Open Problems in the Motivic Stable Homotopy Theory, I.

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1 Introduction

In this paper we discuss a number of conjectures concentrated around the notion of the slice filtration and the related notion of the rigid homotopy groups. Many of the ideas discussed below are in greater or lesser degree the result of conversations I had with Fabien Morel, Mike Hopkins and, more recently, Charles Rezk.

In topology there is a direct connection between the homotopy groups of a spectrum and the Postnikoff tower which describes how one can build this spectrum from the topological Eilenberg-MacLane spectra. On the level of cohomology theories this results in the existence of a spectral sequence which starts from cohomology with coefficients in the homotopy groups of a spectrum and converges to the cohomology theory represented by the spectrum. The connection exists because the Eilenberg-MacLane spectrum corresponding to an abelian group A has only one non trivial homotopy group which equals A.

The motivic Eilenberg-MacLane spectrum corresponding to an abelian group A has many non trivial motivic homotopy groups. As a result, for a motivic spectrum E, one can not recover a Postnikoff tower describing how to bulid E out of motivic Eilenberg-MacLane spectra by looking at the motivic homotopy groups of E. There is a spectral sequence which starts with cohomology with coefficients in the sheaves of motivic homotopy groups of E and converges to the theory represented by E but the cohomology with coefficients in the sheaves of homotopy groups are not ordinary cohomology theories in the sense of the motivic homotopy theory. In particular the trace maps p_* defined by a finite field extension E/F in these theories fail to satisfy the condition $p_*p^* = deg(E/F)Id$ which holds for ordinary motivic cohomology. The problem of constructing the "right" spectral sequence recived a lot of attention in the particular case of algebraic K-theory. Recently S. Bloch and S. Lichtenbaum gave a construction which works for fields in [1] and E.M. Friedlander and A. Suslin generalized it to varieties over a field in [3].

In the first section of this paper we define for any spectrum E a canonical Postnikoff tower (2.1) which we call the slice tower of E. The main conjecture of this paper (Conjecture 10) implies that for any spectrum E its slices $s_i(E)$ have unique and natural module structures over the motivic Eilenberg-MacLane spectrum $H_{\mathbb{Z}}$ and therefore represent ordinary cohomology theories. The main theme of all the conjectures presented here is that the slices $s_i(E)$ play the same role in the motivic homotopy theory as objects of the form $\Sigma^i H_{\pi_i(E)}$ play in topology. In Section 3 we formulate conjectures providing explicit description of the slices of the motivic Eilenberg-MacLane spectrum, motivic Thom spectrum, algebraic K-theory spectrum and the sphere spectrum. The most surprising here is the description of the slices of the sphere spectrum which was first suggested by Charles Rezk.

In section 4 we introduce a class of *slice-wise cellular* spectra whose slices are the motivic Eilenberg-MacLane spectra corresponding to complexes of abelian groups (as opposed to complexes of sheaves with transfers which may appear for a general E). Modulo the conjectures of Section 3 we show that it contains all the standard spectra mentioned above. For spectra of this class the slices are dertmined by actual abelain groups which we call the rigid homotopy groups. An important property of rigid homotopy groups is that for the standard spectra they are expected to be finitely generated abelain groups which do not depend on the base scheme (as long as it is normal and connected). In Section 5 we show (again modulo the conjectures) that rigid homotopy groups have a number of properties which are similar to the properties of the usual stable homotopy groups. In particular rigid homotopy groups are finitely generated if rigid homology are finitely generated and rationally rigid homotopy groups are isomorphic to rigid homology.

The slice tower defines a slice spectral sequence which, for a slice-wise cellular spectrum, starts with motivic cohomology with coefficients in the rigid homotopy groups of the spectrum and tries to converge to the motivic homotopy groups of the spectrum. We conjecture that for the algebraic K-theory the slice spectral sequence coincides with the spectral sequence constructed in [1] and [3] but the precise relation of the two approaches remains to be understood. In general it seems to be hard to figure out whether or not the slice spectral sequence converges. In Section 7 we formulate some conjectures about the convergence of the slice spectral sequence and show how they are related to the convergence problem for the motivic Adams spectral sequence. Unlike all the rest of conjectures of this paper for which a clear strategy exists at least for varieties over a field of characteristic zero the convergence conjectures are simply guesses.

Three other groups of conjectures in motivic homotopy theory, not included in to this paper, seem to be slowly crystallizing. One group describes the behavior of slice filtration with respect to the functors $f_*, f^*, f_!, f^!$ for morphisms of different types. In view of Conjecture 10 one should probably include Conjecture 17 into that group. The second group describes a theory of operadic description of T-loop spaces. The third one concerns explicit constructions of the slice filtration. It seems that something like the construction used by E. M. Friedlander and A. Suslin to get the spectral sequence for algebraic K-theory can be used to produce explicit models for the spectra f_nE for any E. Somehow the third and the second group should be related and should in particular provide a proof of Conjecture 16 but it is still all very murky.

This paper was written during my stay at the Institute for Advanced Study in Princeton. It is a very special place and I am very grateful to all people who make it to be what it is.

2 Slice filtration

Let S be a Noetherian scheme and $SH(S)$ the stable motivic homotopy category defined in [14, §5]. Recall that we denote by $\Sigma_T^\infty(X,x)$ the suspension spectrum of a pointed smooth scheme X over S. The T-desuspensions

$$\Sigma^{\infty-q}(X,x) = \Sigma_T^{-q}\Sigma_T^\infty(X,x)$$

of the suspension spectra for all smooth schemes over S and all $q \geq 0$ form a set of generators of $SH(S)$ i.e. the smallest triangulated subcategory in $SH(S)$ which is closed under direct sums and contains objects of the form $\Sigma^{\infty-q}(X,x)$ coincides with the whole $SH(S)$. Let $SH^{eff}(S)$ be the smallest triangulated subcategory in $SH(S)$ which is closed under direct sums and contains suspension spectra of spaces but not their T-desuspensions. The categories $\Sigma_T^q SH^{eff}(S)$ for $q \in \mathbb{Z}$ form a filtration of $SH(S)$ in the sense that we have a sequence of full embeddings

$$\dots \subset \Sigma_T^{q+1}SH^{eff}(S) \subset \Sigma_T^q SH^{eff}(S) \subset \Sigma_T^{q-1}SH^{eff}(S) \subset \dots$$

and the smallest triangulated subcategory which contains $\Sigma_T^q SH^{eff}(S)$ for all q and is closed under direct sums coincides with $SH(S)$. This filtration is called the *slice filtration*.

Remark 2.1 The intersection of $\Sigma_T^q SH^{eff}(S)$ for all q is non-zero. As an example suppose that $S = Spec(k)$ where k is a field and choose a prime number l not equal to the characteristic of k. Consider the sequence of morphisms between the motivic Eilenberg-MacLane spectra

$$\Sigma^{0,n(l-1)}H_{\mathbb{Z}/l} \to \Sigma^{0,(n+1)(l-1)}H_{\mathbb{Z}/l}$$

given by multiplication with the motivic cohomology class

$$\tau_l \in H^{0,l-1}(Spec(k), \mathbb{Z}/l)$$

Let $H_{et,\mathbb{Z}/l}$ be the homotopy colimit of this sequence. It is clear from the definition that there is a canonical isomorphism $\Sigma^{0,l-1}H_{et,\mathbb{Z}/l} = H_{et,\mathbb{Z}/l}$. Therefore, if this object belongs to $\Sigma_T^q SH^{eff}$ for at least one q then it belongs to the intersection of these subcategories for all q. In fact $H_{et,\mathbb{Z}/l}$ belongs to SH^{eff} since $H_{\mathbb{Z}/l}$ is an effective spectrum (see Conjecture 1 below) and SH^{eff} is closed under formation of homotopy colimits. This example is particularly important because the spectrum $H_{et,\mathbb{Z}/l}$, at least for varieties over a field, represents the etale cohomology with $\mathbb{Z}/l$ coefficients

$$H^{p,q}_{et,\mathbb{Z}/l}(X_+) = H^p_{et}(X, \mu_l^{\otimes q})$$

Since all the triangulated categories we consider have arbitrary direct sums and sets of compact generators a theorem of Neeman [8, Th. 4.1] implies that the inclusions $i_q : \Sigma_T^n SH^{eff} \to SH(S)$ have right adjoints r_q. Since i_q is a full embedding the adjunction $Id \to r_q i_q$ is an isomorphism. Define f_q as $i_q \circ r_q$. Note that we have canonical morphisms $f_{q+1} = f_{q+1} f_q \to f_q$. A standard argument implies the following result.

Theorem 2.2 *There exist unique up to a canonical isomorphism triangulated functors $s_q : SH(S) \to SH(S)$ and natural transformations*

$$\pi_q : Id \to s_q$$

$$\sigma_q : s_q \to \Sigma^{1,0} f_{q+1}$$

satisfying the following conditions:

1. *for any E the sequence*

$$f_{q+1}E \to f_q E \stackrel{\pi_q}{\to} s_q(E) \stackrel{\sigma_q}{\to} \Sigma^{1,0} f_{q+1}E \tag{2.1}$$

is a distinguished triangle

2. *for any* E *the object* $s_q(E)$ *belongs to* $\Sigma_T^q SH^{eff}$

3. *for any* E *the object* $s_q(E)$ *is right orthogonal to* $\Sigma_T^{q+1} SH^{eff}$ *i.e. for any object* X *in* $\Sigma_T^{q+1} SH^{eff}$ *we have* $Hom(X, s_q(E)) = 0$.

For any E in $SH(S)$ the sequence of distinguished triangles (2.1) is called the slice tower of E. The direct sum s_* of functors s_q for $q \in \mathbb{Z}$ is a triangulated functor from SH to SH which commutes with direct sums. This functor does not commute with smash products but for any E and F there is a canonical morphism $s_*(E) \wedge s_*(F) \to s_*(E \wedge F)$. In the following section we will see that in many ways the functor s_* reminds of the functor H_{π_*} from the topological stable homotopy category to itself which takes a spectrum E to $\oplus_{i \in \mathbb{Z}} \Sigma^i H_{\pi_i(E)}$. The main difference bewteen s_* and H_{π_*} is that the former is a triangulated functor while the later is not.

