ON THE LICHTENBAUM-QUILLEN CONJECTURES FROM A STABLE HOMOTOPY-THEORETIC VIEWPOINT

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INTRODUCTION

The original purpose of this paper was to give a leisurely exposition of the author's work [Mitchell 1990a, b], including the philosophy behind it and its connection with the Lichtenbaum-Quillen conjectures. The intended audience included homotopy theorists and algebraic K-theorists. However it soon became clear that this necessitates explaining algebraic K-theory to the former group and stable homotopy theory to the latter; hence the length of the present work. The paper in fact consists of three parts: (1) an exposition of the Lichtenbaum-Quillen conjectures; (2) an introduction to the "chromatic" view of stable homotopy theory, and related topics; and (3) an account of how the first two parts are related, together with an exposition of the author's recent work cited above. We have made an effort to assume as little as possible in the way of background, and wherever it seemed reasonable to do so, we have sketched the proofs of the main results.

The first part (§1–7) is an introduction to the Lichtenbaum-Quillen conjectures on the K-theory of commutative rings and schemes, viewed from a homotopy-theoretic perspective. As this subject is impossibly vast, we have focused on the two cases that are emphasized in the original sources ([Quillen 1974], [Lichtenbaum]): (1) algebraically closed fields, and especially (2) rings of integers in a number field. In case (1) the conjectures say (almost) that the algebraic K-theory *with finite coefficients* of an algebraically closed field is independent of the particular field, and coincides with topological complex K-theory. This case was settled affirmatively by Suslin, and is discussed in detail in §4. In fact we give the complete proof of Suslin's theorem for $\mathbb{C}$, assuming a theorem of Gillet and Thomason. In case (2), the conjectures as formulated by Lichtenbaum relate three very different invariants of a totally real number field F: (a) values of the zeta function $\zeta_F(s)$ at odd negative integers, (b) orders of certain étale cohomology groups attached to the ring of integers $\mathcal{O}_F$ and (c) orders of the K-groups $K_n\mathcal{O}_F$ for $n = 2$ or $3 \mod 4$. The conjecture relating (a) and (b) is now a theorem [Wiles]; see §6. However we will say almost nothing about zeta functions; our main concern is the connection between (b) and (c). Explicit examples can be found in §6, including a complete conjectural description of $K_*\mathbb{Z}$ (assuming Vandiver's conjecture from number theory).

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The general form of the Lichtenbaum-Quillen conjecture asserts that for a nice scheme there is a *descent spectral sequence* with E_2-term given in terms of étale cohomology and converging to the algebraic K-theory of X. This is explained in §5; without assuming any knowledge of schemes or étale cohomology. As usual, our treatment is oriented towards homotopy theorists; we first explain, following [Carlsson] how descent for fields can be viewed as a case of the "homotopy fixed point problem." This is followed by a discussion of étale cohomology. The discussion is necessarily brief, but we hope it is sufficient to make the rest of the paper comprehensible. We then state our first version of the general Lichtenbaum-Quillen conjectures 5.12.

Of course any account of the Lichtenbaum-Quillen conjectures presupposes some familiarity with Quillen's higher K-theory, and higher K-theory depends on lower K-theory, whence §1. The main theme of §1 is that the lower K-groups-K_0, K_1, K_2-of a ring of integers $\mathcal{O}_F$ are closely related to classical number-theoretic invariants: the class group, unit group and Brauer group. it is worth considering these in some detail, since conjecturally all of the K-groups of $\mathcal{O}_F$ are built out of these basic ingredients (see §6). In §2 we give a rapid introduction to higher K-theory. We mention three equivalent constructions of the K-theory of a ring: the plus construction, group completion, and the Q-construction. If one wants to consider vector bundles over schemes, or more general "exact categories," the Q-construction is essential. Furthermore, even for rings, most of Quillen's general theorems use the Q-construction, not the plus construction. However the plus construction provides by far the most elementary definition of higher K-theory, and will be emphasized here. We go on to describe some basic results on the higher K-groups. Some of these are used repeatedly and explicitly in later sections–e.g., Quillen's calculation of the K-theory of finite fields. Others merely lurk in the background–e.g. "devissage." We have included the latter to give the reader a feeling for the remarkable simplicity of Quillen's theory. We also sketch Borel's computation of the rank of $K_n\mathcal{O}_F$. The appearance of the spaces U and U/O in this context is not so surprising to a Lie theorist, but to a homotopy theorist it is rather provocative. These spaces will appear again in §7. We next introduce K-theory *spectra*, and discuss the extremely useful transfer map.

In §3 we show how homotopy-theoretic methods can be used to produce torsion classes in $K_*\mathcal{O}_F$: (1) by considering the natural map from the stable homotopy groups of spheres to $K_*\mathbb{Z}$ and (2) by considering the projection to a residue field. All the classes obtained in this way are closely related to the image of the classical J-homomorphism. This theme is taken up again in §11, 12. The presence of these classes is predicted by the Lichtenbaum-Quillen conjectures; in the mod-ℓ^v theory they correspond to the zero-column of the descent spectral sequence.

In §7 we first discuss the étale K-theory of Dwyer and Friedlander, which leads to a second version of the Lichtenbaum-Quillen conjectures. It also leads to some beautiful, *explicit* conjectures on the nature of algebraic K-theory spectra. Then we state the remarkable theorem of Thomason, which asserts that the Lichtenbaum-Quillen conjectures are true for "Bott-periodic" algebraic K-theory. This theorem plays a crucial role in later sections.

Part 2 (§8–10) is an introduction to some aspects of stable homotopy theory centering around MU (complex cobordism). We would like to have begun with the definition of a spectrum, but that wasn't practical. See [Adams 1974], for further background. In §8 we describe MU, the associated p-local theory BP, and the Morava K-theories. In §9 we discuss some of the remarkable work of Hopkins, Devinatz and J. Smith. The only part of this section that is actually used later is the construction of "v_n-complexes." However the conceptual framework it provides is crucial; among other things, it explains and justifies the emphasis on Morava K-theories in later sections. In §10 we introduce localization with respect to a homology theory, and give some important examples.

One of the main points of Part II can be very vaguely stated as follows: we can associate to each p-local spectrum X its *chromatic tower* $L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow \dots$. Here L_0X is the rationalization of X_1 and L_1X is localization with respect to topological K-theory. For $n \geq 1$ the spectrum L_nX has something to do with "nth order periodicity" in the homotopy groups of X. For example L_1X is related to Adams or Bott periodicity. For $n \geq 2$ there is a more mysterious "higher periodicity." Now in §12 we will show that algebraic K-theory is completely orthogonal to this higher periodicity, so that K-theorists can safely ignore it. At the same time, however, one can't possibly appreciate the significance of §11–12 without a look at the broader picture.

In Part III (§11–13) we apply the stable homotopy theory of Part II to the algebraic K-theory of Part I. In §11 we reformulate the Lichtenbaum-Quillen conjectures in terms of Bousfield localization, following [Waldhausen 1984], and derive some consequences. Some of these consequences are now theorems, and are discussed in §12. In particular we show that the "higher" Morava K-theories of any algebraic K-theory spectrum vanish. Together with Thomason's theorem, this shows that the Lichtenbaum-Quillen conjectures are true after "harmonic" localization. This is a considerable strengthening of Thomason's theorem, and gives some insight into the nature of possible counterexamples to the conjectures. We conclude by indulging in some speculative remarks (§13).

At this point, two apologies are in order. First, I am not an expert on algebraic K-theory, and the reader is given fair warning that the entire paper proceeds from a certain homotopy-theoretic bias. For more expert surveys of various aspects of K-theory, the reader should consult [Soulé 1982] for a discussion of the zeta function aspect, [Grayson 1989] for the K-theory of fields, and [Thomason 1989] for Bott periodic and étale K-theory. In fact one purpose of Part I is to provide a sort of "meta-survey," that will make works such as those just cited accessible to a wider audience. Second, inspite of our relatively narrow focus, it is impossible to cover everything. Among the major omissions I would like to point out the work of Vic Snaith, who was a pioneer in the application of homotopy-theoretic methods to algebraic K-theory (see e.g. [Snaith 1983, 1984]).

Finally, a word on notation: throughout this paper, the letters ℓ and p stand for fixed primes, and q is power of p. With apologies to homotopy theorists, we generally adopt the K-theorist's convention and let ℓ denote the prime at which we are localizing, while p is reserved for the characteristic of a field. The only exception is in §8–10, where only one prime is needed and we use the traditional "p". In addition, ℓ and p *are always distinct* unless stated otherwise. One reason for this is that $K_*\mathbb{F}_{p^m}$ is essentially trivial when localized at p (§2). Another reason is that many of the theorems discussed here are simply false for $\ell = p$. For example, the reader of this paper will learn absolutely nothing about the p-local K-theory of the p-adic integers (contrast §4). In any event, we will frequently localize at ℓ without explicitly saying so. It should be clear from the context when this has been done. Often we will go further and *complete* at ℓ, but this will be explicitly indicated–e.g. $X^\wedge$ denotes the completion at ℓ of the spectrum X (see §10 for the definition). Indeed the reader will also learn absolutely nothing about the uniquely divisible part of K-theory, which is another subject altogether. The notation "+" as a superscript refers to the plus construction; as a subscript it refers to a disjoint basepoint. If in doubt, consider the context. If A is an abelian group, ${}_nA = \{a \in A : na = 0\}$.

