Homology of E_n ring spectra and iterated *THH*

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We describe an iterable construction of *THH* for an E_n ring spectrum. The reduced version is an iterable bar construction and its nth iterate gives a model for the shifted cotangent complex at the augmentation, representing reduced topological Quillen homology of an augmented E_n algebra.

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1 Introduction

Over the past two decades, topological Hochschild homology (*THH*) and its refinement topological cyclic homology (*TC*) have become standard tools in algebraic topology and algebraic K–theory. Waldhausen originally conjectured the theory *THH* and used an ad hoc version of it to split the algebraic K–theory of spaces into stable homotopy theory and stable pseudo-isotopy theory [26]. The theory *TC* provides the key tool in the proof of the K–theoretic Novikov conjecture by Bökstedt, Hsiang, and Madsen [4], and in the algebraic K–theory computations pioneered by Hesselholt and Madsen [12; 13; 14]. K–theory computations in *TC* also led to Ausoni and Rognes' mysterious chromatic red shift phenomenon [1]. Partly because of this, recently interest has grown in iterating *THH* and *TC* and in higher versions of *THH* and *TC* studied by Pirashvili [25] and by Brun, Carlsson, Douglas, and Dundas [5; 7].

Although *THH* makes sense for any ring or A_∞ ring spectrum A, the applications to K–theory computations typically take advantage of the multiplicative structure on $THH(A)$ and in [1] the extended power operation structure on $THH(A)$ that arises only when A has more structure, for example that of an E_∞ ring spectrum. Likewise, to iterate *THH*, $THH(A)$ must have at least an A_∞ multiplication. In [6], Brun, Fiedorowicz, and Vogt showed that when A is an E_n ring spectrum $THH(A)$ is an E_{n-1} ring spectrum; in this case *THH* can be iterated up to n times, and $THH(A)$ admits certain power operations. The construction in [6] requires replacing A by an equivalent ring spectrum over a different operad. As a consequence, to iterate the construction requires re-approximation at each stage.

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This paper describes a construction of *THH* of E_n ring spectra which is iterable without re-approximation. Working in one of the modern categories of spectra, such as the category of EKMM S–modules (see Elmendorf, Kriz, Mandell and May [9]) or the category of symmetric spectra of topological spaces (see Hovey, Shipley and Smith [15] and Mandell, May, Schwede and Shipley [21]), we study algebras over the little n–cubes operad $\mathcal{C}_n$ of Boardman and Vogt [3]. In fact, because the output of our *THH* construction is not quite a $\mathcal{C}_{n-1}$ algebra, we work with a mild generalization called a "partial $\mathcal{C}_n$–algebra", which we review in Section 2. A partial $\mathcal{C}_n$–algebra is a partial $\mathcal{C}_1$–algebra by neglect of structure; we construct a cyclic bar construction version of *THH* for partial $\mathcal{C}_1$–algebras and prove the following theorem.

Theorem 1.1 *For a partial $\mathcal{C}_n$–algebra A satisfying mild technical hypotheses, the cyclic bar construction $THH(A)$ is naturally a partial $\mathcal{C}_{n-1}$–algebra.*

The mild technical hypotheses amount to the partial algebra generalization of the usual hypothesis on algebras that the inclusion of the unit is a cofibration of the underlying S–modules (or symmetric or orthogonal spectra). We write out the hypotheses explicitly as Hypothesis 4.17 below and show in Proposition 4.18 that it is inherited on $THH(A)$, allowing iteration.

When working in the context of symmetric spectra, there are two different constructions of *THH* of an S–algebra: A cyclic bar construction as in the previous theorem and the original construction of Bökstedt. Because only Bökstedt's construction is known to be suitable for constructing *TC*, the previous theorem still leaves open the problem of directly constructing an iterable version of *TC* from an E_n ring structure; the authors plan to return to this question in a future paper. On the other hand, the cyclic bar construction does admit a relative variant for partial $\mathcal{C}_n$ R–algebras (see Definition 2.7), where we use a different base commutative S–algebra R in place of the sphere spectrum S.

Theorem 1.2 *For R a commutative S–algebra, and A a partial $\mathcal{C}_n$ R–algebra satisfying Hypothesis 4.17, $THH^R(A)$ is a partial $\mathcal{C}_{n-1}$ R–algebra.*

The relative construction also admits a reduced version for augmented algebras, which amounts to taking coefficients in R. This is the analogue of the bar construction of an augmented algebra, so we write $B^R(A)$ or BA rather than $THH^R(A;R)$. For a partial augmented $\mathcal{C}_n$ R–algebra A, BA is a partial augmented $\mathcal{C}_{n-1}$ R–algebra, and so we can iterate the bar construction up to n times. Up to a shift, the fiber of the augmentation $B^n A \to R$ is the R–module of $\mathcal{C}_n$ R–algebra derived indecomposables representing reduced topological Quillen homology (see Section 7 for a review of this theory).

Theorem 1.3 *For an augmented $\mathcal{C}_n$ R–algebra A, there is a natural isomorphism in the derived category of R–modules*

$$R \vee \Sigma^n Q^{\mathbf{L}}_{\mathcal{C}_n}(A) \cong B^n A,$$

where $Q^{\mathbf{L}}_{\mathcal{C}_n}$ denotes the $\mathcal{C}_n$ R–algebra cotangent complex at the augmentation (the derived $\mathcal{C}_n$ R–algebra indecomposables).

We regard the previous result as the main theorem in this paper: It allows an inductive approach to obstruction theory for connective E_n ring spectra. We explain this idea and apply it in a future paper to prove that BP is an E_4 ring spectrum. Other non-iterative versions of Theorem 1.3 can be found in the work of Francis [10] and Lurie [18]. An algebraic version can be found in the article by Fresse [11].

Terminology

Because partial $\mathcal{C}$–algebras (for various $\mathcal{C}$) play a fundamental role in the structure and results of the paper, we will sometimes use the terminology "true $\mathcal{C}$–algebra" (for $\mathcal{C}$–algebras in the usual sense) when necessary for emphasis, for contrast, or to avoid confusion with the terminology "partial $\mathcal{C}$–algebra".

Outline

Section 2 reviews the definition of a partial algebra over an operad and the Kriz–May rectification theorem, which gives an equivalence between the homotopy category of partial algebras and the homotopy category of true algebras over an operad. Sections 3 and 4 construct the cyclic nerve of a partial $\mathcal{C}_1$ R–algebra, breaking the construction into two steps. For a $\mathcal{C}_1$ R–algebra A, we construct in Section 3 a closely related partial associative R–algebra, called the "Moore" partial algebra, which has the same relationship to A as the Moore loop space has to the loop space. In Section 4, we construct the cyclic nerve of a partial associative R–algebra and study its multiplicative structure when applied to the Moore algebra of a $\mathcal{C}_n$ R–algebra, proving Theorems 1.1 and 1.2 above. We produce the iterated bar construction for augmented partial $\mathcal{C}_n$ R–algebras in Section 5 and for non-unital $\mathcal{C}_n$ R–algebras in Section 6. Section 7 proves Theorem 1.3. Finally, Section 8 proves two compatibility results for the E_{n-1}–structure on the bar construction: it shows that the usual diagonal map on the bar construction preserves the multiplication (Theorem 8.1) and that the bar construction has the expected behavior with respect to power operations (Theorem 8.2).

2 Partial operadic algebras

In this section, we give a brief review of the definition and basic homotopy theory of partial algebras over an operad. We review the Kriz–May rectification theorem, which shows that any partial algebra over an appropriate operad is naturally weakly equivalent to a true algebra over that same operad. This in particular shows how to recover an operadic algebra from a partial operadic algebra, gives an equivalence of the homotopy theory of partial operadic algebras and of true operadic algebras, and justifies the perspective of the statements of the main theorems in Section 1.

In this section and throughout this paper, we work in one of the modern categories of spectra of [21] or [9]. We write $\mathfrak{M}_S$ for any of these categories, and refer to an object in it as an "S–module"; we write $\wedge_S$ for the smash product, reserving $\wedge$ for the smash product with a based space. For a commutative S–algebra R, we have the category of R–modules $\mathfrak{M}_R$ that has a symmetric monoidal product $\wedge_R$. We avoid needless redundancy by typically working in $\mathfrak{M}_R$; the case of S–modules being precisely the special case $R = S$.

In our cases of interest, we work with the little n–cubes operads $\mathcal{C}_n$. With this in mind, we fix an operad $\mathcal{C}$ in spaces such that each $\mathcal{C}(m)$ is a free Σ_m–CW complex. Then in our context, a $\mathcal{C}$–algebra (or $\mathcal{C}$ R–algebra when R needs to be made explicit) consists of an R–module A and maps

$$\mathcal{C}(m)_+ \wedge_{\Sigma_m} (A \wedge_R \dots \wedge_R A) \longrightarrow A$$

satisfying certain associativity and unit properties. A partial $\mathcal{C}$–algebra replaces the smash powers with an equivalent system of R–modules.

Definition 2.1 An *op-lax power system* of R–modules consists of

(i) A sequence of R–modules $X_1, X_2, \dots,$

(ii) A Σ_m action on X_m, and

(iii) A $\Sigma_m \times \Sigma_n$–equivariant map $\lambda_{m,n}\colon X_{m+n} \to X_m \wedge_R X_n$ for each m, n

such that the following diagrams commute for all m, n, p, where $\tau_{m,n} \in \Sigma_{m+n}$ denotes the permutation that switches the first block of m past the last block of n.

$$\begin{tikzcd} X_{m+n} \arrow[r, "\lambda_{m,n}"] \arrow[d, "\tau_{m,n}"'] & X_m \wedge_R X_n \arrow[d, "\tau"] \\ X_{n+m} \arrow[r, "\lambda_{n,m}"'] & X_n \wedge_R X_m \end{tikzcd} \qquad \begin{tikzcd} X_{m+n+p} \arrow[r, "\lambda_{m,n+p}"] \arrow[d, "\lambda_{m+n,p}"'] & X_m \wedge_R X_{n+p} \arrow[d, "\lambda_{n,p}"] \\ X_{m+n} \wedge_R X_p \arrow[r, "\lambda_{m,n}"'] & X_m \wedge_R X_n \wedge_R X_p \end{tikzcd}$$

We write $\lambda_{m,n,p}$ for the common composite in the righthand diagram, and more generally, $\lambda_{m_1,\ldots,m_r}$ for the iterated composites

$$X_{m_1+\cdots+m_r} \longrightarrow X_{m_1} \wedge_R \ldots \wedge_R X_{m_r}.$$

We write $X_0 = R$ and take $\lambda_{0,n}$ and $\lambda_{n,0}$ to be the (inverse) unit isomorphisms. A map of op-lax power systems $A \to B$ consists of equivariant maps $X_n \to Y_n$, which make the evident diagrams in the structure maps $\lambda_{m,n}$ commute. An op-lax power system is a *partial power system* when the maps $\lambda_{m_1,\ldots,m_r}$ are weak equivalences for all $m_1, \ldots, m_r$.

Example 2.2 For an R–module X, we get a partial power system $X_m = X^{(m)}$ (mth smash power over R for some fixed association) with the usual symmetric group actions and the maps $\lambda_{m,n}$ the associativity isomorphism. We call such a partial power system a *true power system*.

This definition depends strongly on the underlying symmetric monoidal category $\mathfrak{M}_R$: An op-lax power system in R–modules does not have an underlying op-lax power system in S–modules. Definition 2.1 provides the appropriate framework for the cyclic bar construction of *THH*. (A version suitable for the Bökstedt construction of *THH* requires an "external" smash product formulation that is significantly more complex.)

We define a *weak equivalence* of partial power systems as a map $X \to Y$ that is a weak equivalence $X_1 \to Y_1$. This compensates for the awkward fact that the smash product of R–modules does not preserve all weak equivalences. To help alleviate this difficulty, we introduce the following additional terminology.

Definition 2.3 We say that a partial power system X is *tidy* when the canonical maps

$$X_1 \wedge^{\mathbf{L}}_R \ldots \wedge^{\mathbf{L}}_R X_1 \longrightarrow X_1 \wedge_R \ldots \wedge_R X_1$$

from the derived smash powers to the point-set smash powers of X_1 are isomorphisms in the derived category $\mathfrak{D}_R$.

We note that a partial power system X is tidy if and only if the natural maps

$$X_{j_1} \wedge^{\mathbf{L}}_R \ldots \wedge^{\mathbf{L}}_R X_{j_r} \longrightarrow X_{j_1} \wedge_R \ldots \wedge_R X_{j_r}$$

are all isomorphisms in $\mathfrak{D}_R$. To see this, consider the following commutative diagram in $\mathfrak{D}_R$, where $j = \sum j_i$.

$$\begin{tikzcd}
X_{j_1} \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R X_{j_r} \arrow[rr] \arrow[d, "\lambda_{1,\dots,1} \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R \lambda_{1,\dots,1}"', "\sim"] & & X_{j_1} \wedge_R \dots \wedge_R X_{j_r} \arrow[d, "\lambda_{1,\dots,1} \wedge_R \dots \wedge_R \lambda_{1,\dots,1}"] \\
& X_j \arrow[ur, "\lambda_{j_1,\dots j_r}", "\sim"'] \arrow[dr, "\lambda_{1,\dots,1}"', "\sim"] & \\
(X_1 \wedge_R \dots \wedge_R X_1) \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R (X_1 \wedge_R \dots \wedge_R X_1) \arrow[rr] & & X_1 \wedge_R \dots \wedge_R X_1
\end{tikzcd}$$

The maps labelled $\sim$ are isomorphisms in $\mathfrak{D}_R$ by the definition of partial power system, and so we see that the vertical maps are isomorphisms in $\mathfrak{D}_R$. It follows that the top horizontal map is an isomorphism in $\mathfrak{D}_R$ for all $j_1, \dots, j_r$ exactly when the bottom horizontal map is an isomorphism in $\mathfrak{D}_R$ for all $j_1, \dots, j_r$. Looking at the diagram

$$\begin{tikzcd}
(X_1 \wedge_R \dots \wedge_R X_1) \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R (X_1 \wedge_R \dots \wedge_R X_1) \arrow[r] & X_1 \wedge_R \dots \wedge_R X_1 \\
(X_1 \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R X_1) \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R (X_1 \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R X_1) \arrow[r, "\cong"'] \arrow[u] & X_1 \wedge^{\mathbf{L}}_R \dots \wedge^{\mathbf{L}}_R X_1 \arrow[u]
\end{tikzcd}$$

we see that this happens exactly when X is tidy.

Intuitively, tidy means that the mth partial power X_m is equivalent to the derived mth smash power of X_1. True power systems need not be tidy in general, but if X is cofibrant or if we are working in the category of symmetric spectra, orthogonal spectra, or EKMM S–modules and X is cofibrant in the category of $\mathcal{C}$–algebras (for $\mathcal{C}$ as above) or even cofibrant in the category of commutative R–algebras, then the true power system $X_m = X^m$ is tidy.

To define a partial $\mathcal{C}$–algebra structure on a partial power system A, we need slightly more than the sequence of maps

$$\mathcal{C}(m)_+ \wedge_{\Sigma_m} A_m \longrightarrow A_1,$$

that suffices for a true $\mathcal{C}$–algebra; rather, we need maps of the form

$$(\mathcal{C}(j_1) \times \dots \times \mathcal{C}(j_r))_+ \wedge_{\Sigma_{j_1} \times \dots \times \Sigma_{j_r}} A_m \longrightarrow A_r$$

for $j_1 + \dots + j_r = m$. The coherence is most easily expressed by generalizing the monadic description of operadic algebras.

