Invent. math. 135, 743–788 (1999)
DOI 10.1007/s002229900898

Inventiones mathematicae

The Goodwillie tower of the identity functor and the unstable periodic homotopy of spheres

Greg Arone[1], **Mark Mahowald**[2]

[1] Department of Mathematics, University of Chicago, 5734 University Avenue, Chicago, IL 60637, USA (E-mail address: arone@math.uchicago.edu)
[2] Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60288-2730, USA

Oblatum 5-VIII-1996 & 9-VI-1998 / Published online: 14 January 1999

Contents

Abstract. We investigate Goodwillie's "Taylor tower" of the identity functor from spaces to spaces. More specifically, we reformulate Johnson's description of the Goodwillie derivatives of the identity, and prove that in the case of an odd-dimensional sphere the only layers in the tower that are not contractible are those indexed by powers of a prime. Moreover, in the case of a sphere the tower is finite in v_k-periodic homotopy.

Mathematics Subject Classification (*1991*). 05A18; 55P47; 55Q40; 55S12; 55P42; 55R12; 55S10.
The first author was partially supported by the Alexander von Humboldt foundation.

0. Introduction

In this paper we analyze the Goodwillie tower of the identity functor evaluated at spheres. We find that in the case of spheres the tower exhibits a pleasant and surprising behavior. Broadly speaking, we find two new facts that are not consequences of the general theory of calculus. First, in the case of an odd-dimensional sphere localized at a prime p, the only "layers" (homotopy fibers) in the Goodwillie tower of the identity that are not contractible are the ones that are indexed by powers of p. Thus the tower "converges exponentially faster" in this case than it does in general. Second, the stable cohomology of the p^k-th layer is free over $A[k-1]$, where $A[k-1]$ is a certain finite sub-Hopf algebra of the Steenrod algebra (to be defined in section 3.2). This implies, in particular, that in our case all the layers beyond the p^k-th one are trivial in v_k-periodic homotopy (for any reasonable definition of the latter). Thus, in v_k-periodic homotopy the tower has only $k+1$ non-trivial layers, namely $p^0, p^1, \ldots, p^k$.

The two facts imply that the unstable v_k-periodic homotopy of an odd dimensional sphere can be resolved into a tower of fibrations with $k+1$ stages, with infinite loop spaces as fibers. As indicated above, the fibers are analyzed here to a considerable extent. For instance, their stable cohomology is completely calculated.

In the body of the paper we will assume basic familiarity with the notion of "v_k-periodic" homotopy of spaces and spectra. For an informal discussion of the concept, together with references to a more complete discussion, see appendix A. We will also assume familiarity with the basic ideas of Goodwillie's "Calculus of Functors". The basic references for this material are [G90, G92, G3].

We now proceed with a more detailed overview of the paper, its genesis and its goals. The simplest example of periodic homotopy is v_0-periodic homotopy, which is essentially the same as rational homotopy. There is an old theorem of Serre on rational homotopy of spheres, which implies that if X is an odd-dimensional sphere, then the map $X \to \Omega^\infty\Sigma^\infty X$ induces an equivalence in v_0-periodic homotopy. Thus in the v_0-periodic world the unstable homotopy of an odd sphere is the same as its stable homotopy.

In [MT92] Mahowald and Thompson found an analogue of Serre's theorem for v_1-periodic homotopy. Roughly speaking, v_1-periodic homotopy is the homotopy theory one obtains by inverting the maps that induce an isomorphism in K-theory. For a based topological space X, let $P_2(X)$ be the homotopy fiber of the well-known natural map $\Omega^\infty\Sigma^\infty(X) \to \Omega^\infty\Sigma^\infty(X \wedge X \wedge_{\Sigma_2} E\Sigma_{2+})$ which may be defined, at least up to homotopy, as the adjoint of the composed map

$$\Sigma^\infty\Omega^\infty\Sigma^\infty(X) \xrightarrow{\simeq} \bigvee_{i=1}^{\infty} \Sigma^\infty(X^{\wedge i} \wedge_{\Sigma_i} E\Sigma_{i+}) \to \Sigma^\infty(X \wedge X \wedge_{\Sigma_2} E\Sigma_{2+})$$

where the first map is given by the Snaith splitting and the second map is collapsing on the factor corresponding to $i = 2$. Mahowald – Thompson's work implies that if X is an odd sphere (localized at 2) then the natural map $X \to P_2(X)$ induces an equivalence in v_1-periodic homotopy. Using this result, the v_1-periodic homotopy of spheres has actually been computed in [M82] at the prime 2 and in [T90] at odd primes. From this information, it is also possible to recover the "integral" v_1-periodic homotopy. This is done in several places [MT92, T90].

From one point of view, our goal here is to extend the work of Mahowald and Thompson cited above to higher order periodicity. The first technical difficulty with it seemed to be that this work used the existence of maps connected with the Snaith splitting. Such maps are constructed by means of configuration space methods. The homotopy fibers of these maps do not have nice configuration space models and thus do not allow new maps to be constructed in the same way. Instead, we use the Goodwillie tower of the identity functor, which turned out to be the perfect tool for attacking this problem. We will analyze this tower of fibrations in the general case to some extent and apply this understanding to spheres.

The Goodwillie tower of the identity ("the Taylor tower of the identity" in Goodwillie's terminology) is a sequence of functors (from pointed spaces to pointed spaces) $P_n(X)$ and a tower of natural transformations

$$\begin{array}{ccccc}
 & & \vdots & & \\
 & & \downarrow & & \\
X & \longrightarrow & P_n(X) & \longleftarrow & D_n(X) \\
 & \searrow & f_n \downarrow & & \\
 & & P_{n-1}(X) & \longleftarrow & D_{n-1}(X) \\
 & & f_{n-1} \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
 & & P_1(X) \simeq Q(X) := \Omega^\infty\Sigma^\infty(X) & &
\end{array}$$

The functor D_n is the homotopy fiber of the natural transformation $P_n \to P_{n-1}$ and should be thought of as the n-th homogeneous layer, or the n-th differential of the identity. It follows from the general theory of calculus [G3] that for every n there exists a spectrum $\mathbf{C}_n$, endowed with an action of the symmetric group Σ_n, such that

$$D_n(X) \simeq \Omega^\infty((\mathbf{C}_n \wedge X^{\wedge n})_{h\Sigma_n}) := \Omega^\infty((\mathbf{C}_n \wedge X^{\wedge n} \wedge E\Sigma_{n+})_{\Sigma_n}) \ .$$

Here, as well as everywhere else in the paper, $\simeq$ stands for "weakly homotopy equivalent". The spectrum $\mathbf{C}_n$, considered as a spectrum with an action of Σ_n, is the n-th derivative of the identity.

We need to investigate this tower, whose existence derives from the general theory. Some information about it had been available before. As indicated in the diagram above, it is immediate from the definitions that $P_1(X) \simeq Q(X)$, i.e., the linear part of homotopy theory is stable homotopy theory. The description of the second stage is still rather classical: as was indicated above, the second quadratic approximation, $P_2(X)$, is the homotopy fiber of the "stable James-Hopf"map $Q(X) \to Q(X^{\wedge 2}_{h\Sigma_2})$. The second layer of the tower is $D_2(X) \simeq \Omega Q(X^{\wedge 2}_{h\Sigma_2})$.

For a general n, B. Johnson was the first one to provide an explicit closed description of $D_n(X)$ in terms of standard constructions of homotopy theory. In [Jo95] certain spaces Δ_n are constructed, which have the following properties:

(i) the group Σ_n acts on Δ_n,

(ii) non-equivariantly, $\Delta_n \simeq \bigvee_{i=1}^{(n-1)!} S^{n-1}$,

(iii) the n-th derivative of the identity is $\mathrm{Map}_*(\Delta_n, \Sigma^\infty S^0)$, the Spanier-Whitehead dual of Δ_n, considered as a spectrum with an action of Σ_n. Equivalently,

$$D_n(X) \simeq \Omega^\infty(\mathrm{Map}_*(\Delta_n,\, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \ .$$

The description of the space Δ_n is a geometric one, it is defined as a quotient of the $n(n-1)$-dimensional unit cube by a certain subcomplex. In section 1 we reformulate the description of Δ_n. Thus we construct a certain combinatorially defined complex K_n. K_n has a natural action of Σ_n, and we show that for our purposes the suspension of K_n is equivalent to Δ_n. Thus we may write

$$D_n(X) \simeq \Omega^\infty(\mathrm{Map}_*(SK_n,\, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \ .$$

By the spectral sequence for the homology of Borel construction, the stable homology of $D_n(X)$, i.e. the homology of the spectrum $\mathrm{Map}_*(SK_n,\, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$ is essentially given by the homology of the symmetric group with coefficients in the homology module of the (dual of) K_n tensored with the homology of $X^{\wedge n}$. Thus, the simpler the homology of $X^{\wedge n}$ is, the simpler one may expect the layers to be. This, of course, suggests the spheres as candidates for investigation. In the case of an even-dimensional sphere, one is led to investigating $\mathrm{H}_*(\Sigma_n; \mathrm{H}_*(\tilde{K}_n))$, where $\tilde{K}_n$ is the dual of K_n. In the case of an odd-dimensional sphere, one is led to study $\mathrm{H}_*(\Sigma_n; \mathrm{H}_*(\tilde{K}_n) \otimes Z[-1])$, where $Z[-1]$ is the sign representation. Not surprisingly, odd-dimensional spheres turn out to be the more basic case. In section 3, we carry out the homology calculations for the odd sphere case. The following theorem summarizes some of the results in section 3.

Theorem 0.1. Let *X be an odd-dimensional sphere. If n is not a power of a prime, then*

$$D_n(X) \simeq * \ .$$

If $n = p^k$, then $D_n(X)$ has only p-primary torsion.

For a spectrum $\mathbf{E}$, let $\mathrm{H}_*^s(\Omega^\infty \mathbf{E}) = \mathrm{H}_*(\mathbf{E})$ be the stable homology of $\mathbf{E}$. In section 3 we write an explicit basis for $\mathrm{H}_*^s\left(D_{p^k}\left(S^{2s+1}\right); \mathbb{Z}/p\mathbb{Z}\right)$ and investigate the action of the Steenrod algebra on the stable cohomology $\mathrm{H}_s^*\left(D_{p^k}\left(S^{2s+1}\right); \mathbb{Z}/p\mathbb{Z}\right)$. We prove that the stable cohomology of $D_{p^k}(S^{2s+1})$ is $A[k-1]$ free, where $A[k-1]$ is a certain finite subalgebra of the Steenrod algebra. In section 4 we feed this result into the vanishing line theorems of Anderson-Davis [AD73] and Miller-Wilkerson [MW81] to conclude that the v_{k-1} periodic homotopy of $D_{p^j}(S^{2s+1})$ is zero for $j \geq k$ and moreover that the Goodwillie tower converges in v_k periodic homotopy. This implies the main theorem of the paper which is the following:

Theorem 4.1. *Let X be an odd-dimensional sphere localized at a prime p. The map*

$$X \to P_{p^k}(X)$$

is a v_j-periodic equivalence for all $k \geq 0$ and for all $0 \leq j \leq k$.

In the last subsection we formulate and prove the analogue of theorem 4.1 for even dimensional spheres. Basically, the tower is still finite, but it is “twice as long”.

1. The poset of partitions of a finite set

Let n be an integer, $n > 1$. Let $\underline{n} = \{1, \ldots, n\}$. *A partition* λ of $\underline{n}$ is an equivalence relation on $\underline{n}$ (similarly, one defines partitions of an arbitrary finite set). Partitions are ordered by refinements, and may be considered as a category. Let k_n be the category of partitions of $\underline{n}$. Thus, for two partitions λ_1, λ_2, there is a morphism $\lambda_1 \to \lambda_2$ iff λ_1 is a refinement of λ_2. It is clear that k_n has an initial and a final object. Denote these $\hat{0}$ and $\hat{1}$ respectively. Let $\mathrm{N}_\bullet k_n$ be the simplicial nerve of k_n. Since k_n has an initial and a final object, the geometric realization of $\mathrm{N}_\bullet k_n$ is contractible. Let K_n denote the subcomplex of the realization of $\mathrm{N}_\bullet k_n$, whose simplices are those which do not contain the morphism $\hat{0} \to \hat{1}$ as a face. Thus the zero-simplices of K_n are partitions of $\underline{n}$ and i-simplices are increasing chains of partitions

$$(\hat{0} = \lambda_{-1} \leq \lambda_0 < \lambda_1 < \cdots < \lambda_i \leq \lambda_{i+1} = \hat{1})$$

such that not both inequalities $\hat{0} \leq \lambda_0$ and $\lambda_i \leq \hat{1}$ are equalities. Let $\tilde{k}_n$ be the full subcategory obtained from k_n by deleting $\hat{0}$ and $\hat{1}$. Let $\mathrm{N}_\bullet \tilde{k}_n$ be the simplicial nerve of $\tilde{k}_n$ and let $\tilde{K}_n$ be its realization. It is easy to see that K_n is homeomorphic to the unreduced suspension of $\widetilde{K}_n$. Equivalently, there is a cofibration sequence

$$\tilde{K}_{n+} \to S^0 \to K_n \quad .$$

Here by cofibration sequence we mean that K_n is homeomorphic to the homotopy cofiber of the map $\tilde{K}_{n+} \to S^0$. The $+$ subscript stands for an added disjoint basepoint.

For a partition λ let $r(\lambda)$ be the number of its components. Let $S = (S_0, S_1, \ldots, S_i)$ be a sequence of integers such that $n \geq S_0 \geq S_1 \geq \cdots \geq S_i \geq 1$, let $K_n^S \subset \mathrm{N}_i k_n$ be defined as follows

$$K_n^S = \{(\hat{0} \leq \lambda_0 \leq \cdots \leq \lambda_i \leq \hat{1}) \in \mathrm{N}_i k_n \,|\, r(\lambda_j) = S_j \quad \text{for } j = 0, \ldots, i\}$$

Let K_n^i be the set of non-degenerate i-simplices of $\widetilde{k}_n$. Thus

$$\mathrm{N}_i k_n = \coprod_{n \geq S_0 \geq S_1 \geq \cdots \geq S_i \geq 1} K_n^S \quad .$$

$$\mathrm{N}_i \widetilde{k}_n = \coprod_{n > S_0 \geq S_1 \geq \cdots \geq S_i > 1} K_n^S \quad .$$

$$K_n^i = \coprod_{n > S_0 > S_1 > \cdots > S_i > 1} K_n^S \quad .$$

Notice that if $i > n-3$ then $K_n^i = \emptyset$. Therefore, $\tilde{K}_n$ is $n-3$-dimensional. Notice also that K_n^S is defined even if S is empty, and therefore the sets $\mathrm{N}_{-1}k_n$ and K_n^{-1} are defined and have one element each. By our convention, $\mathrm{N}_{-2}k_n = K_n^{-2} = \emptyset$.

