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Hyman Bass

Algebraic K-

"Classical" Algebraic

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Algebraic K-Theory II – "Classical" Algebraic K-Theory, and Connections with Arithmetic

Proceedings of the Conference held at the Seattle Research Center of the Battelle Memorial Institute, Aug. 28–Sept. 8, 1972

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Introduction

A conference on algebraic K-theory was held at the Battelle Seattle Research Center from August 28 to September 8, 1972, with the joint support of the National Science Foundation and the Battelle Memorial Institute. The present volume consists mainly of papers presented at, or stimulated by, that conference, plus some closely related papers by mathematicians who did not attend the conference but who have kindly consented to publish their work here. In addition there are several papers devoted to surveys of subjects treated at the conference, and to the formulation of open research problems. It was our intention thus to present a reasonably comprehensive documentation of the current research in algebraic K-theory, and, if possible, to give this research a greater coherence than it has heretofore enjoyed. It was particularly gratifying to see the latter aim largely achieved already in the course of preparing these Proceedings.

Algebraic K-theory has two quite different historical roots both in geometry. The first is concerned with certain topological obstruction groups, like the Whitehead groups, and the L-groups of surgery theory. Their computation, which is in principle an algebraic problem about group rings, is one of the original missions of algebraic K-theory. It remains a rich source of new problems and ideas, and an excellent proving ground for new techniques.

The second historical source of algebraic K-theory, from which the subject draws its name, is Grothendieck's proof of the Riemann-Roch theorem, and the topological K-theory of Atiyah-Hirzebruch, which has the same point of departure. Starting from the analogy between projective modules and vector bundles one is led to seek a K-theory for rings analogous to that of Atiyah-Hirzebruch for spaces. This enterprise made, at first, only very limited progress. In the few years preceding this conference, however, several interesting definitions of higher K-groups were proposed; the relations between them were far from clear.

Meanwhile the detailed study of K_1 and K_2 had revealed some beautiful arithmetic phenomena within the classical groups. This contact with algebraic number theory had become a major impulse in the subject as well as a theme for conjectures about the significance of the higher K-groups.

More recently there have appeared definitions and potential applications of higher K-theory in the framework of algebraic geometry.

As this brief account suggests, a large number of mathematicians, with quite different motivations and technical backgrounds, had become interested in aspects of algebraic K-theory. It was not altogether apparent whether the assembling of these efforts under one rubric was litte more than an accident of nomenclature. In any case it seemed desireable to gather these mathematicians, some of whom had no other occasion for serious technical contact, in a congenial and relaxed setting, and to leave much of what would ensu.e to mathematical and human chemistry. A consensus of those who were present is that the experiment was enormously successful. Testimony to this is the fact that many of the important new results in these volumes were proved in the few months following the conference, growing out of collaborative efforts and discussions begun there.

One major conclusion of this research is that all of the higher K-theories which give the "classical" K_n's for $n \leq 2$ coincide. Thus, in some sense, the subject of higher algebraic K-theory "exists", an assertion some had begun to depair of making. Moreover one now has, thanks largely to the extraordinary work of Quillen, some very effective tools for calculating higher K-groups in interesting cases.

The papers that follow are somewhat loosely organized under the headings: I. Higher K-theories; II. "Classical" algebraic K-theory, and connections with arithmetic; and III. Hermitian K-theories and geometric applications. Certain papers, as their titles indicate, contain collections of research problems. The reader should be warned, however, that because of the vigorous activity ensu-ing the conference, some of the research problems posed below are in fact resolved elsewhere in these volumes. The editional effort necessary to eliminate such instances would have cost an excessive delay in publication.

I am extremely grateful to the following participants who contributed

to the preparation of the survey and research problem articles:
S. Bloch, J. Coates, Keith Dennis, S. Gersten, M. Karoubi, M.P. Murthy, Ted Petrie, L. Roberts, J. Shaneson, M. Stein, and R. Swan.

On behalf of the participants I express our thanks to the National Science Foundation and the Battelle Memorial Institute for their generous financial support. For the splendid facilities and setting of the Battelle Seattle Research Center, and for the efficient and considerate services of its staff, the conference participants were uniformly enthusiastic in their praise and gratitude.

Finally, I wish to thank Kate March of Columbia University for her invaluable secretarial and administrative assistance in organizing the conference, and Robert Martin of Columbia University for his aid in editing these Proceedings.

H. Bass

Paris, April, 1973

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Table of Contents

ALGEBRAIC K-THEORY III

Lecture Notes in Mathematics, Vol. 343: "Hermitian K-theory and geometric applications"

A. THE FUNCTORS K_0 AND K_1

Some problems in "classical" algebraic K-theory

Hyman Bass

By "classical" we refer to questions about projective modules and their automorphism groups, and, in particular, about K_0 and K_1. In many instances the questions can naturally be posed for K_n for all $n \geq 0$.* When this was the case I have not hesitated to do so, with the result that the discussion below inevitably overlaps with the problem sections on K_2 (Dennis-Stein [D-S]) and on higher K-theory (Gersten [Ger 1]).

The problems are integrated into the text, which furnishes some relevant background. They are designated with Roman numerals, (I), (II),...,(XXV).

I am greatly indebted to several people for their comments and criticisms in drafting this list of problems. I wish particularly to thank M. Pavaman Murthy, Leslie Roberts, Tony Geramita, and David Eisenbud.

*Unless the contrary is indicated K_n here will always denote the functors K_n of Quillen [Q2].

Contents

§1. Serre's problem

Efforts to answer the following question of Serre, posed in his 1955 paper FAC ([Ser 1], p. 243) have generated many of the theorems and problems in algebraic K-theory. Because of its pedigree, and because much that follows consists of variations on the theme of Serre's problem, it seems a good place to begin.

(I) Serre's problem (on projective modules over polynomial rings). *Let* $A = k[t_1,\ldots,t_d]$, *a polynomial ring in* d *variables over a field* k. *Let* P *be a finitely generated projective* A*-module of rank* r. *Is* P *free? I.e. is* P *isomorphic to* A^r?

The moral impulse behind this question arises from the interpretation of P as (the module of sections of) a vector bundle on affine n-space k^n, which should behave like a "contractible" space, and hence have only trivial bundles. To the author's knowledge no confirmed example is yet known for which the answer to (I) is negative.* On the other hand, few people seem willing to vouch with great conviction for an

*See, however, the discussion in (7.3) below, in connection with Segre's paper [Seg].

affirmative response. Some have suggested that the answer may vary with k.

The answer to (I) is known to be affirmative in the following cases:

$\underline{d \le 1 \text{ (all } r)}$ - A is principal.

$\underline{d = 2 \text{ (all } r)}$ - Seshadri's theorem [Sesh].

$\underline{r = 1 \text{ (all } d)}$ - A is factorial

$\underline{r > d}$ - This follows from a theorem of Grothendieck plus stability theorems (see [Ba 4], Cor. (22.4)).

The first unsettled cases are $\underline{d = 3,\ r = 2 \text{ or } 3}$. We remark here that if $d = 3$ and $r = 3$ then $P \cong A \oplus P'$ for some P' of rank 2 (see [Ba 2)]. The analogue of this is not known for $d = 4$, $r = 4$.

Criteria for solving Serre's problem (sometimes in special cases) are discussed below in (4.1), problem (IX); in (4.2), Murthy's proposition; in (5.4), problem (XIV); in (5.5), both of the propositions; in (7.3), problem (XX); and in (8.2), problem $(XXI)_{d,r}$.

§2 Homotopy properties of the functors K_n

2.1. Homotopy functors

Let F be any functor from rings to abelian groups. If A is a ring and t is an indeterminate then the inclusion $A \to A[t]$ and retraction $A[t] \to A$ ($t \mapsto 0$) induces a decomposition

$$F(A[t]) = F(A) \oplus NF(A).$$

We call Fa homotopy functor if $NF(A) = 0$ for all A. In general there is a largest quotient $\bar{F}$ of F which is a homotopy functor, defined by

$$\bar{F}(A) = \operatorname{Coker}(F(A[t]) \xrightarrow{\varepsilon_1 - \varepsilon_0} F(A))$$

where $\varepsilon_i : A[t] \to A$ is the retraction defined by $\varepsilon_i(t) = i$ $(i = 0,1)$. All morphisms of F into a homotopy functor factor through $\bar{F}$ (see [Sw 1], Lem. (4.2)).

For example the functors K_n^{K-V} of Karoubi-Villamayor [K-V] are homotopy functors for $n \geq 1$, whereas $K_0^{K-V} = K_0$ is not a homotopy functor. Moreover Sharma and Strooker [S-S] have shown, curiously enough, that the exact sequence of K_n^{K-V}'s associated to a short exact sequence of rings (wihtout unit) does not remain exact in general if K_0 is replaced by $\bar{K}_0$.

Let n be an integer ≥ 1.

$(II)_n$ Does Gersten's spectral sequence ([Ger 2], Thm. 3.12) induce an isomorphism $\bar{K}_n \longrightarrow K_n^{K-V}$?

The answer to $(II)_n$ is affirmative for n = 1 and, in certain cases, for n = 2 (see Swan [Sw 1], Thm. 4.3).

2.2 (Laurent) K_n-regular rings

Let F as above be a functor from rings to abelian groups. Let A be a ring and let $t_1, t_2, \ldots, t_n, \ldots$ be indeterminantes. We say A is F-regular if $NF(A[t_1, \ldots, t_n]) = 0$ for all $n \geq 0$. We say A is Laurent F-regular if $A[t_1, t_1^{-1}, \ldots, t_n t_n^{-1}]$ is F-regular for all $n \geq 0$.*

Motivation and examples

(1) A ring A is called right regular if (i) A is right neotherian, and (ii) $hd_A(M) < \infty$ for all finitely generated right A-modules M. (Here $hd_A(M)$ denotes the projective

*This terminology relates to some others as follows: Karoubi's "K-regular" [Kl] is our "Laurent K_0-regular," and Gersten's "K-semiregular" is our "K_0-regular." Similarly, putting $P^n(A) = t_1 \ldots t_n \cdot A[t_1, \ldots, t_n]$, we would propose calling a ring homomorphism $A \to B$ a fibration if, as in Gersten [Ger 3], $GL(P^n A) \to GL(P^n B)$ is surjective for all $n > 0$, and a Laurent fibration if $A[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}] \to B[t_1, t_1^{-1}, \ldots, t_n, t_n^{-1}]$ is a fibration for all $n \geq 0$.

homological dimension of M.) Theorems of Hilbert (cf. [Ba 1], Ch XII, Thm. 2.2) imply that both conditions (i) and (ii) on A are inherited by A[t] and $A[t,t^{-1}]$, t an indeterminate. Further, results of Quillen $[Q_2]$, Thm. 11 and [Q3] establish that a right regular ring is Laurent K_n-regular for all n. (The cases n = 0,1 are treated, for example, in [Ba 1], Ch. XII.]

The essential point about right regularity, in deducing results of the above type, is that the category $\underline{\underline{H}}(A)$, of right A-modules having finite resolutions by finitely generated projective A-modules, be an <u>abelian</u> subcategory of the category of all A-modules, i.e. that it be stable under kernels, cokernels, etc. This condition, weaker than right regularity, is equivalent to the following: (i') A is right coherent (i.e. every finitely generated right ideal is finitely presented), and (ii') $hd_A M < \infty$ for all finitely presented right A-modules M. One might call such a ring <u>(right) coherently regular</u> (cf. [Wald], p. 3). Unfortunately the analogue of Hilbert's Basis Theorem fails for coherent rings. Soublin ([Soub], Prop. 18) has even given a commutative coherent A for which A[t] is not coherent, and whose global dimension is finite if one assumes a weak form of the continuum hypothesis. One might thus call a <u>stably right</u>

coherent* if, (i") $A[t_1,\ldots,t_n]$ is right coherent for all $n \geq 0$, and stably (right) coherently regular if $A[t_1,\ldots,t_n]$ is right coherent-regular for all $n \geq 0$. The results of Quillen ([Q 2] and [Q 3]) suggest that a stably right coherently regular ring is K_n-regular for all $n \geq 0$.**

Interesting examples of such rings are furnished by [C-L-L], where it is shown that a free product $A \underset{R}{*} B$ is right coherent whenever R is right noetherian and A and B are "split" R-rings which are free as left R-modules. This implies the stable right coherence of the ring R[G] over R of a free group or monoid G. Since gl dim (R[G]) $\leq$ gl dim (R) + 1, such rings will also be stably right coherently regular whenever gl dim (R) $< \infty$.

(2) Karoubi ([K 1], Part III) has shown that if A is Laurent K_0-regular then so also are ΓA (the path ring), ΩA (the loop ring), CA (the cone), and SA (the suspension). (See [K 1] for these notations.)

(3) That Laurent K_0-regularity is stronger than K_0 regularity may be seen from the following example. Let A be a reduced commutative noetherian ring of dimension one whose integral closure $\bar{A}$ is a finitely generated A-module. Let $C = \mathrm{ann}_A(\bar{A}/A)$, the conductor ideal. Consider the conditions

*Gersten [Ger 1, Prob. 24] uses the term "super-coherent."

**(Added in proof): This has recently been established by Gersten, "Homology of the linear group of free algebras," Theorem 2.10 (to appear).

(a) A/C has zero nil radical

and

(b) $h_0(A) - h_0(A/C) = h_0(\bar{A}) - h_0(\bar{A}/C)$, where, for a commutative noetherian ring B we denote by $h_0(B)$ the number of connected components of spec(B). It follows from Bass-Murthy ([B-M], Thm. 8.1) that

$$A \text{ is } K_0\text{-regular} \Longleftrightarrow \text{(a) holds}$$

and

$$A \text{ is Laurent } K_0\text{-regular} \Longleftrightarrow \text{(a) and (b) hold.}$$

In case $A = \mathbb{Z}\pi$ with π a finite abelian group of order m then ([B-M], Thm. 8.10) (a) holds iff m is square free, and (b) holds iff m is a prime power. Thus if m is square free and not a prime the ring $\mathbb{Z}\pi$ is K_0-regular but not Laurent K_0-regular.

(4) One has a natural decomposition

$$K_n(A[t,t^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus ?(A)$$

for any ring A and $n \geq 1$ (cf. [Ger 3], Thm. (2.9)). From it one deduces a similar decomposition

$$NK_n(A[t,t^{-1}]) = NK_n(A) \oplus NK_{n-1}(A) \oplus N\,?\,(A).$$

In particular

$$NK_n(A[t,t^{-1}]) = 0 \implies NK_{n-1}(A) = 0,$$

and so

Laurent K_n*-regularity of* A *implies*

Laurent K_{n-1}*-regularity of* A.

In connection with the term ?(A) above it is conjecturally explained in [Ger 1], Prob. 3.

(5) If J is a nilpotent ideal in A then $K_0(A) \to K_0(A/J)$ is an isomorphism ([Ba 1], Ch. IX, Prop. 1.3) so $NK_0(A) \to NK_0(A/J)$ is likewise an isomorphism. It follows easily that A is (Laurent) K_0-regular if and only if A/J is so. The analogous assertions for K_1 fail in general. In particular $A = \mathbb{Z}/4\mathbb{Z}$ is Laurent K_0-regular, but not K_1-regular. Apparently no converse example is known, so we ask:

(III) *Does* K_1*-regularity imply* K_0*-regularity?*
More specifically does $NK_1(A) = 0$ *imply* $NK_0(A) = 0$?

This question can be formulated more precisely, as follows: Define $f\colon K_0(A[t]) \to K_1(A[t,t^{-1}])$ by $f[P] = [P[t^{-1}],\ t\cdot 1_{P[t^{-1}]}]$. By considering localisation sequence

$$K_1A[t] \longrightarrow K_1A[t,t^{-1}] \xrightarrow{\partial} K_0A \oplus \mathrm{Nil} \quad (A)$$

(cf. [Ba 1], Ch. XII) we find that $\partial f[P] = [P_0] \in K_0A$, where $P_0 = [P/Pt]$. Further f is compatible with the augmentations $t \mapsto 1$ on $A[t,t^{-1}]$ and on $A[t]$. It follows that, in the decomposition

$$K_1A[t,t^{-1}] = K_1A \oplus N_+K_1A \oplus N_-K_1A \oplus K_0A$$

the image of f lies in $N_+K_1A \oplus K_0A$ and that f decomposes as

$$f = \mathrm{Id} \oplus f' : K_0A[t] = K_0A \oplus N_+K_0A \longrightarrow K_0A \oplus N_+K_1A,$$

whence a natural homomorphism

$$f' : NK_0A \longrightarrow NK_1A.$$

Moreover f is injective if and only if f' is injective. In question (III) we may ask, more precisely, whether f' is injective.

(6) Murthy and Pedrini ([M-P], Cor. 3.4)) have shown that if A is an affine ring over a field k then A is K_0-regular in each of the following cases:

(i) $A = k[X,Y,Z]/(X^n - YZ)$

(ii) A is the homogeneous coordinate ring of an arithmetically normal embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^n_k$.

(iii) k is algebraically closed and A is the coordinate ring of a surface X birationally equivalent to a ruled surface of genus > 0, and such that X has only rational singularities.

They conjecture that A might be K_0-regular whenever A is the coordinate ring of an affine normal surface having only rational singularities. Further, Murthy has asked to:

(IV) Find an example of a noetherian integral domain A which is factorial (or even only normal) for which $NK_0(A) \neq 0$.

In a related vein he asks:

(V) Suppose $A = \coprod_{n\geq 0} A_n$ is a graded normal integral domain finitely generated (as algebra) over a field $k = A_0$. Is $K_0(A) = \mathbb{Z}$?

Murthy remarks that Pic(A) = 0 (cf [Mur 1], Lemma 5.1). Further, put $A_+ = \coprod_{n>0} A_n$ so that the question above asks whether $K_0(A,A_+) = 0$. Taking K_1 of $A \otimes_{A_0} A_0[t,t^{-1}] = A[t,t^{-1}]$ we find

$K_0(A,A_+)$ embedded in $K_1(A[t,t^{-1}], A_+[t,t^{-1}])$. If $char(k) = p > 0$ it follows from [Ba 1], Ch. XII, Cor. 5.3 that the latter group is p-primary, and hence likewise for $K_0(A,A_+)$.

Conceivably it is reasonable in (V) to require only that A_0 be regular, and then ask whether $K_0(A_0) \to K_0(A)$ is an isomorphism.

(7) Traverso [Trav] showed that a reduced commutative noetherian ring A is Pic-regular if and only if it is "semi-normal." This, and criteria for Laurent Pic-regularity, are discussed in Pedrini's article [Ped].

The following question was raised by Sharma and Strooker in [S-S], in the case $n = 0$:

$(VI)_n$ <u>Does</u> $NK_n(A) = 0$ <u>imply that</u> $NK_n(A[t]) = 0$? <u>I.e., if</u> $K_n(A) \longrightarrow K_n(A[t_1])$ <u>is an isomorphism, does it follow that</u> $K_n(A) \longrightarrow K_n(A[t_1,t_2])$ <u>is an isomorphism</u>?

Affirming this (for all A) means that $NK_n(A) = 0$ suffices for K_n-regularity of A.

The analogous question for Pic (in place of K_n) of commutative noetherian rings has an affirmative response [Trav].

§3 Free algebras and free products

3.1 Free algebras (cf. Gersten [Ger 1], Prob. 8)

Here we formulate theorems of Gersten and Stallings about K_0 and K_1, and discuss analogues for K_n.

Let R be a commutative ring. If $R \to A$ is an R-algebra with augmentation $A \to R$ we denote its augmentation ideal by A^a. If F is a functor from rings to abelian groups the maps $R \rightleftarrows A$ furnish a natural decomposition $F(A) = F(R) \oplus F^a(A)$ for augmented R-algebras A. We shall discuss the functors $A \mapsto K_n^a(A)$.

If M is an R-module its tensor algebra $T_R(M)$ is augmented via $M \to 0$. If $M = R^{(X)}$, the free R-module on a set , then $T_R(M)$ is $R\{X\}$, the free (i.e. non commutative polynomial) algebra on the set X.

Let F be a functor as above. We say R is F-freely regular if $F^a(R\{X\}) = 0$ for all sets X. If F commutes with filtered inductive limits (as do all K_n's) then the above condition implies that $F^a(T_R(M)) = 0$ whenever M is a filtered inductive limit of free R-modules. According to D. Lazard [Laz] such inductive limits are precisely the flat R-modules.

THEOREM (Gersten): *If* $NK_1(R) = 0$ *then* R *is* K_1*-freely regular*. This can be found in [Ger 4] or [Ba 1], Ch. XII, Cor. (5.5).

COROLLARY: *Let* M *be a flat* R-*module*.

(a) *If* R *is* K_1*-regular then* $K_1(R) \to K_1(T_R(M))$ *is an isomorphism and* $T_R(M)$ *is* K_1*-regular*.

(b) *If* $R[t,t^{-1}]$ *is* K_1*-regular then* $K_i(R) \to K_i(T_R(M))$ *is an isomorphism and* $T_R(M)$ *is* K_i*-regular for* $i = 0,1$.

(c) *If* R *is Laurent* K_1*-regular then* $T_R(M)$ *is Laurent* K_i*-regular for* $i = 0,1$.

The corollary follows by applying the theorem after the base changes $R \to R[t] \to R[t,t^{-1}]$, using the fact that the tensor algebra commutes with base change, and with the aid of the natural decomposition $K_1(A[t,t^{-1}]) = K_1(A) \oplus K_0(A) \oplus ?(A)$ for any ring A.

$(VII)_n$ *Let* R *be a commutative regular ring. Is* R *then* K_n*-freely regular?* * *More generally, is it true that* R *is* K_n*-freely regular whenever* R *is* K_n*-regular?*

Gersten's theorem affirms this for $n = 0,1$. Further Gersten ([Ger 1], Prob. 8) has announced that $(VII)_n$ holds for all n when $R = \mathbb{Z}$. (Cf. the remarks in (2.2), example (1)

* (Added in proof): This has recently been established by Gersten, "Homology of the linear group of free algebras," Theorem 2.10 (to appear).

above.)

3.2 Free products (cf.Gersten [Ger 1], Prob. 24)

Let A and B be augmented R-algebras. In their free product $A \underset{R}{*} B$ the subalgebra (with unit) generated by $A^a \otimes_R B^a$ can be identified, as Stallings [Stal] has pointed out, with the tensor algebra $T_R(A^a \otimes_R B^a)$ (cf. [Ba 1], Ch. IV, §5).

Let F be a functor from rings to abelian groups. The maps $A \rightleftarrows A \underset{R}{*} B$ and $B \rightleftarrows A \underset{R}{*} B$ furnish a split epimorphism

$$F^a(A \underset{R}{*} B) \longrightarrow F^a(A) \oplus F^a(B)$$

whose kernel contains the image of

$$F^a(T_R(A^a \otimes_R B^a)) \longrightarrow F^a(A \underset{R}{*} B).$$

We shall say R is F-freely additive if the sequence

$$F^a(T_R(A^a \otimes_R B^a)) \longrightarrow F^a(A \underset{R}{*} B) \longrightarrow F^a(A) \oplus F^a(B) \longrightarrow 0$$

is exact for all augmented R-algebras A, B. The following is immediate from the definitions.

PROPOSITION: Suppose F commutes with filtered inductive limits and that R is F-freely regular and F-freely additive. Let A, B be augmented R-algebras such that $A^a \otimes_R B^a$ is a flat

R-*module. Then* $F^a(A \underset{R}{*} B) \to F^a(A) \oplus (B)$ *is an isomorphism*.

THEOREM: (Stallings [Stal]; cf. also [Ba 1], Ch.XII, Thm. 111.) *Every commutative ring* R *is* F-*freely additive for* $F = K_1$, *and hence also for* $F = NK_1, K_0, NK_0, \ldots$

The last assertion follows from the first using the base changes $R \to R[t] \to R[t,t^{-1}]$, the commutativity of free products with base change, and the usual decomposition of $K_1(C[t,t^{-1}])$ for the various rings C above.

COROLLARY: *Let* R *be Laurent* K_1-*regular (e.g. a regular ring)*. *Let* A, B *be augmented* R-*algebras such that* $A^a \otimes_R B^a$ *is a flat* R-*module. Then* $A \underset{R}{*} B$ *is (Laurent)* K_i-*regular if and only if* A *and* B *are (Laurent)* K_i-*regular, for* $i = 0,1$.

Indeed the hypotheses make available the theorem and proposition above, whence $NK_i(A \underset{R}{*} B) = NK_i(A) \oplus NK_i(B)$ and similarly after the base changes $R \to R[t] \to R[t,t^{-1}]$, etc.

$(VIII)_n$ *Is every commutative ring* R K_n-*freely additive? If not is this at least true when* R *is* K_n-*regular, or even regular?*

To allow for rings like group rings $\mathbb{Z}[G_1 \underset{H}{*} G_2]$ of amalgamated free products (cf. [Wald]) one may allow the ring R to be non commutative, and require only that the augmentation $A \to B$ be a homomorphism of R-bimodules. Then analogous questions can be proved.

§4 Projective A[t]-modules

4.1 Extended A[t]-modules

Let A be a ring and t an indeterminate. Right A[t]-modules M which are isomorphic to modules of the form $M_0[t] = M_0 \otimes_A A[t]$, for some A-module M_0, will be called extended; note then that M determines M_0 because $M_0 \cong M/Mt$. Motivated by Serre's problem one is led to ask for general conditions on A which imply that every finitely generated projective right A[t]-module is extended. A necessary condition clearly is that $K_0(A) \to K_0(A[t])$ be an isomorphism, i.e. that $NK_0(A) = 0$. This occurs, for example, if A is right regular. In the converse direction we ask:

(IX) If A is a commutative regular ring is every finitely generated projective A[t]-module extended?

Since an affirmative solution to this problem implies an affirmative solution to Serre's problem, it is perhaps most prudent to approach it by seeking a counterexample.

The need for commutativity is illustrated by the following example, taken from Ojanguren and Sridharan ([O-S], Prop. 1). Let D be a non commutative division ring, or, more generally, any ring for which free modules have invariant basis number and which contains units a, b such that $c = ab - ba$ is a unit. Let $A = D[x,y]$, a polynomial ring in two variables. The homomorphism $p: A^2 \to A$, $p(f,g) = (x + a)f - (y + b)g$, sends $\alpha = (y + b, x + a)$ to $p(\alpha) = c$, so $A^2 = \alpha A \oplus P$, where $P = \mathrm{Ker}(p)$. It is shown in [O-S] that P is not free. It projects isomorphically (in either coordinate) to a right ideal in A. On the other hand $D[x]$ is a principal right ideal domain, so all projective right $D[x]$-modules are free.

Examples. The following are examples where every finitely generated projective $A[t]$-module is known to be extended:

(1) A is a Dedekind domain. More generally, let A be a reduced* commutative noetherian ring of dimension one whose integral closure $\bar{A}$ is finite over A. Let $C = \mathrm{ann}_A(\bar{A}/A)$, the conductor. Then projective $A[t]$-modules are extended $\Leftrightarrow$ $\bar{A}/C$ is reduced. (Cf. [B-M], Cor. 9.2).

(2) A is a regular local ring of dimension ≤ 2 (cf. [Hor] and [Mur 2]).

(3) $A = k[\pi]$, the algebra over a field k of a free non commutative monoid on group π (cf. [Ba 3], or [Ba 1], Ch. IV, Cor. 6.4; to apply these results here one views

*Recall that "reduced" means "with zero nil radical." This assumption is not restrictive since, if J is a nilpotent ideal, the base change $A \to A/J$ induces a bijection on isomorphism classes of projective modules (cf. [Ba 1], Ch. III, Prop. 2.12).

$A[t]$ as $k[t][\pi]$).

4.2 The Horrocks criterion

Let A be a ring. The Laurent polynomial ring $A[t,t^{-1}]$ contains both $A[t]$ and $A[t^{-1}]$. Let P be a finitely generated projective right, $A[t]$-module. We shall say that "P extends to a locally free sheaf on $\mathbb{P}^1(A)$" if there is a finitely generated projective right $A[t^{-1}]$-module P' and an isomorphism

$$P \otimes_{A[t]} A[t,t^{-1}] \cong P' \otimes_{A[t^{-1}]} A[t,t^{-1}]$$

of $A[t,t^{-1}]$-modules. In case P is extended, say $P = P_0[t]$, then one can use $P_0[t^{-1}]$ for P' above. Horrocks [Hor] studied the converse condition:

Hor (A): If P Is a finitely generated projective right $A[t]$-module which extends to a locally free sheaf on $\mathbb{P}^1(A)$ then $P \cong P_0[t]$, where $P_0 = P/Pt$.

He established Hor(A) whenever A is a commutative noetherian local ring. This was used to show that projective $A[t]$-modules are free when A is regular local of dimension 2 (see [Hor], when A contains a field, and [Mur 2] for the general case). In [Ba 1], Ch. XII, Cor. (7.6) it is shown that, for any ring A, Hor (A) is "stably" true, i.e. P and $P_0[t]$ in the definition must be "stably isomorphic." This implies they are isomorphic if A is commutative and P has rank 1 (cf. [B-M], Thm. (6.3)).

(X) Does Hor(A) hold for every commutative noetherian ring A?

An affirmative response would solve Serre's problem, as the following new result communicated by Murthy, illustrates.

PROPOSITION (Murthy): Let k be a field and t an indeterminate. Let A be a k-algebra. Assume Hor(A) and that finitely generated projective $(k(t) \otimes_k A)$-modules are free. Then finitely generated projective A[t]-modules are free.

This follows immediately from the:

LEMMA (Murthy): Let A be any ring, and let f be a central monic polynomial in A[t]. Let P be a finitely generated projective right A[t]-module such that P[1/f] is free over A[t,1/f]. Then P extends to a locally free sheaf on $\mathbb{P}^1(A)$.

Proof of the Lemma. Let $n = \deg(f)$ and write $f(t) = t^n g(t^{-1})$. Since f is monic t^{-1} and $g(t^{-1})$ generate the unit ideal in $A[t^{-1}]$. Moreover $A[t,t^{-1},1/f] = A[t^{-1},t,1/g]$. Since $P[1/f]$ ($= P \otimes_{A[t]} P[t,1/f]$) is $A[t,1/f]$-free we can "glue" $P[t^{-1}]$ with a free $A[t^{-1},1/g]$-module (they are isomorphic over $A[t,t^{-1},1/g]$) to form a projective $A[t^{-1}]$-module P' such that $P'[t] \cong P[t^{-1}]$, whence the lemma.

§5 Stability and indecomposable projective modules

5.1 Terminology

Let A be a commutative* ring. The space spec(A) of prime ideals of A contains the subspace max(A) of maximal ideals; such spaces have dimensions measured by lengths of chains of irreducible closed sets, and we write dim(A) = dim spec(A). We have a (split) exact sequence

$$0 \longrightarrow \tilde{K}_0(A) \longrightarrow K_0(A) \xrightarrow{rk} H_0(A) \longrightarrow 0$$

where $H_0(A)$ is the ring of locally constant functions spec(A) $\to \mathbb{Z}$, and where, for a finitely generated projective module P, rk(P) sends $\wp \in$ spec(A) to the rank of the free $A_\wp$-module $P_\wp$. There is further a natural epimorphism

$$\det: \tilde{K}_0(A) \longrightarrow \mathrm{Pic}(A)$$

induced by sending P to the r^{th} exterior power of P, where $r = rk(P)$.

For each integer $r \geq 0$ let $\underline{\underline{P}}_r(A)$ denote the set of isomorphism classes (P) of finitely generated projective A-modules

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*Many of the problems and results discussed below have interesting non commutative versions; we restrict attention to commutative rings only for ease of exposition. The references cited treat the more general setting.

P of constant rank r. Define

$$s_r: \underline{\underline{P}}_r(A) \longrightarrow \underline{\underline{P}}_{r+1}(A)$$

$$s_r(P) = (P \oplus A),$$

and

$$t_r: \underline{\underline{P}}_r(A) \longrightarrow \tilde{K}_0(A)$$

$$t_r(P) = [P] - [A^r].$$

One checks easily that the maps t_r induce a bijection

$$\varinjlim_r (P_r(A), s_r) \longrightarrow \tilde{K}_0(A)$$

The following notions furnish a measure of the rapidity with which this limit is achieved. We define

(i) surj K_0-range(A)

(ii) inj K_0-range(A)

(iii) stable K_0-range(A)

(iv) ind proj(A)

(v) stable ind proj(A)

to be the least integer $n \geq 0$, or ∞ if none such exists, such that

(i) s_r is surjective for all $r \geq n$

(ii) s_r is injective for all $r > n$

(iii) t_r is surjective for all $r \geq n$

(iv) Every finitely generated projective A-module is isomorphic to a direct sum of modules of rank $\leq n$.

(v) Every finitely generated projective A-module is stably isomorphic to a direct sum of modules of rank $\leq n$,

respectively. Recall that finitely generated projective A-modules P and P' are called "stably isomorphic" if $P \oplus A^m \cong P' \oplus A^m$ for some $m \geq 0$, i.e. if $[P] = [P']$ in $K_0(A)$. Thus condition (v) is equivalent to

(v') The image of t_r additively generates $\tilde{K}_o(A)$ for $r \geq n$.

We further put

$$K_0\text{-range}(A) = \max(\text{surj } K_0\text{-range}(A), \text{ inj } K_0\text{-range}(A)).$$

The following inequalities are immediate.

$$\begin{array}{ccccc} & & K_0\text{-range}(A) & & \\ & \swarrow\!\!\leq & & \geq\!\!\searrow & \\ & \text{surj } K_0\text{-range}(A) & & & \text{inj } K_0\text{-range}(A) \\ \leq & & \geq & & \\ \text{stable } K_0\text{-range}(A) & & \text{ind proj}(A) & & \\ \geq & & \leq & & \\ & \text{stable ind proj}(A) & & & \end{array}$$

<u>Remarks</u>: (1) The choice of inequalities in the above definitions was made so that the K_0-stability theorem (see

(5.2) below) reduces to the assertion that K_0-range(A) $\leq$ d when A is commutative and max(A) is a noetherian space of dimension d.

(2) The quantity surj K_0-range(A) was considered in [B-M] and in [G-R], where it is called "Serre dim(A)," and in [L-M], where it is called the "projective modulus of A."

(3) Evidently the following are equivalent:

(a) surj K_0-range(A) = 0.

(b) Finitely generated projective A-modules of constant rank are free

(c) K_0-range(A) = 0.

(4) For dimension one we have the following equivalent conditions (cf. [Ba 1], Ch. IX, Prop. (3.7) and Cor (3.8)):

(a) surj K_0-range(A) ≤ 1

(b) (rk(P), det(P)) $\in H_0(A) \times$ Pic(A) is a complete isomorphism invariant for finitely generated projective A-modules.

(c) K_0-range(A) ≤ 1.

Further, stable K_0-range(A) ≤ 1 if and only if deg: $\tilde{K}_0(A) \to$ Pic(A) is an isomorphism.

5.2 <u>The K_0-stability theorem</u>

The basic K_0-stability theorem (for commutative rings) is the following.

THEOREM (see [Ba 1], Ch. IV, Cor. (2.7) and Cor. (3.5)): *If* max(A) *is a finite union of noetherian spaces of dimensions* $\leq$ d *then*

$$K_0\text{-range}(A) \leq d.$$

COROLLARY (cf. [Ba 4], Them. 22.1): *Let* A *be a commutative neotherian ring of dimension* d. *Suppose that* A *is* K_0*-regular (e.g. that* A *is regular). If* P *is a finitely generated projective* $A[t_1,..,t_n]$*-module of rank* $> d + n$ *then* $P \cong P_0 \otimes_A A[t_1,...,t_n]$, *where* $P_0 = P/(t_1,...,t_n)P$.

Since $K_0(A) \to K_0(A[t_1,...,t_n])$ is an isomorphism (by K_0-regularity) P is *stably* isomorphic to $P_0 \otimes_A A[t_1,...,t_n]$. Since dim $\max(A[t_1,...,t_n]) = d + n < \text{rank}$ P the theorem implies that inj K_0-range $(A[t_1,...,t_n]) < \text{rank}$ P, whence "stably isomorphic" implies "isomorphic."

In case dim (A/rad A) < d it suffices, for the conclusion of the corollary, that rank $P \geq (d + n)$ (cf. [Ba 4], Cor. 22.4).

COROLLARY: *If* k *is a field then projective* $k[t_1,...,t_n]$*-modules of rank* $> n$ *are free.*

These results suggest that, for fixed A, projective modules are easiest to handle when their ranks are large. This principle is born out by the fact that, if spec(A) is connected and A has only finitely many minimal primes then every non finitely generated projective A-module is free! (cf. [Ba 5]).

For a universal bound the d in the stability theorem is reasonably efficient, as the following examples show (see [G-R] and [Ger 2]): Given $d \geq 1$ let A_d denote the even degree part of $\mathbb{R}[t_0, \ldots, t_d]/(t_0^2 + \ldots + t_d^2 - 1)$, with respect to its natural grading mod 2. Then $\dim A_d = \dim \max (A_d) = \text{gl. dim } (A_d) = d$. Interpreting A_d as the ring of polynomial functions on real projective d-space $\mathbb{P}^d_{\mathbb{R}}$, there is an invertible A_d-module L corresponding to the canonical line bundle on $\mathbb{P}^d_{\mathbb{R}}$. A simple consideration of Stiefel-Whitney classes shows that $L \oplus \ldots \oplus L$ (d terms) is not even stably isomorphic to a module of the form $A \oplus P$. There is further a projective A_d-module T_d corresponding to the tangent bundle to $\mathbb{P}^d_{\mathbb{R}}$, and T_d is indecomposable for <u>even</u> d (see [Gera], Thm. 5). Thus ind $\text{proj}(A_d) = d$ for even d. Further examples can be found in [Sw 2].

To my knowledge, however, the examples in the literature do not yet completely respond to the following problem.

$(XI)_d$ <u>Given $d \geq 2$, exhibit a commutative noetherian ring A of dimension d, and a finitely generated projective A-module</u>

P *of rank* d, *such that* P *is not even stably isomorphic to a module of the form* P' ⊕ P" *with* P' *and* P" *of rank* < d. *In other words find an* A *as above such that stable ind proj*(A) = d.

If A is an affine algebra over a field k the response to $(XI)_d$ might depend on k, for example by being different for k = $\mathbb{R}$ or $\mathbb{C}$.

The discussions that follow are concerned with possible strengthening of the inequalities implied by the stability theorem in special circumstances.

5.3 Indecomposable projective modules

A. Geramita has asked in [Gera] whether (ind proj (A), surj K_0-range(A)) can take any pair (i,s) of values for which $1 \leq i \leq s$ (cf. also [G-R], §7). In particular he has asked:

$(XII)_d$ *Given* $d \geq 2$, *does there exist a commutative noetherian ring* A *of global dimension* d *such that* surj K_0-range (A) = d *and* ind proj (A) < d?

Murthy [Mur 1] has investigated questions germane to this in the following special setting: Let k be an algebraically closed field. Let A be the affine ring of a non singular algebraic surface V over k. Thus A is a regular ring of dimension 2. Murthy asks (cf. [Mur 1], Remark 5.5):

(XIII) *Is* ind proj $(A) \leq 1$?

The answer is negative if we drop the assumption that k is algebraically closed, as the familiar example $A = \mathbb{R}[x,y,z]/(x^2 + y^2 + z^2 - 1)$ and the indecomposable A-module $P = A^3/A\cdot(x,y,z)$ show. Murthy has remarked that if V is a product of two curves then stable ind proj $(A) \leq 1$, while the theorem below shows that stable K_0-range $(A) = 2$ if both curves have genus > 0. Thus if (XIII) is affirmative in the latter case, one has the example sought by $(XII)_2$.

For rings A as above the stability theorem implies that K_0-range $(A) \leq 2$. Murthy ([Mur 1], Thm. (3.2)) shows that K_0-range $(A) \leq 1$ if V is birationally equivalent to a ruled surface (= (a curve) $\times \mathbb{P}^1$). Results of Mumford [Mum] suggest that the converse may also be true.

5.4 Improved stability for polynomial rings

The questions here were first raised in [B-M], §9. They have recently been reconsidered and generalized by Evans and Eisenbud [E-Ei] (see also §7 below).

Let A be a commutative neotherian ring, and let n be an integer ≥ 1.

$(XIV)_n$ *Is* K_0-range $(A[t_1,\ldots,t_n]) \leq \dim A$?

When A is a field this question is equivalent to Serre's problem (I).

Put $d = \dim A$ and $P_n = A[t_1,\ldots,t_n]$. Then $\dim P_n = \dim \max (P_n) = d + n$, even though one might well have $\dim \max (A) < d$ (e.g. when $d > 0$ and A is local). The question $(XIV)_n$ naturally separates into two parts:

$(XIV)_{n,surj}$ *Is* surj K_0-range $(P_n) \leq d$?

$(XIV)_{n,inj}$ *Is* inj K_0-range $(P_n) \leq d$?

One can further ask the less stringent question

$(XV)_n$ *Is* stable K_0-range $(P_n) \leq d$?

Murthy has even asked whether one might replace d by 1 when A is a local ring, in the above questions. Of course $(XV)_n$, even in Murthy's strengthened form, has an affirmative response whenever A is K_0-regular, and the discussion in §2 describes an abundance of K_0-regular rings. The results quoted below affirm $(XIV)_n$ and $(XV)_n$ in other interesting but still quite special cases.

THEOREM: Suppose dim (A/rad A) < d. Then K_0-range (P_n) $\leq d + n - 1$.

This affirms $(XIV)_1$ and $(XV)_1$ for A as in the theorem. The theorem is a corollary of the stability theorem since $\max(P_n)$ is the union of the closed set F consisting of maximal ideals containing rad A (so that F ≅ max $((A/rad)[t_1,\ldots,t_n])$ has dimension < d + n) and the open complement which also has dimension < d + n. (cf. [Ba 1], Ch. IV, Remark after Cor. 2.7.) This result has been generalized by Evans-Eisenbud in [E-E 1].

THEOREM ([B-M], Thms. 7.8 and 9.1). Suppose that $d \leq 1$ and that the integral closure of A_{red} = A/nil rad (A) is a finitely generated A-module. Let B denote either P_n or L_n $A[t_1,t_1^{-1},\ldots,t_n,t_n^{-1}]$.

(a) stable K_0-range (B) ≤ 1

(b) We have K_0-range (B) ≤ 1 if either n = 1, or n = 2 and A is semi-local.

Part (a) affirms $(XV)_n$, and part (b) affirms $(XIV)_n$ for

A and n as in (a), resp. (b). It is very likely, but apparently not yet known, whether $(XIV)_1$ has an affirmative response whenever $d \leq 1$, i.e. without some assumption like the finite generation of the integral closure of A_{red}

5.5 The use of bilinear forms

Let A be a commutative ring. Let P be a finitely generated projective A-module, and let L be an invertible A-module. It is observed in [Ba 2], Prop. 4.1, that if $P \oplus L$ admits a non singular alternating bilinear form then P has a direct summand isomorphic to $L^* = \mathrm{Hom}_A(L,A)$. It follows, in particular that

PROPOSITION: $P \oplus A \cong A^{2n} \Rightarrow P \cong P' \oplus A$ for some P'.

Combining this with the second corollary of the stability theorem above (in (5.2)) we obtain:

COROLLARY: If k is a field and if $n \geq 1$ is an integer then a projective $k[t_1,\ldots,t_{2n-1}]$-module of rank $2n-1$ has a free direct summand of rank 1, whence surj K_0-range $(k[t_1,\ldots,t_{2n-1}])$ $\leq 2n - 2$.

To treat Serre's problem in three variables one can further use symplectic K-theory (see [Ba 6]) as follows.

PROPOSITION: Let A be a commutative noetherian ring of dimension ≤ 3. If $K_0(A) \xrightarrow{rk} \mathbb{Z}$ is an isomorphism then

(a) surj K_0-range (A) ≤ 2; and

(b) All finitely generated projective A-modules are self-dual.

If further $KSp_0(A) \xrightarrow{rk} 2\mathbb{Z}$ is an isomorphism then

(c) All finitely generated projective A-modules are free if and only if $Sp_4(A)$ acts transitively on the set of unimodular elements in A^4.

See [Ba 6] for the notation.

Proof: Let P be a projective A-module of rank r. Then hypotheses and the K_0-stability theorem imply P is free if $r > 3$. The proposition above then implies, if $r = 3$, that $P \cong A \oplus P'$, whence (a). Suppose $r = 2$. Then $\det(P) = \Lambda^2 P$ in Pic(A) is trivial because $\tilde{K}_0(A) = 0$. It follows then from [Ba 2], Prop. 4.4 that P admits a non singular alternating form h. In particular $P \cong P^*$, whence (b). The symplectic module (P,h) is stably hyperbolic if $KSp_0(A) \xrightarrow{\cong} 2\mathbb{Z}$, so it follows from the symplectic stability theorem ([Ba 6] Ch. IV, Cor. 4.15) that $(P,h) \perp H(A) \cong H(A^2) = H(A) \perp H(A)$. If an element σ of $Sp_4(A)$ carries the orthogonal complement of (P,h) to a standard hyperbolic plane then $(P,h) \cong H(A)$ so $P \cong A^2$. Such a σ exists provided $Sp_4(A)$ acts transitively on unimodular elements in A^4 (Cf. [Ba 6], Ch. I, Cor. 5.6), whence one implication of (c). Conversely if A is any commutative ring for which all projective modules are free then all symplectic modules are hyperbolic, clearly, and so $Sp_{2n}(A)$ acts transitively on unimodular elements in A^{2n} for all n. Thus the proposition is proved.

The above proposition applies notably in the following case: Suppose A = B[t] where B is a regular ring of dimension 2 for which all projective modules are free. Then all symplectic B-modules are hyperbolic also. Further $K_0(B) \overset{\cong}{\to} K_0(A)$ and, according to Karoubi [K 2], if 2 is invertible in B, we also have $KSp_0(B) \overset{\cong}{\to} KSp_0(A)$.

Thus, if k is a field of characteristic $\neq 2$ the special case $(I)_{3,r}$ of Serre's problem is equivalent to the problem:

$(I')_{3,r}$ <u>If</u> $A = k[t_1 t_2 t_3]$ <u>does</u> $Sp_4(A)$ <u>act transitively on unimodular elements in</u> A^4?

Another influence of bilinear forms on the structure of projective modules is given by the following consequence of [Ba 2], Cor. 5.2.

PROPOSITION: <u>Let</u> A <u>be a factorial ring in which</u> 2 <u>is a square. Let</u> P <u>be a projective</u> A-<u>module of rank</u> 2. <u>Then</u>

P <u>is free if and only if</u> P <u>supports a non singular symmetric bilinear form</u>.

This applies notably when $A = k[t_1,\ldots,t_n]$ with k an algebraically closed field of characteristic $\neq 2$.

5.6 <u>Lissner-Moore extensions</u>

There is another situation where the surj K_0-range can be significantly improved. It is an algebraic analogue, discovered by Lissner and Moore [L-M], of the fact in topology that the stable range for complex vector bundles is half that for real vector bundles. We indicate here an abstraction of their arguments. (Another has been given by Simis [Sim].)

A Triple (A_0, A, θ) consisting of a commutative ring A_0, a commutative A_0-algebra A, and an element $\theta \in A$, will be called a <u>Lissner-Moore extension of degree</u> d

(i) $1, \theta, \ldots, \theta^{d-1}$ is a free basis of A as A_0-module.

and

(ii) If $b = a_0 + a_1\theta + \ldots + a_{d-1}\theta^{d-1}$ with all $a_i \in A_0$, and if a_{d-1} is invertible in A_0, then b is invertible in A.

<u>Example</u>. If $A = A_0[\theta]$ is a field extension of degree d of a field A_0 then (A_0, A, θ) is a Lissner-Moore extension of degree

d. We shall see less trivial examples below.

THEOREM: Let (A_0, A, θ) be a Lissner-Moore extension of degree d. Then

$$\text{surj } K_0\text{-range } (A) \leq \frac{1}{d} (\text{surj } K_0\text{-range } (A_0))$$

COROLLARY. If surj K_0-range $(A_0) < d$ then projective A-modules of constant rank are free.

The proof of the theorem is based on the lemma below. If M, N are A-modules let M_0, N_0 denote the underlying A_0-modules (restriction of scalars). Suppose $f_0 \in \text{Hom}_{A_0}(M_0, N_0)$. Define $f: M \to N$ by

$$f(m) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d-1}} d_{i+j+1} \, \theta^i \, f_0(\theta^j m),$$

where the $c_h \in A_0$ are defined by the equation

$$c_0 + c_1\theta + \ldots + c_{d-1}\theta^{d-1} + \theta^d = 0,$$

whose existence (and uniqueness) results from (i) above. Allowing ourselves to put scalars on the right in N we have

$$(*) \qquad f(m) = \sum_{i=0}^{d-1} \left(\sum_{j=0}^{d-1-i} c_{i+j+1} \, f_0(\theta^j m) \right) \theta^i,$$

so that the coefficient of θ^{d-1} is just $f_0(m)$.

LEMMA: Assuming only condition (i) above, the map $f: M \to N$ is A-linear.

Evidently f is A_0-linear, so we need only check that $f(\theta m) = \theta f(m)$ for $m \in M$.

$$f(\theta m) = \sum_{\substack{i,j\geq 0 \\ i+j\leq d-1}} c_{i+j+1}\theta^i f_0(\theta^{j+1}m)$$

$$= \sum_{j=0}^{d-1} c_{j+1} f_0(\theta^{j+1}m)) + (\sum_{\substack{u,v>0 \\ u+v\leq d}} c_{u+v}\theta^u f_0(\theta^v m)).$$

Similarly

$$\theta f(m) = \sum_{\substack{i,j\geq 0 \\ i+j\leq d-1}} c_{i+j+1}\theta^{i+1} f_0(\theta^j m)$$

$$= (\sum_{i=0}^{d-1} c_{i+1}\theta^{i+1} f_0(m)) + \sum_{\substack{u,v>0 \\ u+v\leq d}} c_{u+v}\theta^u f_0(\theta^v m)$$

Since $\sum_{i=0}^{d-1} c_{i+1}\theta^{i+1} f_0(m) = -c_0 f_0(m) = f_0(-c_0 m)$ $= f_0(\sum_{j=0}^{d-1} c_{j+1}\theta^{j+1} m) = \sum_{j=0}^{c-1} c_{j+1}f_0(\theta^{j+1}m)$ the lemma follows.

Remark. If f_0 is already A-linear then one can check that

$f = \varphi'(\theta) f_0$, where $\varphi'(\theta) = \sum_{h \geq 1} hc_h \theta^{h-1}$.

Proof of theorem. Let P be a projective A-module of rank r, and suppose surj K_0-range $(A_0) = n$. Assuming $r > \frac{n}{d}$ we must show that there is an $x \in P$ and an A-linear map $f: P \to A$ such that f(x) is invertible. Since A is free of rank d over A_0 the projective A_0-module P_0 has rank $rd > n$. By hypothesis therefore there is an $x \in P$ and an A_0-linear map $f_0: P_0 \to A_0 \subset A$ such that $f_0(x) = 1$. Let $f: P \to A$ be the corresponding A-linear map constructed above. Since $f_0(P) \subset A_0$ the formula (*) above shows that

$$f(x) = a_0 + a_1\theta + \ldots + a_{d-1}\theta^{d-1}$$

with $a_i \in A_0$ and $a_{d-1} = f_0(x) = 1$, whence, by condition (ii) (in the definition of Lissner-Moore extension), f(x) is invertible.

Starting from a Lissner-Moore extension (A_0, A, θ) as above, we can (following the ideas of [L-M]) construct new ones as follows. Let B_0 be a commutative A_0-algebra, and put $B = B_0 \otimes_{A_0} A$, so that $1, \theta, \ldots, \theta^{d-1}$ is a B_0-basis of B. We can identify $\mathrm{Hom}_{A_0\text{-alg}}(B_0, A_0)$ with

$$X = \{x \in \mathrm{Hom}_{A\text{-alg}}(B,A) \mid x(B_0) \subset A_0\}$$

If $x \in X$ and $b \in B$ write b(x) in place of the usual x(b). Fix any non empty subset Y of X and put

$$S_0 = \{b_0 \in B_0 | b_0(y) \text{ is invertible in } A_0 \text{ for all } y \in Y\}$$

$$S = \{b \in B | b(y) \text{ is invertible in } A \text{ for all } y \in Y\}$$

Put $C_0 = B_0[S_0^{-1}]$ and $C = B[S^{-1}]$.

PROPOSITION: <u>We have</u> $C = B[S_0^{-1}]$ <u>and</u> (C_0, C, θ) <u>is a Lissner-Moore extension of degree</u> d.

For the first assertion we need only show that if $b \in S$ then b is invertible in $B[S_0^{-1}]$. Since $B[S_0^{-1}$ is a free $B_0[S_0^{-1}]$-module with basis $1, \theta, \ldots, \theta^{d-1}$ the invertibility of (multiplication by) b in $B[S_0^{-1}]$ is equivalent to that of its determinant, $N(b) \in B_0[S_0^{-1}]$. Now $N(b) = N_{B/B_0}(b) \in B_0$, and if $y \in Y$ we have $N_{B/B_0}(b)(y) = N_{A/A_0}(b(y))$ clearly. By the assumption that $b \in S$, the element $b(y)$ is invertible in A, whence $N_{A/A_0}(b(y))$ is invertible in A_0 (for all $y \in Y$), whence $N_{B/B_0}(b) \in S_0$, whence $N(b)$ is invertible in $B_0[S_0^{-1}]$, as claimed. We now show that (C_0, C, θ) is a Lissner-Moore extension. Condition (i) has already been observed above. To verify (ii) suppose given $c = b_0 + b_1\theta + \ldots + b_{d-1}\theta^{d-1}$ with $b_i \in C_0$ and b_{d-1} invertible in C_0. We must show that c is invertible in C. After multiplying by an element of S_0 we may further assume all $b_i \in B_0$ so $c \in B$. If $y \in Y$ then $c(y) = b_0(y) + b_1(y)\theta + \ldots + b_{d-1}(y)\theta^{d-1}$ and $b_{d-1}(y)$ is invertible in A_0. Hence $c(y)$ is invertible in A by condition (ii) for (A_0, A, θ). Thus $c \in S$, so c is invertible in $C = B[S^{-1}]$, whence the proposition.

To illustrate how these results are applied (as in [L-M] consider the case $(A_0, A, \theta) = (\mathbb{R}, \mathbb{C}, \sqrt{-1})$, and let B_0 be the affine ring of some real algebraic variety, say of dimension n, whose real points may be identified with X. Then $B = \mathbb{C} \otimes_{\mathbb{R}} B_0$ maps to the ring $\mathbb{C}(X)$ of complex valued functions on X, and (taking Y above to be all of X) the set S consists of those $b \in B$ which vanish nowhere on X. It follows from the theorem and proposition above that surj K_0-range $(B[S^{-1}]) \leq \frac{n}{2}$, whereas dim max $(B[S^{-1}]) = n$ in general (cf. [L-M]). As a special case one may take $B = \mathbb{R}[t_1, \ldots, t_n]$, in which case S consists of real polynomials in n variables with no real zeros, e.g. 1 + (a sum of squares).

§6 K_n-stability

6.1 Formulation of the problem

Our discussion here overlaps somewhat with Gersten's ([Ger 1], Prob. 2).

Let A be a ring. Let n be an integer ≥ 3. Then the normal subgroup $E_n(A)$ of $GL_n(A)$ generated by all elementary matrices is perfect. Let $f_n: BGL_n(A) \to BGL_n^+(A)$ be the acyclic map such that Ker $\pi_1(f_n) = E_n'(A)$. Then we have maps

$$s_n: BGL_n^+(A) \longrightarrow BGL_{n+1}^+(A)$$

and

$$t_n: BGL_n^+(A) \longrightarrow BGL^+(A) \quad ,$$

the latter inducing an isomorphism $\varinjlim_n (BGL_n^+(A), s_n) \to BGL^+(A)$.
In analogy with §5, we say

(i) surj K_i-range (A) $\leq$ n

(ii) inj K_i-range (A) $\leq$ n

(iii) stable K_i-range (A) $\leq$ n

if

(i) $\pi_i(s_r)$ is surjective for $r \geq n$

(ii) $\pi_i(s_r)$ is injective for $r > n$

(iii) $\pi_i(t_r)$ is surjective for $r \geq n$,

respectively. By suitably modifying the above constructions one should be able to extend these definitions to the cases $n = 1$ or 2 as well as $n \geq 3$. Then the least n for which the above condition holds defines the corresponding quantity, and we put

$$K_i\text{-range}(A) = \text{mas}(\text{surj } K_i\text{-range }(A),\ \text{inj } K_i\text{-range }(A))$$

The K_1-stability theorem for commutative rings is:

THEOREM (see [Ba 1], Ch. V , and Wasserstein [Was]):
Let A *be a commutative ring such that* max (A) *is a noetherian space. Then*

$$K_1\text{-range }(A) \leq \dim \max\ (A) + 1$$

Moreover the surjective K_2-stability theorem of Dennis implies:

THEOREM (Dennis [Den]): *With* A *as above we have*

$$\text{surj } K_2\text{-range }(A) \leq \dim \max\ (A) + 2.$$

It seems reasonable to *conjecture*, for $i \geq 2$:

$(XVI)_i$ *If* A *is a commutative noetherian ring* dim max (A) = d *then*
K_i-range (A) $\leq d + i$

If a theorem of this type can be established then it would be natural to seek refinements in special cases along the lines of the discussion in §5 for $i = 0$. At the moment $(XVI)_i$ seems rather difficult for large i, though Quillen's results in [Q 4] give some evidence for it in case A is a Dedekind ring.

An alternative, and perhaps more natural, formulation of the stability problem for higher K-functors has been given by Wagoner in [Wag].

6.2 A comparison with topological stability

In topology one has $K^{-n}(X) = \tilde{K}^0(S^n X)$, so one deduces a K^{-n}-stability theorem for X by applying the K^0-stability theorem to $S^n X$. One can imitate this argument using the Nobile-Villamayor suspension SA of a ring A. It is defined by the cartesian square

$$\begin{array}{ccc} SA & \longrightarrow & A[t] \\ \downarrow & & \downarrow p \\ A & \xrightarrow{\ \Delta\ } & A \times A \end{array}$$

where $\Delta(a) = (a,a)$ and $p(f) = (f(0),f(1))$.* Since p is surjective we can apply Milnor's fibre product theorem (cf. [Ba 1], Ch. IX, Thm. (5.1)). It yields the following

---

*In subsequent terminology this has become the "loop ring" ΩA, augmented by the "unit" A.

parametrization of the set G_n of isomorphism classes of projective SA-modules P such that $P \otimes_{SA} A \cong A^n$ and $P \otimes_{SA} A[t] \cong A[t]^n$: Let $GL_n(A[t])$ act on $GL_n(A)$ by

$$\beta * \alpha = \beta(0)\ \alpha\ \beta(1)^{-1}$$

for $\alpha \in GL_n(A)$ and $\beta \in GL_n(A[t])$. Then there is a natural bijection

$$G_n \longrightarrow GL_n(A)/GL_n(A[t])$$

where the quotient is by the action * above. Note that this quotient factors through the quotient group

$$GL_n(A)/U_n(A)$$

where $U_n(A)$ denotes the subgroup (which is normal) in $GL_n(A)$ generated by all unipotent matrices $I + \nu \in GL_n(A)$. (We simply use $\beta = I + t\nu$ to see this.) Since $U_n(A)$ contains $E_n(A)$ the sets above are quotients of the sets $GL_n(A)/E_n(A)$ which converge to $K_1(A)$.

Suppose now that A is commutative. Since the inverse image of maximal ideals by p and Δ are again maximal it results from ([Ba 1], Ch. IX, Prop. 5.11) that

$$\begin{array}{ccc} \max(SA) & \longleftarrow & \max A[t] \\ \uparrow & & \uparrow \\ \max(A) & \longleftarrow & \max(A \times A) \end{array}$$

is cartesian in the category of topological spaces. It follows that max (SA) is noetherian, and that

$$\dim \max(A[t]) = \dim A[t] = 1 + \dim A.$$

Thus we conclude from the K_0-stability theorem for SA: *If* A *is noetherian of dimension* d *then the maps*

$$s_n: GL_n(A)/GL_n(A[t]) \longrightarrow GL_{n+1}(A)/GL_{n+1}(A[t])$$

are surjective for $n \geq d + 1$ *and injective for* $n > d + 1$.

This is weaker than the known K_1-stability theorem above in two respects: (i) the quotient $GL_n(a)/GL_n(A[t])$ is smaller than $GL_n(A)/E_n(A)$; and (ii) $d = \dim A$ is larger, in general, than dim max (A). On the other hand the above arguments presumably give a stability theorem similar to that above for the higher K-functors of Karoubi-Villamayor. We have not attempted to articulate it precisely.

§7 Efficient generation of noetherian modules and ideals

7.1 Basic elements and stability theorems

The stability theorems for projective modules have been extended in various ways to non projective modules. Recently Eisenbud and Evans [E-Ei] have given a coherent and systematic treatment of these results, and raised some questions analogous to some of those in §5 above. We shall summarize here some of these results and questions, referring the reader to Eisenbud-Evans for more details and references.

Let A be a commutative noetherian ring. Let M be a finitely generated A-module. We define

$\mu(A,M)$ = the least cardinal of a generating set of M.

If $x \in M$ and if $\mathfrak{p} \in \operatorname{spec}(A)$ we say x is $\mathfrak{p}$ -basic in M if $\mu(A_{\mathfrak{p}}, (M/Ax)_{\mathfrak{p}}) < \mu(A_{\mathfrak{p}}, M_{\mathfrak{p}})$. By Nakayama's lemma this is equivalent to the condition: $x \notin \mathfrak{p} M_{\mathfrak{p}}$. We call x basic in M (resp., M-basic) if x is $\mathfrak{p}$-basic for all $\mathfrak{p}$ (resp., for all $\mathfrak{p} \in \operatorname{supp}(M)$).

Remarks.

(1) ([E-El], Lem. 1). If M is projective then x is basic if and only if x is unimodular in M, i.e. x

generates a free direct summand of rank 1.

(2) (cf [E-E1], proof of Cor. 7). Suppose I is an ideal in A and $M = I \oplus \ldots \oplus I$ (n terms). Say $x = (a_1, \ldots, a_n) \in M$ and put $I_0 = Aa_1 + \ldots + Aa_n \subset I$. Then

(a) x is M-basic,

is equivalent to,

(b) $I_{0\mathscr{J}} \not\subset \mathscr{J} I_{\mathscr{J}}$ for all $\mathscr{J}$ containing $\mathrm{ann}_A(I)$,

and implies

(c) $\sqrt{I_0} = \sqrt{I}$.

In view of (1) the following result generalizes Serre's theorem (that surj K_0-range (A) $\leq$ dim max (A)).

THEOREM (Eisenbud-Evans [E-E 1], Thm. A): <u>If</u> $\mu(A_{\mathscr{J}}, M_{\mathscr{J}}) >$ dim max (A) <u>for all</u> $\mathscr{J}$ <u>then</u> M <u>contains a basic element</u>.

Actually a stronger result is proved, from which, among others, the following corollaries are deduced.

COROLLARY 1 (Forster-Swan):

$$\mu(A,M) \leq \max_{\mathscr{J} \in \mathrm{supp}(M)} (\mu(A_{\mathscr{J}}, M_{\mathscr{J}}) + \dim\max(A/\mathscr{J}))$$

COROLLARY 2: <u>Let</u> I <u>be an ideal of</u> A. <u>Put</u> $d = \dim\max(A/\mathrm{ann}_A(I))$.

(a) <u>If</u> $\mu(A_{\mathscr{J}}, I_{\mathscr{J}}) \leq m$ <u>for all</u> $\mathscr{J}$ <u>then</u>

$\mu(A,I) \leq \max(d + 1, m + \dim\max(A/I))$

(b) <u>There exist</u> (d+1) <u>elements</u> $a_0, \ldots, a_d \in I$ <u>such that</u>

putting $I' = Aa_0 + \ldots + Aa_d$, we have $I'_{\mathfrak{p}} \not\subset \mathfrak{p} I_{\mathfrak{p}}$ for all $\mathfrak{p}$ containing $\mathrm{ann}_A(I)$. In particular $\sqrt{I'} = \sqrt{I}$.

Part (b) sharpens slightly a classical theorem of Kronecker.

7.2 Conjectural improvements for polynomial rings

Let A be a commutative noetherian ring of dimension **d.** We assume that A is a polynomial ring over some other ring (in at least one variable). In (5.4) we have asked in particular:

$(XIV)_1$ Is K_0-range(A) < d?

In view of their theorem above, Eisenbud-Evans strengthen the condition "surj K_0-range (A) < d" part of $(XIV)_1$ in conjecturing [E-E 3]:

(XVII) If M is a finitely generated A-module such that $\mu(A_{\mathfrak{p}}, M_{\mathfrak{p}}) \geq d$ for all $\mathfrak{p}$ then M contains a basic element.

The following corollary of (XVII) has been proved:

THEOREM ([E-E 2]: _If_ I _is an ideal in_ A _there exist_ d _elements_ $a_1, \ldots, a_d \in I$ _such that, putting_ $I' = Aa_1 + \ldots + Aa_d$, _we have_ $I'_{\mathfrak{p}} \not\subset I_{\mathfrak{p}}$ _for all_ $\mathfrak{p}$ _containing_ $\mathrm{ann}_A(I)$. _In particular_ $\sqrt{I'} = \sqrt{I}$.

Eisenbud-Evans further conjecture the following sharpening of the Forster-Swan Theorem (Cor. 1 above).

(XVIII) _Let_ M _be a finitely generated_ A-_module. Then_

$$\mu(A,M) = \max_{\mathfrak{p}}(\mu(A_{\mathfrak{p}}, M_{\mathfrak{p}}) + \dim \max(A/\mathfrak{p}))$$

where $\mathfrak{p}$ _ranges over all primes for which_ $\dim \max(A/\mathfrak{p}) < d$.

They show in [E-E 3] that (XVIII) is valid if M is a projective module of rank one. They also establish their conjectures in the following case, related to the theorem in (5.4) above.

THEOREM ([E-E 3]).: _Suppose_ $A = B[t_1, \ldots, t_n]$ _with_ $n > 0$ _and_ B _semi-local of dimension_ > 0. _Then_ $(XIV)_1$, (XVII) _and_ (XVIII) _are all affirmed_.

In the case of ideals (XVIII) has the following consequence as one checks easily.

PROPOSITION: *Let* I *be an ideal in* A. *Put* $m(I) = \max_{\wp} \mu(A_{\wp}, I_{\wp})$. *Then* (XVIII) *for* I *implies that*

$$\mu(A,I) \leq \max (d, m(I) + \dim \max (A/I)).$$

If I *is a maximal ideal then* (XVIII) *for* I *is equivalent to the condition* $\mu(A,I) < \max(d, m(I))$.

Some very interesting special cases of (XVIII) have been verified in a sharper form, by Murthy (cf. [Mur 3] and [Mur 1], Prop. (4.1)).

THEOREM (Murthy): *Let* A *be a commutative noetherian ring of global dimension* d. *Assume either* $d = 2$ *and* K_0-range $(A) \leq 1$, *or* $d = 3$ *and* $\tilde{K}_0(A) = 0$. *Then an unmixed ideal of* A *locally generated by* m *elements can globally be generated by* $m + (d - 2)$ *elements*.

Remarks. (1) Murthy's hypotheses are inherited by rings of fractions (of the same dimension as A).

(2) The case $d = 2$ applies notably when $A = D[t]$ with D a Dedekind domain. In the case $d = 2$ the theorem implies that every prime ideal can be generated by ≤ 2 elements.

7.3 Complete intersections in affine 3-space

Let $a = k[t_1,t_2,t_3]$, a polynomial ring in 3 variables over a field k. Let $\mathcal{G}$ be a prime ideal of A such that $A/\mathcal{G}$ is a Dedekind domain, and hence the affine ring of a non singular irreducible algebraic curve C in affine 3-space k^3.

According to Murthy's theorem in (7.2) above, $\mathcal{G}$ can be generated by ≤ 3 elements. In general $\mathcal{G}$ cannot be generated by 2 elements, however, but the following classical problem is still open:

(XIX) <u>Is $\mathcal{G}$ the radical of an ideal with ≤ 2 generators, i.e. is C a set theoretic complete intersection in k^3?</u>

We also have the related question posed by Serre [Ser 3]:

(XX) <u>Suppose k is algebraically closed and that C has genus 0 or 1. Is $\mathcal{G}$ then generated by two elements, i.e. is C then an ideal theoretic complete intersection?</u>

Serre points out that the answer to (XX) is affirmative provided that all projective A-modules of rank 2 are free (in which case all projective A-modules are free, by the results quoted in §1).

Segre in [Seg] claims to furnish a negative solution to (XX), and consequently also to Serre's problem $(I)_{3,2}$. However, Abyankhar has indicated there are some serious deficiencies both in the statements of Segre's results, and in his method of proof. According to Abyankhar's testimony one should not regard [Seg] as essentially altering the open status of (XX).

§8 Symmetric and affine algebras

8.1 Cancellation for affine varieties

Murthy has raised the following general question about affine varieties X, Y over a field k:

(1) Does $X \times k \cong Y \times k$ imply $X \cong Y$?

He has obtained partial affirmative results when X is a non singular surface and k is algebraically closed of characteristic zero.

The cases when Y is an affine space k^r has some formal resemblance to Serre's problem (cf. (8.3) below). Murthy remarks that these cases would be solved affirmatively if k has the property:

$(2)_{s,r}$ Any algebraic action of the torus $(k^*)^s$ on the affine r-space k^r is equivalent to a linear action.

For then the variety of fixed points would again be an affine space. Since $X \times 0$ is the variety of fixed points of the obvious action of k^* on $X \times k$ we thus conclude that

$X \times k \cong k^{r+1} \Rightarrow X \cong k^r$, provided $(2)_{1,r+1}$ holds. This approach to problem (1) is suggested by a result of Byalinicki-Birula [B-B] which establishes $(2)_{r,r+1}$ for all r.

In case $k = \mathbb{C}$ and $Y = \mathbb{C}^2$ a problem related to (1) has been treated by Ramanujam [Ram].

If in problem (1), we denote the affine algebras of X, Y by A,B, respectively, we can rephrase (1) as follows:

(1') Does $A[t] \cong B[t]$ imply $A \cong B$?

Here t is an indeterminate, and the isomorphisms are of k-algebras. Problem (1') motivates the notions discussed next in (8.2).

8.2 Invariance of the coefficient algebras in polynomial algebras.

Let k be a commutative ring. Let A be a k-algebra. We assume all k-algebras here to be commutative, though much of the discussion applies without this restriction (cf. [B-R], for example). One says the k-algebra A is n-invariant if

$$A[t_1,\ldots,t_n] \cong B[t_1,\ldots,t_n] \Longrightarrow A \cong B,$$

whenever B is a k-algebra. Here $t_1,\ldots,t_n$ are indeterminates, and "$\cong$" signifies k-algebra isomorphism.

$(XXI)_{d,r}$ <u>Suppose</u> $k = k_0[s_1,\ldots,s_d]$ <u>is a polynomial algebra in</u> d <u>variables over a field</u> k_0. <u>Let</u> $A = k[t_1,\ldots,t_r]$ <u>be a polynomial algebra in</u> r <u>variables over</u> k. <u>Is the</u> k-<u>algebra</u> A n-<u>invariant for all</u> $n > 0$?

We shall see below in (8.3) Remark (2), that an affirmative solution to $(XXI)_{d,r}$ implies an affirmative solution to Serre's problem $(I)_{d,r}$.

Many interesting examples of k, A for which A is n-invariant for all $n > 0$ can be found in [A-H-E] as well as the several references cited in that paper. In most of their examples A has relative Krull dimension one over k.

8.3 <u>Symmetric algebras</u> (cf. [Hoch])

As above, let k be a commutative ring. Let P be a k-module and $S_k(P)$ its symmetric algebra. The kernel of the augmentation $e_P: S_k(P) \to k$ $e_P(P) = 0$, will be denoted $J(P)$. Evidently the module $J(P)/J(P)^2$ over $S_k(P)/J(P) = k$ is canonically isomorphic to P itself. Let $e: S_k(P) \to k$ be any other augmentation, and put $J = \mathrm{Ker}(e)$. The k-algebra endomorphism α of $S_k(P)$ defined by $\alpha(p) = p - e(p)$ for $p \in P$ is an automorphism (with inverse induced by $p \mapsto p + e(p)$ for $p \in P$). Clearly $\alpha(J(P)) \subset J$, whence $\alpha(J(P)) = J$. It follows that J/J^2 and $J(P)/J(P)^2 \cong P$ are isomorphic k-modules. This observation immediately implies:

PROPOSITION: _Let_ P _and_ Q _be_ k-_modules. Then_ $S_k(P) \cong S_k(Q)$ (_as_ k-_algebras_) $\Leftrightarrow P \cong Q$ (_as_ k-_modules_).

Let P and F be k-modules. We have

$$S_k(P \oplus F) \cong S_k(P) \otimes_k S_k(F) \cong S_{S_k(P)}(S_k(P) \otimes_k F).$$

If F is free with basis $t_1,\ldots,t_n$ then $S_k(F) = k[t_1,\ldots,t_n]$, the polynomial algebra, and similarly $S_k(P \oplus F) = S_k(P)[t_1,\ldots,t_n]$.

COROLLARY: _Let_ P, Q _be_ k-_modules. Assume the_ k-_algebra_ $S_k(Q)$ _is_ n-_invariant. Then_

$$P \oplus k^n \cong Q \oplus k^n \Longrightarrow P \cong Q.$$

For in view of the above remarks an isomorphism $P \oplus k^n \cong Q \oplus k^n$ leads to a k-algebra isomorphism $S_k(P)[t_1,\ldots,t_n] \cong S_k(Q)[t_1,\ldots,t_n]$, whence $S_k(P) \cong S_k(Q)$ if $S_k(Q)$ is n-invariant, and so, by the Proposition, $P \cong Q$.

Remarks. (1) Suppose $Q = k^r$ and $P \oplus k^n \cong k^{r+n}$ whereas $P \not\cong k^r$. Then the argument above shows that $k[t_1,\ldots,t_r]$ is not n-invariant. This is the observation used by Hochster [Hoch] to produce algebras which are not n-invariant.

(2) Suppose $k = k_0[s_1,\ldots,s_d]$, $Q = k^r$, and $A = S_k(Q) = k[t_1,\ldots,t_r]$ as in $(XXI)_{d,r}$. Let P be a projective k-module of rank r. Then it follows from the results cited in (5.2) (Corollary to the K_0-stability theorem) that $P \oplus k^n \cong Q \oplus k^n$ if $n > d - r$. Thus it follows from the corollary above that $P \cong k^r$ provided that A is n-invariant. This explains the relationship $(XXI)_{d,r}$ to Serre's problem $(I)_{d,r}$.

§9 Finiteness questions

9.1 Rings of finite type

If A is a right noetherian ring then $G_n(A) = K_n(\text{Mod } f(A))$, the Quillen K_n-group of the category Mod f(A) of finitely generated right A-modules (cf. [Q. 2] or [Q 3]). There is a canonical "Cartan" homomorphism $K_n(A) \to G_n(A)$ which is an isomorphism if A is right regular (loc. cit.)

We ask here whether the groups $G_n(A)$ are finitely generated* under reasonable finiteness assumptions on A.

$(\text{XXII})_n$ Let A be a finitely generated commutative $\mathbb{Z}$-algebra. Is $G_n(A)$ finitely generated?

(XXIII) Is $G_0(A)$ finitely generated whenever A is a finitely generated commutative R-algebra, where R is either $\mathbb{Z}$ or a field finitely generated (as a field) over its prime field?

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*More generally, we might ask if they are "F-finitely generated," i.e. whether $F \otimes G_n(A)$ is a finitely generated F-module, for $F = \mathbb{Q}, \mathbb{Z}_p, \mathbb{F}_p, \ldots$

$(XXIV)_m$ <u>Let A be a (not necessarily commutative) ring finitely generated as a $\mathbb{Z}$-module. Are $G_n(A)$ and $K_n(A)$ finitely generated? Is the kernel of $K_n(A) \to G_n(A)$ a torsion group?</u>

$(XXV)_{n(>0)}$ <u>Let A be a finite ring. Is $K_n(A)$ finite?</u>

<u>Remarks</u>

(1) <u>Orders</u>

The most far reaching result toward $(XXII)_n$ and $(XXIV)_n$ is Quillen's theorem that $G_n(A)$ is finitely generated when A is the ring of integers in a number field [Q 4]. This relies on work of Borel and Serre on the cohomology of arithmetic groups, which Borel earlier used to calculate $\mathbb{Q} \otimes K_n(A)$. Analogues of the Borel-Serre results in characteristic $p > 0$ would yield the analogue of Quillen's theorem for maximal orders in global fields of characteristic p, though one might here only expect finite generation modulo p-torsion.

(2) <u>Finite rings</u>

If A is a finite ring then $K_n(A)$ is finite for $n > 0$ when A is semi-simple. This reduces, using Morita theorems, to the case of finite fields, where the finite group $K_n(\mathbb{F}_q)$ are known explicitly [Q 1]. If A is not necessarily semi-simple then $G_n(A)$ is finite for n > 0, since Quillen's devissage theorem ([Q 2] or [Q 3]) implies that $G_n(A) \cong G_n(A/\text{rad } A) = K_n(A/\text{rad } A)$. The finiteness of $K_n(A)$ would follow if one had reasonable stability theorems for K_n (cf. §6), as one does for $n \leq 2$. Another approach would be to obtain good control of the kernel of $K_n(A) \to K_n(A/J)$ whenever J is a nilpotent ideal in a ring A.

(3) Use of devissage and localization in $(XXII)_n$

Let A be a commutative finitely generated $\mathbb{Z}$-algebra. Quillen's devissage theorem implies that $A \to A_{red} = A/(\text{nil rad } A)$ induces isomorphisms $G_n(A_{red}) \to G_n(A)$. Thus (for problem (XXII)) we may assume A is reduced. We can then further find a non division of zero s in A such that $A[\frac{1}{s}]$ is a finite product of regular integral domains; this follows from "Closedness of the singular locus." Quillen's localisation and devissage theorems then yield a long exact sequence

$$\cdots \longrightarrow G_n(A/sA) \longrightarrow G_n(A) \longrightarrow G_n(A[\tfrac{1}{s}]) \longrightarrow G_{n-1}(A/sA) \longrightarrow \cdots$$

Since dim (A/sA) < dim A, and since the groups $G_n(A)$ are finitely generated when A is finite, we can argue by induction on dim (A) and so reduce $(XXII)_n$ to the case where

A is a regular integral domain. In this case we further have $K_n(A) \xrightarrow{\cong} G_n(A)$. Thus $(XXII)_n$ is equivalent to:

$(XXII')_n$ <u>Is</u> $K_n(A)$ <u>finitely generated when</u> A <u>is a regular integral domain finitely generated as a</u> $\mathbb{Z}$-<u>algebra</u>?

(4) <u>The Mordell-Weil Theorem</u> (cf. [Roq])

It implies that if A is a normal integral domain finitely generated as a $\mathbb{Z}$-algebra then Pic(A) is finitely generated. If further dim $(A) \leq 1$ then $K_0(A) \cong \mathbb{Z} \oplus \text{Pic}(A)$ is finitely generated. Combining this with the remarks in (3) above one deduces (cf. [Ba 1], Ch. XIII, Cor (3.2)) that $(XXIII)_n$ has an affirmative solution if dim $(A) \leq 1$. A procedure for attacking $(XXIII)_n$ by induction on dim (A) is suggested by Roquette's proof of the Mordel-Weil Theorem [Roq].

9.2 <u>A PID with</u> $SK_1 \neq 0$

Examples showing why problem (XXIII) is formulated only for G_0, and not $G_n (n > 0)$ or $K_n (n \geq 0)$ are given in [Ba 1], Ch. XIII, §3. The constructions used there also furnish the following <u>example of a principal ideal domain</u> B <u>with</u> $SK_1(B) \neq 0$ <u>and not even finitely generated</u>. This responds to a question raised by Swan [(Sw 3], p. 203].

Let k be a field finitely generated over its prime field. Let A be the coordinate ring of an absolutely irreducible and smooth affine curve C of genus $g > 0$ over k. If k' is a k-algebra put $A_{k'} = A \otimes_k k'$. Mordell-Weil implies that Pic(A) is finitely generated. Removing a finite number of points from C we may therefore further impose that Pic(A) = 0, so A is a PID. It follows then that $B = A_{k(t)}$ is likewise a PID, where t is an indeterminate. Now we have from [Ba 1], Ch. XIII, Cor (3.4) an exact sequence

$$SK_1(A) \longrightarrow SK_1(B) \longrightarrow \coprod_x \text{Pic}\ (A_{k(x)}) \longrightarrow 0$$

where k(x) ranges over all residue class fields of k[t]. Since $g > 0$ the groups Pic $(A_{k(x)})$ are $\neq 0$ for infinitely many k(x)'s*, whence $SK_1(B)$ is not finitely generated.

(9.3) Rational varieties

Let k be an algebraically closed field and A the

*Pic$(A_{k(x)})$ is essentially $J(k(x))/(J(k(x)) \cap \Gamma)$ where J(k') denotes k'-rational points on the Jacobian J of the complete non-singular curve containing C, and where Γ denotes the subgroup generated by the (finite number of) points at infinity. If $\bar{k}$ is the algebraic closure of k then the torsion of $J(\bar{k})$ looks like that of $(\mathbb{Q}/\mathbb{Z})^{2g}$ except for p-torsion (p = char(k)); thus J(k') effectively grows in size as k' approaches $\bar{k}$.

coordinate ring of an affine variety X over k. It is unreasonable to expect $K_0(A)$ to be finitely generated unless X is almost rational. Even this does not suffice, as the following example of Murthy shows (cf. [Mur 1], sec. 6).

Example. Let $f \in B = k[t_1,\ldots,t_n]$ define a non-singular hyper-surface in k^n. Put $A = B[x,y] = A[X,Y]/(XY-f)$. Then $A[x^{-1}] = B[x,x^{-1}]$ (Laurent polynomials) and $A/xA \cong (B/fB)[y]$, so A is regular and "birationally equivalent" to $B[x,x^{-1}] = k[t_1,\ldots,t_n,x,x^{-1}]$. Moreover Pic $(A) = 0$, whereas $K_0(A) \cong K_0(B/fB)$. For a suitable choice of f one can make $K_0(B/fB)$ extremely large, whence likewise for $K_0(A)$.

Presumably varieties admitting cell decomposition, e.g. linear algebraic groups, can be shown to have finitely generated K_0's, (cf. [J_0]). Do their K_n's have any similar finiteness properties?

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COMPARISON OF ALGEBRAIC AND TOPOLOGICAL K-THEORY

L. Roberts

Let X be a quasiprojective algebraic variety over the complex numbers C, and let X_C denote the closed points of X, with topology induced by the usual topology on C. (By variety over a field F we mean scheme of finite type over F). To an algebraic vector bundle (locally free sheaf of finite type) on X we can associate a continuous complex vector bundle on X_C. This gives a ring homomorphism

$$\phi_X : K_a(X) \to K(X_C)$$

where K_a denotes the Grothendieck group of algebraic vector bundles and exact sequences while $K(X_C)$ is the Grothendieck group of complex topological vector bundles on X. The problem is to try to understand this homomorphism, with the hope that this will help in computing either $K_a(X)$ or $K(X_C)$. The homomorphism ϕ_X has been studied by J.P. Jouanolou in [6], [7], especially in the cases where X is the complement of a smooth complete intersection in P^r_C, or an affine or projective quadric. It is not an isomorphism in general.

The corresponding problem with real varieties does not seem to have been studied as much. If X is a quasi-projective non-singular algebraic variety of dimension n over the real numbers R, then the set X_R of real points is either empty or an n dimensional real manifold. In the latter case one can define a homomorphism

$$\phi_X : K_a(X) \to KO(X_R) \quad .$$

If X is projective this homomorphism cannot be injective since on X there are line bundles of infinite order (under $\otimes$) but on X_R every line bundle is of order 2. If X is affine, say $X = \operatorname{Spec} A$, then ϕ_X can be obtained as follows: restriction gives a homomorphism $A \to C_R(X_R)$, where C_R = real valued continuous functions. This gives a homomorphism $K_a(X) = K_0(A) \to K_0(C_R(X_R)) = KO(X_R)$. Some examples are the following: If $A = R[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-1)$ then $X_R = S^n$. Fossum has proved in [3] that ϕ_X is surjective. It is known that ϕ_X is an isomorphism for $n \leq 4$, but if $n > 4$ it is not known whether ϕ_X is an isomorphism or not. If A = even part of $R[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-1)$ then $X_R = RP^n$ and it is proved in [5] that ϕ_X is an isomorphism for all n except if $n \equiv 6,7$ or $8 \mod 8$. If $n = 6,7$ or 8 ϕ_X is also an isomorphism, but the cases $n > 13$, $n \equiv 6,7,8$ are not known.

One can try complexifying the real case. For example, if $X = \operatorname{Spec} A$ is affine, then restriction gives a homomorphism $A \otimes_R C \to C_C(X_R)$ where C_C = complex valued continuous functions. This gives a homomorphism $K_0(A\otimes_R C) \to K(X_R)$. If $A = R[X_0,\ldots,X_n]/(X_0^2+\ldots X_n^2-1)$ this was shown to be an isomorphism in [3] and if A = even part of $R[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-1)$ it was shown to be an isomorphism in [5]. However, in the first case $(\text{Spec } A)_C$ is of the same homotopy type as S^n and in the second $(\text{Spec } A)_C$ is of the same homotopy type as RP^n , so both are reduced to a special case of the problem considered by Jouanolou.

If one is allowed to change the algebraic ring much better results have been obtained. Again let X be an affine variety over the reals, $X = \text{Spec } A$. In [2] it is proved that if X_R is compact and $S \subset A$ is the multiplicative set of all elements that vanish nowhere on X_R , then the map $K_0(A_S) \to KO(X_R)$ is a monomorphism but not necessarily a surjection. In [8] it is proved that if one starts with the compact real n-dimensional manifold M , then there exists a non-singular n-dimensional affine variety $X = \text{Spec } A$ such that M is isomorphic to a connected component of X_R and the homomorphisms $K_0(A_S) \to KO(M)$ and $K_0(A_S\otimes_R C) \to K(M)$ are isomorphisms. The rings A_S are no longer algebras of finite type over R , and these results do not seem to help compute $K_0(A)$.

If the real variety X has no real points, $K_a(X)$ is still defined, but few examples seem to be known. One could try extending scalars to C , as in [12], but the 2-torsion gets lost.

One can also consider the relationship between isomorphism classes of algebraic and topological vector bundles. This was done for S^2 in [9]. It follows easily from [9] and [10] that the homomorphism $\phi : \mathbb{C}[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1) \to C_C(S^2)$ induces a bijection on isomorphism classes of projective modules of finite type, and from [9] that the homomorphism $R[X_0,X_1,X_2]/(X_0^2+X_1^2+X_2^2-1) \to C_R(S^2)$ induces a surjection on isomorphism classes. It does not seem to be known if the latter is a bijection. A similar problem for the L-holed torus is considered in [1], but the corresponding problem for other spaces such as S^n , $n \geq 3$ does not seem to have been considered.

In a similar vein, let T be the tangent bundle to S^n . Then the maximum rank of a free direct summand of T is known topologically, and it is shown in [4] that this number arises algebraically, even over $Z[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-1)$. Topological results are also used to obtain non-stable algebraic results in [11], where universal stably free projectives are discussed.

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APPLICATIONS ALGEBRIQUES DU TORE DANS LA SPHERE ET DE $S^p \times S^q$ DANS S^{p+q}

par Jean-Louis LODAY

La sphère S^n est l'ensemble des éléments $x = (x_o, x_1, \ldots, x_n)$ de $\mathbb{R}^{n+1}$ tels que $|x|^2 = x_o^2 + x_1^2 + \ldots + x_n^2 = 1$. Une application algébrique de $S^p \times S^q$ dans S^{p+q} est la donnée de $p+q+1$ polynômes $P_o, P_1, \ldots, P_{p+q}$ en $(p+1)+(q+1)$ variables $x_o, \ldots, x_p$; $y_o, \ldots, y_q$ et à coefficients réels tels que $\sum_{i=o}^{p+q} P_i^2(x,y) = 1$ dès que $|x| = 1$ et $|y| = 1$.

L'étude du cup-produit en K-théorie topologique (cf.[5]) nous amène tout naturellement à la question suivante : existe-t-il une application algébrique de $S^1 \times S^1$ dans S^2 de degré un ?

Le but de cet article est d'étudier plus généralement l'existence d'applications algébriques de $S^p \times S^q$ dans S^{p+q} ou de $T^n = S^1 \times \ldots \times S^1$ dans S^n de degré donné.

On rappelle que les classes d'homotopie d'applications continues d'une variété topologique orientable M de dimension n dans S^n sont classifiées par leur degré $k \in \mathbb{Z}$ (cf.[6]).

Dans le paragraphe 1 on montre que toute application algébrique de $S^p \times S^q$ dans S^{p+q} pour p et q impairs et de T^n dans S^n pour $n \geq 2$ est homotope à une application constante. Ces résultats sont des applications de la K-théorie algébrique. Dans le paragraphe 2 on exhibe plusieurs applications algébriques de $S^p \times S^q$ dans S^{p+q} non homotopiquement triviales. Ces résultats ont été annoncés partiellement dans [4].

1. - Soit X la variété algébrique affine de $\mathbb{R}^{n+1}$ définie par les polynômes $P_o, P_1, \dots, P_k$ de $\mathbb{R}[x_o, \dots, x_n]$. On note $\mathcal{A}(X)$ l'anneau quotient de $\mathbb{C}[x_o, \dots, x_n]$ par l'idéal engendré par les polynômes $P_o, \dots, P_k$. On désignera par $\mathcal{C}(X)$ l'anneau des fonctions continues définies sur X à valeurs dans $\mathbb{C}$. L'homomorphisme d'anneaux de $\mathcal{A}(X)$ dans $\mathcal{C}(X)$ qui, à la classe d'un polynôme Q dans $\mathcal{A}(X)$ fait correspondre sa fonction polynôme, sera noté $\omega(X)$ ou ω s'il n'y a pas d'ambiguité.

THEOREME 1. - <u>Toute application algébrique du tore</u> T^n <u>dans la sphère</u> S^n

$$f : T^n \longrightarrow S^n \quad (n \geq 2)$$

<u>est homotope à une application constante</u>.

DEMONSTRATION. - Soit f une application algébrique de T^n dans S^n. Elle induit deux homomorphismes d'anneaux : l'un f_a de $\mathcal{A}(S^n)$ dans $\mathcal{A}(T^n)$ et l'autre f_t de $\mathcal{C}(S^n)$ dans $\mathcal{C}(T^n)$. Le diagramme (1) est commutatif.

$$\begin{array}{ccc} \mathcal{A}(S^n) & \xrightarrow{\omega(S^n)} & \mathcal{C}(S^n) \\ \downarrow f_a & & \downarrow f_t \\ \mathcal{A}(T^n) & \xrightarrow{\omega(T^n)} & \mathcal{C}(T^n) \end{array} \qquad (1)$$

a) <u>Cas</u> n <u>pair</u> (n=2p). Soit R un anneau unitaire. $K^{\circ}(R)$ est le groupe de Grothendieck de la catégorie des R-modules projectifs de type fini. On pose

$$\widetilde{K}^{\circ}(R) = \mathrm{Coker}(K^{\circ}(\mathbb{Z}) \longrightarrow K^{\circ}(R)).$$

Appliquons le foncteur $\widetilde{K}^{\circ}$ au diagramme (1). On obtient le diagramme (2).

$$\begin{array}{ccc} \widetilde{K}^{\circ}_a(S^{2p}) & \xrightarrow{\omega^*} & \widetilde{K}^{\circ}_t(S^{2p}) \\ \downarrow f^*_a & & \downarrow f^*_t \\ \widetilde{K}^{\circ}_a(T^{2p}) & \longrightarrow & \widetilde{K}^{\circ}_t(T^{2p}) \end{array} \qquad (2)$$

où l'on a posé $\widetilde{K}^{\circ}_a(X) = \widetilde{K}^{\circ}(\mathcal{A}(X))$ et $\widetilde{K}^{\circ}_t(X) = \widetilde{K}^{\circ}(\mathcal{C}(X))$ pour toute variété algébrique X. Le groupe $\widetilde{K}^{\circ}_{t}(X)$ est isomorphe au groupe de Grothendieck de la catégorie des fibrés vectoriels complexes sur l'espace topologique X. Les lemmes 2 et 3 montreront que l'homomorphisme f^{*}_{t} est nul. On en déduira par le lemme 4 que le degré de f est nul.

LEMME 2. - <u>L'homomorphisme</u> $\omega^{*} : \widetilde{K}^{\circ}_{a}(S^{2p}) \longrightarrow \widetilde{K}^{\circ}_{t}(S^{2p})$ <u>est surjectif.</u>

DEMONSTRATION. - Le groupe $\widetilde{K}^{\circ}_{t}(S^{2p})$ est isomorphe à $\mathbb{Z}$. Par conséquent il nous suffit d'exhiber un élément de $\widetilde{K}^{\circ}_{a}(S^{2p})$ dont l'image est un générateur de $\widetilde{K}^{\circ}_{t}(S^{2p})$. Soit C_{n+1} l'algèbre de Clifford de $\mathbb{C}^{n+1}$ muni de la forme quadratique $x_o^2 + x_1^2 + \dots + x_n^2$. C_{n+1} est isomorphe à une sous-algèbre de l'algèbre des matrices d'un certain espace vectoriel de dimension k. On note $\varepsilon_o, \varepsilon_1, \dots, \varepsilon_n$ les images dans C_{n+1} des vecteurs de base de $\mathbb{C}^{n+1}$. On identifie $\varepsilon_o, \varepsilon_1, \dots, \varepsilon_n$ à des $k \times k$ -matrices à coefficients complexes. Ainsi

$$q = \frac{1}{2}\,(\varepsilon_o x_o + \varepsilon_1 x_1 + \dots + \varepsilon_n x_n - \mathrm{id})$$

définit un projecteur $(q^2 = q)$ d'un $\mathcal{C}(S^n)$-module libre de dimension k. L'image de q est un $\mathcal{C}(S^n)$-module projectif de type fini qu'on note $M(q)$. Le projecteur q peut aussi être considéré comme un endomorphisme d'un $C(S^n)$-module libre de dimension k. Il définit alors un $C(S^n)$ -module projectif de type fini $M'(q)$ image de $M(q)$ par $\omega(S^n)$.

Dans le cas de la sphère S^2 on sait (Cf. par exemple [2]) que la classe dans $\widetilde{K}^{\circ}(S^2)$ du projecteur

$$q_2 = \frac{1}{2}\begin{bmatrix} -1 + x_o & x_1 + ix_2 \\ x_1 - ix_2 & -1 - x_o \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x_o + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} x_2 - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

est un générateur de $\widetilde{K}^{\circ}(S^2)$. Le cup-produit d'un générateur q_2 de $\widetilde{K}^{\circ}(S^2)$ par un générateur q_{2p} de $\widetilde{K}^{\circ}(S^{2p})$ est un générateur de $\widetilde{K}^{\circ}(S^{2p+2})$. Le calcul explicite du cup-produit par la formule donnée dans [5] théorème 3, permet de montrer que si on écrit

$$q_2 = \frac{1}{2} \ (\epsilon_o x_o + \epsilon_1 x_1 + \epsilon_2 x_2 - 1), \qquad , \ x \in S^2$$

et

$$q_{2p} = \frac{1}{2} \ (\epsilon'_o x'_o + \epsilon'_1 x'_1 + \dots + \epsilon'_{2p} x'_{2p} - 1), \ x' \in S^{2p}$$

et si on identifie $S^2 \wedge S^{2p}$ avec S^{2p+2} alors $q_2 \cup q_{2p}$ (cup-produit) s'écrit

$$q_2 \cup q_{2p} = \frac{1}{2} \ (\epsilon_o \otimes \epsilon'_o x''_o + \dots + \epsilon_2 \otimes \epsilon'_o x''_2 + 1 \otimes \epsilon'_1 x''_3 + \dots + 1 \otimes \epsilon'_{2p} x''_{2p+2} - 1)$$

avec $x'' \in S^{2p+2}$. D'où le résultat par récurrence.

REMARQUE. - R. FOSSUM a montré que $\omega^*(S^{2n})$ est aussi injectif et donc un isomorphisme. (Cf. [3] Proposition 3.1.).

LEMME 3. - <u>Le groupe</u> $\widetilde{K}^o(\mathbb{G}(T^n)) = \widetilde{K}^o_a(T^n)$ <u>est nul pour</u> $n \geq 1$.

DEMONSTRATION. - $\mathbb{G}(T^n)$ est l'anneau $\mathbb{C}[x_1, x_2, \dots, x_{2n}]/(x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1)$ Posons $u_k = x_{2k-1} + i x_{2k}$ pour $k = 1, \dots, n$; $(i = \sqrt{-1})$. $\mathbb{G}(T^n)$ est alors canoniquement isomorphe à l'anneau $\mathbb{C}[u_1, u_1^{-1}, \dots, u_n, u_n^{-1}]$. R étant un anneau noethérien régulier $\widetilde{K}^o(R[t, t^{-1}])$ est isomorphe à $\widetilde{K}^o(R)$ d'après un théorème de Grothendieck (Cf. [1] p.636). En appliquant n fois ce théorème à l'anneau $\mathbb{G}(T^n)$ on en déduit :

$$\widetilde{K}^o_a(T^n) \xrightarrow{\sim} \widetilde{K}^o(\mathbb{C}) \xrightarrow{\sim} 0$$

LEMME 4. - <u>Soit</u> X <u>une variété topologique de dimension</u> 2p <u>et</u> $f : X \to S^{2p}$ <u>une application continue</u>. <u>Si l'homomorphisme</u> $f^* : \widetilde{K}^o(S^{2p}) \to \widetilde{K}^o(X)$ <u>est nul</u>, <u>alors l'application</u> f <u>est de degré zéro</u>.

DEMONSTRATION. - Dans le diagramme commutatif (3) Ch désigne le caractère de Chern :

$$\begin{array}{ccc} \mathbb{Z} \simeq \tilde{K}^\circ(S^{2p}) & \xrightarrow{\ Ch(S^{2p})\ } & \tilde{H}^{pair}(S^{2p},\mathbb{Q}) \simeq \mathbb{Q} \\ \Big\downarrow f^* & & \Big\downarrow H(f) \\ \tilde{K}^\circ(X) & \xrightarrow{\ Ch(X)\ } & \tilde{H}^{pair}(X,\mathbb{Q}) \end{array} \qquad (3)$$

L'homomorphisme $Ch(S^{2p})$ induit l'inclusion naturelle de $\mathbb{Z}$ dans $\mathbb{Q}$ et $H(f)$ est la multiplication par le degré de f. L'homomorphisme f^* étant nul par hypothèse, on en déduit que le degré de f est zéro.

Terminons la démonstration du cas a) du théorème 1. Dans le diagramme (2) le groupe $\tilde{K}^\circ_a(T^{2p})$ est nul (Lemme 3), et l'homomorphisme $\omega^*(S^{2p})$ est surjectif (Lemme 2), donc $f^*_t = f^*$ est nul. Le degré de f est alors nul (Lemme 4) et par le théorème de Hopf f est homotope à une application constante.

b) <u>Cas n impair</u> : On applique le foncteur K_1 de Bass (Cf.[1]) au diagramme (1). On obtient le diagramme commutatif (4) :

$$\begin{array}{ccc} K_a^{-1}(S^n) & \longrightarrow & K_t^{-1}(S^n) \\ \Big\downarrow f_a^* & & \Big\downarrow f_t^* \\ K_a^{-1}(T^n) & \longrightarrow & K_t^{-1}(T^n) \end{array} \qquad (4)$$

où l'on a posé $K_a^{-1}(X) = K_1(\mathcal{C}(X))$ et $K_t^{-1}(X) = K_1(\mathbb{C}(X))$.

Notons $K^{-1}(X) = [X, GL(\mathbb{C})]$ le groupe de K-théorie topologique. On a une surjection naturelle de $K_t^{-1}(X)$ dans $K^{-1}(X)$; d'où le nouveau diagramme commutatif (5) :

$$\begin{array}{ccc} K_a^{-1}(S^n) & \xrightarrow{\ \omega^*\ } & K^{-1}(S^n) \\ \Big\downarrow f_a^* & & \Big\downarrow f^* \\ K_a^{-1}(T^n) & \longrightarrow & K^{-1}(T^n) \end{array} \qquad (5)$$

Les lemmes 5 et 6 montreront que l'homomorphisme f^* est nul. On en déduira par le lemme 7 que le degré de f est zéro.

LEMME 5. - <u>L'homomorphisme</u> $\omega^* : K_a^{-1}(S^n) \longrightarrow K^{-1}(S^n)$ <u>est surjectif</u>.

DEMONSTRATION. - Si n est pair $K^{-1}(S^n) = 0$. Si n est impair $K^{-1}(S^n)$ est isomorphe à $\mathbb{Z}$. Par conséquent il nous suffit d'exhiber un élément de $K_a^{-1}(S^n)$ dont l'image par ω^* soit un générateur de $K^{-1}(S^n)$. Soit C^n l'algèbre de Clifford de $\mathbb{C}^n$ muni de la forme quadratique $-x_1^2 - x_2^2 \dots - x_n^2$. C^n est isomorphe à une sous-algèbre de $\mathrm{End}(\mathbb{C}^k)$. Notons $e_1, \dots e_n$ les images dans C^n des vecteurs de base de $\mathbb{C}^n$. On identifie $e_1, \dots e_n$ à des matrices à coefficients complexes.

Notons α_x l'automorphisme d'un $\mathcal{G}(S^n)$-module libre de dimension k défini par

$$\alpha_x = \mathrm{id}.\, x_o + e_1 x_1 + \dots + e_n x_n, \qquad x \in S^n$$

Cet automorphisme définit un élément de $K_a^{-1}(S^n)$. On peut aussi le considérer comme une application continue :

$$\alpha : S^n \longrightarrow GL(\mathbb{C})$$
$$x \longmapsto \alpha_x$$

La classe d'homotopie de α est un élément $[\alpha]$ de $[S^n, GL(\mathbb{C})] = K^{-1}(S^n)$. On montre que $[\alpha]$ engendre $K^{-1}(S^n)$ comme dans le lemme 2 .

LEMME 6. - <u>L'homomorphisme</u> $f^* : K^{-1}(S^n) \longrightarrow K^{-1}(T^n)$ <u>induit par l'application algébrique</u> $f : T^n \dashrightarrow S^n (n \geq 2)$ <u>est nul</u>.

DEMONSTRATION. - Le groupe $K_a^{-1}(T^n)$ est isomorphe à $K_a^{-1}(T^{n-1}) \oplus \tilde{K}_a^{o}(T^{n-1}) \oplus K^{-1}(S^1)$ C'est une conséquence immédiate du théorème suivant dû à Bass, Heller et Swan : pour tout anneau régulier A, $K_1(A[t,t^{-1}])$ est isomorphe à $K_1(A) \oplus K^o(A)$. De même en K-théorie topologique le groupe $K^{-1}(T^n)$ est isomorphe à $K^{-1}(T^{n-1}) \oplus \widetilde{K}^{\circ}(T^{n-1}) \oplus K^{-1}(S^1)$.

On va montrer que f^* est nul en prouvant la nullité des trois homomorphismes

$$K^{-1}(S^n) \longrightarrow \widetilde{K}^{\circ}(T^{n-1}),\quad K^{-1}(S^n) \longrightarrow K^{-1}(T^{n-1}),\quad K^{-1}(S^n) \longrightarrow K^{-1}(S^1).$$

i) L'homomorphisme $K_a^{-1}(T^n) \longrightarrow K^{-1}(T^n)$ est la somme directe des homomorphismes

$$K_a^{-1}(T^{n-1}) \longrightarrow K^{-1}(T^{n-1}),\quad \widetilde{K}_a^{\circ}(T^{n-1}) \longrightarrow \widetilde{K}^{\circ}(T^{n-1}) \quad \text{et} \quad \mathbb{Z} \longrightarrow K^{-1}(S^1) \simeq \mathbb{Z}$$

(Cf. Bass [1] p.750 et 751). L'homomorphisme composé

$$K_a^{-1}(S^n) \overset{\omega^*}{\dashrightarrow} K^{-1}(S^n) \overset{f^*}{\dashrightarrow} K^{-1}(T^n) \twoheadrightarrow \widetilde{K}^{\circ}(T^{n-1})$$

se factorise à travers $\widetilde{K}_a^{\circ}(T^{n-1})$. Or on a vu que ce groupe est nul (lemme 3), donc l'homomorphisme composé $K_a^{-1}(S^n) \dashrightarrow \widetilde{K}^{\circ}(T^{n-1})$ est nul. D'où $K^{-1}(S^n) \dashrightarrow \widetilde{K}^{\circ}(T^{n-1})$ est nul puisque ω^* est surjectif (lemme 5).

ii) L'homomorphisme composé $K^{-1}(S^n) \dashrightarrow K^{-1}(T^n) \dashrightarrow K^{-1}(T^{n-1})$ est nul car il est induit par l'application composée $T^{n-1} \hookrightarrow T^n \longrightarrow S^n$, qui est homotope à une application constante.

iii) L'homomorphisme composé $K^{-1}(S^n) \dashrightarrow K^{-1}(T^n) \dashrightarrow\!\!\!\!\!\rightarrow K^{-1}(S^1)$ est nul car il est induit par l'application composée $S^1 \hookrightarrow T^n \overset{f}{\longrightarrow} S^n$ qui est homotopiquement triviale si $n \geq 2$.

Le théorème 1 pour n impair résulte alors du lemme suivant :

LEMME 7. - <u>Soit</u> f <u>une application continue</u> $T^n \longrightarrow S^n$ $(n = 2p+1)$ <u>telle que l'homomorphisme induit</u> $K^{-1}(S^n) \longrightarrow K^{-1}(T^n)$ <u>soit nul, alors</u> f <u>est homotope à une application constante.</u>

DEMONSTRATION. - Comme dans le lemme 4 on compare cet homomorphisme à celui que f induit en cohomologie rationnelle. Ce qui donne le diagramme commutatif (6)

$$\begin{array}{ccc} \mathbb{Z} \simeq K^{-1}(S^n) & \xrightarrow{\ Ch\ } & H^{\text{impair}}(S^n,\mathbb{Q}) \simeq \mathbb{Q} \\ \Big\downarrow f^* & & \Big\downarrow H(f) \\ K^{-1}(T^n) & \xrightarrow{\ Ch\ } & H^{\text{impair}}(T^n,\mathbb{Q}) \end{array} \qquad (6)$$

La flèche horizontale supérieure induit l'inclusion naturelle de $\mathbb{Z}$ dans $\mathbb{Q}$, donc l'homomorphisme $H(f)$ induit par f en cohomologie rationelle est nul. En dimension n cet homomorphisme est la multiplication par le degré de f ; on a donc $\deg(f) = 0$.

Le théorème 1 peut se généraliser partiellement :

THEOREME 8. - Soit X une variété algébrique affine sans singularités de R^k compacte et orientable en tant que variété topologique. Si la dimension de X est impaire $(\dim X = 2n-1)$, alors toute application algébrique $f : S^1 \times X \dashrightarrow S^{2n}$ est homotope à une application constante.

DEMONSTRATION. - On considère le diagramme commutatif (7)

$$\begin{array}{ccc} \tilde{K}^{\circ}_a(S^{2n}) & \longrightarrow & \tilde{K}^{\circ}_t(S^{2n}) \\ \Big\downarrow f^*_a & & \Big\downarrow f^*_t \\ \tilde{K}^{\circ}_a(S^1 \times X) & \longrightarrow & \tilde{K}^{\circ}_t(S^1 \times X) \end{array} \qquad (7)$$

Le groupe $\tilde{K}^{\circ}_t(S^1 \times X)$ est isomorphe à $\tilde{K}^{\circ}_t(S^1 \wedge X) \oplus \tilde{K}^{\circ}_t(X)$. Par un théorème de Grothendieck déjà cité ([1] p.636) $\tilde{K}^{\circ}_a(S^1 \times X)$ est isomorphe à $\tilde{K}^{\circ}_a(X)$ et l'homomorphisme $\omega^*(S^1 \times X)$ est simplement $0 \oplus \omega^*(X)$.

Le diagramme (7) se décompose en les diagrammes commutatifs (8) et (9).

$$\begin{array}{ccc} \widetilde{K}_a^o(S^{2n}) & \longrightarrow & \widetilde{K}_t^o(S^{2n}) \\ \downarrow & & \downarrow \\ \widetilde{K}_a^o(X) & \longrightarrow & \widetilde{K}_t^o(X) \end{array} \quad (8) \qquad \begin{array}{ccc} \widetilde{K}_a^o(S^{2n}) & \longrightarrow & K_t^o(S^{2n}) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{K}_t^o(S^1 \wedge X) \end{array} \quad (9)$$

i) L'homomorphisme $\widetilde{K}_t^o(S^{2n}) \dashrightarrow \widetilde{K}_t^o(X)$ est nul car il est induit par l'application homotopiquement triviale

$$X \hookrightarrow S^1 \times X \xrightarrow{f} S^{2n}$$

ii) L'homomorphisme $\widetilde{K}_t^o(S^{2n}) \dashrightarrow \widetilde{K}_t^o(S^1 \wedge X)$ est nul car sa composition avec l'homomorphisme surjectif $\omega^*(S^{2n})$ est nulle.

Donc

$$f_t^* : \widetilde{K}_t^o(S^{2n}) \dashrightarrow \widetilde{K}_t^o(S^1 \times X)$$

est nul et, par le lemme 4, f est de degré zéro.

THEOREME 9. - <u>Si</u> p <u>et</u> q <u>sont impairs toute application algébrique de</u> $S^p \times S^q$ <u>dans</u> S^{p+q} <u>est homotope à une application constante</u>.

DEMONSTRATION.- Elle est du même type que celle du théorème 1 cas a).
Soit $f : S^p \times S^q \dashrightarrow S^{p+q}$ une application algébrique. Le diagramme (10) est commutatif :

$$\begin{array}{ccc} \widetilde{K}_a^o(S^{p+q}) & \xrightarrow{\omega^*(S^{p+q})} & \widetilde{K}_t^o(S^{p+q}) \\ \downarrow f_a^* & & \downarrow f_t^* \\ \widetilde{K}_a^o(S^p \times S^q) & \longrightarrow & \widetilde{K}_t^o(S^p \times S^q) \end{array} \quad (10)$$

Supposons que $\widetilde{K}_a^o(S^p \times S^q) = 0$. On en déduit alors que l'homomorphisme $f_t^* \circ \omega^*(S^{p+q})$ est nul. Comme $p+q$ est pair, $\omega^*(S^{p+q})$ est surjectif (Lemme 2) et donc $f_t^* : \widetilde{K}_t^o(S^{p+q}) \dashrightarrow \widetilde{K}_t^o(S^p \times S^q)$ est nul. Du lemme 4 on déduit que f est de degré zéro. Il nous reste à démontrer le lemme suivant :

LEMME 10. - Le groupe $\tilde{K}^{o}_{a}(S^{p} \times S^{q})$ est nul lorsque p et q sont impairs.

DEMONSTRATION. - Ce lemme est un corollaire du résultat suivant dû à Jouanolou [3] : soit Q une quadrique lisse sur $\mathbb{C}$ (ici $Q = S^{q}$) et X une variété quasi-projective lisse sur $\mathbb{C}$ (ici $X = S^{p}$) telle que

$$\omega^{*}(X) : \tilde{K}^{o}_{a}(X) \longrightarrow \tilde{K}^{o}_{t}(X)$$

soit un isomorphisme.

Alors la suite

$$0 \dashrightarrow \tilde{K}^{o}_{a}(X \times Q) \dashrightarrow \tilde{K}^{o}_{t}(X \times Q) \dashrightarrow K^{-1}_{t}(X) \dashrightarrow 0$$

est exacte. $\omega^{*}(S^{q})$ est un isomorphisme par la proposition 3.1 de [3]. Dans notre cas particulier la flèche $\tilde{K}^{o}_{t}(X \times Q) \longrightarrow K^{-1}_{t}(X)$ est un isomorphisme de $\mathbb{Z}$ dans $\mathbb{Z}$, d'où le résultat énoncé.

2. - Applications algébriques de $S^{p} \times S^{q}$ dans S^{p+q} non homotopiquement triviales.

DEFINITION. - On appelle multiplication orthogonale toute application bilinéaire $F : \mathbb{R}^{k} \times \mathbb{R}^{\ell} \longrightarrow \mathbb{R}^{m}$ telle que $|F(x,y)| = |x|.|y|$.

Considérons la sphère S^{n} d'équation $x_{o}^{2} + x_{1}^{2} + \ldots + x_{n}^{2} - 1 = 0$ et de point-base $\{*\} = (1,0,\ldots,0)$. Si on pose $x'_{o} = 1 - x_{o}$ son équation devient $x'^{2}_{o} + x_{1}^{2} + \ldots + x_{n}^{2} - 2x'_{o} = 0$.

LEMME 11. - Soit $F : \mathbb{R}^{p+1} \times \mathbb{R}^{q} \longrightarrow \mathbb{R}^{q}$ une multiplication orthogonale, l'application algébrique f: $\begin{cases} S^{p} \times S^{q} \dashrightarrow S^{p+q} \\ (x,y) \mapsto z \end{cases}$ définie par :

$$z'_o = \frac{1}{2} x'_o y'_o$$
$$z_j = \frac{1}{2} x_j y'_o \qquad j = 1,\dots,p .$$
$$z_{p+i} = \frac{1}{2} F_i(x'_o, x_1,\dots,x_p ; y_1,\dots,y_q) \qquad i = 1,\dots,q$$

est de degré un.

DEMONSTRATION. - L'application f envoie $S^p \vee S^q$ sur le point-base de S^{p+q}. De plus par restriction f définit un homéomorphisme de $S^p \times S^q - S^p \vee S^q$ sur $S^{p+q} - \{*\}$, car l'application bilinéaire F est non dégénérée. Un point quelconque de $S^{p+q} - \{*\}$ a donc un seul antécédent ; on en conclut que f est une application de degré un.

THEOREME 12. - Il existe une application algébrique de degré un de $S^p \times S^q$ dans S^{p+q} pour tout couple d'entiers (p,q) tels que

$$q = 2^a.16^b.(2c+1) \qquad 0 \le a \le 3 , \quad b \ge 0 , \quad c \ge 0$$
$$p \le 2^a + 8b - 1$$

DEMONSTRATION. - Grâce au lemme précédent il nous suffit de montrer qu'il existe une multiplication orthogonale de $R^{p+1} \times R^q$ dans R^q. On sait qu'il en existe pour les couples d'entiers (p,q) satisfaisant aux conditions du théorème (Cf. par exemple [2] p.156).

Exemple : La multiplication dans C définit une forme de Hopf de $R^{1+1} \times R^2 \dashrightarrow R^2$ d'où une application algébrique de $S^1 \times S^2$ dans S^3, de degré un :

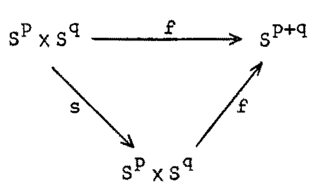
$$Z_o = \frac{1}{2} (1 + x_o + y_o - x_o y_o)$$
$$Z_1 = \frac{1}{2} x_1.(1 - y_o)$$
$$Z_2 = \frac{1}{2} ((1 - x_o)y_1 - x_1 y_2)$$
$$Z_3 = \frac{1}{2} (x_1 y_1 + (1 - x_o)y_2) .$$

COROLLAIRE 13. - *Si en plus des conditions du théorème précédent* $p+q$ *est impair il existe une application algébrique de* $S^p \times S^q$ *dans* S^{p+q} *de degré quelconque.*

DEMONSTRATION .- Etant donnée une application algébrique de degré un de $S^p \times S^q$ dans S^{p+q}, il suffit de la composer avec une application algébrique de S^{p+q} dans S^{p+q} de degré n pour obtenir une application algébrique de $S^p \times S^q$ dans S^{p+q} de degré n. Or Wood a montré que si k est impair toute classe d'homotopie d'applications continues de S^k dans lui-même peut être représentée par une application algébrique (Cf. [7]).

THEOREME 14. - *Si* p *(ou* q*) est pair, il existe une application de degré deux de* $S^p \times S^q$ *dans* S^{p+q}.

DEMONSTRATION. - On considère l'application algébrique $f : S^p \times S^q \longrightarrow S^{p+q}$ définie par $f(x_o,\dots,x_p ; y_o,\dots,y_q) = (x_o y_o, x_1 y_o,\dots,x_p y_o, y_1,\dots,y_q)$. L'image réciproque d'un point N de S^{p+q} est, en général, composée de deux points M et M'. Il suffit donc (Confer par exemple Milnor [6]) de regarder si f conserve ou non l'orientation en M et en M'. Considérons le diagramme suivant :



où $s(x_o,\dots,x_p ; y_o,\dots,y_q) = (-x_o,\dots,-x_p, -y_o, +y_1,\dots,+y_q)$. Ce diagramme est commutatif. L'application s échange les points M et M', et son degré est $(-1)^{q+2}$. Donc si q est pair f conserve l'orientation en M

et en M'. Le degré de f est donc $1+1=2$.

Ces deux théorèmes d'existence et le théorème 1 ne permettent pas de répondre dans tous les cas à la question posée dans l'introduction. Notamment on ne sait pas s'il existe une application algébrique de $S^2 \times S^2$ dans S^4 de degré un.

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On the K_0 of certain polynomial extensions *

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Introduction

It is a well known result of Grothendieck that,if A is a left regular ring and T a finetely generated free abelian monoid,then the inclusion $K_0(A) \to K_0(A[T])$ is an isomorphism.

In this paper we give sufficient conditions for the isomorphism above for certain classes of non-regular commutative rings: in §2 we consider the case of a ring A which is gotten from a regular ring B by glueing two distinct prime ideals p_1 and p_2 (for the definition see §1) and prove that $NK_0(A) \simeq NK_1(B/p_1 \cap p_2)$ (theorem 9).This implies that,if V is an affine non-singular variety and W the variety obtained from V by glueing together two irreducible non-singular subvarieties, which meet transversally at every point, then $K_0(A) \simeq K_0(A[T])$, where $A = k[W]$.(Proposition 2).
I §3 we state an analogous result in the case A is gotten from a regular ring B by glueing one prime to itself via an automorphism (theorem 10): as a consequence of this theorem (Corollary 3) we see that if V is an affine non singular variety and W the variety obtained from V by glueing a non-singular curve to itself then $K_0(A) \simeq K_0(A[T])$.
§4 contains some results which have been obtained jointly with M.Pavaman Murthy.The main result of this section is Corollary 5: if A is a commutative ring containing an algebraically closed field k and $K_0(A) \simeq K_0(k(t) \otimes_k A)$ then $K_0(A) \simeq K_0(A[t])$. Using this we show that, if $A = k[x,y,z]$, $z^n = xy$, then $K_0(A) \simeq K_0(A[T]) \simeq \mathbb{Z}$.

An interesting open problem is to find necessary and sufficient conditions for the isomorphism $K_0(A) \simeq K_0(A[T])$ and relate these conditions,when A is the coordinate ring of an affine va riety V, with the singularities of V. The corresponding problem of the isomorphisms $PicA \simeq PicA[T]$ and $PicA \simeq PicA[T,T^{-1}]$ has been considered by several authors : we record here some of the known results in this direction. C.Traverso (see [11]) has given a definition of seminormal rings (see §1 for more details) and has shown that a ring is seminormal iff $PicA \simeq PicA[T]$. In case A satisfies (S_2) then it is seminormal iff $\sqrt[A]{b} = b$ where b is the conductor from the integral closure $\bar{A}$ to A (see 3 ,Prop.7.12). Salmon (see [10]) has proved that the coordinate ring of a simple algebraic plane curve C is seminormal iff C has at most nodes. His result can be extended to curves in 3-space:such a curve is semi-

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normal iff it has at most nodes or triple points with linearly indipendent tangents. No general result of this to type is known in higher dimension; Bombieri (unpublished) has proved that a surface in $\mathbb{P}_3$, which has only ordinary singularities (i.e. it is a generic projection of a non singular surface in higher projective space) is seminormal. A different geometric characterization of "weakly normal" rings (a class containing the class of seminormal rings and equal to the latter when the base field has characteristic o) has been given by Andreotti-Bombieri (see [1]). A stronger condition than seminormality (but, in general, not equivalent to normality) is the isomorphism PicA $\simeq$ PicA $[T,T^{-1}]$. Bass-Murthy (see [3], th.8.1) proved necessary and sufficient conditions for the isomorphism above, when dimA = 1. If A is the coordinate ring of an irreducible curve C over on algebraically closed field then Pic A $\simeq$ Pic A$[T,T^{-1}]$ iff C is non singular (see [7], th.1). This theorem does not extend to higher dimensional varieties; in §1 (theorems 6 and 8) we recall some results on the isomorphism Pic A $\simeq$ Pic A$[T,T^{-1}]$, when A is obtained from a normal ring by glueing one or two primes.

My thanks are due to H.Bass and M.Pavaman Murthy for many helpful suggestions.

1. In this section we recall some definitions and results which will be used later on. Our notations will be consistent with those in [2]. All rings will always be commutative with identity, and all modules unitary.