Construction 2.4 For an op-lax power system X, let

$$(\mathbb{C}^\sharp X)_m = \bigvee_{j_1,\dots,j_m} (\mathcal{C}(j_1) \times \dots \times \mathcal{C}(j_m))_+ \wedge_{\Sigma_{j_1} \times \dots \times \Sigma_{j_m}} X_{j_1+\dots+j_m}.$$

This obtains a Σ_m–action by permuting the j_i's and performing the corresponding block permutation on $X_{j_1+\cdots+j_m}$. We make $\mathbb{C}^\sharp X$ an op-lax power system using the structure maps λ of X and the associativity isomorphism

$$\bigvee_{j_1,\dots,j_{m+n}} (\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_{m+n}))_+ \wedge_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_{m+n}}} (X_{j_1+\cdots+j_m}\wedge_R X_{j_{m+1}+\cdots+j_{m+n}})$$
$$\cong (\mathbb{C}^\sharp X)_m \wedge_R (\mathbb{C}^\sharp X)_n.$$

We have a natural map of op-lax power systems $X\to\mathbb{C}^\sharp X$ induced by the inclusion of the identity element in $\mathcal{C}(1)$. We have a natural transformation of op-lax power systems

$$\mathbb{C}^\sharp\mathbb{C}^\sharp X \longrightarrow \mathbb{C}^\sharp X$$

induced by the operadic multiplication on $\mathcal{C}$. An easy check of diagrams then proves the following proposition.

Proposition 2.5 *The structure above makes $\mathbb{C}^\sharp$ a monad in the category of op-lax power systems.*

Because we have assumed that each $\mathcal{C}(m)$ is a free Σ_m–CW complex, the smash products

$$(\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_m))_+ \wedge_{\Sigma_{j_1}\times\cdots\times\Sigma_{j_m}} (-)$$

preserve weak equivalences. In particular, we see that when X is a partial power system, so is $\mathbb{C}^\sharp X$. Thus, we obtain the following proposition.

Proposition 2.6 *Under the assumptions on $\mathcal{C}$ above, $\mathbb{C}^\sharp$ is a monad in the category of partial power systems.*

Note that for a true power system $X_m = X^{(m)}$, we have

$$(\mathbb{C}^\sharp X)_m \cong (\mathbb{C}^\sharp X)_1^{(m)} = (\mathbb{C}X)^{(m)},$$

where $\mathbb{C}$ is the usual monad in R–modules associated to the operad $\mathcal{C}$. The monad $\mathbb{C}^\sharp$ therefore generalizes to partial power systems the monad of R–modules associated to the operad $\mathcal{C}$. We now have the following monadic definition of a partial $\mathcal{C}$–algebra, generalizing the monadic definition of a true $\mathcal{C}$–algebra.

Definition 2.7 A *partial $\mathcal{C}$–algebra* (or *partial $\mathcal{C}$ R–algebra* when R needs to be made explicit) is an algebra over the monad $\mathbb{C}^\sharp$ in partial power systems.

In particular, a true $\mathcal{C}$ R–algebra structure on A is precisely a partial $\mathcal{C}$ R–algebra structure on the true power system $A_m = A^{(m)}$. We could define a more general notion of op-lax $\mathcal{C}$–algebra in terms of the op-lax power systems; we learned from Leinster that such an algebra is precisely a strictly unital op-lax symmetric monoidal functor from an appropriate category of operators associated to $\mathcal{C}$. A formulation of the theory of partial $\mathcal{C}$–algebras along these lines as well as generalizations and further examples may be found in Leinster's preprint [17].

The remainder of the section discusses and reviews the proof of the following theorem, the Kriz–May rectification theorem. In it, we write T for the functor from R–modules to partial power systems that takes an R–module X to the true power system $(TX)_m = X^{(m)}$.

Theorem 2.8 (May [24], Kriz–May [16]) *For an operad $\mathcal{C}$ as above, there exists a functor R from partial $\mathcal{C}$–algebras to true $\mathcal{C}$–algebras, an endofunctor $E^\sharp$ on partial $\mathcal{C}$–algebras, and natural weak equivalences of partial $\mathcal{C}$–algebras*

$$\mathrm{Id} \longleftarrow E^\sharp \longrightarrow TR.$$

Moreover, there exists an endofunctor E on $\mathcal{C}$–algebras and natural weak equivalences of $\mathcal{C}$–algebras

$$\mathrm{Id} \longleftarrow E \longrightarrow RT.$$

The functors and natural transformations are natural also in the operad $\mathcal{C}$.

As a consequence, the categories of partial $\mathcal{C}$–algebras and of true $\mathcal{C}$–algebras become equivalent after formally inverting the weak equivalences.

The proof of the theorem is an application of the two-sided monadic bar construction. To avoid confusion, write UX for X_1 for a partial power system X. We note that for any partial power system X, the structure maps λ induce a natural map of partial power systems $X \to TUX$ (in fact, a weak equivalence), which together with the identity $UTY = Y$, identify U and T as adjoints. In this notation, the monad $\mathbb{C}$ in R–modules is $U\mathbb{C}^\sharp T$.

We let $RA = B(\mathbb{C}U, \mathbb{C}^\sharp, A)$ be the geometric realization of the simplicial R–module

$$B_\bullet(\mathbb{C}U, \mathbb{C}^\sharp, A) = \mathbb{C}U \underbrace{\mathbb{C}^\sharp \dots \mathbb{C}^\sharp}_{\bullet \text{ iterates}} A.$$

Here the face maps are induced by the $\mathbb{C}^\sharp$–action on A, the monadic multiplication on $\mathbb{C}^\sharp$, and the multiplication

$$\mathbb{C}U\mathbb{C}^\sharp X \longrightarrow \mathbb{C}\mathbb{C}UX \longrightarrow \mathbb{C}UX.$$

The degeneracy maps are induced by the unit of the monad $\mathbb{C}^\sharp$. Because the monad $\mathbb{C}$ commutes with geometric realization, RA is naturally a $\mathcal{C}$–algebra.

We let $E^\sharp A = B(\mathbb{C}^\sharp, \mathbb{C}^\sharp, A)$ be the geometric realization of the "standard resolution"

$$B_\bullet(\mathbb{C}^\sharp, \mathbb{C}^\sharp, A) = \mathbb{C}^\sharp \underbrace{\mathbb{C}^\sharp \dots \mathbb{C}^\sharp}_{\bullet \text{ iterates}} A$$

(where we understand the geometric realization of a partial power system to be performed objectwise). The augmentation $B_\bullet(\mathbb{C}^\sharp, \mathbb{C}^\sharp, A) \to A$ is a map of simplicial partial $\mathcal{C}$–algebras (where we regard A as a constant simplicial object) and a simplicial homotopy equivalence of partial power systems. Thus, the geometric realization is a natural map of partial $\mathcal{C}$–algebras and a weak equivalence (in fact, a homotopy equivalence of Σ_m–equivariant R–modules in each partial power).

The natural map $\mathbb{C}^\sharp X \to TU\mathbb{C}^\sharp TUX = T\mathbb{C}UX$ is a map of partial $\mathcal{C}$–algebras; it is a weak equivalence since it is induced by the maps $\lambda_{m_1,\dots,m_r}$. Moreover, the map is compatible with the left action of the monad $\mathbb{C}^\sharp$. Thus, we get a map of simplicial partial $\mathcal{C}$–algebras

$$B_\bullet(\mathbb{C}^\sharp, \mathbb{C}^\sharp, A) \longrightarrow B_\bullet(T\mathbb{C}U, \mathbb{C}^\sharp, A)$$

which is a weak equivalence in each simplicial degree. The geometric realization is a weak equivalence and induces the natural weak equivalence of partial $\mathcal{C}$–algebras $E^\sharp \to TR$ in the statement.

On the $\mathcal{C}$–algebra side, we take E to be the geometric realization of the standard resolution $B_\bullet(\mathbb{C}, \mathbb{C}, -)$. The construction and study of the natural transformations are analogous to the partial $\mathcal{C}$–algebra case described in detail above.

3 The Moore partial algebra of a partial $\mathcal{C}_1$–algebra

Moore constructed a variant of the loop space of a topological space where the concatenation of loops is strictly associative and unital and not just associative and unital up to homotopy. This construction easily extends to any $\mathcal{C}_1$ space, and in fact quite generally to $\mathcal{C}_1$–algebras in topological categories. In this section, we generalize Moore's construction to the partial context, constructing a partial associative R–algebra from a partial $\mathcal{C}_1$ R–algebra. In the next section, we use this to construct the cyclic bar construction of a partial $\mathcal{C}_1$ R–algebra.

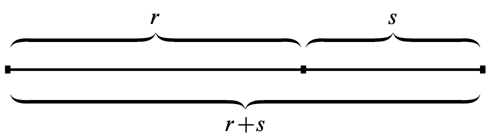
We begin by reviewing the construction of the Moore algebra of a (true) $\mathcal{C}_1$–algebra, before treating the slightly more complicated partial case. Recall that an element of $\mathcal{C}_1(m)$ consists of an ordered list of almost disjoint subintervals of the unit interval, not necessarily in their natural order. The operadic multiplication $a \circ_i b$ replaces the ith subinterval in a with a scaled version of the subintervals in b. The element

$$\gamma = ([0, 1/2], [1/2, 1]) \in \mathcal{C}_1(2)$$

represents the loop multiplication for the action of $\mathcal{C}_1$ on a loop space; the two composites

$$\gamma \circ_1 \gamma = ([0, 1/4], [1/4, 1/2], [1/2, 1]) \quad \text{and} \quad \gamma \circ_2 \gamma = ([0, 1/2], [1/2, 3/4], [3/4, 1])$$

in $\mathcal{C}_1(3)$ represent the two associations of the multiplication of three loops in a loop space. The basic idea of Moore's construction is to add a length parameter to build an associative multiplication from a $\mathcal{C}_1$–multiplication. Specifically, given lengths r and s, we get an element $\gamma_{r,s}$ in $\mathcal{C}_1(2)$ that models the concatenation of a loop of length r with a loop of length s, rescaled to the unit interval.



Namely, $\gamma_{r,s}$ is the element

$$\gamma_{r,s} = ([0, r/(r+s)], [r/(r+s), 1]) \in \mathcal{C}_1(2).$$

Writing $P = (0, \infty) \subset \mathbb{R}$ for the space of positive real numbers, the "concatenation formula"

$$\Gamma \colon P \times P \longrightarrow P \times \mathcal{C}_1(2)$$

sends (r, s) to $(r+s, \gamma_{r,s})$.

For a $\mathcal{C}_1$–algebra A, we get an associative multiplication on $P_+ \wedge A$ using Γ and the $\mathcal{C}_1(2)$–action,

$$\begin{aligned}(P_+ \wedge A) \wedge_R (P_+ \wedge A) \cong (P \times P)_+ \wedge (A \wedge_R A) \longrightarrow \\ (P \times \mathcal{C}_1(2))_+ \wedge (A \wedge_R A) \cong P_+ \wedge (\mathcal{C}_1(2)_+ \wedge (A \wedge_R A)) \longrightarrow P_+ \wedge A.\end{aligned}$$

To make this unital, let $\overline{P} = [0, \infty) \subset \mathbb{R}$ denote the non-negative real numbers, and define the R–module MA by the following pushout diagram.

$$\begin{tikzcd} P_+ \wedge R \arrow[r] \arrow[d] & P_+ \wedge A \arrow[d] \\ \overline{P}_+ \wedge R \arrow[r] & MA \end{tikzcd} \tag{3.1}$$

The multiplication above then extends to an associative multiplication on MA, and is now unital with the unit $R \to MA$ induced by the inclusion of $\{0\}_+ \wedge R$ into $\overline{P}_+ \wedge R$.

Proposition 3.2 *For a $\mathcal{C}_1$–algebra A, MA is naturally an associative R–algebra.*

To relate A and MA, note that forgetting the lengths (collapsing P and $\overline{P}$ to a point), we obtain a map $MA \to A$. This map is a homotopy equivalence of R–modules: The map $A \to MA$ induced by the inclusion of 1 in P provides the homotopy inverse. The composite map on A is the identity and compatible null homotopies on P and $\overline{P}$ induce a homotopy from the identity to the composite on MA. (See [20, Section 6] for a comparison of A and MA as $\mathcal{C}_1$–algebras.)

To extend this to the case of partial algebras, we need to construct an appropriate partial power system with the mth partial power analogous to the mth smash power of the pushout in (3.1). Let $\mathcal{T}$ be the diagram with objects a, b, c and arrows α, β as pictured

$$\begin{tikzcd} & c \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & \\ b & & a \end{tikzcd}$$

so that a pushout is a colimit indexed on $\mathcal{T}$.

Construction 3.3 Let A be a partial $\mathcal{C}_1$–algebra. We construct the op-lax power system MA as follows. Let MA_1 be the pushout

$$MA_1 = (P_+ \wedge A_1) \cup_{P_+ \wedge R} (\overline{P}_+ \wedge R),$$

where the map $R \to A_1$ is induced by the unique element of $\mathcal{C}_1(0)$. Inductively, having defined $MA_1, \dots, MA_{m-1}$, we define MA_m as a colimit over the following diagram D_m indexed on $\mathcal{T}^m$. At a spot indexed by $(x_1, \dots, x_m)$, we put a copy of

$$(P_{x_1} \times \cdots \times P_{x_m})_+ \wedge A_{\#a}$$

where $\#a = \#a(x_1, \dots, x_m)$ is the number of occurrences of a in the x_i's, and

$$P_{x_i} = \begin{cases} P & x_i = a \text{ or } x_i = c \\ \overline{P} & x_i = b. \end{cases}$$

The map

$$(x_1, \dots, x_{i-1}, c, x_{i+1}, \dots, x_m) \longrightarrow (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_m)$$

induced by α (when $y = a$) is induced by the map $A_{\#a-1} \to A_{\#a}$ induced by the element

$$(\mathrm{id}, \dots, \mathrm{id}, 1, \mathrm{id}, \dots, \mathrm{id}) \in \mathcal{C}_1(1)^{\#a(x_1,\dots,x_{i-1})} \times \mathcal{C}_1(0) \times \mathcal{C}_1(1)^{\#a(x_{i+1},\dots,x_n)},$$

where id is the identity element in $\mathcal{C}_1(1)$ and 1 is the unique element in $\mathcal{C}_1(0)$. The map induced by β (when $y = b$) is induced by the inclusion of P in $\overline{P}$ in the appropriate factor. We give MA_m the Σ_m–action induced by permuting the factors of $\mathcal{T}^m$, together with the appropriate re-arrangement of the P_{x_i} factors and the $\Sigma_{\#a}$–action on $A_{\#a}$ (corresponding to restricting a permutation in Σ_m to the rearrangement of the a's in the object of $\mathcal{T}^m$). The structure maps of A induce the structure maps $MA_{m+n} \to MA_m \wedge_R MA_n$ for MA.

Collapsing the P_{x_i} factors to a point defines a map of op-lax power systems $MA \to A$. An argument like the one above shows that each map $MA_m \to A_m$ is a Σ_m–equivariant homotopy equivalence of R–modules.