Definition 1.1. *T_n is the following based simplicial set: The set of i-*simplices, *T_n^i, is*

$$T_n^i \; = \; N_{i-2}k_{n+} \quad \forall i \geq 0 \ .$$

In particular, $T_n^0 = \emptyset_+ = *$ and $T_n^1 = S^0$. The face maps are defined as follows: If $0 < j < i$ then $d_j : T_n^i \longrightarrow {T_n}^{i-1}$ is given by:

$$d_j(\lambda_0, \ldots, \lambda_{i-2}) = (\lambda_0, \ldots, \hat{\lambda}_{j-1}, \ldots, \lambda_{i-2}) \ .$$

For $j = 0, i$ the formulas are

$$d_0(\lambda_0, \ldots, \lambda_{i-2}) = \begin{cases} (\lambda_1, \ldots, \lambda_{i-2}) & \textit{if } \lambda_0 = \hat{0} \\ * & \textit{otherwise} \end{cases}$$

$$d_i(\lambda_0, \ldots, \lambda_{i-2}) = \begin{cases} (\lambda_0, \ldots, \lambda_{i-3}) & \textit{if } \lambda_{i-2} = \hat{1} \\ * & \textit{otherwise} \end{cases}$$

The degeneracy maps are defined similarly. If $0 \leq j \leq i$ then $s_j : {T_n}^i \longrightarrow {T_n}^{i+1}$ is determined by

$$\begin{aligned} &s_j(\hat{0} = \lambda_{-1}, \lambda_0, \ldots, \lambda_{i-2}, \lambda_{i-1} = \hat{1}) \\ &\quad = (\hat{0} = \lambda_{-1}, \lambda_0, \ldots, \lambda_{j-1}, \lambda_{j-1}, \ldots, \lambda_{i-2}, \lambda_{i-1} = \hat{1}) \ . \end{aligned}$$

It is easy to check that T_n is indeed a simplicial set and that its realization is $SK_n = S\Sigma\widetilde{K}_n$, where S and Σ denote reduced and unreduced suspension respectively. (This is, essentially, Milnor's suspension construction [Mi72, page 120], applied to $\tilde{K}_n$ twice).

The symmetric group Σ_n acts on k_n, and therefore on K_n^S, K_n^i, K_n etc. The action of Σ_n on K_n^S is not, in general, transitive. We need to write K_n^S as a union of Σ_n-orbits. The orbits of zero-simplices (partitions of $\underline{n}$) are, simply, partitions of positive integers. A partition P of a positive integer n is a collection $n_1, \ldots, n_k$ of positive integers such that $n_1 \leq \cdots \leq n_k$ and $\sum n_i = n$. We call $\{n_i\}$ the components of P. Such a partition of n is not trivial if $1 < k < n$. We denote the set of partitions of n by $Q(n)$.

Proposition 1.2. *The quotient set K_n^0/Σ_n is naturally isomorphic to the set of non-trivial partitions of n.*

Proof. Every partition λ of $\underline{n}$ induces a partition P of the integer n: the components of P are cardinalities of the components of λ. It is elementary to show that this assignment is surjective and that λ_1 and λ_2 induce the same partition of n if and only if they are in the same orbit of Σ_n. □

If P is as above, we will call P the type of λ. We will sometimes use formal sums $\sum_l n_{i_l} \cdot \underline{i_l}$ to describe partitions of integers. A formal sum as above stands for a partition of $n = \sum_l n_{i_l} i_l$ with n_{i_l} components of cardinality i_l.

Proposition 1.3. *Let λ be a partition of type $P = \sum_l n_{i_l} \cdot \underline{i_l}$. The set of partitions of type P is Σ_n-equivariantly isomorphic to the set of cosets Σ_n/Σ_λ, where Σ_λ is the stabilizer group of λ. There is an isomorphism*

$$\Sigma_\lambda \cong \prod_l \Sigma_{n_{i_l}} \wr \Sigma_{i_l} \quad .$$

Proof. Easy. □

We need to classify orbits of K_n^i, $i > 0$, in a manner similar to the one we have for the orbits of K_n^0. It is sometimes convenient to represent orbits with certain labeled trees. A tree will always have a root r. Distance will mean the number of edges in the unique path between two nodes. Let v be a node.

Definition 1.4. *A tree is balanced if all its leaves have the same distance from the root.*

Given a tree, we define a *height function* on its nodes, which we denote $h(v)$, by letting $h(v)$ be the minimal distance from v to a leaf. Define the height of a tree to be $h(r) - 1$.

Definition 1.5. *A tree is labeled if to every node v there is assigned a positive integer $l(v)$.*

We will make free use of such expressions as sibling nodes, a single child, the subtree spanned by a node, etc. We say that a balanced tree *has no forking on level j* if all the nodes of height j have only one child. We say that two labeled trees T and T' are *isomorphic as labeled trees* (or just isomorphic) if there is an isomorphism of unlabeled trees $\psi : T \to T'$ such that for any node v of T *except possibly the root*, $l(v) = l(\psi(v))$.

Definition 1.6. *A labeled tree is standard if*
1) *it is balanced,*
2) $l(r) = 1$,
3) *no two sibling nodes span isomorphic labeled subtrees.*

Condition (3) implies that every node of height 1 has exactly one child. In other words, in a standard tree there is no forking on level 1.

Given a standard tree, we define the degree function of its nodes as follows: If $h(v) = 0$ then $\deg(v) = l(v)$. If $h(v) > 0$, let $u_1, \ldots, u_k$ be the children of v, then $\deg(v) = l(v)(\deg(u_1) + \cdots + \deg(u_k))$. The degree of a tree is the degree of its root.

Proposition 1.7. *There is a bijective correspondence between orbits of* K_n^i *and standard labeled trees of height* $i+1$ *and degree* n.

Proof. For $i = 0$, let $P = \sum_{l=1}^{L} n_{j_l} \cdot \underline{j_l}$ be an orbit. Then P is represented by the following tree:

$$\begin{array}{ccc} & 1 & \\ \swarrow & & \searrow \\ n_{j_1} & \cdots & n_{j_L} \\ \downarrow & & \downarrow \\ j_1 & & j_L \end{array}$$

For $i > 0$, the assignment of trees to orbits is constructed inductively. But before we describe it, we need some more definitions. For a finite set $\underline{S}$, let $k_{\underline{S}}$ be the category of partitions of $\underline{S}$. Let $\underline{S}_1$ and $\underline{S}_2$ be two finite sets. Let $\Lambda^1 = (\lambda_0^1 \le \cdots \le \lambda_i^1)$ and $\Lambda^2 = (\lambda_0^2 \le \cdots \le \lambda_i^2)$ be i-simplices of $\mathrm{N}_\bullet k_{\underline{S}_1}$ and $\mathrm{N}_\bullet k_{\underline{S}_2}$ respectively. A morphism $\rho : \Lambda^1 \to \Lambda^2$ is a map of sets $\underline{S}_1 \to \underline{S}_2$ which for every $0 \le j \le i$ maps every component of λ_j^1 into a component of λ_j^2. ρ is an isomorphism if it has a two-sided inverse. Λ^1 and Λ^2 are isomorphic if there exists an isomorphism $\Lambda^1 \to \Lambda^2$.

Definition 1.8. *Let* $\Lambda = (\lambda_0 \le \cdots \le \lambda_j)$ *be a chain of partitions. Let* $\underline{S}$ *be a component of* λ_j. *Then* $\lambda_0, \ldots, \lambda_{j-1}$ *determine a* $(j-1)$*-simplex of* $\mathrm{N}_\bullet k_{\underline{S}}$. *We call it the restriction of* Λ *to* $\underline{S}$ *and denote it* $\Lambda|_{\underline{S}}$.

Let $\Lambda = (\lambda_0 \le \cdots \le \lambda_i)$ be an i-simplex of $\mathrm{N}_\bullet k_n$. Thus λ_i is the coarsest partition in the chain. Let $\underline{S_1}$, $\underline{S_2}$ be two components of λ_i. We say that $\underline{S_1}$ and $\underline{S_2}$ induce isomorphic blocks if $\Lambda|_{\underline{S_1}}$ and $\Lambda|_{\underline{S_2}}$ are isomorphic. Of course, a necessary condition for $\underline{S_1}$ and $\underline{S_2}$ to induce isomorphic blocks is that $\underline{S_1}$ and $\underline{S_2}$ are isomorphic sets. The property of inducing isomorphic blocks defines an equivalence relation on the components of λ_i. We consider $\Lambda|_{\underline{S_1}}$ as an element of $\mathrm{N}_{i-1}K_{\underline{S_1}}$ and the orbit of $\Lambda|_{\underline{S_1}}$ under the action of Σ_{S_1} as an element of $(\mathrm{N}_{i-1}K_{S_1})_{\Sigma_{S_1}}$. Two i-simplices Λ^1 and Λ^2 are in the same orbit of Σ_n if and only if they have the same isomorphism classes of blocks, counting with multiplicities. Thus, every orbit B of K_n^i can be written uniquely as a formal sum

$$B = \sum_l n_l \cdot \underline{B_l}$$

where $\underline{B_l}$ are elements of $(K_{k_l}^{i-1})_{\Sigma_{k_l}}$ for some $k_1, \ldots, k_m$ such that

$$\sum_l n_l k_l = n$$

and $\underline{B_1}, \underline{B_2}, \ldots, \underline{B_m}$ are pairwise distinct. Assume by induction that we have assigned to B_l pairwise non-isomorphic standard labeled trees T_l of height i. Now for every T_l replace the label 1 at the root with n_l and join all roots to a common new root. Thus we have constructed a standard tree of height $i+1$, which is the tree assigned to B. It's easy to check that the construction is well-defined, i.e., that two i-simplices are in the same orbit if and only if the above procedure associates to them isomorphic trees. $\square$

To describe the orbit of a given i-simplex Λ of type B as above, we notice, as we did in the case of 0-simplices, that the stabilizer group Σ_Λ of Λ has, up to an isomorphism, the following form:

$$\Sigma_\Lambda \cong (\Sigma_{n_1} \wr \Sigma_{B_1}) \times (\Sigma_{n_2} \wr \Sigma_{B_2}) \times \cdots \times (\Sigma_{n_m} \wr \Sigma_{B_m})$$

where Σ_{B_l} are stabilizers of representatives of B_l. Notice that all groups in sight are naturally subgroups of Σ_n, and the set of partitions of type B can be identified equivariantly with the cosets Σ_n/Σ_Λ. The stabilizer groups of two representatives of a given orbit are conjugate. In the course of the paper we will sometimes confuse between the set of orbits and a set of arbitrarily chosen representatives of orbits.

The group Σ_Λ is isomorphic to a semi-direct product

$$\Sigma_\Lambda \cong (\Sigma_{n_1} \times \Sigma_{n_2} \times \cdots \times \Sigma_{n_m}) \ltimes (\Sigma_{B_1}^{\times n_1} \times \Sigma_{B_2}^{\times n_2} \times \cdots \times \Sigma_{B_m}^{\times n_m})$$

where there is an obvious action of $\Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$ on $\Sigma_{B_1}^{\times n_1} \times \cdots \times \Sigma_{B_m}^{\times n_m}$. Inductively, one may write Σ_Λ in the form

$$G_{i+1} \ltimes (G_i \ltimes (\cdots \ltimes G_0))$$

where each G_l is a product of symmetric groups and there is a "wreath product type" action of G_l on $G_{l-1} \ltimes (G_{l-2} \ltimes \cdots \ltimes G_0)$. As a matter of fact, G_l is isomorphic to the product of powers of symmetric groups indexed by nodes on level l in the tree corresponding to the type B of Λ. The size of each symmetric group is given by the corresponding label and the power to which it is raised is given by the label of the father node.

From here until the end of the section, let p be a fixed prime number.

Definition 1.9. *Let $\Lambda = (\hat{0} = \lambda_{-1} \leq \lambda_0 \leq \cdots \leq \lambda_j)$ be a chain of partitions. We say that λ_j is a p-coarsening of $\hat{0} = \lambda_{-1} \leq \lambda_0 \leq \cdots \leq \lambda_{j-1}$ if for every component $\underline{S}$* of *λ_j the following holds*:

1) *The number of components of λ_{j-1} contained in $\underline{S}$ is a power of p.*

2) *Any two components of λ_{j-1} contained is $\underline{S}$ induce isomorphic blocks.*

We will say that λ is a p-partition if it is a p-coarsening of $\hat{0}$. Obviously, the property of being a p-partition is invariant under the action of Σ_n and therefore we may speak about p-partitions of numbers. A p-partition is simply a partition whose components all have cardinality which is a power of p. We let $\widetilde{P}(\underline{n})$ denote the set of ordered p-partitions of $\underline{n}$ and $P(n)$ denote the set of unordered p-partitions of $\underline{n}$, which is the same as the set of p-partitions of n. We use the following "logarithmic" notation for elements of $P(n)$: a sequence $(n_0, n_1, \ldots)$ denotes the partition with n_j components of cardinality p^j for all $j \geq 0$. Thus $n = \sum_j n_j p^j$.

Definition 1.10. *Let $\Lambda = (\lambda_0 \leq \cdots \leq \lambda_i)$ be an i-simplex of* $\mathrm{N}_\bullet k_n$. *An ordered p-ramification of Λ is a chain of partitions*

$$(\hat{0} = \lambda_{-1} \leq \delta_0 \leq \lambda_0 \leq \delta_1 \leq \lambda_1 \leq \cdots \leq \lambda_i \leq \delta_{i+1} \leq \lambda_{i+1} = \hat{1})$$

such that for all $j = 0, 1, \ldots, i+1$, δ_j is a p-coarsening of $(\hat{0} = \lambda_{-1} \leq \delta_0 \leq \lambda_0 \leq \delta_1 \leq \lambda_1 \leq \cdots \leq \lambda_{j-1})$.

Recall that we denote by Σ_Λ the subgroup of Σ_n which stabilizes Λ. Σ_Λ acts on the set of ordered p-ramifications of Λ. We define an *unordered p-ramification of* Λ to be an orbit of an ordered p-ramification under the action of Σ_Λ. It is clear that if Λ^1 and Λ^2 are two i-simplices in the same orbit of Σ_n then the set of unordered p-ramifications of Λ^1 is isomorphic to the set of unordered p-ramifications of Λ^2.