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13. Concluding remarks.

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1. Lower K-Theory

The main reference for this section is [Milnor 1971]. Let $\mathcal{C}$ be a category with a notion of short exact sequences. Then we can define the Grothendieck group $K_0\mathcal{C}$ as the free abelian group on the objects of $\mathcal{C}$, modulo relations $[M] = [M'] + [M'']$ for every short exact sequence $M' \to M \to M''$. The main examples we have in mind are $\mathcal{C} = \mathbf{PR}$, the category of finitely-generated projective modules over the ring R, and $\mathbf{MR}$, the category of all finitely-generated modules over R. We write $K_0R \equiv K_0\mathbf{PR}$ and $G_0R \equiv K_0\mathbf{MR}$.

1.1. Examples.

(a) If R is a field, a division ring, or a principal ideal domain, then $K_0R \cong G_0R \cong \mathbb{Z}$, generated by the free module of rank one.
(b) (Serre) If R is the coordinate ring of an affine algebraic variety V, then $\mathbf{PR} \cong \mathbf{VectV}$, the category of algebraic vector bundles (locally free sheaves) on V. Hence $K_0R \cong K_0(\mathbf{VectV})$.
(c) (Swan) If R is the ring of continuous functions on a compact Hausdorff space X, $\mathbf{PR} \cong \mathbf{VectX}$, the category of vector bundles on X. Hence K_0R coincides with topological K-theory $K(X)$.
(d) The natural map $K_0R \to G_0R$ need not be an isomorphism–consider, for example, $R = \mathbb{Z}/\ell^2$.

Similarly if X is a scheme we can define K_0X (resp. G_0X) as the Grothendieck group of vector bundles (resp. coherent sheaves) on X. However, without further ado we will move on to our main example.

Let R be an Dedekind domain; for example, the coordinate ring of a smooth affine curve, or the ring of integers in a number field. Here number field means a finite extension F of $\mathbb{Q}$; its ring of integers $\mathcal{O}_F$ is the integral closure of $\mathbb{Z}$ in F. In particular R has Krull dimension 1: every nonzero prime is maximal. If we think of dimension as complex dimension, so that R has real dimension 2, the following fact has a familiar topological analogue:

1.2. Proposition. *Let R be a Dedekind domain with quotient field F, and let* $\operatorname{Pic} R$ *denote the group of rank one projective modules (line bundles) under tensor product. Then there is a short exact sequence*

$$O \to \operatorname{Pic} R \xrightarrow[j]{} K_0 R \to K_0 F \to 0$$

where $j([P]) = [P] - [R]$. □

Any ideal of R–or more generally, any fractional ideal-is a rank one projective module. Conversely any rank one projective is isomorphic to some fractional ideal. Hence there is an isomorphism $ClR \cong \operatorname{Pic} R$, where ClR is the *ideal class group* of fractional ideals modulo principal fractional ideals.

1.3. Theorem. *(Dirichlet) Let F be a number field. Then the ideal class group of $\mathcal{O}_F$ is finite.*

1.4. Corollary. *$K_0\mathcal{O}_F$ is finitely-generated.*

The most important case for us is $F = \mathbb{Q}(\xi_\ell)$, where ℓ is a prime and ξ_ℓ is a primitive ℓth root of unity. Then $\mathcal{O}_F = \mathbb{Z}[\xi_\ell]$. It is known that that $Cl(\mathbb{Z}[\xi_\ell])$ is zero if and only if $\ell < 23$. More critical for K-theory is the question of whether ℓ is *regular*-i.e. $\ell \nmid |Cl\mathbb{Z}[\xi_\ell]|$. The first few irregular primes are 37, 59, 67. It is still unknown whether or not there are infinitely many regular primes. As we will see later, even the K-theory of $\mathbb{Z}$ gets tangled up with the K-theory of $\mathbb{Z}[\xi_\ell]$; hence these class groups can't be avoided. For more information on irregular primes, class groups, etc. see [Washington].

We next turn to the functor K_1R. For any ring R, let $GLR = \cup_n GL_nR$. Thus GLR consists of infinite invertible matrices A that equal the identity matrix I except for a finite number of entries. If $A = I$ except for a single off-diagonal entry, A is called *elementary.* Let $E(R)$ denote the subgroup generated by the elementary matrices. Then a lemma of J. H. C. Whitehead shows that $E(R)$ coincides with the commutator subgroup $[GLR, GLR]$. We define $K_1R = GLR/E(R) = GLR/[GLR, GLR]$. Clearly K_1R is a covariant functor of R. If R is commutative, the determinant induces a surjective homomorphism $\det : K_1R \to R^*$, which is an isomorphism if and only if SLR is generated by elementary matrices: that is, every element of SLR can be reduced to the identity by elementary row and column operations. Thus for example $K_1R \cong R^*$ if R is a field (easy) or a Euclidean domain (harder).

1.5. Theorem. *(Bass-Milnor-Serre) Let $\mathcal{O}_F$ be the ring of integers in a number field. Then $K_1\mathcal{O}_F \cong \mathcal{O}_F^*$.*

Now for a number field F define

$r_1 =$ number of distinct real embeddings of F

$r_2 = \frac{1}{2}$ (number of distinct complex embeddings of F).

Note that if F is Galois over $\mathbb{Q}$, then either $F \subseteq \mathbb{R}$, $r_1 = [F : \mathbb{Q}]$ and $r_2 = 0$, or $F \not\subseteq \mathbb{R}$, $r_1 = 0$ and $r_2 = \frac{1}{2}[F : \mathbb{Q}]$.

1.6. Theorem. *(Dirichlet) $\mathcal{O}_F^*$ is a finitely-generated group, with rank $r_1 + r_2 - 1$.*

1.7. Corollary. *$K_1\mathcal{O}_F$ is finitely-generated with rank $r_1 + r_2 - 1$.*

Of course the torsion subgroup of $\mathcal{O}_F^*$ is just the group of roots of unity in F. We remark that the proof uses the various embeddings to get an embeddings of rings $\mathcal{O}_F \hookrightarrow (\mathbb{R})^{r_1} \times (\mathbb{C})^{r_2}$ with *discrete* image. Note also that $K_1F \cong F^*$ is certainly *not* finitely generated.

Let S be a set of nonzero prime ideals in $\mathcal{O}_F$. If the ideals in question are principal we can invert their generators to obtain a localized ring $S^{-1}\mathcal{O}_F \subseteq F$. In general for any nonzero prime ideal $\mathcal{P}$, $\mathcal{P}^n = (x)$ is principal for some n by 1.3. Hence we can form $x^{-1}\mathcal{O}_F$, with $\operatorname{Spec} x^{-1}\mathcal{O}_F \to \operatorname{Spec} \mathcal{O}_F$ a bijection onto the complement of $\{\mathcal{P}\}$. We reserve the term *ring of S-integers* for the case S finite. Note that if we invert a rational prime $\ell \in \mathbb{Z}$, $\mathcal{O}_F[\frac{1}{\ell}] = S^{-1}\mathcal{O}_F$ where S is the (finite) set of primes over ℓ in $\mathcal{O}_F$. From 1.6 we have at once:

1.8. Theorem. *If S is finite, $(S^{-1}\mathcal{O}_F)^*$ is a finitely-generated group, with rank $r_1 + r_2 - 1 + |S|$.*

The last of the "lower" K-functors is Milnor's K_2R. Consider the subgroup $E(R)$ of GLR. By definition it is generated by the matrices $e_{ij}(r) = I + x_{ij}(r)$, where $x_{ij}(r)$ is the matrix with r as (ij)-th entry and zeros elsewhere $(i \neq j)$, $r \in R$. What are the relations? There are certain "obvious" relations that hold independently of the particular ring R : $e_{ij}(r)$ and $e_{k\ell}(s)$ commute if $j \neq k$ and $i \neq \ell$, the commutator $[e_{ij}(r), e_{jk}(s)]$ equals $e_{ik}(rs)$ for $i \neq k$, and $e_{ij}(r)e_{ij}(s) = e_{ij}(r+s)$. The *Steinberg group StR* is the free group on the symbols $e_{ij}(r)$ modulo these universal relations. By construction StR maps onto $E(R)$ and Milnor defines $K_2R = \operatorname{Ker}(StR \to E(R))$. It turns out that K_2R is a central subgroup of StR and so in particular is abelian. In fact $K_2R \cong H_2(E(R); \mathbb{Z})$, which we can take as the definition. For further details, see [Milnor 1971]. In any case for us K_2 will always be given by theorems 1.9 and 1.16.

