A geometrical proof of Browder's result on the vanishing of the Kervaire invariant

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Abstract

The Kervaire invariant is a $Z/2$-invariant of framed manifolds of dimension $n = 4k + 2$. W. Browder proved [5] that this invariant necessarily vanishes if $n + 2$ is not a power of 2. We give a geometrical proof of this result using a characterization of the Kervaire invariant in terms of multiple points of immersions.

1 Introduction

This note is concerned with the following problem.

Problem 1.1 *Given an immersion $f: N^n \to R^{n+1}$ of a compact closed smooth orientable n-dimensional manifold in Euclidean space, is it cobordant to an immersion $\Sigma^n \to R^{n+1}$ of a homotopy sphere?*

The cobordism of immersions is explained in [19]. Recall that by the work of M.W. Hirsch [10] an immersion $f: N^n \to R^{n+1}$ corresponds to a trivialization of the stable normal bundle of N, in other words a framing of N. In terms of framed manifolds, M.A. Kervaire and J.W. Milnor gave the following partial answer to this problem.

Theorem 1.2 ([14]) *For $n \not\equiv 2 \bmod 4$ every immersion $N^n \to R^{n+1}$ of an orientable manifold is cobordant to an immersion of a homotopy sphere.*

For $n \equiv 2 \bmod 4$ they demonstrated that the Kervaire invariant provides the obstruction to the positive solution of the problem. This case was originally investigated by L.S. Pontrjagin in the simplest case $n = 2$ [17]. Here there is an immersion of the torus $S^1 \times S^1 \to R^3$ which is not cobordant to a homotopy sphere. Kervaire [13] proved that for $n = 10$ every immersion is cobordant to an immersed homotopy sphere. This led to his explicit construction of a closed 10-dimensional PL manifold without a smooth structure (see also the work of S.P. Novikov on the classification of smooth structures on manifolds [16]). Kervaire's result was generalized first of all by E.H. Brown and F.P. Peterson [7] to $n \equiv 2 \bmod 8$ (and $n > 2$) and then by Browder who proved the following.

Theorem 1.3 ([5]) *For $n \equiv 2 \bmod 4$, if $n + 2$ is not a power of 2 then the Kervaire invariant of every immersion $N^n \to R^{n+1}$ of an orientable manifold vanishes and so the immersion is cobordant to an immersion of a homotopy sphere.*

*The first author is supported in part by a grant from the Russian Science Foundation N96-01-00211 and both authors are supported in part by a grant from the Royal Society of London, Great Britain.

The remaining cases are not fully resolved. For $n = 6$ and $n = 14$ there are codimension 1 immersions of $S^3 \times S^3$ and $S^7 \times S^7$ which are not cobordant to homotopy spheres. Such immersions are also known to exist for $n = 30$ [15] and $n = 62$ [4] and an explicit construction of a manifold for $n = 30$ has been given by J.D.S. Jones [11]. It is not known whether higher dimensional examples exist. This is 'the Kervaire invariant problem'.

Kervaire and Milnor tackled Problem 1.1 by using surgery theory to modify the manifold within the same cobordism class so as to increase its connectivity. In the case of Theorem 1.2 there are no obstructions to this and so eventually a homotopy sphere is obtained. For $n \equiv 2 \bmod 4$ the Kervaire invariant of the immersion (or, equivalently, the framed manifold) is the obstruction to completing the final step: a codimension 1 immersion is cobordant to an immersion of a homotopy sphere if and only if its Kervaire invariant vanishes. Their definition of this invariant is in terms of the Arf invariant of a quadratic form defined on the middle dimensional cohomology of a suitable representative of the cobordism class and is quite complicated; Browder's proof of Theorem 1.3 is rather technical, relating the invariant to the homotopy of various Thom complexes. It is closely connected with his proof that in the case $n + 2 = 2^j$ there is an immersion not cobordant to a homotopy sphere (in other words with Kervaire invariant 1) if and only if the element h_{j-1}^2 in the Adams spectral sequence is a permanent cycle. Subsequently Brown [6] has simplified the definition and Jones and E. Rees [12] have given an alternative proof of Browder's result based on this definition and an observation by N. Ray [18].

