New families in the homotopy of the motivic sphere spectrum

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Abstract

In [1] Adams constructed a non-nilpotent map $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$. Using iterates of this map one constructs infinite families of elements in the stable homotopy groups of spheres, the v_1-periodic elements of order 2. In this paper we work motivically over $\mathbb{C}$ and construct a non-nilpotent self map $w_1^4 : \Sigma^{20,12} S/\eta \longrightarrow S/\eta$. We then construct some infinite families of elements in the homotopy of the motivic sphere spectrum, w_1-periodic elements killed by η.

1 Introduction

The chromatic approach to computing the homotopy of a finite 2-local complex X is recursive.

1. Find a non-nilpotent self map $f : X \longrightarrow \Sigma^{-d} X$ and compute $f^{-1}\pi_*(X)$.
2. Attack the problem of computing the f-torsion elements in $\pi_*(X)$ by replacing X with X/f and going back to step 1.

Before the work of DHS in [4] and [6] it was not known that one could always construct the requisite self maps. However, in [1] Adams constructed a non-nilpotent map $v_1^4 : S/2 \to \Sigma^{-8} S/2$ and this gave the first hint that the above procedure is, in fact, implementable. One might say that Adams' work gave birth to chromatic homotopy theory.

The power of Adams' self map is that it gives rise to infinite families in the stable homotopy groups of spheres. Let's recall how one obtains such families. We have elements of order 2

$$\eta \in \pi_1(S^0),\ \eta^2 \in \pi_2(S^0),\ \eta^3 \in \pi_3(S^0),\ \epsilon \in \pi_8(S^0) \text{ and } \epsilon\eta \in \pi_9(S^0),$$

which lift to elements of $\pi_*(S/2)$. We also have an element $i \in \pi_0(S/2)$, the inclusion of the bottom cell of $S/2$. By composing with the maps

$$S/2 \xrightarrow{(v_1^4)^n} \Sigma^{-8n} S/2 \xrightarrow{\text{pinch}} S^{1-8n}$$

we obtain families of elements in the homotopy groups of spheres. These are the v_1-periodic elements of $\pi_*(S^0)$ of order 2.

A corollary of the nilpotence theorem [6] is that the only non-nilpotent self maps that a type n complex admits are v_n-self maps. If we state the nilpotence theorem with motivic BP it is false for there is an element $\eta \in \pi_{1,1}(S^{0,0})$, which is non-nilpotent, and has $BP_{*,*}(\eta) = 0$. One would expect that $S^{0,0}$ should be "type 0" but it admits more than v_0-self maps. However, we can still follow the algorithm with f taken to be η, and it suggests that we try and compute $\eta^{-1}\pi_{*,*}(S^{0,0})$. This computation was carried out in [2] and the description is very simple:

$$\eta^{-1}\pi_{*,*}(S^{0,0}) = \mathbb{F}_2[\eta^{\pm 1}, \sigma, \mu_9]/(\eta\sigma^2).$$

Here, $\eta \in \pi_{1,1}(S^{0,0})$ and $\sigma \in \pi_{7,4}(S^{0,0})$ are motivic Hopf invariant one elements, classes that exist before η is inverted. μ_9 also exists before η is inverted and it can be described by the Toda bracket $\langle 8\sigma, 2, \eta\rangle \in \pi_{9,5}(S^{0,0})$.

The algorithm then suggests that we try and find a non-nilpotent self map of S/η. The main result of this paper is that such a self map exists (proposition 3.3 and theorem 3.4)

$$w_1^4 : S/\eta \longrightarrow \Sigma^{-20,-12}S/\eta$$

and we use this map to construct six infinite families in the homotopy groups of the motivic sphere spectrum (theorem 3.12). The construction of the infinite families is parallel to the story we recalled above. We have elements killed by η

$$\nu \in \pi_{3,2}(S^{0,0}),\ \nu^2 \in \pi_{6,4}(S^{0,0}),\ \nu^3 \in \pi_{9,6}(S^{0,0}),\ \overline{\sigma} \in \pi_{19,11}(S^{0,0}) \text{ and } \overline{\sigma}\nu \in \pi_{22,13}(S^{0,0}),$$

which lift to elements of $\pi_{*,*}(S/\eta)$. We also have an element $i \in \pi_{0,0}(S/\eta)$, the inclusion of the bottom cell of S/η. By composing with the maps

$$S/\eta \xrightarrow{(w_1^4)^n} \Sigma^{-20n,-12n}S/\eta \xrightarrow{\text{pinch}} S^{2-20n,1-12n}$$

we obtain families of elements in the homotopy groups of the motivic sphere spectrum:

$$P^n(\nu) \in \pi_{3+20n,2+12n}(S^{0,0}),\ P^n(\nu^2) \in \pi_{6+20n,4+12n}(S^{0,0}),\ P^n(\nu^3) \in \pi_{9+20n,6+12n}(S^{0,0}),$$

$$P^n(\eta^2\eta_4) \in \pi_{18+20n,11+12n}(S^{0,0}),\ P^n(\overline{\sigma}) \in \pi_{19+20n,11+12n}(S^{0,0}),\ P^n(\overline{\sigma}\nu) \in \pi_{22+20n,13+12n}(S^{0,0}).$$

These are w_1-periodic elements of $\pi_{*,*}(S^{0,0})$ killed by η.

