\{Y, B_n\} \to \{Y, A_n\}) \cong \ker(\{Y, A_n\} \to \{Y, \Omega^{-1}A_{n+1}\}).$$

On the other hand it is known (see e.g. [21]) that the canonical maps

$$\{Y, A_n\} \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(H^*(A_n; \mathbb{F}_p), H^*(Y; \mathbb{F}_p)) \tag{7.5}$$

are isomorphisms; it thus follows that the groups

$$E_2^{s,t}(Y, X) = \frac{\operatorname{im}\left(\{Y, \Omega^{t-s}B_s\} \xrightarrow{\{Y, \Omega^{t-s}i_s\}} \{Y, \Omega^{t-s}A_s\}\right)}{\operatorname{im}\left(\{Y, \Omega^{t-s+1}A_{s-1}\} \xrightarrow{\{Y, \Omega^{t-s+1}d^{s-1}\}} \{Y, \Omega^{t-s}A_s\}\right)}$$

are isomorphic to $\operatorname{Ext}^s_{\mathscr{A}}(H^*(X; \mathbb{F}_p), H^*(Y; \mathbb{F}_p))^t$.

Moreover (see again [21]) the Adams differential $E_2^{s,t} \to E_2^{s+2,t+1}$ is induced by the map

$$\operatorname{im}\left(\{Y, \Omega^{t-s}B_s\} \to \{Y, \Omega^{t-s}A_s\}\right) \to \operatorname{im}\left(\{Y, \Omega^{t-s-1}B_{s+2}\} \to \{Y, \Omega^{t-s-1}A_{s+2}\}\right)$$

which sends the class of a stable map

$$Y \to \Omega^{t-s} B_s \to \Omega^{t-s} A_s$$

to the class of the composite

$$Y \to \Omega^{t-s} B_s \xrightarrow{\Omega^{t-s} d(2)^s} \Omega^{t-s-1} B_{s+2} \to \Omega^{t-s-1} A_{s+2}$$

or, which by 7.4 is the same, the composite

$$Y \to \Omega^{t-s} B_s \to \Omega^{t-s-1} X_{s+2} \to \Omega^{t-s-1} A_{s+2}.$$

To see then that the differential so defined coincides with the secondary differential as constructed in 4.2, 4 and 6.9, let us choose zero tracks α_n for the composites $X_n \to A_n \to \Omega^{-1} X_{n+1}$ and switch from $C_{\mathbf{A}}(\mathscr{R}(\mathscr{T}))$ to the X-coaugmented secondary cochain complex $C_{\mathbf{B}}(\mathscr{R}(\mathscr{T}))$ as defined in 6.2. Then according to 4.2, given an element $\langle c\rangle$ of $\operatorname{Ext}^s_{\mathscr{A}}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p)))^t$, the corresponding element $d_{(2)}\,\langle c\rangle \in \operatorname{Ext}^{s+2}_{\mathscr{A}}(H^*(X;\mathbb{F}_p), H^*(Y;\mathbb{F}_p))^{t+1}$ is constructed in the following way. First represent $\langle c\rangle$ by a cocycle in $C_{\mathbf{A}}(\mathscr{R}(\mathscr{T}))$, i.e., by a homomorphism of $\mathscr{A}$-modules $[c] : H^*(\Omega^{t-s} A_s;\mathbb{F}_p) \to H^*(Y;\mathbb{F}_p)$ with $[c] \circ H^*(\Omega^{t-s} d^s;\mathbb{F}_p) = 0$. By 7.5, this homomorphism is in turn induced by a map $c : Y \to \Omega^{t-s} A_s$ such that $d^s \circ c$ is nullhomotopic. Choosing a homotopy $\gamma : 0 \Rightarrow d^s \circ c$, according to 4.2 the class $d_{(2)}\,\langle c\rangle$ is represented by the map $Y \to \Omega^{t-s-1} A_{s+2} = \Omega\Omega^{t-s-2} A_{s+2}$ which corresponds to the composite homotopy

$$0 \overset{\equiv}{\Longrightarrow} \Omega^{t-s-1} d^{s+1} \circ 0 \overset{\Omega^{t-s-1} d^{s+1}\gamma}{\Longrightarrow} \Omega^{t-s-1} d^{s+1} \circ \Omega^{t-s} d^s \circ c \overset{-\delta c}{\Longrightarrow} 0c \overset{\equiv}{\Longrightarrow} 0$$