Proposition 3.4 *Each map $MA_m \to A_m$ is a Σ_m–equivariant homotopy equivalence of R–modules. In particular, MA is a partial power system and is tidy exactly when A is.*

A partial associative R–algebra structure is a partial $\mathcal{A}$–algebra structure for $\mathcal{A}$ the associative operad. We write $\mu_j \in \mathcal{A}(j)$ for the canonical element, representing the j–fold multiplication (if $j > 1$), the operadic identity (if $j = 1$), or the unit (if $j = 0$). To define a partial R–algebra structure on the partial power system X, we need to specify maps

$$(\mu_{j_1}, \dots, \mu_{j_m})\colon X_{j_1+\cdots+j_m} \longrightarrow X_m$$

for all $j_1, \dots, j_m \geq 0$, satisfying the associativity and identity diagrams implicit in Definition 2.7. These conditions become easier to verify by thinking of the sequence $j_1, \dots, j_m$ as specifying a map of totally ordered sets

$$\{1, \dots, j\} \longrightarrow \{1, \dots, m\}$$

where $j = j_1 + \cdots + j_m$: It specifies the unique weakly increasing map ϕ such that the cardinality of $\phi^{-1}(i)$ is j_i. The following well-known fact is a consequence of an easy check of the diagrams (see Mac Lane [19, Section VII.5]). In it, we write $\underline{m}$ for the totally ordered set $\{1, \dots, m\}$, and $\underline{\boldsymbol{\Delta}}$ for the category whose objects are the totally ordered sets $\underline{0}, \underline{1}, \dots$ and whose morphisms are the weakly increasing maps.

Proposition 3.5 *Let X be a partial power system in the category of R–modules. A partial R–algebra structure on X consists of a map $X_\phi\colon X_j \to X_m$ for each $\phi\colon \underline{j} \to \underline{m}$ in $\underline{\boldsymbol{\Delta}}$, making X a functor from $\underline{\boldsymbol{\Delta}}$ to R–modules, such that the diagram*

$$\begin{array}{ccc} X_{i+j} & \xrightarrow{\lambda_{j,k}} & X_i \wedge_R X_j \\ {\scriptstyle X_{\theta+\phi}}\downarrow & & \downarrow{\scriptstyle X_\theta \wedge_R X_\phi} \\ X_{m+n} & \xrightarrow[\lambda_{m,n}]{} & X_m \wedge_R X_n \end{array}$$

commutes for all $\theta\colon \underline{i} \to \underline{m}$, $\phi\colon \underline{j} \to \underline{n}$.

More intrinsically, the previous proposition states that a partial associative R–algebra is a strictly unital op-lax monoidal functor (X,λ) from $\underline{\mathbf{\Delta}}$ to R–modules together with a Σ_m–action on $X_m = X_{\underline{m}}$ making X into a partial power system; see also Leinster [17, 1.6(a), 2.2.1].

To construct the partial associative R–algebra structure on MA, first we must generalize the elements $\gamma_{r,s}$ above. For $\vec{x} = (x_1,\dots,x_m)$ in $\mathcal{T}^m$ and $\vec{r} = (r_1,\dots,r_m)$ in $P_{x_1}\times\cdots\times P_{x_m}$, let $\gamma^{\vec{x}}_{\vec{r}}$ be the following element of $\mathcal{C}_1(\#a)$, where $\#a = \#a(x_1,\dots,x_n)$, the number of a's among the x_i. If $\#a = 0$, then $\gamma^{\vec{x}}_{\vec{r}}$ is the unique element of $\mathcal{C}_1(0)$. Otherwise, define $k_1 < k_2 < \cdots < k_{\#a}$ by $x_{k_i} = a$, and note that $r_{k_i} > 0$ for all i. Let $\gamma^{\vec{x}}_{\vec{r}}$ consist of the subintervals

$$\left[\frac{r_1+\cdots+r_{k_i-1}}{r_1+\cdots+r_m}, \frac{r_1+\cdots+r_{k_i}}{r_1+\cdots+r_m}\right]$$

(in their natural order) for $i = 1,\dots,\#a$. For example, if $m = \#a$ (i.e., all the $x_i's$ are a's), then $\gamma^{\vec{x}}_{\vec{r}}$ is the element the of $\mathcal{C}_1(m)$ that subdivides the unit interval into m subintervals of lengths the given proportions $r_1,\dots,r_m$; when one of the x_i's is c or b with $r_i > 0$, then $\gamma^{\vec{x}}_{\vec{r}}$ contains a gap proportional to r_i where the subinterval would have been if that x_i were a. An x_i that is b with $r_i = 0$, has neither a subinterval nor a gap corresponding to it; in the multiplication below, it will behave like a unit factor in an mth smash power.

Next, for fixed $\phi\colon \underline{j} \to \underline{m}$ in $\underline{\mathbf{\Delta}}$, let F_ϕ be the functor from $\mathcal{T}^j$ to $\mathcal{T}^m$ that sends $(x_1,\dots,x_j)$ to $(y_1,\dots,y_m)$ where

$$y_i = \begin{cases} a & \text{if at least one } k \in \phi^{-1}(i) \text{ satisfies } x_k = a \\ b & \text{if every } k \in \phi^{-1}(i) \text{ satisfies } x_k = b \text{ (or } \phi^{-1}(i) \text{ is empty)} \\ c & \text{if at least one } k \in \phi^{-1}(i) \text{ satisfies } x_k = c \text{ and none satisfy } x_k = a \end{cases}$$

and does the only possible thing on maps.

We defined MA_m as the colimit of a diagram D_m indexed on $\mathcal{T}^m$; we now define $MA_\phi\colon MA_j \to MA_m$ to be the map on colimits induced by a natural transformation $D_\phi\colon D_j \to F^*_\phi D_m$ as follows. For $(x_1,\dots,x_j)$ in $\mathcal{T}^j$, define J_i and j_i by $\phi^{-1}(i) = \{J_i+1,\dots,J_i+j_i\}$. (In other words, let $j_i = |\phi^{-1}(i)|$ and $J_i = j_1+\cdots+j_{i-1}$.) write $\vec{x}_{\phi^{-1}(i)}$ for $(x_{J_i+1},\dots,x_{J_i+j_i})$, and for $(r_1,\dots,r_j) \in P_{x_1}\times\cdots\times P_{x_j}$, write $\vec{r}_{\phi^{-1}(i)}$ for $(r_{J_i+1},\dots,r_{J_i+j_i})$. Then let D_ϕ be the natural transformation

$$(P_{x_1}\times\cdots\times P_{x_j})_+ \wedge A_{\#a(\vec{x})} \longrightarrow (P_{y_1}\times\cdots\times P_{y_m})_+ \wedge A_{\#a(\vec{y})}$$

(for $(y_1,\dots,y_m)=F_\phi(x_1,\dots,x_j)$ as above), sending $(r_1,\dots,r_j)\in P_{x_1}\times\cdots\times P_{x_j}$ to

$$\Big(\sum\vec{r}_{\phi^{-1}(1)},\dots,\sum\vec{r}_{\phi^{-1}(m)}\Big)\in P_{y_1}\times\cdots\times P_{y_m}$$

(where we understand the sum to be zero when $\phi^{-1}(i)$ is empty), and sending $A_{\#a(\vec{x})}\to A_{\#a(\vec{y})}$ by the map induced by

$$\left(\gamma^{\vec{x}_{\phi^{-1}(k_1)}}_{\vec{r}_{\phi^{-1}(k_1)}},\dots\gamma^{\vec{x}_{\phi^{-1}(k_\ell)}}_{\vec{r}_{\phi^{-1}(k_\ell)}}\right)\in\mathcal{C}_1(\#a(\vec{x}_{\phi^{-1}(k_1)}))\times\cdots\times\mathcal{C}_1(\#a(\vec{x}_{\phi^{-1}(k_\ell)})).$$

where $k_1,\dots,k_\ell$ are the position of the a's in $\vec{y}=F_\phi(\vec{x})$ and $\ell=\#a(\vec{y})$.

The formula in the P_{x_k} factors lands in the appropriate P_{y_i} since $\sum\vec{r}_{\phi^{-1}(i)}$ can only be 0 when every object in the list $\vec{x}_{\phi^{-1}(i)}$ is b.

To see that D_ϕ are natural in $\mathcal{T}^j$, it suffices to check the maps α and β separately. For the maps β, the formulas in the P_{x_k} factors are clearly natural in $\mathcal{T}^n$. For the maps α, we note that α is induced by $1\in\mathcal{C}_1(0)$ and the composition $\gamma^{\vec{x}}_{\vec{r}}\circ_i 1$ drops the ith subinterval.

This completes the construction of the natural transformation D_ϕ and of the map MA_ϕ. A straightforward check of the diagrams now proves the following theorem.

Theorem 3.6 *The maps MA_ϕ above make MA into a partial associative R–algebra.*

4 *THH* of a partial $\mathcal{C}_n$–algebra

In this section, we study the multiplicative structure on $THH^R(A)$ for a partial $\mathcal{C}_n$–algebra A. Starting with the cyclic bar construction of a partial associative R–algebra, we define $THH^R(A)$ as the cyclic bar construction of the Moore partial algebra MA. This depends only on the underlying $\mathcal{C}_1$–structure; we use "interchange" (see Definition 4.4) of the $\mathcal{C}_1$–structure with a $\mathcal{C}_{n-1}$–structure to make $THH^R_\bullet(MA)$ into a partial $\mathcal{C}_{n-1}$–algebra, proving Theorems 1.1 and 1.2 from the introduction. Finally, in Construction 4.16 below, we give a closed description of iterated THH^R.

We begin by reviewing the cyclic bar construction of a partial associative R–algebra.

Construction 4.1 (The cyclic bar construction) For a partial associative R–algebra A, let $THH^R(A)$ be the geometric realization of the cyclic R–module

$$THH^R_p(A)=A_{p+1}$$

with action of the cyclic group C_{p+1} inherited from the action of the symmetric group Σ_{p+1}, with face maps d_i induced by

$$(\mathrm{id}, \dots, \mathrm{id}, \mu, \mathrm{id}, \dots, \mathrm{id}) \in \mathcal{A}(1)^i \times \mathcal{A}(2) \times \mathcal{A}(1)^{p-i-1}$$

for $i = 0, \dots, p-1$ (with d_p induced by the cyclic permutation followed by d_0), and degeneracy maps s_i induced by

$$(\mathrm{id}, \dots, \mathrm{id}, 1, \mathrm{id}, \dots, \mathrm{id}) \in \mathcal{A}(1)^i \times \mathcal{A}(0) \times \mathcal{A}(1)^{p-i}.$$

In the case of a partial $\mathcal{C}_1$ R–algebra, we apply this construction to the Moore partial algebra.

Definition 4.2 For a partial $\mathcal{C}_1$ R–algebra A, we define $\mathit{THH}^R(A) = \mathit{THH}^R(MA)$.

We remark that this point-set construction does not generally represent the correct homotopy type without additional assumptions on A. When A is tidy (Definition 2.3), then MA is tidy, and the simplicial object $\mathit{THH}^R_\bullet(A)$ has the correct homotopy type. For the geometric realization to have the correct homotopy type, it suffices for the simplicial object to be "proper" (for the inclusion of the union of the degeneracies at each stage to be a Hurewicz cofibration). The following proposition usually suffices for most purposes. We provide an iterable generalization in Proposition 4.18 at the end of this section.

Proposition 4.3 *Working in the context of R–modules of symmetric spectra, orthogonal spectra, or EKMM S–modules, assume that A is a true $\mathcal{C}_1$ R–algebra such that the map $R \to A$ is a cofibration of the underlying R–modules. Then MA is tidy and $\mathit{THH}^R_\bullet(MA)$ is a proper simplicial object.*

Proof Under the hypotheses above, each map $A^{m-1} \to A^m$ induced by $\mathcal{C}_1(0)$ is a cofibration of the underlying R–modules, and in the case of symmetric spectra or orthogonal spectra, it follows that (after passing to a retraction if necessary), each degeneracy map from $\mathit{THH}^R_{p-1}(MA)$ is the inclusion of a subcomplex in a fixed relative $\mathcal{I}'$–cell complex structure (see Mandell, May, Schwede and Shipley [21, Section 5.4]) on $\mathit{THH}^R_p(MA)$ (relative to the inclusion of R), where $\mathcal{I}' = \mathcal{I}$ is the set of generating cofibrations in the model structure [21, Section 6.2]. In the context of EKMM S–modules, the same holds, but for $\mathcal{I}'$ the set of generating cofibrations together with the maps $R \wedge S^{j-1}_+ \to R \wedge B^{j-1}_+$ (for $j \geq 0$). It follows that the union (colimit) of these subcomplexes is a subcomplex and its inclusion is a Hurewicz cofibration. □

This completes the generalization of the cyclic bar construction to a partial $\mathcal{C}_1$–algebra A. The remainder of the section studies the multiplicative structure when A is a $\mathcal{C}_n$–algebra. In this case, we understand A to be a $\mathcal{C}_1$–algebra via the *first-coordinate embedding* of $\mathcal{C}_1$ in $\mathcal{C}_n$: This embedding takes the sequence of subintervals

$$([x^1, y^1], \dots, [x^m, y^m]) \in \mathcal{C}_1(m)$$

to the sequence of subrectangles

$$([x^1, y^1] \times [0,1]^{n-1}, \dots, [x^m, y^m] \times [0,1]^{n-1}) \in \mathcal{C}_n(m).$$

We also have a *last coordinates embedding* of $\mathcal{C}_{n-1}$ in $\mathcal{C}_n$, taking the sequence of subrectangles

$$([x_1^1, y_1^1] \times \dots \times [x_{n-1}^1, y_{n-1}^1], \dots, [x_1^m, y_1^m] \times \dots \times [x_{n-1}^m, y_{n-1}^m]) \in \mathcal{C}_{n-1}(m)$$

to the sequence of subrectangles

$$([0,1] \times [x_1^1, y_1^1] \times \dots \times [x_{n-1}^1, y_{n-1}^1], \dots, [0,1] \times [x_1^m, y_1^m] \times \dots \times [x_{n-1}^m, y_{n-1}^m]) \in \mathcal{C}_n(m).$$

Both of these embeddings are special cases of the following "interchange" map.

Definition 4.4 For $\ell, m \geq 0$, the *interchange map* is the map

$$\rho \colon \mathcal{C}_1(\ell) \times \mathcal{C}_{n-1}(m) \longrightarrow \mathcal{C}_n(\ell m)$$

that takes the pair

$$([a^i, b^i] \mid 1 \leq i \leq \ell), ([x_1^j, y_1^j] \times \dots \times [x_{n-1}^j, y_{n-1}^j] \mid 1 \leq j \leq m)$$

to the sequence of subrectangles of $[0,1]^n$,

$$[a^i, b^i] \times [x_1^j, y_1^j] \times \dots \times [x_{n-1}^j, y_{n-1}^j],$$

(for $1 \leq i \leq \ell$, $1 \leq j \leq m$), ordered by lexicographical order in (i, j).

The first coordinate embedding is then

$$\rho_{\mathrm{first}}(-) = \rho(-, ([0,1]^{n-1})) \colon \mathcal{C}_1 \longrightarrow \mathcal{C}_n$$

and the last coordinate embedding is

$$\rho_{\mathrm{last}}(-) = \rho(([0,1]), -) \colon \mathcal{C}_{n-1} \longrightarrow \mathcal{C}_n.$$

Using ρ_{last}, for any element c in $\mathcal{C}_{n-1}(m)$, we get an element $\rho_{\mathrm{last}}(c)$ in $\mathcal{C}_n(m)$ and hence a map $A_m \to A$.