1.9. Theorem. *(Matsumoto) Let F be a field. Then $K_2F \cong (F^* \otimes_{\mathbb{Z}} F^*)/I$, where I is the subgroup generated by all $a \times (1-a)$, $a \in F^* - \{0, 1\}$.*

1.10. Example. *([Milnor 1971]) $K_2\mathbb{F}_q = 0$ for a finite field $\mathbb{F}_q$.*

1.11. Theorem. *(Garland) If F is a number field, $K_2\mathcal{O}_F$ is finite. In particular, $K_2\mathcal{O}_F$ is finitely generated.*

1.12. Example [Milnor 1971] $K_2\mathbb{Z} = \mathbb{Z}/2$.

There is a beautiful relation between K_2 of a field F and the Brauer group $Br\,F$. We first recall the definition of $Br\,F$. A central simple F-algebra A is a finite-dimensional simple F-algebra with center equal to F. If A and B are central simple, so is $A \otimes_F B$. Hence the isomorphism classes of central simples form a commutative monoid with identity F. Now define an equivalence relation $A \sim B$ if $A \otimes_F M_r F \cong B \otimes_F M_s F$ for some r, s–i.e., $M_r A \cong M_s B$. Multiplication is well-defined on the set of equivalence classes $Br\,F$. Furthermore $Br\,F$ is a group, since $A \otimes_F A^{op} \cong \mathrm{End}_F A \sim F$. Note that as a set, $Br\,F \leftrightarrow$ isomorphism classes of F-central division algebras D, since every central simple is isomorphic to some $M_n D$.

1.13. Examples. (a) If F is algebraically closed, $Br\,F = 0$ (obvious).
(b) (Frobenius) $Br\,\mathbb{R} \cong \mathbb{Z}/2$, generated by the quaternions $\mathbb{H}$.
(c) (Wedderburn) $Br\,\mathbb{F}_q = 0$.
(d) (Class field theory) Let F be a number field. Then there is an exact sequence

$$O \to BrF \xrightarrow{i} \oplus_\nu BrF_\nu \xrightarrow{h} \mathbb{Q}/\mathbb{Z} \to 0$$

where F_ν ranges over all completions of F. If ν is a finite prime (i.e. an ordinary prime of $\mathcal{O}_F$) then the *Hasse invariant* $h : BrF_\nu \underset{\cong}{\to} \mathbb{Q}/\mathbb{Z}$. If ν is an infinite prime (i.e. a valuation arising from some real or complex embedding) then $BrF_\nu = \mathbb{Z}/2$ in the real case and is zero otherwise by examples a, b above. Note this says in particular that every central simple F-algebra A becomes a matrix algebra $M_n F_\nu$ at almost all ν. If F has no real embeddings, then $Br\,F$ is a direct sum of a countably infinite number of $\mathbb{Q}/\mathbb{Z}$'s.

When F contains a primitive nth root of unity ξ_n and char $F \nmid n$, there is the following beautiful construction of central simple F-algebras: If $a, b \in F^*$, let $A(a, b)$ denote the F-algebra with generators x, y and relations $x^n = a$, $y^n = b$, $yx = \xi_n xy$. Then one can show $A(a, b)$ is central simple, and moreover (using Matsumoto's theorem):

1.14. Theorem. *The map $F^* \times F^* \to BrF$ given by $(a, b) \mapsto A(a, b)$ factors through a homomorphism $K_2F/n \to_n BrF$.*

The map of 1.14 is called the *power norm residue symbol.* The following remarkable theorem was first proved by [Tate] for number fields and then by Mercurjev and Suslin in general (see [Mercurjev]). The proof of Mercurjev-Suslin uses higher K-theory.

1.15. Theorem. *Suppose F has a primitive nth root of unity, and char $F \nmid n$. Then the power norm residue symbol $K_2F/n \to_n BrF$ is an isomorphism.*

Remark. Tate and Mercurjev-Suslin in fact prove a more general result without nth roots of unity: $K_2F/n \cong H^2(G_F; \mu_n(2))$. See §5 for an explanation of the Galois cohomology on the right. For a summary of the proof, see [Grayson 1989].

To describe K_2 for rings of S-integers we introduce the Brauer group of a commutative ring R. (Alternatively, the reader could take Theorem 1.16 below as the definition.) Now central simple F-algebras are in fact characterized by the property that the natural map $A \otimes_F A^{op} \to \mathrm{End}_F A$ is an isomorphism. Hence the notion of "central simple F-algebra" may be generalized to *Azumaya R-algebras*: that is, an R-algebra Λ such that (a) Λ is an R-order-i.e. finitely-generated and projective as an R-module and (b) $\Lambda \otimes_R \Lambda^{op} \underset{\cong}{\to} \mathrm{End}_R \Lambda$. For example, the ring $M_n R$ of $(n \times n)$-matrices over R is an Azumaya R-algebra. The equivalence relation defining the Brauer group of a field is in fact Morita equivalence (which we won't define here; see e.g. [Reiner]); and in this form it carries over at once to Azumaya algebras. Thus we define the Brauer group $Br\, R$ as the group of Morita equivalence classes of Azumaya R-algebras under tensor project. Now suppose R is a Dedekind domain with quotient field F, F a number field. Let Λ be an R-order and let $A = \Lambda \otimes_R F$. Then if Λ is R-Azumaya, one can show A is central simple and Λ is in fact a maximal R-order in A. Conversely, if Λ is a maximal R-order in A then Λ is an R-Azumaya algebra if and only if for all $p \in \mathrm{Spec}\, R$, $p \neq 0$, the Hasse invariant of A at p is zero. See [Reiner] for details; in the end one finds:

1.16. Theorem. *Let $S^{-1}\mathcal{O}_F$ be a ring of S-integers in a number field, where S is nonempty. Then there is an exact sequence*

$$O \to BrS^{-1}\mathcal{O}_F \to (\bigoplus_{p \in S} \mathbb{Q}/\mathbb{Z}) \bigoplus (\mathbb{Z}/2)^{r_1} \to \mathbb{Q}/\mathbb{Z} \to 0. \qquad \square$$

1.17. Example. If $r_1 = 0$, $Br\, \mathcal{O}_F[\frac{1}{\ell}] \cong \oplus^{k-1}\mathbb{Q}/\mathbb{Z}$, where $k =$ number of primes over ℓ. Similarly for example $Br\mathbb{Z}[\frac{1}{2}] = \mathbb{Z}/2$.

Now Tate also computed $K_2(S^{-1}\mathcal{O}_F)/n$.

1.18. Theorem. *Assume $\xi_{\ell^\nu} \in F$. Then there is a split exact sequence*

$$O \to Cl(S^{-1}\mathcal{O}_F)/\ell^\nu \xrightarrow{i} K_2(S^{-1}\mathcal{O}_F)/\ell^\nu \xrightarrow{j} {}_{\ell^\nu}BrS^{-1}\mathcal{O}_F \to 0$$

Looking ahead a bit, Tate's proof shows that i is in fact multiplication by the "Bott class" $K_0(S^{-1}\mathcal{O}_F; \mathbb{Z}/\ell^\nu) \to K_2(S^{-1}\mathcal{O}_F, \mathbb{Z}/\ell^\nu)$ (§3). The map j arises from $S^{-1}\mathcal{O}_F \hookrightarrow F$ and the power norm residue symbol as in 1.14.

1.19. Example. Let $R = \mathbb{Z}[\xi_\ell, \frac{1}{\ell}]$, where ℓ is an odd regular prime. Then $(K_2R)/\ell = 0$: For ℓ is totally ramified in the extension $\mathbb{Q}(\xi_\ell)/\mathbb{Q}$, and so $k = 1$ in Example (b). In fact by a theorem of Iwasawa, the class group of $\mathbb{Z}[\xi_{\ell^\nu}]$ has no ℓ-torsion if and only if ℓ is regular (ν arbitrary). Hence if ℓ is odd regular, $(K_2\mathbb{Z}[\xi_{\ell^\nu}, \frac{1}{\ell}])/\ell = 0$ for all ν.

As we have presented things so far, the functors K_0, K_1, K_2 may seem rather unrelated. In fact:

1.20. Theorem. *Let F be a number field, $S \subseteq \operatorname{Spec} \mathcal{O}_F - \{0\}$. Then there is an exact sequence*

$$\bigoplus_{\mathcal{P}\in S} K_2\mathcal{O}_F/\mathcal{P} \to K_2\mathcal{O}_F \to K_2S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P}\in S} K_1\mathcal{O}_F/\mathcal{P} \to K_1\mathcal{O}_F \to$$

$$K_1S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P}\in S} K_0\mathcal{O}_F/\mathcal{P} \to K_0\mathcal{O}_F \to K_0S^{-1}\mathcal{O}_F \to 0. \qquad \square$$

Note the special cases: (1) $S = \operatorname{Spec} \mathcal{O}_F - \{0\}$, $S^{-1}\mathcal{O}_F = F$, and (2) $S^{-1}\mathcal{O}_F = \mathcal{O}_F[\frac{1}{\ell}]$, so $S =$ set of primes over ℓ. The usefulness of the sequence is convincingly demonstrated by Bass-Milnor-Serre; their theorem is proved by showing the map $K_2F \to \oplus_{\mathcal{P}\in S} K_1\mathcal{O}_F/\mathcal{P}$ is onto. Naturally one would like to extend the sequence further to the left; this in itself should be sufficient motivation for the higher K-theory of the next section.

