Topology Vol. 22, No. 3, pp. 285-298, 1983
Printed in Great Britain.

0040 9383/83 \$3.00 + .00
Pergamon Press Ltd.

STABLE SPLITTINGS DERIVED FROM THE STEINBERG MODULE

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(*Received* 15 *September* 1982)

IN THIS PAPER we construct a new class of stable splittings for certain classifying spaces, including $B(\mathbb{Z}/p)^k$. Our results involve symmetric products of the sphere spectrum and are based on the fundamental Steinberg module of modular representation theory. Splitting theorems have long played an important role in homotopy theory, see [1–4], one reason being that an equivalence $X \xrightarrow{\cong} \vee X_i$ enables one to construct maps $X_i \to X$ which were *a priori* inaccessible. Examples include Mahowald's maps η_j[5] based on Snaith's splittings and, more recently, certain maps used in Kuhn's proof of the Whitehead conjecture[6, 7]. These latter maps are based on our splitting of $B(\mathbb{Z}/2)^k$.

Our main result shows that the suspension spectrum of a product of lens spaces $B(\mathbb{Z}/p)^k$ can be split using the Steinberg idempotent of $\mathbb{F}_p[GL_k(\mathbb{F}_p)]$. Let $Sp^n(S^\circ)$ denote the n-fold symmetric product of the sphere spectrum. We recall $Sp^\infty(S^\circ) = K(\mathbb{Z})$ by the Dold–Thom theorem. Let $D(k)$ be the cofiber of the diagonal map d: $Sp^{p^{k-1}}(S^\circ) \to Sp^{p^k}(S^\circ)$. Then $D(\infty) = K(\mathbb{Z}/p)$. Let $M(k) = \Sigma^{-k}D(k)/D(k-1)$. In mod-$p$ cohomology $H^*(M(k))$ has a basis consisting of admissible Steenrod operations of length exactly k.

THEOREM A. *Stably*, $B(\mathbb{Z}/p)^k$ *contains* $p^{\binom{k}{2}}$ *summands each equivalent to* $M(k)$. *These summands correspond to the* $p^{\binom{k}{2}}$ *summands of the Steinberg module in* $\mathbb{F}_pGL_k(\mathbb{F}_p)$.

Here and throughout, all spaces are localized at p. Let $L(k) = \Sigma^{-k}Sp^{p^k}(S^\circ)/Sp^{p^{k-1}}(S^\circ)$. A simple argument shows that $L(k)$ is also a summand of $B(\mathbb{Z}/p)^k$; in fact, $M(k) = L(k) \vee L(k-1)$.

Let $\int^k \mathbb{Z}/p$ denote the k-fold wreath product. Using the transfer t: $B(\int^k \mathbb{Z}/p) \to B(\mathbb{Z}/p)^k$ and the double coset formula we prove

THEOREM B. $M(k)$ *is a stable summand in* $B(\int^k \mathbb{Z}/p)$. *Let* $O(k)$ *be the real orthogonal*

Let $O(k)$ be the real orthogonal group. Using Becker–Gottlieb transfer for the fibration $O(k)/(\mathbb{Z}/2)^k \to B(\mathbb{Z}/2)^k \to BO(k)$ we prove

THEOREM C. $M(k)$ *is a stable summand in* $BO(k)$.

Let $T^k = (S^1)^k$ be the k-torus. We construct a spectrum $BP(k)$ such that $H^*BP(k)$ has a basis consisting of admissible Steenrod operations in the reduced powers of length exactly k. Using a lifting of the Steinberg idempotent to $GL_k(\mathbb{Z}_p)$ we show

THEOREM D. *Completed at p*, BT^k *contains* $p^{\binom{k}{2}}$ *stable summands each equivalent to* $BP(k)$. *Further*, $BP(k)$ *is a stable summand of* $BU(k)$.

This paper is organized as follows: The brief §1 contains a few remarks about the length filtration of the Steenrod algebra. In §2 we give an account of those facts about $GL_n(\mathbb{F}_p)$

†Partially supported by an Achievement Rewards for College Scientists Fellowship.
‡Partially supported by NSF Grant number MCS-7827592.

and the Steinberg module needed for our subsequent constructions. Sections 3 and 4 are devoted to the construction and properties of various spectra including Thom spectra and symmetric product spectra. The proof of Theorem A is given in §5. Section 6 contains proofs of Theorems B and C. Finally the construction of $BP(k)$ and the proof of Theorem D is given in §7.

§1. PRELIMINARIES ON THE STEENROD ALGEBRA

Let A denote the Steenrod algebra, and let A_n denote the subalgebra generated by β, $P^1, \dots P^{p^{n-1}}$. (For $p = 2$, $\beta = Sq^1$ and $P^i = Sq^{2i}$.) If I is a finite sequence $(\epsilon_0, r_1, \epsilon_1, r_2, \dots)$, $r_i \geq 0$, $\epsilon_j = 0, 1$, then $\theta^I = \beta^{\epsilon_0} P^{r_1} \beta^{\epsilon_1} P^{r_2} \dots$; if $\epsilon_i = 0$ for all i we write P^I for θ^I. As usual, I is admissible if $r_i \geq pr_{i+1} + \epsilon_i$ for all i. By a classical theorem of Cartan and Serre, the admissible θ^I are a basis for A. The length $l(I)$ is defined by $l(I) = n$ if $r_i = 0$ for $i > n$ and $\epsilon_i = 0$ for $i \geq n$. Thus we obtain vector space filtrations on A defined by $F_n = \langle \theta^I : l(I) \leq n \rangle$ and $G_n = \langle \theta^I : I \text{ admissible}, l(I) > n \rangle$. Finally, we recall that A_n contains an exterior algebra on primitive elements $Q_0, \dots, Q_{n-1}$, where $Q_0 = \beta$ and $Q_{k+1} = [P^{p^k}, Q_k]$.

PROPOSITION 1.1. (a) F_n is spanned by the admissible θ^I, $l(I) \leq n$; (b) F_n is a subcoalgebra of A; (c) F_n is a left A_{n-1} submodule of A. Moreover F_n is free over $E[Q_0, \dots, Q_{n-1}]$ on $\{P^I : I \text{ admissible}, l(I) \leq n\}$; (d) G_n is a left ideal. Moreover $A/G_n = F_n$ as A_{n-1} modules.

Proof: (a) follows from the Adem relations, and (b) is obvious. The first part of (c) also follows from the Adem relations, using induction on n. For the second part, note that the $E[Q_0, \dots, Q_{n-1}]$ submodule of F_n generated by $\{P^I : I \text{ admissible}, l(I) \leq n\}$ is indeed free as claimed; this follows from Milnor[8], Theorem 4(a). Hence this module has Poincaré series $\prod_0^{n-1} (1 + t^{2p^i - 1}) / \prod_1^n (1 - t^{2(p^i - 1)})$, which is precisely the Poincaré series of F_n (by (a)). Finally, (d) also follows from the Adem relations; alternatively, it is a consequence of (3.5) below.

§2. $GL_n F_q$ AND THE STEINBERG MODULE

Let V^n be a vector space over the finite field $\mathbb{F}_q$, $q = p^r$, with basis $y_1, \dots, y_n$. Then $GL_n \mathbb{F}_q$ is the automorphism group of V^n, acting on the right. $GL_n \mathbb{F}_q$ has order $q^{\binom{n}{2}} \prod_{i=1}^{n} (q^i - 1)$, and contains the following distinguished subgroups:

Σ_n = *symmetric group* (permutation matrices).
D_n = *diagonal matrices*.
B_n = *Borel subgroup* = upper triangular matrices.
U_n = *unipotent subgroup* = upper triangular matrices with all diagonal entries equal to 1. (Note U_n is a p-Sylow subgroup.)

In addition we will need to consider

A_n = *top row subgroup* = $\{g \in B_n : y_i g = y_i, \forall i > 1\}$.
T_n = *cyclic subgroup* of Σ_n, of order n generated by $(1, 2, \dots, n)$.