We emphasize here that it is not automatic that these composites are nontrivial and we have to detect this somehow. This should not be unfamiliar. The driving force behind [13] was to detect the nontriviality of the classical γ-family. Our tool for detection is the motivic Adams-Novikov spectral sequence. The homotopy classes are detected by permanent cycles, which cannot be boundaries for degree reasons. We are then left with showing that these elements are nonzero on the E_2-page and we do this by mapping to the classical Adams spectral seqeuence.

One can try to continue with the algorithm. We should compute $w_1^{-1}\pi_{*,*}(S/\eta)$. This is probably of similar difficulty to Mahowald's classical computation of $v_1^{-1}\pi_*(S/2)$, [9, 10, 11]. We should also try and find a self map of $S/(\eta, w_1^4)$. We conjecture that there is a non-nilpotent self map

$$w_2^{32} : S/(\eta, w_1^4) \longrightarrow \Sigma^{-416,-224}S/(\eta, w_1^4).$$

At this point we should explain where our intuition about these self maps comes from. There is a spectral sequence called the algebraic Novikov SS for computing the E_2-page of the ANSS, which is made use of in [2]. It takes the form

$$H(P;Q) \implies H(BP_*BP).$$

Here P is the Hopf subalgebra of squares in the dual Steenrod algebra A, BP_* is filtered using the I-adic filtration, where $I = \ker(BP_* \longrightarrow \mathbb{F}_2)$, and

$$Q = \mathrm{gr}^* BP_* = \mathbb{F}_2[q_0, q_1, \ldots]$$

is the associated graded of BP_*. q_n is the class of the Hazewinkel generator v_n and so Q contains the classical chromatic story, in some sense. In [2] we inverted the element $h_0 = \{[\xi_1^2]\}$ (the index on h is zero because P does not contain ξ_1) to compute $\alpha_1^{-1} H(BP_* BP)$. Using the close the relationship between the ANSS and its motivic analog [7] this enabled our computation of $\eta^{-1}\pi_{*,*}(S^{0,0})$. So ξ_1^2 corresponds to η and Haynes Miller suggested that there may be other non-nilpotent self maps corresponding to $\xi_2^2, \xi_3^2, \ldots \in P$. We call η, w_0 since it corresponds to ξ_{0+1}^2. A self map corresponding to ξ_{1+1}^2 would have motivic degree $(5,3)$, we would call it w_1, and a self map corresponding to ξ_{2+1}^2 would have motivic degree $(13,7)$, we would call it w_2. It is not luck that Adams found v_1^4 and we find w_1^4. Since BHHM constructed a self map v_2^{32} [3], one would guess that we have a self map w_2^{32}.

The results of this paper lead to interesting questions. Adams showed that his map v_1^4 is non-nilpotent by proving that it induces an isomorphism on K-theory. Perhaps there is an analogous spectrum that can detect the self map w_1^4. Motivically, there is not a nilpotence theorem with BP. But perhaps there is a spectrum N, such that $N_{*,*}$ is a $\mathbb{Z}_2[v_1, v_2, v_3, \ldots, w_0, w_1, w_2, \ldots]$ module, with each v_n and w_n acting non-nilpotently, for which a nilpotence theorem does hold. Since $2\eta = 0$, $2w_0$ would have to act as zero on $N_{*,*}$. Maybe, if N exists, then

$$N_{*,*} = \mathbb{Z}_2[v_1, v_2, v_3, \ldots, w_0, w_1, w_2, \ldots]/(2w_0 \text{ and other relations}).$$

The construction of the self map w_1^4 is not difficult; it is similar to the construction of Adams' self map. Adams' map has the the property that the composite

$$S^7 \longrightarrow \Sigma^7 S/2 \xrightarrow{v_1^4} \Sigma^{-1} S/2 \xrightarrow{\text{pinch}} S^0$$

is 8σ. We construct w_1^4 so that $\eta^2\eta_4$ can be factored as

$$S^{18,11} \longrightarrow \Sigma^{18,11} S/\eta \xrightarrow{w_1^4} \Sigma^{-2,-1} S/\eta \xrightarrow{\text{pinch}} S^{0,0}.$$

I would like to thank Haynes Miller for his bold suggestion that such self maps exist. I would like to thank Zhouli Xu for conversations we had at the University of Chicago after the existence of w_1^4 was proved. Upon showing him my result he showed great enthusiasm, and within an hour of talking we had the six infinite familes discussed in this paper. Thank you to Dan Isaksen for his beautiful spectral sequence charts and for a Toda bracket argument he provided. More importantly, we are grateful to him for spotting a flawed argument in what we thought was a simpler proof of these results.

2 The motivic Adams-Novikov spectral sequence

Working motivically requires specifying a ground field; for us, this is always taken to be $\mathbb{C}$. Throughout the paper we work at the prime 2. Our main calculational tool is the motivic Adams-Novikov spectral sequence (MANSS). We need it for the spectra $S^{0,0}$, S/η and $\mathrm{End}(S/\eta)$, that is the motivic sphere spectrum, the cofiber of $\eta : S^{1,1} \longrightarrow S^{0,0}$, and the endomorphism spectrum of S/η.