We call ρ the interchange map because for a (true) $\mathcal{C}_n$–algebra A, for any a in $\mathcal{C}_1(\ell)$ and c in $\mathcal{C}_{n-1}(m)$, both composites in the diagram

(4.5)
$$\begin{tikzcd} (A^{(m)})^{(\ell)} \arrow[r, "(\rho_{\mathrm{last}}(c))^{(\ell)}"] \arrow[d, "a"'] & A^{(\ell)} \arrow[d, "a"] \\ A^{(m)} \arrow[r, "\rho_{\mathrm{last}}(c)"'] & A \end{tikzcd}$$

are the induced map of $\rho(a,c)\colon A^{(\ell m)} \to A$ under the isomorphism $A^{(\ell m)} \cong (A^{(m)})^{(\ell)}$ induced by lexicographical order. In the diagram, the left vertical arrow is the action of a in the "diagonal" $\mathcal{C}_n$–algebra structure on $A^{(m)}$: Its has the structure map

(4.6) $$\mathcal{C}_n(\ell)_+ \wedge (A^{(m)})^{(\ell)} \longrightarrow (\mathcal{C}_n(\ell)_+)^{(m)} \wedge (A^{(m)})^{(\ell)} \cong (\mathcal{C}_n(\ell)_+ \wedge A^{(\ell)})^{(m)} \longrightarrow A^{(m)}$$

where the first map is induced by the diagonal map on $\mathcal{C}_n(\ell)$, the last map is the action map for A on each of the m–factors, and the isomorphism in the middle is the permutation $\sigma_{m,\ell}$ that rearranges the ℓ blocks of m factors of A into m blocks of ℓ factors. The following proposition is an immediate consequence of the commutative diagram.

Proposition 4.7 *Let A be a true $\mathcal{C}_n$–algebra. For any c in $\mathcal{C}_{n-1}(m)$, the map $A^{(m)} \to A$ induced by $\rho_{\mathrm{last}}(c)$ is a map of $\mathcal{C}_1$–algebras.*

The only obstacle to extending the previous proposition to partial $\mathcal{C}_n$–algebras is understanding A_m as a partial $\mathcal{C}_n$–algebra, and the only obstacle here is understanding A_m as a partial power system. We overcome these obstacles in the following definition.

Definition 4.8 For X an op-lax power system and $m > 0$, let $X^{[m]}$ be the op-lax power system with $X^{[m]}_p = X_{pm}$, symmetric group action induced by block permutation (with blocks of size m), and structure maps

$$\lambda_{p,q}\colon X^{[m]}_{p+q} \longrightarrow X^{[m]}_p \wedge_R X^{[m]}_q$$

induced from the structure map $\lambda_{pm,qm}$ for X. Then $X^{[1]} = X$, and we let $X^{[0]} = R$.

We note that when X is a partial power system, $X^{[m]}$ is as well. For a partial $\mathcal{C}_n$–algebra A, the partial power system $A^{[m]}$ obtains a "diagonal" partial $\mathcal{C}_n$–algebra structure from the $\mathcal{C}_n$–action on A: The element

$$(c_1, \ldots, c_j) \in \mathcal{C}_n(\ell_1) \times \cdots \times \mathcal{C}_n(\ell_j)$$

induces the map $A^{[m]}_\ell \to A^{[m]}_j$ (for $\ell = \ell_1 + \cdots + \ell_j$) given by

$$\sigma_{m,j}^{-1} \circ (c_1, \dots, c_j, \dots, c_1, \dots, c_j) \circ \sigma_{m,\ell} \colon A_{\ell m} \longrightarrow A_{jm} \tag{4.9}$$

(with $c_1, \dots, c_j$ repeated m times) where $\sigma_{m,k}$ denotes the permutation in Σ_{km} that rearranges the k blocks of m into m blocks of k, as in (4.6). Given an element c in $\mathcal{C}_n(m)$, we can also make c into a map of partial power systems $A^{[m]} \to A$, by defining the map c_ℓ on the ℓth partial power level to be $(c, \dots, c)$, i.e., c repeated ℓ times, without permutations. Then regarding

$$(a_1, \dots, a_j) \in \mathcal{C}_1(\ell_1) \times \cdots \times \mathcal{C}_1(\ell_j)$$

as a partial $\mathcal{C}_1$–algebra structure map and using the map of partial power systems $\rho_{\mathrm{last}}(c)$ for $c \in \mathcal{C}_{n-1}(m)$, the following interchange diagram commutes.

$$\begin{tikzcd}
A^{[m]}_\ell \arrow[r, "\rho_{\mathrm{last}}(c)_\ell"] \arrow[d, "(a_1,\dots,a_j)"'] & A_\ell \arrow[d, "(a_1,\dots,a_j)"] \\
A^{[m]}_j \arrow[r, "\rho_{\mathrm{last}}(c)_j"'] & A_j
\end{tikzcd}$$

This is the partial analogue of (4.5) and proves the following partial analogue of Proposition 4.7.

Proposition 4.10 *Let A be a partial $\mathcal{C}_n$–algebra. For any c in $\mathcal{C}_{n-1}(m)$, the map $A^{[m]} \to A$ induced by $\rho_{\mathrm{last}}(c)$ is a map of $\mathcal{C}_1$–algebras.*

Returning to the cyclic bar construction, we now have the terminology and notation to describe the multiplicative structure and prove Theorems 1.1 and 1.2. Even in the case when we start with a true $\mathcal{C}_n$–algebra A, $THH^R(A)$ will only have a partial multiplicative structure. We construct the op-lax power system as follows.

Construction 4.11 For a partial $\mathcal{C}_1$ R–algebra A, we define the op-lax power system $THH^R(A)$ by $(THH^R(A))_m = THH^R(M(A^{[m]}))$.

When each of the simplicial objects $THH^R_\bullet(M(A^{[m]}))$ is proper, this defines a partial power system, since geometric realization then preserves the degreewise weak equivalences. With just this mild hypotheses, we can now prove the following version of Theorem 1.2.

Theorem 4.12 *For R a commutative S–algebra, let A be a partial $\mathcal{C}_n$ R–algebra such that the op-lax power system $THH^R(A)$ is a partial power system. Then $THH^R(A)$ is a partial $\mathcal{C}_{n-1}$ R–algebra, naturally in maps of $\mathcal{C}_n$ R–algebra maps in A.*

Proof Applying Proposition 4.10, we see that every element c in $\mathcal{C}_{n-1}(m)$ induces a map of partial associative R–algebras

$$M(A^{[m]}) \longrightarrow MA.$$

More generally, the argument for Propositions 4.7 and 4.10 implies that elements $c_1, \dots, c_r$ of $\mathcal{C}_{n-1}(j_i)$ induce a map of partial $\mathcal{C}_1$–algebras

$$A^{[j_1+\dots+j_r]} \longrightarrow A^{[r]}$$

and hence a map of partial associative R–algebras

$$M(A^{[j_1+\dots+j_r]}) \longrightarrow M(A^{[r]}). \tag{4.13}$$

Restricting to the p power and putting these maps together for all elements of $\mathcal{C}_{n-1}$ at once, we get maps of R–modules

$$(\mathcal{C}_{n-1}(j_1) \times \dots \times \mathcal{C}_{n-1}(j_r))_+ \wedge_{\Sigma_{j_1} \times \dots \times \Sigma_{j_r}} M(A^{[j_1+\dots+j_r]})_p \longrightarrow M(A^{[r]})_p. \tag{4.14}$$

Because the maps (4.13) are maps of partial power systems, the maps (4.14) commute with the Σ_p–action. Because the maps (4.13) are maps of partial associative R–algebras, the maps (4.14) commute with the simplicial face maps when we view $M(A^{[m]})_p$ as $(\mathit{THH}^R_{p-1}(A))_m$ (simplicial degree $p-1$ in the mth power). Likewise, the maps (4.14) commute with the degeneracy operations. Commuting the smash product and geometric realization, we therefore get a map of partial power systems

$$\mathbb{C}^\sharp_{n-1}\mathit{THH}^R(A) \longrightarrow \mathit{THH}^R(A). \tag{4.15}$$

Using the associativity and unity of the $\mathcal{C}_n$–action on A and the fact that ρ_{last} is a homomorphism of operads, a straightforward check shows that (4.15) defines a partial $\mathcal{C}_{n-1}$–algebra structure on $\mathit{THH}^R(A)$. □

As a special case, when $n=2$, $\mathit{THH}^R(A)$ is a $\mathcal{C}_1$–algebra. Iterating THH^R would mean applying THH^R to the Moore partial algebra $M(\mathit{THH}^R(A))$. This partial associative R–algebra admits a closed description in terms of the colimit diagrams that construct the Moore partial algebra. The simplification in terms of the "main" submodules $P^m_+ \wedge A_m \subset MA_m$ captures the main ideas while avoiding the complications. Since geometric realization commutes with the Moore partial algebra construction, it is enough to describe the construction on each simplicial level. In these terms, the main submodule of $M(\mathit{THH}^R_p(A))_m$ (simplicial degree p, partial power m) is

$$P^{(p+1)+m}_+ \wedge A_{(p+1)m}.$$

Instead of having a length for each A power, this rather has a length for each column and each row, where we think of $A_{(p+1)m}$ as organized into m rows of $p+1$ columns. The main submodule of $THH^R(THH^R(A))_1$ in bisimplicial degree p,q then is

$$P_+^{(p+1)+(q+1)} \wedge A_{(p+1)(q+1)}.$$

The face maps in the p–direction add the column lengths and concatenate squares (in $\mathcal{C}_2$) in the horizontal direction, while the face maps in the q–direction add the row lengths and concatenate squares in the vertical direction.

More generally and precisely, for a partial $\mathcal{C}_n$–algebra, we give the following closed construction of the nth iterate of THH^R

Construction 4.16 Let A be a partial $\mathcal{C}_n$–algebra. The nth iterate of THH^R is isomorphic to the geometric realization of the following n–fold simplicial set $T^n(A)$. In multisimplicial degree $p_1,\dots,p_n$, $T^n(A)$ is the colimit of the diagram indexed on $\mathcal{T}^{(p_1+1)+\cdots+(p_n+1)}$ that at the object

$$\vec{x} = (x_1,\dots,x_q) = (x_0^1,\dots,x_{p_1}^1,x_0^2,\dots,x_{p_2}^2,\dots,x_0^n,\dots,x_{p_n}^n)$$

is the R–module

$$(P_{x_1}\times\cdots\times P_{x_q})_+ \wedge A_{m_1\dots m_n},$$

where $m_i = \#a(x_0^i,\dots,x_{p_i}^i)$ (the number of a's among the x_j^i's). The degeneracies are induced by inserting b in the appropriate spot in $\vec{x}$ with zero as the associated length in $P_b=\overline{P}$. In the ith simplicial direction, the jth face map (for $j<p_i$) is induced as follows. The lengths $(r,s)\in P_{x_j^i}\times P_{x_{j+1}^i}$ add. If $x_j^i = x_{j+1}^i = a$, we use the element

$$c = [0,1]^{i-1}\times\gamma_{r,s}\times[0,1]^{n-i}$$

of $\mathcal{C}_n(2)$, applied $m_1\dots\hat{m}_i\dots m_n$ times, with the appropriate permutations (as in (4.9)) to produce the map

$$A_{m_1\dots m_n} \longrightarrow A_{m_1\dots(m_i-1)\dots m_n}.$$

If one of x_j^i, x_{j+1}^i is a, we use the appropriate element

$$\begin{aligned} &([0,1]^{i-1}\times[0,r/(r+s)]\times[0,1]^{n-i}),\\ \text{or}\qquad &([0,1]^{i-1}\times[r/(r+s),1]\times[0,1]^{n-i}) \end{aligned}$$

of $\mathcal{C}_n(1)$. We use the identity on the A's factor if neither of x_j^i, x_{j+1}^i is a. The p_ith face map is the appropriate permutation followed by the zeroth face map.

Using the concrete construction above, we can see that the multisimplicial object $T^n_\bullet(A)$ has the correct homotopy type when A is tidy, and hence in this case a thickened realization will capture the correct homotopy type for iterated THH^R. Moreover, we can see that $T^n_\bullet(A)$ is proper under reasonable hypotheses on A like the one discussed above or its iterable generalization, which we now discuss.

For the iterable generalization of Proposition 4.3, we use the following technical hypothesis. In it, $\mathcal{I}'$ is as in the proof of Proposition 4.3: In the context of symmetric spectra or orthogonal spectra, $\mathcal{I}'$ is the collection of generating cofibrations in the model structure; in the context of EKMM S–modules, $\mathcal{I}'$ also includes the maps $R \wedge S^{j-1}_+ \to R \wedge B^j_+$.

Hypothesis 4.17 (Technical hypothesis on a partial $\mathcal{C}_1$-algebra A) For all m, the maps $R \to A_m$ are relative $\mathcal{I}'$–cell complexes (for $\mathcal{I}'$ as above) such that each of the m maps $A_{m-1} \to A_m$ (induced by the action of $\mathcal{C}_1(0)$) is the inclusion of a relative subcomplex.

Proposition 4.18 *Working in the category of R–modules of symmetric spectra, orthogonal spectra, or EKMM S–modules, let A be a partial $\mathcal{C}_n$ R–algebra satisfying Hypothesis 4.17. Then MA is tidy when A is tidy, $THH^R_\bullet(M(A^{[m]}))$ is a proper simplicial object for all m, and the op-lax power system $THH^R(A)$ is a partial power system. Moreover, if $n \geq 2$, then the partial $\mathcal{C}_{n-1}$ R–algebra $THH^R(A)$ also satisfies Hypothesis 4.17.*

Proof Since $\mathcal{C}_n(0)$ is a point, the maps $A_{m-1} \to A_m$ induced by $\mathcal{C}_1(0)$ (using the first coordinate embedding) coincide with the maps $A_{m-1} \to A_m$ induced by $\mathcal{C}_{n-1}(0)$ (using the last coordinates embedding). The proof is now a straightforward cell argument in terms of the construction of MA and THH^R. □

We now have Theorem 1.2 as an immediate corollary of the previous proposition and Theorem 4.12. Theorem 1.1 is the special case $R = S$.

5 The bar construction for augmented $\mathcal{C}_n$–algebras

In this section we study the reduced version of THH, usually called the bar construction, defined for an augmented partial $\mathcal{C}_n$–algebra. We begin with the definition of augmented partial $\mathcal{C}_n$–algebras and augmented partial associative R–algebras.

Definition 5.1 An *augmented* partial $\mathcal{C}_n$ R–algebra consists of a partial $\mathcal{C}_n$ R–algebra A together with a map of partial $\mathcal{C}_n$ R–algebras $\epsilon\colon A \to R$ called the *augmentation*. Likewise, an augmented partial associative R–algebra consists of a partial associative R–algebra A and a map of partial associative R–algebras $A \to R$.