Let Ψ be an unordered p-ramification of Λ. Consider Ψ as a $2i+2$-simplex of $\mathrm{N}_\bullet k_n$ and consider the orbit of Ψ under the action of Σ_n. This orbit is represented by a standard tree of height $2i+3$. It follows easily from the definitions that all the nodes of even height in this tree are labeled by powers of p and that there is no forking on odd levels. It is also easy to see that the set of orbits of ordered p-ramifications of Λ under the action of Σ_n is isomorphic to the set of orbits under the action of Σ_Λ, which is the set of unordered p-ramifications of Λ. We denote the set of ordered p-ramifications of Λ by $\widetilde{P}(\Lambda)$ and the set of unordered p-ramifications of Λ by $P(\Lambda)$. Thus $P(\Lambda) \cong \widetilde{P}(\Lambda)_{\Sigma_\Lambda}$. We denote by $\widetilde{P}(B_l)^k_{P(B_l)}$ the fibered product of k copies of $\widetilde{P}(B_l)$ over $P(B_l)$. Thus a point in $\widetilde{P}(B_l)^k_{P(B_l)}$ is a k-tuple of elements of $\widetilde{P}(B_l)$ which are all in the same orbit of Σ_Λ. We will need inductive formulae for $\widetilde{P}(\Lambda)$ and $P(\Lambda)$. Suppose that λ_i, the coarsest partition in Λ, has n_l blocks of type B_l for $l = 1, \ldots, L$ where B_l are pairwise non-isomorphic. We denote a generic element of $\prod_{l=1}^L P(n_l)$ by $(n_1^0, \ldots, n_1^{j_1}), (n_2^0, \ldots, n_2^{j_2}), \ldots, (n_L^0, \ldots, n_L^{j_L})$

Proposition 1.11. *There is an isomorphism of* Σ_Λ*-equivariant sets*

$$\widetilde{P}(\Lambda) \cong \coprod_{\prod_l P(n_l)} \left(\frac{\prod_l \Sigma_{n_l}}{\prod_l \left(\prod_{j=1\ldots j_l} \Sigma_{n_l^j} \wr \Sigma_{p^j} \right)} \times \prod_l \prod_j \left(\widetilde{P}(B_l)^{p^j}_{P(B_l)} \right)^{n_l^j} \right)$$

Proof. Fix a p-coarsening δ_{i+1} of Λ. By definition, every component of δ_{i+1} contains a power of p of components of λ_{i+1} of type B_l for some l. Let us say that δ_{i+1} has n_l^j components containing p^j components of type B_l. A p-ramification of Λ whose coarsest partition is δ_{i+1} is determined by a collection of p-ramifications of the blocks B_l such that any two blocks which are in the same component of δ_{i+1} have isomorphic p-ramifications. This set is isomorphic to

$$\prod_l \prod_j \left(\widetilde{P}(B_l)^{p^j}_{P(B_l)} \right)^{n_l^j} .$$

On the other hand, the set of p-coarsenings of Λ which have n_l^j components containing p^j components of type B_l is clearly isomorphic to

$$\frac{\prod_l \Sigma_{n_l}}{\prod_l \left(\prod_{j=1\dots j_l} \Sigma_{n_l^j} \wr \Sigma_{p^j} \right)}$$

The proposition follows by taking union over the set of types of p-coarsenings which is isomorphic to the set $\prod_l P(n_l)$. $\square$

Corollary 1.12. *There is an isomorphism of sets*

$$P(\Lambda) \cong \coprod_{\prod_l P(n_l)} \left(\prod_l \prod_j (P(B_l))^{n_l^j}_{\Sigma_{n_l^j}} \right)$$

Proof. Recall that

$$P(\Lambda) \cong \widetilde{P}(\Lambda)_{\Sigma_\Lambda} \cong \widetilde{P}(\Lambda)_{(\Sigma_{n_1} \wr \Sigma_{B_1} \times \cdots \times \Sigma_{n_L} \wr \Sigma_{B_l})} .$$

Applying proposition 1.11 one readily sees that

$$P(\Lambda) \cong \coprod_{\prod_l P(n_l)} \left(\prod_l \prod_j \left(\left(P(B_l)^{p^j}_{P(B_l)} \right)_{\Sigma_{p^j}} \right)^{n_l^j}_{\Sigma_{n_l^j}} \right) .$$

But $\left(P(B_l)^{p^j}_{P(B_l)} \right) \cong P(B_l)$ where the right hand side can be considered as a set with a trivial action of Σ_{p^j}. The corollary follows. $\square$

2. The layers of the Goodwillie tower of the identity

In this section we will describe $D_n(X)$, the n-th layer of the Goodwillie tower of the identity in terms of the complexes K_n of the previous section. This amounts, basically, to a reformulation of the main result of Johnson in [Jo95]. In [AK97] a different way to derive our description of $D_n(X)$ is presented.

Theorem 2.1.

$$D_n(X) \simeq \Omega^\infty \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} .$$

Proof. By [Jo95, corollary 2.3]

$$D_n(X) \simeq \Omega^\infty \mathrm{Map}_*(\Delta_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

where Δ_n is defined in [Jo95, definition 4.7]. We recall the definition. Let

$$I^{n^2} = \{t = (t_{11}, t_{12}, \ldots, t_{1n}, t_{21}, \ldots, t_{nn}) \in R^{n^2} | 0 \leq t_{ij} \leq 1\}$$

be an n^2-dimensional cube. Let $I^{n(n-1)}$ be the subspace of I^{n^2} defined by $t_{ii} = 0$ for $i = 1, \ldots, n$. Thus $I^{n(n-1)}$ is an $n(n-1)$-dimensional cube. For $1 \leq i < j \leq n$ define

$$W_{ij} = \{t \in I^{n(n-1)} | t_{ik} = t_{jk} \quad \text{for} \quad 1 \leq k \leq n\} \ .$$

Define also

$$Z = \{t \in I^{n(n-1)} | t_{ij} = 1 \quad \text{for some} \quad 1 \leq i, \quad j \leq n\} \ .$$

Then

$$\Delta_n = I^{n(n-1)} \Big/ \left\{ Z \cup \bigcup_{i<j} W_{ij} \right\} \ .$$

Thus to prove the theorem it is enough to prove that there is a Σ_n-equivariant map

$$\Delta_n \simeq SK_n$$

which is a non-equivariant homotopy equivalence. Since $I^{n(n-1)}$ is a Σ_n-equivariantly contractible space, it follows that Δ_n is equivariantly equivalent to the suspension of $Z \cup \bigcup_{i<j} W_{ij}$. Therefore, it is enough to show that there is an equivariant equivalence

$$K_n \simeq Z \cup \bigcup_{i<j} W_{ij} \ .$$

Recall that K_n is itself an unreduced suspension of $\tilde{K}_n$, the geometric realization of the category of non-trivial partitions of $\underline{n}$. On the other hand, we claim that $\bigcup_{i<j} W_{ij}$ and Z are both equivariantly contractible. Indeed, $\bigcup_{i<j} W_{ij}$ is contractible by radial projection on $(0, \ldots, 0)$ and Z is contractible by radial projection on the point t defined by $t_{ii} = 0$ for $i = 1, \ldots, n$ and $t_{ij} = 1$ for $i \neq j$. It follows that $Z \cup \bigcup_{i<j} W_{ij}$ is equivariantly equivalent to the unreduced suspension of $Z \cap \bigcup_{i<j} W_{ij}$. Thus it is enough to prove that there is an equivariant map

$$\tilde{K}_n \to Z \cap \bigcup_{i<j} W_{ij}$$

which is a non-equivariant equivalence. For $1 \leq i < j \leq n$, let $U_{ij} = Z \cap W_{ij}$. The assertion follows from the fact that the spaces U_{ij} cover $Z \cap \bigcup_{i<j} W_{ij}$, all possible intersections of U_{ij} are contractible, and the poset associated with this covering is isomorphic to $\tilde{k}_n^{\mathrm{op}}$. We state it in two propositions.

Proposition 2.2.

$$Z \cap \bigcup_{i<j} W_{ij} = \bigcup U_{ij} \ .$$

Proof. Obvious. □

Let $A = \{(i_1, j_1), (i_2, j_2), \ldots, (i_L, j_L)\}$ be a collection of pairs $1 \leq i_l < j_l \leq n$. Let $U_A = \bigcap_{(i_l, j_l) \in A} U_{i_l, j_l}$. We associate with A a graph on n vertices, labeled $1, \ldots, n$, as follows: There is an edge (i, j) iff $(i, j) \in A$. The connected components of this graph determine a partition of $\underline{n}$.

Proposition 2.3. *U_A depends only on the partition associated with A. Moreover, U_A is empty if the partition associated with A is $\hat{1}$ and is contractible otherwise.*

Proof. In fact, it is easy to see that

$$\begin{aligned} U_A = \{t = (t_{11}, t_{12}, \ldots, t_{1n}, t_{21}, \ldots, t_{nn}) \in R^{n^2} | 0 \leq t_{ij} \leq 1 \\ & t_{ij} = 0 \text{ if } i \text{ and } j \text{ are in the same component of the} \\ & \text{partition associated with } A, \text{ and} \\ & t_{ij} = 1 \text{ for some } \ (i, j)\} \end{aligned}$$

If the partition associated with A is $\hat{1}$ then $t_{ij} = 0$ for all (i, j), contradicting the requirement that $t_{ij} = 1$ for some (i, j), so $U_A = \emptyset$. If the partition is not $\hat{1}$ then U_A is contractible by radial projection on the point given by $t_{ij} = 0$ if i and j are in the same component and $t_{ij} = 1$ otherwise. □

This completes the proof of the theorem. □

$D_n(X)$ is the infinite loop space associated with the spectrum

$$\mathbf{D}_n(X) \simeq \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \ .$$

Since SK_n is the geometric realization of the simplicial set T_n, it follows that $\mathbf{D}_n(X)$ is the total spectrum of a cosimplicial spectrum, which we denote $\mathbf{D}_n^\bullet(X)$. The spectrum of i cosimplices of $\mathbf{D}_n^\bullet(X)$ is $\mathrm{Map}_*(T_n^i, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$. Moreover, recall that SK_n has no non-degenerate simplices in dimensions higher than $n-1$. It follows that the tower of fibrations associated with $\mathbf{D}_n^\bullet(X)$ has n stages, which we denote $\mathrm{Tot}_n^i\big(\mathbf{D}_n^\bullet(X)\big)$. In fact,

$$\mathrm{Tot}_n^i\big(\mathbf{D}_n^\bullet(X)\big) = \mathrm{Map}_*(\mathrm{sk}^i SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

where sk^i stands for the i-th skeleton. The homotopy fiber of the map

$$\mathrm{Tot}_n^i\big(\mathbf{D}_n^\bullet(X)\big) \to \mathrm{Tot}_n^{i-1}\big(\mathbf{D}_n^\bullet(X)\big)$$

is homotopy equivalent to

$$\mathrm{Map}_*((S^i K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

Since this is a tower of fibrations of *spectra*, it yields a spectral sequence calculating the homology of $\mathbf{D}_n(X)$, which is the stable homology of $D_n(X)$. In the next section we will use this spectral sequence to calculate the stable homology of $D_n(X)$ in some interesting special cases.

3. Odd sphere case – the cohomology of the layers

3.1. The homology groups

We now focus our attention on the odd sphere case. Our goal in this section is to study the *stable* homology of $D_n(X)$, the layers of the Goodwillie tower, in this case. Thus, we want to study the homology of the spectra

$$\mathbf{D}_n(X) \simeq \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

where X is an odd-dimensional sphere.

We begin with a proposition which enables us to focus on the torsion part of homology.

Proposition 3.1. *Let X be an odd-dimensional sphere. Let $n > 1$. Rationally*

$$\mathbf{D}_n(X) = \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq *$$

Proof. We saw in the previous section that there exists a tower of fibrations with n stages converging to $\mathbf{D}_n(X)$, in which the fibers are of the form

$$\mathrm{Map}_*((S^i K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}$$

where K_n^{i-2} is the set of non-degenerate $i-2$-chains of partitions. Thus

$$\mathrm{Map}_*((K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq \bigvee_{\Lambda} \Sigma^\infty X^{\wedge n}_{h\Sigma_\Lambda}$$

where the wedge sum on the right hand side is indexed by representatives of orbits of $(K_n^{i-2})_+$. Thus each stabilizer group Σ_Λ can be written as a semi-direct product

$$G_{i-1} \ltimes (G_{i-2} \ltimes (\cdots \ltimes G_0))$$

where each G_l is a product of symmetric groups. Since we consider only non-degenerate orbits, none of the G_ls is trivial. In particular, G_0 is not trivial. The proposition follows since for $k > 1$ and X an odd-dimensional sphere, $X^{\wedge k}_{h\Sigma_k}$ is rationally trivial. □

Notice that proposition 3.1 implies the theorem of Serre that the map $X \to Q(X)$ is a rational equivalence for an odd sphere X.

From now on all spaces considered will be localized at a fixed prime p. All homology groups are taken with $\mathbb{Z}/p\mathbb{Z}$ coefficients. We will calculate $\mathrm{H}_*(\mathbf{D}_n(X))$ explicitly. The case $n = 1$ is trivial. Assume, till the end of the section, that $n > 1$.

The plan is to use the homology spectral sequence associated with the tower of fibrations $\mathrm{Tot}_n^i(\mathbf{D}_n^\bullet(X))$ (as defined on page 14). To see that such a spectral sequence exists, note that since we are dealing with spectra, smashing with a fixed spectrum preserves fibration sequences and finite towers of fibrations. The homology spectral sequence is obtained by smashing our tower of fibrations with the Eilenberg-MacLane spectrum $H\mathbb{Z}/p\mathbb{Z}$ and considering the homotopy spectral sequence of the resulting (finite) tower (see [BK72, page 259] for a reference on the spectral sequence of a tower of fibrations). The first term of the spectral sequence has the following form:

$$\begin{aligned} E_1^{i,t} &= \mathrm{H}_{t-i}(\mathrm{Map}_*((S^i K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \\ &= \mathrm{H}_t(\mathrm{Map}_*((K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \end{aligned}$$

with a differential

$$d_1 : E_1^{i,t} \to E_1^{i+1,t} \ .$$

We may view this E_1 term as a cochain complex $C^\bullet$ of graded $\mathbb{Z}/p\mathbb{Z}$-modules. The module of i-cochains is

$$C^i = \mathrm{H}_*(\mathrm{Map}_*((K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}) \ .$$

The differential $\partial^i : C^i \to C^{i+1}$ is given by the alternating sum $\sum_j (-1)^j d_j^\bullet$, where $d_j^\bullet$ is induced by the face map d_j in $\mathrm{N}_\bullet k_n$. If we write, as we did in the proof of proposition 3.1,

$$\mathrm{Map}_*((K_n^{i-2})_+, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq \bigvee_\Lambda \Sigma^\infty X^{\wedge n}_{h\Sigma_\Lambda} \ ,$$

then for $j = 0$ and $j = i$, d_j^i is the zero homomorphism, and for $1 \le j \le i-1$, d_j^i is a direct sum of the transfer maps associated with the inclusion of stabilizers of representatives of orbits of chains of the form $(\lambda_0, \ldots, \lambda_{i-2})$ into stabilizers of representatives of orbits of chains of the form $(\lambda_0, \ldots, \hat{\lambda}_{j-1}, \ldots, \lambda_{i-2})$. The inclusions are well-defined up to conjugation, and therefore the transfer maps are well-defined.

To study this spectral sequence we will need to study the homology of (reduced) Borel constructions on X with respect to certain subgroups of Σ_n. We recall a few standard facts about the homology of $X^{\wedge n}_{h\Sigma_n} = X^{\wedge n} \wedge_{\Sigma_n} E\Sigma_{n+}$ as described in terms of Dyer-Lashof operations. Let H_* be a graded $\mathbb{Z}/p\mathbb{Z}$ module. Let $\Delta_l(H_*)$ be the free graded $\mathbb{Z}/p\mathbb{Z}$ module generated by allowable Dyer-Lashof words of length l over H_* (see [CLM76, I.2] and [BMMS86, page 298] for details). Thus, if $l > 0$, then $\Delta_l(H_*)$ is generated by the following set

$$\left\{ \begin{array}{ll} \beta^{\epsilon_1} Q^{s_1} \ldots \beta^{\epsilon_l} Q^{s_l} u | & \\ u \in H_*, \ \epsilon_i \in \{0,1\}, \ s_i > 0, \ ps_i - \epsilon_i \ge s_{i-1} & \text{if } p > 2 \\ 2s_1 - \sum_{i=2}^{l} [2s_i(p-1) - \epsilon_i] \ge |u| \right\} & \\ \left\{ Q^{s_1} \ldots Q^{s_l} u | \right. & \\ u \in H_*, \ \ s_i > 0, \ 2s_i \ge s_{i-1} & \text{if } p = 2 \\ s_1 - \sum_{i=2}^{l} s_i \ge |u| \right\} & \end{array} \right.$$

where $u \in H_*$. By convention, $\Delta_0(H_*) = H_*$. The operations Q^s come from elements in the mod p homology of symmetric groups. Q^s raises degree by $2(p-1)s$ if $p > 2$ and by s if $p = 2$ (the βs in the odd-primary case are homology Böcksteins, and thus *lower* the degree by 1). $Q^s u = 0$ if $s < \frac{|u|}{2}$ for $p > 2$ and $|u|$ even (if $s < |u|$ for $p = 2$) and $Q^{\frac{|u|}{2}} u = u^{\otimes p}$ for $p > 2$ ($Q^{|u|} u = u \otimes u$ for $p = 2$). Thus we include powers of elements of H_* in Δ_l. The convenience of this will become clear later – its purpose is to make the "negligible" summands in the proof of lemma 3.11 negligible.