2. Higher K-Theory

In the early 1970's Quillen proposed several equivalent definitions of the higher K-groups of a ring, scheme, or even category with exact sequences. His definition proved spectacularly successful and has been used ever since. We begin by briefly discussing three of these equivalent approaches: the plus construction, group completion, and the Q-construction. The first point to note is that in every case the groups K_*R are by definition the homotopy groups of a space. The second point to note is that the space in question is an infinite loop space, so the groups K_*R are in fact the homotopy groups of a spectrum.

Let X be a space and suppose π_1X has perfect commutator subgroup. Then there is a space X^+ (the plus construction) and a map $f : X \to X^+$ such that (1) the induced map on π_1 is precisely abelianization and (2) f_* is an isomorphism on homology with arbitrary coefficients (including local coefficients, but in the cases of interest X^+ will always be an H-space so we need only consider $H_*(\ ;\mathbb{Z})$). The remarkable fact about this contruction is that it is easy: one simply attaches 2-cells to kill $[\pi_1X, \pi_1X]$ and then 3-cells to eliminate the unwanted homology created in the first step; as a pleasant surprise the process then stops. For a self-contained account see [Adams 1978]. Now take $X = BGLR$. Then $\pi_1X = GLR$ with commutator subgroup $E(R)$. Since $E(R)$ is known to be perfect, $BGLR^+$ is defined and Quillen defines $K_iR = \pi_iBGLR^+$ for $i > 0$. Note this agrees with the classical definition for $i = 1$. For $i = 2$ it is an exercise to show $\pi_2BGLR^+ = H_2(E(R); \mathbb{Z}) = K_2R$. The second approach-group completion–is closely related and includes K_0 in a natural way. Let $M = \coprod_{n \geq 0} BGL_nR$. The natural maps $GL_mR \times GL_nR \to GL_{m+n}R$ give M the structure of a strictly associative and homotopy commutative topological monoid. Any topological monoid M has a classifying space BM and a canonical inclusion $M \xrightarrow{i} \Omega BM$; if M is either connected or a group i is a homotopy equivalence but in general need not be. However under general hypotheses–e.g. if π_0M is central in H_*M–the map i is a *group completion*: $i : H_*M \to H_*\Omega BM$ is precisely the localization formed by inverting a set of generators of π_0M. For example, let $M = \coprod_n BGL_nR$ as above. Then π_0M has a single generator and the localization formed by inverting it is just the direct limit $\varinjlim H_*BGL_nR = H_*BGLR$. It follows easily that $\Omega_0BM = BGLR^+$. To get K_0R into the act we take $M = \coprod_P B \operatorname{Aut} P$, where P ranges over isomorphism classes in $\mathbf{PR}$. Then the group completion ΩBM is $BGLR^+ \times K_0R$, and we can define $K_nR = \pi_n\Omega BM$ for all $n \geq 0$. For further details, see [Adams 1978].

The third and most general construction of higher K-theory is the Q-construction [Quillen 1973a]. Let $\mathcal{C}$ be an *exact category*–that is, an additive category with exact sequences, satisfying a reasonable list of axioms. Quillen constructs out of $\mathcal{C}$ a new category $Q\mathcal{C}$, which has the same objects but in which a morphism $A \to B$ is essentially an isomorphism of A onto a subquotient of B. Thus extensions are destroyed , just as they are in $K_0\mathcal{C}$. Any category $\mathcal{E}$ has a classifying space $B\mathcal{E}$, and Quillen defines $K_i\mathcal{C} = \pi_{i+1}BQ\mathcal{C}$. In particular we can take $\mathcal{C} = \mathbf{PR}$. In this case Quillen (see [Grayson 1976]) showed $\Omega BQ\mathbf{PR} \cong BGLR^+ \times K_0R$; hence the motto "$Q = +$". However we can also take $\mathcal{C}$ to be $\mathcal{M}R$, or vector bundles on a scheme, etc., so the construction is very general. The key technical result that makes the theory work is "Quillen's Theorem B", which allows one to identify, under favorable circumstances, the fibre of a map $B\mathcal{E} \to B\mathcal{E}'$ induced by a functor $\mathcal{E} \to \mathcal{E}'$. Using this result, a remarkable number of classical results on K_0 carry over to higher K-theory–as results about the homotopy type of $BQ\mathcal{C}$. We mention two of these to indicate the general idea:

(1) Devissage. Suppose, to be concrete, that $\mathcal{C} = \mathcal{M}A$ where A is a finite-dimensional algebra over a field. Then every $X \in \mathcal{M}A$ has a finite filtration with simple quotients, unique up to order, and hence $K_0\mathcal{M}A \equiv G_0A$ is free abelian on the simple modules. A better way to say this is as follows: let $\mathcal{SM}A$ denote the full subcategory of completely reducible modules. Then $\mathcal{SM}A \subset \mathcal{M}A$ induces an isomorphism on K_0, and furthermore $\mathcal{SM}A$ is

Morita equivalent to $\Pi_V \mathcal{M} D_V$, where V ranges over the simple modules and $D_V = \mathrm{End}_A V$. This generalizes to higher K-theory in the best way imaginable: there are natural equivalences

$$BQ\mathcal{M}A \cong BQS\mathcal{M}A \cong \Pi_V BQ\mathcal{M}D_V.$$

A similar result holds for any exact category satisfying a suitable Jordan-Hölder theorem.

(2) Reduction by resolution. Let R be a ring. As noted in §1, the natural map $K_0 K \to G_0 R$ need not be an isomorphism. Suppose however that R is Noetherian and regular in the sense that every finitely-generated R-module M has a projective resolution of finite length:

$$O \to P_n \to P_{n-1} \to \ldots \to P_1 \to P_0 \to M \to 0$$

Here we can assume the P_i's are also finitely-generated. For example, if R is a Dedekind domain we can take $n = 1$. Then we can map $G_0 R \xrightarrow{\varphi} K_0 R$ by $[M] \mapsto \Sigma(-1)^i[P_i]$. One can show this map is well-defined, and then it is obvious that $K_0 R \underset{\cong}{\to} G_0 R$ with inverse φ. As the reader will have already guessed, the generalization to higher K-theory is that for R regular $BQ\mathbf{PR} \to BQ\mathcal{M}R$ is a homotopy equivalence. Again Quillen proves a much more general result, valid for suitable exact categories $\mathcal{C}$ with a full subcategory $\mathcal{C}'$ such that every object in $\mathcal{C}$ has a finite-length resolution by objects of $\mathcal{C}'$. For example, one obtains that $K_* X \cong G_* X$ if X is a smooth variety or regular scheme.

For the purposes of this paper, the most important of Quillen's theorems on higher K-theory are probably the following three:

2.1. Theorem. *[Quillen 1972] Let $\mathbb{F}_q$ be a finite field with $q = p^m$ elements. Then $K_{2n}\mathbb{F}_q = 0$ if $n > 0$, and $K_{2n-1}\mathbb{F}_q \cong \mathbb{Z}/(q^n - 1)$. In fact $BGL\mathbb{F}_q^+$ is homotopy equivalent to $F\psi^q$, the fibre of $\psi^q - 1 : BU \to BU$.*

One of the most convincing properties of $K_* R$, extending 1.20:

2.2. Theorem. *[Quillen 1973a] Let F be a number field, $S \subseteq \mathrm{Spec}\,\mathcal{O}_F - \{0\}$. Then there is an exact sequence*

$$\ldots \to \bigoplus_{\mathcal{P}\in S} K_n(\mathcal{O}_F/\mathcal{P}) \to K_n\mathcal{O}_F \to K_n S^{-1}\mathcal{O}_F \to \bigoplus_{\mathcal{P}\in S} K_{n-1}\mathcal{O}_F/\mathcal{P} \to \ldots$$

(More generally, $\mathcal{O}_F$ could be replaced by any Dedekind domain, or a smooth projective curve).