3 Main conjectures

This section contains the main conjectures predicting the structure of the slices of four standard spectra. The first three are the spectra described in [14, §6]. The Eilenberg-MacLane spectra representing motivic cohomology are considered in the first section, the algebraic Thom spectrum representing algebraic cobordisms in the second and the spectrum representing algebraic K-theory in the third. In the last section we consider the sphere spectrum representing the motivic stable (co-)homotopy groups. In the standard topological approach one associates to a ring spectrum E a graded Hopf algebroid whose ring of objects is the ring of homotopy groups of E and the ring of morphisms is a ring of homotopy groups of $E \wedge E$. This algebroid can then be used to compute the Adams spectral sequence build on E and other interesting things. Unfortunately this approach only works for nice enough E which is usually reflected by some "flatness" condition. Already in the case of the ordinary Eilenberg-MacLane spectrum corresponding to integral cohomology it does not work very well. Instead we are going to consider directly the cosimplicial spectrum $N(E)$ with terms of the form $N^i(E) = E^{\wedge(i+1)}$, cofaces given by unit morphisms and codegeneracies given by the multiplication morphisms. Since our goal here is to present some conjectures describing the structure of the motivic stable homotopy category we do not discuss the definition of the homotopy category of cosimplicial spectra of which $N(E)$ is an object. In most examples we deal with below it will be enough to think of $N(E)$ as of a cosimplicial object in SH.

3.1 Motivic Eilenberg-MacLane spectra Our first group of conjectures describes the slices of the motivic Eilenberg-MacLane spectrum $H_{\mathbb{Z}}$ and of the associated standard cosimplicial spectrum $N(H_{\mathbb{Z}})$.

Conjecture 1

$$s_q(H_{\mathbb{Z}}) = \begin{cases} H_{\mathbb{Z}} & \textit{for } q = 0 \\ 0 & \textit{for } q \neq 0 \end{cases} \tag{3.1}$$

This conjecture is equivalent to the combination of two statements. One is that $H_{\mathbb{Z}}$ is an effective spectrum. This seems to be easy enough to prove by showing that the motivic Eilenberg-MacLane spaces $K(\mathbb{Z}(n), 2n)$ can be build out of n-fold T-suspensions. Another one is that $H^{p,q}(X, \mathbb{Z}) = 0$ for $q < 0$. This is currently known for regular schemes S over a field through the comparison of motivic cohomology with the higher Chow groups.

Our next goal is to describe $s_*(N(H_{\mathbb{Z}}))$. Unfortunately we do not know how to formulate the expected answer in one coherent conjecture. Instead we formulate a rather imprecise conjecture about the structure of $s_*(H_{\mathbb{Z}}^{\wedge n})$ and conjectures giving explicit descriptions for $s_*(N(H_k))$ where $k = \mathbb{Q}$ or $k = \mathbb{Z}/l$.

Conjecture 2 *The objects $s_q(H_{\mathbb{Z}} \wedge H_{\mathbb{Z}})$ are isomorphic to direct sums of the form $\oplus_{p \geq 0} \Sigma^{p,q} H_{X_{p,q}}$ where for q or p non zero $X_{p,q}$ is a finite abelian group of the form $\oplus \mathbb{Z}/l_j$ where l_j are prime numbers and $X_{0,0} = \mathbb{Z}$.*

This conjecture is known for $S = Spec(k)$ where k is a field of characteristic zero (see Section 8). Conjecture 2 clearly implies a similar result for all smash powers of $H_{\mathbb{Z}}$ and H_k for $k = \mathbb{Q}$ or $k = \mathbb{Z}/l$ and in combination with Conjecture 1 it implies that all the terms of the cosimplicial spectra $s_q(N(H_k))$ are direct sums of finitely many copies of $\Sigma^{p,q} H_k$.

To describe $s_*(N(H_k))$ explicitly define a Hopf algebra $A^{rig}_{*,*}(k)$ which we call the rigid Steenrod algebra (over k) as follows. For $k = \mathbb{Q}$ we set $A^{rig}_{*,*}(k) = \mathbb{Q}$. Denote by $S_k[x_1, \dots, x_n]$ and $\Lambda_k[x_1, \dots, x_n]$ the symmetric and exterior algebras in variables $x_1, \dots, x_n$ over a field k. For $k = \mathbb{Z}/l$ we set

$$A^{rig}_{*,*}(k) = S_k[\xi_1, \dots, \xi_n, \dots] \otimes \Lambda_k[\tau_0, \dots, \tau_n, \dots]$$

where the bidegree of ξ_i is $(2(l^i - 1), l^i - 1)$, the bidegree of τ_i is $(2l^i - 1, l^i - 1)$ and the comultiplication is given by

$$\Delta(\xi_n) = \xi_n \otimes 1 + \sum_{i=1}^{n-1} \xi_{n-i}^{l^i} \otimes \xi_i + 1 \otimes \xi_n$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{i=0}^{n-1} \xi_{n-i}^{l^i} \otimes \tau_i + 1 \otimes \tau_n$$

For $l \neq 2$ the rigid Steenrod algebra is the usual (dual) Steenrod algebra of topology (see [4, Th.3]). Any Hopf algebra A over a field k defines a cosimplicial algebra $N(A)$ with terms $N(A)^i = A^{\otimes i}$, coface maps given by the unit and comultiplication and codegeneracy maps given by the counit. For a bigraded abelian group $A_{*,*}$ denote by $H_{A_{*,*}}$ the spectrum $\oplus_{p,q}\Sigma^{p,q}H_{A_{p,q}}$. A graded Hopf algebra $A_{*,*}$ defines a cosimplicial spectrum $H_{N(A_{*,*})}$ whose terms are spectra of the form $H_{A_{*,*}^{\otimes i}}$.

Conjecture 3 *For $k = \mathbb{Q}$ or $k = \mathbb{Z}/l$ there is an isomorphism of cosimplicial spectra*

$$s_*(N(H_k)) = H_{N(A^{rig}_{*,*}(k))}$$

such that for any $q \in \mathbb{Z}$

$$s_q(N(H_k)) = \Sigma^{0,q} H_{N(A^{rig}_{*,q}(k))}$$

Using the standard elements in the motivic homology and cohomology of the lense spaces

$$K(\mathbb{Z}/l(1), 1) = (\mathbb{A}^\infty - \{0\})/\mu_l$$

one can assign elements in $\pi_{p,q}(H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l})$ to the generators ξ_n and τ_m and then use the multiplicative structure on $H_{\mathbb{Z}/l}$ to define a homomorphism from $A^{rig}_{*,*}(\mathbb{Z}/l)$ considered as a bigraded vector space generated by the monomials in ξ_n and τ_m to $\pi_{*,*}(H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l})$. The $H_{\mathbb{Z}/l}$-module structure on $H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l}$ allows one to extend any element of the homotopy group $\pi_{p,q}(H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l})$ to a morphism from $\Sigma^{p,q}H_{\mathbb{Z}/l}$ to $H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l}$ and thus gives a morphism

$$H_{A^{rig}_{*,*}(\mathbb{Z}/l)} \to H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l} \tag{3.2}$$

Conjecture 4 *The morphism (3.2) is an isomorphism.*

We know how to prove this conjecture for $S = Spec(k)$ where k is a field of characteristic zero (see [11] and further papers of these series). It is one of the elements of the computation of the algebra of cohomological operations in motivic cohomology needed for the proof of the Milnor conjecture given in [13]. Doing the same thing with the higher smash powers of H_k one can define morphisms

$$H_{(A^{rig}_{*,*}(\mathbb{Z}/l))^{\otimes i}} \to H_{\mathbb{Z}/l}^{\wedge(i+1)} \tag{3.3}$$

and Conjecture 4 implies that they are also isomorphisms. The morphisms (3.3) do not commute with the coface and codegeneracy morphisms and thus do not give a morphism $H_{N(A^{rig}_{*,*}(\mathbb{Z}/l))} \to N(H_{\mathbb{Z}/l})$. For example the two morphisms $H_{\mathbb{Z}/l} \to H_{\mathbb{Z}/l} \wedge H_{\mathbb{Z}/l}$ defined by the unit $\mathbf{1} \to H_{\mathbb{Z}/l}$ which are the zero dimensional coface morphisms in $N(H_{\mathbb{Z}/l})$ do not coincide while the coface morphisms $H_{\mathbb{Z}/l} \to H_{A^{rig}_{*,*}(\mathbb{Z}/l)}$ coincide by construction. When we pass to slices this problem should disappear and explicit computations confirm it. However we do not know an explicit construction of a morphism in either direction required by Conjecture 3 for $k = \mathbb{Z}/l$.

3.2 Motivic Thom spectrum Let MGL be the motivic Thom spectrum representing the algebraic cobordism. An analog of the standard argument from topology should be sufficient to show that

(3.4) $$MGL^{*,*}((\mathbb{P}^\infty)^n) = MGL^{*,*}(\mathbf{1})[[t_1, \dots, t_n]]$$

Together with the obvious properties of the morphism $\mathbb{P}^\infty \times \mathbb{P}^\infty \to \mathbb{P}^\infty$, this formula implies that the image of t_1 under the induced map on algebraic cobordisms is a formal group law. It gives a homomorphism

(3.5) $$MU_* \to MGL_{*,*}(\mathbf{1})$$

from the Lazard ring MU_* to $MGL_{*,*}(\mathbf{1})$ which sends MU_{2q} to $MGL_{2q,q}(\mathbf{1})$.

Conjecture 5 *There exists an isomorphism*

(3.6) $$s_q(MGL) = \Sigma^q_T H_{MU_{2q}}$$

compatible with the homomorphism (3.5)*.*

The compatibility condition in this conjecture means the following. An isomorphism of the form (3.6) defines in particular a homomorphism of abelian groups $MU_{2q} \to \pi_{2q,q}(s_q(MGL))$. On the other hand the definition of s_q's shows that we have a canonical homomorphisms $\pi_{p,q}(E) \to \pi_{p,q}(s_q(E))$ and, therefore, (3.5) also defines a homomorphism $MU_{2q} \to \pi_{2q,q}(s_q(MGL))$. The condition requires the two homomorphisms to be the same. Conjecture 10 discussed in the following section implies that that there exists a unique morphism $H_{MU_{2q}} \to s_q(MGL)$ satisfying the compatibility condition.