We will sometimes abbreviate $\Delta_l(H_*)$ as Δ_l when H_* is clear from the context.

Given a graded vector space D, let $V(D)$ be the augmentation ideal of the free symmetric algebra generated by D. Thus $V(D)$ is the quotient of $\oplus_{k=1}^{\infty} D^{\otimes k}$ by the ideal generated by the relations $a \otimes b - (-1)^{|a||b|} b \otimes a$ where a, b are homogeneous elements. Let $E(D)$ be the quotient of $V(D)$ by the ideal generated by $a^{\otimes p}$. Since the relations are homogeneous, we may write $E(D) \cong \oplus E^k(D)$. By abuse of notation, we will write $E^k(D)$ as $D^{\otimes k}_{\Sigma_k}$. *From now on, whenever we write $D^{\otimes k}_{\Sigma_k}$, where D is a graded $\mathbb{Z}/p\mathbb{Z}$ module, we mean $E^k(D)$.*

The mod p homology of $X^{\wedge n}_{h\Sigma_n}$ is described in terms of the Dyer-Lashof operations. For any based space X, the following is true (the homology is taken with $\mathbb{Z}/p\mathbb{Z}$ coefficients)

$$\mathrm{H}_*(X^{\wedge n}_{h\Sigma_n}) = \bigoplus_{(n_0,\ldots\ldots)\in P(n)} \left(\bigotimes_{l \geq 0} (\Delta_l(\mathrm{H}_*(X))^{\otimes n_l})_{\Sigma_{n_l}} \right) \tag{1}$$

Because of our choice to suppress the p-th powers of elements of $\mathrm{H}_*(X)$, the splitting in (1) is not natural, but depends on a choice of basis of $\mathrm{H}_*(X)$. The point is that the projection map $V(D) \to E(D)$ splits, but not naturally. However, it is easy to see that if we filter $P(n)$ by the number of components, then the splitting is natural up to elements of lower filtration. Thus we get a (more or less natural) expansion of $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_n})$ into a direct sum indexed by p-partitions of n. We will call the terms in the expansion *the standard summands* (or just *summands*) of $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_n})$.

It is well known that the standard summands are detected by certain elementary abelian subgroups of Σ_n. We proceed to recall the basic facts about this. Let us begin with Δ_k as a summand of $\mathrm{H}_*(\Sigma_{p^k})$, or more generally of $\mathrm{H}_*(X^{\wedge p^k}_{\Sigma_{p^k}})$. For $k = 0, 1, \ldots$ let $A_k \cong (\mathbb{Z}/p)^k$. $|A_k| = p^k$, therefore the action of A_k on itself defines an inclusion (up to conjugation) of $A_k \hookrightarrow \Sigma_{p^k}$. We will consider A_k as a subgroup of Σ_k via this inclusion (this is the subgroup that is defined as Δ_k in [KaP78, page 95]). Thus A_k acts transitively on $\underline{p^k}$. A_k is very useful for detecting elements in the homology of Σ_{p^k}: the pure part of $\mathrm{H}_*(\Sigma_{p^k})$ is detected on A_k. This was probably first proved by Kahn and Priddy in [KaP78]. For a more detailed account we recommend [AdMi95]. We need a slightly more general version of this:

Proposition 3.2. *Let X be a based space. Let $H_* = \mathrm{H}_*(X)$. Recall that Δ_k is a summand of $\mathrm{H}_*(X^{\wedge p^k}_{h\Sigma_{p^k}})$. Write $\mathrm{H}_*(X^{\wedge p^k}_{h\Sigma_{p^k}}) \cong \Delta_k \oplus A$. Consider the homomorphism*

$$\mathrm{H}_*(X \wedge BA_{k+}) \to \mathrm{H}_*\left(X^{\wedge p^k}_{h\Sigma_{p^k}}\right)$$

induced by inclusion of subgroups and the diagonal map $X \to X^{\wedge n}$. This map is onto the summand Δ_k and zero on the summand A.

Proof. For $X = S^0$ (a zero-dimensional sphere) this is precisely [KaP78, proposition 3.4]. The proof generalizes straightforwardly. The idea is to reduce the question from Σ_{p^k} to its p-Sylow subgroup and then proceed by direct calculation. □

Now consider a summand on the right hand side of (1) corresponding to a p-partition $(n_0 \ldots, n_l, \ldots) \in P(n)$. This summand is detected, in a suitable sense, by the elementary abelian group $A = \prod_l A_l^{\times n_l}$. Consider the space $X^{\wedge \Sigma_l n_l}$ as a space with a trivial action of A. It is easy to see that there is a diagonal map $X^{\wedge \Sigma_l n_l} \to X^{\wedge n}$ that is equivariant with respect to the subgroup inclusion $A \to \Sigma_n$. We have the following proposition

Proposition 3.3. *With notation as above, consider the homomorphism*

$$\mathrm{H}_*(X^{\wedge \Sigma_l n_l} \wedge BA_+) \to \mathrm{H}_*(X^{\wedge n}_{h\Sigma_n})$$

induced by group inclusion $A \to \Sigma_n$ and the diagonal map $X^{\wedge \Sigma_l n_l} \to X^{\wedge n}$. This homomorphism is onto the summand corresponding to the p-partition $(n_0 \ldots, n_l, \ldots)$ and zero on the other summands of the same filtration and the summands of higher filtration.

Proof. The case $X = S^0$ is well-known. It is largely proved in [KaP78] and in more detail in [AdMi95]. It is a longish, but straightforward, exercise to extend the result to a general X. □

The reason that we need proposition 3.3 is that we have to study the transfer map in the homology of Borel construction. Let $n_0\underline{p^0} + n_1\underline{p^1} + \cdots n_k\underline{p^k}$ be a p-partition of n. Let $\Sigma_P =$

$\Sigma_{n_0} \wr \Sigma_{p^0} \times \cdots \times \Sigma_{n_k} \wr \Sigma_{p^k}$ and $\Sigma'_P = \Sigma_{n_0 p^0} \times \cdots \times \Sigma_{n_k p^k}$. Let X be any based space. The homology groups $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_n})$, $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_P})$ and $\mathrm{H}_*(X^{\wedge n}_{h\Sigma'_P})$ each have a summand isomorphic to $(\Delta_0^{\otimes n_0})_{\Sigma_{n_0}} \otimes \cdots \otimes (\Delta_0^{\otimes n_k})_{\Sigma_{n_k}}$ which we denote simply Δ. Write $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_n}) = \Delta \oplus A$, $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_P}) = \Delta \oplus B$ and $\mathrm{H}_*(X^{\wedge n}_{h\Sigma'_P}) = \Delta \oplus B'$. Consider the homomorphisms $\Delta \oplus A \to \Delta \oplus B$ and $\Delta \oplus A \to \Delta \oplus B'$ induced by the appropriate transfers. These homomorphisms can be represented as two by two matrices of maps

$$\begin{pmatrix} \Delta \to \Delta & \Delta \to B \\ A \to \Delta & A \to B \end{pmatrix} \text{ and } \begin{pmatrix} \Delta \to \Delta & \Delta \to B' \\ A \to \Delta & A \to B' \end{pmatrix}$$

Proposition 3.4. *The map $\Delta \to \Delta$ in both matrices is an isomorphism.*

Proof. The proof is similar to the proof of the main theorem of [KaP78]. To prove that the homomorphism $\Delta \to \Delta$ is an isomorphism it is enough to prove that it is surjective. To do this for the case of the first matrix, it is enough to show that the composite homomorphism

$$\mathrm{H}_*(X^{\wedge \Sigma_l n_l} \wedge (BA_0^{n_0} \times \cdots \times A_k^{n_k})_+) \xrightarrow{i_*} \mathrm{H}_*(X^{\wedge n}_{h\Sigma_n}) \xrightarrow{tr} \mathrm{H}_*(X^{\wedge n}_{h\Sigma_P})$$

is surjective onto Δ. This composed homomorphism can be analyzed by means of a suitable version of the double coset formula. It is not hard to show that the composed map above is the same as the homomorphism induced by the group inclusion $A_0^{n_0} \times \cdots \times A_k^{n_k} \to \Sigma_P$, essentially because of two reasons: the normalizer of $A_0^{n_0} \times \cdots \times A_k^{n_k}$ in Σ_n is the same as in Σ_P and the transfer from an elementary Abelian group to a proper subgroup is zero (see [KaP78]). The argument for the second matrix is similar. $\square$

We will need to consider a slightly more general situation. Let $n = i_0 p^0 + i_1 p^1 + \cdots i_k p^k$ as before. Let $K_1, K_2, \ldots, K_j$ be disjoint subsets of $\{0, 1, \ldots, k\}$ whose union is $\{0, 1, \ldots, k\}$. For $1 \leq l \leq j$, let $m_l = \sum_{t \in K_l} i_t p^t$. Consider the group $\Sigma_{m_1} \times \cdots \Sigma_{m_j}$ as a subgroup of Σ_n. It is easy to see that Δ is a summand of $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_{m_1} \times \cdots \Sigma_{m_j}})$. Write $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_{m_1} \times \cdots \Sigma_{m_j}}) = \Delta \oplus C$ and consider the matrix

$$\begin{pmatrix} \Delta \to \Delta & \Delta \to C \\ A \to \Delta & A \to C \end{pmatrix}$$

describing the transfer map. We have the following proposition

Proposition 3.5. *In the matrix above the map $\Delta \to \Delta$ is an isomorphism.*

Proof. Similar to the proof of the previous proposition. $\square$

Next we need to generalize the formula (1) and proposition 3.3 to $H_*(X^{\wedge n}_{h\Sigma_\Lambda})$, where Σ_Λ is the stabilizer of a chain of partitions of n. More precisely, we will show that $H_*(X^{\wedge n}_{h\Sigma_\Lambda})$ splits as a certain direct sum indexed by unordered p-ramifications of Λ. We first show how to associate a graded $\mathbb{Z}/p\mathbb{Z}$-vector space to a p-ramification of Λ. Indeed, let $\Lambda = (\hat{0} = \lambda_{-1} \le \lambda_0 \le \cdots \le \lambda_i \le \lambda_{i+1} = \hat{1})$ be an i-chain of partitions of $\underline{n}$ and let ϵ be an unordered p-ramification of Λ. Recall that we associate with ϵ a standard tree of height $2i+3$ in which all the nodes of even height are labeled by powers of p and there is no forking on odd levels. Given such a tree, a node v in the tree and a graded $\mathbb{Z}/p\mathbb{Z}$-vector space $H_* = \mathrm{H}_*(X)$, we construct a graded $\mathbb{Z}/p\mathbb{Z}$-vector space H^v_* and, for future use, a detecting elementary abelian group A_v as follows: if the height of v is zero then it is labeled by p^k for some k (since 0 is even) and we define $H^v_* = \Delta_k(H_*)$ and $A_v = A_k$. Assume now that we defined H^v_* and A_v for all v of height $j-1$ or less. Let v be a node of height j. Let l be the label of v. Assume, first, that j is even. Then $l = p^k$ for some k. Let $u_1, \ldots, u_m$ be the children of v. We define

$$H^v_* = \Delta_k(H^{u_1}_* \otimes \cdots \otimes H^{u_m}_*)$$

and

$$A_v = A_k \times (A_{u_1} \times \cdots \times A_{u_k})$$

(A_v should be thought of as the diagonal subgroup of $\Sigma_{p^k} \wr (A_{u_1} \times \cdots \times A_{u_k})$). Now assume j is odd. Then v has only one child u and we define

$$H^v_* = (H^u_*)^{\otimes l}_{\Sigma_l}$$

and

$$A_v = (A_u)^{\times l} \quad .$$

Let H^ϵ_* be the module associated with the root of the tree and similarly let A_ϵ be the elementary abelian group associated with the root of the tree. A_ϵ is in fact a subgroup of Σ_n (determined up to conjugation).

Lemma 3.6. *Let $\Lambda = (\hat{0} = \lambda_{-1} \le \lambda_0 \le \cdots \le \lambda_i \le \lambda_{i+1} = \hat{1})$ be an i-chain of partitions of $\underline{n}$. Recall that $P(\Lambda)$ is the set of unordered p-ramifications of Λ. There is an isomorphism*

$$\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda}) \cong \bigoplus_{\epsilon \in P(\Lambda)} H^{\epsilon}_* \ .$$

Proof. We will prove it by induction on i. The induction starts with $i = -1$. In this case $\Lambda = (\hat{0}, \hat{1})$ and the lemma is given precisely by (1) and proposition 3.3. Assume the lemma holds for $i - 1$. Let $\Lambda = (\hat{0} = \lambda_{-1} \leq \lambda_0 \leq \cdots \leq \lambda_i \leq \lambda_{i+1} = \hat{1})$ be an i-chain of partitions. Consider λ_i, the coarsest partition in the chain. The relation of inducing isomorphic blocks is an equivalence relation on the components of λ_i. Let λ_i have n_l components of type B_l, where the B_ls are pairwise distinct orbits of $i - 1$ chains of partitions of a set with k_l elements and $l = 1, \ldots, L$. Thus, in the tree corresponding to the orbit of Λ, the root has L children labeled $n_1, \ldots, n_L$. The stabilizer group of Λ has the following form

$$\Sigma_\Lambda \cong \Sigma_{n_1} \wr \Sigma_{B_1} \times \cdots \times \Sigma_{n_L} \wr \Sigma_{B_L} \ .$$

Thus

$$\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda}) = \bigotimes_{l=1}^{L} \left(\bigoplus_{(n^l_0, n^l_1, \ldots, n^l_{J_l}) \in P(n_l)} \left(\bigotimes_{j=0}^{J_l} \left(\Delta_j \left(\mathrm{H}_*(X^{\wedge k_l}_{h\Sigma_{B_l}}) \right) \right)^{\otimes n^l_j}_{\Sigma_{n^l_j}} \right) \right)$$

which implies

$$\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda}) = \bigoplus_{\prod_{l=1}^{L} P(n_l)} \left(\bigotimes_{l=1}^{L} \left(\bigotimes_{j=0}^{J_l} \left(\Delta_j \left(\mathrm{H}_*(X^{\wedge k_l}_{h\Sigma_{B_l}}) \right) \right)^{\otimes n^l_j}_{\Sigma_{n^l_j}} \right) \right) \ .$$

We see that $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda})$ splits as a direct sum indexed by $\prod_{l=1}^{L} P(n_l)$. By the induction assumption,

$$\mathrm{H}_*\left(X^{\wedge k_l}_{h\Sigma_{B_l}}\right) \cong \bigoplus_{\epsilon \in P(B_l)} H^{\epsilon}_*$$