Let A be a commutative ring, $\underline{\underline{P}}(A)$ the category of finetely generated projective A-modules with "product" $\oplus$ (in the sense of 2 , chap.VII), $\underline{\underline{\text{Pic}}}$ A the category of finetely generated projective modules of rank 1, with product $\otimes_A$: we will always denote $K_i(\underline{\underline{P}}(A)) = K_iA$ i= 0,1 and $K_o(\underline{\underline{\text{Pic}}}\ A)$ = Pic A. By K_2A we will denote the Milnor's group i.e. the kernel of the homomorphism St(A) $\rightarrow$ GL(A), where St(A) is the Steinberg group (cfr. [5], §5).

Let t be an indeterminate over A, A[t] the polynomial ring. The augmentation A[t]$\rightarrow$A is a left inverse for the inclusion $A \subset A[t]$. Therefore if F:(rings)$\rightarrow$(abelian groups) is a functor we have :

$$F(A[t]) \simeq F(A) \oplus \text{Ker } (F(A[t]) \rightarrow F(A))$$

We will denote by NF the following functor.:

$$NF(A) = \text{Ker } (F(A[t]) \rightarrow F(A))$$

so that we have

$$F(A[t]) \simeq F(A) \oplus NFA$$

T will always denote a finetely generated free abelian monoid, $A[T]$ the polynomial ring and $A[T,T^{-1}]$ the group ring AG, where G is the free abelian group on the generators of T.

Now we state a result of Milnor on cartesian squares:

Theorem 1 : Let

$$\begin{array}{ccc} A & \longrightarrow & A_1 \\ \downarrow & & \downarrow f_1 \\ A_2 & \xrightarrow[f_2]{} & A' \end{array}$$

be a cartesian square of ring homomorphisms. Then : a) *if f_1 or f_2 is surjective there is the following exact Mayer-Vietoris sequence*

$$K_1A \to K_1(A_1) \oplus K_1(A_2) \to K_1(A') \to K_oA \to K_o(A_1) \oplus K_o(A_2) \to K_oA'$$

b) *if all the homomorphisms are surjective the exact sequence above can be extended to the following* :

$$K_2A \to K_2(A_1) \oplus K_2(A_2) \to K_2(A') \to K_1A \to K_1(A_1) \oplus K_1(A_2) \to \dots \to K_oA' \ .$$

Moreover in case a) *we have an exact sequence*

$$NK_1A \to NK_1(A_1) \oplus NK_1(A_2) \to NK_1(A') \to NK_oA \to NK_o(A_1) \oplus NK_o(A_2) \to NK_oA'$$

and in case b).

$$NK_2A \to NK_2(A_1) \oplus NK_2(A_2) \to NK_2(A') \to NK_1(A) \to NK_1(A_1) \oplus NK_1(A_2) \to \dots \to NK_oA' \ .$$

Proof :The first part of the theorem is proved in [5] pp.28 and 55: for the last part note that, if t is an indeterminate, then the diagram

$$\begin{array}{ccc} A[t] & \longrightarrow & A_1[t] \\ \downarrow & & \downarrow f_1[t] \\ A_2[t] & \xrightarrow[f_2[t]]{} & A'[t] \end{array}$$

is again a cartesian square. Therefore we have an epimorphism of exact Mayer-Vietoris sequences

$$\begin{array}{ccccccccccc} K_1(A[t]) & \to & K_1(A_1[t]) \oplus K_1(A_2[t]) & \to & K_1(A'[t]) & \to & K_o(A[t]) & \to & K_o(A_1[t]) \oplus K_o(A_2[t]) & - & K_o(A'[t]) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(A) & \to & K_1(A_1) \oplus K_1(A) & \to & K_1(A') & \to & K_o(A) & \to & K_o(A_1) \oplus K_o(A_2) & \longrightarrow & K_o(A') \end{array}$$

where the vertical arrows are induced by the argumentation $A[t] \xrightarrow{\varepsilon} A$. Since $\varepsilon i = id.$, where $i = A \hookrightarrow A[t]$, all the vertical arrows split and we get an exact sequence of kernels, i.e. of the groups NK_i. In case b) both the Mayer-Vietoris sequences can be extended to the groups K_2 and so does the sequence of kernels. q.e.d.

The following are known results on the vanishing of the groups NK_i.

Proposition 1 (see[2],Corollary 7.3): Let A be a ring and T a finetely generated free abelian monoid. Then the following conditions are equivalent, for $i = 0,1$:

(a) $NK_i A = 0$

(b) $K_i(A) \simeq K_i(A[T])$

(c) $K_i(A) \simeq K_i(A[X])$ where X is an indeterminate over A.

The next is a well known result of Grothendieck for $i = 0$, while the case $i = 1$ is due to Bass-Heller-Swan :

Theorem 2:(see[2],4.3 and 5.4) Let A be regular: then $NK_i A = 0$ for $i = 0,1$.

Theorem 2 can be extended to K_2, thanks to a recent result of D.Quillen (actually Quillen's result is valid for all his higher K_i's) :

Theorem 3 : ([9]): Let A be regular and T a finetely generated free abelian monoid . Then $K_2(A) \simeq K_2(A[T])$.

The following definition of seminormality and the characterization given in theorem 4, are due to Traverso ([11]).

Let $A \subset B$ be rings such that B is integral over A. We define the seminormalization of A in B to be the following ring :

$$ {}_B^{+}A = \{ x \in B \,/\, x \in A_p + \mathrm{Rad}(B_p),\ \forall\ p \in \mathrm{Spec}A \} $$

(where Rad means the Jacobson radical). If $A = {}_B^{+}A$, A is said to be seminormal in B; if B coincideswith the integral closure $\bar{A}$ of A in its total quotient ring and $A = {}_B^{+}A$, then A is said to be seminormal.

Theorem 4 : ([11],3.6) : Let A be a reduced noetherian ring such that $\bar{A}$ is finite over A. Then the canonical homomorphism $\mathrm{Pic}A \to \mathrm{Pic}A[T]$ is an isomorphism if and only if A is seminormal.

Now we recall (see[8]) how given a ring B and two prime ideals P_1, P_2 we can define a ring A in such a way that the conductor from B to A is $p_1 \cap p_2$, B is integral over A and A is seminormal in B.

Let B be a ring p_1, p_2 two distinct primes of B, $\varphi : B/p_1 \to B/p_2$ an isomorphism such that $\varphi(p_1+p_2/p_1) = (p_1+p_2/p_2)$. Then φ induces an automorphism $\tilde{\varphi} : B/p_1+p_2 \leftrightarrow B/p_1+p_2$. Let A be the ring

$$ A = \{ x \in B / x(p_2) = \varphi(x(p_1)) $$

where $x(p_i)$ in the image of x in B/p_i (i=1,2). We say that A is gotten from B by gluenig p_1 and p_2, via φ .

Theorem 5 : ([8],Teorema 1): Let B be a noetherian ring and A the ring gotten from

B by glueing twodistinct prime ideals p_1, p_2 via an is isomorphism φ such that $\tilde{\varphi}$ is the identity. Then :

a) B is integral over A

b) B is finite over A

c) A is noetherian

d) A is seminormal in B

e) The inclusion $A/(p_1 \cap p_2) \rightarrow B/p_i$ is an isomorphism (i=1,2)

Moreover if B is integrally closed and p_i is of height ≥ 1 (i=1,2) then B coincides with the integral closure of A.

The theorem above shows that, given an affine normal variety V and two irreducible subvarieties V_1 and V_2 of codimension ≥ 1, isomorphic under an isomorphsm φ which induces the identity on $V_1 \cap V_2$, we can glue V_1 and V_2 together and get a variety W whose normalization is V . W is always seminormal, hence PicA $\simeq$ PicA[T] if A is the coordinate ring of W. The following theorem gives a necessary and sufficient condition for the isomorphism PicA $\simeq$ PicA$[T,T^{-1}]$.

Theorem 6 :([8],Teorema 6): Let B be a normal ring,and A the ring gotten from B by gluenig p_1 and p_2 via on isomorphism φ such that $\tilde{\varphi}$ is the identity. Then the following conditions are equivalent:

(i) PicA $\simeq$ PicA$[T,T^{-1}]$

(ii) $p_1 + p_2 \neq B$

On analogous construction can be given in the case of a prime p and an automorphism of B/p :more precisely if B is a ring, p a prime ideal of B, φ an automorphism of B|p we define

$$A = \{ b \in B / \ \varphi(\bar{b}) = \bar{b} \}$$

to be the ring gotten from B by glueing p via φ .

We say that φ is locally finite if,for every $x \in B/p$, there exists a positive integer n(x) such that $\varphi^{n(x)}(x) = x$.

Then we have the following result :

Theorem 7. ([8],prop.9) : Let B be a noetherian reduced ring, p a prime ideal of B of height ≥ 1, φ a locally finite automorphism of B/p . Let A be the ring gotten from B by glueing p. Then

a) B is integral over A

b) A is seminormal in B

Moreover if B is integrally closed then B coincideswith the integral closure of A.

Theorem 8. ([8],Teorema 7): Let k be a field, B a finitely generated normal k-algebra, p a prime ideal of height $\geqslant 1$, φ a locally finite k-automorphism of B/p.

Let A be the ring gotten from B by glueing p. Then if B/p is normal we have $\mathrm{Pic}A \simeq \mathrm{Pic}A[T,T^{-1}]$.

2. In this section we give sufficient conditions for $NK_0A = 0$ in the case A is gotten from a regular domain B by glueing two distinct primes p_1, p_2.

Theorem 9: Let B be a noetherian regular ring and A the ring gotten from B by glueing two distinct primes p_1 and p_2 via an isomorphism φ such that $\bar{\varphi}$ is the identity. Then there is a canonical isomorphism :

$$NK_0A \simeq NK_1(B/p_1 \cap p_2)$$

Proof : By theorem 5,B is integral and finite over A and the ideal $b = p_1 \cap p_2$ is the conductor. Therefore the following diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/b & \longrightarrow & B/b \end{array}$$

is a cartesian square and so we get an exact sequence (theorem 1)

$$NK_1A \to NK_1(B) \oplus NK_1(A/b) \to NK_1(B/b) \to NK_0(A) \to NK_0(B) \oplus NK_0(A/b) \to NK_0(B/b)$$

Since B is regular we have (cfr.th.2): $NK_0B = NK_1B = 0$. By theorem 50,e) $A/b \simeq B/p_i$ and B/p_i is regular. This implies: $NK_0(A/b) = NK_1(A/b) = 0$.

So we get

$$NK_1(B/b) \simeq NK_0(A)$$

where the isomorphism is induced by the connecting homomorphism : $K_1(B/b[T]) \to K_0(A[T])$ of the Mayer-Vietoris sequence.

q.e.d.

Corollary 1 : Under the same hypothesis of theorem 9,assume either $p_1+p_2 = B$ or $B/(p_1+p_2)$ is regular. Then $NK_0(A) = 0$, i.e. $K_0(A) \simeq K_0(A[T])$

Proof : If $p_1+p_2 = B$, then $B/(p_1 \cap p_2) \simeq B/p_1 \oplus B/p_2$. Since B/P_i is regular (i=1,2), $NK_1(B/P_i) = 0$. Hence $NK_1(B/p_1+p_2) = 0$.

By theorem 9 we deduce $NK_0(A) = 0$.

Now assume $p_1+p_2 \neq B$ and B/p_1+p_2 regular. In the cartesian square:

$$\begin{array}{ccc} B/(p_1 \cap p_2) & \longrightarrow & B/p_1 \\ \downarrow & & \downarrow \\ B/p_2 & \longrightarrow & B/p_1+p_2 \end{array}$$

all the homomorphisms are surjective. There is the following exact sequence(theorem 1):

$NK_2(B/p_1\cap p_2)\to NK_2(B/p_1)\oplus NK_2(B/p_2)\to NK_2(B/p_1+p_2)\to NK_1(B/p_1\cap p_2)\to NK_1(B/p_1)\oplus NK_1(B/p_2)$

Since B/p_i is regular $NK_1(B/p_i)=0$ (i=1,2). Moreover the regularity of B/p_i and B/p_1+p_2 implies (see theorem 3).

$$NK_2(B/p_1) = NK_2(B/p_2) = NK_2(B/p_1+p_2) = 0$$

Therefore the exact sequence above yields

$$NK_1(B/p_1\cap p_2) = 0$$

From theorem 9 we get $NK_o(A) = 0$.

q.e.d.

The following proposition gives a geometric application of corollary 1.

Proposition 2 : Let V be an irreducible affine non singular variety, V_1 and V_2 two distinct irreducible non singular sub-varieties of V such that there exists an isomorphism φ between V_1 and V_2 which induces the identity on $V_1\cap V_2$. Suppose either $V_1\cap V_2=\emptyset$ or V_1 and V_2 meet transversally atevery point of $V_1\cap V_2$. Then if A is the coordinate ring of the variety W obtained by gluenig V_1 and V_2 via φ ,we have

$$K_o(A[T]) \simeq K_o(A)$$

Proof: Let $B = k[V]$ be the coordinate ring of V, $p_1=\mathfrak{J}(V_1),P_2=\mathfrak{J}(V_2)$. Then $B,B/p_1$ and B/p_2 are all regular. By theorem 9 :

$$NK_o(A) \simeq NK_1(B/p_1\cap p_2)$$

If $V_1\cap V_2 = \emptyset$ then $p_1+p_2 = B$,hence, by corollary 1, $NK_o(A) = 0$.

If $V_1\cap V_2 \neq \emptyset$ and V_1,V_2 meet transversally at every point of $V_1\cap V_2$, then for every maximal ideal p of B containing p_1 and p_2 the local ring $(B/p_1+p_2)_p=B_p/(p_1+p_2)B_p$ is regular. Hence B/p_1+p_2 is regular and by corollary 1, $NK_o(A) = 0$. q.e.d.

Examples : 1) Let k be a field, V the affine plane over k,V_1 the X-axis and V_2 the Y-axis. Define the isomorphism $\varphi:V_1\to V_2$ by sending (X,0) into (0,Y). Then the variety W obtained by glueing V_1 and V_2 is the following surface :

$$Y^3 + Z^2 - XYZ = 0$$

The singular locus of W in the X-axis, i.e. the intersection with the plane Y = 0. The coordinate ring of W is

$$A= k[X,Y,Z]/(Y^3+Z^2-XYZ) = k[x,y,z] \simeq k[u + v, uv, u^2v]$$

where u,v are indeterminates over k. We claim that

$$K_o(A) \simeq K_o(A\ T\ )\simeq \mathbb{Z}$$

By proposition 2 it is enough to show that $K_o(A)\simeq\mathbb{Z}$. Since V is the normalization of W (cfr. th.5)we have $\bar{A} = k[u,v]$and the ideal $b = (uv,u^2v)A = (uv)\bar{A}$ is the conductor. In the exact Mayer-Vietoris sequence:

$K_1A \to K_1(\bar{A}) \oplus K_1(A/b) \to K_1(\bar{A}/b) \to K_oA \to K_o(\bar{A}) \oplus K_o(A/b) \to K_o(\bar{A}/b)$

we have:

$$K_o(\bar{A}) \simeq \mathbb{Z} \ , \ K_o(A/b) \simeq \mathbb{Z}$$
$$K_o(\bar{A}/b) = K_o(k[u,v]/(uv)) = \mathbb{Z} + \mathrm{Pic}(\bar{A}/b) = \mathbb{Z}$$
$$K_1(\bar{A}) = K_1(k[u,v]) = K_1(k) = k^*$$
$$K_1(A/b) = k^*$$

Now we compute $K_1(\bar{A}/b) = K_1(k[u,v]/(uv))$. From the cartesian square of surjective homomorphisms :

$$\begin{array}{ccc} k[u,v]/(uv) & \longrightarrow & k[u] \\ \downarrow & & \downarrow \\ k[v] & \longrightarrow & k \end{array}$$

we deduce the following exact sequence :

$K_2(k[u,v]/(uv)) \to K_2(k[u]) \oplus K_2(k[v]) \to K_2(k) \to K_1(k[u,v]/(uv) \to K_1(k[u]) \oplus K_1(k[v]) \to K_1(k)$

Since k is regular, by theorem 3: $K_2(k[u]) = K_2(k[v]) = K_2(k)$. The exact sequence above yields

$$0 \to K_1(k[u,v]/(uv)) \to k^* \oplus k^* \to k^* \to 0$$

Therefore $K_1(k[u,v]/(uv)) \simeq k^*$, and the Mayer-Vietoris sequence becomes

$$K_1(A) \to k^* \oplus k^* \to k^* \to K_o(A) \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$$

From this we get $K_o(A) \simeq \mathbb{Z}$.

In the case k is algebraically closed it is actually possible to show that every projective A-module is free: this follows from[6]th.3.1 and from the fact that $\mathrm{Pic}A = 0$.

2) The following example shows that proposition 2 fails if V_1 and V_2 don't meet transversally. Let k be a field of characteristic $\neq 2$, $B = k[X,Y]$, $p_1=(Y-X^2)$, $p_2=(Y)$. Define the isomorphism $\varphi: B/p_1 \to B/p_2$ by $\varphi(x) = x$, $\varphi(y) = 0$. Clearly $\bar{\varphi}$ is the identity on $B/p_1 + p_2 = k[X,Y]/(Y,X^2)$. The ring gotten from B by gluenig p_1 and p_2 via φ is $A = k[X, Y(X^2-Y), Y^2(X^2-Y)]$. We want to compute $NK_1(B/p_1 \cap p_2)$ and show it does not vanish: this will imply, by theorem 9, $NK_oA \neq 0$.

$NK_2(B/p_1 \cap p_2) \to NK_2(B/p_1) \oplus NK_2(B/p_2) \to NK_2(B/p_1+p_2) \to NK_1(B/p_1 \cap p_2) \to NK_1(B/p_1) \oplus NK_1(B/p_2)$

B/p_i is regular $(i=1,2)$, hence $NK_1(B/p_i) = NK_2(B/p_i) = 0$ (see th.3). Therefore $NK_1(B/(p_1 \cap p_2)) \simeq NK_2(B/p_1 + p_2)$. Now we compute $NK_2(B/p_1 + p_2)$: we have $B/(p_1+p_2) \simeq k[X]/(X^2) = k[\varepsilon]$, with $\varepsilon^2=0$. By a result of Van der Kallen (see[12]), for any commutative ring R, such that $1/2 \in R$, there is a canonical isomorphism:

$$K_2(R[\varepsilon]) \simeq K_2(R) \oplus \Omega^1_{R/\mathbb{Z}}$$

where $\Omega^1_{R/\mathbb{Z}}$ denotes the module of differentials of A, as a $\mathbb{Z}$-algebra.

Therefore we have, since $K_2(k) \simeq K_2(k[T])$ (see th.3) :

$$K_2(k[\varepsilon]) \simeq K_2(k) \oplus \Omega^1_{k/\mathbb{Z}}$$

$$K_2(k[\varepsilon][T]) \simeq K_2(k[T]) \oplus \Omega^1_{k[T]/\mathbb{Z}} \simeq K_2(k) \oplus \Omega^1_{k[T]/\mathbb{Z}}$$

From the isomorphisms above we get:

$$NK_2(k[\varepsilon]) \simeq \Omega^1_{k[T]/k}$$

where $\Omega^1_{k[T]/k}$ is the module of differentials of $k[T]$ as a k-algebra, i.e. the free abelian group on $dt_1, \dots, dt_n$, if T is generated by $t_1, \dots, t_n$.

In conclusion

$$K_o(A[T]) \simeq K_o(A) \oplus \Omega^1_{k[T]/k}$$

We can actually compute $K_o(A)$ and show :

$$K_o(A) \simeq \mathbb{Z} \oplus \Omega^1_{k/\mathbb{Z}} \quad .$$

To do this observe that $B = k[X,Y]$ is the integral closure of A, $b = p_1 \cap p_2$ the conductor and $A/b = k[X]$, $B/b = k[X,Y]/(Y(X^2-Y))$.

Hence $\mathrm{Pic}\,B = \mathrm{Pic}(A/b) = 0$, $\mathrm{Pic}(B/b) = k$ (as an additive group) and $U(A/b) = U(B/b) = k^*$. These equalities imply $\mathrm{Pic}A = 0$. Moreover we have: $K_o(B) = K_o(A/b) = \mathbb{Z}$, $K_1(B) = k^*$ and $K_1(A/b) = k^*$. Write

$$K_o(A) \simeq H_o(A) \oplus \tilde{K}_o(A) \simeq \mathbb{Z} \oplus \tilde{K}_o(A)$$

where $\tilde{K}_o(A)$ is the kernel of the rank (see [2] p.459). Then we have the following commutative diagram with exact rows and columns (see [2], (5.12) :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & SK_1(B/b) & \longrightarrow & SK_o(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1(B/b) & \longrightarrow & \tilde{K}_o(A) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & k^* & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

So we are left prove $SK_1(B/b) \simeq \Omega^1_{k/\mathbb{Z}}$. In the Mayer-Vietoris sequence:

$$K_2(B/p_1 \cap p_2) \to K_2(B/p_1) \oplus K_2(B/p_2) \to K_2(B/p_1+p_2) \to K_1(B/p_1 \cap p_2) \to K_1(B/p_1) \oplus K_1(B/p_2) \to$$
$$\to K_1(B/p_1 + p_2) \to \dots$$

we have :

$K_2(B/p_1) = K_2(B/p_2) = K_2(k)$; $K_2(B/p_1 + p_2) = K_2(k[\varepsilon]) \simeq K_2(k) + \Omega^1_{k/\mathbb{Z}}$

$K_1(B/p_1) = K_1(B/p_2) = k^*$; $K_1(B/p_1 + p_2) = K_1(k[\varepsilon]) = k^* \oplus k$.

Hence we get the isomorphism

$$K_1(B/b) \simeq k^* \oplus \Omega^1_{k/\mathbb{Z}}$$

which implies, since $U(B/b) = k^*$, $SK_1(B/b) \simeq \Omega^1_{k/\mathbb{Z}}$

3. In this section we compute $NK_0(A)$ in the case A is gotten from a regular domain B by glueing a non-zero prime ideal p via an automorphism φ of B/p.

We will always assume φ is locally finite so that B is integral over A and A is seminormal in B (cfr. th.7)

Theorem 10 : Let B be a noetherian regular ring ,p an non-zero prime ideal of B and φ a locally finite automorphism of B/p . Assume B/p is regular. Then we have a canonical isomorphism :

$$NK_0 A \simeq NK_0(A\ p)$$

where A is the ring gotten from B by glueing p via φ .

Proof: Since p is the conductor from B to A we have the following cartesian square:

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/p & \longrightarrow & B/p \end{array}$$

and so we get an exact sequence (theorem 1)

$$NK_1 A \to NK_1(B) \oplus NK_1(A/p) \to NK_1(B/p) \to NK_0 A \to NK_0 B \oplus NK_0(A/p) \to NK_0(B/p)$$

Since B and B/p are regular, $NK_i(B) = NK_i(B/p) = 0$, $i = 0,1$. Therefore the exact sequence above yields $NK_0 A \simeq NK_0(A/p)$

q.e.d.

Remark : Under the assumptions of theorem 10, A/p is not necessarily regular.

Let $B = k[X,Y,Z]$, $p = (Z)$, $\varphi : k[X,Y] \to k[X,Y]$ defined by $\varphi(X) = -X$, $\varphi(Y) = -Y$. Then $A = k[X^2,Y^2,X,Y,Z,XZ,YZ]$, $A/p = k[X^2,Y^2,XY]$: therefore A/p is not regular.

Now we want to apply theorem 9 in the case p has codimension 1. To do this we need the following lemma :

Lemma 1 : Let R be an integral domain, L its field of fractions, $\overline{R}$ the integral closure of R in L . Let G be a locally finite group of operators on R and let $S = R^G = \{x \in R / g(x) = x \ , \forall\, g \in G\}$. Then

$$\overline{S} = (\overline{R})^G$$

where $\overline{S}$ is the integral closure of S in its field of fractions. In particular, if R is normal, then S is also normal

Proof: Let K be the field of fractions of S : then G acts on L and $L^G = K$ (cfr. [4], p.34). Let $x \in K$ be integral over S: then $x \in K \cap \overline{R} = (L)^G \cap \overline{R} = (\overline{R})^G$. Conversely, if

$x \in (\bar{R})^G$ then x is integral over R : since R is integral over S (cfr.[4],p.33), x is integral over S and $x \in L^G = K$. Therefore $x \in \bar{S}$.

q.e.d.

Corollary 3 : Let V be a non-singular affine variety and C an irreducible non-singular curve on V. Let φ be an automorphism of finite order of C and let W be the variety gotten from V by glueing C via φ .Then $NK_0A = 0$, if A is the coordinate ring of W.

Proof : Let $B = k[V]$, $p = \mathfrak{J}(C)$, $B' = B/p$, $A' = A/p$. Let n be the order of φ and $G = \{1, \varphi, \varphi^2, \dots, \varphi^{n-1}\}$. Then the group G acts on B, is finite and $A' = (B')^G$. Since B' is regular it is also normal. By lemma 1 A' is normal: therefore A' is the coordinate ring of a normal curve, hence non-singular. This implies A' is regular . By theorem 10 $NK_0A = 0$.

q.e.d.

Corollary 4 : Let k be a field of characteristic $\neq$ 2 and let $A = k[x,y,z]$ with $xy^2 - z^2 = 0$. Then $K_0(A) \simeq K_0(A[T]) \simeq \mathbb{Z}$

Proof : Evidently $A \simeq k[X^2, Y, XY]$. Let $B = k[X,Y]$, $p = (Y)$ and define an automorphism φ of $B/p = k[X]$ by $\varphi(X) = -X$. Then A is the ring gotten from B by glueing p via φ and B is the integral closure of A. Thus we have the following exact Mayer-Vietoris sequence :

$$K_1A \to K_1(B) \oplus K_1(A/p) \to K_1(B/p) \to K_0(A) \to K_0(B) \oplus K_0(A/p) \to K_0(B/p)$$

where $A/p = k[X^2]$, $B/p = k[X]$. Computing $K_0(A)$ in the exact sequence above we get $K_0(A) \simeq \mathbb{Z}$. By corollary 3 $NK_0(A) = 0$, hence $K_0(A) \simeq K_0(A[T])$.

q.e.d.

We conclude this section with an example of a glueing over a singular curve(a case where corollary 3 does not apply), such that $K_0(A) \neq K_0(A[T])$.

Let k be a field of characteristic not 2 and let $B = k[X,Y]$, $p = (X^3 - Y^2)$: then $B/p \simeq k[s^2, s^3]$ where s is an indeterminate over k. Define an automorphism φ of B/p by $\varphi(s) = -s$. The ring A gotten from B by glueing p is the following (cfr.[8],§3)):

$$A = k[X, Y^2, Y(X^3 - Y^2)]$$

and B is its normalization. A is the coordinate ring of a surface,whose singular locus is the curve $Y = X^3$ of the plane $Z = 0$. We claim $NK_0A \neq 0$: more precisely we want to show

$$NK_0A \simeq NK_1(k[s^2, s^3]) \neq 0 .$$

From the cartesian square :

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A/p & \longrightarrow & B/p \end{array}$$

we get, as usual, the following exact sequence.

$NK_1A \to NK_1(A/p) \oplus NK_1B \to NK_1(B/p) \to NK_0A \to NK_0(A/p) \oplus NK_0(B) \to NK_0(B/p)$.

Now we have : $A/p \simeq k[s^2]$, $B/p \simeq k[s^2, s^3]$. Thus

$$NK_iB = NK_i(A/p) = 0 \qquad i = 0,1 .$$

So we have an isomorphism $NK_0A \simeq NK_1(k[s^2, s^3])$, and we are left to show $NK_1R \neq 0$, where $R = k[s^2, s^3]$. Let $\bar{R} = k[s]$ be the integral closure of $R, b = (s^2, s^3)R = (s^2)\bar{R}$ the conductor. Consider the split epimorphism of exact sequences induced by the augmentation (see [5], §6)

$$\begin{array}{ccccccccccc} K_2(R/b[T]) & \to & K_1(R[T], bR[T]) & \to & K_1(R[T]) & \to & K_1(R/b[T]) & \to & K_0(R[T], bR[T]) & \longrightarrow & K_0(R[T]) \\ \downarrow\simeq & & \downarrow & & \downarrow & & \downarrow\simeq & & \downarrow & & \downarrow \\ K_2(R/b) & \to & K_1(R,b) & \longrightarrow & K_1(R) & \to & K_1(R/b) & \to & K_0(R,b) & \longrightarrow & K_0(R) \end{array}$$

where the indicated isomorphisms are a consequence of the regularity of R/b (th.2 and 3).

Let $G = \mathrm{Ker}(K_1(R[T], bR[T]) \to K_1(R,b))$: then from the diagram above

$$0 \to G \to NK_1R$$

So if we show $G \neq 0$ we are done. In the commutative diagram

$$\begin{array}{ccc} K_1(R[T],\ bR[T]) & \longrightarrow & K_1(\bar{R}[T],\ b\bar{R}[T]) \\ \downarrow & & \downarrow \\ K_1(R,b) & \longrightarrow & K_1(\bar{R},b) \end{array}$$

the orizontal maps are epimorphisms. For since $GL(R,b)$ and $GL(\bar{R},b)$ both consist of matrices $\alpha \in GL(\bar{R})$ such that $I - \alpha$ and $I - \alpha^{-1}$ have coordinates in b, we have $GL(R,b) = GL(\bar{R},b)$. Thus the map

$$G \to \mathrm{Ker}(K_1(\bar{R}[T], b\,\bar{R}[T]) \to K_1(\bar{R},\bar{b}) = H$$

is an epimorphism. So it is enough to show the group H does not vanish. From the split epimorphism of exact sequences

$$\begin{array}{ccccccccc} K_2(\bar{R}[T]) & \to & K_2(\bar{R}/b[T]) & \to & K_1(\bar{R}[T], b\,\bar{R}[T]) & \to & K_1(\bar{R}[T]) & - & \cdots \\ \downarrow\simeq & & \downarrow & & \downarrow & & \downarrow\simeq & & \\ K_2(\bar{R}) & \longrightarrow & K_2(\bar{R}/b) & \to & K_1(\bar{R},\ b) & \longrightarrow & K_1(\bar{R}) & \to & \cdots \end{array}$$

we deduce, since $\bar{R}$ is regular (cfr.theorem 3)

$$NK_2(\bar{R}/b) \simeq H$$

Now $\bar{R}/b \simeq k[\varepsilon]$,with $\varepsilon^2 = 0$ and ,by[12], $NK_2(k[\varepsilon]) \simeq \Omega^1_{k/\mathbb{Z}} \neq 0$.

4. In this section we prove a sufficient condition (Corollary 5), for $NK_0A = 0$, in the case A is a commutative ring containing an algebraically closed field.

As a corollary of this result we prove (Proposition 5) $K_0(A) \simeq K_0(A[T]) \simeq \mathbb{Z}$ if $A = k[x,y,z]$, $z^n = xy$. Note that $k[x,y,z]$ is normal but not regular, while all the examples considered in the previous sections were seminormal but not normal. The results of this section have been obtained jointly with M.P. Murthy.

Lemma 2 : Let A be a ring, t an indeterminate over A and a an element of A. Then the canonical homomorphism

$$K_0(A[t]) \longrightarrow K_0(A[t,(t-a)^{-1}])$$

is injective

Proof: Let $s = t-a$: then s is an indeterminate over A and $A[t,(t-a)^{-1}] = A[s,s^{-1}]$. Let T be the infinite cyclic group with generator s, T_+ the submonoid generated by s^{-1}. Then the inclusions $f_{\pm}: A[T_{\pm}] \subset A[T]$ induce a homomorphism

$$f: K_0(A[T_+]) \oplus K_0(A[T_-]) \xrightarrow{(f_+, f_-)} K_0(A[T])$$

and the following sequence

$$0 \to K_0(A) \to K_0(A[T_+]) \oplus K_0(A[T_-]) \xrightarrow{f} K_0(A[T])$$

is exact ([2],Corollary 7.6). Thus f_+ and f_- are both monomorphisms.

Since $A[T_+] = A[s]$, $A[T] = A[s,s^{-1}]$ our assertion follows .

q.e.d.

Lemma 3 : Let k be a field, A a ring containing k and t an indeterminate over A: if M is a A[t]-module such that $g(t)M = 0$, $g(t) \in k[t]^*$, then there exist submodules $N_1,\dots,N_n$ of M with the following properties :

1) $M = N_1 \oplus \dots \oplus N_h$

2) $g_i(t)N_i = 0 \qquad (1 \le i \le h)$

where $g_i(t) \in k[t]$ and $g_i(t)/g(t)$.

Proof: Let $g(t) = p_1(t)^{s_1} \cdots p_h(t)^{s_h}$ be the decomposition of g(t) into distinct irreducible factors in k[t]. Let $N_i = f_i(t)M$, where $f_i(t) = \prod_{j \neq i} p_j(t)^{s_j}$. Clearly the N_i's verify 2) with $g_i(t) = p_i(t)^{s_i}$. Since $g.c.d.(f_1, f_2, \dots, f_n) = 1$ in k[t] we have

$$\sum_{i=1}^{h} f_i(t)A[t] = A[t]$$

and

$$N_1 + N_2 + \dots + N_h = M$$

Let $x_i \in N_i$ be such that $x_1 + \dots + x_h = 0$; multiplying by $f_i(t)$ we get $f_i(t)x_i = 0$. On the other hand, since $x_i \in N_i, g_i(t)x_i = 0$. Now g c d $(f_i, g_i) = 1$ in $k[t]$, hence $f_i(t)$ and $g_i(t)$ generate the unit ideal in A t . This implies :

$$\mathrm{Ann}_{A[t]} x_i = A[t]$$

i.e. $x_i = 0$. q.e.d.

<u>Proposition</u> 3: <u>Let</u> k <u>be a field,</u> A <u>a ring containing</u> k <u>and</u> t <u>an indeterminate over</u> A . <u>Set</u> : $k(t) \otimes_k A = k(t)A$. <u>Then the map, induced by</u> $A \to k(t)A$:

$$K_0(A) \xrightarrow[\Phi]{} K_0(k(t)A)$$

<u>is a monomorphism</u> .

<u>Proof</u>: Let P , Q be elements of $K_0(A)$ such that $\Phi([P]) = \Phi([Q])$. We want to show $[P] = [Q]$. We have : $[P \otimes_A k(t)A] = Q \otimes_A k(t)A$ in $K_0(k(t)A)$. Since P and Q are both finitely generated there exists a non-zero polynomial $f(t) \in k[t]$ such that:

$$[P \otimes_A A[t, f^{-1}]] = [Q \otimes_A A[t, f^{-1}]]$$

in $K_0(A[t, f^{-1}])$. Let n be a positive integer and let $g(t) \in k[t]$ be monic and such that g.c.d. $(g, f) = 1$. Then we have

$$A[t, f^{-1}]/(g) = A[t]/(g)$$

Tensoring by $A[t, f^{-1}]/(g)$ gives:

$$[P \otimes_A A[t]/(g)] = [Q \otimes_A A[t]/(g)]$$

Since $A[t]/g(t)$ is a free A-module of rank n the equality above yields : $n[P] = n[Q]$ in $K_0(A)$. But n is an arbitrary positive integer : hence $[P] = [Q]$. q.e.d.

<u>Theorem</u> 10 : <u>Let</u> k <u>be an algebraically closed field and let</u> A <u>be a ring containing</u> k. <u>Set</u> : $k(t)A = k(t) \otimes_k A$, <u>where</u> t <u>is an indeterminate over</u> A. <u>Then the homomorphism</u>

$$K_0(A[t]) \longrightarrow K_0(k(t)A)$$

<u>is injective</u>

<u>Proof</u>: Let $S = \{f(t)/f(t) \in k[t] - 0\}$: S is a moltiplicative set of non-zero divisors in $A[t]$ and $k(t)A \simeq A[t]_S$.

The homomorphism $A[t] \to A[t]_S$ induces the following exact sequence (see [3], th. 4.4) :

$$K_1(k(t)A) \to K_0(\underline{\underline{H}}_S(A[t])_1) \xrightarrow{\Delta} K_0(A[t]) \to K_0(k(t)A)$$

where $\underline{\underline{H}}_S(A[t])_1$ denotes the category of $A[t]$-modules which have a finite resolution of length ≤ 1 by modules in $\underline{\underline{P}}(A)$, and are annihilated by some element of S. We need to show $\mathrm{Im}\,\Delta = 0$.

Let $M \in \underline{\underline{H}}_S(A[t])_1$, $g(t)M = 0$ with $g(t) \in k[t] - (0)$ monic. Since k is algebraically closed there exist $a_1,\dots,a_r$ distinct in k, such that $g(t)=(t-a_1)^{s_1}\cdot\ \dots\ \cdot(t-a_r)^{s_r}$,

By lemma 3 we can find submodules $N_1,\dots,N_r$ of M such that:

$$M = N_1 \oplus \dots \oplus N_r \ ;\ (t-a_i)^{s_i} N_i = 0 \qquad (1 \le i \le r)$$

Let e_{ij} $(1 \le j \le h_i)$ be a set of generators of N_i $(1 \le i \le r)$ and let F be a free module, of rank $m = \sum_{i=1}^{r} h_i$, on the set $\{e_{ij}\}$. Set $P = \operatorname{Ker} f$ where f is the surjection $F \to M$; since $hd_{A[t]}M \le 1$, P is projective. Now define F_i $(1 \le i \le r)$ to be a free module on $e_{i1},\dots,e_{ih_i}$, and let $f_i : F_i \to N_i$. Then $F = \oplus(F_i)$, and $P = \operatorname{Ker} f = \oplus(\operatorname{Ker} f_i)$. This implies $P_i = \operatorname{Ker} f_i$ is projective and

$$0 \to P_i \to F_i \to N_i \to 0$$

is a projective resolution of N_i. So $N_i \in \underline{\underline{H}}_{S_i}(A[t])_1$ where $S_i = \{(t-a_i)^n / n \geq 0\}$.

In the exact sequence, relative to the localization $A[t] \to (A[t])_{S_i} = A[t,(t-a_i)^{-1}]$:

$$K_1(A[t,(t-a_i)^{-1}]) \to K_o(\underline{\underline{H}}_{S_i}(A[t])_1) \xrightarrow{\delta_i} K_o(A[t]) \to K_o(A[t,(t-a_i)^{-1}])$$

we have : $\operatorname{Im}\delta_i = 0$ (lemma 2). So it is enough to show: $\Delta(M) = \sum_{i=1}^{r} \delta_i([N_i])$ in $K_o(A[t])$. $\delta_i[N]$ is defined to be $[P_i]-[F_i]$ (see [3], th.4.4) and we have

$$\Delta([M]) = [P]-[F] = \sum_{i=1}^{r}([P]_i-[F_i]) = \sum_{i=1}^{r}(\delta_i(N_i)) = 0$$

q.e.d.

No we put together proposition 3 and theorem 10 to get our desired result on $N\,K_oA$.

<u>Corollary</u> 5 :<u>Let</u> k <u>be an algebraically closed field</u>, A <u>a ring containing</u> k <u>and</u> t <u>an indeterminate over</u> A. <u>Assume</u> $K_o(A) \to K_o(k(t)A)$ <u>is surjective</u>. <u>Then</u> $NK_o(A) = 0$.

<u>Proof</u>: From the commutative triangle :

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & k(t)A \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle h} & \\ & A[t] & \end{array}$$

where g is the inclusion $A \subset A[t]$, we get

$$\begin{array}{ccc} K_o(A) & \xrightarrow{\ K_o(f)\ } & K_o(k(t)A) \\ & {\scriptstyle K_o(g)}\searrow \quad \nearrow{\scriptstyle K_o(h)} & \\ & K_o(A[t]) & \end{array}$$

By our hypothesis and prop.3 K_o of is an isomorphism. From theorem 10 we deduce $K_o(h)$ is injective. Since the diagram above commutes, $K_o(g)$ is surjective, hence an isomorphism.

q.e.d.

Now we record a result in [6] (corollary 5.3), based upon a theorem of Bass-Murthy (see [3],prop.9.6).

Proposition 4. Let K be a field and let $A = K[x,y,z], z^n = xy$. Then any projective A-module is free.

Proof : For any non-zero element $a \in K$, $(y-a)A$ is an invertible prime ideal and $A/(y-a) \simeq K[Z]$. This implies $(y-a)$ is a special prime ideal (see [6],§1 ;$A/(y-a)$ is generalized euclidean in the terminology of [2],p.197). Let S be the special multiplicative set of ideals generated by the primes $(y-a), a \in K^*$. Evidently

$$S^{-1}A \simeq (K[Y,Z])_{S_o}$$

where S_o is the multiplicative set of A generated by the elements $(y-a)$. $R=K[Y,Z]$ is regular of dimension 2 and every projective R-module is free :therefore every projective module over $R_{S_o} = S^{-1}A$ is free (see[3],lemma 9.8).

By a result of Bass-Murthy (which uses an argument of Seshadri)(see [3],prop.9.6) every projective A-module is a direct sum of a free A-module and a projective module of rank 1. Moreover A is normal and can be made into a graded ring by attaching suitable positive degrees to x and y : thus PicA = 0 (see [6],lemma 5.1).

So every projective A-module is free.

Proposition 5 : Let k be an algebraically closed field and let $A = k[x,y,z], z^n = xy$. Then, if T is a finetely generated free abelian monoid :

$$K_o(A) \simeq K_o(A[T]) \simeq \mathbb{Z}$$

Proof : Let t be an indeterminate over A and let $K = k(t)$.

Then :

$$k(t) \otimes_k A = k(t)A = k(t)[x,y,z] = K[x,y,z], \quad z^n = xy \quad .$$

By proposition 4, every projective A - module is free and every projective k(t)A-module is free. Hence

$$K_o(k(t)A) = K_o(A) \simeq \mathbb{Z}$$

By Corollary 5, we have $NK_oA = 0$ and this is equivalent to our statement (see prop.1).

q.e.d.

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K_0 AND K_1 OF POLYNOMIAL RINGS

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Introduction. Let A be a ring and $f \in A[x]$ a monic polynomial with central coefficients. In §1, we show that the natural map $K_i(A[x]) \to K_i(A[x, 1/f])$ is injective for $i = 0, 1$ (see Th. 1.3). In §2, we apply this to obtain some information about K_0 and K_1 of affine algebras over 'big' algebraically closed fields. For example, we show that for such an algebra A, $SK_1(A)$ is of finite rank implies that $K_0(A)$ is a torsion group. In §3, using Th. 1.3, we produce examples of non-regular normal rings A with $K_0(A) \approx K_0(A[x_1, \ldots, x_n])$.

In this paper, we consider only rings with unit element and finitely generated modules over them. We use freely the notation and results of [1], notably that of Ch. XII. For a ring A and $f \in$ centre (A), we denote by A_f the ring of quotients A_S with $S = \{1, f, f^2, \ldots\}$ and $U(A)$ denotes the group of units of A.

§1. Let F be a functor from rings to abelian groups with the following property: for any ring homomorphism $i: A \to B$ which makes B a free A-module of rank n, there exists a homomorphism ('norm') $N_{B/A}: F(B) \to F(A)$ such that $N_{B/A} F(i)$ is multiplication by n.

Lemma 1.1. Let F be as above and A a ring. Let $h \in A[X]$ be a monic polynomial with coefficients in the centre of A.

(a) The map $F(i): F(A) \to F(A[X, 1/h])$ is injective (i = inclusion $A \subset A[X, 1/h]$).

(b) Let F commute with direct limits. Let k be a field and A a k-algebra. Then the natural map $F(A) \to F(A \otimes_k k(X))$ is injective.

Proof. (b) easily follows from (a). We prove (a). Let h be of degree n. Since $A[X]/(h-1)$ and $A[X]/(Xh-1)$ are A-free of rank n and $n+1$ respectively, the natural maps $A \xrightarrow{i} A[X, 1/h] \longrightarrow A[X]/(h-1)$ and $A \xrightarrow{i} A[X, 1/h] \longrightarrow A[X]/(Xh-1)$ and the existence of 'norm' map for F implies that $\ker F(i)$ has both n-torsion and $(n+1)$-torsion. Hence $\ker F(i) = 0$.

Remark. The lemma above applies notably to K_i, $i = 0, 1, 2$.

Lemma 1.2. Let A be a ring and $a, b \in A$ be non-zero-divisors contained in the center of A. Let $Aa + Ab = A$. Then the natural map

$$\ker(K_1 A \to K_1 A_{ab}) \to \ker(K_1 A_a \to K_1 A_{ab})$$

is surjective.

Proof. For $r \in \text{Centre}(A)$, let $K_0(\underline{\underline{H}}r)$ denote the Grothendieck group of finitely generated A-modules M with finite projective resolutions by finitely generated projective A-modules and $M_r = 0$. Then by [1, p. 494, Th. 6.3], we have the following commutative diagram with vertical rows exact.

$$\begin{array}{ccc}
K_0\underline{\underline{H}}_a & \longrightarrow & K_0\underline{\underline{H}}_{ab} \\
\uparrow & & \uparrow \\
K_1 A_a & \longrightarrow & K_1 A_{ab} \\
\uparrow & & \uparrow \\
K_1 A & = & K_1 A
\end{array}$$

The map $K_1\underline{\underline{H}}_a \to K_1\underline{\underline{H}}_{ab}$ is injective. In fact, since $Aa + Ab = A$, we have a split exact sequence $0 \to K_0\underline{\underline{H}}_a \to K_0\underline{\underline{H}}_{ab} \to K_0\underline{\underline{H}}_b \to 0$. Now the proof of the lemma is immediate.

Theorem 1.3. Let A be a ring and $f \in A[X]$ a monic polynomial with coefficients in the centre of A. Then

$$K_i(A[X]) \to K_i(A[X, 1/f])$$

is injective for $i = 0, 1$.

Proof. Since K_1 is a contracted functor with $LK_1 = K_0$ [1, Ch. XII], it is sufficient to prove the theorem for $i = 1$. Let $f = X^n + a_{n-1}X^{n-1} + \ldots + a_0$. We write $f = g(X^{-1})\cdot X^{-n}$, where $g(X^{-1}) = 1 + a_{n-1}X^{-1} + \ldots + a_0X^{-n}$. Let $\alpha \in \ker(K_1(A[X]) \to K_1(A[X, 1/f]))$ and α' the image of α under the natural map $K_1(A[X]) \to K_1(A[X, X^{-1}])$. Clearly $\alpha' \in \ker(K_1(A[X, X^{-1}]) \to K_1(A[X, X^{-1}, 1/f])$. But $A[X, X^{-1}, 1/f] = A[X^{-1}, 1/X^{-1}g(X^{-1})]$. Also $A[X^{-1}]X^{-1} + A[X^{-1}]g(X^{-1}) = A[X^{-1}]$ and $X^{-1}, g(X^{-1})$ are non-zero-divisors in $A[X^{-1}]$. Hence by Lemma 1.2,

$$\ker(K_1(A[X^{-1}]) \to K_1(A[X^{-1}, 1/X^{-1}g(X^{-1})]))$$
$$\to \ker(K_1(A[X^{-1}, X]) \to K_1(A[X^{-1}, 1/X^{-1}g(X^{-1})]))$$

is surjective. Therefore there is a $\beta \in K_1(A[X^{-1}])$ such that $\beta' = \alpha'$, where β' is the image of β under the natural map $K_1(A[X^{-1}]) \to K_1(A[X^{-1}, X])$. Since K_1 is a contracted functor this implies $\alpha \in K_1(A)$ (we identify $K_1(A)$ as a subgroup of $K_1(A[X])$. Hence $\alpha \in \ker(K_1(A) \to K_1(A[X, 1/f]))$. Now by Lemma 1.1, $\alpha = 0$. This finishes the proof of Theorem 1.3.

Corollary 1.4. Let k be a field and A a k-algebra. The natural map $K_i(A[X_1, \ldots, X_n]) \to K_i(A \otimes_k k(X_1, \ldots, X_n))$ is injective for $i = 0, 1$.

Proof. By induction, we are reduced to the case $n = 1$. Since $K_i(A \otimes_k k(X)) = \varinjlim_{f \in k[X]} K_i(A[X, 1/g])$, the corollary follows from Theorem 1.3.

Corollary 1.5 Let k be a field and A a k-algebra and $f \in k[X]$. Then

$$K_i(A[X, 1/f]) \to K_i(A \otimes_k k(X))$$

is injective $(i = 0, 1)$.

Proof. It is sufficient to prove that for $g \in k[X]$, the map $K_i(A[X, 1/f]) \to K_i(A[X, 1/fg])$ is injective. Also, we may assume f does not divide g so that f, g generate the unit-ideal in $A[X]$. Then by Lemma 1.2,

$$\ker(K_i(A[X]) \to K_i(A[X, 1/fg])) \to \ker(K_i(A[X, 1/f]) \to K_i(A[X, 1/fg]))$$

is surjective. But by Theorem 1.3,

$$K_i(A[X]) \to K_i(A[X, 1/fg])$$

is injective. This proves Corollary 1.5.

Remark 1.6. Let F be a functor from rings to abelian groups. We write $NF(A) = \ker(F(A[X]) \xrightarrow{X \mapsto 1} F(A))$ and $LF(A) = \mathrm{Coker}(F(A[X]) \oplus F(A[X^{-1}]) \to F(A[X, X^{-1}]))$. Using the fact that $L^i N^j K_1$ are contracted functors and L, N commute [1, p. 661, Prop. 7.2], it is easy to see by induction on $i+j$ that Theorem 1.3 and its corollaries remain valid for functors $L^i N^j K_1$. Also they remain valid for SK_1 and $\widetilde{K}_0$ ($\widetilde{K}_0(A) = \ker(K_0(A) \xrightarrow{\text{rank}}$ (continuous functions from Spec A to $\mathbb{Z}$)).

Remark 1.7. With the hypotheses and notation as in Theorem 1.3 we do not know if the $\mathrm{Im}(K_i(A[X]) \to K_i(A[X,1/f]))$ is a direct summand of $K_i(A[X,1/f])$ ($i = 0,1$). Also we do not know a good interpretation for $\mathrm{Coker}(K_i(A[X]) \to K_i(A[X,1/f]))$. But we have the following

Proposition 1.8. Let A be a ring and $a_1, \ldots, a_r$ elements contained in the centre of A. Suppose that $i \neq j$ implies $a_i - a_j$ is a unit in A. Let $g = \prod_{j=1}^{r} (X - a_j)^{m_j}$ with $m_j > 0$ for all j. Then there is a natural split exact sequence

$$0 \to K_i(A[X])) \to K_i(A[X,1/g]) \to (LK_i(A))^r \oplus (NK_i(A))^r \to 0 ,$$

so that

$$K_i(A[X,1/g]) \approx K_i(A) \oplus (NK_i(A))^{r+1} \oplus (LK_i(A))^r .$$

(Here $i = 0$ or 1).

Proof. Again since K_1 is a contracted functor with $LK_1 = K_0$, it is sufficient to prove the proposition for $i = 1$. The hypothesis on a_i means that $X-a_i$ and $X-a_j$ generate a unit-ideal in $A[X]$ for $i \neq j$. Hence

$$K_0(\underline{\underline{H}}_g(A[X])) \approx \sum_{j=1}^{r} K_0(\underline{\underline{H}}_{(X-a_j)}(A[X]) .$$

Since by [1, p. 654, Prop. 6.4], $K_0(\underline{\underline{H}}_{(X-a_j)}(A[X])) \approx K_0(A) \oplus \mathrm{nil}(A)$, we have have, $K_0(\underline{\underline{H}}_g(A[X])) \approx (K_0(A) \oplus \mathrm{nil}(A))^r$. We have exact sequences

$$\begin{array}{ccccccc} K_1(A[X]) & \to & K_1(A[X,1/g]) & \xrightarrow{\partial} & K_0(\underline{\underline{H}}_g(A[X])) & \approx & (K_0A \oplus \mathrm{nil}\, A)^r \\ \| & & \uparrow \varphi_j & & & & \\ K_1 A[X] & \to & K_1 A[X, 1/X-a_j] & \xrightarrow{\partial_j} & K_0(\underline{\underline{H}}_{X-a_j}(A[X])) & \approx & K_0A \oplus \mathrm{nil}\, A \end{array} .$$

By [1,p. 666,Prop. 7.5] we have $h_j: K_0A \oplus \mathrm{nil}(A) \to K_1(A[X,1/X-a_j])$ such that $\partial_j \circ h_j = {}^1(K_0A \oplus \mathrm{nil}(A))$. Let p_j denote the j-th projection $(K_0(A) \oplus \mathrm{nil}(A))^r \to K_0A \oplus \mathrm{nil}\, A$. Define $h: (K_0(A) \oplus \mathrm{nil}(A))^r \to K_1A[X,1/g]$ by $h = \sum_{j=1}^{r} \varphi_j \circ h_j \circ p_j$. It is easy to verify (writing explicitly the maps ∂ and ∂_j) that $\partial_j = p_j \circ \partial \circ \varphi_j$. This implies that $\partial \circ h = \text{identity}$.

Corollary 1.9. Let k be an algebraically closed field and A a k-algebra. If $f \in k[X]$ has r distinct roots, then

$$K_1(A[X, 1/f]) \approx K_1(A) \oplus (\mathrm{nil}(A))^{r+1} \oplus (K_0(A))^r$$

$$K_0(A[X, 1/f]) \approx K_0(A) \oplus (NK_0(A))^{r+1} \oplus (LK_0(A))^r .$$

Remark 1.10. It is easy to see that with the hypothesis as in Corollary 1.9, $K_i(A[X, 1/f])$ is a direct summand of $K_i(A \otimes_k k(X))$, $i = 0, 1$. Also

$$K_i(A \otimes k(X)) = K_i(A[X]) \oplus \sum_{a \in k} M_a ,$$

where each $M_a \approx NK_i(A) \oplus LK_i(A)$, $(i = 0, 1)$.

§2. K_0 and K_1 of affine algebras over big algebraically closed fields

Throughout this section k denotes an algebraically closed field of infinite transcendence degree over its prime field. We apply Theorem 1.3 to obtain some information about K_i $(i = 0, 1)$ of affine algebras over k. Let A be a finitely generated commutative algebra over k. We write

$$A \approx \frac{k[T_1, \ldots, T_m]}{(f_1, \ldots, f_r)} .$$

Let K be the algebraic closure of $k(X_1, \ldots, X_n)$ and let F be a sub-field of k, finitely generated field over the prime field containing all the coefficients of $f_1, \ldots, f_r$. Since k is of infinite transcendence degree over its prime field, there is an F-isomorphism $\sigma: k \to K$ which clearly extends to an isomorphism

$$\bar{\sigma}: \ A \approx \frac{k[T_1, \ldots, T_m]}{(f_1, \ldots, f_r)} \approx \frac{K[T_1, \ldots, T_m]}{(f_1, \ldots, f_r)} \approx A \otimes_k K .$$

Proposition 2.1. Let A and k be as above. Let F denote $SK_1, \widetilde{K}_0$ or $L^i N^j K_1$ $(i \geq 0, j \geq 0)$. If $F(A)$ is of finite rank, then $NF(A)$ and $LF(A)$ are torsion groups.

Proof. By Corollary 1.5 and Remark 1.6,

$$F(A[X, X^{-1}]) \to F(A \otimes_k k(X))$$

is injective. Let K denote the algebraic closure of $k(X)$. Then $\ker(F(A \otimes_k k(X)) \to F(A \otimes_k K))$ is torsion. (This is easily seen using the 'norm' map.) Hence $\ker(F(A[X, X^{-1}]) \to F(A \otimes_k K))$ is torsion. But $F(A[X, X^{-1}]) \approx F(A) \oplus NF(A) \oplus NF(A) \oplus LF(A)$. Since $A \otimes_k K \approx A$ (see above) and $F(A)$ is of finite rank, we see that $NF(A)$ and $LF(A)$ are torsion groups.

Taking $F = SK_1$ and using $LSK_1 = \widetilde{K}_0$ [1, p. 673, Cor. 7.9] we get

Corollary 2.2. $SK_1(A)$ finite rank implies $\widetilde{K}_0(A)$ is a torsion group.

Corollary 2.3.* $NK_i(A)$ torsion $\Longrightarrow$ $K_i(A[X_1,\ldots,X_n])$
$\approx K_i(A) \oplus$ torsion
$\forall\, n$, $(i = 0, 1)$.

In particular, $K_i(A) \approx K_i(A[X]) \Rightarrow K_i(A[X_1,\ldots,X_n]) \approx K_i(A) \oplus$ torsion $(i=0,1)$.

Proof. This follows from Proposition 2.1 immediately, since $NK_i(A) = 0$ and $K_i(A[X_1,\ldots,X_n]) = (1+N)^n K_i(A)$ [1, p. 663, Cor. 7.3].

Corollary 2.4. $K_0(A)$ finite rank $\Rightarrow K_0A[X_1,\ldots,X_n] \approx K_0(A) \oplus$ torsion.

Examples 2.5. a) Let $A = \mathbb{C}[t^2, t^3]$. It is well known that $\tilde{K}_0(A) \approx \mathrm{Pic}(A) \approx \mathbb{C}$. Hence by Corollary 2.2, $SK_1(A)$ is of infinite rank. This was first observed by M.I. Krusemeyer in his Utrecht-thesis.

b) Let k be an algebraically closed field of infinite transcendence degree over its prime field. Let $\mathrm{Char}(k) \neq 2$ and $A_n = k[x_0,\ldots,x_n]$, $\sum_{i=0}^{n} x_i^2 = 1$, n even. It is well known that $\tilde{K}_0(A_n) \approx \mathbb{Z}$. Hence by Corollary 2.2, $SK_1(A_n)$ is of infinite rank. Using Quillen's localization exact sequence for higher K's, it is not hard to show that $K_i(A_n) \approx K_i(k) \oplus K_i(k)$ if n is even and $K_i(A_n) \approx K_i(k)$ if n is odd, (for all $i \geq 0$).

One can generalize the example a) into the following:

Proposition 2.5. Let A be the co-ordinate ring of a reduced irreducible affine curve C over an algebraically closed field k of infinite transcendence degree over $\mathbb{Q}$. Then the following conditions are equivalent.

1. $SK_1(A) = 0$.

2) $SK_1(A)$ is of finite rank.

3) $A \approx k[X, 1/f]$ for some $f \in k[X]$.

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*This corollary was inspired by the following question of J.R.Strooker: If $K_0A \approx K_0A[X]$, does it follow that $K_0A \approx K_0A[X_1,\ldots,X_n]$?

Proof. 1) $\Rightarrow$ 2) is trivial and 3) $\Rightarrow$ 1) is well-known. We prove 2) $\Rightarrow$ 3). By Corollary 2.2, 2) $\Rightarrow$ Pic(A) is torsion. Let $\overline{A}$ be the integral closure of A and I the conductor between A and $\overline{A}$. Then we have the exact sequence [1, p. 481, Th. 5.3]

$$U(\overline{A}) \oplus U(A/I) \to U(\overline{A}/I) \to \text{Pic } A \to \text{Pic } \overline{A} \to 0 \ .$$

Hence Pic $\overline{A}$ is torsion. This implies Pic $\overline{A} = 0$ and $\overline{A}$ is the coordinate ring of a normal rational curve. Hence $\overline{A} \approx k[X, 1/f]$ for some $f \in k[X]$. Also Pic $A \approx \text{Coker}(U(\overline{A}) \oplus U(A/I) \to U(\overline{A}/I))$. Since $U(\overline{A})/k^*$ is finitely generated and Pic A is of finite rank, it follows that $U(\overline{A}/I)/U(A/I)$ is of finite rank. It is easy to see that $U(\overline{A}/I)/U(A/I)$ has a finite filtration with successive quotients isomorphic to k or k^*. Hence $U(\overline{A}/I)/U(A/I)$ is of infinite rank or zero. Hence $U(\overline{A}/I) = U(A/I)$. For $a \in \overline{A}$, there is a $\lambda \in k$ such that the class of $\lambda + a$ is a unit in $\overline{A}/I$. Thus $\lambda + a$ and hence $a \in A$, i.e. $\overline{A} = A$. Hence $A \approx k[X, 1/f]$.

§3. K_0 of polynomial extensions

Lemma 3.1. Let k be a field and A a k-algebra. If the map $K_i(A) \to K_i(A \otimes_k k(X_1, \dots, X_n))$ is an isomorphism, then $K_i(A) \approx K_i(A[X_1, \dots, X_n])$ $(i = 0, 1)$.

Proof. Let $j: K_i(A) \to K_i(A[X_1, \dots, X_n])$ and $\psi: K_i(A[X_1, \dots, X_n]) \to K_i(A \otimes_k k(X_1, \dots, X_n))$ denote the natural maps induced by corresponding inclusions. By Corollary 1.4, ψ is injective. Hence $\psi \circ j$ is an isomorphism implies that j is an isomorphism.

Proposition 3.2. 1) Let k be a field and $A = k[x, y, z]$, $z^n = xy$. Then every projective A-module is free.

2) Let k be a field and A the homogeneous coordinate ring of an arithmetically normal embedding of $\mathbb{P}^1_k$ into some $\mathbb{P}^n_k$, i.e., A is a graded normal ring over k with $\operatorname{Proj}(A) \approx \operatorname{Proj}(k[t_0, t_1])$. Then every projective A-module is free.

3) Let A be the coordinate ring of a normal affine surface X (over an algebraically closed field k) birationally equivalent to $C \times \mathbb{P}^1$, where C is complete non-singular curve of positive genus. Suppose X has only rational singularities. Then every projective A-module is a direct sum of a free module and an ideal.

To prove Proposition 3.1, we need the following

Lemma 3.2. Let A be a Noetherian domain of dimension ≤ 2. Let $F \subset \operatorname{Max}(A)$ ($\operatorname{Max}(A)$ = maximal ideal spectrum of A) be a closed set of dimension ≤ 1. Suppose for every $M \in \operatorname{Max}(A) - F$, there exists an invertible prime ideal $P \subset M$ such that A/P is a principal ideal domain with $SL_n(A/P) = E_n(A/P)$ for all n. Then every projective A-module is a direct sum of a free A-module and an ideal.

(For proof of Lemma 3.2 see [5, Th. 3.1.)

Proof of Proposition 3.2. 1) This is essentially proved in [3, Cor. 5.3]. We reproduce the proof for the sake of completeness. Take $F = V(x)$ in Lemma 3.2. Let M be a maximal ideal of A with $x \notin M$ and $M \cap k[x] = k[x]f$, f an irreducible polynomial in $k[x]$. Then $A/fA \approx k(\alpha)[Y, Z]/(Z^2 - \alpha Y) \approx k(\alpha)[Y]$, where α is a root of f. Hence by Lemma 3.2, every projective A-module is a direct sum of a free module and an ideal. Since A is a graded normal ring (with $\deg z = 1$, $\deg x = 1$, $\deg y = n-1$) over k, we have $\operatorname{Pic}(A) = 0$ [3, Lemma 5.1]. Hence every projective A-module is free.

2) Let $A = k[x_0, \ldots, x_n]$ be a graded normal ring with $\operatorname{Proj}(A) \approx \operatorname{Proj}(k[t_0, t_1])$. In Lemma 3.2, take $F = V(x_0)$. Let M be a maximal ideal such that $x_0 \notin M$. Let $M \cap k[x_0] = k[x_0]f$, f being an irreducible polynomial in $k[x_0]$. Then

$$A/fA = \frac{A[1/x_0]}{(f)} = B[x_0, 1/x_0]/(f) \approx \frac{k[x_0]}{(f)} \otimes_k B ,$$

where $B = k[\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}]$. But $\operatorname{Spec} B = \operatorname{Proj}(A) - V(x_0)$ is an affine open subset of $\mathbb{P}^1_k$. Hence $B \approx k[t, 1/p]$ for some $p \in k[t]$. Hence $A/fA \approx k(\alpha)[t, 1/p]$. The rest of the proof is as in 1).

3) Let $P_1, \ldots, P_r$ be the singular points of X. Since $P_1, \ldots, P_r$ are rational singularities (for generalities on rational singularities see [2]) there is a non-singular surface X' and a proper birational morphism $\pi: X' \to X$ such that all the components of $\pi^{-1}(P_i)$ are rational curves and π induces an isomorphism $X' - \bigcup_{i=1}^{r} \pi^{-1}(P_i) \approx X - \{P_1, \ldots, P_r\}$. Let $\tilde{X}$ be a complete non-singular surface containing X' as an open set. Since $\tilde{X}$ is birationally equivalent to $C \times \mathbb{P}^1$. Since genus $C \geq 1$, it is easy to see by considering the albanese variety of $\tilde{X}$ that we have a commutative diagram

$$\begin{array}{ccc} C \times \mathbb{P}^1 & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle p}\searrow \qquad \swarrow{\scriptstyle \theta} & \\ & C & \end{array}$$

where p is the projection on the first factor θ is a surjective morphism and f a birational transformation. Since the components of $\pi^{-1}(P_i)$ are rational curves, we have $\theta(\pi^{-1}(P_i)) = Q_i$, a point in C. Since f is birational, there is an open set $V \subset C$ such that $Q_i \notin V$, $1 \leq i \leq r$ and $\theta^{-1}(V) \approx V \times \mathbb{P}^1$.

We identify $X - \{P_1, \ldots, P_r\}$ as an open subset of X and set $U = \theta^{-1}(V) \cap (X - \{P_1, \ldots, P_r\})$. Since $\theta^{-1}(V) \cap \pi^{-1}\{P_1, \ldots, P_r\} = \emptyset$, for every $x \in U$, the curve $\Gamma_x = \theta^{-1}(\theta(x)) \cap U$ is closed in X and does not pass through $P_1, \ldots, P_r$. Also Γ_x is isomorphic to an open subset of $\mathbb{P}^1$. Hence taking $F = X - U$ in Lemma 3.1, we see that every projective A-module is a direct summand of a free module and an ideal.

Remark 3.3. It is easy to see that the arguments in 3) remain valid for any base change $L \supset k$. Hence we get that every projective $A \otimes_k L$-module is isomorphic to a direct sum of a free-module and an ideal.

Corollary 3.4. Let A be as in 1), 2) or 3) of Proposition 3.2. Then $K_0(A) \approx K_0(A[X_1, \ldots, X_n])$, for all n.

Proof. By Proposition 3.2 and Remark 3.3, $K_0(A) \approx K_0(A \otimes_k L)$ for any field extension L/k. Hence Corollary 3.4 follows from Lemma 3.1.

Remark 3.5. We do not know any example of a normal ring A such that $K_0(A) \not\approx K_0(A[X])$. Corollary 3.4 suggests the following conjecture.

Conjecture: Let A be the coordinate ring of an affine normal surface having only rational singularities. Then $K_0(A) \approx K_0(A[X_1, \ldots, X_n])$.

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BASE CHANGE FOR K_0 OF ALGEBRAIC VARIETIES

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We consider the effect of a finite normal change of base field on the Grothendieck group K_0 of an algebraic variety. This is first done in the affine case and generalized to schemes. I have tried to give proofs that are valid for K_1 and other groups as well. The essential idea is that the group be defined in term of a category of modules and satisfy certain reasonable properties, rather than merely be a functor from rings to abelian groups. This approach works well with a normal separable extension, but with inseparable extension I had to use special properties of K_0 and K_1 .

Some of the material here is contained in [13]. Throughout, Z = integers, R = real numbers, Q = rational numbers, C = complex numbers. All schemes are separated.

1. Normal Separable Extensions

Let F be a field, and A a commutative algebra over F . If K is an extension field of F , set $B = A \otimes_F K$, and $f: A \to B$ the inclusion $f(a) = a \otimes 1$. In this section we assume that K is a finite normal separable extension of F , and consider inseparable extensions later. Let G be the Galois group of K over F , and $[K:F] = n$. The group G acts on B by $\alpha(a\otimes\lambda) = a \otimes \alpha(\lambda)$ for $\alpha \in G$, $\lambda \in K$. If M is a B-module, we define the B-module $M_\alpha (\alpha\in G)$ by (i) $M_\alpha = M$ as an abelian group (ii) $b \cdot m = \alpha^{-1}(b)m$. Here $\cdot$ denotes the B-action on M_α . If α denotes the ring homomorphism $\alpha: B \to B$ defined above, then $M_\alpha = \alpha^*(M) = (\alpha^{-1})_* M$, where α^* denotes extension of scalars by means of α , and $(\alpha^{-1})_*$ denotes restriction of scalars by

α^{-1} . This terminology agrees with that of Bourbaki [4], but not with that of Milnor [11], p. 137.

If N is an A-module, and M is a B-module, then we have the following:

(1) $f_* f^*(N) \cong nN$ (direct sum of n copies)

(2) $f^* f_*(M) \cong \oplus_{\alpha\in G} M_\alpha$.

The first is obvious. To prove (2) , let $K = F(\mu)$, where μ has minimal polynomial g . We have $B = A[X]/(g(X))$, and $B \otimes_A B = B[X]/(g(X)) = \Pi_{\alpha\in G} B[X]/(X-\alpha(\mu)) = \Pi_{\alpha\in G} B_\alpha$, where $B_\alpha = B[X]/(X-\alpha(\mu)) = B$. There are two homomorphisms $f_1: B \to B \otimes_A B$ defined by $f_1(b) = b \otimes 1$, and $f_2: B \to B \otimes_A B$ defined by $f_2(b) = 1 \otimes b$. If $\pi_\alpha: B \otimes_A B \to B$ denotes projection onto the α^{th} factor, then $\pi_\alpha f_2 = 1_B$ and $\pi_\alpha f_1 = \alpha$. Therefore $f^* f_*(M) =$ $= (f_2)_*(f_1)^*(M) = \oplus_{\alpha\in G} M_\alpha$, as required. Note that both (1) and (2) are natural.

In order to prove (2), B need only be a commutative Galois extension of A .

Now let X_F be a scheme over F , and $X_K = X_F \times_{\mathrm{Spec}F} \mathrm{Spec}\, K$. Let $f: X_K \to X_F$ be projection onto the first factor. Then G acts as a group of automorphism of X_K (α acting via $1 \times \alpha$). If M is a quasicoherent sheaf on X_K , write $M_\alpha = \alpha^*(M)$. This is consistent with the terminology of §1. Suppose $X_F = \bigcup_{i\in I} X_i$, where $X_i = \mathrm{Spec}\, R_i$ is an open covering of X . Then $X_K = \bigcup_{i\in I} X_i'$, where $X_i' = \mathrm{Spec}(R_i \otimes_F K) = f^{-1}(X_i)$ is an open covering of X_K by affines. Over each of the affine open sets X_i we have (2), with compatibility on overlaps by naturality of (2). Therefore we have

$$(2') \quad f^* f_*(M) \cong \oplus_{\alpha\in G} M_\alpha$$

for any quasicoherent sheaf M on X_K .

Of course we have

$$(1') \quad f_* f^*(M) \cong nN$$

for N any quasicoherent sheaf on X_F , for K any field extension of degree n .

The isomorphisms in (1') and (2') are natural so we have a natural equivalence between the functors $f^* f_*$ and $\Sigma_{\alpha \varepsilon G} \alpha^*$, and between $f_* f^*$ and n . By the sum of two functors f_1 and f_2 we mean $(f_1+f_2)(M) = f_1(M) \oplus f_2(M)$ for an object M , and $(f_1+f_2)(\beta) = f_1(\beta) \oplus f_2(\beta)$ for a morphism β .

If X_F is projective over F , the Krull-Schmidt Theorem holds for coherent sheaves on X_F [2]. If M and N are coherent sheaves on X_F , and $f^*M \cong f^*N$ then (1') implies that $nM \cong nN$. By the Krull-Schmidt theorem $M \cong N$. Therefore f^* is an injection on isomorphism classes.

2. The Grothendieck Groups

Define an admissible subcategory $\underline{C}$ of an abelian category $\underline{A}$ as on page 388 of [3] (except that condition (d) might not be needed). Let $\underline{K}$ be a "functor" that assigns to $\underline{C}$ an abelian group $\underline{K}(\underline{C})$. That is, if $f\colon \underline{C} \to \underline{C}'$ is an exact admissible functor in the sense of [3] page 389, then a homomorphism $\overline{f}\colon \underline{K}(\underline{C}) \to \underline{K}(\underline{C}')$ is defined such that $\overline{1} = 1$ and $\overline{gf} = \overline{g}\,\overline{f}$ (with equivalent functors inducing the same homomorphism). Furthermore, if f and g are two exact admissible functors from $\underline{C}$ to $\underline{C}'$, so is $f + g$, and we assume that $\overline{f + g} = \overline{f} + \overline{g}$. To simplify the notation I will usually omit the $\overline{}$.

Now let F be a field, K a normal separable extension of degree n , X_F a noetherian scheme over F , and

$X_K = X_F \times_F K$ as in §1. Let $\underline{A}$ be the category of coherent sheaves on X_F, $\underline{A}'$ the category of coherent sheaves on X_K, $\underline{C}$ the category of locally free sheaves of finite type on X_F, and $\underline{C}'$ the category of locally free sheaves of finite type on X_K. Then f^* is an exact admissible functor from $\underline{A}$ to $\underline{A}'$ that takes $\underline{C}$ to $\underline{C}'$, and f_* is an exact admissible functor taking $\underline{C}'$ into $\underline{C}$. Also the α^* $(\alpha \varepsilon G)$ are exact admissible functors from $\underline{A}'$ to $\underline{A}'$ mapping $\underline{C}'$ into itself. If $\underline{K}$ is as above then (1') and (2') yield equalities

$$(3) \quad f_* f^* = n$$

$$(4) \quad f^* f_* = \Sigma_{\alpha \varepsilon G} \alpha$$

of endomorphisms of $\underline{K}(\underline{A})$ (or $\underline{K}(\underline{C})$) and $\underline{K}(\underline{A}')$ (or $\underline{K}(\underline{C}')$) respectively. I have written simply α instead of $\bar{\alpha}^*$. G acts as a group of automorphisms of $\underline{K}(\underline{A}')$ and $\underline{K}(\underline{C}')$.

In particular, $\underline{K}$ can be the Grothendieck groups K_0 or K_1 as defined in [3], page 389, and perhaps also the groups K_i as defined by Quillen in [12]. For example, $K_0(\underline{A}) = K.(X_F)$, $K_0(\underline{A}') = K.(X_K)$ and the homomorphisms $f_*: K.(X_K) \to K.(X_F)$ and $f^*: K.(X_F) \to K.(X_K)$ **induced by the functors** f_* **and** f^* **respectively** satisfy (3) and (4). If $K.(X_K)^G$ is the subgroup of $K.(X_K)$ consisting of elements fixed by G then f^* maps $K.(X_F)$ into $K.(X_K)^G$. Equations (3) and (4) say that the kernel and cokernel of f^* are killed by n.

By using $\underline{C}$ and $\underline{C}'$ in place of $\underline{A}$ and $\underline{A}'$ we get corresponding statements about $K^{\cdot}(X_F) = K_0(\underline{C})$ and $K^{\cdot}(X_K) = K_0(\underline{C}')$. If $X_F = \text{Spec } A$ is affine, then $K_i(\underline{C})$ is denoted $K_i(A)$, and $K_i(\underline{A})$ is denoted $G_i(A)$ in [3], $i=0,1$.

3. The inseparable case

First assume that K is a purely inseparable extension of F of degree p, that is $\mathrm{char}\, F = p > 0$, and $K = F(\beta)$ where where $\beta^p \in F$, $\beta \notin F$. Let A and B be as in §1. Then $B = A[X]/(X^p-\alpha)$, and $B \otimes_A B = B[X]/(X^p-\alpha) = B[X]/(X-\alpha)^p$. We have a homomorphism $g: B \otimes_A B \to B$ defined by factoring out the nilpotent ideal $(X-\beta)$, and two homomorphisms $f_1, f_2: B \to B \otimes_A B$ defined as before. If M is a projective B-module of finite type, then $f^* f_*(M) = (f_2)_*(f_1)^*(M)$. On the other hand $gf_1 = gf_2 = 1_B$, so $g^*(f_1)^* = g^*(f_2)^*$. But g^* is a bijection on isomorphism classes, by proposition 2.12, page 90 of [3]. Therefore $(f_1)^*(M) \cong (f_2)^*(M)$, so $f^* f_*(M) = (f_2)_*(f_1)^*(M) \cong (f_2)_*(f_2)^*(M) \cong$ $\cong pM$. This isomorphism is not natural (at least, not obviously so) but we still have $f^* f_* = p$ on $K_0(B)$. For the G_i case $(i=0,1)$ we still have $f^* f_* = (f_2)_*(f_1)^*$. From $gf_1 = gf_2 = 1_B$ we get $(f_1)_* g_* = (f_2)_* g_* = 1$. By proposition 2.3, page 454 of [3], $g_*: G_i(B) \to G_i(B \otimes_A B)$ is an isomorphism. Therefore $(f_1)_* = (f_2)_*$ and $f^* f_* = (f_1)_*(f_1)^* = p$ (as endomorphisms of $G_i(B)$).

We can now put these results together to handle the case of an arbitrary normal extension $F \subset K$ of degree n. We can write $F \subset H \subset K$ where H is purely inseparable over F of degree p^d, and K is a separable extension of H. If $i: A \to A \otimes_F H$ and $j: A \otimes_F H \to A \otimes_F K$ are induced by the inclusions of fields and $f = ji$ then we have $f^* f_* = (ji)^*(ji)_* =$ $j^*(i^* i_*) j_* = p^d j^* j_*$ in K_0, G_0 and G_1 cases ($i^* i_* = p^d$ since G can be obtained by adjoining p^{th} roots, one at a time). If M is projective of finite type then $f^* f_*(M) \cong p^d j^* j_*(M)$. The field H is fixed under any automorphism of K over F and restriction gives an isomorphism $G = \mathrm{Gal}(K/F) \to \mathrm{Gal}(K/H)$. Thus we have proved

$$(5)\quad f^*f_*(M) \cong p^d \oplus_{\alpha\in G} M_\alpha \qquad \text{(M projective of finite type).}$$

$$(6)\quad f^*f_* = p^d \Sigma_{\alpha\in G}\, \alpha \qquad (\text{for } K_0, G_0 \text{ and } G_1)\ .$$

The following example shows that (5) is false if M is not assumed to be projective. Let $A = K$, where K/F is purely inseparable with $[K:F] = p$. Then $B = K \otimes_F K$. If M is a B-module, $f_*(M)$ is a free A-module, since A is a field. Therefore $f^*f_*(M)$ is a free B-module. If $f^*f_*(M) \cong pM$ then M is projective. But there are B-modules of finite type which are not projective.

If the extension K/F is normal but not separable, the proof of (6) seems to work in the scheme case for $K_0(\underline{A}')$ and $K_1(\underline{A}')$, but I do not know if the analogues of (5) and (6) hold in the $K^{\cdot}$ case, the problem being the lack of naturality in (5).

4. Some examples

Let S be a graded ring in positive degrees, and let $X = \text{Proj } S$. A homogeneous ideal $I \subset S$ defines a closed subscheme $Y = \text{Proj}(S/I)$ of X . If I is generated by a homogeneous element f , then $X - Y = D_+(f)$ is affine, $D_+(f) = \text{Spec } S_{(f)}$, where $S_{(f)}$ is the degree zero part of S_f . Proj and its properties are discussed in [9] , §2.

If $n = 2r$ is even, write $P^n_K = \text{Proj } K[U_1,V_1,\ldots,U_r,V_r,T]$, and let W_K (or W^n_K if it is necessary to specify n) be the closed subscheme defined by $\Sigma^r_{i=1} U_iV_i + T^2$. That is, $W_K = \text{Proj } K[U_1,V_1,\ldots U_r,V_r,T]/(\Sigma^r_{i=1} U_iV_i+T^2)$. In W_K , $D_+(U_1) = \text{Spec } K[v_1,u_2,v_2,\ldots u_r,v_r,t]/(v_1+\Sigma^r_{i=2} u_iv_i+t^2)$, where

$u_i = U_i/U_1$, $v_i = V_i/U_1$, $t = T/U_1$. The v_1 can be eliminated, so $D_+(U_1) = \mathrm{Spec}\ K[u_2,v_2,\ldots u_r,v_r,t] = A_K^{n-1}$ = affine space over K of dimension $n - 1$. If we let W_i be the closed subscheme defined by the homogeneous ideal $(U_1,\ldots,U_i)$, $1 \leq i \leq r$, then we have $W_K = W_0 \supset W_1 \supset W_2 \supset \cdots\ W_{r-1} \supset W_r$. As above it is seen that $W_{i-1} - W_i = A_K^{n-i}$ $(1 \leq i \leq r)$. The schemes W_i $(0 \leq i \leq r-1)$ are all integral, and $W_r = \mathrm{Proj}\ K[V_1,\ldots V_r,T]/(T^2)$, so $(W_r)_{red} = \mathrm{Proj}\ K[V_1,\ldots,V_r] = P_K^{r-1}$.

In a similar manner, if $n = 2r - 1$ is odd, write $P_K^n = \mathrm{Proj}\ K[U_1,V_1,\ldots,U_r,V_r]$ and let W_K be the closed subscheme defined by $\Sigma_{i=1}^r U_iV_i$. That is, $W_K = \mathrm{Proj}\ S$, where $S = K[U_1,V_1,\ldots U_r,V_r]/(\Sigma_{i=1}^r U_iV_i)$. If we let W_i be the closed subscheme of W_K defined by the homogeneous ideal $(U_1,\ldots,U_i)$ then we have $W_K = W_0 \supset W_1 \supset \cdots \supset W_r$. We have $W_{i-1} - W_i = A_K^{n-i}$, $1 \leq i \leq r$. The schemes W_i are all reduced, all are integral except W_{r-1} , and $W_r = \mathrm{Proj}\ K[V_1,\ldots,V_r] = P_K^{r-1}$.

Let X be a noetherian scheme over K with an ample invertible sheaf, and let Y be a closed subscheme such that $X - Y = A_K^n$. Then we have an exact sequence $0 \to K.(Y) \to K.(X) \to Z \to 0$. This follows from the exact sequence in §5 of [12]. Part of this exact sequence is

$$G_1(X) \xrightarrow{g} G_1(X-Y) \to K.(Y) \to K.(X) \to K.(X-Y) \to 0$$

where G_1 is a group defined by Quillen in [12]. $G_1(X-Y) = G_1(A_K^n) = G_1(K)$, and g is split by the homomorphism $f^*: G_1(K) \to G_1(X)$ induced by the structure morphism $f: X \to \mathrm{Spec}\ K$. Therefore g is onto, and since $K.(X-Y) = K.(A_K^n) = Z$, we have the required short exact sequence. To get g onto, the field K could have been replaced by any commutative noetherian ring, as long as X is of finite tor-dimension over K (this assumption is necessary in order to define the homomorphism f^*). We could also have used corollary 5.7, p. 428 of [3], as was done in [13]. The exact sequence $0 \to K.(Y) \to K.(X) \to Z \to 0$ can be split by sending $1 \in Z$ to $[0_X]$, the class in $K.(X)$ of the structure sheaf 0_X . By proposition 3.3 p. 402 of [3], there is an isomorphism $K.(W_r)_{red} \to K.(W_r)$. Therefore $K.(W_K)$ is free abelian of rank $2r$, with basis $e_0,\ldots e_{r-1}$, $f_1,\ldots f_r$, where $e_i = [0_{W_i}]$, and f_i corresponds to a linear subspace of codimension $i - 1$ in $P_K^{r-1} = (W_r)_{red}$.

Let V_F (or V_F^n if it is necessary to specify n) be a closed subscheme of P_F^n which is defined by a homogeneous polynomial g of degree 2 , and suppose that there exists a finite normal extension K of F such that $V_K = V_F \times_F K$ is isomorphic to W_K . We have an exact sequence

$$K.(V_F) \to K.(P_F^n) \to K.(P_F^n - V_F) \to 0 \quad .$$

By the corollary p. 299 of [8], rank $K.(P_F^n - V_F) = 1$. Rank $K.(P_F^n) = n + 1$. Therefore rank $K.(V_F) \geq n$. Also, by $(1')$ rank $K.(V_F) \leq$ rank $K.(V_K)$. If n is even we have proved that rank $K.(V_K) = n$. Therefore rank $K.(V_F)$ = rank $K.(V_K) = n$, or equivalently, every element of $G = Gal(K/F)$ acts trivially on $K.(V_K)$. Therefore we need consider only odd n . If n is odd, rank $K.(V_K) = n + 1$, so rank $K.(V_F) = n$ if some element of $G = Gal(K/F)$ acts non-trivially on $K.(V_K)$, and rank $K.(V_F) = n + 1$ otherwise.

If char $F \neq 2$, we may assume $g = \Sigma_{i=1}^r (a_i S_i^2 + b_i T_i^2)$ $(n+1=2r)$, where $a_i, b_i \neq 0$ and the S_i, T_i are $n + 1$ indeterminants defining the homogeneous co-ordinate ring of P_F^n . Then we can obtain a suitable (separable) extension K by adjoining to F a finite number of square roots $\alpha_i = \sqrt{(-b_i)/a_i}$. In K we can make the change of variable $U_i = a_i(S_i - \alpha_i T_i)$ and $V_i = S_i + \alpha_i T_i$, so that $g = \Sigma_{i=1}^r U_i V_i$. The effect of an automorphism σ of K over F is to interchange α_i and $-\alpha_i$ for $i \in I$, where I is some subset of the integers from 1 to r . That is $\sigma(U_i) = a_i V_i$ and $\sigma(V_i) = (1/a_i)U_i$ if $i \in I$. The automorphism μ of W_K defined by $\mu(U_i) = a_i U_i$, $\mu(V_i) = (1/a_i)V_i$ induces the identity on $K.(W_K)$ because it leaves fixed the homogeneous ideals defining the basis for $K.(W_K)$. Therefore the automorphism of $K.(W_K)$ produced by σ is the same as that produced by interchanging U_i and V_i , $i \in I$.

If char $F = 2$, we may assume $g = \Sigma_{i=1}^r a_i S_i^2 + S_i T_i + b_i T_i^2$ by [1] . Then a suitable (separable) extension K can be obtained by adjoining to F the roots of the polynomials $a_i x^2 + x + b_i$, and as above an automorphism of K over F will produce the same automorphism of $K.(W_K)$ as interchanging U_i and V_i for $i \in I$, I defined as above.

Let t_j be the automorphism of W_K defined by interchanging U_j and V_j , and $\tau_j = t_j^*$, the automorphism induced by t_j on $K.(W_K)$. I claim that $\tau_j(e_i) = e_i$, $0 \le i \le r-1$, and $\tau_j(f_i) = f_i$, $2 \le i \le r$. This was proved in [13] by using the ring structure on $K.(W_K)$ $(=K^{\cdot}(W_K))$. However, one can also give the following more elementary proof. For $2 \le i \le r$, we have $f_i = [0_Y]$, where Y is the closed subscheme defined by the homogeneous ideal $(U_1,\ldots U_r, V_j, V_{k_2},\ldots V_{k_{i-2}})$, where the integers $j, k_2, \ldots k_{i-2}$ are all distinct. The ideal is fixed by t_j , so $\tau_j(f_i) = f_i$, $2 \le i \le r$. Similarly $\tau_j(e_i) = e_i$ if $i < j$. Write $S = K[U_1,V_1,\ldots U_r,V_r]/(\Sigma_{i=1}^r U_i V_i)$ as before. If $j \le i$, set $I = (U_1,\ldots U_i)$, $J = (U_1,\ldots,U_{j-1},U_{j+1},\ldots,U_i)$, and $I' = (U_1,\ldots U_{j-1},V_j,U_{j+1},\ldots U_i)$. We have the following exact sequences of graded S-modules:

$$0 \to J \to I \to I/J \to 0$$

$$0 \to J \to I' \to I'/J \to 0$$

$$0 \to S/J \xrightarrow{U_i} I/J \to 0$$

$$0 \to S/J \xrightarrow{V_i} I'/J \to 0$$

From this it follows that in $K.(W_K)$, $[\tilde{I}] = [\tilde{I}']$ ($\sim$ as in [9], page 30), and therefore $e_i = \tau_j e_i$. Now we consider f_1 . Let $J = (U_1,\ldots U_{j-1},U_{j+1},\ldots U_r)$, $I = (U_1,\ldots,U_r)$, $I' = (U_1,\ldots,U_{j-1},V_j,U_{j+1},\ldots U_r)$, $L = (U_1,\ldots,U_r,V_j)$ and let Y_1,Y_2,Y_3,Y_4 be the closed subschemes defined respectively by these homegeneous ideals. We have $I \cap I' = J$, and $I + I' = L$. There is an exact sequence of graded S-modules

$$0 \to S/J \to S/I \oplus S/I' \to S/L \to 0$$

and hence (applying $\sim$) an exact sequence

$$0 \to O_{Y_1} \to O_{Y_2} \oplus O_{Y_3} \to O_{Y_4} \to 0 \quad .$$

But $[O_{Y_2}] = f_1$, $[O_{Y_3}] = \tau_j(f_1)$, $[O_{Y_4}] = f_2$, and an argument similar to that used to prove that $\tau_j(e_i) = e_i$ for $j \leq i$ shows that $[O_{Y_1}] = e_{r-1}$. Therefore we have $\tau_j(f_1) = e_{r-1} - f_1 + f_2$. Thus the τ_j are all equal, say $\tau_j = \tau$. Therefore $\sigma \in G = Gal(K/F)$ acts trivially on $K.(W_K)$ if σ acts as an even number of transpositions, and non-trivially if σ acts as an odd number of transpositions.

As an example, let $F = R$ and let $V_R \subset P^n_R$ be defined by $X_0^2 + \ldots + X_n^2$, $K = C$ so that $G = Z/2Z$. We may make the following table:

	rank $K.(W^n_C)$	number of transpositions	action of G	rank $K.(V^n_R)$
n even	n	-----	trivial	n
$n\equiv 1 \bmod 4$	n+1	odd	non-trivial	n
$n\equiv 3 \bmod 4$	n+1	even	trivial	n+1

We can also give some affine examples. Suppose that char $F \neq 2$, and that $A_n = F[X_0,\ldots,X_{n-1}]/(a_0X_0^2+\ldots+a_{n-1}X_{n-1}^2+a_n)$, where $a_i \neq 0$, $a_i \in F$. We can adjoin a finite number of square roots (including $\sqrt{-1}$) to F to obtain K so that $A_n \otimes_F K \cong K[X_0,\ldots,X_{n-1}]/(X_0^2+\ldots+X_{n-1}^2-1) = B_n$. By [6] , p. 252, $K_0(B_n) = Z \oplus Z$ if n is odd and Z if n is even. Therefore, if n is even, rank $K_0(A_n) = 1$, and if n is odd, rank $K_0(A_n)$ is either 1 or 2 . Suppose n is odd. Spec A_n is the open subset $D_+(X_n)$ of $V^n_F = \text{Proj } F[X_0,\ldots,X_n]/(a_0X_0^2+\ldots+a_nX_n^2)$. Furthermore, $V^n_F \times_F K = V^n_K \cong W^n_K$, where W^n_K is as previously defined. If every element of $G = \text{Gal}(K/F)$ produces an even number of transpositions, then G acts trivially on $K.(V^n_K)$, and hence also acts trivially on $K_0(B_n)$. In this case rank $K_0(A_n) = 2$. If some element of G produces an odd number of transpositions, then G acts non-trivially on $K.(V^n_K)$. If $V^{n-1}_F = \text{Proj } F[X_0,\ldots,X_{n-1}]/(a_0X_0^2+\ldots+a_{n-1}X_{n-1}^2)$ then $V^{n-1}_K \cong W^{n-1}_K$ (if K has been made big enough). G acts trivially on $K.(V^{n-1}_K)$ since $n - 1$ is even. Therefore we have an exact sequence of free abelian groups

$$0 \to \text{image } K.(V^{n-1}_K) \to K.(V^n_K) \to K_0(B_n) \to 0 \quad .$$

The first homomorphism is obtained from the inclusion $V^{n-1}_K \subset V^n_K$. The group G acts as an automorphism of this exact sequence, trivially on image $K.(V^{n-1}_K)$, and non-trivially on $K.(V^n_K)$. From this, using the fact that G is finite and the groups are free abelian, it is readily seen that G acts non-trivially on $K_0(B_n)$. Therefore, in this case rank $K_0(A_n) = 1$.

Some examples are as follows:

(1) Let $A_n = R[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2+1)$. Then rank $K_0(A_n) = 2$ if $n \equiv 2 \bmod 4$, and rank $K_0(A_n) = 1$ otherwise.

(2) Let $A_n = R[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-1)$. Then rank $K_0(A_n) = 2$ if $n \equiv 0 \bmod 4$ and rank $K_0(A_n) = 1$ otherwise. This proves that the homomorphism $K_0(A_n) \to KO(S^n)$ considered in [7] is an isomorphism mod torsion, since the groups have the same rank and Fossum has shown that the map is onto. (The kernel is of course killed by 2).

(3) Let $A_n = Q[X_0,\ldots,X_n]/(X_0^2+\ldots+X_n^2-2)$. Then rank $K_0(A_n) = 1$ for all n since the 2 always makes an odd number of transpositions possible.

I have not been able to say anything in general about the 2-torsion part of $K.(V_F^n)$. The cokernel of $f^*: K.(V_F) \to K.(V_K)^G$ $(V_K \cong W_K)$ also seems difficult to compute, but at least it is clearly finitely generated. Examples in [13] show that the cokernel can be non-zero.

5. Further remarks on K_1

A Brauer-Severi variety is a variety over a field F which becomes isomorphic to P_K^{n-1} after a finite separable extension K/F . There is a bijection between Brauer-Severi varieties of dimension n - 1 and central simple algebras over F of rank n^2 . The quadrics W_F^2 considered in section 4 are examples, with n = 2 . In [13] I proved that $K_1(W_F^2) = K_1(F) \oplus K_1(D)$, where D is the central simple algebra corresponding to W_F^2 . Quillen has obtained the same result, using the definition of K_1 proposed in [12].

Gersten has shown, however, that if X is a complete elliptic curve over C, there is a naturally occuring homomorphism $K_1(X) \to K_1^Q(X)$ (Q denoting Quillen's definition) which is onto but not injective.

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ON FREE PRODUCT OF RINGS AND THE COHERENCE PROPERTY

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§1. Introduction

A unital ring R is said to be (right) coherent, if every homomorphism $f: R^n \to R^m$ of (right) R-modules has finitely generated kernel. Standard references for such rings are Chase [3], Bourbaki [2] and Soublin [7]. Of course, any right Noetherian ring is right coherent, but there are important examples of coherent rings which are not Noetherian. Indeed the integral group ring of a non-cyclic free group is one such example.

The importance of coherence in Algebraic K-theory can be traced back to the following (cf. [1]) :

Proposition (1.1) If R is a coherent ring of finite right global dimension, then the inclusion map $R \to R[t]$ induces an isomorphism $K_1(R) \tilde{\to} K_1(R[t])$, where $R[t]$ denotes the polynomial ring over R.

This proposition has been used by various people [1], [5] in computing the K-groups of polynomial extensions.

The purpose of this paper is, roughly, to establish the coherence property for the free product of two coherent rings. The precise statement is given in Theorem 2.1. This theorem can be applied to yield certain vanishing theorems of Whitehead groups and projective class groups, see [4].

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It should be pointed out that Waldhausen in [9] established, among other things, that if two groups G and H have coherent group rings, then so does the almagamated product $G *_C H$, where C is a common subgroup with Noetherian group ring. Waldhausen's proof depends heavily on his machinery of "surgeries" and "Mayer-Vietoris presentations" of chain complexes. Our proof of Theorem 2.1 is a drastic simplification of his ideas, and at the same time constitutes an extension of these ideas from group rings to arbitrary rings.

§2. Statement of the Main Theorem

Let R be a unital ring. By a R-ring we mean a unital ring A containing R as subring, such that there is an augmentation homomorphism $\varepsilon_A : A \to R$ satisfying $\varepsilon_A(r) = r$ for all r in R. We call $\bar{A} = \operatorname{Ker} \varepsilon_A$ the augmentation ideal of A, and note the following split exact sequence :

$$0 \longrightarrow \bar{A} \longrightarrow A \underset{i}{\overset{\varepsilon_A}{\rightleftarrows}} R \longrightarrow 0 .$$

If A and B are R-rings, then we can form their free product over R, denoted by $A *_R B$. A good description of this free product can be found in Stallings [7]. We only record that, as bimodule over R,

$$(1) \qquad A *_R B = R \oplus \bar{A} \oplus \bar{B} \oplus \bar{A}\bar{B} \oplus \bar{B}\bar{A} \oplus \bar{A}\bar{B}\bar{A} \oplus \bar{B}\bar{A}\bar{B} \oplus \ldots \quad ,$$

where $\bar{A}\bar{B}$ is an abbreviation for $\bar{A} \otimes_R \bar{B}$, etc. The multiplication in this free product can be illustrated by the following typical examples : if $\alpha_i \in \bar{A}$, $\beta_j \in \bar{B}$, then

$$(\alpha_1 \otimes \beta_1)(\alpha_2 \otimes \beta_2) = \alpha_1 \otimes \beta_1 \otimes \alpha_2 \otimes \beta_2 \in \bar{A}\bar{B}\bar{A}\bar{B} ,$$

$$(\alpha_1 \otimes \beta_1 \otimes \alpha_2)(\alpha_3 \otimes \beta_2) = \alpha_1 \otimes \beta_1 \otimes (\alpha_2\alpha_3) \otimes \beta_2 \in \bar{A}\bar{B}\bar{A}\bar{B} \cdot$$

The main purpose of this paper is to prove :

Theorem (2.1) Let R be a right Noetherian ring. Let A, B be right coherent R-rings such that the augmentation ideals $\bar{A}$, $\bar{B}$ are free as left R-modules. Then the free product $A *_R B$ is right coherent.

Corollary (2.2) If R is a right Noetherian ring and X is a set, then the free ring R{X} generated by X over R is right coherent.

This corollary is an immediate consequence of Theorem 2.1 when X is a finite set. If X is infinite, we can use a direct limit argument to complete the proof. Compare [2, p.63].

§3. Some Technical Lemmas

We begin with some notations and terminology. A homomorphism $f: R^n \to R^m$ of right R-modules can be represented by an associated $m \times n$ matrix Q over R, such that it maps a column vector $x \in R^n$ to $Qx \in R^m$. We call Q a (right) coherent matrix if its "solution space" $\{ x \mid Qx = 0 \}$ is finitely generated as right R-module. If Q_1 (resp. Q_2) is the matrix obtained from Q by an elementary row (resp. column) operation(*), and if Q_3 is the extended matrix $\left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right]$, then the following lemma is easy to prove :

Lemma (3.1) For each i, Q_i is coherent if and only if Q is coherent.

Another easy lemma is :

(*) In performing an elementary operation, we multiply rows by scalars from the left, and columns by scalars from the right.

Lemma (3.2) Let A' be a ring containing A such that A' is free when considered as a left A-module. If Q is a right coherent matrix over A, then it is also right coherent when considered as a matrix over A'.

Let A, B be R-rings as in Theorem 2.1. Let us fix left bases $\{\alpha_i\}_{i\in I}$, $\{\beta_j\}_{j\in J}$ for the (left) R-modules $\bar{A}$ and $\bar{B}$. Then $\{\alpha_i \otimes \beta_j\}_{i\in I, j\in J}$ form a left bases of $\bar{A}\bar{B}$. In this way, we can assign a left basis to each direct summand of $A *_R B$ appearing in the right hand side of (1). Furthermore, each basis element w has an obviously defined length $|w|$. For example, $|1| = 0$, $|\alpha_i \otimes \beta_j| = 2$, etc. If $w = 1$, or if $w = \alpha_i \otimes \beta_j \otimes \ldots$, then we say that w is a basis element of A-type. Similarly, we can define a basis element of B-type.

Consider now the following diagram of natural inclusions of right modules:

$$\begin{array}{ccccc} & \nearrow & A^m & \searrow & \\ R^m & & & & (A *_R B)^m = D^m, \\ & \searrow & B^m & \nearrow & \end{array}$$

where for brevity we have used D to denote the free product $A *_R B$. Our next lemma is the key step towards the proof of Theorem 2.1 :

Lemma (3.3) Let M_A be a submodule of D^m generated by certain elements in A^m, and let M_B be another submodule of D^m generated by certain elements in B^m. Let $K = (M_A + M_B) \cap R^m$. Then

$$(2) \qquad (M_A + K\cdot D) \cap (M_B + K\cdot D) = K\cdot D ,$$

where $K\cdot D$ denotes the right D-module generated by K.

Proof : It suffices to show that an arbitrary element d in the left hand side of (2) belongs to the right hand side. Let M_A^o (resp. M_B^o) be the A-submodule of A^m (resp. B-submodule of B^m) generated by the same set of elements which by hypothesis generate M_A (resp. M_B). Then $M_A = M_A^o \cdot D$ and $M_B = M_B^o \cdot D$. Considering d as an element in $D^m = R^m \otimes_R D$, we can express it uniquely as

(3) $$d = \Sigma_i \, c_i w_i \, ,$$

with each $c_i \in R^m$, and each w_i a left basis element of D, satisfying $|w_1| \geq |w_2| \geq |w_3| \geq \ldots \geq 0$. On the other hand, by considering D^m as $A^m \otimes_A D$ or as $B^m \otimes_B D$, we can express d uniquely in each case as

(4) $$d = \Sigma_j \, a_j u_j \, ,$$

or

(5) $$d = \Sigma_k \, b_k v_k \, ,$$

respectively, where $a_j \in M_A^o + K \cdot A$, $b_k \in M_B^o + K \cdot B$; u_j is a basis element of B-type and v_k is a basis element of A-type.

We now assert $c_i \in K$ for each i. Without loss of generality, we can suppose w_1 is a basis element of B-type. Then, in the expression (4), there must be a j such that $u_j = w_1$. Let's say $j = 1$ so that $u_1 = w_1$. We claim that $c_1 = a_1$. For this purpose, observe that $a_1 \in A^m = R^m \otimes_R A$, so that one can write

$$a_1 = c_1' + \Sigma_\ell \, c_\ell'' \alpha_\ell$$

where c_1', $c_\ell'' \in R^m$ and α_ℓ is a left basis element of $\bar{A}$ for each ℓ. If $c_\ell'' \neq 0$ for some ℓ, then $c_\ell'' \alpha_\ell \otimes w_1$ must appear in the expression (3), contradicting the fact that w_1 is of maximal length. Hence all $c_\ell'' = 0$ so that $a_1 = c_1' = c_1$, implying $c_1 \in M_A^o + K\cdot A \subset M_A + M_B$. Since c_1 is already in R^m, this proves $c_1 \in K$. By repeating the same argument to $d - c_1 w_1$, we deduce inductively that $c_i \in K$ for all i. Hence $d \in K\cdot D$, as is to be proved.

§4. Proof of Theorem 2.1.

It suffices to show that any rectangular matrix Q over D is (right) coherent. By Lemma 3.1, we can first change Q by elementary row and column operations, or by extensions of the type $Q \longmapsto \left[\begin{array}{c|c} Q & 0 \\ \hline 0 & 1 \end{array}\right]$, until finally Q takes the following form:

$$Q = [\, Q_A \mid Q_B \,],$$

where Q_A, Q_B are $m \times p$ and $m \times q$ matrices over A and B respectively, with $p + q = n$, for some integers m and n. (This procedure is known as "Higman's trick").

Let $a_1, \ldots, a_p$ be the column vectors of Q_A and $b_1, \ldots, b_q$ be those of Q_B. Let M_A, M_B be D-submodules of D^m generated by $\{a_1, \ldots, a_p\}$ and $\{b_1, \ldots, b_q\}$ respectively. If $f : D^n \to D^m$ is the homomorphism associated with Q, then we have the following presentation of $M_A + M_B$:

$$(6) \qquad 0 \longrightarrow \ker f \hookrightarrow D^n \xrightarrow{f} M_A + M_B \longrightarrow 0 .$$

Our objective is to show that $\ker f$ is a finitely generated D-module.

Let $K = (M_A + M_B) \cap R^m$. Since R is right Noetherian, K is finitely generated over R, say, by elements $c_1, \ldots, c_r \in R^m$. We use these elements as column vectors to form an $m \times r$ matrix Q_R, and consider the $m \times (p+r+q)$ matrix

$$\bar{Q} = [\ Q_A \mid Q_R \mid Q_B\ ] .$$

Notice that the two submatrices $\bar{Q}_A = [\ Q_A \mid Q_R\ ]$ and $\bar{Q}_B = [\ Q_R \mid Q_B\ ]$ of $\bar{Q}$ have entries entirely in A and B respectively, and are hence right coherent over D according to Lemma 3.2.

Since $K \subset M_A + M_B$, the column vectors of $\bar{Q}$ still generate $M_A + M_B$. If $\bar{f} : D^{n+r} \to D^m$ is the homomorphism associated with $\bar{Q}$, then we have another presentation of $M_A + M_B$:

(7) $$0 \longrightarrow \ker \bar{f} \hookrightarrow D^{n+r} \xrightarrow{\bar{f}} M_A + M_B \longrightarrow 0 .$$

Applying Schanuel's lemma [6, Theorem 3.41] to (6) and (7), we obtain

$$\ker f \oplus D^{n+r} \tilde{\to} \ker \bar{f} \oplus D^n ,$$

so that $\ker f$ is finitely generated over D if and only if $\ker \bar{f}$ is. To see the finite generation of $\ker \bar{f}$, let

$$(x_1, \ldots, x_p, z_1, \ldots, z_r, y_1, \ldots, y_q) \in \ker \bar{f} ,$$

which is to say that x_i, z_k, y_j are elments in D satisfying

(8) $$a_1x_1 + \ldots + a_px_p + c_1z_1 + \ldots + c_rz_r + b_1y_1 + \ldots + b_qy_q = 0 .$$

Write $d = -(b_1y_1 + \ldots + b_qy_q)$. Then (8) implies that d is an element in $(M_A + K\cdot D) \cap (M_B + K\cdot D)$, and so $d \in K\cdot D$ by Lemma 3.3. Thus

(9) $$d = c_1z_1' + \ldots + c_rz_r' ,$$

for some $z_1', \ldots, z_r'$ in D . From (8) and (9), we easily obtain

(10) $$a_1x_1 + \dots + a_px_p + c_1(z_1-z_1') + \dots + c_r(z_r-z_r') = 0 ,$$

and

(11) $$c_1z_1' + \dots + c_rz_r' + b_1y_1 + \dots + b_qy_q = 0 .$$

Now, since $(x_1, \dots, x_p, z_1, \dots, z_r, y_1, \dots, y_q)$ can be written as

(12) $$(x_1, \dots, x_p, z_1-z_1', \dots, z_r-z_r', 0, \dots, 0) + (0, \dots, 0, z_1', \dots, z_r', y_1, \dots, y_q) ;$$

and since $\bar{Q}_A$ and $\bar{Q}_B$ are right coherent matrices <u>over D</u>, we easily conclude from (10), (11) and (12) that $\ker \bar{f}$ is a finitely generated D-module, thereby completing the proof.

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WHITEHEAD GROUPS OF FREE PRODUCTS WITH AMALGAMATION

by A.J.Casson

Introduction

We use the notation of Milnor's survey [4]. Stallings [5] has shown that, if A and B are augmented algebras, then (under certain conditions) $K_1(A*B) = K_1(A) \oplus K_1(B)$. We aim to generalise this result to deal with free products with amalgamation.

Given rings A,B,C and homomorphisms $\alpha:C\to A$, $\beta:C\to B$, we construct a group $K_1(\alpha,\beta)$ which fits into an exact sequence

$$K_1(C) \longrightarrow K_1(A)\oplus K_1(B) \longrightarrow K_1(\alpha,\beta) \longrightarrow K_0(C) \longrightarrow K_0(A)\oplus K_0(B) .$$

We say that a subring C of A is <u>pure</u> if A admits a decomposition $A = C\oplus A'$ as C-bimodule. (For example, if A is a group ring $Z[G]$ and E is a subgroup of G, then $Z[E]$ is pure in $Z[G]$.) Suppose C is also pure in a ring $B = C\oplus B'$. Then one can form the amalgamated free product $A*_C B$; it contains the tensor algebra $T = T_C(A'\otimes_C B')$ of the C-bimodule $A'\otimes_C B'$. We construct a homomorphism

$$\theta:K_1(\alpha,\beta)\longrightarrow K_1(A*_C B)$$

(where $\alpha:C\to A$, $\beta:C\to B$ are the inclusions) and our main result (Theorem 2) states that

$$K_1(T)\oplus K_1(\alpha,\beta)\longrightarrow K_1(A*_C B)$$

is surjective. If $A'\otimes_C B'$ is a "free" C-bimodule (that is, a direct sum of copies of C), then

$$\mathrm{im}(K_1(T))\subseteq \mathrm{im}(\theta)$$

so θ is already surjective (Theorem 3). It would be interesting to know whether θ is actually an isomorphism in this case. When applied to a group ring $A = Z[G*_E H] = Z[G]*_{Z[E]}Z[H]$ the freeness hypothesis in Theorem 3 is satisfied if G and H are generated by E together with the respective centralizers of E, but not apparently in general. One can thus obtain (from Theorem 3) the vanishing of the Whitehead groups of groups built up from infinite cyclic groups by finitely many direct and free products (and even "central amalgamations", i.e. those of the type $G*_E H$ with E central in G and in H.)

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§1. Generalities

Let A,B,C be rings with 1 and let $\alpha: C \to A$, $\beta: C \to B$ be ring homomorphisms respecting 1. Define a group $K_1(\alpha,\beta)$ as follows. A triple (P,X,Y) consists of a finitely generated projective right C-module P, an A-basis $X = (x_1,\ldots,x_n)$ of $P\otimes_C A$ and a B-basis $Y = (y_1,\ldots,y_n)$ of $P\otimes_C B$. Note that X,Y are required to have the same number of elements. The sum of two triples is defined by

$$(P,X,Y)\oplus(P',X',Y') = (P\oplus P',X\oplus X',Y\oplus Y') .$$

For each integer $n \geq 0$ there is a standard triple

$$S_n = (C^n, Z^n\otimes 1_A, Z^n\otimes 1_B)$$

where Z^n denotes the standard C-basis of C^n.

Triples (P,X,Y),(P',X',Y') are equivalent if there exist a C-isomorphism $\gamma: P \to P'$ and elements M,N in the commutator subgroups of $\mathrm{Aut}_A(P\otimes_C A)$, $\mathrm{Aut}_B(P\otimes_C B)$ respectively such that

$$X' = (\gamma\otimes 1_A)MX \ , \ Y' = (\gamma\otimes 1_B)NY .$$

Triples (P,X,Y),(P',X',Y') are stably equivalent if there exist integers r,r' such that $(P,X,Y)\oplus S_r$ is equivalent to $(P',X',Y')\oplus S_{r'}$. It is easily checked that equivalence and stable equivalence are equivalence relations. Moreover, if δ,δ' are the stable equivalence classes of (P,X,Y),(P',X',Y'), then the stable equivalence class $\delta+\delta'$ depends only on δ and δ'.

<u>Lemma 1 Stable equivalence classes of triples form an Abelian group</u> $K_1(\alpha,\beta)$.

<u>Proof</u>. Addition is clearly associative and commutative. All standard triples are stably equivalent, and represent the zero element of $K_1(\alpha,\beta)$. It remains to produce an inverse for the class (P,X,Y). There is a finitely generated projective C-module P' such that $P\oplus P' \cong C^m$ for some m. If X,Y each have n elements, then $(P'\oplus C^n)\otimes_C A$, $(P'\oplus C^n)\otimes_C B$ are free on m generators, with bases X',Y'. Then $(P,X,Y)\oplus(P'\oplus C^n,X',Y')$ is equivalent to (C^r,X'',Y'') for some bases X'',Y'' and $r = m + n$. Let M,N be the unique elements of $\mathrm{Aut}_A(A^r)$, $\mathrm{Aut}_B(B^r)$ such that

$$X'' = M(Z^r\otimes 1_A) \ , \ Y'' = N(Z^r\otimes 1_B) \ .$$

Let

$$X^* = M^{-1}(Z^r\otimes 1_A) \ , \ Y^* = N^{-1}(Z^r\otimes 1_B) \ ;$$

then

$$(C^r,X'',Y'')\oplus(C^r,X^*,Y^*) = (C^{2r},(M\oplus M^{-1})(Z^{2r}\otimes 1_A),(N\oplus N^{-1})(Z^{2r}\otimes 1_B))$$

But $M\oplus M^{-1}$, $N\oplus N^{-1}$ belong to the commutator subgroups of $\mathrm{Aut}_A(A^{2r})$, $\mathrm{Aut}_B(B^{2r})$ respectively, so $(C^r,X'',Y'')\oplus(C^r,X^*,Y^*)$ is equivalent to S_{2r}. Therefore $(P'\oplus C^n,X',Y')\oplus(C^r,X^*,Y^*)$ represents an inverse to (P,X,Y), as required.

<u>Theorem 1</u> There is an exact sequence

$$K_1(C)\xrightarrow{i}K_1(A)\oplus K_1(B)\xrightarrow{j}K_1(\alpha,\beta)\xrightarrow{\partial}K_0(C)\xrightarrow{i}K_0(A)\oplus K_0(B).$$

<u>Proof</u>. First we define the maps. For $r = 0,1$ let

$$i = \alpha_*\oplus\beta_* : K_r(C)\longrightarrow K_r(A)\oplus K_r(B) \ .$$

If (P,X,Y) is a triple and X,Y each have n elements, let $P - C^n$ represent $\partial(P,X,Y)$. If $\mu\in K_1(A)$, $\nu\in K_1(B)$, then for large n there exist $M\in \mathrm{Aut}_A(A^n)$, $N\in\mathrm{Aut}_B(B^n)$ representing μ,ν respectively. Let $(C^n,M(Z^n\otimes 1_A),N(Z^n\otimes 1_B))$ represent $j(\mu\oplus\nu)$. It is not hard to show that j,∂ are well-defined homomorphisms, and that the composites $i\partial$, ∂j, ji are zero.

Let $\sigma \in K_0(C)$ be such that $i(\sigma) = 0$, so $\alpha_*(\sigma) = 0$ and $\beta_*(\sigma) = 0$. Then σ is represented by $P - C^n$, where P is a finitely generated projective C-module and $n \geq 0$. Since $\alpha_*(\sigma) = 0$ and $\beta_*(\sigma) = 0$, there is an integer r such that $(P \oplus C^r) \otimes_C A$, $(P \oplus C^r) \otimes_C B$ are both free on n+r generators. Let X,Y be bases of $(P \oplus C^r) \otimes_C A$, $(P \oplus C^r) \otimes_C B$, each containing n+r elements. Then $\partial(P \oplus C^r, X, Y)$ is represented by $P \oplus C^r - C^{n+r}$, so $\sigma = \partial(P \oplus C^r, X, Y)$. This proves exactness at $K_0(C)$.

If $\partial(P,X,Y) = 0$ and X,Y each have n elements, then there is an integer r such that $P \oplus C^r \cong C^{n+r}$. Therefore $(P,X,Y) \oplus S_r$ is in the image of j, so the sequence is exact at $K_1(\alpha,\beta)$.

Suppose $\mu \in K_1(A)$, $\nu \in K_1(B)$ are such that $j(\mu \oplus \nu) = 0$. Let $M \in Aut_A(A^n)$, $N \in Aut_B(B^n)$ represent μ, ν respectively. Then $(C^n, M(Z^n \otimes 1_A), N(Z^n \otimes 1_B))$ is stably equivalent to S_n, so there is an integer r such that $(C^{n+r}, (M \oplus I_r)(Z^{n+r} \otimes 1_A), (N \oplus I_r)(Z^{n+r} \otimes 1_B))$ is equivalent to $(C^{n+r}, Z^{n+r} \otimes 1_A, Z^{n+r} \otimes 1_B)$. There exist a C-isomorphism $\gamma : C^{n+r} \longrightarrow C^{n+r}$ and elements M',N' in the commutator subgroups of $Aut_A(A^{n+r})$, $Aut_B(B^{n+r})$ respectively, such that

$$(M \oplus I_r)(Z^{n+r} \otimes 1_A) = (\gamma \oplus 1_A) M' (Z^{n+r} \otimes 1_A) ,$$

$$(N \oplus I_r)(Z^{n+r} \otimes 1_B) = (\gamma \oplus 1_B) N' (Z^{n+r} \otimes 1_B) .$$

Therefore μ, ν are represented by $\gamma \otimes 1_A$, $\gamma \otimes 1_B$ respectively, so $\mu \oplus \nu$ belongs to the image of i. This completes the proof of exactness.

Suppose now that R is a ring with 1 and that $\varphi : A \longrightarrow R$, $\psi : B \longrightarrow R$ are homomorphisms respecting 1 such that $\varphi\alpha = \psi\beta$. Define a map $\theta : K_1(\alpha,\beta) \longrightarrow K_1(R)$ as follows. If (P,X,Y) is a triple, then $X \otimes 1_R$ is an R-basis of $(P \otimes_C A) \otimes_A R = P \otimes_C R$. Similarly, $Y \otimes 1_R$ is an R-basis of $P \otimes_C R$ having the same number of elements as $X \otimes 1_R$. Let $\theta(P,X,Y)$ be represented by the unique automorphism of $P \otimes_C R$ carrying $X \otimes 1_R$ onto $Y \otimes 1_R$. It is easy to check that θ is a well-defined homomorphism.

Now we give a way of recognising elements in the image of θ. Let us identify $C^n \otimes_C A$, $C^n \otimes_C B$, $C^n \otimes_C R$ with A^n, B^n, R^n respectively by making the standard bases correspond.

<u>Lemma 2</u> <u>Let</u> P, Q <u>be right C-submodules of</u> C^{2n} <u>with</u> $C^{2n} = P \oplus Q$. <u>Let</u> $M_A : A^n \longrightarrow A^{2n}$, $M_B : B^n \longrightarrow B^{2n}$ <u>be monomorphisms such that</u> $\text{im}(M_A) = P \otimes_C A$, $\text{im}(M_B) = Q \otimes_C B$. <u>Then</u>

$$M = (M_A \otimes 1_R) \oplus (M_B \otimes 1_R) \; : \; R^n \oplus R^n \longrightarrow R^{2n}$$

<u>represents an element in the image of</u> θ.

<u>Proof</u>. Define $N_A : A^n \oplus A^n \longrightarrow (P \otimes_C A) \oplus A^n$ by $N_A(u,v) = M_A u + v$, and define $N_B : (P \otimes_C B) \oplus B^n \longrightarrow B^{2n}$ by $N_B(x,y) = x + M_B y$. Then N_A, N_B are isomorphisms with $M = (N_B \otimes 1_R)(N_A \otimes 1_R)$. Take $X = N_A(Z^{2n} \otimes 1_A)$ as basis of $(P \oplus C^n) \otimes_C A$ and $Y = N_B^{-1}(Z^{2n} \otimes 1_B)$ as basis of $(P \oplus C^n) \otimes_C B$. Then $(N_B^{-1} \otimes 1_R)(N_A^{-1} \otimes 1_R)$ is the automorphism taking X to Y ; but this represents the same element of $K_1(R)$ as $(N_A^{-1} \otimes 1_R)(N_B^{-1} \otimes 1_R) = M^{-1}$. Therefore M represents $-\theta(P \oplus C^n, X, Y)$, and the lemma is proved.

§2. <u>Free products with amalgamation</u>

Let A be a ring with 1. A subring C of A is called <u>pure</u> if it contains 1_A and there is a C-bimodule homomorphism $\rho : A \longrightarrow C$ with $\rho|_C = 1$. Let A,B be rings with 1, each containing C as a pure subring. Cohn [2] gives the following description of the <u>free product with amalgamation</u> $A *_C B$.

Let $A' = \ker(\rho : A \longrightarrow C)$, $B' = \ker(\rho : B \longrightarrow C)$, so A' and B' are C-bimodules. Following Stallings[5], we consider the semigroup G on two generators a,b with relations $a^2 = a$, $b^2 = b$. If $\gamma \in G$, let $|\gamma|$ denote the number of symbols in the reduced word for γ. Define a C-bimodule R_γ for each $\gamma \in G$ by $R_1 = C$, $R_{\gamma a} = R_\gamma \otimes_C A'$ if $|\gamma a| > |\gamma|$ and $R_{\gamma b} = R_\gamma \otimes_C B'$ if $|\gamma b| > |\gamma|$. Let $R = \sum_{\gamma \in G} R_\gamma$ as a C-bimodule, so

$$R = C \oplus A' \oplus B' \oplus (B' \otimes_C A') \oplus (A' \otimes_C B') \oplus (A' \otimes_C B' \otimes_C A') \oplus \ldots \quad .$$

To make R into a ring, it suffices to define associative and distributive products $\pi_{\gamma,\delta} : R_\gamma \otimes_C R_\delta \longrightarrow R$. We do this by induction on $|\gamma| + |\delta|$.

If $|\gamma\delta| = |\gamma| + |\delta|$, let $\pi_{\gamma,\delta}$ be the inclusion map $R_\gamma \otimes_C R_\delta = R_{\gamma\delta} \subset R$. Define $\pi_{a,a} : A' \otimes_C A' \longrightarrow A = C \oplus A' \subset R$ by multiplication in A, and similarly define $\pi_{b,b}$. Suppose $|\gamma\delta| < |\gamma| + |\delta|$ so $\gamma = \gamma' x$, $\delta = x\delta'$ with $x = a$ or b and $|\gamma'| < |\gamma|$, $|\delta'| < |\delta|$. Then $R_\gamma \otimes_C R_\delta = R_{\gamma'} \otimes_C R_x \otimes_C R_x \otimes_C R_{\delta'}$, and $\pi_{\gamma',\delta'}$ is already constructed, so we may define $\pi_{\gamma,\delta}$ by the following diagram.

$$\begin{array}{ccc} R_{\gamma'} \otimes_C R_x \otimes_C R_x \otimes_C R_{\delta'} & \xrightarrow{\pi_{\gamma,\delta}} & R \\ {\scriptstyle 1 \otimes \pi_{x,x} \otimes 1} \downarrow & & \downarrow {\scriptstyle 1 \oplus \pi_{\gamma',\delta'}} \\ (R_{\gamma'} \otimes_C R_x \otimes_C R_{\delta'}) \oplus (R_{\gamma'} \otimes_C R_{\delta'}) & = & R_{\gamma' x \delta'} \oplus (R_{\gamma'} \otimes_C R_{\delta'}) \end{array}$$

Clearly $\pi_{\gamma,\delta}$ is distributive; an inductive proof that $\pi_{\gamma,\delta}$ is associative is not too hard. One can also show that R has the universal mapping property which characterises free products with amalgamation. If S is a ring and $\xi : A \longrightarrow S$, $\eta : B \longrightarrow S$ are ring homomorphisms such that $\xi|_C = \eta|_C$, then there is a unique ring homomorphism $\zeta : R \longrightarrow S$ with $\xi = \zeta|_A$, $\eta = \zeta|_B$. We shall define $A *_C B$ to be R.