Generalizing Dirichlet's theorems, we have:

2.3. Theorem. *[Quillen 1973b] Let F be a number field. Then $K_n\mathcal{O}_F$ is finitely-generated for all n.*

Observe that, except for $K_0\mathbb{F}_q \cong \mathbb{Z}$, the groups $K_n\mathbb{F}_q$ are all torsion groups and furthermore have order prime to the characteristic p. In particular $BGL\mathbb{F}_q^+$ is trivial when localized at p. Hence if we are working at a fixed prime ℓ, $K_*\mathbb{F}_q$ is interesting only when char $\mathbb{F}_q = p \neq \ell$. Note also the effect of this phenomenon on the localization sequence

$$\cdots \to \bigoplus_{\mathcal{P} \text{ over } \ell} K_n(\mathcal{O}_F/\mathcal{P}) \to K_n\mathcal{O}_F \to K_n\mathcal{O}_F[\frac{1}{\ell}] \to \cdots$$

After localizing at ℓ, we get isomorphisms $K_n\mathcal{O}_F \cong K_n\mathcal{O}_F[\frac{1}{\ell}]$ for $n \geq 2$, and an exact sequence

$$O \to K_1\mathcal{O}_F \to K_1\mathcal{O}_F[\frac{1}{\ell}] \to \bigoplus_{\mathcal{P} \text{ over } \ell} \mathbb{Z} \to Cl\mathcal{O}_F \to Cl\mathcal{O}_F[\frac{1}{\ell}] \to 0.$$

Thus if $\mathcal{P}_1, \dots, \mathcal{P}_k$ are the primes over ℓ, the rank of K_1 increases by k; while $Cl\mathcal{O}_F[\frac{1}{\ell}] = Cl\mathcal{O}_F/\langle P_1, \dots, P_k\rangle$. For example, the only difference between $K_*\mathbb{Z}$ and $K_*\mathbb{Z}[\frac{1}{\ell}]$ at ℓ is that $K_1\mathbb{Z}[\frac{1}{\ell}] = K_1\mathbb{Z} \oplus \mathbb{Z}$.

We comment on the proofs of these theorems, beginning with 2.1. The first step is to produce a map $\theta : BGL\mathbb{F}_q^+ \to BU$. Fix an embedding $\overline{\mathbb{F}}_q^* \underset{\varphi}{\subseteq} \mathbb{C}^*$, and let $\rho : G \to GL_n\mathbb{F}_q$ be a representation of a finite group G. The eigenvalues of each $\rho(g)$ are now complex numbers and can be summed to produce a complex-valued class function χ on G: the Brauer character. Brauer showed that χ is a virtual character, and hence we obtain a homomorphism $R_{\mathbb{F}_q}G \to R_{\mathbb{C}}G$ of representation rings–the Brauer lifting. Hence ρ determines a map $BG \to BU$. Taking $G = GL_n\mathbb{F}_q$ and ρ the identity, we get maps $BGL_n\mathbb{F}_q \to BU$ which assemble into a single map $BGL\mathbb{F}_q \to BU$. By a universal property of the plus construction, this determines a map $\theta : BGL\mathbb{F}_q^+ \to BU$. This is Quillen's Brauer lifting; it depends on the choice of embedding φ, but any φ will do. Now the Adams operations ψ^k are defined on class functions f by $(\psi^k \cdot f)(g) = f(g^k)$, and the map $R_{\mathbb{C}}G \to K^0BG$ commutes with ψ^k operations. If χ is the Brauer character of a representation over $\mathbb{F}_q$ then clearly $\psi^q\chi = \chi$. It follows that θ lifts to a map $BGL\mathbb{F}_q^+ \xrightarrow{\widetilde{\theta}} F\psi^q$ (we continue to ignore various technical problems, such as limits). One then shows that $\widetilde{\theta}_*$ is an isomorphism on mod ℓ and rational homology: The rational homology is trivial. Now suppose $\ell \neq p$. $H^*(F\psi^q; \mathbb{Z}/\ell)$ is easily computed from the Eilenberg-Moore spectral sequence. The crux of the matter is of course to compute $H^*(BGL_n\mathbb{F}_q;\ \mathbb{Z}/\ell)$, at least for $n = \infty$. The key point is that $H^*(\quad, \mathbb{Z}/\ell)$ is detected by maximal ℓ-tori, and these are all conjugate.

One can then show $\widetilde{\theta}_*$ is an isomorphism, by explicit computation. Finally, suppose $\ell = p$. Obviously $\widetilde{H}_*(F\psi^q, \mathbb{Z}/p) = 0$ since the p-local homotopy vanishes. That $\widetilde{H}_*(BGL\mathbb{F}_q, \mathbb{Z}/p) = 0$ is a somewhat surprising fact, since this is certainly false for $BGL_n\mathbb{F}_q$, $n < \infty$. Here is a heuristic argument: think of elements $\theta \in H^*(BGL\mathbb{F}_q, \mathbb{Z}/p)$ as stable characteristic classes for representations over $\mathbb{F}_q$. Suppose we knew all such θ were additive on short exact sequences–i.e., whenever $V' \to V \to V''$ is a short exact sequence of $\mathbb{F}_qG$-modules, $\theta(V) = \theta(V') + \theta(V'')$. Then if G is a p-group, $\theta(V) = 0$ for all θ, V, since all the composition factors of V are trivial. But then the same is true for all G, since restriction to a p-Sylow subgroup is injective on $H^*(\quad, \mathbb{Z}/p)$. Taking $G = GL_n\mathbb{F}_q$ completes the proof. The additivity on short exact sequences is true in a much more general setting-see e.g. [Quillen 1976a]. It is also enlightening to see why the mod p homology of the Q-construction of $\mathbb{F}_q$ is zero–see [Mitchell 1989].

Theorem 2.2 is a special case of a much more general result on localization of *abelian* categories. In particular the general result applies only to $\mathcal{M}R$, not to **PR**–note the latter is not abelian; for instance, it doesn't have cokernels. However if R is regular we can appeal to "reduction by resolution". In any case the main point is that the sequence is the exact homotopy sequence of a fibration: Suppose for instance R is Noetherian and $S \subset R$ is a central subset. Then, using "Theorem B", the fibre of $BQ\mathcal{M}R \to BQ\mathcal{M}(S^{-1}R)$ is identified as $BQ\mathcal{M}^{S-\text{ tor}}R$, where $\mathcal{M}^{S-\text{ tor}}R$ is the full subcategory of $\mathcal{M}R$ consisting of the S-torsion modules. In the situation of 2.2, taking S finite for simplicity, it follows by "devissage" that $BQ\mathcal{M}^{S-\text{ tor}}R \cong \Pi_{p \in S} BQ\mathcal{M}(R/p)$. Hence the exact sequence. To further illustrate the fantastic generality of the theorem, let X be an algebraic variety, Z a closed subvariety. Then a similar analysis of the category of coherent sheaves on X leads to an exact sequence

$$\dots \to G_nZ \to G_nX \to G_n(X - Z) \to G_{n-1}Z \to \dots$$

Here G-theory can be replaced by K-theory if X and Z are smooth, using "reduction by resolution".

Finally, consider 2.3. From the point of view of the plus construction, clearly the following would suffice to prove K_nR finitely generated:

(i) K_0R is finitely generated
(ii) $H_*(BGL_nR; \mathbb{Z})$ has finite type
(iii) (Stability) $H_*(BGL_nR, \mathbb{Z}) \to H_*(BGL_{n+1}R, \mathbb{Z})$

is an isomorphism in a range of dimensions that tends to ∞ as $n \to \infty$. In particular $H_k(BGL_nR; \mathbb{Z}) \to H_k(BGLR; \mathbb{Z})$ is an isomorphism in some "stable range", $n \gg k$.

For $R = \mathcal{O}_F$, (i) is Dirichlet's theorem; (ii) was first proved by [Raghunathan] and later in a much stronger form by [Borel-Serre]. Homological stability for R a Dedekind domain was proved by [Charney 1980]. However Quillen's original proof of 2.3 was based on the Q-construction. Essentially, Quillen proved stability for a natural rank filtration on the Q-construction, and showed that the homology of the filtration quotients is the homology of $GL_n\mathcal{O}_F$ with coefficients in the "Steinberg representation". Since the latter has finite type by Borel-Serre, the theorem follows. A variant of this proof is given in [Mitchell 1989]. Actually Quillen's proof applies equally well to $S^{-1}\mathcal{O}_F$ if S finite. Alternatively, note that Theorem 2.1, 2.2 and 2.3 together show $K_*S^{-1}\mathcal{O}_F$ has finite type if S finite.

Theorem 2.3 raises the obvious question: what is the rank?

2.4. Theorem. *[Borel] Let $n > 1$. Then*

$$\operatorname{rank} K_n\mathcal{O}_F = \begin{cases} r_1 + r_2 & \text{(n=1 mod 4)} \\ r_2 & \text{(n=3 mod 4)} \\ 0 & \text{(n even).} \end{cases} \qquad \square$$

2.5. Remark. Recall from §1 that $K_1\mathcal{O}_F$ has rank $r_1 + r_2 - 1$. Hence the rank is not quite periodic. Note however that for $n > 0$ rank $K_n\mathbb{Z}[\frac{1}{\ell}]$ has period 4 and rank $K_n\mathbb{Z}[\xi_\ell, \frac{1}{\ell}]$ has period 2.