The motivic Adams-Novikov spectral sequence [7] is a convergent spectral sequence of the form

$$H^{s,t,w}(BP_*BP; BP_*(X)) \overset{s}{\Longrightarrow} \pi_{t-s,w}(X_2^\wedge).$$

Here, BP is the motivic Brown-Peterson spectrum at 2, we write BP_* for $BP_{*,*}$ to save space, and $H(BP_*BP; BP_*(X))$ is the cohomology of the Hopf algebroid BP_*BP with coefficients in $BP_*(X)$, otherwise known as $\mathrm{Cotor}_{BP_*BP}(BP_*, BP_*(X))$. We recall that differentials in the MANSS interact with the s and t gradings as in the classical case, and they preserve the *weight* w.

The $w = t/2$ slice of the motivic Adams-Novikov E_2-page for the sphere spectrum $S^{0,0}$ is plotted in the range $15 < t - s < 24$ in figure 1. We can deduce the E_2-pages for S/η and $\mathrm{End}(S/\eta)$, up to extensions, in a smaller range, using the cofibration sequences of (2.1) below.

Figure 1 is due Ravenel [15]. Ignoring the naming of elements, his chart agrees with Isaksen's charts [8] in the plotted range. We have chosen only to label the two elements which we will need to consider. We have labelled the first element $\beta_{4/3}$ as Ravenel does; it is a short calculation (lemma 4.1) to verify that this is the correct name. We have labelled the second element z_{19} in accordance with Isaksen, [8].

Notation 2.1. We use i for "include" and c for "collapse" throughout this paper.

$$\begin{gathered} S^{0,0} \xrightarrow{\quad i \quad} S/\eta \xrightarrow{\quad c \quad} S^{2,1} \\ \Sigma^{-2,-1}S/\eta \xrightarrow{i=c^*} \mathrm{End}(S/\eta) \xrightarrow{c=i^*} S/\eta. \end{gathered}$$

We never have to worry about differentials in our computations. For the most part, this is due to the existence of the following vanishing line and the corollary that follows.

Lemma 2.2. *When $X = S^{0,0}$, S/η or $\mathrm{End}(S/\eta)$ we have $H^{s,t,w}(BP_*BP; BP_*(X)) \neq 0$ only when t is even and $w \leq t/2$.*

Proof. The result is true for $X = S^{0,0}$ by [7, (36)]. We have an element $\alpha_1 \in H^{1,2,1}(BP_*BP)$ and so multiplication by α_1 gives a map $H(BP_*BP) \longrightarrow \Sigma^{-1,-2,-1}H(BP_*BP)$.

The first cofibration sequence of (2.1) gives a short exact sequence

$$0 \longrightarrow \mathrm{coker}\,\alpha_1 \longrightarrow H(BP_*BP; BP_*(S/\eta)) \longrightarrow \Sigma^{0,2,1}\mathrm{ker}\,\alpha_1 \longrightarrow 0$$

Since $(\mathrm{coker}\,\alpha_1)^{s,t,w} \neq 0$ only when t is even and $w \leq t/2$, and the same is true for $\Sigma^{0,2,1}\mathrm{ker}\,\alpha_1$, the result holds when $X = S/\eta$. Similarly, we use the second cofibration of (2.1) sequence to show the result for $\mathrm{End}(S/\eta)$. $\square$

Corollary 2.3. *Given an element of $H^{s,2w,w}(BP_*BP; BP_*(X))$, where $X = S^{0,0}$, S/η or $\mathrm{End}(S/\eta)$, it cannot be the target of a differential in the MANSS.*

Proof. The differentials with the given group as the target can be enumerated:

$$d_{2r+1} : E_{2r+1}^{s-2r-1,2(w-r),w} \longrightarrow E_{2r+1}^{s,2w,w},\ r > 0.$$

But $E_{2r+1}^{s-2r-1,2(w-r),w} = 0$ since $w > w - r$. $\square$

In this paper, we show that many elements of the motivic Adams-Novikov E_2-page are nonzero by mapping to the classical Adams E_2-page. To define the so-called *detection map* we need to recall the structure of the Hopf algebroid (BP_*BP, BP_*) and the dual Steenrod algebra $(A, \mathbb{F}_2)$.

Notation 2.4. Recall that $BP_* = \mathbb{Z}_2[\tau, v_1, v_2, v_3 \ldots]$. Here $\mathbb{Z}_2$ denotes the 2-adics, τ has bigrading $(0, -1)$ and v_n has bigrading $(2^{n+1} - 2, 2^n - 1)$. $BP_*BP = BP_*[t_1, t_2, t_3, \ldots]$ where $|t_n| = |v_n|$ and there are structure maps making the pair (BP_*BP, BP_*) into a Hopf algebroid.

The dual Steenrod algebra is given as an algebra by $\mathbb{F}_2[\zeta_1, \zeta_2, \zeta_3, \ldots]$ where $|\zeta_n| = 2^n - 1$. Here ζ_n is the Hopf conjugate of the Milnor generator ξ_n and the diagonal is given by the Milnor diagonal. We write $H(A; M)$ for $\mathrm{Cotor}_A(\mathbb{F}_2, M)$ when M is an A-comodule.

We can then define a map of Hopf algebroids.

Definition 2.5. Define $(BP_*BP, BP_*) \longrightarrow (A, \mathbb{F}_2)$ by demanding that τ, v_n, and t_n are mapped to 0, 0, and ζ_n, respectively. If we choose only to remember the weight of elements in (BP_*BP, BP_*), then this map preserves degree.

We also need maps between various homology groups, compatible with the map just defined. We note that $BP_*(S/\eta) = BP_*\langle 1, t_1\rangle$ and that $BP_*(\mathrm{End}(S/\eta)) = BP_*(\Sigma^{-2,-1}S/\eta) \otimes^{\Delta}_{BP_*} BP_*(S/\eta)$.