Consider the graded cosimplicial ring $\pi_*(N(MU))$ where MU is the complex cobordisms spectrum and π_* refer to topological homotopy groups. Alternatively it can be defined as the ring of functions on the simplicial scheme which represents the functor sending an affine scheme $Spec(R)$ to the nerve of the groupoid whose objects are the formal group laws of dimension one over R and morphisms are changes of the generator.

Conjecture 6 *There is an isomorphism*

$$s_q(N(MGL)) = \Sigma_T^q H_{\pi_{2q}(N(MU))}$$

which coincides with the isomorphism (3.6) on the zero term.

It seems to be possible to repeat the argument used to compute oriented cohomology of classifying spaces for algebraic cobordism which leads to a canonical isomorphism

$$MGL \wedge MGL = MGL[b_1, \ldots, b_n, \ldots]$$

where

$$E[x_1, \ldots, x_n] = \oplus_{i_1,\ldots,i_n \geq 0} E$$

In particular it seems that Conjecture 6 is relatively simple modulo Conjecture 5. Thus the situation here is different from the case of Eilenberg-MacLane spectra where Conjecture 1 is relatively easy while Conjecture 3 is hard.

3.3 Algebraic K-theory spectrum In this paper we denote the algebraic K-theory spectrum by KGL to distinguish it from the space BGL. It is (2,1)-periodic that is we have a canonical isomorphism $T \wedge KGL = KGL$. This immediately implies that $s_q(KGL) = \Sigma_T^q s_0(KGL)$ for all q.

Conjecture 7

$$s_0(KGL) = H_{\mathbb{Z}}$$

The slices of the standard cosimplicial spectrum associated with the algebraic K-theory are described by the following analog of Conjecture 6.

Conjecture 8

$$s_q(N(KGL)) = \Sigma_T^q H_{\pi_{2q}(N(KU))}$$

3.4 Sphere spectrum

Conjectures 1 and 6 lead to a complete computation of the slices of the sphere spectrum. Observe first that the cone of the unit morphism $\mathbf{1} \to MGL$ belongs to $\Sigma_T^1 SH^{eff}$. It can be seen from the fact that this cone is built out of the n-fold T-desuspensions of the suspension spectra of the Thom spaces $MGL(n)$ and each

$MGL(n)$ can be built out of the n-fold T-suspensions of open subsets of $BGL(n)$. This observations implies that for $n > q$ the morphisms

$$s_q(cosk_{n+1}N(MGL)) \to s_q(\mathbf{1})$$

are isomorphisms and in particular that $s_q(\mathbf{1}) = Tot(s_q(N(MGL)))$. The right hand side can be computed from Conjecture 6. The n-th term of $s_q(N(MGL))$ is just

$\Sigma_T^q H_{\pi_{2q}(N^n(MU))}$. Correspondingly the total object $Tot(s_q(N(MGL)))$ is nothing but the Elenberg-MacLane spectrum of the form $\Sigma_T^q H_{\pi_{2q}N(MU)}$ where $H_{\pi_{2q}N(MU)}$ is the Eilenberg-MacLane spectrum corresponding to the complex of abelian groups associated with the cosimlicial abelian group $\pi_{2q}(N(MU))$. The cohomology groups of $\pi_*(N(MU))$ are denoted in topology by

$$H^n(\pi_{2q}(N(MU))) = Ext^n_{MU_*(MU)}(MU_*, MU_*)_{2q}.$$

They form the E_2-term of the Adams-Novikov spectral sequence (see [10]). Summarizing we have.

Conjecture 9

$$s_q(\mathbf{1}) = \Sigma_T^q H_{\pi_{2q}(N(MU))}$$

The particular case of this conjecture for $q = 0$ looks as follows.

Conjecture 10

$$s_0(\mathbf{1}) = H_{\mathbb{Z}}$$

Consider the canonical morphism $\mathbf{1} \to H_{\mathbb{Z}}$ and let $\tilde{H}_{\mathbb{Z}}$ be its fiber (the desuspension of its cone). Conjecture 10 is equivalent to the combination of two statements. One is that $\tilde{H}_{\mathbb{Z}}$ belongs to $\Sigma_T^1 SH^{eff}$ and another one is that $H_{\mathbb{Z}}$ is right orthogonal to $\Sigma_T^1 SH^{eff}$. Let us call the first one the divisibility conjecture and the second one the T-rigidity conjecture. T-rigidity conjecture is also a part of Conjecture 1 and was discussed there. The divisibility part so far is unknown even over a field of characteristic zero. This conjecture seems to be very important and more fundamental that the rest of the conjectures of this paper. In particular it provides the only way we know to characterize the Eilenberg-MacLane spectra without giving an explicit definition. One of the implications of Conjecture 10 is that for any spectrum its slices have unique and natural module structures over the Eilenberg-MacLane spectrum which explains that all our conjectures predict that different objects of the form $s_*(-)$ are generalized Eilenberg-MacLane spectra.

4 Slice-wise cellular spectra

Many important spectra including the algebraic cobordism spectrum and the algebraic K-theory spectrum are T-cellular that is they belong to the smallest triangulated subcategory closed under direct sums which contains the spheres T^i for $i \in \mathbb{Z}$. Unfortunately we do not know whether or not the Eilenberg-MacLane spectrum is T-cellular. In this section we will describe another class of spectra which contains $H_\mathbb{Z}$ and such that its objects have many of the nice properties of T-cellular spectra. Using Conjectures 10 and Conjecture 2 we will show that it contains T-cellular spectra.

Definition 4.1 *An object E of $SH(S)$ is called slice-wise cellular if for any $q \in \mathbb{Z}$ the slice $s_q(E)$ of E belongs to the smallest triangulated subcategory of $SH(S)$ closed under direct sums which contains the Eilenberg-MacLane spectrum $\Sigma_T^q H_\mathbb{Z}$.*

Our definition immediately implies that the subcategory of slice-wise cellular objects is a triangulated subcategory closed under direct sums and direct summands. Conjecture 1 implies that the Eilenberg-MacLane spectrum $H_\mathbb{Z}$ is slice-wise cellular.

Lemma 4.2 *The subcategory of slice-wise cellular spectra is closed under smash product.*

Proof. The proof is modulo Conjecture 2. Let E and F be slice-wise cellular spectra. We need to show that $s_q(E \wedge F)$ is in the smallest triangulated subcategory of $SH(S)$ closed under direct sums which contains the Eilenberg-MacLane spectrum $\Sigma_T^q H_\mathbb{Z}$. Replacing E or F by an appropriate T-suspention we may assume that $q = 0$. For any F we have $F = hocolim f_{-n}F$ and both the smash product and the functor s_0 commute with homotopy colimits. Thus $s_0(E \wedge F) = hocolim s_0(f_{-n}E \wedge f_{-n}F)$ and it is enough to prove that $s_0(f_{-n}E \wedge f_{-n}F)$ is of the required form for any n. For any $m > n$ the smash product $f_m E \wedge f_{-n}F$ is in $\Sigma_T^1 SH^{eff}$. Thus the slice tower of E gives a finite sequence of distinguished triangles of the form

$$0 \to s_0(f_n E \wedge f_{-n}F) \cong s_0(s_n E \wedge f_{-n}F)$$

$$s_0(f_n E \wedge f_{-n}F) \to s_0(f_{n-1}E \wedge f_{-n}F) \to s_0(s_{n-1}E \wedge f_{-n}F)$$

$$\dots$$

$$s_0(f_{1-n}E \wedge f_{-n}F) \to s_0(f_{-n}E \wedge f_{-n}F) \to s_0(s_{-n}E \wedge f_{-n}F)$$

It is enough to check now that $s_0(s_{-m}E \wedge f_{-n}F)$ are in the smallest triangulated subcategory closed under direct sums which contains $H_{\mathbb{Z}}$. Our assumption on E implies that it is enough to check that $s_0(\Sigma^{p,q}H_{\mathbb{Z}} \wedge f_{-n}F)$ is of the required form. Using the slice tower of F we reduce the problem to the case of $s_0(\Sigma^{p,q}H_{\mathbb{Z}} \wedge H_{\mathbb{Z}})$ where our result follows from Conjectures 2 and 1. □

Definition 4.3 *An object is called T-connective or just connective if it belongs to* $\Sigma_T^q SH^{eff}$ *for some* q.

Proposition 4.4 *A connective spectrum* E *is slice-wise cellular if and only if* $E \wedge H_{\mathbb{Z}}$ *is slice-wise cellular.*

Proof. The proof requires Conjectures 10, 1, 2. We will only prove the "only if" part. We may clearly assume that E belongs to SH^{eff}. Consider the Adams tower for the Eilenberg-MacLane spectrum

$$
\begin{array}{ccccc}
 & & \dots & & \\
\bar{H}_{\mathbb{Z}}^{\wedge n} & \to & \bar{H}_{\mathbb{Z}}^{\wedge n-1} & \to & \bar{H}_{\mathbb{Z}}^{\wedge n-1} \wedge H_{\mathbb{Z}} \\
 & & \dots & & \\
\bar{H}_{\mathbb{Z}}^{\wedge 2} & \to & \bar{H}_{\mathbb{Z}} & \to & \bar{H}_{\mathbb{Z}} \wedge H_{\mathbb{Z}} \\
\bar{H}_{\mathbb{Z}} & \to & \mathbf{1} & \to & H_{\mathbb{Z}}
\end{array} \tag{4.1}
$$