Observe that $\sum_{n=0}^{\infty} R_{(ab)^n}$ is a subring of R, isomorphic to the tensor ring $T(A' \otimes_C B')$ of the C-bimodule $A' \otimes_C B'$. Let $V = \sum_{\gamma \in G} R_{\gamma a}$, $W = \sum_{\gamma \in G} R_{\gamma b}$; these are both C-bimodules, and $R = C \oplus V \oplus W$.

We shall often use the relations

$$AV \subset C \oplus V \ , \ BV = V \ , \ AW = W \ , \ BW \subset C \oplus W \ .$$

Observe also that

$$V = A' \oplus (W \otimes_C A') \ , \ W = B' \oplus (V \otimes_C B') \ .$$

§3. Main theorem

Let A,B be rings with 1, each containing C as a pure subring, and let $\alpha:C\longrightarrow A$, $\beta:C\longrightarrow B$ be the inclusion maps. Then the inclusions $\varphi:A\longrightarrow A*_C B$, $\psi:B\longrightarrow A*_C B$ define a map $\theta:K_1(\alpha,\beta)\longrightarrow K_1(A*_C B)$. The inclusion $\lambda:T(A'\otimes_C B')\longrightarrow A*_C B$ induces a map

$$\lambda_*:K_1(T(A'\otimes_C B'))\longrightarrow K_1(A*_C B) .$$

Theorem 2 $K_1(A*_C B)$ is generated by the images of $K_1(\alpha,\beta)$ and $K_1(T(A'\otimes_C B'))$.

Proof. Let τ be any element of $K_1(A*_C B)$. By Higman's trick (explained in [5,§4]), τ is represented by some invertible $(n\times n)$ matrix $T_A + T_B$, where T_A, T_B have entries in A,B respectively. Now make the further simplification

$$T_A + T_B \sim \begin{pmatrix} T_A + T_B & 0 \\ 0 & 1_n \end{pmatrix} \sim \begin{pmatrix} T_A + T_B & 0 \\ 1_n & 1_n \end{pmatrix} \sim \begin{pmatrix} T_A & -T_B \\ 1_n & 1_n \end{pmatrix} .$$

Write M_A, M_B for the $(2n\times n)$ matrices $\begin{pmatrix} T_A \\ 1_n \end{pmatrix}$, $\begin{pmatrix} -T_B \\ 1_n \end{pmatrix}$ respectively, and let $M = (\,M_A \quad M_B\,)$. Then M is an invertible $(2n\times 2n)$ matrix representing τ, and M_A, M_B have entries in A, B respectively. Let the inverse N of M be partitioned as $\begin{pmatrix} N^1 \\ N^2 \end{pmatrix}$, where N^1, N^2 are $(n\times 2n)$ matrices.

Recall that, in the notation of §2,

$$A*_C B = R = C\oplus V\oplus W .$$

Write

$$N^i = N^i_C + N^i_V + N^i_W \quad (i=1,2) ,$$

where N^i_C, N^i_V, N^i_W have entries in C,V,W respectively. Let

$$K = M_A N^1_C + M_A N^1_V + M_B N^2_V$$

$$L = M_B N^2_C + M_B N^2_W + M_A N^1_W .$$

<u>Lemma 3</u> K,L <u>have entries in</u> C <u>and</u>

$$K + L = 1 \ , \ K^2 = K \ , \ L^2 = L \ , \ KL = LK = 0 \ .$$

<u>Proof</u>.

$$K + L = (M_A \quad M_B) \begin{pmatrix} N^1 \\ N^2 \end{pmatrix} = 1 \ .$$

$M_A N_C^1$, $M_A N_V^1$, $M_B N_V^2$ have entries in $C \oplus V$, and $M_B N_C^2$, $M_B N_W^2$, $M_A N_W^1$ have entries in $C \oplus W$. But K + L has entries in C, so K,L both have entries in C.

The equation NM = 1 implies that

$$N^1 M_A = 1 \ , \ N^1 M_B = 0 \ , \ N^2 M_A = 0 \ , \ N^2 M_B = 1 \ .$$

Therefore

$$NK = \begin{pmatrix} N^1 K \\ N^2 K \end{pmatrix} = \begin{pmatrix} N_C^1 + N_V^1 \\ N_V^2 \end{pmatrix} \quad ,$$

so $N_C^1 K + N_V^1 K + N_W^1 K = N_C^1 + N_V^1$, and $N_C^2 K + N_V^2 K + N_W^2 K = N_V^2$.

But K has entries in C ; it follows that

$$N_C^1 K = N_C^1 \ , \ N_V^1 K = N_V^1 \ , \ N_W^1 X = 0 \ ,$$

$$N_C^2 K = 0 \ , \ N_V^2 K = N_V^2 \ , \ N_W^2 X = 0 \ .$$

Therefore $NK^2 = NK$; since N is invertible, $K^2 = K$. It follows that $L^2 = L$, $KL = LK = 0$, as required.

Now write $V = A' \oplus (W \otimes_C A')$ and $N_V^i = N_{A'}^i + N_{WA'}^i$ (i = 1,2), where $N_{A'}^i$, $N_{WA'}^i$ have entries in A' , $W \otimes_C A'$ respectively. Similarly write $W = B' \oplus (V \otimes_C B')$, $N_W^i = N_{B'}^i + N_{VB'}^i$. Let

$$E = M_A(N_C^1 + N_{A'}^1) \ , \ F = M_B(N_C^2 + N_{B'}^2) \ .$$

<u>Lemma 4</u> K − E , L − F <u>have entries in</u> A' , B' <u>respectively, and</u>

$$E^2 = EK = E \ , \ KE = K \ , \ EM_A = M_A \ ,$$

$$F^2 = FL = F \ , \ LF = L \ , \ FM_B = M_B \ .$$

<u>Proof</u>. By definition of E, K − E has entries in A. But

$$K - E = M_A N_{WA'}^1 + M_B N_{WA'}^2 + M_B N_{A'}^2 \ ,$$

and all terms on the right have entries in $A' \oplus (W \otimes_C A')$. Therefore

K-E has entries in A'.

$$(N^1_C + N^1_{A'} + N^1_{WA'} + N^1_W)M_A = N^1 M_A = 1 .$$

But $N^1_C M_A$, $N^1_{A'}M_A$ have entries in $C \oplus A'$, and $N^1_{WA'}M_A$, $N^1_W M_A$ have entries in $(W \otimes_C A') \oplus W$. Therefore

$$(N^1_C + N^1_{A'})M_A = 1 \ ;$$

it follows that $EM_A = M_A$ and $E^2 = E$. The argument used in Lemma 3 to prove $N^1_C K = N^1_C$ also proves $N^1_{A'}K = N^1_{A'}$, so $EK = E$. Similarly, $L - F$ has entries in B' and $F^2 = FL = F$, $FM_B = M_B$. It remains to prove that $KE = K$ and $LF = L$.

Observe that $R = (C \oplus W) \otimes_C (C \oplus A') = (C \oplus W) \otimes_C A$. Thus $R^{2n} = (C \oplus W)^{2n} \otimes_C A$ (as C-bimodule), and the columns of $K - E$ are in $C^{2n} \otimes_C A'$. The columns of $M_A N^1_{WA'} + M_B N^2_{WA'} + M_B N^2_{A'}$ are in $H \otimes_C A'$, where

$$H = M_B C^n \oplus MW^{2n} \subset (C \oplus W)^{2n} .$$

Now $LC^{2n} \subset H$ and $KC^{2n} \subset M_A C^n \oplus MV^{2n}$. But

$$R^{2n} = MC^{2n} \oplus MV^{2n} \oplus MW^{2n} = M_A C^n \oplus M_B C^n \oplus MV^{2n} \oplus MW^{2n},$$

so $KC^{2n} \cap H = \{0\}$. Since $C^{2n} = KC^{2n} \oplus LC^{2n}$, it follows that $LC^{2n} = C^{2n} \cap H$ Moreover, $H = LC^{2n} \oplus \{H \cap (XC^{2n} \oplus W^{2n})\}$. So all the inclusion maps in the diagram

$$\begin{array}{ccc} LC^{2n} & \longrightarrow & H \\ \downarrow & & \downarrow \\ C^{2n} & \longrightarrow & (C \oplus W)^{2n} \end{array}$$

are split; it follows that $K - E$ has columns in

$$(H \otimes_C A') \cap (C^{2n} \otimes_C A') = (LC^{2n}) \otimes_C A' \subset LR^{2n} .$$

But $L|_{LR^{2n}} = 1$, so $L(K - E) = K - E$. Therefore $KE = E - LE = K$. Similarly $LF = L$, so Lemma 4 is proved.

Since $EK = E$ and $KE = K$, $\ker E = \ker K$. Since $EM_A = M_A$ and $E = M_A(N^1_C + N^1_{A'})$, $\operatorname{im}E = \operatorname{im}M_A$. Similarly, $\ker F = \ker L$ and $\operatorname{im}F = \operatorname{im}M_B$.

Lemma 5 $E + F$ is invertible, and represents an element in the image of $K_1(T(A'\otimes_C B'))$.

Proof. Since $(E + F)K = E$, $(E + F)L = F$,

$$(E + F)R^{2n} \supset ER^{2n} + FR^{2n} = M_A R^n + M_B R^n = R^{2n} .$$

If $u \in \ker(E + F)$, then $Eu + Fu = 0$ with $Eu \in M_A R^n$, $Fu \in M_B R^n$. It follows that $Eu = Fu = 0$, so $u \in \ker E \cap \ker F = \ker K \cap \ker L = \{0\}$. Therefore $E + F$ is invertible.

Now $(1 + E - K)L = L$ and $K(1 + E - K) = K$, so $1 + E - K$ is an elementary matrix. Similarly $1 + F - L$ is an elementary matrix. Since

$$(E - K)^2 = (F - L)^2 = 0 ,$$

$$E + F = (1 + E - K)(1 - (E - K)(F - L))(1 + F - L) .$$

Therefore $1 + (E - K)(F - L)$ is invertible; since its entries lie in $T(A'\otimes_C B')$, $E + F$ represents an element in the image of $K_1(T(A'\otimes_C B'))$, as required. (Recall that a similar trick was used in [5].)

Now $(E + F)(\; M_A \quad M_B \;) = (\; EM_A \quad FM_B \;) = M$.

Let $P = KC^{2n}$, $Q = LC^{2n}$; then $C^{2n} = P\oplus Q$ as right C-modules. Since

$$(KM_A)A^n = (KE)A^{2n} = KA^{2n} = P\otimes_C A \; , \; (LM_B)B^n = Q\otimes_C B \; ,$$

Lemma 2 shows that $(\; KM_A \quad LM_B \;)$ represents an element in the image of $K_1(\alpha,\beta)$. Therefore the element τ represented by M is in the group generated by the images of $K_1(\alpha,\beta)$ and $K_1(T(A'\otimes_C B'))$. This completes the proof of Theorem 2.

Bass [1] has defined Nil(C) to be the cokernel of the map $K_1(C) \longrightarrow K_1(C[t])$ induced by inclusion. Stallings [5] uses a method of Gersten [3] to prove the following result.

Theorem If $A'\otimes_C B'$ is a direct limit of free C-bimodules, and Nil(C) = 0, then the map $K_1(C) \longrightarrow K_1(T(A'\otimes_C B'))$ is surjective.

(Here, "free C-bimodule" means the direct sum of copies of C).

Theorem 3 If $A'\otimes_C B'$ is a direct limit of free C-bimodules, and $Nil(C) = 0$, then $\theta: K_1(\alpha,\beta) \longrightarrow K_1(A*_C B)$ is surjective.

Proof. Observe that the image of the map

$$K_1(C) \longrightarrow K_1(T(A'\otimes_C B')) \longrightarrow K_1(A*_C B)$$

is already contained in the image of θ. Theorem 3 now follows immediately from Theorem 2 and the Theorem of Gersten and Stallings.

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WHITEHEAD GROUPS OF GENERALIZED FREE PRODUCTS

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The purpose of these notes is to describe a splitting theorem for the Whitehead group. Its application is in vanishing theorems of the sort that Wh(G) = 0 if G is a classical knot or link group.

An example of such a link group is the group with generators a, b, c, and relators

$$[a,[b,c^{-1}]]\ ,\quad [b,[c,a^{-1}]]\ ,\quad [c,[a,b^{-1}]]$$

where $[x,y]$ denotes the commutator $xyx^{-1}y^{-1}$. This group may look complicated, but it happens to be the group of one of the simplest links (the 'Borromean rings').

It is not their presentations that make knot groups tractable. What makes them tractable is the fact that they can be built up out of nothing by iterating a construction that I call 'generalized free product'. As this construction (or at least the motivation to look at it) is of topological origin, I will start by giving the topology flavored description.

Let X be a 'nice' topological space, e.g., a CW complex (or, if the reader prefers, a simplicial complex, or even a smooth manifold; all that matters for our purpose, is the global picture), and let Y be a closed 'nice' subspace, e.g., a subcomplex. We assume Y is bicollared in X, this means there exists an open embedding $i\colon Y\times R \to X$ (where R is the euclidean line) so that $i(Y\times 0) = Y$. We do not ask that Y be connected, in fact, Y may have infinitely many components.

A recipe says that in this situation, the fundamental groupoid of X can be calculated as the colimit of certain other groupoids.

Now assume that for every component Y_j of Y, the inclusion induced homomorphism of fundamental groups, $\pi_1 Y_j \to \pi_1 X$, is a monomorphism. Then the diagram obtained is called a generalized free product (g.f.p.) structure on $\pi_1 X$.

Let us denote X_i, $i \in I$, the components of X - Y, and Y_j, $j \in J$, the components of Y. The groups $\pi_1 X_i$ are called the building blocks of the g.f.p. structure, and the groups $\pi_1 Y_j$ are called the amalgamations. For the sake of uniform notation, we write

$$G = \pi_1 X \ , \quad B = \bigcup_{i \in I} \pi_1 X_i \ , \quad A = \bigcup_{j \in J} \pi_1 Y_j \ ,$$

where '$\cup$' denotes the sum ('disjoint union') in the category of groupoids.

As Y_j locally dissects X, we may pick one of its sides (arbitrarily, but forever) and denote it 'left', and the other one 'right'. There are injections of groups (well-determined up to inner automorphisms)

$$l_j: \pi_1 Y_j \to \pi_1 X_{l(j)} \quad \text{and} \quad r_j: \pi_1 Y_j \to \pi_1 X_{r(j)} \ .$$

Let F be a functor from groups to abelian groups which sends inner automorphisms to identities. Letting

$$F(B) = \bigoplus_{i \in I} F(\pi_1 X_i)$$

and similarly with F(A), we have well defined maps $F(l)$: $F(A) \to F(B)$, $F(r)$: $F(A) \to F(B)$, and $F(\iota)$: $F(B) \to F(G)$, satisfying $F(\iota) \circ F(l) = F(\iota) \circ F(r)$.

Examples of such functors F are

(1) $H_0(G)$, the integral homology in dimension 0

(2) $K_0(RG)$, the projective class group of the group algebra of G over R, and in particular, $K_0(G) := K_0(ZG)$

(3) $\tilde{K}_0(G) = \text{coker}(H_0(G) \to K_0(G))$

(4) $Z_2 \oplus H_1(G)$

(5) $K_1(RG)$

(6) $Wh(G) = \text{coker}(Z_2 \oplus H_1(G) \to K_1(G))$, this map being induced from $GL(Z,1) \times G \to GL(ZG,1)$

We can now formulate the splitting theorem.

Proposition. There is an abelian group $\mathfrak{N}$ and a map δ so that the following sequence is exact

$$Wh(A) \xrightarrow{1_*-r_*} Wh(B) \xrightarrow{\iota_*} Wh(G) \xrightarrow{\delta} \mathfrak{N} \oplus \tilde{K}_0(A) \xrightarrow{(0,1_*-r_*)} \tilde{K}_0(B)$$

There is a similar sequence for the unreduced functors; the one with integral coefficients maps onto the one given, and the kernel is the Mayer Vietoris sequence of homology (as indicated in (3) and (6)). One can continue the sequence to the right (by Bass' 'contracted functor' argument).

The splitting theorem contains as special cases both the splitting theorem for a free product of groups, and the Künneth formula for extensions of the integers.

In order to deduce vanishing results from the splitting theorem, one uses the five lemma and some a priori information about the vanishing of the exotic term $\mathfrak{N}$. The trick here is not to work with an individual group G, but with the totality of groups $G \times F$, where F is a free abelian group. One can thus exploit the fact that $\tilde{K}_0(G \times F)$ is a direct summand of $Wh(G \times F \times Z) = Wh(G \times F')$. The trick works well since a g.f.p. structure on G (with building blocks B and amalgamation A, say) induces a g.f.p. structure on $G \times F$ (with building blocks $B \times F$ and amalgamation $A \times F$, and the obvious maps).

The next proposition describes such a vanishing result for the exotic term.

Proposition. In order that $\mathfrak{N} = 0$, it is sufficient that for any component A_j of A, the group algebra ZA_j be regular coherent.

Note that no condition is asked of the building blocks or the structure maps. In the case of the more general splitting theorem with R coefficients, one would correspondingly ask that RA_j be regular coherent.

(A ring is called coherent if its finitely presented modules form an abelian category; it is called regular coherent if, in addition, each finitely presented module has a finite dimensional projective resolution).

The sort of arguments used in deriving the splitting theorem , also gives information on this type of structure of rings:

<u>Proposition</u>. Let G have a g.f.p. structure with building blocks B and amalgamations A. For RG to be regular coherent, it is sufficient that the group algebras RB_i be regular coherent and that the group algebras RA_j be regular noetherian.

The proposition says, for example, if G is a free group, or a 2-manifold group, then ZG is regular coherent.

I will now indicate how g.f.p. structures occur in nature. This necessitates the notion of <u>iterated</u> g.f.p. structure. The main point in the definition is an appropriate transfinite recursion.

Notationally, it is convenient to introduce classes of groups, $C_{m,n}$, indexed by pairs of non-negative integers in lexicographical ordering. Each class contains the preceding ones. We abbreviate

$$C_m = \bigcup_n C_{m,n} \,, \quad C = \bigcup_m C_m \,.$$

<u>Definition</u>. (1) $C_{0,0}$ contains only the trivial group

(2) $G \in C$ if and only if G has a g.f.p. structure with all building blocks, B, and all amalgamations, A, in C_m, for some fixed m

(3) if $G \in C$, then $G \in C_m$ if and only if

all $B_i \in C_{m,n}$, for some fixed n, and

all $A_j \in C_{m-1}$

(4) if $G \in C_m$, then $G \in C_{m,n}$ if and only if all $B_i \in C_{m,n-1}$ (here $C_{m,-1}$ is to be interpreted as C_{m-1}) .

<u>Examples</u>. (1) $C_{m,n}$ is closed under taking subgroups.

(2) C is closed under extensions. (Proof: Let $1 \to \ker(p) \to F \xrightarrow[p]{} G \to 1$ be exact, with ker(p), G $\in$ C. Let $G \in C_{m,n}$. The proof is by induction on (m,n). Let G have a g.f.p. structure with building blocks B_i, and amalgamations A_j. Then F has a g.f.p. structure with building blocks $p^{-1}(B_i)$ and amalgamations $p^{-1}(A_j)$.)

(The assertions under (1) and (2) will be obvious from the definition of g.f.p. structure to be given in the next section).

(3) $C_1 = C_{1,0}$ is the class of free groups.

(4) If M is a closed 2-manifold other than the projective plane, then $\pi_1 M \in C_{2,0}$.

(5) There is a large class of 3-dimensional manifolds (e.g., all compact submanifolds of the 3-sphere) whose fundamental groups are in C_3 (and even in C_2 if the manifold has non-empty boundary), however, the 'n' may be quite large.

(6) A one-relator-group is in C_2 if (and only if) the relator is not a proper power. This can be checked from Magnus' analysis of these groups (note that the groups encountered on the way as building blocks, need not be one-relator-groups). Consequently, if G is a one-relator-group, and its relator is not a proper power, then $Wh(G) = \tilde{K}_0(G) = 0$.

To conclude this section, we exploit the geometric picture to see that the general type of g.f.p. structure can be reduced, in a sense, to two rather special types. For, let X and Y be as in the beginning. We can break X at Y, and can then reconstruct X, by glueing, one by one, at the components of Y, and eventually taking a direct limit.

Each of the steps in the above procedure corresponds to a g.f.p. structure in which (by abuse of the old notation) the subspace Y is connected. There are two cases left, according to whether X - Y is connected or not.

Denote by G, A, B (resp. B_1, B_2) the fundamental groups of X, Y, and X-Y (or its components), respectively.

In the case where X - Y has two components , G is the pushout in the diagram

$$\begin{array}{ccc} A & \longrightarrow & B_1 \\ \downarrow & & \downarrow \\ B_2 & \longrightarrow & G \end{array}$$

In a classical terminology, G is the 'free product of B_1 and B_2, amalgamated at A ', $G = B_1 \star_A B_2$ in customary notation.

There is yet another description available, namely G is also the pushout in the category of groupoids in the diagram

$$\begin{array}{ccc} A \cup A & \longrightarrow & B_1 \cup B_2 \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & G \end{array}$$

Here '$\cup$' is the sum in the category of groupoids, and I is the connected groupoid with two vertices and trivial vertex groups.

In the case where X - Y is connected, let $\alpha, \beta : A \to B$ denote the two inclusion maps. Then G is the pushout in the category of groupoids in the diagram

$$\begin{array}{ccc} A \cup A & \xrightarrow{\alpha \cup \beta} & B \\ \downarrow & & \downarrow \\ A \times I & \longrightarrow & G \end{array}$$

A classical terminology is not available for this construction. Logicians have used it to construct groups with weird properties (unsolvable word problem, etc.). They sometimes refer to it (and also to a more general construction) as the 'Higman-Neumann-Neumann-Britton-extension', cf. Miller's book. It can be checked, incidentally, that for quite a few of the weird groups in this book, our method shows their Whitehead group is trivial.

An explicit description of G is this. Let T be a free cyclic group, with generator t. Then G is isomorphic to the quotient of the free product $B \star T$ by the normal subgroup generated by

$$t\, \alpha(a)\, t^{-1} \, (\beta(a))^{-1} \ , \quad a \in A \ .$$

In the next section, I will give the definition of g.f.p. structures which is the most useful one to actually work with. The subsequent section is mostly devoted to a discussion of the exotic term in the splitting theorem. In the final section, some indication of proof is given for the splitting theorem itself.

Up to reformulation of some parts, essentially all of the present material has been taken from a preliminary report which was issued in fall '69 in mimeographed form. I have not included here the full proof of the splitting theorem, as I doubt if those details have any relevance to the conjecture described in the appendix.

2. Generalized free product structures, revisited.

Let the spaces X and Y be as in the preceding section. Denote $\tilde{X}$ the universal covering space of X, and $\tilde{Y}$ the induced covering space over Y. Identify G ($\approx \pi_1 X$) to the covering translation group of $\tilde{X}$, acting from the right.

The subspace $\tilde{Y}$ induces on $\tilde{X}$ a certain decomposition whose nerve is a graph, Γ, on which G acts. By a 'graph' we mean here a certain combinatorial device, consisting of its set of vertices, Γ^0, set of segments, Γ^1, and incidence relations ('initial vertex' and 'terminal vertex' of a segment, denoted $v_i(s)$ and $v_t(s)$, respectively). The elements of Γ^0 correspond to the components of $\tilde{X} - \tilde{Y}$, and the orbits Γ^0/G correspond to the components of X - Y. Similarly, the elements of Γ^1 correspond to the components of $\tilde{Y}$, and the orbits Γ^1/G correspond to the components of Y.

As the realization $|\Gamma|$ of Γ can be embedded as a retract in $\tilde{X}$, Γ must be a tree (i.e., the 1-complex $|\Gamma|$ is connected and simply connected).

Another property is obtained from the 'two-sidedness' of Y in X, namely the action of G on Γ preserves local orientations. By this we mean if $g \in G$ and $s \in \Gamma^1$, then $(s)g = s$ implies that g preserves the initial vertex of s. Consequently we can assume the segments of Γ are oriented in such a way that G preserves all orientations. We now define

Definition. A generalized free product structure on a group G consists of a tree Γ and an action (from the right) of G on Γ, preserving local orientations.

Remarks. (1) This is of course equivalent to our original definition. To recover that one, we need only construct Eilenberg-MacLane spaces $K(G_s,1)$ and $K(G_v,1)$ (corresponding to the stability groups of segments and vertices, one for each orbit), construct mapping cylinders and glue as prescribed by the quotient graph Γ/G. Since for the component Y_0 of Y, the map $\pi_1 Y_0 \to \pi_1 X$ is a monomorphism, $\pi_1 Y_0$ is indeed detected as the stability group of a certain segment.

(2) By our definition of g.f.p. structure, the 'set of g.f.p. structures on a group' is a certain contravariant functor, indeed a sum of representable ones. There is no corresponding assertion if we restrict attention to the two special types of g.f.p. structure considered at the end of the previous section.

We will now analyse g.f.p. structures a bit. By a basic tree in Γ we shall mean a subtree with the property that its set of vertices contains one and only one representative of every orbit Γ^0/G. A basic tree exists, e.g., one can lift a maximal tree from Γ/G. We choose a basic tree and keep it fixed henceforth, it will be denoted $\Gamma_\$$.

A segment in Γ is called non-recurrent if it is equivalent, under the action of G, to a segment in $\Gamma_\$$ (this notion depends on the choice of the basic tree, in general). Otherwise, it will be called recurrent. There exists a basic set of recurrent segments, denoted Γ^1_r. This means, Γ^1_r contains one and only one representative of any orbit of recurrent segments, and if $s \in \Gamma^1_r$, then the initial vertex of s is in $\Gamma_\$$ (the terminal vertex of s is then necessarily not in $\Gamma_\$$). We fix a group element, denoted t_s, with the property that t_s^{-1} carries the terminal vertex of s into $\Gamma_\$$.

The element t_s just described, acts necessarily without fixed points on Γ. This can easily be seen from the existence of the distance function on Γ which associates to any pair of vertices the number of segments in a shortest path joining them.

If $x \in \Gamma^0$ or $x \in \Gamma^1$, we let G_x denote the stability group of x,

$$G_x = \{ g \in G \mid (x)g = x \} .$$

The condition involved in the definition of a g.f.p. structure, is equivalent to: For any segment s, and its end points $v_i(s)$ and $v_t(s)$, we have the relation of stability groups

$$G_{v_i(s)} \cap G_{v_t(s)} = G_s .$$

We let $\Gamma_£$ denote the tree whose set of segments is

$$\Gamma^1_£ = \Gamma^1_\$ \cup \Gamma^1_r \cup \{ (s)t_s^{-1} \mid s \in \Gamma^1_r \} .$$

For any subtree Δ of Γ, and any vertex v of Δ, we let $\Delta^1(v)$ denote the set of those segments in Δ which are incident to v. Then clearly, for any $v \in \Gamma^0_\$$, the set $\Gamma^1(v)$ is in one-one correspondence to the union of cosets

$$\bigcup_s G_s \backslash G_v , \quad s \in \Gamma^1_£ .$$

From this follows by an inductive argument involving distance, that G is generated by

$$G_v , \quad v \in \Gamma^0_\$, \quad \text{and} \quad t_s , \quad s \in \Gamma^1_r .$$

3. Modules over generalized free product structures.

The central notion is that of a certain diagram which I call a Γ-object, and which I will now describe, after some preliminaries.

Following the notation set up before, we denote building blocks of the g.f.p. structure the groupoid

$$B = \bigcup_v G_v , \quad v \in \Gamma^0_\$,$$

and amalgamation the groupoid

$$A = \bigcup_s G_s \ , \quad s \in \Gamma^1_\$ \cup \Gamma^1_r \ .$$

Let Mod_{RG_v} be the category of modules over the group algebra RG_v, where R is some fixed ring with unit. We define Mod_B to be the restricted product

$$Mod_B = X_v \ Mod_{RG_v} \ , \quad v \in \Gamma^0_\$ \ ,$$

and similarly

$$Mod_A = X_s \ Mod_{RG_s} \ , \quad s \in \Gamma^1_\$ \cup \Gamma^1_r \ .$$

If $M \in Mod_B$, then $M \otimes_B G$ is defined: If, say, $M = X_v \ M_v \ , \ M_v \in Mod_{RG_v}$, $v \in \Gamma^0_\$$, then

$$M \otimes_B G = \bigoplus_v M_v \otimes_{RG_v} RG \ , \quad v \in \Gamma^0_\$ \ .$$

It is clear from the definition that, as an abelian group, $M \otimes_B G$ is a direct sum, indexed by <u>all</u> of Γ^0,

$$M \otimes_B G = \bigoplus_v M_v \ , \quad v \in \Gamma^0 \ .$$

If $g \in G$ is such that $(v_0)g = v$, where $v_0 \in \Gamma^0_\$$, we can write

$$M_v = M_{v_0} \otimes_{RG_{v_0}} RG_{v_0} \cdot g \ .$$

We can also consider M_v as a module over RG_v.

Similarly, if $N \in Mod_A$, then $N \otimes_A G$ is defined, and there is a direct sum decomposition of abelian groups,

$$N \otimes_A G = \bigoplus_s N_s \ , \quad s \in \Gamma^1 \ .$$

<u>Definition</u>. A <u>Γ-object</u> consists of modules $N \in Mod_A$ and $M \in Mod_B$, and a map over G,

$$\iota : M \otimes_B G \to N \otimes_A G$$

satisfying: if (for any v and s) the restriction of ι to M_v has a non-zero projection to N_s, then the segment s is incident to the vertex v.

A <u>map</u> of Γ-objects is a pair of maps, one in Mod_B and one in Mod_A, so that the obvious diagram commutes. The resulting category is abelian since the functors $\otimes_B G$ and $\otimes_A G$ are exact.

Dually, a Γ^*-object consists of modules, and a map

$$M \otimes_B G \leftarrow N \otimes_A G$$

satisfying the same sort of condition. The duality functor $\mathrm{Hom}_{RG}(\ ,RG)$ maps Γ-objects to Γ^*-objects, and vice-versa (however, in order to stay with right modules, we may have to replace the coefficient ring by its opposite).

We can be somewhat more explicit about the structure map

$$\iota\colon M \otimes_B G \to N \otimes_A G$$

in a Γ-object. Let us write

$$\iota_{v,s}$$

for the composition

$$M_v \to \bigoplus_{v'} M_{v'} \to \bigoplus_{s'} N_{s'} \to N_s \ .$$

Then ι is of course determined by its components $\iota_{v,s}$, $v \in \Gamma^0_\$$, $s \in \Gamma^1_£$; and for fixed v, those components assemble to an (arbitrary) RG_v-map

$$M_v \to \bigoplus_s N_s \ , \quad s \in \Gamma^1(v) \ .$$

<u>Definition</u>. A Γ-<u>module</u> is a Γ-object $\iota\colon M \otimes_B G \to N \otimes_A G$ satisfying that ι is an isomorphism. The resulting category is denoted Mod_Γ; it is abelian.

A Γ-module is called <u>elementary</u> if N is finitely generated projective and, in addition, at most one of the component maps $\iota_{v,s}$, $v \in \Gamma^0_\$$, $s \in \Gamma^1_£$, is not the zero map; this $\iota_{v,s}$ must then itself be an isomorphism.

A Γ-module is called <u>triangular</u> if it has a finite filtration with elementary subquotients.

We denote $K_0(\mathrm{Mod}_\Gamma,R)$ the class group of those objects in Mod_Γ which are made up of finitely generated projective modules, the relations coming from all exact sequences (not just split ones). Using elementary Γ-modules, we obtain a map

$$j\colon K_0(RA) \oplus K_0(RA) \to K_0(\mathrm{Mod}_\Gamma,R)$$

which is a split injection by an argument below (the construction of the modules denoted P(s,v)). The cokernel of j is denoted $\mathfrak{N}$. This is the $\mathfrak{N}$ that appears in the splitting theorem. The definition of $\mathfrak{N}$ is related to maps which are 'nilpotent' if this term is taken in a suitable sense. The vanishing theorem for $\mathfrak{N}$ will come in in somewhat disguised form: under the hypothesis that RA is regular coherent, the proposition below implies that the above map j is an isomorphism.

We now proceed to the analysis of Γ-modules. Let s be a segment of Γ, and v a vertex incident to s. Define $\Gamma_{s,v}$ to be the maximal subtree of Γ which contains v but not s. Given s, there are two such trees, $\Gamma_{s,v_i(s)}$ and $\Gamma_{s,v_t(s)}$.

Given $M \in \mathrm{Mod}_B$, then $M \otimes_B G$, considered as a module over RG_s, splits naturally as a direct sum

$$\bar{M}(s,v_i(s)) \oplus \bar{M}(s,v_t(s))$$

where, as an abelian group,

$$\bar{M}(s,v_i(s)) = \bigoplus_v M_v \ , \quad v \in \Gamma^0_{s,v_i(s)} \ .$$

Similarly, if $N \in \mathrm{Mod}_A$, then $N \otimes_A G$, considered as a module over RG_s, splits as

$$\bar{N}(s,v_i(s)) \oplus N_s \oplus \bar{N}(s,v_t(s))$$

where, as an abelian group,

$$\bar{N}(s,v_i(s)) = \bigoplus_{s'} N_{s'} \ , \quad s' \in \Gamma^1_{s,v_i(s)} \ .$$

If now $\iota\colon M \otimes_B G \to N \otimes_A G$ is a Γ-module, then

$$\iota(\bar{M}(s,v_i(s))) \subset \bar{N}(s,v_i(s)) \oplus N_s$$

and

$$\iota^{-1}(\bar{N}(s,v_i(s))) \subset \bar{M}(s,v_i(s)) \ .$$

Whence the canonical splitting

$$N_s = P(s,v_i(s)) \oplus P(s,v_t(s))$$

where
$$P(s,v_i(s)) = \mathrm{Im}(\bar{M}(s,v_i(s)) \to \bar{N}(s,v_i(s)) \oplus N_s \to N_s)$$
$$\approx \ker(\bar{M}(s,v_i(s)) \to \bar{N}(s,v_i(s))) \ ,$$

and analogously with $P(s,v_t(s))$.

On the other hand, if v is a fixed vertex, and s a segment incident to v, let us denote $\Gamma_{v,s}$ the maximal subtree of Γ which is incident to s, but does not contain v. We have $\Gamma_{v,s} = \Gamma_{s,\tilde{v}}$ where $\tilde{v}$ is the other end point of s. As before, let us denote $\Gamma^1(v)$ the set of segments of Γ which are incident to v. Let $\Gamma^1_{rep}(v)$ denote a set of representatives for the quotient set $\Gamma^1(v)/G_v$; e.g., if $v \in \Gamma^0_{\$}$, then $\Gamma^1_{£}(v)$ is such a set of representatives.

Given $M \in Mod_B$, then $M \otimes_B G$, considered as a module over RG_v, splits naturally as a direct sum

$$M_v \oplus \bigoplus_s \widetilde{M}(v,s) \ , \quad s \in \Gamma^1_{rep}(v)$$

where, as RG_v-module,

$$\widetilde{M}(v,s) = \overline{M}(s,\tilde{v}) \otimes_{RG_s} RG_v \ ,$$

$\overline{M}(s,\tilde{v})$ is defined as above, and $\tilde{v}$ is the other end point of s.

Similarly, if $N \in Mod_A$, then $N \otimes_A G$, considered as a module over RG_v, splits as

$$\bigoplus_s N_s \otimes_{RG_s} RG_v \oplus \bigoplus_s \overline{N}(s,\tilde{v}) \otimes_{RG_s} RG_v \ , \quad s \in \Gamma^1_{rep}(v) \ .$$

If again $\iota: M \otimes_B G \to N \otimes_A G$ is a Γ-module, we can write ι as a map of RG_v-modules in the form

$$M_v \oplus \bigoplus_s \overline{M}(s,\tilde{v}) \otimes_{RG_s} RG_v \to \bigoplus_s N_s \otimes_{RG_s} RG_v \oplus \bigoplus_s \overline{N}(s,\tilde{v}) \otimes_{RG_s} RG_v \ ,$$

$$s \in \Gamma^1_{rep}(v) \ .$$

Now the restriction to the second summand is of a type considered before. Hence we obtain a map

$$M_v \oplus \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v \to \bigoplus_s N_s \otimes_{RG_s} RG_v =$$

$$\bigoplus_s P(s,v) \otimes_{RG_s} RG_v \oplus \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v$$

whose restriction to the second summand is the obvious identity. Therefore the restriction to the first summand is the sum of an isomorphism

$$\varkappa_v : M_v \to \bigoplus_s P(s,v) \otimes_{RG_s} RG_v$$

and some map

$$\lambda_v : M_v \to \bigoplus_s P(s,\tilde{v}) \otimes_{RG_s} RG_v .$$

For fixed $s \in \Gamma^1_{rep}(v)$, the composition $\lambda_v \circ \kappa_v^{-1}$ induces an RG_v-map

$$P(s,v) \otimes_{RG_s} RG_v \to \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v , \quad s' \in \Gamma^1_{rep}(v)$$

which in turn is determined by the induced RG_s-map

$$\mu_{s,v} : P(s,v) \to \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v , \quad s' \in \Gamma^1_{rep}(v).$$

The target of this latter map is in fact slightly smaller since the composition of $\mu_{s,v}$ with the projection to $P(s,\tilde{v})$ is zero (inspection of the definitions shows that this composition can be factored through $\bar{M}(s,v)$).

The map now reads

$$\nu_{s,v} : P(s,v) \to P(s,\tilde{v}) \otimes_{RG_s} \widehat{RG}_v \oplus \bigoplus_{s'} P(s',\tilde{v}) \otimes_{RG_s} RG_v ,$$

$$s' \in \Gamma^1_{rep}(v) , \quad s' \neq s ,$$

where $\widehat{RG}_v(s)$ is the summand in the canonical splitting of RG_s-bi-modules

$$RG_v = RG_s \oplus \widehat{RG}_v(s) .$$

It is clear now that there is an (exact) functor

$$F : Mod_A \times Mod_A \to Mod_A \times Mod_A$$

which depends only on the g.f.p. structure (in particular it does not depend on the choice of the sets $\Gamma^1_{rep}(v)$) so that the collection of maps

$$\nu_{s,v} , \quad s \in \Gamma^1_{\$} \cup \Gamma^1_r ,$$

assembles to a map

$$\nu : P \to F(P)$$

where the first component of $P \in Mod_A \times Mod_A$ is given by the collection $P(s,v_i(s))$, $s \in \Gamma^1_{\$} \cup \Gamma^1_r$.

The original Γ-module is determined by the pair (P,ν). Conversely, a necessary and sufficient condition for (P,ν) to arise from a Γ-module, is that the map ν be nilpotent in the following sense.

Define a filtration $0 = P_0 \subset P_1 \subset \dots \subset P_j \subset \dots \subset P$ by the rule

$$P_{j+1} = \nu^{-1}(F(P_j)) .$$

Then we call ν <u>nilpotent</u> if $\bigcup P_j = P$.

<u>Remark</u>. If the g.f.p. structure comes from a product with the integers (so that we are in the situation of the classical Künneth formula) then a nilpotent ν in our sense is just a pair of nilpotent maps in the usual sense.

We will not prove here that ν is nilpotent as this follows directly from the lemma below. We note the following interpretation of ν. If $x \in P(s,v)$ then $x \in P_1$ (the first term of the filtration) if and only if there exists $y \in M_v$ so that $\iota(y) = x$.

Given ν: $Q \to F(Q)$, it is convenient to consider a more general type of filtration, $0 \subset Q_1 \subset \dots \subset Q_j \subset \dots \subset Q$, which we call a <u>nil-filtration</u> if

$$\nu(Q_{j+1}) \subset F(Q_j) , \quad \text{and} \quad \bigcup Q_j = Q .$$

We say it is of <u>finite length</u>, q, if $Q_q = Q$, and we say it is <u>finitely generated</u>, if all the Q_j are.

The filtration originally derived from a Γ-module, denoted $.. \subset P_j \subset ..$ above, will certainly be of finite length if N is finitely generated, but it need not itself be finitely generated. It is clear nevertheless that there exists some finitely generated nil-filtration which is a subfiltration of the original one, and is of the same length.

We will now describe our resolution argument. Let $.. \subset Q_j \subset ..$ be a finitely generated nilfiltration of length q, associated to a Γ-module. Pick finitely generated projectives U_j in $\mathrm{Mod}_A \times \mathrm{Mod}_A$, and surjections

$$U_j \to Q_j , \quad j \leq 1 .$$

Then we can find maps u_j: $U_j \to F(U_{j-1})$ so that the diagrams

$$\begin{array}{ccc} U_j & \to & F(U_{j-1}) \\ \downarrow & & \downarrow \\ Q_j & \to & F(Q_{j-1}) \end{array}$$

commute. Define a filtration $0 \subset V_1 \subset \ldots \subset V_q = V$, by

$$V_i = U_1 \oplus \ldots \oplus U_i \ .$$

It is a nil-filtration for the map

$$v\colon V \to F(V) \ , \quad v = \Sigma_j \ u_j \ .$$

This map is associated to a certain triangular Γ-module in which the A-module is V, considered as an A-module via $\oplus$: $Mod_A \times Mod_A$. Furthermore there is a surjection of Γ-modules, compatible with the surjection of nil-filtrations, $V_j \to Q_j$. Define $.. \subset W_j \subset ..$ to be the kernel filtration, it is a nil-filtration for the map $w = v|W$, where $W = W_q$. If Q_1 was projective to begin with, we could have chosen $V_1 = Q_1$, and the new filtration would be of shorter length.

Now assume the amalgamation A is coherent, and Q is finitely presented. Then, as ${}_{f.p.}Mod_A$ is an abelian category, it follows that Q_j and W_j are finitely presented. Therefore we can repeat our construction <u>using the filtration</u> W_j .

On iterating the procedure we are building up, in particular, a projective resolution of Q_1. Therefore, if A is regular coherent, we can eventually reduce the length of the filtration, and so, by induction on this length, we have proved:

<u>Proposition</u>. If A is regular coherent, then any finitely presented Γ-module has a resolution by triangular Γ-modules.

(By abuse of language, we have called a Γ-module 'finitely presented' if the A-module involved is. Note that the main interest of the proposition is in the case where this A-module is actually projective).

Above we referred to the following lemma. The above application of the lemma just exploits the obvious fact that a nil-filtration does exist for a triangular Γ-module. The lemma says that there are as many maps from triangular Γ-modules as we can expect at all.

Lemma. Let ι: $M \otimes_B G \to N \otimes_A G$ be any Γ-object.

(1) Let $y \in N_s$, $s \in \Gamma^1$, and $y \in \mathrm{Im}(\iota)$. Then y is in the image of some map from a triangular Γ-module.

(2) Let $x \in M_v$, $v \in \Gamma^0$. Then x is in the image of some map from a triangular Γ-module.

Proof. Ad (1). Let $y = \Sigma_v\ \iota(z_v)$, $z_v \in M_v$, $v \in \Delta^0$, where Δ is some finite subtree of Γ. The sought for triangular Γ-module is made up of rank-one free modules over the appropriate rings. There is one basis element for each vertex and segment in Δ, and there is an additional basis element for the segment s. Each of the components of the structure map is an 'identity' (i.e., it sends the basis element to the basis element), and there is one such for each incidence relation in Δ, and one additional one into the extra component. The definition of the map is automatic.

Ad (2). This follows from (1) by the same sort of splicing argument.

4. Mayer Vietoris presentations of G-modules.

Let L be a G-module (more precisely, an RG-module). A left Mayer Vietoris presentation of L is a short exact sequence

$$0 \to L \to M \otimes_B G \to N \otimes_A G \to 0$$

the right part of which is a Γ-object, as defined in the previous section.

Dually, a right Mayer Vietoris presentation is a short exact sequence

$$0 \to N \otimes_A G \to M \otimes_B G \to L \to 0$$

involving a Γ*-object.

A left or right Mayer Vietoris presentation is called f.g.p. if all the modules involved are finitely generated projective. F.g.p. left and right Mayer Vietoris presentations are interchanged by the duality map $\mathrm{Hom}_{RG}(\ ,RG)$ (with the usual proviso on the coefficient ring R). Hence it is sufficient to concentrate on either one. For us this will be the left Mayer Vietoris presentations, abbreviated MV presentations henceforth.

Remark. The concept of MV presentation is an axiomatization of a Mayer Vietoris type situation that occurs if one looks at chain complexes in the universal cover of a pair X,Y as considered in the introductory section.

Namely, if L is a chain complex over $G \approx \pi_1 X$, then 'subdividing at Y ' produces an MV presentation of chain complexes

$$0 \to L \to M \otimes_B G \underset{\iota}{\to} N \otimes_A G \to 0 .$$

After the subdivision, L will have been replaced (up to a dimension shift) by the mapping cone $C(\iota)$. And the Mayer Vietoris sequence of chain complexes that one is accustomed to read off, now appears as the right Mayer Vietoris presentation which is the sequence of cones

$$0 \to C(\iota_1) \to C(\iota_2) \to C(\iota) \to 0$$

where ι_1 is the trivial inclusion $0 \to N \otimes_A G$, and

$$\iota_2 \colon M \otimes_B G \to N \otimes_A G \oplus N \otimes_A G$$

is the map whose components are ι_i and ι_t in the canonical sum decomposition of ι. The B-structures on the two copies of $N \otimes_A G$ come, respectively, from the two natural maps $A \to B$. The proposition below is the 'subdivision lemma' that one would naturally expect.

We will now verify that there exist quite a few MV presentations, and maps thereof. Our main tool will be certain 'standard' MV presentations, defined for a free G-module; part of the data will be a basis of the G-module, in the description we will assume that it has cardinality one. (Inspection shows that the construction below can actually be carried through for any

G-module equipped with a reduction to Mod_A). In describing free modules of the type $M \otimes_B G$, it is sometimes convenient to use a basis which does not come from Mod_B.

Definition. Let F be a free G-module, with basis element f. Let Δ be a finite subtree of Γ. Then the standard MV presentation of F,f, associated to Δ, is the following

(1) $M \otimes_B G$ is the free G-module on basis elements $\bar{m}_v$, $v \in \Delta^0$

(2) $N \otimes_A G$ is the free G-module on basis elements $\bar{n}_s$, $s \in \Delta^1$

(3) the G-structure on $M \otimes_B G$ is such that $\bar{m}_v$ generates a free RG_v-module; similarly with $N \otimes_A G$

(4) the structure map $\varkappa: F \to M \otimes_B G$ is given by $\varkappa(f) = \Sigma_v \bar{m}_v$, $v \in \Delta^0$

(5) the structure map $\iota: M \otimes_B G \to N \otimes_A G$ is given in terms of its components $\iota_{v,s}: M_v \to N_s$ by

$$\iota_{v,s}(\bar{m}_v) = \bar{n}_s \ , \quad \text{if } v = v_i(s), \text{ the initial vertex}$$
$$\iota_{v,s}(\bar{m}_v) = -\bar{n}_s, \quad \text{if } v = v_t(s), \text{ the terminal vertex}$$
$$\iota_{v,s}(\bar{m}_v) = 0 \ , \quad \text{if } v \text{ is not incident to } s$$

(6) in order to describe the reduction of $M \otimes_B G$ to Mod_B, i.e., to define M, we must pick representatives of cosets for the various inclusions involved in the g.f.p. structure, so we assume this has been done once and forever. It is crucial here that we need only choose representatives of cosets for the inclusions of amalgamation groups in building block groups, and the elements denoted t_s in section 2, and that this choice determines representatives of all the cosets in G (this statement is the general version of the existence of the usual normal form for an element of a free product with amalgamation, it is easily proved by the use of the distance function on Γ). In particular then, we have picked for every $v \in \Delta^0$ an $x_v \in G$ so that $(v)x_v^{-1} \in \Gamma^0_\$$, the basic tree. By definition now, M is the B-module whose component at $v' \in \Gamma^0_\$$ is the direct sum $\bigoplus_v M_v \cdot x_v^{-1}$, taken over those $v \in \Delta^0$ for which $(v)x_v^{-1} = v'$.

In terms of the basis elements $m_v = \bar{m}_v \cdot x_v^{-1}$ (which live in M), we could now redefine $\varkappa(f) = \Sigma_v\ m_v \cdot x_v$

(7) the reduction of $N \otimes_A G$ to Mod_A is described similarly.

Before proceding, let us note that for any MV presentation (or even Γ-object), there is a canonical decomposition

$$\iota = \iota_i - \iota_t$$

where ι_i is defined so that its non-zero components are those $\iota_{v,s}$ for which $v = v_i(s)$, the initial vertex (this decomposition was used in the remark above). For the standard MV presentation just described, we have the important property

$$\iota_i(\varkappa(f)) = \Sigma_s\ \bar{n}_s \ , \quad s \in \Delta^1 \ .$$

<u>Proposition</u>. Let $0 \to L \to M' \otimes_B G \to N' \otimes_A G \to 0$ be any MV presentation. Let F be the free G-module on the basis element f, and let g: $F \to L$ be any G-map. Then for suitable Δ, the standard MV presentation of F,f, associated to Δ, admits a map of MV presentations, inducing g. Moreover, this map is uniquely determined by g.

<u>Proof</u>. By definition, $M' \otimes_B G$ is a direct sum

$$\bigoplus_v\ M'_v \otimes_{RG_v} RG \ , \quad v \in \Gamma^0_\$ \ .$$

Let $\tilde{g}_v$ denote the projection of $\varkappa' \circ g$ to $M'_v \otimes_{RG_v} RG$. Then we can write

$$\tilde{g}_v(f) = \Sigma_w\ a_w \cdot x_w$$

where $a_w \in M'_w$, $x_w \in G$ is a representative of a coset $G_v \backslash G$ as chosen before, and $w \in \Gamma^0$ runs through the vertices with $(w)x_w^{-1} = v$. From this formula and the fact that

$$\varkappa(f) = \Sigma_w\ m_w \cdot x_w \ , \quad w \in \Delta^0 \ ,$$

it is clear that the required B-map can be defined as soon as the finite tree Δ has been chosen so large that it contains all the vertices w for which $a_w \neq 0$.

Next we define the required A-map, g_A, directly, by decomposing similarly the map

$$\iota_i^! \circ \varkappa' \circ g : F \to N' \otimes_A G$$

using

$$\iota_i(\varkappa(f)) = \Sigma_s \bar{n}_s = \Sigma_s n_s \cdot x_s , \quad s \in \Delta^1 .$$

The sum decompositions involved in our construction were canonical, and it is now easily seen that the maps g, g_B, g_A are compatible as required. We record the uniqueness part in a separate lemma.

Lemma. If in the above proposition, g is the zero map, then g_B and g_A must be zero maps, too.

Proof. It is enough to treat g_A. Since the source MV presentation is standard, we have

$$\iota_i(\varkappa(f)) = \Sigma_s n_s \cdot x_s ,$$

and on application to this element of the map $g_A \otimes G$, no cancellation is possible between the individual summands.

I will now indicate how the splitting theorem can be obtained. Following Whitehead's original treatment, a torsion element can be represented by a based free acyclic chain complex. The relations come from certain short exact sequences, called elementary expansions.

Using our machinery of MV presentations, we can now say that any chain complex over G comes, via the forgetful map, from a chain complex of MV presentations (with bases suitably). And we can also say what, in the framework of MV presentations, corresponds to elementary expansions.

Technically, the analysis boils down to situations which are blown up versions of the following simple prototype. If we have a chain complex which on the G-level (i.e., apply the forgetful map to Mod_G) is acyclic, there is still no reason that it be acyclic on the A-level (a Γ-module is an example for this). So we can try to make it acyclic on the A-level as well, using simple operations. The details are standard and there are no surprises: one just goes on killing homology groups, working up in dimension. It turns out that there is a global obstruction, and this gives the connecting map.

To illustrate the technique, we prove

<u>Proposition</u>. Let G have a g.f.p. structure with building blocks B and amalgamation A.

(1) If $\mathrm{gl.dim.Mod}_A \leq n-1$, and $\mathrm{gl.dim.Mod}_B \geq n$, then $\mathrm{gl.dim.Mod}_G \leq n$.

(2) If the building blocks are coherent, and the amalgamations noetherian, then G is coherent.

<u>Proof</u>. Ad (1). Let L. be a free (n-1)-dimensional resolution of $\mathrm{coker}(L_1 \to L_0)$. By the subdivision lemma, there is a complex of standard MV presentations over L.,

$$0 \to L. \to M. \otimes_B G \to N. \otimes_A G \to 0 \quad .$$

Since no conditions had to be met in dimension 0, we can assume $N_0 = 0$. Now the last lemma of the previous section tells us that we can add a triangular Γ-module (or maybe a big sum of such) to the 2-chains to kill

$$\mathrm{Im}(H_1(M. \otimes_B G) \to H_1(N. \otimes_A G))$$

and hence $H_1(M. \otimes_B G)$. Again it tells us that we can kill $H_2(N. \otimes_A G)$, and so on. But once we killed $H_{n-2}(N. \otimes_A G)$, we know that (using $H_*(N. \otimes_A G) \approx H_*(N.) \otimes_A G$, etc.) $\ker(N_{n-1} \to N_{n-2})$ must be projective since we resolved $H_1(N.)$. Similarly, $\ker(M_{n-1} \to M_{n-2})$ is projective, and we are done.

Ad (2). By a bit of diagram chasing, the assertion is reduced to proving that $\ker(L_1 \to L_0)$ is finitely generated once L_1 and L_0 are finitely generated free RG-modules. Again the subdivision lemma gives us a map of standard MV presentations over $L_1 \to L_0$. We regard it as a complex in dimensions 1 and 0, and can assume as before that $N_0 = 0$. Arguing as before, we can introduce a big sum of triangular Γ-modules into the 2-chains in order to kill

$$\mathrm{Im}(H_1(M. \otimes_B G) \to H_1(N. \otimes_A G)) \; .$$

This time we would like to have N_2 finitely generated. But $\mathrm{Im}(N_2 \to N_1)$ is finitely generated by the noetherian hypothesis. Therefore some finite part of the big sum is already sufficient for our purpose. We have achieved now that the sequence

$$H_2(N_\bullet \otimes_A G) \to H_1(L_\bullet) \to H_1(M_\bullet \otimes_B G)$$

is short exact. But the base changes are exact. So the extreme terms can be rewritten $H_2(N_\bullet) \otimes_A G$ and $H_1(M_\bullet) \otimes_B G$, respectively. So they are finitely generated by the coherence hypothesis, and we are done.

5. Appendix.

Let $\underline{K}(C)$ denote Quillen's K-theory associated to the category-with-exact-sequences C. Here C is assumed to be equivalent to a small category, and, by definition, $\underline{K}(C) \cong$ (homotopy equivalent to) $\Omega\, Q'(C)$, the loop space of the nerve of the category $Q'(C)$, where $Q'(C)$ is small and equivalent to $Q(C)$, and $Q(C)$ is constructed from certain diagrams in C, involving the notions of 'admissible monomorphism' and 'admissible epimorphism'.

If $\underline{MV}$ denotes the category of MV presentations over a g.f.p. structure (of a group G, with building blocks B, and amalgamations A), we define $Q(\underline{MV})$ by the rule

(1) an identity map is admissible if all the modules involved in the object are finitely generated projective

(2) an epimorphism is admissible if its source and target are

(3) a monomorphism is admissible if its source, target, and cokernel are.

Similarly, we define $Q(\mathrm{Mod}_\Gamma)$.

There is a natural embedding

$$\underline{K}(\mathrm{Mod}_\Gamma) \to \underline{K}(\underline{MV})$$

whose composition with the natural projection, induced from the forgetful map,

$$\underline{K}(\underline{MV}) \rightarrow \underline{K}(\mathrm{Mod}_G)$$

is trivial.

There is evidence that the following should be true

Conjecture 1. The sequence

$$\underline{K}(\mathrm{Mod}_\Gamma) \rightarrow \underline{K}(\underline{MV}) \rightarrow \underline{K}(\mathrm{Mod}_G)$$

has the homotopy type of a fibration, or equivalently, the long sequence of homotopy groups is exact.

(It is not conjectured that the map $\underline{K}(\underline{MV}) \rightarrow \underline{K}(\mathrm{Mod}_G)$ is locally fiber homotopy trivial: indeed this is almost certainly not the case. Similarly below).

For the amalgamation A, define

$$\underline{K}(\mathrm{Mod}_A) = \bigtimes_j \underline{K}(\mathrm{Mod}_{A_j}) ,$$

the restricted product (the direct limit over the finite products) over the component groups. Similarly with $\underline{K}(\mathrm{Mod}_B)$.

There is a natural embedding

$$\underline{K}(\mathrm{Mod}_B) \rightarrow \underline{K}(\underline{MV})$$

so that the composition with the natural projection

$$\underline{K}(\underline{MV}) \rightarrow \underline{K}(\mathrm{Mod}_A)$$

is trivial. The latter map has a section (in fact, there are three obvious such).

Conjecture 2. The sequence

$$\underline{K}(\mathrm{Mod}_B) \rightarrow \underline{K}(\underline{MV}) \rightarrow \underline{K}(\mathrm{Mod}_A)$$

is a homotopy fibration. Consequently

$$\underline{K}(\underline{MV}) \cong \underline{K}(\mathrm{Mod}_A) \times \underline{K}(\mathrm{Mod}_B) .$$

From the retraction $\mathrm{Mod}_\Gamma \rightarrow \mathrm{Mod}_A \times \mathrm{Mod}_A$, we can conclude that

$$\underline{K}(\mathrm{Mod}_\Gamma) \cong \underline{K}(\mathrm{Mod}_A) \times \underline{K}(\mathrm{Mod}_A) \times \underline{N} ,$$

defining $\underline{N}$. (And $\pi_0\underline{N} = \mathfrak{N}$, our old exotic term). Combining conjectures 1 and 2, and noting that two terms cancel, we obtain

Conjecture 3. There is a homotopy fibration

$$\underline{K}(\text{Mod}_A) \times \underline{N} \rightarrow \underline{K}(\text{Mod}_B) \rightarrow \underline{K}(\text{Mod}_G) .$$

Concerning the exotic space $\underline{N}$, there is the vanishing

Conjecture 4. If A is regular coherent, then $\underline{N}$ is contractible.

Conjecture 4 happens to be true, for under the regular coherence hypothesis, we can replace in the definitions of both $\underline{K}(\text{Mod}_A \times \text{Mod}_A)$ and $\underline{K}(\text{Mod}_\Gamma)$, respectively, finitely generated projectives by finitely presented modules, and can then conclude that the two spaces are equivalent. This uses the resolution of Γ-modules by triangular ones, and Quillen's theorems on reduction by resolution and devissage, respectively.

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B. REPRESENTATION THEORY

Contributions to the theory of induced representations

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Contents

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Introduction

The theory of induced representations took its origin in the work of Frobenius on complex representationtheory as a tool to relate problems, concerning complex characters of a given group, e.g. their decomposition into irreducibel characters, with the corresponding question for one or several of its subgroups. A classical example for the utility of this approach is for instance the orginal proof of the Frobeniustheorem (see [38],§63), but of course there is a wide range of further good examples in that direction. Still a rather different point of view emerged from E.Artin's idea, to consider induced representations on the level of virtual representations (i.e. generalized characters), where he was able to prove, that a certain multiple of any rational generalized character is a sum of characters, which are induced from generalized characters of cyclic subgroups, and to use this fact in an essential way in his study of generalized L-functions (cf. [1]). The next milestone in that direction was - no doubt - the paper of R. Brauer " On Artin's L-series with general group characters" ([3]), which - based on an improvement of Artin's inductiontheorem - solved quite a number of classical problems in a surprisingly simple way and - at the same time - stimulated a series of further investigations in that direction by Roquette ([31]), Berman ([2]), Witt ([36]), probably several others and Brauer himself. The next essential step was probably taken by R. Swan, who - elaborating on the ideas and techniques of R-Brauer - used this technique very successfully in his study of Grothendieck- and classgroups of integral representations (e.g. [34] and [35]). The wide range of possible further exploitation of these ideas then led T.Y. Lam (see [28]) to a first attempt of an axiomatic formulation of the techniques, in which way induced representations, especially the Frobenius-reciprocity-law were used in the study of the structure of "virtual representations" in various situations, i.e. of various Grothendieckgroups and -rings.

The usefullness of this axiomatic approach was demonstrated not only by a number of new and important examples (e.g. the Whiteheadgroup of a finite group) in T.Y. Lam's thesis itself and several other papers in that direction, but also for instance by its surprising use, made by W. Scharlau (cf. [32],[33]) to simplify considerably the proofs of several theorems concerning the structure of the Wittring of quadratic forms.

Still - further investigations in that direction and especially the central rôle of the Mackey-theorems (cf. [7], §44, p.323-27) in J.A. Greens study of modular representations (cf. [21],[22]) suggested a modification of T.Y. Lam's approach, taking into account not only the Frobenius-reciprocity-law, but also the Mackey-subgroup-theorem, which resulted in two rather similar approaches to an axiomatic treatment of induction-theory, one developed by J.A. Green in [23] and [24], the other one by myself ([13],[14],[16]).

The first part of this paper now contains a new version of my own axiomatic theory: As before it is based on the notion of Mackey-functors, but whereas in [16] the approach took its bearing from the theory of Burnsiderings, this time I have tried to develop the theory using its close relations to certain aspects of relative homological algebra.

Thus §1 contains a short outline of some basic notions and constructions of relative homological algebra, put in a way, which is convenient for our later purposes. Especially we define a co-, resp. contravariant functor M from a category A with finite products into an abelian category B to be X-projective, resp. X-injective for some object X in A, if the canonical natural transformation $M_X \to M$: $M(X\times Y) \to M(Y)$, resp. $M \to M_X$: $M(Y) \to M(Y\times X)$ is split-surjective, resp. split-injective (with $M_X(Y) := M(X\times Y)$ of course for any object Y in A), which turns out to be the proper definition to understand the homological significance of the Amitsur-complex, associated with X (Prop. 1.2). Additionally-generalizing a concept of J.A. Green - one can define vertices of such functors under appropriate assumptions on A.

An example to have in mind is the following: Let G be a finite group and A the category $\hat{G}$ of finite G-sets. Let M be a $\mathbb{Z}$G-module and define $M_M(S) = \mathrm{Hom}_G(S,M)$ the set=abelian group of G-maps from S to M for any G-set S, thus $M_M(G/U) \simeq M^U = \{m \in M \mid u\cdot m = m \text{ for any } u \in U\}$ for $U \leq G$.
M_M is in an obvious way a contravariant functor on $\hat{G}$ and one can show, that it is S-injective, if and only if M is relatively U-injective for $U=\{U \leq G \mid S^U \neq \emptyset\}$ in the sense of [12], i.e. M is a direct summand in $\bigoplus_{U\in U} \mathbb{Z}G \otimes_{\mathbb{Z}U} M = \bigoplus_{U\in U} (M|_U)^{U \to G}$.

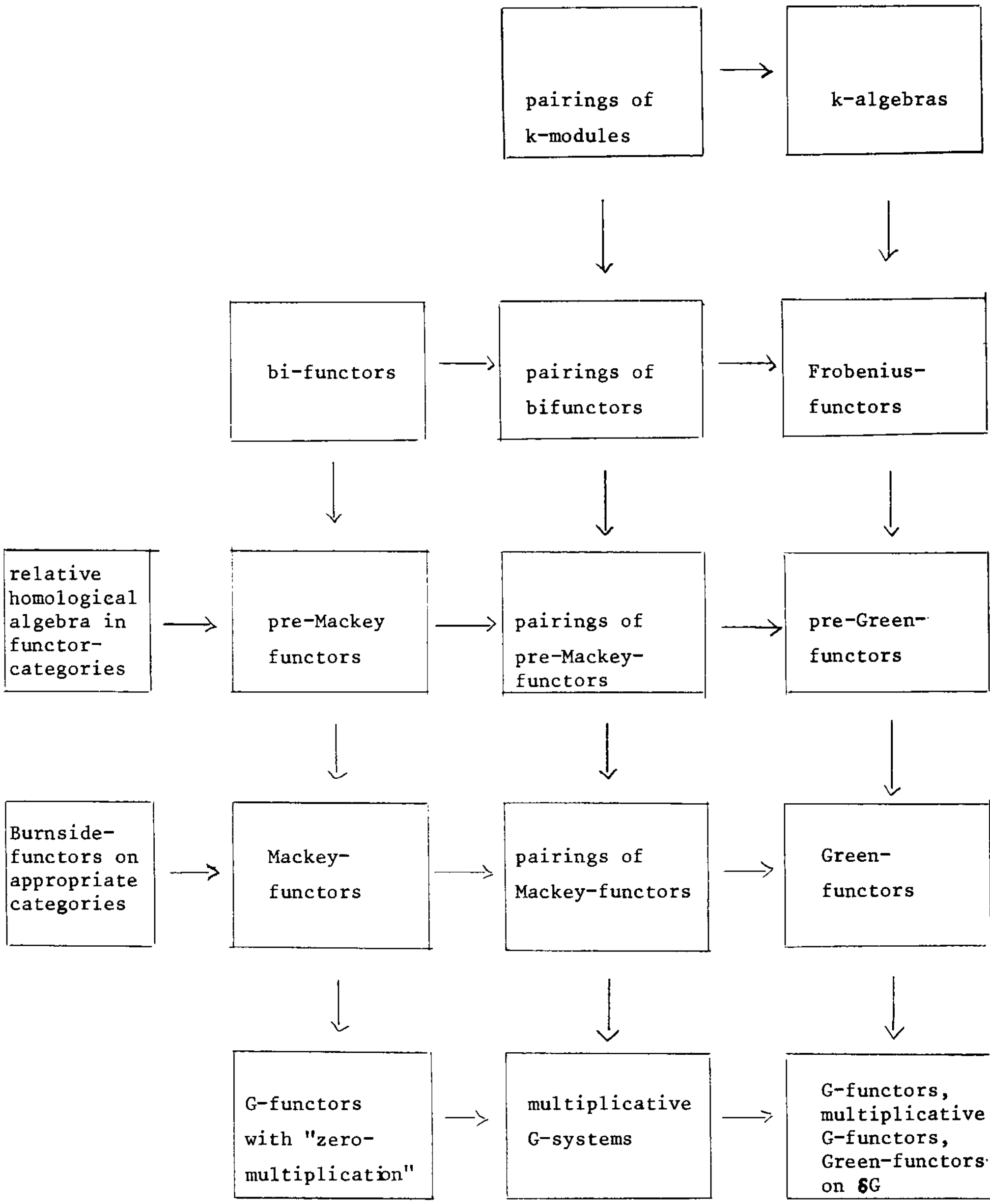
Moreover one can also make M_M a covariant functor by associating to any G-map $\varphi: S \to T$ between two G-sets S and T the map
$\varphi^*: \mathrm{Hom}_G(S,M) \to \mathrm{Hom}_G(T,M): f \mapsto \varphi^*(f)$ with $\varphi^*(f)(t) = \sum_{s\in\varphi^{-1}(t)} f(s)$, $t \in T$ and again one has M_M S-projective as a covariant functor if and only if M is relatively U-projective for $U=\{U \leq G \mid S^U \neq \emptyset\}$ in the sense of [12]. But by Gaschütz-Higman U-projectivity of M is equivalent to U-injectivity. To obtain something equivalent in the abstract theory we then define bi-functors in §2 as a pair of functors $M=(M_*,M^*)$ from A to B, one contravariant, the other one covariant, which coincide on the objects: $M_*(X) = M^*(X) = M(X)$.

To develop some relative homological algebra of bifunctors analogously to the theory of co- or contravariant functors in §1, one has to restrict oneself to such - so to say "admissible" - bi-functors M, for which the family of maps $M_X \to M$: $M^*(X\times Y) \to M^*(Y)$ as well as the family of maps $M \to M_X$: $M_*(Y) \to M_*(X\times Y)$ are natural transformations of bi-functors. This is indeed the case, if M satisfies the "Mackey-property" for pull-back-diagramms as defined in the beginning of §2, i.e. if M is a "Pre-Mackey-functor", and for such bi-functors X-projectivity is indeed

equivalent to X-injectivity.
Things get more interesting once one starts to consider also pairings of bi-functors, which allows to introduce an axiomatic formulation of the Frobenius-reciprocity-law. Especially considering such pre-Mackey-functors G with an "inner composition", i.e. a pairing $G \times G \to G$, such that G_* becomes a contravariant functor into the category of rings with a unit, which I tend to call "pre-Green-functors" and which are studied in §3, one can articulate the basic formal connection between induction-theory and the special form of relative homological algebra developed before:
Theorem 1: A pre-Green-functor G is X-projective, if and only if the covariant map $G^*(X) \to G^*(\bullet)$ ("$\bullet$" the final object in A) is surjective.
This connects especially on a rather abstract level and in a surprisingly simple and obvious way the notions of defectbases and vertices, both introduced by J.A. Green (see [21], [12] and [23]).
Only in §4 we begin to put further restrictions on A, so as to be able to develop the theory of Burnsiderings and to connect it with the theory of "Mackey-functors", i.e. pre-Mackey-functors, whose contravariant part transforms finite sums into products. More precisely it is shown, that for any "based category" A one can define the "Burnside-functor" Ω-being a canonically defined Mackey-functor from A into the category of abelian groups-, which plays more or less the same rôle in the category of all such Mackey-functors as the integers in the category of abelian groups (actually this is just the special case one gets for A the (based) category of finite sets).
Thus any information about Ω immediately implies corresponding and sometimes rather basic results for any Mackey-functor M, defined on A. This is illustrated in some detail in Theorem 2 and 3 and their Corollaries, which deal with the computation of the defect base (vertex) of certain Green-functors (i.e. pre-Green-functors, whose underlying pre-Mackey-functor actually is a Mackey-functor) associated with Ω.
In §5 finally the relation with G-functors as defined and studied by J.A. Green in [23] and [24] is explained and a number of consequences is stated. §5 and Part I closes with a reformulation of the transfer-theorem of J.A. Green (see [23], [24]) in the language of pre-Mackey-functors.
Part I altogether thus could be considered as a general framework for induction-theory, mainly concerned with the wealth of formal consequences, which can be drawn once some kind of induction-theorem is established. Consequently the second part of this paper is concerned with developing certain methods on how to prove induction-theorems in the framework of equivariant K-Theory with a rather general type of "coefficients" (§6-§8), giving detailed applications for linear representations (§9), where the "coefficients" are just finitely generated, projective R-modules for some commutative ring R with a unit, and only prospects of further applications (§10), but leaving it mostly to the reader, to draw all the consequences explicitly, which can be drawn according to Part I.

There may be special interest in the way, composition in a category is defined in §6, and in further applications of the technique of "multiplicative induction", which plays a central rôle in §8.
It just should be mentioned, that "equivariant K-Theories" and its derivatives are not the only field, in which the general abstract nonsense of Part I can make sense, but that relative cohomology of G-modules, equivariant Homology-theories (see [8], [26], [27]), Galoiscohomology (see [17]) and perhaps still further theories can make profitable use of this language.

Tabulation of Definitions



Part I

Inductiontheory and Homological Algebra

§1 On relative homological algebra in functor-categories.

The material of this section is basically well known. Indications of proofs, when given, are just for the convenience of the reader. Let A be a small category with finite products, especially a final object $\bullet \in |A|$ ($|A|$ the class of objects in A) and let B be an abelian category. With $[A^o, B]$, resp. $[A, B]$ we denote the abelian category of contravariant, resp. covariant functors from A to B. For an object $X \in |A|$ and $M \in |[A^o, B]|$, resp. $\in |[A, B]|$ define $M_X: A \to B: Y \mapsto M(Y \times X)$. One has an obvious natural transformation $M \to M_X$, resp. $M_X \to M$, more generally $X \mapsto M_X$ defines a contravariant functor $A \to [A^o, B]$, resp. a covariant functor $A \to [A, B]$. A sequence $M' \overset{\phi'}{\to} M \overset{\phi''}{\to} M''$ is said to X-split (at M) if the associated sequence $M'_X \overset{\phi'_X}{\to} M_X \overset{\phi''_X}{\to} M''_X$ splits (i.e. if their exist $\psi': M_X \to M'_X$ and $\psi'': M''_X \to M_X$ with $\phi'_X \psi' + \psi'' \phi''_X = \mathrm{Id}_{M_X}$).

Lemma 1.1: (a) $0 \to M \to M_X$ (resp. $M_X \to M \to 0$) is X-split.

(b) If $M' \to M \to M''$ is X-split and $Y \in |A|$ with $Y \prec X$ (i.e. $\mathrm{Hom}_A(Y,X) \neq \emptyset$), then it is Y-split (since M_Y is a direct summand in $M_{X \times Y} = (M_X)_Y$).

Proposition 1.1: Let $M \in |[A^o, B]|$ and $X \in |A|$. Then the following statements are equivalent:

(i) $0 \to M \to M_X$ splits

(ii) There exists a contravariant functor $N: A/X \to B$ (A/X the category of objects over X, i.e. of morphisms into X), such that M is a direct summand in $N^X: A \to A/X \overset{N}{\to} B$, where $A \to A/X$ is defined by $Y \mapsto Y \times X/X$ (right-adjoint to the forgetfull functor $A/X \to A$).

(iii) For any diagramm
$$\begin{array}{ccc} 0 \to M' & \to & M'' \\ \downarrow & \swarrow & \\ M & & \end{array}$$
with an X-split line one has a morphism $M'' \to M$, which makes the diagramm commutative.

(iv) Any X-split sequence $0 \to M \to M'$ splits.

In this case we call M X-injective. One has corresponding statements for covariant functors, defining X-projectivity.

Corollary 1: M_X is X-injective (X-projective).

Corollary 2: If M is X-injective (-projective) and $Y \in |A|$, $X \prec Y$, then M is Y-injective (-projective).

Corollary 3: If $X, Y \in |A|$, then M is X- and Y-injective(-projective), if and only if it is $X \times Y$-injective (-projective).

Especially if any set of $\succ\!\prec$-equivalence-classes ($X \succ\!\prec Y \iff X \prec Y$ and $Y \prec X$) of objects in A contains minimal elements (i.e. if any sequence $X_1 \leftarrow X_2 \leftarrow \dots$ in A finally

contains only ☆-equivalent objects), e.g. if there are only finitely many ☆-equivalence-classes, then there exists for any M an object X-unique up to ☆-equivalence-such that M is Y-injective (Y-projective) for some $Y \varepsilon |A|$ if and only if $X \prec Y$. Any such object may be called a vertex of M (cf. [21], [22], [12]). Roughly speaking inductiontheory can be understood as one possibel method of computing vertices of various functors M by extending such functors to bi-functors as will be seen in the next sections. But before let us put together some basic facts on the homological algebra, associated to X-injectivity, resp. X-projectivity.
By the above statements we have for any $M \varepsilon |[A^o,B]|$ an X-split map into an X-injective functor $0 \to M \to M_X$ and thus we can always construct resolutions, whose cohomology-"groups" are denoted by $H^n_X(M)$, resp. by $H^n_X(M,Y)$ if evaluated at some $Y \varepsilon |A|$, $(n \geq 0)$.
Correspondingly one has for any $M \varepsilon |[A,B]|$ homology-"groups" $H_n^X(M)$, resp. $H_n^X(M,Y)$.
Canonical resolutions are given by
<u>Proposition 1.2 (Amitsur):</u> For any $X \varepsilon |A|$ consider the semisimplicial complex in A:

$$\mathrm{Am}\,(X): X \underset{p_1^1}{\overset{p_0^1}{\leftleftarrows}} X \times X \;\overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}}\; X \times X \times X \;\overset{\leftarrow}{\underset{\leftarrow}{\vdots}}\; \ldots$$ (with $\mathrm{Am}\,(X)_n = X^{n+1}$ and $\mathrm{Am}\,(X,\phi)$: $X^{n+1} \to X^{m+1}$ for any $\phi: \{0, \ldots, m\} \to \{0, \ldots, n\}$ given by the commutativity of

$$\begin{array}{ccc} X^{n+1} & \longrightarrow & X^{m+1} \\ & \searrow^{\pi_{\phi(\mu)}} \quad \swarrow_{\pi_\mu} & \\ & X & \end{array},$$

π_μ the projection onto the μ-th factor, $\mu = 0,\ldots,m$). Applying $M \varepsilon |[A^o,B]|$ to this complex, one gets a complex of X-injective functors:

$\mathrm{Am}\,(X,M): 0 \to M_X \overset{\partial^1}{\to} M_{X^2} \overset{\partial^2}{\to} M_{X^3} \to \ldots,\ \partial^n = \sum_{\nu=0}^{n} (-1^\nu)\, M\,(p_\nu^n)$ together with an augmentation $M \to M_X$, such that the augmented complex is X-split. Thus $H^i_X(M) = \mathrm{Ke}\,\partial^{i+1}/\mathrm{Im}\,\partial^i$. One has corresponding statements for covariant functors $A \to B$.

To prove, that the augmented complex is X-split, one has to observe that $0 \to M_X \to (M_X)_X \to (M_{X^2})_X \to \ldots$ is just $\mathrm{Am}\,(X,M)$ with precisely the last face-operator missing everywhere. Thus one can use the corresponding degeneracy-operators, to construct a homotopy from zero to the identity on this complex, which proves, that it is X-split everywhere.
We give some applications
<u>Proposition 1.3:</u> If M is X-injective, then $0 \to M \to M_X \to M_{X^2} \to \ldots$ is exact everywhere. If M is X-projective, then $\ldots \to M_{X^2} \to M_X \to M \to 0$ is exact everywhere.

Corollary 1: If $0 \to M_1 \to M_2 \to M_3 \to \ldots$ is a sequence of X-injective contravariant functors from A to B, which is exact at any $Y \prec X$, then it is exact. Correspondingly any sequence $\ldots \to M_3 \to M_2 \to M_1 \to 0$ of X-projective covariant functors, which is exact at any $Y \prec X$, is exact.

Corollary 2: If M is X-injective, then $M(\bullet)$ is isomorphic to the difference kernel of the two maps from $M(X)$ to $M(X \times X)$, thus it is determined by its behavior on X and $X \times X$. (This is precisely the point, why one wants to prove X-injectivity: it allows to reduce the computation of $M(\bullet)$ to the computation of $M(X)$, $M(X \times X)$ and the two maps from $M(X)$ to $M(X \times X)$.)

Proposition 1.4: Let $0 \to M' \to M \to M'' \to 0$ be a sequence of functors from A to B, which is exact at every $Y \prec X$. Then one has a long exact sequence

$$0 \to H_X^o(M') \to H_X^o(M) \to H_X^o(M'') \to H_X^1(M') \to \ldots\ldots$$

resp. $\ldots \to H_1^X(M'') \to H_o^X(M') \to H_o^X(M) \to H_o^X(M'') \to 0.$

Remark: The general constructions of homological algebra would only give such long exact sequences for X-split exact sequences $0 \to M' \to M \to M'' \to 0$.

Proposition 1.5: Let $X, Y \in |A|$ with $Y \prec X$ and $M \in |[A^o, B]|$, resp. $\in |[A, B]|$. Then one has a spectral sequence

$$E_2^{p,q} = H_X^p(H_Y^q(M)) \Longrightarrow H_Y^{p+q}(M), \text{ resp. } E_{p,q}^2 = H_p^X(H_q^Y(M)) \Longrightarrow H_{p+q}^Y(M).$$

Proof: Consider the diagramm

$$\begin{array}{ccccc}
X \times Y & \leftleftarrows & X^2 \times Y & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & \ldots \\
\uparrow\uparrow & & \uparrow\uparrow & & \\
X \times Y^2 & \leftleftarrows & X^2 \times Y^2 & \overset{\leftarrow}{\underset{\leftarrow}{\leftarrow}} & \ldots \\
\uparrow\uparrow\uparrow & & \uparrow\uparrow\uparrow & & \\
\vdots & & & &
\end{array}$$

Applying M one gets a double-complex. One of its two spectral sequences collapses by Prop. 1.2, giving the (co-) homology of the total complex, the other one is just the one mentioned.

Corollary: If $Y, X \in |A|$ and $\alpha, \beta: Y \to X$ two morphisms, then both induce the same homomorphisms $H_X^i(M) \to H_Y^i(M)$ (resp. $H_i^Y(M) \to H_i^X(M)$), especially any endomorphism $X \to X$ induces the identity on $H_X^i(M)$, resp. $H_i^X(M)$ and any $\alpha: Y \to X$ a canonical isomorphism $H_X^i(M) \to H_Y^i(M)$, resp. $H_i^Y(M) \to H_i^X(M)$, whenever $Y \prec X$.

Proposition 1.6: Let $M, N, L \in |[A^o, B]|$ with B the category k-mod of k-left-modules for a commutative ring k with $1 \in k$ (or any abelian category with an internal tensor-product) and let $<,>: M \times N \to L$ be a pairing, i.e. a family of k-bilinear maps $<,>_X: M(X) \times N(X) \to L(X)$ $(X \in |A|)$ such that for any $\alpha: Y \to X$ one has

$\alpha(\langle a,b\rangle_X) = \langle\alpha(a),\alpha(b)\rangle_Y$ ($a \in M(X)$, $b \in N(X)$). Then this pairinginduces pairings $\langle,\rangle: H^p_X(M) \times H^q_X(N) \to H^{p+q}_X(L)$ $(p,q > 0)$.

Proof: $\langle,\rangle$ induces a map from the double-complex $M(X^{p+1}) \times N(X^{q+1})$ into the double-complex $L(X^{p+1} \times X^{q+1})$ and thus a pairing from $H^p_X(M) \times H^q_X(N)$ into the cohomology of the associated total complex of the latter, which by prop. 1.4 is just $H^{p+q}_X(L)$. (An explicit isomorphism of course is induced by the usual map.

$\bigoplus_{p+q=n} L(X^{p+1} \times X^{q+1}) \to L(X^{p+q+1})$, whose components come from mapping the first p+1 factors onto the first p+1 factors and the last q+1 factors onto the last q+1 factors.)

Remark: There is no equivalent statement for covariant functors in this setting.

§2 Homological algebra of bifunctors

A bifunctor $M: A \to B$ from a category A to a category B is defined to be a pair of functors (M_*, M^*) from A to B, such that M_* is contravariant, M^* is covariant and both coincide on the objects: thus for any $X \in |A|$ we have one object $M_*(X) = M^*(X) =: M(X) \in |B|$ and for any morphism $\alpha: Y \to X$ in A we have two morphisms $M(Y) \underset{\alpha^*}{\overset{\alpha_*}{\leftrightarrows}} M(X)$. A natural transformation $\Theta: M \to N$ of bifunctors is a family of morphisms $\Theta_X: M(X) \to N(X)$, such that Θ is a natural transformation as well for M_* as for M^*.

Obviously if A is small, then we have the category $Bi(A,B)$ of bifunctors from A to B, which as usual inherits most of the usual formal properties of B, e.g. $Bi(A,B)$ is abelian if B is so.

Now assume A to be small and to contain finite products. For any $X \in |A|$ and any $M \in Bi(A,B)$ again one has $M_X \in Bi(A,B)$ $(M_X(Y) =: M(X \times Y))$, and one can also define X-split sequences $M' \to M \to M''$ as sequences, for which $M'_X \to M_X \to M''_X$ splits, but since generally neither of the two families

$$p_*: M \to M_X: M(Y) \xrightarrow{p(Y)_*} M(Y \times X)$$

and

$$p^*: M_X \to M: M(X \times Y) \xrightarrow{p(Y)^*} M(Y)$$

($p(Y): Y \times X \to Y$ the projection) are natural transformations of bi-functors, we cannot develop a relative homological algebra of arbitrary bi-functors similarly to the above theory of co- or contravariant functors. Thus we restrict ourselves to the more convenient class of pre-Mackey-functors: a bi-functor $M: A \to B$ is called a pre-Mackey-functor, if for any pull-back-diagramm

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y_2 \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y_1 & \xrightarrow[\varphi]{} & X \end{array}$$

in A the diagramm

$$\begin{array}{ccc} M(Y) & \xrightarrow{\Phi^*} & M(Y_2) \\ {\scriptstyle\Psi_*}\uparrow & & \uparrow{\scriptstyle\psi_*} \\ M(Y_1) & \xrightarrow{\varphi^*} & M(X) \end{array}$$

commutes.

A first consequence of this definition is

<u>Lemma 2.1:</u> If $\alpha: Y \to X$ is a monomorphism in A and $M: A \to B$ a pre-Mackey-functor, then $M_*(\alpha) \circ M^*(\alpha): M(Y) \to M(Y)$ is the identity. Especially if α is an isomorphism, then $M_*(\alpha^{-1}) = M^*(\alpha)$.

Proof: Just apply M to the pull-back-diagramm

$$\begin{array}{ccc} Y & \xrightarrow{Id} & Y \\ {\scriptstyle Id}\downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow[\alpha]{} & X \end{array} .$$

Now for pre-Mackey-functors we have indeed natural transformations of bi-functors $M \to M_X$, $M_X \to M$ or more generally: Any pre-Mackey-functor $M: A \to B$ defines a pre-Mackey-functor from A into the full subcategory $Bi'(A,B)$ of pre-Mackey-functors in $Bi(A,B)$ by $X \mapsto M_X$, $(\alpha: Y \to X) \mapsto (\alpha_*: M_X \to M_Y,\ \alpha^*: M_Y \to M_X)$.

Moreover for B abelian $0 \to M \to M_X$ and $M_X \to M \to 0$ are both X-split and any X-split sequence $M' \to M \to M''$ of pre-Mackey-functors is also Y-split for any $Y \in |A|$ with $Y \prec X$.

We can define $M \in |Bi'(A,B)|$ to be X-injective, if $0 \to M \to M_X$ splits, and X-projective, if $M_X \to M \to 0$ splits, and have - analogously to Prop. 1.1 - all the equivalent conditions for X-injectivity, resp. X-projectivity now in the category of pre-Mackey-functors. Especially M_X is both X-injective and X-projective for any $M \in |Bi'(A,B)|$. But then both X-injectivity and X-projectivity of M are equivalent to M being a direct summand in M_X, thus a pre-Mackey-functor is X-injective if and only if it is X-projective, which generalizes a well known result of Gaschütz-Higman (see [20], [25], [7], [12]).
Therefore we will only use the term "X-projective", but keep in mind, that for pre-Mackey-functors this means "X-injective" as well.

As before we get, that any X-projective pre-Mackey-functor M is also Y-projective for any $Y \in |A|$ with $X \prec Y$, and that M is X- and Y-projective, if and only if it is $X \times Y$-projective. Especially we can again define the vertex of a pre-Mackey-functor as the smallest $X \in |A|$ - with respect to "$\prec$" and thus up to $\ast$-equivalence - such that M is X-projective, whenever such an X exists (e.g. A contains only finitely many $\ast$-equivalence-classes).
Again $0 \to M \to M_X \to M_{X^2} \to \ldots.$ and $\ldots \to M_{X^2} \to M_X \to M \to 0$ are X-split and thus (without the augmentation) can be used to define (and perhaps compute) the (co-) homology"groups" $H^n_X(M)$ and $H^X_n(M)$ for any $M \in |Bi'(A,B)|$.
We have $H^n_X(M) = H^X_n(M) = 0$ $(n > 0)$ and $H^o_X(M) = M = H^X_o(M)$ whenever M is X-projective.
Moreover we can splice together the two complexes to just one doubly-infinite complex

$$\ldots \to M_{X^2} \xrightarrow{\partial^{-1}} M_X \xrightarrow{\partial^{o}} M_X \xrightarrow{\partial^{1}} M_{X^2} \to \ldots.$$
$$M_X \searrow M \nearrow M_X$$

with ∂^n $(n \geq 1)$ as in §1 for M_*, ∂^o the composition $M_X \to M \to M_X$ and ∂^{-n} $(n \geq 1)$ as ∂_n in §1 for M^*. We define $\hat{H}^n_X(M) = \mathrm{Ke}\ \partial^{n+1}/\mathrm{Im}\ \partial^n$ $(n \in \mathbb{Z})$ to be the Tate-cohomology of M. Obviously $\hat{H}^n_X(M) = H^n_X(M)$ and

$$\hat{H}^{-n-1}_X(M) = H^X_n(M) \text{ for } n > 0,$$

wheras for n = 0 the map ∂^o induces a map $H^X_o(M) \overset{\partial^o}{\to} H^o_X(M)$ and $\hat{H}^{-1}_X(M) = \mathrm{Ke}(\partial^o)$, $\hat{H}^o_X(M) = \mathrm{Coke}(\partial^o)$.

One can characterize $\hat{H}^o_X(M)$ also as the cokernel of the natural map $H^o_X(M_X) \to H^o_X(M)$, since in the diagramm

$$\begin{array}{ccccc} M & \to & M_X & \to & M_{X^2} \\ \uparrow & & \uparrow & & \uparrow \\ M_X & \to & (M_X)_X & \to & (M_X)_{X^2} \end{array}$$

the lower left horizontal arrow maps M_X isomorphically onto $H^o_X(M_X)$.

Again any sequence $0 \to M' \to M \to M'' \to 0$ of pre-Mackey-functors from A to B, which is exact on any $Y \prec X$, gives rise to a long exact sequence

$\ldots \to \hat{H}^n_X(M') \to \hat{H}^n_X(M) \to \hat{H}^n_X(M) \to \hat{H}^{n+1}_X(M') \to \ldots$. and we have $\hat{H}^n_X(M) = 0$ whenever M is X-projective. Thus if $M \in \mathrm{Bi}'(A,B)$ and

$\mathrm{Ke}(M_X \to M) =: M': A \to B: Y \mapsto \mathrm{Ke}(M(X\times Y) \overset{p_*}{\to} M(Y))$,

$\mathrm{Coke}(M \to M_X) =: M'': A \to B: Y \mapsto \mathrm{Coke}(M(Y) \overset{p^*}{\to} M(X\times Y))$, (p: $X \times Y \to Y$ the projection)

then $\hat{H}^n_X(M) \simeq \hat{H}^{n+1}_X(M') \simeq \hat{H}^{n-1}_X(M'')$, i.e. we can shift dimensions as usal in Tate-cohomology.

The spectral sequences from §1 of course now have pre-Mackey-functors as term whenever applied to a pre-Mackey-functor M, and again any morphism $\alpha: X \to X$ induces the identity on $\hat{H}^n_X(M)$.

Finally to define cup-products of pre-Mackey-functors we first have to define pairings: so assume B=<u>k-mod</u> (as in §1) and let M, N, L: $A \to$ <u>k-mod</u> be three bi-functors. A pairing $<,>: M \times N \to L$ is then a family:

$<,>_X: M(X) \times N(X) \to L(X)$ $(X \in |A|)$ of k-bilinear maps, such that for any $\alpha: Y \to X$ in A we have

(P1) $\alpha_*(<a,b>_X) = <\alpha_*(a),\alpha_*(b)>_Y$ $(a \in M(X),\ b \in N(X))$,

(P2) $\alpha^*(<\alpha_*(a),b>_Y) = <a,\alpha^*(b)>_X$ $(a \in M(X),\ b \in N(Y))$,

(P3) $\alpha^*(<a,\alpha_*(b)>_Y) = <\alpha^*(a),b>_X$ $(a \in M(Y),\ b \in N(X))$.

Remark: (P2) and (P3) can be considered as some kind of an axiomatic Frobenius-reciprocity-law (see [29], [23] ....).

A straight-forward consequence of these definitions is

<u>Lemma 2.2 (cf. [29]):</u> Let $<,>: M \times N \to L$ be a pairing of bi-functors M, N, L: $A \to$ <u>k-mod</u> and $\alpha: Y \to X$ a morphism in A. For any bi-functor $X: A \to$ <u>k-mod</u> write $K_\alpha X = \mathrm{Ke}(\alpha_*\ X(X) \to X(Y))$ and $I_\alpha X = \mathrm{Im}(\alpha^*: X(Y) \to X(X))$.

Then one has: $< K_\alpha M,\ N(X) >_X \subseteq K_\alpha L$,

$< M(X),\ K_\alpha N >_X \subseteq K_\alpha L$,

$< I_\alpha M,\ N(X) >_X \subseteq I_\alpha L$,

$< M(X),\ I_\alpha N >_X \subseteq I_\alpha L$,

$< K_\alpha M,\ I_\alpha N >_X = < I_\alpha N,\ K_\alpha M >_X = 0$.

Now let M, N, L: $A \to$ k-mod be pre-Mackey-functors and $<,>: M \times N \to L$ a pairing of bifunctors.

Proposition 2.1: For any $X \in |A|$ one has an induced pairing of bifunctors $M \times N_X \to L_X$ (and of course $M_X \times N \to L_X$) defined by $M(Y) \times N(Y{\times}X) \to L(Y{\times}X)$: $(a,b) \mapsto < p_*(Y)\ (a), b >_{Y\times X}$ with $p(Y)$: $Y{\times}X \to Y$ the projection. For any morphism α: $Z \to X$ one has commutative diagramms:

$$\begin{array}{ccc} M \times N_X & \to & L_X \\ \downarrow \mathrm{Id}\times\alpha_* & & \downarrow \alpha_* \\ M \times N_Z & \to & L_Z \end{array} \qquad \begin{array}{ccc} M \times N_Z & \to & L_Z \\ \downarrow \mathrm{Id}\times\alpha^* & & \downarrow \alpha^* \\ M \times N_X & \to & L_X \end{array} .$$

Proof: direct verification.

An immediate consequence is

Proposition 2.2: The induced pairings $H^p_X(M_*) \times H^q_X(N_*) \to H^{p+q}_X(L_*)$ as defined in §1 actually are pairings of bi-functors.

Especially for $p = 0$ one gets pairings $H^o_X(M) \times H^q_X(N) \to H^q_X(L)$ and one checks easily, that there are corresponding well defined pairings $H^o_X(M) \times H^X_q(N) \to H^X_q(L)$. (Just extend the obvious pairing $M \times H^X_q(N) \to H^X_q(L)$ to $H^o_X(M)$).

But for α: $X \to \bullet$ and $q \neq 0$ we have $H^q_X(N_X) = H^X_q(N_X) = 0$, thus $K_\alpha(H^q_X(N)) = H^q_X(N)$, $K_\alpha(H^X_q(N)) = H^X_q(N)$ and therefore by Lemma 2.2 $< I_\alpha(H^o_X(M)), H^q_X(N) > = < I_\alpha H^o_X(M), H^X_q(N) > = 0$, i.e. the above pairing induces well defined pairings of

$$\hat{H}^o_X(M) = H^o_X(M)/I_\alpha(H^o_X(M)) \text{ with } H^q_X(N), \text{ resp. } H^X_q(N)$$

into $H^q_X(L)$, resp. $H^X_q(L)$.

Using dimension-shifting together with Prop. 2.1 (or any other appropriate technique) this can be generalized to

Proposition 2.3: Any pairing $M \times N \to L$ of pre-Mackey-functors $A \to$ k-mod induces pairings $\hat{H}^p_X(M) \times \hat{H}^q_X(N) \to \hat{H}^{p+q}_X(L)$ $(p,q \in \mathbb{Z})$, which have all usual properties of cup-products for Tate-cohomology-groups.

Remark: It might be a usefull exercise for the reader to show, that already to get a well defined cup-product of zero-dimensional Tate-cohomology

$$\hat{H}^{0}_{X}(M) \times \hat{H}^{0}_{X}(N) \to \hat{H}^{0}_{X}(L)$$

one is forced to define pairings of bi-functors using the properties (P2) and (P3) (together with (P1), the multiplicativity of the contravariant part of course) instead of postulating analogously to (P1) multiplicativity of the covariant part as well.

§3 pre-Green-functors

At first let A be an arbitrary category. Following T.Y. Lam (see [28]) we define a Frobenius-functor F: $A \to$ k-mod to be a bi-functor together with a pairing $F \times F \to F$, such that for any $X \in |A|$ the k-bilinear map $F(X) \times F(X) \to F(X)$ makes $F(X)$ into a k-algebra with a unit $1_{F(X)} \in F(X)$ and with $\alpha_*(1_{F(X)}) = 1_{F(Y)}$ for any $\alpha: Y \to X$ in A. A left, resp. right F-module M is a bi-functor $A \to$ k-mod together with a pairing $F \times M \to M$, resp. $M \times F \to M$, such that for any $X \in |A|$ $M(X)$ becomes a left, resp. right unitary $F(X)$-module.

Lemma 3.1 (T.Y. Lam): Let F: $A \to$ k-mod be a Frobenius-functor, M a left (or right) F-module and α: $Y \to X$ a morphism in A.

(a) $K_\alpha M$ and $I_\alpha M$ are $F(X)$-submodules of $M(X)$, especially $I_\alpha F = \alpha^*(F(Y))$ is a two-sided ideal in $F(X)$.

(b) If $\alpha^*(F(Y)) = F(X)$, then α^*: $M(Y) \to M(X)$ is split-surjective .

Especially

(i) $M(Y) = 0 \Rightarrow M(X) = 0$

(ii) If Θ: $M \to N$ is a natural transformation of F-modules(i.e. compatibel with the F-module-structure), then Θ_X: $M(X) \to N(X)$ is surjective (resp. split-surjective, injective, split-injective or bijective) if Θ_Y is so.

(iii) If $M' \to M \to M''$ is a sequence of F-modules, then $M'(X) \to M(X) \to M''(X)$ is (split-) exact, if $M'(Y) \to M(Y) \to M''(Y)$ is so.

Proof:(a) follows immediately from Lemma 2.2; a right inverse of α^*: $M(Y) \to M(X)$ is given by $\hat{\alpha}$: $M(X) \to M(Y)$: $x \mapsto r \cdot \alpha_*(x)$ with $r \in F(Y)$ such that $\alpha^*(r) = 1_{F(X)}$, since $\alpha^*(\hat{\alpha}(x)) = \alpha^*(r \cdot \alpha_*(x)) = \alpha^*(r)x = 1 \cdot x = x$.

Now assume A to contain finite products. We define a pre-Green-functor G: $A \to$ k-mod to be a Frobenius-functor, which is a pre-Mackey-functor as well. A G-module is then also supposed to be a pre-Mackey-functor, too. In this case we can interpret the surjectivity-condition in Lemma 3.1 (b), as follows:

Theorem 1: Let G: $A \to$ k-mod be a pre-Green-functor and $X \in |A|$. Then the following statements are equivalent:

(i) The natural map $G(X) \to G(\bullet)$ (associated to $X \to \bullet$) is surjective

(ii) G is X-projective

(iii) Any G-module M is X-projective.

Proof: (iii) $\to$ (ii) $\to$ (i) is trivial; for (i) $\to$ (iii), i.e. to construct a splitting map $M \to M_X$ one just uses the maps $\hat{\alpha}_Y$: $M(Y) \to M(Y \times X)$ as defined in the proof of Lemma 3.1 with α_Y: $Y \times X \to Y$ the projection and with $r = r_Y = \beta_{Y*}(r_1)$ for a fixed preimage r_1 of $1 \in G(\bullet)$ taken in $G(X)$ and β_Y: $Y \times X \to X$ the other projection.

Remark: This theorem states the essential connection between inductiontheory and (relative) homological algebra and perhaps - in a rather formal way - the real motive for proving induction-theorems: one just wants to prove X-injectivity of certain contravariant functors M: $A^o \to$ k-mod and may do so by 1. extending M to a pre-Mackey-functor, 2. constructing a pre-Green-Functor G , which acts unitary on M, and 3. proving the surjectivity of $G(X) \to G(\bullet)$, i.e. an inductiontheorem for G.

Corollary 1:Let G: $A \to$ k-mod be a pre-Green-functor, M a G-module and $X \varepsilon |A|$ with $G(X) \to G(\bullet)$ surjective. Then $\hat{H}^n_X(M) = 0$ for all $n \varepsilon \mathbb{Z}$ and the augmented Amitsur-complexes $0 \to M \to M_X \to M_{X^2} \to$ and $\ldots \to M_{X^2} \to M_X \to M \to 0$ are split-exact.

It should be remarked, that for G and M as in Cor. 1 and X an arbitrary object in A one also has pairings $H^p_X(G) \times H^q_X(M) \to H^{p+q}_X(M)$ $(p,q \geq 0)$ and $\hat{H}^p(G) \times \hat{H}^q(M) \to \hat{H}^{p+q}_X(M)$, $(p,q \varepsilon \mathbb{Z})$ especially for $M = G$ and $p = q = 0$ one gets, that $H^o_X(G)$ and $\hat{H}^o_X(G)$ are pre-Green-functors, $H^q_X(M)$ and $\hat{H}^q_X(M)$ are modules with respect to these pre-Green-functors respectively, and the natural transformations $G \to H^o_X(G) \to \hat{H}^o_X(G)$ are natural transformations of pre-Green-functors and thus make $H^o_X(G)$ and $\hat{H}^o_X(G)$ into "G-algebras", whenever G is commutative.
Especially all $H^q_X(M)$ and $\hat{H}^q_X(M)$ are G-modules. Moreover the "graded cohomology-rings" $H^*_X(G)$ and $\hat{H}^*_X(G)$ are"graded pre-Green-functors" and $H^*_X(M)$, resp. $\hat{H}^*_X(M)$ is a graded $H^*_X(G)$-, resp. $\hat{H}^*_X(G)$ -module.

Corollary 2 (cf. Green, [23]) If G: $A \to$ k-mod is a pre-Green-functor and $X,Y \varepsilon |A|$, then $G(X) \to G(\bullet)$ and $G(Y) \to G(\bullet)$ are surjective if and only if $G(X \times Y) \to G(\bullet)$ is surjective.

A direct proof for this may also be based on considering the pull-back-diagramm

$$\begin{array}{ccc} X \times Y & \xrightarrow{\Phi} & Y \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow[\phi]{} & \bullet \end{array}$$

and either using the argument: "ϕ^*: $G(X) \twoheadrightarrow G(\bullet)$ surjective $\Longleftrightarrow$ there exists $x \varepsilon G(X)$ with $\phi^*(x) = 1_{G(\bullet)} \Rightarrow 1_{G(Y)} = \psi^*(1_{G(\bullet)}) = \psi^*(\phi^*(x))$
$= \Phi^*(\Psi_*(x)) \varepsilon \operatorname{Im} \Phi^* \Rightarrow \Phi^*$: $G(X \times Y) \twoheadrightarrow G(Y)$ is surjective" or the "Mackey-tensor-product-theorem":

Lemma 3.2: If $<,>$: $M \times N \to L$ is a pairing of pre-Mackey-functors $A \to$ k-mod,

$$\begin{array}{ccc} Y & \xrightarrow{\Phi} & Y_2 \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ Y_1 & \xrightarrow[\phi]{} & X \end{array}$$

a pull back with $\phi \circ \Psi = \psi \circ \Phi = \alpha$: $Y \to X$, $a \varepsilon M(Y_1)$, $b \varepsilon N(Y_2)$, then

$\langle\phi^*(a), \psi^*(b)\rangle_X = \alpha^*(\langle\Psi_*(a), \Phi_*(b)\rangle_Y)$.

Proof: $\langle\phi^*(a),\psi^*(b)\rangle_X = \phi^*(\langle a,\phi_*\psi^*(b)\rangle_{Y_1}) = \phi^*(\langle a,\Psi^*\Phi_*(b)\rangle_{Y_1}) = \phi^*\Psi^*(\langle\Psi_*(a),\Phi_*(b)\rangle_Y)$
$= \alpha^*(\langle\Psi_*(a),\Phi_*(b)\rangle_Y)$.

Remark: Lemma 3.2 shows, that "$G(X) \twoheadrightarrow G(\bullet)$ and $G(Y) \twoheadrightarrow G(\bullet) \Longleftrightarrow G(X\times Y) \twoheadrightarrow G(\bullet)$" holds already if G is a pre-Mackey-functor with an arbitrary inner composition $G \times G \to G$ such that $G(\bullet) \times G(\bullet) \to G(\bullet)$ is surjective.

Thus if any set of objects in A contains minimal objects with respect to $\prec$, one can again find for any such G an object $X \in |A|$ such that $G(Y) \twoheadrightarrow G(\bullet)$ is surjective for some $Y \in |A|$ if and only if $X \prec Y$. Following Green, [23] we may call any such object a defect-object for G and get, that for a pre-Green-functor G defect-objects and vertices coincide. In the following we will follow Green, [23] (instead of Green, [21]) and mainly use the term "defect-object" for pre-Green-functors.

§4 Mackey-functors

Let A and B at first be arbitrary categories. A Mackey-functor M: $A \to B$ is a pre-Mackey-functor with the additional property, that M_* transforms finite sums in A into finite products in B. Of course for a small A we have the full subcategory $Mc(A,B)$ of Mackey-functors in $Bi'(A,B) \subseteq Bi(A,B)$ which again is abelian if B is.
For B = k-mod we define Green-functors G: $A \to B$ to be pre-Green-functors, which are also Mackey-functors.
We want to study Green- and Mackey-functors $A \to$ k-mod on categories A satisfying the following properties:
(M1) A is small, contains finite sums ("X∪Y"), products ("X×Y") and pullbacks, especially an initial object $\emptyset \in |A|$ and a final object $\bullet \in |A|$.
(M2) The two squares in a commutative diagramm

$$\begin{array}{ccccc} X' & \longrightarrow & Z' & \longleftarrow & Y' \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & X\cup Y & \longleftarrow & Y \end{array}$$

are pull backs if and only if the upper line represents Z' as a sum of X' and Y'.

Lemma 4.1: Let A satisfy (M1) and (M2). Then

(a)
$$\begin{array}{ccc} X & \xrightarrow{Id} & X \\ {\scriptstyle Id}\downarrow & & \downarrow \\ X & \longrightarrow & X\cup Y \end{array} \quad \text{and} \quad \begin{array}{ccc} X & \longleftarrow & \emptyset \\ \downarrow & & \downarrow \\ X\cup Y & \longleftarrow & Y \end{array} \quad \text{are}$$

pull-backs

(b) The natural map $(Z \times X) \cup (Z \times Y) \to Z \times (X\cup Y)$ is an isomorphism.

(c) The category A/X of morphisms into X satisfies (M1) and (M2) for any $X \in |A|$.

Proof: (a): Choose X' = Z' = X, Y' = $\emptyset$ in (M2).
(b): Choose X' = Z×X, Y' = Z×Y, Z'=Z×(X∪Y) in (M2).
(c): Direct verification.

Next we have

Lemma 4.2: If A satiesfies (M1) and (M2) and if M: $A \to B$ is a Mackey-functor into an abelian category B, then M^* transforms finite sums into finite sums.
Proof: Since M_* transforms finite sums into finite products, we have $M(\emptyset) = 0$. Thus applying M to the diagramms in Lemma 4.1 we get a diagramm

$$\begin{array}{ccccc} M(X) & \xrightarrow{M^*} & & \xleftarrow{M^*} & M(Y) \\ & & M(X\cup Y) & & \\ {\scriptstyle Id}\downarrow & \swarrow{\scriptstyle M_*} & & {\scriptstyle M_*}\searrow & \downarrow{\scriptstyle Id} \\ M(X) & & & & M(Y) \end{array}$$

with zero-diagonals $\searrow$, $\swarrow$. Since $\mathcal{B}$ is abelian and $M_* \times M_*: M(X \cup Y) \to M(X) \times M(Y)$ an isomorphism, this implies, that $M^* \oplus M^*: M(X) \oplus M(Y) \to M(X \cup Y)$ is an isomorphism as well.

Now let us observe, that because of Lemma 4.1, (b) the isomorphism-classes of objects in A form a halfring $\Omega^+(A)$ with respect to sum and product with $\emptyset$ representing $0 \in \Omega^+(A)$ and $\bullet$ representing $1 \in \Omega^+(A)$. Let $\Omega(A)$ be the associated Grothendieck-ring. Since by Lemma 4.1, (c) A/X satisfies (M1) and (M2) for any $X \in |A|$ we can also define $\Omega(X) = \Omega(A/X)$.
Since any morphism $\alpha: Y \to X$ induces functors $\alpha_*: A/X \to A/Y$: $(Z \xrightarrow{\beta} X) \mapsto (Z_\beta \times_{X^\alpha} Y \to Y)$ and $\alpha^*: A/Y \to A/X$: $(Z \xrightarrow{\beta} Y) \mapsto (Z \xrightarrow{\alpha\beta} X)$, both of which are additive, the first one even multiplicative., we get induced maps $\alpha_*: \Omega(X) \to \Omega(Y)$, $\alpha^*: \Omega(Y) \to \Omega(X)$.
One verifies easily:

Proposition 4.1: The above definitions make $\Omega: A \to \underline{\underline{\mathbb{Z}\text{-mod}}}$ and thus also $\Omega^k = k \otimes_{\mathbb{Z}} \Omega: A \to \underline{\underline{k\text{-mod}}}$ into a commutative Green-functor.

We call Ω the Burnside-functor, associated to A. Note that $1_{\Omega(\bullet)} = 0_{\Omega(\bullet)}$ can happen, for instance if A is the category of at most countable sets.
Still one can prove:

Proposition 4.2: Any Mackey-functor $M: A \to \underline{\underline{k\text{-mod}}}$ is in a natural way a $k \otimes_{\mathbb{Z}} \Omega$-module and any Green-functor $G: A \to \underline{\underline{k\text{-mod}}}$ a $k \otimes_{\mathbb{Z}} \Omega$-algebra. The action of $k \otimes_{\mathbb{Z}} \Omega$ on M is induced by $\Omega^+(X) \times M(X) \to M(X): (Z \xrightarrow{\beta} X, a) \mapsto \beta^*(\beta_*(a))$.
Especially the action of Ω on Ω is just multiplication.
Proof: Lemma 4.2 guarantees linearity with resprct to β. (P2) follows just from functoriality, (P1) and (P3) from the fact, that M is a pre-Mackey-functor, applied to the pullback

$$\begin{array}{ccc} Y_\alpha \times_\beta Z & \to & Z \\ \downarrow \alpha_*(\beta) & & \downarrow \beta \\ Y & \xrightarrow{\alpha} & X \end{array} .$$

In case $1_{G(\bullet)} = 0_{G(\bullet)}$ this just says, that any Mackey-functor $M: A \to \underline{\underline{k\text{-mod}}}$ is identically zero. To make a more proper use of the Burnside-functor we have to impose some further restrictions on A, which allow to get some more information on Ω.
For a start just let us observe, that for an indecomposable object $Z \in |A|$, i.e. an object with "$Z \simeq Z_1 \cup Z_2 \Rightarrow Z_1 = \emptyset$ or $Z_2 = \emptyset$", the natural map $\mathrm{Hom}_A(Z,X) \cup \mathrm{Hom}_A(Z,Y) \to \mathrm{Hom}_A(Z, X \cup Y)$ is an isomorphism by (M1). Since anyway $\mathrm{Hom}_A(Z,X) \times \mathrm{Hom}_A(Z,Y) \to \mathrm{Hom}_A(Z, X \times Y)$ is an isomorphism, the assumption, that $\mathrm{Hom}_A(Z,X)$

is finite for any X, implies, that we have a well defined ringhomomorphism: $\varphi_Z: \Omega(A) \to \mathbb{Z}: X \mapsto |\mathrm{Hom}_A(Z,X)|$.

Morover if Z' is another such object and $\varphi_Z = \varphi_{Z'}$, then especially $Z \prec Z' \prec Z$ (evaluate at Z and Z'!); thus if we assume that any endomorphism of Z and Z' is an Automorphism, we get $Z \cong Z'$.

These considerations lead to the following definition: a category A is a based category, if it satisfies (M1) and (M2) and moreover:

(M3) There is precisely a finite number of isomorphismclasses of indecomposable objects in A and any object in A is isomorphic to finite sum of indecomposable objects.
(M4) If $Z,Z' \in |A|$ are indecomposable, then $\mathrm{Hom}_A(Z,Z')$ is finite and $\mathrm{End}_A(Z) = \mathrm{Aut}_A(Z)$.

Any set T of representatives of the isomorphism-classes of indecomposable objects in A is called a basis of A. Observe that by (M4) $Z \prec Z' \prec Z$ for $Z,Z' \in T$ implies $Z \cong Z'$, thus $Z=Z'$, if T contains precisely one object out of any isomorphismclass of indecomposable objects.

Moreover already by (M3) we have for any $X,Y \in |A|$: "$X \prec Y$" $\Longleftrightarrow$ "$Z \prec X$ implies $Z \prec Y$ for all $Z \in T$", especially one has at most $2^{|T|}$ -equivalence-classes in A.

Thus any pre-Mackey-functor $M: A \to B$ has a vertex and especially any pre-Green-functor $G: A \to$ k-mod a defect-object X.
Moreover the $\prec$-equivalence-class of X is uniquely determined by the finite set $D(G) = \{Z \in T \mid Z \prec X\}$, which is then also called the defect-set of G.

Examples: The category of finite sets is based with basis just the final object. If A and A' is based, then also $A \times A'$. If A is based with basis T and $X \in |A|$, then A/X is based with basis $T/X = \{\varphi: Z \to X \mid Z \in T, \varphi \in \mathrm{Hom}_A(Z,X)\}$ (modulo isomorphisms in A/X). For any finite group G the category $\hat{G}$ of finite left G-sets is based with basis $T = \{G/U \mid U \leq G\}$ (modulo isomorphisms); more generally: if A is based and G finite, then the category of G-objects in A is based.

Now let A be based with basis T. Let $\mathbb{Z}[T]$ be the free abelian group generated by T and $\mathbb{Z}^+[T] \subseteq \mathbb{Z}[T]$ the free abelian semigroup generated by T. Then one has a commutative diagramm:

$$\begin{array}{ccccc} \mathbb{Z}^+[T] & \longrightarrow & \mathbb{Z}[T] & & \\ \downarrow & & \downarrow & & \\ \Omega^+(A) & \longrightarrow & \Omega(A) & \xrightarrow[Z \in T]{\prod \varphi_Z} & \prod_{Z \in T} \mathbb{Z} = \tilde{\Omega}(A) \end{array}$$

The vertical arrows are surjective by (M3). Since all φ_Z are different ringhomomorphisms into $\mathbb{Z}$ by (M4), they are linearly indepedent over $\mathbb{Z}$. Thus the image of $\prod_{Z \in T} \varphi_Z$ has $\mathbb{Z}$-rank precisely $|T| = \mathrm{rk}_{\mathbb{Z}} \tilde{\Omega}(A)$, which implies, that all arrows must be injective.

This proves

Proposition 4.3: Let A be a based category with basis T. Then

(a) $\Omega^+(A)$, resp. $\Omega(A)$ is a free abelian semigroup, resp. group with basis represented by T and $\Omega^+(A)$ maps injectively into $\Omega(A)$, i.e. $X \cup Y \simeq X' \cup Y \Rightarrow X \simeq X'$.

(b) $\prod_{Z \in T} \varphi_Z : \Omega(A) \to \prod_{Z \in T} \mathbb{Z} = \tilde{\Omega}(A)$ is injective and has finite cokernel.

(c) In other words: for $X \simeq \sum_{Z \in T} n_Z Z$ and $X' \simeq \sum_{Z \in T} n'_Z Z$ we have

$$X \simeq X' \iff \varphi_Z(X) = \varphi_Z(X') \text{ for all } Z \in T \iff n_Z = n'_Z \text{ for all } Z \in T.$$

Remark: For $A = \hat{G}$ this last statement is a well known theorem of Burnside.

Since $\prod_{Z \in T} \varphi_Z : \Omega(A) \to \tilde{\Omega}(A)$ is injective, we may identify $\Omega(A)$ with its image in $\tilde{\Omega}(A)$, which itsself can be identified with the integral closure of $\Omega(A)$ in its total quotientring. Since $\tilde{\Omega}(A)$ is finite, it has a well-defined exponent $\|A\| \in \mathbb{N}$, which we define to be the Artin-index of A; thus $n \cdot \tilde{\Omega}(A) \subseteq \Omega(A) \iff \|A\|$ divides n.

Proposition 4.4: For a finite group G one has $\|\hat{G}\| = |G|$.

Proof: An easy inductionargument with respect to $|U|$ $(U \leq G)$ shows, that for any $U \leq G$ there exists $x_U \in \Omega(\hat{G})$ with $\varphi_{G/U}(x_U) = |G|$, $\varphi_{G/V}(x_U) = 0$ for $G/V \neq G/U$, using the fact, that $\varphi_{G/U}(G/U) = |\mathrm{Aut}(G/U)| = (N_G(U):U)$ divides $\varphi_{G/V}(G/U)$ for any $V \leq G$. Thus $|G| \cdot \tilde{\Omega}(\hat{G}) \subseteq \Omega(\hat{G})$. On the other hand if $x \in \Omega(\hat{G})$ with $\varphi_{G/U}(x) = 0$ for all $U \leq G$; $U \neq E$, then $x = n \cdot G/E$ for some $n \in \mathbb{Z}$ and $\varphi_{G/E}(x) = n \cdot |G|$. Thus $\|\hat{G}\| = |G|$.

For details see [16], § 5 . More generally $|A|$ is the smallest common multiple of $|\mathrm{Aut}(Z)|$, $Z \in T$, if all maps $Z \to Z'$ $(Z, Z' \in T)$ are surjective.

Theorem 2: If A is a based category and $M: A \to$ k-mod a Mackey-functor, then $|A|$ annihilates all cohomology-groups $\hat{H}^n_X(M, Y)$ $(X, Y \in |A|)$. Especially

(1) $\|A\| \cdot M(Y) \subseteq \mathrm{Ke}(M(Y) \to M(X \times Y)) + \mathrm{Im}(M(X \times Y) \to M(Y))$ and

(2) $\|A\| \cdot (\mathrm{Ke}(M(Y) \to M(X \times Y)) \cap \mathrm{Im}(M(X \times Y) \to M(Y))) = 0$.

Proof: Since the canonical map $M(Y) \to \hat{H}^0_X(M, Y)$ has kernel precisely the right side of (1) and since $\hat{H}^{-1}_X(M, Y) \to M(Y)$ has image precisely $\mathrm{Ke}(M(Y) \to M(X \times Y)) \cap \mathrm{Im}(M(X \times Y) \to M(Y))$, (1) and (2) are indeed corollaries of $\|A\| \cdot \hat{H}^n_X(M) = 0$. On the other hand by Prop. 4.2 it is enough to show, that $\|A\| \cdot 1 = 0$ in $\hat{H}^0_X(\Omega, \bullet)$, which of course follows from $\|A\| \cdot 1_{\Omega(\bullet)} \in \mathrm{Ke}(\Omega(\bullet) \to \Omega(X)) + \mathrm{Im}(\Omega(X) \to \Omega(\bullet))$. But obviously

$K = \mathrm{Ke}(\Omega(\bullet) \to \Omega(X)) = \{x \in \Omega(\bullet) \mid \varphi_Z(x) = 0 \text{ for all } Z \in T \text{ with } Z \prec X\}$ and

$I = \mathrm{Im}(\Omega(X) \to \Omega(\bullet)) = \{\sum_{Z \in T, Z \prec X} n_Z Z \mid n_Z \in \mathbb{Z}\} = \{x \in \Omega(\bullet) \mid \varphi_Z(x) = 0 \text{ for all } Z \in T \text{ with } Z \not\prec X\}$ (the last equation holds, since $x = \sum_{Z \in T} n_Z Z \in \Omega(\bullet)$ and $\varphi_Z(x) = 0$ for all $Z \in T$ with $Z \not\prec X$ implies $n_Z = 0$ for all $Z \not\prec X$, -otherwise choose a $Z_0 \in T$ with $Z_0 \not\prec X$, $n_{Z_0} \neq 0$ and Z_0 maximal with respect to $\prec$, then $\varphi_{Z_0}(x) = n_{Z_0} \cdot \varphi_{Z_0}(Z_0) \neq 0$, a

contradication).
Now consider $e = (e_Z)_{Z\varepsilon T} \varepsilon\ \tilde{\Omega}(A)$ with $e_Z = 0$ for $Z \not\prec X$ and $e_Z = 1$ for $Z \prec X$, $f = 1-e$. Then $\|A\|\cdot e$, $\|A\|\cdot f\ \varepsilon\ \Omega(A) = \Omega(\bullet)$ by definition of $\|A\|$ and thus $\|A\|\cdot e\ \varepsilon\ I$, $\|A\|\cdot f\ \varepsilon\ K$ by the above remarks, which yields $\|A\|\cdot 1_{G(\bullet)} = \|A\|\cdot e + \|A\|\cdot f\ \varepsilon\ I + K$, q.e.d.

Remark: As shown below, Theorem 2 can be considered as a generalization of Artin's inductiontheorem as well as of the fact, that $|G|$ annihilates all cohomology-groups $\hat{H}^n(G,M)$, M a $\mathbb{Z}$G-module. Now assume $\|A\|\cdot 1_k$ to be invertibel in k. Then (1) and (2) in Thm 2 imply $M = \mathrm{Ke}(M \to M_X) \oplus \mathrm{Im}(M_X \to M)$, especially $M(Y) \to M(Y\times X)$ is injective for some $Y\ \varepsilon |A|$ if and only if $M(Y\times X) \to M(Y)$ is surjective.