Obviously, if H^1_* and H^2_* are two graded modules then $\Delta_l(H^1_* \oplus H^2_*) \cong \Delta_l(H^1_*) \oplus \Delta_l(H^2_*)$, therefore

$$\Delta_j \mathrm{H}_*\left(X^{\wedge k_l}_{h\Sigma_{B_l}}\right) \cong \bigoplus_{\epsilon \in P(B_l)} \Delta_j H^{\epsilon}_*$$

Thus

$$\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda}) = \bigoplus_{\prod_{l=1}^{L} P(n_l)} \left(\bigotimes_{l=1}^{L} \left(\bigotimes_{j=0}^{J_l} \left(\bigoplus_{\epsilon \in P(B_l)} \Delta_j H_*^{\epsilon} \right)^{\otimes n_j^l}_{\Sigma_{n_j^l}} \right) \right).$$

By multiplying out, we see that $H_*(X^{\wedge n}_{h\Sigma_\Lambda})$ splits as a direct sum indexed by the set

$$\coprod_{\prod_l P(n_l)} \left(\prod_l \prod_j (P(B_l))^{n_l^j}_{\Sigma_{n_l^j}} \right)$$

which, by corollary 1.12 is isomorphic to $P(\Lambda)$. It is tedious, but entirely straightforward to verify that the summand corresponding to $\epsilon \in P(\Lambda)$ is indeed H_*^{ϵ}. $\square$

Remark 3.7. Recall that given an unordered p-ramification ϵ of Λ we constructed an elementary abelian group A_ϵ (just before lemma 3.6). It is not hard to show that A_ϵ detects the summand H_*^{ϵ} of the homology of $X^{\wedge n}_{h\Sigma_\Lambda}$ in the sense of proposition 3.3. If ϵ_1 and ϵ_2 are different p-ramifications of Λ then A_{ϵ_1} and A_{ϵ_2} are non-conjugate in Σ_Λ. The map on the homology of Borel constructions induced by the inclusion (defined up to conjugation) $A_\epsilon \hookrightarrow \Sigma_n$ is non-zero only on the summand H_*^{ϵ} and summands corresponding to elementary abelian groups with strictly fewer components (the number of components of a subgroup G of Σ_n is the number of components of the induced partition of n).

Definition 3.8. *An integer n is pure if $n = p^k$ for some integer k. If n is pure, an i-chain of partitions of $\underline{n}$, $\Lambda = (\hat{0} = \lambda_{-1} \le \lambda_0 \le \cdots \le \lambda_i \le \lambda_{i+1} = \hat{1})$, is pure if λ_j is a p-coarsening of $\lambda_{-1} \le \lambda_0 \le \cdots \le \lambda_{j-1}$ for all $j = 0, \ldots, i+1$.*

Clearly, purity is preserved by the action of Σ_n, hence we may speak about pure orbits of chains of partitions. It is easy to see that an orbit is pure iff the corresponding tree has one branch and has all labels powers of p. Given a pure Λ, the corresponding stabilizer group has the form $\Sigma_\Lambda \cong \Sigma_{p^{k_0}} \wr \Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_i}}$ for some $(k_0, \ldots, k_i)$ such that $\sum k_j = k$. Also, consider the chain of partitions $(\hat{0} = \lambda_{-1} \le \delta_0 \le \lambda_0 \le \cdots \le \delta_i \le \lambda_i \le \delta_{i+1} \le \lambda_{i+1} = \hat{1})$ in which $\delta_j = \lambda_j$ for all $j = 0, \ldots, i+1$. Since Λ is pure, it is easy to see from definition that it is a p-ramification of Λ. The corresponding summand of $\mathrm{H}_*(X^{\wedge n}_{h\Sigma_\Lambda})$ is of the form $\Delta_{k_0}(\Delta_{k_1}\dots(\Delta_{k_i}))$. We call it the pure summand associated with Λ. All other summands are impure.

Let C^i be the graded $\mathbb{Z}/p\mathbb{Z}$-module of i-cochains in the complex $C^\bullet$ defined above. C^i can be written as a direct sum $C^i \cong P^i \oplus I^i$ where P^i and I^i are the pure and impure summands of C^i (P^i is often trivial). The coboundary map $\partial^i : P^i \oplus I^i \to P^{i+1} \oplus I^{i+1}$ can be represented by a matrix of matrices as follows

$$\begin{pmatrix} P^i \to P^{i+1} & P^i \to I^{i+1} \\ I^i \to P^{i+1} & I^i \to I^{i+1} \end{pmatrix} .$$

Proposition 3.9. *$P^i \to I^{i+1}$ are zero matrices for all i.*

Proof. It is not hard to show (similar to proposition 3.2) that the pure summands are detected by the elementary abelian group A_k (the transitive elementary abelian subgroup of Σ_{p^k}). More precisely, the homomorphism

$$\mathrm{H}_*(X \wedge BA_{k+}) \to \mathrm{H}_*\left(X^{\wedge p^k}_{h\Sigma_{p^{k_0}}\wr\Sigma_{p^{k_1}}\wr\cdots\wr\Sigma_{p^{k_i}}}\right)$$

is onto the pure summand and zero on the impure summands. The proposition follows, using the double coset formula. □

Corollary 3.10. *The pure summands span a subcomplex of $C^\bullet$, which we denote $P^\bullet$. $P^\bullet$ is non-trivial only if n is a power of p. There is a short exact sequence of cochain complexes*

$$0 \to P^\bullet \to C^\bullet \to I^\bullet \to 0$$

where $I^\bullet$ is the complex of impure summands.

The following lemma is important:

Lemma 3.11. *$I^\bullet$ is acyclic.*

Proof. We will use the following evident proposition

Proposition 3.12. *Let*

$$C_0^1 \oplus C_0^2 \to C_1^1 \oplus C_1^2 \to C_2^1 \oplus C_2^2 \to \cdots \to C_j^1 \oplus C_j^2 \to \cdots$$

be a cochain complex of graded $\mathbb{Z}/p\mathbb{Z}$-vector spaces, where C_0^1 is the trivial module. Suppose that for all $j \geq 0$ the differential $C_j^1 \oplus C_j^2 \to C_{j+1}^1 \oplus C_{j+1}^2$ is given by a matrix

$$\begin{pmatrix} C_j^1 \to C_{j+1}^1 & C_j^1 \to C_{j+1}^2 \\ C_j^2 \to C_{j+1}^1 & C_j^2 \to C_{j+1}^2 \end{pmatrix}$$

where $C_j^2 \to C_{j+1}^1$ is an isomorphism. Then the complex is acyclic.

Consider now the cochain complex $I^\bullet$. Obviously, I^0 is the trivial module (since $n > 1$). For $j \geq 1$ we will write I^j as a direct sum of two modules $I_1^j \oplus I_2^j$, which we now proceed to define. Recall that I^j is the direct sum of the impure summands of $\oplus_\Lambda \mathrm{H}_*(X_{h\Sigma_\Lambda}^{\wedge n})$ where Λ ranges through a set of representatives of orbits of non-degenerate $(j-2)$-chains of partitions $(\hat{0} = \lambda_{-1} < \lambda_0 < \cdots < \lambda_{j-2} < \lambda_{j-1} = \hat{1})$. Moreover by lemma 3.6 we know that given $\Lambda = (\hat{0} = \lambda_{-1} < \lambda_0 < \cdots < \lambda_{j-2} < \lambda_{j-1} = \hat{1})$, the impure summands of $\mathrm{H}_*(X_{h\Sigma_\Lambda}^{\wedge n})$ are indexed by unordered p-ramifications

$$\epsilon = (\hat{0} = \lambda_{-1} \leq \delta_0 \leq \lambda_0 < \cdots \leq \delta_{j-2} \leq \lambda_{j-2} \leq \delta_{j-1} \leq \lambda_{j-1} = \hat{1})$$

such that not for all i $\delta_i = \lambda_i$. We say that an unordered p-ramification ϵ of Λ is *admissible* if there exists $0 \leq l \leq j-2$ such that $\delta_m = \lambda_m$ for all $0 \leq m \leq l$ and $\lambda_m = \delta_{m+1}$. If ϵ is not admissible then we say it is *unadmissible*. Let $P_a(\Lambda)$ and $P_u(\Lambda)$ be the set of admissible, impure and unadmissible, impure p-ramifications of Λ. We define

$$I_1^j = \bigoplus_{\Lambda \in (K_n^{j-2})_{\Sigma_n}} \left(\bigoplus_{\epsilon \in P_a(\Lambda)} H_*^\epsilon \right)$$

and

$$I_2^j = \bigoplus_{\Lambda \in (K_n^{j-2})_{\Sigma_n}} \left(\bigoplus_{\epsilon \in P_u(\Lambda)} H_*^\epsilon \right) .$$

By lemma 3.6, $I^j \cong I_1^j \oplus I_2^j$ for all j. Clearly, I_1^1 is the trivial module. It remains to prove that the map $I_2^j \to I_1^{j+1}$ induced by the coboundary map in $I^\bullet$ is an isomorphism for all j. Then the lemma will follow from proposition 3.12.

First we establish that I_2^j and I_1^{j+1} are abstractly isomorphic. Let

$$\Lambda = (\hat{0} = \lambda_{-1} < \lambda_0 < \cdots < \lambda_{j-2} < \lambda_{j-1} = \hat{1})$$

be a $(j-2)$-chain and let

$$\epsilon = (\hat{0} = \lambda_{-1} \le \delta_0 \le \lambda_0 < \cdots \le \delta_{j-2} \le \lambda_{j-2} \le \delta_{j-1} \le \lambda_{j-1} = \hat{1})$$

be an unadmissible p-ramification of Λ. Let l be the smallest index such that $\delta_{l+1} \ne \lambda_{l+1}$. Such an l exists because otherwise ϵ would be pure. Call l the *level* of ϵ. Suppose first that $l = -1$. We claim that if $\lambda_{-1} = \delta_0$ then H_*^ϵ is the trivial module. Indeed, in this case it is easy to see that the tree corresponding to ϵ has all the nodes on level 0 labeled by 1, but not all the nodes on level 1 labeled by 1, since $\lambda_{-1} \ne \lambda_0$. It follows that H_*^ϵ has a tensor factor of the form $(\Delta_0(H_*))_{\Sigma_k}^{\otimes k}$, where $k > 1$. But it is easy to see that $(\Delta_0(H_*))_{\Sigma_k}^{\otimes k}$ is the trivial module if H_* has exactly one generator of odd degree, which it does if X is an odd-dimensional sphere. Thus if $l = -1$ and $\lambda_{-1} = \epsilon_0$ we say that ϵ is a negligible p-ramification and H_*^ϵ is a negligible summand of I_j^2. We denote the set of non-negligible unadmissible impure p-ramifications of Λ by $P_u^n(\Lambda)$. We proceed to establish an isomorphism between the sum of non-negligible summands of I_j^2 and I_{j+1}^1. We may assume now that $\lambda_l \ne \delta_{l+1}$ because otherwise ϵ is either negligible (if $l = -1$) or admissible (if $l > -1$). It follows that the sequence

$$\Lambda' = (\hat{0} = \lambda_{-1} < \lambda_0 < \cdots < \lambda_l < \delta_{l+1} < \lambda_{l+1} < \cdots < \lambda_{j-2} < \lambda_{j-1} = \hat{1})$$

is a non-degenerate $(j-1)$-chain of partitions. It is obvious by inspection that

$$\begin{aligned}\epsilon' = (\hat{0} = \lambda_{-1}, \delta_0, \lambda_0, \delta_1, \ldots, \delta_l, \lambda_l, \delta_{l+1}, \delta_{l+1}, \delta_{l+1}, \\ \lambda_{l+1}, \ldots, \lambda_{j-2}, \delta_{j-1}, \lambda_{j-1} = \hat{1})\end{aligned}$$

is an admissible, impure p-ramification of Λ' and thus $H_*^{\epsilon'}$ is a summand of I_{j+1}^1. It is also obvious by inspection that H_*^ϵ is isomorphic to $H_*^{\epsilon'}$ and that the above procedure establishes an abstract isomorphism between the sum of non-negligible summands of I_2^j and I_1^{j+1}. It remains to prove that the coboundary homomorphism of $I^\bullet$ induces an isomorphism between the two. We now may write this map as follows:

$$\bigoplus_{\Lambda \in (K_n^{j-2})_{\Sigma_n}} \left(\bigoplus_{\epsilon \in P_u^n(\Lambda)} H_*^\epsilon \right) \to \bigoplus_{\Lambda \in (K_n^{j-2})_{\Sigma_n}} \left(\bigoplus_{\epsilon \in P_u^n(\Lambda)} H_*^{\epsilon'} \right)$$

where ϵ' is obtained from ϵ by the procedure described above. This map can be described as a matrix $\mathscr{M}$ of maps $H_*^{\epsilon_1} \to H_*^{\epsilon_2'}$. To show that this map is an isomorphism it is enough to show that the matrix is block upper triangular with respect to a certain ordering of the indexing set and that all the diagonal blocks are isomorphisms. To show that all the diagonal blocks are isomorphisms we need to show that for any $(j-2)$-chain Λ as above and for any $\epsilon \in P_u^n(\Lambda)$ the map $H_*^{\epsilon} \to H_*^{\epsilon'}$, induced by the transfer map from Σ_Λ to $\Sigma_{\Lambda'}$, is an isomorphism. We may write

$$\Sigma_\Lambda \cong G_{j-1} \ltimes (\cdots G_{l+1} \ltimes (G_l \ltimes (\cdots G_0)))$$

where all G_i are products of symmetric groups. Now consider $\Sigma_{\Lambda'}$. It is not difficult to see that for $i = l+1, \ldots, j-1$

$$\Sigma_{\Lambda'} \cong G'_{j-1} \ltimes (\cdots G''_{l+1} \ltimes G'_{l+1} \ltimes (G_l \ltimes (\cdots G_0)))$$

where $G''_{l+1} \ltimes G'_{l+1}$ is a subgroup of G_{l+1} of the form $\prod_i \Sigma_{m_i} \wr \Sigma_{p^i}$ and G'_i is a subgroup of G_i of the form required for corollary 3.5. The fact that the map $H_*^{\epsilon} \to H_*^{\epsilon'}$ is an isomorphism follows from propositions 3.4 and 3.5.

It remains to show that the matrix $\mathscr{M}$ is equivalent to a block upper triangular one with respect to some ordering of the indexing set. It is easy to see, using remark 3.7 and the double coset formula, that if Λ is a $(j-2)$-chain of partitions, and ϵ is an unadmissible p-ramification of Λ (so $H_*^{\epsilon} \in I_2^j$), then the only summands of I_1^{j+1} that H_*^{ϵ} maps non-trivially on are $H_*^{\epsilon'}$ and summands whose detecting elementary abelian group has strictly fewer components than the elementary abelian group detecting H_*^{ϵ}. This completes the proof of lemma 3.11. □

An immediate consequence of lemma 3.11 is the following theorem:

Theorem 3.13. *Let X be an odd-dimensional sphere localized at a prime p. Assume n is not a power of p. Then*

$$D_n(X) \simeq \Omega^\infty \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq * \ .$$

Proof. Let E_1 be the first term of the spectral sequence associated with the skeletal filtration of K_n abutting to

$$\mathrm{H}_*\Big(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\Big) \ .$$

We saw that E_1 can be identified with the cochain complex $C^\bullet$ of graded $\mathbb{Z}/p\mathbb{Z}$-vector spaces. Moreover, there is a short exact sequence $P^\bullet \to C^\bullet \to I^\bullet$. It is obvious that since n is not a power of a prime, $P^\bullet$ is trivial. By lemma 3.11, $I^\bullet$ is acyclic. It follows that E_2 is zero. Therefore, $\mathrm{H}_*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$ is zero and the proposition follows. $\square$

Thus if X is an odd-dimensional sphere localized at a prime p then the only interesting values of n are powers of p. If $n = p^k$ then the E_2 term of the spectral sequence computing

$$\mathrm{H}_*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$$

may be identified with the cohomology of the cochain complex $P^\bullet$. So we proceed to analyze the complex $P^\bullet$.