In general for an op-lax power system X, given a map $X \to R$, we get a pair of maps $X_m \to X_{m-1}$ as composites

$$\begin{aligned} e_1\colon X_m &\longrightarrow X_1 \wedge_R X_{m-1} \longrightarrow R \wedge_R X_{m-1} \cong X_{m-1} \\ e_m\colon X_m &\longrightarrow X_{m-1} \wedge_R X_1 \longrightarrow X_{m-1} \wedge_R R \cong X_{m-1} \end{aligned}$$

using $\lambda_{1,m-1}$ and $\lambda_{m-1,1}$. We think of e_1 and e_m as applying the augmentation to the first and last spots in X_m. Note that e_m can be obtained from e_1 and the permutation actions on X_m and X_{m-1}, and using permutations like this, we can define analogous maps $e_2, \dots, e_{m-1}$ that apply the augmentation to an arbitrary spot in X_m.

In the case of an augmented partial associative R–algebra A, the maps e_j make the following diagram commute.

$$\begin{tikzcd} A_m \arrow[r, "e_1"] \arrow[d, "{(\mu,\mathrm{id},\dots,\mathrm{id})}"'] & A_{m-1} \arrow[d, "e_1"] \\ A_{m-1} \arrow[r, "e_1"'] & A_{m-2} \end{tikzcd} \qquad \begin{tikzcd} A_m \arrow[r, "e_m"] \arrow[d, "{(\mathrm{id},\dots,\mathrm{id},\mu)}"'] & A_{m-1} \arrow[d, "e_{m-1}"] \\ A_{m-1} \arrow[r, "e_{m-1}"'] & A_{m-2} \end{tikzcd}$$

We use these maps in the following construction.

Construction 5.2 For an augmented partial associative R–algebra A, let BA be the geometric realization of the simplicial R–module $B_\bullet A$ that is A_m in simplicial degree m, with degeneracy maps s_i induced by the action of $\mathcal{A}(0)$, with face maps d_i for $1 \le i \le m-1$ induced by the action of $\mu \in \mathcal{A}(2)$, and with face maps $d_0 = e_1$ and $d_m = e_m$ as defined above.

As in the previous section, the construction may not have the correct homotopy type without the additional assumption that A is tidy and an additional assumption ensuring that the simplicial object is proper.

For A a partial augmented $\mathcal{C}_n$–algebra, we set $BA = BMA$, and we extend BA to a partial power system by setting $BA_m = B(A^{[m]}) = B(M(A^{[m]}))$. The trick used in the proof of Theorems 1.1 and 1.2 now constructs the $\mathcal{C}_{n-1}$–structure in the following theorem.

Theorem 5.3 *Let A be an augmented partial $\mathcal{C}_n$–algebra A such that $B_\bullet A^{[m]}$ is a proper simplicial object for all m. Then the bar construction BA is naturally an augmented partial $\mathcal{C}_{n-1}$–algebra.*

The properness hypothesis holds in particular when A satisfies Hypothesis 4.17 or the hypothesis of Proposition 4.3. The augmentation in the theorem is the map $BA \to R$ induced by the augmentations $M(A^{[m]})_p \to R$. We now give a detailed description of the iterated bar construction along the lines of Construction 4.16.

Construction 5.4 Let A be an augmented partial $\mathcal{C}_n$–algebra. The nth iterate of the bar construction is isomorphic to the geometric realization of the following n–fold simplicial set $B^n(A)$. In multisimplicial degree $p_1, \dots, p_n$, $B^n(A)$ is the colimit of the diagram indexed on $\mathcal{T}^{p_1+\cdots+p_n}$ that at the object

$$\vec{x} = (x_1, \dots, x_q) = (x_1^1, \dots, x_{p_1}^1, x_1^2, \dots, x_{p_2}^2, \dots, x_1^n, \dots, x_{p_n}^n)$$

is the R–module

$$(P_{x_1} \times \cdots \times P_{x_q})_+ \wedge A_{m_1 \dots m_n},$$

where $m_i = \#a(x_1^i, \dots, x_{p_i}^i)$ (the number of a's among the x_j^i's). The degeneracies are induced by inserting b in the appropriate spot in $\vec{x}$ with zero as the associated length in $P_b = \bar{P}$. In the ith simplicial direction, the jth face map (for $0 < j < p_i$) is induced as follows. The lengths $(r, s) \in P_{x_j^i} \times P_{x_{j+1}^i}$ add. If $x_j^i = x_{j+1}^i = a$, we use the element

$$c = [0,1]^{i-1} \times \gamma_{r,s} \times [0,1]^{n-i}$$

of $\mathcal{C}_n(2)$, applied $m_1 \dots \widehat{m}_i \dots m_n$ times, with the appropriate permutations (as in (4.9)) to produce the map

$$A_{m_1 \dots m_n} \longrightarrow A_{m_1 \dots (m_i - 1) \dots m_n}.$$

If one of x_j^i, x_{j+1}^i is a, we use the appropriate element

$$\begin{aligned} &([0,1]^{i-1} \times [0, r/(r+s)] \times [0,1]^{n-i}), \\ \text{or} \qquad &([0,1]^{i-1} \times [r/(r+s), 1] \times [0,1]^{n-i}) \end{aligned}$$

of $\mathcal{C}_n(1)$. We use the identity on the A's factor if neither of x_j^i, x_{j+1}^i is a. The 0th and p_ith face maps are obtained by application of the appropriate maps e_j.

The category of R–modules under and over R has R as both an initial and final object. As a consequence, this category admits a tensor with based spaces: For M an

R–module under and over R and X a based space, the tensor of M with X, $M \mathbin{\widehat{\otimes}} X$, is formed as a pushout

$$\begin{tikzcd} R \wedge X_+ \cup_R M \arrow[r] \arrow[d] & M \wedge X_+ \arrow[d] \\ R \arrow[r] & M \mathbin{\widehat{\otimes}} X. \end{tikzcd}$$

In particular, we have a suspension in the category of R–modules under and over R that we denote as Σ_R. For an augmented partial R–algebra A, the first partial power A_1 is an R–module under and over R, as is the first partial power MA_1 of the Moore partial algebra. The unit maps $MA_1 \to MA_m$ induce maps

$$\underbrace{MA_1 \cup_R \ldots \cup_R MA_1}_{m \text{ factors}} \longrightarrow MA_m,$$

which together induce a map of simplicial objects

$$MA_1 \mathbin{\widehat{\otimes}} S^1_\bullet \longrightarrow B_\bullet A,$$

which on geometric realization induces a map $\Sigma_R MA_1 \to BA$, natural in A. Using the explicit description of the multisimplicial object $B^n A$ above, the simplicial map above generalizes to a multi-simplicial map

$$MA_1 \mathbin{\widehat{\otimes}} (S^1_\bullet \wedge \ldots \wedge S^1_\bullet) \longrightarrow B^n_{\bullet,\ldots,\bullet} A$$

as follows. Thinking of an element of S^1_p as an element of the based set $\{0, \ldots, p\}$, a non-basepoint element of $S^n_{p_1,\ldots,p_n}$ is an n–tuple $\vec{j} = (j_1, \ldots, j_n)$ with $1 \leq j_i \leq p_i$. We have one copy of MA_1 in $MA_1 \mathbin{\widehat{\otimes}} S^n_{p_1,\ldots,p_n}$ for each $\vec{j}$, which we map into $B^n_{p_1,\ldots,p_n} A$ by a map induced by a map of diagrams. For fixed $\vec{j}$, we have the functor $\mathcal{T} \to \mathcal{T}^{p_1+\cdots+p_n}$ sending x to the object $\vec{x}$ where $x^i_{j_i} = x$ and $x^i_k = b$ for $k \neq j_i$. We use the map of diagrams compatible with this functor sending $P_+ \wedge A_1$ (or $\overline{P}_+ \wedge R$ or $P_+ \wedge R$, for $x = a$, b, or c, respectively) into $(P_{x_1} \times \cdots \times P_{x_q})_+ \wedge A_1$ (or $(P_{x_1} \times \cdots \times P_{x_q})_+ \wedge R$) induced by the identity on A_1 (or R) and sending r in P to $(r^1_1, \ldots, r^n_{p_n})$ in $P_{x^1_1} \times \cdots \times P_{x^n_{p_n}}$ with $r^i_{j_i} = r$ and $r^i_k = 0$ for $k \neq j_i$. This then describes a map from $MA_1 \mathbin{\widehat{\otimes}} S^n_{p_1,\ldots,p_n}$ to $B^n_{p_1,\ldots,p_n} A$ that respects the face and degeneracy maps in all simplicial direction. On geometric realization, we get a map

$$\Sigma^n_R MA_1 \longrightarrow B^n A, \tag{5.5}$$

natural in A. The map of R–modules under and over R from MA_1 to A_1 (obtained by forgetting length coordinate) induces a map

$$\Sigma^n_R MA_1 \longrightarrow \Sigma^n_R A_1$$

which is a weak equivalence when the unit $R \to A_1$ is a Hurewicz cofibration, so in particular, when the hypothesis of Theorem 5.3 holds.

6 The bar construction for non-unital partial $\mathcal{C}_n$–algebras

In the next section, we study the reduced André–Quillen cohomology of a $\mathcal{C}_n$ R–algebra; as we will see there, this is easiest when we work with "non-unital" $\mathcal{C}_n$ R–algebras in place of augmented $\mathcal{C}_n$ R–algebras. For this reason, we take this section to redo the work of the previous section in the non-unital context. We begin with the definition of a non-unital partial $\mathcal{C}_n$ R–algebra.

Definition 6.1 Let $\widetilde{\mathcal{C}}_n$ be the operad with $\widetilde{\mathcal{C}}_n(0)$ empty and $\widetilde{\mathcal{C}}_n(m) = \mathcal{C}_n(m)$ for $m > 0$. A *non-unital* partial $\mathcal{C}_n$–algebra is a partial $\widetilde{\mathcal{C}}_n$–algebra.

If N is a non-unital partial $\mathcal{C}_n$–algebra, then we can form an associated augmented partial $\mathcal{C}_n$–algebra by formally adding a unit. Noting that for an R–module X,

$$(X \vee R)^m = \bigvee_{s \subset \underline{m}} X^{(s)},$$

(smash power indexed on the set s) where $\underline{m} = \{1, \dots, m\}$, the R–modules

$$KN_m = \bigvee_{s \subset \underline{m}} N_{|s|}$$

naturally form a partial power system with $KN_1 = N_1 \vee R$. The partial $\widetilde{\mathcal{C}}_n$–algebra structure on N extends to a partial $\mathcal{C}_n$–algebra structure on KN, with the missing elements in the subsets acting like place-holders for the unit $1 \in \mathcal{C}_n(0)$ and with the action of $\mathcal{C}_n(0)$ manipulating the sets $\underline{m} = \{1, \dots, m\}$ and their subsets. Specifically, on the summand corresponding to $s \subset \underline{m}$, the element

$$(c_1, \dots, c_j) \in \mathcal{C}_n(m_1) \times \cdots \times \mathcal{C}_n(m_j)$$

(for $m_1 + \cdots + m_j = m$) induces the map

$$(c'_1, \dots, c'_{j'})\colon N_{|s|} \longrightarrow N_{|t|} \tag{6.2}$$

into the $t \subset \underline{j} = \{1, \dots, j\}$ summand, as follows. Noting that $(c_1, \dots, c_j)$ has m total inputs, we form

$$(c'_1, \dots, c'_{j'}) \in \mathcal{C}_n(m'_1) \times \cdots \times \mathcal{C}_n(m'_{j'})$$

(with $m'_1 + \cdots + m'_{j'} = |s|$) by plugging $1 \in \mathcal{C}_n(0)$ into each input that corresponds to an element not in s, and then dropping any c_i whose inputs all become plugged. The subset t then consists of those elements i in $\underline{j}$ where c_i was not dropped.

In the case of true algebras, we can also go from augmented $\mathcal{C}_n$–algebras to non-unital $\mathcal{C}_n$–algebras: For a true $\mathcal{C}_n$–algebra A, we can form the true non-unital $\mathcal{C}_n$–algebra N as the homotopy pullback of the augmentation, $N = R^I \times_R A$. In the next section, we will see that for true $\mathcal{C}_n$–algebras, up to homotopy, working with non-unital $\mathcal{C}_n$–algebras is equivalent to working with augmented $\mathcal{C}_n$–algebras (Theorem 7.1).

We have a corresponding notion of non-unital partial associative R–algebra, and the construction K above also defines a functor from non-unital partial associative R–algebras to augmented partial associative R–algebras. We can generalize the Moore algebra to this context. For a non-unital partial $\mathcal{C}_1$–algebra N, the power system

$$MN_m = P^m_+ \wedge N_m$$

forms a non-unital partial associative R–algebra using the length concatenation construction in the Moore algebra. To compare this with $M(KN)$, note that

$$KMN_m = \bigvee_{s \subset \underline{m}} P^s_+ \wedge N_{|s|},$$

(the cartesian product of copies of P indexed on s), while

$$M(KN)_m = \bigvee_{s \subset \underline{m}} (P^m_s)_+ \wedge N_{|s|},$$

where P^m_s is the subset of $\bar{P}^m = [0, \infty)^m$ of points $(r_1, \dots, r_m)$ with $r_i > 0$ if $i \in s$. The inclusion of P^s in P^m_s as the subset where $r_i = 0$ for $i \notin s$ induces a map KMN to $M(KN)$ that is a map of partial associative R–algebras and a Σ_m–equivariant homotopy equivalence of R–modules in each partial power.

We can construct a version of the bar construction for a non-unital $\mathcal{C}_n$–algebra, taking

$$B_m N = MN_m$$

for the non-unital Moore algebra construction above. Since N and MN do not have units, this collection of R–modules does not admit degeneracy maps, but the face maps in Construction 5.2 still make sense (using the trivial map for the zeroth and last face map in place of the augmentation). Then $B_\bullet N$ forms a Δ–object (a simplicial object without degeneracies), and we can form BN as the geometric realization (by gluing $B_m \wedge \Delta[m]_+$ along the face maps). We compare BN and BKN in the following proposition.

Proposition 6.3 *The R–modules BN and $B(KMN)$ are canonically isomorphic and the map of R–modules $B(KMN) \to B(M(KN)) = BKN$ is a homotopy equivalence.*

Proof Associated to any Δ–object we obtain a simplicial object by formally attaching degeneracies; the geometric realization of the Δ–object is canonically isomorphic to the geometric realization of the associated simplicial object. In this case, the simplicial object associated to BN is in simplicial degree m the R–module

$$\bigvee_{j \le m} \bigvee_{f:\mathbf{m}\to\mathbf{j}} MN_j,$$

where the maps $f\colon \mathbf{m}\to\mathbf{j}$ in the inner wedge are the iterated degeneracies in the category $\mathbf{\Delta}$, i.e., the weakly increasing epimorphisms of totally ordered sets $\{0,\dots,m\} \to \{0,\dots,j\}$. We have a one-to-one correspondence between the set of such epimorphisms f and the set of j element subsets s of $\{1,\dots,m\}$ (where the elements in s are the first elements of $\{0,\dots,m\}$ that f sends to each of the elements $1,\dots,j$ of $\{0,\dots,j\}$), and this defines an isomorphism between the R–module above and KMN_m. As m varies, these isomorphisms preserve the face and degeneracy maps in the simplicial objects. This proves the first statement. For the second statement, we note that the homotopy equivalences $KMN_p \to M(KN)_p$ (using linear homotopies) commute with the face and degeneracy maps in the bar construction and so extend to homotopy equivalences on the geometric realizations. □

We note that the properness issues that plague the discussion of the cyclic bar construction and the bar construction for augmented algebras disappear for non-unital algebras: For the simplicial version of the construction BN, each degeneracy is the inclusion of a wedge summand and the inclusion of the union of degeneracies likewise is the inclusion of a wedge summand. For the construction BKN, each degeneracy is induced by smashing with the inclusion of $\{0\}_+$ in $\overline{P}_+ = [0,\infty)_+$ followed by the inclusion of a wedge summand, and the inclusion of the union of the degeneracies admits a similar description on each of its wedge summands. Thus, in particular, we have the following version of Theorem 5.3.