Notation 2.6. Write $S/2$ for the classical mod 2 Moore spectrum and $H_*(-)$ for mod 2 homology.

We note that $H_*(S/2) = \mathbb{F}_2\langle 1, \zeta_1\rangle$ and that $H_*(\mathrm{End}(S/2)) = H_*(\Sigma^{-1}S/2) \otimes^{\Delta}_{\mathbb{F}_2} H_*(S/2)$.

Definition 2.7. Define $BP_*(S/\eta) \to H_*(S/2)$ and $BP_*(\mathrm{End}(S/\eta)) \to H_*(\mathrm{End}(S/\eta))$ by demanding that τ, v_n, 1 and t_1 are mapped to 0, 0, 1 and ζ_1, respectively.

We are now ready to define our detection maps.

Definition 2.8 (The detection maps)**.** The maps of definition 2.5 and definition 2.7 induce maps

$$d : H(BP_*BP) \longrightarrow H(A), \quad d : H(BP_*BP; BP_*(S/\eta)) \longrightarrow H(A; H_*(S/2)),$$

$$d : H(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \longrightarrow H(A; H_*(\mathrm{End}(S/2)))$$

We label each map by d for "detection."

3 The self map, the homotopy classes, and the main results

In this section we state our main results which are proposition 3.3, theorem 3.4 and theorem 3.12. We define the homotopy classes which appear in theorem 3.12. Doing so requires defining a number of auxilary homotopy classes. In section 4 we prove theorem 3.12 by working at the algebraic level with the elements detecting these classes. For this reason we keep track of all the elements detecting our homotopy classes.

The first elements that one encounters in homotopy theory are the Hopf invariant one elements.

Definition 3.1. We write $\eta \in \pi_{1,1}(S^{0,0})$, $\nu \in \pi_{3,2}(S^{0,0})$ and $\sigma \in \pi_{7,4}(S^{0,0})$ for the motivic Hopf invariant one elements, [5]. These elements are detected by α_1, $\alpha_{2/2}$ and $\alpha_{4/4}$, respectively.

Mahowald discovered the η_j-family, [12]. These are classes which are detected by h_1h_j in the Adams spectral sequence and, thus, they are defined up to higher Adams filtration. We need the motivic analog of η_4 but we are more precise, defining it without any indeterminancy.

Definition 3.2. In the motivic Adams-Novikov spectral sequence $\beta_{4/3} \in H^{2,18,9}(BP_*BP)$ detects a unique homotopy class. We call this homotopy class $\eta_4 \in \pi_{16,9}(S^{0,0})$.

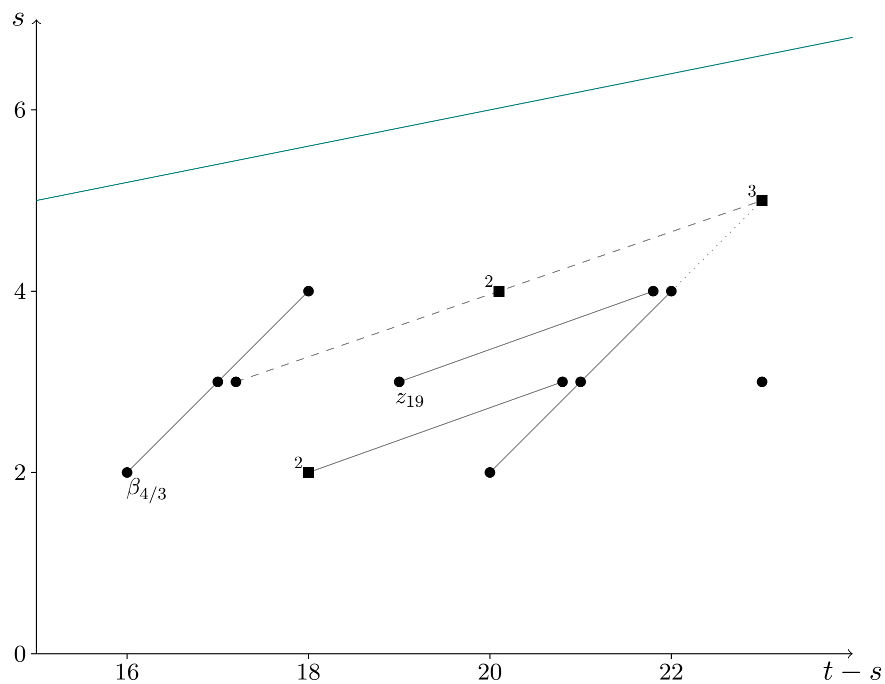

Figure 1: $H^{s,t,t/2}(BP_*BP)$ in the range $15 < t - s < 24$, minus the algebra generated by the α's. Round nodes indicate copies of $\mathbb{Z}/2$. Square nodes labelled with an n indicate copies of $\mathbb{Z}/2^n$. Lines indicate multiplication by α_1 and $\alpha_{2/2}$. Dashed lines indicate hitting twice a generator. Dotted lines indicate hitting four times a generator. Above the teal line, there are only α_1-free elements.

The main result of this paper is that we have a non-nilpotent self map $w_1^4 : \Sigma^{20,12} S/\eta \longrightarrow S/\eta$. The next proposition gives a map; non-nilpotence is left for theorem 3.4. We will use this map to construct the infinite familes of theorem 3.12.