By the divisibility part of Conjecture 10, $\bar{H}_{\mathbb{Z}}^{\wedge n}$ belongs to $\Sigma_T^n SH^{eff}$ and since E is assumed to be effective so does $E \wedge \bar{H}_{\mathbb{Z}}^{\wedge n}$. Therefore applying the functor s_q to the tower (4.1) smashed with E we get a finite sequence of distinguished triangles of the form

$$
\begin{array}{ccccc}
0 & \to & s_q(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q}) & \cong & s_q(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q} \wedge H_{\mathbb{Z}}) \\
 & & \dots & & \\
s_q(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge 2}) & \to & s_q(E \wedge \bar{H}_{\mathbb{Z}}) & \to & s_q(E \wedge \bar{H}_{\mathbb{Z}} \wedge H_{\mathbb{Z}}) \\
s_q(E \wedge \bar{H}_{\mathbb{Z}}) & \to & s_q(E) & \to & s_q(E \wedge H_{\mathbb{Z}})
\end{array} \tag{4.2}
$$

Therefore it is enough to show that objects $s_q(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge n} \wedge H_{\mathbb{Z}})$ are of the required form. The multiplication morphism allows one to make $\bar{H}_{\mathbb{Z}}^{\wedge n} \wedge H_{\mathbb{Z}}$ into a direct summand of $\Sigma^{-n,0}H_{\mathbb{Z}}^{\wedge(n+1)}$. Our result follows now from Conjectures 2 and 1. □

Corollary 4.5 *The category of slice-wise cellular spectra contains the sphere spectrum* $\mathbf{1}$ *and therefore all T-cellular spectra.*

The algebraic cobordism spectrum MGL is a slice-wise cellular spectrum. There are two ways to see it. One is to use the fact that

$$MGL \wedge H_{\mathbb{Z}} = H_{\mathbb{Z}[b_1,\dots,b_n,\dots]}$$

where b_i are of bidegree $(2i, i)$ and Proposition 4.4. Another one is to note that the Thom spaces $MGL(n)$ out of which the motivic Thom spectrum is built can be built in turn out of spheres and then use Corollary 4.5. Similarly the spectrum KGL is a slice-wise cellular spectrum. We can not use Proposition 4.4 directly to prove it since it is not connective but we can use the fact that it is built out of suspension spectra of the space BGL and these spaces are T-cellular.

5 Reformulations in terms of rigid homotopy groups

Definition 5.1 *The rigid homotopy groups of an object E in $SH(S)$ are given by*

$$\pi^{rig}_{p,q}(E) = \pi_{p,q}(s_q(E))$$

More generally we define the presheaves of rigid homotopy groups setting

$$\underline{\pi}^{rig}_{p,q}(E) : U/S \mapsto Hom(\Sigma^{p,q}\Sigma^{\infty}_T U_+, s_q(E)).$$

Note that by definition the rigid homotopy groups are the values of the presheaves of rigid homotopy groups on the base scheme S. Conjecture 10 implies that a slice of any spectrum has a unique structure of a module over the Eilenberg-MacLane spectrum. This leads to the following conjecture (for the definition of a presheaf with transfers over a general base scheme S see [11]). indextermspresheaves!with transfers

Conjecture 11 *The presheaves of rigid homotopy groups have canonical structures of presheaves with transfers.*

We have canonical homomorphisms from the motivic stable homotopy groups $\pi_{p,q}(E) = Hom(\Sigma^{p,q}1, E)$ to the rigid homotopy groups. For any E the group $\pi_{*,*}(E)$ is a module over the ring of motivic homotopy groups of spheres $\pi_{*,*}(1)$ and one can easily see that the submodule $\pi_{*,<0}(1)\pi_{*,*}(E)$ goes to zero in the rigid homotopy groups. In general it seems that nothing else can be said about this homomorphism.

Conjectures of the previous sections allow us to compute the rigid homotopy groups of the standard spectra explicitly provided we know homotopy groups of the form $\pi_{p,0}H_{\mathbb{Z}}$ i.e. the motivic cohomology of weight zero.

Conjecture 12 *For a normal connected scheme S one has*

$$\pi_{p,0}H_{\mathbb{Z}} = \begin{cases} \mathbb{Z} & \text{for } p = 0 \\ 0 & \text{for } p \neq 0 \end{cases} \tag{5.1}$$

For a regular S over a field this conjecture is known. It follows from the comparison of motivic cohomology with the higher Chow groups. For any S of characteristic zero it can be proved using resolution of singularities and the blow-up long exact sequence in generalized cohomology established in [12]. In this case one can prove more namely that for any Noetherian S of characteristic zero which is of finite dimension and any $p \in \mathbb{Z}$ one has

$$\pi_{p,0}(H_{\mathbb{Z}}) = H^{-p}_{cdh}(S, \mathbb{Z}).$$

The same should be true for all Noetherian S of finite dimension.

Conjecture 10 implies that the map

$$Hom(H_{\mathbb{Z}}, \Sigma^{p,0}H_{\mathbb{Z}}) \to \pi_{p,0}(H_{\mathbb{Z}})$$

defined by the unit map $\mathbf{1} \to H_{\mathbb{Z}}$ is an isomorphism for any $n \in \mathbb{Z}$. Together with Conjecture 12 it implies that for a normal connected scheme S the functor

$$D(Ab) \to SH(S) \tag{5.2}$$

which sends a complex of abelian groups C to the Eilenberg-MacLane spectrum H_C is a full embedding and its image coincides with the smallest triangulated subcategory of $SH(S)$ closed under direct sums which contains the Eilenberg-MacLane spectrum $H_{\mathbb{Z}}$. Thus in the case of a normal connected scheme S a spectrum E is slice-wise cellular if and only if there exist complexes of abelian groups $\Pi_q(E)$ such that $s_qE = \Sigma^{0,q}H_{\Pi_q(E)}$.

Combining Conjecture 12 with Conjectures 1, 5, 7 and 9 we get that for a normal connected S the rigid homotopy groups of the standard spectra are given by the following formulae:

$$\pi^{rig}_{p,q}(H_{\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{for } p = q = 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

$$\pi^{rig}_{p,q}(MGL) = \begin{cases} MU_{2q} & \text{for } p = 2q \\ 0 & \text{for } p \neq 2q \end{cases} \tag{5.4}$$

$$\pi^{rig}_{p,q}(KGL) = \begin{cases} \mathbb{Z} & \text{for } p = 2q \\ 0 & \text{for } p \neq 2q \end{cases} \tag{5.5}$$

$$\pi^{rig}_{p,q}(1) = Ext^{2q-p}_{MU_*(MU)}(MU_*, MU_*)_{2q} \tag{5.6}$$

In particular these groups do not depend on S which is one of the reasons we call them "rigid". Note that in the first three cases the rigid homotopy groups of a motivic spectrum coincide with the homotopy groups of its topological counterpart but in the last case they do not.

6 Rigid homology and rigid Adams spectral sequence

Define the rigid homology of a spectrum with coefficients in a commutative ring R setting

$$H^{rig}_{p,q}(E, R) = \pi^{rig}_{p,q}(E \wedge H_R).$$

The unit map $\mathbf{1} \to H_R$ defines the rigid analog of the Hurewicz map

$$\pi^{rig}_{p,q}(E) \to H^{rig}_{p,q}(E, R)$$

These homomorphisms have a number of useful properties analogous to the properties of the usual topological Hurewicz map which are missing for the motivic Hurewicz homomorphisms $\pi_{p,q} \to H_{p,q}$.

Lemma 6.1 *Let E be a connective spectrum such that the rigid homology groups $H^{rig}_{p,q}(E, \mathbb{Z})$ are finitely generated. Then the rigid homotopy groups $\pi^{rig}_{p,q}(E)$ are finitely generated.*

Proof. Conjecture 10 implies that smashing the Adams tower (4.1) with E and applying s_q one gets a finite Postnikoff tower for $s_q(E)$ whose quotients are direct summands of objects of the form $s_q(E \wedge H_{\mathbb{Z}}^{\wedge n})$. Conjecture 2 implies that they are finite direct sums of objects of the form $s_q(E \wedge H_A)$ for finitely generated abelian groups A. Together with our condition on E it implies that $\pi^{rig}_{p,q}(E) = \pi_{p,q}(s_q(E))$ are finitely generated. ∎

Lemma 6.2 *For any spectrum E the homomorphism*

$$\pi^{rig}_{p,q}(E) \otimes \mathbb{Q} \to H^{rig}_{p,q}(E, \mathbb{Q})$$

is an isomorphism.

Proof. Note first that both sides as functors in E take filtered homotopy colimits to filtered colimits. Thus since any spectrum E is a filtered homotopy colimit of its connective parts $f_{-n}E$ we may assume that E is connective and thus that it is in SH^{eff}. The rational version of Conjecture 10 implies that smashing the Adams tower with the rational Moore spectrum $\mathbf{1}_{\mathbb{Q}}$ and with E and applying functor s_q we get a finite Postnikoff tower for $s_q(E \wedge \mathbf{1}_{\mathbb{Q}})$ and the rational version of Conjecture 3 implies that this tower degenerates providing an isomorphism $s_q(E \wedge \mathbf{1}_{\mathbb{Q}}) \to s_q(E \wedge H_{\mathbb{Q}})$. Finally since $\mathbf{1}_{\mathbb{Q}}$ is a filtered homotopy colimit of sphere spectra we have $s_q(E \wedge \mathbf{1}_{\mathbb{Q}}) = s_q(E) \wedge \mathbf{1}_{\mathbb{Q}}$ and the homotopy groups of this spectrum are isomorphic to $\pi^{rig}_{p,q}(E) \otimes \mathbb{Q}$. □

Remark 6.3 The statement of Lemma 6.2 does not hold for motivic homotopy groups. In particular while $s_q(\mathbf{1}_{\mathbb{Q}})$ should be isomorphic to $s_q(H_{\mathbb{Q}})$ that is should be zero for $q \neq 0$ and $H_{\mathbb{Q}}$ for $q = 0$ the morphism $\mathbf{1}_{\mathbb{Q}} \to H_{\mathbb{Q}}$ is not an isomorphism at least in some cases.

Multiplication on H_R defines for any E and F a homomorphism

$$H^{rig}_{*,*}(E, R) \otimes_R H^{rig}_{*,*}(F, R) \to H^{rig}_{*,*}(E \wedge F, R)$$

Charles Rezk pointed out that using Conjectures 2, 1 and 10 one can prove the following "Künneth Theorem".