Throughout this paper, we regard V^k as the subspace $\langle y_{n-k+1}, \dots, y_n \rangle$ of V^n; this convention determines inclusions $GL_k \mathbb{F}_q \subseteq GL_n \mathbb{F}_q$, etc. Note that many of our subgroups fit together as semi-direct products, e.g. $\Sigma_n \tilde{\times} D_n$, $D_n \tilde{\times} U_n = B_n$, and the "maximal parabolic subgroup" $GL_{n-1} \tilde{\times} A_n$.

We digress briefly to review some general facts from representation theory (see [9]). All modules are understood to be right modules. Let R be any finite-dimensional algebra over a field K. Then there is a unique set of indecomposable two-sided ideals $B_1, \ldots, B_r$, called *blocks*, such that $R = \Pi B_i$ (as algebras). Each B_i corresponds to a central idempotent f_i such that $B_i = Rf_i = f_iR_i$; the f_i are orthogonal and $\Sigma f_i = 1$. A nonzero right R-module M is said to *belong* to the block B_i (alternatively, B_i "contains M") if $Mf_j = 0$ $\forall j \neq i$. If M is indecomposable, then obviously M belongs to a unique block. Now if R is semisimple, then each block is a matrix algebra. More generally, suppose R is a "quasi-Frobenius" algebra, i.e. every projective over R is injective. (For example, group algebras are quasi-Frobenius). Then:

PROPOSITION 2.1. *If R is quasi-Frobenius, a block B of R is a matrix algebra if and only if B contains a projective irreducible module.*

Proof. First recall (see [9], p. 378) that two indecomposables U, V are *linked* if there is a finite sequence $U = U_0, U_1, \ldots, U_n = V$ of indecomposables such that U_i and U_{i+1} have a common irreducible constituent (i.e. composition factor) for each i. (Curtis and Reiner use only "principal" indecomposables, but this makes no difference.) This defines an equivalence relation on the set of indecomposable modules. Moreover it is true (over any finite-dimensional algebra) that U and V are linked if and only if they belong to the same block ([9], Theorem 55.2).

Now suppose B contains a projective irreducible N. Since N is also injective, it is a direct summand of any module in which it occurs as a composition factor. Hence the linking class of N consists solely of N itself. But this means every B-module is a direct sum of copies of N, and the classical Artin–Wedderburn theory then implies B is a matrix algebra over some K-central division algebra.

The converse is a standard fact.

Now take $R = \mathbb{F}_p[GL_n\mathbb{F}_q]$. If H is a subgroup of $GL_n\mathbb{F}_q$, we let $\bar{H} = \sum_{h\in H} h$ (if $H \not\subseteq \Sigma_n$) and $\bar{H} = \sum_{h\in H} \epsilon(h)h$ (if $H \subseteq \Sigma_n$); here $\epsilon\colon \Sigma_n \to \{\pm 1\}$ is the usual map.

Definition 2.2. The *Steinberg idempotent e_n* is defined by $e_n = \bar{B}_n\bar{\Sigma}_n/[GL_n\colon U_n]$; the corresponding module $St = e_nR$ is called the *Steinberg module.*

THEOREM 2.3. (Steinberg[10]), *(a) e_n is idempotent; (b) St is projective and absolutely irreducible; (c) as a U_n-module, St is the regular representation. In particular* dim $St = q^{\binom{n}{2}}$ *with basis* $\{e_nu\colon u \in U_n\}$.

Remark 2.4. By Proposition 2.1, the block B_{St} containing St is a matrix algebra over $\mathbb{F}_p$ of degree $q^{\binom{n}{2}}$.

Remark. Steinberg originally defined St as a certain composition factor of the permutation representation obtained from the action of GL_n on the flag complex $F(V^n)$. Later, Solomon and Tits showed that $F(V^n)$ has the homotopy type of a wedge of $q^{\binom{n}{2}}(n-2)$-spheres, and that St is the representation of GL_n on the cohomology group $N^{n-2}(V^n)$. Yet another description of St is given in (5.12) below.

Now suppose $K \subseteq H \subseteq GL_n$, $H \not\subseteq \Sigma_n$, and let $H = \bigcup_i h_iK$ (left coset decomposition). Then clearly $\bar{H} = (\Sigma h_i)\bar{K}$. If K is normal in H, then also $(\Sigma h_i)\bar{K} = \bar{K}(\Sigma h_i)$. Similar remarks apply if $H \subseteq \Sigma_n$. For example, $\bar{B}_n = \bar{A}_n\bar{B}_{n-1} = \bar{B}_{n-1}\bar{A}_n$, $\bar{\Sigma}_n = \bar{\Sigma}_{n-1}\bar{T}_n$, $\bar{B}_n = \bar{D}_n\bar{U}_n = \bar{U}_n\bar{D}_n$, and $\bar{A}_n\bar{\Sigma}_{n-1} = \bar{\Sigma}_{n-1}\bar{A}_n$. The following inductive formula is then immediate:

PROPOSITION 2.5. $e_n = e_{n-1}\bar{A}_n\bar{T}_n/(q^n - 1)$.

Our last proposition will be needed in Section 6. Let $e'_n = \bar{\Sigma}_n\bar{B}_n/[GL_n\colon U_n]$.

PROPOSITION 2.6. (a) e'_n is a primitive idempotent belonging to the Steinberg block B_{St}. For any M belonging to B_{St}, $Me'_n = M^{B_n}$; (b) for any GL_n-module W, there are vector space isomorphisms $We_n \xrightarrow{\cong} We'_n$ and $We'_n \xrightarrow{\cong} We_n$ given by $\bar{B}_n$, $\bar{\Sigma}_n$ (respectively).

Proof. Since e'_n is the conjugate of e_n in the Hopf algebra $\mathbb{F}_p[GL_n]$, e'_n is a primitive idempotent. Now by Theorem 2.3, St^{B_n} is equal to $St\bar{B}_n = \langle e_n\bar{B}\rangle$ and has dimension one. Thus e'_n is the identity on St^{B_n}. In particular $Ste'_n \neq 0$, so e'_n belongs to B_{St}. This also shows $Me'_n = M^{B_n}$ for M belonging to B_{St}, since such an M is just a direct sum of copies of St (by Remark 2.4). (b) is obvious.

§3. $B(\mathbb{Z}/p)^n$ AND ASSOCIATED SPECTRA

Let L^{2n+1} denote the lens manifold $S^{2n+1}/(\mathbb{Z}/p)$. We identify $B\mathbb{Z}/p$ with $L^\infty = \lim_{\vec{n}} L^{2n+1}$ and $B(\mathbb{Z}/p)^n$ with $\prod_1^n B\mathbb{Z}/p$. The canonical complex line bundle λ over $B\mathbb{Z}/p$ is $S^\infty \times_{\mathbb{Z}/p}\mathbb{C}$, where $\mathbb{Z}/p$ acts on $\mathbb{C}$ via the standard inclusion $\mathbb{Z}/p \subset S^1$. Let $P_n = H^*B(\mathbb{Z}/p)^n$. Then, at odd primes, $P_1 = E[x]\otimes \mathbb{Z}/p[y]$, where $y = c_1(\lambda)$ and $\beta x = y$. From the Künneth theorem we then have

$$P_n = E[x_1, \ldots, x_n]\otimes \mathbb{Z}/p[y_1, \ldots, y_n]. \tag{3.1}$$

For $p = 2$, $P_n = \mathbb{Z}/2[x_1, \ldots, x_n]$. However, in order to avoid separating cases, we will make use of the following device: Let $y_i = x_i^2$, and replace P_n by the quotients of the filtration

$$0\to P_n^2\to P_n\to P_n/P_n^2\to 0$$

where P_n^2 denotes the subring of squares. Then (3.1) becomes valid for all primes. In particular (3.1) describes P_n as a module over the Steenrod algebra.