The second author has obtained a geometric definition of the Kervaire invariant of a codimension 1 orientable immersion $f: N^n \to R^{n+1}$ where $n \equiv 2 \bmod 4$ as follows. By use of a regular homotopy if necessary such an immersion may be supposed to be in general position. In this case, there is an immersion of an $(n-1)$-dimensional manifold $\Delta_2(f) \to R^{n+1}$ whose image is the double point set of f. This immersion has a non-vanishing normal vector field given by taking the average of the two normal vectors provided at each point by an orientation of N. So by Hirsch theory this gives a codimension 1 immersion $g: \Delta_2(f) \to R^n$ in general position. The set of n-fold points of this immersion is a finite set of points.

Theorem 1.4 ([8]) *For $n \equiv 2 \bmod 4$, the Kervaire invariant of an orientable immersion $f: N^n \to R^{n+1}$ is given by the parity of the number of n-fold points of the immersion $g: \Delta_2(f) \to R^n$.*

This process of taking the double point manifold can be iterated. The immersion $g: \Delta_2(f) \to R^n$ has itself a double point manifold $h: \Delta_2(g) = M^{n-2} \to R^n$. The following result is implicit in the calculations in [8].

Theorem 1.5 *For $n = 4k + 2$, the parity of the number of n-fold points of a self-transverse immersion $g: L^{n-1} \to R^n$ is given by the normal Stiefel-Whitney number $\langle \overline{w}_2^{2k}; [\Delta_2(g)] \rangle$.*

This means that Theorem 1.3 is an immediate corollary of the following result.

Theorem 1.6 *For $n = 2m$, if $n+2$ is not a power of 2 then for every immersion $h: M^{n-2} \to R^n$, the normal Stiefel-Whitney number $\langle \overline{w}_2^{m-1}; [M] \rangle$ vanishes.*

2 A proof of Theorem 1.5

We first of all recall some results which are described in more detail in [8].

R. Wells has shown [19] that the cobordism group of codimension 1 immersions $L^{n-1} \to R^n$ is isomorphic to the homotopy group $\pi_n^S MO(1) = \pi_n QMO(1)$. Here QX is the direct limit $\lim \Omega^n \Sigma^n X$ and $MO(1)$ is the Thom complex of the universal line bundle. Recall that we can take $MO(1)$ to be RP^∞, infinite dimensional real projective space. For $i \geq 1$ write $a_i \in H_iRP^\infty \cong Z/2$ for the generator. (Throughout this note homology and cohomology groups have $Z/2$ coefficients.) Then the Pontrjagin ring H_*QRP^∞ has a basis provided by monomials in certain elements of the form $Q^{i_1}Q^{i_2}\dots Q^{i_r}a_s$, where Q^i denotes a Kudo-Araki operation, known as the Dyer-Lashof basis. There is a height function on the Dyer-Lashof basis determined by $\text{height}(a_i) = 1$, $\text{height}(\xi\eta) = \text{height}(\xi) + \text{height}(\eta)$, and $\text{height}(Q^i\xi) = 2\text{height}(\xi)$.

Given a self-transverse immersion $g: L^{n-1} \to R^n$ this corresponds to an element $\alpha \in \pi_nQMO(1)$. It is shown in [9] how the elements of height r in the Hurewicz image $h(\alpha) \in H_nQMO(1)$ determine the normal characteristic numbers and so the cobordism class of the r-fold self-intersection manifold of g. (Further details of how these calculations may be carried out may be found in [2] or [3].) In particular, the parity of the number of n-fold points is $\lambda \in Z/2$ if and only if, when $h(\alpha)$ is written in terms of the Dyer-Lashof basis, the coefficient of a_1^n is λ (a special case of [8] Theorem 2.3). For $n = 4k+2$, then by [8] Lemma 3.1 this means that the coefficient of a_{2k+1}^2 is also λ. It follows, by the method of [9] Theorem 3.2, that the double point manifold $\Delta_2(g) \to R^n$ represents an element β of $\pi_n^SMO(2)$ such that when $h(\beta) \in H_nMO(2)$ is written in terms of the basis $\{a_ia_{n-i}\}$ the coefficient of a_{2k+1}^2 is λ. Here $a_ia_{n-i} \in H_nMO(2)$ denotes the image of $a_i \otimes a_{n-i} \in \tilde{H}_*MO(1) \otimes \tilde{H}_*MO(1) \cong \tilde{H}_*(MO(1) \wedge MO(1))$ under the homomorphism induced by $MO(1) \wedge MO(1) \to MO(2)$, the map of Thom complexes corresponding to the Whitney sum map $BO(1) \times BO(1) \to BO(2)$. In cohomology, $\tilde{H}^*MO(2) \cong \tilde{H}^*BO(2)/BO(1) \cong w_2Z/2[w_1, w_2]$, where w_1 and w_2 are the universal Stiefel-Whitney classes, $\tilde{H}^*(MO(1) \wedge MO(1)) \cong x_1x_2Z/2[x_1, x_2]$, where $\dim x_1 = \dim x_2 = 1$, and $m^*(w_2) = x_1x_2$. It follows that the coefficient of a_{2k+1}^2 in $h^S(\beta)$ is λ if and only if $\langle w_2^{2k+1}, h^S(\beta)\rangle = \lambda$. Finally, by the analogue of [9] Lemma 2.2 for immersions, this is equivalent to the statement $\langle \overline{w}_2^{2k}, [\Delta_2(g)]\rangle = \lambda$ as required.