As a first consequence we get

Corollary 1: If $\|A\|\cdot k = k$, $G: A \to$ k-mod a Green-functor and M a G-module, such that $M(\bullet)$ is a faithfull $G(\bullet)$-module. Then the following statements are equivalent:
(i) M is X-projective
(ii) $M(X) \twoheadrightarrow M(\bullet)$ is surjective
(iii) $M(\bullet) \hookrightarrow M(X)$ is injective
(iv) $G(\bullet) \hookrightarrow G(X)$ is injective
(v) $G(X) \twoheadrightarrow G(\bullet)$ is surjective
(vi) G is X-projective
Proof: (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (vi) ⇒ (i).
This implies especially that $\Omega^k/\mathrm{Ke}(\Omega^k \to \Omega^k_X) \simeq \mathrm{Im}(\Omega^k \to \Omega^k_X)$ is X-projective (choose $M = \Omega^k_X$, $G = \mathrm{Im}(\Omega^k \to \Omega^k_X)$!).

Thus we get:

Corollary 2: If $\|A\|\cdot k = k$ and $M: A \to$ k-mod a Mackey-functor, then the following statements are equivalent:
(i) M is X-projective
(ii) $M(X\times Y) \twoheadrightarrow M(Y)$ is surjective for all $Y\ \varepsilon |A|$
(iii) $M(Y) \hookrightarrow M(X\times Y)$ is injective for all $Y\ \varepsilon |A|$.

Especially any subfunctor and any quotient functor of an X-projective Mackey-functor $M: A \to$ k-mod is X-projective.
Proof: (i) ⇒ (ii) ⇔ (iii) is clear. (iii) ⇒ (i) holds, since (iii) implies, that M as an Ω^k-module even is an $\Omega^k/\mathrm{Ke}(\Omega^k \to \Omega^k_X)$-module, which is an X-projective Green-functor. (iii) holds for any subfunctor of M, if it holds for M, (ii) holds for any quotient-functor of M, if it holds for M.
Especially $\mathrm{Im}(N \to N_X)$ and $H^o_X(N)$ are X-projective as subfunctors of N_X for any Mackey-functor $N: A \to$ k-mod and $\mathrm{Im}(N_X \to N)$ and $H^X_o(N)$ are X-projective as quotients of N_X. Also a Green-functor $G: A \to$ k-mod is X-projective, if and only if the image of Ω^k in G is X-projective, which illuminates perhaps a bit the rôle of permutation-representations (the image of Ω^k in G!) in inductiontheory.
Corollary 3 (cf Conlon [4]): Assume $\|A\|\cdot k = k$ and let $M: A \to$ k-mod be a Mackey-

functor.

Let T be a bais of A and define $M^Z =: \mathrm{Im}(M_Z \to M) \cap \bigcap_{Z' \varepsilon T, Z' \neq Z} \mathrm{Ke}(M \to M_{Z'})$

for any $Z \varepsilon T$. Then $M = \bigoplus_{Z \varepsilon T} M^Z$.

M^Z can be characterized as the largest Z-projective subfunctor of M, all of whose Z'-projective subfunctors are zero for $Z' \prec Z (Z, Z' \varepsilon T)$.

For a Green-functor G: $A \to$ k-mod one has $G = \prod_{Z \varepsilon T} G^Z$ as a direct product of Green-functors.

Proof: By definition of $\|A\|$ and because $\|A\| \cdot k = k$ one has

$\Omega^k(A) = k \otimes \Omega(A) \cong k \otimes \tilde{\Omega}(A) = \prod_{Z \varepsilon T} k.$

Thus one has a set $e_Z (Z \varepsilon T)$ of pairwise orthogonal idempotents in $\Omega^k(A) = \Omega^k(\bullet)$ with $\sum_{Z \varepsilon T} e_Z = 1$. The statements then follow from $M^Z(Y) = e_Z|_Y \cdot M(Y)$ for any $Y \varepsilon |A|$ (i.e. $M^Z = e_Z \cdot M$).

In the rest of this section we want to compute the defectset of $\mathrm{Im}(\Omega^k \to \Omega^k_X)$ without any additional assumption of k and state some important consequences. For this purpose one has to consider primeideals $p \subseteq \Omega(A) = \Omega(\bullet)$.

By Cohen-Seidenberg any $p \subseteq \Omega(A)$ can be lifted to some $\tilde{p} \subseteq \tilde{\Omega}(A) = \prod_{Z \varepsilon T} \mathbb{Z}$ and thus is of the form $p = p(Z,p) = \{x \varepsilon \Omega(A) \mid \varphi_Z(x) \equiv 0 \bmod p\}$ for $p = \mathrm{char}\, \Omega(A)/p$. (0 or a prime).

More explicitly let $Z \varepsilon T$ be a minimal element (w.r.t. $\prec$), such that $Z \notin p$ (since $1 = \bullet \notin p$ such minimal elements always exist!).

Since $Z \times X \cong \varphi_Z(X) \cdot Z + \sum_{Z' \varepsilon T, Z' \precneqq Z} n_{Z'} Z'$ (apply φ_Z to both sides) one gets

$Z \times X \equiv \varphi_Z(X) \cdot Z \bmod p$, thus dividing by $Z \notin p$: $X \equiv \varphi_Z(X) \cdot 1 \bmod p$ and $p = p(Z,p)$ with $p = \mathrm{char}\, \Omega/p$. Moreover we have $Z \prec X$ for all X with $X \notin p$, especially Z is the smallest object in T with $Z \notin p$ and therefore uniquely determined by p. One can also characterize Z as the only element in T with $p = p(Z,p)$ and $\varphi_Z(Z) \not\equiv 0 \bmod p$ ($p = \mathrm{char}\, \Omega/p$), since these two properties at least hold for Z and on the other hand $p = p(T,p)$ and $\varphi_T(T) \not\equiv 0 \bmod p$ for some $T \varepsilon T$ implies $\varphi_Z(T) \equiv \varphi_T(T) \not\equiv 0 \bmod p$ and $\varphi_T(Z) \equiv \varphi_Z(Z) \not\equiv 0 \bmod p$, i.e. $Z \prec T \prec Z$ and therfore $Z = T$.

Thus for any $T \varepsilon T$ and any characteristic p we have a unique element $T_p \varepsilon T$ with $p(T,p) = p(T_p,p)$ and $\varphi_{T_p}(T_p) \not\equiv 0 \bmod p$. Obviously $p(T,p) = p(T',p) \iff T_p = T'_p$

$\iff \varphi_T \equiv \varphi_{T'} \bmod p$ and $T_0 = T$, since $\varphi_T(T) \neq 0$.

Proposition 4.5: For a finite group G, $A = \hat{G}$ and $T = G/U \varepsilon T$ for some subgroup $U \leq G$ one has $T_p = G/V$ with V maximal such that $U \leq V \leq G$ and $v^{p^n} \varepsilon U$ for all $v \varepsilon V$ and an appropriate power p^n of p (e.g. the p-part of $|G|$).

Proof (see also [9] and [16], § 5): Since $v^{p^n} \varepsilon U$ for all $v \varepsilon V$, we have a sequence of subgroups

$U = U_o \overset{p}{\trianglelefteq} U_1 \overset{p}{\trianglelefteq} U_2 \overset{p}{\trianglelefteq} \ldots \overset{p}{\trianglelefteq} U_m = V$ with $U_{\mu-1}$ normal in U_μ with p-power-index $(\mu=1,\ldots,m)$. But this implies $p_{U_{\mu-1}}(S) \equiv p_{U_\mu}(S) \bmod p$ for all G-sets S, thus $p(U,p)=p(U_1,p)=\ldots=p(U_1,p)=p(V,p)$. On the other hand $\varphi_V(G/V)=(N_G(V):V)\not\equiv 0(p)$, since V is maximal with $V^{p^n} \subseteq U$, thus $g^p \in V$ for some $g \in N_G(V)$ implies $g \in V$.

Theorem 3: Let A be a based category with basis T, $X \in |A|$, Ω^k: $A \to$ k-mod the Burnside-functor. Then

$D(\mathrm{Im}(\Omega^k \to \Omega^k_X)) = \{T_p | T \in T,\ T \prec X,\ p\cdot k \neq k\}$ (where p runs through all possible characteristics).

Proof: Let $K^k_X = \mathrm{Ke}(\Omega^k(\bullet) \to \Omega^k(X))$ and $I^k_Y = \mathrm{Im}(\Omega^k(Y) \to \Omega^k(\bullet))$. Then we have to show $\Omega^k(\bullet) = K^k_X + I^k_Y$ if and only if $T_p \prec Y$ for all $T_p \in T$ with $T\prec X$ and $p\cdot k \neq k$.

But $\Omega^k(\bullet) \neq K^k_X + I^k_Y$ if and only if there exists some maximal ideal $m \subseteq \Omega^k(\bullet)$ with $K^k_X + I^k_Y \subseteq m$. Let $p \subseteq \Omega(\bullet)$ be the preimage of m with respect to the canonical map $\Omega(\bullet) \to \Omega^k(\bullet)$ and p=char $\Omega^k(\bullet)/m$=char $\Omega(\bullet)/p$, thus $p\cdot k \neq k$.

Now $K_X \subseteq p$ if and only if $p=p(T,p)$ for some $T\prec X$ (even $K_X = \bigcap_{T\prec X} p(T,0)$, see above) and $I_Y \subseteq p(T,p)$ if and only if $T_p \not\prec Y$. Thus $K^k_X + I^k_Y \neq \Omega^k(\bullet)$ if and only if there exists $T\prec X$ and p with $p\cdot k \neq k$, such that $T_p \not\prec Y$, q.e.d..

Now define X(k) to be the sum of all T_p with $T \in T$, $T\prec X$ and $p\cdot k \neq k$, thus X(k) is a defect-object of $\mathrm{Im}(\Omega^k \to \Omega^k_X)$.

Then we have:

Corollary 1: For any Mackey-functor M: $A \to$ $\mathbb{Z}$-mod we have

$$\|A\|_k \cdot M(\bullet) \subseteq \mathrm{Ke}(M(\bullet) \to M(X)) + \mathrm{Im}(M(X(k)) \to M(\bullet)) \quad (\text{with } \|A\|_k = \prod_{p\cdot k=k} p^{\alpha_p} \text{ if } \|A\| = \prod p^{\alpha_p})$$

Proof: Let $\mathbb{Z}' = \mathbb{Z}[\frac{1}{p} | p\cdot k=k] \subseteq \mathbb{Q}$ and $M' = \mathbb{Z}' \otimes M$. Then Thm 3 implies $M'(\bullet) = \mathrm{Ke}(M'(\bullet) \to M'(X)) + \mathrm{Im}(M'(X(k)) \to M'(\bullet))$, since $X(k) = X(\mathbb{Z}')$. This together with Thm 2 implies the result.

Corollary 2: Let G, G': $A \to$ k-mod be Green-functors with G' X-projective, and θ: $G \to G'$ a homomorphism (natural transformation) of Green-functors, such that $\mathrm{Ke}(\theta_\bullet : G(\bullet) \to G'(\bullet)) \cap \mathrm{Im}(\Omega^k(\bullet) \to G(\bullet)) \subseteq \mathrm{Rad}(G(\bullet))$ (e.g. $k=\mathbb{Z}$, $G' = \mathbb{Q}\otimes G$ and all torsion-elements in $G(\bullet)$ nilpotent), then G is X(k)-projective.

Proof: We have $\Omega^k(\bullet)=K^k_X+I^k_{X(k)}$, thus $1_{\Omega^k(\bullet)} = x+y$ with $x \in K^k_X$, $y \in I^k_{X(k)}$. Applying the canonical map $\Omega^k \to G$ we get $1_{G(\bullet)} = x_1+y_1$ with

$x_1 \in \mathrm{Ke}(G(\bullet) \to G(X)) \cap \mathrm{Im}(\Omega^k(\bullet) \to G(\bullet))$ and $y_1 \in \mathrm{Im}(G(X(k)) \to G(\bullet))$.

But $\mathrm{Ke}(G(\bullet) \to G(X)) \subseteq \mathrm{Ke}(G(\bullet) \to G'(\bullet))$, since G' is X-projective, thus $x_1 \in \mathrm{Rad}(G(\bullet))$ and $y_1 = 1 - x_1$ is a unit in $G(\bullet)$, which implies the surjectivity of $G(X(k)) \twoheadrightarrow G(\bullet)$, i.e. the X(k)-projectivity of G.

I still want to give another application of our despriction of primeideals in $\Omega(\bullet)$:

so let $p = p(T,p) \in \Omega(\bullet)$ be a primeideal.
Since any Mackey-functor M: $A \to$ Z-mod is an Ω-module, thus any $M(X)$ an $\Omega(\bullet)$-module via the canonical ring-homomorphism $\Omega(\bullet) \to \Omega(X)$, we can form the localization $M_p(X)$ and check easily, that this way we get a "localized" Mackey-functor $A \to \mathbb{Z}_p$-mod ($\mathbb{Z}_p = \mathbb{Z}[\frac{1}{q} | q \neq p]$), especially G_p is a Green-functor for any Green-functor G.

Proposition 4.6 (cf. [26], [29]): T_p is a defect-object of Ω_p, thus any M_p is T_p-projective.

Proof: We have $\Omega_p(X) \to \Omega_p(\bullet)$ surjective

$\iff$ there exists $Y \prec X$ with $Y \notin p$

$\iff$ there exists $Y \prec X$ with $\varphi_{T_p}(Y) \notin O(p)$

$\iff$ $T_p \prec X$, q.e.d.

§5 Mackey-functors and G-functors

In this section I want to discuss the relations of the above theory and J.A. Green's axiomatic representationtheory as given in [23]. So let G be a finite group and $A = \hat{G}$ the category of (left finite) G-sets. In [23] Green defines the subgroup-category $\delta(G)$ of G, whose objects are just the subgroups H,F,... with morphisms $Hom_{\delta(G)}(H,F)=\{(H,g,F)\,|\,g\in G, g^{-1}Hg\subseteq F\}$.

One has a natural functor $\eta: \delta(G) \to \hat{G}: H \mapsto G/H, (H,g,F) \mapsto (\eta_g: G/H \to G/F$ with $\eta_g(x\cdot H)=x\cdot g\cdot F$ (which is well defined if $g^{-1}Hg\subseteq F$!).

Now let $M: \hat{G} \to$ k-mod be a Mackey-functor and consider $M\circ\eta: \delta(G) \to$ k-mod. One checks easily, that $M\circ\eta$ satisfies the axioms G1 - G4 in [23], p44 (with $R = M_*\circ\eta$ and $T = M^*\circ\eta$), thus any Mackey-functor M determines a G-functor "with zero multiplication".

We note, that M is uniquely determined by $M\circ\eta$, since any G-set S is a disjointunion of transitive G-sets of type G/H: $S\simeq \dot{\bigcup}_{i=1}^{n} G/H_i$ and thus $M(S) = \bigoplus_{i=1}^{n} M(G/H_i) = M\circ\eta(H_i)$, and any map $\dot{\bigcup}_{i=1}^{n} G/H_i \to \dot{\bigcup}_{j=1}^{m} G/F_j$ uniquely composed out of maps $\eta_{g_i}: G/H_i \mapsto G/F_{j(i)} \subseteq \dot{\bigcup}_{j=1}^{m} G/F_j$, thus $M(\dot{\bigcup}_{i=1}^{n} G/H_i) \overset{\leftarrow}{\to} M(\dot{\bigcup}_{j=1}^{m} G/F_j)$ uniquely determined by $M_*\circ\eta(H_i,g_i,F_{j(i)})$ and $M^*\circ\eta(H_i,g_i,F_{j(i)})$ $(i=1,\dots,n)$.

Now assume M is given together with a pairing $M\times M \to M$ which satisfies (P2) and (P3). Then $M\circ\eta$ can be considered as a functor into "A_k" (the category of k-modules P together with a k-bilinear pairing $P\times P \to P$, see [23], p.43) and (P2) and (P3) just assure the validity of G5, i.e. make $M\circ\eta$ a G-functor in the sense of [23], whereas additionally (P1) assures, that $M\circ\eta$ is a multiplicative G-functor.

This leads to

Proposition 5.1: Restricting Mackey-functors from $\hat{G}$ to $\delta(G)$ via η (resp. Mackey-functors with an inner composition satisfying (P2) and (P3) [and (P1)]) sets up a one-one correspondence between isomorphy-classes of (such) Mackey-functors and G-functors with zero-multiplication (resp. [multiplicative] G-functors).

Proof: One just has to check, that any such G-functor is of the type $M\circ\eta$ for some such Mackey-functor M, which follows easily from the axioms G1 - G4, resp. G5 along the same lines as the fact, that $M\circ\eta$ already determines M.

As an application one gets from Prop.4.4, Prop.4.5, Thm 2 and 3:

Theorem 4: Let G be a finite group, $\mathcal{U}$ a set of subgroups of G and M: $\delta(G) \to$ Z-mod a G-functor.

Then

(A) $|G|\cdot M(G) \subseteq \sum_{U\in\mathcal{U}} Im(M(U) \to M(G)) + \bigcap_{U\in\mathcal{U}} Ke(M(G) \to M(U))$.

(B) If π is a set of primes, $H_\pi U = \{V \leq G \mid \text{ex } N \trianglelefteq V,\ U \in U \text{ and } p \in \pi \text{ with } V/N \text{ a p-group and } N \overset{G}{\lesssim} U\}$ and $|G| = |G|_\pi \cdot |G|_{\pi'}$, the decomposition of $|G|$ into its π- and π'-part, then $|G|_{\pi'} \cdot M(G) \in \sum_{V \in H_\pi U} \operatorname{Im}(M(V) \to M(G)) + \bigcap_{U \in U} \operatorname{Ke}(M(G) \to M(U))$.

There is a similar correspondence between triples of Mackey-functor M, N, L together with a pairing $M \times N \to L$ and G-systems as defined by J.A. Green in [24], §2.
One can also identify Green-functors G: $\hat{G} \to$ k-mod with such multiplicative G-functors $G' = G \circ \eta$ on $\delta(G)$, for which multiplication makes the k-modules $G'(H)$ $(H \leq G)$ into rings (even k-algebras!) with a unit, such that restriction sends units onto units. We call such G-functors also Green-functors, defined on $\delta(G)$.
For any G-functor G': $\delta(G) \to A_k$ with a surjective bilinear pairing $G'(G) \times G'(G) \twoheadrightarrow G'(G)$ J.A. Green has defined its defect-basis as the smallest set $D(G')$ of subgroups of G, which is subconjugately closed (i.e. $gVg^{-1} \leq U$ for some $g \in G$, $V \leq G$, $U \in D(G')$ implies $V \in D(G')$), such that the inductionmap $\sum_{U \in D(G')} G'(U) \to G'(G)$ is surjective. Thus if $G' = G \circ \eta$ for some Green-functor G: $\hat{G} \to$ k-mod, if X is a defect-object of G and $T = \{G/H \mid H \lesssim G\}$ a basis of $\hat{G}$ (modulo isomorphisms), then $D(G') = \{U \leq G \mid X^U \neq \emptyset\}$ (with $X^U = \{x \in X \mid u \cdot x = x \text{ for all } u \in U\}$) $= \{U \leq G \mid G/U \in D(G)\}$, $D(G) = \{G/U \mid U \in D(G')\}$ and G is Y-projective for some $Y \in |\hat{G}|$ if and only if $Y^U \neq \emptyset$ for all $U \in D(G')$.
Thus as an application of the results of §4 we get:

Proposition 5.2: Let G': $\delta G \to$ $\mathbb{Z}$-mod be a Green-functor and assume

(i) all torsionelements in $G'(G)$ are nilpotent (e.g. $G'(G)$ is torsionfree!).

(ii) The product of the restriction-maps $\mathbb{Q} \otimes G'(G) \to \prod_{C \leq G, C \text{ cyclic}} \mathbb{Q} \otimes G'(C)$ is injective.

Then the defect-set of G' is contained in the set of hyperelementary subgroups, i.e. subgroups H with a cyclic normal subgroup $C \trianglelefteq H$ and H/C a p-group for some p.
More generally if π is a set of primes, $\mathbb{Z}_\pi = \mathbb{Z}[\frac{1}{q} \mid q \notin \pi]$ and if

(i)' all π-torsionelements in $G'(G)$ are nilpotent,

(ii)' the product of the restriction-maps $\mathbb{Q} \otimes G'(G) \to \prod_{C \in C} \mathbb{Q} \otimes G'(C)$ is injective for some set C of subgroups of G, then the defect-set of $\mathbb{Z}_\pi \otimes G'$ is contained in $H_\pi C = \{H \leq G \mid \text{ex. } N \trianglelefteq H,\ p \in \pi \text{ and } C \in C \text{ with } H/N \text{ a p-group and } N \overset{G}{\lesssim} C\}$.

Proof: By Cor. 1 to Thm. 2 the defect-set of $\mathbb{Q} \otimes G'$ is contained in $C = \{C' \leq G \mid \text{ex. } C \in C \text{ with } C' \overset{G}{\lesssim} C\}$, thus by Cor. 2 to Thm. 3 and by Prop. 4.5 $\mathbb{Z}_\pi \otimes G'$ has a defect-set contained in $H_\pi C$.
As an application one gets for instance Swan's induction-theorem:
For a commutative ring Λ let $X(G,\Lambda)$ be the Grothendieckring of finitely generated

Λ-projective ΛG-modules with respect to exact sequences. Then restriction and induction of modules defines a Green-functor-structure on $X(-,\Lambda)$: $\delta G \to \underline{\mathbb{Z}\text{-mod}}$ and one has $\mathcal{D}(\mathbb{Q}\otimes X(-,\Lambda)) \subseteq \{C \le G | C \text{ cyclic}\}$, $\mathcal{D}(X(-,\Lambda)) \subseteq \{H \le G | H \text{ hyperelementary}\}$.
Proof: Since $X(-,\Lambda)$ is an $X(-,\mathbb{Z})$-module one may assume w.l.o.g. $\Lambda = \mathbb{Z}$.
But then all torsion-elements in $X(G,\mathbb{Z})$ are nilpotent (see [35],) and $\mathbb{Q}\otimes X(G,\mathbb{Z}) \simeq \mathbb{Q}\otimes X(G,\mathbb{Q})$ (see [35]), which maps injectively into $\prod_{C\le G, C \text{ cyclic}} \mathbb{Q}\otimes X(C,\mathbb{Q})$,

since a $\mathbb{Q}G$-module is determined by its character, thus a fortiori by its restriction to cyclic subgroups. (Later we will come along still another proof of this last fact, which doesn't even use character-theory).
Using Thm 4 we can also get the wellknown more precise statements on the cokernel of the induction map $\sum_{C\le G,\ C \text{ cyclic}} X(C,\Lambda) \to X(G,\Lambda)$: if Λ is a field, injectivity of the

restriction maps $X(G,\Lambda) \to \prod_{C\le G, C \text{ cyclic}} X(G,\Lambda)$ together with Thm 4, (A) immediately

implies Artin's Inductiontheorem $|G|\cdot X(G,\Lambda) \subseteq \text{Im}(\sum_{C\le G, C \text{ cyclic}} X(C,\Lambda) \to X(G,\Lambda))$.

In general we may as well restrict again to $\Lambda=\mathbb{Z}$, in which case we even know, that any two torsion-elements in $X(G,\mathbb{Z})$ annihilate each other (see [35], §11). Since $n\cdot 1 \in \text{Im}(\sum_{C\le G, C \text{ cyclic}} X(C,\mathbb{Z}) \to X(G,\mathbb{Z}))$ for some $n \in \mathbb{N}$, we know that any element in

$\bigcap_{C\le G, C \text{ cyclic}} \text{Ke}(X(G,\mathbb{Z}) \to X(C,\mathbb{Z}))$ is a torsion-element (annihilated by n).

By Thm 4 we have $|G|\cdot 1 = x+y$ with $x \in \text{Im}(\sum_{\dots} X(C,\mathbb{Z}) \to X(G,\mathbb{Z})) = I$ and $y \in \bigcap_{\dots} \text{Ke}(X(G,Z) \to X(C,Z)) = K$.

Thus we get at first:
$|G|^2\cdot 1 = (x+y)\ (x+y) = x^2+2xy \in I$ (since $y^2=0$), which is due to Swan.
Moreover we get, that any torsionelement $z \in X(G,\mathbb{Z})$ is annihilated by $|G|\cdot$ g.c.m.{order of $z_{|C}$ in $X(C,\mathbb{Z}) | C \le G, C$ cyclic}, not only by $|G|^2$ g.c.m. {...} as would follow just from Swan's result. Especially if z is a virtual permutation-representation, i.e. in the image of $\Omega(G) \to X(G,\mathbb{Z})$, we have $|G|\cdot z=0$.
For G abelian I can show that even $z=0$ then holds; for arbitrary G its seems to be an interesting question as to wether or not the image $\Omega(G)$ in $X(G,\mathbb{Z})$ contains torsion-elements.
With similar arguments one can show, that any element t in the projective class-group $C_0(G,\mathbb{Z})$ is annihilated by $|G|\cdot$g.c.m. {order of $t_{|C}$ in $C_0(C,\mathbb{Z}) | C \le G, C$ cyclic}. Moreover one always can replace $|G|$ by the Artinindex $A(G)$ of G as defined by T.Y. Lam in [29] in these considerations.
To indicate just one further application let $\Lambda=\mathbb{F}$, a field of characteristic $p \neq 0$. We know by Brauer, that $X(G,\mathbb{F})$ is torsion-free and that $X(G,\mathbb{F}) \to \prod_{C\in C_{p'}} X(C,\mathbb{F})$ with

C_p, the set of p-regular cyclic subgroups is injective, thus the inductionmap $\sum_{H \in H_p, C_p} \mathbb{Z}[\frac{1}{p}] \otimes X(H,\mathbb{F}) \to \mathbb{Z}[\frac{1}{p}] \otimes X(G,\mathbb{F})$ is surjective. But since $|H|$ is prime to p for $H \in H_p, C_p$, the image of the inductionmap $X(H,\mathbb{F}) \to X(G,\mathbb{F})$ is contained in the ideal of $\mathbb{F}G$-projective modules (the image of the Cartan-map) thus the above formula implies, that the Cartan-map has a p-torsion-cokernel.

Now let $G: \hat{G} \to \underline{k\text{-mod}}$ be an X-projective Green-functor and $M: \hat{G} \to \underline{\mathbf{k}\text{-mod}}$ a G-module. Putting $D = \{H \leq G \mid X^H \neq \emptyset\}$ ($\supseteq D(G')$) we know that restriction maps $M(\bullet) = M \circ \eta(G)$ injectively into $\prod_{H \in D} M \circ \eta(H) = M(\dot{\bigcup}_{H \in D} G/H)$ and that the image is precisely the differencekernel of the two maps

$$M(\dot{\bigcup}_{H \in D} G/H) \rightrightarrows M(\dot{\bigcup}_{H \in D} G/H \times \dot{\bigcup}_{H \in D} G/H)$$ defined by the two projections. In the terminology of G-functors this is equivalent to

$$M \circ \eta(G) = \{(x_H)_{H \in D} \in \prod_{H \in D} M \circ \eta(H) \mid \eta_{g^*}(x_{H_1}) = x_{H_2} \text{ whenever } g^{-1} H_2\, g \subseteq H_1\} = \varprojlim_{D} M \circ \eta$$

where D stands for the full subcategory of δG with objects just in D.

As an example let us consider $G = A_4$, the alternating group on 4 elements, with subgroups $V_4 \trianglelefteq A_4$, the Klein-four-group, $A_3 \leq A_4$ and $E \leq A_4$.

If $M: \hat{G} \to \underline{k\text{-mod}}$ is $(G/V_4 \cup G/A_3)$ projective, then we have a pull-back of restriction-maps

$$\begin{array}{ccc} M \circ \eta(A_4) & \to & M \circ \eta(A_3) \\ \downarrow & & \downarrow \\ M \circ \eta(V_4)^{A_3} & \to & M \circ \eta(E), \end{array}$$

i.e. the value of $M \circ \eta$ on A_4 is completely determined by the behaviour of $M \circ \eta$ on its proper subgroups.

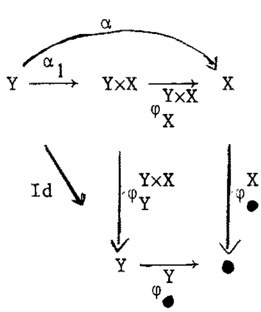
I want to point out, that this way - using not only an axiomatic formulation of the Frobenius-reciprocity-law (as T.Y. Lam did), but also of the Mackey-subgroup-theorem (as already done by J.A. Green) as well - we do not only get "upper bounds", i.e. conclusions like "$M \circ \eta(G)$ is zero or finite or finitely generated, if all $M \circ \eta(H)$, $H \in D$ are so", but we get an explicit description of $M \circ \eta(G)$ in terms of the $M \circ \eta(H)$, $H \in D$ and the way, the subgroups in D are imbedded into G. In some way this generalizes Brauer's characterization of generalized characters by their restrictions to elementary subgroups. Thus our theory can be used for instance for the explicit calculation of the Whiteheadgroup or some Wallgroups of a finite group G, once these groups are known for all hyperelementary subgroups of G together with the way, they restrict to each other, and the way, G acts on them by conjugation.

Let us just remark, that there is still another way to apply our techniques: if M is a covariant functor on the category of commutative rings (or any appropriate subcategory) into the category of abelian groups (or any abelian category), it may sometimes be possible to extend this functor to a bi-functor, defined on some subcategory (e.g. étale R-algebras with étale morphisms) by using some kind of norm- or trace-construction. Generally such a functor then turns out to be a Mackey-functor (on the dual category of affine spectrums, of course!) and proving it to be R_1-projective for some R-algebra R_1 can lead to rather interesting results on M, for instance its Galois-(or Amitsur-)cohomology. E.g. see [46], App A & B and [14] for the case of Wittrings.

Finally let us shortly discuss the transfer-theorem of Green (cf [23], p 61). This can be done even in the context of pre-Mackey-functors: So let A be a categroy with finite products and pull-backs and $M: A \to B$ a pre-Mackey-functor into an abelian category B.

By Lemma 2.1 we have for any injective morphism $\alpha: Y \to X$ in A the formula $\alpha_* \alpha^* = Id_{M(Y)}$, thus if $\zeta: X \to Y$ is a left-inverse of α(i.e $\zeta\alpha = Id_Y$), we get $\alpha_* \alpha^* = \alpha_* \zeta_* = Id_Y$, i.e. $\alpha^* \equiv \zeta_*$ mod Ke α_*.

Especially if $\alpha: Y \to X$ is any morphism and if we consider $\alpha_1 = Id_Y \times \alpha: Y \to Y \times X$, we get $\alpha_1^* \equiv \varphi_{Y*}^{Y\times X}$ mod Ke α_{1*} (with $\varphi_{T_i}^{T_1\times T_2}: T_1 \times T_2 \to T_i$ the projection onto T_i), thus applying $\varphi_X^{Y\times X\,*}$ we get $\alpha^* = \varphi_X^{Y\times X\,*} \cdot \alpha_1^* \equiv \varphi_X^{Y\times X\,*} \circ \varphi_{Y\,*}^{Y\times X} = \varphi_{\bullet\,*}^{X} \circ \varphi_{\bullet}^{Y\,*}$ mod $\varphi_X^{Y\times X\,*}(\text{Ke } \alpha_{1*})$:

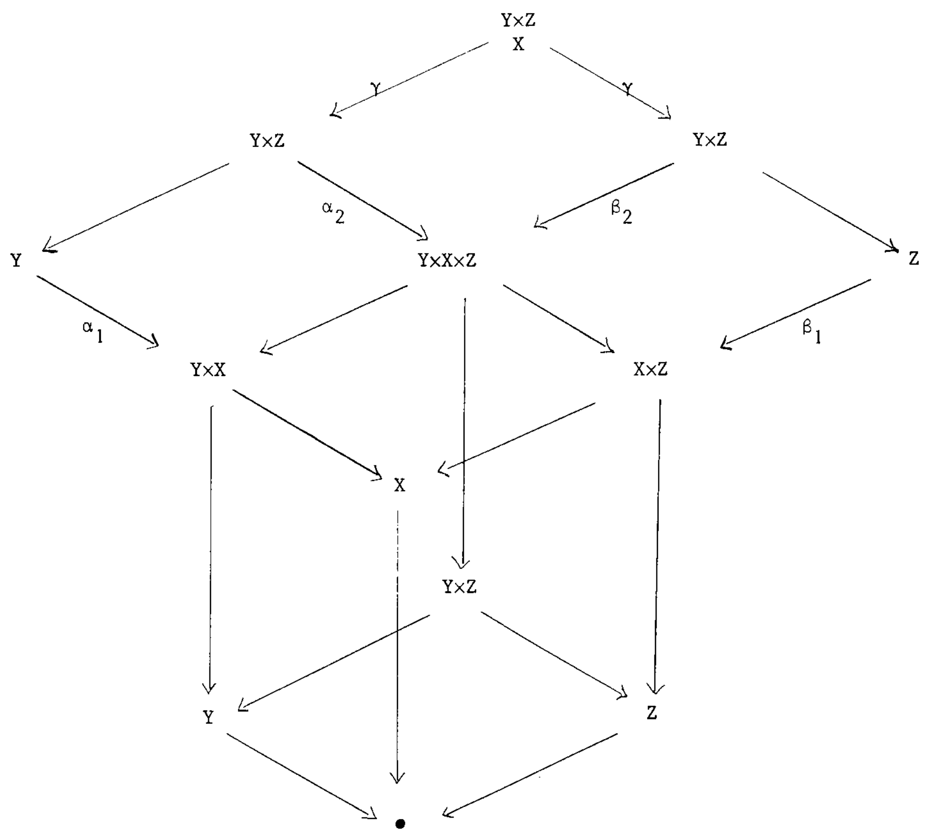


; in other words

we have a commutative diagramm

$$\begin{array}{ccc} \text{Im } \alpha^* \subseteq \text{Im } \varphi_X^{Y\times X\,*} & \subseteq & M(X) \\ \cap & & \downarrow \\ M(X) \longrightarrow M(\bullet) & \longrightarrow & M(X)/\varphi_X^{Y\times X\,*}(\text{Ke } \alpha_{1*}). \end{array}$$

Now let $Z \in |A|$ be a further object with a map $\beta: Z \to X$ and consider the diagramm of pull-backs



We claim

(i) $\varphi^{X}_{\bullet *}(\varphi^{Y\times Z*}_{\bullet}(\mathrm{Ke}\ \gamma_*)) \subseteq \varphi^{Y\times X*}_{X}(\mathrm{Ke}\ \alpha_{1*}) + \varphi^{X\times Z*}_{X}(\mathrm{Ke}\ \beta_{1*})$

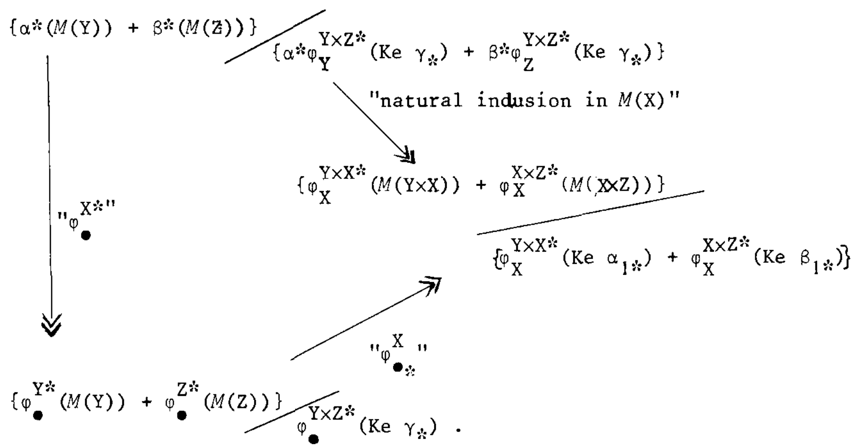
and

(ii) $\alpha^* \varphi^{Y\times Z*}_{Y}(\mathrm{Ke}\ \gamma_*) \subseteq \varphi^{X\times Z*}_{X}(\mathrm{Ke}\ \beta_{1*})$.

Proof: (ii) follows immediately from $\alpha\ \varphi^{Y\times Z}_{Y} = \varphi^{X\times Z}_{X}(\varphi^{Y\times X\times Z}_{X\times Z}\alpha_2)$ and the above pull-back-diagramm,

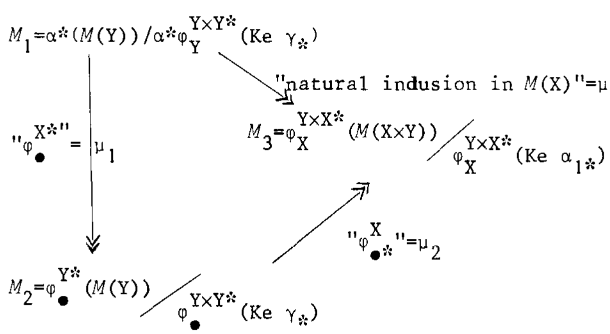
(i) from $\varphi^{X}_{\bullet *}(\varphi^{Y\times Z*}_{\bullet}(\mathrm{Ke}\ \gamma_*)) = \varphi^{Y\times X\times Z*}_{X}\ \varphi^{Y\times X\times Z}_{Y\times Z*}(\mathrm{Ke}\ \gamma_*) \equiv \varphi^{Y\times X\times Z*}_{X}\ \alpha_2^*(\mathrm{Ke}\ \gamma_*)$ modulo $\varphi^{Y\times X\times Z*}_{X}(\mathrm{Ke}\ \alpha_{2*})$.

But $\varphi^{Y\times X\times Z*}_{X}\ \alpha_2^*(\mathrm{Ke}\ \gamma_*) = \alpha^*\ \varphi^{Y\times Z*}_{Y}(\mathrm{Ke}\ \gamma_*) \subseteq \varphi^{X\times Z*}_{X}(\mathrm{Ke}\ \beta_{1*})$ by (ii) and $\varphi^{Y\times X\times Z*}_{X}(\mathrm{Ke}\ \alpha_{2*}) = \varphi^{Y\times X*}_{X}\ \varphi^{Y\times X\times Z*}_{Y\times X}(\mathrm{Ke}\ \alpha_{2*}) \subseteq \varphi^{Y\times X*}_{X}(\mathrm{Ke}\ \alpha_{1*})$. Thus alltogether we get a commutative diagramm of well defined surjective maps:



Let us just note, that the surjectivity of these three maps implies, that all are isomorphisms in case the upper right map is.

Especially for Y=Z, α=β symmetry implies, that in each term the summands conicide, thus one gets the simplified diagramm:



,which for $A=\hat{G}$, X=G/H, Y=G/D with D<H and $G/D=Y \overset{\alpha}{\to} G/H=X$ the natural map $gD \to gH$ and M any Mackey-functor on $\hat{G}$ just is the first part of the transfertheorem of Green. The other parts deal with multiplication, which can always be replaced by pairings $M \times N \to L$ (see also [24], §2). The results then are, that such a pairing induces pairings $M_i \times N_i \to L_i$ of the corresponding terms in the above triangel taken for M, N and L respectively, which are compatibel with the maps in the triangel (i.e. these maps are multiplicative), and that $M_1 \times N_1 \to L_1$ vanishes on Ke $\mu_1 \times N_1$ and $M_1 \times$ Ke ν_1, whereas $M_2 \times N_2 \to L_2$ vanishes on Ke $\mu_2 \times N_2$ and $M_2 \times$ Ke ν_2.

Part II

Representations of finite groups and K_G-theories

§6 Distributive categories

In §4 we have considered categories A satisfying the properties (M1) and (M2) and shown, that the isomorphism-classes of objects in such a category form a commutative half-ring $\Omega^+(A)$, with addition and multiplication in $\Omega^+(A)$ defined by categorical sum and product. If one wants to define something similar for - say - the category $P(\mathbf{k})$ of finitely generated, projective k-modules (k comm. with $1 \in k$ as above) of course one has to replace the categorical product, which in this case coincides with the categorical sum, by the tensorproduct over k, to define multiplication. And in case one wants to consider the category $L(k)$ of k-lattices, i.e.of finitely generated, projective k-modules M together with a nonsingular symmetric bilinear form f: $M \times M \to k$, one has neither categorical sum nor product, but still can define a half-ring-structure on the set of isomorphism-classes of k-lattices using orthogonal sum and tensorproduct.

To handle all three cases at the same time one may define the concept of a distributive category as a category C together with two "compositions", which behave-say-like direct sum and tensorproduct in $P(k)$.

Because later on we will have to take "sum" and "product" of any finite family $(X_i | i \in I)$ of objects in C, indexed by an arbitrary finite set I, it seems appropiate to define such a "composition" as a covariant functor Σ, (resp. Π) from the category $F(C)$ of finite families $(X_i | i \in I)$ of objects in C (with morphisms $(X_i | i \in I) \to (Y_j | j \in J)$) pairs consisting of a bijective map $\mu: I \to J$ and a family $(\varphi_i: X_i \to Y_{\mu(i)} | i \in I)$ of morphisms in C and obvious compositions) back into C, such that in case I contains exactly one element, e.g. $I=\{i_o\}$, $X_{i_o}=X$ one has $\Sigma(X_i | i \in I)=X$ independently of I, i.e. for $\mu: I \to J, i_o \mapsto j_o$ and $\varphi_{i_o}=Id_X: X_{i_o}=X \to X_{j_o}=X$ one has $\Sigma(\mu,(\varphi_i | i \in I))=Id_X: X \to X$.

Associativity then can be expressed as saying, that one has a natural equivalence between the two functors from $F(F(C))$ into C, defined by $F(F(C)) \to F(C) \xrightarrow{\Sigma} C$: $((X_i | i \in I_j) | j \in J) \mapsto (X_i | i \in \bigcup_{j \in J} I_j = L) \mapsto \Sigma(X_i | i \in L)$ and

$$F(F(C)) \xrightarrow{F\Sigma} F(C) \xrightarrow{\Sigma} C: ((X_i | i \in I_j) | j \in J) \mapsto (\Sigma(X_i | i \in I_j) | j \in J) \mapsto \Sigma(\Sigma(X_i | i \in I_j) | j \in J)$$

Associativity especially implies, that for $X_o=\Sigma(X_i | i \in \emptyset)$ one has a natural isomorphism $\Sigma(X_o,X) \simeq \Sigma(X,X_o) \simeq X$ $(X \in |C|)$ (with $\Sigma(X,Y)=:\Sigma(X_i | i \in I)$ with $I=\{1,2\}$, $X_1=X$, $X_2=Y$!), i.e. X_o is a "natural object" w.r.t. Σ.

Now we define a category C or rather a category C together with two associative compositions Σ,Π: $F(C) \to C$ to be distributive, if we have a functorial isomorphism $\Pi(\Sigma(X,Y),Z) \simeq \Sigma(\Pi(X,Z),\Pi(Y,Z))$.

Here "functoriality" shall mean to imply, that for any finite family $(X_i | i \in I)$ and any map μ: $I \to J$ (J finite set, μ not necessarily bijective) we have a natural isomorphism $\Pi(\Sigma(X_i | i \in \mu^{-1}(j)) | j \in J) \simeq \Sigma(\Pi(X_{\gamma(j)} | j \in J) | \gamma \in \Gamma)$ with Γ the (possibly empty!) set of sections (i.e. right-inverses) $\gamma: J \to I$ of $\mu: I \to J$.

Of course any category A with (M1) and (M2) as well as $P(k)$ or $L(k)$ are distributive as explained above. Moreover if C is distributive and A any small category, then the category of (covariant) functors from A to C is distributive as well. All our examples arise essentially that way from the above three cases, thus a reader who (as myself) does not like the above rather abstract and involved definitions might just restrict himself to those cases.
Anyway we can associate to any small category C with just one associative composition Σ its "Grothendieckgroup" $K(C) = K(C,\Sigma)$: the universal abelian group associated with the abelian semigroup $K^+(C)$ of isomorphism-classes $[X]$ of objects X in C with addition defined by Σ(i.e. $[X]+[Y]=: [\Sigma(X,Y)]$).

If moreover there exists a second associative composition Π on C, such C with Σ and Π becomes a distributive category, then we can use Π to define a multiplication on $K(C)$ by $[X]\cdot[Y]=: [\Pi(X,Y)]$, such that $K(C)=K(C,\Sigma,\Pi)$ becomes a commutative ring with a unit (represented by $X_1=\Pi(X_i | i \in \emptyset)$!).

§7 Construction of K_G-theories.

Now let G be a finite group and S a finite G-set. To S we associate the category $\underline{\underline{S}}$, whose objects are precisely the element in S with morphisms
$[s,s']_{\underline{\underline{S}}} = \{(g,s,s') | g \in G,\ gs=s'\}$ $(s,s' \in S)$ and obvious composition of morphisms,

e.g. $\underline{\underline{\bullet}} = \underline{G/G}$ is just the category usually associated with the group G. Now let C be a small category at first with just one composition Σ and consider the category $[\underline{\underline{S}},C]$ of covariant functors from $\underline{\underline{S}}$ to C. An object $\zeta \in |[\underline{\underline{S}},C]|$ will also be called a "G-equivariant C-bundle over S", since it associates to any $s \in S = |\underline{\underline{S}}|$ the fiber $\zeta(s) = \zeta_s \in |C|$ and to any $g \in G$ a morphism $\zeta_s \to \zeta_{gs}$ with compositions compatibel with the group-structure.
Especially for $S = \bullet = G/G$ the category $[\underline{G/G}, C]$ is just the category of "G-objects in C".
For any G-map $\varphi: S \to T$ between finite G-sets we have obviously an associated functor $\underline{\underline{\varphi}}: \underline{\underline{S}} \to \underline{\underline{T}}$ and thus a functor $\varphi_*: [\underline{\underline{T}},C] \to [\underline{\underline{S}},C]$, defined by $\zeta \mapsto \zeta\underline{\underline{\varphi}}$. Moreover we can also define a functor $\varphi^*: [\underline{\underline{S}},C] \to [\underline{\underline{T}},C]$, which maps any G-equivariant C-bundle ζ over S onto the C-bundle $\varphi^*(\zeta) = \zeta'$ over T with fibers $\zeta'_t = \Sigma(\zeta_s | s \in \varphi^{-1}(t))$ $(t \in T)$ and correspondingly defined G-actions and so on. (In other words: $\varphi^*: [\underline{\underline{S}},C] \to [\underline{\underline{T}},C]$ is defined as the composition of $[\underline{\underline{S}},C] \to [\underline{\underline{T}},F(C)]: \zeta \mapsto (\zeta_s | s \in \varphi^{-1}(t))_{t \in T}$ and the functor $[\underline{\underline{T}},F(C)] \to [\underline{\underline{T}},C]$, induced by Σ. It is easily checked, that this way one defines something like a Mackey-functor on $\hat{G}$, the category of finite G-sets, with values in the "category of categories with an associative composition", especially φ_* and φ^* commute (the latter one at least up to canonical isomorphisms) with the associative composition defined on $[\underline{\underline{S}},C]$ and $[\underline{\underline{T}},C]$ by Σ.
Thus taking Grothedieckgroups we get a Mackey-functor
$K_G(-,C): \hat{G} \to \underline{\underline{Z\text{-mod}}}: S \mapsto K_G(S,C) =: K([\underline{\underline{S}},C])$ which defines K_G-theory on $\hat{G}$ with C-coefficients.
If moreover C is distributive with respect to Σ and a further associative composition Π, then Π induces a multiplicative structure, which makes $K_G(S,C)$ to a commutative ring with a unit and $K_G(-,C)$ to a Green-functor. (Proofs for these facts are straight-forward and left to the reader).
Now let H be another finite group and $\theta: H \to G$ a group-homomorphism. Restricting the action of G on a G-set S, resp. on a G-equivariant C-bundle ζ over S to H via θ defines a functor $\hat{\theta}: \hat{G} \to \hat{H}: S \mapsto S_{|H}$, resp. a natural transformation of Green-functors from $K_G: \hat{G} \to \underline{\underline{Z\text{-mod}}}$ to $K_H \circ \hat{\theta}: \hat{G} \to \hat{H} \to \underline{\underline{Z\text{-mod}}}$.
Especially if $H \leq G$, T an H-set and $G \underset{H}{\times} T$ the induced G-set (defined as set of H-orbits (g,t) in $G \times T$ w.r.t. the H-action $h(g,t) = (gh^{-1},ht)$, $h \in H$, $g \in G$, $t \in T$), we get a homomorphism $K_G(G \underset{H}{\times} T, C) \to K_H(G \underset{H}{\times} T|_H, C) \to K_H(T,C)$, where the second map is defined by the H-map $T \to G \underset{H}{\times} T$: $t \to (e,t)$ (e the trivial element in G); e.g. for $T = H/U$ for some

$U \leq H$ we have $G \underset{H}{\times} T = G/U$ and the above homomorphism is just the obvious map

$K_G(G/U,C) \to K_H(H/U,C)$, defined by restricting a C-bundle over G/U to H/U and the action of G to H at the same time, i.e. by the obvious functor $\underline{\underline{H/U}} \to \underline{\underline{G/U}}$

Lemma 7.1: The above homomorphism $K_G(G \underset{H}{\times} T,C) \to K_H(T,C)$ is an isomorphism.

Proof: W.l.o.g. we may restrict to $T=H/U$ a transitive G-set and because of the commutative triangel

$$\begin{array}{ccc} & & K_H(H/U,C) \\ & \nearrow & \downarrow \\ K_G(G/U,C) & & \\ & \searrow & \\ & & K_U(U/U,C) \end{array}$$

even to $H = U$. But in this case it is obvious, that $\underline{\underline{U/U}} \to \underline{\underline{G/U}}$ is an equivalence of categories (any object gU in $\underline{\underline{G/U}}$ is isomorphic to $U \in \mathrm{Im}(|\underline{\underline{U/U}}| \to |\underline{\underline{G/U}}|)$, which has the same endo-(-auto-)morphisms in $\underline{\underline{U/U}}$ and $\underline{\underline{G/U}}$!), thus $[\underline{\underline{G/U}},C] \to [\underline{\underline{U/U}},C]$ is an equivalence of categories.

Remark: of course $\underline{\underline{T}} \to \underline{\underline{G \underset{H}{\times} T}}$ is always an equivalence of categories, thus

$|\underline{\underline{G \underset{H}{\times} T}},C| \xrightarrow{\sim} |\underline{\underline{T}},C|$ as well for arbitrary H-sets T. Especially for C the category of finite sets one can identify on the one hand $[\underline{\underline{S}},C]$ (S a G-set) with the category $\hat{G}/S$ of G-sets over S, on the other hand for $S=G/U$ one has a natural equivalence of $[\underline{\underline{G/U}},C]$ with $[\underline{\underline{U/U}},C] \cong \hat{U}$, thus we have also a natural equivalence between the category of G-sets over G/U and the category of U-sets.

One may formalize the above considerations by introducing the concept of a universal family of (Mackey - or) Green-functors as a family of Green-functors $\mathcal{G}_G : \hat{G} \to \underline{\underline{k\text{-mod}}}$, one for each finite group G, together with natural transformations of Green-functors: $\Theta_\theta : \mathcal{G}_G \to \mathcal{G}_H\hat{\theta}$, one for any grouphomomorphism $\theta: H \to G$, such that $\Theta_{Id}=Id$, $\Theta_{\theta_1\theta_2} = \Theta_{\theta_1} \circ (\Theta_{\theta_2}\hat{\theta}_1)$:

$$\begin{array}{ccc} \mathcal{G}_G & \longrightarrow & \mathcal{G}_U\,\widehat{\theta_1\theta_2} = \mathcal{G}_U\hat{\theta}_2\hat{\theta}_1 \\ & \searrow & \nearrow \\ & \mathcal{G}_H\hat{\theta}_1 & \end{array}$$

for any $\theta_1: H \to G$, $\theta_2: U \to H$ and $\mathcal{G}_G(G/U) \xrightarrow{\Theta_{\iota_U}} \mathcal{G}_U(G/U|_U) \to \mathcal{G}_U(U/U)$ an isomorphism for any imbedding $\iota_U: U \to G$.

In other words $\mathcal{G}$ is determined by its values $\mathcal{G}(U) = \mathcal{G}_U(U/U)$ together with the maps $\mathcal{G}_*(\theta): \mathcal{G}(G) \to \mathcal{G}(H)$, defined for any $\theta: H \to G$, and the maps $\mathcal{G}^*(\iota_U): \mathcal{G}(U) \to \mathcal{G}(G)$ defined for any injective homomorphism $\iota_U: U \to G$, which are such that $\mathcal{G}$ restricted to the subgroupcategory δG of any finite group G becomes a Green-functor on δG.

It should be remarked , that whereas the second description might be simpler to work with the first one is generally more easily verified, as in the case of K_G-theories. Anyway we have

<u>Proposition 7.1:</u> Any small distributive category C defines a universal family of Green-functors $K_G(-,C): \hat{G} \to \underline{\underline{Z\text{-mod}}}$, such that $K_G(G/G,C) =: K(G,C) = K([\underline{\underline{G/G}},C])$ is the Grothendieckring of G-objects in C.

§8 Defect-groups of K_G-functors.

Again let C be a small distributive category. We want to determine the defect-basis of the associated Green-functors $K_G(-,C)$. Of course this will be impossible without additional assumptions on C. But still we can prove a general result on these defect bases, which will be rather helpfull in the explicit determination for various categories C later on.

At first we have

Proposition 8.1: Let $\mathcal{G}$ be a universal family of Green-functors with values in k-mod (as defined in §7). Define $D'(\mathcal{G})$ to be the class of all finite groups H, such that H/H is contained in the defect-set of $\mathcal{G}_H$(i.e. such that $\sum_{U \subsetneq H} \mathcal{G}_H(H/U) \to \mathcal{G}_H(H/H)$ is not surjective, resp. such that $\mathcal{G}_H(S) \to \mathcal{G}_H(\bullet)$ is surjective if and only if $S^H=\{s\in S \mid hs=s \text{ for all } h \in H\} \neq \emptyset$).

Then (i) $D(\mathcal{G}_G) = \{G/U \mid \text{ex.} H\in D'(\mathcal{G}) \text{ with } U \overset{G}{\lesssim} H<G\}$, i.e. $D(\mathcal{G}_G')=\{U<G \mid \text{ex.} H\in D'(\mathcal{G}) \text{ with } U \overset{G}{\lesssim} H<G\}$.

(ii) $D'(\mathcal{G})$ is closed with respect to epimorphic images, i.e. if $\theta\colon H \twoheadrightarrow H'$ is surjective and $H \in D'(\mathcal{G})$, then $H' \in D'(\mathcal{G})$.

Proof: (i) To show $D'(\mathcal{G}_G) \subseteq \{U \leq G \mid \text{ex. } H \in D'(\mathcal{G}) \text{ with } U \overset{G}{\lesssim} H \leq G\}$, i.e. $\sum_{H\leq G, H\in D'(\mathcal{G})} \mathcal{G}(H) = \sum_{H\leq G, H\in D'(\mathcal{G})} \mathcal{G}_G(G/H) \to \mathcal{G}_G(G/G) = \mathcal{G}(G)$ surjective, we use induction w.r.t. $|G|$: For $|G|=1$ or more generally for $G \in D'(\mathcal{G})$ surjectivity obviously holds. For $G \notin D'(\mathcal{G})$ one has by definition of $D'(\mathcal{G})$ a surjective map $\sum_{U \subsetneq G} \mathcal{G}(U) \twoheadrightarrow \mathcal{G}(G)$ and for $U \subsetneq G$, thus $|U| < |G|$ one has $\sum_{H\leq U, H\in D'(\mathcal{G})} \mathcal{G}(H) \twoheadrightarrow \mathcal{G}(U)$, thus we get $\sum_{U \subsetneq G} \sum_{H\leq U, H\in D'(\mathcal{G})} \mathcal{G}(H) \twoheadrightarrow \mathcal{G}(G)$ which implies $\sum_{H\leq G, H\in D'(\mathcal{G})} \mathcal{G}(H) = \sum_{H\leq G, H\in D'(G)} \mathcal{G}_G(G/H)$

$$\twoheadrightarrow \mathcal{G}_G(G/G) = \mathcal{G}(G).$$

On the other hand, if $\sum_{V\in D} \mathcal{G}_G(G/V) \twoheadrightarrow \mathcal{G}_G(G/G)$ is surjective for some set D of subgroups of G, we have to show, that for any $H \leq G$ with $H \in D'(\mathcal{G})$ there exists $V \in D$ with $H \overset{G}{\lesssim} V$, i.e. $G/V^H \neq \emptyset$.

But restricting the above formula to H via θ_{ι_H} we get a diagramm

$$\begin{array}{ccc}
\sum_{V\in D} \mathcal{G}_G(G/V) = \mathcal{G}_G\left(\bigcup_{V\in D} G/V\right) & \longrightarrow & \mathcal{G}_G(G/G) \\
\downarrow & & \downarrow \\
\mathcal{G}_H\left(\bigcup_{V\in D} G/V|_H\right) & \longrightarrow & \mathcal{G}_H(H/H)
\end{array}$$

Since θ_{ι_H} maps the unit 1_G in $\mathcal{G}_G(G/G)$ onto the unit 1_H in $\mathcal{G}_H(H/H)$ and the upper arrow is surjective, we see, that 1_H is contained in the image of the lower arrow,

which on the other hand is an ideal, thus the lower arrow is surjective.
By definition of $D'(G)$ and because $H \in D'(G)$ this implies $(\bigcap_{V \in D} G/V)^H \neq \emptyset$, q.e.d.

(ii) For any $\theta: H \to H'$ and any H'-set S consider the diagramm

$$\begin{array}{ccc} G_{H'}(S) & \longrightarrow & G_{H'}(H'/H') \\ \downarrow \theta_\theta & & \downarrow \theta_\theta \\ G_H(S|_H) & \longrightarrow & G_H(H/H) \end{array}$$

Again surjectivity of the upper arrow implies surjectivity of the lower arrow. Thus if $H \in D'(G)$ and θ surjective we get: $G_{H'}(S) \twoheadrightarrow G_{H'}(H'/H') \Longrightarrow G_H(S|_H \twoheadrightarrow G_H(H/H))$ $\Rightarrow (S|_H)^H \neq \emptyset \Rightarrow S^{H'} \neq \emptyset$, since $S^{H'} = (S|_H)^H$ by the surjectivity of θ.

We now define a universal family of Green-functors G to be saturated, if $D'(G)$ is also closed with respect to subgroups.

In this case the first part of Prop. 8.1. can be written even in the form $D(G_G') = \{H \leq G | H \in D'(G)\}$, but what is more important: whenever we have an explicit inductiontheorem for one particular group G we immediately get induction theorems for all groups G' which contain G as a "section" (i.e. $G \cong V/U$ for some $U \trianglelefteq V \leq G'$), e.g. if we can exhibit for $G = V_4$ (the Klein 4-group) elements $x_U \in G_G(G/U)$ for any $U \lneqq V_4 = G$ such that the sum of the induced elements

$\sum_{U \lneqq V_4} x_U^{G/U \to G/G} = 1$, then we have an induction-theorem for any group with a non-cyclic 2-Sylow-subgroup.

Unfortunately universal families of Green-functors are not nessarily saturated. Thus it is worthwhile to realize, that we still have:

<u>Theorem 5:</u> Let $C = (C, \Sigma, \Pi)$ be a distributive category and $K_G(-,C)$ the associated universal Green-functor. Then $k \otimes K_G(-,C)$ is saturated for any k.

We write $D_k(C)$ for $D'(k \otimes K(-,C))$.

Proof: For any universal Green-functor G define $\overline{G(G)} = G(G)/\mathrm{Im}(\sum_{U \lneqq G} G(U) \to G(G))$, thus $\overline{G(G)} \neq 0 \Longleftrightarrow G \in D'(G)$. Now consider $G = K(-,C)$. We have to show $\overline{k \otimes G(H)} = k \otimes \overline{G(H)} \neq 0 \Rightarrow \overline{k \otimes G(G)} \neq 0$ whenever $H \leq G$ and for that purpose it is enough to construct a ringhomomorphism $\overline{G(H)} \to \overline{G(G)}$.

At first let us interpret $G(H) = K_H(H/H,C)$ as $K_G(G/H,C) = K([\underline{G/H},C])$.

To the map $\varphi: G/H \to G/G$ we have associated already two functors: $\varphi_*: [\underline{G/G},C] \to [\underline{G/H},C]$ and $\varphi^* = \varphi^*_\Sigma: [\underline{G/H},C] \to [\underline{G/G},C]$, for the second one using the composition Σ in C. Thus we can as well define another functor $\varphi^*_\Pi: [\underline{G/H},C] \to [\underline{G/G},C]$, which associates to any G-equivariant C-bundle ζ over G/H the G-object (i.e. G-equivariant C-bundle over G/G) $\Pi(\zeta) = \Pi(\zeta_x | x \in G/H)$ (note that ζ can be considered as a G-object in $F(C)$, that $\Pi(\zeta)$ is a G-object in C).

This functor defines a Π-multiplicative map from isomorphy-classes in $[\underline{G/H},C]$ into isomorphy-classes in $[\underline{G/G},C]$, thus we get a diagramm

$$\begin{array}{ccc}
K_H^+(H/H,C) = K_G^+(G/H,C) & \longrightarrow & K_G^+(G/G,C) \\
\downarrow & & \downarrow \\
K_H(H/H,C) & & K_G(G/G,C) \\
\downarrow & & \downarrow \\
\overline{K_H(H/H,C)} & \cdots\to & \overline{K_G(G/G,C)} .
\end{array}$$

Our claim now is, that the lower arrow $\cdots\to$ exists as a ringhomomorphism. This follows obviously from

Lemma 8.1: (a) For any two bundles ζ^1 and ζ^2 over G/H we have
$\Pi(\Sigma(\zeta^1,\zeta^2)) \equiv \Sigma(\Pi(\zeta^1),\Pi(\zeta^2))$ modulo $\mathrm{Im}(\sum_{U\subsetneqq G} K_G(G/U,C) \to K_G(G/G,C))$.

(b) Whenever $\xi = \varphi_\Sigma^*(\zeta)$ for some C-bundle ζ over some G-set S with $S^H=\emptyset$ with respect to some G-map $\varphi: S \to G/H$ (e.g. $S=G/V \to G/H$ with $V\subsetneqq H<G$), then $\Pi(\xi) \in \mathrm{Im}(\sum_{U\subsetneqq G} K_G(G/U,C) \to K_G(G/G,C))$.

Proof: At first let us remark, that $\Sigma(\eta_t | t\varepsilon T) \varepsilon I = \mathrm{Im}(\sum_{U\subsetneqq G} K_G(G/U,C) \to K_G(G/G,C))$ whenever η is a G-equivariant C-bundle over T with $T^G=\emptyset$. Now we have $\Pi(\Sigma(\zeta^1,\zeta^2)) = \Pi(\Sigma(\zeta_x^1,\zeta_x^2)|x \varepsilon G/H) \simeq \Sigma(\Pi(\zeta_x^{\alpha(x)}|x \varepsilon G/H)|\alpha \varepsilon \mathrm{Hom}(G/H,\{1,2\}))$ with $\mathrm{Hom}(G/H,\{1,2\})$ the G-set of all maps from G/H into $\{1,2\}$ - identified with the set of all sections of the projection $G/H \times \{1,2\} \to G/H$. Here we may consider $\Pi(\zeta_x^{\alpha(x)}|x \varepsilon G/H)_{\alpha\varepsilon \mathrm{Hom}(G/H,\{1,2\})}$ as a G-equivariant C-bundle over $\mathrm{Hom}(G/H,\{1,2\})$.
But $\mathrm{Hom}(G/H,\{1,2\})$ is a disjoint union of $T_1=\mathrm{Hom}(G/H,\{1\})\simeq G/G$, $T_2=\mathrm{Hom}(G/H,\{2\})\simeq G/G$ and $T=\{\alpha \varepsilon \mathrm{Hom}(G/H,\{1,2\})|\alpha \text{ not constant}\}$, thus $T^G=\emptyset$, and the above bundle restricted to T_i has fiber just $\Pi(\zeta^i)$ $(i=1,2)$. Thus $\Sigma(\Pi(\zeta_x^{\alpha(x)}|x \varepsilon G/H)|\alpha \varepsilon \mathrm{Hom}(G/H,\{1,2\}))$ $\equiv \Sigma(\Pi(\zeta^1),\Pi(\zeta^2))$ mod I, since by $T^G=\emptyset$ $\Sigma(-)$ applied to any bundle over T is contained in I.

(b) We have $\Pi(\xi) = \Pi(\varphi_\Sigma^*(\zeta)) \simeq \Sigma(\Pi(\zeta_{\gamma(x)}|x \varepsilon G/H)|\gamma \varepsilon \Gamma)$ with Γ the G-set of all sections γ: $G/H \to S$ of φ: $S \to G/H$. Since $S^H=\emptyset$ we have $\Gamma^G=\emptyset$ and thus $\Pi(\varphi_\Sigma^*(\zeta))\varepsilon I$.

Now to prove induction-theorems for $k\otimes K_G(-,C)$ we just have to compute $\mathcal{D}_k(C)$ and we know, that this class of finite groups is closed with respect to epimorphic images and subgroups. In the next section we will show, how this fact can be used to reduce the proof of rather general inductiontheorems to the consideration of rather special and simple cases.

§9 Applications to linear representations

We start with the purely group-theoretic

Lemma 9.1: Let $\mathcal{D}$ be a class of finite groups, which is closed with respect to epimorphic images and subgroups, and let p be a prime. If the elementary abelian group of order p^2: $Z_p \times Z_p$ and any nonabelian group of order $p \cdot q$ with $q | p-1$ another prime is not contained in $\mathcal{D}$, then any group in $\mathcal{D}$ has a cyclic p-Sylow-subgroup and is p-nilpotent.

Proof: If $G \in \mathcal{D}$ and G_p a p-Sylowsubgroup of G, then any factorgroup of G_p is in $\mathcal{D}$. But $Z_p \times Z_p \notin \mathcal{D}$. Thus G_p is cyclic. If G would not be p-nilpotent, then by a well-known transferargument there would exists an element $g \in G$ with $g \in N_G(G_p)$, but $g \notin C_G(G_p)$; since the p-part of g is necessarily contained in $G_p \subseteq C_G(G_p)$, we may even assume g to be p-regular and then as well $g^q \in C_G(G_p)$ for some prime $q \neq p$. But then with $G_p = \langle h \rangle$ the group $\langle h,g \rangle / \langle h^p, g^q \rangle$ is non abelian of order p q with $q | p-1$ a contradiction to: $G \in \mathcal{D} \Rightarrow \langle h,g \rangle \in \mathcal{D} \Rightarrow \langle h,g \rangle / \langle h^p, g^q \rangle \in \mathcal{D}$.

This Lemma will be used together with

Lemma 9.2: If $p \cdot R = R$ for some prime p and some commutative ring R with $1 \in R$, then $\mathcal{D}_{\mathbb{Q}}(\mathcal{P}(R)) =: \mathcal{D}_{\mathbb{Q}}(R)$ contains neiter $Z_p \times Z_p$ nor any non abelian group of order $p \cdot q$ with $q | p-1$.

Proof: Let us first fix some notations: For $U \leq G$ and N an RU-module we write $N^{U \to G}$ for the induced RG-module $RG \otimes_{RU} N$, i.e. the RG-module, which is induced from G-equivariant $\mathcal{P}(R)$-bundle $G \times_U N$ over G/U; for a G-set S we write R[S] for the associated permutation representation, i.e. the RG-module which is induced from the trivial G-equivariant $\mathcal{P}(R)$-bundle $R \times S/S$ over S. Thus $R[G/U] \cong R^{U \to G}$, where $R = R[U/U]$ is the trivial RU-module. Now Lemma 9.2 is a more or less direct consequence of the more explicit

Lemma 9.2': a) If pR=R, $G = Z_p \times Z_p$ and if $U_o, \ldots, U_p$ are the p+1 subgroups of order p in G, then $\underbrace{R \oplus \ldots \oplus R}_{p \text{ times}} \oplus R[G/E] \cong \bigoplus_{i=o}^{p} R[G/U_i]$.

(Here R of course means the trivial RG-module, representing 1 in $K(G,R)$)

b) Let $R = \mathbb{Z}(\frac{1}{p}, \zeta)$ with $\zeta \in \mathbb{C}$ a primitive p^{th} root of unity and let G be the semidirect product $Z_p \circledS A$ with $A = Aut(Z_p)$ cyclic of order p-1.

Let $\tilde{R}$ be R considered as a Z_p-module with $z_i \cdot r \mapsto \zeta^i \cdot r$ ($r \in R, i \in \mathbb{F}_p$ and the elements $z = z_i \in Z_p$ indexed by the elements $i \in \mathbb{F}_p$, such that $z_i \cdot z_j = z_{i+j}$). Then

$$R[G/A] \cong R \oplus \tilde{R}^{Z_p \to G}$$

Lemma 9.2', a) shows directly, that $p \cdot 1_{K(Z_p \times Z_p, R)}$ is induced from proper subgroups, thus $Z_p \times Z_p \notin \mathcal{D}_{\mathbb{Q}}(R)$.

To get also $H = Z_p \circledS Z_q \notin \mathcal{D}_{\mathbb{Q}}(R)$ whenever $Z_q \leq A = Aut(Z_p)$, we restrict the RG-isomorphism in Lemma 9.2, b) to R'H ($R' = \mathbb{Z}[\frac{1}{p}]$, $H = Z_p \circledS Z_q \leq G = Z_p \circledS A$), to get $R' \oplus \underset{p-1}{\cdots} \oplus R' \oplus N^{Z_q \to H} \cong M^{Z_p \to H}$ for some appropriate $R'Z_q$-module N and $R'Z_p$-module M, which shows that in this case $(p-1)\cdot 1_{K(H,R')}$ is induced from proper subgroups, thus $H \notin \mathcal{D}_{\mathbb{Q}}(R')$. The same holds then for any R'-algebra, i.e. for any ring, in which p is invertibel.

Proof of Lemma 9.2': a): Quite generally let us define for any finite group G,G-setS and ring R: $I_R[S] = I[S] = \mathrm{Ke}(R[S] \to R)$, where $R[S] \to R$ is defined by $s \mapsto 1 (s \in S)$.

Then $p\cdot R=R$ implies

$$R[G/E] \cong R \oplus I[G/E],\ R[G/U_i] \cong R \oplus I[G/U_i]$$

and it is enough to show

$$I[G/E] \cong \bigoplus_{i=0}^{p} I[G/U_i].$$

An explicit isomorphism is given by first restricting the canonical maps $R[G/E] \to R[G/U_i]: g\cdot E \mapsto gU_i$ to $I[\ldots]$ and then taking their product, its inverse by the sum of the restriction to $I[\ldots]$ of the maps $R[G/U_i] \to R[G/E]$: $gU_i \mapsto \frac{1}{p} \sum_{x \in gU_i} x\cdot E$.

b) We also index the elements in A by the elements in $\mathbb{F}_p^{x}$: $a=a_j (j \in \mathbb{F}_p, j \neq 0)$, such that $a_j^{-1}\, z_i\, a_j = a_j(z_i) = z_{ij}$.

$R[G/A]$ has an R-basis $x_i = z_i A (i \in \mathbb{F}_p)$ such that $z_j\, x_i = x_{i+j},\ a_j\, x_i = x_{i/j}$.

Consider $y_j = \sum_{i \in \mathbb{F}_p} \zeta^{-ji}\, x_i\ (j \in \mathbb{F}_p)$.

since the determinant
$$\begin{vmatrix} 1 & 1 \ldots\ldots & 1 \\ 1 & \zeta \ldots\ldots & \zeta^{p-1} \\ 1 & \zeta^{p-1} \ldots & \zeta^{(p-1)^2} \end{vmatrix} = \prod_{0 \le i<j \le p-1} (\zeta^j - \zeta^i) \text{ is}$$

invertibel in R ($p = \prod_{i=1}^{p-1} (1-\zeta^i)$ is a unit in R!), the set $\{y_j | j \in \mathbb{F}_p\}$ is also an R-basis of $R[G/A]$. But $z_t y_j = \zeta^{jt} y_j$, $a_t y_j = y_{jt}$, thus Ry_o is a trivial RG-Module, whereas the sub-R-modules $Ry_j (j \in \mathbb{F}_p^{x})$ are blocks of imprimitivity with Z_p being the stabilizer-subgroup of the first (and - being normal - of any) block and $Ry_1|_{Z_p} \cong \tilde{R}$,

thus $R[G/A] \cong \bigoplus_{j \in \mathbb{F}_p} Ry_j \cong R \oplus \tilde{R}^{Z_p \to G}$, q.e.d.

As a consequence of Lemma 9.1, 9.2 and Theorem 5 we get

Proposition 9.1 (cf. [5],[10]): If any prime p is invertibel in R, i.e. if R is a $\mathbb{Q}$-algebra, then $\mathcal{D}_{\mathbb{Q}}(R) \subseteq C = \{H | H \text{ cyclic}\}$. If any prime except one, say ℓ, is invertibel in R, e.g. R is a local ring with residue-class-characteristic ℓ, then $\mathcal{D}_{\mathbb{Q}}(R) \subseteq C_\ell = \{H | H \text{ cyclic mod } \ell\}$, where a group H is called cyclic mod ℓ, if the ℓ-Sylow-subgroup H_ℓ is normal in H and H/H_ℓ cyclic.

Proof: If $p\cdot R=R$ for any p, then any group in $\mathcal{D}_{\mathbb{Q}}(R)$ is p-nilpotent and has a cyclic p-Sylow-subgroup for any p, thus it is nilpotent with only cyclic Sylow-subgroups, thus it is cyclic.

If $p\cdot R=R$ for any $p \neq \ell$, then any group H in $\mathcal{D}_{\mathbb{Q}}(R)$ has a normal p-complement for any $p \neq \ell$, thus the intersection of all these normal p-complements, i.e. the ℓ-Sylow-subgroup H_ℓ of H is normal. Moreover H/H_ℓ is p-nilpotent with a cyclic p-Sylow-subgroup for any $p \,|\, |H/H_\ell|$, thus by the above argument it is cyclic.
To get also results for arbitrary R one has to use

Lemma 9.3: If R is a Dedekindring, then $\mathcal{D}_k(R) = \bigcup_{m} \mathcal{D}_k(R_m)$, where m runs through all maximal ideals in R and k is an arbitrary commutative ring with $1 \in k$ as above.

Remark: Actually the proof below is valid for any Prüferring R, i.e. any ring, for which any finitely generated torsionfree R-module is projective. I do not know, wether the above statement is true for any R, but its analog with $P(R)$ replaced by the also distributive category $P'(R)$ if finitely presented R-modules is true, i.e. for $\mathcal{D}'(k\otimes K(-,P'(R)))$, which is a bit more technical to prove. On the other hand - as we will see below - the computation of $\mathcal{D}_k(R)$ can anyway always more or less be reduced to Dedekindrings R.

Proof: Obviously $\mathcal{D}_k(R_m) \subseteq \mathcal{D}_k(R)$ for any m, since $K_G(-,R_m)$ is a $K_G(-,R)$-algebra. Now assume $G \in \mathcal{D}_k(R)$, but $G \notin \bigcup_m \mathcal{D}_k(R_m)$. For any m we thus have elements $x_V \in k\otimes K(V,R_m)$ $(V \lneq G)$, such that $1_{k\otimes K(G,R_m)} = \sum_{V\lneq G} x_V^{V\to G}$ (with $x^{V\to G}$ the image of $x \in G(V)$ in $G(G)$ with respect to the inductionmap: $G(V) \to G(G)$ $(V \leq G)$ for any universal Green-functor G). Since only finitely many R_mV-modules and only finitely many isomorphisms are involved in this equation, it is obvious, that it can be realized already in a finite subextension of R in R_m, thus we can find an element $s_m \in R-m$, such that the above situation can be realized already over $R_{\{s_m^n | n\in\mathbb{N}\}} = R_{s_m}$, especially $G \notin \mathcal{D}_k(R_{s_m})$. Thus it is enough to show, that the set $\mathfrak{s}=\{s \in R \,|\, s = 0 \text{ or } G \notin \mathcal{D}_k(R_s)\}$ is an ideal in R - since $s_m \in \mathfrak{s}$ would imply $\mathfrak{s} \nsubseteq m$ for all m, thus $\mathfrak{s} = R \ni 1$ and $G \notin \mathcal{D}_k(R)$, a contradiction. So assume $s,t \in \mathfrak{s}$. W.l.o.g. we may assume $s + t \neq 0$ and even $s + t = 1$, since $R_s \subseteq (R_{s+t})_{\frac{s}{s+t}}$, $R_t \subseteq (R_{s+t})_{\frac{t}{s+t}}$. Now we use

Lemma 9.4: Let $C \subseteq R$ be a multiplicatively closed subset of a Dedekindring R with $0 \notin C$ and R_C the associated ring of C-quotients of R. Let $i_C \subseteq k\otimes K(G,R)$ be the ideal, generated by $\{[M]-[N] \in k\otimes K(G,R) \,|$ there exists $\varphi\colon M \to N$ and $\psi\colon N \to M$ with $\varphi\circ\psi = c\cdot Id_N$, $\psi\circ\varphi = c\cdot Id_M$ for some $c \in C\}$. Then the canonical map $k\otimes K(G,R) \to k\otimes K(G,R_C)\colon [M] \to [R_C\otimes M]$ induces an isomorphism

$k\otimes K(G,R)/i_C \xrightarrow{\sim} k\otimes K(G,R_C)$.

Proof: Obviously i_C is in the kernel of $k\otimes K(G,R) \to k\otimes K(G,R_C)$. To construct an inverse of $k\otimes K(G,R)/i_C \to k\otimes K(G,R_C)$ choose for any finitely generated R_C-projective R_CG-module M' a finitely generated R-projective RG-module M with $R_C\otimes M \cong M'$, which is possible, since R is a Dedekindring, and define $k\otimes K(G,R_C) \to k\otimes K(G,R)/i_C$ by $[M'] \to [M] + i_C$, which is welldefined, since $R_C\otimes M \cong R_C\otimes N$ easily implies $[M]-[N] \in i_C$, and obviously is an inverse.

Using this Lemma we get, that there exist elements $x_V, y_V \in k \otimes K(V,R)$ $(V \lneqq G)$ with

$$x = 1 - \sum_{V \lneqq G} x_V^{V \to G} \in i_{\{s^n | n \in \mathbb{N}\}} = i_s$$

and

$$y = 1 - \sum_{V \lneqq G} y_V^{V \to G} \in i_{\{t^n | n \in \mathbb{N}\}} = i_t$$

Multiplying we get $x \cdot y = 1 - \sum_{V \lneqq G} z_V^{V \to G} \in i_s \cdot i_t$ for appropriate $z_V \in k \otimes K(V,R)$ and thus our result (i.e. $G \notin \mathcal{D}_k(R_s)$, $G \notin \mathcal{D}_k(R_t)$ and $s + t = 1$ implies $G \notin \mathcal{D}_k(R)$) follows from

Lemma 9.5: If $C_1, C_2 \subseteq R$ are multiplicatively closed subsets of R and $c_1R + c_2R = R$ for any $c_1 \in C_1$, $c_2 \in C_2$, then $i_{C_1} \cdot i_{C_2} = 0$.

Proof: If $[M_\nu] - [N_\nu] \in i_{C_\nu}$ with maps $\varphi_\nu : M_\nu \to N_\nu$, $\psi_\nu : N_\nu \to M_\nu$,

$\varphi_\nu \psi_\nu = c_\nu \cdot Id_{N_\nu}$, $\psi_\nu \varphi_\nu = c_\nu \cdot Id_{M_\nu}$ $(c_\nu \in C_\nu)$ and $r_1c_1 + r_2c_2 = 1$, then we have an isomorphism from $M_1 \otimes M_2 \oplus N_1 \otimes N_2$ into $M_1 \otimes N_2 \oplus N_1 \otimes M_2$, given by the matrix

$$\begin{pmatrix} Id_{M_1} \otimes \varphi_2 & \psi_1 \otimes r_1 Id_{N_2} \\ \varphi_1 \otimes Id_{M_2} & -r_2 Id_{N_1} \otimes \psi_2 \end{pmatrix}$$

whose inverse is given by

$$\begin{pmatrix} r_2 Id_{M_1} \otimes \psi_2 & \psi_1 \otimes r_1 Id_{M_2} \\ \varphi_1 \otimes Id_{N_2} & -Id_{N_1} \otimes \varphi_2 \end{pmatrix}.$$

Thus $([M_1] - [N_1])([M_2] - [N_2]) = 0$, q.e.d.

As an application we get

Proposition 9.2 (cf. [6], [10]): For any commutative ring R with $1 \in R$ we have $\mathcal{D}_{\mathbb{Q}}(R) = \bigcup_{\ell R \neq R} C_\ell = \{H | H$ cyclic mod ℓ for some characteristic ℓ with $\ell R \neq R\}$[1]).

Proof: Define $R' = \mathbb{Z}[\frac{1}{p} | p \cdot R = R]$. Then R' is a Dedekindring and R an R'-algebra, thus $\mathcal{D}_{\mathbb{Q}}(R) \subseteq \mathcal{D}_{\mathbb{Q}}(R') \subseteq \bigcup_m \mathcal{D}_{\mathbb{Q}}(R'_m)$.

Moreover $\mathcal{D}_{\mathbb{Q}}(R'_m) \subseteq C_\ell$, $\ell = \text{char } R'/m$ by Prop. 9.1 and $\ell = \text{char } R'/m$ obviously implies

1) $C_o = C = \{H | H$ cyclic $\}$!

$\ell R' \neq R'$, thus $\ell R \neq R$, so we get $D_{\mathbb{Q}}(R) \subseteq \bigcup_{\ell R \neq R} C_\ell$.

For the opposite inclusion, i.e. $C_\ell \subseteq D_{\mathbb{Q}}(R)$ whenever $\ell \cdot R \neq R$ choose a maximal ideal $m \subseteq R$ with char $R/m=\ell$, resp. with arbitrary residue-class-characteristic if $\ell=0$. In any case we have $D_{\mathbb{Q}}(R/m) \subseteq D_{\mathbb{Q}}(R)$ and thus it is enough to show $C_\ell \subseteq D_{\mathbb{Q}}(R)$, whenever R is a field of characteristic ℓ. So let G be cyclic mod ℓ, G_ℓ its ℓ-Sylow-subgroup (resp. E, if $\ell=0$) and $G=G_\ell \cdot \langle g \rangle$ for some appropriate $g \in G$. We construct a non-zero linear map $K(G,R) \to \mathbb{C}$, which vanishes on $\mathrm{Im}(\sum_{V \subsetneqq G} K(V,R) \to K(G,R))$ (and thus proves $G \in D_{\mathbb{Q}}(R)$), by associating to any RG-module M with a direct decomposition $M = \bigoplus_{\nu=1}^{n} M_i$ into indecomposable RG-modules the (sum) $\Sigma' \chi_{M_i}(g)$ of the Brauer-characters[1]) of g on those direct summands M_i, which have the vertex G_ℓ in the sense of Green, [21], i.e, are not a direct summand in any $N^{U \to G}$ with $U \subsetneqq G_\ell$, N any RU-module.

This is well defined and additive by the Krull-Remak-Schmidt-Theorem, nonzero since the trivial RG-module R is mapped onto 1 and vanishes on any M, which is induced from a proper subgroup V: if $M=N^{V \to G}$ for some RV-module N, which w.l.o.g. may be assumed to be indecomposable, then either the vertex of N and thus the vertex of any indecomposable summand of M is properly contained in G_p and thus $0=\Sigma' \chi_{M_i}(g)$, an empty sum, or $G_\ell \leq V$ and N is a direct summand in $N_1^{G_\ell \to V}$ for some indecomposable RG_ℓ-module N_1 with vertex G_ℓ and then any indecomposable summand M_1 of M, restricted to G_ℓ is isomorphic to a direct sum of copies of G-conjugates of N_1 and thus has vertex G_ℓ, too, in which case we get

$$\Sigma' \chi_{M_i}(g) = \Sigma \chi_{M_i}(g) = \chi_M(g) = 0,$$

since $G_\ell \leq V \subsetneqq G$ implies $g \notin V$.

To get results on $D_k(R)$ for arbitrary k, especially $k=\mathbb{Z}$, let us first observe

Lemma 9.6 (G. Segal): Let K be an arbitrary (i.e. not necessarily special) λ-ring $(\lambda^0(x)=1, \lambda^1(x) = x, \ldots)$.

Then any torsion-element in K is nilpotent.

Proof: At first let us state:

(*) If K is a λ-ring and $x \in K$,

then

$$\lambda^n(mx) = \sum_{(j_0,\ldots,j_n)}' \frac{m!}{j_0! \cdots j_n!} \prod_{\nu=0}^{n} (\lambda^\nu(x))^{j_\nu} ,$$

where the sum is taken over all (n+1)-tupels $(j_0,\ldots,j_n)$ of non negative integers j_ν with $\sum_{\nu=0}^{n} j_\nu = m$, $\sum_{\nu=0}^{n} \nu\, j_\nu = n$.

This is a straight-forward consequence of the formula $\lambda^n(x+y) = \sum_{a+b=n} \lambda^a(x)\lambda^b(y)$.

Especially if $m=n=p^t$ for some prime p, then $\frac{m!}{j_0! \cdots j_m!} \not\equiv 0(p)$ if and only if

[1]) Taken w.r.t. some fixed imbedding of the roots of unity in some algebraic closure $\tilde{R}$ of R into $\mathbb{C}$.

$j_o=j_2=\ldots=j_m=0$, $j_1=m$, thus $\lambda^{p^t}(p^t x)=x^{p^t}+py$ for some appropriate $y \in K$. Thus if $p^t x=0$ and if we assume by induction, that all $z \in K$ with $p^{t-1}\cdot z=0$ are nilpotent ($t \geq 1$), then
$0=x\cdot\lambda^{p^t}(p^t x)=x^{p^{t+1}}+p\,x\cdot y=x^{p^{t+1}}+z$ with $p^{t-1}\cdot z=0$, thus $(x^{p^{t+1}})^n=(-z)^n=0$ for some appropiate $n \in \mathbb{N}$. But if any p-torsion-element in K is nilpotent for any p, then of course any torsion-element is nilpotent, too, q.e.d.

Now it is not difficult to check, that exterior powers define a λ-ring-structure on any K(G,R) for any R (which isn't special unless $|G|\cdot R=R$, by the way), thus as an application of the results of Part I together with Prop. 9.2 we get
Proposition 9.3: Let k and R be two commutative rings with a unit. Then
$\mathcal{D}_k(R) \subseteq \{H \mid H$ q-hyperelementary mod ℓ for some q with $qk \neq k$ and some ℓ with $\ell R \neq R\}$,
where H being q-hyperelementary mod ℓ means, that there exists a normal series $E \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq H$ with N_1 an ℓ-group, $N_2|N_1$ cyclic and H/N_2 a q-group.

It is natural to expect even better upper bounds for $\mathcal{D}_k(R)$, once one makes additional assumptions on the existence of roots of unity in R. The following result for instance generalizes Brauer's classical inductiontheorem for complex characters:
Proposition 9.4: If R contains a primitive p^{th} root of unity ζ (i.e. R is $Z[\zeta]$-algebra with $\zeta \in \mathbb{C}$ a primitive p^{th} root of unity) and $H \in \mathcal{D}_k(R)$, then there exists a normal series $E \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq H$ as in Prop. 9.3 with the additional condition, that H/N_2 acts trivial on the p-part of N_2/N_1.
Proof: R is an R'-algebra now with $R'=\mathbb{Z}[\zeta,\frac{1}{r} \mid r\cdot R=R,\ r \in \mathbb{N}]$, a Dedekind-ring. Thus $H \in \mathcal{D}_k(R) \subseteq \mathcal{D}_k(R') = \bigcup_{m} \mathcal{D}_k(R'_m)$, so we may already assume R to be a local Dedekind-ring with residue-class-characteristic ℓ (possibly 0). Thus H has a normal series $E \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq H$ with N_1 an ℓ-group (i.e. $N_1=E$ for $\ell=0$), N_2/N_1 cyclic and H/N_2 a q-group for some q with $qk \neq k$. If $\ell = p$ or $q = p$, we may put any possible p-part of N_2/N_1 into N_1 or H/N_2 and thus can assume N_2/N_1 p-regular, in which case our statement is trivial. If $\ell \neq p \neq q$, we use, that $\mathcal{D}_k(R)$ is closed with respect to subgroups and quotients, so if H/N_2 does not act trivially on the p-part of N_2/N_1 we may even assume H to be nonabelian of order $p\cdot q$ with $q|p-1$. But the isomorphism in Lemma 9.2', b) of course holds for any $\mathbb{Z}(\frac{1}{p},\zeta)$-algebra, thus especially for a local ring R of residue-class-characteristic $\ell \neq p$, and restricting this isomorphism to $H=Z_p \circledS Z_q \leq Z_p \circledS A$ we get

$$R[H/Z_q] \cong R \oplus \underbrace{\tilde{R}^{Z_p \to H} \oplus \ldots \oplus \tilde{R}^{Z_p \to H}}_{\frac{p-1}{q} \text{ times}}$$

thus $1 \in K(H,R)$ is induced from proper subgroups and $H \notin \mathcal{D}(R)$, a fortiori $H \notin \mathcal{D}_k(R)$, a contradiction.
Proposition 9.4 implies, that for a finite group G and a ring R, which contains a p^{th} root of unity for any prime p dividing $|G|$, $k \otimes K_G(-,R)$ has a defect-basis contained in $C_k^R(G) = \{H \leq G \mid H$ q-elementary mod ℓ for some characteristic q with

$q\cdot k \neq k$ and some characteristic ℓ with $\ell R \neq R\}$, where a group H is called q-elementary mod ℓ, if the ℓ-Sylow-subgroup H_ℓ of H is normal and H/H_ℓ a direct product of a cyclic group and a q-group. For $q = 0$ or $\ell = 0$ a q-group, resp. an ℓ-group is always the trivial group. We show a little bit more precise:

Proposition 9.5: Let G be a finite group and R a commutative ring with $1 \in R$, such that for any prime p dividing $|G|$ the ring R contains a primitive p^{th}-root of unity. Then the defect-basis of $k \otimes K_G(-,R): \hat{G} \to \underline{k\text{-mod}}$ is precisely $C_k^R(G)$ (for any commutative ring k with $1 \in k$).

Proof: We have to show, that for any subgroup $H \in C_k^R(G)$ of G we have $k \otimes \overline{K(H,R)} \neq 0$; thus if H is q-elementary mod ℓ with $q\cdot k \neq k$, $\ell R \neq R$ and w.l.o.g. $q \neq \ell$ unless $q = \ell = 0$ we may already assume k and R to be algebraically closed fields of characteristic q and ℓ respectively and it will be enough, to construct a nonzero linear map $K(H,R) \overset{\chi}{\to} k$, which vanishes on $\mathrm{Im}(\sum_{V \lneq H} K(V,R) \to K(H,R))$. So let H_ℓ be the ℓ-Sylow-subgroup of H. By our assumption we have $H_\ell \trianglelefteq H$ and $H/H_\ell \cong H_q \times \langle g\rangle$ for some appropriate $g \in H$ of order say n. Choose a fixed isomorphism of the group of n^{th}-roots of unity in R onto the same group in k ($(n,q)=(n,\ell)=1$!), so that for any RH-module M we have a well defined Brauer character $\chi_M(g)$ with values in k. Now define again $\chi(M) = \Sigma'\chi_{M_i}(g)$, where $M = \bigoplus_i M_i$ is a decomposition of M into indecomposable RH-modules and the sum $\Sigma'\chi_{M_i}(g)$ is taken over all M_i with vertex H_ℓ. χ is nonzero, since it maps the trivial representation onto 1, but it vanishes on any $M = \oplus M_i \cong N^{V \to H}$ if $V \neq H$, since otherwise N must have vertex H_ℓ, especially $H_\ell \leq V$, in which case all M_i have vertex H_ℓ (as above, since H_ℓ is normal in H!), thus $\Sigma'\chi_{M_i}(g) = \chi_M(g) = 0$ unless also $g \in V$, in which case $\chi_M(g) = (H:V)\chi_N(g)$, since H_q acts trivial on $\langle g\rangle$. But then again $\chi_M(g) = 0$, since $(H:V)$ is a power of q, thus zero in k, unless $H = V$, which was excluded.

One can also generalize the induction-theorems of Berman-Witt as follows:
For any pair of primes p and q consider the q-Sylow-subgroup A_q of $A=\mathrm{Aut}(Z_p)$ $= \mathrm{Gal}(\mathbb{Q}(\zeta_p):Q)$ ($\zeta_p \in \mathbb{C}$ a primitive p^{th} root of unity). Since A is cyclic (of order p-1), we have $A=A_q \times A_{q'}$ with both factors cyclic. Thus for any ring R we have a unique smallest subgroup $A(p,q,R)$ of A_q, such that there exists a ring-homomorphism $Z(\zeta_p)^{A(p,q,R) \times A_{q'}} \to R$. We define a group H to be (R,q)-hyperelementary, if it is cyclic for $q = 0$, resp. has a cyclic normal subgroup $N \trianglelefteq H$ with H/N a q-group, such that for any p dividing $|N|$ the action of H/N on $N/N^p \cong Z_p$ – defining a homomorphism $H/N \to A_q \subsetneq A$ – maps H/N into $A(p,q,R)$ for $q \neq 0$.
We define H to be (R,q)-hyperelementary mod ℓ for some characteristic ℓ, if it has a normal ℓ-group $N_1 \trianglelefteq H$ (for $\ell = 0$ this means $N_1 = E$), such that H/N_1 is (R,q) hyperelementary .
Then we have finally:

Theorem 6: For R and k commutative rings with a unit one has $\mathcal{D}_k(R) \subseteq \{H | H$ (R,q)-hyperelementary mod ℓ for some characteristics q and ℓ with $\ell R \neq R$, $qk \neq k\}$.

Proof: Assume $H \in \mathcal{D}_k(R)$, then $H \in \mathcal{D}_k(R_m)$ for some maximal ideal m and thus we have a normal series $E \trianglelefteq N_1 \trianglelefteq N_2 \trianglelefteq H$ with N_1 an ℓ-group for ℓ=char R/m, N_2/N_1 cyclic, H/N_2 a q-group for some characteristic q with $qk \nmid k$ and w.l.o.g. $|N_2/N_1|$ prime to ℓ and q. Assume p divides $|N_2/N_1|$. Then we have a homomorphism $\mathbb{Z}(\frac{1}{p},\zeta_p)^{A(p,q,R) \times A_{q'}} \to R_m$ and thus $H \in \mathcal{D}_k(\mathbb{Z}(\frac{1}{p},\zeta_p)^{A(p,q,R) \times A_{q'}})$, so w.l.o.g. $R = \mathbb{Z}(\frac{1}{p},\zeta_p)^{A(p,q,R) \times A_{q'}}$.

Now if H/N_1 is not (R,q)-hyperelementary, it is easy to construct a surjective homomorphism $H/N_1 \twoheadrightarrow Z_p \circledS Z_{q^i}$ with $Z_{q^i} \leq A_q \subsetneq A = \mathrm{Aut}(Z_p)$, $|Z_{q^i}| = q^i$, but $Z_{q^i} \nleq A(p,q,R)$, thus $Z_{q^i} \nleq A(p,q,R) \times A_{q'} = B \subsetneq A$. Since $\mathcal{D}_k(R)$ is closed w.r.t. epimorphic images, we may therefore assume $H = Z_p \circledS Z_{q^i} \leq G = Z_p \circledS A, Z_{q^i} \nleq B \leq A$ and $R = \mathbb{Z}[\frac{1}{p},\zeta_p]^B$. Now consider the isomorphism $R'[G/A] \cong R' \oplus \tilde{R}'^{Z_p \to G}$ as constructed in Lemma 9.2', b) with $R' = \mathbb{Z}[\frac{1}{p},\zeta_p]$: with $y_o = 1 \in R'$, and $y_j = a_j \otimes 1 \in R'[G] \otimes_{R'[Z_p]} \tilde{R}'$ $(j \in \mathbb{F}_p^\times)$ an R'-basis of R', resp. $\tilde{R}'^{Z_p \to G}$ this was given explicitly by $y_j \mapsto \sum_{i \in \mathbb{F}_p} \zeta^{-ji} x_i$ $(j \in \mathbb{F}_p)$ with $x_i = z_i \cdot A \in R'[G/A]$ an R'-basis of $R'[G/A]$. We now define an action of B on $R'[G/A]$, R' and $\tilde{R}'^{Z_p \to G}$, which is compatibel with this isomorphism, commutes with the action of G and satisfies $\beta(rm) = \beta(r) \cdot \beta(m)$ for $\beta \in B \subseteq \mathrm{Aut}(\mathbb{Q}(\zeta_p) : \mathbb{Q})$, $r \in R'$, $m \in R'[G/A]$, resp. $\in R'$, resp. $\in \tilde{R}'^{Z_p \to G}$: for $\beta \in B$ and $m = \sum_{i \in \mathbb{F}_p} r_i x_i$ we define $\beta(m) = \sum_{i \in \mathbb{F}_p} \beta(r_i) x_i$, for $m = r \cdot y_o \in R'$ of course $\beta(m) = \beta(r) \cdot y_o$ and for $m = \sum_{j \in \mathbb{F}_p^\times} r_j y_j \in \tilde{R}'^{Z_p \to G}$ finally $\beta(m) = \sum_{j \in \mathbb{F}_p} \beta(r_j) y_{j \cdot \beta}$ (identifying $\beta \in B \subseteq A$ with the corresponding element in $\mathbb{F}_p^\times \cong A$), Then we get for the B-invariant elements an $R'^B[G]$-, i.e. RG-isomorphism $(R'[G/A])^B \cong (R')^B \oplus (\tilde{R}'^{Z_p \to G})^B$. But obviously $(R'[G/A])^B = R[G/A]$ and $(R')^B = R$. Moreover $(\tilde{R}'^{Z_p \to G})^B = \{\sum_{j \in \mathbb{F}_p^\times} r_j y_j \mid \beta(r_j) = r_{j \cdot \beta}, \beta \in B\}$ can be decomposed into blocks of imprimitivity
$(\tilde{R}'^{Z_p \to G})^B = \bigoplus_{aB \in A/B} \{\sum_{j \in aB \subseteq \mathbb{F}_p} r_j y_j \mid \beta(r_j) = r_{j \cdot \beta}, \beta \in B\}$, such that the stabilizer-group of the first one is just $Z_p \circledS B \leq G$, Thus $(\tilde{R}'^{Z_p \to G})^B$ is of the form $M^{Z_p \circledS B \to G}$ for some $R[Z_p \circledS B]$-module M (actually $M = \{\sum_{j \in B} r_j y_j \mid \beta(r_j) = r_{j \cdot \beta}, \beta \in B\}$ is an $R[Z_p \circledS B]$-module isomorphic to $\tilde{R}'$ considered as an $R[Z_p \circledS B]$-module by first restricting the $R'[Z_p]$-action to $R[Z_p]$ and then extending it to an $R[Z_p \circledS B]$-action by using the Galois-group-action of B on R', an explicit isomorphism being given by

$r\cdot 1 \mapsto \sum_{j\in B} j(r)\cdot y_j)$, thus we get: $R[G/A] \cong R \oplus M^{Z_p \circledS B \to G}$.

Restricting this to $H = Z_p \circledS Z_q i \lneqq Z_p \circledS B$ we get

$R[H/Z_q i] \cong R \oplus N^{Z_p \circledS (Z_q i \cap B) \to H}$ for some $Z_p \circledS (Z_q i \cap B)$-module N, thus

$1 \in \sum_{V \lneqq H} K(V,R)^{V\to H}$ and $H \notin \mathcal{D}(R) \subseteq \mathcal{D}_k(R)$, a contradiction, which proves the theorem.

Remark: The inclusion in Thm 6 actually is an equality, if R is a field or a complete discrete valuation-ring, which can be proved, using similar ideas as in the proofs of Prop. 9.2 and Prop. 9.5. But I do not know, wether it is an equality for arbitrary - or, what is essentially the same, for any local-Dedekindring R. Even if this is not the most important question, it might give some more insight into the structure of RG-modules for R a local, but not necessarily complete Dedekindring, to try to determine $\mathcal{D}_k(R)$ precisely for such R.

As a final application I want to prove a result, which I understand happens to be usefull in the study of conjugation of maximal tori in algebraic groups over not necessarily algebraically closed fields (see [37]): For any G-set S (G and S finite, of course) let $I[S] = \mathrm{Ke}(\mathbb{Z}[S] \to \mathbb{Z}: s \mapsto 1)$ and $J[S] = \mathrm{Coke}\ (\mathbb{Z} \to \mathbb{Z}[S]: 1 \mapsto \sum_{s\in S} s)$.

<u>Proposition 9.6:</u> For a finite group G the following statements are equivalent:

(i) G is cyclic mod p for some prime p;

(ii) $G \in \mathcal{D}_{\mathbb{Q}}(\mathbb{Z})$;

(iii) The homomorphism $\Omega(G) \to K(G,\mathbb{Z}): S \mapsto \mathbb{Z}[S]$ is injective;

(iv) For any two G-sets S, T we have "$\mathbb{Z}[S] \cong \mathbb{Z}[T] \iff S \cong T$"

(v) For any two G-sets S,T we have "$I[S] \cong I[T] \iff S \cong T$";

(vi) For any two G-sets S,T we have "$J[S] \cong J[T] \iff S \cong T$".

Proof: (i) $\iff$ (ii) is contained in Prop. 9.2; (ii) $\Rightarrow$ (iii): Assume $x = \sum n_U G/U \in \Omega(G)$ has image 0 in $K(G,\mathbb{Z})$. We have to show $\varphi_V(x) = 0$ for all $V \leq G$. But restricting to V in case $V \lneqq G$ we have w.l.o.g. $V = G$ (using that any subgroup of G is again cyclic mod p, resp. contained in $\mathcal{D}_{\mathbb{Q}}(\mathbb{Z})$). But $n_G = \varphi_G(x) \neq 0$ would imply $n_G \cdot 1_{K(G,\mathbb{Z})} \in \sum_{U \lneqq G} K(U,\mathbb{Z})^{U\to G}$, thus $G \notin \mathcal{D}_{\mathbb{Q}}(\mathbb{Z})$, q.e.d.. (iii) $\Rightarrow$ (iv) is obvious, using the fact (Prop. 4.3), that two G-sets represent the same element in $\Omega(G)$, if and only if they are isomorphic.

(iv) $\Rightarrow$ (ii): Assume $G \notin \mathcal{D}_{\mathbb{Q}}(\mathbb{Z})$. By Cor. 2 to Thm 2 (§4) this implies $G/G \notin \mathcal{D}(\mathrm{Im}(\mathbb{Q} \otimes \Omega \to \mathbb{Q} \otimes K_G(-,\mathbb{Z})))$, thus we have $n \in \mathbb{N}$ and G-sets S and T with $S^G = T^G = \emptyset$, such that $\mathbb{Z}[\underbrace{G/G \,\dot\cup \ldots \dot\cup\, G/G}_{n\ \text{times}} \dot\cup S]$ and $\mathbb{Z}[T]$ represent the same element in $\mathbb{Q} \otimes K_G(-,\mathbb{Z})$. So the result follows from the wellknown

<u>Lemma 9.7:</u> If two ZG-modules M and N represent the same element in $\mathbb{Q} \otimes K(G,\mathbb{Z})$, then there exist natural numbers r and s with

$$\underbrace{N\oplus\ldots\oplus N}_{r\text{-times}} \oplus \underbrace{M\oplus\ldots\oplus M}_{s\text{-times}} \cong \underbrace{N\oplus\ldots\oplus N}_{r+s\ \text{times}}$$

Proof: Since they represent the same element in $\mathbb{Q}\otimes K(G,\mathbb{Z})$, they do so for any localization $\mathbb{Z}_p$ of $\mathbb{Z}$ and its completion $\hat{\mathbb{Z}}_p$. But over $\hat{\mathbb{Z}}_p$ the Krull-Remak -Schmidt Theorem then implies $\hat{\mathbb{Z}}_p\otimes M \cong \hat{\mathbb{Z}}_p\otimes N$ and this in turn by a wellknown density-argument $\mathbb{Z}_p\otimes M \cong \mathbb{Z}_p\otimes N$, thus for any p we have $\mathbb{Z}G$-homomorphisms $\varphi_p: M \to N, \psi_p: N \to M$ with $\varphi_p\psi_p = c_p\cdot Id_N$, $\psi_p\varphi_p = c_p Id_M$ for some $c_p \in \mathbb{Z}$ with $(p,c_p) = 1$. Moreover using the same density-argument with respect to a finite number of primes (i.e. some kind of weak approximation, resp. the chinese remainder theorem) we can make c_p relatively prime to any given finite number of primes. Thus starting with some $c = c_p$ we can find some c' prime to c, so that there exists homomorphisms $\varphi,\varphi': M \to N$; $\psi,\psi': N \to M$ with $\varphi\psi = c\cdot Id_N$, $\varphi'\psi' = c' Id_N$, $\psi\varphi = c\cdot Id_M$, $\psi'\varphi' = c' Id_M$. But then the "diagonal" $M \xrightarrow{\varphi\oplus\varphi'} N\oplus N$ is split-injective, a left inverse being given by $N\oplus N \xrightarrow{d\psi\oplus d'\psi'} M$ with $dc + d'c' = 1$, thus we have $N\oplus N \cong M\oplus M'$ for some $\mathbb{Z}G$-module M'. But again the Krull-Remak -Schmidt-Theorem implies $\mathbb{Z}_p\otimes M' \cong \mathbb{Z}_p\otimes M \cong \mathbb{Z}_p\otimes N$, so using the same argument we can find M" with $M\oplus M'' \cong M'\oplus N$ and so on $M^{(r)}$ with $M\oplus M^{(r)} \cong M^{(r-1)}\oplus N$ $(r \in \mathbb{N})$, thus $\underbrace{N\oplus\ldots\oplus N}_{r+1} \cong \underbrace{M\oplus\ldots\oplus M}_{r}\oplus M^{(r)}$.

But now the Jordan-Zassenhaus-Theorem implies $M^{(r)} \cong M^{(r+s)}$ for some natural numbers r,s and thus

$$\underbrace{N\oplus\ldots\oplus N}_{r+s+1} \cong \underbrace{M\oplus\ldots\oplus M}_{r+s}\oplus M^{(r+s)} \cong \underbrace{M\oplus\ldots\oplus M}_{s}\oplus\underbrace{M\oplus\ldots\oplus M}_{r}\oplus M^{(r)} \cong \underbrace{M\oplus\ldots\oplus M}_{s}\oplus\underbrace{N\oplus\ldots\oplus N}_{r+1}$$

Remark: Another way, to prove this implication would have been to consider only permutationrepresentations and their Grothendieck-rings with respect to various coefficient-rings R. Since all the basic constructions map permutationrepresentations allways onto permutationrepresentations and since the basic isomorphisms in Lemma 9.2 are also those of permutationrespresentations (one has to check this for $H = Z_p \circledS Z_q$: here one has the explicit isomorphism

$$\underbrace{\mathbb{Z}\oplus\ldots\oplus\mathbb{Z}}_{p-1}\oplus\underbrace{\mathbb{Z}[H/E]\oplus\ldots\oplus\mathbb{Z}[H/E]}_{\frac{p-1}{q}} \cong \underbrace{\mathbb{Z}[H/Z_q]\oplus\ldots\oplus\mathbb{Z}[H/Z_q]}_{p-1}\oplus\underbrace{\mathbb{Z}[H/Z_p]\oplus\ldots\oplus\mathbb{Z}[H/Z_p]}_{\frac{p-1}{q}}.),$$

one gets again that the defectgroups of the Grothendieckring of permutationrepresentations over $\mathbb{Z}$, tensored with $\mathbb{Q}$, are cyclic mod p, thus for any other group G one allways has G-sets S,T,X with $S^G=T^G=\emptyset$, but
$\mathbb{Z}[\underbrace{G/G\,\dot\cup\ldots\dot\cup\,G/G}_{n}\dot\cup S\,\dot\cup\,X] \cong \mathbb{Z}[T\,\dot\cup\,X]$ for some $n \in \mathbb{N}$.

(v) $\Longleftrightarrow$ (vi) is obvious, since $I[S]$ and $J[S]$ are $\mathbb{Z}$-duals for each other.

(v) $\Rightarrow$ (iv): For any G-set S we have an isomorphism $\mathbb{Z}[S] \xrightarrow{\sim} I[S\,\dot\cup\,G/G]: s \mapsto s-G/G$. Thus $\mathbb{Z}[S] \cong \mathbb{Z}[T] \Rightarrow I[S\,\dot\cup\,G/G] \cong I[T\,\dot\cup\,G/G] \overset{(v)}{\Rightarrow} S\,\dot\cup\,G/G \cong T\,\dot\cup\,G/G \Rightarrow S \cong T$.

(iv) $\Rightarrow$ (v): By (iv) $\Longleftrightarrow$ (i) we know that G is cyclic mod p. We use induction on G, so for $I[S] \cong I[T]$ we get $S|_U \cong T|_U$ for all $U \lneqq G$, especially $\varphi_U(S) = \varphi_U(T)$, $U \lneqq G$. If morover $\varphi_G(S) = \varphi_G(T) = 0$, we get $S \cong T$ by Prop. 4.3. If $\varphi_G(S) \neq 0 \neq \varphi_G(T)$, we have

$S \cong S' \dot{\cup} G/G$, $T \cong T' \dot{\cup} G/G$ and get $\mathbb{Z}[S'] \cong I[S] \cong I[T] \cong \mathbb{Z}[T']$, thus $S' \cong T'$, $S \cong T$.
So there remains the case $\varphi_G(S) \neq 0$, $\varphi_G(T) = 0$.
Since $\varphi_U(T) = \varphi_U(S) \geq \varphi_G(S) > 0$ for any $U \lneq G$, we get g.c.d. $\{(G:U) | T^U \neq \emptyset\} = 1$ unless G is a p-group. But $\mathbb{Z}[T] \to \mathbb{Z} : t \mapsto 1$ maps the G-invariant part of $\mathbb{Z}[T]$ onto the ideal, generated by $(G:G_t)$ $(t \in T, G_t = \{g \in G | gt = t\})$, which contains $\{(G:U) | T^U \neq \emptyset\}$. Thus if G is not a p-group, the map $\mathbb{Z}[T] \to \mathbb{Z}$ is split-surjective, i.e. we have $\mathbb{Z}[T] \cong \mathbb{Z} \oplus I[T] \cong \mathbb{Z} \oplus I[S] \cong \mathbb{Z}[S] \overset{(iv)}{\Rightarrow} T \cong S$, q.e.d., resp. a contradiction to $\varphi_G(T) = 0 \neq \varphi_G(S)$.
For G a p-group, let U be a maximal subgroup, thus U is normal of index p. We get $0 < \varphi_G(S) \equiv \varphi_U(S) = \varphi_U(T) \equiv \varphi_G(T) = 0(p)$, thus if $S = S' \dot{\cup} G/G$, then $\varphi_G(S') > 0$ and $I[S] \cong \mathbb{Z}[S']$ contains a direct summand isomorphic to $\mathbb{Z}$. So it remains to show:
If G is a p-group, T a G-set and $\varphi_G(T) = 0$, then $I[T]$ contains no direct summand isomorphic to $\mathbb{Z}$. But this follows from $p^{n-1} \cdot \hat{H}^0(G, I[T]) = 0$ and $\hat{H}^0(G,\mathbb{Z}) \cong \mathbb{Z}/p^n\mathbb{Z}$, if $|G| = p^n$, the first fact following from
$0 = \hat{H}^{-1}(G,\mathbb{Z}) \to \hat{H}^0(G, I[T]) \to \hat{H}^0(G,\mathbb{Z}[T])$, $\hat{H}^0(G,\mathbb{Z}[T]) = \bigoplus_i \hat{H}^0(G,\mathbb{Z}[G/U_i]) \cong \bigoplus_i \hat{H}^0(U_i,\mathbb{Z})$
annihilated by p^{n-1}, if $T = \dot{\bigcup_i} G/U_i$ and $U_i \lneq G$ (by $\varphi_G(T) = 0$).

§10 Prospects of further applications

In this last section of this paper I want to indicate several further possible applications of the above methods. Detailed versions will appear elsewhere.

At first we may try to study the equivariant K-theory associated to the distributive category $L(R)$ of "R-lattices": the objects in $L(R)$ are pairs (M,f), where M is a finitely generated, projective R-module and $f: M \times M \to R$ a nonsingular symmetric, bilinear form on M (where nonsingularity means, that the associated map $\hat{f}: M \tilde{\to} \mathrm{Hom}_R(M,R): \hat{f}(m)\ (m') = f(m,m')$ is an isomorphism), the morphisms $\varphi:(M,f) \to (M',f')$ R-linear maps from M to M' with $f(m_1,m_2)=f'(\varphi(m_1),\varphi(m_2))$. As allready observed in §6 this category is distributive with respect to orthogonal sum and tensor product.

Analogously to $P(R)$ one has

<u>Theorem 7:</u> a) $D_{\mathbb{Q}}(L(R))$ $[=: D'(\mathbb{Q}\otimes K(-,L(R)))]$ = {H|H cyclic mod p for some characteristic p with $pR \neq R$}

b) $D_k(L(R)) \subseteq$ {H|H q-hyperelementary mod p for some characteristics p and q with $pR \neq R$, $qk \neq k$}.

<u>Outline of proof:</u> a) implies obviously b), since we may assume w.l.o.g. $k \subseteq \mathbb{Q}$ and then use - as before in the linear case - the fact, that exterior powers of R-lattices define a λ-ring-structure on $K(G,L(R))$, thus torsion-elements are nilpotent and we can use Prop. 5.2'.

So it remains to prove a) and this is done just as in the linear case: At first one proves, that $Z_p \times Z_p$ and $Z_p \circledS Z_q (Z_q \leq \mathrm{Aut}(Z_p))$ are not contained in $D_{\mathbb{Q}}(L(R))$ whenever $pR = R$, using similar isomorphisms as in Lemma 9.2', which establishes the result for local rings. For arbitrary R again one can at first replace R by $R'=\mathbb{Z}[\frac{1}{p}|pR=R] \subseteq Q$, thus w.l.o.g. $R \subseteq \mathbb{Q}$ and then has to delocalize, which can be done essentially as in the linear case, only the isomorphism constructed in the proof of Lemma 9.5 has to be replaced by the following observation:

<u>Lemma 10.1:</u> Let (M_o^ν,f_o^ν), (M_1^ν,f_1^ν) $(\nu=1,\dots,n)$ be RG-lattices (R any commutative ring with $1 \in R$) and assume that for any $\nu \in \{1,\dots,n\}$ there exists $\varphi_o^\nu: M_o^\nu \to M_1^\nu$, $\varphi_1^\nu: M_1^\nu \to M_o^\nu$, $c_\nu \in R$ and $\varepsilon_\nu \in \mathbb{F}_2$ with

(1) $f_o^\nu(m_o^\nu,\varphi_1^\nu(m_1^\nu)) = f_1^\nu(\varphi_o^\nu(m_o^\nu),m_1^\nu)$ for all $m_o^\nu \in M_o^\nu$, $m_1^\nu \in M_1^\nu$,

(2) $\varphi_1^\nu\varphi_o^\nu = c_\nu^2 \cdot \mathrm{Id}_{M_o^\nu}$, $\varphi_o^\nu\varphi_1^\nu = c_\nu^2 \cdot \mathrm{Id}_{M_1^\nu}$,

(3) $\sum_{\nu=1}^{n} (-1)^{\varepsilon_\nu} c_\nu^2 = 1$.

(An RG-lattice of course is a G-object in $L(R)$.)

Then one has an RG-Isomorphism

$$\perp_\alpha M_\alpha \to \perp_\beta M_\beta$$

where α, resp. β runs through all maps $\alpha,\beta: \{1,\dots,n\} \to \mathbb{F}_2$ with $\sum_{\nu=1}^{n} \alpha(\nu) = 0$, resp. $\sum_{\nu=1}^{n} \beta(\nu) = 1$ and

$$M_\alpha = \bigotimes_{\nu=1}^{n} (M^\nu_{\alpha(\nu)}, (-1)^{\alpha(\nu)\varepsilon_\nu} f^\nu_{\alpha(\nu)})$$

resp.

$$M_\beta = \bigotimes_{\nu=1}^{n} (M^\nu_{\beta(\nu)}, (-1)^{\beta(\nu)\varepsilon_\nu} f^\nu_{\beta(\nu)}),$$

given by

$$M_\alpha \ni x^1_{\alpha(1)} \otimes \dots \otimes x^n_{\alpha(n)} \to$$

$$\sum_{k=1}^{n} (-1)^{\eta_k} x^1_{\alpha(1)} \otimes \dots \otimes x^{k-1}_{\alpha(k-1)} \otimes \varphi^k_{\alpha(k)} (x^k_{\alpha(k)}) \otimes x^{k+1}_{\alpha(k+1)} \otimes \dots \otimes x^n_{\alpha(n)}$$

with $\eta_k = \sum_{i<k} \alpha(i) + \alpha(k)\varepsilon_k$.

This together with the fact, that for $R \subsetneq \mathbb{Q}$ any element in R is a sum or difference of finitely many squares in $R (R \ni r = a_1^2 + \dots + a_n^2 - b_1^2 - \dots - b_m^2)$ allows then to delocalize (i.e. to prove $\mathcal{D}_{\mathbb{Q}}(R) = \bigcup_m \mathcal{D}_{\mathbb{Q}}(R_m)$), establishing the theorem.

Remark: Especially for $R \subsetneq \mathbb{Q}$ it may make sense, to consider the distributive subgategory $\mathcal{L}^+(R)$ of positive definite R-lattices.
Here one can show the perhaps surprising result $\mathcal{D}_{\mathbb{Q}}(\mathcal{L}^+(R)) = \mathcal{D}_{\mathbb{Q}}(\mathcal{L}(R))$, whenever $R \neq \mathbb{Z}$, whereas $\mathcal{D}_{\mathbb{Q}}(\mathcal{L}^+(\mathbb{Z}))$ is the class of all finite groups.

Finally I want to discuss relative K_G-theories: Let G be a fixed finite group and S and T G-sets.
A sequence $0 \to \zeta_1 \to \zeta_2 \to \zeta_3 \to 0$ of $\mathcal{P}(R)$-bundles over S is called T-split, if the restricted sequence $0 \to T \times \zeta_1 \to T \times \zeta_2 \to T \times \zeta_3 \to 0$ over $T \times S$ is split. Define

$K_G(S,R;\,T) = K_G(S,R) / \langle \zeta_1 - \zeta_2 + \zeta_3 \mid 0 \to \zeta_1 \to \zeta_2 \to \zeta_3 \to 0 \text{ T-split} \rangle$.

One verifies easily, that restriction and induction are well-defined on $K_G(-,R;T)$, thus $K_G(-,R;T)$ is a Green-functor. Especially for $T = G/E$ the ring $K_G(G/U,R;G/E)$ is just the Grothendieck-ring $G_o(R,U)$ of RU-modules as defined by Swan. One can apply the above methods to compute the defect-sets of $K_G(-,R;T)$ and this way get simple proofs (cf. [18]) of the results announced in [11], [14] and [15], which will be done in some detail and together with applications on the structure of the relative Grothendieckgroups in another paper.

Finally one may also define relative K_G-theories with coefficients in $\mathcal{L}(R)$. Of course one cannot use exact sequences. Instead - exploiting an idea of D.Quillen (cf. [30], §5) - one can define a "T-Quillenpair" (ζ,ξ) to be a G-equivariant $\mathcal{L}(R)$-bundle ζ over some G-set S together with an $\mathcal{P}(R)$-subbundle ξ, such that the exact sequence $0 \to \xi \to \zeta \to \zeta/\xi \to 0$ of $\mathcal{P}(R)$-bundles is T-split and furthermore any fiber of ξ is an isotropic submodule in the corresponding fiber in ζ, i.e. $\xi \subseteq \xi^{\perp}$. One may then define $U_G(S,R;T) = K_G(S,L(R))/I_T$ with I_T the ideal generated by $<\zeta - \xi^{\perp}/\xi - H(\xi) \mid (\zeta,\xi)$ a T-Quillenpair over S> with $\xi^{\perp}/\xi$ the obvious well defined (!) G-equivariant L(R)-bundle and H(ξ) the "hyperbolic" L(R)-bundle, associated to ξ. It should be remarked, that in general even $I_{G/G} \neq 0$, i.e. $U_G(S,R;G/G) \neq K_G(S,L(R))$, but $I_{G/U}=0$ if $2 \cdot R=R$ and $(G:U) \cdot R=R$.

I guess, that corresponding inductiontheorems hold as in the linear case. In the most important special case T=G/E, which especially applies to the computation of L-groups, they are allready proved and have been announced in [19].

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C. THE FUNCTOR K_2 OF MILNOR

THE FUNCTOR K_2: A SURVEY OF COMPUTATIONS AND PROBLEMS

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In the past few years there has been a great deal of research on the functor K_2 and it would appear that now is an appropriate time to give a survey of these results. Several different definitions have been proposed for K_2 and it is now known that those given by Gersten-Swan, Keune, Milnor, Strooker-Villamayor, and Quillen all agree (see [41] and [94]). It is also known that these agree with that of Karoubi-Villamayor if the ring in question is regular [73]. However, we give only Milnor's definition as it easily adapts to define "unstable" K_2's and as many results of a computational nature have been derived with it.

The first section of this paper gives a brief list of known properties and computations of K_2 with references for further information. The second section gives a list of research problems, and the final section is a bibliography. We would like to take this opportunity to thank everyone who sent suggestions and research problems. Any changes or omissions in the problems reflect the interests and prejudices of the authors.