from the zero map $Y \to \Omega^{t-s-2} A_{s+2}$ to itself, as in

$$\begin{tikzcd}
Y \arrow[d, "c"'] \arrow[dr, "0"] \arrow[d, phantom, shift left=4ex, "\Downarrow\gamma"] & & \\
\Omega^{t-s} A_s \arrow[r, "\Omega^{t-s} d^s"'] \arrow[rr, bend right, "0"'] & \Omega^{t-s-1} A_{s+1} \arrow[r, "\Omega^{t-s-1} d^{s+1}"] \arrow[d, phantom, "\Downarrow\delta"] & \Omega^{t-s-2} A_{s+2}. \\
 & {} &
\end{tikzcd}$$

Now according to the construction of $C_{\mathbf{B}}(\mathscr{R}(\mathscr{T}))$ given in 6.2, this diagram reduces to the following diagram:

$$\begin{tikzcd}
Y \arrow[d, "c"'] \arrow[drr, "0"] \arrow[dr, phantom, "\Downarrow\gamma"] & & & & \\
\Omega^{t-s} A_s \arrow[r] & \Omega^{t-s-1} X_{s+1} \arrow[r] \arrow[rr, bend right, "0"'] & \Omega^{t-s-1} A_{s+1} \arrow[r] \arrow[d, phantom, "\Downarrow\alpha"] & \Omega^{t-s-2} X_{s+2} \arrow[r] & \Omega^{t-s-2} A_{s+2}. \\
 & & {} & &
\end{tikzcd} \tag{7.6}$$

Next note that because of the fibre sequence

$$\Omega^{t-s} B_s \to \Omega^{t-s} A_s \to \Omega^{t-s-1} A_{s+1} \to \Omega^{t-s-1} B_s,$$

choosing $\gamma : 0 \Rightarrow \Omega^{t-s}d^s \circ c$ is equivalent to choosing a lift of c to a map $Y \to \Omega^{t-s}B_s$. Similar correspondences between homotopies and liftings of maps take place further along the sequence, as can be summarized in the following diagram

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d, equal] & \Omega^{t-s}B_s \arrow[r] \arrow[d] & \Omega^{t-s-1}X_{s+2} \arrow[r, equal] \arrow[d] & \Omega^{t-s-1}X_{s+2} \arrow[r] \arrow[d] & \Omega^{t-s-1}A_{s+2} \arrow[d] \\
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\gamma\Nearrow"] & \Omega^{t-s}A_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\equiv"] & \Omega^{t-s-1}X_{s+1} \arrow[r] \arrow[d] \arrow[dr, phantom, "\alpha\Nearrow"] & * \arrow[r, equal] \arrow[d] \arrow[dr, phantom, "\equiv"] & * \arrow[d] \\
* \arrow[r] & \Omega^{t-s-1}A_{s+1} \arrow[r, equal] & \Omega^{t-s-1}A_{s+1} \arrow[r] & \Omega^{t-s-2}X_{s+2} \arrow[r] & \Omega^{t-s-2}A_{s+2},
\end{tikzcd}
$$

in which the columns form fiber sequences and the upper horizontal maps are liftings corresponding to the homotopies indicated in lower squares. That the resulting upper horizontal composite is indeed the lifting corresponding to the composite homotopy in 7.6 now follows from the following standard homotopy-theoretic lemma which can be found e.g. in [2, (2.9) on p. 263]:

7.7. Lemma. *Given a diagram*

$$
\begin{tikzcd}
F \arrow[r, dashed] \arrow[d] & F' \arrow[r, dashed] \arrow[d] & F'' \arrow[d] \\
E \arrow[r, "e"] \arrow[d] \arrow[dr, phantom, "\gamma\Nearrow"] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\delta\Nearrow"] & E'' \arrow[d] \\
B \arrow[r] & B' \arrow[r, "b"] & B''
\end{tikzcd}
$$

whose columns are fiber sequences and upper horizontal maps are liftings corresponding to the indicated homotopies, then the composite $F \to F''$ is the lifting corresponding to the composite homotopy $\delta e \square b\gamma$. □

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References

[1] J.F. Adams, On the non-existence of elements of Hopf invariant one, Ann. Math. 72 (2) (1960) 20–104.

[2] H.-J. Baues, Algebraic homotopy, Cambridge Studies in Advanced Mathematics, vol. 15, Cambridge University Press, Cambridge, 1989, ISBN 0-521-33376-8.