The proof of 2.4 is provocative. Since $BGL\mathcal{O}_F^+$ is an H-space the rational Hurewicz map is an isomorphism onto the homology primitives. Hence it is sufficient to compute the rational cohomology. As spaces, $BGL\mathcal{O}_F^+ \cong BSL\mathcal{O}_F^+ \times BGL_1\mathcal{O}_F$, so it will be enough to compute $H^*(BSL\mathcal{O}_F, \mathbb{Q})$. Consider first the case $\mathcal{O}_F = \mathbb{Z}$. Let $X = SL_n\mathbb{R}/SO(n)$, and let $\Gamma = SL_n\mathbb{Z}$. Then X is contractible and Γ acts on X with compact and hence finite isotropy groups. Let $\Gamma' \subset \Gamma$ be a torsion-free normal subgroup of finite index, with quotient G. For example if $m > 2$ the subgroup of matrices congruent to the identity mod m is torsion-free; this is an exercise, originally due to Minkowski. Then Γ' acts freely on X so $X/\Gamma' = B\Gamma'$. Let Ω^*X denote the complex of differential forms on X. Then $H^*(B\Gamma', \mathbb{R}) = H((\Omega^*X)^{\Gamma'})$ and $H^*(B\Gamma, \mathbb{R}) = (H^*(B\Gamma', \mathbb{R}))^G$, so $H^*(B\Gamma, \mathbb{R}) = H((\Omega^*X)^{\Gamma})$. Now comes the hard part of the proof: the inclusion $(\Omega^*X)^{SL_n\mathbb{R}} \subset (\Omega^*X)^{\Gamma}$ is a cohomology isomorphism in a range of dimensions that tends to ∞ with n. Assuming this, we have only to compute $H((\Omega^*X)^{SL_n\mathbb{R}})$. By a classical theorem, this is the relative Lie algebra cohomology $H^*_{\mathrm{Lie}}(\mathfrak{sl}_n\mathbb{R}, \mathfrak{so}(n))$. Since Lie algebra cohomology obviously commutes with extension of scalars from $\mathbb{R}$ to $\mathbb{C}$, and $\mathfrak{su}(n) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_n\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, this in turn is the same as $H^*_{\mathrm{Lie}}(\mathfrak{su}(n), \mathfrak{so}(n))$. From the same classical theorem this is $H^*(SU(n)/SO(n); \mathbb{R})$. We conclude that $H^*(BSL\mathbb{Z}, \mathbb{Q}) \cong H^*(SU/SO, \mathbb{Q}) \cong \mathbb{Q}\langle x_1, x_5, \dots \rangle$, where $|x_k| = 4k + 1$. The general case follows the same pattern. First we need to embed $SL_n\mathcal{O}_F$ as a discrete subgroup of a semisimple real Lie group. Each real embedding of F determines a group monomorphism $SL_n\mathcal{O}_F \to SL_n\mathbb{R}$; similarly each complex embedding yields $SL_n\mathcal{O}_F \to SL_n\mathbb{C}$. As in the proof of Dirichlet's theorem 1.6, if we take one complex embedding from each conjugate pair we get a monomorphism $SL_n\mathcal{O}_F \to (\Pi_1^{r_1} SL_n\mathbb{R}) \times \Pi_1^{r_2} SL_n\mathbb{C}$ with *discrete* image. The space X is replaced by $X_1^{r_1} \times X_2^{r_2}$ where $X_1 = SL_n\mathbb{R}/SO(n)$, $X_2 = SL_n\mathbb{C}/SU(n)$. A similar argument then leads to the conclusion that $H^*(BSL\mathcal{O}_F, \mathbb{Q}) \cong H^*((SU/SO)^{r_1} \times (SU)^{r_2}; \mathbb{Q})$, whence the theorem. Here the SU factors arise from the isomorphisms

$$(\mathfrak{sl}_n\mathbb{C} \otimes_{\mathbf{R}} \mathbb{C}, \mathfrak{su}(n) \otimes_{\mathbf{R}} \mathbb{C}) \cong (\mathfrak{sl}_n\mathbb{C} \oplus \mathfrak{sl}_n\mathbb{C}, \Delta(\mathfrak{sl}_n\mathbb{C})) \cong \\ ((\mathfrak{su}(n) \oplus \mathfrak{su}(n)) \otimes_{\mathbf{R}} \mathbb{C}, \Delta(\mathfrak{su}(n)) \otimes_{\mathbf{R}} \mathbb{C})$$

and the obvious identification $(SU(n) \times SU(n))/\Delta(SU(n)) = SU(n)$.

The space $BGLR^+ \times K_0R$ is a loop space, and hence an H-space, by "$Q = +$".

In fact much more is true, which brings us to one of the main themes of this paper.

2.6. Theorem. *$BGLR^+ \times K_0R$ is an infinite loop space, and so defines a spectrum KR.* □

More generally we get a spectrum KX for X a scheme or even an exact category; thus for example there is a spectrum GR with $\pi_*GR = G_*R$. Furthermore an exact functor $\mathcal{C} \to \mathcal{C}'$ between exact categories induces a map of spectra $K\mathcal{C} \to K\mathcal{C}'$. The infinite loop space structure comes from the general machinery of [May 1974] or [Segal 1974], although there are also approaches more specific to K-theory, such as [Wagoner 1972]. A theorem of [May-Thomason] shows that the infinite loop space structure, and hence the spectrum KX, is unique. All of which raises the question: What manner of spectrum is it? Since the spectrum is produced from a black box this question isn't so easy to answer. However we at least have:

2.7. The spectrum KX is connective.

2.8. [May 1980] If X is a commutative ring or scheme, KX is a commutative associative ring spectrum. The ring multiplication arises from tensor product of projective modules or vector bundles. Similarly, virtually any natural multiplication on K_0 leads to a corresponding structure on the spectrum level. For example, if R is commutative G_0R is a K_0R-module, and GR is a KR-module spectrum. Or if Λ is a possibly noncommutative R-algebra, $K_0\Lambda$ is a K_0R-module and $K\Lambda$ is a KR-module spectrum. In fact if X is an arbitrary ring or scheme KX is a $K\mathbb{Z}$-module spectrum.

Theorems 2.1 and 2.2 have spectrum level analogues: Let $J(q)$ denote the fibre of $\psi^q - 1 : KU \to KU$ (here, as usual, we have localized at ℓ). Let $j(q)$ denote the connective cover of $J(q)$-i.e. the fibre of $\psi^q - 1 : bu \to \Sigma^2 bu$.

2.9. Theorem. *(see [May 1977]) The ℓ-adic completion of the Brauer lift is an infinite loop map and so determines a map of spectra $K\mathbb{F}\hat{}_q \xrightarrow{\theta\hat{}} bu\hat{}$. Moreover $\theta\hat{}$ factors through an equivalence $K\mathbb{F}\hat{}_q \cong j(q)\hat{}$.* □

Remark. A quick proof of 2.9 can be given using Suslin's theorem 4.7b and Bousfield's theorem 10.8).

Remark. If R is a topological ring the hom-sets in **PR** are topological spaces and we obtain a topological exact category $\mathbf{PR}^{\text{top}}$. Applying the machinery above, we obtain a spectrum KR^{top} and a natural map $KR \to KR^{\text{top}}$, which is a map of ring spectra if R is commutative. For example, by [May 1977]–see p.214–$K\mathbb{R}^{\text{top}} \cong bo$, the connective real K-theory spectrum. Similarly $K\mathbb{C}^{\text{top}} \cong bu$. It follows e.g. that there is a natural map of ring spectra $K\mathbb{Z} \to bo$; this will be used frequently below.

2.10. Theorem. *Let F, S be as in 2.2. Then there is a fibre sequence of spectra*

$$\bigvee_{\rho \in S} K(\mathcal{O}_F/\rho) \to K\mathcal{O}_F \to KS^{-1}\mathcal{O}_F. \qquad \square$$

Similarly, all of the equivalences and fibrations of [Quillen 1973a] are valid on the spectrum level, because the maps involved always arise from exact functors between exact categories. We conclude this section with another very important example of such functors: the *transfer*. Suppose $\varphi : R \to S$ is a ring homomorphism which is *finite* in the sense that S is a finitely-generated left R-module. Then we obtain a functor $\varphi^* : \mathcal{M}S \to \mathcal{M}R$ which is obviously exact and hence a map of spectra $GS \to GR$. If R is a regular Noetherian ring this yields a map $t_\varphi : KS \to KR$, called the transfer. The induced map on homotopy is usually written φ^*.

2.11. Example. R is a Dedekind domain and φ is reduction modulo a maximal ideal $\mathcal{P}$. One can easily check that the first map in the sequence 2.2 is the wedge of the transfers $KR/\mathcal{P} \to KR$.