Definition 3.14. *An ordered partition of a positive integer k is an ordered sequence $K = (k_1, \ldots, k_j)$ of positive integers with $k_1 + \cdots + k_j = k$.*

For future use, we denote by 2_l the ordered partition

$$\underbrace{(1, \ldots, 1, 2, 1, \ldots, 1)}_{2 \text{ at place } l} \ .$$

Ordered partitions of k are partially ordered by refinement, we write $K \leq J$ if K is a refinement of J. Moreover, ordered partitions form a lattice: any collection of partitions $\{K_m\}_m$ has well behaved greatest common refinement and least common coarsening (denoted by $\cap_m(K_m)$ and $\cup_m(K_m)$ respectively). In fact, the lattice of ordered partitions of k is isomorphic to the Boolean lattice of subsets of $\underline{k-1}$ ordered by inclusion. Given $K = (k_1, \ldots, k_j)$, let Σ_K be the group $\Sigma_{p^{k_1}} \wr \cdots \wr \Sigma_{p^{k_j}}$ and let Δ_K be the summand $\Delta_{k_1}\Delta_{k_2} \cdots \Delta_{k_j}$ of $\mathrm{H}_*(\Sigma_K)$ or more generally of $\mathrm{H}_*(X^{\wedge p^k}_{h\Sigma_K})$ depending on the context. We denote by $N^j(k)$ the set of ordered partitions of k with j components. The following is obvious by inspection:

$$P^0 \cong 0$$

For $j > 0$

$$P^j \cong \bigoplus_{K \in N^j(k)} \Delta_K \ .$$

In particular, if $j > k$ then $P^j \cong 0$.

In the following definition, the underlying assumption is that X is a $2s+1$-dimensional sphere and u is a generator of $\mathrm{H}_{2s+1}(X)$.

Definition 3.15. *For a fixed k, let CU_* be the free graded $\mathbb{Z}/p\mathbb{Z}$ module on the following generators:*
if $p > 2$

$$\{\beta^{\epsilon_1}Q^{s_1}\dots\beta^{\epsilon_k}Q^{s_k}u|s_k \geq s,\ s_i > ps_{i+1} - \epsilon_{i+1}\forall i\}\ ,$$

if $p = 2$

$$\left\{Q^{s_1}\dots Q^{s_k}u|s^k \geq 2s+1,\ s^i > 2s^{i+1}\right\}\ .$$

Thus CU_* is generated by the "completely unadmissible" words of length k (hence the notation).

Theorem 3.16. *Let $n = p^k$. The cohomology of $P^\bullet$ is concentrated in degree k. Moreover there are isomorphisms of modules over the Steenrod algebra*

$$\mathrm{H}^k(P^\bullet) \cong CU_* \cong \Sigma^k\mathrm{H}_*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$$

where the action of the Steenrod algebra on CU_* is given by the Nishida relations ([CLM76]).

Proof. The results about the cohomology of $P^\bullet$ are known, and are more or less implicit in [Ku85] (see also [Ku82] and [KuP85]), although the language there is somewhat different from ours. First of all, let us see that the claim is plausible by counting dimensions. Consider a pure summand of a form $\Delta_{k_1}\Delta_{k_2}$. The module $\Delta_{k_1+k_2}$ is a submodule of $\Delta_{k_1}\Delta_{k_2}$ (however, this obvious inclusion is not the same as the transfer map in homology – if it was, the theorem would be easier to prove). When we consider $\Delta_{k_1+k_2}$ as a subobject of $\Delta_{k_1}\Delta_{k_2}$ we will denote it $\Delta^a_{k_1}\Delta_{k_2}$– the module generated by words which are admissible at place k_1 (from the left). Let $\Delta^u_{k_1}\Delta_{k_2}$ be the quotient of $\Delta_{k_1}\Delta_{k_2}$ by $\Delta^a_{k_1}\Delta_{k_2}$. Thus $\Delta^u_{k_1}\Delta_{k_2}$ is generated by words which are unadmissible at place k_1. By a slight abuse of notation, we will write

$$\Delta_{k_1}\Delta_{k_2} = \Delta^a_{k_1}\Delta_{k_2} \oplus \Delta^u_{k_1}\Delta_{k_2}\ \ .$$

The splitting is valid on the level of vector spaces, and is valid up to filtration on the level of A-modules. More generally, given an ordered partition $K = (k_1, \dots, k_j)$ of k, we may write Δ_K as a direct sum of 2^{j-1} modules. These 2^{j-1} "subsummands" are indexed by sequences

$(s_1, \ldots, s_{j-1})$ where each s_i stands for either the letter a or the letter u. The subsummand corresponding to a sequence $S = (s_1, \ldots, s_{j-1})$ is generated by the words which are admissible (resp. unadmissible) at the place $k_1 + \cdots + k_i$ if s_i is a (resp. s_i is u). We denote this subsummand by Δ_K^S. Let $\underline{u}$ stand for the sequence $(u, u, \ldots, u)$. Clearly, if s_i is a for some i, then

$$\Delta_K^S \cong \Delta_{(k_1,\ldots,k_i+k_{i+1},\ldots k_j)}^{(s_1,\ldots,\hat{s}_i,\ldots,s_{j-1})} .$$

Thus every "subsummand" is canonically isomorphic to $\Delta_K^{\underline{u}}$ for some K. It is easy to see that for any $K \in N^j(k)$, the summand $\Delta_K^{\underline{u}}$ occurs in P_{j+i} $(i \geq 0)$ with multiplicity

$$\sum_{(i_1,\ldots,i_j)} \binom{k_1 - 1}{i_1} \cdots \binom{k_j - 1}{i_j}$$

where the summation is over j-tuples $(i_1, \ldots, i_j)$ of non-negative integers whose sum is i. It is also easy to see that

$$\sum_i (-1)^i \sum_{(i_1,\ldots,i_j)} \binom{k_1 - 1}{i_1} \cdots \binom{k_j - 1}{i_j} = (1-1)^{k_1-1} \ldots (1-1)^{k_j-1}$$

where $0^0 = 1$. Thus the alternating sum of multiplicities of all subsummands is 0 except for the subsummand $\Delta_{(1,1,\ldots,1)}^{\underline{u}}$, for which the "total multiplicity" is 1. Also, it is obvious that $\Delta_{(1,1,\ldots,1)}^{\underline{u}} \cong CU_*$. Thus CU_* concentrated in dimension k is a "lower bound" for $\mathrm{H}^*(P^\bullet)$. Thus we have to show that the rank of the coboundary map in $P^\bullet$ is as large as possible. This boils down to analyzing the effect of the transfer map on the pure part of the homology of spaces of the form $X_{h\Sigma_{k_1}\wr\cdots\wr\Sigma_{k_j}}^{\wedge p^{k_1+\cdots+k_j}}$ and showing that the intersection of the images of such various transfer maps is as small as possible (this, and much more, was done in [Ku85]). It is helpful to consider the chain complex $P_\bullet$ which is the "reverse" of $P^\bullet$. $P_k \cong P^k$ for all k and the boundary maps in $P_\bullet$ are induced by inclusion of groups where the coboundary maps in $P^\bullet$ are induced by transfer maps. Indeed, given two ordered partitions $K \leq K'$ of k, the map $\Delta_{K'} \to \Delta_K$ induced by the transfer has a retraction induced by inclusion of subgroups. It is a retraction (up to multiplication by a unit in $\mathbb{Z}/p\mathbb{Z}$) because all the groups in sight contain a common p-Sylow subgroup. We let $e_{K,K'}$ denote the idempotent (up to a unit in $\mathbb{Z}/p\mathbb{Z}$) homomorphism given by the composition $\Delta_K \xrightarrow{i_*} \Delta_{K'} \xrightarrow{tr_*} \Delta_K$. In the special case $K = (1, \ldots, 1)$, $K' = 2_l$, we denote $e_{K,K'}$ simply e_l. The following crucial properties of these idempotents are proved in [Ku85]:

1) $e_{l_1}e_{l_2} = e_{l_2}e_{l_1}$ if $|l_1 - l_2| \geq 2$

2) $e_l e_{l+1} e_l = e_{l+1} e_l e_{l+1}$

Moreover, for any ordered partition K of k and a collection $\{K_i'\}_{i\in I}$ of ordered partitions such that $K \leq K_i'$ for all $i \in I$ the following holds:

3) $\mathrm{Im}(e_{K,\cup_{i\in I}K_i'}) = \cap_{i\in I}\mathrm{Im}(e_{K,K_i'})$

4) $\ker(e_{K,\cup_{i\in I}K_i'}) = \sum_{i\in I}\ker(e_{K,K_i'})$.

The basic reason that properties (1)–(4) hold is that the (dual of the) summand Δ_K of the cohomology of Σ_K is detected by the ring of invariants $\mathrm{H}^*(A_k)^{P_K}$, where P_K is the parabolic subgroup of $\mathrm{GL}_k(\mathbb{F}_p)$ associated with the partition K, and thus propreties (1)–(4) can be read off the structure of the Hecke algebra of endomorphisms of $\mathbb{Z}/p\mathbb{Z}[\mathrm{GL}_k(\mathbb{F}_p)/B]$ where B is the Borel subgroup of $\mathrm{GL}_k(\mathbb{F}_p)$. As a matter of fact, (3) and (4) are only proved in [Ku85, theorem 4.11 (2) and (3)] for the special case $K_i' = 2_i$, $I = \{1, \ldots, k-1\}$, but the general case can be deduced from it quite easily.

Property (3) implies, by the inclusion-exclusion principle, that the rank of the coboundary maps in $P^\bullet$ is as large as it can be. Therefore $\mathrm{H}^i(P^\bullet) \cong 0$ for $i < k$ and $\mathrm{H}^*(P^\bullet)$ is concentrated in degree $* = k$, moreover, $\mathrm{H}^k(P^\bullet)$ is abstractly isomorphic to CU_*, at least as a graded vector space. Property (4) implies the same for $\mathrm{H}_*(P_\bullet)$. It remains to show that the isomorphisms are isomorphisms of Steenrod algeba modules, and not only of graded vector spaces.

The graded vector space $\mathrm{H}^k(P^\bullet)$ can be identified with the cokernel of the coboundary homomorphism $P_{k-1} \to P_k$. The maps $i_* : \Delta_{(1,\ldots,1)} \to \Delta_{2_l}$ $l = 1, \ldots, k-1$ assemble to the boundary homomorphism $P_k \to P_{k-1}$ in $P_\bullet$. $\mathrm{H}_k(P_\bullet)$ is the kernel of this map

$$\mathrm{H}_k(P_\bullet) = \bigcap_{l=1}^{k-1} \ker\{\Delta_{(1,1,\ldots,1)} \to \Delta_{2_l}\} \ .$$

Obviously, $\mathrm{H}_k(P_\bullet) = \bigcap_{l=1}^{k-1}\ker(e_l)$ and $\mathrm{H}_k(P^\bullet) = \mathrm{coker}\{\oplus_{l=1}^{k-1}\mathrm{Im}\ (e_l) \to \Delta_{(1,\ldots,1)}\}$. There is a homomorphism of Steenrod algebra modules $CU_* \to \mathrm{H}_k(P_\bullet)$, given by the Adem relations, which is clearly injective and thus is an isomorphism. On the other hand, there is a homomorphism of Steenrod algebra modules $\mathrm{H}_k(P_\bullet) \to \mathrm{H}^k(P^\bullet)$ given by the composition $\mathrm{H}_k(P_\bullet) \to \Delta_{(1,\ldots,1)} \to \mathrm{H}^k(P^\bullet)$. We claim that this homomorphism is surjective, and therefore is an isomorphism. To prove that the map is surjective, we need to show that for any element of $u \in \Delta_{(1,\ldots,1)}$ there exists an element $v \in \Sigma_{l=1}^{k-1}\mathrm{Im}(e_l)$ such that $u + v \in \mathrm{H}_k(P_\bullet)$ (we consider $\mathrm{H}_k(P_\bullet)$ as a subspace of $\Delta_{(1,\ldots 1)}$). To see this, let

$$\begin{aligned} w = &(1-e_1)(1-e_2)\dots(1-e_{k-1})(1-e_1)(1-e_2)\dots(1-e_{k-2})\\ &\times(1-e_1)\dots(1-e_{k-3})(1-e_1)\dots(1-e_1)(1-e_2)(1-e_1)u\ . \end{aligned}$$

Let $v = w - u$. It is easy to see that $v \in \sum_{l=1}^{k-1} \mathrm{Im}(e_l)$. It is also easy to see that since the idempotents e_i satisfy the braid relations, so do the idempotents $1 - e_i$ and that as a consequence $(1 - e_l)w = w$ for all $l = 1, \dots, k-1$, and thus $w = u + v \in \mathrm{H}_k(P_\bullet)$. It follows that $\mathrm{H}^k(P^\bullet)$ is isomorphic to CU_* as a module of the Steenrod algebra.

Once we know that the cohomology of $P^\bullet$ is concentrated in degree k, it follows that the spectral sequence collapses at E_2 for dimensional reasons. Thus $E_2 \cong E_\infty$. Since E_∞ has only one column, there is an isomorphism

$$E_\infty^{*,k} \cong \mathrm{H}_{*-k}\left(\mathrm{Map}_*(SK_{p^k}, \Sigma^\infty X^{\wedge p^k})_{h\Sigma_{p^k}}\right)\ . \qquad \square$$

3.2. Action of the Steenrod algebra.

Let $n = p^k$. Our goal in this subsection is to study the action of the Steenrod algebra on

$$\mathrm{H}^*\left(\mathrm{Map}_*(K_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$$

where X is an odd-dimensional sphere localized at a prime p.

Let A be the mod-p Steenrod algebra. Let $A[k]$ be the subalgebra of A, generated by the Milnor basis (see [Mar, ch. 15] for notation and basic definitions) elements $P_1^0, P_1^1, \dots, P_1^k$ and (if $p > 2$) by $Q_0, \dots, Q_k$.

Theorem 3.17. *Let X be a $2s+1$-dimensional sphere localized at a prime p. Let $n = p^k$. The module*

$$\mathrm{H}^*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$$

is free over $A[k-1]$.

Proof. Our situation is very similar to that of [W81, theorem 2.1]. The idea of the proof is taken from there entirely.