Theorem 6.4 *Let E be a slice-wise cellular spectrum, F any spectrum and k a field. Then the multiplication homomorphism*

$$H^{rig}_{*,*}(E, k) \otimes_k H^{rig}_{*,*}(F, k) \to H^{rig}_{*,*}(E \wedge F, k)$$

is an isomorphism.

For any commutative ring R and any spectrum E we can apply the functor s_q to the Adams tower (4.1) based on H_R smashed with E. This gives a sequence of distinguished triangles of the form

$$
(6.1)\qquad
\begin{array}{ccccc}
 & & \dots & & \\
s_q(E\wedge \bar{H}_R^{\wedge(n+1)}) & \to & s_q(E\wedge \bar{H}_R^{\wedge n}) & \to & s_q(E\wedge \bar{H}_R^{\wedge n}\wedge H_R) \\
 & & \dots & & \\
s_q(E\wedge \bar{H}_R^{\wedge 2}) & \to & s_q(E\wedge \bar{H}_R) & \to & s_q(E\wedge \bar{H}_R\wedge H_R) \\
s_q(E\wedge \bar{H}_R) & \to & s_q(E) & \to & s_q(E\wedge H_R)
\end{array}
$$

which defines a spectral sequence whose E_1-term consists of the rigid homology groups

$$E^1_{p,n} = H^{rig}_{p,q}(E\wedge \bar{H}_R^{\wedge n}) = \pi_{p,q}(s_q(E\wedge \bar{H}_R^{\wedge n}\wedge H_R))$$

and the r-th differential is of the form

$$H^{rig}_{p,q}(E\wedge \bar{H}_R^{\wedge n}) \to H^{rig}_{p-1,q}(E\wedge \bar{H}_R^{\wedge(n+r)})$$

This spectral sequence is called the rigid Adams spectral sequence with coefficients in R. The complexes

$$E\wedge \bar{H}_R \to \Sigma^1_s(E\wedge \bar{H}_R\wedge H_R) \to \Sigma^2_s(E\wedge \bar{H}_R^{\wedge 2}\wedge H_R) \to \dots$$

which define the E_1-term of the Adams spectral sequence are isomorphic to the normalizied chain complexes of the cosimplicial objects $E\wedge N(H_R)$ (this is actually true for the Adams spectral sequence based on any commutative ring spectrum). Thus the E_1-term of the rigid Adams spectral sequence with coefficients in R can be identified with the collection of complexes of abelian groups of the form $\pi^{rig}_{p,q}(E\wedge N(H_R))$. If k is a field we have by Theorem 6.4

$$\pi^{rig}_{p,q}(E\wedge H_k^{\wedge(n+1)}) = H^{rig}_{p,q}(E\wedge H_k^{\wedge n}) = (H^{rig}_{*,*}(E)\otimes H^{rig}_{*,*}(H_k)^{\otimes n})_{p,q}$$

Together with Conjecture 3 this implies that the rigid homology with coefficients in $\mathbb{Z}/l$ are comodules over the rigid Steenrod algebra $A^{rig}_{*,*}(k)$ and that the E_2-term of the rigid Adams spectral sequence with coefficients in $\mathbb{Z}/l$ can be identified with Ext-groups from k to $H^{rig}_{*,*}(E,k)$ in the category of comodules over $A^{rig}_{*,*}$.

According to Conjecture 10 for any connective spectrum E and any q there exists n such that $s_q(E\wedge \bar{H}^{\wedge(n+1)}) = 0$. Thus if $R=\mathbb{Z}$ and E is connective the triangles (6.1) give a finite Postnikoff tower for $s_q(E)$ and we conclude that the rigid Adams spectral sequence with intergral coefficients converges for any connective spectrum E. Unfortunately this spectral sequence is not very convenient in practice since we do not know how to describe its E_2-term. To describe the convergence properties of the rigid Adams spectral sequence with coefficients in $\mathbb{Z}/l$ denote the associated filtration on rigid homotopy groups by

$$a^i_l\pi^{rig}_{p,q}(E) = Im(\pi^{rig}_{p,q}(E\wedge \bar{H}^{\wedge i}_{\mathbb{Z}/l}) \to \pi^{rig}_{p,q}(E))$$

Proposition 6.5 *Let E be an effective spectrum. Then one has*

$$a_l^{q+1}\pi_{p,q}^{rig}(E)\subset l\pi_{p,q}^{rig}(E) \tag{6.2}$$

Proof. Denote by M_l the Moore spectrum $cone(\mathbf{1}\stackrel{l}{\to}\mathbf{1})$. To show that the inclusion (6.2) holds we have to show that the composition

$$\pi_{p,q}^{rig}(E\wedge\bar{H}_{\mathbb{Z}/l}^{\wedge(q+1)})\to\pi_{p,q}^{rig}(E)\to\pi_{p,q}^{rig}(E\wedge M_l)$$

is zero. This composition factors through the morphism

$$\pi_{p,q}^{rig}(E\wedge M_l\wedge\bar{H}_{\mathbb{Z}/l}^{\wedge(q+1)})\to\pi_{p,q}^{rig}(E\wedge M_l)$$

which by Lemma 6.6 factors through the morphism

$$\pi_{p,q}^{rig}(E\wedge M_l\wedge\bar{H}_{\mathbb{Z}}^{\wedge(q+1)})\to\pi_{p,q}^{rig}(E\wedge M_l)$$

Since E is effective so is $E\wedge M_l$ and thus by Conjecture 10 we have

$$\pi_{p,q}^{rig}(E\wedge M_l\wedge\bar{H}_{\mathbb{Z}}^{\wedge(q+1)})=\pi_{p,q}(s_q(E\wedge M_l\wedge\bar{H}_{\mathbb{Z}}^{\wedge(q+1)}))=0.$$

□

Lemma 6.6 *For any $i\geq 0$ there exists a morphism*

$$M_l\wedge\bar{H}_{\mathbb{Z}/l}^{\wedge i}\to M_l\wedge\bar{H}_{\mathbb{Z}}^{\wedge i} \tag{6.3}$$

such that the diagram

$$\begin{array}{ccc} M_l\wedge\bar{H}_{\mathbb{Z}/l}^{\wedge i} & \to & M_l\wedge\bar{H}_{\mathbb{Z}}^{\wedge i}\\ \downarrow & & \downarrow\\ M_l & = & M_l\end{array}$$

commutes.

Proof. Proceed by induction on i. To prove the lemma for $i=1$ it is sufficient to show that the composition

$$M_l\wedge\bar{H}_{\mathbb{Z}/l}\to M_l\to M_l\wedge H_{\mathbb{Z}} \tag{6.4}$$

is zero. Consider the morphism of distinguished squares

$$\begin{array}{ccccc} M_l\wedge\bar{H}_{\mathbb{Z}} & \to & M_l & \to & M_l\wedge H_{\mathbb{Z}}\\ \downarrow & & \downarrow & & \downarrow\\ M_l\wedge\bar{H}_{\mathbb{Z}/l} & \to & M_l & \to & M_l\wedge H_{\mathbb{Z}/l}\end{array}$$

The composition of (6.4) with the right vertical arrow is the composition of the two lower arrows which is zero. On the other hand the right vertical arrow is a split monomorphism. Therefore (6.4) is zero. For $i > 1$ one defines the morphism (6.3) inductively as the composition

$$M_l \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge i} = M_l \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge(i-1)} \wedge \bar{H}_{\mathbb{Z}/l} \to M_l \wedge \bar{H}_{\mathbb{Z}}^{\wedge(i-1)} \wedge \bar{H}_{\mathbb{Z}/l} \to$$

$$\to M_l \wedge \bar{H}_{\mathbb{Z}}^{\wedge(i-1)} \wedge \bar{H}_{\mathbb{Z}} = M_l \wedge \bar{H}_{\mathbb{Z}}^{\wedge i}$$

□

Proposition 6.5 easily implies the following convergence result.

Proposition 6.7 *Let E be a connective spectrum and l a prime number such that the rigid homology groups $H^{rig}_{p,q}(E, \mathbb{Z}_{(l)})$ are finitely generated. Then the rigid Adams spectral sequence for E with coefficients in $\mathbb{Z}/l$ converges to $\pi^{rig}_{p,q}(E) \otimes \mathbb{Z}_{(l)}$, that is*

$$\cap_{i \geq 0} a_l^i \pi^{rig}_{*,*}(E) \otimes \mathbb{Z}_{(l)} = 0$$

and the canonical homomorphisms

$$(a_l^i \pi^{rig}_{p,q}(E)/a_l^{i+1} \pi^{rig}_{p,q}(E)) \otimes \mathbb{Z}_{(l)} \to H^{rig}_{p,q}(E \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge i})_\infty$$

where the subscript ∞ denotes the infinite term of the spectral sequence, are isomorphisms.

Proof. A four term exact sequence similar to the one used in the proof of Lemma 7.2 implies that it is sufficient to prove that for any p, q and n one has

$$\cap_{i \geq 0} Im(\pi^{rig}_{p,q}(E \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge(n+i)}) \to \pi^{rig}_{p,q}(E \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge n})) = 0 \tag{6.5}$$

We can invert all the primes but l and assume that the rigid homology of E are finitely generated $\mathbb{Z}_{(l)}$-modules. The same argument as the one used in the proof of Lemma 6.1 shows that then the rigid homotopy groups of E are finitely generated $\mathbb{Z}_{(l)}$-modules. Conjecture 2 implies that the same holds for the spectra $E \wedge \bar{H}_{\mathbb{Z}/l}^{\wedge n}$ and thus (6.5) follows from Proposition 6.5. □

The E_2-term of the rigid Adams spectral sequence with $\mathbb{Z}/l$-coefficients for $E = \mathbf{1}$ consists of the Ext-groups $Ext_{A_*(l)}(\mathbb{Z}/l, \mathbb{Z}/l)$ in the category of comodules over the rigid Steenrod algebra and according to (5.6) and Proposition 6.7 the E_∞-term gives the quotients of a filtration on the groups $Ext_{MU_*(MU)}(MU_*, MU_*) \otimes$

$\mathbb{Z}_{(l)}$. For $l > 2$ the rigid Steenrod algebra coincides with the topological Steenrod algebra and therefore, as was pointed out by Charles Rezk, the rigid Adams spectral sequence in this case looks exactly like the algebraic Adams-Novikov spectral sequence [9], [10]. There is no doubt that these two spectral sequences are indeed isomorphic.