[3] H.-J. Baues, Combinatorial homotopy and 4-dimensional complexes, de Gruyter Expositions in Mathematics, vol. 2, Walter de Gruyter & Co., Berlin, 1991, with a preface by Ronald Brown.

[4] H.-J. Baues, Homotopy type and homology, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1996 (Oxford Science Publications).

[5] H.-J. Baues, Secondary cohomology and the Steenrod square, Homol. Homotopy Appl. 4 (2002) 29–62 (electronic) (The Roos Festschrift volume, 1).

[6] H.-J. Baues, The Algebra of Secondary Cohomology Operations, Progress in Mathematics, vol. 247, 700 pp, Birkhuser 2006.
[7] H.-J. Baues, W. Dreckmann, The cohomology of homotopy categories and the general linear group, K-Theory 3 (1989) 307–338.
[8] H.-J. Baues, M. Jibladze, Suspension and loop objects and representability of tracks, Georgian Math. J. 8 (2001) 683–696.
[9] H.-J. Baues, M. Jibladze, Suspension and loop objects in theories and cohomology, Georgian Math. J. 8 (2001) 697–712.
[10] H.-J. Baues, M. Jibladze, Classification of abelian track categories, K-Theory 25 (2002) 299–311.
[11] H.-J. Baues, M. Jibladze, The Steenrod algebra and theories associated to Hopf algebras, Appl. Categ. Structures 12 (2004) 109–126 (Homotopy theory).
[12] H.-J. Baues, M. Jibladze, Computation of the E_3-term of the Adams spectral sequence, arXiv:math. AT/0407045.
[13] H.-J. Baues, M. Jibladze, The algebra of secondary cohomology operations and its dual, Preprint MPIM2004-111, Max-Planck-Institut für Mathematik, Available online at http://www.mpim-bonn.mpg.de/preprints/send?bid = 2531.
[14] H.-J. Baues, M. Jibladze, T. Pirashvili, Strengthening track theories, arXiv:math.CT/0307185.
[15] H.-J. Baues, E.G. Minian, Track extensions of categories and cohomology, K-Theory 23 (2001) 1–13.
[16] H.-J. Baues, T. Pirashvili, Shukla cohomology and additive track theories, arXiv:math.KT/0401158.
[17] A.I. Bondal, M.M. Kapranov, Enhanced triangulated categories, Math. USSR-Sb. 70 (1991) 93–107.
[18] S. Eilenberg, S. MacLane, Acyclic models, Amer. J. Math. 75 (1953) 189–199.
[19] P. Gabriel, M. Zisman, Calculus of fractions and homotopy theory, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 35, Springer, New York, Inc., New York, 1967. MR0210125, 35 #1019.
[20] J.-P. Meyer, Acyclic models for multicomplexes, Duke Math. J. 45 (1978) 67–85.
[21] R.M.F. Moss, On the composition pairing of Adams spectral sequences, Proc. London Math. Soc. 18 (3) (1968) 179–192.
[22] C. Nassau, Ein neuer Algorithmus zur Untersuchung der Kohomologie der Steenrod-Algebra, Frankfurt am Main, 2001, Thesis, available online at http://www.nullhomotopie.de/papers/diss.ps.gz.
[23] A. Neeman, Triangulated categories, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.
[24] N.R. O'Brian, D. Toledo, Y.L.L. Tong, The trace map and characteristic classes for coherent sheaves, Amer. J. Math. 103 (1981) 225–252.
[25] T. Pirashvili, Models for the homotopy theory and cohomology of small categories, Soobshch. Akad. Nauk Gruzin. SSR 129 (1988) 261–264 (Russian).
[26] T. Pirashvili, H^3 and models for the homotopy theory, Trudy Tbiliss. Mat. Inst. Razmadze Akad. Nauk Gruzii 94 (1991) 73–85.
[27] T. Pirashvili, F. Waldhausen, Mac Lane homology and topological Hochschild homology, J. Pure Appl. Algebra 82 (1992) 81–98.
[28] B. Richter, A proof that Adams resolutions form a secondary complex, personal communication, February 2004.
[29] G.S. Rinehart, Singular homology as a derived functor, Trans. Amer. Math. Soc. 174 (1973) 243–256.
[30] M. Takeuchi, K.-H. Ulbrich, Complexes of categories with abelian group structure, J. Pure Appl. Algebra 27 (1983) 61–73.
[31] M. Tierney, W. Vogel, Simplicial resolutions and derived functors, Math. Z. 111 (1969) 1–14.