

ENDOTRIVIAL MODULES FOR THE QUATERNION GROUP AND ITERATED JOKERS IN CHROMATIC HOMOTOPY THEORY

ANDREW BAKER

ABSTRACT. The algebraic Joker module was originally described in the 1970s by Adams and Priddy and is a 5-dimensional module over the subHopf algebra $\mathcal{A}(1)$ of the mod 2 Steenrod algebra. It is a self-dual *endotrivial module*, i.e., an invertible object in the stable module category of $\mathcal{A}(1)$. Recently it has been shown that no analogues exist for $\mathcal{A}(n)$ with $n > 1$. In previous work the author used doubling to produce an ‘iterated double Joker’ which is an $\mathcal{A}(n)$ -module but not stably invertible. We also showed that for $n = 1, 2, 3$ these iterated doubles were realisable as cohomology of CW spectra, but no such realisation existed for $n > 3$.

The main point of the talk is to show that in the height 2 chromatic context, the Morava K -theory of double Jokers realise an exceptional endotrivial module over the quaternion group of order 8 that only exists over a field of characteristic 2 containing a primitive cube root of unity. This has connections with certain Massey products in the cohomology of the quaternion group.

INTRODUCTION

Following Adams & Priddy [AP76], in [Bak18, BB20] we considered the Joker $\mathcal{A}(1)$ -module and its iterated doubles over the finite subHopf algebras $\mathcal{A}(n) \subseteq \mathcal{A}$. We showed that for small values of n , there were spectra and spaces realising these. From an algebraic point of view, the original $\mathcal{A}(1)$ Joker module was important because it gave a self inverse stably invertible module, i.e., an element of order 2 in the Picard group of the stable module category of $\mathcal{A}(1)$. More recently, Bhattacharya & Ricka [BR17] and Pan & Yan have claimed that no such exotic elements can exist for $\mathcal{A}(n)$ when $n \geq 2$; their proof makes use of ideas found in the related study of endotrivial modules for group algebras, conveniently described in the recent book of Mazza [Maz19].

The main aim of this paper is to show that at least some of our geometric Joker spectra have Lubin-Tate cohomology which realises a certain lifting of a 5-dimensional endotrivial module over the quaternion group Q_8 and the field \mathbb{F}_4 . Here Q_8 is realised as a subgroup of the second Morava stabilizer group chromatic. This example suggests that in the chromatic setting there may be other interesting endotrivial modules associated with finite subgroups of Morava stabilizer groups; Lennart Meier has pointed out that this fits well with results in [CMNN20, appendix B].

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Conventions and notation: We will work at the prime $p = 2$ and chromatic height 2 when considering stable homotopy theory.

1. HOMOTOPY FIXED POINTS FOR FINITE SUBGROUPS OF MORAVA STABILIZER GROUPS

We briefly recall the general set-up for homotopy fixed point spectra of Lubin-Tate spectra, where the group involved is finite, although work of Devinatz & Hopkins [DH04] allows for more general subgroups of Morava stabilizer groups to be used. We will adopt the notation of Henn [Hen19]; in particular, \mathbb{G}_n is the *extended Morava stabilizer group*

$$\mathbb{G}_n = \mathbb{D}_n^\times / \langle S^n \rangle \cong \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \ltimes \mathcal{O}_n^\times,$$

where $S \in \mathbb{D}_n$ is the uniformizer satisfying $S^n = p$.

Example 1.1. For any prime p and $n \geq 1$, there is a unique central subgroup of order 2, namely $C_2 = \{\pm 1\} \triangleleft \mathbb{G}_n$. When $n = 1$ and $p = 2$, it is well known that $E_1^{C_2} \sim KO_2$.

For p odd, there is a unique central cyclic subgroup $C_{p-1} \triangleleft \mathbb{G}_n$ of order $p-1$, and when $n = 1$ $E_1^{hC_{p-1}}$ is the Adams summand of KU_p .

Example 1.2. When $p = 2 = n$, \mathbb{G}_2 contains a subgroup G_{24} of order 24 whose unique 2-Sylow subgroup is isomorphic to the quaternion group Q_8 . In fact this group is the semidirect product $C_3 \rtimes Q_8$ and there is also a split extension

$$G_{48} = \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \ltimes G_{24} \leq \mathbb{G}_2$$

of order 48 in the extended Morava stabilizer group. The fixed point spectrum $E_2^{hG_{48}}$ is an avatar of the spectrum of topological modular forms; see the article by Hopkins & Mahowald in [DFHH14, part III]. A subgroup $H \leq G_{48}$ gives rise to extensions $E_2^{hG_{48}} \rightarrow E_2^{hH} \rightarrow E_2$ where the latter is a faithful H -Galois extension in the sense of Rognes [Rog08]; this depends on work of Devinatz & Hopkins [DH04].

2. TWISTED GROUP RINGS AND THEIR MODULES

The results in this section are aimed at the specific circumstances that occur in chromatic homotopy theory. More general statements on twisted (or skew) group rings can be found in Passman [Pas, section 4]; Lam [Lam01, chapter 7] is a good source on local and semilocal rings. We begin with a result which includes both [Lam01, theorem 20.6] and [Pas, theorem 4.2] as special cases.