7 Slice spectral sequence and convergence problems

For any $n \in \mathbb{Z}$ the slice tower (2.1) of a spectrum E defines in the usual way a spectral sequence of abelian groups which starts with groups of the form $\pi_{p,n}(s_q(E))$. We call these spectral sequences the slice spectral sequences for E. The r-th differential in the n-th slice spectral sequence goes from $\pi_{p,n}(s_q(E))$ to $\pi_{p-1,n}(s_{q+r}(E))$ which suggests that one can visualize it in the same way as one visualizes the Adams spectral sequence in topology. One considers p as the horizontal and q as the vertical index. The differentials then go from a given column ("stem") to the previous one reaching higher and higher in the vertical direction. For any n one has $\pi_{p,n}(s_q(E)) = 0$ for $q < n$ therefore each of this spectral sequences is zero below the horizontal line $n = q$. In particular for each term there are only finitely many incoming differentials. Let $f_q\pi_{p,n}(E)$ be the image of $\pi_{p,n}(f_qE)$ in $\pi_{p,n}(E)$. These subgroups form a filtration on $\pi_{p,n}(E)$ and one verifies in the standard manner that one gets canonical monomorphisms

$$f_q\pi_{p,n}(E)/f_{q+1}\pi_{p,n}(E) \to \pi_{p,n}(s_q(E))_\infty \tag{7.1}$$

where the subscript ∞ indicates that we consider the infinite term of the spectral sequence.

Definition 7.1 *A spectrum E is called convergent with respect to the slice filtration if for any $p, n, q \in \mathbb{Z}$ one has*

$$\cap_{i\geq 0} f_{q+i}\pi_{p,n}(f_qE) = 0 \tag{7.2}$$

Lemma 7.2 *Let E be a spectrum convergent with respect to the slice filtration. Then the homomorphisms (7.1) are isomorphisms.*

Proof. A standard argument shows that the homomorphisms (7.1) fit into exact sequences of the form

$$0 \to f_q\pi_{p,n}(E)/f_{q+1}\pi_{p,n}(E) \to \pi_{p,n}(s_q(E))_\infty \to$$

$$\to \cap_{i\geq 1} f_{q+i}\pi_{p-1,n}(f_{q+1}E) \to \cap_{i\geq 0} f_{q+i}\pi_{p-1,n}(E)$$

which implies the statement of the lemma. □

For E as in Lemma 7.1 $f_q\pi_{p,n}(E)$ is a nondegenerate filtration on $\pi_{p,n}(E)$ and its quotients are subquotients of the groups $\pi_{p,n}(s_q(E))$. We say that E is bounded with respect to the slice filtration if for any p, n there exists q such that $\pi_{p,n}(f_{q+i}E) = 0$ for $i > 0$. Any bounded E is clearly convergent. If in a distinguished triangle two out of three terms are bounded then so is the third. If one term is bounded and another one is convergent then the third one is convergent. Any direct sum of convergent spectra is convergent. Conjecture 1 implies that any E of the form $\Sigma^{p,q}H_C$ where C is a complex of abelian groups or more generally any spectrum which belongs to the smallest triangulated subcategory which contains objects of this form is bounded. We will see below that the intersection (7.2) is non zero for the rational Moore spectrum $\mathbf{1}_{\mathbb{Q}}$ and thus one can not expect any spectrum to be convergent. We say that E is a finite spectrum if it belongs to the smallest triangulated subcategory which contains T-desuspensions of suspension spectra of smooth schemes over S and is closed under direct summands.

Conjecture 13 *Any finite spectrum is convergent with respect to the slice filtration.*

The convergence property for the slice spectral sequence is closely related to the convergence property for the motivic analog of the classical Adams spectral sequence which is defined by taking motivic homotopy groups of the Adams tower (4.1) smashed with E. We will show this modulo the following conjecture.

Conjecture 14 *Let (X, x) be a smooth pointed scheme of dimension d over S. Then*

$$\pi_{p,q}(\Sigma^\infty_T(X, x) \wedge H_{\mathbb{Z}}) = 0$$

for $q > d$.

The existing techniques imply this conjecture for $S = Spec(k)$ where k is a field of characteristic zero. Using resolution of singularities and the blow-up long exact sequence in generalized homology (see [12]) one reduces the problem to a smooth proper X of pure dimension $d' \leq d$. The Spanier-Whitehead duality (loc.cit) and the Thom isomorphism for motivic cohomology imply that for such X one has

$$\pi_{p,q}(\Sigma^\infty_T X \wedge H_{\mathbb{Z}}) = H^{2d'-p,d'-q}(X, \mathbb{Z})$$

The right hand side is zero for $d' < q$ by the rigidity part of Conjecture 1 which is known for varieties over a field.

Assuming Conjecture 14 we immediately see that for any finite spectrum E the spectrum $E \wedge H_{\mathbb{Z}}$ is bounded and in particular convergent. Conjectures 2 and 4 imply that $H_{\mathbb{Z}} \wedge H_{\mathbb{Z}}$ splits into a direct sum of the form $\Sigma^{0,q} H_{C_q}$ where C_q is a complex of abelian groups which is bounded in both directions and has no homology groups in dimensions less than $2q$. Together with the fact that for any smooth scheme X one has

$$\pi_{p,q}(\Sigma_T^\infty X \wedge H_{\mathbb{Z}}) = 0$$

for $2q - p > dim(S)$ it implies that for a finite E only finitely many summands of $H_{\mathbb{Z}}^{\wedge i}$ will contribute to $\pi_{p,n}(f_q(E \wedge H_{\mathbb{Z}}^{\wedge i}))$ for any given p. This implies in turn that for a finite spectrum E and any $i \geq 0$ the spectrum $E \wedge H_{\mathbb{Z}}^{\wedge i}$ is bounded. Thus a finite spectrum E is convergent with respect to the slice filtration if and only if all the spectra $E \wedge \bar{H}^{\wedge i}$ forming the Adams tower for E are.

Lemma 7.3 *If E is a finite spectrum convergent with respect to the slice filtration then E is convergent with respect to the Adams filtration i.e. for any p, n and q one has*

$$\cap_{i \geq 0} Im(\pi_{p,n}(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge(q+i)}) \to \pi_{p,n}(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q})) = 0 \qquad (7.3)$$

Proof. We may assume that E is effective. If E is convergent with respect to the slice filtration then as was shown above the spectra $E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q}$ are convergent with respect to the slice filtration. On the other hand Conjecture 10 implies that the morphism

$$E \wedge \bar{H}_{\mathbb{Z}}^{\wedge(q+i)} \to E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q}$$

factors through the morphism

$$f_i(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q}) \to E \wedge \bar{H}_{\mathbb{Z}}^{\wedge q}$$

Combining we conclude that (7.3) holds. □

Lemma 7.4 *Let E be a connective spectrum convergent with respect to the Adams filtration (in the sense of Lemma 7.3) and such that $E \wedge H_{\mathbb{Z}}$ is bounded with respect to the slice filtration. Then E is convergent with respect to the slice filtration.*

Proof. Conjecture 10 implies that for any E and any q the morphism $s_q(E) \to s_q(E) \wedge H_{\mathbb{Z}}$ is a split monomorphism and thus the morphism $\bar{H}_{\mathbb{Z}} \wedge s_q(E) \to$

$s_q(E)$ is zero. It implies easily that a connective spectrum E is convergent with respect to the Adams filtration if and only if f_qE are convergent with respect to the Adams filtration for all q. Therefore, it is sufficient to show that for E satisfying the conditions of the lemma the intersection $\cap_{q\geq 0} f_q\pi_{p,n}(E)$ is zero. Our assumption on E together with Conjecture 2 implies that all the spectra $E \wedge \bar{H}_{\mathbb{Z}}^{\wedge i} \wedge H_{\mathbb{Z}}$ are bounded with respect to the slice filtration. An inductive argument shows now that any element in this intersection will lie in the image of the homomorphism

$$\pi_{p,n}(E \wedge \bar{H}_{\mathbb{Z}}^{\wedge(i+1)}) \to \pi_{p,n}(E)$$

for any i and thus is zero by the assumption. □

Combining Lemmas 7.3 and 7.4 we see that, modulo the rest of the conjectures, Conjecture 13 is equivalent to the conjecture predicting that finite spectra are convergent with respect to the Adams filtration.

Consider the slice spectral sequence for a slice-wise cellular spectrum E. For simplicity let us assume that S is normal and connected such that $s_q(E) = \Sigma^{0,q} H_{\Pi_q(E)}$ where Π_q is a complex of abelian groups whose homology are the rigid homotopy groups of E and thus

$$\pi_{p,n}(s_q(E)) = H^{-p,q-n}(S, \Pi_q(E)).$$

Remark 7.5 If E is not a slice-wise cellular spectrum the groups $\pi_{p,n}(s_q(E))$ should still be isomorphic to motivic cohomology with coefficients in some complexes of sheaves with transfers whose cohomology presheaves are the presheaves of rigid homotopy groups of E. Thus for an arbitrary E the slice spectral sequence starts with motivic cohomology and approximates the motivic homotopy groups of the spectrum.

The canonical filtration

$$\dots \subset \tau^{\geq 2}\Pi_q(E) \subset \tau^{\geq 1}\Pi_q(E) \subset \tau^{\geq 0}\Pi_q(E) \subset \dots$$

on the complex $\Pi_q(E)$ gives a sequence of distinguished triangles of the form

(7.4)
$$\Sigma^{0,q} H_{\tau \geq (p+1)\Pi_q(E)} \to \Sigma^{0,q} H_{\tau \geq p \Pi_q(E)} \to \Sigma^{p,q} H_{H^p(\Pi_q(E))} \to \Sigma^{1,q} H_{\tau \geq (p+1)\Pi_q(E)}$$

and since $s_q(E) = hocolim_{p<0} \Sigma^{0,q} H_{\tau \geq p \Pi_q(E)}$ this sequence defines a spectral sequence which starts with the groups $H^{p-p',q-n}(S, \pi^{rig}_{p,q}(E))$ and converges to

$\pi_{p',n}(s_q(E))$ (the fact that it really converges requires some extra work based on the vanishing of $H^{i,j}(S, A)$ for $i > j + dimS$). We see that in the case of a slice-wise cellular spectrum the combination of the slice spectral sequence with the spectral sequences generated by the towers (7.4) provides an "approximation" of the motivic homotopy groups of E by motivic cohomology of S with coefficients in the rigid homotopy groups of E.