Now $GL_n = GL_n\mathbb{Z}/p$ acts on $(\mathbb{Z}/p)^n$ and hence on the homotopy type $B(\mathbb{Z}/p)^n$ (on the left). The resulting right action on P_n is then the obvious one implied by (3.1) (with our usual proviso for $p = 2$). As explained in [11, §1], for each idempotent $e \in \mathbb{Z}/p[GL_n]$ we obtain a stable summand X of $B(\mathbb{Z}/p)^n$ with cohomology P_ne. We will use the notation $e\cdot B(\mathbb{Z}/p)^n$ for X. For example, let $d_n = \bar{D}_n/(p-1)^n$, where D_n is the diagonal subgroup and $\bar{D}_n = \sum_{g\in D_n} g$. Then d_n is idempotent and we have the following well known fact:

PROPOSITION 3.2. *The map $B(\mathbb{Z}/p)^n\to B(\Sigma_p)^n$ induced by the inclusion $(\mathbb{Z}/p)^n \subset (\Sigma_p)^n$ restricts to an equivalence $d_n\cdot B(\mathbb{Z}/p)^n \xrightarrow{\cong} B(\Sigma_p)^n$.*

The transfer provides an explicit inverse. Note that

$$H^*B(\Sigma_p)^n = P_n^{D_n} = E[x_1y_1^{p-2}, \ldots, x_ny_n^{p-2}]\otimes \mathbb{Z}/p[y_1^{p-1}, \ldots, y_n^{p-1}].$$

3.3 *Thom spectra*

We will need to consider various Thom spectra, and quotients of Thom spectra, over these classifying spaces. The following notation is very convenient: For any finite group G and representation θ of G, we use the same letter θ to denote the corresponding vector bundle over BG. In fact in place of θ, we could take any element of the complex representation ring

$R_{\mathbb{C}}(G)$. For example, if α is the reduced regular representation of $(\mathbb{Z}/p)^n$ (i.e. the regular representation minus a trivial one-dimensional representation), then $(B(\mathbb{Z}/p)^n)^\alpha$ is the Thom spectrum of the sum of all the nontrivial line bundles over $B(\mathbb{Z}/p)^n$. When $n = 1$ and λ: $\mathbb{Z}/p \to S^1$ is the standard representation mentioned above, we write L_{2k}^∞ for $(B\mathbb{Z}/p)^{k\lambda}$ $(k \in \mathbb{Z})$ and L_{2k+1}^∞ for L_{2k}^∞/S^{2k}. When β: $\Sigma_p \to U(p-1)$ is the reduced standard representation; we write P_{kq}^∞ for $(B\Sigma_p)^{k\beta}$ $(q = 2(p-1))$ and $P_{(k+1)q-1}^\infty$ for P_{kq}^∞/S^{kq}. (Note that P_{kq}^∞ has cells only in dimensions congruent to 0 or $-1 \bmod q = 2(p-1)$. Note also that for $p = 2$, this definition of P_k^∞ agrees with the usual one based on the canonical real line bundle.) In this notation, we have $B(\mathbb{Z}/p)_+^n = \Lambda^n L_0^\infty$, $B(\Sigma_p)_+^n = \Lambda^n P_0^\infty$, etc.

The cohomology of these spectra is very easy to describe. Let S_n denote the localization of P_n obtained by inverting all nonzero linear forms in $y_1, \ldots, y_n$ (i.e. all elements of $V^n - 0$). By a theorem of Wilkerson[12], S_n has a unique A-module structure extending that of P_n. Then the cohomology of virtually every spectrum considered in this paper can be regarded in a natural way as an A-submodule of S_n. For example, if $\theta \in R_{\mathbb{C}}((\mathbb{Z}/p)^n)$ then θ has an "Thom class' $e(\theta) \in S_n$, and $H^*(B(\mathbb{Z}/p)^n)^\theta$ is the (free) P_n-submodule of S_n generated by $e(\theta)$. (Note this is also an A-submodule). We list here a few explicit descriptions that we will need; further examples are left to the reader.

Examples 3.4

(a) $H^*L_{2k+\epsilon}^\infty$ $(\epsilon = 0$ or $1)$ is the P_1-submodule of S_1 generated by y^k (or xy^k, y^{k+1} if $\epsilon = 1$).

(b) $H^*(B(\mathbb{Z}/p)^n)^{-\alpha}$ is the P_n-submodule of S_n generated by l_n^{-1}, where $l_n = \prod_{a \in V^n - 0} a$ is the Euler class of α.

(c) $H^*\Lambda^n P_{-1}^\infty$ is the $H^*B(\Sigma_p)^n$-submodule of S_n generated by $X_n Y_n^{-1}$, where $X_n = x_1 \ldots x_n$ and $Y_n = y_1 \ldots y_n$.

We emphasize that in all of these examples the A-module structure follows from the Cartan formula together with the action of A on the "Thom class" in the lowest dimension. This in turn is determined by the standard formulas, $P^i y^k = \binom{k}{i} y^{k+i(p-1)}$, $\beta x = y$ where $\dim x = 1$, $\dim y = 2$ and k is allowed to be negative. In fact, we make no essential use of Wilkerson's result, since all of our A-modules will actually be submodules of the A-module of example (b).

Of particular importance for us is the A-submodule M_n of $H^*\Lambda^n P_{-1}^\infty$ generated by $X_n Y_n^{-1}$.

PROPOSITION 3.5. *$M_n = \Sigma^{-n} A/G_n$. Moreover $M_n \cap P_n$ has basis $\{\theta^I(X_n Y_n^{-1})$: I admissible, $l(I) = n\}$.*

Proof. Define a filtration ω on $H^*\Lambda^n P_{-1}^\infty$ as follows: given an n-tuple $(a_1, \ldots, a_n)$, $a_i \geq -1$, let $z \in \omega(a_1, \ldots, a_n)$ iff

(1) $z = x_1^{e_1} \ldots x_n^{e_n} y_1^{f_1} \ldots y_n^{f_n}$, $e_i \in \{0, 1\}$, $f_i \geq -1$ and $(f_1, \ldots, f_n) \leq (a_1, \ldots, a_n)$ in the lexicographical order (starting at the left), or

(2) z is a linear combination of monomials, each of which is in $\omega(a_1, \ldots, a_n)$.

Then $\omega(a_1, \ldots, a_n) \subset \omega(a_1', \ldots, a_n')$ if $(a_1, \ldots, a_n) \leq (a_1', \ldots, a_n')$.

Now for $I = (\epsilon_0, r_1, \epsilon_1, r_2, \ldots, \epsilon_{n-1}, r_n)$ define $Y_n^I = y_1^{k_1}, \ldots, y_n^{k_n}$, where $k_i = r_i(p-1) + \epsilon_{i-1}$, and $X_n^I = x_1^{1-\epsilon_0}, \ldots, x_n^{1-\epsilon_{n-1}}$.

LEMMA 3.6. *If I is admissible and $l(I) \leq n$, $\theta^I(X_n Y_n^{-1}) = \pm X_n^I Y_n^I$ modulo terms of lower filtration $(X^I Y^I \in \omega(k_1, \ldots, k_n))$.*