Proposition 3.3. *There's an element $x \in H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$ which maps to $\alpha_1^2\beta_{4/3}$ under the collapse maps of (2.1).*

$$\begin{array}{ccccc} H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta))) & \to & H^{4,24,12}(BP_*BP; BP_*(S/\eta)) & \to & H^{4,22,11}(BP_*BP; BP_*(S^{0,0})) \\ x & \longmapsto & y & \longmapsto & \alpha_1^2\beta_{4/3} \end{array}$$

x is a permanent cycle in the motivic ANSS for $\mathrm{End}(S/\eta)$ *detecting a map $w_1^4 : \Sigma^{20,12} S/\eta \longrightarrow S/\eta$. Moreover, $\eta^2\eta_4$ is the composite*

$$S^{18,11} \xrightarrow{i} \Sigma^{18,11} S/\eta \xrightarrow{w_1^4} \Sigma^{-2,-1} S/\eta \xrightarrow{c} S^{0,0}.$$

Proof. We find that $\alpha_1^3\beta_{4/3} = 0$ and so there exists y mapping to $\alpha_1^2\beta_{4/3}$. We also find that $\alpha_1 y = 0$ and so there exists an x mapping to y.

All the targets groups of the differentials emanating from $H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$ in the motivic Adams-Novikov spectral sequence are zero and so x is a permanent cycle. Because x maps to $\alpha_1^2\beta_{4/3}$ and there are no elements of higher Novikov filtration in that stem and weight, we obtain the factorization of $\eta^2\eta_4$ in the proposition statement. $\square$

One of our main results is the following theorem. We will postpone the proof until section 4.

Theorem 3.4. *$w_1^4 : \Sigma^{20,12} S/\eta \longrightarrow S/\eta$ is non-nilpotent.*

We will define one of the classes that we need by a Toda bracket. We must recall some relations in homotopy.

Lemma 3.5 (Isaksen)**.** *We have the following relations: $\eta\nu = 0$, $\nu\sigma = 0$, and $\eta\sigma^2 = 0$.*

Proof. In [5] it is proved that that $\eta\nu = 0$ and $\nu\sigma = 0$. Moreover, $\eta\sigma^2 = 0$ holds classically [16], and this immediately implies the motivic version because there are no "exotic" classes in the 15-stem with weight 9, [8]. $\square$

Definition 3.6. We define $\overline{\sigma} \in \pi_{19,11}(S^{0,0})$ by the Toda bracket $\langle \nu, \sigma, \eta\sigma \rangle$. There is no indeterminancy in this Toda bracket and the class is nonzero, [8].

To see the element which detects $\overline{\sigma}$ in the motivic Adams-Novikov spectral sequence we make note of the following property.

Lemma 3.7 (Isaksen)**.** *We have the following relation: $\eta\overline{\sigma} = 0$.*

Proof. $\langle \eta, \nu, \sigma \rangle$ is defined and seen to be 0. Thus $\eta\overline{\sigma} = \eta\langle \nu, \sigma, \eta\sigma \rangle = \langle \eta, \nu, \sigma \rangle \eta\sigma = 0$. $\square$

Corollary 3.8. *$\overline{\sigma}$ is the unique element detected by z_{19}.*

To define the infinite families of theorem 3.12 we need to lift some homotopy classes in $\pi_{*,*}(S^{0,0})$ to $\pi_{*,*}(S/\eta)$. We also need to keep track of the elements detecting these classes. That is the purpose of the next two definitions.

Definition 3.9. We write $\widetilde{\nu} \in \pi_{5,3}(S/\eta)$ for a fixed choice of lift of $\nu \in \pi_{3,2}(S^{0,0})$ under the map $c : S/\eta \longrightarrow S^{2,1}$. This element has Novikov filtration one; we write $\widetilde{\alpha}_{2/2} \in H^{1,6,3}(BP_*BP; BP_*(S/\eta))$ for the element which detects it. $\widetilde{\alpha}_{2/2}$ lifts $\alpha_{2/2} \in H^{1,4,2}(BP_*BP)$.

Definition 3.10. We fix a lift $\widetilde{z}_{19} \in H^{3,24,12}(BP_*BP; BP_*(S/\eta))$ of $z_{19} \in H^{3,22,11}(BP_*BP)$ under the map $c : S/\eta \longrightarrow S^{2,1}$. This is a permanent cycle and detects a homotopy class which we call $\widetilde{\overline{\sigma}} \in \pi_{21,12}(S/\eta)$. $\widetilde{\overline{\sigma}}$ lifts $\overline{\sigma} \in \pi_{19,11}(S^{0,0})$.

We are ready to construct the homotopy classes of interest. We let Φ_n be the following composite

$$S/\eta \xrightarrow{(w_1^4)^n} \Sigma^{-20n,-12n} S/\eta \xrightarrow{\quad c \quad} S^{2-20n,1-12n}.$$

and recall that $\pi_{*,*}(S/\eta)$ is a $\pi_{*,*}(S^{0,0})$-module.