Recall from Lam [Lam01, §20] that a ring A is *semilocal* if $A/\text{rad } A$ is semisimple, where $\text{rad } A$ is the Jacobson radical of A .

Proposition 2.1. *Suppose that $A \subseteq B$ be a semilocal subring where B is finitely generated as a left A -module. Let $\mathfrak{a} \triangleleft A$ be a radical ideal, $\mathfrak{b} = \text{rad } B \triangleleft B$ and $B\mathfrak{a} \subseteq \mathfrak{a}B$. Then*

- (a) $\mathfrak{a} \subseteq \mathfrak{b}$;
- (b) B is semilocal;
- (c) *There is a $k \geq 1$ such that $\mathfrak{b}^k \subseteq B\mathfrak{a}$.*

Proof. (a) Let M be a simple left B -module. Then M is cyclic and so is finitely generated over B and therefore over A . Also,

$$B(\mathfrak{a}M) = (B\mathfrak{a})M \subseteq (\mathfrak{a}B)M = \mathfrak{a}(BM)\mathfrak{a}M,$$

so $\mathfrak{a}M \subseteq M$ is a B -submodule. If $\mathfrak{a}M \neq 0$ then the A -module M satisfies $\mathfrak{a}M = M$, so by Nakayama's Lemma, $M = 0$. So we must have $\mathfrak{a}M = 0$.

Since \mathfrak{a} annihilates every simple B -module, $\mathfrak{a} \subseteq \mathfrak{b}$.

(b) The finitely generated left A/\mathfrak{a} -module $B/B\mathfrak{a}B$ is also a ring which is left Artinian with radical $\mathfrak{b}/B\mathfrak{a}B$. This implies that the quotient ring B/\mathfrak{b} is left semisimple.

(c) The finitely generated left A/\mathfrak{a} -module $B/B\mathfrak{a}$ is left Artinian. The B -submodules $\mathfrak{b}^k/B\mathfrak{a}$ form a decreasing chain which must stabilize, so for some $k \geq 1$,

$$\mathfrak{b}^k/B\mathfrak{a} = \mathfrak{b}^{k+1}/B\mathfrak{a} = \mathfrak{b}(\mathfrak{b}^k/B\mathfrak{a}),$$

By Nakayama's Lemma $\mathfrak{b}^k/B\mathfrak{a}B = 0$, hence $\mathfrak{b}^k \subseteq B\mathfrak{a}$. □

The special case $\mathfrak{a} = \text{rad } A$ is particularly important. In practise we will consider the case where $B\mathfrak{a} = \mathfrak{a}B$ so $B\mathfrak{a} \triangleleft B$. This is true when R is a ring with a finite group acting on it by automorphisms; then the radical $\mathfrak{r} \triangleleft R$ is necessarily invariant so we can apply our results with $A = R$ and $B = R\langle G \rangle$, the twisted/skew group ring. This recovers [Pas, theorem 4.2]. We will discuss this special case in detail, making additional assumptions relevant in chromatic stable homotopy theory.

Let (R, \mathfrak{m}) be a complete and Hausdorff (i.e., $\bigcap_{r \geq 1} \mathfrak{m}^r = 0$) Noetherian commutative local ring with residue field $\kappa = R/\mathfrak{m}$ of positive characteristic p . Let G be a finite group which acts on R by (necessarily local) automorphisms, so that G also acts on κ by field automorphisms.

We can form the *twisted group rings* $R\langle G \rangle$ and $\kappa\langle G \rangle$; if the action of G on R or κ is trivial then we have the ordinary group ring $R[G]$ or $\kappa[G]$. The subset

$$\mathfrak{M} = R\langle G \rangle \mathfrak{m} = \mathfrak{m} R\langle G \rangle = \left\{ \sum_{g \in G} x_g g : x_g \in \mathfrak{m} \right\} \subseteq R\langle G \rangle$$

is a two-sided ideal with quotient ring $R\langle G \rangle / \mathfrak{M} \cong \kappa\langle G \rangle$. There is a maximal ideal

$$\mathfrak{n} = \left\{ \sum_{g \in G} y_g (g - 1) : y_g \in \kappa \right\} \triangleleft \kappa\langle G \rangle$$

with quotient ring $\kappa\langle G \rangle / \mathfrak{n} \cong \kappa$ defining the trivial $\kappa\langle G \rangle$ -module as well as the trivial $R\langle G \rangle$ -module $R\langle G \rangle / \mathfrak{M}$ where

$$\mathfrak{N} = \mathfrak{M} + \left\{ \sum_{g \in G} z_g (g - 1) : z_g \in R \right\} \triangleleft R\langle G \rangle.$$

Our next two results follow from our Proposition 2.1 as well as being special cases of [Pas, theorem 4.2].

Lemma 2.2.

- (a) $\kappa\langle G \rangle$ is semilocal;
- (b) The ideal $\mathfrak{M} \triangleleft R\langle G \rangle$ is a radical ideal and $R\langle G \rangle$ is semilocal.
- (c) The simple $R\langle G \rangle$ -modules are obtained by pulling back the simple modules of $\kappa\langle G \rangle$ along the quotient homomorphism $R\langle G \rangle \rightarrow \kappa\langle G \rangle$.