Proof of Lemma. For $n = 1$, the lemma is clear; suppose inductively it is true for $n - 1$. By the Cartan formula $\theta^I(X_n Y_n^{-1}) = \Sigma \pm \theta^{J_1}(x_1 y_1^{-1}), \ldots, \theta^{J_n}(x_n y_n^{-1})$ where the sum is taken over all sequences $J_1, \ldots, J_n$ with $\Sigma J_i = I$. Those terms with $J_1 = (\epsilon_0, r_1)$ can be grouped as $\theta^{(\epsilon_0, r_1)}(x_1 y_1^{-1})\theta^{I'}(X_{n-1}Y_{n-1}^{-1})$ where $I' = (\epsilon_1, r_2, \ldots, \epsilon_{n-1}, r_n)$. By induction the sum of these terms equals $\pm X_{n-1}^{I'} Y_{n-1}^{I'}$ mod elements of lower filtration. It remains to consider terms with $l(J_1) > 1$. For such admissible J_1, $\theta^{J_1}(x_1 y_1^{-1}) = 0$ for dimensional reasons. For such inadmissible J_1 write $J_1 = (\epsilon_0', r_1', \ldots, \epsilon_{n-1}', r_n')$. The Adem relations show that the only admissible summand of length 1 in θ^{J_1} is $c\beta^{\epsilon}P^{r}$ where $c \in \mathbb{Z}/p$, $\epsilon = 1$ if $\Sigma\epsilon_i' = 1$, $\epsilon = 0$ if $\Sigma\epsilon_i' = 0 \bmod 2$ and $r = \Sigma r_i'$ (note that $c = 0$ if $\Sigma\epsilon_i' > 1$). If $\Sigma\epsilon_i' = 0$ then

$$c = \binom{(r_2' + \cdots + r_n')(p-1) - 1}{r_1'}\binom{(r_3' + \cdots + r_n')(p-1) - 1}{r_2'} \cdots \binom{r_n'(p-1) - 1}{r_{n-1}'}$$

and $c \neq 0$ implies $r_i' < (r_{i+1}' + \cdots + r_n')(p-1)$. Hence $\Sigma r_i' < p^{n-1} r_n' \leq p^{n-1} r_n \leq r_1$ and so $\theta^{J_1}(x_1 y_1^{-1}) \ldots \theta^{J_n}(x_n y_n^{-1})$ has filtration less than that of $X^I Y^I$. The case of $\Sigma\epsilon_i' = 1$ is similar. This completes the proof of the lemma.

From the lemma it is immediate that the set $\{\theta^I(X_n Y_n^{-1}),\ I \text{ admissible and } l(I) \leq n\}$ is independent. Moreover it is easy to see that $l(I) = n$ iff $\theta^I(X_n Y_n^{-1}) \in P_n$. It then follows for dimensional reasons that the ideal G_n annihilates $X_n Y_n^{-1}$.

3.7 *Transfer*

We conclude this section with a discussion of the various transfer maps that we will need. Suppose X is a CW-complex, η is an n-dimensional complex vector bundle over X and ξ is a stable complex vector bundle over X (i.e. a map to BU). Then the inclusion of ξ in the Whitney sum $\eta \oplus \xi$ induces a map of Thom spectra $X^{\xi} \xrightarrow{t} X^{\eta\oplus\xi}$; this is the *transfer* associated to η, ξ. (A quite general discussion of transfer maps can be found in [13]. We leave it to the reader to discover in what sense the construction described here is a special case of that of [13].) The following is well known:

PROPOSITION 3.8. *The following diagram commutes*

$$\begin{array}{ccc} H^*X^{\xi} & \xleftarrow{t^*} & H^*X^{\eta\oplus\xi} \\ \cong\uparrow & & \uparrow\cong \\ H^*X & \xleftarrow{\cup e(\eta)} & H^*X \end{array}$$

where $\cup e(\eta)$ *denotes cup product with the mod p Euler class* $e(\eta)$ *and the vertical maps are Thom isomorphisms.*

Remark 3.9. The proposition is in fact true for any cohomology theory E such that η and ξ are E-oriented.

Example 3.10. There is a transfer $(B(\mathbb{Z}/p)^n)^{-\alpha} \xrightarrow{t} B(\mathbb{Z}/p)^n_+$. The map t^*: $P_n \to P_n \cdot l_n^{-1}$ is the obvious one, by (3.8).

Example 3.11. $(B\mathbb{Z}/p)^{-\alpha} \xrightarrow{t} (B\mathbb{Z}/p)^{-\lambda} = L_{-2}^{\infty}$. Again the map t^*: $P_1 \cdot y^{-1} \to P_1 \cdot y^{-(p-1)}$ is the obvious one.

Composing with the quotient map $L_{-2}^{\infty} \to L_{-1}^{\infty}$ in example (3.11), we obtain a map $(B\mathbb{Z}/p)^{-\alpha} \to L_{-1}^{\infty}$. Maps of this type will also be referred to as "transfers".

Note that $GL_1\mathbb{Z}/p$ acts on $(B(\mathbb{Z}/p))^{-\alpha}$, and that β (of 3.3) restricted to $\mathbb{Z}/p$ is α. The final result of this section is straightforward; its proof will be left to the reader.

PROPOSITION 3.12. *The induced map of Thom spectra* ϕ: $(B\mathbb{Z}/p)^{-\alpha} \to (B\Sigma_p)^{-\beta} = P_{-q}^{\infty}$

restricts to an equivalence $d_1 \cdot (B\mathbb{Z}/p)^{-\alpha} \cong P^{\infty}_{-q}$. *Moreover there is a commutative diagram*

$$\begin{array}{ccc}
(B\mathbb{Z}/p)^{-\alpha} & \xrightarrow{\phi} & P^{\infty}_{-q} \\
t\downarrow & & \psi\downarrow \\
L^{\infty}_{-1} & \to & P^{\infty}_{-1} \\
\downarrow & & \bar{\psi}\downarrow \\
L_0^{\infty} & \to & P_0^{\infty}
\end{array}$$

where t *is the transfer and the unlabelled maps are the obvious ones. Moreover* ψ *and* $\bar{\psi}$ *are stable retractions; in particular* P^{∞}_{-1} *is a summand of* L^{∞}_{-1}.

Of course $\bar{\psi}$ is just the retraction of (3.2).

§4. SYMMETRIC PRODUCT SPECTRA

If X is a space and H is a subgroup of Σ_n, Sp^HX is the orbit space X^n/H. If $H = \Sigma_n$, we write Sp^n in place of Sp^H. If $X = \{X_k, \epsilon_k\}$ is a spectrum with structure maps $\epsilon_k: S^1 \wedge X_k \to X_{k+1}$, then Sp^HX is the spectrum $\{Sp^HX_k, Sp^H(\epsilon_k)\circ f_k\}$, where f_k: $S^1 \wedge Sp^HX_k \to Sp^H(S^1 \wedge X_{k+1})$ is defined by $f_k(t \wedge (x_1 \dots x_n)) = (t \wedge x_1 \dots t \wedge x_n)$. Thus Sp^H becomes a functor on the stable category; for further details the reader may consult [14].

The natural inclusions $Sp^nX \subseteq Sp^{n+1}X$ allow us to define $Sp^{\infty}X = \varinjlim Sp^nX$ for a spectrum X. By the Dold–Thom theorem, $Sp^{\infty}S^0 = K\mathbb{Z}$; in particular $H^*Sp^{\infty}S^0 = A/A\beta$.

THEOREM 4.1. (*Nakaoka* [15]). *The inclusions* $Sp^nS^0 \to Sp^{\infty}S^0$ *are surjective on cohomology. Moreover* $H^*Sp^{p^n}S^0$ *has basis* $\{\theta^I$: *I admissible,* $l(I) \le n$, $\theta^I \notin A\beta\}$.

4.2 *The spectrum* $D(n)$

If $M\mathbb{Z}/p$ is the mod p Moore spectrum, then $Sp^{\infty}M\mathbb{Z}/p = K\mathbb{Z}/p$. In view of Theorem (4.1) it is natural to ask whether the finite symmetric products $Sp^{p^n}M\mathbb{Z}/p$ realize the Cartan–Serre filtration G_n on $A = H^*K\mathbb{Z}/p$. The answer is no; it can easily be seen from Remark (4.5) that the filtration provided by the $Sp^{p^n}M\mathbb{Z}/p$ is slightly different. Instead we use the following construction: On the space level there are obvious p-fold diagonal maps $Sp^{p^{n-1}}S^k \to Sp^{p^n}S^k$; these induce maps of spectra $Sp^{p^{n-1}}S^0 \xrightarrow{d} Sp^{p^n}S^0$. Let $D(n)$ denote the cofibre of d. Now clearly d^* is zero on $H^0Sp^{p^n}S^0$; hence by (4.1) d^* is zero on all of H^*. In other words, the cofibration $Sp^{p^n}S^0 \to D(n) \to \Sigma Sp^{p^{n-1}}S^0$ has a short exact cohomology sequence. Letting $u_n \in H^0D(n)$ denote a generator, the following proposition is now evident:

PROPOSITION 4.3. *There are commutative diagrams*

$$\begin{array}{ccc}
D(n-1) & \xrightarrow{j_n} & D(n) \\
& {\scriptstyle i_{n-1}}\searrow \quad \swarrow{\scriptstyle i_n} & \\
& K\mathbb{Z}/p &
\end{array}$$

such that i_n^* *is surjective with kernel* G_n *for all* n. *In particular,* $H^*D(n)$ *has basis* $\{\theta^I(u_n)$: *I admissible,* $l(I) \le n\}$.