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PROPERTIES AND COMPUTATIONS OF K_2

All rings are associative with 1. If R is a ring, R^* denotes its group of units. If G is a group and $\sigma, \tau \in G$, we write

$$[\tau,\sigma] = \tau\sigma\tau^{-1}\sigma^{-1}$$

If G is finite, $|G|$ denotes its order. The rational integers are denoted by $\mathbb{Z}$, the rational numbers by $\mathbb{Q}$, and a finite field with q elements by $\mathbb{F}_q$. $H_i(G) = H_i(G;\mathbb{Z})$ will denote the i-th homology group of G with coefficients in $\mathbb{Z}$ where G acts trivally on $\mathbb{Z}$.

For $n \geq 2$ we denote by $E(n,R)$ the subgroup of the general linear group $GL(n,R)$ generated by the elementary matrices $E_{ij}(r)$, $r \in R$. The Steinberg group, $St(n,R)$, is the group with generators $x_{ij}(r)$, where $r \in R$ and i,j are distinct integers between 1 and n, subject to the Steinberg relations

(R1) $$x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$$

(R2) $$[x_{ij}(r),x_{k\ell}(s)] = \begin{cases} 1 & \text{if } i \neq \ell,\ j \neq k \\ x_{i\ell}(rs) & \text{if } i \neq \ell,\ j = k \end{cases}$$

(R3) $$w_{ij}(u)x_{ji}(r)w_{ij}(u)^{-1} = x_{ij}(-uru) \quad \text{for any unit } u$$

$$\text{where } w_{ij}(u) = x_{ij}(u)x_{ji}(-u^{-1})x_{ij}(u).$$

It should be noted that for $n = 2$, (R2) is vacuous and for $n \geq 3$, (R3) is a consequence of (R1) and (R2). As the generators $E_{ij}(r)$ of $E(n,R)$ satisfy relations analogous to (R1) - (R3), there is a surjective homomorphism $St(n,R) \twoheadrightarrow E(n,R)$ defined by $x_{ij}(r) \mapsto E_{ij}(r)$. We define $K_2(n,R)$ to be the kernel of this homomorphism. For every $n \geq 2$, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_2(n,R) & \longrightarrow & St(n,R) & \longrightarrow & E(n,R) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & K_2(n+1,R) & \longrightarrow & St(n+1,R) & \longrightarrow & E(n+1,R) & \longrightarrow & 1 \end{array}$$

where the vertical maps are defined by sending the generators $x_{ij}(r)$ and $E_{ij}(r)$ in the top row to the element of the same name in the bottom row. Passing to the direct limit as $n \to \infty$ yields the definitions

$$St(R) = \varinjlim St(n,R)$$

$$E(R) = \varinjlim E(n,R)$$

$$K_2(R) = \varinjlim K_2(n,R)$$

It is clear from the definitions that the sequence

$$(*) \qquad 1 \longrightarrow K_2(R) \longrightarrow St(R) \longrightarrow E(R) \longrightarrow 1$$

is exact. It should be noted that $St(n,R)$ and $K_2(n,R)$ are denoted $St(A_{n-1},R)$ and $L(A_{n-1},R)$, respectively, in [88] and [89]. In the following α will denote a pair of indices ij, $i \neq j$, and $-\alpha$, the reversed pair, ji.

1. Central extensions and homology.

In [69, §5] it is shown that $K_2(R)$ is precisely the center of the Steinberg group $St(R)$. The extension $(*)$ above is a universal central extension and it follows that $K_2(R) \approx H_2(E(R))$ ([56]; [92]).

2. The exact sequence of an ideal.

Let I be a 2-sided ideal in the ring R. Then there is an exact sequence

$$K_2(I) \longrightarrow K_2(R) \longrightarrow K_2(R/I) \longrightarrow K_1(I) \longrightarrow \cdots$$

(see [69, §6] for a definition of $K_2(I)$ and a proof).

3. <u>The Mayer-Vietoris exact sequence</u>.

(a) If the commutative square of surjective ring homomorphisms

$$\begin{array}{ccc} R & \longrightarrow & R' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

is cartesian, there is an exact sequence

$$K_2(R) \longrightarrow K_2(S) \oplus K_2(R') \longrightarrow K_2(S') \longrightarrow K_1(R) \longrightarrow \cdots$$

[69, p. 55].

(b) Let R be a commutative noetherian regular ring and let $(f,g) = R$. Then

$$\cdots \longrightarrow K_2(R_{fg}) \longrightarrow K_2(R) \longrightarrow K_2(R_f) \oplus K_2(R_g) \longrightarrow K_1(R_{fg}) \longrightarrow \cdots$$

is exact [41, Theorem 2.19].

(c) Let $R \longrightarrow R' = \Pi\, T_i$ be an inclusion of rings with the maps $R \longrightarrow T_i$ surjective. If I is a 2-sided ideal of R' contained in R, the square of part (a) is cartesian for $S = R/I$ and $S' = R'/I$. Moreover, if the term $K_2(R)$ is deleted, the sequence in part (a) is exact [1].

4. <u>The exact sequence of a localization</u>.

If A is a Dedekind domain with fraction field F, then there is an exact sequence

$$\cdots \longrightarrow \coprod_m K_2(A/m) \longrightarrow K_2(A) \longrightarrow K_2(F) \longrightarrow \coprod_m K_1(A/m) \longrightarrow K_1(A) \longrightarrow \cdots$$

where m runs over the set of maximal ideals of A [73].

A simple example of the use of this sequence is mentioned in Problem 17 of the second section: If S is an arbitrary set of rational primes and $\mathbb{Z}_S$ is the localization of $\mathbb{Z}$ at the monoid generated by S, then

$$K_2(\mathbb{Z}_S) \approx \{\pm 1\} \oplus \coprod_{p \in S} (\mathbb{Z}/p\mathbb{Z})^*$$

5. The product structure.

If A is a commutative ring there are pairings (see [41, §2], [69, §8])

$$K_i(A) \times K_j(A) \longrightarrow K_{i+j}(A)$$

such that $x \cdot y = (-1)^{i+j} y.x$ for $x \in K_i(A)$, $y \in K_j(A)$. In particular, under this product $K_0(A)$ becomes a commutative ring and $K_i(A)$ becomes a $K_0(A)$-module. It should be noted that the map is not surjective in general.

6. The transfer homomorphism.

If $f\colon R \longrightarrow S$ is an inclusion of rings and S is a finitely generated projective module over R, there is a transfer homomorphism

$$f^*\colon K_i(S) \longrightarrow K_i(R)$$

(see [69, §14] and [41, §2]). Moreover, if the rings are commutative the projection formula

$$f^*(x \cdot f_*(y)) = (f^*(x)) \cdot y$$

is valid for $x \in K_i(S)$, $y \in K_j(R)$. Here $\cdot$ denotes the product given in 5 and f_* is the homomorphism from $K_i(R)$ to $K_i(S)$ induced by f. If S is a free R-module of rank n over R, then $f^* \circ f_*$ is multiplication by n. In case R and S are local fields, the transfer homomorphism is surjective for $i = 2$ [69, Corollary A.15].

7. Differential "symbols".

If A is a commutative ring and $\Omega^2_{A/\mathbb{Z}}$ denotes the second exterior power of the module of absolute differentials $\Omega_{A/\mathbb{Z}}$, there is a homomorphism

$$K_2(A) \longrightarrow \Omega^2_{A/\mathbb{Z}}$$

[40, Remark 6 in §7]. In case A is a field, this agrees with Tate's differential symbol

$$\{a,b\} \longmapsto \frac{da}{a} \wedge \frac{db}{b}$$

[104, p. 202] (see 9 and 11 below).

8. Technical computations in $St(n,R)$.

A large number of formulas, normal forms and other computational conveniences are now available for the Steinberg group. We only give two examples and the reader is advised to consult [25], [27], [69, §§5, 9, 10, 12], [77, §1], [82], [86], [88], [89], [100], [105], and [107] for further information.

(a) For any $z \in St(n,R)$ define $I(z)$ to be the minimal number of indices involved in any expression for z. Assume $I(z) < n$ and the image of z in $E(n,R)$ can be written as PD where P is a permutation matrix corresponding to the permutation π and $D = \mathrm{diag}(v_1,\ldots,v_n)$ is a diagonal matrix. Then

$$z\, x_{ij}(r)\, z^{-1} = x_{\pi(i),\pi(j)}(v_i r v_j^{-1})$$

for any $x_{ij}(r) \in St(n,R)$ [25]. It easily follows that the image of $K_2(n,R)$ in $St(n+1,R)$ is central and hence that $K_2(R)$ is in the center of $St(R)$.

(b) Let R be an arbitrary ring. Then every element of $St(R)$ can be represented as a product $LPL'U$ where L,L' are products of elements of the form $x_{ij}(r)$ with $i > j$, U is a product of elements of the form $x_{ij}(r)$ with $i < j$, and P is in the subgroup of $St(R)$ generated by the elements $w_{ij}(1)$. This was proved by R. Sharpe using an argument similar to that in [77, §5] (see Problem 25 below).

9. Elements of $K_2(n,R)$.

(a) For units u,v of R, define

$$w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$$

$$h_\alpha(u) = w_\alpha(u)w(-1)$$

$$\{u,v\}_\alpha = h_\alpha(uv)h_\alpha(u)^{-1}h_\alpha(v)^{-1}.$$

If u and v commute then $\{u,v\}_\alpha \in K_2(n,R)$ and lies in the center of $St(n,R)$ for any n. If $n \geq 3$, it follows from the formula in 8 (a) that this element does not depend on α. Deleting the α, we obtain the Steinberg symbol $\{u,v\}$. If R is a commutative ring and $n \geq 3$, these symbols satisfy the identities listed below. For $n = 2$ more complicated identities exist (see [67], [88]).

(S1) $\{uv,w\} = \{u,w\}\,\{v,w\}$

$\{u,vw\} = \{u,v\}\,\{u,w\}$

(S2) $\{u,v\} = \{v,u\}^{-1}$

(S3) $\{u,-u\} = 1$

(S4) $\{u,1-u\} = 1$

(S5) $\{v,\ 1 - pqv\} = \left\{-\dfrac{1-qv}{1-p}, \dfrac{1-pqv}{1-p}\right\} \left\{-\dfrac{1-pv}{1-q}, \dfrac{1-pqv}{1-q}\right\}$

(S6) $\left\{-\dfrac{1-qr}{1-p}, \dfrac{1-pqr}{1-p}\right\} \left\{-\dfrac{1-pr}{1-q}, \dfrac{1-pqr}{1-q}\right\} \left\{-\dfrac{1-pq}{1-r}, \dfrac{1-pqr}{1-r}\right\} = 1$

(S7) $\displaystyle\prod_{i=1}^{s} \left\{\frac{u_i}{1+qy_{i-1}}, \frac{1+qy_i}{1+qy_{i-1}}\right\} = \prod_{j=1}^{t} \left\{\frac{v_j}{1+qz_{j-1}}, \frac{1+qz_j}{1+qz_{j-1}}\right\}$

where $q,\ u_1,\ldots,u_s,v_1,\ldots v_t \in R$ and $y_0 = z_0 = 0$,

$$y_k = \sum_{i=1}^{k} u_i, \qquad z_k = \sum_{j=1}^{k} v_j \quad \text{with } y_s = z_t.$$

In all of the above identities, it is assumed that the elements involved are all defined (i.e. $1 - u$, $1 - pq$, $1 + qy_i$, etc. are all units). Proofs of (S1) - (S4) can be found in [69, p. 74] and proofs of the others can be found in [27, §1]. These identities are not independent. For example, if u and $1 - u$ are both units, then (S3) is a consequence of (S1) and (S4). In case R is local all of the identities of (S7) are consequences of the identity where $s = t = 2$ together with (S1) - (S4) [27, Proposition 1.5].

(b) Let $a, b \in R$ be any two elements such that $1+ab \in R^*$. For each α, define

$$H_\alpha(a,b) = x_{-\alpha}(-b(1+ab)^{-1})x_\alpha(a)x_{-\alpha}(b)x_\alpha(-(1+ab)^{-1}a)$$

and set

$$\langle a,b\rangle_\alpha = H_\alpha(a,b)h_\alpha(1+ab)^{-1}.$$

If a and b commute, then $\langle a,b\rangle_\alpha \in K_2(n,R)$ for all n and for $n \geq 3$ $\langle a,b\rangle_\alpha$ is a central element that does not depend on α. We denote it simply $\langle a,b\rangle$. If R is a commutative ring and $n \geq 3$, the following identities hold:

(H1) $\langle a,b\rangle = \langle -b,-a\rangle^{-1}$

(H2) $\langle a+b,c\rangle = \langle a,c\rangle \langle b,\frac{c}{1+ac}\rangle \left\{\frac{1+(a+b)c}{1+ac}, 1+ac\right\}$

$\langle a,b+c\rangle = \langle a,b\rangle \langle \frac{a}{1+ab},c\rangle \left\{1+ab, \frac{1+a(b+c)}{1+ab}\right\}$

(H3) $\langle a+b,c\rangle = \langle a,c\rangle \langle b,c\rangle \langle \frac{b}{1+bc},\frac{-ac^2}{1+ac}\rangle \{-1,1+ac\} \left\{\frac{1+(a+b)c}{1+bc},\frac{1+ac}{1+bc}\right\}$

$\langle a,b+c\rangle = \langle a,b\rangle \langle a,c\rangle \langle \frac{-a^2b}{1+ab}, \frac{c}{1+ac}\rangle \{1+ab,-1\} \left\{\frac{1+ab}{1+ac}, \frac{1+a(b+c)}{1+ac}\right\}$

(H4) $\langle a,bc\rangle \langle b,ac\rangle \langle c,ab\rangle = 1$

$\langle a,bc\rangle = \langle ab,c\rangle \langle ac,b\rangle$

As in part (a), it is assumed that the elements above are all defined. Proofs of these identities can be found in [90, Proposition 1.1].

(c) These elements of $St(n,R)$ are related to each other and to other elements defined in the literature as follows:

(i) $\langle a,b\rangle = \{-a,1+ab\}$ if $a \in R^*$

$\langle a,b\rangle = \{1+ab,b\}$ if $b \in R^*$

(ii) If $ab = 0$, then $\langle a,b\rangle = c(a,b)$ where $c(a,b)$ was defined by Swan in [100, §6].

(iii) The generators given by Van der Kallen [105] are related to these elements as follows:

$$f_\alpha(a,b) = \langle a\epsilon, b\epsilon\rangle = \{1+a\epsilon, 1+b\epsilon\}$$

$$H_\alpha(a,b) = \langle b,a\epsilon\rangle\, h_\alpha(1+ab\epsilon) = \text{the } H_\alpha(b,a\epsilon) \text{ defined above}$$

$$N_\alpha(a,b) = \langle b,a\epsilon\rangle \langle ab\epsilon, abc\rangle = \langle b,a\epsilon\rangle \{1+ab\epsilon, 1+ab\epsilon\}.$$

(d) Cohn [18] and Silvester [83] defined the concepts "R is universal for GE_n" and "R is quasi-universal for GE_n". These definitions are statements that $GE_n(R)$ (the subgroup of $GL(n,R)$ generated by $E(n,R)$ together with the diagonal matrices) has a certain presentation. Let $W(R)$ be the subgroup of R^* generated by the elements of the form $(1+ab)(1+ba)^{-1}$ for $1+ab \in R^*$. Let $V_n(R)$ be the subgroup of R^* generated by all elements $u \in R^*$ such that $diag(u,1,\ldots,1)$ is in $E(n,R)$. It is shown in [25] that the definitions mentioned above are related to $K_2(n,R)$ as follows:

(i) If $n \geq 2$, R is universal for GE_n if and only if $K_2(n,R)$ is contained in the subgroup of $St(n,R)$ generated by the elements $h_\alpha(u)$, $u \in R^*$, and $V_n(R) = [R^*,R^*]$ (the commutator subgroup of R^*). If R is commutative and $n \geq 2$, then R is universal for GE_n if and only if $K_2(n,R)$ is generated by the Steinberg symbols.

(ii) If $n \geq 3$, R is quasi-universal for GE_n if and only if $K_2(n,R)$ is contained in the subgroup of $St(n,R)$ generated by the elements $H_\alpha(a,b)$ and $V_n(R) = W(R)$. If R is commutative and $n \geq 3$, then R is quasi-universal for GE_n if and only if $K_2(n,R)$ is generated by the elements $\langle a,b\rangle$.

10. Complete sets of generators for $K_2(n,R)$.

(a) (i) The Steinberg symbols generate $K_2(n,R)$ for $n \geq 3$ if R is a commutative semi-local ring [90, Theorem 2.7].

(ii) The Steinberg symbol $\{-1,-1\}$ generates $K_2(n,\mathbb{Z})$ for all $n \geq 2$ [69, §10].

(b) In this section only, if J is an ideal of R let $K_2(n,J)$ be defined by the exact sequence

$$1 \longrightarrow K_2(n,J) \longrightarrow K_2(n,R) \longrightarrow K_2(n,R/J).$$

If J is an ideal contained in the Jacobson radical of the commutative ring R, then $K_2(n,J)$ is generated by the elements $\langle a,q\rangle$, $a \in R$, $q \in J$, for all $n \geq 3$ [90, Theorem 2.1]. Note that if R is additively generated by its units, then it follows from (H2) and (c) (i) of 9 that $K_2(n,J)$ is actually generated by Steinberg symbols of the form $\{u,1+q\}$, $u \in R^*$, $q \in J$, a result proved earlier by Stein [89].

Let $R = W_2(\mathbb{F}_q)$ denote the ring of Witt vectors of length two over $\mathbb{F}_q$, $q = p^n$. The preceding result together with the techniques of [27] yield the following: $K_2(R[X])$ is an elementary abelian p-group of countably infinite rank. It should be noted that if p is odd all Steinberg symbols in $K_2(R[X])$ are trivial. This gives an example of a ring where $K_2(R[X])$ is not isomorphic to $K_2(R)$ [90, Theorem 2.8].

11. K_2 for fields.

Matsumoto [67] (cf. [69, §§11, 12]) proved that K_2 of a field F is presented by the generators $\{u,v\}$, $u,v \in F^*$, subject to the relations (S1) and (S4) (given in 9 (a) above). If a symbol is defined to be a bimultiplicative function

$$(\ , \ ): F^* \times F^* \longrightarrow C$$

taking values in an abelian group C and which satisfies $(u,1-u) = 1$, then Matsumoto's theorem can be rephrased to say that the function

$$\{\ ,\ \}: F^* \times F^* \longrightarrow K_2(F)$$

is the universal symbol. Thus any symbol (,) defines a homomorphism from $K_2(F)$ to C. Examples of such symbols are the tame symbol [69, p. 98], the power norm residue symbol [69, §15], the norm residue symbol [69, p. 151], and the differential symbol of Tate [104, p. 202].

Matsumoto's presentation of $K_2(F)$ yields many properties and computations of $K_2(F)$:

(i) K_2 of a finite field is trivial [91,3.3] (cf. [69, p. 78]).

(ii) If $X^m - a$ splits into linear factors for all $a \in F$, then $K_2(F)$ is uniquely divisible by m. Hence K_2 of an algebraically closed field is a torsion free divisible group, K_2 of a perfect field of characteristic $p > 0$ is uniquely p-divisible, and the only torsion in K_2 of the real numbers is 2-torsion (in fact, just $\{-1,-1\}$) [5, (1.2)].

(iii) $K_2(Q) = \{\pm 1\} \oplus \coprod_p (Z/pZ)^*$ [69, p. 101].

(iv) $K_2(F(X)) = K_2(F) \oplus \coprod_{\underline{p}} (F[X]/\underline{p})^*$ [69, p. 106].

(v) If F is a local field and μ_F denotes the group of roots of unity in F, then Moore [70] (cf. [69, Theorem A.14]) has proved that $K_2(F) \approx D \oplus \mu_F$ where D is a divisible group. Let q be the order of the residue field of F. J. Carroll has proved that D is uniquely p-divisible if p does not divide $q(q-1)$ (see Problem 12 in the next section).

12. K_2 for some local rings.

If A is a discrete valuation ring or a homomorphic image thereof, then $K_2(A)$ and $K_2(n,A)$ for $n \geq 3$ are presented by the generators $\{u,v\}$, $u,v \in A^*$, subject to the relations (S1) - (S7) [27, Theorems 2.3, 2.5].

If A is a discrete valuation ring with field of fractions F and residue field $\underline{k}$, then there is an exact sequence

$$1 \longrightarrow K_2(A) \longrightarrow K_2(F) \longrightarrow K_1(\underline{k}) \longrightarrow 1$$

[27, Theorem 2.2] which is split exact if A is complete. In case F is a local field and $\underline{k}$ has characteristic p, it follows that $K_2(A) \approx D \oplus \mu_p$ where D is the group given in 11 (v) and μ_p is the p-component of the roots of unity in F.

Let A be a discrete valuation ring with finite residue field of characteristic p and whose maximal ideal P is generated by the element π. Write $p = \omega\pi^e$ for some $\omega \in A^*$ (let $e = \infty$ in case A has characteristic p). Then $K_2(A/P^m)$ is a cyclic p-group of order p^t where

$$t = \left[\frac{m}{e} - \frac{1}{p-1}\right]_{[0,r]}$$

with p^r denoting the order of the p-component of the roots of unity in the completion of A in the P-adic topology [27, Theorem 4.3]. (For any real number x and any integer $r \geq 0$, $[x]_{[0,r]}$ denotes the nearest integer in the interval $[0,r]$ to the largest integer $\leq x$.) Moreover, $K_2(A/P^m)$ is generated by any symbol of the form

$$\{1+u\pi,\ 1+\pi^{\ell-1}\}$$

where $\ell = \frac{pe}{p-1}$ and u is any unit of A for which there is no solution z to the congruence

$$u \equiv \omega z + z^p \mod P.$$

In particular, any finite local principal ideal ring is the homomorphic image of a discrete valuation ring in a local field [27, §4] and hence its K_2 can be computed by the above formula. For example, if

$W_m(\mathbb{F}_q)$ denotes the ring of Witt vectors of length m over $\mathbb{F}_q$, $q = p^n$, then

(i) $K_2(\mathbb{F}_q[X]/(X^m)) = 1$ for all $m \geq 1$

(ii) $K_2(W_m(\mathbb{F}_q)) = 1$ if p is odd or if $m = 1$

(iii) $K_2(W_m(\mathbb{F}_q)) = \mathbb{Z}/2\mathbb{Z}$ if $p = 2$ and $m \geq 2$.

13. K_2 for some radical ideals.

Let A be a commutative ring and let $A[\epsilon]$, $\epsilon^2 = 0$, denote the dual numbers over A. Then Van der Kallen [105] has given a presentation for the kernel of the map $K_2(A[\epsilon]) \longrightarrow K_2(A)$ induced by $\epsilon \mapsto 0$. If 2 is an invertible element of A, then this kernel is isomorphic to the module of absolute differentials $\Omega_{A/\mathbb{Z}}$ (see [105] for a presentation in the general case). It should be noted that Van der Kallen's generators and relations are special consequences of those given in 9 above.

Using Van der Kallen's result together with a result of Stein (see 10 (b) above), it is possible to compute K_2 of some other rings. For example, if F is a perfect field of characteristic $p > 0$ (including $p = 2$), then

$$K_2(F[X,Y]/(X^2,XY,Y^2)) \approx K_2(F) \oplus F^+$$

where F^+ denotes the additive group of F. It then follows that

$$K_2(F[X_1,\ldots,X_m]/(X_iX_j \mid \text{all } i,j)) \approx K_2(F) \oplus (F^+)^k$$

where k is the binomial coefficient $\binom{m}{2}$. It should be noted that the generators not coming from $K_2(F)$ are of the form $\{1+X_i, 1+uX_j\}$, $i \neq j$, $u \in F$. If $u \neq 0$, these generators are non-trivial. Taking F a finite field, this answers a question of Swan [100, the end of §6].

14. Stability results.

We now make a list of some of the properties of the groups $K_2(n,R)$

and $St(n,R)$ and describe how they vary with n.

(a) $H_1(St(n,R))$ is trivial if $n \geq 3$ or if $n = 2$ and the elements u^2-1, $u \in R^*$, generate the unit ideal [88, (4.4)].

(b) $H_2(St(n,R))$ is trivial if $n \geq 5$; if $n = 4$ and u^2-1, $u \in R^*$, generate the unit ideal; or if $n = 2, 3$ and R is a K algebra over a field K such that $card(K) > 5$, $card(K) \neq 9$ [88, (5.3) and following remarks].

(c) If R is a ring which satisfies the stable range condition SR_m (see H. Bass, Algebraic K-Theory, p. 231), then

(i) The homomorphisms $K_2(n,R) \longrightarrow K_2(n+1,R)$ are surjective for all $n \geq m+1$,

(ii) $K_2(n,R)$ is in the center of $St(n,R)$ for all $n \geq m+2$,

(iii) The central extension

$$1 \longrightarrow K_2(n,R) \longrightarrow St(n,R) \longrightarrow E(n,R) \longrightarrow 1$$

is a universal central extension for all $n \geq \max(m+2,5)$,

(iv) $K_2(n,R) \approx H_2(E(n,R))$ for all $n \geq \max(m+2,5)$.

These results can be strengthened under special hypotheses on R (see [24], [25] and 15 below). These maps are known to be isomorphisms in only a few cases:

(i) $R = \mathbb{Z}$ and $n \geq 3$ [69, §10].

(ii) R is a field and $n \geq 3$ (see 11 above).

(iii) R is a discrete valuation ring or a homomorphic image thereof and $n \geq 3$ (see 12 above).

(iv) R is any semi-simple artinian ring or the polynomial ring in one indeterminant over such and $n \geq 3$ (see [24] and [25]).

(v) A few other simple cases can be derived from Van der Kallen's theorem which actually implies that the groups $K_2(n,(\epsilon))$ (as defined in 10 (b) above) are all isomorphic for $n \geq 3$. Since $K_2(n,(\epsilon))$ is a direct summand of $K_2(n,A[\epsilon])$, the maps will be isomorphisms if and only if the corresponding maps are isomorphisms on the complementary summand $K_2(n,A)$.

15. Rings of algebraic integers.

If $\underline{\underline{O}}$ is the ring of integers in an algebraic number field F, then the maps

$$K_2(n,\underline{\underline{O}}) \longrightarrow K_2(n+1,\underline{\underline{O}}) \longrightarrow K_2(\underline{\underline{O}})$$

are surjective for all $n \geq 3$ (see [24], [25]). It thus follows from a result of Garland [34] that $K_2(\underline{\underline{O}})$ is a finite group (in fact, that $K_2(n,\underline{\underline{O}})$ is finite for $n \geq 7$). Several other proofs of this result are now known. In particular, Quillen's localization exact sequence [73] yields

$$1 \longrightarrow K_2(\underline{\underline{O}}) \longrightarrow K_2(F) \xrightarrow{\lambda} \coprod_{\underline{\underline{p}}} (\underline{\underline{O}}/\underline{\underline{p}})^* \longrightarrow 1$$

and hence $K_2(\underline{\underline{O}}) = \operatorname{Ker} \lambda$ which is known to be finite by Garland [34].

An explicit computation of $K_2(\underline{\underline{O}})$ is known in very few cases. If $\underline{\underline{O}}$ is the ring of integers in a Euclidean quadratic imaginary number field $\mathbb{Q}(\sqrt{d})$, then Tate (unpublished computation) has shown that, $K_2(\underline{\underline{O}})$ is trivial unless $d = -7$ in which case it is cyclic of order 2 generated by the symbol $\{-1,-1\}$.

The results given in 12 above allow one to compute K_2 of any proper homomorphic image of a ring of integers $\underline{\underline{O}}$ since K_2 preserves finite products and since $\underline{\underline{O}}$ modulo a power of any maximal ideal is a finite local principal ideal ring. This computation, the exact sequence associated to an ideal, and the computation of $SK_1(\underline{\underline{O}},\underline{\underline{q}})$ by Bass-Milnor-Serre combine to give an estimate on the order of $K_2(\underline{\underline{O}})$. If F has more than one real embedding, the reciprocity uniqueness exact sequence of Moore [70, Theorem 7.4] (*cf.* [69, Theorem 16.1]) gives a better estimate on the order of $K_2(\underline{Q})$: If F_v denotes the completion of F with respect to v and $\mu(K)$ denotes the roots of unity in the field K, the sequence

$$K_2(F) \longrightarrow \coprod_v \mu(F_v) \longrightarrow \mu(F) \longrightarrow 1$$

is exact, where the sum is taken over all discrete or real archimedean valuations v. It is conjectured ([6], [65], [104]) that the order of the group $K_2(\underline{Q})$ is given by an explicit formula involving the zeta function of F. This has been proved in some cases by Coates and Lichtenbaum [17]. It should be noted that the analogous formula in the case of function fields has been proved [104, p. 206].

16. Free rings and polynomial extensions.

(a) Let X be any set and let $F\langle X\rangle$ be the free associative algebra over the division ring F. Then $K_2(F\langle X\rangle) = K_2(F)$ [82]. Using this result and a generalization of Quillen's localization exact sequence, Swan was able to prove that $K_2(\underline{Z}\langle X\rangle) = K_2(\underline{Z})$. This result is also true if $\underline{Z}$ is replaced by any left noetherian ring of finite global dimension (and 2 by i) [41, Theorem 2.8].

(b) If R is any regular ring, then Quillen [73, Theorem 11] has shown that

$$K_2(R[X]) = K_2(R),$$

and

$$K_2(R[X,X^{-1}]) = K_2(R) \oplus K_1(R).$$

17

PROBLEMS ON K_2

We have restricted this list of research problems to those which are only concerned with K_2. As there are many interesting problems dealing with the relationships of K_2 to other areas of mathematics a brief list of references appears at the end of this section. The conjectures of Lichtenbaum do not appear as they are discussed elsewhere in this volume [65]. It should be noted that several of the problems appearing below are special cases of those considered for higher K-functors [42].

We would like to thank H. Bass, S. Bloch, S. U. Chase, J. N. Graham, A. E. Hatcher, S. Lichtenbaum, R. W. Sharpe, R. G. Swan and J. Tate for suggesting problems. Any problems not attributed to one of the aforementioned are due to the authors of this note.

Problem 1. Is the "fundamental theorem of K-theory" valid for the functor K_2? As a discussion of this problem for the functors K_n appears in [42, Problem 3], we confine our remarks to the case where R is a commutative ring. Let C denote the kernel of the map $K_2(R[X]) \longrightarrow K_2(R)$ given by $X \mapsto 0$. If the product map [69, p. 67] $K_1(R[X]) \times K_1(R[X]) \longrightarrow K_2(R[X])$ is surjective, it follows from [95, Theorem 16.1] that C is generated by the symbols $\{A, I + XN\}$ where A is any element of $GL(R[X])$ and N is a nilpotent matrix with entries in R. Is it true that C is generated by these symbols for any commutative ring R?

Problem 2. Keeping the notation of the previous problem, we now assume that R has prime characteristic p. Is every element of C p-torsion? An affirmative answer to the last question of the previous problem would imply an affirmative answer to this question as the symbols of Milnor are bimultiplicative. (S.U.C.)

Problem 3. Do Milnor's elements $\alpha \star \beta$ (α, β commutating elements of $E(A)$; see [69, p. 63]) generate $K_2(A)$ for any ring A? Equivalently, given a central extension $1 \to C \to S \to E(A) \to 1$ such that commuting elements of $E(A)$ lift to commuting elements of S, is the extension trivial? (H.B.)

Problem 4. Let R be a ring which satisfies the stable range condition SR_m (see H. Bass, Algebraic K-Theory, p. 231). Prove that the maps

$$K_2(n,R) \longrightarrow K_2(n+1,R) \longrightarrow K_2(R)$$

are isomorphisms for $n \geq m+1$. Is this true for $n = m+1$? It is known that the maps are surjective for $n \geq m+2$ [25].

Problem 5. For each integer $n \geq 3$ give an example of a ring for which the map $K_2(n,R) \longrightarrow K_2(n+1,R)$ is not surjective. Do there exist rings for which this map is not injective? The case $n = 2$ is quite different from $n \geq 3$ as information about the multiplicative structure of R is not reflected in the structure of $St(2,R)$. The ring of integers $\mathbb{Z}$ gives an example where the map is not injective for $n = 2$ [69, p. 82]. In fact, $R = \mathbb{Z}[\sqrt{-17}]$ is an example for which the map is neither injective nor surjective for $n = 2$.

Problem 6. For each integer $n \geq 3$, is there an example of a ring for which $K_2(n,R)$ is not contained in the center of $St(n,R)$? Such a ring will have the property that $K_2(n,R) \longrightarrow K_2(n+1,R)$ is not injective as the image of $K_2(n,R)$ in $St(n+1,R)$ is always central (see [25] or [69, the proof of Theorem 5.1]). For $n = 2$, $R = \mathbb{F}_2 \times \mathbb{F}_2$, $\mathbb{Z}/6\mathbb{Z}$ give examples [90, Appendix].

Problem 7. If R is a Euclidean ring, the maps $K_2(n,R) \longrightarrow K_2(n+1,R)$ are surjective for all $n \geq 3$ [23], [25]. Is the map $K_2(2,R) \longrightarrow K_2(3,R)$ surjective? The answer is "yes" in case R is $\mathbb{Z}$, the ring of integers in a Euclidean quadratic imaginary number field, or F[X].

Problem 8. Let F be a field. Quillen [73, Theorem 11] has proven that $K_2(F) = K_2(F[X_1,..,X_m])$. How large must n be in order that $K_2(n,F) \longrightarrow K_2(n,F[X_1,...,X_m])$ be an isomorphism? For m = 1, using the results of Silvester [82] it can be shown that these maps are isomorphisms for $n \geq 2$ [25].

Problem 9. Let F be a field of characteristic $p > 0$. Does $K_2(F)$ have any p-torsion? If F is perfect $K_2(F)$ has no p-torsion as it is uniquely p-divisible [5, (1.4)]. It should be noted that if $K_2(F)$ has no p-torsion, then the same is true for any pure transcendental extension of F in view of the exact sequence

$$1 \longrightarrow K_2(F) \longrightarrow K_2(F(X)) \longrightarrow \coprod_{\underline{p}} (F[X]/\underline{p})^* \longrightarrow 1$$

[69, p. 106]. (S.U.C.)

Problem 10. If F is a subfield of L which is algebraically closed in L, is the homomorphism $K_2(F) \longrightarrow K_2(L)$ injective? An interesting special case of this is the following: Let $\underline{O}$ be a ring of integers in the number field K and let $\underline{p}$ be a prime of $\underline{O}$. Now take F to be the henselization of K at $\underline{p}$ and L to be the completion of K at $\underline{p}$. (S.L.)

Problem 11. Let F be a field with a primitive p-th root of unity ζ. Is every element of $K_2(F)$ of order p of the form $\{a,\zeta\}$ for some $a \in F$? If not, find conditions on F so that this will be true. This result holds for many fields if p = 2 by a result of Tate [104, Theorem 6] (cf. [6]). (S.L.)

Problem 12. Let F be a local field. By a theorem of Moore [69, Theorem A14] $K_2(F) \approx D \oplus \mu_F$ where μ_F is the group of roots of unity in F and D is a divisible group. Is D uniquely divisible? J. Carroll has proved that $K_2(F)$ is uniquely p-divisible provided that p does not divide $q(q-1)$ where q is the order of the residue field. A computation of Tate based on the solution of the previous problem for $p = 2$ gives the result for the 2-adic numbers $\mathbb{Q}_2$.

(J.T.)

Problem 13. Are the relations (S1) - (S7) listed in the previous section sufficient to present K_2 of a local ring? In view of [27, Lemma 2.4], it suffices to find a presentation for a local domain since any local ring is the homomorphic image of a local domain. In fact, it is possible to further assume that the ring is a noetherian unique factorization domain.

Problem 14. Let A be a discrete valuation ring with field of fractions F. In [27] (S1) - (S7) were shown to give a presentation for $K_2(A)$ by showing that they forced the map $K_2(A) \longrightarrow K_2(F)$ to be injective. Is this map injective for any local domain A? If not, is it injective if A is also regular?

Problem 15. If the last question has an affirmative answer when A is regular, does it follow that

$$K_2(A) = \bigcap K_2(A_{\underline{p}})$$

where the intersection is taken over all primes of height 1 ?

(S.B.)

Problem 16. If J is an ideal contained in the radical of the commutative ring R, it is known that the elements $\langle a,q\rangle$, $a \in R$, $q \in J$, generate $K_2(n,J)$ for all $n \geq 3$. Do the relations (H1) - (H4) given in the first section suffice to present $K_2(n,J)$?

Problem 17. Let S be an arbitrary collection of rational primes and let $\mathbb{Z}_S$ denote $\mathbb{Z}$ localized at the monoid generated by S. It follows from the exact sequence of Quillen [73] that

$$1 \longrightarrow K_2(\mathbb{Z}_S) \longrightarrow K_2(\mathbb{Q}) \longrightarrow \coprod_{p \notin S} (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow 1$$

is exact as $K_2(\mathbb{Z}/p\mathbb{Z})$ is trivial. If S is the set of all primes, a result of Tate [69, Theorem 11.6] shows that the sequence is split exact and it follows that the sequence is split exact for any set of primes S. Hence $K_2(Z_S) \approx \{\pm 1\} \oplus \coprod_{p \in S} (Z/pZ)^*$. Tate's argument also shows that there is an exact sequence

$$1 \longrightarrow K_2(F[X]) \longrightarrow K_2(F(X)) \longrightarrow \coprod_{\underline{p}} (F[X]/\underline{p})^* \longrightarrow 1.$$

If S is now an arbitrary set of primes from F[X], is it true that

$$1 \longrightarrow K_2(F[X]_S) \longrightarrow K_2(F(X)) \longrightarrow \coprod_{\underline{p} \notin S} (F[X]/\underline{p})^* \longrightarrow 1$$

is exact?

Problem 18. Let $F = K((t))$ be the field of Laurent series over a field K. If F has the (t) - adic topology, J. Graham [44], [45] has constructed a continuous symbol

$$F^* \times F^* \longrightarrow K_2(K) \oplus K^* \oplus \Omega_K[[t]]$$

where the first two factors have the discrete topology and where $\Omega_K[[t]]$ (the module of formal power series over the module of absolute differentials Ω_K) has the (t) - adic topology. If K has characteristic 0, the above symbol is universal for continuous symbols with values in the projective limit of discrete groups. Find the universal continuous symbol in case the characteristic of K is non-zero. (J.N.G.)

Problem 19. Let A be a commutative ring. Compute K_2 of the ring $R = A[X]/(X^n)$. As there is a split exact sequence

$$1 \longrightarrow K \longrightarrow K_2(R) \longrightarrow K_2(A) \longrightarrow 1,$$

it suffices to compute the kernel (assuming $K_2(A)$ to be known). If $n = 2$, this has been done by van der Kallen [105] for any commutative ring. If $A = F$ is a field, a presentation for this group can be found for any n as it was for $n = 2$ in [27].

In the case F has characteristic 0, Graham [44] has identified the kernel as the direct sum of $n-1$ copies of the absolute differentials Ω_F.

Problem 20. What is the relation between $K_2(R)$ and $K_2(R/I)$ where I is a nilpotent ideal? Note that the previous problem is a special case of this question. In particular, if I is any abelian group, make I a ring by $I^2 = 0$ and adjoin a unit getting $I^+ = \mathbb{Z} \times I$ with the obvious multiplication. Compute $K_2(I^+)$ (cf. [42, Problem 22]).

(R.G.S.)

Problem 21. Let $\underline{\underline{O}}$ be the ring of integers in an algebraic number field F. The exact sequence

$$1 \longrightarrow K_2(\underline{\underline{O}}) \longrightarrow K_2(F) \xrightarrow{\lambda} \coprod_{\underline{\underline{p}}} (\underline{\underline{O}}/\underline{\underline{p}})^* \longrightarrow 1$$

due to Quillen [73, Theorem 8] shows that the computations of $\operatorname{Ker} \lambda$ by Coates and Lichtenbaum [17] sometimes give the precise order of $K_2(\underline{\underline{O}})$. In particular, they obtain the following:

$$F = Q(\sqrt{11}) \qquad |K_2(\underline{\underline{O}})| = 28$$

$$F = Q(\sqrt{14}) \qquad |K_2(\underline{\underline{O}})| = 40$$

$$F = Q(\sqrt{19}) \qquad |K_2(\underline{\underline{O}})| = 76.$$

As all symbols in $K_2(\underline{\underline{O}})$ for a real quadratic field are generated by $\{\varepsilon,-1\}$ and $\{-1,-1\}$ where ε is the fundamental unit, it is clear that $K_2(\underline{\underline{O}})$ is not generated by symbols. Explicitly exhibit the generators of $K_2(\underline{\underline{O}})$. It is known that the maps $K_2(n,\underline{\underline{O}}) \longrightarrow K_2(n+1,\underline{\underline{O}})$ are surjective for $n \geq 3$ but are not surjective in general for $n = 2$ [25], [27, Theorem 5.3]. In particular, examples of elements that lie in $K_2(3,\underline{\underline{O}})$ but not in $K_2(2,\underline{\underline{O}})$ would be interesting.

Problem 22. Let π be a finite group. Can the results of Garland [34] be extended to prove that $K_2(\mathbb{Z}\pi)$ is a finite group? Is $Wh_2(\pi)$

(a certain quotient of $K_2(Z\pi)$; see [48], [50], [108]) a finite group? A character on π will induce a homomorphism $K_2(\mathbb{Z}\pi) \longrightarrow K_2(\mathbb{Q}(\zeta))$ for some root of unity ζ. By completing $\mathbb{Q}(\zeta)$ at an appropriate prime and then applying the norm residue symbol, Milnor (unpublished) was able to show that for π cyclic of order 20, $Wh_2(\pi)$ and $K_2(\mathbb{Z}\pi)$ have at least 5 elements. An equivalent computation based on the results of [27] was made by Dennis (also unpublished) for π cyclic of order 21. In this case it follows that there are at least 7 elements. This method fails to detect any elements of $Wh_2(\pi)$ if π is cyclic of prime-power order. Is $Wh_2(\pi)$ trivial in this case? (A.E.H.)

Problem 23. Can generators and relations for K_2 of a division ring be given as in Matsumoto's presentation for K_2 of a field? (R.G.S.)

Problem 24. Compute K_2 of a finite ring. (R.G.S.)

Problem 25. Can Sharpe's LPLU form in the Steinberg group (see the first section) be used to compute K_2? The analogous normal form for unitary K_2 can be used to make such computations [79].

(R.W.S.)

Related Areas of Interest

(1) The functors K_n defined for fields by Milnor are intimately related to K_2. Several problems concerning them are discussed in this volume [5], [33] (see also [31], [32], [84]).

(2) It is possible to define functors analogous to K_2 by using groups other than the elementary group. Many of the questions asked above for K_2 can also be asked for these functors. The interested reader should consult [27], [52], [53], [58], [59], [67], [77], [79], [85], [86], [88], [89], and [90]. It is known [67], [27] that all of the K_2-like functors defined by using a non-symplectic Chevalley group agree for fields and discrete valuation rings. Is this true for all rings?

BIBLIOGRAPHY FOR K_2

The basic material listed in this bibliography consists of books and papers that fall into two categories: 1) those that deal primarily with the functor K_2 and 2) those that might be of use in computing K_2 (i.e. those that deal with the presentation of linear groups). Also included are some papers dealing with applications or relationships of K_2 to number theory, topology, or other parts of K-theory. We have not attempted to give a complete listing in these areas. In particular, those readers interested in higher K-theory should also consult the survey article of Gersten [41] which appears in this volume.

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t Topology, relationships and applications.

28, 47, 48, 49, 50, 51, 52, 55, 77, 78, 79, 80, 106, 108, 109, 110.

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K_2 OF RADICAL IDEALS AND SEMI-LOCAL RINGS REVISITED

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Quite general surjective stability theorems are now known for the functor K_2 [D]. These imply, in particular, that for a semi-local ring R, the maps

$$K_2(n,R) \longrightarrow K_2(n+1,R) \longrightarrow K_2(R)$$

are surjective for all $n \geq 2$. This special case was first proved for most commutative semi-local rings by showing that $K_2(n,R)$ was generated by the Steinberg symbols $\{u,v\}_R$, $u,v \in R^*$ [St2, Theorem 2.13]. This method had the advantage of exhibiting an explicit set of generators for $K_2(R)$, but suffered from the restriction that it was necessary to assume that R was additively generated by its group of units, R^*.

In this note we shall outline a method of constructing elements of $K_2(R)$ for any commutative ring R which in the semi-local case provides a set of generators for $K_2(R)$ and removes the restriction mentioned above. In the case of commutative semi-local rings which are generated by their units, these new generators are related in an explicit way to Steinberg symbols, but in the general case they provide elements of $K_2(R)$ which need not be products of such symbols. Moreover, these elements satisfy certain identities analogous to those satisfied by Steinberg symbols which allow one to

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compute effectively with them. In particular, we will show that for a commutative semi-local ring R, $K_2(n,R)$ is always generated by Steinberg symbols when $n \geq 3$. This settles certain outstanding cases of finding generators and relations for SL_n of a semi-local ring which were left open in [Si2] and [St2, Corollary 2.14]. However, we have been unable to decide the one remaining case, namely under what conditions will $K_2(2,R)$ be generated by symbols when R has one residue class with exactly 2 elements.

The construction and theorems which we present in this note are not peculiar to K_2, but are valid for any of the functors $L(\Phi, \ )$ introduced in [St1], provided that Φ is a non-symplectic root system with only one root length and the Chevalley group in question is assumed to be universal (see [St1, (3.3)] and [St2, Notation and Terminology]). The interested reader may make the necessary translations according to the usual dictionary.

Throughout this note, R is a commutative ring with 1, α,β denote pairs of indices ij, $1 \leq i, j \leq n$, and $-\alpha,-\beta$ denote the reversed pairs ji. Unexplained notation and terminology is that of [D-S, Section 0].

1. The elements $\langle a,b\rangle$ and some relations they satisfy.

Let $a,b \in R$ by any two elements such that $1+ab \in R^*$. For each pair of indices α, define

$$H_\alpha(a,b) = x_{-\alpha}(-b(1+ab)^{-1})x_\alpha(a)x_{-\alpha}(b)x_\alpha(-(1+ab)^{-1}a)$$

and set

$$\langle a,b\rangle_\alpha = H_\alpha(a,b)h_\alpha(1+ab)^{-1}.$$

Clearly for all $n \geq 2$, $\langle a,b\rangle_\alpha \in K_2(n,R)$, and it follows immediately from the definition that

(1) $$^{x_\alpha(a)}x_{-\alpha}(b) = x_\alpha(a)x_{-\alpha}(b)x_\alpha(-a)$$

$$= x_{-\alpha}(b(1+ab)^{-1})\langle a,b\rangle_\alpha h_\alpha(1+ab)x_\alpha(-a^2b(1+ab)^{-1}).$$

1.1 PROPOSITION. <u>For all $n \geq 3$, the elements $\langle a,b\rangle_\alpha$ are independent of the pair of indices α and satisfy the following relations</u>:

(H1) $\langle a,b\rangle = \langle -b,-a\rangle^{-1}$

(H2) $\langle a,b\rangle = \{-a,1+ab\}$ <u>if</u> $a \in R^*$

$\langle a,b\rangle = \{1+ab,b\}$ <u>if</u> $b \in R^*$

(H3) $\langle a+b,c\rangle = \langle a,c\rangle\langle b,\frac{c}{1+ac}\rangle\left\{\frac{1+(a+b)c}{1+ac},\ 1+ac\right\}$

$\langle a,b+c\rangle = \langle a,b\rangle\langle\frac{a}{1+ab},c\rangle\left\{1+ab,\ \frac{1+a(b+c)}{1+ab}\right\}$

(H4) $\langle a+b,c\rangle = \langle a,c\rangle\langle b,c\rangle\langle\frac{b}{1+bc},\ \frac{-ac^2}{1+ac}\rangle\{-1,1+ac\}\left\{\frac{1+(a+b)c}{1+bc},\ \frac{1+ac}{1+bc}\right\}$

$\langle a,b+c\rangle = \langle a,b\rangle\langle a,c\rangle\langle -\frac{a^2b}{1+ab},\ \frac{c}{1+ac}\rangle\{1+ab,-1\}\left\{\frac{1+ab}{1+ac},\ \frac{1+a(b+c)}{1+ac}\right\}$

(H5) $\langle a,bc\rangle\langle b,ac\rangle\langle c,ab\rangle = 1$

$\langle a,bc\rangle = \langle ab,c\rangle\langle ac,b\rangle$

Since $n \geq 3$, it follows from any one of [Mi, the proof of Theorem 5.7], [D] or [St1, Theorem 5.1] that $\langle a,b\rangle_\alpha$ is central in $St(n,R)$ for any α. In particular, if $\alpha = (ij)$ and β is any other pair of indices, we may find (since $n \geq 3$) a $w \in St(n,R)$ such that $\varphi(w) = PD$ is the product of a permutation matrix P carrying α to β and a diagonal matrix $D = \mathrm{diag}(v_1,\ldots,v_n)$ with $v_i = v_j = 1$ (cf. [Mi, Corollary 9.4]). It is then clear that

$$\langle a,b\rangle_\alpha = w\langle a,b\rangle_\alpha w^{-1} = \langle a,b\rangle_\beta ,$$

which proves the first statement of the Proposition.

Identities (H1)-(H5) are proved using the centrality of $\langle a,b\rangle$, Equation (1), and the usual Steinberg relations and their consequences ([St1, (3.8)], [Mi, Corollary 9.4]). Moreover it is clear that either of the parts of (H2)-(H5) can be deduced immediately from the other part using (H1).

To prove (H1) we evaluate the extreme left and right sides of the equalities

$$\begin{aligned} {}^{x_\alpha(a)}x_{-\alpha}(b) &= x_{-\alpha}(b)\ {}^{x_{-\alpha}(-b)}x_\alpha(a)\ \ x_\alpha(-a) \\ &= x_{-\alpha}(b)\left({}^{x_{-\alpha}(-b)}x_\alpha(-a)\right)^{-1}\ x_\alpha(-a) \end{aligned}$$

using Equation (1). Identity (H2) is an immediate consequence of [St2, Proposition 2.7c]. To prove the first statement of (H3), we evaluate the two sides of

$${}^{x_\alpha(a+b)}x_{-\alpha}(c) = {}^{x_\alpha(b)x_\alpha(a)}x_{-\alpha}(c),$$

and the second part of (H4) is proved by similarly evaluating

$$\begin{aligned} {}^{x_\alpha(a)}x_{-\alpha}(b+c) &= {}^{x_\alpha(a)}(x_{-\alpha}(b)x_{-\alpha}(c)) \\ &= {}^{x_\alpha(a)}x_{-\alpha}(b)\ \ {}^{x_\alpha(a)}x_{-\alpha}(c). \end{aligned}$$

Finally, (H5) is proved by evaluating the Philip Hall identity

$$^{y}[x,[y^{-1},z]]\ ^{z}[y,[z^{-1},x]]\ ^{x}[z,[x^{-1},y]] = 1$$

as in [Sw, Lemma 7.7] or [D-S, Proposition 1.1] with $x = x_{12}(-a)$, $y = x_{23}(-b)$, $z = x_{31}(-c)$, and then applying (H1).

REMARKS. 1. For $n = 2$, the elements $\langle a,b\rangle_\alpha$ are not necessarily central in $St(n,R)$ as is shown in the Appendix. It is still possible to carry through the computations indicated in the proof of Proposition 1.1, but is not clear what value the more complicated identitites thus proved have. An example of such a calculation can be found in the next section (Lemma 2.3).

2. There are many other identities satisfied by the elements $\langle a,b\rangle$ which may be deduced from Proposition 1.1. Here are some examples.

(a) If $ab = 0$, (H5) implies

$$\langle a,bc\rangle = \langle ac,b\rangle.$$

(b) If $ab = 0$ and $1+a, 1+b \in R^*$, (H2) and (H3) imply

$$\begin{aligned}\{1+a,1+b\} &= \{1+a(b+1),1+b\}\\ &= \langle a,b+1\rangle\\ &= \langle a,b\rangle.\end{aligned}$$

(c) It follows from (H1), (H2) and (H5) that

$$\langle a,b\rangle\langle b,a\rangle = \{1+ab,-1\}.$$

(d) Equating the second parts of (H3) and (H4), then applying (H5) and (H2) yields

$$\langle \tfrac{a}{1+ab}, c\rangle = \langle a,c\rangle\langle \tfrac{-a^2b}{(1+ab)(1+ac)}, c\rangle.$$

Applying (H1) to this, then replacing a, b and c by their negatives and interchanging a and c yields

$$\langle a, \tfrac{c}{1+bc}\rangle = \langle a,c\rangle\langle a, \tfrac{-bc^2}{(1+ac)(1+bc)}\rangle.$$

(e) Let $a_1,\dots,a_n \in R$ and set $a = \Pi a_i$, $\hat{a}_i = \prod_{j\neq i} a_j$. Then if $1+a$ is a unit,

$$\prod_{i=1}^{n} \langle \hat{a}_i, a_i \rangle = \langle 1, a \rangle = \{-1, 1+a\}$$

which follows by induction from (H1), (H2) and (H5).

(f) Let $q, a_1, \ldots, a_s, b_1, \ldots, b_t \in R$. Define $y_0 = z_0 = 0$; $y_k = \sum_{i=1}^{k} a_i$, $z_k = \sum_{j=1}^{k} b_j$. Then if $y_s = z_t$ and if $1+qy_i$, $1+qz_j \in R^*$, $i = 1, \ldots, s$, $j = 1, \ldots, t$, we have

$$\prod_{i=1}^{s} \langle a_i, \frac{q}{1+qy_{i-1}} \rangle \left\{ \frac{1+qy_i}{1+qy_{i-1}}, 1+qy_{i-1} \right\}$$

$$= \prod_{j=1}^{t} \langle b_j, \frac{q}{1+qz_{j-1}} \rangle \left\{ \frac{1+qz_j}{1+qz_{j-1}}, 1+qz_{j-1} \right\} .$$

These identities are all consequences of the special case $s = 1$, $t = 2$, (that is, of (H3)). Moreover if $a_1, \ldots, a_s, b_1, \ldots, b_t \in R^*$, replacing each of them and q by their negatives yields the (s,t)-identities of [D-S, Proposition 1.5].

3. The generators given by Van der Kallen [V] for $K_2(R[\epsilon], (\epsilon))$ are related to the elements $\langle a, b \rangle$ as follows:

$$f_{ij}(a,b) = \langle a\epsilon, b\epsilon \rangle = \{1+a\epsilon, 1+b\epsilon\}$$

$$H_\alpha(a,b) = \langle b, a\epsilon \rangle h_\alpha(1+ab\epsilon)$$

$$N_\alpha(a,b) = \langle b, a\epsilon \rangle \langle ab\epsilon, ab\epsilon \rangle = \langle b, a\epsilon \rangle \{1+ab\epsilon, 1+ab\epsilon\} .$$

It is easy to derive Van der Kallen's relations from this list and Proposition 1.1. Van der Kallen, of course, proves the deep result that these relations suffice to present $K_2(R[\epsilon], (\epsilon))$. In Section 2 we will show that if J is an ideal contained in the radical of some commutative ring R, $K_2(R,J)$ is generated by the elements $\langle a, q \rangle$, $a \in R$, $q \in J$. Based on the evidence of Van der Kallen's theorem and the results of [D-S, Section 2], we conjecture that the relations of Proposition 1.1 suffice to present $K_2(R,J)$ in the general case.

2. Surjective stability for radical ideals and semi-local rings.

Suppose J is an ideal in the Jacobson radical of the commutative ring R. Since $1+q \in R^*$ for every $q \in J$, we may define for any $n \geq 3$ a pairing

$$\langle\ ,\ \rangle : R \times J \longrightarrow K_2(n,J)$$

by $(a,q) \longmapsto \langle a,q\rangle$. The subgroup of $K_2(n,J)$ generated by the image of this pairing will be denoted by $D_n(J)$. We extend this definition to the case $n = 2$ by letting $D_2(J)$ be the subgroup of $K_2(2,J)$ generated by all $\langle a,q\rangle_\alpha$ and $\langle a,q\rangle_{-\alpha}$, $\alpha = (12)$, $a \in R$, $q \in J$.

The main results of this section are the following Theorem and Corollary.

2.1 THEOREM. *Let J be an ideal contained in the Jacobson radical of the commutative ring R. Then $D_n(J) = K_2(n,J)$ for all $n \geq 2$, and consequently the maps*

$$K_2(n,J) \longrightarrow K_2(n+1,J) \longrightarrow K_2(J)$$

are surjective for all $n \geq 2$.

2.2 COROLLARY. *Let R be a commutative semi-local ring. If $n \geq 3$, $K_2(n,R)$ is generated by the elements $\langle a,b\rangle$, $a,b \in R$, $1+ab \in R^*$. Moreover, $K_2(2,R)$ is normally generated by the elements $\langle a,b\rangle_{12}$, $\langle a,b\rangle_{21}$. Consequently, for all $n \geq 2$, the maps*

$$K_2(n,R) \longrightarrow K_2(n+1,R) \longrightarrow K_2(R)$$

are surjective.

The proofs of these two results are almost exactly the same as those of [St2, Theorems 2.5 and 2.13]. We define

$U_n^-(J)$ = the subgroup of $St(n,J)$ generated by all $x_{ij}(q)$, $q \in J$, $i > j$,

$U_n(J)$ = the subgroup of $St(n,J)$ generated by all $x_{ij}(q)$, $q \in J$, $i < j$,

$H_n(J)$ = the subgroup of $St(n,J)$ generated by all $h_\alpha(1+q)$, $q \in J$,

and set

$$M_n(J) = U_n^-(J)D_n(J)H_n(J)U_n(J).$$

According to [Mi, Lemma 9.14] the projection map $St(n,J) \twoheadrightarrow E_n(J)$ restricts to an isomorphism on each of $U_n^-(J)$ and $U_n(J)$. Moreover it follows exactly as in [St2, Theorem 2.3b] that

$$M_n(J) \cap K_2(n,J) = D_n(J).$$

Thus to complete the proof of Theorem 2.1, it will suffice to prove that $M_n(J) = St(n,J)$.

It is clear, however, that $M_n(J) \subset St(n,J)$; moreover $x_\alpha(q) \in M_n(J)$ for each $q \in J$ and all α. Thus it will suffice to show that $M_n(J)$ is a normal subgroup of $St(n,R)$ (since $St(n,J) = Ker(St(n,R) \twoheadrightarrow St(n,R/J))$ is the smallest such normal subgroup). The proof now proceeds by a series of reductions as in [St2, Theorem 2.5]. The only possible source of difficulty occurs when $n = 2$, for then $\langle a,b\rangle_\alpha$ is not necessarily central. We first deal with this problem.

2.3 LEMMA. $D_2(J)$ _is a normal subgroup of_ $St(2,R)$.

Let $a, b \in R$ and write $\alpha = (12)$. We begin by using Equation (1) to compute the two sides of the equality

$$^{x_\alpha(a+b)}x_{-\alpha}(q) = {}^{x_\alpha(b)x_\alpha(a)}x_{-\alpha}(q),$$

taking care not to assume that the elements $\langle a,q\rangle_\alpha$ are central in $St(n,R)$. After simplifying the resulting equation, we obtain

$$x_\alpha\left(\frac{b(1+(a+b)q)}{1+aq}\right)\left\langle\frac{a(1+(a+b)q)^2}{(1+aq)^2}, \frac{q(1+aq)^2}{(1+(a+b)q)^2}\right\rangle_\alpha$$

$$= \left\langle b, \frac{q}{1+aq}\right\rangle_\alpha^{-1} \left\langle a+b, q\right\rangle_\alpha \left\{\frac{1+(a+b)q}{1+aq}, 1+aq\right\}_\alpha .$$

For $c, d \in R$, $p \in J$, the above equation allows us to show that

$$x_\alpha(c)^{\langle d,p\rangle_\alpha} \in D_2(J)$$

provided that we can solve the equations

$$c = \frac{b(1+(a+b)q)}{1+aq},$$

$$d = \frac{a(1+(a+b)q)^2}{(1+aq)^2},$$

$$p = \frac{q(1+aq)^2}{(1+(a+b)q)^2}$$

for some $a,b \in R$, $q \in J$. It is easily checked that

$$a = \frac{d(1+(d-c)p)^2}{(1+dp)^2},$$

$$b = \frac{c(1+(d-c)p)}{1+dp}$$

$$c = \frac{p(1+dp)^2}{(1+(d-c)p)^2}$$

satisfy the above equations. Moreover a simple computation shows that

$$w_\alpha(1)^{\langle a,q\rangle_\alpha\{-1,1+aq\}_{-\alpha}} = \langle -a,-q\rangle_{-\alpha} .$$

Since the elements $x_\alpha(c)$, $w_\alpha(1)$ generate $St(2,R)$, this completes the proof of the lemma.

We now outline the series of reductions which prove Theorem 2.1.

(2.4) If the set $M_n(J)$ is normalized by $St(n,R)$, then $M_n(J)$ is a normal subgroup of $St(n,R)$ (proof as in [St2, proof of Proposition 2.10]).

(2.5) The set $M_n(J)$ is normalized by $St(n,R)$ if and only if

$$^{x_\alpha(a)}x_{-\alpha}(q) \in M_n(J)$$

for all α and all $a \in R$, $q \in J$ (proof as in [St2, Lemma 2.6]).

(2.6) Equation (1) holds; i.e.

$$^{x_\alpha(a)}x_{-\alpha}(q) \in M_n(J).$$

Let us now pass to the proof of Corollary 2.2. We now take J to be the whole Jacobson radical of our semi-local ring R and we consider $\bar{R} = R/J$, a finite product of fields. We see from the proof of [St2, Theorem 2.13] that $K_2(n,\bar{R})$ is generated by the Steinberg symbols $\{\bar{u},\bar{v}\}$ together with all conjugates of the elements (if $n = 2$)

$$[x_\alpha(0,\ldots,\bar{a}_i,\ldots,0), x_{-\alpha}(0,\ldots,\bar{a}_j,\ldots,0)],\ i \neq j,$$

where $\bar{a}_k$ occurs in the k-th factor of $\bar{R}$ and the component in all other factors is 0. Since

$$1 + (0,\ldots,\bar{a}_i,\ldots,0)(0,\ldots,\bar{a}_j,\ldots,0) = 1,$$

it follows immediately from the definition that these additional generators are conjugates of the elements $\langle\bar{a},\bar{b}\rangle_\alpha$ for $\bar{a},\bar{b} \in \bar{R}$. Thus $K_2(n,\bar{R})$ is generated by the conjugates of the elements $\langle\bar{a},\bar{b}\rangle_{\pm\alpha}$, $\bar{a},\bar{b} \in \bar{R}$, $1+\bar{a}\bar{b} \in \bar{R}^*$. But units in $\bar{R}$ can be lifted to units of R; hence the Corollary follows from the Theorem and the exact sequence

$$1 \longrightarrow K_2(n,J) \longrightarrow K_2(n,R) \longrightarrow K_2(n,\bar{R}) \longrightarrow 1.$$

REMARK. In the Appendix it is shown that the word "normally" cannot be deleted from the statement of the Corollary in case $\bar{R}$ has two or more $\mathbb{F}_2$ factors. If R is local or $\bar{R}$ has no $\mathbb{F}_2$ factors, $K_2(2,R)$ is actually generated by Steinberg symbols [St2, Theorem 2.13].

We will now give two applications of these results. The first is to the problem of finding generators and relations for $SL_n(R) = E_n(R)$ when R is a commutative semi-local ring. Partial solutions to this problem were given by Silvester [Si2] in terms of the concepts "universal and quasi-universal for GE_n, $n \geq 2$"; a partial solution simultaneously was found in [St2, Theorem 2.14] as a Corollary to work on $K_2(R)$. The connection between these two papers is given succinctly by the result of [D] that for commutative rings R, the statement "R is universal for GE_n, $n \geq 2$ (resp. quasi-universal for GE_n, $n \geq 3$)" is equivalent to the statement "$K_2(n,R)$ is generated by Steinberg symbols (resp. by the elements $\langle a,b \rangle$, $a,b \in R$, $1+ab \in R^*$)." For commutative semi-local rings R, with $\bar{R} = R/J$, the situation until now may be conveniently summarized in the following table:

R is \ $\bar{R}$ has	no $\mathbb{F}_2$ factor	1 $\mathbb{F}_2$ factor	2 or more $\mathbb{F}_2$ factors
quasi-universal for GE_n, $n \geq 2$	Yes [Si2, Theorem 14]	Yes [Si2, Theorem 14]	Yes [Si2, Theorem 14]
universal for GE_n, $n \geq 3$	Yes [Si2, Theorem 14], [St2, Corollary 2.14]	Yes [Si2, Theorem 14], [St2, Corollary 2.14]	? (See below)
universal for GE_2	Yes [Si2, Theorem 14], [St2, Corollary 2.14]	? (See Appendix, Example 2)	No [Si, Corollary 28] (see Appendix, Example 1)

We will now show that, in fact, all commutative semi-local rings are universal for GE_n, $n \geq 3$. Thus there is only one outstanding case: Is a semi-local ring R such that $\bar{R}$ has exactly one direct factor isomorphic to $\mathbb{F}_2$, universal for GE_2? We do not know the answer in general; however, J. Silvester has proved that $\mathbb{Z}/6\mathbb{Z}$ is <u>not</u> universal for GE_2. A proof of this appears in the Appendix, Example 2.

2.7 THEOREM. <u>Let</u> R <u>be a commutative semi-local ring and let</u> J <u>be an ideal contained in the Jacobson radical</u> $J(R)$ <u>of</u> R. <u>Then for all</u> $n \geq 3$, $K_2(n,J)$ <u>and</u> $K_2(n,R)$ <u>are generated by Steinberg symbols</u>.

Since $K_2(n,R/J(R))$ is generated by symbols for all $n \geq 3$ [St2, Theorem 2.13], it will suffice to prove that $K_2(n,J)$ is generated by symbols. According to Theorem 2.1, $K_2(n,J)$ is generated by the elements $\langle a,q\rangle$, $a \in R$, $q \in J$. It follows from Proposition 1.1 and the remarks following it that modulo the subgroup of $K_2(n,J)$ generated by symbols, the following identities hold:

1) $\langle a,q\rangle \equiv 1$ if $a \in R^*$ (H2)

2) $\langle a+b,q\rangle \equiv \langle a,q\rangle\langle b,\frac{q}{1+aq}\rangle$ (H3)

3) $\langle ab,q\rangle \equiv \langle a,bq\rangle\langle b,aq\rangle$ (H5)

4) $$\langle b,\frac{q}{1+aq}\rangle \equiv \langle b,q\rangle\langle b,\frac{-aq^2}{(1+aq)(1+bq)}\rangle$$
$$\equiv \langle b,q\rangle\langle abq,\frac{q}{(1+aq)(1+bq)}\rangle\langle bq,\frac{-aq}{(1+aq)(1+bq)}\rangle .$$
(Remark 2d and (H5))

Moreover, it follows from 1) and 2) that if $u \in R^*$,

5) $\langle a+u,q\rangle \equiv \langle a,q\rangle$

and that if $p \in J(R)$,

6) $$\langle p,q\rangle = \langle (1+p)-1,q\rangle$$
$$\equiv \langle 1+p,q\rangle\langle -1,\frac{q}{1+(1+p)q}\rangle$$
$$\equiv 1.$$

It then follows from 2), 4) and 6) that

7) $\langle a+b,q\rangle \equiv \langle a,q\rangle\langle b,q\rangle$.

Let us now write $R/J(R) = \mathbb{F}_2^k \times S$, where S is a product of fields all different from $\mathbb{F}_2$. Then given any $a \in R$, there exist units $u_1,\ldots,u_n \in R^*$, such that

$$\overline{a+u_1+\ldots+u_n} = (x,0) \in \mathbb{F}_2^k \times S.$$

Hence it follows from 5) that $K_2(n,J)$ modulo symbols is generated by the elements

$$\langle a,q\rangle, a \in R,\ \bar{a} = (x,0),\ q \in J.$$

But if $\bar{a} = (x,0) \in \mathbb{F}_2^k \times S$, we must have $2\bar{a} = \bar{a}^2 + \bar{a} = 0$; that is

8) $2a \in J(R)$,

9) $a^2 + a \in J(R)$.

It then follows from 6), 7) and 8) that

$$\begin{aligned} 1 &\equiv \langle 2a,aq\rangle \\ &\equiv \langle a,aq\rangle^2. \end{aligned}$$

On the other hand, it follows from 9), 6), 7) and 3) that

$$\begin{aligned} 1 &\equiv \langle a^2+a,q\rangle \\ &\equiv \langle a^2,q\rangle\langle a,q\rangle \\ &\equiv \langle a,aq\rangle^2\langle a,q\rangle. \end{aligned}$$

Thus

$$\begin{aligned} \langle a,q\rangle &\equiv (\langle a,aq\rangle^{-1})^2 \\ &\equiv 1 \end{aligned}$$

for all generators of $K_2(n,J)$ modulo the symbols, and $K_2(n,J)$ is generated by symbols, as asserted.

Let $W_2(\mathbb{F}_q)$ denote the ring of Witt vectors of length two over the finite field $\mathbb{F}_q$, $q = p^n$. The second application of Theorem 2.1 is

2.8 THEOREM. Let p be a rational prime and let $R = W_2(\mathbb{F}_q)$, $q = p^n$. Then $K_2(R[X])$ is an elementary abelian p-group of countably infinite rank.

It follows from results of Silvester [Sil] and Steinberg [Stb, 3.3] that $K_2(\mathbb{F}_q[X]) \approx K_2(\mathbb{F}_q) = 1$. Hence if $J = \text{rad } R[X] = pR[X]$, we deduce from the exact sequence

$$1 \longrightarrow K_2(R[X],J) \longrightarrow K_2(R[X]) \longrightarrow K_2(\mathbb{F}_q[X])$$

and Theorem 2.1 that $K_2(R[X])$ is generated by the elements

$$\langle f,pg\rangle, f,g \in R[X],\ g \notin J.$$

If p is odd, note first that any symbol of the form $\sigma = \{1+\alpha p, 1+\beta p\}$ is trivial, since by (a) and (b) of Remark 2 in §1

$$\sigma = \{1+\alpha\beta p, 1+p\} = \{1+p, 1+\alpha\beta p\}$$

which implies $\sigma^2 = 1$. But clearly $\sigma^p = 1$ as well.

It follows, therefore, from (b) and (H4) that we may assume $f \notin J$, and that

$$1 = \langle pf,pg\rangle = \langle f,pg\rangle^p.$$

Thus $K_2(R[X])$ is generated by the elements

$$\langle f,pg\rangle, f,g \in R[X], f,g \notin J,$$

each of which has order p.

If $p = 2$, $K_2(R[X])$ is generated by

$$\langle f,2g\rangle$$

$$\langle 2f,2g\rangle = \{1+2f,1+2g\} = \{-1,1+2fg\}$$

for $f,g \in R[X]$, $f,g \notin J$. It is clear that the elements $\langle 2f,2g\rangle$ have order 2. However we also have by (H4)

$$\{-1,1+2fg\} = \langle 2f,2g\rangle$$
$$= \langle f,2g\rangle^2\{-1,1+2fg\}$$

which shows that $\langle f,2g\rangle^2 = 1$. Thus $K_2(R[X])$ is an elementary abelian p-group in this case as well.

For a given finite field $\mathbb{F}_q$, we choose an element $u \in W_2(\mathbb{F}_q)$ for which there is no solution $z \in \mathbb{F}_q$ to the congruence

$$-u \equiv -z + z^p \bmod p.$$

To complete the proof we will show that the infinite set of generators

$$\langle uX, pX^{k_i}\rangle,\ k_i = p^i-1$$

are non-trivial and distinct from each other, using the techniques of [D-S].

Write $A = W(\mathbb{F}_q)$, the ring of infinite Witt vectors over $\mathbb{F}_q$, and let $A_j = A[\zeta_j]$, where ζ_j is a primitive p^jth root of unity. Then

$$A_j \approx A[X]/(\Phi_{p^j}(X)) \approx A[Y]/(\Phi_{p^j}(Y+1))$$

where $\Phi_{p^j}(X)$ is the usual cyclotomic polynomial. Since $\Phi_{p^j}(Y+1)$ is an Eisenstein polynomial, it follows from [S, Chapitre 1, Proposition 17] that A_j is a discrete valuation ring for all $j \geq 1$, whose maximal ideal is generated by $\pi_j = \zeta_j-1$. We define

$$e_j = p^{j-1}(p-1)$$
$$r_j = \frac{pe_j}{p-1} = p^j$$

and set

$$R_j = A_j/(\pi_j^{r_j}).$$

We define a homomorphism

$$A[X] \longrightarrow A_j = A[\zeta_j]$$

by sending X to $\pi_j = \zeta_j - 1$. This induces a homomorphism

$$R[X] = A[X]/p^2A[x] \longrightarrow R_j$$

which in turn induces a map

$$\psi_j : K_2(R[X]) \longrightarrow K_2(R_j)$$

such that

$$\psi_j(\langle uX, pX^m\rangle) = \langle u\pi_j, p\pi_j^m\rangle .$$

Since $p = \omega_j \pi_j^{e_j}$ for some $\omega_j \equiv -1 \bmod \pi_j$, we see that $p\pi_j^{k_i} = 0$ in R_j if $j \leq i$. In particular,

$$\psi_j(\langle uX, pX^{k_i}\rangle) = 1 \quad \text{for} \quad j \leq i.$$

However if $j = i+1$,

$$\begin{aligned} \psi_j(\langle uX, pX^{k_i}\rangle) &= \langle u\pi_j, p\pi_j^{k_i}\rangle \\ &= \{1+u\pi_j, 1+p\pi_j^{k_i}\} \\ &= \{1+u\pi_j, 1-\pi_j^{r_j-1}\} \\ &= \{1-u\pi_j, 1+\pi_j^{r_j-1}\} \end{aligned}$$

which is different from 1 by [D-S, Theorems 3.8e and 4.3].

REMARKS. 1. In particular, taking $q = p$ this shows that $K_2(R[X])$ is an elementary abelian p-group of countably infinite rank in case $R = \mathbb{Z}/p^2\mathbb{Z}$.

2. If R is a left regular ring, Quillen [Q, Theorem 11] has shown that the map $K_2(R) \longrightarrow K_2(R[X])$ is an isomorphism. The rings of

the preceding theorem give examples for which $K_2(R) \longrightarrow K_2(R[X])$ is not an isomorphism. These rings are not regular as their residue fields have infinite projective dimension.

Appendix: Non Steinberg symbols in $K_2(n,R)$

It was shown by Cohn [C2] that for $d \neq -1, -3$, the rings of integers in the Euclidean imaginary quadratic number fields $\mathbb{Q}(\sqrt{d})$ are not universal for GE_2, i.e. the $K_2(2, \ )$ of these rings are not generated by Steinberg symbols. In a similar vein, Silvester [Si2, Corollary 28] has shown that the element

$$\langle (1,0),(0,1)\rangle_\alpha \in K_2(2,\mathbb{F}_2 \times \mathbb{F}_2)$$

is not expressible as a product of Steinberg symbols.

Recall that the Steinberg group, $St(2,R)$, is the group with generators $x_{12}(r)$, $x_{21}(r)$, $r \in R$, subject to the relations

$$x_\alpha(r)x_\alpha(s) = x_\alpha(r+s)$$

$$w_\alpha(u)x_{-\alpha}(r)w_\alpha(u)^{-1} = x_\alpha(-uru)$$

where $w_\alpha(u) = x_\alpha(u)x_{-\alpha}(-u^{-1})x_\alpha(u)$ for any unit u of R and $\alpha = (12), (21)$. If R and S are rings and $f: R \longrightarrow S$ is an additive homomorphism which also satisfies

(i) $f(1) = 1$,

(ii) $f(uru) = f(u)f(r)f(u)$, $r \in R$, $u \in R^*$,

then f induces a homomorphism

$$f^*: St(2,R) \longrightarrow St(2,S)$$

defined by $x_\alpha(r) \longmapsto x_\alpha(f(r))$. If $f(uv)=f(u)f(v), u,v \in R^*$, then

$$f^*(\{u,v\}_\alpha) = \{f(u),f(v)\}_\alpha$$

and hence $f^*(K_2(2,R)) \subset K_2(2,S)$ if R is universal for GE_2 (i.e. $K_2(2,R)$ is generated by the elements $\{u,v\}_\alpha$). In this case, f also induces a map

$$E_2(R) \longrightarrow E_2(S)$$

That the analogous result for the elements $\langle a,b\rangle_\alpha$ is not true will be exploited below in Example 1. The first example is a variation on Silvester's proof that $\underset{\sim}{F}_2 \times \underset{\sim}{F}_2$ is not universal for GE_2 [Si2, Corollary 28]. The second example is an adaptation of Silvester's proof[1] that $\underset{\sim}{Z}/6\underset{\sim}{Z}$ is not universal for GE_2.

EXAMPLE 1. Let $\underset{\sim}{F}_4 = \underset{\sim}{F}_2[x]$ be the field with four elements which is obtained from $\underset{\sim}{F}_2$ by adjoining an element x with $1+x+x^2 = 0$. We define

$$f\colon \underset{\sim}{F}_2 \times \underset{\sim}{F}_2 \longrightarrow \underset{\sim}{F}_4$$

by $0 \longmapsto 0$, $1 \longmapsto 1$, $(1,0) \longmapsto x$ and $(0,1) \longmapsto 1+x$. It is clear that f is an additive homomorphism which satisfies conditions (i) and (ii). Let h denote the composition of the map induced by f followed by the projection to $E_2(\underset{\sim}{F}_4)$:

$$St(2,\underset{\sim}{F}_2\times\underset{\sim}{F}_2) \longrightarrow St(2,\underset{\sim}{F}_4) \longrightarrow E_2(\underset{\sim}{F}_4).$$

The elements $\langle(0,1),(1,0)\rangle_{\pm\alpha}$ and $\langle(1,0),(0,1)\rangle_{\pm\alpha}$ are the only non-trivial elements of the form $\langle a,b\rangle_{\pm\alpha}$ in $St(2,\underset{\sim}{F}_2 \times \underset{\sim}{F}_2)$. A computation yields

$$h(\langle 0,1),(1,0\rangle_\alpha) = \begin{pmatrix} 0 & 1+x \\ x & 1 \end{pmatrix} = A,$$

$$h(\langle (1,0),(0,1)\rangle_\alpha) = \begin{pmatrix} 0 & x \\ 1+x & 1 \end{pmatrix} = B,$$

$$h(\langle (0,1),(1,0)\rangle_{-\alpha}) = B^2,$$

$$h(\langle (1,0),(0,1)\rangle_{-\alpha}) = A^2.$$

Now letting $C = AB$ we see that $A^3 = C^2 = (AC)^3 = 1$. It thus follows that A and B generate a subgroup of $E_2(\underset{\sim}{F}_4)$ isomorphic to the alternating group A_4 [C-M, p.134]. As $E_2(\underset{\sim}{F}_4) = PSL(2,4)$ is a simple group of order 60 and as h is surjective, it follows

1. Private correspondence.

that the elements $\langle (0,1),(1,0)\rangle_{\pm\alpha}$, $\langle (1,0),(0,1)\rangle_{\pm\alpha}$ do not generate a normal subgroup of $St(2,\mathbb{F}_2\times\mathbb{F}_2)$.

If R is any commutative semi-local ring for which $\overline{R}$ has 2 or more $\mathbb{F}_2$ factors, there is a surjective homomorphism

$$St(2,R) \longrightarrow St(2,\mathbb{F}_2\times\mathbb{F}_2)$$

and it follows that the subgroup of $St(2,R)$ generated by the elements $\langle a,b\rangle_{\pm\alpha}$ is not normal as its image in $St(2,\mathbb{F}_2\times\mathbb{F}_2)$ is not a normal subgroup. In particular, the elements $\langle a,b\rangle_{\alpha}$ are not central.

EXAMPLE 2. Let $\theta = e^{i\pi/6}$ be a primitive 12-th root of unity. Then there is a homomorphism

$$St(2,\mathbb{Z}/6\mathbb{Z}) \longrightarrow GL_2(\mathbb{C})$$

defined by

$$x_\alpha(-1) \longmapsto \begin{pmatrix} \theta^2 & 0 \\ -i\theta & 1 \end{pmatrix}$$

$$x_{-\alpha}(1) \longmapsto \begin{pmatrix} 1 & -i\theta \\ 0 & \theta^2 \end{pmatrix}$$

(cf. [Cx, p.112]). Letting $R_1 = x_\alpha(-1)$ and $R = x_\alpha(-1)x_{-\alpha}(1)$, it is easy to check that $St(2,\mathbb{Z}/6\mathbb{Z})$ has the presentation

$$R_1^6 = 1,\ R^3 = (RR_1)^2$$

(see [Cx, §3], [C-M, pp. 73-78]). Hence the center of $St(2,\mathbb{Z}/6\mathbb{Z})$ is generated by the element $R^3 = (RR_1)^2 = w_\alpha(-1)^2$ [Cx, p.101]. Under the given homomorphism every element of the center, including the only symbol $\{-1,-1\}_\alpha = w_\alpha(-1)^4 = \{-1,-1\}_{-\alpha}^{-1}$, becomes trivial. However, the element $\langle 3,2\rangle_\alpha$ does not vanish under this homomorphism. Hence $\langle 3,2\rangle_\alpha$ is not central and $\mathbb{Z}/6\mathbb{Z}$ is not universal for GE_2. Using the computations of Miller [M] it is possible to show that the subgroup of $St(2,\mathbb{Z}/6\mathbb{Z})$ generated by all elements of the form $\langle a,b\rangle_{\pm\alpha}$ is normal. In fact, this subgroup is generated by the three elements $\{-1,-1\}_{\alpha}$, $\langle 3,2\rangle_{\alpha}$ and $\langle 2,3\rangle_{\alpha}$, $\alpha = (12)$.

For all $n \geq 2$, examples of rings of algebraic integers $\mathcal{O}$ for which $K_2(n,\mathcal{O})$ is not generated by Steinberg symbols can be constructed as in [D-S, Section 5, Example].

Suppose $\mathcal{O}$ is the ring of integers in an algebraic number field F and let ϵ be some unit of $\mathcal{O}$. Suppose further that $\epsilon-1 = ab$, $a,b \notin \mathcal{O}^*$. Then we may form the element $\langle a,b\rangle$. The techniques of [D-S] often allow one to pass modulo some ideal of $\mathcal{O}$ to show that $\langle a,b\rangle$ is non-trivial and has order divisible by some integer m. The final step of the argument is to show that in $K_2(n,\mathcal{O})$ there are no Steinberg symbols whose orders are divisible by m.

It should be noted that the elements $\langle a,b\rangle$ all exist in $K_2(2,R)$. Hence they do not account for the appearance in $K_2(3,R)$ of elements which do not come from $K_2(2,R)$ [D-S, Theorem 5.3].

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Variations on Milnor's Computation of $K_2 Z$

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Milnor's computation of $K_2 Z$ [4, §10] yields an explicit finite presentation of $SL(n,Z)$, $n \geq 2$. (Z denotes the rational integers, R the field of real numbers.) The method, based on a lemma of Silvester, involves finding the kernel of the canonical map $St(n,Z) \to SL(n,Z)$, where $St(n,Z)$ is the Steinberg group. This is simpler than the earlier approach of Nielsen and Magnus [2], although the ideas are similar. The kernel in question is Z (resp. $Z/2Z$) when $n = 2$ (resp. $n > 2$), and in fact arises from the restriction to $SL(n,Z)$ of the universal topological covering $St(n,R) \to SL(n,R)$.

In this note we sketch an analogous argument for arbitrary Chevalley groups other than G_2; full details will appear elsewhere. In the case of Siegel's modular group $Sp(2n,Z)$ $(n \geq 2)$, the result is simpler than those obtained by Klingen and by Birman [1] (moreover, the latter author has pointed out that [1] rests in part on an erroneous argument in one of her sources).

G will denote a simply connected Chevalley group scheme over Z of simple type, Φ its (irreducible) root system (e.g., $G = SL_n$). For background material consult [5, §3] and [3, No. 2]. If A is any commutative ring with 1, $E(\Phi,A)$ denotes the elementary subgroup of $G(A)$, generated by unipotents $e_\alpha(t)$ $(\alpha \in \Phi,\ t \in A)$. When $A = Z$ or A = field, it is known that $E(\Phi,A) = G(A)$ (cf. [3, Thm. 12.7]). Let $St(\Phi,A)$ be the Steinberg group, generated by elements $x_\alpha(t)$ $(\alpha \in \Phi,\ t \in A)$, subject to the usual relations, and let $\pi_A\colon St(\Phi,A) \to E(\Phi,A)$ be the canonical epimorphism.

<u>Theorem</u>. Let Φ be not of type G_2. Ker π_Z is central in $St(\Phi,Z)$, and is generated by the symbol $\{-1,-1\} = (x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1))^4$,

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where α is any fixed long root. Moreover, Ker $\pi_Z = Z$ (resp. $Z/2Z$) when Φ is of symplectic type C_ℓ, $\ell \geq 1$ (resp. when Φ is non-symplectic).

Corollary. Let rank $\Phi \geq 2$. Then $G(Z)$ is generated by the $e_\alpha(1)$ ($\alpha \in \Phi$) subject only to the commutator relations [5, (3.7)] and the relation $(e_\alpha(1)\, e_{-\alpha}(1)^{-1}\, e_\alpha(1))^4 = 1$, α any fixed long root.

(For Φ of type G_2, this is probably true, but some details remain to be checked.)

As in the special case $G = SL_n$, the proof amounts to showing that the middle vertical arrow in the following diagram is injective:

$$\begin{array}{ccccccccc} 1 & \to & \text{Ker}\ \pi_R & \to & St(\Phi,R) & \to & G(R) & \to & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \to & \text{Ker}\ \pi_Z & \to & St(\Phi,Z) & \to & G(Z) & \to & 1 \end{array}$$

This in turn rests upon showing that Ker π_Z comes from the (generalized) Weyl group, as Ker π_R does. Denote by W the subgroup of $St(\Phi,Z)$ generated by the elements $x_\alpha(1)\, x_{-\alpha}(-1)\, x_\alpha(1)$ ($\alpha \in \Phi$).

The proof of the theorem involves a reduction of rank, as follows. G has at least one "basic representation" [3, No. 2] (which in the case $G = SL_n$ can be taken to be the standard representation), containing an "admissible" lattice L on which $G(Z)$ acts. Since the nonzero weights all occur with multiplicity one, there is an almost canonical basis for L, relative to which the action of $e_\alpha(t)$ ($t \in Z$) can be described very explicitly. Let the first basis vector v^+ be of highest weight. The stabilizer of the line through v^+ is a parabolic subgroup $P = (G'H)\cdot U$ of G, with unipotent radical U, reductive part $G'H$, and semisimple part G'. The basic representation can be chosen so that G' is again of simple type (i.e., has irreducible root system Φ'), e.g., for $G = SL_n$, $G' = SL_{n-1}$. Since G' is in any case simply connected and of smaller rank than G, induction can be used, starting either with the trivial group (rank 0) or the known case $G = SL_2$ (rank 1).

The action of G(Z) on L (written on the right for convenience) induces an action of $St(\Phi,Z)$, via π_Z. For $v \in L$, let $\|v\|$ be the sum of absolute values of the coordinates of v relative to our chosen basis, e.g., $\|v^+\| = 1$. Then the key lemma (analogous to Silvester's lemma [4, 10.6]) is the following:

Lemma. Each $g \in St(\Phi,Z)$ can be written as $g_1 \cdots g_r w$, where $w \in W$, each g_i is a generator $x_\alpha(\pm 1)$, and $\|v^+ \cdot g_1\| \leq \|v^+ \cdot g_1 g_2\| \leq \cdots \leq \|v^+ \cdot g_1 \cdots g_r\|$.

We apply this lemma to an element $g \in \operatorname{Ker} \pi_Z$, for which all terms become equal to $1 = \|v^+ \cdot g\|$. By further manipulation (using commutator relations) g can be forced, modulo a factor in $W \cap \operatorname{Ker} \pi_Z$, into the canonical image of $St(\Phi',Z)$ in $St(\Phi,Z)$, where by induction we have an element of the image of the analogous group W', which in turn lies in W. From this we obtain $\operatorname{Ker} \pi_Z \subset W$; in particular, $\operatorname{Ker} \pi_Z$ is central. The proof is now easily completed by means of [3, Thm. 6.3].

Problems. (1) Devise a more conceptual proof that the canonical map $St(\Phi,Z) \to St(\Phi,R)$ is injective.

(2) Treat rings of algebraic integers other than Z. The fact (observed by Dennis and Stein) that K_2 of such a ring need not be generated by symbols seems to present a serious obstacle.

Remark. After formulating the above approach I learned of the 1966 U.C.L.A. thesis written by W. P. Wardlaw, "Defining relations for integrally parametrized Chevalley groups," in which essentially the same presentations are obtained (in cases other than G_2). However, in treating types B, C, F_4, Wardlaw first reduces the problem to Sp(4,Z) and then appeals to the same faulty reference used by Birman [1].

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DECOMPOSITION FORMULA OF LAURENT EXTENSION IN ALGEBRAIC K-THEORY AND THE ROLE OF CODIMENSION 1 SUBMANIFOLD IN TOPOLOGY

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I. Introduction. Let $\mathbb{A}$ be a ring with 1 . $K_n\mathbb{A}$ $(n \in Z)$ was introduced in [1] [9] [16]. Suppose that t is an indeterminate. We have the ring of finite Laurent series $\mathbb{A}[t,t^{-1}]$. Following [1] [10] [21], we have the decomposition formula (1)

$$(1) \qquad K_n\mathbb{A}[t,t^{-1}] = K_n\mathbb{A} + K_{n-1}\mathbb{A} + Nil_n\mathbb{A} \quad .$$

$K_{n-s}\mathbb{A}$ is naturally embedded in $K_n\mathbb{A}[t_1,t_1^{-1};\dots;t_s,t_s^{-1}]$ as a direct summand and the original definition of $K_{-s}\mathbb{A}$, s=1,2.... was gotten from this embedding [1].

Now, suppose that $\mathbb{A} = Z\pi_1M^m$ with M^m a manifold. Let S^1 denote the circle and let $\mathbb{A}[t,t^{-1}]$ be identified as $Z\pi_1M^m \times S^1$ with t identified to a preferred generator of π_1S^1 . There are geometric interpretations for $K_n\mathbb{A}$ for n=0, 1,2 [22] [14] [11] and there is also a geometric interpretation of the decomposition formula (1) for n=1 [7].

In the first part (2) of the note, we shall give a description of $Nil_2\mathbb{A}$ and identify this description with the geometric obstruction to a codim 1 isotopy problem. We recast a geometric version of a Quillen's theorem that $Nil_2\mathbb{A} = 0$ for $\mathbb{A}$ left regular [17].

In the second part, we discuss some joint work with Douglas R. Anderson[3]. Let $X = S^s M^m$ be the s-fold suspension of a closed manifold M^m $(m \geq 5)$ such that M^m is not a homology sphere. Let $\mathcal{R} = S^s M^m - S^{s-1}$, $\mathcal{S} = S^{s-1}$ be the regular set and the singular set respectively. Suppose that τ_1, τ_2 are two triangulations of X such that the induced triangulations on $\mathcal{R}$ and $\mathcal{S}$ are combinatorial. Let $f: X \to X$ be a homeomorphism of X onto itself. We say that f is an 'isotopic isomorphism' from τ_1 to τ_2 if f is (topologically) isotopic to a PL homeomorphism g . We shall describe sequences of elements in

$$K_{-\ell+1}\mathbb{A}, \ldots\ldots\ldots\ldots\ldots, K_1\mathbb{A} \quad (\ell=t,\ t-1,\ldots,1,\ \text{and}\ t \leq s-1)$$

as different level of obstructions to 'isotopic isomorphism'. In particular, if $\pi_1 M^m$ is a torsion-free solvable group, then Hauptvermutung for X is practically true. Roughly speaking, we view τ_1 , τ_2 as combinatorial compactification of $R^s \times M^m$ and these sequences of elements are different level of obstructions to make f isotopically isomorphic when we add different pieces of S^{s-1} to $R^s \times M^m$. The order of the sequence will exactly correspond to the iterated formula of (1) as we adjoin the indeterminates $t_1, \ldots\ldots, t_\ell$. This result gives an explanation of the counter-examples to Hauptvermutung [15] [20].

II. $\mathrm{Nil}_2\mathbb{A}$ and Codim 1 Isotopy.

In this section, we shall give an algebraic description of $\mathrm{Nil}_2\mathbb{A}$ and interpret it as the obstruction to a codim 1 isotopy problem. Let us first define a category $\mathcal{N}il_2\mathbb{A}$. Let $C_*^{(1)}$, $C_*^{(2)}$ be two chain complexes and let $f: C_*^{(i)} \to C_*^{(2)}$ be a degree-1 chain map. We can form the mapping cylinder of f [4,p.159] M(f) with $M(f)_i = C_i^{(1)} \oplus C_i^{(2)}$ and $\partial_f(x^{(1)}, x^{(2)}) = (\partial^{(1)} x^{(1)}, f(x^{(1)}) + \partial^{(2)} x^{(2)})$. Suppose that $f^{(i)}: C_*^{(i)} \longrightarrow C_*^{(i+1)}$ (i=0, 1,..., N-1) are degree

-1 chain maps with $f^{(i+1)} \cdot f^{(i)} = 0$. In an obvious way, we can form the mapping tower $M = M(f^{(o)}, \ldots, f^{(N-1)})$.

An object in $\mathcal{N}il_2\mathbb{A}$ is an acyclic finite dimensional free chain complex over $\mathbb{A}$

$$C_* : 0 \longrightarrow C_\ell \xrightarrow{d} C_{\ell-1} \xrightarrow{d} \ldots\ldots \longrightarrow C_1 \xrightarrow{d} C_o \longrightarrow 0$$

satisfying the following conditions:

(A) There is a filtration of subcomplexes

$$o \subset C_*^{(o)} \subset C_*^{(1)} \subset \ldots\ldots \subset C_*^{(N)} \subset C_*^{(N+1)} = C_*$$

(2) such that both $C_*^{(i-1)}$ and $C_*^{(i)}/C_*^{(i-1)}$ $(i=1,\ldots,N+1)$ are free chain complexes over $\mathbb{A}$.

(B) There are degree-1 chain maps

$$f^{(i)} : C_*^{(i)} \longrightarrow C_*^{(i+1)} \quad (i=0,\ldots,N-1)$$

such that $f^{(i-H)} \cdot f^{(i)} = 0$ and the mapping tower M is acyclic.

We can define morphisms and exact sequences in $\mathcal{N}il_2\mathbb{A}$ in the usual way. A 'trivial object' in $\mathcal{N}il_2\mathbb{A}$ is a chain complex

$$0 \longrightarrow C_\ell \cong \xrightarrow[\cong]{d} C_{\ell-1} \cong \mathbb{A}^n \longrightarrow 0 \qquad (n \geq 1)$$

(3) with $o \subset o \subset \ldots\ldots \subset C_*^{(i)} = C_* \subset C_* \subset \ldots\ldots$

$$\ldots\ldots \subset C_*^{(N)} = C_* \subset C_*^{(N+1)} = C_*$$

and $f^{(j)} = o$ $(j=0,\ldots,N-1)$.

An 'elementary object' in $\mathcal{N}il_2\mathbb{A}$ is a chain complex

$$\begin{array}{ccc} & & C_\ell^{(i+2)} \cong \mathbb{A} \xrightarrow[\cong]{d} C_{\ell-1}^{(i+1)} \cong \mathbb{A} \longrightarrow 0 \\ & & \oplus \\ 0 \longrightarrow C_{\ell+1}^{(i+1)} \cong \mathbb{A} \xrightarrow[\cong]{d} & C_\ell^{(i)} \cong \mathbb{A} & \end{array}$$

satisfying the following conditions:

$$(A)\quad 0 \subset 0 \subset \dots \subset C_*^{(i)} = C_\ell^{(i)} \subset C_*^{(i+1)} = \{C_{\ell+1}^{(i+1)},$$

$$(4)\qquad C_\ell^{(i)} \oplus C_\ell^{(i+1)},\ C_{\ell-1}^{(i+1)}\} \subset C_*^{(i+2)} = C_* \subset C_*^{(i+3)} \subset \dots$$

$$\dots \subset C_*^{(N)} \subset C_*^{(N+1)} = C_* .$$

$$(B)\quad f^{(j)} = 0 \ \text{ for } \ j \neq i \ \text{ and}$$

$$f^{(i)} = C_\ell^{(i)} \cong \mathbb{A} \xrightarrow{\cong} C_{\ell-1}^{(i+1)} \cong \mathbb{A}$$

Let us denote the Grothendieck group of the isomorphism classes of the objects of $\mathcal{N}il_2\mathbb{A}$ with respect to the exact sequences modulo the subgroup generated by trivial objects and elementary objects by $\mathrm{Nil}_2\mathbb{A}$.

Theorem 2.1 Let $\mathbb{A} = Z\pi$ be the integral group ring of a finitely presented group π . Then,

(A) $K_2\mathbb{A}[t,t^{-1}] = K_2\mathbb{A} + K_1\mathbb{A} + 2\mathrm{Nil}_2\mathbb{A}$;

(B) for $\mathbb{A}$ is (left) regular, $\mathrm{Nil}_2\mathbb{A} = 0$[17] ;

(C) $\mathrm{Nil}_2\mathbb{A}[t,t^{-1}] \supset \mathrm{Nil}_1\mathbb{A}$.

In particular, if $\mathbb{A} = Z(\mathbb{Z}_{p^2} \times \mathbb{Z}^3)$, then $\mathrm{Nil}_2(\mathbb{A})$ is not finitely generated.

Actually, we do not need the assumption that $\mathbb{A}$ is a group ring at all, but since we are only interested in the geometric interpretations of Theorem 2.1, we leave it in. Let us now consider an orientable closed manifold M^m $(m\geq 5)$ with $\pi_1 M^m = \pi$. Identify $Z\pi_1 M^m \times S^1$ (S^1 = the circle) with $\mathbb{A}[t,t^{-1}]$ such that t is a preferrred generator of $\pi_1 S^1 \subset \pi_1 M^m \times S^1$. Let us now follow the geometric interpretation of $K_2\mathbb{A}[t,t^{-1}]$. For $\xi \in K_2\mathbb{A}[t,t^{-1}]$, there is a generic map

$$M^m \times S^1 \times I \times I \xrightarrow{F} I \times I$$

satisfying the following conditions:

(5)
(A) $F|M^m \times S' \times \partial(I \times I)$ has no critical point.

(B) $F|M^m \times S' \times 0 \times I$ is the standard projection onto the last factor.

(C) The graphic of F has no vertical tangent.

We refer to [11] for details. F determines a pseudo-isotopy

(6) $$f : M^m \times S^1 \times I \longrightarrow M^m \times S^1 \times I$$

such that $f|M^m \times S^1 \times 0 = id$. f induces a psuedo-isotopy of a codim 1 embedding

(7) $$g : M^m \times p_o \times I \longrightarrow M^m \times S^1 \times I$$

with $g|M^m \times p_o \times 0 = id$ where p_o denotes the base point of S^1 . Then, the component η of ξ in $Nil_2\mathbb{A}$ of the decomposition (1) has the following geometric interpretation: With possibly adding a second obstruction which is of order 2 [12], η is the obstruction to finding an embedding

(8) $$h : M^m \times p_o \times I \longrightarrow M^m \times S^1 \times I$$

isotopic to g of (7) such that

(9) $$h(M^m \times p_o \times I) \cap M^m \times p_o \times I = \phi \quad .$$

For such an embedding g , the corresponding object $\hat{\eta} \in \mathcal{N}il_2\mathbb{A}$ (i.e., $\hat{\eta}$ is a representative of η) may be constructed as follows. Let

(10) $$q : M^m \times R \times I \longrightarrow M^m \times S^1 \times I$$

be the infinitely cyclic covering space of $M^m \times S^1 \times I$ corresponding to the subgroup $\pi_1 M^m$ of $\pi_1 M^m \times S^1$ such that $M^m \times p_o \times I$ is lifted to $M^m \times 0 \times I$. Let us lift $g(M^m \times p_o \times I)$ into $M^m \times R \times I$ such that $t^{-1}g(M^m \times p_o \times I) \subset M^m \times (-\infty,0] \times I$ and $g(M^m \times p_o \times I) \cap M^n \times (0,1) \times I \neq \phi$, where t denotes the preferred generator of the covering transformation of (10).

There is a large positive integer N such that $t^N g(M^m \times p_o \times I) \subset M^m \times (0,\infty) \times I$ but $t^{N-1} g(M^m \times p_o \times I) \not\subset M^m \times (0,\infty) \times I$. Let

(11) $$L_i = (M^m \times [0,\infty] \times I) \cap (t^i f(M^m \times (-\infty,0] \times I)$$

for $i = 0,1,\ldots, N$. (See Figure 1.)

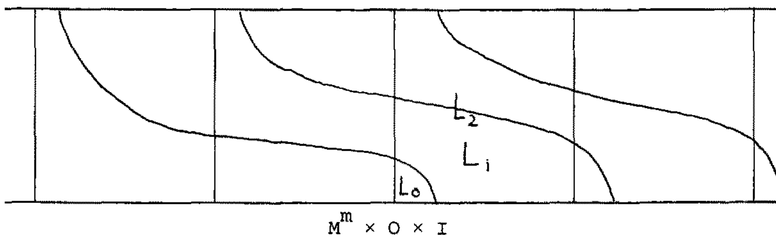

Figure 1.

put

(12)
$$\begin{aligned} R_o &= L_o \cup M^m \times 0 \times I \\ R_1 &= (\overline{L_1 - tL_o}) \cup M^m \times 0 \times I \\ &\vdots \\ R_N &= (\overline{L_N - tL_{N-1}}) \cup M^m \times 0 \times I \\ R_{N+1} &= M^m \times [0,1] \times I \ . \end{aligned}$$

Let us now consider the chain complex

(13) $$C_* = C_*(M^m \times [0,1] \times I,\ M^m \times 0 \times I\ ;\ \mathbb{A})$$

with the filtration

(14) $$C_*^{(i)} = C_*(R_i,\ M^m \times 0 \times I\ ;\ \mathbb{A})$$

$i = 0,\ldots, N+1$. (The chain complexes are gotten from the handles on $M^m \times 0 \times I$).

Let us consider the composite map

$$(15) \qquad f^{(i)} : C_*^{(i)} = C_*(R_i, M \times 0 \times I ; \mathbb{A}) \xrightarrow[\cong]{t_*} C_*(tR_i, M \times 1 \times I ; \mathbb{A}) \xrightarrow{\partial} C_*(R_{i+1}, M \times 0 \times I ; \mathbb{A})$$

$i = 0,\dots,N-1$. It is easy to see that the mapping tower is acyclic. Therefore, it is an object $\hat{\eta}$ of $\mathcal{N}il_2\mathbb{A}$. The trivial object is essentially represented by an h-cobordism on $M \times 0 \times I$ inside of $M \times [0,1) \times I$. The geometric model of an elementary object may be described as follows. Add a complementary pair of handles $h^{(i+1)}$, $h^{(i)}$ to $M^m \times 0 \times I$. Drag $h^{(i+1)}$ in the direction of t and let it go across $M \times 1 \times I$ such that the tip of $h^{(i+1)}$ is trivially embedded in a ball contained in the translated region of the cobordism. (See Figure 2).

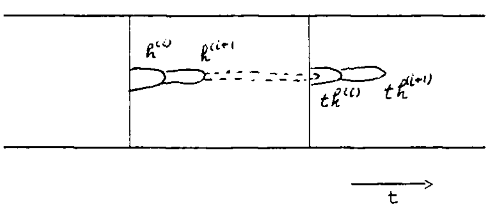

Figure 2.

Using these geometric interpretations, we see that different representatives of η are gotten from isotopies of g with possibly adding elements of second obstructions of [12]. From these observations, we may deduce (A) of Theorem 2.1.

Let us now indicate a geometric proof of (B) of Theorem 2.1. We can use the geometric models for trivial objects and elementary objects to perform isotopy of g . After a finite number such isotopies with possibly adding second obstructions of [12], we may assume that R_i is gotten from R_{i-1} by adding $k-1$, k , $k+1$ handles. We may assume that $3 \leq k-1$ and $k+1 << {}^m/2$ without loss of generality. We can actually write

(16) $$R_i = R_{i-1} \cup S_i \ , \quad T_i = R_{i-1} \cap S_i$$

$(i=1,\ldots,N+1)$ where T_i is a codim 1 submanifold of $M^m \times [0,1] \times I$ separating R_{i-1} from S_i . Set $S_o = R_o$. We have

(17) $$R_i = S_o \cup_{T_1} S_1 \cup_{T_2} S_2 \cup \ldots \cup_{T_i} S_i$$

Put

(18) $$D_*^{(i)} = C_*(S_i \ , \ S_i \cap M^m \times 0 \times I \ ; \mathbb{A})$$
$$E_*^{(i)} = C_*(T_i \ , \ T_i \cap M^m \times 0 \times I \ ; \mathbb{A}) \ .$$

There are monomorphic chain mappings

(19) $$\rho^{(i)} : E_*^{(i)} \longrightarrow D_*^{(i)}$$
$$\lambda^{(i)} : E_*^{(i)} \longrightarrow D_*^{(i-1)}$$

of degree 0 . We can use $\rho^{(i)}$ and $\lambda^{(i)}$ to form the Meyer-Vietoris sum of $D_*^{(i-1)}$ and $D_*^{(i)}$ in the usual way, and $C_*^{(i)}$ becomes the repeated Meyer-Vietoris sum of $D_*^{(0)},\ldots,D_*^{(i)}$ along $E_*^{(1)},\ldots,E_*^{(i)}$. Under the assumption $3 \leq k-1$ and $k + 1 << {}^m/2$, we may assume that the homomorphisms

(20) $$\mu^{(i)} : H_j(E_*^{(i)}) \longrightarrow H_j(C_*^{(i-1)})$$
$$\nu^{(i)} : H_j(D_*^{(i)}) \longrightarrow H_j(C_*^{(i)})$$

$(i=0,\ldots,N)$ are monomorphic for $j < {}^m/2$ where $\mu^{(i)}, \nu^{(i)}$ are induced by inclusions. After some diagram chasing, we find that

(A) $H_j(D_*^{(i)}) = 0$ for ${}^m/2 > j \neq k-1,k$

where $0 \leq i \leq N$ and $3 \leq k-1$, $k+1 << {}^m/2$;

(B) $H_{k-1}(D_*^{(0)}) = 0$, $H_k(D_*^{(N)}) = 0$.

Let us now consider the following inclusion

(22) $$K^{(i,j)} : D_*^{(i)} \longrightarrow D_*^{(i)} \oplus tD_*^{(i-1)} \oplus \ldots \oplus t^{i-j}D_*^{(j)}$$

$(j < i)$ where $D_*^{(i)} \oplus \ldots \oplus t^{i-j}D_*^{(j)}$ denotes a suitable mapping tower which may be identified with the chain complex of

(23) $$(S_i \cup tS_{i-1} \cup \ldots \cup t^{i-j}S_j \ , \ S_i \cap M^m \times 0 \times I).$$

Consider the filtration

(24) $$0 \subset \ker K^{(i,i-1)} \subset \ldots \subset \ker K^{(i,j)} \subset \ldots \subset H_{k-1}(D_*^{(i)}) \ .$$

We can use the geometric model of the elementary object to exchange cycles of $D_*^{(i-1)}$ to $D_*^{(i)}$. The effect is killing some element of ker $K^{(i,i-1)}$ at the expenses of possibly creating elements in $H_k(D_*^{(i)})$ and $H_{k+1}(D_*^{(i-1)})$. When we apply this procedure successively and carefully and denote the new chain complexes by $D'^{(i)}_*$, we would have

(A) $H_j(D'^{(i)}_*) = 0$ for $\frac{m}{2} > j \neq k,k+1$

where $0 \leq i \leq N$;

(B) $H_k(D'^{(0)}_*) = 0$ and $H_{k+1}(D'^{(N)}_*) = 0$;

(C) There is a filtrated free modules

(25) $$0 \subset F^{(i,i-1)} \subset \ldots \subset F^{(i,j)} \subset \ldots \subset F$$

with $F^{(i,j)}/F^{(i,j-1)}$ free and there are short exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & \cdots & & 0 & & & 0 \\
 & & \downarrow & & & \downarrow & & & \downarrow \\
0 & \subset & \mathrm{Ker}\, K'^{(i,i-1)} & \subset \cdots \subset & & \mathrm{Ker}\, K'^{(i,j)} & \subset \cdots \subset & & H_k(D'^{(i)}_*) \\
 & & \downarrow & & & \downarrow & & & \downarrow \\
0 & \subset & F^{(i,i-1)} & \subset \cdots \subset & & F^{(i,j)} & \subset \cdots \subset & & F \\
 & & \downarrow & & & \downarrow & & & \downarrow \\
0 & \subset & \mathrm{Ker}\, K^{(i,i-1)} & \subset \cdots \subset & & \mathrm{Ker}\, K^{(i,j)} & \subset \cdots \subset & & H_{k-1}(D^{(i)}_*) \\
 & & \downarrow & & & \downarrow & & & \downarrow \\
 & & 0 & \cdots\cdots & & 0 & \cdots\cdots & & 0
\end{array}$$

where $K'^{(i,j)}$ is defined as $K^{(i,j)}$. Next, we observe that we may move the indices k , $k+1$ back to $k-1$, k with all the properties of (25) retained.

Since $\mathbb{A}$ is (left) regular, we can finally eliminate all $H_*(D^{(i)}_*)$. Modifying by 'trivial objects' , we would have (B) of Theorem 2.1.

Let us now indicate a geometric construction of the embedding $\mathrm{Nil}_2\mathbb{A} \subset \mathrm{Nil}_2\mathbb{A}[t,t^{-1}]$. (It was pointed out to me by A. Hatcher that one can construct $\mathrm{Nil}_i\mathbb{A} \subset \mathrm{Nil}_{i+1}\mathbb{A}[t,t^{-1}]$ directly from [10]). Let ξ be an element in $\mathrm{Nil}_1\mathbb{A}$. Consider the embedding $\mathrm{Nil}_1\mathbb{A} \subset K_1\mathbb{A}[t_1,t_1^{-1}]$. Following [10] [21], there is an embedding $K_1\mathbb{A}[t_1 t_1^{-1}] \subset K_2\mathbb{A}[t_1,t_1^{-1};t_2,t_2^{-1}]$ and let us denote its image by $\hat{\xi}$. Using $\hat{\xi}$, there is a pseudo-isopotopy on $M^m \times S^1_1 \times S^1_1$ such that t_1 , t_2 are identified to the preferred generators of $\pi_1 S^1_1$, $\pi_1 S^1_2$ respectively. Using the geometric interpretation of Nil_2 at the beginning of this section and the interpretation of Nil_1[7], we see that $\hat{\xi}$ has non-trivial component in $\mathrm{Nil}_2\mathbb{A}[t_2,t_2^{-1}]$ and $\mathrm{Nil}_1\mathbb{A} \subset \mathrm{Nil}_2\mathbb{A}[t_2,t_2^{-1}]$. By [3], $\mathbb{A}$ is not generally finitely generated for $\mathbb{A}$ (commutative) Noetherian and $\mathbb{A} = Z(\mathbb{Z}_{p^2} \times \mathbb{Z}^3)$ (p odd) is such an example.

III. $K_{-i}\mathbb{A}$ and obstructions to Hauptvermutung of iterated suspensions of a manifold.

In this section, we shall discuss some joint work with Douglas R. Anderson. Let $M^m (m \geq 5)$ be a closed manifold which is not a homology sphere. Let $X = S^s M^m$ $(m \geq 5,\ s \neq 5)$ be the s-fold suspension of M^m. Then, X is a topological stratified space with 2 strata : $\mathcal{S} = S^{s-1}$ is the singular set and $\mathcal{R} = X - S^{s-1}$ is the regular set. For any triangulation of X, $\mathcal{S}$ is always a subcomplex and it also induces an infinite triangulation on $\mathcal{R}$. We say that a triangulation τ on X is 'admissible' if the induced triangulations of τ on $\mathcal{S}$ and $\mathcal{R}$ are combinatorial. We shall only consider admissible triangulations and when we say 'triangulation' we shall always mean 'admissible triangulation'. Let τ_1, τ_2 be two triangulations of X and let

(26) $$f : X \longrightarrow X$$

be a homeomorphism of X onto itself. We say that f is an 'isotopic isomorphism' from τ_1 to τ_2 if f is topologically isotopic to a PL homeomorphism g from τ_1 to τ_2, i.e. g is an isomorphism from a subdivision of τ, to a subdivision of τ_2. The obvious necessary conditions for f to be an 'isotopic isomorphism' are:

(27)
- (A) The induced triangulations $\tau_1|\mathcal{S}$, $\tau_2|\mathcal{S}$ are isotopically isomorphic. Since $s \neq 5$, this is always true.
- (B) The induced triangulations $\tau_1|\mathcal{R}$, $\tau_2|\mathcal{R}$ are ε-isotopic. According to Kirby-Siebenmann, this depends on an obstruction in $H^3(\mathcal{R};Z_2)$.

We shall always assume that the obstruction of (27,B) vanishes. By (27,A), we shall also assume that f identifies the triangulation of $\tau_1|S^{s-1}$ with that of $\tau_2|S^{s-1}$ where $S^{s-1} = \mathcal{S}$, and f is PL from the induced (infinite) triangulation $\tau_1|\mathcal{R}$ to that of $\tau_2|\mathcal{R}$. For notational simplicity, we shall assume that $\tau_i|S^{s-1}$ are triangulated into cubes instead of simplices. We shall study the obstructions to extending f to a isotopically isomorphic PL homeomorphism of $f|\mathcal{R}$ to $\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots\ldots \cup \{\square^{\ell}\}$ assuming that we have the extension to $\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots\ldots \cup \{\square^{\ell+1}\}$ where $\square^{i}$ denotes an ith dimensional cube in the triangulation $\tau_1|S^{s} = \tau_2|S^{s-1}$. So, the obstructions may be viewed as the obstacles to making f compatible with the fitting in of the cubes according to the triangulations τ_1 and τ_2 . We shall discuss the obstruction to extending $f|\mathcal{R}$ to $\mathcal{R} \cup \{\square^{s-1}\}$ with a little detail but only sketch briefly the obstruction to extending $f|\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots \{\square^{\ell+1}\}$ to $f|\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots\ldots \cup \{\square^{\ell}\}$. We shall publish a detailed proof with further results in this direction on a future occasion.

Let $\square^{s-1}$ be a cube of the top dimension of the triangulation $\tau_1|S^{s-1} = \tau_2|S^{s-1}$. Let us first identify $\square^{s-1}$ with the standard cube $I_1 \times \ldots \times I_{s-1}$ in R^{s-1} with $I_i = [-1, 1]$ $(i=1,\ldots,s-1)$. Denote the variable in I_i by t_i . Let us consider the hyperplanes defined by $t_i = \pm \sum_{j=1}^{\ell} \frac{1}{2^j}$ and $t_i = 0$ of R^{s-1} . These hyperplanes together cut Int $\square^{s-1}$ into a lattice. (See Figure 3 for $s-1 = 2$).

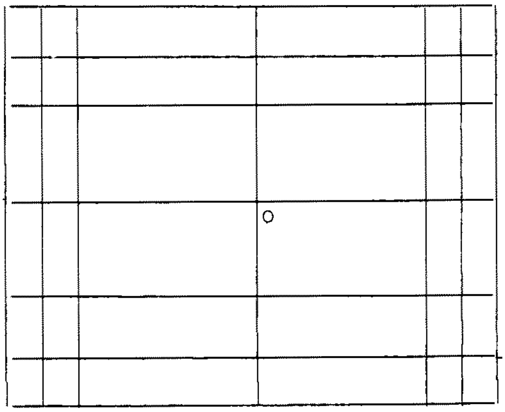

Figure 3

We next identify the induced lattice structure of Int $\square^{s-1}$ with the standard lattice structure of R^{s-1} by making to hyperplane defined by $t_i = \sum_{j=1}^{\ell} \frac{1}{2^j}$ corresponding to the standard hyperplane $(t_1,\ldots,t_{i-1},\ell,t_{i+1},\ldots,t_{s-1})$ and the hyperplane $t_i = 0$ to itself. Let N_1 , N_2 be spindle neighborhoods of $\square^{s-1}$ with respect to the triangulations τ_1 , τ_2 respectively. There are natural projections

$$p_1 : N_1 \longrightarrow \square^{s-1}$$
(28)
$$p_2 : N_2 \longrightarrow \square^{s-1}$$

gotten from τ_1 , τ_2 respectively. Let us call the inverse images of the hyperplanes in $\square^{s-1}$ hyperplanes in N_1 , N_2 and denote the inverse image of the hyperplane corresponding to $t_i = \ell$ (i=1,...,s-1) of R^{s-1} by $N_1(t_1,\ldots,t_{i-1},\ell,t_{i+1},\ldots,t_{s-1})$, $N_2(t_1,\ldots,t_{i-1},\ell,t_{i+1},\ldots,t_{s-1})$ respectively. $N_j(t_1,\ldots,t_{i-1},\ell,t_{i+1},\ldots,t_{s-1})$ (j=1,2) are PL homeomorphic to $(M_j^m \times R^{s-2}) \times R^1$ (j=1,2) where M_j^m(j=1,2) denotes the link of $\square^{s-1}$ in τ_j (j=1,2) respectively, and the positive direction of R^1 corresponds to the compactification of $\mathcal{R}$ by $\square^{s-1}$. See Figure 4 for s-1 = 1).

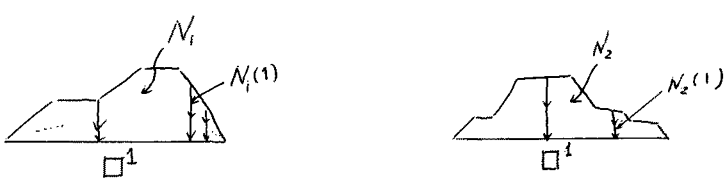

Figure 4

We can also give sequences of hyperplanes in N_j (j=1,2) parallel to $\square^{s-1}$ corresponding to $(M_j^m \times R^{s-2}) \times \ell$ for $\ell \in Z \subset R^1$. See Figure 5 for s-1 = 1).

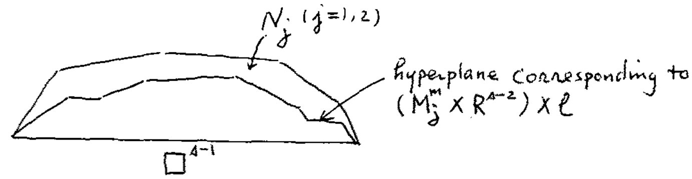

Figure 5

Using these hyperplanes, we have sequences of spindle neighborhoods with respect to τ_i (i=1,2)

(29)
$$\ldots\ldots,N_1^{-i},\ldots\ldots,N_1^{o},\ldots\ldots,N_1^{j},\ldots\ldots$$
$$\ldots\ldots,N_2^{-i},\ldots\ldots,N_2^{o}\ldots\ldots,N_2^{j},\ldots\ldots$$

such that $N_i^j \supset N_i^k$ for $j < k$, $\bigcup_{j=\ell}^{-\infty} N_i^j = N_i$ and $\bigcap_{j=\ell}^{\infty} N_i^j = \square^{s-1}$ (i=1,2) .

Using the fact that $f|\mathcal{R}$ is ε-isotopic to a PL homeomorphism, we may assume that

(30)

(A) $f|\mathcal{R}$ is PL (with respect to the induced infinite triangulations $\tau_1|\mathcal{R}$ and $\tau_2|\mathcal{R}$) .

(B) $\ldots\ldots N_2{}^i \subset f(N_1^i) \subset N_2^{i+1} \subset f(N_1^{i+1}) \subset \ldots\ldots$

(C) $\ldots\ldots N_2(t_1,\ldots,t_{i-1},\ell-1,t_{i+1},\ldots,t_{s-1})$

$\subset f(N_1(t_1,\ldots,t_{i-1},\ell-1,t_{i+1},\ldots,t_{s-1})$

$\subset N_2(t_1,\ldots,t_{i-1},\ell,t_{i+1},\ldots,t_{s-1}) \subset f(N_1(t_1,\ldots,t_{i-1},\ell,$

$t_{i+1},\ldots,t_{s-1}) \subset \ldots\ldots$

for $-\infty < \ell < \infty$.

Let us now consider the opposite sides of $\square^{s-1}$ as pairs of ideal points $\varepsilon(t_1,\ldots,t_{i-1},+\infty,t_{i+1},\ldots,t_{s-1})$ and $\varepsilon(t_1,\ldots,-\infty,\ldots,t_{s-1})$. There are (s-1) such pairs. There is also a pair ε_+, ε_- corresponding to the direction R^1 of the compactification by $\square^{s-1}$ and the sequence of embeddings of (30,B).