Proof. (a) This follows from Artin-Wedderburn theory since $\kappa\langle G \rangle$ is a finite dimensional κ -vector space and hence Artinian.

(b) Use Proposition 2.1.

(c) This follows from (b). □

A detailed discussion of lifting of idempotents and results on Krull-Schmidt decompositions for complete local Noetherian rings can be found in Lam [Lam01, section 21].

Now we can deduce an important special case.

Lemma 2.3. *Suppose that G is a p -group. Then*

- (a) $\kappa\langle G \rangle$ is local with unique maximal left/right ideal \mathfrak{n} equal to the radical $\text{rad } \kappa\langle G \rangle$;
- (b) $R\langle G \rangle$ is local with unique maximal left/right ideal \mathfrak{N} .

Hence $R\langle G \rangle$ and $\kappa\langle G \rangle$ each have the unique simple module κ .

Proof. (a) Suppose that S is a (non-trivial) simple left $\kappa\langle G \rangle$ -module. For $0 \neq s \in S$, consider the finite dimensional \mathbb{F}_p -subspace $\mathbb{F}_p[G]s \subseteq S$ whose cardinality is a power of p . It is also a non-trivial finite $\mathbb{F}_p[G]$ -module, so the p -group G acts linearly with 0 as a fixed point. Since every orbit has cardinality equal to a power of p there must be at least one other fixed point $v \neq 0$ and this spans a $\kappa\langle G \rangle$ -submodule $\kappa v \subseteq S$. It follows that $S = \kappa v \cong \kappa$. Of course if the G -action on κ is trivial, $\kappa\langle G \rangle = \kappa[G]$ and this argument is well-known.

(b) This is immediate from (a) together with parts (b) and (c) of Lemma 2.2. \square

Corollary 2.4. *If G is a p -group, then $\mathfrak{N} \triangleleft R\langle G \rangle$ is the unique maximal ideal and $R\langle G \rangle$ is \mathfrak{N} -adically complete and Hausdorff.*

Proof. This follows from Proposition 2.1(c): some power of \mathfrak{N} is contained in $\mathfrak{M} = R\langle G \rangle \mathfrak{m}$, and for $k \geq 1$, $\mathfrak{M}^k = R\langle G \rangle \mathfrak{m}^k \subseteq \mathfrak{N}^k$. Therefore the \mathfrak{N} -adic, \mathfrak{M} -adic and \mathfrak{m} -adic topologies agree. \square

We recall that for a local ring, every projective module is free by a theorem of Kaplansky [Kap58, theorem 2], so in statements involving local rings, projective modules can be taken to be free.

Lemma 2.5.

- (a) *Let P be a projective $R\langle G \rangle$ -module. Then P is a projective R -module.*
- (b) *Let Q be a finitely generated projective $\kappa\langle G \rangle$ -module. Then there is a projective $R\langle G \rangle$ -module \tilde{Q} for which $\kappa\langle G \rangle \otimes_{R\langle G \rangle} \tilde{Q} \cong Q$.*

Proof. (a) Every projective module is a retract of a free module and $R\langle G \rangle$ -module is a free R -module.

(b) By the Krull-Schmidt theorem, we may express Q as a coproduct of projective indecomposable $\kappa\langle G \rangle$ -modules, so it suffices to assume Q is a projective indecomposable, hence cyclic. Viewing Q as an $R\langle G \rangle$ -module we can choose a cyclic projective module \tilde{Q} with an epimorphism $\pi: \tilde{Q} \rightarrow Q$. \square

We will make use of the following result.

Lemma 2.6. *Suppose that M is an $R\langle G \rangle$ -module which is finitely generated free as an R -module. If $\kappa \otimes_R M$ is an endotrivial $\kappa\langle G \rangle$ -module, then M is an endotrivial $R\langle G \rangle$ -module.*

Proof. Let $\text{End}_R(M) = \text{Hom}_R(M, M)$ with its usual left $R\langle G \rangle$ -module structure. If $\kappa \otimes_R M$ is endotrivial then as $\kappa\langle G \rangle$ -modules,

$$\kappa \otimes_R \text{End}_R(M) \cong \text{End}_{\kappa}(\kappa \otimes_R M, \kappa \otimes_R M) \cong \kappa \oplus P$$

where P is a projective $\kappa\langle G \rangle$ -module. Recall that the units give monomorphisms $R \rightarrow \text{End}_R(M)$ and $\kappa \rightarrow \kappa \otimes_R \text{End}_R(M)$, where the latter is split.