Frequently, the generator u_n will be omitted from the notation. Note that $H^*D(1) = H^*\Sigma P^{\infty}_{-1}$ as A-modules. In fact:

PROPOSITION 4.4 $\Sigma P^{\infty}_{-1} \cong D(1)$.

Proof. Note that it is enough to exhibit a map $\Sigma L^{\infty}_{-2} \xrightarrow{f} D(1)$ which is nonzero on H^0, since we can then use the following composite g:

$$\Sigma P^{\infty}_{-q} \xrightarrow{i} \Sigma B(\mathbb{Z}/p)^{-\alpha} \xrightarrow{t} \Sigma L^{\infty}_{-1} \xrightarrow{f} D(1).$$

Here i is the inclusion of ΣP^{∞}_{-q} as a stable summand, as in (3.12) and t is the transfer. The induced map $\bar{g}$: $\Sigma P^{\infty}_{-1} = \Sigma P^{\infty}_{-q}/S^{-q+1} \to D(1)$ is then clearly an equivalence.

Now let $-\lambda_n$ denote the complement in $\mathbb{C}^{n+1}$ of the canonical complex line bundle λ_n over L^{2n+1}. Thus $-\lambda_n$ has total space $\{([x], v)$: $\langle x, v \rangle = 0\}$ where $x \in S^{2n+1}$, [ ] denotes equivalence class in L^{2n+1}, and $\langle , \rangle$ is the usual Hermitian inner product on $\mathbb{C}^{n+1}$. Now if L_x is the complex line spanned by x, and $|v| \le 1$, then $L_x + v$ intersects S^{2n+1} in a circle of radius $\sqrt{(1-|v|^2)}$. Hence we may define a map $\tilde{f}_n$ from the unit disc bundle $D(-\lambda_n)$ to Sp^pS^{2n+1} by $\tilde{f}_n([x], v) = (\sqrt{(1-|v|^2)}x + v,\ \sqrt{(1-|v|^2)}ax + v, \ldots,\ \sqrt{(1-|v|^2)}a^{p-1}x + v,$ where $a = \exp(2\pi i/p)$. (In fact $\tilde{f}_n$ is well defined as a map into the cyclic product $Sp^{\mathbb{Z}/p}S^{2n+1}$). Moreover, if $S(-\lambda_n)$ is the unit sphere bundle, we have a commutative diagram of cofibrations:

$$\begin{array}{ccc}
S(-\lambda_n) & \to & S^{2n+1} \\
\downarrow & & \downarrow d \\
D(-\lambda_n) & \xrightarrow{\tilde{f}_n} & Sp^pS^{2n+1} \\
\downarrow & & \downarrow \\
(L^{2n+1})^{-\lambda_n} & \xrightarrow{f_n} & Sp^pS^{2n+1}/d(S^{2n+1}).
\end{array}$$

The maps f_n fit together to yield a map of spectra $\Sigma(B\mathbb{Z}/p)^{-\lambda} = \Sigma L^{\infty}_{-2} \xrightarrow{f} M(1)$. To show f^* is an isomorphism on H^0, it is enough to show $(f_n)_*$ is an isomorphism on H_{2n+1}. Consider the restriction of $\tilde{f}_n$ to the zero section L^{2n+1}: $\tilde{f}_n([x]) = (x, ax, \ldots, a^{p-1}x)$. There is a commutative diagram

$$\begin{array}{ccc}
S^{2n+1} & \xrightarrow{F} & Sp^pS^{2n+1} \\
& \pi\searrow \quad \nearrow\tilde{f}_n & \\
& L^{2n+1} &
\end{array}$$

where F is the composite

$$S^{2n+1} \xrightarrow{\Delta} (S^{2n+1})^p \xrightarrow{1 \times a \times \cdots \times a^{p-1}} (S^{2n+1})^p \to Sp^pS^{2n+1}.$$

Now π_* is multiplication by p on $H_{2n+1}(\cdot; \mathbb{Z})$. Since a^k: $S^{2n+1} \to S^{2n+1}$ has degree one, F_* is also multiplication by p. Hence $(\tilde{f}_n)_*$ is an isomorphism on H_{2n+1}, and the proposition follows.

Remark 4.5. By a theorem of Kan and Whitehead ([16], see also [14]) the functors Sp^H preserve cofibrations in the category of spectra. An equivalent statement is that the natural map $Sp^HS^0 \wedge X \to Sp^HX$ is an equivalence. Now if H is a wreath product $K \int L$, it is easy to see that $Sp^HX \cong Sp^K(Sp^LX)$ (on the space level, this is actually a homeomorphism). Combining these remarks, we see that if $H_n = \int^n \Sigma_p$, then $Sp^{H_n}S^0 \cong \Lambda^n Sp^pS^0$.

If $D'(n)$ is the cofibre of the diagonal $Sp^{H_{n-1}}S^0 \to Sp^{H_n}S^0$, as in the definition (4.2) of $D(n)$, there is an analogous equivalence $D'(n) \cong \Lambda^n D'(1) \cong \Lambda^n \Sigma P^\infty_{-1}$ (by 4.4). Although we make no essential use of these facts, they are very helpful for understanding symmetric product spectra.

§5. PROOF OF THEOREM A

Let $M(n) = \Sigma^{-n}(D(n)/D(n-1))$. It follows from Proposition 4.3 that $H^*(M(n)$ has basis $\{\theta^I: I \text{ admissible}, l(I) = n\}$.

Theorem A is a consequence of the following:

THEOREM 5.1. *There is a map g: $(B(\mathbb{Z}/p)^n \to M(n)$ such that on mod p cohomology, g^* is an isomorphism onto $[H^*B(\mathbb{Z}/p)^n]e_n$.*

For it follows that g restricts to an equivalence $e_n \cdot B(\mathbb{Z}/p)^n \xrightarrow{\cong} M(n)$.

Since the Steinberg block B_{St} is a matrix algebra of degree $p^{\binom{n}{2}}$ over $\mathbb{F}_p$, the corresponding central idempotent decomposes into the sum of $p^{\binom{n}{2}}$ primitive orthogonal idempotents one of which is e. The corresponding summands of $B(\mathbb{Z}/p)^n$ are equivalent [11, 1.6]. Thus Theorem A follows from Theorem 5.1.

In fact the map is a very natural one, as we proceed to explain (see [11]). There are maps (of spaces) $Sp^iS^m \wedge Sp^jS^n \to Sp^{ij}S^{m+n}$ defined by $(x_1 \cdot x_2 \ldots x_i) \wedge (y_1 \cdot y_2 \ldots y_j) \to (x_1 \wedge y_1 \cdot x_1 \wedge y_2 \ldots x_i \wedge x_j)$. These yield a map of spectra $Sp^iS^0 \wedge Sp^jS^0 \to Sp^{ij}S^0$ and by iteration a map μ_0: $\Lambda^n Sp^pS^0 \to Sp^{p^n}S^0$. As noted in [11], by factoring out the appropriate subspectra we obtain a commutative diagram

$$\begin{array}{ccc} \Lambda^n Sp^pS^0 & \xrightarrow{\mu_0} & Sp^{p^n}S^0 \\ \downarrow & & \downarrow \\ \Lambda^n \overline{Sp^pS^0} & \xrightarrow{\bar{\mu}_0} & \overline{Sp^{p^n}S^0} \end{array} \tag{5.2}$$

where $\overline{Sp^{p^n}S^0} = Sp^{p^n}S^0/Sp^{p^n-1}S^0$.