Theorem 6.4 *For a non-unital $\mathcal{C}_n$–algebra N, BKN is a naturally an augmented partial $\mathcal{C}_{n-1}$–algebra.*

We have a variant of BN where we use the trivial R–module in place of R at the zero level. We denote this as $\widetilde{B}N$, and we identify the partial power system $B(KMN)$ as $K\widetilde{B}N$. Specifically, $(\widetilde{B}N)_m$ is the geometric realization of the simplicial object which in degree p is

$$(\widetilde{B}_p N)_m = \bigvee_{s_1,\dots,s_m} P_+^{|s_1\cup\ldots\cup s_m|} \wedge N_{|s_1|+\dots+|s_m|},$$

with the wedge over the m–tuples of non-empty subsets of $\underline{p} = \{1,\dots,p\}$. We can identify $(\widetilde{B}_p N)_m$ as a submodule of

$$(B_p KN)_m = \bigvee_{s_1,\dots,s_m} (P^p_{s_1\cup\ldots\cup s_m})_+ \wedge N_{|s_1|+\dots+|s_m|}$$

with the wedge over the m–tuples of all subsets of $\underline{p}$. From the work above we have a partial $\mathcal{C}_{n-1}$–algebra structure on the partial power system $B_p KN$; this restricts to a non-unital partial $\mathcal{C}_{n-1}$–algebra structure on the partial power system $\widetilde{B}_p N$ as follows.

The action becomes easier to describe when we re-index the summands by writing

$$(B_p KN)_m = \bigvee_{s_1,\dots,s_p} (P(s_1)\times\dots\times P(s_p))_+ \wedge N_{|s_1|+\dots+|s_p|}$$

where the sum is over the p–tuples of subsets of $\underline{m} = \{1,\dots,m\}$ and $P(s) = \overline{P}$ if s is empty and $P(s) = P$ if s is non-empty; the relationship between these two indexings of the wedge sum corresponds to arranging $\{1,\dots,pm\}$ into the p blocks of m elements $(1,\dots,m), (m+1,\dots,2m),\dots,((p-1)m+1,\dots,pm)$ in the first description or the m blocks of p elements $(1, m+1, 2m+1,\dots,(p-1)m+1),\dots,(m, 2m,\dots,pm)$ in the new description. Letting $P'(s) = \{0\}$ if s is empty and $P'(s) = P(s) = P$ if s is non-empty, then in this formulation,

$$(\widetilde{B}_p N)_m = \bigvee_{s_1,\dots,s_p} (P'(s_1)\times\dots\times P'(s_p))_+ \wedge N_{|s_1|+\dots+|s_p|},$$

where the sum is over the p–tuples of subsets of $\{1,\dots,m\}$ such that $s_1\cup\ldots\cup s_p = \{1,\dots,m\}$. In the $\mathcal{C}_{n-1}$–action, for $m_1+\dots+m_j = m$, $m_i > 0$, an element

$$(c_1,\dots,c_j) \in \mathcal{C}_{n-1}(m_1)\times\dots\times\mathcal{C}_{n-1}(m_j)$$

acts on the $s_1,\dots,s_p$ summand by the identity map on the P factors and by the map

$$(c^1_1,\dots,c^1_{k_1},\dots,c^p_1,\dots,c^p_{k_p}) \in \mathcal{C}_{n-1}(m^1_1)\times\dots\times\mathcal{C}_{n-1}(m^p_{k_p})$$

from $N_{|s_1|+\dots+|s_p|}$ to $N_{|t_1|+\dots+|t_p|}$ (using the last coordinates embedding of $\mathcal{C}_{n-1}$ in $\mathcal{C}_n$), where t_i and $(c^i_1,\dots,c^i_{k_i})$ are as in (6.2) above: $(c^i_1,\dots,c^i_{k_i})$ is formed by plugging $1 \in \mathcal{C}_{n-1}(0)$ into the input corresponding to elements of $\{1,\dots,m\}$ not in s_i, dropping any c_r where all inputs are plugged; t_i consists of the indexes r where c_r is not dropped when forming $(c^i_1,\dots,c^i_{k_i})$. To see that this action restricts to $\widetilde{B}_pN$, we just need to observe that when $s_1\cup\ldots\cup s_p=\{1,\dots,m\}$, then $t_1\cup\ldots\cup t_p=\{1,\dots,j\}$.

Just as for B_pKN, the $\mathcal{C}_{n-1}$–action on $\widetilde{B}_pN$ is compatible with the face and degeneracy maps in the simplicial object $\widetilde{B}N$. This makes $\widetilde{B}N$ into a non-unital partial $\mathcal{C}_{n-1}$–algebra with the map $K\widetilde{B}N \to BKN$ a map of partial $\mathcal{C}_{n-1}$–algebras.

Theorem 6.5 *For a non-unital partial $\mathcal{C}_n$–algebra N, the bar construction $\widetilde{B}N$ is naturally a non-unital $\mathcal{C}_{n-1}$–algebra and the weak equivalence $K\widetilde{B}N \to B(KN)$ is a natural map of partial $\mathcal{C}_{n-1}$–algebras.*

Next we describe the iterated bar construction $\widetilde{B}^nN$. Using the first description of the partial power system $\widetilde{B}_pN$ above, we see that $\widetilde{B}^2N$ is the Δ–object in simplicial R–modules which in degree p,q is

$$M(\widetilde{B}_pN)_q = P^q_+ \wedge \Bigg(\bigvee_{s_1,\dots,s_q} P_+^{|s_1\cup\ldots\cup s_q|} \wedge N_{|s_1|+\dots+|s_q|}\Bigg),$$

with the wedge over the q–tuples of non-empty subsets of $\underline{p}=\{1,\dots,p\}$. In bidegree p,q, the associated bisimplicial R–module is then

$$\widetilde{B}^2_{p,q}N = \bigvee_{\substack{t\subset\{1,\dots,q\}\\ t\neq\{\}}} \ \bigvee_{s_1,\dots,s_{|t|}} P_+^{|s_1\cup\ldots\cup s_{|t|}|+|t|} \wedge N_{|s_1|+\dots+|s_{|t|}|},$$

with the inside wedge over the $|t|$–tuples of non-empty subsets of $\underline{p}$. If we re-index the subsets s_i by the elements of t and set $s_i=\{\}$ for $i\notin t$, then we get

$$\widetilde{B}^2_{p,q}N = \bigvee_{\substack{s_1,\dots,s_q\subset\underline{p}\\ s_1\cup\ldots\cup s_q\neq\{\}}} P_+^{|s_1\cup\ldots\cup s_q|+n(s_1,\dots,s_q)} \wedge N_{|s_1|+\dots+|s_q|},$$

where $n(s_1,\dots,s_q)$ denotes the number of $s_1,\dots,s_q$ that are non-empty. Finally, the collections $s_1,\dots,s_q\subset\{1,\dots p\}$ satisfying $s_1\cup\ldots\cup s_q\neq\{\}$ are in one to one correspondence with the non-empty subsets of

$$\underline{p}\times\underline{q}=\{1,\dots,p\}\times\{1,\dots,q\}.$$

For $s\subset\underline{p}\times\underline{q}$, write $\nu_1^{p,q}(s)$ for the subset of $\underline{p}$ of elements i such that $s\cap(\{i\}\times\underline{q})$ is non-empty and likewise write $\nu_2^{p,q}(s)$ for the subset of $\underline{q}$ of elements i such that

$s \cap (\underline{p} \times \{i\})$ is non-empty. In other words, $\nu_1^{p,q}$ and $\nu_2^{p,q}$ are the images of the projection maps from $\underline{p} \times \underline{q}$ to $\underline{p}$ and $\underline{q}$, respectively. Now we can identify $\widetilde{B}^2_{p,q}N$ as

$$\widetilde{B}^2_{p,q}N = \bigvee_{\substack{s \subset \underline{p} \times \underline{q} \\ s \neq \{\}}} \big(P^{\nu_2^{p,q}(s)} \times P^{\nu_1^{p,q}(s)}\big)_+ \wedge N_{|s|}.$$

Working inductively, we see that in multisimplicial degree $p_1, \dots, p_n$,

$$\widetilde{B}^n_{p_1,\dots,p_n}N = \bigvee_{\substack{s \subset \underline{p}_1 \times \cdots \times \underline{p}_n \\ s \neq \{\}}} \big(P^{\nu_n^{p_1,\dots,p_n}(s)} \times \cdots \times P^{\nu_1^{p_1,\dots,p_n}(s)}\big)_+ \wedge N_{|s|}, \tag{6.6}$$

where $\nu_i^{p_1,\dots,p_n}(s)$ is the subset of elements j in $\underline{p}_i$ such that

$$s \cap \big(\underline{p}_1 \times \cdots \times \underline{p}_{i-1} \times \{j\} \times \underline{p}_{i+1} \times \cdots \times \underline{p}_n\big)$$

is non-empty.

For $B^n KN$, we obtain a completely analogous description, but with $s = \{\}$ in the wedge sum and with different length coordinates. Recall that for $u \subset \underline{p}$, P^p_u denotes the subset of $\bar{P}^p = [0,\infty)^p$ of elements $(r_1, \dots, r_p)$ where $r_j > 0$ for all $j \in u$. Then the length coordinates on the summand indexed by s is $P^{p_i}_{\nu_i^{p_1,\dots,p_n}(s)}$. In other words,

$$B^n_{p_1,\dots,p_n}KN = \bigvee_{s \subset \underline{p}_1 \times \cdots \times \underline{p}_n} \Big(P^{p_n}_{\nu_n^{p_1,\dots,p_n}(s)} \times \cdots \times P^{p_1}_{\nu_1^{p_1,\dots,p_n}(s)}\Big)_+ \wedge N_{|s|}.$$

To complete the closed description of the iterated bar construction, we describe the face and degeneracy maps. The degeneracy map s_j in the ith simplicial direction is induced by the map $\underline{p}_i$ to $\{1, \dots, p_i + 1\}$ sending $1, \dots, j-1$ by the identity and $j, \dots, p_i$ by $m \mapsto m+1$.

For $0 < j < p_i$ the face map d_j adds the appropriate pair $r, s \mapsto r + s$ in the P factors (corresponding to $j, j+1 \in \underline{p}_i$), and performs the action

$$c = \mathrm{id}^{i-1} \times \gamma_{r,s} \times \mathrm{id}^{n-i} \in \mathcal{C}_n(2)$$

on the corresponding spots in $N_{|s|}$: We first use a permutation to rearrange from lexicographical order on s to the lexicographical order where the ith index is least significant. Then for fixed $a_k \in \underline{p}_k$, $k \neq i$, in this order, the elements

$$(a_1, \dots, a_{i-1}, j, a_{i+1}, \dots, a_n) \qquad \text{and} \qquad (a_1, \dots, a_{i-1}, j+1, a_{i+1}, \dots, a_n)$$

are adjacent when they both appear in s. For such elements, we apply c on the appropriate spot on $N_{|s|}$, but when one is missing, we plug $1 \in \mathcal{C}_n(0)$ in that input of c and apply that element of $\mathcal{C}_n(1)$ to the spot in $N_{|s|}$. (When neither element is in s, no action needs to be taken for that pair.) We then use the permutation to rearrange back to the natural lexicographical order.

The zeroth face map d_0 is the trivial map on the summands indexed by those subsets s where $1 \in \nu_i^{p_1,\dots,p_n}(s)$. Fixing s with $1 \notin \nu_i^{p_1,\dots,p_n}(s)$, the action of d_0 sends this summand to summand indexed by s', where s' is obtained by subtracting 1 from the ith coordinate of each element of s. On the length factors, the first factor gets dropped and for $j \geq 1$ the $j+1$ coordinate becomes the new j coordinate. On the N factor, $|s| = |s'|$ and $N_{|s|}$ maps by the identity to $N_{|s'|}$. The last face map d_{p_i} has a similar description: it is the trivial map on the summands indexed by those subsets s where $p_i \in \nu_i^{p_1,\dots,p_n}(s)$ and on those summands where $p_i \notin \nu_i^{p_1,\dots,p_n}(s)$, it lands in $s \subset \underline{p}_1 \times \cdots \times \underline{p'_i} \times \cdots \times \underline{p}_n$ (for $p'_i = p_i - 1$), dropping the last length coordinate and acting by the identity on $N_{|s|}$.

We note from the description above that we have a canonical inclusion of $\Sigma^n N_1$ into $\widetilde{B}^n N$: Using the singleton subsets of $\underline{p}_1 \times \cdots \times \underline{p}_n$ and the element $1 \in P$, we get a map of multisimplicial R–modules $N_1 \wedge S^1_\bullet \wedge \ldots \wedge S^1_\bullet \to \widetilde{B}^n_{\bullet,\dots,\bullet} N$, which on geometric realization induces a map

$$\Sigma^n N_1 \longrightarrow \widetilde{B}^n N.$$

We have an inclusion of $KN_1 = R \vee N_1$ in $M(KN)_1$ where we send the R factor in as length zero and the N_1 factor in as length one. This defines a map in the category of R–modules under and over R, splitting the usual map $M(KN)_1 \to KN_1$ (induced by forgetting lengths). Noting that $\Sigma^n_R KN_1 = R \vee \Sigma^n N_1$, we have the following commutative diagram, relating the map above to the map (5.5).

(6.7)
$$\begin{tikzcd}
 & R \vee \Sigma^n N_1 \arrow[r] \arrow[dl, "="'] \arrow[d] & R \vee \widetilde{B}^n N \arrow[d] \\
\Sigma^n_R KN_1 & \Sigma^n_R MKN_1 \arrow[l, "\simeq"'] \arrow[r] & B^n KN
\end{tikzcd}$$

7 Reduced topological Quillen homology and the iterated bar construction

In this section we relate the iterated bar construction for augmented $\mathcal{C}_n$–algebras of the previous section to reduced topological Quillen homology, proving Theorem 1.3. Quillen homology theories are defined in terms of derived indecomposables, and (as shown in Basterra [2]) work best in the context of non-unital algebras. We begin by reviewing the Quillen equivalence of augmented and non-unital algebras.

Recall from the previous section the functor K from (true) non-unital $\mathcal{C}_n$–algebras to (true) augmented $\mathcal{C}_n$–algebras that wedges on a unit, $KN = R \vee N$. This functor is left adjoint to the functor I from augmented $\mathcal{C}_n$–algebras to non-unital $\mathcal{C}_n$–algebras that takes the (point-set) fiber of the augmentation map, $IA = * \times_R A$. We have a Quillen closed model structure on each of these categories where the fibrations and weak equivalences are the underlying fibrations and weak equivalences of R–modules; the cofibrations are the retracts of $\widetilde{\mathbb{C}}_n\mathcal{I}$–cell complexes (in the non-unital case) and of $\mathbb{C}_n\mathcal{I}$–cell complexes (in the augmented case) where $\mathcal{I}$ is the set of generating cofibrations. The functor I preserves fibrations and acyclic fibrations, and so the adjoint pair K, I forms a Quillen adjunction. Since we can calculate the effect on homotopy groups of K on arbitrary non-unital $\mathcal{C}_n$–algebras and of I on fibrant augmented $\mathcal{C}_n$–algebras, we see that when A is fibrant, a map of augmented $\mathcal{C}_n$–algebras $KN \to A$ is a weak equivalence if and only if the adjoint map $N \to IA$ is a weak equivalence; it follows that K, I is a Quillen equivalence.