Now choose a projective $R\langle G \rangle$ -module \tilde{P} with an epimorphism $\pi: \tilde{P} \rightarrow P$ and $\kappa \otimes_R \tilde{P} \cong P$. There is a commutative diagram of solid arrows with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & R & \longrightarrow & \text{End}_R(M) & \xrightarrow{\pi''} & \tilde{P} \\
& & \downarrow & & \downarrow & \swarrow \pi' & \downarrow \pi \\
0 & \longrightarrow & \kappa & \longrightarrow & \kappa \otimes_R \text{End}_R(M) & \xrightarrow{\sigma} & P \longrightarrow 0
\end{array}$$

and the composition $\sigma \circ \pi$ lifts to $\pi'': \tilde{P} \rightarrow \text{End}_R(M)$. On applying $\kappa \otimes_R (-)$ to the composition

$$\tilde{P} \xrightarrow{\pi''} \text{End}_R(M) \xrightarrow{\pi} P$$

we obtain the composition

$$P \xrightarrow{\sigma} \kappa \otimes_R \text{End}_R(M) \rightarrow P$$

which is an epimorphism. Using Nakayama's Lemma we now see that $\text{End}_R(M) \rightarrow \tilde{P}$ is an epimorphism, hence $\text{End}_R(M) \cong R \oplus \tilde{P}$ and so M is endotrivial. \square

Although we don't really make use of this, we note that an appropriate dual of a twisted group ring over a commutative ring admits the structure of a Hopf algebroid; a discussion of this appears in the appendix of [Bak95].

3. A FINITE GROUP OF OPERATIONS IN LUBIN-TATE THEORY OF HEIGHT 2

Our work requires an explicit realisation of Q_8 as a subgroup of the height 2 Morava stabilizer group. We follow the account and notation of Henn [Hen19, section 2], especially lemma 2.1.

The ring of *Hurwitz quaternions* \mathcal{H} is the subdomain of \mathbb{H} additively generated by the elements

$$\frac{(\pm 1 \pm i \pm j \pm k)}{2}.$$

It has a unique completely prime maximal ideal \mathcal{M} which contains 2 as well as $i+1, j+1, k+1$. The quotient ring is a field with 4 elements,

$$\mathbb{F}_4 = \mathcal{H}/\mathcal{M} = \mathbb{F}_2(\omega),$$

where ω denotes (the residue class of) the primitive cube root of unity

$$\omega = -\frac{(1+i+j+k)}{2}.$$

Routine calculations show that

$$i\omega i^{-1} = \omega + j + k \equiv \omega \pmod{\mathcal{M}}$$

and also

$$j\omega j^{-1} \equiv \omega \equiv k\omega k^{-1} \pmod{\mathcal{M}},$$

therefore the quaternion subgroup $Q_8 = \langle i, j \rangle \leq \mathcal{H}^\times$ acts trivially of \mathbb{F}_4 and we may form the (trivially twisted) group ring $\mathbb{F}_4\langle Q_4 \rangle = \mathbb{F}_4[Q_4]$.

We can complete \mathcal{H} with respect to \mathcal{M} or equivalently 2, to obtain a model for the maximal order \mathcal{O}_2 in the division algebra $\mathbb{D}_2 = \mathcal{H}_{\mathcal{M}}$. In fact

$$\mathbb{D}_2 = \mathbb{Z}_4\langle S \rangle / (S^2 - 2),$$

where $\mathbb{Z}_4 = W(\mathbb{F}_4) = \mathbb{Z}_2(\omega)$ is the ring of Witt vectors for \mathbb{F}_4 and the uniformizer S intertwines with \mathbb{Z}_4 so that $S(-)S^{-1}$ is the lift of Frobenius (and so S^2 acts trivially). The quotient group

$$\mathbb{G}_2 = \mathbb{D}_2^\times / \langle S^2 \rangle \cong \text{Gal}(\mathbb{F}_4/\mathbb{F}_2) \ltimes \mathcal{O}_2^\times$$

is the *extended Morava stabilizer group*.

Here is an explicit description for elements of Q_8 in terms of Teichmüller expansions as in [Hen19, lemma 2.1]:

$$(3.1) \quad i = \frac{1}{3}(1 + 2\omega^2)(1 - aS), \quad j = \frac{1}{3}(1 + 2\omega^2)(1 - a\omega^2 S), \quad k = \frac{1}{3}(1 + 2\omega^2)(1 - a\omega S),$$

where we choose $\sqrt{-7} \in \mathbb{Z}_2$ to be the square root of -7 satisfying $\sqrt{-7} \equiv 5 \pmod{8}$ and set

$$a = \frac{1 - 2\omega}{\sqrt{-7}} \in \mathbb{Z}_4.$$

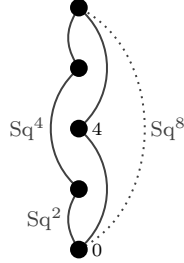
Notice that working modulo $S^2 = 2$ in \mathcal{O}_2 ,

$$(3.2) \quad i \equiv 1 + S, \quad j \equiv 1 + \omega^2 S, \quad k \equiv 1 + \omega S.$$

Of course there is a twisted group ring $(E_2)_0 \langle Q_8 \rangle$ which has $\mathbb{F}_4[Q_4]$ as a quotient ring.

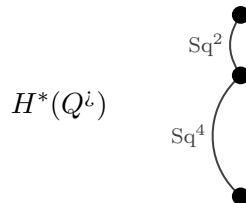
4. LUBIN-TATE THEORY FOR DOUBLE JOKER SPECTRA

Let $J = J(2)$ be one of the finite CW spectra constructed in [Bak18]. Its mod 2 cohomology is the cyclic $\mathcal{A}(2)$ -module $H^*(J) \cong \mathcal{A}(2)/\mathcal{A}(2)\{Q^0, Q^1, Q^2, \text{Sq}^6\}$ (here the Q^i are the Milnor primitives), and there are two possible extensions to an \mathcal{A} -module with trivial or non-trivial Sq^8 -action giving dual \mathcal{A} -modules.