From the definition of $M(n)$, it is clear on inspection that (5.2) yields a further commutative diagram

$$\begin{array}{ccc} \Lambda^n D(1) & \xrightarrow{\mu} & D(n) \\ \downarrow & & \downarrow \\ \Lambda^n(\Sigma M(1)) & \xrightarrow{\bar{\mu}} & \Sigma^n M(n). \end{array} \tag{5.3}$$

Remark 5.4. In view of the Dold–Thom theorem, the maps μ_0, μ can be viewed as filtrations of the ring spectrum multiplication on $K\mathbb{Z}$, $K\mathbb{Z}/p$. For another interpretation, see Remark (5.7) below.

Finally, from the results of §3 we obtain our main commutative diagram

$$\begin{array}{ccccc} \Lambda^n L^\infty_{-1} & \xrightarrow{\Lambda^n\psi} & \Lambda^n P^\infty_{-1} & \xrightarrow{\mu} & \Sigma^{-n}D(n) \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda^n L^\infty_0 & \xrightarrow{\Lambda^n\bar{\psi}} & \Lambda^n P^\infty_0 & \xrightarrow{\bar{\mu}} & M(n). \end{array} \tag{5.5}$$

Here we recall that

$$\Lambda^n L^\infty_0 = B(\mathbb{Z}/p)^n_+, \quad P^\infty_{-1} \cong \Sigma^{-1}D(1), \quad \Lambda^n P^\infty_0 = B(\Sigma_p)^n_+, \quad \text{and } P^\infty_0 \cong M(1).$$

Let $f = \mu(\Lambda^n\psi)$, $g = \bar{\mu}(\Lambda^n\bar{\psi})$; we will show that g is the required map of Theorem 5.1. Let $R_n = H^*\Lambda^n L_{-1}^{\infty} \subseteq S_n$, $P_n = H^*\Lambda^n L_0^{\infty} \subseteq R_n$, and $M_n = A$-submodule of R_n generated by the bottom class $X_n Y_n^{-1}$.

LEMMA 5.6. *(a) f^* is an isomorphism $H^*\Sigma^{-n}D(n) \to M_n$. (b) g^* is an isomorphism $H^*M(n) \to M_n \cap P_n$.*

Proof. Since $f^*(u_n) = X_n Y_n^{-1}$, (a) is immediate from (3.5) and (4.3). Moreover we have seen in (3.5) that $M_n \cap P_n$ is precisely $\langle \theta^I(X_n Y_n^{-1})$: I admissible and $l(I) = n\rangle$. (b) then follows from (a), using (5.5).

Remark 5.7. Since our proof of (5.6) relies on Nakaoka's calculation of $H^*Sp^{p^n}S^0$, in a sense it puts the cart before the horse. In fact one can show directly that μ_0: $\Lambda^n Sp^p S^0 \to Sp^{p^n}S^0$ is injective in cohomology, and indeed this is essentially equivalent to a key step in Nakaoka's original proof: As remarked in Section 4, $\Lambda^n Sp^p S^0 \cong Sp^p(Sp^p(\ldots Sp^p S^0))\ldots) \cong Sp^H S^0$, where $H = \int^n \Sigma_p$. Moreover, it is easy to see that the resulting map $Sp^H S^0 \xrightarrow{\pi} Sp^{p^n}S^0$ corresponding to μ_0 is the obvious "projection" associated to the inclusion $H \subseteq \Sigma_{p^n}$. Now algebraically one can define a transfer t^*: $H^*Sp^H S^0 \to H^*Sp^{p^n}S^0$ enjoying the usual properties, e.g. $t^*\pi^*$ = multiplication by the index $[\Sigma_p n: H]$. But $[\Sigma_p n: H]$ is prime to p, which shows μ_0^* is injective.

Lemma (5.6) reduces Theorem (5.1) to the following purely algebraic result:

THEOREM 5.8. $R_n e_n = M_n$.

For then $P_n e_n = P_n \cap R_n e_n = P_n \cap M_n = Img^*$ by 5.6b. (As usual, we are regarding R_n as embedded in S_n). The proof of Theorem (5.8) is based on the following curious lemma, which relates the action of the Steenrod algebra on R_n to the action of $GL_n\mathbb{F}_p$.

LEMMA 5.9. *Let $J = (j_0, \ldots, j_{n-1})$, $j = \Sigma j_i$, $j_i = 0$ or 1, and let I be any multiindex of length $\leq n-1$. Then*

$$(x_1 y_1^{-1} Q^J P^I(X_{n-1} Y_{n-1}^{-1}))e_n = (-1)^j Q^J P^I(X_n Y_n^{-1}).$$

Proof of Theorem 5.8. Taking $Q^J P^I = 1$ in the lemma we have $(X_n Y_n^{-1})e_n = X_n Y_n^{-1}$, so $M_n \subseteq R_n e_n$. To show $R_n e_n = M_n$ we use induction on n. For $n = 1$ this is clear (see 3.4). Now suppose we have shown $R_{n-1}e_{n-1} = M_{n-1}$. From (2.5) we have $e_n = -e_{n-1}\bar{A}_n\bar{T}_n$ and hence

$$R_n e_n = (R_1 \otimes R_{n-1}e_{n-1})\bar{A}_n\bar{T}_n = (R_1 \otimes M_{n-1})\bar{A}_n\bar{T}_n = (R_1 \otimes M_{n-1})e_n.$$

Let $R_1' = H^*P_{-1}^{\infty} = H^*(\Sigma^{-1}D(1))$ (see Prop. 4.3, Ex. 3.4 (iii)). Since A_n contains the diagonal matrices $F_p^* \times I_{n-1}$ we have $(R_1 \otimes M_{n-1})e_n = (R_1' \otimes M_{n-1})e_n$. Further for any A-module N, $R_1' \otimes N$ is generated by $x_1 y_1^{-1} \otimes N$. Hence it is enough to show $(x_1 y_1^{-1} \otimes M_{n-1})e_n \subseteq M_n$. But this is immediate from the lemma together with (1.1(c)).

Recall V^n is the vector space $\langle y_1, \ldots, y_n \rangle$. To prove the lemma we will need:

PROPOSITION 5.10. Let $\alpha_{n,k} = \sum_{a \in V^n} a^k$. *Then if $k = ip^r$ with $0 \leq i < p^n - p^{n-1}$, $\alpha_{n,k} = 0$.*

Proof. It is enough to prove the case $r = 0$. Clearly $\alpha_{n,k}$ is a GL_n invariant. But by a classical theorem of Dickson[17], the smallest nonzero dimension in which such an invariant occurs is $2(p^n - p^{n-1})$ for p odd and 2^{n-1} for $p = 2$.

Proof of Lemma 5.9. Fix $p > 2$. For $n = 1$ the lemma merely states that $(x_1 y_1^{-1})e_1 = x_1 y_1^{-1}$; this is clear since $e_1 = d_1$. From now on we assume $n > 1$. We consider first the case $Q^J = 1$. Suppose inductively we have shown the special case $(X_{n-1} Y_{n-1}^{-1})e_{n-1} = X_{n-1} Y_{n-1}^{-1}$. Let π_i: $V^n \to F_p$ denote the coordinate projections. Then

$$\begin{aligned}
x_1y_1^{-1}P^l(X_{n-1}Y_{n-1}^{-1})e_n &= -x_1y_1^{-1}P^l(X_{n-1}Y_{n-1}^{-1})\bar{A}_n\bar{T}_n \text{ (by (2.5) and inductive hypothesis)}\\
&= -X_ny_1^{-1}P^l(Y_{n-1}^{-1})\bar{A}_n\bar{T}_n\\
&= -X_n\sum_{i=1}^{n}\sum_{a\in V^n-0}\pi_i(a)a^{-1}P^l(y_1^{-1}\ldots\hat{y}_i^{-1}\ldots y_n^{-1})\\
&= -X_n\sum_{a\in V^n-0}a^{-1}\sum_{i=1}^{n}\pi_i(a)P^l(y_1^{-1}\ldots\hat{y}_i^{-1}\ldots y_n^{-1})\\
&= -X_n\sum_{a\in V^n-0}a^{-1}P^l(aY_n^{-1}).
\end{aligned}$$