Let us look in more detail on the slice spectral sequences for MGL, KGL and $\mathbf{1}$. Conjecture 5 predicts that $s_q(MGL) = \Sigma^{2q,q}H_{MU_{2q}}$ and in particular the complex Π_q in this case has only one nontrivial cohomology group in dimension $-2q$. Thus the tower (7.4) degenerates and we have

$$\pi_{p,n}(s_q MGL) = H^{2q-p,q-n}(S, MU_{2q}). \tag{7.5}$$

The slice spectral sequence starts with groups $H^{2q-p,q-n}(S, MU_{2q})$ and the r-th differential is of the form

$$H^{2q-p,q-n}(S, MU_{2q}) \to H^{2q-p+2r+1,q+r-n}(S, MU_{2(q+r)}).$$

If we reindex it we get the "motivic Atiyah-Hirzebruch spectral sequence" for MGL.

Conjecture 15 *The spectrum MGL is convergent with respect to the slice filtration.*

For $q > dim(S) + p - n$ the group (7.5) is zero and thus Conjecture 15 implies that MGL is bounded with respect to the slice filtration. If S is regular and local over a field one has $H^{p,q}(S, \mathbb{Z}) = 0$ for $p > 2q$ or $p = 2q$ and $q \neq 0$ and $H^{0,0}(S, \mathbb{Z}) = \mathbb{Z}$. The same is expected to be true for any regular local S. Thus in this case the only nontrivial groups in the slice spectral sequence contributing to $\pi_{2q,q}MGL$ are $H^{0,0}(S, MU_{2q}) = MU_{2q}$ and all outgoing differentials are zero. Togther with the convergence conjecture it implies that the homomorphism (3.5) maps MU_{2q} surjectively to $MGL_{2q,q}$ which implies the surjectivity part of [14, Conj.1 p.601]. Since MU_* has no torsion the injectivity part can be proved by considering the rational coefficient case where everything splits.

For the algebraic K-theory spectrum Conjecture 7 implies that the slice spectral sequence has the same form as the spectral sequence constructed in [1] and [3]. We expect that these two spectral sequences are isomorphic. K-theory

Consider now the case of the sphere spectrum 1. By Conjecture 9 $s_q(1)$ is of the form $\Sigma_T^q H_{N_{2q}}$ where N_{2q} is a complex of abelian groups whose cohomology groups are given by

$$H^r(N_{2q}) = Ext^r_{MU_*(MU)}(MU_*, MU_*)_{2q} \tag{7.6}$$

These groups are zero for $r < 0$ and $r > q$ and thus the sequences (7.4) in this case give finite Postnikoff towers for $s_q(\mathbf{1})$ and we have a strongly convergent spectral sequence which starts with the groups

$$H^{r-p,q-n}(S, \pi^{rig}_{r,q}(\mathbf{1})) = H^{r-p,q-n}(S, Ext^{2q-r}_{MU_*(MU)}(MU_*, MU_*)_{2q}) \tag{7.7}$$

and converges to $\pi_{p,n}(s_q(\mathbf{1}))$. Let us consider two particular cases. First assume that $S = Spec(k)$ where k is an algebraically closed field of characteristic zero. Then for a torsion abelian group A we have $H^{i,j}(k, A) = 0$ for $i \neq 0$ or $j < 0$ and $H^{0,j}(k, A) = A(j)$ for $j \geq 0$ where $A(j)$ denotes the twisting by the j-th power of roots of unity. For $q \neq 0$ the groups (7.6) are torsion and thus the spectral sequence computing $\pi_{p,n}(s_q(\mathbf{1}))$ degenerates and for $q \neq 0$ we get

$$\begin{aligned}\pi_{p,n}(s_q(\mathbf{1})) &= H^{0,q-n}(S, \pi^{rig}_{p,q}(\mathbf{1})) \\ &= H^{0,q-n}(S, Ext^{2q-p}_{MU_*(MU)}(MU_*, MU_*)_{2q}) \\ &= Ext^{2q-p}_{MU_*(MU)}(MU_*, MU_*)_{2q}(q-n)\end{aligned}$$

For $n = 0$ the slice spectral sequence in this case becomes isomorphic to the usual Adams-Novikov spectral sequence in topology which shows that our conjectures predict that for $S = Spec(k)$ as above one has

$$\pi_{p,0}(\mathbf{1}) = \pi^s_p(S^0)$$

Let now k be any field and consider the part of the slice spectral sequence which contributes to $\pi_{0,0}(\mathbf{1})$. The picture one gets here is very similar to the picture obtained by Fabien Morel in his work in [6] and [5] on $\pi_{0,0}$ based on the motivic Adams spectral sequence. The groups which contribute to $\pi_{0,0}(\mathbf{1})$ are $\pi_{0,0}(s_q(\mathbf{1}))$ and the groups which contribute to $\pi_{0,0}(s_q(\mathbf{1}))$ in the spectral sequence, defined by the tower (7.4), are

$$H^{r,q}(S, \pi^{rig}_{r,q}(\mathbf{1})) = H^{r,q}(S, Ext^{2q-r}_{MU_*(MU)}(MU_*, MU_*)_{2q})$$

For any field k we have

$$H^{i,j}(Spec(k), A) = 0$$

for $i > j$ thus the only nontrivial contributions come from cohomology with coefficients in $Ext^{2q-r}_{MU_*(MU)}(MU_*, MU_*)_{2q}$ for $r \leq q$. All such groups are zero except for the ones with $r = q$ which are equal to $\mathbb{Z}$ for $r = q = 0$ and to $\mathbb{Z}/2$ for $r = q > 0$ ([10]). This is consistent with the conjecture of Fabien Morel predicting that $\pi_{0,0}(\mathbf{1})$ for any field k is isomorphic to its Grothendieck-Witt ring of quadratic forms. In terms of the Grothendieck-Witt ring the f_q filtration is then expected to coincide with the filtration by the powers of the ideal of forms of even dimension and the Milnor conjecture becomes a degeneracy result for the spectral sequence in this range. The results of [6] and [5] imply that in this case the slice filtration $f_n\pi$ coincides with the Adams filtration. In general it is not so since the Adams filtration on any spectrum which is a module over the Eilenberg-MacLane spectrum is trivial while the slice filtration may be not.

Finally consider the slice spectral sequence for the rational Moore spectrum $\mathbf{1}_{\mathbb{Q}}$ such that $\pi_{p,q}(\mathbf{1}_{\mathbb{Q}}) = \pi_{p,q}(\mathbf{1}) \otimes \mathbb{Q}$. We have $s_q(\mathbf{1}_{\mathbb{Q}}) = 0$ for $q \neq 0$ and $s_0(\mathbf{1}_{\mathbb{Q}}) = H_{\mathbb{Q}}$. Thus

$$\pi_{p,n}(s_q(\mathbf{1}_{\mathbb{Q}})) = \begin{cases} H^{-p,-n}(S,\mathbb{Q}) & \text{for } q = 0 \\ 0 & \text{for } q \neq 0 \end{cases}$$

The slice spectral sequence in this case degenerates and the intersection of all the terms of the filtration $f_q\pi_{p,n}(\mathbf{1}_{\mathbb{Q}})$ equals to the kernel of the motivic Hurewicz homomorphism

$$\pi_{p,n}(\mathbf{1}_{\mathbb{Q}}) \to H^{-p,-n}(S,\mathbb{Q}).$$

If $k = \mathbb{Q}$ the group $\pi_{0,0}(\mathbf{1}) \otimes \mathbb{Q}$ contains at least two linearly independent elements while

$$H^{0,0}(Spec(\mathbb{Q}),\mathbb{Q}) = \mathbb{Q}$$

which implies that in this case the intersection (7.2) is not zero.

8 Possible strategies of the proof

We know of two strategies which can be used to prove the conjectures of this paper. The first one looks as follows. One starts with Conjecture 1. The existing techniques are sufficient to prove it in the case when S is a regular scheme over a field. For any such S one has $p^*(H_{\mathbb{Z}}) = H_{\mathbb{Z}}$ where $p : S \to Spec(k)$ is the canonical morphism. An unstable version of this fact is proved in [11]. A stable version which easily follows will be done in one of the later papers of the series. The inverse image functor p^* obviously takes effective objects to effective objects and, for p of the form we consider here, it also takes rigid objects to rigid objects (see

[12]). This argument show that it is enough to consider the case $S = Spec(k)$. The effectiveness part of the conjecture in this case should follows from the description of the Eilenberg-MacLane spaces in terms of effective cycles given in [11] and will be considered in the next paper of the series. The rigidity part follows from the comparison between motivic cohomology defined in terms of motivic complexes in [17] and motivic cohomology defined in terms of SH since for the former one has $H^{p,q} = 0$ for $q < 0$ by definition. The s-stable form of this comparison result is proved in [11]. The T-stable form will be proved in one of the later papers of the series. It requires the cancellation theorem proved through comparison with higher Chow groups in [15]. The same argument proves Conjecture 12 for regular schemes S over a field.