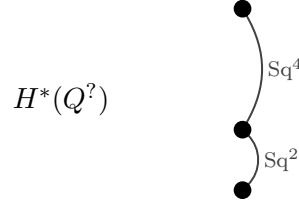


The attaching maps in such a CW spectrum are essentially suspensions of η and ν . Up to homotopy equivalence there are two such spectra which are Spanier-Whitehead dual to each other and realise the two \mathcal{A} -module extensions.

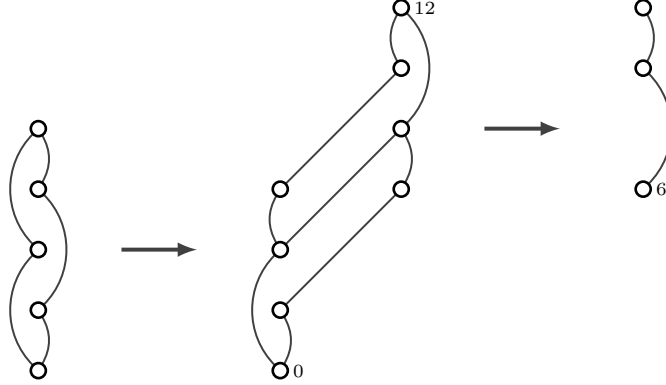
There is a CW spectrum $dA(1)$ known as ‘the double of $\mathcal{A}(1)$ ’ whose cohomology as an $\mathcal{A}(2)$ -module is $H^*(dA(1)) \cong \mathcal{A}(2)//\mathcal{E}(2)$; for a detailed discussion see Bhattacharya et al [BEM17]. In [Bak18, remark 5.1] we outlined how to construct such a spectrum starting with a double Joker and attaching cells. By construction, $dA(1)$ contains J as a subcomplex with cofibre a suspension of the ‘upside-down double question mark’ complex Q^i whose cohomology is 3-dimensional and has a non-trivial action of $\text{Sq}^2 \text{Sq}^4$.



This is stably Spanier-Whitehead dual to the ‘double question mark’ complex $Q^?$ whose cohomology has a non-trivial action of $\text{Sq}^4 \text{Sq}^2$.



Thus we have the cofibre sequence $J \rightarrow dA(1) \rightarrow \Sigma^6 Q^i$ as shown.



We can apply a complex oriented homology theory to this cofibre sequence, thus obtaining a short exact sequence; in particular we will apply $BP_*(-)$, $(E_2)_*(-)$ or $(K_2)_*(-)$. Our goal is to understand the Lubin-Tate cohomology $E_2^*(J)$ as a left $E_2^*\langle Q_8 \rangle$ -module where $Q_2 \leq \mathbb{G}_2$ is a quaternion subgroup. Since $E_2^*(J)$ is a finitely generated free module and J is dualizable, we can instead work with right module $(E_2)_*(J)$ in terms of the corresponding $(E_2)_*(E_2)$ -comodule structure. Actually we prefer to work directly with the smaller complex Q^i and use the fact $(E_2)_*(dA(1))$ and $(E_2)^*(dA(1))$ are free $E_2^*\langle Q_8 \rangle$ -modules of rank 1: this is well-known and appears in Hopkins & Mahowald [DFHH14, part III], but a detailed discussion also occurs in [BR21, lemma 1.42]. The key point is to use the equivalence $E_2 \wedge dA(1) \sim E_2$ together with results of Devinatz & Hopkins [DH04]. So our main calculational result identifies the right $E_2^*\langle Q_8 \rangle$ -module $(E_2)_*(Q^i)$; we will do this by first describing the $K_2^*[Q_8]$ -module $(K_2)_*(Q^i)$.

Here is our main result.

Theorem 4.1. *The $E_2^*\langle Q_8 \rangle$ -module $E_2^*(J)$ is stably invertible and self dual, and its reduction to $K_2^*(J)$ is a 5-dimensional stably invertible $K_2^*[Q_8]$ -module.*

Of course we can reduce to studying $K_2^0(J)$ as a $K_2^0[Q_8] = \mathbb{F}_4[Q_8]$ -module. The 5-dimensional stably invertible $\mathbb{F}_4[Q_8]$ -module W_5 is that of [Maz19, theorem 3.8(1)] and this is ΩW_3 for a 3-dimensional stably invertible $\mathbb{F}_4[Q_8]$ -module W_3 which we will show is isomorphic to $(K_2)_0(Q^i)$. The lifting of results to the Lubin-Tate setting uses the algebra discussed in Section 2. Of course we need to do some topological calculations to obtain these results and these are outlined in the next section.

5. CALCULATIONS

Homological algebra conventions. Before describing the calculations required, we explain our notational conventions for homological algebra.