Now $\Delta P^l = \Sigma\theta_j'\otimes\theta_j''$ with $l(\theta_j')$, $l(\theta_j'')\le n-1$. Hence $P^l(aY_n^{-1}) = \sum_{k=0}^{n-1}a^{p^k}\theta_k(Y_n^{-1})$ for certain θ_k independent of a, with $\theta_0 = P^l$. We then have

$$\begin{aligned}
-X_n\sum_{a\in V^n-0}a^{-1}P^l(aY_n^{-1}) &= -X_n\sum_{a\in V^n-0}a^{-1}\sum_{k=0}^{n-1}a^{p^k}\theta_k(Y_n^{-1})\\
&= -X_nP^lY_n^{-1}\left(\sum_{a\in V^n-0}a^0\right) - X_n\sum_{k=1}^{n-1}\theta_kY_n^{-1}(\alpha_{n,p^k-1})\\
&= P^l(X_nY_n^{-1}). \text{ (Using 5.10).}
\end{aligned}$$

For the general case consider the equation

$$(x_1y_1^{-1}\theta(X_{n-1}Y_{n-1}^{-1}))\bar{A}_n\bar{T}_n = \pm\,\theta(X_nY_n^{-1}),\quad \theta\in A. \tag{5.11}$$

Then it is enough to show that if (5.11) holds for θ, then it holds for $Q_i\theta$ if $0\le i\le n-2$ (but with opposite sign). Since Q_i is primitive, by applying Q_i to both sides of 5.11 we are reduced to showing $((Q_ix_1y_1^{-1})\theta(X_{n-1}Y_{n-1}^{-1}))\bar{A}_n\bar{T}_n = 0$. But in fact

$$\begin{aligned}
((Q_ix_1y_1^{-1})\theta(X_{n-1}Y_{n-1}^{-1}))\bar{A}_n = (y_1^{p^i-1}\theta(X_{n-1}Y_{n-1}^{-1}))\bar{A}_n &= (\alpha_{n,p^i-1}-\alpha_{n-1,p^i-1})\theta(X_{n-1}Y_{n-1}^{-1})\\
&= 0.
\end{aligned}$$

By (5.10). This completes the proof if $p>2$. The proof for $p=2$ is similar but easier if we use the elements Sq^l. Then

$$\begin{aligned}
x_1^{-1}Sq^l(X_{n-1}^{-1})e_n = x_1^{-1}Sq^l(X_{n-1}^{-1})\bar{A}_n\bar{T}_n &= \sum_{i=1}^{n}\sum_{a\in W^n-0}\pi_i(a)a^{-1}Sq^l(x_1\ldots\hat{x}_i^{-1}\ldots x_n)\\
&= \sum_{a\in W^n-0}a^{-1}Sq^l(aX_n^{-1}) = Sq^l(X_n^{-1})
\end{aligned}$$

where W^n is the vector space $\langle x_1,\ldots,x_n\rangle$. This finishes the proof of the lemma, and the proof of Theorem 5.8.

Remark 5.12. Lemma (5.9) shows $(X_nY_n^{-1})$ is fixed by e_n (over any finite field $\mathbb{F}_q$). It follows that the Steinberg module can be described as the $GL_n\mathbb{F}_q$ submodule of $E[x_1,\ldots,x_n]\otimes\mathbb{Z}/p(y_1,\ldots,y_n)$ generated by $X_nY_n^{-1}$.

Remark 5.13. Theorem 5.8 determines the multiplicity of the Steinberg module in $E[x_1,\ldots,x_n]\otimes\mathbb{Z}/p[y_1,\ldots,y_n]$. Let $f(s,t) = \Sigma a_{ij}s^it^j$ where a_{ij} is the multiplicity of St in

$E[x_1, \dots, x_n]_{n-i} \otimes \mathbb{Z}/p[y_1, \dots, y_n]_j$. Then using (5.8) we obtain

$$f(s,t) = t^{-n} \frac{\prod_{i=0}^{n-2} (1 + st^{2p^i - 2})}{\prod_{i=1}^{n} (1 - t^{2(p^i - 1)})} (st^{2p^{n-1}-2} + t^{2(p^n - 1)}).$$

Remark 5.14. Since $B(\Sigma_p)^n \simeq d_n \cdot B(\mathbb{Z}/p)^n$, and d_n commutes with $\bar{U}_n$ and $\bar{\Sigma}_n$, e_n restricts naturally to a self-map of $B(\Sigma_p)^n$. Hence $M(n)$ is a stable summand of $B(\Sigma_p)^n$.

Let $L(n) = \Sigma^{-n}\overline{Sp}^{p^n}(S^0)$. We conclude this section by proving

PROPOSITION 5.15. $M(n) \simeq L(n) \vee L(n-1)$.

Proof. By definition, there is a cofibration $L(n) \to M(n) \to L(n-1)$, with the resulting cohomology sequence short exact (§4). Hence it will be enough to produce a map h: $M(n) \to L(n)$ such that h^* is an isomorphism onto $\langle \theta^I : \epsilon_{n-1} = 0 \rangle$. Let H be the composite $\Lambda^n \Sigma P_0^\infty \to \Lambda^n \Sigma P_1^\infty \xrightarrow{\bar{\mu}_0} \overline{Sp}^{p^n} S^0$, where $\bar{\mu}_0$ is as in (5.2) (recall $\Sigma P_1^\infty \simeq \overline{Sp}^p S^0$). By Theorem (5.1) and Remark 5.14, $M(n)$ is a stable summand of $\Lambda^n \Sigma P_0^\infty$. From diagram (5.2), it is clear that a map h with the desired property is obtained by restricting H to $M(n)$.

§6. SPLITTING $B(\int^n \mathbb{Z}/p)$ AND $BO(n)$

Regarding Σ_{p^n} as the permutation group of the set $\mathbb{F}_p{}^n$, one obtains an embedding of the affine group $Aff_n(\mathbb{F}_p) = GL_n\mathbb{F}_p \tilde{\times} \mathbb{F}_p{}^n$ in Σ_{p^n}. In particular this defines an inclusion j: $\mathbb{F}_p{}^n \to \Sigma_{p^n}$ (as the group of translations) with Weyl group $W_{\Sigma_{p^n}}(\mathbb{F}_p{}^n) = GL_n$. Now the wreath product embeds $\int^n \mathbb{Z}/p \subset \Sigma_{p^n}$ as a p-Sylow subgroup and factors j

$$\begin{tikzcd} \mathbb{F}_p{}^n \arrow[r, "j"] \arrow[dr, "j'"'] & \Sigma_{p^n} \\ & \int^n \mathbb{Z}/p. \arrow[u, phantom, "\cup"] \end{tikzcd}$$

This embedding can be chosen so that $Aff_n \cap \int^n \mathbb{Z}/p = U_n \tilde{\times} \mathbb{F}_p{}^n$, and $W_{\int^n \mathbb{Z}/p}(\mathbb{F}_p{}^n) = U_n$. Similarly, $\int^n \Sigma_p \subset \Sigma_{p^n}$ and $Aff_n \cap \int^n \Sigma_p = B_n \tilde{\times} \mathbb{F}_p{}^n$; then $W_{\int^n \Sigma_p}(\mathbb{F}_p{}^n) = B_n$. Letting t: $B \int^n \mathbb{Z}/p \to B(\mathbb{Z}/p)^n$ denote the transfer associated to j' we then have the following easy consequence of the double coset formula (see [11], Proposition 1.4).

LEMMA 6.1. $j'^* t^* = \bar{U}_n$.

Proof of Theorem B. From the lemma and (2.6(b)), we see that tj' restricts to an equivalence $e'_n \cdot B(\mathbb{Z}/p)^n \xrightarrow{\simeq} e_n \cdot B(\mathbb{Z}/p)^n$. Hence Theorem B follows from Theorem A.

Remark. Since e_n and e'_n commute with d_n, it follows that the summand $M(n)$ of $B \int^n \mathbb{Z}/p$ actually is a summand of $B \int^n \Sigma_p$.