Let us now apply the operation of "gluing" to these pairs of ideal points [19][5]. We see that N_i^j (i=1,2) are glued together to give us manifolds PL homeomorphic to $(M_i^m \times T^{s-2}) \times (j,\infty)$ (i=1,2 and j=1,2,.....) . By (30,B), f induces PL embeddings

$$
\begin{aligned}
&\cdots \subset M_2^m \times T^{s-1} \times (j,\infty) \subset f(M_1^m \times T^{s-1} \times (j,\infty)) \\
(31)\qquad &\subset M_2^m \times T^{s-1} \times (j+1,\infty) \subset f(M_1^m \times T^{s-1} \times (j+1,\infty)) \\
&\subset \cdots.
\end{aligned}
$$

for j=1,2,..., and the embeddings are proper in the direction toward ∞ . So we have an h-cobordism

$$(32)\qquad (W_j;\ M_2^m \times T^{s-1} \times (j+\tfrac{1}{2}),\ f(M_1^m \times T^{s-1} \times (j+1+\tfrac{1}{2})) \quad \text{for } j=1,2....$$

The hyperplanes of (30,C) are glued together to become codimension 1 subtori of $M^m \times T^{s-1} \times (j+\frac{1}{2})$ and of $f(M_1^m \times T^{s-1} \times (j+1+\frac{1}{2}))$. Their intersections give us nests of codim 1 subtori in $M_2^m \times T^{s-1} \times (j+\frac{1}{2})$ and $f(M_1^m \times (j+1+\frac{1}{2})$ respectively. When we take a finite cover of W_j corresponding to a normal subgroup of $\pi_1 W_j$ which contains $\pi_1 M_2^m = \pi_1(f(M_1^m))$, the nests of subtori lift to nests of subtori in the covering It is not all that difficult to see that the PL homeomorphism $f|\mathcal{R}$ may be isotopically extended to a PL homeomorp hism to $\mathcal{R} \cup \square^{s-1}$ if and only if there is a finite cover of the above such that the lifted h-cobordism becomes an s-cobordism.

Let us now recall the fundamental decomposition formula of [1,Chap.XII] . Set $Z\pi_1 M^m = \mathbb{A}$. We have

$$
\begin{aligned}
&K_1\mathbb{A}\,[t_1, t_1^{-1}; \ldots\ldots; t_{s-1}, t_{s-1}^{-1}] \\
(33)\qquad &= K_1\mathbb{A} + \sum_{i=1}^{s-1} t_i K_0\mathbb{A} + \cdots + \sum_{\substack{i_1,\ldots,i_\ell \\ i_1 \neq i_2 \neq \cdots \neq i_\ell}}^{s-1} t_{i_1}\cdots t_{i_\ell} K_{-\ell+1}\mathbb{A}
\end{aligned}
$$

$$+ \dots\dots + t_1 t_2 \dots t_{s-1} K_{-s+2} \mathbb{A}$$

mod Nil groups

where $t_{i_1} \dots t_{i_2}$ means 'applying the projection operator L of [1] in the directions $t_{i_1}, \dots, t_{i_\ell}$ successively'. If we consider Wh_1 as a quotient group K_1, we have a decomposition formula corresponding to (33). But we shall abuse our language for simplicity and consider K_1 as Wh_1. Let us observe that $\tau(W_j) \in K_1\mathbb{A}[t_1, t_1^{-1}; \dots; t_{s-1}, t_{s-1}^{-1}]$ are all equal for $j=1,2,\dots$. Denote it by $\tau(W)$, and decompose into the components

(34)
$$a' + \sum_{i=1}^{s-1} t_i a_i^o + \dots + \sum_{\substack{i_1,\dots,i_\ell \\ i_1 \neq i_2 \neq \dots \neq i_\ell}}^{s-1} t_{i_1} \dots t_{i_\ell} a_{i_1 \dots i_\ell}^{-\ell+1} + \dots\dots + t_1 \dots t_{s-1} a^{-s+2}$$

according to (33). For different cubes of the top dimension, we take disjoint spindle neighborhoods and apply our procedure separately. The obstructions to extending to these different cubes are not independent, but actually satisfy a 'cycle condition'.

Now, suppose that we have extended our PL homeomorphism to

(35)
$$\mathcal{R} \cup \{\square^{s-1}\} \cup \dots \cup \{\square^{\ell+1}\} .$$

Let $\square^\ell$ be an ℓ-dim cube in S^{s-1}. We can find relative spindle neighborhoods of $\square^\ell$ with respect to τ_1, τ_2 and apply a relative version of the above construction. Then we may use a decomposition formula

(36)
$$K_1\mathbb{A}[t_1, t_1^{-1}; \dots; t_\ell, t_\ell^{-1}] = K_1\mathbb{A} + \sum_{i=1}^{\ell} t_i K_o\mathbb{A} + \dots\dots + t_1 \dots t_\ell K_{-\ell+1}\mathbb{A}$$

mod Nil groups

of [1, Chap.XII] again such that the total obstruction to extending f to

(37) $$\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots \cup \{\square^{\ell+1}\} \cup \square^{\ell}$$

isotopically is an element

(38) $$a' + \sum_{i=1}^{\ell} t_i a_i^o + \ldots\ldots\ldots + t_1 \ldots\ldots t_\ell \, a_{1\ldots\ell}^{-\ell+1}$$

corresponding to (37). For different ℓ-dim cubes, the obstruction is again related by a 'cycle condition'. Let us summarize it into the following theorem.

Theorem 3.1 Let τ_1, τ_2 be two (admissible) triangulations of X and let $f : X \longrightarrow X$ be a homeomorphism of X onto itself such that $f|\mathcal{R}$ is a properly isotopic isomorphism of $\tau_1|\mathcal{R}$ to $\tau_2|\mathcal{R}$. Suppose that f extends to an isotopic isomorphism from $\tau_1|\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots \cup \{\square^{\ell+1}\}$ to $\tau_2|\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots \cup \{\square^{\ell+1}\}$. Let $\square^{\ell}$ be an ℓ-dim cube of S^{s-1}. Then, the obstruction to extending f to an isotopic isomorphism to

$$\mathcal{R} \cup \{\square^{s-1}\} \cup \ldots \cup \{\square^{\ell+1}\} \cup \square^{\ell}$$

is an element of the form of (38) in the decomposition (37). (Moreover, the obstructions to extending to different ℓ-dim cubes satisfy a 'cycle condition').

Following from [6], we have the following corollary.

Corollary 3.2 Suppose that $\pi_1 M^m$ is a torsion-free solvable group. Let τ_1, τ_2 be two (admissible) triangulations of X, and let $f : X \longrightarrow$ be a homeomorphism. Then, the only obstruction to making of f isotopically isomorphic lies in $H^3(\mathcal{R}; Z_2)$.

Footnotes

(1) For n=1, we actually have $Nil_1\mathbb{A} = Nil_1^+\mathbb{A} \oplus Nil_1^-\mathbb{A}$ with $Nil_1^+\mathbb{A} \cong Nil_1\mathbb{A}$. See [1] [6] for details. Cf. Theorem 2.1.

(2) I am grateful to R. Sharpe for many useful discussions about this part of the paper.

(3) We are grateful to R. Edwards for many useful discussions about this part of the paper.

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Pseudo-Isotopy and K_2

Allen E. Hatcher

This paper is a brief expository account of an application of the functor K_2 to a problem in differential topology, the so-called pseudo-isotopy problem. In fact, with a little hindsight one can see that the geometric problem completely determines K_2. Attempting to turn hindsight to foresight, I propose at the end of the paper a definition of higher K_n's which may be suitable for higher-order pseudo-isotopy problems.

Our starting point is the h-cobordism theorem for smooth manifolds. Recall that an h-cobordism is a (connected) compact manifold W whose boundary is the disjoint union of two closed manifolds M and M' such that each inclusion $M \subset W$ and $M' \subset W$ is a homotopy equivalence. Thus W looks homotopically like the product of M or M' with the closed interval $I = [0,1]$. Recall also the definition of the Whitehead group $Wh_1(\pi_1 M)$ as $K_1\mathbb{Z}[\pi_1 M]$ modulo 1×1 matrices (σ) for $\sigma \in \pm \pi_1 M \subset \mathbb{Z}[\pi_1 M]$.

<u>h-Cobordism Theorem</u>. Provided the dimension of W is at least six, W is diffeomorphic to $M \times I$ if and only if an obstruction $\tau(W,M) \in Wh_1(\pi_1 M)$ vanishes. Moreover, for a given M of dimension at least five each $\tau \in Wh_1(\pi_1 M)$ is realized as the obstruction $\tau(W,M)$ for some h-cobordism W.

Having settled the existence question for product structures on W, one asks about uniqueness: If $F_1, F_2: W \to M \times I$ are two diffeomorphisms, can F_1 be isotoped (i.e., connected by a path of such diffeomorphisms) to F_2? Since we are not interested in the internal structure of M we may as well assume $F_1|M = F_2|M$. Then $F_2 \circ F_1^{-1}$ belongs to $\mathcal{P}(M)$ = {diffeomorphisms $F: M \times I \to M \times I$ such that $F|M \times \{0\}$ = identity}, the topological group of "pseudo-isotopies" on M, and the uniqueness problem becomes to compute $\pi_0\mathcal{P}(M)$.

<u>Pseudo-Isotopy Theorem</u>. There is a homomorphism

$$\pi_0\mathcal{P}(M) \to Wh_2(\pi_1 M) \oplus Wh_1(\pi_1 M; \mathbb{Z}_2 \times \pi_2 M)$$

which is surjective if dim $M \geq 5$ and injective if dim $M \geq 7$.

To define $Wh_2(\pi)$ for a group π we use the definition of $K_2\mathbb{Z}[\pi]$ as the kernel of the natural map $\varphi: St(\mathbb{Z}[\pi]) \longrightarrow GL(\mathbb{Z}[\pi])$ which takes the Steinberg generator x_{ij}^{a} to the elementary matrix e_{ij}^{a} for $a \in \mathbb{Z}[\pi]$ and $i \neq j$. In $St(\mathbb{Z}[\pi])$ let $W\pi$ be the subgroup generated by the words
$w_{ij}^{\sigma} = x_{ij}^{\sigma} x_{ji}^{-\sigma^{-1}} x_{ij}^{\sigma}, \sigma \in \pm\pi$.

Definition. $Wh_2(\pi) = K_2\mathbb{Z}[\pi]/K_2\mathbb{Z}[\pi] \cap W\pi$.
If π is abelian, so that Milnor's symbol pairing is defined, then $K_2\mathbb{Z}[\pi] \cap W\pi$ is just the subgroup of $K_2\mathbb{Z}[\pi]$ generated by the symbols $\{\sigma,\tau\}$ for $\sigma,\tau \in \pm\pi$.

Here is a list of computations of Wh_2 groups:

π	$Wh_2\pi$	
0	0	Milnor [M1]
free	0	Gersten [Ge]
free abelian	0	Quillen [Q]
$G \times \mathbb{Z}$	$Wh_2G \oplus Wh_1G \oplus (?)$	Wagoner [W1]
finite	finite	Garland[Ga], Dennis [D]
$\mathbb{Z}_{20}$	at least 5 elements	Milnor [M2]

Recent work of Dennis and Stein should produce more examples like the last one.

Although the rest of this paper will be about the Wh_2 invariant, for completeness we will now give the definition of $Wh_1(\pi_1M;\mathbb{Z}_2 \times \pi_2M)$. Let the group π act on the abelian group Γ, denoted a^{σ} for $a \in \Gamma$ and $\sigma \in \pi$. In the case at hand $\pi = \pi_1M$ and $\Gamma = \mathbb{Z}_2 \times \pi_2M$ with the usual action of π_1 on π_2 and the trivial action on $\mathbb{Z}_2$, the integers mod 2. Giving Γ trivial multiplication, form the group ring $\Gamma[\pi]$. This is an ideal in the twisted product $\Gamma[\pi] \times \mathbb{Z}[\pi]$, with the twisting given by $\sigma(a\tau) = a^{\sigma}\sigma\tau$.

Proposition. $K_1(\Gamma[\pi] \times \mathbb{Z}[\pi], \Gamma[\pi]) \approx \Gamma[\pi]/(a\sigma - a^{\tau}\tau\sigma\tau^{-1})$.

Definition-Corollary. $Wh_1(\pi;\Gamma) \approx \Gamma[\pi]/(a\sigma - a^{\tau}\tau\sigma\tau^{-1}, b\cdot 1)$. Here $(x,y,\cdots)$ denotes the additive subgroup generated by the elements $x,y,\cdots$.

Oddly enough, the ideal $\Gamma[\pi]$ is of the sort concocted by Swan [S] to show the failure of excision for the relative K_1 functor. Thus $K_1(\Gamma[\pi] \times \mathbb{Z}[1], \Gamma[\pi]) \approx \Gamma[\pi]$ may not equal $K_1(\Gamma[\pi] \times \mathbb{Z}[\pi], \Gamma[\pi])$.

Remarks. The pseudo-isotopy theorem was proved first when M is simply-connected by Cerf [C], who showed in fact that $\pi_0 \mathcal{P}(M) = 0$ if $\dim M \geq 5$ and $\pi_1 M = 0$. The Wh_2 obstruction was discovered independently by J. B. Wagoner [W2] and myself [H1], after which I went on to compute the second obstruction. A write-up of the whole theorem will appear in [H-W] and [H2]. For an exposition of matters relating to the second obstruction, see [H3].

Defining the Wh_2 Invariant

An h-cobordism W is a product $M \times I$ if and only if there exists a smooth map $(W,M,M') \longrightarrow (I,0,1)$ having no critical points. This functional approach carries over to the pseudo-isotopy theorem. Let $\mathcal{F} = \{$smooth maps $(M \times I, M \times \{0\}, M \times \{1\}) \longrightarrow (I,0,1)\}$ and let $\mathcal{E} \subset \mathcal{F}$ be the subspace of maps with no critical points. It is not hard to see that $\pi_{k-1}\mathcal{P}(M) \approx \pi_{k-1}\mathcal{E} \approx \pi_k(\mathcal{F},\mathcal{E})$ for $k \geq 1$. Thus, computing the homotopy groups of $\mathcal{P}(M)$ is parametrized h-cobordism theory.

The main technique for computing $\pi_k(\mathcal{F},\mathcal{E})$, as in so many other places in geometric topology, is "transversality" or "general position". One approximates a given problem by a "generic" problem, reads off some algebraic data from this generic problem, and then factors the data by the generic changes which result from passing from one generic approximation to another. (For example, an early application of this method was the identification of the stable homotopy groups of spheres with framed cobordism.)

A single function $f : W \longrightarrow I$ is generic if and only if it is a morse function, i.e., has only nondegenerate critical points. With the aid of a "gradient-like vector field" for f, the algebraic data one gets from f is a certain exact chain complex over $\mathbb{Z}[\pi], \pi = \pi_1 W = \pi_1 M$, which is free with a (finite) basis in one-to-one correspondence with the critical points of f. Moreover, after some preliminary geometric modification of f we can assume that this based exact chain complex is non-zero only in two dimensions i and i + 1, and hence can be identified with an invertible matrix A over $\mathbb{Z}[\pi]$.

To get an invariant of W we must consider a different choice of f. This can always be connected to f by a generic path f_t, $0 \leq t \leq 1$, which also involves only the two dimensions i and i + 1, and so that the associated matrix A changes only in the following three ways:

(1) Left (right) multiplication by an elementary matrix e^{σ}_{jk}, $\sigma \in \pm \pi$, corresponding to a "handle addition", i.e., an isolated trajectory of the gradient-like vector field connecting two critical points of dimension i (respectively, i+1).

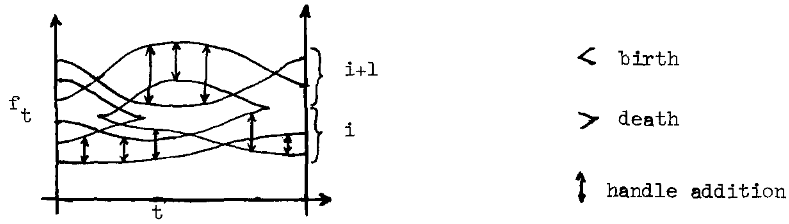
(2) Stabilizing the standard way $A \longrightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, corresponding to the "birth" of a complementary pair of nondegenerate critical points of dimension i and i + 1.

(3) Destabilizing in a non-standard way by cancelling a row and column of A which consist of zeros except for an entry $\sigma \in \pm \pi$ where the row and column meet. This corresponds to the "death" of a critical point pair.

A convenient way of visualizing a one-parameter family is by its <u>graphic</u>, which is the set

$$\{(t, f_t(x)) | x \text{ is a critical point of } f_t\}.$$

For example:



In view of (1) and (2) we should first consider A as lying in $K_1\mathbb{Z}[\pi]$. Then to account for (3) we should factor out further by matrices in $PD\pi = \{(\text{permutation}) \times (\text{diagonal with entries in } \pm\pi)\} \subset GL(\mathbb{Z}[\pi])$. The resulting quotient of $K_1\mathbb{Z}[\pi]$ is just $Wh_1(\pi)$, according to (a) of the following easy lemma.

Lemma. (a) $PD\pi = \varphi(W\pi) \times (\pm\pi)$, where $(\pm\pi)$ denotes the set of 1×1 matrices (σ) for $\sigma \in \pm\pi$.

(b) $\varphi(W\pi) = PD\pi \cap E(\mathbb{Z}[\pi])$.

Thus the class of A in $Wh_1(\pi)$ is an invariant of the h-cobordism W. This is usually proved by identifying this class with the Whitehead torsion of the pair (W,M), which is an invariant of the underlying cell structure of W. However, with the present approach we are all set to define the Wh_2 invariant.

If the generic path $f_t : M \times I \longrightarrow I$ has f_0 and f_1 without critical points, then the product Π of the elementary matrices in (1) above, taken in order as t goes from 0 to 1, is a matrix in $PD\pi$. (We can imagine all the stabilizations in (2) as occurring first, before the type (1) changes, and all the destabilizations in (3) as occurring last.) Part (b) of the preceding lemma implies that such representations of matrices in $PD\pi$ as products of elementary matrices, modulo the Steinberg relations and multiplication by products $e^{\sigma}_{jk}e^{-\sigma^{-1}}_{kj}e^{\sigma}_{jk}$ for $\sigma \in \pm\pi$, form the group $Wh_2(\pi)$. The element of $Wh_2(\pi)$ determined by the product Π is by definition the Wh_2 invariant of f_t.

To show that this association gives rise to a well-defined map $\pi_1(\mathcal{F},\mathcal{E}) \longrightarrow Wh_2(\pi)$ we look at a generic deformation of f_t through a second parameter. Again we can do preliminary geometric work permitting us to restrict to critical points of dimension i and i+1 throughout the two-parameter family, so it suffices to examine the possible changes in the product Π. These are of two types.

(I) The Steinberg relations within Π. These correspond to cancelling or introducing a pair of consecutive handle additions (the relation $e_{jk}^{\sigma}e_{jk}^{-\sigma} = 1$, which for an integral group ring is the only interesting case of the relation $e_{jk}^{a}e_{jk}^{b} = e_{jk}^{a+b}$) and permuting two consecutive handle additions (the relation for a commutator $[e_{jk}^{a}, e_{\ell m}^{b}]$ when $k \neq \ell$ or $j \neq m$). Actually there is another kind of relation coming from an exchange of i/i handle additions for i+1/i+1 handle additions. To state this for an arbitrary ring R with identity, let $(a_{jk}) = \prod_n e_{r_n s_n}^{b_n} \in E(R)$ have an entry $a_{\ell m} = 0$, and let $x \in R$.

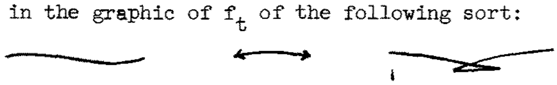
Lemma (Exchange Relation). The relation

$$\prod_{j\neq \ell} e_{j\ell}^{a_{jm}x} \prod_n e_{r_n s_n}^{b_n} = \prod_n e_{r_n s_n}^{b_n} \prod_{k\neq m} e_{mk}^{xa_{\ell k}}$$

is a consequence of the Steinberg relations.

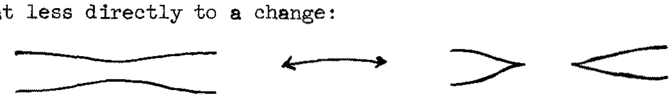
This is a rather interesting relation. Taking $(a_{jk}) = I$, for example, it shows that $K_2(R)$ is the center of $St(R)$. Also, the Steinberg commutator relations are special cases of the exchange relation.

(II) Multiplying Π by an element of $\varphi(W\pi)$. This corresponds directly to changes in the graphic of f_t of the following sort:



or

and somewhat less directly to a change:



The geometric changes in (I) and (II) are the only changes in the one-parameter family f_t which affect Π in any significant way. So we have in fact a well-defined map $\pi_1(\mathcal{F},\mathcal{E}) \longrightarrow Wh_2(\pi)$.

Higher K_n's and More Parameters

In the preceding, K_1 appears as "$\pi_0 GL$" and K_2 as "$\pi_1 GL$." There is an easy way to make this precise which works for any ring R with identity. Consider the cover $\{\alpha pTp^{-1}\}$ of GL(R) by cosets αpTp^{-1} where $\alpha \in GL(R)$, T is the subgroup of (upper) triangular matrices having ones on the diagonal, and p ranges over the permutation matrices in GL(R). Define a simplicial structure $\check{GL}(R)$ on GL(R) by saying that an n-simplex of $\check{GL}(R)$ is a set of n+1 elements of GL(R) lying in one of the cosets αpTp^{-1}. It is not hard to see that $\pi_0\check{GL}(R) \approx K_1R$ and $\pi_1\check{GL}(R) \approx K_2R$. Tentatively then we make the following:

<u>Definition</u>. $K_nR = \pi_{n-1}\check{GL}(R)$ for $n \geq 1$.

I. A. Volodin [V] has also given a definition of algebraic K-theory which seems to be equivalent to this definition But the real precedence belongs to Cerf who in [C] considered a space homotopy equivalent to $\check{GL}(\mathbf{Z})$ (the nerve of the cover $\{\alpha pTp^{-1}\}$, in fact), although he did not call its homotopy groups the K-theory of $\mathbb{Z}$. For more on this K-theory see the paper of Wagoner in these proceedings.

The definition of $\check{GL}(R)$ is based on the behavior of k-parameter families of Morse functions $f:M \times I \longrightarrow I$ (with gradient-like vector fields) for which "all the action is restricted to critical points of a single dimension i," for example by the requirement that f(x) equal a constant c_j for each critical point x of dimension $j \neq i$. I would consider the definition of K_n,s above less tentative if dropping this "single dimension" restriction lead to a space homotopy equivalent to $\check{GL}(R)$. One would also like to drop the requirement that f have only nondegenerate critical points, since this is what must be done to compute $\pi_k(\mathcal{F},\mathcal{E})$. This should correspond to passing from $K_*\mathbf{Z}[\pi]$ to the as yet undefined groups "$Wh_*(\pi)$."

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SUSPENSION, AUTOMORPHISMS, AND DIVISION ALGEBRAS

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The Bott suspension map $\pi_i(GL(\mathbb{C})/GL(\mathbb{R})) \to \pi_{i+1}(GL(\mathbb{H})/GL(\mathbb{C}))$ and in fact all the suspension isomorphisms leading to the periodicity of order 8 in real K-theory can be obtained from the following data: let $R \subset S \subset T$ be rings, σ an automorphism of S which is the identity on R and is inner in T: i.e., $\sigma(s) = jsj^{-1}$ for all $s \in S$, where j is an element of T in the centralizer of R. The Bott maps use Clifford algebras for R, S, T: for example $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, $\sigma(z) = \bar{z} = jzj^{-1}$.

For general R, S, T, σ one would like to define homomorphisms $E: K_i(S,R) \to K_{i+1}(T,S)$, where $K_i(S,R)$ for instance is the (i-1) homotopy group of the fibre of the map $BGL(R)^+ \to BGL(S)^+$ so that these groups fit into a long exact sequence:

$$\to K_i(R) \to K_i(S) \to K_i(S,R) \xrightarrow{\partial} K_{i-1}(R).$$

We will give a somewhat weaker construction, namely homomorphisms Σ, q_* giving a commutative diagram

$$\begin{array}{ccccc} \to K_i(R) & \to & K_i(S) & \to & K_i(S,R) \\ \downarrow\Sigma & & \downarrow\Sigma & & \downarrow q_* \\ K_{i+1}(T) & \to & K_{i+1}(T,S) & \xrightarrow{\partial} & K_i(S) \end{array}$$

such that $\partial\Sigma = \sigma_* - 1$ (σ the given automorphism). In the first part of this paper we construct Σ and give some examples of its non-triviality. In the second part, which is only rather loosely related to the first, we make some computations involving K_2 where R is a local field, T a central division algebra over R and S a splitting field.

I. Construction of q_* and Σ.

For any ring R the space $BGL(R)^+$ may be defined as $\Omega B(\coprod_n BGL_n(R))$, where $\coprod_{n\geq 0} BGL_n(R)$ is a (topological) monoid under the "Whitney sum" operation induced by the inclusions $GL_m(R) \times GL_n(R) \to GL_{m+n}(R)$, and $B(\ )$ denotes classifying space, Ω denotes loop space. The groups $K_i(R)$ are defined to be $\pi_i(BGL(R)^+)$ for $i > 0$. To define a map of $BGL(R)^+$ it suffices to define a monoid homomorphism of $\coprod_n BGL_n(R)$ (with respect to the Whitney sum operation). We may also consider $BGL_n(R)$ as the classifying space of a category (the group $GL_n(R)$), as in [3].

Denote by i the inclusion $GL_n(S) \to GL_n(T)$, σ the automorphism of $GL_n(S)$ induced by that of S, and J conjugation by jI_n in $GL_n(T)$. We have a commutative diagram

$$\begin{array}{ccc} GL_n(S) & \xrightarrow{\ i\ } & GL_n(T) \\ & \underset{i\circ\sigma}{\searrow} & \downarrow J \\ & & GL_n(T) \end{array}$$

which may be regarded as exhibiting jI_n as a natural transformation between the functors i and $i\sigma$ from $GL_n(S)$ to $GL_n(T)$. It is clear that these functors and transformations preserve Whitney sum. According to [3] we thus have an induced homotopy $h_t\colon BGL_n(S) \to BGL_n(T)$ which at $t = 0$ and $t = 1$ lies in $BGL_n(S)$. Because of the proper behavior for Whitney sums we also have a homotopy $h_t^+\colon BGL(S)^+ \to BGL(T)^+$, which has image in $BGL(S)^+$ at $t = 0,1$; in fact $h_o^+ = i^+$ and $h_1^+ = i^+\circ\sigma^+$ (i^+,σ^+ induced by i,σ on $BGL(S)^+$). Furthermore, the restrictions of h_o^+,h_1^+ to $BGL(R)^+$ are just the map $BGL(R)^+ \to BGL(T)^+$ induced by the inclusion $R \to T$. However, we have not shown that the homotopy is constant on $BGL(R)^+$. We may form the space $BGL(T)^+/BGL(S)^+$ which fits into the fibration sequence

$$BGL(S)^+ \xrightarrow{i^+} BGL(T)^+ \longrightarrow BGL(T)^+/BGL(S)^+ \longrightarrow B(\coprod_n BGL_n(S)) \longrightarrow B(\coprod_n BGL_n(T)).$$

The homotopy h_t^+ may be multiplied by the map $x \mapsto i^+(x)^{-1}$, as $BGL(T)^+$ is an H-space: thus let $\phi_t \colon BGL(S)^+ \to BGL(T)^+$

$$\phi_t(x) = h_t^+(x)i^+(x)^{-1}$$

then ϕ_o is a map into the base point and $\phi_1(x)$ is the map $x \mapsto \sigma^+(x)x^{-1} \mapsto i^+(\sigma^+(x))i^+(x)^{-1}$. ϕ_t gives us a map $\overline{\phi}\colon BGL(S)^+ \to \Omega(BGL(T)^+/BGL(S)^+)$ which composed with the natural map

$$\Omega(BGL(T)^+/BGL(S)^+) \to BGL(S)^+$$

is the map previously used by E. Cartan and S. Lang $x \mapsto \sigma^+(x)x^{-1}$ of $BGL(S)^+$ into itself. ϕ_t restricted to the image of $BGL(R)^+$ defines a map ϕ of this space into $BGL(T)^+$. The map $x \mapsto \sigma^+(x)x^{-1}$ of $BGL(S)^+$ into itself takes the image of $BGL(R)^+$ into a point and further factors through a map $q\colon BGL(S)^+/BGL(R)^+ \to BGL(S)^+$. ($q$ may be described also by saying that a point in $BGL(S)^+/BGL(R)^+$ is a path ω in $B(\coprod_n BGL_n(S))$ from the base point to a point in $B(\coprod_n BGL_n(R))$ if this latter is regarded as a subspace. Then $q(\omega)$ is the closed path consisting of $\sigma(\omega)$ followed by the inverse of ω).

We now have the needed maps Σ, q_* if we let Σ on $K_i(S)$ be defined by $\overline{\phi}$, and on $K_i(R)$ by ϕ.

As the first example consider finite fields $\mathbb{F}_q \subset \mathbb{F}_{q^r}$ with Frobenius automorphism σ on $\mathbb{F}_{q^r}$: $\sigma(x) = x^q$. Let $R = \mathbb{F}_q \subset S = \mathbb{F}_{q^r} \subset T = \mathbb{F}_{q^r}\cdot G$: here G is the group generated by σ, and $\mathbb{F}_{q^r}\cdot G$ is the "twisted group algebra" $\{\Sigma\, x\cdot g \mid x \in \mathbb{F}_{q^r},\ g \in G\}$ with multiplication defined by $gx = g(x)\cdot g$. $\mathbb{F}_{q^r}\cdot G$ is a "trivial crossed product" and is isomorphic to the ring $M_r(\mathbb{F}_q)$ of $r \times r$ matrices over $\mathbb{F}_q$. The homomorphism $i_*\colon K_*(\mathbb{F}_{q^r}) \to K_*(\mathbb{F}_{q^r}\cdot G)$ may be identified with the corestriction or transfer $u^*\colon K_*(\mathbb{F}_{q^r}) \to K_*(\mathbb{F}_q)$, where $u\colon \mathbb{F}_q \to \mathbb{F}_{q^r}$ is the inclusion. The results of Quillen [2] on the groups $K_*(\mathbb{F}_q)$ show that we have exact rows in the diagram:

$$\begin{array}{ccccccccc}
0 \to & K_{2n-1}(\mathbb{F}_q) & \xrightarrow{u_*} & K_{2n-1}(\mathbb{F}_{q^r}) & \longrightarrow & K_{2n-1}(\mathbb{F}_{q^r},\mathbb{F}_q) & \to 0 & & \\
 & \downarrow\Sigma & & \downarrow\Sigma & & \downarrow q_* & & & \\
 & 0 & \longrightarrow & K_{2n}(\mathbb{F}_q,\mathbb{F}_{q^r}) & \xrightarrow{\partial} & K_{2n-1}(\mathbb{F}_{q^r}) & \xrightarrow{u^*} & K_{2n-1}(\mathbb{F}_q) & \to 0
\end{array}$$

Further, from Quillen's computation of the groups and the effect of σ_*, we deduce that Σ is surjective and its kernel is $\mathrm{Im}\, u_*$. Σ thus induces an isomorphism $E\colon K_{2n-1}(\mathbb{F}_{q^r},\mathbb{F}_q) \to K_{2n}(\mathbb{F}_q,\mathbb{F}_{q^r})$ as discussed in the introduction.

As another example (discussed in more detail in the second part of this paper), let $R = F$, a local field with residue field $\mathbb{F}_q$ and p a prime distinct from the characteristic of $\mathbb{F}_q$ such that p does not divide $q - 1$. Let r be a positive integer such that p divides $q^r - 1$, and let E be the unramified extension of F of degree r. E is cyclic Galois over F with generating automorphism σ that induces $\sigma(x) = x^q$ on $\mathbb{F}_{q^r}$, the residue field of E. Finally, let $S = E$, T = a central division algebra of degree r^2 over F. The groups K_2F, K_2E are the direct sum of a divisible subgroup and the group of roots of unity $\mu(F)$, respectively $\mu(E)$. Now consider the p-primary subgroup $\mu(E)_{(p)}$ which is a direct summand of K_2E. The map $\sigma_* - 1$ on K_2E induces the automorphism $x \to x^{q-1}$ on $\mu(E)_{(p)}$ (since $(p,q-1) = 1$.) The factorization $\sigma_* - 1 = \partial\Sigma$:

$$\begin{array}{ccccccc} K_2(F) & \longrightarrow & K_2(E) & \longrightarrow & K_2(E,F) & \longrightarrow & 0 \\ & & \downarrow \Sigma & \searrow^{\sigma_*-1} & \downarrow q_* & & \\ & & K_3(D,E) & \xrightarrow{\partial} & K_2(E) & & \end{array}$$

shows that Σ maps $\mu(E)_{(p)}$ isomorphically onto a direct summand of $K_3(D,E)$.

II. K_2 of local division algebras.

Let F be a local field, namely the completion of a global field with respect to a discrete valuation. Let D be a finite dimensional division algebra over F with center F - in short a central division algebra over F (see [4]). It is natural to compare $K_2(D)$ and $K_2(F)$. We prove:

Theorem. K_2D has a direct summand isomorphic to K_2F, under the following additional assumption: if F has characteristic 0 and residual characteristic p and if p divides $[D: F] = n^2$ say $n = p^m n'$, $(p,n') = 1$ then we assume F contains the $(p^m)^{th}$ roots of unity and also that if $p = 2$, F contains the 4^{th} roots of unity.

Proof. We will make considerable use of the transfer (or corestriction) homomorphism. Let $u: F \to D$ be the inclusion, and u_* the corresponding homomorphism on K_2. The inclusion $v: D \to \mathrm{Hom}_F(D,D) = M_{n^2}(F)$ induces $u^*: K_2D \to K_2F$. The composite

$uv\colon D \to \mathrm{Hom}_F(D,D) \to \mathrm{Hom}_D(D \underset{F}{\otimes} D,\ D \underset{F}{\otimes} D) = M_{n^2}(D)$ induces $u_* u^*$. The inclusion $D \to M_{n^2}(D)$ is by means of the left action of D on the right D-module $D \underset{F}{\otimes} D$; however, every 2-sided D module (or $D \otimes D^o$ module) is a direct sum of copies of D, so that $D \underset{F}{\otimes} D = D^{n^2}$ as $D \otimes D^o$-module, and so $D \to M_{n^2}(D)$ is equivalent to the diagonal inclusion. Consequently, $u_* u^*$ on K_2D is multiplication by $n^2 = [D\colon F]$. Similarly $u^* u_*$ on K_2F is multiplication by n^2.

It is known that $K_2F =$ (divisible group) $\oplus\ \mu(F)$, $\mu(F)$ = group of roots of unity in F (a finite abelian group). Consideration of u_*, u^* shows easily that the maximal divisible subgroups of K_2F, K_2D are isomorphic and $K_2D/(\text{Max. div.})$ is a torsion group which differs from $K_2F/(\text{Max. div.})$ at most for the primes dividing n.

Next, we consider the class of D in the Brauer group of F: this is an element of order n. If $n = p_1^{m_1} \cdots p_r^{m_r}$ then $D = \underset{i=1}{\otimes} D_i$, D_i central division algebras over F of degrees $p_i^{2m_i}$. For each i, $D = D_i \otimes D_i'$, D_i' a central division algebra of degree $(n_i')^2$ relatively prime to p_i. Let $w_i\colon D_i \to D$ be the inclusion. We claim $w_i^* w_{i*}$ on $K_2(D_i)$ and $w_{i*} w_i^*$ on $K_2(D)$ are both multiplication by $(n_i')^2$ which is prime to p_i: in fact $w_{i*} w_i^*$ is given by the inclusions $D \to \mathrm{Hom}_{D_i}(D,D) \to \mathrm{Hom}_D(D \underset{D_i}{\otimes} D,\ D \underset{D_i}{\otimes} D)$. The 2-sided D module $D \underset{D_i}{\otimes} D \cong D \underset{F}{\otimes} D'$ is the direct sum of $[D'\colon F]$ copies of D, which proves the statement about $w_{i*} w_i^*$, and the statement about $w_i^* w_{i*}$ is proved in a similar way.

Finally, let E be a Galois extension field of F of degree n, $i\colon F \to E$ the inclusion. Then $i^* i_*$ is multiplication by n on K_2F, but $i_* i^*$ on K_2E is $\sum_{\sigma \in G} \sigma_*$, G being the Galois group of E over F: this follows from the fact that $E \underset{F}{\otimes} E \to \oplus E$ (G copies of E) given by $x \otimes y \mapsto (\ldots, \sigma(x)y, \ldots)$ is an isomorphism of

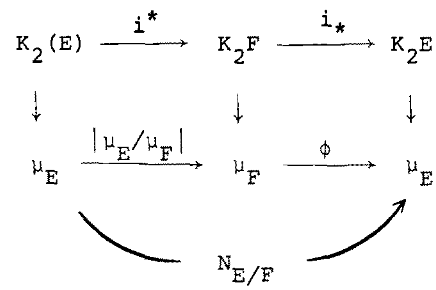
2-sided E-modules, and the corresponding map of E into $\mathrm{Hom}_E(E \otimes_F E, E \otimes_F E) = M_n(E)$ is equivalent to $x \mapsto$ diagonal matrix $(\ldots,\sigma(x),\ldots)$. Suppose now that $F \subset E \subset D$ and E is a maximal subfield of D; let $j: E \to D$ be the inclusion. Then the composite inclusion $E \to D \to \mathrm{Hom}_E(D,D) = M_n(E)$ is the <u>same</u> as the one just considered above, since D is isomorphic to $E \otimes_F E$ as 2-sided E-module. We thus have a commutative diagram (where $N_{E/F}$ denotes $\sum_{\sigma\in G} \sigma_*$):

$$\begin{array}{ccc} K_2E & \xrightarrow{j_*} & K_2D \\ {\scriptstyle i^*}\downarrow & \searrow^{N_{E/F}} & \downarrow{\scriptstyle j^*} \\ K_2F & \xrightarrow[i^*]{} & K_2E \end{array}$$

We can now proceed to the proof of the theorem. We start by considering p-primary components of the groups $K_2/(\text{Max. div.})$, which we will abbreviate as $K_2(\ \ )/\text{Div.}$, where p is the residue characteristic and F has characteristic 0. By using the transfer to a division algebra factor, we may assume $n = p^m$. The isomorphism $K_2F/\text{Div.} \to \mu(F)$ is given by the norm residue symbol. If E is a Galois extension of F, we will need the fact that the following diagram commutes if i denotes the inclusion $F \to E$, and the vertical map is the norm residue symbol:

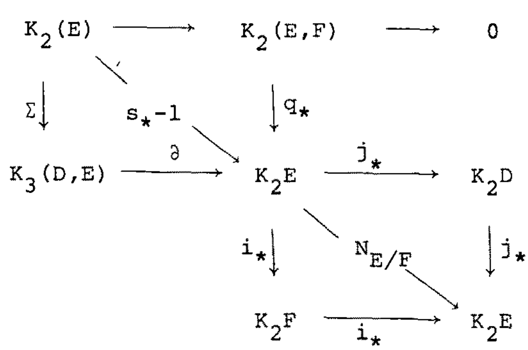
$$\begin{array}{ccc} K_2E & \xrightarrow{i_*i^*} & K_2E \\ \downarrow & & \downarrow \\ \mu_E & \xrightarrow{N_{E/F}} & \mu_E \end{array}$$

We will assume this (presumably well-known fact) without proof. In fact, although we do not need it, the following diagram commutes:



here $|\mu_E/\mu_F|$ is multiplication by the order of this group, this map together with $N_{E/F}$ determining ϕ.

We are assuming that the p-part of μ_F is cyclic of order p^h, $h \geq m$, $[D: F] = p^{2m}$, and $h \geq 2$ if $p = 2$. Let E be obtained from F by adjoining the p^{h+m} roots of unity. It is easy to show that E is a cyclic Kummer extension of F of degree p^m, and the p-part of $\mu(E)$ has order p^{m+h}; if ω generates it so that $\omega^{p^m} = \zeta$ generates the p-part of $\mu(F)$ then the Galois group of E over F has generator, s, $s(\omega) = \omega\zeta^{p^{h-m}}$. Further $N_{E/F}(\omega) = \zeta$ if p is odd, $-\zeta$ if $p = 2$. Thus on the p-parts, $N_{E/F}:(\mu_E)_{(p)} \to (\mu_E)_{(p)}$ has image $(\mu_F)_{(p)}$ and kernel generated by $\omega^{p^h} = \zeta^{p^{h-m}} = s(\omega)\omega^{-1}$, thus the kernel of $N_{E/F}$ is the image of $s_* - 1$ on $K_2E/(\mathrm{div.})$. Consider now the following commutative diagram, in which the rows are exact sequences of the pairs (E,F), (D,E):



We have $j_*(s_*-1) = 0$ since s is induced by an inner automorphism of D. It follows that j_* maps the cokernel of $s_* - 1$ (or of q_*) isomorphically into K_2D (considering p-primary parts of the groups $K_2/(\mathrm{Div.})$) and j^* maps this subgroup isomorphically onto $\mu(F)_{(p)}$. This gives the desired direct summand in $K_2(D)$.

The remaining case, that of p-primary components where p is distinct from the residue field characteristic, can be done in a similar way but without any assumption on roots of unity. We choose E to be the unramified extension of F of degree n, $[D: F] = n^2$. If the residue fields of F,E are $\mathbb{F}_q, \mathbb{F}_{q^n}$ and $\zeta,$ are generators of $\mathbb{F}_q^{\cdot}, \mathbb{F}_{q^n}^{\cdot}$ such that $\omega^{q^{n-1}/q-1} = \zeta$, the Frobenius automorphism is $s(\omega) = \omega^q$ and $N(\omega) = \zeta$. The rest of the proof is the same as in the previous case, completing the proof of the theorem.

Note that the assumption on roots of unity only was used for p-primary components if char. $F = 0$, residue characteristic $= p$ and p divides $[D: F]$.

The theorem is also valid with D,F replaced by their maximal orders $\mathcal{O}_D, \mathcal{O}_F$, since $K_2(\mathcal{O}_F)$ is the direct summand of $K_2(F)$ which is the kernel of the tame symbol, according to a theorem of Dennis and Stein. In other words, $K_2(\mathcal{O}_F) =$ (Divisible group) $\oplus \ (\mu(F))_{(p)}$ where p is the residue characteristic. The proof can now be extracted from the preceding calculations.

It should be noted that the direct summand of $K_2(D)$ isomorphic to $K_2(F)$ is not necessarily the image of $u_*: K_2F \to K_2D$: in fact this homomorphism can be zero modulo divisible subgroups.

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D. K_2 OF FIELDS VIA SYMBOLS

The Milnor ring of a global field

H. Bass and J. Tate

Introduction

The Milnor ring $K_*F = \coprod_{n\geq 0} K_nF$ of a field F was introduced (but not so christened) by Milnor in [8]. He showed there how a discrete valuation v on F with residue class field $k(v)$ gives rise to a homomorphism $\partial_v: K_*F \to K_*k(v)$ of degree -1 of graded abelian groups. The basic result proved here is that if F is a global field then the kernel of

$$K_nF \xrightarrow{\lambda = (\partial_v)} \coprod_v K_{n-1}k(v),$$

where v ranges over all finite places of F, is a finitely generated abelian group.

This "finiteness theorem" leads to a determination of K_nF for $n \geq 3$, viz. $K_nF \cong (\mathbf{Z}/2\mathbf{Z})^{r_1}$, where r_1 is the number of real places of F. The main step in proving this is the determination of $K_{n/p}F = K_nF/pK_nF$ for all primes p and all $n \geq 2$. If $p \neq \mathrm{char}(F)$ and if F contains the group μ_p of p^{th} roots of unity then $K_{2/p}F$ is known from results of Tate [14]. From this information one can compute $K_{n/p}F$ for $n \geq 3$ by the argument reproduced for $p = 2$ in Milnor [8]. The cases when $\mu_p \not\subset F$

and when $p = \mathrm{char}(F)$ are then handled easily with the aid of so called "transfer maps," $N: K_*E \to K_*F$ defined for finite field extensions E/F. These have so far been defined only for K_n with $n \leq 2$. Such transfer maps, with the properties necessary for the above arguments, are constructed here for all $n \geq 0$.

Concerning the finitely generated group $\mathrm{Ker}(K_2F \xrightarrow{\lambda} \coprod_v K_1k(v))$, the transfer arguments show that it is finite of order prime to p if $\mathrm{char}(F) = p > 0$. Indeed its structure has been completely determined in this case by Tate (see [2] and [14]). When F is a number field its finiteness follows from results of Garland [5] and Dennis [4], and conjectures on its structure and order have been formulated by Birch and Tate (cf. [13] and [14]). These have been partially confirmed in special cases by Coates [3], and spectacularly generalized by Lichtenbaum [7].

This paper consists of two chapters, the second one being devoted to the finiteness theorem and its applications described above. The finiteness theorem for K_2 was among the results announced in [1] and [13].

Chapter I contains some general remarks, partly of an expository nature, on the Milnor ring of a general field. Much of this is a review and retreatment of material in Milnor [8], in particular the construction of the maps ∂_v. In §5 we use

Milnor's description of $K_*k(t)$ (a rational function field) to construct the transfer maps. Some typical applications of them are derived. In an appendix by the second named author, $\mathrm{Ker}(\lambda)$ is computed for the imaginary quadratic fields of discriminants -3, -4, -7, -8, -11, and -15.

Contents

Notation. The group of units of a ring A is denoted $A^{\bullet}$.

Chapter I

Some general remarks on the Milnor ring

§1. Definition and first properties of K_*F (cf. [8]).

Let F be a field, and $F^\cdot$ its multiplicative group.

In the tensor algebra $T(F^\cdot) = \coprod_{n\geq 0} T^n(F^\cdot)$ of the $\mathbb{Z}$-module $F^\cdot$ we denote the isomorphism $F^\cdot \to T^1(F^\cdot)$ by $a \mapsto [a]$. If $a \neq 0,1$ then $r_a = [a] \otimes [1 - a] \in T^2(F^\cdot)$. The two sided ideal R generated by such elements r_a is graded, and we put

$$K_*F = T(F^\cdot)/R = \coprod_{n\geq 0} K_nF.$$

The image of $[a] \in T^1(F^\cdot)$ in K_1F will be denoted $\ell(a)$. Thus K_*F is presented, as a ring, by generators $\ell(a)$ $(a \in F^\cdot)$ subject to the relations:

$$(R_1) \quad \ell(ab) = \ell(a) + \ell(b)$$

$$(R_2) \quad \ell(a)\ell(b) = 0 \quad \text{if} \quad a + b = 1.$$

The identity $-a = (1 - a)/(1 - a^{-1})$ implies that

$$(1) \qquad r_a + r_{a^{-1}} = [a] \otimes [-a]$$

for $a \neq 0,1$, whence

$$(R_3) \quad \ell(a)\ell(-a) = 0$$

for $a \in F^\cdot$, or, equivalently,

$$(R_3')\quad \ell(a)^2 = \ell(a)\ell(-1)$$

The formula

$$(2)\qquad [ab]\otimes[-ab] = ([a]\otimes[-a) + [b]\otimes[-b]) + ([a]\otimes[b] + [b]\otimes[a])$$

then further implies that

$$(R_4)\quad \ell(a)\ell(b) = -\ell(b)\ell(a).$$

Since K_1F generates the graded ring K_*F it follows ([8], Lemma 1.1) from (R_4) that

$$(R_4')\ K_*F \text{ is anticommutative.}$$

Further ([8], Lemma 1.3) we have

$$(R_5)\quad \ell(a_1)\ldots\ell(a_n) = 0 \quad \text{if} \quad a_1+\ldots+a_n = 1 \text{ or } 0$$

This is established by induction on n, the case $n = 2$ being (R_2) and (R_3).

(1.1) Remark. Suppose $d\colon F^{\bullet} \to A$ is a homomorphism into the additive group of a ring A, and we wish to show that d induces a homomorphism $K_*F \to A$ $(\ell(a) \mapsto d(a))$. We must verify (R_2) for d, i.e. $d(a)d(1-a) = 0$ for $a \neq 0,1$. If we know (R_3) for d then, by (1), we see that we are free to replace a by a^{-1} in verifying (R_2), and also to replace a by $1 - a$, in view of (R_4).

Further, if we have (R_4), then to verify (R_3) it suffices, by (2), to do so when a ranges over a set of generators of $F^{\cdot}$.

Since R is generated by elements r_a of degree 2, we have $R = \coprod_{n\geq 2} R_n$ with $R_n = \sum_{p+q=n-2} T^p R_2 T^q$, where we write T^p for $T^p(F^{\cdot})$. It follows that

$$\mathbb{Z} \xrightarrow{\cong} K_0\mathbb{Z} \quad \text{and} \quad \ell: F^{\cdot} \xrightarrow{\cong} K_1F$$

are isomorphisms, and that, for $n \geq 2$, K_nF is presented, as abelian group, by generators $\ell(a_1)\ldots\ell(a_n)$ $(a_1,\ldots,a_n \in F^{\cdot})$ subject to the relations:

$(R_1)_n$ $(a_1,\ldots,a_n) \longmapsto \ell(a_1)\ldots\ell(a_n)$ is a multilinear function $F^{\cdot}\times\ldots\times F^{\cdot} \longrightarrow K_nF$;

and

$(R_2)_n$ $\ell(a_1)\ldots\ell(a_n) = 0$ if $a_i + a_{i+1} = 1$ for some $i < n$.

Thus the homomorphisms from K_nF to a (multiplicative) abelian group C are equivalent to multilinear functions $f: F^{\cdot}\times\ldots\times F^{\cdot} \to C$ of n variables on $F^{\cdot}$ such that $f(a_1,\ldots,a_n) = 1$ if $a_i + a_{i+1} = 1$ for some $i < n$. Such a function f will be called a (C-valued) n-symbol on F.

The relations in K_*F derived above imply that f is anti-symmetric and that $f(a_1,\ldots,a_n) = 1$ if $a_i+\ldots+a_j = 1$ or 0 for some $1 \leq i \leq j \leq n$.

(1.2) PROPOSITION. *Let* m *be an integer* ≥ 1. *Assume that each polynomial* $X^m - a$ $(a \in F)$ *splits into linear factors in* $F(X)$; *thus* $F^{\cdot}$ *is divisible by* m. *Then* K_nF *is uniquely divisible by* m *for* $n \geq 2$.

Consider the exact commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R_n & \longrightarrow & T^n & \longrightarrow & K_n & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle m} & & \\
0 & \longrightarrow & R_n & \longrightarrow & T^n & \longrightarrow & K_n & \longrightarrow & 0
\end{array}$$

where $T^n = T^n(F^{\cdot})$ and $K_n = K_nF$. If we show that (i) $T^n \overset{m}{\to} T^n$ is bijective for $n \geq 2$, and (ii) $R_n \overset{m}{\to} R_n$ is surjective, then the bijectivity of $K_n \overset{m}{\to} K_n$ (for $n \geq 2$) will follow from the Snake Lemma. Assertion (i) results from:

If A *and* B *are abelian groups divisible by* m *then* $A \otimes B$ *is uniquely divisible by* m.

In fact let $A_m = \bigcup_{r \geq 1} \mathrm{Ker}(A \xrightarrow{m^r} A)$, the "m-primary part of A." Clearly $A_m \otimes B = 0$ (B is divisible by m) so $A \otimes B \to (A/A_m) \otimes B$ is an isomorphism. Multiplication by m is an isomorphism on A/A_m, hence also on $(A/A_m) \otimes B$.

To prove (ii), i.e. that $R_n = \sum_{p+q=n-2} T^p R_2 T^q$ is divisible by m it suffices to treat the case n = 2, and even to show that $r_a \in mR_2$ for each $a \neq 0,1$. By hypothesis $X^m - a = \prod_{i=1}^{m} (X-b_i)$ where each $b_i \in F$, and $b_i^m = a$. Then $r_a = [a]\otimes[1-a] = [a]\otimes[\prod_i (1-b_i)]$ $= \sum_i [a]\otimes[1-b_i] = \sum_i [b_i^m]\otimes[1-b_i] = m(\sum_i r_{b_i})$. This completes the proof of (1.2).

(1.3) COROLLARY. *If F is algebraically closed then K_nF is torsion free and divisible for* $n \geq 2$.

(1.4) COROLLARY. *If F is a perfect field of characteristic* $p > 0$ *then K_nF is uniquely divisible by p for* $n \geq 2$.

§2. κ-Algebras.

The graded ring $\kappa = \coprod_{n\geq 0} \kappa_n$ is defined by

$$\kappa = \mathbf{Z}[X]/2X\mathbf{Z}[X] = \mathbf{Z}[\epsilon]$$

where X is an indeterminate with image ϵ (of degree 1) in κ. Thus $\kappa_0 = \mathbf{Z}$ and $\kappa_n = \mathbf{F}_2\epsilon^n$ for $n \geq 1$; κ is the ring of polynomials in a variable ϵ with constant term in $\mathbf{Z}$ and higher degree terms in $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$.

A graded κ-algebra is a graded ring $A = \coprod_{n\geq 0} A_n$ equipped with a homomorphism $\kappa \to A$ of graded rings, defined by $\epsilon \mapsto \epsilon_A \in A_1$, such that $\epsilon_A \in$ Center (A). We call A a κ-Algebra if further A_1 generates A as a κ-algebra and

(1) $$a^2 = \epsilon_A a \quad \text{for all } a \in A_1.$$

(2.1) EXAMPLE. Let F be a field. Then $\kappa \to K_*F$, $\epsilon \mapsto \ell(-1)$, gives K_*F the structure of a κ-Algebra. Indeed $\ell(-1)$ is central because K_*F is anticommutative and $2\ell(-1) = 0$, and (1) above follows from relation (R_3') in §1.

Other examples include $A = \Lambda(M)$, the exterior algebra of a $\mathbf{Z}$-module M, with $\epsilon_A = 0$.

(2.2) PROPOSITION. Let A be a $\varkappa$-Algebra.

(a) A is anticommutative.

(b) If $\epsilon_A = 0$ then the inclusion $A_1 \to A$ induces an epimorphism $\Lambda(A_1) \to A$ from the exterior algebra of the $\mathbb{Z}$-module A_1.

(c) If J is a finitely generated ideal contained in $A_+ = \coprod_{n\geq 1} A_n$ then some power of J lies in $\epsilon_A A$. If further $J \subset 2A_+$ then J is nilpotent.

(d) A_+ is a nil ideal, i.e. its elements are all nilpotent, if and only if ϵ_A is nilpotent.

To prove (a) it suffices to show that $ab = -ba$ for $a,b \in A_1$. This follows, using (1), from the calculation

$$\epsilon_A(a+b) = (a+b)^2 = a^2 + b^2 + ab + ba$$

$$= \epsilon_A(a+b) + (ab+ba).$$

Assertion (b) is immediate from the definition of a $\varkappa$-Algebra. To prove the first part of (c) we may pass to $A/\epsilon_A A$ and then apply (b) in order to reduce to the case $A = \Lambda(A_1)$. To show then that a finitely generated ideal $J \subset A_+$ is nilpotent it suffices to treat the case $J = EA$ for some finitely generated subgroup E of A_1, since any J as above is clearly contained in such an ideal EA. Since A is anticommutative we have $(E\cdot A)^n = E^n \cdot A$. If E has < n generators then $\Lambda^n(E) = 0$ and so $(EA)^n = 0$. This proves the first part of (c).

To prove the second part of (c) we first note (as just proved) that $J^n \subset \epsilon_A A$ for some $n > 0$. Now if $J \subset 2A$ then $J^{n+1} \subset 2\epsilon_A A = 0$. This proves (c), and (d) is immediate from (c).

Since $2A_+$ is a nil ideal, and since the ring $A/2A_+$ is commutative it is natural to call an ideal of A _prime_ if it is the inverse image of a prime ideal of $A/2A_+$. In the graded $\mathbb{F}_2$-algebra $A/2A$ the set of homogeneous prime ideals not containing $(A/2A)_+$ is denoted Proj $(A/2A)$.

Since $A/A_+ = A_0$ is a quotient of $\mathbb{Z}$ it is easy to determine the prime ideals of A containing A_+.

(2.3) PROPOSITION. _Let_ A _be a_ $\varkappa$_-Algebra. Let_ $\mathscr{G}$ _be a graded prime ideal of_ A _not containing_ A_+. _Then_ $\mathscr{G} = 2A + (\mathscr{G} \cap A_1)A$, _and_ $A/\mathscr{G} \cong \varkappa/2\varkappa = \mathbb{F}_2[\epsilon]$, _a polynomial ring over_ $\mathbb{F}_2$ _in one variable. The map_ $\mathscr{G} \mapsto \mathscr{G}/2A$ _is a bijection from the set of such prime ideals_ $\mathscr{G}$ _to_ Proj$(A/2A)$.

Passing to $A/2A_+$ we may assume $2A_+ = 0$, whence A is commutative. We may further factor out $(\mathscr{G} \cap A_1)A$ to achieve the condition $\mathscr{G} \cap A_1 = 0$. Then the equation $a(a - \epsilon_A) = 0$ for $a \in A_1$ implies that $a - \epsilon_A \in \mathscr{G} \cap A_1 = 0$ for any $a \neq 0$ in A_1. Since $A_+ \not\subset \mathscr{G}$ there exists an $a \neq 0$ in A_1, whence

$A_1 = \mathbb{F}_2\epsilon$. It follows that $\varkappa \to A$ is surjective, with kernel a graded ideal containing no power of ϵ. It follows easily that $A \cong \varkappa/2n\varkappa$ for some integer n. Since $2\epsilon = 0$ we have $2 \in \mathfrak{q}$; thus $A/\mathfrak{q}$ is a quotient of $\varkappa/2\varkappa = \mathbb{F}_2[\epsilon]$ by a graded ideal containing no power of ϵ. Clearly the only such ideal is zero, so $A/\mathfrak{q} \cong \mathbb{F}_2[\epsilon]$.

Since all primes $\mathfrak{q}$ as above contain $2A$, they are precisely the inverse images of the elements of $\mathrm{Proj}(A/2A)$.

(2.4) PROPOSITION. *Let A be a $\varkappa$-Algebra such that $A_0 \cong \mathbb{Z}$. The map $\rho \mapsto (\mathrm{Ker}(\rho) + 2A)/2A$ is a bijection from $\mathrm{Hom}_{\varkappa\text{-Alg}}(A,\varkappa)$ to $\mathrm{Proj}(A/2A)$. The nil radical of A is given by,*

$$\mathrm{nil}(A) = \bigcap_{\rho} \mathrm{Ker}(\rho)$$

where ρ varies over $\mathrm{Hom}_{\varkappa\text{-Alg}}(A,\varkappa)$.

If $\rho: A \to \varkappa$ then $A/(\mathrm{Ker}(\rho) + 2A) \cong \varkappa/2\varkappa = \mathbb{F}_2[\epsilon]$ is an integral domain, so $\mathrm{Ker}(\rho) + 2A$ is a graded-prime ideal of A not containing A_+. Conversely if $\mathfrak{q}$ is such a prime ideal then it follows easily from Prop. (2.3) and the fact that $A_0 = \mathbb{Z}$ that $A/(\mathfrak{q} \cap A_+) \cong \varkappa$. Moreover this isomorphism is unique since $\varkappa$ has no non identity graded ring automorphisms. Therefore $\mathfrak{q} \cap A_+ = \mathrm{Ker}(\rho)$ for a unique $\varkappa$-Algebra homomorphism $\rho: A \to \varkappa$, and $\mathfrak{q} = \mathrm{Ker}(\rho) + 2A$ by Prop. (2.3).

The nil radical of the graded ring A is the intersection of the graded prime ideals. Those containing A_+ intersect in A_+ since $A/A_+ \cong \mathbf{Z}$. The others we have seen to be of the form $\mathrm{Ker}(\rho) + 2A$ $(\rho: A \to \kappa)$, and $(\mathrm{Ker}(\rho) + 2A) \cap A_+ = \mathrm{Ker}(\rho)$. It follows that $\mathrm{nil}(A) = \bigcap_\rho \mathrm{Ker}(\rho)$.

§3. Real fields.

Let F be a field. An ordering of F is a subset P of F such that $a,b \in P \Rightarrow ab$ and $a+b \in P$, and such that $F^{\bullet}$ is the disjoint union of P and $-P$. A field which admits an ordering is called formally real.

Let $\rho: K_*F \to \varkappa$ be a homomorphism of $\varkappa$-Algebras (see §2). Put

$$P_\rho = \{a \in F^{\bullet} \mid \rho(\ell(a)) = 0\}.$$

(3.1) THEOREM. *The map $\rho \mapsto P_\rho$ is a bijection from $\mathrm{Hom}_{\varkappa\text{-Alg}}(K_*F,\varkappa)$ to the set of orderings of F.*

In view of Prop. (2.4) this yields the:

(3.2) COROLLARY. *If $a \in F^{\bullet}$ then $\ell(a)$ is nilpotent if and only if a is positive under every ordering of F. Hence $(K_*F)_+$ is a nil ideal if and only if F is not formally real.*

(3.3) *Remark.* It is known that the "totally positive" elements of $F^{\bullet}$ are the sums of squares. In case $a = b_1^2 + \ldots + b_n^2$ one can prove the nilpotence of $\ell(a)$ directly as follows (cf. [8], Thm. 1.4): From (R_5) one has $\ell(a)\ell(-b_1^2)\ldots\ell(-b_n^2) = 0$. Since $\ell(-b^2) = \ell(-1) + 2\ell(b)$ one obtains the congruence modulo $2K_*F$, $0 \equiv \ell(a)\ell(-1)\ldots\ell(-1)$ $= \ell(a)\ell(-1)^n = \ell(a)^{n+1}$, whence $\ell(a)^{n+2} = \ell(a)^{n+1}\ell(-1) = 0$.

(3.4) Remark. From Prop. (2.4) we have a bijection $\mathrm{Hom}_{\varkappa\text{-Alg}}(K_*F,\varkappa) \to \mathrm{Proj}(K_*F/2K_*F)$. The latter has a natural topology in which closed sets consist of those primes containing a given subset S of $K_*F/2K_*F$. Since these primes are generated by their degree 1 components (c.f. Prop. (2.3)) one can restrict attention to sets S of elements of degree 1. Pulling this description back to $\mathrm{Hom}_{\varkappa\text{-Alg}}(K_*F,\varkappa)$ and then, via Thm. (3.1), to the set $O(F)$ of orderings of F, we deduce a homeomorphism $O(F) \to \mathrm{Proj}(K_*F/2K_*F)$, where closed sets in $O(F)$ consist of all orderings containing a given subset $T \subset F^{\bullet}$.

Proof of Thm. (3.1). Since the composite $F^{\bullet} \xrightarrow{\ell} K_1F \xrightarrow{\rho} \mathbb{F}_2\epsilon$ is a surjection with kernel $P = P_\rho$ and $\rho(\ell(-1)) = \epsilon$ we see that $F^{\bullet} = \{\pm 1\} \times P$ (direct product). To see that P is an ordering it remains to show that if $a,b \in P$ then $a + b = c \in P$. We have $c \neq 0$ for otherwise $a = -b \in P \cap -P = \emptyset$. From $\frac{a}{c} + \frac{b}{c} = 1$ we conclude that $(\ell(a) - \ell(c))(\ell(b) - \ell(c)) = 0$. Applying ρ we have $\rho(\ell(c))^2 = 0$, whence $\rho(\ell(c)) = 0$ since $\mathrm{nil}(\varkappa) = 0$. Thus $c \in P$, as claimed.

Suppose now that P is a given ordering of F. Define $s\colon F^{\bullet} \to \varkappa$ by $s(a) = 0$ if $a \in P$ and $s(a) = \epsilon$ if $-a \in P$. By well known properties of orderings s is a homomorphism. Moreover

$s(a)s(1-a) = 0$ for $a \neq 0,1$ since a and $1 - a$ cannot both be negative for P; otherwise $1 = a + (1-a) \in -P$. Thus s induces a homomorphism $\rho: K_*F \to \varkappa$ and evidently $P = P_\rho$. It is clear that this construction is inverse to the map $\rho \mapsto P_\rho$ above, thus proving Theorem (3.1).

§4. Discrete valuations.

(4.1) Constructions on $\varkappa$-Algebras. Let A and B be $\varkappa$-Algebras. Let $A \otimes_{\mathbb{Z}} B$ denote the graded ring with $\coprod_{p+q=n} A_p \otimes_{\mathbb{Z}} B_q$ in degree n, and with product defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{\deg(b)\deg(a')} aa' \otimes bb'$$

for homogeneous elements $a, a' \in A$, $b, b' \in B$. The elements $a\epsilon_A \otimes b - a \otimes \epsilon_B b$, for homogeneous $a \in A$ and $b \in B$, generate a graded ideal, modulo which we obtain a graded anticommutative ring

$$A \otimes_{\varkappa} B$$

with $(A \otimes_{\varkappa} B)_n = \sum_{p+q=n} A_p \otimes B_q$. The latter sum is not direct since $A_p\epsilon_A \otimes B_q = A_p \otimes \epsilon_B B_q$ is contained in $(A_{p+1} \otimes B_q) \cap (A_p \otimes B_{q+1})$.

If $c = a \otimes 1 + 1 \otimes b \in (A \otimes_{\varkappa} B)_1$ then
$c^2 = a^2 \otimes 1 + a \otimes b - a \otimes b + 1 \otimes b^2 = \epsilon_A a \otimes 1 + 1 \otimes \epsilon_B b = \epsilon c$, where $\epsilon = \epsilon_A \otimes 1 = 1 \otimes \epsilon_B$. Therefore putting $\epsilon_{A \otimes_{\varkappa} B} = \epsilon$ gives $A \otimes_{\varkappa} B$ the structure of a $\varkappa$-Algebra. We shall understand $A \otimes_{\varkappa} B$ to denote this $\varkappa$-Algebra, called the tensor product of A and B. It is the coproduct of $\varkappa$-Algebras.

The free $\varkappa$-Algebra on a generator Π is the $\varkappa$-Algebra

$$\varkappa\langle \Pi \rangle = \varkappa[X]/(X^2 - \epsilon X)$$

where X is an indeterminate of degree 1 with image Π modulo

$X^2 - \epsilon X$. Evidently $\varkappa\langle\Pi\rangle$ is a free $\varkappa$-module with basis $1, \Pi$.

For any $\varkappa$-Algebra A we put

$$A\langle\Pi\rangle = A \otimes_{\varkappa} \varkappa\langle\Pi\rangle = A \oplus A\Pi.$$

This is a free left (or right) A-module with basis $1, \Pi$: $A\langle\Pi\rangle_p = A_p \oplus A_{p-1}\Pi$. If $a + b\Pi \in A\langle\Pi\rangle_p$ and $c + d\Pi \in A\langle\Pi\rangle_q$ then

$$\begin{aligned}&(a + b\Pi)(c + d\Pi)\\ &= ac + ad\Pi + (-1)^q bc\Pi + (-1)^{q^{-1}} bd\Pi^2\\ &= ac + (ad + (-1)^q bc + bd\epsilon)\Pi\end{aligned}$$

We shall consider below the map $\partial: A\langle\Pi\rangle \to A$, $\partial(a + b\Pi) = b$; it is an epimorphism of degree -1 of graded abelian groups. It is also an antiderivation, in the following sense: There are $\varkappa$-Algebra retractions $\lambda, \rho: A\langle\Pi\rangle \to A$ defined by $\lambda(\Pi) = 0$ and $\rho(\Pi) = \epsilon$. Then for $x, y \in A\langle\Pi\rangle$ we have

$$\partial(xy) = \lambda(x)\partial(y) + (-1)^{\deg(y)}\partial(x)\rho(y).$$

Writing $x = a + b\Pi$ and $y = c + d\Pi$, this follows from the formula derived above for xy.

(4.2) <u>Discrete valuations</u>. Let v be a discrete valuation on a field F, i.e. an epimorphism $v: F^{\cdot} \to \mathbb{Z}$ such that, putting $v(0) = \infty$, we have $v(a + b) \geq \min(v(a), v(b))$. Then

$\mathcal{O} = \mathcal{O}_v = \{a \mid v(a) \geq 0\}$ is a ring, the valuation ring of v. Choose a local parameter π of v, i.e. $\pi \in F^{\cdot}$ and $v(\pi) = 1$. Then $F^{\cdot}$ is the direct product of $\mathcal{O}^{\cdot} = \mathrm{Ker}(v)$ and the infinite cyclic group $\pi^{\mathbb{Z}}$. In particular all non zero ideals of $\mathcal{O}$ are of the form $\pi^n \mathcal{O}$ $(n \geq 0)$. The unique maximal ideal is $\pi\mathcal{O}$ and $k = k(v) = \mathcal{O}/\pi\mathcal{O}$ is called the residue class field of v. The canonical map $\mathcal{O} \to k$ will be denoted $a \mapsto \bar{a}$. It induces an exact sequence of groups

$$1 \longrightarrow (1 + \pi\mathcal{O}) \longrightarrow \mathcal{O}^{\cdot} \longrightarrow k^{\cdot} \longrightarrow 1$$

Define

$$d = d_{\pi} : F^{\cdot} \longrightarrow (K_* k)\langle \Pi \rangle$$

$$d(u\pi^i) = \ell(\bar{u}) + i\Pi$$

for $u \in \mathcal{O}^{\cdot}$, $i \in \mathbb{Z}$.

(4.3) PROPOSITION. *The homomorphism d_{π} induces a homomorphism*

$$\partial_{\pi} : K_* F \longrightarrow (K_* k(v))\langle \Pi \rangle$$

of $\varkappa$-Algebras. The latter is surjective, and

$$\mathrm{Ker}(\partial_{\pi}) = \ell(1 + \pi\mathcal{O})K_* F.$$

If $u_1, \ldots, u_n \in \mathcal{O}^{\cdot}$ then

$$\partial_{\pi}(\ell(u_1)\ldots\ell(u_n)) = \ell(\bar{u}_1)\ldots\ell(\bar{u}_n)$$

and

$$\partial_\pi(\ell(u_1)\ldots\ell(u_{n-1})\ell(\pi)) = \ell(\bar{u}_1)\ldots\ell(\bar{u}_{n-1})\Pi$$

We must verify

$$(R_2)\quad d(a)\ d(1-a) = 0$$

for $a \neq 0,1$. If $a \in \mathcal{O}^\cdot$ then either $1 - a \in \mathcal{O}^\cdot$ also and $d(a)d(1-a) = \ell(\bar{a})\ell(\bar{1} - \bar{a}) = 0$ or $1 - a \notin \mathcal{O}^\cdot$ and $\bar{a} = 1$ so $d(a) = \ell(\bar{a}) = 0$. Thus (R_2) holds for $a \in \mathcal{O}^\cdot$. For any a we have $a \in \mathcal{O}$ or $a^{-1} \in \mathcal{O}$, and if $a \in \mathcal{O}$ then $a \in \mathcal{O}^\cdot$ or $1 - a \in \mathcal{O}^\cdot$. Hence, by Remark (1.1), (R_2) will follow once we verify

$$(R_3)\quad d(a)d(-a) = 0.$$

Since $(K_*k)\langle\Pi\rangle$ is anticommutative it suffices to verify (R_3) for generators of $F^\cdot$, so we may assume $a \in \mathcal{O}^\cdot$ or $a = \pi$. If $a \in \mathcal{O}^\cdot$ then $d(a)d(-a) = \ell(\bar{a})\ell(-\bar{a}) = 0$. Finally $d(\pi)d(-\pi) = \Pi(\ell(-1) + \Pi) = \Pi\epsilon + \Pi^2 = 0$. Thus ∂_π exists, and the formulas in the Proposition are immediate since ∂_π is an algebra homomorphism.

If $a \in 1 + \pi\mathcal{O}$ than $\bar{a} = 1$ so $\partial_\pi(\ell(a)) = 0$. Hence $J = \ell(1 + \pi\mathcal{O})K_*F \subset \mathrm{Ker}(\partial_\pi)$. To show that this is an equality denote by $\bar{x}$ the class modulo J of $x \in K_*F$. Define $s\colon (K_*k)\langle\Pi\rangle \to K_*F/J$ by $\ell(\bar{a}) \mapsto \overline{\ell(a)}$ for $a \in \mathcal{O}^\cdot$ and $\Pi \mapsto \overline{\ell(\pi)}$. Since Π is a free κ-Algebra generator we need only check, in order to show that the definition of s is legitimate, that $s(\ell(\bar{a}))s(\ell(\bar{1} - \bar{a})) = 0$ for $\bar{a} \neq 0,1$ in k. If $a \in \mathcal{O}^\cdot$ represents

$\bar{a}$ then $1 - a \in \mathcal{O}^{\cdot}$ and we have $s(\ell(\bar{a}))s(\ell(\bar{1} - \bar{a})) = \overline{\ell(a)}\ \overline{\ell(1-a)}$ $= \overline{\ell(a)\ell(1-a)} = 0$. The image of s contains $\overline{\ell(\mathcal{O}^{\cdot})}$ and $\overline{\ell(\pi)}$; the latter generate K_*F/J, so s is surjective. Further it is clear that $s(\partial_\pi(x)) = \bar{x}$ for $x \in \ell(\mathcal{O}^{\cdot})$ or $x = \ell(\pi)$ Thus s is an inverse to the map $K_*F/J \to (K_*k)\langle\Pi\rangle$ induced by ∂_π. This proves that $J = \mathrm{Ker}(\partial_\pi)$ and so completes the proof of Prop. (4.3).

We define maps

$$\partial_\pi^0,\ \partial_v\colon K_*F \longrightarrow K_*k(v)$$

by

$$\partial_\pi(x) = \partial_\pi^0(x) + \partial_v(x)\Pi$$

(4.4) PROPOSITION. ∂_π^0 _is an epimorphism of_ κ-_Algebras with kernel_ $\mathrm{Ker}(\partial_\pi) + \ell(\pi)K_*F$. _If_ $u \in \mathcal{O}^{\cdot}$ _then_

$$\partial_{u\pi}^0(\ell(a)) = \partial_\pi^0(\ell(a)) - v(a)\ell(\bar{u})$$

for $a \in F^{\cdot}$.

The first assertion is immediate from Prop. (4.3) and the fact that, for any κ-Algebra A, $a + b\Pi \mapsto a$ is a κ-Algebra eipmorphism $A\langle\Pi\rangle \to A$ with kernel $\Pi A\langle\Pi\rangle$. If $a = a_0\pi^i = a_0u^{-i}(u\pi)^i$ then $\partial_\pi^0(\ell(a)) = \ell(\bar{a}_0)$ while $\partial_{u\pi}^0(\ell(a)) = \overline{\ell(a_0u^{-i})} = \ell(\bar{a}_0) - i\ell(\bar{u})$. This completes the proof of Prop. (4.4).

(4.5) PROPOSITION.

(a) ∂_v _is an epimorphism of degree -1 of graded abelian groups._

(b) One has $\mathrm{Ker}(\partial_v) = \mathbb{Z}[\ell(\mathcal{O}^\bullet)]$, *where* $\mathbb{Z}[\ell(\mathcal{O}^\bullet)]$ *denotes the subring of* K_*F *generated by* $\ell(\mathcal{O}^\bullet)$.

(c) *If* $u_1,\ldots,u_{n-1} \in \mathcal{O}^\bullet$ *and* $a \in F^\bullet$ *then*

$$\partial_v(\ell(u_1)\ldots\ell(u_{n-1})\ell(a)) = \ell(\bar{u}_1)\ldots\ell(\bar{u}_{n-1})v(a)$$

(d) ∂_v *depends only on* v *and not on* π.

(e) *The following diagrams commute:*

$$\begin{array}{ccc} K_1F & \xrightarrow{\partial_v} & K_0k \\ \ell\uparrow & & \| \\ F^\bullet & \xrightarrow[v]{} & \mathbb{Z} \end{array} \qquad\qquad \begin{array}{ccc} K_2F & \xrightarrow{\partial_v} & K_1k \\ \ell\times\ell\uparrow & & \uparrow\ell \\ F^\bullet\times F^\bullet & \xrightarrow[(\ ,\ )_v]{} & k^\bullet \end{array}$$

Here $(a,b)_v = \bar{c}$ *where* $c = (-1)^{v(a)v(b)}\dfrac{a^{v(b)}}{b^{v(a)}}$

(a) and (c) are immediate from Prop. (4.3) and the fact that $\partial_\pi(\ell(\mathcal{O}^\bullet)) = \ell(k^\bullet)$. Part (d) follows from (c), which characterizes ∂_v on generators $\ell(u_1)\ldots\ell(u_{n-1})\ell(a)$ of K_nF in terms of v alone.

It is clear from Prop. (4.3) that $\mathrm{Ker}(\partial_v) = \mathrm{Ker}(\partial_\pi) + \mathbb{Z}[\ell(\mathcal{O}^\bullet)]$. To prove (b) therefore it suffices to show that $\mathrm{Ker}(\partial_\pi) = \ell(1+\pi)K_*F$ is contained in $\mathbb{Z}[\ell(\mathcal{O}^\bullet)]$. The elements $1 - u\pi\,(u\in\mathcal{O}^\bullet)$ generate $1 + \pi\mathcal{O}$. We have $0 = \ell(1-u\pi)\ell(u\pi) = \ell(1-u\pi)\ell(\pi)+(1-u\pi)\ell(u)$, whence the assertion.

To prove (e) let $a = a_0\pi^\alpha$ and $b = b_0\pi^\beta \in F^\bullet$ with $c_0, b_0 \in \mathcal{O}^\bullet$. Then $\partial_\pi(\ell(a)) = \ell(\bar{a}_0) + \alpha\Pi$ so $\partial_v(\ell(a)) = \alpha = v(a)$. Further

$$\partial_\pi(\ell(a)\ell(b)) = (\ell(\bar{a}_0) + \alpha\Pi)(\ell(\bar{b}_0) + \beta\Pi)$$

$$= \ell(\bar{a}_0)\ell(\bar{b}_0) + (\ell(\bar{a}_0)\beta - \alpha\ell(\bar{b}_0) + \alpha\beta\ell(-1))\Pi$$

(see (4.1)). If $c = (-1)^{\alpha\beta}\dfrac{a_0^\beta}{b_0^\alpha} = (-1)^{v(a)v(b)}\dfrac{a^{v(b)}}{b^{v(a)}}$ then $\ell(\bar{c}) = \ell(c_0)\beta - \alpha\ell(\bar{b}_0) + \alpha\beta\ell(-1)$, so (e) is established, thus completing the proof of Prop. (4.5).

(4.6) Remarks. There are $\varkappa$-Algebra homomorphisms $\lambda, \rho: K_*F \to K_*k$ defined by $\lambda(\ell(u\pi^i)) = \ell(\bar{u})$ and $\rho(\ell(u\pi^i)) = \ell(\bar{u}) + i\varepsilon$ for $u \in \mathcal{O}^\cdot$. Indeed $\lambda = \partial_\pi^0$ and $\rho = \partial_{-\pi}^0$ (Prop. (4.4)). It follows from the last part of (4.1) that

$$\partial_v(xy) = \lambda(x)\partial_v(y) + (-1)^{\deg(y)}\partial_v(x)\rho(y)$$

for x,y homogeneous elements of K_*F.

If there is a splitting $s: k \to \mathcal{O}$ of $a \mapsto \bar{a}$ it induces a splitting $(K_*k)\langle\Pi\rangle \to K_*F$ by $\ell(\bar{a}) \mapsto \ell(s(\bar{a}))$ and $\Pi \mapsto \ell(\pi)$.

Suppose F is complete with respect to the topology defined by v. Then the exact sequence

$$1 \longrightarrow (1 + \pi\mathcal{O}) \longrightarrow \mathcal{O}^\cdot \longrightarrow k^\cdot \longrightarrow 1$$

splits. If char(k) = p > 0 moreover then $1 + \pi\mathcal{O}$ is uniquely divisible by any integer m prime to p. It follows that $\mathrm{Ker}(\partial_\pi) = \ell(1 + \pi\mathcal{O})K_*F$ is also divisible by m, whence:

(4.7) COROLLARY. *Suppose* F *is complete and* $\mathrm{char}(k) = p > 0$. *Then if* m *is prime to* p *the homomorphism*

$$K_*F/mK_*F \longrightarrow (K_*k/mK_*k)\langle \Pi \rangle$$

induced by ∂_π *is an isomorphism.*

Let w be a discrete valuation on an extension field E of F. Assume that $\mathcal{O}_v \subset \mathcal{O}_w$, whence a homomorphism $\mathcal{O}_v \to k(w)$. Either (i) this is injective, or (ii) it induces a homomorphism $j_{w/v}: k(v) \to k(w)$. Let π_v be a local parameter of v and put $e = e(w/v) = w(\pi_v)$. Thus $w(a) = v(a)^e$ for $a \in F^\cdot$. Then $e = 0$ in case (i) and $e > 0$ in case (ii).

In case (i) we have $F^\cdot \subset \mathcal{O}_w^\cdot$ so the composite $K_*F \to K_*E \xrightarrow{\partial_w} K_*k(w)$ is zero.

(4.8) PROPOSITION. *Suppose* $e = e(w/v) > 0$. *Then the diagram*

$$\begin{array}{ccc} K_*F & \longrightarrow & K_*E \\ \Big\downarrow \partial_v & & \Big\downarrow \partial_w \\ K_*k(v) & \xrightarrow[e \cdot j_{w/v}]{} & K_*k(w) \end{array}$$

is commutative.

Let $u_1, \ldots, u_{n-1} \in \mathcal{O}_v^\cdot$ and $a \in F^\cdot$. Then by Prop. (4.5) part (c) we have $\partial_w(\ell(u_1)\ldots\ell(u_{n-1})\ell(a)) = \ell(\bar{u}_1)\ldots\ell(\bar{u}_{n-1})w(a)$ $= e\ \ell(\bar{u}_1)\ldots\ell(\bar{u}_{n-1})v(a) = e \cdot j_{w/v}\partial_v(\ell(u_1)\ldots\ell(u_{n-1})\ell(a))$.

Since the elements $\ell(u_1)\ldots\ell(u_{n-1})\ell(a)$ as above generate K_nF this proves Prop. (4.8).

§5. Rational function fields; the transfer $N_v: K_*k(v) \to K_*k$.

Let $F = k(t)$, the field of rational functions in a variable t over a field k. Then

$$v_\infty(f) = -\deg(f)$$

is a discrete valuation of F, trivial on k, for which $1/t$ is a local parameter. For each remaining discrete valuation v on F, trivial on k, there is a unique monic irreducible polynomial $\pi_v \in k[t]$ which is a local parameter for v, and each monic irreducible polynomial so occurs. We have $k(v) = k[t]/(\pi_v)$, and we put $\deg(v) = [k(v):k] = \deg(\pi_v)$. For $f \in F^\cdot$ we have, by unique factorization,

(1) $$f = (\prod_{v \neq v_\infty} \pi_v^{v(f)}) \cdot \mathrm{lead}(f),$$

where lead (f) is the leading coefficient of f if $f \in k[t]$, and lead (f/g) = lead (f)/lead (g) in general.

(5.1) THEOREM (thm. (2.3) of [8]). The homomorphisms ∂_v yield a split exact sequence

$$0 \longrightarrow K_*k \longrightarrow K_*F \xrightarrow{\partial=(\partial_v)} \coprod_{v \neq v_\infty} K_*k(v) \longrightarrow 0.$$

The proof shows, more precisely, the following: Let U_d denote the subgroup of $F^\cdot$ generated by all non zero polynomials of degree $\leq$ d and put $L_d = \mathbb{Z}[\ell(U_d)]$, the subring of K_*F generated by $\ell(U_d)$. Then $\partial L_d \subset \coprod_{\substack{v \neq v_\infty \\ \deg(v) \leq d}} K_*k(v)$ and ∂ induces, for each $d > 0$, an isomorphism from L_d/L_{d-1} to $\coprod_{\substack{v \neq v_\infty \\ \deg(v) = d}} K_*k(v)$.

The proof uses the following useful fact (cf. Springer [12]):

(5.2) LEMMA. L_d *is generated as a left* (K_*k)-*module by the elements* $\ell(\pi_1)\ldots\ell(\pi_r)$ *where the* π_i *are monic irreducible polynomials and* $0 < \deg(\pi_1) < \ldots < \deg(\pi_r)$; *in particular* $r \leq d$.

It suffices to show that if π and π' are monic irreducible polynomials of degree d then

(2) $$L_{d-1}\ell(\pi)\ell(\pi') \subset (L_{d-1}\ell(\pi) + L_{d-1}\ell(\pi')).$$

For then $L_{d-1} + \sum_{\pi} L_{d-1}\ell(\pi)$, where π ranges over monic irreducible polynomials of degree d, is a subring of K_*F containing L_{d-1} and all such $\ell(\pi)$, whence it equals L_d; the lemma then follows by induction on d. To prove (2) write $\pi = \pi' + f$ with $\deg(f) < d$. If $f = 0$ then $\ell(\pi)\ell(\pi') = \ell(-1)\ell(\pi)$. If $f \neq 0$ then from $1 = \frac{\pi'}{\pi} + \frac{f}{\pi}$ we have $(\ell(f) - \ell(\pi))(\ell(\pi') - \ell(\pi)) = 0$, whence $\ell(\pi)\ell(\pi') = \ell(f)\ell(\pi') - \ell(f)\ell(\pi) + \ell(-1)\ell(\pi)$ $\in L_{d-1}\ell(\pi') + L_{d-1}\ell(\pi)$.

Let $x = \ell(\pi_1)\ldots\ell(\pi_r)$ be as in Lemma (5.2). Suppose deg(v) = d. Then it is clear that $\partial_v(x) = 0$ unless $\pi_r = \pi_v$, in which case $\ell(x) = \ell(\bar{\pi}_1)\ldots\ell(\bar{\pi}_{r-1})$. Since $\partial_v L_d = K_*k(v)$ we therefore obtain the:

(5.3) COROLLARY. _Suppose_ deg(v) = d. _Let_ α _denote the image of_ t _in_ $k(v) = k[t]/(\pi_v)$. _Then_ $K_*k(\alpha)$ _is generated as a left_ (K_*k)-_module by the elements_ $\ell(\pi_1(\alpha))\ell(\pi_2(\alpha))\ldots\ell(\pi_r(\alpha))$ _with each_ π_i _a monic irreducible polynomial and_ $0 < \deg(\pi_1) < \ldots < \deg(\pi_r) < d$. _In particular_ $\coprod_{i<d} K_i k(\alpha)$ _generates_ $K_*k(\alpha)$ _as a left_ (K_*k)-_module_.

This is of particular interest when d = 2, in which case 1 and $K_1k(\alpha)$ generate the (K_*k)-module $K_*k(\alpha)$. For example each element of $K_rk(\alpha)$ is then a sum of elements $\ell(a_1)\ldots\ell(a_{n-1})\ell(b)$ with $a_1,\ldots,a_{n-1} \in k^{\cdot}$ and $b \in k(\alpha)^{\cdot}$.

(5.4) _The transfer_ $N_v : K_*k(v) \to K_*k$. The inclusions $k \to k(t)$ and $k \to k(v)$ induce homomorphisms $j : K_*k \to K_*k(t)$ and $j_v : K_*k \to K_*k(v)$ of $\varkappa$-Algebras. These permit us to view $K_*k(t)$ and $K_*k(v)$ as (left or right) (K_*k)-modules.

If $c \in k^{\cdot}$ then $v(c) = 0$ for all valuations v in Thm. (5.1). It follows that $\partial_v : K_*k(t) \to K_*k(v)$ is a homomorphism of degree - 1 of graded (K_*k)-modules, and ∂_v vanishes in jK_*k. These remarks apply also to v_∞. Since

$$K_*k(t)/jK_*k \xrightarrow{\partial = (\partial_v)} \coprod_{v \neq v_\infty} K_*k(v)$$

is an isomorphism of (K_*k)-modules it follows that there is a unique homomorphism N of degree 0 of graded (K_*k)-modules making the following diagram commutative:

$$(3) \qquad \begin{array}{ccc} K_*k(t) & \xrightarrow{\partial = (\partial_v)} & \coprod_{v \neq v_\infty} K_*k(v) \\ \Big\downarrow \partial_{v_\infty} & & \Big\downarrow N \\ K_*k(v_\infty) & \xleftarrow[-j_\infty]{\cong} & K_*k \end{array}$$

We shall view j_∞ as an identification and put $N_{v_\infty} = \mathrm{Id}$: $K_*k(v_\infty) \to K_*k$. For $v \neq v_\infty$ let N_v denote the v-component of N. Then the commutativity of (3) translates as follows:

$$(4) \qquad \sum_v N_v(\partial_v(x)) = 0 \ \text{for all} \ x \in K_*k(t).$$

Moreover *the homomorphisms* $N_v : K_nk(v) \to K_nk$ *are uniquely characterized by* (4) *and the fact that* $N_{v_\infty} = \mathrm{Id}$. The fact that the N_v are (K_*k)-linear translates into

$$(5) \qquad N_v(j_v(x)y) = xN_v(y) \ \text{for} \ x \in K_*k, \ y \in K_*k(v)$$

Taking $y = 1 \in K_0k(v)$ this yields:

(6) $N_v \circ j_v: K_*k \to K_*k$ *is multiplication by* $N_v(1) \in K_0k = \mathbb{Z}$.

Finally Theorem (5.1) and diagram (3) furnish an exact sequence

$$(7) \qquad 0 \longrightarrow K_*k \xrightarrow{j} K_*F \xrightarrow{(\partial_v)} \coprod_{\text{all } v} K_*k(v) \xrightarrow{(N_v)} K_*k \longrightarrow 0$$

(5.5) PROPOSITION. $N_v: K_0k(v) = \mathbb{Z} \to K_0k = \mathbb{Z}$ *is multiplication by* $\deg(v) = [k(v):k] = N_v(1)$. *Hence* $N_v \circ j_v: K_nk \to K_nk$ *is multiplication by* $[k(v):k]$ *for all* $n \geq 0$.

The last assertion follows from the first in view of (6) above. To prove the first assertion we recall from Prop. (4.5) part (e) that $\partial_v(\ell(f)) = v(f)$ for $f \in k(t)^{\bullet}$. In view of the uniqueness of the N_v's the first assertion is thus equivalent to:

$$(7) \qquad \sum_v \deg(v)\ v(f) = 0 \quad \text{for all } f \in k(t)^{\bullet}$$

Since $v_\infty(f) = -\deg(f)$ and, by (1), $\sum_{v \neq v_\infty} \deg(v)\ v(f) = \deg(f)$, (7) is indeed valid.

COROLLARY. *Let* $j:k \to L$ *be a finite field extension of degree* d *of* k. *Then* $\mathrm{Ker}(j: K_*k \to K_*L)$ *is annihilated by* d. *Moreover* j *induces an injection* $K_*k/mK_*k \to K_*L/mK_*L$ *for all* m *prime to* d. *If* L *is only an algebraic extension of* k *then* $\mathrm{Ker}(j)$ *is a torsion group*.

The last assertion follows fromthe first one since K_*L

is the direct limit of K_*L' where L' varies over finite sub-extensions of k in L.

If $L = k(v)$ as in Prop (5.5) the first assertions follow from the last part of Prop (5.5). Any simple extension $L = k(\alpha)$ is isomorphic to some $k(v)$, whence the corollary in this case. In general we write L/k as a finite tower of simple extensions and note that the conclusions follow formally for a tower if they hold in each layer.

(5.6) THEOREM. <u>The following diagram commutes:</u>

$$\begin{array}{ccc} K_1k(v) & \xrightarrow{N_v} & K_1k \\ \uparrow{\scriptstyle \ell} & & \uparrow{\scriptstyle \ell} \\ k(v)^{\cdot} & \xrightarrow[N_{k(v)/k}]{} & k^{\cdot} \end{array}$$

In view of Prop. (4.5) part (e) and the uniqueness property of the N_v's Thm. (5.6) is equivalent to:

(5.6)' THEOREM (Weil, Cf. [11], Ch. III, n°4). <u>If</u> $f,g \in k(t)^{\cdot}$ <u>then</u>

(8) $$\prod_v N_{k(v)/k}(f,g)_v = 1.$$

The left side of (8) is bimultiplicative in (f,g), and

$(f,f)_v = (-1,f)_v$ for all v. Hence it suffices to verify (8) when f and g are relatively prime polynomials in $k[t]$. In this case we have, since $(f,g)_v = 1$ whenever $v(f) = v(g) = 0$,

$$(9)\quad \prod_v N_{k(v)/k}(f,g)_v = (f,g)_{v_\infty}\Big(\prod_{v(g)>0} N_{k(v)/k}(f,g)_v\Big)\Big(\prod_{v(f)>0} N_{k(v)/k}(f,g)_v\Big)$$

$$= (f,g)_{v_\infty}\left(\frac{f}{g}\right)\left(\frac{g}{f}\right)^{-1},$$

where

$$\left(\frac{f}{g}\right) = \prod_{v(g)>0} N_{k(v)/k}(f,g)_v = \prod_{g(\alpha_v)=0} N_{k(v)/k}(f(\alpha_v)^{v(g)})$$

Let $\bar{k}$ be an algebraic closure of k. In $\bar{k}[t]$ we can write $f = a(t-\alpha_1)\ldots(t-\alpha_n)$ and $g = b(t-\beta_1)\ldots(t-\beta_m)$. We claim:

$$(10)\qquad \left(\frac{f}{g}\right) = \prod_{j=1}^{m} f(\beta_j) = a^m \prod_{j=1}^{m}\prod_{i=1}^{n} (\alpha_i - \beta_j).$$

The second equality is clear. To prove the first we may assume g is constant, in which case both terms equal 1, or $g = \pi_v$ for some v. In the latter case we have $\left(\frac{f}{\pi_v}\right) = N_{k(\alpha_v)/k}(f(\alpha_v))$, where α_v is the image of t in $k(v) = k[t]/(\pi_v)$. The images of α_v under the different embeddings of $k(v)$ in $\bar{k}$ are $\beta_1,\ldots,\beta_m$, whence $N_{k(\alpha_v)/k}(f(\alpha_v)) = \prod_{j=1}^{m} f(\beta_j)$, as claimed.

It follows from (10) that $\left(\frac{f}{g}\right)\left(\frac{g}{f}\right)^{-1} = (-1)^{nm}\frac{a^m}{b^n}$. Since

$v_\infty(f) = -n$ and $v_\infty(g) = -m$ we have $(f,g)_\infty = (-1)^{nm} \frac{a^{-m}}{b^{-n}}$. In view of (9) this establishes (8), whence Thm. (5.6)'.

(5.7) <u>An inductive formula for</u> N_v. Say $[k(v):k] = d$. Then by Cor. (5.3) $K_*k(v)$ is generated as a (K_*k)-module by elements $x = \ell(\pi_1(\alpha_v))\ldots\ell(\pi_{r-1}(\alpha_v))$ where α_v is the image of t in $k(v) = k[t]/(\pi_v)$ and where the π_i are monic irreducible polynomials, say $\pi_i = \pi_{v_i}$, with $0 < \deg(\pi_1) < \ldots < \deg(\pi_{r-1}) < 0$. Put $\pi_r = \pi_v$ and $y = \ell(\pi_1(t))\ldots\ell(\pi_r(t))$; then $\partial_v(y) = x$. Hence $N_v(x)$ is a term in the equation

$$\sum_w N_w(\partial_w(y)) = 0.$$

We have $\partial_w(y) = 0$ unless $w =$ some v_i or v_∞, and $\partial_{v_i}(y) = (-1)^{r-i} x_i$ where

(11) $$x_i = \ell(\pi_1(\alpha_i))\ldots\ell(\pi_{i-1}(\alpha_i))\ell(\pi_{i+1}(\alpha_i))\ldots\ell(\pi_r(\alpha_i))$$

and $\alpha_i = \alpha_{v_i}$. Since the π_i are all monic one has $\partial_\infty(y) = (-1)^r \deg(\pi_1)\ldots\deg(\pi_r)\ell(-1)^{r-1}$. It follows that

(12) $$N_v(x) = (-1)^{r-1}\deg(\pi_1)\ldots\deg(\pi_r)\ell(-1)^{r-1} - \sum_{i=1}^{r-1} (-1)^{r-1} N_{v_i}(x_i).$$

Since $\deg(v_i) < d$ for $i = 1,\ldots,r-1$ we can, in some sense, regard N_{v_i} as known by induction on d. Note that $N_v = \mathrm{Id}$ if $d = 1$. If $d = 2$ then (12) determines N_v since each $x_i \in K_1 k(v_i)$ and $N_{v_i} = N_{k(v_i)/k}$ on $K_1 k(v_i)$ (Thm. (5.6)).

(5.8) <u>Changing the constant field</u>. Let L be an algebraic field extension of k, and put $E = L(t)$. The valuations w of E which are trivial on L each "lie over" some such valuation v of $F = k(t)$, a condition we shall denote by writing w/v. The valuation w_∞ with local parameter t^{-1} lies over v_∞. If $v \neq v_\infty$ then

$$\pi_v = \prod_{w/v} \pi_w^{e(w/v)} \tag{13}$$

is the factorization of $\pi_v \in k[t]$ in $L[t]$. This yields the embeddings $j_{w/v}\colon k(v) = k[t]/(\pi_v) \to k(w) = L[t]/(\pi_w)$.

PROPOSITION. <u>The following is a commutative exact diagram</u>

$$\begin{array}{ccccccccc}
0 \longrightarrow & K_*L & \longrightarrow & K_*L(t) & \xrightarrow{(\partial_w)} & \coprod_v \coprod_{w/v} K_*k(w) & \xrightarrow{(N_w)} & K_*L & \longrightarrow 0 \\
& \uparrow j_{L/k} & & \uparrow j_{L(t)/k(t)} & & \uparrow \coprod_v (e(w/v)\cdot j_{w/v}) & & \uparrow j_{L/k} & \\
0 \longrightarrow & K_*k & \longrightarrow & K_*k(t) & \xrightarrow[(\partial_v)]{} & \coprod_v K_*k(v) & \xrightarrow[(N_v)]{} & K_*k & \longrightarrow 0
\end{array}$$

The commutativitiy of the left hand square is just the functionality of K_*. That of the middle square follows from the commutativity of the diagrams

$$\begin{array}{ccc}
K_*L(t) & \xrightarrow{\partial_w} & K_*k(w) \\
\uparrow & & \uparrow {\scriptstyle e(w/v)\,j_{w/v}} \\
K_*k(t) & \xrightarrow[\partial_v]{} & K_*k(v)
\end{array}$$

for each w/v (Prop. (4.8)). The rows are the exact sequences of (7) above for L and k, respectively. It follows therefore that there is a unique homomorphism $h: K_*k \to K_*L$ which, in place of $j_{L/k}$, will make the right hand square commute. In particular, since w_∞ is the only w lying over v_∞ and $e(w_\infty/v_\infty) = 1$ the diagram

$$\begin{array}{ccccc}
K_*k(w_\infty) & = & K_*L & \xrightarrow{N_{w_\infty}} & K_*L \\
{\scriptstyle j_{w_\infty/v_\infty}} \uparrow & & \uparrow {\scriptstyle j_{L/k}} & & \uparrow {\scriptstyle h} \\
K_*k(v_\infty) & = & K_*k & \xrightarrow[N_{v_\infty}]{} & K_*k
\end{array}$$

commutes. But N_{v_∞} and N_{w_∞} are the identity maps, whence $h = j_{L/k}$. This proves the proposition.

(5.9) A problem. One would like to be able to define a "transfer map"

$$N = N_{L/k}: K_*L \longrightarrow K_*k$$

for any finite field extension L/k. Beyond being a homomorphism of degree zero of graded groups it should satisfy the following conditions.

Tr 1). <u>The projection formula</u>:

$$N(jx \cdot y) = x \cdot N(y)$$

for $x \in K_*k$, $y \in K_*L$.

Here $j = j_{L/k}: K_*k \to K_*L$ is induced by $k \to L$, and Tr 1) can be read as saying that N is a homomorphism of (K_*k)-modules. Taking $y = 1$ it implies that

(14) $j \circ N: K_*k \longrightarrow K_*k$ is multiplication by $N(1) \in K_0k = \mathbb{Z}$.

Tr 2). <u>Functoriality</u>: $N_{k/k} = \mathrm{Id}$ and $N_{L/k} \circ N_{E/L} = N_{E/k}$ if L/k and E/L are finite field extensions.

In view of (5.4) we might further require:

Tr 3). <u>Reciprocity</u>:

$$\sum_v N_{k(v)/k}\,(\partial_v(x)) = 0$$

for all $x \in K_*k(t)$.

It would then follow from the uniqueness property of the N_v's that $N_{k(v)/k} = N_v$ for all v. Conversely this suggests a method for defining the maps $N_{L/k}$ in general.

First suppose $L = k(\alpha)$, a simple extension, and put $\pi = Irr(t,\alpha/k)$, the irreducible monic polynomial in $k[t]$ of which α is a root. Then $\pi = \pi_v$ for some v, whence a k-isomorphism $k(\alpha) \to k(v)$, and a map $N_{\alpha/k}: K_*k(\alpha) \to K_*k$ obtained from $N_v: K_*k(v) \to K_*k$.

If $L = k(\alpha_1,\ldots,\alpha_n)$ we can put $k_i = k(\alpha_1,\ldots,\alpha_i)$ and $N_{(\alpha_1,\ldots,\alpha_n)/k} = N_{\alpha_1/k} \circ N_{\alpha_2/k_1} \circ \cdots \circ N_{\alpha_n/k_{n-1}}$.

Note ((5.4), formula (5)) that each $N_{\alpha/k}$ satisfies Tr 1) so it follows that each $N_{(\alpha_1,\ldots,\alpha_n)/k}$ does likewise. Furthermore $N_{(\alpha_1,\ldots,\alpha_n)/k} = Id$ if $L = k$.

The <u>problem</u> in general of course is to show that $N = N_{(\alpha_1,\ldots,\alpha_n)/k}$ depends only on L/k and not on the choice of generating sequence $(\alpha_1,\ldots,\alpha_n)$. This is true on K_0, where, by Prop. (5.5), N is multiplication by $[L:k]$, and on K_1, where by Theorem (5.6), N is the field norm $N_{L/k}$. For K_i $(i \geq 2)$, however, the invariance of $N_{(\alpha_1,\ldots,\alpha_n)/k}$ is not at all clear already for $n = 1$. If this problem has an affirmative response then functoriality (Tr 2)) follows immediately.

The $N_{\alpha/k}$'s have one naturality property which we may deduce from Prop. (5.8): Suppose $k(\alpha)$ is a simple algebraic extension of k and L/k is any algebraic extension. Then $L \otimes_k k(\alpha)$ modulo its radical is a product $\prod_i L(\alpha_i)$ of simple extensions $L(\alpha_i)$ of L, where α_i denotes the projection of α into the factor $L(\alpha_i)$. We have $k(\alpha) = k[t]/(\pi)$, where $\pi = \mathrm{Irr}(t,\alpha/k) = \prod_i \pi_i^{e_i}$ in $L[t]$, and $L(\alpha_i) = L[t]/(\pi_i)$. Then the diagram

$$\begin{array}{ccc} \coprod_i K_*L(\alpha_i) & \xrightarrow{(N_{\alpha_i/L})} & K_*L \\ \Big\uparrow{\scriptstyle (e_i \cdot j_i)} & & \Big\uparrow{\scriptstyle j} \\ K_*k(\alpha) & \xrightarrow[N_{\alpha/k}]{} & K_*k \end{array} \tag{15}$$

commutes, where $j = j_{L/k}$ and $j_i = j_{L(\alpha_i)/k(\alpha)}$. This furnishes a method for showing that $N_{\alpha/k}$ is independent of α, by induction on $\deg_k(\alpha) = [k(\alpha):k]$. For suppose $k(\alpha) = k(\beta)$ and $(L \otimes_k k(\beta))/\text{radical} = \prod_i L(\beta_i)$ as above. Then we have a diagram analogous to (15) for β. If the degrees of the $L(\alpha_i) = L(\beta_i)$ over L are $< [k(\alpha):k]$ then we may assume inductively that $N_{\alpha_i/L} = N_{\beta_i/L}$ for all i. The commutativity of (15) and its analogue for β then implies that $N_{\alpha/k} - N_{\beta/k}$ maps $K_*k(\alpha)$ into $\mathrm{Ker}(j:K_*k \to K_*L)$. By the Corollary to Prop. (5.5) $\mathrm{Ker}(j)$ is a torsion group; in fact it is killed by $[L:k]$ when the latter is finite. Taking for L an algebraic closure of k we conclude: If $k(\alpha) = k(\beta)$ then $N_{\alpha/k}$ and $N_{\beta/k}$ agree modulo torsion.

It therefore suffices to show that, for each prime p, the p-primary part of $\mathrm{Im}(N_{\alpha/k} - N_{\beta/k})$ is zero. To check this we can take for L the fixed field in $\bar{k}$ of a Sylow p-subgroup of $\mathrm{Gal}(\bar{k}/k)$. Here we take $\bar{k}$ to be an algebraic closure of k if $p \neq \mathrm{char}(k)$ and a separable closure if $p = \mathrm{char}(k)$. Then L is a limit of finite extensions of k of degrees prime to p, so $j: K_*k \to K_*L$ is injective on p-torsion (Cor. to Prop. (5.5)), and all finite extensions of L have p-power degree. After replacing k by L therefore, and using (15), we reduce the problem to the following case:

Every finite extension of k is of degree a power of p. In particular every irreducible polynomial of degree $< p$ is linear. It follows therefore from Cor. (5.3) that if $[k(\alpha):k] = p$ then $K_0k(\alpha)$ and $K_1k(\alpha)$ generate $K_*k(\alpha)$ as a (K_*k)-module. By Prop. (5.5) and Theorem (5.6) $N_{\alpha/k}$ is characterized on K_0 and on K_1 independently of α. Hence we conclude from the projection formula in this case that $N_{\alpha/k} = N_{\beta/k}$ if $k(\alpha) = k(\beta)$.

It is not yet clear how to handle the case $[k(\alpha):k] = p^n$ with $n > 1$.

However the above arguments can be used to prove the following: If transfer maps $N_{E/F}$ (satisfying Tr 1) and Tr 2)) are defined so that $N_{E/F}(1) = [E:F]$ and $K_1E \to K_1F$ $N_{E/F}$ corresponds to the field theoretic norm $E^\cdot \to F^\cdot$, then the $N_{E/F}$'s are unique.

We conclude this section now with some simple applications of the transfer maps.

Let F be a field and k_0 its prime field. The Kronecker dimension $\delta(F)$ of F is $\text{tr. deg}_{k_0}(F)$ if $k_0 = \mathbb{F}_p$ $(p > 0)$ and $1 + \text{tr deg}_{k_0}(F)$ if $k_0 = \mathbb{Q}$. The following result was proved more directly by Springer in [12].

(5.10) PROPOSITION. <u>If</u> $1 \leq n \leq \delta(F)$ <u>then the rank of the abelian group</u> K_nF <u>is</u> $\text{Card}(F)$.

We argue by induction on $d = \delta(F)$. If $d = 0$ then F is algebraic over a finite field, so $K_1F = F^{\cdot}$ is torsion, whence K_nF is torsion for all $n \geq 1$. (In fact $K_nF = 0$ for $n \geq 2$, by Steinberg.)

If $d = 1$ then F is algebraic over $F_1 = \mathbb{Q}$ or $\mathbb{F}_p(t)$. Therefore F is countable, and $F^{\cdot}$ contains $F_1^{\cdot}$ which, modulo torsion, is free abelian of infinite rank. (There are infinitely many primes (Euclid).)

If $d \geq 2$ we can choose a subfield F_1 of F of Kronecker dimension $d - 1$ and a $t \in F$ transcendental over F_1, such that F is algebraic over $F_1(t)$. Then since F_1 is infinite, it is easily seen that $\text{Card } F_1 = \text{Card } F_1(t) = \text{Card } F$. By Thm. (5.1) we have an epimorphism $K_nF_1(t) \to \coprod_v K_{n-1}F_1(v)$, and, by induction, each $K_{n-1}F_1(v)$ has rank equal to $\text{Card } F_1(v) = \text{Card } F$. Thus $K_nF_1(t)$ has rank $\geq \text{Card } F$. According to the Corollary to Prop. (5.5) the kernel of $K_nF_1(t) \to K_nF$ is torsion, so rank $K_nF \geq$ Card F. Finally the reverse inequality follows since K_nF is a quotient of $F^{\cdot}\otimes\ldots\otimes F^{\cdot}$ (n factors).

Question. It is tempting to conjecture that K_nF is torsion for $n > \delta(F)$. This is trivially so for $d = 0$. For $d = 1$ it is also true, thanks to a theorem of Garland [5] in the number field case.

(5.11) PROPOSITION. Let m be an integer ≥ 1. Suppose that for all finite extensions E of a field F we have $F^{\cdot} = N_{E/F}(E^{\cdot})\cdot F^{\cdot m}$. Then K_nF is divisible by m for all $n \geq 2$.

Suppose $x,y \in K_*F$. Let $j: F \to E$ be a finite extension. Let $N: K_*E \to K_*F$ be some transfer map as in (5.9). Suppose we can find $x',y' \in K_*E$ such that

$$jx = mx' \quad , \quad y = Ny'$$

Then we have $x\cdot y = x\cdot Ny' = N(jx\cdot y') = N(mx'\cdot y') = mN(x'\cdot y')$, so

$$x\cdot y = mN(x'\cdot y') \in m\,K_*F \ .$$

We apply this now to $x = \ell(a)$, $y = \ell(b)$ with $a,b \in F^{\cdot}$. We wish to show that $\ell(a)\ell(b) \in mK_2F$. Choose $E = F(\alpha)$ with $\alpha^m = a$. By hypothesis we can, after modifying b by an m^{th}

power, which is harmless, solve $b = N_{E/F}(\beta)$. Then the calculation above shows that $\ell(a)\ell(b) = mN(\ell(\alpha)\ell(\beta))$. This shows that K_2F is divisible by m, so K_nF is divisible by m for $n \geq 2$.

(5.12) COROLLARY. *If the norm is surjective in all finite extensions of* F *then* K_nF *is a divisible group for all* $n \geq 2$.

This applies notably to finite fields, where it yields Steinberg's theorem:

$$K_n\mathbf{F}_q = 0 \quad \text{for all} \quad n \geq 2.$$

It also applies to C_1 (quasi-algebracially closed) fields, examples of which are furnished by theorems of Tsen and Lang.

(5.13) PROPOSITION. *Suppose* $\mathrm{char}(F) = p > 0$ *and* $[F:F^p] = p^d$. *Then for* $n > d$,

$$p^{d-1}K_nF \text{ is divisible by } p$$

and

$$p^dK_nF \text{ is uniquely divisible by } p.$$

This proposition applies notably to an algebraic function field in d variables over a perfect field.

Let $j: F \to F$, $j(a) = a^p$, and let $N: K_*F \to K_*F$ be a transfer map for j as in (5.9). Since $j: K_*F \to K_*F$ is multiplication by p on K_1F, it is multiplication by p^n on K_nF. On the other hand, $N \circ j$ is multiplication by $[F:jF] = p^d$ (see (14) in (5.9)). Thus on K_nF we have $p^d = N \circ j = N \circ p^n$. If $n > d$ this gives $p^d = f \circ p^d$ where $f = N \circ p^{n-d} = p^{n-d} \circ N$. It follows that multiplication by p is invertible on p^dK_nF if $n > d$.

To show that $p^{d-1}K_nF$ is divisible by p for $n > d$ consider an element $x = \ell(a_1)\ldots\ell(a_n) \in K_nF$. It suffices to show that $p^{d-1}x \in p^{n-1}K_nF$. Put $E = F^{1/p}$ and $b_i = a_i^{1/p} \in E$. Let $N:K_*E \to K_*F$ be a transfer map for $j: F \to E$. Then $N_{E/F}(b_n) = b_n^{p^d} = a_n^{p^{d-1}}$ so

$$\begin{aligned} p^{d-1}x &= \ell(a_1)\ldots\ell(a_{n-1})\ell(N_{E/F}b_n) \\ &= N(j(\ell(a_1)\ldots\ell(a_{n-1}))\ell(b_n)) \\ &= p^{n-1}\, N(\ell(b_1)\ldots\ell(b_{n-1})\ell(b_n)) \\ &\in p^{n-1}\, K_nF. \end{aligned}$$

This completes the proof of Prop. (5.13).

Chapter II

The Milnor ring of a global field

§1. A finiteness theorem.

Let F be a global field, i.e. a finite extension of $\mathbb{Q}$ (a number field) or a finitely generated extension of transcendence degree 1 over a finite field (a function field). Let S_∞ denote the set or archimedean places of F. Thus $S_\infty = \emptyset$ if F is a function field; if F is a number field then Card $S_\infty = r_1 + r_2$ where $\mathbb{R} \otimes_{\mathbb{Q}} F \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$. A finite place can be identified with a discrete valuation v of F. If S is a non empty set of places containing S_∞ we put

$$A_S = \{a \in F \mid v(a) \geq 0 \text{ for all } v \notin S\},$$

the ring of "S-integers." It is a Dedekind ring, with field of fractions F, whose maximal ideals P correspond to the places $v \notin S$ so that $k(v) = A_S/P$. We shall put

$$\begin{aligned} K_*^S F &= \mathbb{Z}[\ell(A_S^\bullet)] \\ &= \text{the subring of } K_*F \\ &\quad \text{generated by } \ell(A_S^\bullet) \end{aligned}$$

If $v \notin S$ the homomorphism $\partial_v : K_*F \to K_*k(v)$ of Ch. I, Prop. (4.5) vanishes on $K_*^S F$ since A_S is contained in the valuation ring of v. Thus we have a homomorphism

$$K_*F/K_*^S F \xrightarrow{\partial^S = (\partial_v)} \coprod_{v \notin S} K_* k(v)$$

The norm of a finite place v is defined to be $N(v) = \mathrm{Card}\ k(v)$. We can list the finite places of F,

$$v_1, v_2, \ldots, v_m, \ldots$$

so that $N(v_i) \leq N(v_{i+1})$ for all i. This done we put

$$S_m = S_\infty \cup \{v_1, \ldots, v_m\}$$

Our main objective is the following theorem

(1.1) THEOREM. <u>For all sufficiently large</u> m <u>the homomorphism</u>

$$K_*F/K_*^{S_m} F \xrightarrow{\partial^{S_m} = (\partial_v)} \coprod_{v \notin S_m} K_* k(v)$$

<u>is an isomorphism</u>.

This will be proved in §3-5. The reason for calling it a finiteness theorem is the next corollary, and its consequences drawn in §2.

(1.2) COROLLARY. <u>For all</u> $n \geq 0$ <u>the kernel</u> H_n <u>of</u>

$$K_n F \xrightarrow{\partial^{S_\infty} = (\partial_v)} \coprod_{v \notin S_\infty} K_{n-1} k(v)$$

<u>is a finitely generated abelian group</u>.

In fact $H_n \subset L_n = \mathrm{Ker}(K_nF \xrightarrow{\partial^{S_m}} \coprod_{v \notin S_m} K_{n-1}k(v))$,

and Thm. (1.1) says L_n is the n^{th} degree term of the ring $\mathbf{Z}[\ell(A^{\bullet}_{S_m})]$. Hence L_n is a quotient of the n-fold tensor product of $A^{\bullet}_{S_m}$ with itself. Since $A^{\bullet}_{S_m}$ is finitely generated (Dirichlet) it follows that L_n and hence also H_n are finitely generated.

§2. Applications of the finiteness theorem.

As in §1, F is a global field. Its completion at a place v is denoted F_v. The group of roots of unity in F is denoted $\mu(F)$.

We put

$$H_n = \mathrm{Ker}(K_nF \xrightarrow{(\partial_v)} \coprod_{v \notin S_\infty} K_{n-1}k(v))$$

for each $n \geq 0$. By Cor. (1.4) H_n is a finitely generated abelian group. Clearly $H_0 = K_0F = \mathbb{Z}$. If k is a finite field then $K_nk = 0$ for $n \geq 2$ (cf. Cor. (5.12) of Ch. I). It follows that $H_n = K_nF$ for $n \geq 3$.

(2.1) THEOREM.

1) (Dirichlet) H_1 _is a finitely generated group of rank_ $r_1 + r_2 - 1$ _and torsion subgroup isomorphic to_ $\mu(F)$.

2) H_2 _is a finitely generated group._ _If_ $\mathrm{char}(F) = p > 0$ _then_ H_2 _is finite and of order prime to_ p.

3) _If_ $n \geq 3$ _then_ $H_n = K_nF$ _and the natural homomorphism_

$$K_nF \longrightarrow \coprod_{v \text{ real}} K_nF_v/2K_nF_v$$

is an isomorphism. _In particular_

$$K_nF \cong (\mathbb{Z}/2\mathbb{Z})^{r_1}$$

<u>Remark</u> . It follows from results of Garland [5] and Dennis [4] that H_2 is finite also in the number field case.

<u>Proof of 1)</u>. The map $(\partial_v): K_1F \to \coprod_{v \notin S_\infty} K_0 k(v)$ is, by Prop. (4.5) (part (e)) of Ch. I, equivalent to the map $F^{\cdot} \xrightarrow{(v)} \coprod_{v \notin S_\infty} \mathbb{Z}$. The kernel is therefore $A^{\cdot}_{S_\infty}$ in the number field case, and the non zero constants, i.e. $\mu(F)$, in the function field case. The announced description of $A^{\cdot}_{S_\infty}$ follows from the Dirichlet Unit Theorem.

We next prove:

(1) <u>If</u> char $(F) = p > 0$ <u>and if</u> $n \geq 2$ <u>then</u> H_n <u>is finite and of order prime to</u> p.

We know that H_n is finitely generated (Cor. (1.4)) so it suffices to show that H_n is divisible by p. Consider the exact sequence

(2) $$0 \longrightarrow H_n \longrightarrow K_nF \longrightarrow \coprod_v K_{n-1}k(v)$$

Since $k(v)$ is a finite field of characteristic p and $n \geq 2$ the group $K_{n-1}k(v)$ is finite of order prime to p (for this is true of $K_1k(v) = k(v)^{\cdot}$). Hence the right hand term of (2) is uniquely divisible by p. Since $[F:F^p] = p$ it follows from Prop. (5.13) of Ch. I that K_nF is divisible by p. The exact sequence (2) thus implies H_n is divisible by p, whence (1).

Note that (1) also completes the proof of part 2) of Theorem (2.1).

Proof of 3). For any prime p and field E we shall put

$$K_{n/p}E = K_nE/pK_nE$$

We propose to prove, for $n \geq 3$:

a) If char (F) = p then $K_{n/p}F = 0$.

b) If $p \neq 2$ and $p \neq$ char(F) then $K_{n/p}F = 0$.

c) If char(F) $\neq 2$ then $K_nF \to \coprod_{v \text{ real}} K_{n/2}F_v$ is a split epimorphism inducing an isomorphism

$$K_{n/2}F \to \coprod_{v \text{ real}} K_{n/2}F_v.$$

Since, as we noted above, $H_n = K_nF$ is a finitely generated group, it is clear that a), b), and c) imply 3). Furthermore a) follows from (1) above, so it remains only to prove b) and c). The proof below is an elaboration of the argument reproduced in the appendix of [8], which computes $K_{n/2}F$.

Suppose $p \neq$ char(F). Let $E = F(\mu_p)$, the field obtained by adjoining to F the group μ_p of p^{th} roots of unity. Then $[E:F] = d \leq p - 1$ so d is prime to p. It follows therefore from the corollary to Prop. (5.5) of Ch. I that $K_{n/p}F \to K_{n/p}E$

is injective. Therefore to prove b) we may assume $\mu_p \subset F$. In case $p = 2$ this is automatic. Thus to prove both b) and c) we may assume

$$\mu_p \subset F$$

For each non complex place v of F let $[\ ,\ ]_v : F_v^{\cdot} \times F_v^{\cdot} \to \mu_p$ denote the p^{th} power norm residue symbol in F_v (see, e.g., [9], §15). Let $d_v : K_{2/p}F_v \to \mu_p$ denote the corresponding homomorphism; it is an isomorphism (Moore [10]).

The exactness of

$$(3) \qquad K_{2/p}F \longrightarrow \coprod_{\substack{v \text{ non} \\ \text{complex}}} K_{2/p}F_v \xrightarrow{(d_v)} \mu_p \longrightarrow 0$$

is classical, and can be deduced also from theorems of C. Moore [10] (see also Milnor [9], Thm. A.14 and Thm. 16.1). In fact it follows further from [14] that

$$(4) \qquad 0 \longrightarrow K_{2/p}F \longrightarrow \coprod_{\substack{v \text{ non} \\ \text{complex}}} K_{2/p}F_v$$

is exact. For Thm.2 of [14] permits one to replace $K_{2/p}E$ by $Br(E)_p \otimes \mu_p$ for each field E above. Here $Br(E)_p$ is the kernel of multiplication by p on the Brauer group $Br(E)$ of E. The exactness of (4) then results from the Hasse prinicple, i.e. the injectivity of $Br(F) \to \coprod_v Br(F_v)$.

With the aid of the exact sequences (3) and (4) we shall now compute $K_{3/p}F$. It suffices to describe all homomorphisms $\varphi\colon K_3F \to \mu_p$. Put $\varphi(a,b,c) = \varphi(\ell(a)\ell(b)\ell(c))$. For fixed c we obtain a 2-symbol $(a,b) \mapsto \varphi(a,b,c)$ with values in μ_p. The exact sequences above then permit us to write

$$\varphi(a,b,c) = \prod_v{}' [a,b]_v^{\varepsilon_v(c)}$$

where $0 \leq \varepsilon_v(c) < p$ and Π' signifies that v ranges over non complex places. Further the $\varepsilon_v(c)$'s are unique up to addition of the same constant (modulo p) to each of them, i.e. modulo the product formula $\prod_v{}' [a,b]_v = 1$. Since $\varphi(a,b,c) = \varphi(a,c,b)^{-1}$ we also have, for b and c fixed,

$$\varphi(a,b,c) = \prod_v{}' [a,c]_v^{-\varepsilon_v(b)}$$

whence

$$\prod_v{}' [a,d_v]_v = 1,$$

where $d_v = b^{\varepsilon_v(c)} c^{\varepsilon_v(b)}$. Thus the idele $\underline{\underline{d}} = (d_v)$ is orthogonal to all $a \in F^{\cdot}$ in the product formula. It follows therefore from Weil ([15], Ch. XIII, §5, Prop. 8) that $\underline{\underline{d}} = d\,\underline{\underline{e}}^p$ for some idele $\underline{\underline{e}}$ and some $d \in F^{\cdot}$.