2.12. Example. $\varphi : F \to E$ is a finite field extension of degree d. One can show that $\Omega_0^\infty t : BGLE^+ \to BGLF^+$ is induced by the evident maps $BGL_nE \to BGL_{dn}F$. It is also clear that the composite $KF \xrightarrow{\varphi} KE \xrightarrow{t} KF$ is induced by the functor $PF \to PF : V \mapsto E \otimes_F V$, which is isomorphic to the functor $V \mapsto \oplus_{i=1}^d V$. Thus $t\varphi = d$ as maps of spectra; in particular $(t\varphi)_*$ is multiplication by $d : K_*F \to K_*F$. Now suppose E/F is Galois with group G. Clearly φt corresponds to the functor $PE \to PE : V \mapsto (E \otimes_F E) \otimes_F V$. A fundamental theorem of Galois theory says that the map $E \otimes_F E \to \Pi_{g\in G} E$ given by $a \otimes b \mapsto \Pi a(gb)$ is an isomorphism of F-algebras. It follows that $\varphi t = \sum_{g\in G} g$ as maps of spectra. If $\ell \nmid d$ and we localize at ℓ, by combining the above remarks we see that (i) $\frac{1}{d} \cdot t$ is a retraction with right inverse φ, and hence KF is a wedge summand of KE, and (ii) the wedge summand in question is the "fixed point" spectrum of G, by which we simply mean the mapping telescope of the idempotent $\frac{1}{d} \sum_{g\in G} g$. In particular $K_*F = (K_*E)^G$.

2.13. Example. $\varphi : R \to S$ is a finite extension of Dedekind domains, of degree d. As in the previous example, $t\varphi$ is multiplication by $[S] \in K_0R$. If S is R-free, we get multiplication by d as before. If the extension is Galois and unramified we again have $\varphi t = \sum_{g\in G} g$. For example, take $R = \mathbb{Z}[\frac{1}{\ell}]$, $S = \mathbb{Z}[\xi_\ell, \frac{1}{\ell}]$. Then after localization at ℓ, KR is a wedge summand of KS and $(K_*R) = (K_*R)^G$-exactly as in 2.12.

2.14. Example. Suppose B_1, B_2 are commutative finite A-algebras, A commutative and all three are regular Noetherian. Then inspection of the various functors shows there is a commutative diagram of spectra

$$\begin{array}{ccc} KA & \longrightarrow & KB_1 \\ \uparrow & & \uparrow \\ KB_2 & \longrightarrow & K(B_1 \otimes_A B_2) \end{array}$$

where the vertical maps are transfers. Those familiar with the "pullback" property of the transfer in stable homotopy theory should note this is quite analogous, since Spec $(B_1 \otimes_A B_2)$ is the pullback in the category of schemes.

2.15. Example. Suppose $\varphi : E \to F$ is a purely inseparable field extension of degree p^d. As an amusing exercise the reader can show directly from 2.14 and devissage that not only $t\varphi$, but also φt, is multiplication by p^d (or see [Quillen 1973a]). In fact we will never consider such extensions in this paper, but it's nice to know we don't need to: for it follows from the exercise that any purely inseparable extension $E \to F$ induces an equivalence $KE \to KF$ after localization at ℓ, $\ell \neq p$.

3. Torsion in the K-Theory of a Ring of Algebraic Integers

Throughout this section, F is a number field with ring of integers $\mathcal{O}_F$. By Quillen's theorem 2.3 $K_*\mathcal{O}_F$ has finite type, and Borel 2.4 computed the ranks. In this section we take some first steps toward computing the torsion subgroups, or at least exhibiting some systematic families of torsion classes. We also take the opportunity to introduce K-theory with finite coefficients, and the Bott element. There are two obvious places to look for torsion classes: (1) consider the projection to a residue field $\mathcal{O}_F \to \mathcal{O}_F/\mathcal{P}$. Does this map detect torsion classes? (2) consider the unit map $S^0 \xrightarrow{i} K\mathcal{O}_F$. For $n > 0$, $\pi_n S^0$ is a torsion group. What is the image of i?

Consider first the unit map i. We may as well take $\mathcal{O}_F = \mathbb{Z}$. As an element of $\pi_0 K\mathbb{Z} = K_0\mathbb{Z}$, i corresponds to the free module of rank one. Applying Ω^∞ we get a map $QS^0 \xrightarrow{\Omega^\infty i} BGL\mathbb{Z}^+ \times \mathbb{Z}$, or $\Omega_0^\infty i : Q_0 S^0 \to BGL\mathbb{Z}^+$. Now by the Barratt-Priddy-Quillen theorem [Priddy], $Q_0 S^0 = B\Sigma_\infty^+$, or equivalently $QS^0 = \Omega B(\coprod_{n\geq 0} B\Sigma_n)$. In fact the infinite loop space QS^0 can be obtained by applying the May/Segal machinery to the category of finite sets, and the map i is induced by the obvious functor (finite sets) $\to$ free $\mathbb{Z}$-modules. Hence the slogan "stable homotopy groups of spheres = K-theory of the category of finite sets." In any case we have the natural question of describing the map $i_* : \pi_n S^0 \to K_n\mathbb{Z}$. Obviously $\pi_0 S^0 \cong K_0\mathbb{Z}$. We have $\pi_1 S^0 \cong \mathbb{Z}/2$, generated by the Hopf map η, and one can show in a number of ways (cf. below) that $i_*\eta \neq 0$. Since i is a ring map it follows that $i_*\eta^2 \neq 0$. Thus i_* is an isomorphism for $n \leq 2$. For $n = 3$ we have $\pi_3 S^0 \cong \mathbb{Z}/24$, and it was shown early on by Quillen that i_* is injective for $n = 3$ (see below). However after a period of some confusion, judging by the literature, [Lee-Szczarba] showed $K_3\mathbb{Z} \cong \mathbb{Z}/48$, so i_* is not onto. And Mahowald (see [Browder]) observed that the map $\mathbb{Z}/2 \cong \pi_6 S^0 \to K_6\mathbb{Z}$ is zero. Thus i_* is not injective either, which is certainly a great relief to K-theorists. The complete answer is now known, and can be described as follows: Let $(\operatorname{Im} J)_{(\ell)}$ denote the ℓ-component of the image of the J-homorphism $\pi_* O \to \pi_* S^0$.

3.1. Theorem.

(a) *[Quillen 1976b]* $(\operatorname{Im} J)_{(\ell)}$ *injects into* $K_n\mathbb{Z}$ *if* ℓ *is odd or* $n = 3, 7 \mod 8$, *and onto a direct summand if* ℓ *is odd or* $n = 7 \mod 8$.

(b) *[Quillen 1976b] The Adams elements* μ_{8k+1}, μ_{8k+2} *of order 2 generate direct summands* $\mathbb{Z}/2$ *in* $K_{8k+1}\mathbb{Z}$, $K_{8k+2}\mathbb{Z}$.

(c) *[Lee-Szczarba]* $K_3\mathbb{Z} \cong \mathbb{Z}/48$ *and [Browder] the* $\mathbb{Z}/16$ *occurs as a direct summand in* $K_{8k+3}\mathbb{Z}$ *for all* k. *Hence* $(\operatorname{Im} J)_{(2)}$ *is not a direct summand in* $K_{8k+3}\mathbb{Z}$.

(d) *[Waldhausen 1982] If* $n = 0$ *or 1 mod 8;* $i_* : \operatorname{Im} J \cong \mathbb{Z}/2 \to K_n\mathbb{Z}$ *is zero.*

3.2. Corollary (of (a) and (b)). *$K_{4_{n}-1}\mathbb{Z}$ contains a cyclic subgroup of order d_n = denominator of $\frac{B_n}{4n}$, where B_n is the nth Bernoulli number (see [Milnor–Stasheff, Appendix B]). This subgroup is a direct summand if n is even.*

The author has recently shown that i_* factors through $\operatorname{Im} J$ (provided $\operatorname{Im} J$ is interpreted as including the Adams μ-family) [Mitchell 1990a]. Hence Theorem 3.1 gives a complete description of the map i_*. This will be discussed further in §10.

Some remarks on the proof of 3.1: (a) Consider the Chern character as a map $\mathrm{ch} : BO \to \Pi_{n\geq 1}K(4n,\mathbb{Q})$, and let F denote the fibre. The natural map $Q_0S^0 \to BO$ lifts uniquely to F. Since $\pi_{4_{n}-1}F \cong \mathbb{Q}/\mathbb{Z}$, we get homomorphisms $\pi^s_{4n-1} \to \mathbb{Q}/\mathbb{Z}$ which in fact correspond to Adams' e-invariant $e_{\mathbf{R}}$. On the other hand the natural map $BGL\mathbb{Z}^+ \to BO$ also lifts to F: for it is enough to lift $BGL\mathbb{Z} \to BO$, and the real Pontrjagin classes of a flat bundle vanish since they can be defined in terms of the curvature. Thus we have a commutative diagram

$$\begin{tikzcd} & & F \arrow[d] \\ Q_0S^0 \arrow[r] \arrow[urr, dashed] & BGL\mathbb{Z}^+ \arrow[r] & BO \end{tikzcd}$$

Since $e_{\mathbf{R}}$ detects the image of J in dimensions $n = 3 \mod 4$, this yields the injectivity in (a). Now fix an odd prime ℓ and choose p as in 3.3a below. Then the unit map $S^0 \to K\mathbb{F}_p$ induces homomorphisms $\pi^s_{2n+1} \to \mathbb{Z}/(p^n-1)_{(\ell)}$ that can be identified with the ℓ-part of the complex e-invariant (of course this is zero unless $\ell - 1$ divides n). Since the unit map factors through the reduction map $K\mathbb{Z} \to K\mathbb{Z}/p$, this yields the splitting for ℓ odd. If $\ell = 2$ we can take $p = 3$. Again, the maps $\pi^s_{2n-1} \to \mathbb{Z}/(3^n - 1)_{(2)}$ can be identified with the 2-primary complex e-invariant $e_{\mathbf{C}}$. However if n is even $e_{\mathbf{C}} = ke_{\mathbf{R}}$, where $k = 1$ if $n = 0 \mod 4$ and $k = 2$ otherwise. Hence, we only obtain the splitting when $n = 0 \mod 4$. Note also that for ℓ odd, the Chern character argument can be replaced by the reduction argument just given.