Proof of Theorem C. The inclusion of $(\mathbb{Z}/2)^n$ in $O(n)$ as the diagonal matrices yields a map $B(\mathbb{Z}/2)^n \xrightarrow{i} BO(n)$ with fibre the flag manifold $O(n)/(\mathbb{Z}/2)^n$. Let t: $BO(n) \to B(\mathbb{Z}/2)^n$ be the associated Becker–Gottlieb transfer.

LEMMA 6.2. $i^* t^* \doteq \bar{\Sigma}_n$[18].

As in the proof of Theorem B, it follows that ti restricts to an equivalence $e_n \cdot B(\mathbb{Z}/2)^n \xrightarrow{\simeq} e'_n \cdot B(\mathbb{Z}/2)^n$. Hence Theorem C follows from Theorem A.

§7. SPLITTING BT^n AND $BU(n)$

In this section all spectra are completed at p. We begin by observing that $GL_n(\mathbb{Z}_p)$ acts on $T^n = B(\mathbb{Z}_p)^n$, BT^n, and hence diagonally on $T^n_+ \wedge BT^n_+ = (T^n \times BT^n)_+$. In mod-$p$ cohomology

$$H^*(T^n \times BT^n)_+ = E[x_1, \ldots, x_n] \otimes \mathbb{Z}/p[y_1, \ldots, y_n]$$

where $x_i = 1 \otimes \ldots \otimes x \otimes \ldots \otimes 1 \in H^1 T^n$ and $y_i = c_1(\pi_i)$ where π_i: $T^n \to S^1$ is the i-th projection map. This notation is chosen to agree with that of (3.1) since $H^*(T^n \times BT^n)_+ \approx H^*(B\mathbb{Z}/p)^n_+$ as $GL_n(\mathbb{F}_p)$ modules. Here $GL_n(\mathbb{Z}_p)$ acts via mod p reduction $GL_n(\mathbb{Z}_p) \to GL_n(\mathbb{F}_p)$.

Since mod p reduction is surjective, we can choose $\hat{e}_n \in \mathbb{Z}_p[GL_n(\mathbb{Z}_p)]$ which projects to the Steinberg idempotent $e_n \in \mathbb{F}_p[GL_n(\mathbb{F}_p)]$; hence $\hat{e}_n$ defines a map

$$\hat{e}_n\colon (T^n + BT^n)_+ \to (T^n \times BT^n)_+$$

which induces action by e_n on $H^*(T^n \times BT^n)_+$. As explained in §3, $\hat{e}_n$ splits $(T^n \times BT^n)_+$; however, we wish to split BT^n at least up to suspension. Hence we define

$$\tilde{e}_n\colon S^n \wedge BT^n_+ \xrightarrow{i \wedge 1} T^n_+ \wedge BT^n_+ \xrightarrow{\hat{e}_n} T^n_+ \wedge BT^n_+ \xrightarrow{p \wedge 1} S^n \wedge BT^n_+$$

where i and p are inclusion and projection on the top cell. We shall see that $\tilde{e}_n$ is an idempotent in cohomology and hence splits $S^n \wedge BT^n_+$.

Definition. $BP(n) = \Sigma^n \tilde{e}_n(S^n \wedge BT^n_+)$.

Proof of Theorem D. First we show that $BP(n)$ has the correct cohomology. We proceed to consider some complex analogues of our previous constructions. Let η be the canonical line bundle over BS^1 and write $\mathbb{C}P^\infty_{2k}$, $k \in \mathbb{Z}$ for the Thom spectrum $(BS^1)^{k\eta}$. Then $\Lambda^n \mathbb{C}P_0 = BT^n_+$ and we let $P_n = H^*(T^n \wedge BT^n)_+$, $S_n = P_n[l_n^{-1}]$ where l_n is the product of all non-zero linear forms in $y_1, y_2, \ldots, y_n$. Let $R_n = H^*(T^n_+ \wedge \Lambda^n \mathbb{C}P^\infty_{-2}) \subset S_n$ and let M be the $P = A/(\beta)$ module generated by $X_n Y_n^{-1}$ where $X_n = x_1 \ldots x_n$, $Y_n = y_1 \ldots y_n$. Then $M_n \cong \Sigma^n(P/P \cap G_n)$ as in Prop. 3.5. Further, $R_n e_n = M_n$ as in Theorem 5.8. Thus $P_n e_n = P_n \cap R_n e_n = P_n \cap M_n$ which has the required basis $\Sigma^n\{P^I(X_n Y_n^{-1})\colon I \text{ admissible}, l(I) = n\}$ as in Prop. 3.5.

It is now clear that $\tilde{e}_n$ is an idempotent in cohomology since X_n represents the top cell in S^n.

To see that BT^n contains $p^{\binom{n}{2}}$ copies of $BP(n)$ we note that lifting orthogonal idempotents of $\mathbb{F}_p[GL_n(\mathbb{F}_p)]$ to $\mathbb{Z}_p[GL_n(\mathbb{Z}_p)]$ results in self maps of $(T^n \wedge BT^n)_+$ which give orthogonal idempotents in cohomology.

The proof that $BU(n)$ splits is analogous to that of $BO(n)$, Theorem C (6.1). One uses the fibration $U(n)/T^n \to BT^n \to BU(n)$ and Becker–Gottlieb transfer.

REFERENCES

1. J. F. Adams, Lectures on generalized cohomology. *Lecture Notes in Math.* Vol. 99, pp. 1–138. Springer-Verlag (1969).
2. D. Quillen, On the formal group laws of unoriented and complex cobordism theory. *Bull. Am. Math. Soc.* **75**, 1293–1298 (1969).
3. D. Kahn and S. Priddy, The transfer and stable homotopy theory. *Math. Proc. Camb. Math. Soc.* **83**, 102–111 (1978).
4. V. Snaith, A stable decomposition for $\Omega^n\Sigma^n X$. *J. London Math. Soc.* **7**, 577–583 (1974).
5. M. Mahowald, A new infinite family in ${}_2\pi^S$. *Topology* **16**, 249–254 (1971).

6. N. J. KUHN, A Kahn–Priddy sequence and a conjecture of G. W. Whitehead. *Math. Proc. Camb. Phil. Soc.* **92**, 467–483 (1982).
7. N. J. KUHN, S. MITCHELL and S. PRIDDY. The Whitehead conjecture and splitting $B(\mathbb{Z}/2)^k$. *Bull. Amer. Math. Soc.* **7**, 255–258 (1982).
8. J. MILNOR, The Steenrod algebra and its dual. *Annals of Math.* 150–171 (1958).
9. C. CURTIS and I. REINER, *Representation Theory of Finite Groups and Associative Algebras*. Interscience Publishers, New York (1962).
10. R. STEINBERG, Prime power representations of finite linear groups II. *Can. J. Math.* **18**, 580–591 (1956).
11. S. MITCHELL and S. PRIDDY, Symmetric product spectra and splittings of classifying spaces. *Am. J. Math.* (to appear).
12. C. WILKERSON, Classifying spaces, Steenrod operations and algebraic closure. *Topology* **16**, 227–237 (1977).
13. B. MANN, E. MILLER and H. MILLER, S^1-equivariant function spaces and characteristic classes. *Trans. Am. Math. Soc.* (to appear).
14. P. WELCHER, Symmetric products and the stable Hurewicz homomorphism. *Ill. J. Math.* **24**, 527–542 (1980).
15. M. NAKAOKA, Cohomology mod p of symmetric products of spheres. *J. Inst. Poly., Osaka City Univ.* **9**, 1–18 (1958).
16. D. KAN and G. WHITEHEAD, The reduced join of two spectra. *Topology* **3**, Suppl. 2, 239–261 (1965).
17. L. DICKSON, A fundamental system of invariants of the general modular linear group with a solution of the form problem. *Trans. Am. Math. Soc.* **12**, 75–98 (1911).
18. M. FESHBACH, The transfer and compact Lie groups. *Trans. Am. Math. Soc.* **251**, 139–169 (1979).

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