Definition 3.11. For $n \geq 0$, we define

$$P^n(\nu) \in \pi_{3+20n,2+12n}(S^{0,0}),\ P^n(\nu^2) \in \pi_{6+20n,4+12n}(S^{0,0}),\ P^n(\nu^3) \in \pi_{9+20n,6+12n}(S^{0,0}),$$
$$P^n(\eta^2\eta_4) \in \pi_{18+20n,11+12n}(S^{0,0}),\ P^n(\overline{\sigma}) \in \pi_{19+20n,11+12n}(S^{0,0}),\ P^n(\overline{\sigma}\nu) \in \pi_{22+20n,13+12n}(S^{0,0})$$

by

$$P^n(\nu) = (\Phi_n)_*(\widetilde{\nu}),\ P^n(\nu^2) = (\Phi_n)_*(\widetilde{\nu}\nu),\ P^n(\nu^3) = (\Phi_n)_*(\widetilde{\nu}\nu^2),$$
$$P^n(\eta^2\eta_4) = (\Phi_{n+1})_*(i),\ P^n(\overline{\sigma}) = (\Phi_n)_*(\widetilde{\overline{\sigma}}),\ P^n(\overline{\sigma}\nu) = (\Phi_n)_*(\widetilde{\overline{\sigma}}\nu).$$

One of our main results is the following theorem. We will postpone the proof until section 4.

Theorem 3.12. *The homotopy classes $P^n(\nu)$, $P^n(\nu^2)$, $P^n(\nu^3)$, $P^n(\eta^2\eta_4)$, $P^n(\overline{\sigma})$, $P^n(\overline{\sigma}\nu)$ are non trivial. i.e. they are w_1-periodic.*

Proving the theorem comes down to algebra and we can make the analogous construction algebraically. We have a map $\mathrm{End}(S/\eta) \wedge S/\eta = \mathrm{Hom}(S/\eta, S/\eta) \wedge \mathrm{Hom}(S^0, S/\eta) \longrightarrow \mathrm{Hom}(S^0, S/\eta) = S/\eta$, given by composition. Let φ be the composite

$$\begin{array}{ccc} H(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \otimes H(BP_*BP; BP_*(S/\eta)) & \xrightarrow{\text{composition}} & H(BP_*BP; BP_*(S/\eta)) \\ & & \downarrow{\scriptstyle c} \\ & & H(BP_*BP; BP_*(S^{0,0})) \end{array}$$

and recall that $H(BP_*BP; BP_*(S/\eta))$ is a $H(BP_*BP)$-module. Recall the element x of proposition 3.3, too.

Definition 3.13. For $n \geq 0$, we define

$$P^n(\alpha_{2/2}) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}),\ P^n(\alpha_{2/2}^2) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}\alpha_{2/2}),\ P^n(\alpha_{2/2}^3) = \varphi(x^n \otimes \widetilde{\alpha}_{2/2}\alpha_{2/2}^2),$$
$$P^n(\alpha_1^2\beta_{4/3}) = \varphi(x^{n+1} \otimes 1),\ P^n(z_{19}) = \varphi(x^n \otimes \widetilde{z}_{19}),\ P^n(z_{19}\alpha_{2/2}) = \varphi(x^n \otimes \widetilde{z}_{19}\alpha_{2/2}).$$

The construction of these elements together with Moss's convergence theorem [14], tells us that we have the following result.

Lemma 3.14. *The elements*

$$P^n(\alpha_{2/2}),\ P^n(\alpha_{2/2}^2),\ P^n(\alpha_{2/2}^3),\ P^n(\alpha_1^2\beta_{4/3}),\ P^n(z_{19}),\ \textit{and } P^n(z_{19}\alpha_{2/2})$$

detect $P^n(\nu)$, $P^n(\nu^2)$, $P^n(\nu^3)$, $P^n(\eta^2\eta_4)$, $P^n(\overline{\sigma})$, and $P^n(\overline{\sigma}\nu)$, respectively.

4 Proof of main results

In this section we prove theorem 3.4 and theorem 3.12. This comes down to analysing the effect of the detection map (2.8) on the class x of proposition 3.3 and the effect of the detection map on the classes of definition 3.13.

First, for completeness, we prove the following lemma, which was referred to in section 2.

Lemma 4.1. *We have $\alpha_1^3\beta_{4/3} = 0$.*

Proof. We have the following short exact sequences of BP_*BP-comodules.

$$\begin{array}{c} 0 \longrightarrow BP_* \xrightarrow{2} BP_* \longrightarrow BP_*/2 \longrightarrow 0 \\ 0 \longrightarrow BP_*/2 \xrightarrow{v_1^3} BP_*/2 \longrightarrow BP_*/(2, v_1^3) \longrightarrow 0 \end{array}$$

By definition $\beta_{4/3}$ is the image of v_2^4 under the composite

$$H^{0,24,12}(BP_*BP; BP_*/(2, v_1^3)) \xrightarrow{\delta_1} H^{1,18,9}(BP_*BP; BP_*/2) \xrightarrow{\delta_0} H^{2,18,9}(BP_*BP; BP_*).$$

From [2, 7.2.1] there exists an $N > 0$ with $\alpha_1^N\delta_1(v_2^4) = 0$. Thus $\alpha_1^N\beta_{4/3} = 0$. Since $\alpha_{8/5}$ is α_1-free, this means we cannot have $\alpha_1^3\beta_{4/3} = \alpha_1^4\alpha_{8/5}$ and so we must have $\alpha_1^3\beta_{4/3} = 0$. ☐

In order to prove theorem 3.4 we need the following two lemmas.