Theorem 3.16 gives us a basis for

$$\mathrm{H}_*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)\ .$$

We dualize this basis to get a basis for the cohomology groups

$$\mathrm{H}^*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right) .$$

The action of A is given by dualizing the homology Nishida relations as described in [CLM76, page 6]. To make the connection with [W81] explicit, we will rewrite the basis in terms of Steenrod operations rather then the Dyer-Lashof operations. We define a correspondence between the two kind of operations as follows: If $p = 2$ then $(Q^i_*) \leftrightarrow P^{i+1} := Sq^{i+1}$, and if $p > 2$ then $(Q^i)_* \leftrightarrow \beta P^i$ and $(\beta Q^i)_* \leftrightarrow P^i$ (we remind the reader that on the left hand side β stands for the homology Böckstein and thus lowers degree by 1 while on the right hand side it stands for the cohomology Böckstein and hence raises degree by 1.) By comparing the dualized Nishida relations with the Adem relations in the Steenrod algebra, it is not hard to see that this correspondence establishes an isomorphism (up to a dimension shift) of A-modules between $\mathrm{H}^*\left(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n}\right)$ and the module generated by admissible (in the sense of the Steenrod algebra) words $P^{s_1} \dots P^{s_k}$ such that if $p = 2$ then $s_k \geq 2s + 2$ and if $p > 2$ then $s_k \geq s + 1$ (in the case $p > 2$ there are also Böcksteins which we omitted). It is interesting to notice that in case $X = S^1$, the cohomology that we get is isomorphic, as a module over the Steenrod algebra, to the cohomology of certain subquotients of symmetric product of the sphere spectrum which was first computed in [N58] and further studied in [W81]. These subquotients of the symmetric product spectra play a key role in [Ku82, KuP85]. We conjecture that the spectrum

$$\mathrm{Map}_*(SK_{p^k}, \Sigma^\infty S^{\wedge p^k})_{h\Sigma_{p^k}}$$

i.e. the p^k-th layer of the Goodwillie tower of the identity evaluated at S^1 is homotopy equivalent (up to a suitable suspension) to the spectrum denoted $L(k)$ in [Ku82, KuP85][1]. In any case, when $X = S^1$, our statement is equivalent on the level of cohomology to [W81, theorem 2.1]. We sketch Welcher's proof, and indicate the required very minor generalization. Given a sequence $I = (s_1, \dots, s_k)$ we denote by $P^I u$ the element $P^{s_1} \dots P^{s_k} u$, where $P^i = Sq^i$ if $p = 2$. Suppose

[1]Added in revision: since this paper was written, W. Dwyer, jointly with the first-named author, proved this conjecture. Details will appear in [AD97]. The overall connection of the material in this paper with the work of Kuhn, Mitchell and Priddy is made clear and explicit in [AD97]. As a byproduct, this leads to a substantial simplification of some of the proofs in this paper (especially those in section 3).

first that $p = 2$. Following [W81] we define B_k^s to be the vector space generated by the set $\{P^I | I = (2^k j_1, \ldots, 4j_k, 2j_k), \text{where } j_1 \geq \cdots \geq j_k \geq s+1\}$. By computing the Poincare series, one can easily show that $B_k^s \otimes A[k-1] \cong \mathrm{H}^*(\mathrm{Map}_* (SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n})$ as graded $\mathbb{Z}/p\mathbb{Z}$ vector spaces. The calculation is exactly as in [W81] and we omit it. It follows that if the $A[k-1]$ module generated by B_n is free, then it must be $H^*(\mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n})$. This part of the proof again carries over from [W81]. If $p > 2$, the same strategy applies with

$$B_k^s = \{P^I | I = (p^{k-1} j_1, \ldots, p j_{k-1}, j_k), \quad \text{where} \quad j_1 \geq \cdots \geq j_k \geq s+1\} \ .$$

$\square$

4. The v_k-periodic homotopy of the tower

4.1. The case of an odd-dimensional sphere.

Let p be a fixed prime. All spaces in this section are automatically localized at p. In the previous section we saw that in the Goodwillie tower of the identity evaluated at an odd dimensional sphere, the only layers that are non-trivial are those indexed by powers of p. So, there exists a tower of fibrations converging to the homotopy type of S^{2s+1}

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
S^{2s+1} \arrow[r] \arrow[dr] & R_k \arrow[d, "f_k"] & D_{p^k}(S^{2s+1}) \arrow[l] \\
 & R_{k-1} \arrow[d, "f_{k-1}"'] & D_{p^{k-1}}(S^{2s+1}) \arrow[l] \\
 & \vdots \arrow[d] & \\
 & R_0 = Q(S^{2s+1}) &
\end{tikzcd}$$

where $R_k = P_{p^k}(S^{2s+1})$.

Moreover, $D_{p^k}(S^{2s+1})$ is an infinite loop space, and we saw in theorem 3.17 that the cohomology of the associated spectrum is free over A_{k-1}. This implies that the $D_{p^k}(S^{2s+1})$ is trivial in v_{k-1}-periodic homotopy and so are all the higher layers. In other words, in v_k-periodic homotopy, the tower has only $k+1$ non-trivial layers $(D_{p^0}, \ldots, D_{p^k})$. We would like to conclude that the map $S^{2s+1} \rightarrow R_k$ is an equivalence in v_k-periodic homotopy. Apriori, it is not clear that the tower converges in v_k-periodic homotopy. Consider, for instance, the Postnikov resolution of a space X. The layers in this resolution are trivial in v_k-periodic homotopy, but X need not be, because the tower does not converge after inverting v_k. Thus our goal in this section is to study the convergence of this tower in v_k-periodic homotopy. It turns out that since the connectivity of the layers grows so fast, the tower converges in the sense that we need. The main theorem of this paper is the following

Theorem 4.1. *The map*

$$S^{2s+1} \to R_k$$

is a v_i-periodic equivalence for all $k \geq 0$ and all $i \leq k$.

Proof. Let k be fixed all along. We are going to use theorem 3.17 in conjunction with the "vanishing line" theorems in [AD73, MW81]. For the rest of the proof we assume, for simplicity, that $p = 2$, the odd primary case is only marginally more complicated. Recall the following theorem:

Theorem 4.2. [AD73, *theorem* 1.1] *If M is an A-module and $P_{t_0}^{s^0}$ is the lowest degree p_t^s with $s < t$ such that $H(M; P_t^s) \neq 0$, then $Ext^{s,t}(M, \mathbb{Z}/2\mathbb{Z}) = 0$ for $ds > t + c$, where $d = \deg(P_{t_0}^{s_0})$ and $\frac{c}{d-1}$ is approximately $t_0 - 2$.*

Let

$$M = \mathrm{H}^*\left(\mathrm{Map}_*\left(SK_{2^k}, \Sigma^\infty (S^{2s+1})^{\wedge 2^k}\right)_{h\Sigma_{2^k}}\right)$$

By theorem 3.17, M is free over $A[k-1]$, and since $P_t^s \in A[k-1]$ if $s + t \leq k$ it follows that the lowest degree P_t^s with $s < t$ s.t. $H(M, P_t^s) \neq 0$ is at least $P_{t_0}^{s_0}$, where

$$s_0 = \begin{cases} \frac{k-1}{2} & \text{if } k-1 \text{ is even} \\ \frac{k}{2} & \text{if } k-1 \text{ is odd} \end{cases}$$

$$t_0 = \begin{cases} \frac{k-1}{2} + 2 & \text{if } k-1 \text{ is even} \\ \frac{k}{2} + 1 & \text{if } k-1 \text{ is odd.} \end{cases}$$

Thus $\left|P_{t_0}^{s_0}\right| = 2^{s_0}(2^{t_0} - 1) = 2^{k+1} - 2^{s_0} > 2^k - 1$.

Corollary 4.3. *The Adams Spectral Sequence converging to the homotopy of $D_{2^{k+i}}$ has an $(s, t-s)$ vanishing line of slope which is smaller than $\frac{1}{2^{k+i}-2} = \frac{1}{|v_{k+i-1}|}$. It also has a vertical intercept smaller than $k+i$.*

Since v_i acts on the level of the Adams spectral sequence as multiplication by an element on a line of slope $\frac{1}{|v_i|}$, it follows that D_{2^k} is v_{k-1}-trivial and more generally, if $i > 0$ then $D_{2^{k+i}}$ is v_k-trivial.

We need to prove that the Goodwillie tower converges to S^{2s+1} in v_k-periodic homotopy. Till the end of this section, let $\pi_*(-)$ denote $\pi_*(-; V_{k-1})$, where V_{k-1} is a finite space (not a spectrum) of type k with a v_k self map (see appendix). Since $S^{2s+1} = \operatorname{holim} R_j$ we have to show that

$$v_k^{-1}\pi_*(\operatorname{holim} R_j) \cong \lim_{\leftarrow} v_k^{-1}\pi_*(R_j) \ .$$

Let $Q_j = \operatorname{fiber}(R_{k+j} \to R_k)$. There is a tower of fibrations

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ Q_j & \longleftarrow & D_{p^{k+j}} \\ g_j \downarrow & & \\ Q_{j-1} & \longleftarrow & D_{p^{k+j-1}} \\ g_{j-1} \downarrow & & \\ \vdots & & \\ \downarrow & & \\ Q_1 = D_{p^{k+1}} & & \end{array}$$

Our statement is equivalent to showing that the v_k-periodic homotopy of the inverse limit of this tower is trivial. In other words, we want to show that

$$v_k^{-1}\pi_*(\operatorname{holim} Q_j) \cong 0$$

or equivalently

$$v_k^{-1}(\lim_{\leftarrow} \pi_*(Q_j)) \cong 0 \ .$$

Let $\alpha = (\ldots, \alpha_2, \alpha_1) \in \lim_{\leftarrow} \pi_*(Q_j)$. Then $\alpha_j \in \pi_d(Q_j)$, $g_j(\alpha_j) = \alpha_{j-1}$, $d = \deg(\alpha)$. We identify an element of $\pi_d D_{p^{k+j}}$ with its pullback at the E_∞ term of the corresponding ASS. (We will also assume that such an element has $(s, t-s)$ bidegree $(0, d)$. It will be clear that from our point of view it is a harmless assumption, it amounts to taking the worst possible case.)

Suppose that $\alpha = (\ldots, \alpha_{j+1}, \alpha_j, 0, \ldots)$, where $j > 1$ and $\alpha_j \neq 0$. Since $\alpha_{j-1} = 0$, α_j can be thought of as an element of $\pi_{d_j}(D_{2^{k+j}})$. Let k_j be the maximal integer such that $v_k^{k_j}(\alpha_j) \neq 0$. Let $d_{j+1} = |v_k^{k_j}(\alpha_j)| =$

$d_j + \underbrace{k_j}(2^{k+1} - 1)$. It follows from corollary 4.3 that d_{j+1} is bounded by $\widetilde{d_{j+1}}$, which is determined by the following equations

$$\begin{cases} \frac{y-(k+j)}{\widetilde{d_{j+1}}} = \frac{1}{2^{k+j}-2} \\ \frac{y}{\widetilde{d_{j+1}}-d_j} = \frac{1}{2^{k+1}-2} \end{cases}$$

Here $(\widetilde{d_{j+1}}, y)$ are the coordinates of the intersection of the line passing through $(0, k+j)$ and having slope $\frac{1}{2^{k+j}-2}$ (the "vanishing line") and the line passing through $(d_j, 0)$ and having slope $\frac{1}{2^{k+1}-2}$ (the line along which v_k moves α_j). Solving for $\widetilde{d_{j+1}}$ we obtain the following bound

$$(2) \qquad \widetilde{d_{j+1}} = \frac{k+j}{\frac{1}{2^{k+1}-2} - \frac{1}{2^{k+j}-2}} + \frac{d_j}{1 - \frac{2^{k+1}-2}{2^{k+j}-2}} < 2^{k+1}(k+j) + \frac{3}{2} d_j \ .$$

Now let $\alpha = (\ldots, \alpha_2, \alpha_1)$ be any element of $\lim \pi_*(Q_j)$. We may assume $\alpha_1 = 0$ (by applying v_k enough to annihilate α_1). Let d_j be the maximal d such that $d = \left| v_k^{k_j}(\alpha_j) \right|$ and $v_k^{k_j}(\alpha_j) \neq 0$. It follows from (2) that the sequence d_j has the rate of growth of at most $\left(\frac{3}{2}\right)^j$ and thus it grows slower than the connectivity of $D_{2^{k+j}}$ (the connectivity of $D_{2^{k+j}}$ has the rate of growth 2^j), which proves the theorem. □

4.2. The case of an even-dimensional sphere.

Throughout this subsection, let X denote an *even*-dimensional sphere (possibly localized at a prime p). In this case the tower is still finite in v_k-periodic homotopy, but it is "twice as long" as in the odd sphere case. More precisely, there is the following version of our main theorems.

Theorem 4.4. *If n does not equal p^k or $2p^k$ for some prime p, then*

$$D_n(X) \simeq \Omega^\infty \mathrm{Map}_*(SK_n, \Sigma^\infty X^{\wedge n})_{h\Sigma_n} \simeq * \ .$$

If $n = p^k$ or $n = 2p^k$ then $D_n(X)$ has only p-primary torsion.

Thus, if X is an even sphere localized at p, there is a regraded Goodwillie tower, which, in the case $p > 2$, looks as follows

$$\begin{array}{ccccc} & & \vdots & & \\ & & \downarrow & & \\ X & \longrightarrow & R_k^2 & \longleftarrow & D_{2p^k} \\ & \searrow & \downarrow & & \\ & & R_k^1 & \longleftarrow & D_{p^k} \\ & & \downarrow & & \\ & & \vdots & & \\ & & \downarrow & & \\ & & R_0^2 & \longleftarrow & D_{2p^0} \\ & & \downarrow & & \\ & & R_0^1 = Q(S^{2s+1}) & & \end{array}$$

where $R_k^1 = P_{p^k}$ and $R_k^2 = P_{2p^k}$. If $p = 2$ then the tower looks just as in the odd-sphere case.

Theorem 4.5. *If $p > 2$ then the map*

$$X \to R_k^2$$

is a v_k-periodic equivalence for all $k \geq 0$. If $p = 2$ then the map

$$X \to R_{k+1}$$

is a v_k-periodic equivalence for all $k \geq 0$.

Proof of theorems 4.4 and 4.5. Rather than adapting the calculations of section 3 to this case, we make use of the Goodwillie calculus and the James fibration. Consider the sequence of natural maps

$$X \xrightarrow{s} \Omega\Sigma X \xrightarrow{j} \Omega\Sigma X^{\wedge 2}$$

where s and j are the suspension map and the James map respectively. If X is an odd-dimensional sphere localized at a prime then this is a fibration sequence. For a general X, this is a fibration sequence in the meta-stable range [Ja53]. Let $F(X)$ be the homotopy fiber of j. Since the composition $j \circ s$ is trivial, there is a natural map

$$f : X \to F(X)$$

which is a homotopy equivalence for odd spheres localized at a prime. We want to conclude that the Taylor polynomials of $F(X)$ are the same as of the identity when evaluated at odd spheres.

Proposition 4.6. *Let*

$$f : G(X) \to F(X)$$

be a natural transformation of reduced analytic functors. Suppose there exists a space K such that f induces an equivalence

$$G(S^{2i}K) \to F(S^{2i}K)$$

for all $i \geq 0$. Then the map

$$P_nf : P_nG(S^{2i}K) \to P_nF(S^{2i}K)$$

is an equivalence for all i and n.