The next step in this strategy is to prove Conjecture 2. The existing techniques are sufficient to do it for a regular scheme S of characteristic zero. As before one first reduces the problem to $S = Spec(\mathbb{Q})$ by showing that $p^*(H_{\mathbb{Z}}) = H_{\mathbb{Z}}$. The Eilenberg-MacLane spectrum is built out of the suspension spectra of the Eilenberg-MacLane spaces $K(\mathbb{Z}(q), 2q)$ and thus it is sufficient to prove an analog of Conjecture 2 for $\Sigma_T^\infty K(\mathbb{Z}(q), 2q) \wedge H_{\mathbb{Z}}$ and show that the corresponding direct sum decompositions are compatible with the assembly morphisms of the Eilenberg-MacLane spectrum. To do it one constructs a functor ρ from DM, the stable version of the category DM_-^{eff} of [17], to SH right adjoint to the functor $\lambda : SH \to DM$ which takes the suspension spectrum of a smooth scheme to its "motive". The unstable version of this construction is described in [11]. For any E one gets a natural morphism of the form

$$E \wedge H_{\mathbb{Z}} \to \rho\lambda(E) \tag{8.1}$$

The morphism (8.1) is an isomorphism if E is a sphere. Using the Spanier-Whitehead duality established in [12] one can show that it is an isomorphism if E is the suspension spectrum of a scheme which is smooth and proper over S. If S is the spectrum of a field of characteristic zero the resolution of singularities implies that the suspension spectra of smooth projective varieties generate SH and thus (8.1) is an isomorphism for any E. In particular one concludes that

$$\Sigma_T^\infty K(\mathbb{Z}(q), 2q) \wedge H_{\mathbb{Z}} \cong \rho\tilde{M}(K(\mathbb{Z}(q), 2q))$$

where $\tilde{M}$ is the functor which takes a pointed space to its "motive" in DM. The spaces $K(\mathbb{Z}(q), 2q)$ for $q > 0$ can be represented by spaces of effective cycles (see [11]) on T^q and thus in the case of characteristic zero by infinite symmetric powers of T^q. In the second paper of the series started by [11] we analyze the structure of the motives of infinite symmetric powers. We first show that they admit an analog of the Steenrod splitting such that the motive of an infinite symmetric product splits as a direct sum of motives of "reduced" finite symmetric products. For any prime l the motive of each finite symmetric product localized in l can be represented as a direct summand of the iterated circle powers (the μ_n-version of the cyclic power). This construction uses the fact that motives are functorial with respect to "correspondences" given by finite relative cycles. Finally the motives of iterated circle powers of spheres are explicitly computed and showed to be isomorphic to direct sums of Tate motives of the form required by Conjecture 2.

The next step which is again possible to do over a general base scheme is to construct reduced power operations and use them to prove Conjecture 4. Together with explicit computations of the action of reduced power operations in motivic cohomology of the lense spaces this should lead to a proof of Conjecture 3.

The next step is to prove Conjecture 10. We do not know how to deal with the divisibility part of it yet but one may hope to prove it by looking at the geometry of the symmetric products of spheres. Over a field of characteristic zero $K(\mathbb{Z}(q), 2q)$ is built out of q-fold T-suspensions of $\Sigma^1_s(\mathbb{A}^{(m-1)q} - \{0\})/S_m$ where S_m is the symmetric group and $\mathbb{A}^{(m-1)q}$ is identified with the the q-th power of the subspace V of $\mathbb{A}^m$, on which S_m acts by permutation of cooordinates, given by the equation $\sum_{i=1}^m x_i = 0$. What one has to show is that the suspension spectrum of $(\mathbb{A}^{(m-1)q} - \{0\})/S_m$ considered as a pointed space belongs to $\Sigma^1_T SH^{eff}$. It can probably be done by explicit computaion using some resolution of singularities for these spaces.

Assuming Conjecture 10 one can try to prove Conjecture 5 as follows. Fisrt note that Conjecture 10 shows that there exists a unique homomorphism $MU_* \to \pi_{*,*}(MGL)$ compatible with the homomorphism (3.5). To verify that it is an isomorphism it is sufficient to check that its analogs with coefficients in $\mathbb{Q}$ and $\mathbb{Z}/l$ are isomorphisms. The usual approach to the compuation of the homology of MU as a module over the Steenrod algebra should work with no problem in the rigid setting which together with this identification of the E_2-term of the rigid Adams spectral sequence and the convergence theorem 6.7 should lead to a proof of Conjecture 5. Conjecture 6 seem to be easy modulo Conjecture 5 and how one proves Conjecture 9 using Conjecture 6 is explained in Section 3.2.

Conjectures 7, 8 about algebraic K-theory can not be proved by means of the Adams spectral sequence since KGL is not a connective spectrum. There are several things one can do about it. One is to try to prove an analog of Conner and Floyd formula [2]

$$K^{*,*}(X) = MGL^{*,*}(X) \otimes_{MGL^{*,*}(S)} K^{*,*}(S)$$

and use it together with Conjecture 5 to get Conjecture 7. Alternatively, one can use the Adams spectral sequence to approximate the slices of the suspension spectrum of BGL and use the fact that $KGL = hocolim_{n\geq 0} \Sigma_T^{-n} \Sigma_T^{\infty} BGL$. One can also use the following approach[1]. Define the s-stable homotopy category $SH_s(S)$ starting from the unstable $\mathbb{A}^1$-homotopy category and inverting S^1_s but not S^1_t. We get the s-suspenison spectrum functor

$$\Sigma_s^\infty : \mathcal{H}_{\mathbb{A}^1,\bullet}(S) \to SH_s(S)$$

and the t-suspension spectrum functor

$$\Sigma_t^\infty : SH_s(S) \to SH(S)$$

such that $\Sigma_T^\infty = \Sigma_t^\infty \Sigma_s^\infty$. The functor Σ_t^∞ has a right adjoint which we denote by Ω_t^∞. The definition of the slice filtration can be given in the context of SH_s in the same way as we did for $SH(S)$ except that all s-spectra are effective.

Conjecture 16

$$\Omega_t^\infty(\Sigma_T^n SH^{eff}(S)) \subset \Sigma_T^n SH_s(T)$$

This conjecture says that for a space (X,x) the space $\Omega_T^\infty \Sigma_T^\infty(\Sigma_T^n(X,x))$ can be bulit, at least s-stably, from n-fold T-suspensions. It connects the theme of this paper to another bunch of conjectures describing the hypothetical theory of operadic description of T-loop spaces. Any such theory should provide a model for $\Omega_T^\infty \Sigma_T^\infty$ which could then be used to prove Conjecture 16.

Let us show how Conjecture 16 can be used to prove Conjecture 7. The unit $\mathbf{1} \to KGL$ of the ring structure of KGL defines by Conjecture 10 a morphism $H_{\mathbb{Z}} \to s_0(KGL)$. The functor Ω_t^∞ clearly reflects isomorphisms between effective spectra. Thus to prove Conjecture 7 it is sufficient to show that

$$\Omega_t^\infty H_{\mathbb{Z}} \to \Omega_t^\infty s_0(KGL) \tag{8.2}$$

is an isomorphism. Consider the distinguished triangle

$$f_1 KGL \to f_0 KGL \to s_0(KGL)$$

Applying to it the functor Ω_t^∞ we get a triangle of the form

$$\Omega_t^\infty f_1 KGL \to \Omega_t^\infty f_0 KGL \to \Omega_t^\infty s_0(KGL) \tag{8.3}$$

It is easy to see that $\Omega_t^\infty f_0(KGL) = \Omega_t^\infty(KGL)$ and that $\Omega_t^\infty s_0(KGL)$ is orthogonal to $\Sigma_T^1 SH_s$. Conjecture 16 implies that $\Omega_t^\infty f_1 KGL$ belongs to $\Sigma_T^1 SH_s$ and therefore the triangle (8.3) is isomorphic to the triangle

$$f_1 \Omega_t^\infty KGL \to \Omega_t^\infty KGL \to s_0 \Omega_t^\infty (KGL)$$

[1] This approach is further elaborated in [16].

Since the space $BGL \times \mathbb{Z}$ represents algebraic K-theory in the unstable category at least for a regular S (see [7]) the s-spectrum $\Omega_t^\infty KGL$ can be represented by the sequence of spaces which consists of s-deloopings of $BGL \times \mathbb{Z}$. Since BGL is build out of spheres in a rather explicit way it seems to be easy to show that $s_0\Omega_t^\infty(BGL) = H_\mathbb{Z}$ where $H_\mathbb{Z}$ is considered as the Eilenberg-MacLane s-spectrum i.e. just the sequence of usual simplicial $K(\mathbb{Z}, n)$'s. Finally Conjectures 1 and 12 imply that for a normal connected S one has $\Omega_t^\infty(H_\mathbb{Z}) = H_\mathbb{Z}$. Thus we see that (8.2) is an endomorphism of $H_\mathbb{Z}$ and one verifies immediately that it takes unit to the unit which in turn implies that it is an identity.

This is how the first strategy looks like. It can be extended to prove the conjectures for all schemes S of characteristic zero but we have no idea how to extend it to positive or mixed characterstic. The bottleneck of this approach is the method used to prove Conjecture 2. One problem is that in positive characteristic the spaces of effective cycles are not representable by symmetric products and the argument used to establish the fact that their motives are direct sums of Tate motives does not work. The other problem is that without resolution of singularities I do not know how to prove that the morphism (8.1) is an isomorphism for $E = H_\mathbb{Z}$.

The second strategy is much less detailed than the first one but it may offer a way to prove conjectures of this paper in their full generality. I learned the ideas on which it is based from Mike Hopkins, Fabien Morel and Markus Rost. This strategy takes Conjecture 5 and closely related to it Conjecture 6 as the starting point. Over a general base scheme it is much easier to work with cobordisms than with motivic cohomology since the Thom spectrum is built directly from the suspension spectra of smooth varieties while the Eilenberg-MacLane spectrum is not. One can then attempt to prove the torsion part of Conjecture 6 by some analog of Quillen's argument. We do not know what to do with the rational part but may be one can prove Lemma 6.2 directly. This would imply in particular that $s_0(MGL) = H_\mathbb{Z}$ and thus we get Conjecture 10 since $s_0(MGL) = s_0(\mathbf{1})$ for simple geometric reasons. On the other hand knowing Conjecture 6 it seems possible to show that $H_\mathbb{Z}$ is the homotopy colimit of an explicit diagram built out of many copies of suspended MGL's and since the rigid homology of MGL are easy to compute we can compute rigid homology of $H_\mathbb{Z}$ thus solving Conjecture 3. This approach would also imply another result which deserves a separate formulation.

Conjecture 17 *For any morphism of schemes $f : S' \to S$ the natural morphism $f^*H_\mathbb{Z} \to H_\mathbb{Z}$ is an isomorphism.*

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