Lemma 4.2. *We have $d(\alpha_1) = h_0$, $d(\alpha_{2/2}) = h_1$, and $d(\alpha_{4/4}) = h_2$, where d is the detection map of definition 2.8.*

Proof. One can compute directly with cocycle representatives in $\Omega(BP_*BP)$. For instance, $\alpha_{4/4}$ is represented by $5[t_1^4] - 2[t_1t_2] + 9v_1[t_1^3] - v_1[t_2] + 7v_1^2[t_1^2] + 2v_1^3[t_1] - v_2[t_1]$. ☐

Lemma 4.3. *Under the detection map we have $d(\beta_{4/3}) = h_0h_3$.*

Proof. One can compute directly with cocycle representatives in $\Omega(BP_*BP; BP_*/2)$ and $\Omega(BP_*BP)$. The differential on v_2^4 is $v_1^4[t_1^8] + v_1^8[t_1^4] \pmod 2$ and so $\delta_1(v_2^4)$ is represented by $v_1[t_1^8] + v_1^5[t_1^4]$. If one applies the differential to $v_1[t_1^8] + v_1^5[t_1^4]$, divides by 2 and evaluates mod $(2, v_1)$, one obtains $[t_1|t_1^8]$. Thus $d(\beta_{4/3})$ is represented by $[\zeta_1|\zeta_1^8]$ and we are done. ☐

Now we address theorem 3.4 even though it will follow, independently, as a corollary of theorem 3.12. The key is to recall how Adams' self map $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$ is detected in the classical ASS. We have the following cofibration sequences.

$$\begin{array}{c} S^0 \xrightarrow{i} S/2 \xrightarrow{c} S^1 \\ \Sigma^{-1}S/2 \xrightarrow{i=c^*} \mathrm{End}(S/2) \xrightarrow{c=i^*} S/2. \end{array} \tag{4.4}$$

Proposition 4.5. *There exists a unique nonzero element $\overline{x} \in H^{4,12}(A; H_*(\mathrm{End}(S/2)))$. It maps to $h_0^3h_3$ under the collapse maps of (4.4) and is a permanent cycle in the ASS for $\mathrm{End}(S/2)$ detecting $v_1^4 : \Sigma^8 S/2 \longrightarrow S/2$.*

$$\begin{array}{c} H^{4,12}(A; H_*(\mathrm{End}(S/2))) \xrightarrow{c=i^*} H^{4,12}(A; H_*(S/2)) \xrightarrow{c} H^{4,11}(A; H_*(S^0)) \\ \overline{x} \longmapsto h_0^3h_3 \end{array}$$

Moreover, $\overline{x}$ is non-nilpotent.

We now prove theorem 3.4 by proving the following corollary.

Corollary 4.6. *The* $x \in H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta)))$ *of lemma 3.3 is non-nilpotent and so* w_1^4 *is non-nilpotent.*

Proof. We consider the following diagram in which the horizontal maps are obtained by applying the appropriate two collapse maps ((2.1) and (4.4)) and the vertical maps are detection maps (2.8). It is straightforward to see this diagram commutes.

$$\begin{tikzcd}
H^{4,24,12}(BP_*BP; BP_*(\mathrm{End}(S/\eta))) \arrow[r] \arrow[d, "d"'] & H^{4,22,11}(BP_*BP) \arrow[d, "d"] \\
H^{4,12}(A; H_*(\mathrm{End}(S/2))) \arrow[r] & H^{4,11}(A)
\end{tikzcd}$$

Start with x. We chose x so that it maps right to $\alpha_1^2\beta_{4/3}$ and we know $d(\alpha_1^2\beta_{4/3}) = h_0^3h_3$ by lemmas 4.2 and 4.3. So $d(x)$ gives a lift of $h_0^3h_3$ but, by proposition 4.5, $\overline{x}$ is the unique such lift, so $d(x) = \overline{x}$. Since $\overline{x}$ is non-nilpotent, x is non-nilpotent. Moreover, corollary 2.3 tells us that no power of x can ever be hit by a differential. We deduce that w_1^4 is non-nilpotent. □

In order to prove theorem 3.12 we need the following lemma.

Lemma 4.7. *Under the detection map of definition 2.8 we have* $d(z_{19}) = c_0$.

Proof. We note the Massey product $\langle h_1, h_2, h_0h_2\rangle$ has zero indeterminancy and defines c_0, [8]. Since $\nu\sigma = 0$ and $\eta\sigma^2 = 0$, we have $\alpha_{2/2}\alpha_{4/4} = 0$ and $\alpha_1\alpha_{4/4}^2 = 0$. This means that $\langle \alpha_{2/2}, \alpha_{4/4}, \alpha_1\alpha_{4/4}\rangle$ is defined and its elements give a lift for c_0. The only elements in the correct trigrading to lift c_0 are linear combinations of $\alpha_1^2\alpha_9$ and z_{19}. Since α_9 maps to zero, z_{19} must map to c_0. □

We are now ready to prove theorem 3.12.

Proof of theorem 3.12. By lemma 3.14 and corollary 2.3 we see that it is enough to prove that each of the following elements is nonzero in $H(BP_*BP)$:

$$P^n(\alpha_{2/2}),\ P^n(\alpha_{2/2}^2),\ P^n(\alpha_{2/2}^3),\ P^n(\alpha_1^2\beta_{4/3}),\ P^n(z_{19}),\ \text{and } P^n(z_{19}\alpha_{2/2}). \tag{4.8}$$

We do this by mapping to $H(A)$, using the detection map $d : H(BP_*BP) \longrightarrow H(A)$ of definition 2.8. In the case $n = 0$, they map to

$$h_1, h_1^2, h_1^3, h_0^3h_3, c_0, c_0h_1$$

by lemmas 4.2, 4.3, and 4.7, and so we're done.