Given a flat Hopf algebroid (A, Γ) which might be graded and two left Γ -comodules L, M , we denote by $\text{Cohom}_\Gamma(L, M)$ the set of comodule homomorphisms $L \rightarrow M$, and $\text{Coext}_\Gamma^{s,*}(L, -)$ for the s -th right derived functor of $\text{Cohom}_\Gamma^*(L, -)$ where $*$ indicates the internal degree shift. When the grading is trivial (i.e., concentrated in degree 0) we write $\text{Cohom}_\Gamma^s(L, -)$ and $\text{Coext}_\Gamma^s(L, -)$.

If G is a finite group and R is a (possibly graded) commutative ring, then the group ring $R[G]$ is a Hopf algebra over R and its dual $R(G) = \text{Hom}_R(R[G], R)$ forms a commutative Hopf algebra $(R(G), R)$. Every left $R(G)$ -comodule becomes a right $R[G]$ -module in a natural way, and vice versa. Moreover, for a left $R(G)$ -comodule L there is a natural isomorphism

$$\text{Cohom}_{R(G)}^*(L, -) \cong \text{Hom}_{R[G]}^*(L, -)$$

and this induces natural isomorphisms of right derived functors

$$(5.1) \quad \text{Coext}_{R(G)}^{s,*}(L, -) \cong \text{Ext}_{R[G]}^{s,*}(L, -).$$

Finally, if R is a graded ring of the form $R = \mathbb{k}[w, w^{-1}]$ where w has even positive degree, then $R[G] = R \otimes_{\mathbb{k}} \mathbb{k}[G]$ and $R(G) = R \otimes_{\mathbb{k}} \mathbb{k}(G)$. When L has the form $L = R \otimes_{\mathbb{k}} L_0$, then

$$(5.2) \quad \text{Ext}_{R[G]}^{s,*}(L, -) \cong \text{Ext}_{\mathbb{k}[G]}^{s,*}(L_0, -).$$

Comodules for some iterated mapping cones. We begin by recalling that the mapping cones of the Hopf invariant 1 elements have the following BP -homology as $BP_*(BP)$ -comodules, where x_k has degree k and x_0 is coaction primitive. Here

$$BP_*(C(\eta)) = BP_*\{x_0, x_2\}, \quad BP_*(C(\nu)) = BP_*\{x_0, x_4\}, \quad BP_*(C(\sigma)) = BP_*\{x_0, x_8\},$$

with

$$(5.3a) \quad \psi(x_2) = t_1 \otimes x_0 + 1 \otimes x_2,$$

$$(5.3b) \quad \psi(x_4) = (v_1 t_1 + t_1^2) \otimes x_0 + 1 \otimes x_4.$$

$$(5.3c) \quad \begin{aligned} \psi(x_8) &= (v_2 t_1 - 3t_1^4 - v_1^3 t_1 - 4v_1^2 t_1^2 - 5v_1 t_1^3 + v_1 t_2 + 2t_1 t_2) \otimes x_0 + 1 \otimes x_8 \\ &\equiv (v_2 t_1 + t_1^4) \otimes x_0 + 1 \otimes x_8 \pmod{(2, v_1)}. \end{aligned}$$

Such formulae are well-known and follow from the fact that these homotopy elements are detected by elements that originate in the chromatic spectra sequence on

$$\begin{aligned} v_1/2 &\in \text{Coext}_{BP_*(BP)}^{0,2}(BP_*, BP_*/2^\infty), \quad v_1^2/4 \in \text{Coext}_{BP_*(BP)}^{0,4}(BP_*, BP_*/2^\infty), \\ (v_1^4 + 8v_1 v_2)/16 &\in \text{Coext}_{BP_*(BP)}^{0,8}(BP_*, BP_*/2^\infty); \end{aligned}$$

see [MRW77, Rav78] for details.

Now given a map of ring spectra $BP \rightarrow E$, where E is Landweber exact, these $BP_*(BP)$ -comodules map to $E_*(E)$ -comodules $E_*(C(\eta))$ and $E_*(C(\nu))$. Our main interest will focus on the examples $E = E_2$ and $E = K_2$. In the latter case we have $(K_2)_* = \mathbb{F}_4[u, u^{-1}]$ where $u \in (K_2)_2$ and we set $\bar{x}_{2k} = u^{-k} x_{2k} \in (K_2)_0(C(\gamma))$ when $\gamma = \eta, \nu$. The Hopf algebroid here is

$$(K_2)_*(E_2) = (K_2)_*[\alpha_r : r \geq 0]/(\alpha_0^3 - 1, \alpha_r^4 - \alpha_r : r \geq 1),$$

where the right unit on u is $\eta_*(u) = u\alpha_0$ and the image of $t_k \in BP_{2k+1-2}(BP)$ is

$$u^{2^k-1} \alpha_k \in (K_2)_{2k+1-2}(E_2).$$

It is standard that every element of \mathcal{O}_2^\times has a unique series expansion as $\sum_{r \geq 0} a_r S^r$, where the Teichmüller representatives a_r satisfy

$$a_0^3 = 1, \quad a_r^4 = a_r \quad (r \geq 1).$$

Then we may identify $(K_2)_0(E_2)$ with the algebra of continuous maps $\mathcal{O}_2^\times \rightarrow \mathbb{F}_4$ and then α_k is identified with the locally constant function given by