Theorem 7.1 *The functors K and I form a Quillen equivalence between the category of non-unital $\mathcal{C}_n$–algebras and the category of augmented $\mathcal{C}_n$–algebras.*

Given an R–module M, we can make M into a non-unital $\mathcal{C}_n$–algebra by giving it the trivial $\mathcal{C}_n$–action, letting

$$\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} M^{(m)} \longrightarrow M$$

be the trivial map for $m > 1$ and the composite $\mathcal{C}_n(1)_+ \wedge M \to *_+ \wedge M \cong M$ for $m = 1$. This defines a functor Z (the "zero multiplication" functor) from R–modules to non-unital R–algebras. The functor Z has a left adjoint "indecomposables" functor Q, which can be constructed as the coequalizer

$$\bigvee_{m>0} \mathcal{C}_n(m)_+ \wedge_{\Sigma_m} N^{(m)} \rightrightarrows N \longrightarrow QN,$$

where one map is the action map for N and the other map is the zero multiplication action map. Since the functor Z preserves fibrations and weak equivalences, we see that the pair Q, Z forms a Quillen adjunction.

Theorem 7.2 *The functors Q and Z form a Quillen adjunction between the category of non-unital $\mathcal{C}_n$–algebras and the category of R–modules.*

For an augmented $\mathcal{C}_n$–algebra A and an R–module M, one can define the reduced topological Quillen cohomology groups in terms of derivations of A with coefficients in M, i.e., as maps in the homotopy category of augmented $\mathcal{C}_n$–algebras from A

to KZM (or $KZ\Sigma^* M$). Applying the Quillen equivalence of Theorem 7.1 and the Quillen adjunction of Theorem 7.2, we can identify this as maps in the derived category of R–modules from QN to M (or $\Sigma^* M$), where N is a cofibrant non-unital $\mathcal{C}_n$–algebra with KN equivalent to A. In other words, the left derived functor $Q^{\mathbf{L}}$ of Q produces an object representing topological Quillen homology. We write $Q^{\mathbf{L}}_{\mathcal{C}_n}$ for the composite of $Q^{\mathbf{L}}$ with the right derived functor of I.

Definition 7.3 For an augmented $\mathcal{C}_n$–algebra A, the $\mathcal{C}_n$–algebra cotangent complex at the augmentation is the R–module of derived indecomposables $Q^{\mathbf{L}}_{\mathcal{C}_n}$.

We have a canonical natural map $\widetilde{B}^n N \to \Sigma^n QN$ defined as follows. Thinking of $\Sigma^n QN$ as the geometric realization of the multisimplicial R–module $QN \wedge (S^1_\bullet)^n$, we send the summand indexed by $s \subset \underline{p}_1 \times \cdots \times \underline{p}_n$ in (6.6) by the counit of the Q, Z adjunction $N \to QN$ (and dropping the P factors) when $|s| = 1$ and by the trivial map when $|s| > 1$. This clearly commutes with the degeneracy maps and commutes with the face maps since every face map that changes the cardinality of the indexing set is either the trivial map or lands in the decomposables.

Using the constructions above, Theorem 1.3 becomes the following theorem stated in terms of non-unital $\mathcal{C}_n$–algebras.

Theorem 7.4 *The natural map $\widetilde{B}^n N \to \Sigma^n QN$ is a weak equivalence when N is a cofibrant non-unital $\mathcal{C}_n$–algebra.*

To prove this theorem, we use the monadic bar construction trick from Basterra [2, Section 5]. The key observation is that both the functors $\widetilde{B}^n$ and Q commute with geometric realization. This is clear from the construction for $\widetilde{B}^n$, but follows for Q because Q is a topological left adjoint and because geometric realization of simplicial operadic algebras can be formed as a topological colimit in the category of algebras [9, Section VII.3]. In particular, applied to the monadic bar construction $B(\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N)$, we get isomorphisms

$$\begin{aligned} \widetilde{B}^n B(\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N) &\cong B(\widetilde{B}^n \widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N) \\ \text{and} \qquad QB(\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N) &\cong B(Q\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N). \end{aligned}$$

The argument now simplifies from [2] since in our context a cofibrant non-unital $\mathcal{C}_n$–algebra is cofibrant as an R–module. Regarding $B(\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N)$ as a topological colimit in non-unital $\mathcal{C}_n$–algebras, it follows that $B(\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N)$ is cofibrant as a non-unital $\mathcal{C}_n$–algebra when N is. Since the simplicial objects

$$B_\bullet(\widetilde{B}^n \widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N) \qquad \text{and} \qquad B_\bullet(Q\widetilde{\mathbb{C}}_n, \widetilde{\mathbb{C}}_n, N).$$

are always proper (the inclusion of the union of the degeneracies in each simplicial degree is induced by the inclusion of id in $\mathcal{C}_n(1)$ and the inclusion of a wedge summand), Theorem 7.4 now reduces to the following lemma.

Lemma 7.5 *For any cofibrant R–module X, the natural map $\widetilde{B}^n\widetilde{\mathbb{C}}_nX \to \Sigma^n Q\widetilde{\mathbb{C}}_nX$ is a weak equivalence.*

The functor $\Sigma^n Q\widetilde{\mathbb{C}}_n$ is canonically isomorphic to the nth suspension functor Σ^n; moreover, the natural transformation $\widetilde{B}^n\widetilde{\mathbb{C}}_n \to \Sigma^n$ is split by the natural transformation

$$\Sigma^n \longrightarrow \Sigma^n\widetilde{\mathbb{C}}_n \longrightarrow \widetilde{B}^n\widetilde{\mathbb{C}}_n,$$

induced by the inclusion of the singleton subsets in (6.6). Thus, to prove Lemma 7.5, it suffices to prove the following lemma.

Lemma 7.6 *For any cofibrant R–module X, the natural map $\Sigma^n X \to \widetilde{B}^n\widetilde{\mathbb{C}}_nX$ is a weak equivalence.*

We can make a further reduction by identifying $\widetilde{B}^n\widetilde{\mathbb{C}}_nX$ as the functor associated to a symmetric sequence. Applying the explicit description of $\widetilde{B}^n$ in the previous section, we note that the face and degeneracies in $\widetilde{B}^n\widetilde{\mathbb{C}}_nX$ preserve homogeneous degrees in X. We can therefore decompose $\widetilde{B}^n\widetilde{\mathbb{C}}_n$ naturally into a wedge sum of its homogeneous pieces,

$$\widetilde{B}^n\widetilde{\mathbb{C}}_nX = \bigvee_{m>0} \mathcal{B}(m) \wedge_{\Sigma_m} X^{(m)},$$

where each $\mathcal{B}(m)$ is a based Σ_m–space; in fact, each $\mathcal{B}(m)$ is a free based Σ_m–cell complex since it is the geometric realization of a multisimplicial Σ_m–space that in each multisimplicial degree is a wedge of pieces of the form

$$\big((\overline{P}^i \times P^j) \times (\mathcal{C}_n(m_1) \times \cdots \times \mathcal{C}_n(m_r)) \times_{\Sigma_{m_1}\times\cdots\times\Sigma_{m_r}} \Sigma_m\big)_+.$$

The natural map $\Sigma^n X \to \widetilde{B}^n\widetilde{\mathbb{C}}_nX$ is induced by the inclusion of S^n in $\mathcal{B}(1)$. Thus, it suffices to show that the map $S^n \to \mathcal{B}(1)$ induces a weak equivalence on R–homology and that each $\mathcal{B}(m)$ has trivial R–homology.

Although it is not hard to show directly that the map $S^n \to \mathcal{B}(1)$ is a weak equivalence, analysis of the construction of $\mathcal{B}(m)$ is rather complicated for a direct argument (see Fresse [11, Section 8]) and we take a shorter oblique approach in terms of the functor $\widetilde{B}^n\widetilde{\mathbb{C}}_n$. Let R^0_c be a cofibrant R–module equivalent to R, and consider the set with m elements, $\underline{m} = \{1, \dots, m\}$. We note that $\mathcal{B}(m) \wedge_{\Sigma_m} (\underline{m}_+)^{(m)}$ contains $\mathcal{B}(m)$ as a wedge summand, and so

$$\pi_*\big(\widetilde{B}^n\widetilde{\mathbb{C}}_n(R^0_c \wedge \underline{m}_+)\big)$$

contains the R–homology $R_*\mathcal{B}(m)$ as a direct summand. Rewriting in terms of the natural transformation $\Sigma^n \to \widetilde{B}^n\widetilde{\mathbb{C}}_n$, we have reduced Lemma 7.6 to the following lemma.

Lemma 7.7 *The natural map $\Sigma^n(R^0_c \wedge X_+) \to \widetilde{B}^n\widetilde{\mathbb{C}}_n(R^0_c \wedge X_+)$ is a weak equivalence for every finite set X.*

Lemma 7.7 follows from the analogous statement in terms of augmented algebras, that the map

$$R \vee \Sigma^n(R^0_c \wedge X_+) \longrightarrow B^n\mathbb{C}_n(R^0_c \wedge X_+)$$

is a weak equivalence for all finite sets X. We have an isomorphism of $\mathcal{C}_n$–algebras

$$\mathbb{C}_n(R \wedge X_+) \cong R \wedge (\mathbb{C}_n X)_+$$

where $\mathbb{C}_n X$ is the free $\mathcal{C}_n$–space on X. The weak equivalence $R^0_c \to R$ induces a weak equivalence of $\mathcal{C}_n$–algebras

$$\mathbb{C}_n(R^0_c \wedge X_+) \longrightarrow \mathbb{C}_n(R \wedge X_+) \cong R \wedge (\mathbb{C}_n X)_+,$$

but this is not a map of augmented $\mathcal{C}_n$–algebras: The augmentation on the left is induced by the trivial map $X_+ \to *$, but the augmentation on the right is induced by the trivial map $\mathbb{C}_n X \to *$. In terms of $\mathbb{C}_n(R \wedge X_+)$, the right-hand augmentation is induced by $X_+ \to S^0$ and the $\mathcal{C}_n$–action $\mathbb{C}_n R \to R$. Since we are free to choose any cofibrant model R^0_c, choosing one that is a suspension, we can construct a map

$$\alpha\colon R^0_c \wedge X_+ \longrightarrow R^0_c \vee R^0_c \wedge X_+$$

that represents the sum of the map $X_+ \to S^0$ and the identity on X_+ smashed with R^0_c. Write ϵ for the composite map

$$R^0_c \wedge X_+ \longrightarrow R^0_c \longrightarrow R,$$

which is homotopic to (but probably not equal to) the map induced by $X_+ \to S^0$, and choose a homotopy h. Using α, we get a map of $\mathcal{C}_n$–algebras

$$\overline{\alpha}\colon \mathbb{C}_n(R^0_c \wedge X_+) \longrightarrow \mathbb{C}_n(R^0_c \wedge X_+),$$

which respects augmentations when we give the copy on the left the augmentation induced by ϵ; an easy filtration argument shows that this map is a weak equivalence.

Now we get a diagram of weak equivalences of augmented $\mathcal{C}_n$–algebras

$$\begin{tikzcd}
\mathbb{C}_n(R^0_c \wedge X_+ \wedge \{0\}_+) \arrow[r] \arrow[d, "\bar\alpha"'] & \mathbb{C}_n(R^0_c \wedge X_+ \wedge I_+) & \mathbb{C}_n(R^0_c \wedge X_+ \wedge \{1\}_+) \arrow[l] \arrow[d] \\
\mathbb{C}_n(R^0_c \wedge X_+) & & R \wedge (\mathbb{C}_n X)_+
\end{tikzcd}$$

which respects augmentations when we use the augmentation induced by $X_+ \to S^0$ on the right and the augmentation induced by h in the middle (on $\mathbb{C}_n(R^0_c \wedge X_+ \wedge I_+)$). Since applying B^n preserves these weak equivalences, we get that

$$B^n\mathbb{C}_n(R^c_0 \wedge X_+) \qquad \text{and} \qquad B^n(R \wedge (\mathbb{C}_n X)_+).$$

are weakly equivalent.

The map in Lemma 7.7 is induced by the inclusion of $R^0_c \wedge X_+$ in $\widetilde{\mathbb{C}}_n(R^0_c \wedge X_+)$ and the section (6.7) of the natural map (5.5)

$$\Sigma^n_R M(\mathbb{C}_n(R^0_c \wedge X_+)) \longrightarrow B^n\mathbb{C}_n(R^0_c \wedge X_+).$$

We can follow the natural map (5.5) along the diagram of weak equivalences between $B^n\mathbb{C}_n(R^c_0 \wedge X_+)$ and $B^n(R \wedge (\mathbb{C}_n X)_+)$ above, and lift the map from

$$R \vee \Sigma^n(R^0_c \wedge X_+) = \Sigma^n_R(R^0_c \wedge X_+)$$

up to homotopy all the way around to a map

$$R \vee \Sigma^n(R^0_c \wedge X_+) \longrightarrow \Sigma^n_R M(R \wedge (\mathbb{C}_n X)_+) \longrightarrow B^n(R \wedge (\mathbb{C}_n X)_+).$$

Specifically, the map from $R \vee \Sigma^n(R^0_c \wedge X_+)$ to each of

$$\begin{tikzcd}
\Sigma^n_R M\mathbb{C}_n(R^0_c \wedge X_+ \wedge \{0\}_+) \arrow[r] \arrow[d] & \Sigma^n_R M\mathbb{C}_n(R^0_c \wedge X_+ \wedge I_+) & \Sigma^n_R M\mathbb{C}_n(R^0_c \wedge X_+ \wedge \{1\}_+) \arrow[l] \arrow[d] \\
\Sigma^n_R M\mathbb{C}_n(R^0_c \wedge X_+), & & \Sigma^n_R M(R \wedge (\mathbb{C}_n X)_+)
\end{tikzcd}$$

induces a weak equivalence on the submodules in homogeneous filtration one (and below). In particular, on the bottom right, looking at the map

$$R\vee\Sigma^n(R^0_c\wedge X_+) \longrightarrow \Sigma^n_R M(R\wedge(\mathbb{C}_n X)_+) \simeq \Sigma^n_R(R\wedge(\mathbb{C}_n X)_+) = R\wedge(\Sigma^n(\mathbb{C}_n X))_+,$$

we now see that the map in Lemma 7.7 is a weak equivalence if and only if the map

$$R \wedge \Sigma^n(X_+)_+ \longrightarrow B^n(R \wedge (\mathbb{C}_n X)_+)$$

induced by the inclusion of X in $\mathbb{C}_n X$ is a weak equivalence. This reduces Lemma 7.7 to the following lemma.