Put $E = F(b^{1/p}, c^{1/p})$. Then, since $d \equiv b^{\varepsilon_v(c)} c^{\varepsilon_v(b)} \mod F_v^{\cdot p}$ for all v, we see that d is a p^{th} power everywhere locally, and hence globally, in E. Kummer theory then implies that $d \equiv b^r c^s \bmod F^{\cdot p}$ for some integers r,s. Then we have

$$(5) \qquad b^{r-\epsilon_v(c)} \; c^{s-\epsilon_v(b)} \in F_v^{\cdot p}$$

for all v.

<u>Claim</u>. $\epsilon_v(c) = \epsilon_w(c)$ <u>for all finite</u> v <u>and</u> w.

The fact that $\mu_p \subset F$ implies that $\mathrm{Card}(F_v^{\cdot}/F_v^{\cdot 2}) \geq p^2$ for all finite v. Hence, given c, we can choose b outside the cyclic group generated by c modulo $F_v^{\cdot p}$ and modulo $F_w^{\cdot p}$. Then the condition (5) above for v and w implies that $\epsilon_v(c) \equiv r \equiv \epsilon_w(c) \bmod p$, whence $\epsilon_v(c) = \epsilon_w(c)$, as claimed.

Now multiplying $\varphi(a,b,c)$ by $1 = (\prod_v{}'[a,b]_v)^{-r}$ we reduce to the case $\epsilon_v(c) = 0$ for all finite v. If all non complex places are finite this shows that $\varphi = 1$, so $K_{3/p}F = 0$, and hence $K_{n/p}F = 0$ for $n \geq 3$. This applies notably when F is a function field and when F is a number field and $p \geq 3$; for in the latter case, since $\mu_p \not\subset \mathbb{R}$, F must be totally imaginary. Note that these conclusions imply b). They further imply in general that, for $n \geq 3$, K_nF is a finite 2-primary group.

It remains to treat the case when F is a number field and p = 2. The arguments above then show that

(6) $$K_{3/2}F \longrightarrow \coprod_{v \text{ real}} K_{3/2}F_v \quad \text{is injective.}$$

Let $v_1,\ldots,v_{r_1}$ denote the real places of F and put $F_i = F_{v_i}$. Choose $e_1,\ldots,e_{r_1} \in F^{\cdot}$ so that e_i is negative in F_i and positive in F_j for $j \neq i$. Then $F^{\cdot}$ is generated by $e_1,\ldots,e_{r_1}$ together with the totally positive elements of $F^{\cdot}$. Hence K_nF is generated additively by elements $x = \ell(a_1)\ldots\ell(a_n)$ where each a_i is either totally positive or equals some e_j. It is then clear that x goes to zero in $K_{n/2}F_h$ unless all a_i equal e_h, i.e. unless $x = x_h = \ell(e_h)^n = \ell(-1)^{n-1}\ell(e_h)$. It follows therefore from (6) that for $n \geq 3$ the element x lies in $2K_nF$ unless $x = x_h$ for some h.

We have $K_{*/2}F_h = \mathbb{Z}_2[\varepsilon_h]$ where $\varepsilon_h = \ell_{F_h}(-1)$, and x_h maps to ε_h^n. Since $2x_h = 0$ we obtain a section $\coprod_i K_{n/2}F_i \to K_nF$, $\varepsilon_h^n \mapsto x_h$, of $K_nF \to \coprod_i K_{n/2}F_i$. It follows that $K_nF \cong (\coprod_i K_{n/2}F_i) \oplus 2K_nF$. Since K_nF is a finite 2-primary group for $n \geq 3$ we must further have $2K_nF = 0$. This proves c), and so completes the proof of 3), and of Thm. (2.1).

§3. Proof of the finiteness theorem: reduction to Lemma (3.5).

Recall that F is a global field with archimedean places S_∞ and finite places $v_1, v_2, v_3, \ldots$ with $N(v_i) \leq N(v_{i+1})$. We put $S_m = S_\infty \cup \{v_1, \ldots, v_m\}$ and $K_*^{S_m}(F) = \mathbb{Z}[\ell(A_{S_m}^\cdot)] \subset K_*F$. It is clear that Thm. (1.1) results from the following more precise statement.

(3.1) THEOREM. *For all sufficiently large* m

$$K_*^{S_{m+1}}(F)/K_*^{S_m}(F) \xrightarrow{\partial_{v_{m+1}}} K_*k(v_{m+1})$$

is an isomorphism.

To prove this we fix an m whose (large)size will be determined by the requirements of the arguments to follow. Put

$$S = S_m$$

$$v = v_{m+1} \notin S$$

$$S' = S_{m+1} = S \cup \{v\}$$

Note that, for any finite place w,

$$w \in S \Longrightarrow N(w) \leq N(v)$$

$$w \in S \Longleftarrow N(w) < N(v)$$

Put

$$A = A_S$$

$$U = A_S^\cdot$$

$$k = k(v) = A/P,$$

where P is the maximal ideal such that A_P is the valuation ring of v. The natural map $A_P \to k$ will be denoted $a \mapsto \bar{a}$.

(3.2) LEMMA. The following conditions on A and v imply that

$$K_*^{S'}F/K_*^S F \xrightarrow{\partial_v} K_*k$$

is an isomorphism:

a) The ideal P is principal; say $P = \pi A$.

b) The group $(1 + P)^{\cdot} = \mathrm{Ker}(U \to k^{\cdot})$ is generated by the elements $1 + a \in U$ such that $Aa = P$.

c) There is a subset E of U such that

c_1) The map $E \times E \times E \to k^{\cdot} \times k^{\cdot}$ sending (a,b,c) to $(\bar{b}/\bar{a}, \bar{c}/\bar{a})$ is surjective.

c_2) If $e_1, e_2, e_3 \in E$ and $\bar{e}_1 = \bar{e}_2 + \bar{e}_3$ then $e_1 = e_2 + e_3$.

Condition a) clearly implies that $A_S{}' = A[\frac{1}{\pi}]$ and that $U' = A_{S'}^{\cdot}$ is the direct product of U with the cyclic group generated by π. Since $v(U) = 0$ and $v(\pi) = 1$ it follows that v induces an isomorphism $U'/U \to \mathbb{Z}$. But (see Ch. I, Prop. (4.5), part e)) this last arrow is equivalent to $\partial_v: K_1^{S'}F/K_1^S F \to K_0 k$.

We now treat $\partial_v : K_n^{S'}F/K_n^S F \to K_{n-1}k$ for $n > 1$. Denote the image modulo $K_*^S F$ of $x \in K_*F$ by $[x]$. Then (Ch. I, Prop. (4.5), part c)) the following diagram commutes for each $n > 1$:

$$\begin{array}{ccc} & U \times \cdots \times U & \quad (n-1 \text{ factors}) \\ {}^{\alpha'}\swarrow & & \searrow^{\beta'} \\ K_n^{S'}F/K_n^S F & \xrightarrow{\partial_v} & K_{n-1}k \end{array}$$

where $\alpha'(u_1,\ldots,u_{n-1}) = [\ell(u_1)\ldots\ell(u_{n-1})\ell(\pi)]$ and $\beta'(u_1,\ldots,u_{n-1}) = \ell(\bar{u}_1)\ldots\ell(\bar{u}_{n-1})$. Both α' and β' are evidently multilinear, so they induce homomorphisms α and β making the diagram

$$\begin{array}{ccc} & U \otimes \cdots \otimes U & \\ {}^{\alpha}\swarrow & & \searrow^{\beta} \\ K_n^{S'}F/K_n^S F & \xrightarrow{\partial_v} & K_{n-1}k \end{array}$$

commutative. To prove that ∂_v is an isomorphism it therefore suffices to show that:

(i) α is surjective

and

(ii) β is surjective and $\operatorname{Ker}(\beta) \subset \operatorname{Ker}(\alpha)$.

<u>Proof of</u> (i). As noted above $U' = U \times \pi^{\mathbb{Z}}$ where $U' = A_{S'}^{\cdot}$. Since K_*F is anticommutative and since $\ell(\pi)^2 = \ell(-1)\ell(\pi)$ it follows that $K_*^{S'}F = \mathbb{Z}[\ell(U')]$ is generated as a left $(K_*^S F)$-module by 1 and $\ell(\pi)$. In particular $K_n^{S'}F$ is generated additively by

elements $x = \ell(u_1)\ldots\ell(u_n)$ and $y = \ell(u_1)\ldots\ell(u_{n-1})\ell(\pi)$ with $u_1,\ldots,u_n \in U$. Since $[x] = 0$ and $[y] \in \mathrm{Im}(\alpha)$ it follows that α is surjective, as claimed.

Proof of (ii). Conditions b) and c_1) imply the exactness of

$$1 \longrightarrow U_1 \longrightarrow U \longrightarrow k^{\cdot} \longrightarrow 1$$

where U_1 denotes the subgroup of U generated by all elements $1 - u\pi \in U$ with $u \in U$. It follows from this that β is surjective and that $\mathrm{Ker}(\beta)$ is generated by elements $x = u_1 \otimes \ldots \otimes u_{n-1}$ of the following types: (I) $u_i = 1 - u\pi$ with $u \in U$ for some $i \leq n-1$; (II) $\bar{u}_i + \bar{u}_{i+1} = \bar{1}$ for some $i \leq n - 2$. It remains to show that $\alpha(x) = 0$ in each of these two cases.

Type (I): $\alpha(x) = [\ell(u_1)\ldots\ell(u_{n-1})\ell(\pi)]$. Put $y = (-1)^{n-i+1}\ell(u_1)\ldots\ell(u_{i-1})\ell(u_{i+1})\ldots\ell(u_{n-1})$, so that $\alpha(x) = [y\ \ell(u_i)\ell(\pi)]$. We have $0 = \ell(1 - u\pi)\ell(u\pi) = \ell(1 - u\pi)\ell(u) + \ell(1 - u\pi)\ell(\pi) = \ell(u_i)\ell(u) + \ell(u_i)\ell(\pi)$. Hence $\alpha(x) = -[y\ \ell(u_i)\ell(u)] = 0$ because $y\ \ell(u_i)\ell(u) \in K_n^S F$.

Now that $\alpha(x) = 0$ for x of type (I) it follows that

(*) $$\alpha(u_1 \otimes \ldots \otimes u_{n-1}) = \alpha(u'_1 \otimes \ldots \otimes u'_{n-1})$$

whenever $\bar{u}_j = \overline{u'_j}$ $(1 \leq j \leq n - 1)$

Type (II): Assume $\bar{u}_i + \bar{u}_{i+1} = \bar{1}$. Condition c_1)

furnishes elements $e_1, e_2, e_3 \in E$ such that $\bar{u}_i = \bar{e}_2/\bar{e}_1$ and $\bar{u}_{i+1} = \bar{e}_3/\bar{e}_1$. In view of (*) above there is no loss in assuming $u_i = e_2/e_1$ and $u_{i+1} = e_3/e_1$. We have then

$\bar{e}_2 + \bar{e}_3 = \bar{e}_1$ so

condition c_2) implies that $e_2 + e_3 = e_1$, i.e. that $u_i + u_{i+1} = 1$. It follows that $\ell(u_i)\ell(u_{i+1}) = 0$, so $x = 0$ and $\alpha(x) = 0$.

This completes the proof of Lemma (3.2).

(3.3) Norms. Before going further we introduce some additional notation. Put

$$A_\infty = \begin{cases} A_{S_\infty} & \text{if } F \text{ is a number field} \\ A_{S_1} & (S_1 = \{v_1\}) \text{ if } F \text{ is a function field} \end{cases}$$

We define a multiplicative function $\mathbf{N}(\mathfrak{a}) \in \mathbb{Q}^\cdot$ for fractional A_∞-ideals $\mathfrak{a}$ of A_∞ so that, when $\mathfrak{a} \subset A_\infty$, $N(\mathfrak{a}) = \mathrm{Card}(A_\infty/\mathfrak{a})$. Thus if P_w is the prime ideal of A_∞ corresponding to a finite place w ($\neq v_1$ if F is a function field) then $N(P_w) = \mathrm{Card}\ k(w) = N(w)$. If $a \in F^\cdot$ we put $N(a) = N(A_\infty a)$. If F is a number field then $N(a) = |N_{F/\mathbb{Q}}(a)|$. We agree to put $N(0) = 0$.

(3.4) LEMMA. Suppose we are given subsets $D \subset A_\infty$ and $W \subset (A_\infty \cap U)$. Put

$E = \{d - d' \mid d,d' \in D,\ d \neq d'\}.$

Then A,v, and E satisfy conditions b) and c) of Lemma (3.2) provided that D and W satisfy the following conditions:

1) $(\text{Card } D)^3 > N(v)^2$.

2) $E \subset U$.

3) $1 \in W$ and W generates U.

4) If $e_1,e_2,e_3,e_4 \in E$ and $w \in W$ then

(i) $N(e_1 + e_2 + e_3) < N(v)$

(ii) $N(e_1e_2 - e_3e_4) < N(v)^2$

(iii) $N(e_1w - e_2) < N(v)^2$.

If $A \neq A_\infty$ these conditions further imply condition a) of Lemma (3.2)

The proof will be carried out in several steps.

4) (i) $\Rightarrow c_2$) Since $E = -E$ it follows from 4)(i) that for $e_1,e_2,e_3 \in E$ we have $N(e_1 - e_2 - e_3) < N(v)$. But if $\bar{e}_1 = \bar{e}_2 + \bar{e}_3$ we have $e_1 - e_2 - e_3 \in P_v$, so the inequality above is possible only if $e_1 - e_2 - e_3 = 0$.

1) and 2) $\Rightarrow c_1$) Given $x_2,x_3 \in k^{\cdot}$ we must solve $\bar{x}_i = \bar{e}_i/\bar{e}_1$ $(i = 2,3)$ for $e_1,e_2,e_3 \in E$. Define

$$L: A_\infty \times A_\infty \times A_\infty \longrightarrow k \times k$$

$$L(a,b,c) = (\bar{b} - \bar{a}x_2, \ \bar{c} - \bar{a}x_3).$$

Condition 1) implies that L can't be injective on $D \times D \times D$, i.e. $L(d) = L(d')$ for some $d = (d_1,d_2,d_3) \neq d' = (d_1', d_2', d_3')$ in $D \times D \times D$. Put $e = d - d' = (e_1,e_2,e_3) \neq (0,0,0)$. Since L is additive we have $L(e) = L(d) - L(d') = 0$, i.e. $\bar{e}_2 = \bar{e}_1 x_2$ and $\bar{e}_3 = \bar{e}_1 x_3$. Since $e \neq 0$ some $e_i \neq 0$, so, by 2), some $\bar{e}_i \neq 0$. Since $x_2,x_3 \neq 0$ it then follows that $\bar{e}_i \neq 0$ for all i, whence $e_1,e_2,e_3 \in E$. Clearly $x_j = \bar{e}_j/\bar{e}_1$ $(j = 2,3)$; this proves $c_1)$.

<u>Claim 1. Conditions</u> 3) <u>and</u> $c_1)$ <u>imply that</u> $(1 + P)^\cdot = \mathrm{Ker}(U \to k^\cdot)$ <u>is generated by its elements of the following types:</u>

$$\text{(I)} \quad \frac{e_1 e_2}{e_3 e_4} \qquad (e_1,e_2,e_3,e_4 \in E)$$

$$\text{(II)} \quad \frac{e_1 w}{e_2} \qquad (e_1,e_2 \in E, \ w \in W).$$

Let H be the subgroup of $(1 + P)^\cdot$ generated by its elements of types (I) and (II). If $x,y \in U$ write $x \sim y$ if $x \equiv y \bmod H$. We must show that

$$(*) \qquad \bar{x} = 1 \implies x \sim 1.$$

If x is of type I or II this follows from the definition of H. Condition c_1) implies each element of $k^\cdot$ is of the form $\bar{e}_1/\bar{e}_2$ with $e_1, e_2 \in E$. If $w \in W$ and $\bar{w} = \bar{e}_1/\bar{e}_2$ then $w \sim e_1/e_2$ since $\frac{e_2 w}{e_1} \in H$. Condition 3) asserts that $1 \in W$ and W generates U. It follows that **for** any $x \in U$ we have

$$x \sim \frac{e_1 \ldots e_n}{e'_1 \ldots e'_n}$$

for suitable $e_i, e'_i \in E$ $(1 \leq i \leq n)$. We claim we can even take $n = 1$. For if $n > 1$ then c_1) furnishes elements $a,b,c \in E$ such that $\bar{e}_1/\overline{e'_1} = \bar{b}/\bar{a}$ and $\overline{e'_2}/\overline{e_2} = \bar{c}/\bar{a}$. Hence

$\frac{e_1 e_2}{e'_1 e'_2} = (\frac{e_1 a}{e'_1 b})(\frac{ce_2}{ae'_2})(\frac{b}{c}) \sim \frac{b}{c}$ because the first two factors are elements of type I in H. Thus $x \sim \frac{be_3 \ldots e_n}{ce'_3 \ldots e'_n}$ and we finish by induction on n.

Now if $x = e_1/e'_1$ and $\bar{x} = \bar{1}$ then x is of type I in H (with $w = 1 \in W$) so $x \sim 1$, whence the claim.

Let U_1 denote the subgroup of U generated by all elements $1 + a \in U$ such that $Aa = P$. Note that $U_1 \subset (1 + P)^\cdot$ and $U_1 = \{1\}$ unless P is principal. If $P = A\pi$ then $U \cap (1 + U\pi)$ generates U_1.

<u>Claim 2</u>. <u>Suppose</u> $a,b \in A_\infty \cap U$ <u>satisfy</u> $\bar{a} = \bar{b}$ and $N(a-b) < N(v)^2$. <u>Then</u> $a/b \in U_1$.

We may assume $a \neq b$. Then $A_\infty(a-b) = \mathfrak{A} P_v$ for some ideal $\mathfrak{A}$ with $N(\mathfrak{A}) < N(v)$. It follows that for all prime divisors P_w of $\mathfrak{A}$ we have $N(w) < N(v)$, whence $w \in S$. Thus $\mathfrak{A} A = A$ and so $A(a-b) = P_v A = P$. Finally $\frac{a}{b} = 1 + \frac{a-b}{b} \in U_1$, as claimed.

<u>c_1), 3), 4) (ii), and 4) (iii) ⇒ b).</u> With the notation above condition b) says that $U_1 = (1 + P)^{\cdot}$. Using claim 1 above it suffices to show that U_1 contains the elements of types I and II in that claim. In view of claim 2 condition 4) (ii) implies this for type I and 4) (iii) does so for type II.

<u>If $A \neq A_\infty$ then b) ⇒ a).</u> For $A \neq A_\infty \Rightarrow U$ is infinite ⇒ $(1 + P)^{\cdot} \neq \{1\}$. In this case b) implies $U_1 \neq \{1\}$ so there is an $a \in A$ such that $Aa = P$; this is condition a).

The implications proved above together establish Lemma (3.4).

In view of Lemmas (3.2) and (3.4) we see that Theorem (3.1) follows from:

(3.5) LEMMA. *If m is sufficiently large then there exist sets D and W satisfying conditions 1), 2), 3), and 4) of Lemma (3.4).*

It will be convenient here to separate the arguments for number fields and for function fields.

§4. Proof of Lemma (3.5) for number fields.

(4.1) Absolute values. We keep the notation of §3 and assume further that F is a number field, say $[F:\mathbb{Q}] = n = r_1 + 2r_2$. If $w \in S_\infty$ then $| \ |_w$ denotes the usual absolute value on $F_v = \mathbb{R}$ or $\mathbb{C}$. If w is p-adic then $| \ |_w$ denotes the absolute value on F_v normalized so that $|p|_w = p^{-1}$. We put $n_w = [F_w: \mathbb{Q}_{w_0}]$ if w lies over the place w_0 in $\mathbb{Q}$.

For any $t > 0$ we put

$$L_t = \{a \in A_\infty | \ |a|_w \leq t \quad \text{for all } w \in S_\infty\}$$

Clearly $A_\infty = \bigcup_{t>0} L_t$. Further it is clear that $L_t = - L_t$ and

$$(1) \qquad \begin{aligned} L_s L_t &\subset L_{st} \\ L_s + L_t &\subset L_{s+t} \end{aligned}$$

for $s,t > 0$. If $a \in F$ then $N(a) = |N_{F/\mathbb{Q}}(a)| = \prod_{w \in S_\infty} |a|_w^{n_w}$. Since $\sum_{w \in S_\infty} n_w = n$ we have

$$(2) \qquad a \in L_t \implies N(a) \leq t^n.$$

(4.2) PROPOSITION. There exist constants $C,\gamma > 0$ depending only on F such that if $t > 0$ satisfies

(3) $$C \leq 3^n t^{5n/4} < N(v) < \gamma\ t^{3n/2}$$

then $D = L_{t/2}$ and $W = L_{t^{3/2}} \cap U$ satisfy conditions 1), 2), 3) and 4) of Lemma (3.4).

It is clear that this proposition implies Lemma (3.5). In fact making m large is equivalent to making $N(v)$ large, and, for sufficiently large values of t we have $3^n t^{5n/4} < \gamma t^{3n/2}$ so that a t satisfying (3) can be found provided that $N(v)$ is sufficiently large.

The rest of this § is devoted to the proof of Prop. (4.2).

(4.3) Parallelotopes; the constants C and γ. We recall some classical facts (see Lang, [6], Ch. V). If $\alpha = (\alpha_w)$ is an idèle of F we put

$$\|\alpha\| = \prod_w |\alpha_w|_w^{n_w},$$

$$\mathfrak{A}(\alpha) = \prod_{w \notin S_\infty} P_w^{w(\alpha_w)}$$

a fractional A_∞-ideal of norm $N(\mathfrak{A}(\alpha)) = \prod_{w \notin S_\infty} N(w)^{w(\alpha_w)}$ $= \prod_{w \notin S_\infty} |\alpha_w|_w^{-n_w}$. Thus

(4) $$\|\alpha\| = \|\alpha\|_\infty \cdot N(\mathfrak{A}(\alpha))^{-1}$$

where $\|\alpha\|_\infty = \prod_{w \in S_\infty} |\alpha_w|_w^{n_w}$. The parallelotope defined by α is

$$L(\alpha) = \{a \in F \mid |a|_w \leq |\alpha_w|_w \text{ for all } w\}.$$

For example if $s > 0$ then $L_s = L(\alpha)$ where $\alpha_w = s$ if $w \in S_\infty$ and $\alpha_w = 1$ otherwise. In this case $\|\alpha\| = s^n$. Put

$$B = \frac{2^{r_1} (2\pi)^{r_2}}{|d|^{1/2}}$$

where d is the discriminant of F. Then (Lang [6], Ch. V, §2, Thm. 1)

(5) $$\text{Card } L(\alpha) = B\|\alpha\| + O(\|\alpha\|^{1-1/n})$$

as $\|\alpha\| \to \infty$. Fix some constant C_1 so that $C_1 B > 1$. Then (5) implies that there is a constant $C_2 > 0$ such that

(6) Card $L(\alpha) > C_1^{-1}\|\alpha\|$ _whenever_ $\|\alpha\| > C_2$. _In particular_ Card $L_s > C_1^{-1} s^n$ _if_ $s^n > C_2$.

Put

(7)
$$C_3 = \max (C_1, C_2)$$
$$T = S_\infty \cup \{w \notin S_\infty \mid N(w) \leq C_3\}$$
$$U_T = A_T^{\cdot}$$

Since U_T is a finitely generated group there is an $s_0 > 0$ such that

(8) U_T <u>is contained in the group</u>
<u>generated by</u> $L_{s_0} - \{0\}$.

We can now introduce the constants C and γ to be used for Prop. (4.2):

(9) $$C = \max(3^n(2^n C_2)^{5/4},\ C_3,\ s_0^n)$$

(10) $$\gamma = (2^n C_3)^{-3/2}$$

(4.4) LEMMA. *Let $\mathfrak{a} \neq 0$ be an ideal in A_∞. Put $t = (N(\mathfrak{a})\cdot C_3)^{1/n}$. Then there is an $a \neq 0$ in $L_t \cap \mathfrak{a}$. Writing $A_\infty a = \mathfrak{a}\mathfrak{b}$, the ideal $\mathfrak{b}$ in A_∞ is in the ideal class of $\mathfrak{a}^{-1}$ and has norm $N(\mathfrak{b}) \leq C_3$.*

Choose an idèle α such that $\alpha_w = t$ for $w \in S_\infty$ and $\mathfrak{a}(\alpha) = \mathfrak{a}$. Then it is clear that $L_t \cap \mathfrak{a} = L(\alpha)$. Moreover we have from (4) that $\|\alpha\| = t^n N(\mathfrak{a})^{-1} = C_3 = \max(C_1, C_2)$. It follows therefore from (6) that Card $(L_t \cap \mathfrak{a}) > \|\alpha\| C_1^{-1}$ $= C_3 C_1^{-1} \geq 1$, whence the existence of $a \neq 0$ in $L_t \cap \mathfrak{a}$. We then have, by (2) $N(\mathfrak{a})\ C_3 = t^n \geq N(a) = N(\mathfrak{a})N(\mathfrak{b})$, whence the other assertions of the Lemma.

Since every ideal class of A_∞ has an integral representative of norm $\leq C_3$ it follows that A_T <u>is principal</u>, and hence $A = A_S$ is principal if $S \supset T$, for example if

(11) $$C_3 < N(v).$$

We record this conclusion

(12) <u>condition</u> (11) <u>implies</u> <u>that</u> A <u>is principal</u>.

(4.5) LEMMA. <u>Assume</u> (11) <u>and</u>

(13) $$s_0^n < N(v).$$

<u>Let</u> t <u>satisfy</u>

(14) $$N(v) \leq C_3^{-1}\, t^{3n/2}$$

<u>Then</u> $W = L_{t^{3/2}} \cap U$ <u>contains</u> 1 <u>and generates</u> U.

The non zero elements of L_{s_0} have norm $\leq s_0^n < N(v)$ and hence belong to U. Since $t^{3n/2} \geq C_3 N(v) \geq N(v) > s_0^n$ we have $t^{3/2} > s_0$ so the group V generated by W contains that generated by $L_{s_0} - \{0\}$ which, by construction of s_0, contains the group U_T. Recall from above that A_T is principal. Moreover condition (11) implies $A_T \subset A$ so that U is generated by U_T together with generators π_w of the principal ideals $P_w A_T$ $(w \in S-T)$. It remains therefore to find such generators π_w in W.

Let $w \in S - T$ and put $r = (N(w)C_3)^{1/n}$. Then Lemma (4.4) suppli es an element $\pi_w \neq 0$ in $L_r \cap P_w$. We claim $\pi_w A_T$, and hence $\pi_w \in U$. Once this is shown, the inequalities $r \leq (N(v)C_3)^{1/n} \leq t^{3/2}$

(see (13)) further imply that $\pi_w \in W$, so the proof of Lemma (4.5) will be complete.

Put $\pi_w A_\infty = P_w \mathfrak{A}$. Since $N(\pi_w) \leq r^n = N(w)C_3$, we have $N(\mathfrak{A}) \leq C_3$, whence $\mathfrak{A} A_T = A_T$, so $\pi_w A_T = P_w A_T$, as claimed.

Proof of Prop. (4.2). With $C = \mathrm{Max}(3^n(2^n C_2)^{5/4}, C_3, s_0^n)$ as in (9), and $\gamma = (2^n C_3)^{-3/2}$ as in (10), condition (3) of Prop. (4.2) implies the following inequalities:

(a) $C < N(v)$

(b) $3^n\, t^{5n/4} < N(v)$

(c) $N(v) < \gamma\, t^{3n/2}$

We shall prove Prop. (4.2) by deducing conditions 1), 2), 3) and 4) of Lemma (3.4) from (a), (b), and (c).

(a) and (c) $\Rightarrow$ 1). We must show that $(\mathrm{Card}\ D)^{3/2} > N(v)$ where $D = L_{t/2}$. Conditions (a) and (c) easily imply that $(t/2)^n > C_2$. It follows therefore from (6) that $\mathrm{Card}\ D > C_1^{-1}(t/2)^n$. The latter dominates $C_3^{-1}(t/2)^n = \gamma^{2/3} t^n$. Thus (a) and (c) imply $(\mathrm{Card}\ D)^{3/2} > \gamma t^{3n/2} > N(v)$, which proves 1).

(b) $\Rightarrow$ 2). We must show that $E = \{d - d' \mid d, d' \in D,\ d \neq d'\}$ is contained in U. It suffices to show that, for $e \in E$, $N(e) < N(v)$. In fact $E \subset L_{t/2} + L_{t/2} \subset L_t$ so $N(e) \leq t^n$ which, by (b), is $< N(v)$.

<u>(a) and (c) ⇒ 3)</u>. Condition 3) is just the conclusion of Lemma (4.5). The hypotheses of Lemma (4.5) are (11) and (13), which both result from (a), and (14), which is a consequence of (c).

<u>(b) ⇒ 4)</u>. Let $e_1, e_2, e_3, e_4 \in E \subset L_t$ and $w \in W \subset L_{t^{3/2}}$. Then $x = e_1 + e_2 + e_3 \in L_{3t}$, $y = e_1e_2 - e_3e_4 \in L_{2t}$ and $z = e_1w - e_2 \in L_{t^{5/2}+t}$. It follows that

$$N(x) \leq 3^n t^n \ , \quad N(y) \leq 2^n t^{2n} \ , \quad N(z) \leq (t^{5/2} + t)^n$$

Condition 4) follows therefore if we know that $3^n t^n < N(v)$, $2^n t^{2n} < N(v)^2$, and $(t^{5/2} + t)^n < N(v)^2$. The first two inequalities are immediate from (b). Since (for $t \geq 1$) we have $t^{5/2} + t \leq (2t)^{5/2}$ the third inequality results also from (b).

§5. Proof of Lemma (3.5) for function fields.

(5.1) Degrees. Let F be a function field with finite constant field $k = \mathbb{F}_q$, and genus g. For each place w of F we put

$$\deg(w) = [k(w):k]$$

so that

$$N(w) = q^{\deg(w)} = \text{Card } k(w).$$

Changing notation slightly from §3 we shall write v_∞ in place of v_1, so that

$$A_\infty = \{a \in F \mid w(a) \geq 0 \quad \text{for all } w \neq v_\infty\}.$$

The place v_∞ has smallest possible degree.

$$d_\infty = \deg(v_\infty).$$

The w's different from v_∞ correspond to the prime ideals P_w of A_∞. We define $\deg(\mathfrak{a})$ for a fractorial A_∞-ideal $\mathfrak{a}$ so that

$$N(\mathfrak{a}) = q^{\deg(\mathfrak{a})}$$

In particular this defines $\deg(aA_\infty)$ for $a \in F^\bullet$. If $t \in \mathbb{R}$ we put

(1) $\qquad L_t = \{a \in A_\infty \mid a = 0 \quad \text{or} \quad \deg(aA_\infty) \leq td_\infty\}.$

Note that $A_\infty = \bigcup_{t>0} L_t$.

The notation $v, S, A = A_S$, $U = A_S^\bullet$, $S' = S \cup \{v\}$, etc. retains the meaning given it in §3.

(5.2) PROPOSITION. *There is an integer* $s_0 \in \mathbb{Z}$ *depending only on* F *such that if*

(2) $$\deg(v) \geq s_0 d_\infty$$

and if $t \in \mathbb{Z}$ *satisfies*

(3) $$\frac{5}{4} td_\infty - \frac{1}{4}(g-1) + \frac{d_\infty}{2} < \deg(v) < \frac{3}{2}(td_\infty - (g-1))$$

then $D = L_t$ *and* $W = L_s \cap U$ *satisfy conditions* 1), 2), 3) *and* 4) *of Lemma* (3.4), *where* s *is defined by*

(4) $$sd_\infty = \frac{3}{2} td_\infty - \frac{1}{2}(g-1) + d_\infty.$$

To deduce Lemma (3.5) from this proposition we need only verify that, when N(v), or, equivalently, deg(v), is sufficiently large, then a $t \in \mathbb{Z}$ satisfying (3) can be found. Condition (3) can be transformed into

(5) $$\frac{2}{3} \deg(v) + (g-1) < td_\infty < \frac{1}{5}(4 \deg(v) - 2d_\infty + (g-1))$$

Putting $\deg(v) = 6(g-1) + 3d_\infty + e$ condition (5) takes the form

(6) $$5(g-1) + 2d_\infty + \frac{2}{3}e < td_\infty < 5(g-1) + 2d_\infty + \frac{4}{5}e$$

Therefore there is a real solution for t as soon as $e > 0$, i.e. as soon as $\deg(v) > 6(g-1) + 3d_\infty$. To obtain an integer solution, however, we require the difference, $\frac{2}{15}$ e, of the right and left sides of (6) to be $\geq d_\infty$, i.e. $e \geq \frac{15}{2} d_\infty$, i.e.

(7) $$\deg(v) \geq 6(g-1) + 11d_\infty .$$

Thus Prop. (5.2) implies:

(5.3) THEOREM. *Assuming*

(2) $$\deg(v) \geq s_0 d_\infty$$

and

(7) $$\deg(v) \geq 6(g-1) + 11d_\infty$$

the homomorphism

$$\partial_v : K_*^{S'}F/K_*^S F \longrightarrow K_*k(v)$$

is an isomorphism.

(5.4) *Divisors and Riemann-Roch.* The degree of a divisor $D = \sum_w n_w w$ of F is $\sum_w n_w \deg(w)$. The divisor $(a) = \sum_w w(a)w$ of an $a \in F^\cdot$ has degree zero. Since $aA_\infty = \prod_{w \neq v_\infty} P_w^{w(a)}$ we see therefore that

(8) $$\deg(aA_\infty) = -v_\infty(a)d_\infty$$

For any divisor $D = \sum_w n_w w$

$$L(D) = \{a \in F^\cdot \mid (a) \geq -D\} \cup \{0\}$$

$$= \{a \in F \mid w(a) \geq -n_w \text{ for all } w\}$$

is a k-module whose dimension

$$\ell(D) = \dim_k L(D),$$

is finite, and zero if $\deg(D) < 0$. Note that $L(D)\cdot L(D') \subset L(D + D')$.

The Riemann-Roch Theorem (see, for example, Serre [11], Ch. II, n°9, Thm. 3) asserts that

(9) $$\ell(D) - \ell(K-D) = \deg(D) + 1 - g,$$

where K is the canonical divisor of F. Setting $D = 0$, and noting that $L(0) = k$, one finds that $\ell(K) = g$. Then taking $D = K$ one finds thag $\deg(K) = 2g - 2$. It follows that:

(10) <u>One has</u> $\ell(D) \geq \deg(D) + 1 - g$, <u>with equality if</u> $\deg(D) > 2g - 2$.

It is known (cf. [15], XIII, 12, Cor. of Thm. 12) that there exists a divisor D of degree 1. Then $\ell(gD) \geq g + 1 - g = 1$, so there is an $a \neq 0$ in $L(gD)$. Then $(a) + (gD)$ is a positive divisor of degree g, so there exists a place w (in its support) of degree $\leq g$. It follows that

$$d_\infty \leq g.$$

Let $t \in \mathbb{R}$ have integral part $[t]$. Then it follows from (1) and (8) that

(11) $$L_t = \{a \in A_\infty \mid v_\infty(a) \geq -t\} = \{a \in A_\infty \mid v_\infty(a) \geq -[t]\} = L([t]v_\infty)$$

Putting

$$\ell_t = \dim_k L_t = \ell([t]v_\infty),$$

it follows therefore from (9) and (10) that

(12) $\ell_t \geq [t]d_\infty + 1 - g$, _with equality if_ $[t]d_\infty > 2(g-1)$.

(5.5) LEMMA. _Let_ $\mathfrak{A} \neq 0$ _be an ideal of_ A_∞. _Let_ s _be the least integer such that_ $sd_\infty > \deg(\mathfrak{A}) + g - 1$. _Then there is an_ $a \neq 0$ _in_ $L_s \cap \mathfrak{A}$. _We then have_ $aA_\infty = \mathfrak{A}\mathfrak{b}$, _where_ $\mathfrak{b}$ _is in the ideal class of_ $\mathfrak{A}^{-1}$, _and_ $\deg(\mathfrak{b}) \leq g - 1 + d_\infty$.

Clearly $L_s \cap \mathfrak{A} = L(D)$ where $D = sv_\infty - \sum_{w \neq v_\infty} n_w w$ with $\mathfrak{A} = \prod_{w \neq v_\infty} P_w^{n_w}$. We have $\deg(D) = sd_\infty - \deg(\mathfrak{A}) > g - 1$, so $\ell(D) \geq \deg(D) + 1 - g > 0$, whence the existence of a. We then have $sd_\infty \geq -v_\infty(a)d_\infty = \deg(aA_\infty) = \deg(\mathfrak{A}) + \deg(\mathfrak{b})$, so $\deg(\mathfrak{b}) \leq sd_\infty - \deg(\mathfrak{A}) \leq g - 1 + d_\infty$. This proves the lemma.

We now introduce

$$T = \{v_\infty\} \cup \{w \neq v_\infty \mid \deg(w) \leq g - 1 + d_\infty\}$$

$$A_T = \{a \in F \mid w(a) \geq 0 \text{ for all } w \notin T\}$$

$$U_T = A_T^{\bullet} \ .$$

The group U_T is finitely generated so there is a constant $s_0 \in \mathbf{Z}$ such that

(13) U_T *is contained in the group generated by* $L_{s_0} - \{0\}$.

This is the constant s_0 which appears in Prop. (5.2) and Thm. (5.3).

Lemma (5.5) implies that A_T *is principal*, and hence that A is principal if $T \subset S$, for example if

(14) $$g - 1 + d_\infty < \deg(v).$$

We record this conclusion:

(15) *Condition* (14) *implies that* A *is principal*.

(5.6) LEMMA. *Assume*

(2) $$s_0 d_\infty < \deg(v)$$

and

(14) $$g - 1 + d_\infty < \deg(v)$$

Let $t \in \mathbf{Z}$ *satisfy*

$$(16) \qquad \deg(v) \leq \frac{3}{2}\,(td_\infty - (g - 1))$$

and define s *by*

$$(4) \qquad sd_\infty = \frac{3}{2}\,td_\infty - \frac{1}{2}(g - 1) + d_\infty.$$

Then $W = L_s \cap U$ *contains* 1 *and generates the group* U.

Condition (14) implies that $T \subset S$, and A_T is principal. Hence U is generated by U_T together with elements $\pi_w \in A_T$ such that $\pi_w A_T = P_w A_T$ one for each $w \in S - T$. In view of (13) it suffices therefore to show that (i) $L_{s_0} - \{0\} \subset W$, and (ii) the elements π_w above can be chosen from W.

Proof of (i). If $a \in L_{s_0} - \{0\}$ then $\deg(aA_\infty) \leq s_0 d_\infty < \deg(v)$, by (14), so $a \in U$. It further follows from (16) and (4) that

$$(17) \qquad sd_\infty = \frac{3}{2}(td_\infty - (g - 1)) + (g - 1) + d_\infty$$

$$\geq \deg(v) + g - 1 + d_\infty \geq \deg(v)$$

so that $s_0 d_\infty < sd_\infty$, whence $a \in L_s$. Thus $a \in L_s \cap U = W$.

Proof of (ii). Let $w \in S - T$, and define $s_w \in \mathbb{Z}$ by the inequalities

$$\deg(w) + g - 1 < s_w d_\infty \leq \deg(w) + g - 1 + d_\infty.$$

Then Lemma (5.5) furnishes an element $\pi_w \neq 0$ in $L_{s_w} \cap P_w$, and

$\pi_w A_\infty = P_w \mathfrak{A}$ with $\deg(\mathfrak{A}) \leq g - 1 + d_\infty$. The latter inequality implies that $\mathfrak{A} A_T = A_T$ and so $\pi_w A_T = P_w A_T$. Since $w \in S$ we have $\pi_w \in U$. Finally $\deg(\pi_w A_\infty) \leq s_w d_\infty \leq \deg(w) + g - 1 + d_\infty$ $\leq \deg(v) + g - 1 + d_\infty \leq s d_\infty$, by (17). Thus $\pi_w \in L_s \cap U = W$, so (ii) is proved.

Proof of Prop. (5.2). We assume (2), that $t \in \mathbb{Z}$ satisfies (3), and that s is defined by (4). Note that (3) is the conjunction of

(16)' $$\deg(v) < \frac{3}{2}(td_\infty - (g - 1))$$

and of

(18) $$\deg(v) > \frac{5}{4}\, td_\infty - \frac{1}{4}(g - 1) + \frac{d_\infty}{2}$$

Put $D = L_t$, $E = \{d - d' \mid d, d' \in D \text{ and } d \neq d'\} = L_t - \{0\}$, and $W = L_s \cap U$. We must verify the conditions of Lemma (3.5):

1) $3 \dim D > 2 \deg(v)$

2) $E \subset U$

3) $1 \in W$ and W generates U.

4) If $e_1, e_2, e_3, e_4 \in E$ and $w \in W$ then

(i) $N(e_1 + e_2 + e_3) < N(v)$

(ii) $N(e_1 e_2 - e_3 e_4) < N(v)^2$

(iii) $N(e_1 w - e_2) < N(v)^2$.

(16)' $\Rightarrow$ 1). We have $\dim D = \ell_t$. By Riemann-Roch (see (12))

$\ell_t \geq td_\infty + 1 - g$, since $t \in \mathbb{Z}$. Thus 1) follows from (16)'.

<u>(3) ⇒ 2).</u> Comparing (16)' and (18) one obtains $td_\infty > 5(g - 1) + 2d_\infty$. This together with (18) yields

(19) $$\deg(v) > td_\infty + g - 1 + d_\infty.$$

Let $e \in E \subset L_t$. Then $\deg(eA_\infty) \leq td_\infty$ so (19) implies $\deg(eA_\infty) < \deg(v)$. Thus $e \in U$ as claimed.

<u>(2) and (3) ⇒ 3).</u> By Lemma (5.6) above 3) results from (2), (14), and (16). But (3) ⇒ (16)' ⇒ (16), and (3) ⇒ (19), as we saw above, and (19) ⇒ (14) clearly.

<u>(3) ⇒ 4).</u> Since $e_1 + e_2 + e_3 \in L_t$, $e_1e_2 - e_3e_4 \in L_2t$, and $e_1w + e_2 \in L_{s+t}$ it suffices, in order to prove 4), to verify

(i)' $td_\infty < \deg(v)$

(ii)' $2td_\infty < 2\deg(v)$

(iii)' $(s+t)d_\infty < 2\deg(v)$.

Now (i)' and (ii)' follow from (19) which, as we've seen, follows from (3). By (4) we have $\frac{1}{2}(s + t)d_\infty = \frac{5}{4}td_\infty - \frac{1}{4}(g-1) + \frac{d_\infty}{2}$, and (18) asserts this is $< \deg(v)$.

This completes the proof of Proposition (5.2).

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Appendix

by John Tate

In this appendix we compute the "tame kernel" H_2F, i.e., the kernel of the map

$$K_2F \xrightarrow{(\partial_v)} \coprod_{v \notin S_\infty} k^{\cdot}(v),$$

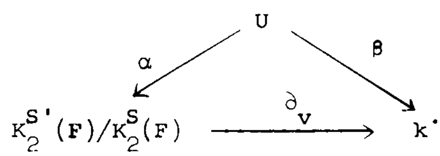
for the first six imaginary quadratic fields F, i.e., those with discriminants $d = -3,-4,-7,-8,-11$, and -15. For these d's, the result is that $H_2F = 0$ for $d \not\equiv 1 \pmod 8$, and H_2F is of order 2, generated by $\ell(-1)^2$, for $d \equiv 1 \pmod 8$.

The proof of finite generation of H_2F given in Ch. II gives a method for computing generators for it in a finite number of steps, but the number of steps is quite large because the actual value of the m in Theorem 1.1 which one gets by the general methods of §4 is large. But for the fields considered here one can use Euclidean Algorithm type techniques to get a reasonably low value of m in Theorem 1.1. For whatever value of m is obtained, we have

$$H_2F = \operatorname{Ker}\left(K_2^{S_m}F \xrightarrow{(\partial_v)} \coprod_{v \in S_m - S_\infty} k^{\cdot}(v)\right),$$

and we can make a list of generators (approximately $\frac{1}{2}m^2$ of them) for $K_2^{S_m}F$, and then try to find relations among them. If we find enough relations, we are done (using the "wild" 2-adic Hilbert symbol to show that $\ell(-1)^2 \neq 0$ when 2 splits, i.e., when $d \equiv 1 \pmod 8$). This is our approach, except that we quote a theoretical result, Proposition 3 below, which can be used to cut down on the amount of computation needed. However, except the last case, $d = -15$, we include computations which make Proposition 3 superfluous.

Our assumptions and notations are as in §3 of Ch. II. The first result concerns an arbitrary global field F. Suppose the ideal P is principal; say P = πA. We can then consider (for n = 2) the commutative triangle on p. 58:



where $\alpha(u) = \ell(u)\ell(\pi) \pmod{K_2^S(F)}$, and $\beta(u) = u \pmod{\pi}$ for $u \in U$, the group of S-units.

Let U_1 denote the subgroup of U generated by $(1+\pi U) \cap U$.

<u>Proposition 1</u>: <u>Suppose</u> W, C, <u>and</u> G <u>are subsets of</u> U <u>such that</u>

(1) $W \subset CU_1$ and W generates U.

(2) $CG \subset CU_1$ and $\beta(G)$ generates $k^\cdot$.

(3) $1 \in C \cap \operatorname{Ker} \beta \subset U_1$.

<u>Then</u> ∂_v <u>is bijective</u>.

Since $\beta(G)$ generates $k^\cdot$, the map β is surjective. As proved on pp. 58, 59, the map α is surjective, and $U_1 \subset \operatorname{Ker} \alpha \subset \operatorname{Ker} \beta$. Hence it will suffice to show that $U_1 = \operatorname{Ker} \beta$. Since $U_1 \subset \operatorname{Ker} \beta$, condition (3) implies $CU_1 \cap \operatorname{Ker} \beta \subset U_1$, and so we will be done if we show $U = CU_1$. By (1) this will follow if CU_1 is a subgroup of U. Hence we are reduced to proving $(CU_1)(CU_1)^{-1} \subset CU_1$, i.e., $CC^{-1} \subset CU_1$. By induction from (2) we have $CG^n \subset CU_1$ for $n \geq 0$, hence we have only to show that for any $c \in C$ there is an n such that $c^{-1} \in G^n U_1$. Let $c \in C$. Choose $g_1, \ldots, g_n \in G$ such that $\beta(c)^{-1} = \beta(g_1)\cdots\beta(g_n)$. Choose $c' \in C$ such that $cg_1\cdots g_n \in c'U_1$. Then by construction, $c' \in \operatorname{Ker} \beta$, so $c' \in U_1$, and so $c^{-1} \in g_1\cdots g_n U_1$ as was to be shown.

Now suppose F is an imaginary quadratic number field. Choose an embedding $F \subset \mathbb{C}$, and for each $a \in F$, let $\bar{a}$ denote the conjugate of a, and $|a| = (a\bar{a})^{1/2} = (Na)^{1/2}$ its absolute value. Recall that A_∞ denotes the lattice of integers in F.

<u>Lemma</u> 1. <u>Suppose</u> $a, b \in U \cap A_\infty$ <u>and</u> $|a| + |b| < Nv$. <u>If</u> $\beta(a) = \beta(b)$, <u>then</u> $a \equiv b \pmod{U_1}$.

This is just a special case of Claim 2 on p. 63. For each $t \geq 0$, let $B_t = \{a \in A_\infty \,\big|\, |a| \leq t\}$.

<u>Proposition 2</u>: <u>Let</u> $r, s, t \geq 1$. <u>Suppose</u>

(a) $B_r \cap U$ <u>generates</u> U.

(b) $\beta(B_s \cap U) = k^\bullet$.

(c) $\beta(B_t \cap U)$ <u>generates</u> $k^\bullet$.

(d) $s + r < Nv$.

(e) $s + st < Nv$.

<u>Then</u> ∂_v <u>is bijective</u>.

Let $W = B_r \cap U$, $C = B_s \cap U$, $G = B_t \cap U$ and apply Proposition 1. Given $w \in W$, choose $c \in C$ with $\beta(c) = \beta(w)$. Then $w \in cU_1$ by Lemma 1, because $|c| + |w| \leq s+r < Nv$. Given $c \in C$, $g \in G$, choose $c' \in C$ with $\beta(c') = \beta(cg)$. Then $cg \in c'U_1$ by Lemma 1, because $|cg| + |c'| \leq st + s < Nv$. Given $c \in C$ such that $\beta(c) = 1$, then $c \in U_1$ by Lemma 1, because $|c| + |1| \leq s+1 \leq s + st < Nv$.

Let d be the discriminant of F. The ring of integers A_∞ is a lattice in $\mathbb{C}$ with $\mathbb{Z}$-base $1,\Theta$, where

$$\Theta = \begin{cases} \dfrac{\sqrt{|d|}}{2} & \text{if } d \text{ even,} \\ \dfrac{1}{2} + \dfrac{\sqrt{|d|}}{2} & \text{if } d \text{ odd.} \end{cases}$$

A point in $\mathbb{C}$ at maximum distance from A_∞ is

$$\gamma = \begin{cases} \dfrac{1}{2} + \dfrac{\sqrt{|d|}}{4} = \dfrac{1}{2} + \dfrac{1}{2}\Theta \ , & \text{if } d \text{ even,} \\ \dfrac{1}{2} + \dfrac{|d|-1}{2|d|}\Theta \ , & \text{if } d \text{ odd.} \end{cases}$$

Let δ denote the distance from γ to A_∞. Then

$$\delta^2 = \begin{cases} \dfrac{|d|+4}{16} \ , & \text{if } d \text{ even,} \\ \dfrac{(|d|+1)^2}{16|d|} \ , & \text{if } d \text{ odd.} \end{cases}$$

As the following table indicates,

d	-3	-4	-7	-8	-11	-15	-19	-20
δ^2	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{3}{4}$	$\frac{9}{11}$	$\frac{16}{15}$	$\frac{25}{19}$	$\frac{3}{2}$

,

there are five fields for which $\delta < 1$. These are the imaginary quadratic fields in which the norm furnishes a Euclidean Algorithm, i.e., in which, for given $a,b \in A_\infty$ with $b \neq 0$, there exists $q \in A_\infty$ such that $\left|\frac{a}{b} - q\right| < 1$, hence $|a - qb| < |b|$.

Lemma 2. *Suppose* $\delta < 1$. *Then* ∂_v *is bijective if either one of the following two conditions holds*

(i) $(Nv)^{1/2} > 1 + \delta$ *and* $(Nv)^{1/2} > \frac{\delta}{1-\delta^2}$

(ii) $(Nv)^{1/2} > 1 + \delta$ *and* $(Nv)^{1/2} > (1+|g|)\delta$ *for some primitive root* $g \in A_\infty$ *for* v.

Apply Proposition 2 with $r = (Nv)^{1/2}$, $s = \delta(Nv)^{1/2}$, and with $t = s$ (resp. $t = |g|$) in case (i) (resp. in case (ii)). Since F is Euclidean, A_∞ is a P.I.D. Let π be a prime element in A_∞ corresponding to the place v. Then $|\pi| = (Nv)^{1/2}$. Division by π with remainder of absolute value $\leq \delta|\pi| = s$ shows that the residue classes (mod π) are represented by elements of B_s, and any non-zero element of B_s is in U, because $s < |\pi|$. Also, U is generated by roots of unity and by prime elements u_i of A_∞ such that $|u_i| \leq \pi$, i.e., such that $u_i \in B_r$.

We are now ready to compute the tame kernel H_2F for some imaginary quadratic fields F with low discriminant. In several cases, relatively little computation is needed to show that H_2F has no elements of odd order, whereas to analyse the 2-primary part of H_2F by the same direct methods is a more tedious job. Thus the following fact saves some computational effort.

<u>Proposition 3</u>: <u>Suppose</u> F <u>is an imaginary quadratic field of discriminant</u> d, <u>with</u> $|d| < 35$. <u>If</u> $d \not\equiv 1 \pmod 8$, <u>then</u> $H_2(F)$ <u>is of odd order</u>. <u>If</u> $d \equiv 1 \pmod 8$, <u>then the</u> 2-<u>primary part of</u> $H_2(F)$ <u>is of order</u> 2, <u>generated by</u> $\ell(-1)^2$, <u>and is mapped isomorphically onto the group</u> (± 1) <u>by the "wild" Hilbert symbol at one of the primes above</u> 2.

Every ideal class of F contains an ideal of norm $\leq \sqrt{|d|/3}$. Hence, if $|d| < 27$, or if $d \equiv -1 \pmod 3$ and $|d| < 75$, then the primes above 2 generate the ideal class group of F. We now show that Proposition 3 holds even with the hypothesis $|d| < 35$ replaced by the hypothesis that the primes above 2 generate a subgroup of odd index in the ideal class group of F.

An element of order 2 in K_2F is of the form $\ell(-1)\ell(a)$, with $a \in F^\cdot$, and the a's for which $\ell(-1)\ell(a) = 0$ form a subgroup Δ of $F^\cdot$ in which $(F^\cdot)^2$ is of index 2^{1+r_2}, where r_2 is the number of complex places of F. This much is true for any global field; for a discussion, unfortunately without complete proofs, see [14, pp. 209-211]. For $\ell(-1)\ell(a)$ to be in the tame kernel is equivalent to $v(a)$ being even at all finite places v not above 2, for at such a place we have $\partial_v(\ell(-1)\ell(a)) = (-1)^{v(a)}$. From our hypothesis on the ideal class group, it follows that if $\ell(-1)\ell(a)$ is in the tame kernel, then $a \in UF^{\cdot 2}$, where U is the group of S(2)-units, S(2) denoting the set of primes above 2. Thus the map $u \longmapsto \ell(-1)\ell(u)$ is a homomorphism of U/U^2 <u>onto</u> the group $(H_2F)_2$

of elements of order 2 (or 1) in H_2F, and its kernel is of order 2^{1+r_2}. The order of U/U^2 is $2^{r_1+r_2+m}$, where r_1 is the number of real places above 2. Hence, under our hypothesis on the ideal class group, $(H_2F)_2$ is of order 2^{r_1+m-1}, for any global F.

In case of an imaginary quadratic F, this order is 2^{m-1} and is 1 unless 2 splits, in which case it is 2. Suppose 2 splits (i.e., $d \equiv 1 \pmod 8$). Then the completion of F at a prime above 2 is isomorphic to $\mathbb{Q}_2$, the field of 2-adic numbers, and the Hilbert symbol on $\mathbb{Q}_2$ gives a homomorphism $K_2\mathbb{Q}_2 \longrightarrow (\pm 1)$ carrying $\ell(-1)^2$ to -1. Thus, $\ell(-1)^2 \neq 0$, and hence $\ell(-1)^2$ generates $(H_2F)_2$. Moreover, since $2K_2F$ is killed by the 2-adic Hilbert symbol, there is no element $x \in K_2F$ such that $\ell(-1)^2 = 2x$; in particular, the 2-primary part of H_2F has no element of order 4.

<u>Remark</u>: For d = -35, the situation is definitely different. The elements $-1,2,5 \in F^{\cdot}$ are independent $\mathrm{mod}(F^{\cdot})^2$ so they cannot all belong to the group Δ, in which $(F^{\cdot})^2$ is of index 4. Of course $2 \in \Delta$. Hence, two of the three elements $\ell(-1)^2$, $\ell(-1)\ell(5)$, and $\ell(-1)\ell(-5)$ are non-zero, and one of them is zero, in K_2F. (<u>Exercise</u>: which one?). But those elements are in H_2F. Hence $H_2F \neq 0$ for d = -35, even though there is no wild local symbol showing this; $\mathbb{Q}(\sqrt{-35})$ is a field with an "exotic" symbol. The case d = -35 is almost certainly the first such case occurring among imaginary quadratic fields.

Let us now treat some individual imaginary quadratic fields, in order of increasing size of the discriminant, d.

$\underline{d = -3}$

Here the smallest value of Nv is 3, and $\delta = 1/\sqrt{3}$. By Lemma 2, ∂_v is therefore bijective for every v, because

$$1 + \delta = \frac{\sqrt{3}+1}{\sqrt{3}} < \sqrt{3} \qquad \text{and} \qquad \frac{\delta}{1-\delta^2} = \frac{\sqrt{3}}{2} < \sqrt{3}\ .$$

It follows that the tame kernel H_2 is equal to $K_2^{S_\infty}$ and is generated by $\ell(\zeta)^2$, where ζ is a primitive 6-th root of unity. Since $\zeta+\zeta^{-1} = 1$, we have $0 = \ell(\zeta)\ell(\zeta^{-1}) = -\ell(\zeta)^2$. Hence $H_2 = 0$.

$\underline{d = -4}$

Here $\delta = 1/\sqrt{2}$. By Lemma 2, ∂_v is bijective for $Nv > 2$, because after 2 the smallest value of Nv is 5, and

$$1 + \delta = \frac{\sqrt{2}+1}{\sqrt{2}} < \sqrt{5}, \qquad \text{and} \qquad \frac{\delta}{1-\delta^2} = \sqrt{2} < \sqrt{5}\ .$$

Hence the tame kernel H_2 is generated by the following three elements, each of which is 0.

$$\ell(i)^2 = \ell(-1)\ell(i) = \ell(i^2)\ell(i) = 2\ell(i)^2$$

$$\ell(i)\ell(1-i) = 0$$

$$\ell(1-i)^2 = \ell(-1)\ell(1-i) = \ell(i^2)\ell(1-i) = 2\ell(i)\ell(1-i) = 0.$$

Thus $H_2 = 0$.

$d = -7$

In $\mathbb{Q}(\sqrt{-7})$ the primes 3 and 5 are undecomposed. Hence the smallest value of Nv after $Nv = 2$ is $Nv = 7$. Trivial calculation with $\delta = 2/\sqrt{7}$ shows $1+\delta < \sqrt{7}$ and $\delta/(1-\delta^2) < \sqrt{7}$. Hence, by Lemma 2, ∂_v is bijective for $Nv > 2$. There are two places v with $Nv = 2$, corresponding to the prime elements

$$u = \frac{1+\sqrt{-7}}{2} \qquad \text{and} \qquad \bar{u} = \frac{1-\sqrt{-7}}{2} = 1-u \ .$$

Hence the tame kernel H_2 is generated by the elements $\ell(a)\ell(b)$ for a and b running through the set $\{-1,u,\bar{u}\}$. But $\ell(u)\ell(\bar{u}) = 0$, because $u+\bar{u} = 1$, and

$$\ell(-1)^2, \qquad \ell(u)^2 = \ell(-1)\ell(u), \qquad \text{and} \qquad \ell(\bar{u})^2 = \ell(-1)\ell(\bar{u})$$

are all killed by 2, since $(-1)^2 = 1$. This shows the tame kernel H_2 is killed by 2, and is therefore of order 2, generated by $\ell(-1)^2$, by Proposition 3. Of course the fact that H_2 is not trivial follows from the "wild" 2-adic Hilbert symbol; it is mainly to show that H_2 is not of order greater than 2 that we are appealing to the Proposition 3. However in this case it is not too difficult to give a direct proof of the latter fact, as follows.

The equation $1 = -u-\bar{u}^2$ shows

$$0 = \ell(-u)\ell(-\bar{u}^2) = \ell(-1)^2 + \ell(u)\ell(-1) + 2\ell(-u)\ell(\bar{u})$$

and since $2\ell(-u)\ell(\bar{u}) = 0$, we conclude that

$$\ell(-1)^2 = \ell(u)\ell(-1).$$

Since -1+2 = 1, we have

$$0 = \ell(-1)\ell(2) = \ell(-1)\ell(u\bar{u}) = \ell(-1)\ell(u) + \ell(-1)\ell(\bar{u})$$

and consequently

$$\ell(u)^2 = \ell(-1)\ell(u) = \ell(-1)\ell(\bar{u}) = \ell(\bar{u})^2$$

and this element is equal to $\ell(-1)^2$ by the preceding relation. Thus, H_2 is indeed generated by one element.

d = -8

Here $A_\infty = \mathbb{Z}[\sqrt{-2}]$. A list of prime elements of A_∞ in order of non-decreasing norm begins

$$u_1 = \sqrt{-2}, \qquad u_2 = 1+\sqrt{-2}, \qquad u_3 = 1-\sqrt{-2}.$$

Since 5 and 7 are undecomposed, the next value of Nv is $11 = N(3+\sqrt{-2})$. Using $\delta = \frac{\sqrt{3}}{2}$, we find by Case (i) of Lemma 2 that ∂_v is bijective for $Nv > 12$, and by Case (ii), with the primitive root $g = 2$, that it is also bijective for $Nv = 11$. Using Proposition 1 with the sets

$$W = \{-1,u_1\} \quad \text{or} \quad \{-1,u_1,u_2\}$$

$$C = \{1,-1\}$$

$$G = \{-1\}$$

one can show that ∂_v is also bijective for $Nv = 3$. For example, if S consists of S_∞ together with the two finite places corresponding to the prime elements u_1 and u_2, and if v is the place corresponding to $u_3 = \pi$, then

$$U \text{ is generated by } -1,\ u_1, \text{ and } u_2.$$

The set U_1 contains $u_1 = 1+u_0\pi$ and $-u_2 = 1+u_0u_1\pi$. Hence, the generators for U are clearly in CU_1, if $C = \{1,-1\}$. And with $G = \{-1\}$ we have $CG \subset CU_1$ (even $CG \subset C$). Also $(\text{Ker } \beta) \cap C = \{1\} \subset U_1$.

It follows that H_2 is generated by the elements $\ell(-1)^2$ and $\ell(-1)\ell(u_1)$. Consequently $2H_2 = 0$ and we can use Proposition 3 to conclude that $H_2 = 0$.

Of course, a direct proof can also be made, and we shall give one below. For such computations we have found it convenient to use a shorthand notation which we now explain. We let

$$-1 = u_0, u_1, u_2, u_3, \cdots$$

be a sequence of elements such that, for each m, the set (u_i), $0 \le i \le m$, generates the group of S_m-units, where S_m consists of $v_\infty, v_1, \cdots, v_m$, the v_i being a list of all finite places, with $Nv_i \le Nv_{i+1}$ as in §1. These generators u_i determine elements $\ell(u_i)\ell(u_j)$ in K_2F which we abbreviate as follows.

$$(ij) = \ell(u_i)\ell(u_j)\ , \qquad \text{and}$$

$$(i) = (ii) = \ell(u_i)^2 = \ell(-1)\ell(u_i) = (oi).$$

We shall use without comment the obvious relations

$$2(i) = 0 \qquad \text{and} \qquad (ji) = -(ij) .$$

Thus, for each m, $K_2^{S_m}(F)$ is generated by the elements

$$(ij) \qquad 1 \leq i < j \leq m,$$

and

$$(i) \qquad 0 \leq i \leq m ,$$

the last m+1 of which are of order 1 or 2.

For example, in $\mathbb{Q}(\sqrt{-8})$, with

$$u_0 = -1, \quad u_1 = \sqrt{-2}, \quad u_2 = 1+\sqrt{-2}, \quad u_3 = 1-\sqrt{-2} ,\ldots,$$

as above, we have shown via Propositions 1 and 2 that the tame kernel H_2 is $K_2^{S_1}(F)$ and is therefore generated by (0) and (1). From Proposition 3 we know that these elements are 0. We now prove this directly.

$$u_1 + u_3 = 1 \Longrightarrow (13) = 0$$

$$u_0u_1 + u_2 = 1 \Longrightarrow (2) + (12) = 0, \qquad \text{i.e.,} \quad (12) = (2)$$

$$u_0 + u_0u_1^2 = 1 \Longrightarrow (0) + 2(1) = 0, \qquad \text{i.e.,} \quad \boxed{(0) = 0}$$

$$u_0u_2u_1^{-2} + u_0u_3u_1^{-2} = 1 \Longrightarrow (0) + (3) + (2) + (23) + 2(12)-2(13) = 0.$$

Combining this last relation with those previously obtained, we find

$$(23) = (2) + (3) .$$

Finally,

$$u_0u_3 + u_0u_1u_2 = 1 \Longrightarrow (0) + (1) + (2) - (3) - (13) - (23) = 0,$$

which, combined with what we had before, shows $\boxed{(1) = 0}$.

Incidentally, the relations we have just obtained show that $K_2^{S_3}(F)$ is generated by (2) and (3) and that $K_2^{S_2}(F)$ is generated by (2). This gives another proof of the fact that ∂_v is bijective for $v = v_2, v_3$, i.e., for $Nv = 3$.

d = -11

Here we can take

$$u_o = -1,\ u_1 = \frac{1+\sqrt{-11}}{2},\quad u_2 = 1-u_1 = \bar{u}_1,\ u_3 = 2,\ u_4 = 1+u_1,$$
$$u_5 = 2-u_1 = \bar{u}_4,\ \cdots$$

with

$$Nu_o = 1,\quad Nu_1 = 3 = Nu_2,\quad Nu_3 = 4,\quad Nu_4 = 5 = Nu_5.$$

We claim ∂_v is bijective for every v! For $Nv \geq 25$ this follows from Case (i) of Lemma 2, because $\delta = 3/\sqrt{11}$ and $\delta/(1-\delta^2) = \sqrt{99/4} < 5$. The only values of Nv such that $5 < Nv < 25$ are $Nv = 11$ and $Nv = 23$. Case (ii) of Lemma 2 handles these cases, because 2 (resp. -2) is a primitive root for 11 (resp. 23) and $3\delta = \sqrt{81/11} < \sqrt{11}$. For $v = v_5$ with $Nv = 5$ we use Proposition 1 with

$$W = \{u_o,u_1,u_2,u_3,u_4\},\quad C = \{u_ou_1,u_o,1,u_1\},\quad G = \{u_1\}\ .$$

We have $W \subset CU_1$ because the elements $u_2-u_o = u_3-u_1 = u_5$ and $u_4-u_ou_1 = 2+\sqrt{-11}$ have norms 5 and 15 whose prime factorizations involve only primes < 5 and one 5. Similarly, $GC \subset CU$, because $u_1^2-u_o = u_ou_5$ has the same property. For $v = v_4$ we just conjugate the above, after dropping u_4 from W. For $v = v_3$ we use

$$W = \{u_o, u_1, u_2\}, \quad C = \{1, u_1, u_o u_2\}, \; G = \{u_1\}$$

and have only to observe that $u_1^2 - u_o u_2 = u_o u_3$. For $v = v_2$, we use Proposition 1 again, with

$$W = \{u_o, u_1\}, \quad C = \{1, -1\}, \quad G = \{-1\}$$

and for $v = v_1$ the same, after dropping u_1 from W.

It follows that the tame kernel H_2 is $K_2^{S_\infty}(F)$ and is therefore generated by $\ell(-1)^2$, the element which is denoted by (0) in our shorthand notation. Thus $2H_2 = 0$, and, by Proposition 3, $H_2 = 0$.

To show (0) = 0 directly is tedious, but it can be done as follows:

$$1 = \frac{u_o}{u_2^2} + \frac{u_o u_4}{u_2^2} \Longrightarrow (4) = 2(24) + (0) \Longrightarrow 4(24) = 0$$

$$1 = \frac{u_o}{u_1} + \frac{u_4}{u_1} \Longrightarrow (14) = (4)$$

$$1 = \frac{u_2}{u_3} + \frac{u_4}{u_3} \Longrightarrow (34) = (24) - (23) + (3)$$

$$1 = u_o + u_3 \Longrightarrow (3) = 0$$

$$1 = u_1 u_2 + u_o u_3 \Longrightarrow (23) = -(13) + (1) + (2)$$

$$1 = \frac{u_o u_2}{u_1^2} + \frac{u_o u_3}{u_1^2} \Longrightarrow (23) = 2(13) - 2(12) + (0) + (2) + (3).$$

Subtracting, we get $3(13) = 2(12) + (0) + (1) + (3)$

$$1 = u_1 + u_2 \Longrightarrow (12) = 0 .$$

Simplifying, we have

$$(34) = (24) - 2(13) + (2) + (0) \qquad 4(24) = 0$$

$$(4) = (14) = 2(24) + (0)$$

$$(23) = 2(13) + (2) + (0) \qquad 3(13) = (1) + (0)$$

$$(3) = 0 \qquad 2(2) = 0$$

$$(12) = 0 \qquad 2(1) = 0$$

$$2(0) = 0$$

Finally,

$$1 = \frac{u_1^2}{u_4} + \frac{u_3^2}{u_4} \Longrightarrow 4(13) - 2(14) + 2(34) + (4) = 0,$$

is a relation which, together with those already obtained, implies $(0) = 0$.

<u>$d = -15$</u>

Here the class number is 2. We take

$$u_0 = -1, \quad u_1 = \frac{1+\sqrt{-15}}{2}, \quad u_2 = 2, \quad u_3 = \frac{3+\sqrt{-15}}{2}, \quad u_4 = \frac{5+\sqrt{-15}}{2}, \ldots$$

We claim that $4H_2F = 0$, and hence, by Proposition 3, H_2F is of order 2, generated by $\ell(-1)^2$. Since

$$u_1 + u_1^{-1}u_2^{\,2} = 1 \Longrightarrow (1) + 2(12) = 0 \Longrightarrow 4(12) = 0,$$

we have $4K_2^SF = 0$ if S consists of the two primes of norm 2. To prove our claim, by showing $H_2F \subset K_2^SF$, we have only to show that ∂_v is bijective for $Nv > 2$. For $Nv = 3$, use Proposition 1 with $\pi = u_3$ and

$$W = \{u_o, u_1, u_2 u_1^{-1}\}, \quad C = \{u_o, 1\}, \quad G = \{u_o\}.$$

For Nv = 5, use $\pi = u_4$ and

$$W = \{u_o, u_1, u_2, u_3\}, \quad C = \{1, u_1, u_o, u_o u_1\}, \quad G = \{u_1\}.$$

After 5, the next values of Nv are 17,19,23,··· and we look for a general method to handle them.

Lemma 15.1. *Let* $Q = (2, u_1)$ *be the prime ideal such that* $NQ = 2$ *and* $(u_1) = Q^2$. *Given any* $z \in \mathbb{C}$, *there exists an element* $q \in Q$ *such that* $|z-q|^2 \leq 8/5$.

Indeed it is easy to see that the point $1 + \frac{\sqrt{-15}}{5}$ is maximally distant from Q, and that its distance is $\sqrt{8/5}$.

Lemma 15.2. *If* M *is a non-principal ideal every residue class* (mod M) *is represented by an integer* c *such that* $Nc < (4/5)NM$.

Let $M = bQ$. Let $a \in A_\infty$. Let $q \in Q$ such that $\left|\frac{a}{b} - q\right|^2 \leq 8/5$. Then $bq \in M$, and $|a-bq|^2 \leq (8/5)|b|^2 = (4/5)NM$, so $c = a-bq$ is the desired representative of the residue class of a.

Using Proposition 2, with $s^2 = (4/5)Nv$ and $r^2 = 2Nv$, we can now show ∂_v bijective for all v with Nv > 5 such that the corresponding prime ideal P is non-principal. Indeed, U is generated by integers u such that $|u|^2 < 2Nv$, because as we choose generators $u_1, u_2, \cdots$ corresponding to primes $P_1 = Q$, $P_2 = \overline{Q}$, $P_3, P_4, \cdots$ we can take u_i such that $(u_i) = P_i$ if P_i is principal and such that $(u_i) = QP_i$ if P_i is not principal. Condition (d) of Prop. 2 is satisfied if $(Nv)^{1/2} > (4/5)^{1/2} + 2^{1/2}$, so for $Nv > 10$. Condition (e) is satisfied with $t = s$ for $Nv > 20$. For $Nv = 17$, we can take $t = 3$, using the primitive root $g = 3$ for 17.

To treat the v corresponding to principal P we use

Lemma 15.3. *If* (b) *is a principal ideal prime to* Q, *then every residue class* (mod (b)) *is represented by an element* $c \in Q$ *such that* $|c|^2 \leq (8/5)Nb$.

The proof is the same as for the preceding lemma, but starting with an $a \in Q$.

Suppose v corresponds to a prime ideal P which is principal in A_∞, say $P = (\pi)$. Let us try to apply Proposition 1 with

$$W = \{u \in U \cap A_\infty \,\big|\, |u|^2 \leq 2Nv\}$$

$$C = \{c \in Q \,\big|\, |c|^2 \leq (8/5)Nv\} .$$

As discussed above, W generates U. We will have $W \subset CU_1$ by Lemma 1, if $(\sqrt{2} + \sqrt{8/5}) < \sqrt{Nv}$, so certainly if $Nv > 16$. Also by Lemma 1, we will have $C \cap \operatorname{Ker} \beta \subset U_1$ if $1 + \sqrt{(8/5)Nv} < Nv$, which holds for $Nv > 4$. To continue, we need a slight generalization of Lemma 1.

Lemma M1. *Let* F *be an imaginary quadratic field. Let* M *be an ideal in the ring of integers of* F, *the prime factorization of which involves only primes in* S. *Suppose* $a,b \in U \cap M$ *and* $|a| + |b| < Nv(NM)^{1/2}$. *If* $\beta(a) = \beta(b)$, *then* $a \in bU_1$.

Let P be the prime ideal corresponding to v. We have $a-b \in MP$ and $N(a-b) \leq (|a|+|b|)^2 < (NP)^2 NM$. Consequently $(a-b) = MPL$ where L is an ideal with $NL < NP$, whose prime factors are therefore in S. It follows that $a-b = \pi u$ with $u \in U$, hence $(a/b) = 1 + \pi(u/b) \in (1+\pi U) \cap U \subset U_1$.

Using Lemma M1, we see that <u>if</u> $g \in U$ <u>and</u> $Nv > (4/5)(|g|+1)^2$, <u>then</u> $gC \subset CU_1$; indeed, given any $c \in C$ we can choose a $c' \in C$ such that $\beta(c') = \beta(gc)$, and then $gc \in c'U$, by Lemma M1 because $gc-c' \in Q$ and $|gc| + |c'| \leq (|g|+1)\sqrt{(8/5)Nv} < \sqrt{2}\, Nv$. This takes care of the cases $Nv = 19$ and $Nv = 31$ because 2 (resp. 3) is a primitive root for 19 (resp. 31). The remaining principal prime ideals have $Nv > 40$ (the next two cases being $Nv = 49, 61$), and they are all taken care of by the fact that $C^2 \subset CU_1$ if $Nv > 40$. Let $c_1, c_2 \in C$. Choose $c \in C$ such that $\beta(2)\beta(c) = \beta(c_1 c_2)$. By Lemma M1, with $a = c_1 c_2$, $b = 2c$, $M = Q^2$, we conclude $c_1 c_2 \in 2cU_1$, if $Nv > 40$; and we have seen just above (with $g = 2$), that $2cU_1 \subset CU_1$.

On the Quaternion Symbol Homomorphism
$g_F\colon k_2F \longrightarrow B(F)$

Richard Elman and T. Y. Lam[1]

1. Introduction and terminology

In this short note, several sufficient conditions are obtained for the map g_F in the title to be injective.

Throughout this work, F denotes a field of characteristic not 2; B(F) denotes the Brauer group of F, and k_2F denotes Milnor's K_2F modulo 2 (see [9]). The pairing

$$(a,b) \longmapsto \text{the quaternion algebra } \left(\frac{a,b}{F}\right) \quad (a,b \in \dot{F}=F-\{0\})$$

is clearly a Steinberg symbol $\dot{F}\times\dot{F} \longrightarrow B(F)$, so it induces a homomorphism $g_F\colon k_2F \longrightarrow B(F)$, by the universal property of k_2F. The following question then arises naturally:

Q1: *Is g_F a monomorphism ?*

After a slight reformulation, it will turn out that Q1 is completely equivalent to a question in the theory of quadratic forms over fields. Let W(F) be the Witt ring of (non-singular) quadratic forms over F, and IF be the ideal in W(F) consisting of all even-dimensional forms. In [9], Milnor has shown that there exists a natural isomorphism $k_2F = I^2F/I^3F$. Under this isomorphism, a 'generator'

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$\ell(a)\ell(b) \in k_2F$ (in the notation of [9]) corresponds to a coset $<1,-a,-b,ab> + I^3F$. Here, the 4-dimensional form $<1,-a,-b,ab>$ is precisely the norm form of the quaternion algebra $\left(\frac{a,b}{F}\right)$, and is a '2-fold Pfister form' in the terminology of [5]. Recall that, in [5], we have introduced the notation $<<a_1,\cdot\cdot,a_n>>$ for the n-fold Pfister form $\varphi = \bigotimes_{i=1}^{n} <1,a_i>$. This notation will be used freely in the sequel (though only for $n \leqslant 3$). Also, following [2],[5], we shall always write φ' for the 'pure subform' of the Pfister form φ; it is the unique form for which $<1> \perp \varphi' \cong \varphi$.

From here on, we shall identify k_2F with I^2F/I^3F, using Milnor's isomorphism mentioned above. Under this identification, the map g_F: $I^2F/I^3F \longrightarrow B(F)$ is easily checked to be just the 'Witt invariant' c in [10]. Thus, Q1 is completely equivalent to the following basic question investigated in [10]:

Q2: <u>If a form</u> $q \in I^2F$ <u>has Witt invariant</u> $c(q) = 1 \in B(F)$, <u>does it follow that</u> $q \in I^3F$?

In this note, we obtain some evidence for the apparent truth of Q1 and Q2. In Section 2, we establish a necessary and sufficient condition for the sum of four 2-fold Pfister forms to lie in I^3F (Theorem 2.2). From this, we show that g_F <u>is injective if every element in</u> k_2F <u>is a sum of at most five generators</u> (Theorem 2.6). A consequence of this result is Pfister's Satz 14 of [10] about Q2 (see Corollary 2.8). The theorem is also applicable to local, global, and C_3-fields, as well as fields F with tr. $d._{\mathbb{R}}$ $F \leqslant 3$ (Proposition 2.9). In Section 3, we investigate the behaviour of the ideals I^n (mainly for $n \leqslant 3$) under a quadratic extension $F \subset K = F(\sqrt{a})$.

It is shown that $I^3F = 0$ implies $I^3K = 0$ (Corollary 3.5). This, together with an inductive argument, shows that if $I^3F = 0$, then g_F is indeed injective (Theorem 3.10). It follows that, for a field F, quadratic forms are classified by dimension, discriminant, and the Hasse invariant iff $I^3F = 0$, i.e. iff four-dimensional forms of determinant 1 are all universal over F (Theorem 3.11). In Section 4, we obtain some necessary conditions for $\alpha = \sum_{i=1}^{r} \ell(a_i)\ell(b_i)$ to lie in ker (g_F) — namely, we must have $2^{r-1} \sum <<-a_i,-b_i>> \in I^{r+2}F$, and $\ell(-1)^{t-2}.\alpha=0 \in k_tF$, where $t=2^r$ (Theorem 4.1). In particular, if $\ell(-1)^m: k_2F \to k_{m+2}F$ is injective for all $m \geq 1$, then g_F is indeed a monomorphism (Corollary 4.2).

The beginning point of our investigation is the following well-known result, which answers Q1 affirmatively in case every element in k_2F is a sum of three generators. Our theorems in Section 2 are, therefore, all generalizations of this result.

Theorem 1.1. Suppose $\prod_{i=1}^{3} \left(\frac{-x_i,-y_i}{F}\right) = 1 \in B(F)$. Then,

(1) The form $q = <<x_1,y_1>>' \perp <-1><<x_2,y_2>>'$ is isotropic over F.

(2) $\left(\frac{-x_i,-y_i}{F}\right)$, $1 \leq i \leq 3$, have a common splitting field L such that $[L:F] \leq 2$.

(3) $\sum_{i=1}^{3} \ell(-x_i)\ell(-y_i) = 0 \in k_2F$.

This result was first proved by Pfister [10, P.124,Zusatz]. In [5,Theorem 6.1(2)], we gave a slightly different proof. Recently, a third proof using only the theory of algebras appeared in A.A. Albert's posthumous work [1]. For the sake of completeness, we sketch below a quick proof of 1.1.

Proof. Assume that q is anisotropic over F. Let $K = F(\sqrt{-x_3})$. Since $\left(\frac{-x_1,-y_1}{K}\right) \cong \left(\frac{-x_2,-y_2}{K}\right)$, q_K is clearly hyperbolic over K (in particular, $[K:F] = 2$). By [11,P.52], we have $q \cong \langle 1,x_3\rangle\cdot\varphi$, where φ is a ternary form over F. Equating determinants, we get $-1 = \det q = x_3 \in \dot{F}/\dot{F}^2$, a contradiction to $[K:F] = 2$. This proves (1), and (2), (3) follow immediately.

2. Sums of 4 or 5 Pfister forms

In this Section, we shall

(A) establish some criteria for the sum of four 2-fold Pfister forms to lie in I^3F (Theorem 2.2).

(B) show that, if every element in k_2F is a sum of five generators, then g_F is injective (Theorem 2.6).

These results depend on the following lemma, which will also be crucial for Section 3.

Lemma 2.1. *If φ and τ are 2-fold Pfister forms over F such that $q = \varphi' \perp \langle -a\rangle\tau'$ becomes isotropic over $K = F(\sqrt{a})$, then there exist $z,b,c,d \in \dot{F}$ such that $\varphi \perp \langle -a\rangle\tau \cong \langle\langle -a,z\rangle\rangle \perp \langle b\rangle\langle\langle c,d\rangle\rangle$.*

Proof. CASE 1. *q is isotropic over F.*

In this case, φ' and $\langle a\rangle\tau'$ represent some common element $c \in \dot{F}$. Write $\varphi \cong \langle\langle c,b\rangle\rangle$, $\tau \cong \langle\langle ac,z\rangle\rangle$, where $b,z \in \dot{F}$. Thus,

$$\begin{aligned}\varphi \perp \langle -a\rangle\tau &\cong \langle 1,b,cb,-a,-az,-cz\rangle \perp H \quad (\text{ H=hyperbolic plane})\\ &\cong \langle 1,-a,z,-az\rangle \perp \langle b,-z,cb\ ,-cz\rangle\\ &\cong \langle\langle -a,z\rangle\rangle \perp \langle b\rangle\langle\langle c,d\rangle\rangle \qquad \text{where } d = -bz.\end{aligned}$$

CASE 2. *q is anisotropic over F.*

In this case, we must have $[K:F] = 2$, and, by [11,P.52], $q \cong \langle z\rangle\langle\langle -a\rangle\rangle \perp q_1$, where $z \in \dot{F}$, and q_1 is a 4-dimensional form over F. Since det $q = -a$, we have det $q_1 = 1$, so we may write $q_1 \cong \langle b\rangle\langle\langle c,d\rangle\rangle$, where $b,c,d \in \dot{F}$. We now conclude that

$$\varphi \perp \langle -a\rangle\tau \cong q \perp \langle\langle -a\rangle\rangle \cong \langle\langle -a,z\rangle\rangle \perp \langle b\rangle\langle\langle c,d\rangle\rangle. \qquad \text{Q.E.D.}$$

<u>Theorem</u> 2.2. <u>Let</u> $\varphi_i = \langle\langle x_i,y_i\rangle\rangle$, $1 \leq i \leq 4$, <u>and</u> $\sigma = \varphi_1 \perp \langle x_3\rangle\varphi_2 \perp \langle -1\rangle\varphi_3 \perp \langle y_3\rangle\varphi_4$. <u>Then, the following statements are equivalent</u>:

(1) $\sigma = \langle b\rangle\cdot\beta \in W(F)$, <u>where</u> $b \in \dot{F}$, <u>and</u> β <u>is a</u> 3-<u>fold Pfister form over</u> F.

(2) $\varphi_1+\varphi_2+\varphi_3+\varphi_4 \in I^3F$.

(3) $\prod_{i=1}^{4}\left(\frac{-x_i,-y_i}{F}\right) = 1 \in B(F)$, i.e. $\left(\frac{-x_1,-y_1}{F}\right)\otimes\left(\frac{-x_2,-y_2}{F}\right) \cong \left(\frac{-x_3,-y_3}{F}\right)\otimes\left(\frac{-x_4,-y_4}{F}\right)$

<u>Proof</u>. (1) $\Longrightarrow$ (2) is trivial, since $\varphi_1+\varphi_2+\varphi_3+\varphi_4 \equiv \sigma \pmod{I^3F}$.
(2) $\Longrightarrow$ (3). Identifying k_2F with I^2F/I^3F after Milnor [9], (2) implies that $\ell(-x_1)\ell(-y_1) + \ell(-x_2)\ell(-y_2) = \ell(-x_3)\ell(-y_3) + \ell(-x_4)\ell(-y_4) \in k_2F$. Therefore, (3) follows by applying the homomorphism g_F: $k_2F \longrightarrow B(F)$.
(3) $\Longrightarrow$ (1). Let $K = F(\sqrt{-x_3})$. Then, by (3), the K-algebra $\left(\frac{-x_1,-y_1}{K}\right)\otimes\left(\frac{-x_2,-y_2}{K}\right)\otimes\left(\frac{-x_4,-y_4}{K}\right)$ splits. By Theorem 1.1, this implies that, over K, $\varphi_1' \perp \langle x_3\rangle\varphi_2' \cong \varphi_1' \perp \langle -1\rangle\varphi_2'$ is isotropic. Therefore, by Lemma 2.1, there exists an F-isometry $\varphi_1 \perp \langle x_3\rangle\varphi_2 \cong \langle\langle x_3,z\rangle\rangle \perp \langle b\rangle\langle\langle c,d\rangle\rangle$, where $z,b,c,d \in \dot{F}$. We have then

$$\begin{aligned}\sigma &= \langle b\rangle\langle\langle c,d\rangle\rangle + \langle\langle x_3\rangle\rangle(\langle\langle z\rangle\rangle - \langle\langle y_3\rangle\rangle) + \langle y_3\rangle\langle\langle x_4,y_4\rangle\rangle \\ &= \langle b\rangle\langle\langle c,d\rangle\rangle + \langle y_3\rangle(\langle\langle x_4,y_4\rangle\rangle - \langle\langle x_3,-y_3z\rangle\rangle) \in W(F).\end{aligned}$$

Applying g_F and using (3), we get $\left(\frac{-c,-d}{F}\right)\otimes\left(\frac{-x_4,-y_4}{F}\right)\otimes\left(\frac{-x_3,y_3z}{F}\right) = 1 \in B(F)$. Therefore, again by 1.1, we can write $\langle y_3\rangle(\langle\langle x_4,y_4\rangle\rangle - \langle\langle x_3,-y_3z\rangle\rangle)$ as $\langle b'\rangle\langle\langle c',d'\rangle\rangle$, where b',

$c', d' \in \dot{F}$. Repeating the same argument, we have $\left(\frac{-c,-d}{F}\right) \cong \left(\frac{-c',-d'}{F}\right)$, $\langle\langle c,d\rangle\rangle \cong \langle\langle c',d'\rangle\rangle$, so $\sigma = \langle b,b'\rangle\langle\langle c,d\rangle\rangle = \langle b\rangle\cdot\beta \in W(F)$, where $\beta = \langle\langle bb',c,d\rangle\rangle$. Q.E.D.

Theorem 2.3. *Let* $\varphi_i = \langle\langle x_i,y_i\rangle\rangle$, $1\leqslant i\leqslant 5$, *and assume that* $\prod_{i=1}^{5}\left(\frac{-x_i,-y_i}{F}\right) = 1 \in B(F)$. *Then, there exists an equation*

$$(2.4)\qquad \varphi_1 \perp \langle x_3\rangle\varphi_2 \perp \langle -1\rangle\varphi_3 \perp \langle y_3\rangle\varphi_4 \perp \langle -b\rangle\varphi_5 = \langle\langle x_5\rangle\rangle\mu + q$$

in $W(F)$, *where* $b \in \dot{F}$, $q \in I^2F$, $\dim \mu =$ *even* $\leqslant 4$ *and* $\dim q = 8$.

Proof. Let $\sigma = \varphi_1 \perp \langle x_3\rangle\varphi_2 \perp \langle -1\rangle\varphi_3 \perp \langle y_3\rangle\varphi_4$ (as in 2.2), and let $L = F(\sqrt{-x_5})$. We have $\prod_{i=1}^{4}\left(\frac{-x_i,-y_i}{L}\right) = 1 \in B(L)$, so, by 2.2, $\sigma_L = \langle b\rangle\cdot\beta$ where $b \in \dot{L}$, and β is a 3-fold Pfister form over L. Observe that $\dim \sigma_L = 16$, and $\dim \beta = 8$. By [11,P.52], we may then write $\sigma = \langle\langle x_5\rangle\rangle\gamma + q \in W(F)$, where γ, q are forms over F, with $\dim q = 8$, $\dim \gamma \leqslant 4$. We may assume that γ is even-dimensional. [Indeed, suppose not (in particular $\dim \gamma \leqslant 3$). Write $q \cong \langle a\rangle \perp q_1$, $\dim q_1 = 7$. Then, in $W(F)$,

$$\begin{aligned}\sigma &= \langle\langle x_5\rangle\rangle\gamma + \langle a,ax_5\rangle + (\langle -ax_5\rangle + q_1)\\ &= \langle\langle x_5\rangle\rangle\bar{\gamma} + \bar{q}\ ,\end{aligned}$$

where $\bar{\gamma} = \gamma \perp \langle a\rangle$, $\dim \bar{\gamma} \leqslant 4$, and $\bar{q} = \langle -ax_5\rangle \perp q_1$, $\dim \bar{q} = 8$]. Write $\gamma \cong \langle b\rangle \perp \gamma_1$, $\dim \gamma_1 =$ odd, $b \in \dot{F}$. Then, in $W(F)$,

$$\begin{aligned}\sigma - \langle b\rangle\varphi_5 &= \langle\langle x_5\rangle\rangle(\langle b\rangle \perp \gamma_1) + q - \langle b\rangle\langle\langle x_5,y_5\rangle\rangle\\ &= \langle\langle x_5\rangle\rangle\mu + q\ ,\end{aligned}$$

where $\mu = \langle -by_5\rangle \perp \gamma_1$ has even dimension $\leqslant 4$. Since σ, φ_5 and $\langle\langle x_5\rangle\rangle\mu$ all belong to I^2F, it follows that $q \in I^2F$. Q.E.D.

Lemma 2.5. *Suppose a* 2n-*dimensional form* η *lies in* I^2F. *Then,* *there exist 2-fold Pfister forms* $\eta_1,\cdots,\eta_{n-1}$ *and scalars* $a_1,\cdots,a_{n-1}\in\dot{F}$ *such that* $\eta = \sum_{i=1}^{n-1} \langle a_i\rangle\eta_i \in W(F)$.

Proof. Induction on n. We may assume $n \geq 2$, since the case $n = 1$ is trivial. Write $\eta = \langle a,b,c\rangle \perp \tau$, where $a,b,c\in\dot{F}$ and $\dim \tau = 2n-3$. Then $\eta = \eta\perp H \cong \langle a,b,c,abc\rangle \perp (\langle -abc\rangle \perp \tau)$ in $W(F)$. Since $\langle a,b,c,abc\rangle \cong \langle a\rangle\langle\langle ab,ac\rangle\rangle$, and $\dim (\langle -abc\rangle\perp\tau) = 2(n-1)$, the induction proceeds. Q.E.D.

Theorem 2.6. *If* α *is a sum of five generators in* k_2F, *and* $g_F(\alpha) = 1 \in B(F)$, *then* $\alpha = 0$. *In particular, if every element in* k_2F *is a sum of at most five generators, then* g_F *is injective*.

Proof. Write $\alpha = \sum_{i=1}^{5} \ell(-x_i)\ell(-y_i)$, $\varphi_i = \langle\langle x_i,y_i\rangle\rangle$, $1\leq i\leq 5$. Then, we can apply the conclusion of Theorem 2.3. The 8-dimensional form q there can be written as $\sum_{i=1}^{3} \langle a_i\rangle\langle\langle b_i,c_i\rangle\rangle$, according to Lemma 2.5. Reading the equation (2.4) in $k_2F \cong I^2F/I^3F$, we see that $\alpha = \ell(-x_5)\ell(z) + \sum_{i=1}^{3}\ell(-b_i)\ell(-c_i)\in k_2F$ for a suitable $z\in\dot{F}$. Since α is now a sum of just *four* generators, the desired conclusion follows from Theorem 2.2. Q.E.D.

Corollary 2.7. *If* $|k_2F| \leq 2^{10}$, *then* g_F *is injective*.

Proof. Every element of k_2F is a sum of 5 generators, by [5,Corollary 5.7]. Q.E.D.

Theorem 2.6 also includes the following result of Pfister:

Corollary 2.8.(= [10,Satz 14]). *Let* q *be a form of dimension* ≤ 12 *such that* $q\in I^2F$, *and* q *has Witt invariant* $c(q) = 1\in B(F)$. *Then* $q\in I^3F$.

Proof. Let α be the element in k_2F which corresponds to q under the identification $I^2F/I^3F \cong k_2F$. By Lemma 2.5, α is a sum of five generators in k_2F. Since $g_F(\alpha) = c(q) = 1$, Theorem 2.6 applies. Q.E.D.

For non-real fields F, <u>let</u> u(F) <u>denote the maximum dimension of anisotropic</u> (<u>quadratic</u>) <u>forms over</u> F. The above Corollary, therefore, implies that g_F is injective for any non-real field F with $u(F) \leqslant 12$. Explicit examples are: fields F such that $\text{tr.d.}_{\mathbb{C}} F \leqslant 3$, or $\text{tr.d.}_{\mathbb{F}_q} F \leqslant 2$ (both are C_3-fields). We note also that Theorem 2.6 applies to fields like $F = \mathbb{Q}_p((t_1))((t_2))$ —— every $\alpha \in k_2F$ is a sum of at most 4 generators. For more examples, we record:

<u>Proposition</u> 2.9. <u>Suppose</u> $F(\sqrt{a})$ <u>is a non-real field such that</u> $u(F(\sqrt{a})) \leqslant 8$. <u>Then</u>, g_F <u>is injective</u>. (<u>This applies</u>,<u>for instance</u>,<u>to any field</u> F <u>with</u> $\text{tr.d.}_{\mathbb{R}} F \leqslant 3$, <u>on taking</u> $a = -1$).

<u>Proof</u>. We claim that any anisotropic form $\varphi \in I^2F$ can be expressed as

$$(2.10) \qquad \varphi = \sum_{i=1}^{m} \langle x_i\rangle\langle\langle -a, y_i\rangle\rangle \perp \mu \in W(F)$$

where $m \geqslant 0$, and μ is some form (clearly in I^2F) of dimension $\leqslant 8$. By Lemma 2.5, this implies that any element in k_2F is a sum of four generators, and hence Theorem 2.2 applies. Since $u(F(\sqrt{a})) \leqslant 8$, we have an isometry $\varphi \cong \langle\langle -a\rangle\rangle\tau \perp \mu$ with dim $\mu \leqslant 8$, by repeated applications of [11,P.52]. We may assume, as in the proof of Theorem 2.3,that dim τ = even. This proves (2.10). Q.E.D.

<u>Proposition</u> 2.11. <u>Suppose</u> F <u>is a non-real field such that</u> $u(F) \leqslant 8$. <u>Then</u> $g_{F(\sqrt{a})}$ <u>is injective for all</u> $a \in \dot{F}$.

Proof. By [7,Theorem 4.3], $u(F(\sqrt{a})) \leqslant \frac{3}{2}\cdot u(F) \leqslant 12$. Therefore, the result follows from Corollary 2.8.

3. *Quadratic extensions*

In this Section, we study the behaviour of the ideals I^n under a quadratic extension $K = F(\sqrt{a}) \supset F$. Let r^* denote the functorial map $W(F) \longrightarrow W(K)$, and let s_* denote the transfer map $W(K) \longrightarrow W(F)$ induced by the F-linear functional $s: K \longrightarrow F$ where $s(1) = 0$, $s(\sqrt{a}) = 1$. We record the following two known facts:

Proposition 3.1 (see [8,P.201]) *If* q *is an anisotropic form over* F, *then* $r^*(q)$ *is hyperbolic over* K *iff* $q \cong \langle\langle -a\rangle\rangle\cdot q_1$ *for some form* q_1 *over* F. *If* γ *is any form over* K, *then* $s_*(\gamma)$ *is hyperbolic over* F *iff* $\gamma \cong r^*(q)$ *for some form* q *over* F. *In particular, the following sequence is exact*:

$$0 \longrightarrow \langle\langle -a\rangle\rangle\cdot W(F) \longrightarrow W(F) \xrightarrow{r^*} W(K) \xrightarrow{s_*} W(F).$$

Theorem 3.2. (special case of [4,Theorem A2.9]) *For any* $n \geqslant 1$, $s_*(I^nK) \subset I^nF$.

Putting together these results, we shall prove

Theorem 3.3. *For any* $n \geqslant 1$, *we have a zero sequence*

$$0 \longrightarrow \langle\langle -a\rangle\rangle\cdot I^{n-1}F \longrightarrow I^nF \xrightarrow{r^*} I^nK \xrightarrow{s_*} I^nF.$$

For $n = 1,2$, *this sequence is exact* ($I^0F = W(F)$ *by definition*). *For* $n = 3$, *it is exact except possibly at* I^3K.

Proof. The zero sequence is clear from 3.1 and 3.2 above. For $n = 1$, the exactness follows trivially from 3.1. Suppose $n = 2$, and, say, q is an anisotropic form in I^2F, $r^*(q) = 0$. By 3.1, $q \cong \langle\langle -a\rangle\rangle \cdot q_1$ for some form q_1 over F. If dim q_1 is odd, then det q = det $\langle\langle -a\rangle\rangle \cdot q_1 = -a$, contradicting $q \in I^2F$. Consequently, dim q_1 is even, and $q \in \langle\langle -a\rangle\rangle \cdot IF$. Next, suppose $\gamma \in I^2K$ and $s_*(\gamma) = 0$. By 3.1, $\gamma \cong \langle a_1, \cdots, a_{2m}\rangle$ for suitable $a_j \in \dot{F}$. Since det $\gamma = (-1)^m$ over K, we must have $(-1)^m a_1 \cdots a_{2m} = 1$ or a, up to square classes in $\dot{F}$. In the first case, clearly $\gamma \in r^*(I^2F)$. In the second case, $\gamma \cong r^*(\langle aa_1, a_2, \cdots, a_{2m}\rangle) \in r^*(I^2F)$. Suppose now $n = 3$, and q is an anisotropic form in I^3F, $r^*(q) = 0$. Then, $q \cong \langle\langle -a\rangle\rangle \cdot q_1$ where dim $q_1 = 2m$ for some m. Write $q_1 = \langle\langle (-1)^m d\rangle\rangle + q_2$ in W(F), where d = det q_1 and $q_2 \in I^2F$. Then $q = \langle\langle -a, (-1)^m d\rangle\rangle + \langle\langle -a\rangle\rangle \cdot q_2 \in I^3F$ implies that $\langle\langle -a, (-1)^m d\rangle\rangle \cong 2\mathbb{H}$, by the Hauptsatz of [2]. Now we have $q = \langle\langle -a\rangle\rangle \cdot q_2 \in \langle\langle -a\rangle\rangle \cdot I^2F$. Q.E.D.

Proposition 3.4. *If $\gamma \in I^3K$ is 8-dimensional and $s_*(\gamma) = 0$, then there exists $q \in I^3F$ such that $r^*(q) = \gamma$. (In particular, if K is non-real and $u(K) \leq 8$, then the sequence in 3.3 is exact also for $n = 3$).*

Proof. By the proof of 3.3, there exists an 8-dimensional form $q_1 \in I^2F$ such that $r^*(q_1) \cong \gamma$. According to Lemma 2.5, we may write $q_1 = \sum_{i=1}^{3} \langle x_i\rangle\langle\langle a_i, b_i\rangle\rangle$, $x_i, a_i, b_i \in \dot{F}$, $1 \leq i \leq 3$. Let $q_2 = \langle\langle a_1, b_1\rangle\rangle \perp \langle -a\rangle\langle\langle a_2, b_2\rangle\rangle \perp \langle e\rangle\langle\langle a_3, b_3\rangle\rangle$, where $e \in \dot{F}$ is to be specified. Since $q_2 \equiv q_1 \pmod{I^3F}$, we have $r^*(q_2) \equiv r^*(q_1) \equiv 0 \pmod{I^3K}$. Therefore, the form $\langle\langle a_1, b_1\rangle\rangle' \perp \langle -a\rangle\langle\langle a_2, b_2\rangle\rangle'$ must become isotropic over K, by Theorem 1.1. Using Lemma 2.1, we may write $q_2 \cong \langle\langle -a, z_1\rangle\rangle \perp \langle b\rangle\langle\langle c, d\rangle\rangle \perp \langle e\rangle\langle\langle a_3, b_3\rangle\rangle$, where $z_1, b, c, d \in \dot{F}$. Let $e = -ab$. Then, as before, $\langle\langle c, d\rangle\rangle' \perp \langle -a\rangle\langle\langle a_3, b_3\rangle\rangle'$ becomes isotropic over K. Consequently,

$$q_2 \cong \langle\langle -a, z_1\rangle\rangle \perp \langle b\rangle\langle\langle -a, z_2\rangle\rangle \perp \langle t\rangle\langle\langle u,v\rangle\rangle \quad (z_2, t, u, v \in \dot{F}).$$

This gives $r^*(q_2) = \langle t\rangle\langle\langle u,v\rangle\rangle$ in $W(K)$. But $r^*(q_2) \in I^3K$, so $\langle t\rangle\langle\langle u,v\rangle\rangle = 0 \in W(K)$, by [2]. In particular, $r^*(q_2) = 0$. Setting $q = q_1 - q_2 \in I^3F$, we then have $r^*(q) = r^*(q_1) = \gamma$, as required. Q.E.D.

Corollary 3.5. _If_ $I^3F = 0$, _then_ $I^3K = 0$.

Proof. If γ is any 3-fold Pfister form over K, then, by Theorem 3.2, $s_*(\gamma) \in I^3F = 0$. The Proposition above implies that $\gamma = 0 \in W(K)$. Q.E.D.

Remark 3.6. Corollary 3.5 is peculiar to _quadratic_ extensions. In fact, take two fields $F \subset F(\alpha)$ where F is quadratically closed but $F(\alpha)$ is _not_ quadratically closed. Let $E = F((t_1))((t_2))$ and $L = F(\alpha)((t_1))((t_2)) = E(\alpha)$. _Then_, $I^3E = 0$, _but_ $I^3L \neq 0$.

Proposition 3.7. _The following are equivalent_:

(1) $\ell(a)\cdot k_1F \longrightarrow k_2F \xrightarrow{r^*} k_2K$ _is exact_.

(2) $I^3F \xrightarrow{r^*} I^3K \xrightarrow{s_*} I^3F$ _is exact_.

(_If either condition holds, we shall say that the quadratic extension_ $K = F(\sqrt{a}) \supset F$ _is exact_).

Proof. (1)$\Longrightarrow$(2). Suppose $s_*(\gamma) = 0$ where $\gamma \in I^3K$. Then there exists $q \in I^2F$ such that $r^*(q) = \gamma$, by 3.3. Identifying I^2/I^3 with k_2 after Milnor, (1) implies that $q \in \langle\langle -a\rangle\rangle\cdot IF + I^3F$. Therefore, $\gamma = r^*(q) \in r^*(I^3F)$.

(2)$\Longrightarrow$(1). Suppose $\alpha \in k_2F$ and $r^*(\alpha) = 0$. Let $q \in I^2F$ be such that its class in I^2F/I^3F corresponds to α. Then $r^*(q) \in I^3K$. Since $s_*r^*(q) = 0$, (2) implies that $r^*(q) = r^*(q_1)$ where

$q_1 \in I^3F$. Thus, $r^*(q - q_1) = 0$, and so $q - q_1 \in \langle\langle -a\rangle\rangle \cdot IF$ by Theorem 3.3. Going back to k_2, we get $\alpha \in \ell(a)\cdot k_1F$, since $q_1 \in I^3F$. Q.E.D.

Our interest in the notion of 'exactness' stems from the following properties:

<u>Proposition</u> 3.8. (1) <u>If</u> g_F <u>is injective, then any quadratic extension</u> $K = F(\sqrt{a}) \supset F$ <u>is exact</u>. (2) <u>Suppose</u> $K=F(\sqrt{a}) \supset F$ <u>is exact. Then,</u> g_K <u>injective</u> $\Longrightarrow g_F$ <u>injective</u>. (3) <u>If all quadratic extensions of all fields are exact, then</u> g_F <u>is injective for all fields</u> F.

<u>Proof</u>. (1) For $\alpha = \sum_{i=1}^{n} \ell(a_i)\ell(b_i) \in k_2F$, consider the F-algebra $A = \bigotimes_{i=1}^{n} \left(\frac{a_i, b_i}{F}\right)$. By the Wedderburn theorems, $A \cong M_m(D)$ for some integer m and some F-central division algebra D. Suppose $r^*(\alpha) = 0 \in k_2K$. Then, D splits over K. This implies that $\dim_F D$ divides $[K:F]^2 = 4$ (see, for instance, [12,Corollaire 2 of Théorème 10]). Therefore, either $D \cong F$, or $D \cong \left(\frac{a,b}{F}\right)$ for some $b \in \dot{F}$. If $D \cong F$, we have $g_F(\alpha) = 1$. If $D \cong \left(\frac{a,b}{F}\right)$, we have $g_F(\alpha) = g_F(\ell(a)\ell(b))$. Since g_F is injective by hypothesis, we conclude, in either case, that $\alpha \in \ell(a)\cdot k_1F$.

(2) Take $\alpha \in \ker(g_F)$. Then $r^*(\alpha) \in \ker(g_K) = 0$. Since $K \supset F$ is exact, $\alpha = \ell(a)\ell(b)$ for some $b \in \dot{F}$. But then clearly $\alpha = 0$ in k_2F.

(3) Suppose $\alpha \in \ker(g_F)$, where α is a sum of n generators in k_2F. We shall show, by induction on n (for all fields F) that $\alpha = 0 \in k_2F$. The case $n = 1$ is trivial, so we proceed to

any $n \geq 2$. Write $\alpha = \ell(a)\ell(b) + \alpha' \in k_2F$, where α' is a sum of n-1 generators. We may assume that $K = F(\sqrt{a})$ is a quadratic extension of F. Since $g_K(r^*(\alpha')) = 1$, our inductive hypothesis implies that $r^*(\alpha') = 0 \in k_2K$. But $K \supset F$ is exact (by hypothesis), so $\alpha' = \ell(a)\ell(c)$ for some $c \in \dot{F}$. We now have $\alpha = \ell(a)\ell(bc)$, and clearly $g_F(\alpha) = 1 \Longrightarrow \alpha = 0$. Q.E.D.

Corollary 3.9. If every element of k_2F is a sum of five generators, then any quadratic extension $K = F(\sqrt{a}) \supset F$ is exact.

Proof. Under the given hypothesis, we know that g_F is indeed injective, by Theorem 2.6. Thus, the desired conclusion follows from part (1) of the Proposition. Q.E.D.

Theorem 3.10. (1) If $I^3F = 0$, then g_F is injective.
(2). For $K = F(\sqrt{a})$, if $I^3K = 0$, then g_F is injective.

Proof. (1) By 3.5 and 3.7(2), all quadratic extensions $K \supset F$ are exact, and share the common property that $I^3K = 0$. Thus, (1) follows by repeating the same inductive proof in 3.8(3), for the class of fields with $I^3F = 0$. After proving (1), (2) follows from 3.8(2). Q.E.D.

Theorem 3.11. $I^3F = 0$ iff quadratic forms over F are completely classified by dimension, discriminant, and the Hasse invariant.

(The Hasse invariant of a quadratic form $\langle a_1, \cdots, a_n \rangle$ is defined to be the algebra class $\bigotimes_{i<j} \left(\frac{a_i, a_j}{F}\right)$ in the Brauer group B(F)).

Proof. By [6,Theorem 2.15], dimension and Milnor's total Stiefel-Whitney class w classify quadratic forms over F iff I^3F is torsion-free. Assume that $I^3F = 0$. Then, dimension, w_1 and w_2 classify quadratic forms (since $w_i \equiv 0$ for $i \geq 3$). By 3.10(1), w_2 is equivalent to the Hasse invariant. This proves the 'only if' part of the theorem. The 'if' part is trivial and well-known. Q.E.D.

Corollary 3.12. *If dimension, discriminant, and the Hasse invariant classify quadratic forms over* F, *then they also classify quadratic forms over any quadratic extension* $K \supset F$.

Proof. Clear from 3.5 and 3.11.

Remark 3.13. By 3.6, we see that the last corollary is peculiar to *quadratic* extensions. We also note the following example. Let $F = \mathbb{R}((t_1))\cdots((t_n))$, $K = \mathbb{C}((t_1))\cdots((t_n)) = F(\sqrt{-1})$. Then, F is pythagorean; and, in particular, dimension and w classify quadratic forms over F. However, if $n \geq 3$, $I^3K \neq 0$ and $W(K)$ is torsion, so dimension and w *do not* suffice to classify quadratic forms over K !

4. Necessary conditions for $\alpha \in \ker(g_F)$

In this Section, we shall provide further sufficient conditions for the map g_F to be injective. The main result is as follows.

Theorem 4.1. *Suppose* $\alpha = \sum_{i=1}^{r} \ell(a_i)\ell(b_i) \in \ker(g_F)$. *Then*,

(1) $2^{r-1}. \sum_{i=1}^{r} \langle\langle -a_i, -b_i \rangle\rangle \in I^{r+2}F$.

(2) $\ell(-1)^{t-2}.\alpha = 0 \in k_tF$, *where* $t = 2^r$.

Corollary 4.2. g_F is injective if either of the following holds:
(A) In $W(F)$, $2x \in I^{n+1}F \Rightarrow x \in I^nF$ whenever $n \geq 3$.
(B) For all $m \geq 1$, $\ell(-1)^m$: $k_2F \longrightarrow k_{m+2}F$ is injective.

The main work in this section will be to establish 4.1(1). This part (1) implies part (2), by the following argument with Stiefel-Whitney classes (see [9]). Lifting (1) to the Witt-Grothendieck ring $\hat{W}(F)$, we have

$$\sum_{i=1}^{r} (<-1> - <1>)^{r-1}\cdot(<a_i> - 1)(<b_i> - 1) \in \hat{I}^{r+2}F,$$

where $\hat{I}F$ denotes the augmentation ideal of $\hat{W}(F)$. Applying the $t^{\underline{th}}$ Stiefel-Whitney class, $t = 2^r$, we obtain, according to [9,Corollary 3.2], the equation $\sum_{i=1}^{r} \ell(-1)^{t-2}\ell(a_i)\ell(b_i)=0 \in k_tF$. This is precisely 4.1(2).

The proof of 4.1(1) will be based on the construction of a 'trace form' on an arbitrary central simple algebra. For any F-central simple algebra A, let T_{rd}: $A \longrightarrow F$ denote the reduced trace on A (see [3, 12,No.3]). We define the trace form on A to be the pairing $(a,b) \longmapsto T_{rd}(ab)$, which is easily seen to be symmetric, bilinear, and non-degenerate. We shall denote this pairing by $< , >_A$.

Lemma 4.3. If A, B are F-central simple algebras, then $< , >_{A\otimes B}$ is isometric to $< , >_A \otimes < , >_B$.

This follows easily by working over a common splitting field for A, B, and observing that, for square matrices X, Y, one has $tr(X\otimes Y) = tr(X)\cdot tr(Y)$.

Since we assume that F has characteristic not 2, the symmetric bilinear form $< , >_A$ may be identified with its associated quadratic form $x \longmapsto \langle x,x\rangle_A$. We shall need the explicit calculation of this quadratic form in two important cases, as follows.

Lemma 4.4. (1) For $A = \left(\frac{a,b}{F}\right)$, $\langle\ ,\ \rangle_A \cong \langle 2\rangle\langle 1,a,b,-ab\rangle$ $= \langle 2\rangle\cdot(\ 2 - \langle\langle -a,-b\rangle\rangle\ ) \in W(F)$.
(2) For $A = M_n(F)$, $\langle\ ,\ \rangle_A \cong n\langle 1\rangle \perp \frac{n(n-1)}{2}\cdot\mathbb{H} = n\langle 1\rangle \in W(F)$.

The proofs are straightforward, and will be left to the reader.

We are now ready to prove 4.1(1). By hypothesis, there exists an F-algebra isomorphism $\bigotimes_{i=1}^{r} \left(\frac{a_i,b_i}{F}\right) \cong M_n(F)$, for some n. By a simple dimension count, we have $4^r = n^2$, hence $n = 2^r$. Using the two preceding lemmas, we obtain an equation:

$$\prod_{i=1}^{r} (2 - \langle\langle -a_i,-b_i\rangle\rangle) = \langle 2\rangle^r\cdot 2^r\langle 1\rangle \in W(F).$$

The RHS is just $2^r\langle 1\rangle$ since $\langle 2\rangle\cdot\langle 1,1\rangle \cong \langle 1,1\rangle$. Therefore, in expanding this product, the first term $2^r\langle 1\rangle$ cancels. The next term is $\pm\ 2^{r-1}\cdot \sum_{i=1}^{r} \langle\langle -a_i,-b_i\rangle\rangle$. If we multiply s factors of the form $\langle\langle -a_i,-b_i\rangle\rangle$ and r-s factors of 2, the resulting form lies in $(I^2F)^s\cdot(IF)^{r-s} = I^{r+s}F$. Thus,

$$2^{r-1}\cdot \sum_{i=1}^{r} \langle\langle -a_i,-b_i\rangle\rangle = (\ \pm \text{ terms with } s \geq 2) \in I^{r+2}F.$$

Q.E.D.

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On The Torsion in K_2 of Local Fields

Joseph E. Carroll

For all that follows let us assume that F is a local field with finite residue field of order q. Moore has proved (c.f. Milnor, Introduction to Algebraic K-theory, p. 175) that K_2F is the direct product of a cyclic group whose order is the same as the order of the group of roots of 1 in F, and a divisible group which is the kernel of the Hilbert symbol on F. John Tate has raised the question (c.f. Proceedings of the International Congress of Mathematicians, 1970, Vol. 1, p. 203) of whether or not the divisible group is torsion free.

Let π be a fixed prime of F. In this paper we prove that the map from the group of roots of 1 of order prime to q to the torsion in K_2F of order prime to q given by $\eta \mapsto \{\eta, \pi\}$ is an isomorphism onto. As an easy corollary, we prove that the tame kernel in K_2F, which contains the kernel of the Hilbert symbol, has no non-trivial m-torsion for $(m,q) = 1$.

I would like to thank Professor John Tate for making many suggestions for smoothing out my proofs.

<u>Theorem 1</u>: Let η be a q-1 root of 1 in F. Let $x \in F^*$ and suppose $\langle \eta, x \rangle_F = 1$, where $\langle\ ,\ \rangle_F$ denotes the tame symbol on F. Then $\{\eta, x\} = 1$ in K_2F.

<u>Proof</u>: Let U_1 denote the group of units in F congruent to 1 (mod π). Write $x = \pi^n \zeta u$ where $u \in U_1$, $\zeta^{q-1} = 1$

$$\{\eta, x\} = \{\eta, \pi^n\}\{\eta, \zeta\}\{\eta, u\}$$

But u has a q-1 root in U_1 and $1 = \langle \eta, x \rangle_F = \eta^n$ so,

$$\{\eta, x\} = \{\eta^n, \pi\}\{\eta, \zeta\}\{\eta^{q-1}, u^{1/q-1}\} = \{\eta, \zeta\}$$

So we must show that $\{\eta, \zeta\} = 1$ in K_2F. To this end we prove:

<u>Lemma 1</u>: Let E be any field and m a positive integer such that E contains μ_m, the m^{th} roots of 1. Let A be the subgroup of K_2E generated by elements of the form $\{\eta_1, \eta_2\}$ where $\eta_1, \eta_2 \in \mu_m$. Then if m is odd or $4|m$, $A = 0$. Otherwise A is generated by $\{-1, -1\}$.

<u>Proof</u>: Let $m = 2^t s$ where s is odd and let η generate μ_m. If $\eta_1, \eta_2 \in \mu_m$ we can write $\eta_1 = \eta^j$, $\eta_2 = \eta^k$.
$\{\eta_1, \eta_2\} = \{\eta^j, \eta^k\} = \{\eta, \eta\}^{jk}$, so $\{\eta, \eta\}$ generates A.
$\{\eta, \eta\} = \{\eta, -1\} = \{\eta, (-1)^s\} = \{\eta^s, -1\}$ and

If $t = 0$, then $\{\eta^s, -1\} = \{\eta^m, -1\} = 1$

If $t = 1$, then $\{\eta^s,-1\} = \{-1,-1\}$

If $t \geq 2$, then $\{\eta^s,-1\} = \{\eta^s,(\eta^s)^{2^{t-1}}\} = \{\eta^s,\eta^s\}^{2^{t-1}} = \{\eta^s,-1\}^{2^{t-1}} = 1$

We apply Lemma 1 to F where $m = q - 1$. The only difficulty in deducing Theorem 1 arises when $2|q-1$ and $4\nmid q-1$. Suppose this is the case. If F is a local function field over a finite field k, $\{-1,-1\} = 1$ in K_2k (since $K_2k = 0$) so $\{-1,-1\} = 1$ in K_2F. If F is a local number field, then we may assume $F \supset \mathbb{Q}_p$ where $p \equiv 3 (\mathrm{mod}\ 4)$. Therefore, to finish off Theorem 1, the following lemma, which was proved by Alan Waterman, suffices:

<u>Lemma 2 (Waterman)</u>: If $p \equiv 3\ (\mathrm{mod}\ 4)$, then $\{-1,-1\} = 1$ in $K_2\mathbb{Q}_p$.

<u>Proof</u>: First we mimic the proof that $K_2\mathbb{F}_p = 0$. Since the norm map $\mathbb{F}_p(\sqrt{-1}) \to \mathbb{F}_p$ is surjective, we can find $x,y \in \mathbb{Z} - \{0\}$ such that

$$x^2 + y^2 \equiv -1 (\mathrm{mod}\ p)$$

Let ζ be a $p - 1$ root of 1 in $\mathbb{Q}_p$ such that $\zeta \equiv x\ (\mathrm{mod}\ p)$. Let $\gamma \in \mathbb{Q}_p$ such that $\gamma^2 = -1-\zeta^2$ (by Hensel's Lemma there is such a γ). Then $-\zeta^2 - \gamma^2 = 1$, so $\{-\zeta^2,-\gamma^2\} = 1$ in $K_2\mathbb{Q}_p$. So, $1 = \{-\zeta^2,-\gamma^2\}^{p-1/2} = \{(-\zeta^2)^{p-1/2},-\gamma^2\} = \{-1,-\gamma^2\}$ since $\frac{p-1}{2}$ is odd. But $\{-1,\gamma^2\} = 1$, so $\{-1,-1\} = \{-1,-\gamma^2\}/\{-1,\gamma^2\} = 1$ in $K_2\mathbb{Q}_p$.

Now let us fix some more notation.

Let ℓ be a fixed prime number with $(\ell,q) = 1$.

Let U be the group of units of F.

Let C be the group of roots of 1 of ℓ-power order in F.

Let V be the product of U_1 and the group of roots of 1 in F whose order is prime to ℓ.

If A is any abelian group and m is any positive integer, let A_m be the kernel of the m^{th} power map $A \overset{m}{\to} A$.

Let $A(\ell) = \bigcup_{n=0}^{\infty} A_{\ell^n}$, the ℓ-primary part of A.

<u>Remark 1</u>: We have $F^* = \pi^{\mathbb{Z}}CV \approx \mathbb{Z} \times C \times V$. V is uniquely divisible by ℓ. Since $CV = U$, if $x \in F^*$, then x, $1 - x$, or $1 - x^{-1} \in CV$.

<u>Lemma 3</u>: Let $b \in C$, $w \in V$. Then $\{1 - bw^{\ell},w\} = 1$ in K_2F.

<u>Proof</u>: We divide the proof into three cases:

Case (i), $C \neq 0$ and b does not generate C:

Let $c \in C$ such that $c^{\ell} = b$. Let ζ be a primitive ℓ^{th} root of 1 in C. Then

$$\{1 - bw^{\ell},w\} = \{1 - c^{\ell}w^{\ell},w\} = \{\prod_{i=0}^{\ell-1}(1 - \zeta^{i}cw),w\}$$

$$= \prod_{i=0}^{\ell-1}\{1 - \zeta^{i}cw,w\} = \prod_{i=0}^{\ell-1}\{\zeta^{i}c,1 - \zeta^{i}cw\}$$

This element is easily seen to be of the form $\{a,x\}$ with $a \in C$, $x \in F^*$, so to show that it is trivial, it suffices, by

Theorem 1, to show that its tame symbol is 1. But $\langle 1 - bw^{\ell}, w\rangle_F = 1$, because $w \in U$, and if $1 - bw^{\ell} \notin U$, then $w \in U_1$.

Case (ii), $C \neq 0$ and b generates C:

Consider the extension field $F(c)$ where $c^{\ell} = b$. Let $Tr_{F(c)/F}$ denote the transfer homomorphism $K_2F(c) \to K_2F$. Then

$$\{1 - bw^{\ell}, w\} = \{N_{F(c)/F}(1 - cw), w\} = Tr_{F(c)/F}(\{1 - cw, w\})$$
$$= Tr_{F(c)/F}(\{c, 1 - cw\})$$

It is, then, enough to show that $\{c, 1 - cw\} = 1$ in $K_2F(c)$ and as in case (i) we need only show that $\langle 1 - cw, w\rangle_{F(c)} = 1$, and the reasoning is the same as in case (i).

Case (iii), $C = 0$, and so $b = 1$:

Consider the extension field $F(\zeta)$ where ζ is a primitive ℓ^{th} root of 1.