$$\alpha_k \left(\sum_{r \geq 0} a_r S^r \right) = a_k.$$

The left $(K_2)_*(E_2)$ -coaction on a comodule M_* induces an adjoint right action of \mathcal{O}_2^\times . For any finite subgroup $G \leq \mathcal{O}_2^\times$ there is an induced action of the twisted group ring $(K_2)^* \langle G \rangle$. This also gives a right action of $\mathbb{F}_4 \langle G \rangle$ on each M_k . Of course we are using the $(K_2)_*$ -linear pairing $M_* \otimes_{(K_2)_*} M^* \rightarrow (K_2)_*$ to define this. Standard linear algebra says that when M_* is finite dimensional over $(K_2)_*$, given a basis for M_* and the dual basis for $M^* = \text{Hom}_{(K_2)_*}(M_*, (K_2)_*)$, the matrices for expressing the action on M_* and its adjoint action on M^* are mutually transpose.

$(K_2)_0(\mathbf{C}(\eta))$. Here we have the coaction formulae

$$\bar{x}_0 \mapsto 1 \otimes \bar{x}_0, \quad \bar{x}_2 \mapsto \alpha_1 \otimes \bar{x}_0 + \alpha_0 \otimes \bar{x}_2,$$

and the right action Q_8 has matrix representations with respect to the basis \bar{x}_0, \bar{x}_2 obtained from the above discussion together with (3.1) and (3.2).

$$i: \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad j: \begin{bmatrix} 1 & \omega^2 \\ 0 & 1 \end{bmatrix}, \quad k: \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}.$$

$(K_2)_0(\mathbf{C}(\nu))$. The coaction is

$$\bar{y}_0 \mapsto 1 \otimes \bar{y}_0, \quad \bar{y}_4 \mapsto \alpha_1^2 \otimes \bar{y}_0 + \alpha_0^2 \otimes \bar{y}_4,$$

and the matrix representation is

$$i: \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad j: \begin{bmatrix} 1 & \omega \\ 0 & 1 \end{bmatrix}, \quad k: \begin{bmatrix} 1 & \omega^2 \\ 0 & 1 \end{bmatrix}.$$

$(K_2)_0(S^0 \cup_\nu e^4 \cup_\eta e^6)$. Here we have a basis $\bar{z}_0, \bar{z}_4, \bar{z}_6 \in (K_2)_0(S^0 \cup_\nu e^4 \cup_\eta e^6)$ and

$$\bar{z}_0 \mapsto 1 \otimes \bar{z}_0, \quad \bar{z}_4 \mapsto \alpha_1^2 \otimes \bar{z}_0 + \alpha_0^2 \otimes \bar{z}_4, \quad \bar{z}_6 \mapsto * \otimes \bar{z}_0 + \alpha_0^2 \alpha_1 \otimes \bar{z}_4 + 1 \otimes \bar{z}_6,$$

$$i: \begin{bmatrix} 1 & 1 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad j: \begin{bmatrix} 1 & \omega & * \\ 0 & 1 & \omega^2 \\ 0 & 0 & 1 \end{bmatrix}, \quad k: \begin{bmatrix} 1 & \omega^2 & * \\ 0 & 1 & \omega \\ 0 & 0 & 1 \end{bmatrix}.$$

$K_2^0(\mathbf{C}(\sigma))$. Here the relation $\alpha_1^4 + \alpha_1 = 0$ gives

$$\bar{x}_0 \mapsto 1 \otimes \bar{x}_0, \quad \bar{x}_8 \mapsto (\alpha_1^4 + \alpha_1) \otimes \bar{x}_0 + \alpha_0 \otimes \bar{x}_8 = \alpha_0 \otimes \bar{x}_8,$$

so i, j, k all act trivially.

$(K_2)_0(S^0 \cup_\sigma e^8 \cup_\nu e^{12})$. Here we have a basis $\bar{z}_0, \bar{z}_8, \bar{z}_{12} \in (K_2)_0(S^0 \cup_\sigma e^8 \cup_\nu e^{12})$ and

$$\bar{z}_0 \mapsto 1 \otimes \bar{z}_0, \quad \bar{z}_8 \mapsto \alpha_0 \otimes \bar{z}_8, \quad \bar{z}_{12} \mapsto * \otimes \bar{z}_0 + \alpha_1^2 \otimes \bar{z}_8 + \alpha_0^2 \otimes \bar{z}_{12},$$

$$i: \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad j: \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & \omega \\ 0 & 0 & 1 \end{bmatrix}, \quad k: \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & \omega^2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here $Z(Q_8)$ acts trivially so the representation factors through the abelianisation, hence this does not give a stably invertible Q_8 -module.

6. GROUP COHOMOLOGICAL INTERPRETATION

The reader may find it useful to relate the results in this section to Ravenel [Rav77, proposition 3.5].

Recall that for any field \mathbf{k} of characteristic 2, the cohomology of $\mathbf{k}[Q_8]$ has the form

$$(6.1) \quad \text{Ext}_{\mathbf{k}[Q_8]}^*(\mathbf{k}, \mathbf{k}) = \mathbf{k}[u, v, w]/(u^2 + uv + v^2, u^2v + uv^2, u^3, v^3),$$

where u, v have degree 1 and w has degree 4. This result can be found in [AM04, lemma IV.2.10] for example.