Lemma 7.8 *The natural map $R \wedge \Sigma^n(X_+)_+ \to B^n(R \wedge (\mathbb{C}_n X)_+)$ is a weak equivalence for every finite set X.*

The point of this lemma is that it lets us compare with the classical bar construction on spaces. The construction B^n in the previous section used little about the category of R–modules and generalizes to an iterated bar construction on the category of partial $\mathcal{C}_n$–spaces (where we use the cartesian product to define partial power systems). In fact, in spaces, it is much easier to describe because we can talk in terms of elements. For a $\mathcal{C}_n$–space A, each element of the Moore construction MA has a length, and we make the bar construction

$$B_\bullet A = B_\bullet(MA) = MA \times \cdots \times MA$$

into a partial power system by insisting that the lengths match up: We take the mth partial power $(B_p A)_m$ to be the subset of $(B_p A)^m = (MA^p)^m$ where the length vectors for each of the m copies of MA^p all agree. We have an entirely similar description when A is a partial $\mathcal{C}_n$–space (noting that elements of $MA_\bullet$ also have sequences of lengths). We also note that when $A = \Omega Z$ with its $\mathcal{C}_1$–structure coming from the standard $\mathcal{C}_1$–structure on the loop space, then by construction, MA is the Moore loop space $\Omega_M Z$.

The functor $R \wedge (-)_+$ from unbased spaces to R–modules takes partial power systems to partial power systems. By inspection, for any partial $\mathcal{C}_n$–space Z, we have an isomorphism of partial $\mathcal{C}_{n-1}$–algebras

$$B(R \wedge Z_+) \cong R \wedge (BZ)_+,$$

and iterating, an isomorphism $B^n(R \wedge Z_+) \cong R \wedge (B^n Z)_+$. Finally, Lemma 7.8 is a consequence of the following proposition.

Proposition 7.9 *For any finite set X, the map $\Sigma^n X_+ \to B^n \mathbb{C}_n X$ is a weak equivalence.*

To prove the proposition, we inductively analyze $B^i \mathbb{C}_n X$. The inclusion $X \to \Omega^n \Sigma^n X_+$ induces a map of $\mathcal{C}_n$–spaces $\mathbb{C}_n X \to \Omega^n \Sigma^n X_+$. By the group completion theorem, this map induces a weak equivalence

$$B\mathbb{C}_n X \longrightarrow B\Omega^n \Sigma^n X_+ \simeq \Omega^{n-1} \Sigma^n X_+. \tag{7.10}$$

Since up to homotopy this map is compatible with the inclusion of ΣX_+ in $B\mathbb{C}_n X$, this completes the argument in the case $n = 1$. For $n > 1$, we need the reduced free $\mathcal{C}_j$–space functor C_j from May [22, Section 2.4]; its fundamental formal property is that it gives an adjunction between *based* maps from a based spaced Y to a $\mathcal{C}_j$–space Z

and maps of $\mathcal{C}_j$–spaces from $C_j Y$ to Z. Its fundamental homotopical property is that when Y is connected and nondegenerately based, the universal map $C_j Y \to \Omega^j \Sigma^j Y$ is a weak equivalence. The fundamental formal property also holds when the target Z is a partial $\mathcal{C}_j$–space: For a based space Y, based maps of partial power systems $Y \to Z$ are in bijective correspondence with maps of partial $\mathcal{C}_j$–spaces $C_j Y \to Z$.

To be specific about the weak equivalence $B\Omega Z \simeq Z$ in (7.10), we use the zigzag in May [23, Section 14.3]

$$B(\Omega Z) = B(\Omega_M Z) \xleftarrow{\simeq} B(P_M Z, \Omega_M Z, *) \xrightarrow{\simeq} Z, \tag{7.11}$$

which is a weak equivalence whenever Z is connected. Here $P_M Z$ denotes the Moore based path space (the space of positive length paths ending at the base point), the middle term the classical two-sided bar construction for the action of the Moore loop space on the Moore path space, and the maps are induced by the trivial map $P_M Z \to *$ (on the left) and the start point projection $P_M Z \to Z$ on the right. When Z is a $\mathcal{C}_j$–space, we can make

$$B_\bullet(P_M Z, \Omega_M Z, *) = P_M Z \times \Omega_M Z \times \cdots \times \Omega_M Z \times *$$

a partial power system by taking the mth partial power to be the subset of the mth power where the lengths match up, as for B above. Then both maps in the zigzag become maps of partial $\mathcal{C}_j$–spaces (with the true power system for Z).

Returning to (7.10), we have a zigzag of weak equivalences of partial $\mathcal{C}_{n-1}$–spaces

$$B\mathbb{C}_n X \longrightarrow B\Omega^n \Sigma^n X_+ \longleftarrow B(P_M \Omega^{n-1}\Sigma^n X_+, \Omega_M \Omega^{n-1}\Sigma^n X_+, *) \longrightarrow \Omega^{n-1}\Sigma^n X_+.$$

We can now see that the inclusion of ΣX_+ into $B\mathbb{C}_n X$ induces a weak equivalence of partial $\mathcal{C}_{n-1}$–spaces

$$C_{n-1}\Sigma X_+ \longrightarrow B\mathbb{C}_n X$$

(for the true $\mathcal{C}_{n-1}$–space $C_{n-1}\Sigma X_+$). We use this as the base case of an inductive argument: Assume by induction that the natural map $\Sigma^i X_+ \to B^i \mathbb{C}_n X$ induces a weak equivalence of partial $\mathcal{C}_{n-i}$–spaces

$$C_{n-i}\Sigma^i X_+ \longrightarrow B^i \mathbb{C}_n X.$$

Applying B, the weak equivalence of $\mathcal{C}_{n-i}$–spaces $C_{n-i}\Sigma^i X_+ \to \Omega^{n-i}\Sigma^n X_+$ and the zigzag (7.11) give us a zigzag of weak equivalences of partial $\mathcal{C}_{n-(i+1)}$–spaces

$$\begin{aligned}\Omega^{n-(i+1)}\Sigma^n X_+ \longleftarrow B(P_M \Omega^{n-(i+1)}\Sigma^n X_+, \Omega_M \Omega^{n-(i+1)}\Sigma^n X_+, *) \\ \longrightarrow B\Omega^{n-i}\Sigma^n X_+ \longleftarrow BC_{n-i}\Sigma^i X_+ \longrightarrow B^{i+1}\mathbb{C}_n X.\end{aligned}$$

The inclusions of $\Sigma^{i+1}X_+$ into each of these spaces agree up to homotopy under these maps, and so the induced map of partial $\mathcal{C}_{n-(i+1)}$–spaces $C_{n-(i+1)}\Sigma^{i+1}X_+ \to B^{i+1}\mathbb{C}_n X$ is a weak equivalence. This completes the proof of Proposition 7.9, which completes the proof of Theorem 7.4.

Remark 7.12 For $Z = \Omega^j Y$ for a $(j-1)$–connected space Y, the identification of (7.11) as a zigzag of weak equivalences of partial $\mathcal{C}_j$–spaces implies by induction that B^n is an "n–fold de-looping machine". As an alternative argument, it should be possible to deduce Proposition 7.9 from a uniqueness theorem for n–fold de-looping machines as in Dunn [8]; however, translating the problem to the context in which such a theorem applies is more complicated than the direct argument above.

8 Further structure on the bar construction

With an eye to using Theorem 1.3 for computations, we take this final section to verify two of the expected properties of the multiplication on the bar construction. We begin by studying the diagonal map on the bar construction, and we show that it commutes with the $\mathcal{C}_{n-1}$–multiplication constructed in Section 5. We then study power operations, showing that the (dimension shifting) map on homotopy groups from a non-unital $\mathcal{C}_n$–algebra N to its bar construction BN preserves power operations in the expected way. In this section, we work in the context of true algebras since that is where these remarks are of primary interest.

Given an augmented R–algebra A, it is well-known that the bar construction BA admits a diagonal map

$$BA \longrightarrow BA \wedge_R BA$$

that is associative up to homotopy, even up to coherent homotopy. The best construction of this map uses "edgewise subdivision" (see Bökstedt, Hsiang and Madsen [4, Section 1]). For a simplicial object, $X_\bullet$, the edgewise subdivision is the object $\mathrm{sd}_2 X_\bullet$ where $\mathrm{sd}_2 X_n = X_{2n+1}$. The argument for [4, 1.1] shows that just as in the context of simplicial sets or simplicial spaces, in the context of simplicial R–modules, we have a natural isomorphism between the geometric realization of $X_\bullet$ and the geometric realization of the edgewise subdivision $\mathrm{sd}_2 X_\bullet$. We get the diagonal map on the bar construction as the composite

$$BA \cong |\,\mathrm{sd}_2 B_\bullet A| \longrightarrow |B_\bullet A \wedge_R B_\bullet A| \cong BA \wedge_R BA$$

for a particular simplicial map $\mathrm{sd}_2 B_\bullet A \to B_\bullet A \wedge_R B_\bullet A$. This map in degree m is the map

$$(\mathrm{sd}_2 B_\bullet A)_m = A^{(2m+1)} \longrightarrow A^{(m)} \wedge_R A^{(m)} = B_m A \wedge_R B_m A$$

that performs the augmentation $A \to R$ on the $(m+1)$st factor of A. We prove the following theorem.

Theorem 8.1 *Let N be a non-unital $\mathcal{C}_n$–algebra. The diagonal map $BKN \to BKN \wedge_R BKN$ above is a map of $\mathcal{C}_{n-1}$–algebras.*

Proof The edgewise subdivision functor and isomorphism on geometric realization preserve smash products of R–modules in the sense that the diagram of natural isomorphisms

$$\begin{tikzcd}
{|X_\bullet \wedge_R Y_\bullet|} \arrow[rr] \arrow[d] & & {|X_\bullet| \wedge_R |Y_\bullet|} \arrow[d] \\
{|\operatorname{sd}_2(X_\bullet \wedge_R Y_\bullet)|} \arrow[r] & {|(\operatorname{sd}_2 X_\bullet) \wedge_R (\operatorname{sd}_2 Y_\bullet)|} \arrow[r] & {|\operatorname{sd}_2 X_\bullet| \wedge_R |\operatorname{sd}_2 Y_\bullet|}
\end{tikzcd}$$

commutes. It therefore suffices to check that the map from $\operatorname{sd}_2 B_\bullet A$ to $B_\bullet A \wedge_R B_\bullet A$ is a map of simplicial $\mathcal{C}_{n-1}$–algebras (for $A = MKN$), and this is clear from the construction of the $\mathcal{C}_{n-1}$–structure. □

We close with a remark on power operations. For technical reasons about homotopy groups, we restrict to the context of R–modules of orthogonal spectra or EKMM S–modules for this discussion. Consider a non-unital true $\mathcal{C}_n$–algebra N and choose a representative map of R–modules $R^q_c \to N$ where R^q_c is some cofibrant version of the q–sphere R–module. We then get a map of non-unital true $\mathcal{C}_n$–algebras $\widetilde{\mathbb{C}}_n R^q_c \to N$, where (as above) $\widetilde{\mathbb{C}}_n$ denotes the free non-unital $\mathcal{C}_n$–algebra functor in R–modules. The induced map

$$\bigoplus_{m>0} R_*(\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} S^{(mq)}) \cong \pi_*(\widetilde{\mathbb{C}}_n R^q_c) \longrightarrow \pi_* N$$

depends only on the original $x \in \pi_q N$ and not on the choice of representative. Restricting to the mth homogeneous piece, we get the map

$$\mathcal{P}^m(x) \colon R_*(\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} S^{(mq)}) \longrightarrow \pi_* N,$$

which we think of as the total m–ary $\mathcal{C}_n$–algebra power operation of x; we think of $R_*(\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} S^{(mq)})$ as parametrizing the m–ary power operations on $\pi_q N$. We relate the power operations on π_q to the power operations on π_{q+1} using the suspension sequence

$$\widetilde{\mathbb{C}}_n R^q_c \longrightarrow \widetilde{\mathbb{C}}_n C R^q_c \longrightarrow \widetilde{\mathbb{C}}_n R^{q+1}_c.$$

The composite map is the trivial map and the middle term has a canonical contraction; this then defines a map

$$\pi_*(\widetilde{\mathbb{C}}_n R^q_c) \longrightarrow \pi_{*+1}(\widetilde{\mathbb{C}}_n R^{q+1}_c)$$

and in particular a map

$$\sigma\colon R_*(\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} S^{(mq)}) \longrightarrow R_{*+1}(\mathcal{C}_n(m)_+ \wedge_{\Sigma_m} S^{(m(q+1))})$$

In terms of our work above, we have the following result.

Theorem 8.2 *For a non-unital true $\mathcal{C}_n$–algebra N, the canonical map*

$$\sigma\colon \pi_* N \longrightarrow \pi_{*+1} \widetilde{B} N$$

preserves m–ary $\mathcal{C}_{n-1}$–algebra power operations for all m, meaning that the diagrams

$$\begin{tikzcd} R_*(\mathcal{C}_{n-1}(m)_+ \wedge_{\Sigma_m} S^{(mq)}) \arrow[r, "\sigma"] \arrow[d, "\mathcal{P}^m(x)"'] & R_{*+1}(\mathcal{C}_{n-1}(m)_+ \wedge_{\Sigma_m} S^{(m(q+1))}) \arrow[d, "\mathcal{P}^m(\sigma x)"] \\ \pi_* N \arrow[r, "\sigma"'] & \pi_{*+1} \widetilde{B} N \end{tikzcd}$$

commute for all $x \in \pi_q N$. Here we regard N as a non-unital $\mathcal{C}_{n-1}$–algebra via the last coordinates embedding of $\mathcal{C}_{n-1}$ in $\mathcal{C}_n$.

Proof We have an "E" version of the bar construction where $E_0 N = KMN$ and

$$E_\bullet N = \underbrace{KMN \wedge_R \ldots \wedge_R KMN}_{\bullet \text{ factors}} \wedge_R KMN$$

for $\bullet > 0$. Likewise, we have a $\widetilde{E}$ version such that $EN = K\widetilde{E}N$. Constructions analogous to those above make these into partial non-unital $\mathcal{C}_{n-1}$–algebras, and the inclusion of N in MN in $\widetilde{E}_0 N$ (as, say, $\{1\}_+ \wedge N \subset P_+ \wedge N$) induces a map of partial $\mathcal{C}_{n-1}$–algebras $N \to \widetilde{E}N$. The trivial map $MN \to *$ induces a map of partial $\mathcal{C}_{n-1}$–algebras $\widetilde{E}N \to \widetilde{B}N$, giving us a sequence of maps of partial $\mathcal{C}_{n-1}$–algebras

$$N \longrightarrow \widetilde{E}N \longrightarrow \widetilde{B}N$$

with the composite map $N \to \widetilde{B}N$ the trivial map. The usual simplicial contraction argument shows that $\widetilde{E}$ is contractible. Choosing a contraction, any map of R–modules $R^q_c \to N$ gives us a map of partial power systems $CR^q_c \to \widetilde{E}N$ and hence a map of partial power systems $\Sigma R^q_c \to \widetilde{B}N$. This correspondence lifts the map $\pi_* N \to \pi_{*+1} \widetilde{B}N$ to representatives. Applying the free functor $\widetilde{\mathbb{C}}^\sharp_{n-1}$ and unwinding the definition of

$$\sigma\colon R_*(\mathcal{C}_{n-1}(m)_+ \wedge_{\Sigma_m} S^{(mq)}) \longrightarrow R_{*+1}(\mathcal{C}_{n-1}(m)_+ \wedge_{\Sigma_m} S^{(m(q+1))})$$

above, the result follows. □

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