Proof. By induction on n. The case $n = 0$ is trivial. Indeed, since we assume G and F are reduced

$$P_0G(K) \simeq P_0F(K) \simeq * \ .$$

Assume that the proposition is true for $n - 1$. It is clear that it is enough to show that

$$D_nf : D_nG(K) \to D_nF(K)$$

is an equivalence. Recall that the maps $G(S^{2i}K) \to P_nG(S^{2i}K)$ and $F(S^{2i}K) \to P_nF(S^{2i}K)$ are $(n+1)(k+2i)+c$ connected, where k is the connectivity of K. It follows that the map $P_nf(S^{2i}K)$ is $(n+1)(k+2i)+c$ connected. Using our induction assumption, it follows that the map $D_nf(S^{2i}K)$ is $(n+1)(k+2i)+c$ connected. By Goodwillie's classification of homogeneous functors, there exist spectra G_n and F_n with an action of Σ_n which represent D_nG and D_nF. Thus, the map

$$D_nf : (G_n \wedge (S^{2i}K)^{\wedge n})_{h\Sigma_n} \to (F_n \wedge (S^{2i}K)^{\wedge n})_{h\Sigma_n}$$

is $(n+1)(k+2i)+c$ connected. By the Thom isomorphism, this implies that the map

$$D_nf : (G_n \wedge K^{\wedge n})_{h\Sigma_n} \to (F_n \wedge K^{\wedge n})_{h\Sigma_n}$$

is $(n+1)k+2i+c$ connected for all i. The proposition follows. $\square$

The following proposition is an easy consequence of the general theory of calculus

Proposition 4.7. (1) *The operator P_n commutes up to natural equivalence with finite homotopy inverse limits of functors. In particular*

$$P_n(\Omega F) \simeq \Omega P_n F \ .$$

(2) *Let $Sq(X) = X \wedge X$. Then*

$$P_n F(\Sigma X \wedge X) \simeq P_n(F \circ \Sigma \circ Sq)(X) \ .$$

Returning to the notation of our main text, it follows from the two propositions that if X is an odd sphere localized at p, then there is a fibration sequence

$$P_n(X) \to \Omega P_n(\Sigma X) \to \Omega P_n(\Sigma X \wedge X)$$

where $P_n(X)$ is really $P_n(\mathrm{Id})(X)$. Taking $X = S^{2k-1}_{(p)}$, the fibration sequence becomes

$$P_n\left(S^{2k-1}_{(p)}\right) \to \Omega P_n\left(S^{2k}_{(p)}\right) \to \Omega P_n\left(S^{4k-1}_{(p)}\right) \ .$$

Thus, we have a resolution of the Goodwillie tower for an even sphere by towers for odd spheres and theorems 4.4 and 4.5 readily follow. □

Appendix A. Background on v_k-periodic homotopy

In this appendix we collect some material from [MS95] concerning the definition of v_k^{-1} homotopy and L_k^f localization.

Let M be a finite complex, endowed with a map $v : \Sigma^d M \to M$ such that $MU_*(v)$ is not zero. This implies, in particular, that all iterates of v are essential. We can consider the homotopy theory which results from looking at the homotopy classes of maps from M to a space X. We will write $\pi_i(X;M) = [\Sigma^i M, X]$. We can consider this as a $Z[v]$ module. The periodic homotopy of X defined by v is $\pi_*(X;M) \otimes_{Z[v]} Z[v, v^{-1}]$. The simplest case is obtained by letting $M = S^1$, $k = 0$ and v be a map of degree two. Then the periodic theory is obtained by tensoring the homotopy with $Z[1/2]$. This is an example of a v_0-periodic homotopy.

Higher order periodicity is defined in terms of a family of finite complexes which are detected in BP_* by some power of v_n (the idea of v_1-periodic homotopy goes back to Adams – it can be defined using the Adams map $\Sigma^{13} RP^2 \to \Sigma^5 RP^2$). These complexes are not unique and there does not seem to be a canonical choice, but such complexes do exist and the choices do not matter much. That's the point of the forthcoming discussion.

Definition A.1. *We take $M(p^{i_0}, v_1^{i_1}, \ldots, v_k^{i_k})$ to be any choice of a finite spectrum such that*

$$BP_*(M(p^{i_0}, v_1^{i_1}, \ldots, v_k^{i_k})) = BP^*/(p^{i_0}, v_1^{i_1}, \ldots, v_k^{i_k}) \ .$$

As shorthand, we write I for $(i_0, \ldots, i_k)$, and $M(I)$ for $M(p^{i_0}, \ldots, v_k^{i_k})$. We also write $I \leq J$ if $i_l \leq j_l$ for $0 \leq l \leq k$, and $I \prec J$ if $i_l < j_l$ for $0 \leq l \leq k$. Below we collect some facts about the spectra $M(I)$.

Proposition A.2. (1) *Given a multi-index I, $M(I)$ need not exist, but $M(J)$ exists for some $J \geq I$.*

(2) *There may be more than one possible choice of homotopy type for $M(I)$, but there are at most finitely many choices.*

(3) *Given $M(I)$, there is a $J \succ I$ and a map*

$$f_J^I : M(I) \to \Sigma^{(i_1-j_1)(2p-2)+(i_2-j_2)(2p^2-2)+\cdots+(i_k-j_k)(2p^k-2)} M(J)$$

commuting with projection to the top cell. (*Note that the top cell of $M(I)$ is in dimension $k+1+i_1(2p-2)+\cdots+i_k(2p^k-2)$, so the suspension is just the difference in dimension between the top cells of $M(I)$* and *$M(J)$. To spare notation, we will frequently omit the suspension.*) *The map f_J^I induces the obvious map on BP_* – multiplication by $p^{j_0-i_0} v_1^{j_1-i_1} \cdots v_k^{j_k-i_k}$.*

(4) *For each I, J there are at most finitely many choices of homotopy classes for f_J^I.*

(5) *Given $M(I), M(J), M(K)$ with $J \geq I$ and $K \geq I$, and maps f_J^I, f_K^I as above, there exists $L \geq J, K$, $M(L)$* and *f_L^J, f_L^K so that*

$$\begin{array}{ccc} M(I) & \stackrel{f_J^I}{\to} & M(J) \\ \downarrow f_K^I & & \downarrow f_L^J V \\ M(K) & \stackrel{f_L^K}{\to} & M(K) \end{array}$$

commutes.

(6) *One can choose a sequence of spectra $M(I_l)$ and maps $f_{I_{l+1}}^{I_l}$ so that given any $M(I)$ there is an $f_{I_l}^I$ for l sufficiently large. If F is a specific finite type k complex, then one can choose the $M(I_l)$ so that $M(I_l) \wedge F$ is a wedge of 2^{k+1} copies of F* (*one for each cell in $M(I_l)$*), *and so that $f_{I_{l+1}}^{I_l}$ factors*

$$M(I_l) \wedge F \xrightarrow{g} F \xrightarrow{h} M(I_{l+1}) \wedge F$$

where g is projection to the top cell of $M(I_l)$ smashed with F and h is inclusion of the wedge factor of F associated to the top cell of $M(I_{l+1})$ (once again we've neglected suspensions here).

(7) *The Spanier-Whitehead dual of an $M(I)$ is also an $M(I)$. The Spanier-Whitehead dual of f_J^I gives the obvious projection*

$$BP_* \Big/ \left(p^{j_0}, \ldots, v_k^{j_k}\right) \to BP_* \Big/ \left(p^{i_0}, \ldots, v_k^{i_k}\right) \ .$$

The finiteness results are consequences of the fact that a finite torsion spectrum has finite homotopy groups in every dimension. The existence results are all applications of the Nilpotence and Periodicity theorems.

We will make use of the direct system one can form by using the spectra $M(I)$ and the maps f_J^I. Let $\bar{M}(I)$ be the fiber of the projection to the top cell

$$M(I) \xrightarrow{\pi} S^{k+1+i_1 2(p-1)+i_2 2(p^2-1)+\cdots+i_k 2(p^k-1)} \ .$$

Then there is a cofiber sequence

$$S^{k+i_1 2(p-1)+i_2 2(p^2-1)+\cdots+i_k 2(p^k-1)} \xrightarrow{g_I} \bar{M}(I) \to M(I) \ . \tag{3}$$

Since the f_J^I have been chosen to commute with the projections to the top cell, we get induced maps (of positive degree which we omit from our notation)

$$\bar{M}(I) \xrightarrow{\bar{f}_J^I} \bar{M}(J)$$

such that $\bar{f}_J^I g_I = g_J$.

Corresponding to the direct system of $M(I)$'s and f_J^I's, we get a direct system of $\bar{M}(I)$'s and $\bar{f}_J^I$.

Proposition A.3. *The map*

$$S^0 \to \operatorname*{hocolim}_I [\Sigma^{-k-i_1 2(p-1)-\cdots-i_k 2(p^k-1)} \bar{M}(I)] \tag{4}$$

induced by the $\{g_I\}$ is L_k localization.

The next proposition gives a functorial description of v_k-torsion generalizing the usual definition when X has a v_k-map.

Proposition A.4. *Let X be a spectrum and $f \in \pi_*(X)$. The following are equivalent*:

i) *f factors as*

$$S^0 \xrightarrow{\tilde{f}} M \xrightarrow{g} X$$

where M is a complex with a v_k-map v such that $v^j \tilde{f} \simeq *$ for some j.

ii) *f factors through a finite complex in C_{k+1}.*

iii) *f is in the kernel of*

$$\pi_* X \longrightarrow \pi_*(L_k^f X) \ .$$

If X is a finite complex of type k, the above conditions are equivalent to

iv) *If v is any v_k-map of X, then $v^j f \simeq *$ for j sufficiently large.*

Here is the definition of v_k-periodic homotopy with integral coefficients.

Definition A.5.

$$v_k^{-1}\pi_k(X) = \mathrm{dirlim}(i_0, \dots, i_{k-1}) v_k^{-1}[M_l(p^{i_0}, \dots, v_{k-1}^{i_{k-1}}), X] \ .$$

Here the subscript l indicates the dimension of the bottom cell of the coefficient spectrum.

Note that this definition also makes sense unstably for l sufficiently large: suppose for some $(i_0, \dots, i_{k-1})$, $M_k(p^{i_0}, \dots, v_{k-1}^{i_{k-1}})$ exists unstably, and supports a $v_k^{i_k}$ self map. Then after inverting $v_k^{i_k}$ we can still form the direct limit over $(j_0, \dots, j_{k-1})$ by noting that the stable map

$$M_l\left(p^{i'_0}, \dots, v_{k-1}^{i'_{k-1}}\right) \rightarrow M_l(p^{i_0}, \dots, v_{k-1}^{i_{k-1}})$$

is the stabilization of some unstable map

$$M_{l+r|v_k^{i_k}|}\left(p^{i'_0}, \dots, v_{k-1}^{i'_{k-1}}\right) \rightarrow S^{r|v_k^{i_k}|} M_l(p^{i_0}, \dots, v_{k-1}^{i_{k-1}}) \ .$$

We also need to know that a v_k-map of a spectrum can be represented on the level of a (perhaps suitably modified) Adams spectral sequence by multiplication by an element on the line of slope $\frac{1}{|v_k|}$ passing through the origin.

Acknowledgements. The first named author would like to offer a special thanks to T. Kashiwabara for many helpful conversations on this material. We also are grateful to N. Kuhn for many in-depth comments on some of the early versions of the manuscript.

References

[AD73] Anderson, D.W., Davis, D.W.: A vanishing theorem in homological algebra. Comment. Math. Helv. **48**, 318–327 (1973)

[AdMi95] Adem, A., Milgram, J.: Cohomology of finite groups. Grundlehren der mathematischen Wissenschaften 309, Springer-Verlag (1995)

[AK97] Arone, G., Kankaanrinta, M.: A functorial model for iterated Snaith splitting with applications to calculus of functors. Stable and Unstable Homotopy, Fields Institute Communications, Vol. **19**, 1–30 (1998)

[AD97] Arone, G., Dwyer, W.: Partition posets, Tits buildings and symmetric product spectra (tentative title). in preparation.

[BK72] Bousfield, A.K., Kan, D.M.: Homotopy limits, completions and localizations. Springer Lecture Notes, Vol. 304 (1972)

[BMMS86] Bruner, P.R., May, J.P., McClure, J.E., Steinberg, M.: H_∞ ring spectra and their applications. Springer Lecture Notes, Vol. 1176 (1986)

[CLM76] Cohen, F.R., Lada, T.J., May, J.P.: The homology of iterated loop spaces. Springer Lecture Notes, Vol. 533 (1976)

[G90] Goodwillie, T.G.: Calculus I: the first derivative of pseudoisotopy theory. K-Theory **4**, 1–27 (1990)

[G92] Goodwillie, T.G.: Calculus II: analytic functors. K-Theory **5**, 295–332 (1992)

[G3] Goodwillie, T.G.: Calculus III: the Taylor series of a homotopy functor. in preparation

[Ja53] James, I.M.: Reduced product spaces. Annals of Math. **62**, 170–197 (1953)

[Jo95] Johnson, B.: The derivatives of homotopy theory. Trans. Amer. Math. Soc. **347**, Number 4, 1295–1321 (1995)

[KaP78] Kahn, D.S., Priddy, S.B.: On the transfer in the homology of symmetric groups. Math. Proc. Cambridge Philos. Soc. **83**, 91–101 (1978)

[Ku82] Kuhn, N.J.: A Kahn-Priddy sequence and a conjecture of G.W. Whitehead. Math. Proc. Cambridge Philos. Soc. **92**, 467–483 (1982)

[Ku85] Kuhn, N.J.: Chevalley group theory and the transfer in the homology of symmetric groups. Topology **24**, 247–264 (1985)

[KuP85] Kuhn, N.J., Priddy, S.B.: The transfer and Whitehead conjecture. Math. Proc. Cambridge Philos. Soc. **98**, 459–480 (1985)

[M82] Mahowald, M.: The image of J in the EHP sequence. Annals Math **116**, 65–112 (1982)

[MS95] Mahowald, M., Sadofsky, H.: v_n telescopes and the Adams spectral sequence. Duke Math J. **78**, 101–129 (1995)

[MT92] Mahowald, M., Thompson, R.: The K-theory localization of an unstable sphere. Topology **31**, 133–141 (1992)

[MT94] Mahowald, M., Thompson, R.: On the secondary suspension homomorphism. Amer. J. of Math. **116**, 179–206 (1994)

[Mar] Margolis, H.R.: Spectra and the Steenrod algebra. North Holland Mathematical Library

[MW81] Miller, H., Wilkerson, C.: Vanishing lines for modules over the Steenrod algebra. J. Pure Appl. Algebra **22**, 293–307 (1981)

[Mi72] Milnor, J.: On the construction FK. in "Algebraic Topology: A Student's Guide" by J.F. Adams, London Math. Soc. Lecture Note Series 4, Cambridge Univ. Press, Cambridge 119–135 (1972)

[N58] Nakaoka, M.: Cohomology mod p of symmetric products of spheres. J. Inst. Poly., Osaka City Univ. 9 1–18 (1958)

[T90] Thompson, R.: The v_1-periodic homotopy groups of an unstable sphere at odd primes. Trans. Am. Math. Soc. **319**, 535–559 (1990)

[W81] Welcher, P.J.: Symmetric fiber spectra and K(n)-homology acyclicity. Indiana Univ. Math. J. **30**, 801–812 (1981)