Of course $\text{Ext}_{\mathbf{k}[Q_8]}^1(\mathbf{k}, \mathbf{k})$ can be identified with the group of all homomorphisms $Q_8 \rightarrow \mathbf{k}$ into the additive group of \mathbf{k} . We will need to make explicit choices for the generators and we define them to be the homomorphisms $u, v: Q_8 \rightarrow \mathbf{k}$ given by

$$u(i) = 1, \quad u(j) = 0, \quad v(i) = 0, \quad v(j) = 1.$$

The functions $\alpha_1, \alpha_1^2: Q_8 \rightarrow \mathbf{k}$ are also homomorphisms and can be expressed as

$$\alpha_1 = u + \omega^2 v, \quad \alpha_1^2 = u + \omega v.$$

Notice that when \mathbf{k} does not contain a primitive cube root of unity, $u^2 + uv + v^2$ does not factor, but if $\omega \in \mathbf{k}$ is a primitive cube root of unity then

$$u^2 + uv + v^2 = (u + \omega v)(u + \omega^2 v).$$

This means that the Massey product $\langle u + \omega^2 v, u + \omega v, u + \omega^2 v \rangle \subseteq \text{Ext}_{\mathbf{k}[Q_8]}^2(\mathbf{k}, \mathbf{k})$ is defined and this has indeterminacy $\mathbf{k}\{u^2 + \omega v^2\}$.

The Massey product

$$\langle [t_1], [v_1 t_1 + t_1^2], [t_1] \rangle = \{[v_1 t_1 + t_1^2]^2\} \subseteq \text{Coext}_{BP_*(BP)}^{2,8}(BP_*, BP_*)$$

corresponds to the Toda bracket

$$\langle \eta, \nu, \eta \rangle = \{\nu^2\} \subseteq \pi_6(S).$$

We can exploit naturality in cohomology of Hopf algebroids together with (5.1) and (5.2) to obtain an algebra homomorphism

$$\text{Coext}_{BP_*(BP)}^*(BP_*, BP_*) \rightarrow \text{Ext}_{(K_2)_*[Q_8]}^*((K_2)_*, (K_2)_*) \xrightarrow{\cong} (K_2)_* \otimes_{\mathbb{F}_4} \text{Ext}_{\mathbb{F}_4[Q_8]}^*(\mathbb{F}_4, \mathbb{F}_4).$$

Our calculations show that under this

$$[t_1] \mapsto u(u + \omega^2 v), \quad [v_1 t_1 + t_1^2] \mapsto u^2(u + \omega v),$$

hence $\langle u + \omega^2 v, u + \omega v, u + \omega^2 v \rangle$ must contain $(u + \omega v)^2 = u^2 + \omega^2 v^2$. It follows that for any extension field \mathbf{k} of \mathbb{F}_4 ,

$$\langle u + \omega^2 v, u + \omega v, u + \omega^2 v \rangle = \mathbf{k}\{u + \omega v\} + (u^2 + \omega^2 v^2) \subsetneq \text{Ext}_{\mathbf{k}[Q_8]}^2(\mathbf{k}, \mathbf{k}).$$

Of course this could also be verified directly using a good choice of resolution of \mathbf{k} over $\mathbf{k}[Q_8]$.

We remark that the Massey product

$$\langle [v_1 t_1 + t_1^2], [t_1], [v_1 t_1 + t_1^2] \rangle \subseteq \text{Coext}_{BP_*(BP)}^{2,10}(BP_*, BP_*)$$

corresponds to the Toda bracket

$$\langle \nu, \eta, \nu \rangle = \{\eta\sigma + \varepsilon\} \subseteq \pi_8(S),$$

and is related to the Massey product

$$\langle u + \omega v, u + \omega^2 v, u + \omega v \rangle = \mathbf{k}\{u + \omega^2 v\} + (u^2 + \omega v^2) \subsetneq \text{Ext}_{\mathbf{k}[Q_8]}^2(\mathbf{k}, \mathbf{k}).$$

CONCLUSION

The main import of this paper is the appearance of unexpected relationships between seemingly disparate topics. It has long been hinted at that there may be some connections between the cohomology of the $\mathcal{A}(n)$ and that of finite groups, although few direct connections have been found. The case of Q_8 is one where such connections have been observed and we provide further evidence of this. But it is unclear whether there are other examples, perhaps in higher chromatic heights.

We also remark that the actions of the groups G_{24} and G_{48} mentioned in Example 1.2 on $K_2^*(J)$ have fixed points $\mathbb{F}_4 \otimes K(2)^*(J)$ and $K(2)^*(J)$ with coefficient rings $\mathbb{F}_4[v_2, v_2^{-1}]$ and $\mathbb{F}_2[v_2, v_2^{-1}]$. Neither of these supports the endotrivial module W_5 because of the way the gradings spread out the action.

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SCHOOL OF MATHEMATICS & STATISTICS, UNIVERSITY OF GLASGOW, GLASGOW G12 8QQ, SCOTLAND.

Email address: `andrew.j.baker@glasgow.ac.uk`

URL: `http://www.maths.gla.ac.uk/~ajb`