3 A proof of Theorem 1.6

The first author and A. Szűcs have given a geometric proof of the following analogue of Theorem 1.6 for codimension 1 immersions.

Theorem 3.1 ([1]) *If $n+1$ is not a power of 2 then for every immersion $h : M^{n-1} \to R^n$, the characteristic number $\langle w_1^{n-1}; [M]\rangle$ vanishes, where w_1 is the orientation class.*

This result is equivalent to the easy part of the solution to the Hopf invariant 1 problem, first obtained by J. Adem using the Adem relations for compositions of Steenrod squares.

In this note we prove Theorem 1.6 using the method of [1].

Suppose that n is even and $n+2$ is not a power of 2. Then we may suppose that $n+2 = (2p+1)2^{k+l} + 2^l$ where $k > 0$, $l > 0$ and $p \geq 0$. Let $h: M^{n-2} \to R^n$ be an immersion with normal bundle ν. Let the total Stiefel-Whitney class of ν be $1 + w_1 + w_2$. Then $w_1 \in H^1M$ is the orientation class of M and $w_2 \in H^2M$ is the mod 2 reduction of the Euler class of ν. We can construct a submanifold of M representing the Poincaré dual of the cohomology class w_2 by taking the transverse self-intersection of the zero section of the normal bundle ν. Similarly we can construct a submanifold representing the Poincaré dual of w_1 by using the orientation bundle of M, ξ. This process may be iterated to construct a submanifold $K(w_1^aw_2^b)$ of M (of dimension $n - a - 2b - 2$) representing the Poincaré dual of $w_1^aw_2^b$. The normal bundle of $K = K(w_1^aw_2^b)$ in R^n is the pull-back of $a\xi \oplus (b+1)\nu$ with total Stiefel-Whitney class

$$\overline{w}(K) = (1+w_1)^a(1+w_1+w_2)^{b+1},$$

where w_1 and w_2 are the restrictions to K of the Stiefel-Whitney classes of ν.

Our aim is to prove that $\langle w_2^{(2p+1)2^{k+l-1}+2^{l-1}-2}, [M]\rangle = 0$. To do this, we take the submanifold $K = K(w_2^{p2^{k+l}+2^{k+l-2}+2^{l-1}-2})$ so that, by Poincaré duality,

$$\langle w_2^{(2p+1)2^{k+l-1}+2^{l-1}-2}, [M]\rangle = \langle w_2^{2^{k+l-2}}, [K]\rangle. \tag{1}$$

The total normal Stiefel-Whitney class of K is

$$\begin{aligned} \overline{w}(K) &= (1+w_1+w_2)^{p2^{k+l}+2^{k+l-2}+2^{l-1}-1} \\ &= (1+w_1+w_2)^{p2^{k+l}}(1+w_1^{2^{k+l-2}}+w_2^{2^{k+l-2}})(1+w_1+w_2)^{2^{l-1}-1}. \end{aligned}$$

Now suppose that $k > 1$. For these values of k the top class

$$\overline{w}_{2^{k+l-1}}(K) = w_2^{2^{k+l-2}}.$$

By the Whitney immersion theorem this top normal class vanishes. Thus by equation (1) the required result follows in these cases.