In $K_2F(\zeta)$, $\{1 - w^{\ell}, w\} = 1$ by case (i). Therefore, in K_2F we have:

$$\{1 - w^{\ell}, w\}^{[F(\zeta):F]} = Tr_{F(\zeta)/F}(\{1 - w^{\ell}, w\}) = 1$$

But also, $\{1 - w^{\ell}, w\}^{\ell} = \{1 - w^{\ell}, w^{\ell}\} = 1$ in K_2F and $([F(\zeta):F], \ell) = 1$. Therefore, $\{1 - w^{\ell}, w\} = 1$ and this completes the proof of Lemma 3.

<u>Theorem 2</u>: Let M be the subgroup of K_2F consisting of all elements of the form $\{a,x\}$ where $a \in C$ and $x \in F^*$. Then $(K_2F)(\ell) = M$.

<u>Proof</u>: First of all we wish to construct an endomorphism, β, of K_2F which is close to being an inverse of the ℓ^{th}-power map. We treat the case of $\ell = 2$ slightly differently from that of ℓ odd. If $x \in F^*$ we can, by Remark 1, write uniquely

$$x = \pi^m av \qquad a \in C,\ v \in V$$

Define $B: F^* \times F^* \to K_2F$ by

$$B(\pi^m av, \pi^n bw) = \{\pi, (-1)^{mn}(w^m/v^n)^{1/\ell}\}\{v, w^{1/\ell}\} \text{ if } \ell \text{ is odd}$$

$$B(\pi^m av, \pi^n bw) = \{\pi, (w^m/v^n)^{1/2}\}\{v, w^{1/2}\} \text{ if } \ell = 2$$

We claim that B is a symbol. It is easy to see that B is bimultiplicative. Also $B(y,x) = (B(x,y))^{-1}$ because

$$\{w, v^{1/\ell}\} = \{w^{1/\ell}, v^{1/\ell}\}^{\ell} = \{w^{1/\ell}, v\} = \{v, w^{1/\ell}\}^{-1}$$

Since B is bimultiplicative, we have, for all $x \in F^*$

$$B(1 - x, x)\cdot B(1 - x^{-1}, x^{-1}) = B\left(\frac{1 - x}{1-x^{-1}}, x\right) = B(-x,x)$$

Thus, by Remark 1, to show that B is a symbol we need only show:

(a) $B(1 - bw, bw) = 1$ for all $b \in C$; $w \in V$

(b) $B(-x,x) = 1$ for all $x \in F^*$

Let $1 - bw = \pi^m av \quad a \in C,\ v \in V$

$$B(1 - bw, bw) = B(\pi^m av, \pi^0 bw) = \{\pi, w^{m/\ell}\}\{v, w^{1/\ell}\} = \{\pi^m v, w^{1/\ell}\}$$

Now, by Theorem 1, $\{a, w^{1/\ell}\} = 1$, so

$$B(1 - bw, bw) = \{\pi^m v, w^{1/\ell}\}\{a, w^{1/\ell}\} = \{\pi^m av, w^{1/\ell}\} = \{1 - bw, w^{1/\ell}\}$$

$$= 1 \quad \text{by Lemma 3.}$$

Let $x \in F^*$. Write $x = \pi^m av$.

Suppose, first, that ℓ is odd. Then $-1 \in V$, so we write $-x = \pi^m a(-v)$ and

$$\begin{aligned} B(-x,x) &= B(\pi^m a(-v), \pi^m av) \\ &= \{\pi, (-1)^{m^2} (v^m/(-v)^m)^{1/\ell}\}\{-v, v^{1/\ell}\} \\ &= \{\pi, (-1)^{m^2+m}\}\{-v^{1/\ell}, v^{1/\ell}\}^\ell \quad \text{since } \ell \text{ is odd} \\ &= 1 \end{aligned}$$

If $\ell = 2$, then $-1 \in C$, so we write $-x = \pi^m(-a)v$ and

$$\begin{aligned} B(-x,x) &= B(\pi^m(-a)v, \pi^m av) \\ &= \{\pi, (v^m/v^m)^{1/2}\}\{v, v^{1/2}\} \\ &= \{v^{1/2}, v^{1/2}\}^2 \\ &= \{v^{1/2}, -1\}^2 \\ &= 1 \end{aligned}$$

Thus B is a symbol, as claimed, and so induces a map β,

$$\beta\colon K_2F \longrightarrow K_2F$$

We claim that for all $\alpha \in K_2F$,

$$(\beta \circ \ell)(\alpha) \equiv \alpha (\text{mod } M)$$

Clearly it is enough to demonstrate the congruence for α an arbitrary generator of K_2F. Let $x = \pi^m av$, $y = \pi^n bw$. Suppose, first, that ℓ is odd.

$$\begin{aligned}
\beta \circ \ell(\{x,y\}) &= B(\{x,y^{\ell}\}) = B(\pi^m av, \pi^{n\ell} b^{\ell} w^{\ell}) \\
&= \{\pi, (-1)^{mn\ell}(w^{\ell m}/v^{n\ell})^{1/\ell}\}\{v,w\} \\
&= \{\pi,-1\}^{mn}\{\pi, w^m/v^n\}(v,w\} \quad \text{since } \ell \text{ is odd} \\
&= \{\pi^m,\pi^n\}\{\pi^m,w\}\{v,\pi^n\}\{v,w\} \\
&= \{\pi^m v, \pi^n w\}
\end{aligned}$$

But $\{a,\pi^n w\}\{\pi^m av,b\} \in M$ so

$$\begin{aligned}
(\beta \circ \ell)(\{x,y\}) &\equiv \{\pi^m v,\pi^n w\}\{a,\pi^n w\}\{\pi^n av,b\} \quad (\text{mod } M) \\
&= \{\pi^m av, \pi^n bw\} \\
&= \{x,y\} \text{ as claimed.}
\end{aligned}$$

If $\ell = 2$, the argument is exactly the same except that we must use the fact that $\{\pi,(-1)^{mn}\} \in M$.

In order to use all this to prove the theorem we make one more observation, namely that $M \subset \ker \beta$, for if $a,b \in C$, $w \in V$ we have

$$\beta(\{a,\pi^n bw\}) = B(\pi^0 a1,\pi^n bw) = 1$$

Now we shall finish up. Of course $(K_2F)_{\ell^0} = \{1\} \subset M$. Assume inductively that $(K_2F)_{\ell^r} \subset M$, and let $\alpha \in (K_2F)_{\ell^{r+1}}$. Then $\alpha^\ell \in (K_2F)_{\ell^r}$. Modulo M we can write

$$\alpha \equiv (\beta \circ \ell)(\alpha) = \beta(\alpha^\ell) = 1 \text{ since } \alpha^\ell \in M.$$

So $\alpha \in M$, and by mathematical induction $(K_2F)(\ell) \subset M$. But $M \subset (K_2F)(\ell)$ trivially, so $(K_2F)(\ell) = M$ and Theorem 2 is proved.

Now we shall examine M a little more closely. First, we claim that every element of M is actually of the form $\{a,\pi\}$ where $a \in C$. Let $\{b,x\} \in M$ where $b \in C$, $x \in F^*$. Write $x = \pi^n u$ with $u \in U$. Then

$$\{b,x\} = \{b,\pi^n\}\{b,u\} = \{b^n,\pi\}\{b,u\}$$

But $\{b,u\} = 1$ by Theorem 1. In fact, the proof of Theorem 1 was essentially a proof that $\{b,u\} = 1$. So

$$\{b,x\} = \{b^n,\pi\} \quad \text{and, of course,} \quad b^n \in C.$$

We have a map $\varphi: C \to M$

$$\varphi: a \longmapsto \{a,\pi\}$$

which is onto by the above reasoning. It is also one to one, since

$$\{a,\pi\} = 1 \Longrightarrow 1 = \langle a,\pi\rangle_F = a$$

C is trivial for all ℓ except those dividing q - 1, so by taking the direct sum over all ℓ noting $C = F^*(\ell)$ and $M = K_2F(\ell)$, we get

<u>Theorem 3</u>: The map

$$\Phi: (F^*)_{q-1} \longrightarrow K_2F$$

given by

$$\Phi: \eta \longmapsto \{\eta,\pi\}$$

is an isomorphism onto the torsion in K_2F of order prime to q.

<u>Corollary 1</u>: The tame kernel in K_2F has no non-trivial torsion elements of order prime to q.

<u>Proof</u>: Suppose α is tamely trivial and $\alpha^m = 1$ where $(m,q) = 1$. Since $\alpha^m = 1$, $\alpha = \{\eta,\pi\}$ where $\eta^m = 1$, by Theorem 3, but then

$$1 = \langle\eta,\pi\rangle_F = \eta \quad \text{so} \quad \alpha = 1.$$

CONTINUOUS SYMBOLS ON FIELDS OF FORMAL POWER SERIES

by

Jimmie Graham

1. Introduction

Let $F = k((t))$ denote the field of formal power series in one indeterminant over an arbitrary field k, and let G be any abelian group. A *symbol on F with values in G* is an antisymmetric, bimultiplicative function, $b : F^* \times F^* \longrightarrow G$, that satisfies the following identity $\forall\ \beta \neq 1$ in $F^* = F - \{0\}$:

$$b(\beta, 1-\beta) = 0 . \qquad (1)$$

It is well known that $K_2(F)$ is the value group of the *universal symbol* b_F on F, i.e. every symbol on F factors uniquely through $b_F : F^* \times F^* \longrightarrow K_2(F)$. The purpose of this paper is to construct a continuous symbol

$$B : F^* \times F^* \longrightarrow K_2(k) \quad \oplus \quad k^* \quad \oplus \quad \Omega_k[[t]]$$

and to show that if $char(k) = 0$, then B is universal for a certain class of continuous symbols on F, where $\Omega_k[[t]]$ denotes the group of formal power series over the module of absolute differentials on k.

We first define symbols $\tilde{b}_k$ and b_t on F with values in $K_2(k)$ and k^* respectively. For each integer $n \geq 1$, let $U_n = 1 + t^n \cdot k[[t]]$. Then $F^* = k^* \cdot (t) \cdot U_1$, where (t) denotes the subgroup of F^* generated by t; and each $\beta \in F^*$ can be uniquely written as $\beta = xt^n u$ with x in k^*, $n \in Z$ and $u \in U_1$. We reserve the letters x, y and z for elements of k^* and u, v and w for elements of U_1. One easily verifies that any symbol d on k can be extended to a symbol $\tilde{d}$ on F by defining

$$\tilde{d}(xt^n u, yt^m v) = d(x,y).$$

In particular, the universal symbol b_k on k can be extended in this way to a symbol $\tilde{b}_k$ on F with values in $K_2(k)$.

Next we have the well known tame symbol on F, $b_t : F^* \times F^* \longrightarrow k^*$, defined by

$$b_t(xt^n u, yt^m v) = (-1)^{nm} y^n x^{-m} .$$

Definition 1. For any abelian group G and any symbol b on F with values in G define functions b_1, b_2 and b_3 from $F^* \times F^*$ to G as follows:

$$b_1(xt^n u, yt^m v) = b(x,y)$$

$$b_2(xt^n u, yt^m v) = b(t,(-1)^{nm} y^n x^{-m})$$

$$b_3(xt^n u, yt^m v) = b(xt^n, v) + b(u, yt^m) + b(u,v) .$$

It is easy to verify that each b_i is a symbol on F with values in G, and that

$$b = b_1 + b_2 + b_3 . \tag{2}$$

And moreover, it is clear that b_1 factors through $\tilde{b}_k$ (i.e. there exists $g \in \mathrm{Hom}(K_2(k), G)$ such that $b_1 = g \circ b_k$) and that b_2 factors through b_t; and these factorizations are unique because $\tilde{b}_k$ and b_t generate their value groups.

We have now proved that every symbol b on F is a sum of three symbols, $b = b_1 + b_2 + b_3$, and that $\tilde{b}_k$ and b_t completely determine b_1 and b_2, respectively. The remaining symbol, b_3, lives on $U_1 \times F^*$ by definition, and the problem of completely describing all such symbols on F has not yet been solved. In section 5 below we show that if b satisfies a certain continuity condition, then b_3 is completely determined by some finite number of derivations on k. Then in section 6 we apply these results to compute K_2 of certain rings of truncated polynomials.

2. Continuous Symbols

Put the valuation topology on F^* (i.e. take the subgroups $U_1, U_2, \ldots$ as a system of basic open neighbourhoods of 1 in F^*) and let G be any Hausdorff commutative topological group. We denote by $S_F(G)$ the group of continuous symbols on F with values in G (i.e. $b \in S_F(G)$ means that $b : F^* \times F^* \longrightarrow G$ is both a symbol on F and a continuous function). Let R/Z denote the circle group with its usual topology. It is well known that R/Z has no small subgroups, that is, there is a neighbourhood (nbd.) N of 0 in R/Z such that N contains only one subgroup of R/Z, the trivial subgroup. Clearly, discrete groups have no small subgroups. We show (lemma 1) that if $b \in S_F(G)$ and G has no small subgroups, then b must vanish on some $U_m \times F^*$. For this result we require the fact that $\forall\ x \in k^*\ \forall\ n,m \geq 1$ and $\forall\ u \in U_n$

$$\frac{1 - xt^m u}{1 - xt^m} \quad \in \quad U_{n+m} \,. \tag{3}$$

To prove this write $u = 1 + \beta t^n \in U_n$ for some integral element β in F (i.e. $1 + \beta t \in U_1$) and set $w = 1 - xt^m \in U_m$. Then $(1 - xt^m u)w^{-1} = (w + x\beta t^{m+n})w^{-1} = 1 + \sigma t^{m+n} \in U_{n+m}$, where $\sigma = x\beta w^{-1}$ is integral. As an application of (3), assume $b(U_m,\beta) = \{0\}$ for some symbol b and some $\beta \in F^*$. Then for all $x \in k$, $u \in U_m$ and $1 \leq i < m$

$$b(1 - xt^{m-i}u,\beta) \quad = \quad b(1 - xt^{m-i},\beta)\,. \tag{4}$$

Another useful consequence of (3) is

$$U_{m-1} \quad = \quad \bigcup_{x \in k} (1 - xt^{m-1})\cdot U_m \,. \tag{5}$$

From (5) it follows that if $b(U_m,\beta) = \{0\}$ for some $\beta \in F^*$ and some symbol b, and if $b(1 - xt^{m-1},\beta) = 0\ \ \forall\ x \in k$, then $b(U_{m-1},\beta) = \{0\}$.

<u>Lemma</u> 1. If G has no small subgroups and $b \in S_F(G)$, then $\exists\ m \geq 1$ such that $b(U_m,F^*) = \{0\}$.

Proof. Fix arbitrary $b \in S_F(G)$ and choose a nbd. N of 0 in G such that N contains only one subgroup of G. We first find m such that $b(U_m,k^*\cdot(t)) = \{0\}$. By continuity of b, there is a nbd. $U_i \times U_j$ of $(1,1)$ in $F^* \times F^*$ such that $b(U_i,U_j) \subset N$ since $b(1,1) = 0$. Fix arbitrary $v_0 \in U_j$ and map U_i homomorphically into G via $u \longmapsto b(u,v_0)$. Then $b(U_i,v_0)$ is a subgroup of G contained in N, so $b(U_i,v_0) = \{0\} = b(U_i,U_j)$. Let $n = \max(i,j)$, then $b(U_n,U_n) = \{0\}$. Likewise, $b(U_r,t) = \{0\}$ for some $r \geq 1$ since $b(1,t) = 0$. Take $m = \max(2n,r)$ and note that $b(U_m,(t)) = \{0\}$.

Now choose any $v \in U_m$ and any $x \in k^*$. We may write $v = 1 + \beta t^{2n}$ for some integral $\beta \in F$ and solve for u in $v = \dfrac{1 - xt^n u}{1 - xt^n}$ getting $u = 1 - \beta x^{-1} t^n + \beta t^{2n} \in U_n$. We have $0 = b(1 - xt^n u, xt^n u)$

$= b(1 - xt^n u, xt^n) + b(1 - xt^n u, u)$

$$= b\left(\frac{1-xt^n u}{1-xt^n}, xt^n\right) + 0 \qquad \text{by (1) and fact that } b(U_n,U_n) = \{0\}$$

$$= b(v,xt^n) = b(v,x) + 0 \qquad \text{since } v \in U_m \subset U_r .$$

Hence, $b(U_m,k^*) = \{0\}$, so $b(U_m,k^*\cdot(t)) = \{0\}$.

It remains to descend from $b(U_m,U_m) = \{0\}$ to $b(U_m,U_1) = \{0\}$. Choose any $u \in U_m$ and any $x \in k^*$. Then

$$0 = b(1-xt^{m-1}u, xt^{m-1}u) = b(1-xt^{m-1}u, xt^{m-1}) + b(1-xt^{m-1}u, u)$$

$$= b\left(\frac{1-xt^{m-1}u}{1-xt^{m-1}}, xt^{m-1}\right) + b(1-xt^{m-1}, u) \qquad \text{by (1) and (4)}$$

$$= 0 + b(1-xt^{m-1}, u) . \qquad \text{by (3).}$$

It follows from (5) that $b(U_{m-1},u) = \{0\}$. Keep $u \in U_m$ fixed and repeat the computation:

$$0 = b(1-xt^{m-2}u, xt^{m-2}) + b(1-xt^{m-2}u, u)$$

$$= b\left(\frac{1-xt^{m-2}u}{1-xt^{m-2}}, xt^{m-2}\right) + b(1-xt^{m-2}, u) \qquad \text{by (1) and (4)}$$

$$= 0 + b(1-xt^{m-2}, u) . \qquad \text{by (3).}$$

Therefore, $b(U_{m-2},u) = \{0\}$ by (5), and it is clear that we can repeat this process until we arrive at $b(U_1,u) = \{0\}$, because $\frac{1-xt^{m-i}u}{1-xt^{m-i}} \in U_m$ implies that $b\left(\frac{1-xt^{m-i}u}{1-xt^{m-i}}, xt^{m-i}\right) = 0$ for $1 \leq i < m$.

Therefore, $b(U_1,U_m) = \{0\}$ since $u \in U_m$ was arbitrary. //

We use this lemma in two ways. First, it guarantees that every continuous symbol on F with values in any discrete group or in R/Z must vanish on some $U_m \times k^*\cdot(t)$, and this will be explored in the next section. The second application is the following corollary that states that for certain symbols b on F, the action of b on $U_1 \times F^*$ is completely determined by the action of b on $U_1 \times k^*\cdot(t)$.

Corollary 1. If G is locally compact and $b \in S_F(G)$ vanishes on $U_1 \times k^*\cdot(t)$, then $b(U_1,F^*) = \{0\}$.

Proof. Suppose $b \in S_F(G)$ vanishes on $U_1 \times k^*\cdot(t)$ but not on $U_1 \times F^*$, say $b(u,\beta) \neq 0$ for some $(u,\beta) \in U_1 \times F^*$. We use the well known fact every locally compact group has enough characters, that is, there exists a continuous homomorphism $g : G \longrightarrow R/Z$ such that $g(b(u,\beta)) \neq 0$. Then $g \circ b \in S_F(R/Z)$ vanishes on some $U_m \times F^*$ by the lemma, and we can descend from $g \circ b(U_m, U_1) = \{0\}$ to $g \circ b(U_1, U_1) = \{0\}$ just as in the proof of the lemma because $g \circ b(U_1, k^* \cdot (t)) = \{0\}$. Hence, $g \circ b(U_1, F^*) = \{0\}$, contradicting the assumption that $b(u,\beta) \neq 0$. //

3. Derivations On The Ground Field

Let Ω_k denote the module of absolute differentials on k, that is, Ω_k is the k-module generated over k by elements dy $\forall\, y \in k$ subject to the relations $d(x+y) = dx + dy$ and $d(xy) = xdy + ydx$ $\forall\, x, y \in k$. Let G be any abelian group and suppose that b is any symbol on F with values in G that vanishes on $U_m \times F^*$ for some $m \geq 1$. We find that the action of b on $U_{m-1} \times k^*$ is completely determined by some derivation on k, and that the map $xdy \longmapsto b(1+xyt^{m-1}, y)$ defines a homomorphism $\Omega_k \longrightarrow G$.

Keep b and m fixed, where $b(U_m, F^*) = \{0\}$, and consider the homomorphism $U_{m-1} \otimes_Z k^* \longrightarrow G$ defined on generators by sending $u \otimes y$ to $b(u,y)$. By the condition on m, this map factors through $(U_{m-1}/U_m) \otimes_Z k^* \cong k^+ \otimes_Z k^*$ (see (5)), where k^+ denotes the additive group of k. We now have a homomorphism

$$g : k^+ \otimes_Z k^* \longrightarrow G \tag{6}$$

defined by $g(x \otimes y) = b(1+xt^{m-1}, y)$. There is also a homomorphism $h : k^+ \longrightarrow G$ defined by $h(x) = b(1+xt^{m-1}, t^{m-1})$, since b vanishes on $U_m \times (t)$. Note that $b(U_1, -1) = \{0\}$ because U_1 is 2-divisible unless $\mathrm{char}(k) = 2$, in which case $-1 = 1$. Therefore, $\forall\, x \in k^*$ we have

$$\begin{aligned} 0 &= b(1+xt^{m-1}, -xt^{m-1}) = b(1+xt^{m-1}, xt^{m-1}) \\ &= b(1+xt^{m-1}, x) + b(1+xt^{m-1}, t^{m-1}) \\ &= g(x \otimes x) + h(x) . \end{aligned}$$

It follows that $\forall\, x \in k^*$

$$g(x \otimes x) = -h(x) \tag{7}$$

From (7), we have $\forall\, x, y \in k^*$ such that $x+y \in k^*$

$$g((x+y) \otimes (x+y)) = g(x \otimes x) + g(y \otimes y) \tag{8}$$

<u>Definition</u> 2. Let $D_k = (k^+ \otimes_Z k^*) / J$ where J denotes the subgroup of the tensor product generated by all elements of the form

$$(x+y) \otimes (x+y) - (x \otimes x) - (y \otimes y)$$

such that x, y and $x+y \in k^*$.

We denote generators of D_k by $[x,y)$ and give this group a k-module structure by defining $z[x,y) = [zx,y)$ $\forall\ z, x \in k$ and $y \in k^*$. We verify that this action of k on D_k is well defined. If $z \neq 0$ we have $z[x+y,x+y)$

$$= [zx+zy,x+y) = [zx+zy, \frac{zx+zy}{z})$$

$$= [zx+zy,zx+zy) - [zx+zy,z)$$

$$= [zx,zx) + [zy,zy) - [zx,z) - [zy,z)$$

$$= [zx,x) + [zy,y) = z[x,x) + z[y,y) .$$

<u>Lemma</u> 2. $D_k \cong \Omega_k$ (as k-modules)

Proof. The maps are $[x,y) \longmapsto x\frac{dy}{y} = \frac{x}{y}dy$ and $xdy \longmapsto [xy,y)$. //

Let b and g be as in (6). Then g factors through D_k by (8) giving a homomorphism $g : D_k \longrightarrow G$ defined by sending $[x,y)$ to $b(1+xt^{m-1},y)$. We therefore have a homomorphism

$$f : \Omega_k \longrightarrow G \tag{9}$$

defined by $f(xdy) = b(1+xyt^{m-1},y)$.

Suppose that f is trivial (for example, if $\Omega_k = 0$) so that $b(1+zt^{m-1},y) = 0$ $\forall\ z, y \in k^*$. This implies that $b(U_{m-1},k^*) = \{0\}$ by (5), and that $b(U_{m-1},t^{m-1}) = \{0\}$ by (5) and (7). Suppose further that $m-1$ is prime to the characteristic of k (for example, if $char(k) = 0$ or if $m \leq char(k)$). Then U_{m-1} is $(m-1)$-divisible, so $\forall\ u \in U_{m-1}$ $\exists\ v \in U_{m-1}$ such that $b(u,t) = b(v^{m-1},t) = b(v,t^{m-1}) = 0$. Hence, $b(U_{m-1},t) = \{0\}$ in this case.

<u>Lemma</u> 3. If b is any symbol on F that vanishes on $U_m \times k^* \cdot (t)$ and on every pair $(1+xt^{m-1},y) \in U_{m-1} \times k^*$, then b vanishes on $U_{m-1} \times k^* \cdot (t^{m-1})$. Moreover, if $m-1$ is prime to $char(k)$ or if $\Omega_k = 0$, then b vanishes on $U_{m-1} \times k^* \cdot (t)$.

Proof. It remains to show that $b(U_{m-1},t) = \{0\}$ in the case where $char(k) = p > 0$, p divides $m-1$, and $\Omega_k = 0$. Then k is perfect because $\Omega_k = 0$. Write $m-1 = p^s n$, where $s > 0$ and n is prime to p, and choose any $x \in k^*$. Then $\exists\ y \in k^*$ such that $y^q = x$, where $q = p^s$, and we have

$$\begin{aligned} 0 &= b(1-xt^n,x) + b(1-xt^n,t^n) = b(1-xt^n,y^q) + b(1-xt^n,t^n) \\ &= b((1-xt^n)^q,y) + b(1-xt^n,t^n) \\ &= 0 + b(1-xt^n,t^n) \qquad\qquad \text{since } (1-xt^n)^q \in U_{m-1}. \end{aligned}$$

This shows that $b(1-xt^n,t)$ has order dividing n, but its order also divides pq because $(1-xt^n)^{pq} \in U_m$ implies $b((1-xt^n)^{pq},t) = 0$ by hypothesis. Therefore, $b(1-xt^n,t) = 0\ \forall\ x \in k^*$ since pq and n are relatively prime.

Now consider $b(1-xt^{m-1},t)$ for arbitrary $x \in k^*$. Let $y^q = x$ as above, and write $b(1-xt^{m-1},t) = b((1-yt^n)^q,t) = b(1-yt^n,t^q)$ which equals 0 since $b(1-yt^n,t) = 0\ \forall\ y \in k$. Thus, $0 = b(1-xt^{m-1},t)\ \forall\ x \in k$, so $b(U_{m-1},t) = \{0\}$ by (5). //

4. Russell's Continuous Tate Symbol

Let $\Omega_k[[t]]$ denote the group of formal power series over Ω_k. Then $\Omega_k[[t]]$ is the projective limit of the discrete groups $\Omega_k[[t]]/t^m\cdot\Omega_k[[t]]$. The purpose of this section is to construct a symbol $\mathcal{b} \in S_F(\Omega_k[[t]])$. We begin by extending the derivation $d : k \longrightarrow \Omega_k$ to a derivation $D : F \longrightarrow \Omega_k((t))$ (= group of formal Laurent series over Ω_k) via

$$D(\Sigma x_i t^i) = \Sigma (dx_i) t^i.$$

Denote a typical element of $\Omega_k((t))$ by $\Sigma \gamma_j t^j$ and give this group an F-module structure by defining

$$(\Sigma x_i t^i)(\Sigma \gamma_j t^j) = \Sigma \delta_n t^n$$

where $\delta_n = \Sigma x_i \gamma_{n-i}$.

For each element $\beta = \Sigma x_i t^i \in F$, let $\beta' = \Sigma i x_i t^{i-1} \in F$ be the usual formal derivative of β. Note that $\forall\ \beta,\sigma \in F^*$, $\frac{\beta'}{\beta}\cdot\frac{D\sigma}{\sigma}$ lies in $t^{-1}\cdot\Omega_k[[t]] \subset \Omega_k((t))$. Define

$$\mathcal{b}_d : F^* \times F^* \longrightarrow t^{-1}\cdot\Omega_k[[t]]$$

by

$$\mathcal{b}_d(\beta,\sigma) = \frac{\beta'}{\beta}\cdot\frac{D\sigma}{\sigma} - \frac{\sigma'}{\sigma}\cdot\frac{D\beta}{\beta}.$$

The function $\mathcal{b}_d$ is bimultiplicative because the maps $\beta \longmapsto \frac{\beta'}{\beta}$ and

$\beta \longmapsto \frac{D\beta}{\beta}$ are homomorphisms from F^*; and $\mathcal{E}_d$ satisfies (1) because $(1-\beta)' = -\beta'$ and $D(1-\beta) = -D\beta$.

To obtain a symbol $\mathcal{E} \in S_F(\Omega_k[[t]])$, we write $\mathcal{E}_d = (\mathcal{E}_d)_1 + (\mathcal{E}_d)_2 + (\mathcal{E}_d)_3$ as in (2), and set $\mathcal{E} = (\mathcal{E}_d)_3$. It is easy to check that $(\mathcal{E}_d)_1 = 0$ and that $(\mathcal{E}_d)_2 = f \circ b_t$, where f is defined by $f(x) = \frac{1}{t}\frac{dx}{x} = \mathcal{E}_d(t,x)$.

From the definition of $\mathcal{E}$, we have $\mathcal{E}(xt^n u, yt^m v)$

$$= \mathcal{E}_d(xt^n, v) + \mathcal{E}_d(u, yt^m) + \mathcal{E}_d(u,v)$$

$$= \mathcal{E}(xt^n, v) + \mathcal{E}(u, yt^m) + \mathcal{E}(u,v).$$

We compute $\mathcal{E}(xt^n, v)$ as follows: write $v = 1 + \Sigma c_i t^i \in U_r$, for some $r \geq 1$, and $c_i \in k$ for $i = 1,2,\ldots$, then

$$\mathcal{E}(xt^n, v) = \frac{nxt^{n-1}}{xt^n} \cdot \frac{Dv}{v} - \frac{v'}{v} \cdot \frac{(dx)t^n}{xt^n}$$

$$= \left(ndc_r - rc_r \frac{dx}{x} \right) t^{r-1} + \cdots \tag{10}$$

Note that we have computed only the first coefficient of the power series $\mathcal{E}(xt^n, v)$. For future reference, we take $n = 0$ and $v = 1 + zt^r$ in (10) to obtain

$$\mathcal{E}(1 + zt^r, x) = rz\frac{dx}{x} t^{r-1} + \cdots \tag{11}$$

From (10) and the fact that $\mathcal{E}(u,v) \in \Omega_k[[t]]$ $\forall\, u, v \in U_1$, it follows that $\mathcal{E}$ takes values in $\Omega_k[[t]]$; and it is easy to show that

$$\mathcal{E}(U_m, U_r) \subset t^{m+r-1} \cdot \Omega_k[[t]]$$

so that $\mathcal{E}$ is continuous, i.e. $\mathcal{E} \in S_F(\Omega_k[[t]])$.

Assume for the moment that $\mathrm{char}(k) = 0$ and choose any element α in $\Omega_k[[t]]$. From (11) it follows that there is an element α_1 in $\mathrm{Im}(\mathcal{E})$ (= group generated by $\mathcal{E}(F^*, F^*)$) such that $\alpha - \alpha_1$ lies in $t^1 \cdot \Omega_k[[t]]$ (i.e. α and α_1 have the same first coefficient). By induction, $\forall\, n \geq 1$ $\exists\, \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Im}(\mathcal{E})$ such that $\alpha - (\sum_{i=1}^{n} \alpha_i) \in t^n \cdot \Omega_k[[t]]$. Therefore, $\mathcal{E}$ generates $\Omega_k[[t]]$ topologically (i.e. generates a dense subgroup) when $\mathrm{char}(k) = 0$.

Let k be arbitrary again and define, for each positive m prime to $\mathrm{char}(k)$, the projection $h_m : \Omega_k[[t]] \longrightarrow \Omega_k$ and the symbol $\mathcal{E}_m \in S_F(\Omega_k)$ as follows:

$h_m(\Sigma \gamma_i t^i) = \frac{1}{m}\gamma_{m-1}$

and $\ell_m = h_m \circ \ell$.

$$\begin{array}{ccc} F^* \times F^* & \xrightarrow{\ \ell\ } & \Omega_k[[t]] \\ & \searrow^{\ell_m} & \downarrow h_m \\ & & \Omega_k \end{array}$$

From the definition of ℓ_m it follows that

$$\ell_m(U_{m+1}, F^*) = \{0\} \qquad (12)$$

and $\forall\ z, x \in k^*$

$$\ell_m(1 + zxt^m, x) = z dx . \qquad (13)$$

<u>Remark</u>. For any field E , the <u>Tate symbol</u> on E with values in $\Omega_E \wedge \Omega_E$ (= alternating product) is defined by $(x,y) \longmapsto \frac{dx}{x} \wedge \frac{dy}{y}$. In our case, $F = k((t))$, we have a derivation $F \longrightarrow \Omega_k((t)) \oplus F$ defined by $\beta \longmapsto (D\beta, \beta')$, and Peter Russell constructed the symbol ℓ_d by wedging this "continuous Omega" ($\Omega_k((t)) \oplus F$) with itself.

<u>5. Proofs Of Main Results</u>

Recall our notation: $F = k((t))$ with k arbitrary; for each topological group G, $S_F(G)$ denotes the group of continuous symbols on F with values in G; and $\mathrm{Hom}_c(\ ,\ )$ denotes the group of continuous homomorphisms. Define

$$M_k = K_2(k) \oplus k^* \oplus \Omega_k[[t]] .$$

Then M_k is clearly a projective limit of discrete groups, and we have the symbol

$$B = (\tilde{b}_k, b_t, \ell) \in S_F(M_k) .$$

<u>Theorem</u> 1. If $\mathrm{char}(k) = 0$ and G is any projective limit of discrete groups, then there is a natural isomorphism $S_F(G) \cong \mathrm{Hom}_c(M_k, G)$.

Proof. We first prove this for arbitrary discrete group G . Fix arbitrary $b \in S_F(G)$ and write $b = b_1 + b_2 + b_3$ as in (2). Then b_1 and b_2 factor uniquely through $\tilde{b}_k$ and b_t , respectively, by section 1 , so we must show that b_3 factors uniquely through ℓ . This factorization is unique if it exists because ℓ generates a dense subgroup of the Hausdorff topological group $\Omega_k[[t]]$ when $\mathrm{char}(k) = 0$.

By lemma 1, $\exists\ m \geq 1$ such that $b(U_m, F^*) = \{0\}$; and $b_3 = 0$ if $m = 1$. Assume $m > 1$ and define $f_{m-1} : \Omega_k \longrightarrow G$ by $f(xdy) = b(1 + xyt^{m-1}, y)$ as in (9) . By (13) we have $\forall\ x, y \in k^*$

$$f_{m-1}\circ \mathcal{B}_{m-1}(1+xyt^{m-1},y) = f_{m-1}(xdy)\ .$$

Therefore, $f_{m-1}\circ \mathcal{B}_{m-1}$ and b both vanish on $U_m \times k^*\cdot(t)$ and agree on all pairs $(1+zt^{m-1},y) \in U_{m-1}\times k^*$. It follows from lemma 3 that the symbol $(b - f_{m-1}\circ \mathcal{B}_{m-1})$ vanishes on $U_{m-1}\times k^*\cdot(t)$. If $m>2$ we apply the same reasoning to the symbol $(b - f_{m-1}\circ \mathcal{B}_{m-1}) \in S_F(G)$ and obtain a homomorphism $f_{m-2} : \Omega_k \longrightarrow G$ such that the symbol $(b - f_{m-1}\circ \mathcal{B}_{m-1} - f_{m-2}\circ \mathcal{B}_{m-2})$ vanishes on $U_{m-2}\times k^*\cdot(t)$. In this way we construct $f_{m-1}, f_{m-2}, \ldots, f_1 \in \mathrm{Hom}(\Omega_k, G)$ such that the symbol

$$b - \sum_{i=1}^{m-1} f_i\circ \mathcal{B}_i = b - \Big(\sum_{i=1}^{m-1} f_i\circ h_i\Big)\circ \mathcal{B}$$

vanishes on $U_1\times k^*\cdot(t)$.

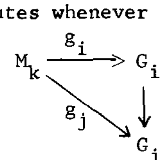
Set

$$f = \sum_{i=1}^{m-1} f_i\circ h_i \in \mathrm{Hom}_c(\Omega_k[[t]], G).$$

Then $b - f\circ\mathcal{B}$ vanishes on $U_1\times F^*$ by corollary 1, so $b_3 = f\circ\mathcal{B}$.

Now suppose that G is a projective limit of discrete groups $\{G_i\}_{i\in I}$, and choose arbitrary $b \in S_F(G)$. For each $i \in I$, the projection $\pi_i : G \longrightarrow G_i$ determines a continuous symbol $b^{(i)} = \pi_i\circ b$ with values in the discrete group G_i. Hence, $\forall\ i \in I$ there exists $g_i \in \mathrm{Hom}_c(M_k, G_i)$ such that $b^{(i)} = g_i\circ B$.

It is easy to verify that the following diagram commutes whenever $i \geq j$ in I. Hence, by the universal property of projective limits, $\exists!\ g = \varprojlim g_i : M_k \longrightarrow G$ such that for each $i \in I$, $g_i = \pi_i\circ g$. To verify now that $b = g\circ B$, we check that the i^{th} components of $b(\beta,\sigma)$ and $g\circ B(\beta,\sigma)$ agree $\forall\ i\in I,\ \forall\ \beta,\sigma \in F^*$:



$\pi_i(b(\beta,\sigma))$	$= b^{(i)}(\beta,\sigma)$	by definition of $b^{(i)}$
	$= g_i\circ B(\beta,\sigma)$	since $b^{(i)} = g_i\circ B$
	$= \pi_i(g\circ B(\beta,\sigma))$	since $g_i = \pi_i\circ g$.

Therefore, $b = g\circ B$.

//

In the first part of the proof of the theorem we needed $\mathrm{char}(k) = 0$ in order to guarantee existence of the symbols $\mathcal{B}_i\ \ \forall\ i\geq 1$. Now, if b is any symbol on F that vanishes on $U_m\times F^*$ for some $m\geq 1$, then $\mathrm{char}(k) \geq m$ will guarantee existence of $\mathcal{B}_i$, for $1\leq i<m$, and it is

clear that we can again construct $f : \Omega_k \longrightarrow G$ such that $b_3 = f \circ \mathfrak{b}$. Also, it follows from the definition of $h_1, \dots, h_{m-1}$ that f vanishes on $t^{m-1} \cdot \Omega_k[[t]]$, so that $f = \tilde{f} \circ \pi_{m-1}$, where $\tilde{f}$ denotes the obvious map (see adjacent triangle). Therefore, b_3 factors through $\pi_{m-1} \circ \mathfrak{b}$, where π_{m-1} denotes the natural projection. We record this in the following:

$$\begin{array}{ccc} \Omega_k[[t]] & \xrightarrow{\ f\ } & G \\ \Big\downarrow{\scriptstyle \pi_{m-1}} & \nearrow{\scriptstyle \tilde{f}} & \\ \Omega_k[[t]]\,/\,t^{m-1} \cdot \Omega_k[[t]] & & \end{array}$$

<u>Corollary</u> 2. If b is any symbol on F that vanishes on $U_m \times F^*$ and if $m \leq \mathrm{char}(k)$ or if $\mathrm{char}(k) = 0$, then b factors uniquely through $(\tilde{b}_k, b_t, \pi_{m-1} \circ \mathfrak{b})$.

The next result is a generalization of a theorem of Calvin Moore [M] that states that $S_F(G) \cong \mathrm{Hom}(k^*, G)$ for every locally compact G in case k is finite (i.e. the tame symbol is universal in this case). In general, $\tilde{b}_k \neq 0$ and does not factor through b_t.

<u>Theorem</u> 2. For every field k and every locally compact G

$$S_F(G) \cong \mathrm{Hom}(K_2(k) \oplus k^*, G) \iff \Omega_k = 0 .$$

Proof. If $\mathfrak{b}_1 \in S_F(\Omega_k)$ factors through $(\tilde{b}_k, b_t) \in S_F(K_2(k) \oplus k^*)$, then $\mathfrak{b}_1 = 0$ since $(\tilde{b}_k, b_t)$ vanishes on $U_1 \times F^*$. Hence, $\Omega_k = 0$ by (13).

Conversely, suppose $\Omega_k = 0$. We first prove the assertion for $G = R/Z$. Fix arbitrary $b \in S_F(R/Z)$ and choose smallest $m \geq 1$ such that $b(U_m, k^* \cdot (t)) = \{0\}$ (see lemma 1). If $m = 1$, then b factors through $(\tilde{b}_k, b_t)$ by section 1. On the other hand, if $m > 1$, then b vanishes on $U_{m-1} \times k^* \cdot (t)$ by lemma 3 since $\Omega_k = 0$ implies (see (9)) that b vanishes on all pairs $(1 + xt^{m-1}, y) \in U_{m-1} \times k^*$. This contradicts mimimality of m, so $m = 1$. Therefore, every $b \in S_F(R/Z)$ must vanish on $U_1 \times F^*$.

Now let G be any locally compact group, and choose any $b \in S_F(G)$. If $b(u, \beta) \neq 0$ for some $(u, \beta) \in U_1 \times F^*$, then $\exists\, g \in \mathrm{Hom}_c(R/Z, G)$ such that $g(b(u, \beta)) \neq 0$. But this contradicts the fact that the symbol $g \circ b \in S_F(R/Z)$ must vanish on $U_1 \times F^*$. Therefore, b vanishes on $U_1 \times F^*$, and if follows from section 1 that b factors through $(\tilde{b}_k, b_t)$.

6. K_2 Of Rings Of Truncated Polynomials

Keith Dennis and Michael Stein have given presentations (i.e. generators and relations) for K_2 of the discrete valuation ring $L = k[[t]]$ and its homomorphic images $L_m = k[t] / t^m \cdot k[t]$, where $m \geq 1$ and k is arbitrary. They prove [D-S;§2] that the tame symbol on $F = k((t))$ induces a split exact sequence

$$1 \longrightarrow K_2(L) \underset{\sigma}{\overset{\longrightarrow}{\longleftarrow}} K_2(F) \longrightarrow k^* \longrightarrow 1$$

and that, for each $m \geq 1$, there is a natural surjection $\delta_m : K_2(L) \longrightarrow K_2(L_m)$ defined by sending a typical generator $\{xu, yv\}_L$ to a generator $\{x\bar{u}, y\bar{v}\}_{L_m}$ of $K_2(L_m)$, where $\bar{u}$ denotes the obvious truncated power series.

Then $d_m = \delta_m \circ \sigma \circ b_F$ is a symbol on F with values in $K_2(L_m)$; and d_m vanishes on $k^* \cdot (t) \times k^* \cdot (t)$ because the tame symbol induced the above split exact sequence. This means that $(d_m)_2 = 0$, where $d_m = (d_m)_1 + (d_m)_2 + (d_m)_3$ as in (2).

$$\begin{array}{ccc} F^* \times F^* & \xrightarrow{b_F} & K_2(F) \\ \downarrow d_m & & \downarrow \sigma \\ K_2(L_m) & \xleftarrow{\delta_m} & K_2(L) \end{array}$$

Also, $d_m(U_m, k^* \cdot U_1) = \{0\}$ by definition of δ_m. We claim that d_m must also vanish on $U_{m+1} \times (t)$. To prove this, we choose arbitrary $u = 1 + \beta t^{m+1} \in U_{m+1}$ and use the following identity due to Dennis and Stein (see the proof of Theorem 2.5 in [D-S]) :

$$b_F(u,t) = b_F(- \frac{1+\beta t^m}{1 - t}, \frac{u}{1-t}).$$

It follows from the def$^{\underline{n}}$. of δ_m that $d_m(u,t) = d_m(-(1-t)^{-1}, (1-t)^{-1})$ since $1 + \beta t^m, u \in U_m$; and it is not difficult to show that every symbol vanishes on all pairs $(-\beta, \beta) \in F^* \times F^*$. Hence, $d_m(U_{m+1}, (t)) = \{0\}$.

The following theorem was first proved in the case $m = 2$ by Wilberd Van Der Kallen [V]. Dennis and Stein have also proved this result in this case.

Theorem 3. If $1 \leq m < char(k)$, or if $char(k) = 0$, then

$$K_2(k[t] / t^m \cdot k[t]) \cong K_2(k) \oplus \Omega_k[t] / t^{m-1} \cdot \Omega_k[t] .$$

Proof. For brevity, we set $A = \Omega_k[t] / t^{m-1} \cdot \Omega_k[t]$, and $b = (b_k, \pi_{m-1} \circ \mathcal{b}) \in S_F(K_2(k) \oplus A)$ since m is now fixed. From the above arguments it follows that d_m vanishes on $U_{m+1} \times F^*$ and on $U_m \times k^* \cdot U_1$. Then $d_m(U_m, k^* \cdot (t)) = \{0\}$ by lemma 3. Now, $(d_m)_2 = 0$, so d_m factors uniquely through b by corollary 2, say $d_m = f \circ b$, where $f : K_2(k) \oplus A \longrightarrow K_2(L_m)$. We will show that f is an isomorphism.

Next we define a map $K_2(L) \longrightarrow K_2(k) \oplus A$ by sending a typical generator $\{xu, yv\}_L$ to $b(xu, yv)$. It follows from the above exact sequence that this map is a homomorphism. To define a map $g : K_2(L_m) \longrightarrow K_2(k) \oplus A$, we choose any generator $\{x\bar{u}, y\bar{v}\}_{L_m}$ of $K_2(L_m)$ and lift it to a generator $\{xu, yv\}_L \in K_2(L)$ and define $g(\{x\bar{u}, y\bar{v}\}_{L_m}) = b(xu, yv)$. The choice of u and $v \in U_1$ doesn't matter because b vanishes on $U_m \times F^*$. Therefore, g is a homomorphism.

To check that f and g are inverses, choose any $\{x\bar{u}, y\bar{v}\}_{L_m}$ and compute:

$$f \circ g(\{x\bar{u}, y\bar{v}\}_{L_m}) = f(b(xu, yv)) = d_m(xu, yv) = \{x\bar{u}, y\bar{v}\}_{L_m} .$$

Now $K_2(k) \oplus A$ is clearly generated by elements $b(xu, yv)$ (see (13)), and we have

$$g \circ f(\, b(xu, yv)\,) = g(\, d_m(xu, yv)\,) = g(\{x\bar{u}, y\bar{v}\}_{L_m})$$
$$= b(xu, yv) .$$

Therefore, f and g are inverses. //

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E. ARITHMETIC ASPECTS OF K-THEORY

Values of zeta-functions, étale cohomology, and algebraic K-theory

by Stephen Lichtenbaum

In this paper we give various conjectures expressing values of zeta-functions in terms of the orders of étale cohomology groups and algebraic K-groups, together with a description of the relationships between the conjectures and some indication of why one might believe them to be true. In order partly to make up for the great profusion of conjectures that will occur at the end of this paper, we begin with some results that are well-known and undeniably true.

Let F be an algebraic number field of finite degree n over the rationals, with ring of integers $\mathcal{O}_F$. We define the zeta-function of F, $\zeta(F,s)$, to be $\sum_{\mathfrak{a}} \frac{1}{(N\mathfrak{a})^s}$. This series converges if $\mathrm{Re}(s) > 1$, and can be extended to a function meromorphic in the whole plane, and satisfying a simple functional equation which we shall now describe.

As usual, let r_1 be the number of real places of F, r_2 the number of complex places of F, d the discriminant of F, and define

$$\Phi(F,s) = \Gamma(s/2)^{r_1}\,\Gamma(s)^{r_2}\left(\frac{|d|}{4^{r_2}\pi^n}\right)^{s/2}\zeta(F,s).$$

Then

$$\Phi(F,s) = \Phi(F,1-s). \tag{1}$$

Also, the zeta-function is analytic except when $s = 1$, and has a simple pole with residue given by

$$\lim_{s\to 1}(s-1)\zeta(F,s) = \frac{hR}{w}\cdot\frac{2^{r_1}(2\pi)^{r_2}}{|d|^{1/2}}$$

where h is the class number of F, w is the number of roots of unity in F, and R is the regulator of F. For the purposes of comparison with analogues of the regulator which will occur in later conjectures, we recall its definition. Let $t = r_1 + r_2 - 1$. Then the group of units of F is, by the Dirichlet unit theorem, a finitely-generated abelian group U of rank t, and we choose a basis $u_1,\dots u_t$ for U modulo torsion. Pick any t infinite places $v_1 \dots v_t$, and define R to be $|\det(|u_i|_{v_j})|$. Then R is independent of the choice of basis and of the one omitted infinite place.

We also recall a result of Siegel, [13, v.I, p. 545-546] to the effect that if F is totally real and m is an odd positive integer, then $\pi^{-n(m+1)}|d|^{1/2}\,\zeta(F,m+1)$ is a rational number.

It is an immediate and well-known consequence of applying the functional equation to Siegel's result that $\zeta(F,-m)$ is a non-zero rational number if F is totally real and m is odd and positive. It is only slightly less immediate that if we apply the functional equation to the formula for the residue of the zeta-function at $s = 1$ we obtain the following result:

<u>Proposition 1</u>. The zeta-function $\zeta(F,s)$ has a zero of order (r_1+r_2-1) at $s = 0$, and we have the formula

$$\lim_{s\to 0} \zeta(F,s)s^{-(r_1+r_2-1)} = -hR/w.$$

The details of the proof will appear in [9].

We are now faced with the problem of giving an interpretation of the rational numbers $\zeta(F,-m)$. We begin with the special case $m = 1$. In this case Birch and Tate ([1], [14]) have made a very striking conjecture. We begin with some notation.

Let W denote the group of roots of unity in the algebraic closure $\overline{F}$ of F, and G the Galois group of $\overline{F}$ over F. Then G acts on W through an abelian quotient, and so we may define for any integer m a new action of G on W by $\sigma \underset{(m)}{*} x = \sigma^m x$, where juxtaposition denotes the usual action. We define W_m to be W together with this G-action, $W_m(F)$ to be W_m^G and $w_m(F)$ to be the order of $W_m(F)$. We can then state the Birch-Tate conjecture as follows:

<u>Conjecture 2</u>. $|\zeta_F(-1)| = \#(K_2(\mathcal{O}_F))/w_2(F)$.

We should observe here that this is not the original form of the conjecture; in the original version ([1], [14]) $K_2(\mathcal{O}_F)$ is replaced by $\operatorname{Ker} \lambda$, where $\lambda: K_2(\mathcal{O}_F) \to \coprod (\tilde{F}_v)^*$ is the map induced by the tame symbols. However, Quillen ([11]) has recently shown that for Dedekind domains A with quotient field L there exists an exact sequence

$$\cdots \coprod K_i(\tilde{L}_v) \to K_i(A) \to K_i(L) \to \coprod K_{i-1}(\tilde{L}_v) \to \cdots .$$

In view of the fact that K_2 of a finite field is zero this establishes the isomorphism of $\operatorname{Ker} \lambda$ with $K_2(\mathcal{O}_F)$.

We now want to restate Conjecture 2 in cohomological terms, making use of the following theorem of Tate [15]:

<u>Theorem 3</u>. Let F be a totally real number field. Then $K_2(F)$ is naturally isomorphic to $H^1(G,W_2)$.

This is only a special case of the actual theorem of Tate, which gives a cohomological characterization of $K_2(F)$ valid for all number fields F, but it will suffice for our purposes.

Now let ℓ be a fixed prime number, and S the set of primes of F lying over ℓ. Let $\mathcal{O}_{F,S}$ be the set of S-integers of F, $X_S = \operatorname{Spec} \mathcal{O}_{F,S}$ and j the natural inclusion of Space F in X_S. If we endow F and X_S with the étale topology, then, for each m, W_m amy be viewed as a sheaf on Spec F, and we may take the direct image sheaves $R^q j_* W_m$ on X_S. We then ([8], [15]) have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \to H^1(X_s,j_*W_2) & \to & H^1(G_F,W_2) & \to & H^0(X_s,R^1j_*W_2) & \to & H^2(X_s,j_*W_2) & \to & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & & & \\ 0 \to \text{Ker } \lambda & \to & K_2(F) & \xrightarrow{\lambda} & \coprod_v \tilde{F}_v^{\,*} & \longrightarrow & 0 & & \end{array}$$

where α and β are isomorphisms and the top row is the exact sequence of terms of low degree coming from the Leray spectral sequence for the map j_* and the sheaf W_2, namely:

$H^p(X_s, R^qj_*W_2) \Longrightarrow H^{p+q}(G_F,W_2)$.

From this we see that $\text{Ker } \lambda \simeq H^1(X_s,j_*W_2)$ and that $H^2(X_s,j_*W_2) = 0$. In view of this, the ℓ-part of the Birch-Tate conjecture may be restated as

Conjecture 1.4. If F is totally real, then the ℓ-part of $\zeta(F,-1)$ is equal to $\#H^1(X_s,j_*W_2)/\#H^0(X_s,j_*W_2)$, and one is naturally led to more general conjecture ([8]):

Conjecture 1.5. If F is totally real and m is any odd positive integer, then the ℓ-part of $\zeta(F,-m)$ is equal to $\# H^1(X_s,j_* W_{m+1})/\#H^0(X_s,j_*W_{m+1})$. Also, $H^p(X_s,j_*W_{m+1}) = 0$ for $p \geq 2$.

This conjecture has been verified in many special cases, by the use of the theory of p-adic L-functions developed by Leopoldt and Kubota ([7]) and extended by Iwasawa ([6]) and Coates ([5]). The strongest positive result is as follows:

Let F_0 be the field obtained from F by adjoining the ℓ-th roots of unity, and F_0^+ the maximal real subfield of F_0. Let A_0 be the ℓ-component of the class group of F_0, and $A_0^- = \{x \in A_0 : \sigma x = -x\}$, where σ denotes complex conjugation. Let π be the Galois group of F_0 over $\mathbb{Q}$.

Theorem 1.6 [5]. Assume (i) that ℓ is odd,

(ii) π is abelian of order prime to ℓ

(iii) no prime of F_0^+ lying over ℓ splits in F_0

(iv) A_0^- is a cyclic $\mathbb{Z}[\pi]$-module.

Then Conjecture 1.5 is true for F, ℓ and any m.

We remark here that it is almost certain that the methods of [8] would prove Theorem 1.6 for any real subfield of the field obtained by adjoining the ℓ-power roots of unity to F, if F satisfies the hypotheses of Theorem 1.6. Also, if ℓ is regular or properly irregular (the second factor of the class number of $\mathbb{Q}_0$ is not divisible by ℓ), then any subfield of $\mathbb{Q}_0^+$ satisfies the hypotheses of Theorem 1.6.

We next wish to point out that Conjecture 1.5 of course implies the following result:

Conjecture 1.7. (Serre, [12, p. 164]). If F is totally real and m is an odd negative integer, then $w_{m+1}(F)\,\zeta(F,-m)$ is an integer.

Serre has proved Conjecture 1.7 in [12] for the case $m = 1$, and, more generally, has shown there that the product over the first k odd integers m of $w_{m+1}(F)\zeta(F,-m)$ is an integer for any k. Extensions of Conjecture 1.6 to L-functions are discussed and special cases are proved in [5] and [9].

2. Algebraic K-theory.

We now return to the point of view of algebraic K-theory, which was left aside in Section 1 with the interpretation of $K_2(\mathcal{O}_F)$ as an étale cohomology group. We begin by discussing finite fields. First recall that if k is a finite field with q elements, then the zeta function of k is defined by $\zeta(k,s) = (1 - q^{-s})^{-1}$. The Quillen [10] has proved the following suggestive result:

<u>Theorem 2.1</u>. Let k be a finite field, and i a positive integer. Then $K_{2i}(k)$ is equal to zero, and $K_{2i-1}(k)$ is a finite group of order equal to $|\zeta(k,-i)|^{-1}$.

In the number field case, Quillen has recently proved that $K_i(\mathcal{O}_F)$ is a finitely-generated abelian group for any i and any number field F. The ranks of these groups are determined by the following theorem of Borel:

<u>Theorem 2.2</u>. (Borel [2]). For any non-negative integer i, the rank of $K_{2i}(\mathcal{O}_F)$ is equal to zero, and the rank of $K_{2i+1}(\mathcal{O}_F)$ is equal to r_2 if i is odd, to r_1+r_2 if i is even, and positive, and to r_1+r_2-1 if $i = 0$.

The significance of this result for us is that it can be stated more simply as follows:

<u>Corollary 2.3</u>. The rank of $K_{2i+1}(\mathcal{O}_F)$ is equal to the order of the zero of $\zeta(F,s)$ at $s = -i$.

(The order of the zero of the zeta-function at $s = -i$ may easily be computed from the functional equation, together with a knowledge of the poles of the gamma function.)

Now that we have seen that some connection exists between algebraic K-groups and zeta-functions, we state the following conjecture:

Conjecture 2.4. Let F be a totally real number field, and m an odd positive integer. Then $|\zeta(F,-m)| = \#K_{2m}(\mathcal{O}_F)/\#K_{2m+1}(\mathcal{O}_F)$, up to 2-torsion.

We note that the groups involved in the conjecture are finite by the theorems of Borel and Quillen referred to above. It is clear that there ought to be a relation between Conjectures 1.5 and 2.4; the missing link is provided by a conjecture of Quillen which we will proceed to describe.

Let ℓ be an odd prime, as in Section 1, and m a positive integer. Let $W_m^{(n)}$ be the kernel of the map from W_m to W_m consisting of multiplication by ℓ^n (in additive relation). Let F again be an arbitrary number field, and let S be a finite set of primes of F which contain all primes of F lying over ℓ. Let $\mathcal{O}_S$ be the ring of S-integers of F. Then Quillen conjectures:

Conjecture 2.5.

a) $K_{2m}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell \simeq \varprojlim_n H^2(X_S, j_* W_{m+1}^{(n)})$

b) $K_{2m+1}(\mathcal{O}_S) \otimes \mathbb{Z}_\ell \simeq \varprojlim_n H^1(X_S, j_* W_{m+1}^{(n)})$,

with the isomorphisms being given by a generalized Chern character.

If $m = 1$, a) is equivalent to the theorem of Tate referred to earlier, and proved in his talk at this conference. If $\mathcal{O}_S$ is replaced by a finite field k, and X_S by Spec k, then the analogue to Conjecture 2.5 follows easily from the computation of the K-groups of a finite field, done by Quillen in [10].

We now suppose again that m is odd positive and F is totally real. Then $K_{2m}(\mathcal{O}_F)$ and $K_{2m+1}(\mathcal{O}_F)$ are finite, by Theorem 2.2. It follows from the exact sequence of a localization that $K_{2m}(\mathcal{O}_S)$ and $K_{2m+1}(\mathcal{O}_S)$ are finite for any finite set of primes S. If we assume in addition Conjecture 1.5, then $H^i(X_s, j_* W_m)$ is finite for all i, which implies that $\varprojlim_n H^{i+1}(X_s, j_*, W^{(n)}_{m+1}) \simeq H^i(X_s, j_* W_{m+1})$. In view of this isomorphism, we see that Conjecture 2.5 and Conjecture 1.5 imply Conjecture 2.4.

2.6. There does not seem to be any a priori reason why Conjecture 2.4 should not also include 2-torsion, but this does not seem to be the case. Using his Hermitian K-theory, Karoubi has indicated an argument which shows that the 2-torsion part of $K_3(\mathbb{Z})$ is not equal to $\mathbb{Z}/8\mathbb{Z}$, as would be predicted by the extended form of Conjecture 2.4, but is at least big enough to map surjectively onto $\mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. It would be very desirable to have an exact description of the whole of $K_3(\mathbb{Z})$.

2.7. It seems also likely that the strange-looking quantity $\#K_{2i+1}(\mathcal{O}_F)/\#K_{2i}(\mathcal{O}_F)$ should also be interpreted as an Euler characteristic. Namely, if we let $\widetilde{K}_n(\mathcal{O}_F)$ be the sheaf associated to the obvious étale presheaf defined by the functor K_n, then it seems possible that $K_{2i+1}(\mathcal{O}_F) \approx H^0(\mathrm{Spec}\,\mathcal{O}_F, \widetilde{K}_{2i+1}(\mathcal{O}_F))$ and $K_{2i}(\mathcal{O}_F) \simeq H^1(\mathrm{Spec}\,\mathcal{O}_F, \widetilde{K}_{2i+1}(\mathcal{O}_F))$ with $H^p(\mathrm{Spec}\,\mathcal{O}_F, \widetilde{K}_{2i+1}(\mathcal{O}_F)) = 0$ for $p > 1$. These isomorphisms would come from the degeneration of a fourth-quadrant spectral sequence going (approximately) from the cohomology of the sheaves $\widetilde{K}_i$ to the groups K_i, which would be the analogue for the étale topology of the Zariski-topology spectral sequences described by Bloch and Gersten elsewhere in this volume. The possibility of the existence of such a spectral sequence has been investigated (in the case of a field) by K. Brown, among others.

3. The case when $F = \mathbb{Q}$.

There is some additional evidence for the conjectures in the case when $F = \mathbb{Q}$ and $\mathcal{O}_F = \mathbb{Z}$. Let i be a positive integer of the form $4n-1$. Quillen has shown that there is always a map from the stable i-stem to $K_i(\mathbb{Z})$, which is injective when restricted to the image of the J-homomorphism and whose image when so restricted is a direct summand of $K_i(\mathbb{Z})$. Furthermore the order of this image is then (by results of Adams, Quillen and Sullivan) equal to twice the denominator $\alpha(2n)$ of the Bernoulli number $B_{2n}/2n$, where we fix our notation by the formula

$$\frac{X}{e^X-1} = \sum_{n=0}^{\infty} B_n \, X^n/n! \ .$$

It is also well-known that $\zeta(1-2n) = -B_{2n}/2n$.

Furthermore, for a fixed prime ℓ, the order of $H^0(X_s, j_* W_{m+1})$ may be computed if $X = \operatorname{Spec} \mathbb{Z}$, by using Von-Staudt's Theorem and Kummer's Congruence ([3], pp. 384-385) to be also equal to the ℓ-part of $\alpha(2n)$ if $m = 2n-1$. So at least $K_{2m+1}(\mathbb{Z})(\ell)$ contains a cyclic direct summand whose order is equal to the order of the cyclic group $H^0(X_s, j_* W_{m+1})$, in support of Conjecture 2.5.

4. Generalizations of the regulator.

We conclude with some guesses as to what might happen in the cases where the zeta-function does have a zero. We must first define analogues of the regulator.

Let i be an odd integer > 1. Let F be any number field. If $i \equiv 1 \pmod 4$ we are going to define $r_1 + r_2$ maps φ_j^i, $j = 1,\ldots,r_1 + r_2$ of $K_i(\mathcal{O}_F)$ to $\mathbb{R}$. If $i \equiv 3 \pmod 4$, there will be r_2 such maps. Let $g = g_i$ be the rank of $K_i(\mathcal{O}_F)$ and note that by Theorem 2.2, g_i is also equal to $r_1 + r_2$ if $i \equiv 1 \pmod 4$, and to r_2 if $i \equiv 3 \pmod 4$. Let $\beta_1 \ldots \beta_g$ be a basis for $K_i(\mathcal{O}_F)$.

Definition 4.1. We define the m-th regulator of F, $R_m(F)$, to be $|\det|\varphi_k^{2m+1}(\beta_j)||$ as j and k both range from 1 to $g = g_{2m+1}$. Then, inspired by the classical Proposition 1.1, we ask the following question.

Question 4.2. When is it true that

$$\lim_{s\to -m} \zeta(F,s)(s+m)^{-g} = \pm \frac{\# K_{2m}(\mathcal{O}_F)}{\# K_{2m+1}(\mathcal{O}_F)_{tor}} \cdot R_m(F)?$$

It remains for us to define the φ_i's. We proceed as follows: By a result of Quillen's [10], $K_i(\mathcal{O}_F) \otimes \mathbb{Q}$ is naturally isomorphic to the space of primitive elements in $H_i(GL(\mathcal{O}_F),\mathbb{Q})$. If $i > 1$ this is the same as $H_i(SL(\mathcal{O}_F),\mathbb{Q})_{prim}$. Now, $H_i(SL(\mathcal{O}_F),\mathbb{Q})_{prim} \otimes \mathbb{R}$ $H_i(SL(\mathcal{O}_F), \mathbb{R})_{prim}$, which by a result of Borel [2], is naturally isomorphic to $H_i((SU)^{r_2} \times (SU/SO)^{r_1}, \mathbb{R})_{prim}$. We have the natural projection maps to $H_i(SU,\mathbb{R})_{prim}$, and $H_i(SU/SO, \mathbb{R})_{prim}$. If i is odd, $\pi_i(SU) \simeq \mathbb{Z}$ (mod torsion) by the Bott periodicity theorem, and the image of a generator by the Hurewicz map gives a primitive homology class in $H_i(SU,\mathbb{R})$. We then use this element to give us a natural identification of $H_i(SU,\mathbb{R})_{prim}$ with $\mathbb{R}$. Similarly, if

$i \equiv 1 \pmod 4$, $\pi_i(SU/SO) \simeq \mathbb{Z}$ (mod torsion) and we get a canonical identification of $H_i(SU/SO, \mathbb{R})_{prim}$ with $\mathbb{R}$. Putting all these isomorphisms together, we get the desired maps φ_i.

Since these higher regulators have not been computed in any single example, it is not at all clear that we have chosen the correct normalization of the φ_i's. We may, for instance, want to take a generator of $H_i(SU, \mathbb{Z})_{prim}$ instead of a spherical class to get the identification of $H_i(SU, R)_{prim}$ with $\mathbb{R}$. Also, the identifications themselves might need to be adjusted by suitable powers of π, presumably depending only on i and not on the field F.

Finally, I should say that the definition of the φ_i's is essentially due to Borel, with some modifications by Bott and Milnor, although the actual words here, and the responsibility for any errors in my interpretation of their work, are my own.

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"K-Theory and Iwasawa's Analogue of the Jacobian"

by

John Coates

Introduction. Following the initial idea of Birch and Tate, Lichtenbaum has made a remarkable conjecture relating the values of the zeta function of a totally real number field F at the odd negative integers to the orders of certain K-groups of the ring of integers of F (see [11] and his article in this volume). In the present paper, we begin by indicating the connection between this conjecture and Iwasawa's theory of Z_ℓ-extensions of number fields, and, in particular, his proposed analogue of the Jacobian for F (most of what we say is already contained in [2] and [11]). It turns out that Lichtenbaum's conjecture is very closely related to the assertion that the characteristic polynomial of the Γ-module in Iwasawa's analogue is essentially the ℓ-adic zeta function of F as constructed by Leopoldt-Kubota [10] when F is abelian over Q and by Serre [15] for all F. Unfortunately, this latter fact is still only known for a very restricted class of fields. Nevertheless, by employing some of Iwasawa's ideas, one can prove it, and thereby also Lichtenbaum's conjecture, for a class of abelian extensions of Q. We indicate some of the main points involved in such a proof. The reader interested in the full details of the proof, as well as some related material, is referred to [3] and [11]. In conclusion, it is a pleasure to express my thanks to J. Tate, both for introducing me to the subject, and for many helpful suggestions.

Notation. Throughout we use the following notation. We write Q, $\mathbb{C}$, Q_ℓ, Z_ℓ for the rational field, the complex field, the field of ℓ-adic numbers (ℓ a prime), and the ring of ℓ-adic integers, respectively. Λ will denote the ring of formal power series in an indeterminate T with coefficients in Z_ℓ, and $W^{(\ell)}$ the group of all ℓ-power roots of unity. If m is an integer ≥ 1, μ_m will signify the group of m^{th} roots of unity. The cardinality of a finite set M will be denoted by $\#(M)$. Finally, if E/F is a Galois extension of fields, $G(E/F)$ will denote the Galois group of E over F.

1. Iwasawa's Analogue. In this section, we briefly describe Iwasawa's proposed analogue of the Jacobian for totally real number fields, and indicate its connection with one form of Lichtenbaum's conjecture about the values of the complex zeta function of the field at the odd negative integers.

Let F be a totally real number field of finite degree over Q. Let l be an odd prime number, and let $F_o = F(\mu_\ell)$, $F_\infty = F(W^{(\ell)})$. Then, of course, $\Gamma = G(F_\infty/F_o)$ is non-canonically isomorphic to the additive group of Z_ℓ. For each $n \geq 0$, let F_n be the unique sub-extension of F_∞/F_o of degree l^n over F_o, and let A_n be the l-primary subgroup of the ideal class group of F_n. If $n \leq m$, the natural inclusion of the divisor group of F_n in the divisor group of F_m induces a homomorphism $A_n \to A_m$, and we let $A = \varinjlim A_n$. Let J denote complex conjugation. Since F is totally real, there is a natural action of J on A, which is easily seen to be independent of the particular embedding of F_∞ into $\mathbb{C}$. If B is any Z_ℓ-module on which J operates, we put $B^+ = (1+J)B$, $B^- = (1-J)B$. Now, for reasons which will become clear in the next paragraph, we shall only be concerned with the $G(F_\infty/F)$-module A^-. Let χ be the character of $G(F_\infty/F)$ with values in the group of units of Z_ℓ, defined by $\sigma(\zeta) = \zeta^{\chi(\sigma)}$ for all $\zeta \in W^{(\ell)}$. Plainly, $G(F_o/F) = H \times \Gamma$, where H is canonically isomorphic to $G(F_\infty/F)$. We denote the restriction of χ to H by θ, and the restriction of χ to Γ by κ. Since $d = [F_o : F]$ is prime to l, the orthogonal idempotent e_{θ^i} associated with each power of θ lies in the group ring $Z_\ell[H]$. For each odd integer i with $1 \leq i \leq d-1$, put ${}^iA = e_{\theta^{-i}}A^-$, so that

$$A^- = \bigoplus_{\substack{i=1 \\ i \text{ odd}}}^{d-1} {}^iA .$$

Let $\widehat{{}^iA} = \mathrm{Hom}({}^iA, Q_\ell/Z_\ell)$ be the Pontrjagin dual of the discrete group ${}^iA$. We define an action of Γ on $\widehat{{}^iA}$ by specifying that $(\sigma\phi)(a) = \phi(\sigma a)$ for all $\sigma \in \Gamma$, $\phi \in \widehat{{}^iA}$, and $a \in {}^iA$. Fix a topological generator γ_o of Γ. Then as is well known, the Γ-structure on $\widehat{{}^iA}$ gives rise to a unique Λ-module structure on

$\widehat{{}^iA}$ such that $\gamma_0\phi = (1+T)\phi$ for all $\phi \in \widehat{{}^iA}$. Iwasawa [9] has proven the following basic facts about this Λ-structure, by using arguments from class field theory. Firstly, $\widehat{{}^iA}$ is a finitely generated Λ-torsion Λ-module, and secondly, $\widehat{{}^iA}$ has no Λ-submodule of finite cardinality. Thus the structure theory of finitely generated Λ-modules implies that there exists an integer $r_i \geq 1$ and non-zero power series $f_{1i}(T), \ldots, f_{r_i i}(T)$ in Λ such that we have an exact sequence

$$(1) \qquad 0 \to \widehat{{}^iA} \to \bigoplus_{j=1}^{r_i} \Lambda/(f_{ji}(T)) \to D_i \to 0 ,$$

where D_i is a Λ-module of finite cardinality. Moreover, assuming the choice of γ_0 fixed, the power series $f_{ji}(T)$ are uniquely determined by $\widehat{{}^iA}$ up to units in Λ. We often call, by a slight abuse of language, $f_i(T) = \prod_{j=1}^{r_i} f_{ji}(T)$ the characteristic polynomial of $\gamma_0 - 1$ acting on ${}^iA$.

Let C be a complete, non-singular curve of genus ≥ 1 defined over a finite field k, and let $\mathcal{J}$ be the Jacobian variety of C. Assume that l is distinct from the characteristic of k, and let $\mathcal{J}_\ell$ be the l-primary subgroup of the group of points of $\mathcal{J}$ defined over the algebraic closure $\bar{k}$ of k. The Frobenius automorphism of $\bar{k}/k$ induces an endomorphism of $\mathcal{J}_\ell$, and a fundamental theorem of Weil asserts that the characteristic polynomial of this endomorphism is essentially the zeta function of the curve C. Iwasawa has proposed that, in the number field case, the $G(F_\infty/F)$-module A^- should provide an analogue of $\mathcal{J}_\ell$. The basic conjecture underlying such an analogy is that the characteristic polynomials $f_i(T)$ of the $\widehat{{}^iA}$ $(1 \leq i \leq d-1,\ i$ odd) should be very closely related to the l-adic zeta functions of F in the sense of Leopoldt-Kubota [10], thereby giving a result for number fields parallel to Weil's theorem. From our point of view, the most natural way to formulate this conjecture precisely is in terms of the $G(F_\infty/F)$-invariants of certain twisted versions of A^-. Let $\mathfrak{J}$ denote the $G(F_\infty/F)$-module $\varprojlim \mu_{\ell^n}$. If B is a discrete l-primary $G(F_\infty/F)$-module, and n is a positive integer, $B(n)$ will denote the tensor product of B over $\mathbb{Z}_\ell$ with the n-fold tensor product of $\mathfrak{J}$ with itself over $\mathbb{Z}_\ell$. Of course, since $\mathfrak{J}$ is a free Z_ℓ-module of rank 1, $B(n)$ is isomorphic to B as an abelian group. However, they are definitely not isomorphic as $G(F_\infty/F)$-modules, since we shall always view $B(n)$ as a $G(F_\infty/F)$-module via the diagonal action on the tensor product. For each integer $r \geq 1$, let $w_r(F)$ denote the largest integer m such that $G(F(\mu_m)/F)$ is annihilated by r. Finally, let $\zeta(F,s)$ be the complex zeta function of F. We recall that Siegel [16] has proven that, for each odd positive integer n, $\zeta(F,-n)$ is a non-zero rational number.

Conjecture 1. For each odd positive integer n, $(A^-(n))^{G(F_\infty/F)}$ is finite, and its order is equal to the l-part of $w_{n+1}(F)\zeta(F,-n)$.

Special cases of this conjecture have already been proven. We discuss these, as well as other evidence for the conjecture, in §2 and §3. For the moment, we simply translate the conjecture into several equivalent forms. If B is a Γ-module, let $(B)_\Gamma$ denote $B/(\gamma_0-1)B$. Also, let $|\ |_\ell$ be the valuation of Q_ℓ, normalized as usual so that $|l|_\ell = l^{-1}$.

Lemma 2. For all $n \geq 0$, the following assertions are equivalent:

i) $({}^iA(n))^\Gamma$ *is finite,*

ii) $({}^iA(n))_\Gamma = 0$,

iii) $f_i(\kappa(\gamma_0)^{-n}-1) \neq 0$.

If these assertions do hold, the order of $({}^iA(n))^\Gamma$ is $|f_i(\kappa(\gamma_0)^{-n}-1)|_\ell^{-1}$.

This lemma is quite elementary, and we refer the reader to §7 of [3] for its proof.

Proposition 3. Let i be a fixed odd integer with $1 \leq i \leq d-1$. Then, for all integers $n \geq 0$ with $n \equiv i \bmod d$, we have

i) $(A^-(n))^{G(F_\infty/F)}$ *is finite if and only if* $f_i(\kappa(\gamma_0)^{-n}-1) \neq 0$, *and*

ii) *if* $(A^-(n))^{G(F_\infty/F)}$ *is finite, then its order is* $|f_i(\kappa(\gamma_0)^{-n}-1)|_\ell^{-1}$.

Recall that the action of H on $\mathfrak{J}$ is given by $\sigma t = \theta(\sigma)t$ for $\sigma \in H$, whence it is easily seen that $(A^{-}(n))^{H} = {}^{i}A(n)$ for all integers n with $n \equiv i \bmod d$. Thus Proposition 3 follows immediately from Lemma 2. Note that the finite Λ-modules D_i do not appear in Proposition 3.

In view of Proposition 3, we see that Conjecture 1 is equivalent to the following statement. Fix an odd integer i with $1 \le i \le d-1$. Then, for all positive integers n with $n \equiv i \bmod d$, we have

$$f_i(\kappa(\gamma_o)^{-n}-1) \neq 0 \quad \text{and} \quad |f_i(\kappa(\gamma_o)^{-n}-1)|_{\ell} = |w_{n+1}(F)\zeta(F,-n)|_{\ell} .$$

This suggests that the power series $f_i(T)$ are very closely related to the l-adic zeta functions of F constructed by Leopoldt-Kubota [10] when F is abelian over Q, and recently by Serre [15] for all totally real F. However, we cannot be more precise at this point because the $\widehat{{}^{i}A}$ do not provide us with a canonical choice of the undetermined unit in Λ, which is implicit in our definition of the $f_i(T)$.

Finally, following Lichtenbaum [11], we give an equivalent form of Conjecture 1 in terms of étale cohomology. We refer the reader to [1] for the basic facts about étale cohomology. Let $\mathcal{O}$ be the ring of integers of F, and X the spectrum of the ring $\mathcal{O}[\frac{1}{\ell}]$. Let $j : \text{Spec}(F) \to X$ be the natural inclusion. Let $\overline{F}$ denote the algebraic closure of F. For each $n \ge 0$, we can view the $G(\overline{F}/F)$-module $W^{(\ell)}(n)$ as a sheaf for the étale topology of $\text{Spec}(F)$, and we may take its direct image $j_*W^{(\ell)}(n)$ on X. By definition, $H^0(X, j_*W^{(\ell)}(n)) = (W^{(\ell)}(n))^{G(\overline{F}/F)}$, and it is easily seen that the order of this latter group is the l-part of $w_{n+1}(F)$.

<u>Proposition</u> 4. <u>For all odd positive integers</u> n, <u>we have</u>

i) $H^1(X, j_*W^{(\ell)}(n))$ <u>is canonically isomorphic to</u> $(A^{-}(n))^{G(F_\infty/F)}$, <u>and</u>

ii) $H^1(X, j_*W^{(\ell)}(n))$ <u>is finite if and only if</u> $H^i(X, j_*W^{(\ell)}(n)) = 0$ <u>for all</u> $i \ge 2$.

The proposition follows from Lemma 2 on noting that, on the one hand, it is shown in § 9 of [11] that we have canonical isomorphisms

$$H^1(X, j_* W^{(\ell)}(n)) \tilde{\to} ({}^iA(n))^{\Gamma}, \quad H^2(X, j_* W^{(\ell)}(n)) \tilde{\to} ({}^iA(n))_{\Gamma}$$

for all n with $n \equiv i \bmod d$, and, on the other hand, that we always have $H^k(X, j_* W^{(\ell)}(n)) = 0$ for $k \geq 3$, by a general theorem on cohomological dimension.

We conclude from Proposition 4 that Conjecture 1 is valid if and only if $H^k(X, j_* W^{(\ell)}(n)) = 0$ for all $k \geq 2$, and

$$|\zeta(F,-n)|_{\ell}^{-1} = \frac{\#(H^1(X, j_* W^{(\ell)}(n)))}{\#(H^0(X, j_* W^{(\ell)}(n)))}.$$

The beauty of this formulation of the conjecture is that it gives some indication of why the factor $w_{n+1}(F)$ arises naturally in the theory.

2. The Analytic Theory. In this section, we indicate a proof of Conjecture 1 for a class of abelian extensions of Q. We only sketch some of the arguments involved, and the reader is referred to [3] and [11] for full details. The method of proof is based on the important ideas introduced by Iwasawa in [8]. These, in turn, have their origins in a classical theorem of Stickelberger [17], and the classical analytic class number formula [6].

We use the notation of §1, the prime number l being odd, as before. Also, F_o^+ will denote the maximal totally real subfield of F_o, so that $[F_o : F_o^+] = 2$. We assume throughout this section that F is an *abelian* extension of Q. We first establish the following rather weak consequence of Conjecture 1.

Theorem 5. Assume that (i) l does not divide $[F:Q]$, and (ii) no prime of F_o^+ lying above l splits in F_o. Then, for each odd positive integer n, we have $(A^-(n))^{G(F_\infty/F)} = 0$ if l does not divide $w_{n+1}(F)\zeta(F,-n)$.

The special role that the primes l not satisfying (ii) play in the theory will be explained in §3. For the present, we simply note that (ii) excludes only finitely many primes since l must certainly ramify in F if (ii) is not valid.

Theorem 5 is quite useful for studying particular fields, For example, if we take the two quadratic fields $F_1 = Q(\sqrt{11})$, $F_2 = Q(\sqrt{19})$, it is easily seen that (i) and (ii) exclude no primes l (except $l = 2$) . Since $w_2(F_1) = w_2(F_2) = 2^3 \cdot 3$, and $\zeta(F_1,-1) = \pm 7/(2.3)$, $\zeta(F_2,-1) = \pm 19/(2.3)$, we conclude from Theorem 5 that $(A^-(l))^{G(F_\infty/F)} = 0$ for all primes $l \neq 7$ for F_1 , and for all primes $l \neq 19$ for F_2 .

<u>Proof of Theorem</u> 5. Let χ be a primitive Dirichlet character satisfying $\chi(-1) = -1$. We view the values of χ as lying in the algebraic closure of Q_ℓ , and let $\mathcal{O}_\chi$ be the ring generated over Z_ℓ by the values of χ . Let Λ_χ be the ring of formal power series in T with coefficients in $\mathcal{O}_\chi$. In [8], Iwasawa has associated with χ an element $g(T;\chi)$ of the quotient field of Λ_χ . Define $f(T;\chi)$ to be either $g(T;\chi)$ or $(T-1)g(T;\chi)$, according as $\chi \neq \tilde{\omega}$ or $\chi = \tilde{\omega}$; here $\tilde{\omega}$ is the Dirichlet character modulo l satisfying $\tilde{\omega}(a) \equiv a \bmod l\, Z_\ell$ for all integers a . We shall only consider those χ which have order prime to l , and, in this case, $f(T;\chi)$ is an element of Λ_χ . Also, it is not difficult to see (cf. [7]) that $f(T;\tilde{\omega})$ is in fact a unit in Λ . Finally, for each positive integer n , let B_χ^n be the n^{th} Bernoulli number associated with χ in the sense of Leopoldt [12].

Now $F_o = F(\mu_\ell)$ is abelian over Q . Thus we can associate with each absolutely irreducible character ϕ of $G(F_o/Q)$ a primitive Dirichlet character $\tilde{\phi}$ in the usual way. In particular, if ω is the character of $G(Q(\mu_\ell)/Q)$ given by $\sigma\zeta = \zeta^{\omega(\sigma)}$ for all $\zeta \in \mu_\ell$, then $\tilde{\omega}$ is just the character described in the last paragraph. If Φ is the character of a representation of $G(F_o/Q)$ irreducible over Q_ℓ , let e_Φ be the associated orthogonal idempotent in the group ring $Z_\ell[G(F_o/Q)]$. Let I denote the set of characters of representations of $G(F/Q)$ which are irreducible over Q_ℓ . Fix, for the rest of the proof, an odd positive integer n . Then, with H defined as in §1, we see easily that

(2) $$(A^-(n))^H = \bigoplus_{\Phi\in I} (e_{\Phi\omega^{-n}}A^-)(n) .$$

For each $\Phi \in I$, put $\Phi^* = \Phi\omega^{-n}$. Note that, since Φ is real and n is odd, Φ^* is imaginary. Let ϕ be an absolutely irreducible component of Φ , and $\phi^* = \phi\omega^{-n}$ the corresponding component of Φ^* . Then, if q_o denotes the least common multiple of l and the conductor of ϕ^* , it is shown in [8] that

$$g(0; \widetilde{\phi^*}) = (1-\widetilde{\phi^*}^{-1}(l))B^1_{\widetilde{\phi^*}} , \quad g((1+q_o)^{-n}-1; \widetilde{\phi^*}) = (1-\widetilde{\phi}^{-1}(l)l^n)B^{n+1}_{\widetilde{\phi^{-1}}}/(n+1) .$$

We denote by $\mathfrak{I}$ the set of absolutely irreducible characters of $G(F/Q)$ which are distinct from ω^{n+1} (observe that ω^{n+1} is a character of $G(F/Q)$ if and only if $[F_o:F]$ divides $n+1$) . Now assume that ϕ is any element of $\mathfrak{I}$. Since $\phi \neq \omega^{n+1}$, we have $\phi^* \neq \omega$, and thus $g(T;\widetilde{\phi^*})$ is in Λ_ϕ . Consequently, $g(0;\widetilde{\phi^*}) \equiv g((1+q_o)^{-n}-1; \phi^*)$ mod $l\mathcal{O}_\phi$, and both values lie in $\mathcal{O}_\phi$. Further, it is easy to see using class field theory that our hypothesis that no prime of F_o^+ above l splits in F_o implies that $\widetilde{\phi^*}(l) \neq 1$, whence $1 - \phi^{*^{-1}}(l)$ is a unit in $\mathcal{O}_\phi$ because $(l, [F_o:Q]) = 1$. Thus we conclude that

(3) $$B^1_{\widetilde{\phi^*}} \equiv u \frac{B^{n+1}_{\widetilde{\phi^{-1}}}}{n+1} \bmod l\mathcal{O}_\phi ,$$

where u is a unit in $\mathcal{O}_\phi$.

Next we show that

(4) $$w_{n+1}(F)\zeta(F,-n) = v \prod_{\phi\in\mathfrak{I}} \frac{B^{n+1}_{\widetilde{\phi^{-1}}}}{n+1} ,$$

where v is a unit in Z_l . For, by the decomposition of $\zeta(F,s)$ into a product of L-series, we have $\zeta(F,n) = \pm \prod_\phi B^{n+1}_{\widetilde{\phi^{-1}}}/n+1$, where the product is taken over all absolutely irreducible characters ϕ of $G(F/Q)$. The proof of (4) divides into two cases according as ω^{n+1} is not or is a character of $G(F/Q)$. If ω^{n+1} is not a character of $G(F/Q)$, (4) is clear because $\mathfrak{I}$ contains all characters of $G(F/Q)$ and $w_{n+1}(F)$ is not divisible by l since $[F_o:F]$ does not divide $n+1$.

On the other hand, if $\phi = \omega^{n+1}$ is a character of $G(F,Q)$, then $\phi^* = \omega$, and, as $(T-1)g(T;\tilde{\omega})$ is a unit in Λ, it follows that its value at $(1+\ell)^{-n}-1$, namely, $\{(1+\ell)^{-n} - (1+\ell)\}B^{n+1}_{\tilde{\omega}^{-(n+1)}}/(n+1)$, is a unit in Z_ℓ. But then, since $[F_0:F]$ divides $n+1$, it is not difficult to prove (see §6 of [11]) that the power of l dividing both $(1+\ell)^{-(n+1)}-1$ and $w_{n+1}(F)$ is the same, as required.

Now assume that l does not divide $w_{n+1}(F)\zeta(F,-n)$. Since each term in the product on the right of (4) is integral at l, it follows that $B^{n+1}_{\widetilde{\phi^{-1}}}/(n+1)$ is a unit in $\mathcal{O}_\phi$ for all $\phi \in \mathcal{S}$. We then conclude from (3) that $B^1_{\widetilde{\phi^*}}$ is a unit in $\mathcal{O}_\phi$ for all $\phi \in \mathcal{S}$. Let ϕ be any element of $\mathcal{S}$, and let K be the fixed field of the kernel of ϕ^*. We write ψ for the character of $G(K/Q)$ induced by ϕ^*. Let Φ^*, Ψ be the sum of the conjugates of ϕ^*, ψ over Q_ℓ, and let e_Ψ be the orthogonal idempotent corresponding to Ψ in the group ring $R = Z_\ell[G(K/Q)]$. Now, if f denotes the conductor of ψ, let α be the element of $Q_\ell[G(K/Q)]$ defined by

$$\alpha = \frac{1}{f} \sum_{\substack{a=1 \\ (a,f)=1}}^{f} a\left(\frac{K}{a}\right)^{-1} ;$$

here $\left(\frac{K}{a}\right)$ is the restriction to K of the automorphism of $Q(\mu_f)$, which raises each element of μ_f to the a^{th} power. It is easily seen that $e_\Psi\alpha$ is in R, and it is plain that $e_\Psi\alpha$ is mapped to $B^1_{\tilde{\psi}}$ under the ring isomorphism $e_\Psi R \tilde{\to} \mathcal{O}_\psi$ which is induced by the map $g \mapsto \psi(g)$. Thus $e_\Psi\alpha$ is a unit in the ring $e_\Psi R$. But, by a classical theorem of Stickelberger [17], $e_\Psi\alpha$ annihilates $e_\Psi\mathfrak{A}$, where $\mathfrak{A}$ denotes the l-primary subgroup of the ideal class group of K, whence we conclude that $e_\Psi\mathfrak{A} = 0$. Now, on the one hand,,the natural map from $\mathfrak{A}$ to A_0 induces an isomorphism $e_\Psi\mathfrak{A} \tilde{\to} e_{\Phi\omega^{-n}}A_0^-$ because $(\ell, [F_0:K]) = 1$, and, on the other hand, it can be shown (see §2 of [3]), that our hypothesis that no prime of F_0^+ lying above l splits in F_0 implies that $e_{\Phi\omega^{-n}}A_0^- = (e_{\Phi\omega^{-n}}A^-)^\Gamma$. Hence $(e_{\Phi\omega^{-n}}A^-)^\Gamma = 0$, whence, by a basic property of discrete Γ-modules, $e_{\Phi\omega^{-n}}A^- = 0$.

This argument applies to all characters Φ of $G(F_0/Q)$, which are irreducible over Q_ℓ, except $\Phi = \omega^{n+1}$. However, a similar argument to the above, using the fact that $(T-1)g(T;\tilde{\omega})$ is a unit in Λ, shows that we always have $e_\omega A^- = 0$. Thus, in view of (2), we have certainly shown that $(A^-(n))^{G(F_\infty/F)} = 0$ if l does not divide $w_{n+1}(F)\zeta(F,-n)$.

Much of the above proof is classical and well known. In particular, the congruence (3) was pointed out several years ago in letters of Iwasawa and Brumer to Tate, and special cases of it are probably very old. The reader should also note that the above argument could be considerably simplified, and the conclusion of Theorem 5 strengthened, if the following unknown assertion could be proven in general. For each character $\Phi \neq \omega^{n+1}$ of an imaginary representation of $G(F_0/Q)$ irreducible over Q_ℓ, the order of $e_\Phi A_0^-$ is the exact power of l dividing $\prod_\phi B^1_{\tilde{\phi}}$, where the product is taken over all absolutely irreducible components ϕ of Φ.

We next discuss a general conjecture, in the spirit of the proof of Theorem 5, from which we can derive the full conclusion of Conjecture 1. Let F be a totally real abelian extension of Q, and let l be an odd prime number which does not divide $[F:Q]$. Let Φ be the character of an imaginary representation of $G(F_0/Q)$ irreducible over Q_ℓ, ϕ an absolutely irreducible component of Φ, and let $f(T;\tilde{\phi})$ be the associated power series in Λ_ϕ, which is defined at the beginning of the proof of Theorem 5. Let $A_\Phi = e_\Phi A^-$, and let $\hat{A}_\Phi =$ Hom $(A_\Phi, Q_\ell/Z_\ell)$ be the Pontrjagin dual of A_Φ, endowed with a Γ-module structure in the same way as described in §1. Let $q_0(\phi)$ be the least common multiple of l and the conductor of $\tilde{\phi}$, and let γ_0 be the unique topological generator of Γ such that $\kappa(\gamma_0) = 1 + q_0(\phi)$.

<u>Conjecture</u> 6. <u>For each character</u> Φ <u>of an imaginary representation of</u> $G(F_0/Q)$, <u>irreducible over</u> Q_ℓ, <u>there is an exact sequence of</u> Λ-<u>modules</u>

$$0 \to \hat{A}_\Phi \to \Lambda_\phi/(f(T;\tilde{\phi})) \to D_\Phi \to 0 \ ,$$

where D_Φ *is a finite* Λ-*module*.

Theorem 7. *If Conjecture* 6 *is valid for* F *and* l , *then Conjecture* 1 *is valid for* F , l , *and all odd positive integers* n .

Proof. By (2) above, we have

$$(5) \qquad (A^-(n))^{G(F_\infty/F)} = \bigoplus_{\Phi \in I} (A_{\Phi\omega^{-n}}(n))^\Gamma \ ,$$

where, as before, I denotes the set of characters of representations of $G(F/Q)$ which are irreducible over Q_ℓ . To compute the order of the Γ-invariants on the right, we first note the following facts about Γ-modules. If B is a discrete Γ-module, and $C = \text{Hom}\,(B, Q_\ell/Z_\ell)$ is its Pontrjagin dual, we always assume that the Γ-structure on C is given by $(\gamma c)(b) = c(\gamma b)$, where $\gamma \in \Gamma$, $c \in C$, and $b \in B$. Thus, in particular, it follows that $(B)^\Gamma$ is dual to $(C)_\Gamma$. Also, let $B[n]$ denote the Γ-module having the same underlying group as B , but with a new action of Γ given by $\gamma \circ b = \kappa(\gamma)^n \gamma b$, the latter action being the original one. We define $C[n]$ in the same way. It is therefore clear that $C[n]$ can be identified with the Pontrjagin dual of $B[n]$. Note also that $B[n]$ is non-canonically Γ-isomorphic to $B(n)$. Now, applying these remarks to our particular situation, we conclude that $(A_{\Phi^*}(n))^\Gamma$ is dual to $(\hat{A}_{\Phi^*}[n])_\Gamma$, where, as before, $\Phi^* = \Phi\omega^{-n}$. Further, if $C = \Lambda_\phi/(f(T;\phi^*))$, then it is easily seen that $C[n]$ is Λ-isomorphic to $\Lambda_\phi/(f_n(T;\phi^*))$, where

$$f_n(T;\phi^*) = f((1+q_o(\phi^*))^{-n}(1+T)-1) \ .$$

Writing $E = \hat{A}_{\Phi^*}$, $D = D_{\Phi^*}$, the validity of Conjecture 6 implies that we have an exact sequence

$$(6) \qquad 0 \to E[n] \to C[n] \to D[n] \to 0 \ .$$

Note that, in view of the explicit formula for $f_n(0;\phi^*)$ derived in [8], we have $f_n(0;\phi^*) \neq 0$. It follows easily that $(C[n])^{\Gamma} = 0$ and $(C[n])_{\Gamma}$ is finite of order $|f_n(0;\phi^*)|_{\ell}^{-1}$. Hence, applying the snake lemma to (6), we obtain the exact sequence

$$0 \to (D[n])^{\Gamma} \to (E[n])_{\Gamma} \to (C[n])_{\Gamma} \to (D[n])_{\Gamma} \to 0 .$$

But, as $D[n]$ is finite, $(D[n])^{\Gamma}$ and $(D[n])_{\Gamma}$ have the same order, whence $(E[n])_{\Gamma}$ and $(C[n])_{\Gamma}$ also have the same order, namely $|f_n(0;\phi^*)|_{\ell}^{-1}$. Recalling that we always have $A_{\omega} = 0$, the conclusion of Conjecture 1 follows from (4) and (5).

By using Iwasawa's methods [7], we have been able to prove Conjecture 6 in some cases.

Theorem 8. Assume that l is an odd prime number such that (i) l does not divide $[F:Q]$, (ii) no prime of F_0^+ lying above l splits in F_0, and (iii) A_0^- is cyclic as a module over $Z_{\ell}[G(F_0/Q)]$. Then Conjecture 2 is valid for F and l.

For the proof of Theorem 8, which involves similar ideas to those given above in the proof of Theorem 5, we refer the reader to [3]. Unfortunately, hypothesis (iii) is very restrictive, and difficult to verify for any particular field. Nevertheless, it can sometimes be verified by using tables of class numbers [13]. For example, if we take $F_1 = Q(\sqrt{11})$, $l = 7$, or $F_2 = Q(\sqrt{19})$, $l = 19$, we conclude easily from the tables [13] that (iii) is valid. Hence, in view of the remarks after Theorem 5, we see that $(A^-(1))^{G(F_{\infty}/F)}$ has order 7 in the first example, and order 19 in the second.

3. Divisibility Assertions. Let F be any totally real finite extension of Q. A particular consequence of Conjecture 1 would be that, for each odd positive integer n, $w_{n+1}(F)\zeta(F,-n)$ is integral at l for all primes l

(although $\ell = 2$ has been excluded in our discussion, it can be included if one uses a different formulation of Conjecture 1, cf. [11]). Such an integrality result was first conjectured by Serre [14], who proved it for $n = 1$. It is still unknown for $n > 1$. However, it is shown in [3] that the validity of Conjecture 1 would imply an even stronger result than this integrality assertion. Assume again that ℓ is odd. If $\wp$ is a prime of F, let $F_{\wp}$ denote the completion of F at $\wp$. Also, if K is any field, let $w_n^{(\ell)}(K)$ be the largest power of ℓ, say ℓ^r, such that $G(K(\mu_{\ell^r})/K)$ has exponent n.

Theorem 9 (Lichtenbaum). Let n be an odd positive integer, and assume that $(A^-(n))^{G(F_\infty/F)}$ is finite. Then the order of $(A^-(n))^{G(F_\infty/F)}$ is divisible by $\prod_{\wp/\ell} w_n^{(\ell)}(F_{\wp})$, where the product is taken over all primes $\wp$ of F lying above ℓ.

Note that, since n is odd, the term $\prod_{\wp/\ell} w_n^{(\ell)}(F_{\wp})$ is greater than 1 for some $n > 1$ if and only if at least one prime of F_o^+ lying above ℓ splits in F_o.

Conjecture 10. Let n be an odd positive integer. Then $w_{n+1}(F)\zeta(F,-n)$ is an ℓ-integer, which is divisible by $\prod_{\wp/\ell} w_n^{(\ell)}(F_{\wp})$, where the product is taken over all primes $\wp$ of F lying above ℓ.

It is not difficult to see that Theorem 9 and Conjecture 10 are very closely related to the existence of a zero at $T = 0$ of a certain order for the various power series discussed in §1 and §2. For example, using the existence of this zero for certain of the Iwasawa power series $g(T;\chi)$, the following result is proven in [3].

Theorem 11. Assume that F is a totally real abelian extension of Q. Then Conjecture 10 is true for F and all odd primes ℓ.

On the other hand, Theorem 9 implies the following result about the power series $f_i(T)$ $(1 \le i \le d-1$, i odd, $d = [F_o:F])$ introduced in §1.

Theorem 12. *Assume that* F *is any totally real finite extension of* Q . *Then, for each odd integer* i *with* $1 \le i \le d-1$, $f_i(T)$ *has a zero at* $T = 0$ *of order greater than or equal to* $s(i)$, *where* $s(i)$ *denotes the number of primes* $\mathfrak{P}$ *of* F *lying above* l *such that* $[F_{\mathfrak{P}}(\mu_\ell) : F_{\mathfrak{P}}]$ *divides* i .

Proof. Let $\mathfrak{P}$ be any prime of F lying above l such that $[F_{\mathfrak{P}}(\mu_\ell) : F_{\mathfrak{P}}]$ divides i . It is plain that, for all integers $m \ge 0$, $[F_{\mathfrak{P}}(\mu_{\ell^{m+1}}) : F_{\mathfrak{P}}]$ divides $l^m i$, or equivalently that $w^{(\ell)}_{\ell^m i}(F_{\mathfrak{P}})$ is divisible by l^{m+1} . Now, since d divides $l-1$, it is also clear that the integers $l^m i$ $(m = 0,1,\dots)$ are all congruent modulo d . Further, as $f_i(T)$ has only finitely many zeros, we have $f_i(\kappa(\gamma_o)^{-\ell^m i}-1) \ne 0$ for all sufficiently large m . It then follows from Proposition 3 that $(A^-(l^m i))^{G(F_\infty/F)}$ is finite for all sufficiently large m , whence, again by Proposition 3 and Theorem 9, we conclude that $f_i(\kappa(\gamma_o)^{-\ell^m i}-1)$ is divisible by $l^{(m+1)s(i)}$. Letting m tend to infinity, we easily see that $f_i(T)$ must have a zero at $T = 0$ of order $\ge s(i)$.

Recently, R. Greenberg [5] has shown that, when F is a totally real abelian extension of Q , and l is any odd prime number, then the order of the zero of $f_i(T)$ at $T = 0$ is exactly $s(i)$ for all odd i with $1 \le i \le d-1$. His proof makes essential use of the p-adic analogue of Baker's theorem on linear forms in the logarithms of algebraic numbers.

So far, no proof of Conjecture 10 has been found for non-abelian extensions F of Q , although we have verified special cases of it for many particular fields by direct computations. We mention two examples. Let $F_1 = Q(\theta_1)$, $F_2 = Q(\theta_2)$, where θ_1 is a root of $X^3 - 9X + 1$, and θ_2 is a root of $X^3 - 6X + 2$. The discriminant of F_1 is 3.107 and that of F_2 is $2^2 \cdot 3^3 \cdot 7$. It is readily verified that Conjecture 10 predicts that $w_2(F_1)\zeta(F_1,-1)$, $w_4(F_1)\zeta(F_1,-3)$,

$w_4(F_2)\zeta(F_2,-3)$ should be integers divisibly by 3, 3^2, and 7, respectively. This is indeed the case, because direct computations show that $w_2(F_1) = 2^3\cdot 3$, $w_4(F_1) = w_4(F_2) = 2^4\cdot 3\cdot 5$, and $\zeta(F_1,-1) = \pm 1$, $\zeta(F_1,-3) = \pm (3\cdot 5\cdot 37)/2$, $\zeta(F_2,-3) = \pm (7^2\cdot 3589)/(2\cdot 3\cdot 5)$.

4. Connection with K-theory. In this last section, we briefly discuss the relationship of Conjecture 1 with K-theory. We use the notation of §1. Thus F is any totally real finite extension of Q, l is an odd prime number, $F_o = F(\mu_\ell)$, etc. Let $\mathcal{O}$ denote the ring of algebraic integers in F.

Theorem 13. The l-primary subgroup of $K_2\mathcal{O}$ is canonically isomorphic to $(A^-(1))^{G(F_\infty/F)}$.

Conjecture 14. For each odd positive integer n, the l-primary subgroup of $K_{2n}\mathcal{O}$ is canonically isomorphic to $(A^-(n))^{G(F_\infty/F)}$.

Note the following consequences of Theorem 13 and our earlier results.

Corollary 15. $(A^-(1))^{G(F_\infty/F)}$ is finite, or equivalently $f_1(\kappa(\gamma_0)^{-1}-1) \neq 0$.

For, by Garland's theorem [4], $K_2\mathcal{O}$ is a finite group.

Corollary 16. Let F be a totally real abelian extension of Q. Let $\mathcal{S}$ be the finite set of rational primes consisting of $l = 2$, and all l such that either l divides $[F:Q]$, or at least one prime of F_o^+ lying above l splits in F_o. Then, if $l \notin \mathcal{S}$, l divides the order of $K_2\mathcal{O}$ only if l divides $w_2(F)\zeta(F,-1)$. Further, if $l \notin \mathcal{S}$, and A_o^- is cyclic over the group ring $Z_\ell[\![G(F_o/Q)]\!]$, the order of the l-primary subgroup of $K_2\mathcal{O}$ is the exact power of l dividing $w_2(F)\zeta(F,-1)$.

This is clear from Theorem 13 and Theorems 5 and 8. In particular, if we consider the two examples mentioned before, namely $F_1 = Q(\sqrt{11})$, $F_2 = Q(\sqrt{19})$, then, in both cases, $\mathcal{S} = \{2\}$, and we conclude that (writing $\mathcal{O}_1$, $\mathcal{O}_2$ for the rings of integers of F_1 , F_2) $\#(K_2\mathcal{O}_1) = 4\cdot 7$, $\#(K_2\mathcal{O}_2) = 4\cdot 19$, except perhaps for the 2-primary subgroups. In fact, a simple direct argument enables us to verify that the above orders are correct even for the 2-primary subgroup.

Sketch of the proof of Theorem 13. We first remark that, by Quillen's long exact sequence [18], the inclusion of $\mathcal{O}$ in F induces an isomorphism from $K_2\mathcal{O}$ onto $\operatorname{Ker}\lambda_F$, where $\lambda_F : K_2F \to \bigoplus_{\wp} k_{\wp}^{\times}$ is the homomorphism induced by the tame symbols (here $\wp$ runs over all finite primes of F , and $k_{\wp}^{\times}$ denotes the multiplicative group of the residue field of $\wp$) . Let I'_∞ be the free abelian group generated by the non-archimedean primes of F_∞ which do not lie above l . Since only the primes above l are ramified in the extension F_∞/F , and since there are only finitely many primes of F_∞ lying above each finite rational prime, we have the natural map from $F_\infty^{\times}$ to I'_∞ which associates to a field element its divisor outside l . This gives rise to a homomorphism $(Q_\ell/Z_\ell)\otimes_Z F_\infty^{\times} \to (Q_\ell/Z_\ell)\otimes_Z I'_\infty$, and we define $\mathfrak{M}$ to be the kernel of this homomorphism. Now $\mathfrak{M}$ is a discrete l-primary $G(F_\infty/F)$-module, and so, in particular, it has the decomposition $\mathfrak{M} = \mathfrak{M}^+ \oplus \mathfrak{M}^-$. It is shown in [2] (see Theorems 6 and 11) or [11] (see §7), and we do not repeat the arguments here, that Tate's cohomological description of K_2F (see his article in this volume) implies that, since F is totally real, the l-primary subgroup of $\operatorname{Ker}\lambda_F$ is canonically isomorphic to $(\mathfrak{M}^-(1))^{G(F_\infty/F)}$. Theorem 13 then follows immediately from this result and the corollary of the following lemma. Let $\mathcal{O}_\infty$ be the ring of algebraic integers in F_∞ , and let E_∞ be the group of units of $\mathcal{O}_\infty$ (note that we are not taking the group of units of the ring $\mathcal{O}_\infty[1/l]$) . It is very easy to see that the inclusion of E_∞ in F_∞ induces an injection $(Q_\ell/Z_\ell)\otimes_Z E_\infty \to \mathfrak{M}$.

Lemma 17. *There is a canonical* $G(F_\infty/F)$-*homomorphism* $\phi : \mathfrak{M} \to A$ *such that the sequence*

$$0 \to (Q_\ell/Z_\ell) \underset{Z}{\otimes} E_\infty \to \mathfrak{M} \overset{\phi}{\to} A \to 0$$

is exact.

Corollary. $\mathfrak{M}^-$ is canonically isomorphic to A^- as a $G(F_\infty/F)$-module.

To deduce the corollary from the lemma, let F_n^+ be the maximal totally real subfield of F_n, E_n the units of F_n, E_n^+ the units of F_n^+, and Ω_n the group of roots of unity of F_n. Then it is well known that $\Omega_n E_n^+$ is a subgroup of E_n of index at most 2. Hence, since 1 is odd and $E_\infty = \cup_{n=0}^\infty E_n$, we deduce easily that $((Q_\ell/Z_\ell) \underset{Z}{\otimes} E_\infty)^- = 0$.

Proof of Lemma 17. The proof is entirely elementary, and is based on the fact that there exists an integer $n_0 \geq 0$ such that the extension F_∞/F_{n_0} is totally ramified at all primes of F_{n_0} lying above 1 (we do not include a proof of this since it is both easy to prove and very well known). Let s denote the number of primes of F_∞ lying above 1, and, for each $n \geq n_0$, let $\mathfrak{P}_j(n)$ $(1 \leq j \leq s)$ denote the primes of F_n lying above 1, our notation being chosen so that, for $m \geq n$, we have $\mathfrak{P}_j(n) = \mathfrak{P}_j(m)^{\ell^{m-n}}$ when $\mathfrak{P}_j(n)$ is viewed as an ideal of F_m. Now let x be any element of $\mathfrak{M}$, say $x = \alpha \otimes (1^{-a} \bmod Z_\ell)$. Choose $n \geq n_0$ so large that $\alpha \in F_n$ and $\alpha \mathfrak{O}_n' = \mathfrak{w}_n'^{\ell^a}$ with $\mathfrak{w}_n' \in I_n'$ (here $\mathfrak{O}_n'$ denotes the ring generated by the algebraic integers of F_n and $1/\ell$, and I_n' denotes the free abelian group generated by the primes of $\mathfrak{O}_n'$). Now, if $\mathfrak{O}_n$ denotes the ring of algebraic integers of F_n, we have $\alpha \mathfrak{O}_n =$ $\mathfrak{w}_n'^{\ell^a} \mathfrak{P}_1(n)^{j_1} \cdots \mathfrak{P}_s(n)^{j_s}$, for certain integers $j_1, \ldots, j_s$ (of course, $j_1, \ldots, j_s$ are not necessarily divisible by 1^a). Now $\alpha \mathfrak{O}_{n+a} =$ $(\mathfrak{v}_n' \mathfrak{P}_1(n+a)^{j_1} \cdots \mathfrak{P}_s(n+a)^{j_s})^{\ell^a}$, where $\mathfrak{v}_n'$ is the image of $\mathfrak{w}_n'$ under the natural inclusion of I_n' in I_{n+a}'. We define $\phi(x)$ to be the image in A under the canonical map $A_{n+a} \to A$ of the class of $\mathfrak{v}_n' \mathfrak{P}_1(n+a)^{j_1} \cdots \mathfrak{P}_s(n+a)^{j_s}$ in A_{n+a}. It is trivial to verify that ϕ does not depend on any of the choices made in the above definition, that it is a $G(F_\infty/F)$-homomorphism, and that its kernel is $(Q_\ell/Z_\ell)\underset{Z}{\otimes}E_\infty$. To prove ϕ surjective, let ξ be any element of A, and pick an integer $n \geq n_o$ such that ξ is the image under the canonical map $A_n \to A$ of the class of an ideal $\mathfrak{b}$ of F_n. Thus there exists an integer $b \geq 0$ such that $\mathfrak{b}^{\ell^b} = \beta\mathfrak{O}_n$ for some β in F_n, and it is then plain that $\xi = \phi(\beta \otimes 1^{-b} \bmod Z_\ell)$. This completes the proof.

Finally, as was remarked by Tate several years ago, Theorem 13 shows that the divisibility assertion of Theorem 9 for $n = -1$ has a simple interpretation in terms of K-theory. For each finite or real prime $\mathfrak{p}$ of F, let $\mu_{\mathfrak{p}}$ be the group of all roots of unity in the completion of F at $\mathfrak{p}$, and let $\nu_{\mathfrak{p}} : K_2F \to \mu_{\mathfrak{p}}$ be the homomorphism induced by the Hilbert norm residue symbol relative to the whole of $\mu_{\mathfrak{p}}$. By using Moore's theorem, a simple computation shows that the kernel of the homomorphism $\nu_F = \oplus \nu_{\mathfrak{p}} : K_2F \to \oplus \mu_{\mathfrak{p}}$ is a subgroup of $\operatorname{Ker} \lambda_F$ of index $2^{r_1-1} \prod_{\ell} \prod_{\mathfrak{p}/\ell} w_1^{(\ell)}(F_{\mathfrak{p}})$, where the product is taken over all primes l, including $l = 2$. Granted Conjecture 14, Theorem 9 presumably has a similar interpretation for all odd $n > 1$.

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Research Problems: Arithmetic Questions in K-theory

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Throughout F will denote a finite extension of the rational field $\mathbb{Q}$, $\mathcal{O}$ will be the ring of integers of F, l will be any prime number, and, for each integer $m \geq 1$, μ_m will be the group of m-th roots of unity.

1. Is the natural map from $K_n \mathcal{O}$ to $K_n F$ injective for all odd positive integers n ? (If n is even, it is injective, as is immediately seen by looking at the long exact sequence of localization and using the fact that K_n of a finite field is zero for n even).

2. Assume F is totally real. Then $K_n \mathcal{O}$ is finite for all positive integers n with $n \not\equiv 1 \bmod 4$. Determine the orders of these groups. What relation do these orders have to the values of the zeta function of F at the negative integers ? (See Lichtenbaum's article in this volume for some more detailed possibilities on this subject). In particular, determine the order (and structure, if possible) of $K_3 \mathbb{Z}$.

3. Let $\mathfrak{p}$ be a finite prime of F, and let $\widetilde{F}_{\mathfrak{p}}$ be the Henselization of F at $\mathfrak{p}$ (the algebraic closure of F in the completion of F at $\mathfrak{p}$). What is $K_n(\widetilde{F}_{\mathfrak{p}})$? In particular, is $K_2(\widetilde{F}_{\mathfrak{p}})$ naturally isomorphic to the group of roots of unity in $\widetilde{F}_{\mathfrak{p}}$.

4. Assume l is odd, and let F_∞ be the field obtained by adjoining to F all l-power roots of unity. Let $K_2F(l)$, $K_2F_\infty(l)$

denote the l-primary subgroups of K_2F, K_2F_∞, respectively, and let $j: K_2F(l) \longrightarrow K_2F_\infty(l)$ be the natural map. Determine the kernel of j. (If F is totally real, j is injective; on the other hand, examples are known where j is not injective, e.g. $l = 3$, and $F = \mathbb{Q}(\sqrt{257}, \sqrt{-3})$ or $\mathbb{Q}(\sqrt{993}, \sqrt{-3})$). In particular, determine the kernel of j when $F = \mathbb{Q}(\mu_l)$. (If the class number of the maximal real subfield of $\mathbb{Q}(\mu_l)$ is prime to l, j is injective, e.g. for $l \leq 4001$).

5. Assume that $\mu_l \subset F$, and let $F^\times$ denote the multiplicative group of F. Let Δ be the kernel of the map from $\mu_l \otimes_{\mathbb{Z}} F^\times$ to K_2F given by $\varsigma \otimes a \longmapsto \{\varsigma, a\}$. If $\varsigma \otimes a$ is in Δ, and a is not an l-th power in $F^\times$, is it true that $F(\sqrt[l]{a})$ is always the first layer of a $\mathbb{Z}_l$-extension above F in the sense of Iwasawa (if F is totally real, whence $l = 2$, this is true) ? Note that, by a result of Tate (see his article in this volume), the order of Δ is l^{1+r_2}, where r_2 is the number of pairs of complex conjugate embeddings of F in $\mathbb{C}$.

6. Let $\overline{F}$ be the algebraic closure of F, and let G_F be the Galois group of $\overline{F}$ over F. Let $T = \varprojlim \mu_{l^n}$, and write $T^{\otimes 2}$ for the tensor product of T with itself over $\mathbb{Z}_l$, viewed as a G_F-module via the diagonal action. Excluding perhaps the prime $l = 2$, is it true that $K_3F \otimes_{\mathbb{Z}} \mathbb{Z}_l$ is isomorphic to $H^1(G_F, T^{\otimes 2})$, the latter cohomology group being formed with continuous cochains ? (See Tate's article in this volume).

7. If X = Spec(A) is a non-singular affine curve defined over a finite field, and n is greater than 2, is K_nA finite ? (It does not even seem to be known that K_nA is finitely generated). By results of Bass and Tate, K_2A is finite.

8. Let L be any field, and l a prime, distinct from the characteristic of L, such that $\mu_\ell \subset L$. Is it true that every element of K_2L of order l is of the form $\{\zeta, a\}$ with $\zeta \in \mu_\ell$ and $a \in L$? (When L is a global field, this has been proven by Tate; see his article in this volume). Also, do there exist fields of characteristic l such that their K_2's have non-trivial l-primary subgroups ?

Letter from Tate to Iwasawa on a relation between K_2 and Galois cohomology

The text below is a copy of a letter I wrote Iwasawa in January 1971. It contains a sketch of the proof of what is called the "Main Conjecture" on page 210 of my talk in the Proceedings of the International Congress of Mathematicians at Nice. The letter below and the Nice talk, taken together, provide an outline of the proof for number fields of the fundamental isomorphism between K_2 and Galois cohomology (formula (42) on page 210 of the Nice talk). I hope to publish the details sometime soon.

The notes of Iwasawa referred to below will appear shortly as a paper in the Annals.

J. Tate

Dear Iwasawa,

Thank you for sending me the notes of your course. They have been very helpful to me.

I am enclosing a copy of the manuscript which I am submitting to the Nice volume, because I think that now I can prove what I there called the Main Conjecture, and this result is equivalent to the following statement about your Γ-module $X = \mathrm{Gal}(M/K)$, by your theorem that $X/X_{\mathrm{tors}} \sim \Lambda^{r_2}$.

THEOREM: The character $\varkappa^2$ does not occur in X; more precisely, the module $T^{(-2)} \otimes_{\mathbb{Z}_\ell} X = \mathrm{Hom}_{\mathbb{Z}_\ell}(T^{(2)}, X)$ contains no non-zero element fixed by Γ.

Here I am using without explanations notations from your notes (except I use X instead of your $\mathfrak{X}$), and also the notation $T^{(r)}$ from my manuscript. In order that this combination of documents (i.e. my Nice talk and this letter) will be self-contained, let me review your notation:

ℓ is a prime number.

k is a finite extension of $\mathbb{Q}$ containing the ℓ-th roots of unity, and containing $\sqrt{-1}$ if $\ell = 2$.

$K = k(W)$, where W is the group of ℓ^n-th roots of 1, all n, in some algebraic closure of k.

$\Gamma = \mathrm{Gal}(K/k) = \gamma_0^{\mathbb{Z}_\ell} \approx \mathbb{Z}_\ell$.

$\varkappa : \Gamma \longrightarrow \mathcal{U} = \mathbb{Z}_\ell^{\cdot}$ via $\gamma(\zeta) = \zeta^{\varkappa(\gamma)}$ for $\zeta \in W$, $\gamma \in \Gamma$.
(In other words, $\gamma t = \varkappa(\gamma) t$ for $t \in T$.)

M the maximal abelian ℓ-extension of K which is unramified outside ℓ.

$X = \mathrm{Gal}(M/K)$.

$I' =$ the group of ℓ-ideals of K = free abelian group generated by discrete valuations of K (i.e. by the non-archimedean valuations not dividing ℓ).

$\mathfrak{M}$ is defined by the exactness of the sequence.

(1) $$0 \longrightarrow \mathfrak{M} \longrightarrow (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes K^{\cdot} \longrightarrow (\mathbb{Q}_\ell/\mathbb{Z}_\ell) \otimes I' \longrightarrow 0 \quad .$$

By Kummer theory we have your theorem 2, namely

$$X = \mathrm{Hom}(\mathfrak{M}, W) \quad ,$$

and the resulting pairing $X \times \mathfrak{M} \longrightarrow W$ is a Γ-pairing, i.e. satisfies $\langle \gamma x, \gamma m \rangle = \gamma \langle x, m \rangle$, for $\gamma \in \Gamma$, $x \in X$, $m \in \mathfrak{M}$. Hence

$$(T^{(-2)} \underset{\mathbb{Z}_\ell}{\otimes} X)^{\Gamma} = \mathrm{Hom}_\Gamma(T^{(2)} \otimes \mathfrak{M}, W) = \mathrm{Hom}_\Gamma(T \otimes \mathfrak{M}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$$

$$= \mathrm{Hom}((T \otimes \mathfrak{M})/(\gamma_0 - 1)(T \otimes \mathfrak{M}), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \approx \mathrm{Hom}(H^1(\Gamma, T \otimes \mathfrak{M}), \mathbb{Q}_\ell/\mathbb{Z}_\ell) \; ,$$

so our theorem is equivalent to

$$H^1(\Gamma, T \otimes \mathfrak{M}) = 0 \quad .$$

Tensoring the exact sequence (1) with T , and then passing to cohomology, we get an exact sequence

$$(2) \qquad (W \otimes K^{\cdot})^{\Gamma} \xrightarrow{\alpha} (W \otimes I')^{\Gamma} \longrightarrow H^1(\Gamma, T \otimes \mathfrak{M}) \longrightarrow H^1(\Gamma, W \otimes K^{\cdot})$$

So our theorem is now reduced to two statements:

PROPOSITION : <u>The map</u> α <u>in</u> (2) <u>is surjective</u> , and

LEMMA : $H^1(\Gamma, W \otimes K^{\cdot}) = 0$.

The lemma is trivial. In fact, if N is any discrete module on which Γ operates continuously, then $H^1(\Gamma, W \otimes N) = 0$.

<u>Proof.</u> $N = \varinjlim N_\alpha$, N_α finitely generated, so we can assume N is finitely generated and fixed by $\gamma_0^{\ell^n}$ for some n (since N is discrete, a finitely generated Γ-module is a finitely generated abelian group). Now $W \otimes N$ is a quotient of the finite-dimensional $\mathbb{Q}_\ell$-vector space $V \otimes N$, where $V = T \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. The eigenvalues of γ_0 on $\mathbb{Q}_\ell \otimes N$ are ℓ^n-th roots of unity, so on $V \otimes N$, γ_0 has eigenvalues which are not roots of unity (since $\varkappa(\gamma_0)$ is not a root of unity). Thus $\gamma_0 - 1$ operates bijectively on $V \otimes N$, hence surjectively on $W \otimes N$, Q.E.D.

To prove the proposition we use non-trivial facts from K_2-theory, namely Moore's theorem on Coker λ, Garland's theorem that Ker λ is finite, and Matsumoto's theorem that a symbol gives a homomorphism of $K_2 k$. Garland's theorem implies that $K_2 k$ is a torsion group. This, the discussion on pages 208, 209 of my Nice talk (with F = k), and the isomorphisms

$$H^1(k, W^{(2)}) = H^1(K, W^{(2)})^{\Gamma} = (W \otimes K^{\cdot})^{\Gamma} ,$$

give a diagram

$$
\begin{array}{ccc}
\ell PP(K_2 k) & \xrightarrow{\ h\ } & H^2(k, T^{(2)})_{tors} = (W \otimes K^{\cdot})^{\Gamma} / ((W \otimes K^{\cdot})^{\Gamma})_{div} \\
\Big\downarrow \lambda_{tame} & & \Big\downarrow \text{induced by } \alpha \text{ of (2)} \\
\ell PP \coprod_{v \nmid \ell \infty} (\mu_v) & \xrightarrow{\ \sim\ } & (W \otimes I^{\cdot})^{\Gamma}
\end{array}
$$

where ℓPP denotes ℓ-primary part.

Local considerations show that the diagram commutes, and that the lower horizontal arrow is bijective. The arrowed marked λ_{tame} is surjective by Moore's theorem. The map h is defined via Matsumoto's theorem and has values in the torsion subgroup of H^2 by Garland's theorem. Hence α is surjective.

Best regards,

J.Tate