We prove that each $P^n(\alpha_{2/2}) \neq 0$ by induction on n. We take as the inductive hypothesis that $P^{n-1}(\alpha_{2/2})$ maps to $P^{n-1}(h_1) \neq 0$, where, by abuse of notation, we also use P to denote the Adams periodicity operator $P = \langle h_0^3h_3, h_0, -\rangle$. The definition of $P^n(\alpha_{2/2})$ (3.13) gives

$$P^n(\alpha_{2/2}) \in \langle \alpha_1^2\beta_{4/3}, \alpha_1, P^{n-1}(\alpha_{2/2})\rangle.$$

Using lemmas 4.2 and 4.3, we see that $P^n(\alpha_{2/2})$ maps to $P^n(h_1) \neq 0$, which completes the induction.

Similarly, applying d to the other elements of (4.8) gives the elements

$$P^n(h_1^2),\ P^n(h_1^3),\ P^n(h_0^3h_3),\ P^n(c_0),\ P^n(c_0h_1),$$

which are all nonzero. This completes the proof. □

5 Observations and conjectures

From definition 3.11 we immediately see that

$$P^n(\nu^2) = P^n(\nu)\nu,\ P^n(\nu^3) = P^n(\nu)\nu^2,\ \text{and } P^n(\overline{\sigma}\nu) = P^n(\overline{\sigma})\nu.$$

Moreover, $P^n(\nu)$, $P^n(\nu^2)$, $P^n(\nu^3)$, $P^n(\overline{\sigma})$, and $P^n(\overline{\sigma}\nu)$ are detected by

$$h_2 g^n,\ h_2^2 g^n,\ h_2^3 g^n,\ c_1 g^n,\ \text{and } c_1 h_2 g^n,$$

respectively, in the motivic Adams spectral sequence. The naming of elements in the motivic Adams spectral sequence is fairly unsystematic, since the naming conventions follow those for the classical Adams spectral sequence. We content ourselves with noting that $P^{2^n-1}(\eta^2\eta_4)$ is detected by

$$h_1^{2^{n+2}-1} h_{n+4},\ n \geq 0.$$

We know that $\overline{\sigma}\nu^2 \neq 0$.

Conjecture 5.1. *We think that $P^n(\overline{\sigma})\nu^2 \neq 0$ for all $n \geq 0$.*

We know the ν^3 is divisible by η: $\nu^3 = \eta(\eta\sigma + \epsilon)$.

Conjecture 5.2. *We think that $P^n(\nu^3)$ is divisible by η for all $n \geq 0$.*

We do have the following proposition, which proves the conjecture up to higher Adams filtration, since $h_1^2 h_3 g^n = h_2^3 g^n$.

Proposition 5.3. *$h_1 h_3 g^n$ is a permanent cycle in the motivic Adams spectral sequence detecting a nonzero homotopy class, for all $n \geq 0$.*

Proof. In this proof we write P for $\langle h_1^2 h_4, h_1^2, -\rangle$, an operator on the motivic Adams E_2-page, and $\overline{P}$ for $\langle h_0^2 h_3, h_0^2, -\rangle$, an operator on the classical Adams E_2-page.

From [8, theorem 2.1.12] and the classical result that $\overline{P}^n(h_0h_2)$ is well-defined with no indeterminancy, we see that $P^n(h_1h_3)$ is well-defined with no indeterminancy. Moreover, this is exactly what $h_1h_3g^n$ means. Since this element lies in a tridegree with $t = 2w$ one can verify the hypothesis of [8, theorem 3.1.1] to see that $h_1h_3g^n$ is a permanent cycle for all $n \geq 0$. It cannot be a boundary because of the analog of lemma 2.2 and corollary 2.3 for the motivic Adams spectral sequence, [8, remark 2.1.13]. □

When one looks at the motivic Adams spectral sequence [8], one sees other potential w_1-periodic elements. We suggest the following program to find them.

We have the algebraic Novikov spectral sequence

$$H(P; Q \otimes H_*(S/\eta))[\tau] \implies H(BP_*BP; BP_*(S/\eta)).$$

We have a periodicity operator w_1^4 lying in $H(P; H_*(\mathrm{End}(S/\eta)))$, which detects the element x of proposition 3.3. We rename x as w_1^4. We should attempt to compute the localized algebraic Novikov spectral sequence

$$w_1^{-1}H(P; Q \otimes H_*(S/\eta)) \implies w_1^{-1}H(BP_*BP; BP_*(S/\eta)),$$

and then run the η-Bockstein spectral sequence to recover the w_1-periodic elements of $H(BP_*BP)$.

6 Nomenclature

This paper suggests that there should be many new periodicities arising in motivic homotopy theory over $\mathbb{C}$. In the motivic Adams spectral sequence charts the conjectured periodicity associated with w_n should occur according to a slope of $1/(2^{n+2}-3)$. The classical periodicity associated with v_n occurs according to a slope of $1/(2^{n+1}-2)$. Since the new conjectured periodicity slopes lie between the classical chromatic ones and chromatic means "using notes not belonging to the diatonic scale of the key" we suggest that these new periodicities are *microtonal*.

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