The case $k = 1$ is a little more complicated. Here an induction on l gives the result. Consider first the case $l = 1$ so that $n = 4(2p+1) = 8p+4$. In this case we have an immersion $h\colon M^{8p+2} \to R^{8p+4}$. If we take $K = K(w_2^{4p})$ then $\overline{w}(K) = (1+w_1+w_2)^{4p+1}$ and the top class $\overline{w}_2(K) = w_2$. The result follows by the Whitney theorem and equation (1). This gives the base of the induction.

For $l = 2$, $n = 8(2p+1)+2 = 16p+10$ and so we have an immersion $h\colon M^{16p+8} \to R^{16p+10}$. If we take $K = K(w_2^{8p+2})$ then $\overline{w}(K) = (1+w_1+w_2)^{8p+3}$ and the top class $\overline{w}_4(K) = w_1^2w_2 + w_2^2$. By the Whitney theorem this vanishes and so $w_2^2 = w_1^2w_2$. We now take the Poincaré dual of w_1^2 in K which is $K_1 = K(w_1^2w_2^{8p+2})$. The total normal class is given by $\overline{w}(K_1) = (1+w_1+w_2)^{8p+3}(1+w_1)^2$ and the top class is w_2 which vanishes by the Whitney theorem. Hence, by Poincaré duality,

$$\langle w_2^{8p+4}, [M]\rangle = \langle w_2^2, [K]\rangle = \langle w_1^2w_2, [K]\rangle = \langle w_2, [K_1]\rangle = 0.$$

We illustrate the inductive step by considering $l = 3$. In this case $n = 16(2p+1)+6 = 32p+22$ and we have an immersion $h\colon M^{32p+20} \to R^{32p+22}$. This time we take $K = K(w_2^{16p+6})$ and calculation of the top normal class gives

$$0 = \overline{w}_8(K) = w_1^6w_2 + w_1^4w_2^2 + w_2^4.$$

We prove that the first two terms on the right hand side of this equation vanish by further Poincaré duality. For $w_1^6w_2$ we use the submanifold $K_1 = K(w_1^6w_2^{16p+6})$ which has top class $\overline{w}_2(K_1) = w_2$ so that

$$\langle w_1^6w_2, [K]\rangle = \langle w_2, [K_1]\rangle = 0.$$

For $w_1^4w_2^2$ we use $K_2 = K(w_1^4w_2^{16p+6})$ which has top class $w_1^2w_2 + w_2^2$. By the method of the case $l = 2$ it follows that $\langle w_2^2, [K_2]\rangle = 0$. Hence

$$\langle w_1^4w_2^2, [K]\rangle = \langle w_2^2, [K_2]\rangle = 0.$$

It follows that

$$\langle w_2^{16p+10}, [M]\rangle = \langle w_2^4, [K]\rangle = \langle w_1^6w_2, [K]\rangle + \langle w_1^4w_2^2, [K]\rangle = 0.$$

The argument for general values of l is a similar two stage process with the second stage leading to cases which can be treated using the methods used for lower values of l.

This proves the theorem.

4 Remarks

We have presented above a geometric proof of Theorem 1.6. However the proof that Browder's Theorem 1.3 is a consequence of this is not geometric but makes use of the homology of Thom complexes. A fully geometric proof of Browder's result along the present lines would include a geometric proof of Theorems 1.4 and 1.5. This is work in progress.

Given an n-dimensional vector bundle ξ over the space B with Thom class $t \in H^n(T\xi)$, then Thom's identity tells us that the Stiefel-Whitney class $w_i(\xi) \in H^i(B)$ corresponds to $\mathrm{Sq}^i t \in H^{i+n}T\xi$ (where Sq^i is the Steenrod squaring operation) under the Thom isomorphism. This shows that the above geometric proof Theorem 1.6 using normal characteristic numbers corresponds to calculations using Steenrod squares in $H^*MO(2)$. Another proof of Browder's theorem may be given using the action of the Steenrod algebra in $H_*QMO(1)$ and using the fact that a spherical class in this Pontrjagin ring is necessarily annihilated by all Steenrod homology operations.

To appear in the Proceedings of the Steklov Institute.

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July 1998

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