(b) The Lee-Szczarba theorem involves a delicate analysis of the cohomology of the Q-construction for $\mathbb{Z}$ in low degrees. We remark that in fact the extra factor of two arises for the "usual reason"; see §7. The propagation of $K_3\mathbb{Z}$ into higher degrees is an instance of Bott periodicity; see §7.

(c) The μ-family was once described by Frank Adams as a family that "homotopy theorists know and love, but need not concern anyone else." Quillen's theorem shows that Adams' assertion was too modest. The proof is easy: the μ-family is a periodic family of elements μ_{8k+1}, μ_{8k+2} of order two in $\pi^s_{8k+1}, \pi^s_{8k+2}$, and is detected by the bo-Hurewicz map $S^0 \xrightarrow{i} bo$. Since i factors through $K\mathbb{Z}$, the theorem follows.

(d) Waldhausen's proof involves his "algebraic K-theory of spaces." It would be nice to have a more elementary argument.

Remark. Given a general $\mathcal{O}_F$, one can consider the composite $S^0 \to K\mathbb{Z} \to K\mathcal{O}_F$. For example, if F has a real embedding then (obviously) the μ-family produces direct summands $\mathbb{Z}/2$ in $K_*\mathcal{O}_F$.

Next we consider the residue field projections. Fix a prime ℓ and define $w(F) = [F(\xi_\ell) : F]$, $a(F) =$ maximal a such that $F(\xi_\ell)$ contains ξ_{ℓ^a}. If $\mathbb{F}_q$ is a finite field of characteristic not ℓ, define $w(\mathbb{F}_q)$ and $a(\mathbb{F}_q)$ the same way. If $\mathcal{P}$ is a nonzero prime of $\mathcal{O}_F$, with char $(\mathcal{O}_F/\mathcal{P}) \neq \ell$, call $\mathcal{P}$ *retractible* if $w(F) = w(\mathcal{O}_F/\mathcal{P})$ and $a(F) = a(\mathcal{O}_F/\mathcal{P})$. If $\ell = 2$ we assume in addition that $a \geq 2$ – i.e. $\sqrt{-1} \in F$.

3.3. Remarks. (a) Take $F = \mathbb{Q}$, ℓ odd. Then $w(\mathbb{Q}) = \ell - 1$, $a(\mathbb{Q}) = 1$. A prime $p \in \mathbb{Z}$ is retractible precisely when p has order $\ell - 1$ in $\mathbb{Z}/\ell^*$ and $\ell^2 \nmid (p^{\ell-1} - 1)$ – in other words, p generates $(\mathbb{Z}/\ell^2)^*$, or equivalently p generates $\mathbb{Z}_\ell^*$. There are infinitely many such p, by Dirichlet's theorem on arithmetic progressions. The corresponding spectra $K\mathbb{F}_p$ are all ℓ-adically equivalent to the connective "Image of J" spectrum j–in fact we can even take this as the definition of j.

(b) Take $F = \mathbb{Q}(\xi_\ell)$, ℓ odd. Then $w(F) = 1 = a(F)$. There are various kinds of retractible primes. Fix a rational prime $p \neq \ell$ and a prime $\mathcal{P}$ in $\mathbb{Z}[\xi_\ell]$ lying over p. At one extreme, we could take p to be "completely split"; i.e. p splits into $\ell - 1$ distinct primes in $\mathbb{Z}[\xi_\ell]$. This is true precisely when $p = 1 \bmod \ell$, and then $\mathbb{Z}[\xi_\ell]/\mathcal{P} = \mathbb{F}_p$. Such a $\mathcal{P}$ will be retractible precisely when $p \neq 1 \bmod \ell^2$ – in other words, p topologically generates the kernel of $\mathbb{Z}_\ell^* \to (\mathbb{Z}/\ell)^*$. At the opposite extreme, we could take p to be "inert" – i.e. p remains prime in $\mathbb{Z}[\xi_\ell]$, so $\mathcal{P} = p\mathbb{Z}[\xi_\ell]$. Then $\mathbb{Z}[\xi_\ell]/\mathcal{P} = \mathbb{F}_q$, where $q = p^{\ell-1}$. In this case $\mathcal{P}$ will be retractible precisely when p is as in (a).

(c) The Cebotarev density theorem, which is a generalization of Dirichlet's theorem on arithmetic progressions, guarantees the existence of infinitely many retractible primes. The existence of *infinitely* many such $\mathcal{P}$ is useful since for any ring of S-integers $S^{-1}\mathcal{O}_F$, we can then find primes which are simultaneously retractible for $\mathcal{O}_F$ and $S^{-1}\mathcal{O}_F$.

(d) Observe that the ℓ-primary part of $K_*\mathbb{F}_q$ is uniquely determined by the numbers $w(\mathbb{F}_q)$ and $a(\mathbb{F}_q)$. In fact one can show that the ℓ-adic homotopy type of the spectrum $K\mathbb{F}_q$ is uniquely determined by $w(\mathbb{F}_q)$, $a(\mathbb{F}_q)$.

3.4. Theorem. *[Harris-Segal] Fix ℓ, and if $\ell = 2$ assume $\sqrt{-1} \in F$. Let $\mathcal{P}$ be a retractible prime of $\mathcal{O}_F$. Then after localization at ℓ, the reduction map $BGL\mathcal{O}_F^+ \to BGL(\mathcal{O}_F/\mathcal{P})^+$ is a homotopy retraction.*

3.5. Corollary. *If ℓ is odd, the space* $\operatorname{Im} J$ *is a retract of* $K\mathbb{Z}$.

3.6. Remarks. (a) conjecturally this is true as spectra; see §11.

(b) A different proof of 3.4 was given in [Browder]. See also [Cohen-Peterson].

(c) The assumption $\sqrt{-1} \in F$ when $\ell = 2$ can be weakened slightly [Harris-Segal]. The essential point is that the Galois groups $G(F(\xi_{2^n})/F)$ should be cyclic, as opposed to $\mathbb{Z}/2\times$ (cyclic). Harris and Segal also get some weaker conclusions in the general case. However there is no odd prime p such that the reduction map $f : BGL\mathbb{Z}^+ \to BGL\mathbb{F}_p^+$ is a retraction at 2: for if $f_{(2)}$ is a retraction, the Lee-Szczarba theorem would imply f_* is an isomorphism on $\pi_3 \cong \mathbb{Z}/16$. Let η denote the nonzero element of $K_1\mathbb{Z}$. Since f_* is a ring homomorphism and $\eta^3 \neq 0$, we conclude $f_*(\eta^3) \neq 0$. This is a contradiction since $K_2\mathbb{F}_p = 0$. Note also that the natural map $BGL\mathbb{Z}^+ \to J_2$ can't be a retraction, by Waldhausen's theorem 3.1d.

(d) The factors obtained for different choices of $\mathcal{P}$ are essentially identical. This is because the equivalences of Remark 3.3d lead to commutative diagrams (at least on the space level)

$$\begin{tikzcd} BGL\mathcal{O}_F^+ \arrow[d] \arrow[dr] & \\ BGL(\mathcal{O}_F/\mathcal{P}')^+ \arrow[r, "\cong"'] & BGL(\mathcal{O}_F/\mathcal{P})^+ \end{tikzcd}$$

(e) It seems very likely that the converse of 3.4 is also true.

(f) As a corollary we obtain systematic families of cyclic summands in $K_*\mathcal{O}_F$ – for example, 3.2. For another example, take $\mathcal{O}_F = \mathbb{Z}[\xi_\ell]$, ℓ odd. Let s denote a number prime to ℓ. We see that $K_{2s\ell^n-1}\mathbb{Z}[\xi_\ell]$ contains a cyclic summand of order ℓ^{n+1}. In general we obtain summands in $K_{2sw\ell^n-1}\mathcal{O}_F$ of order ℓ^{n+a}, where $w = w(F), a = a(F)$.

Here is a quick sketch of the proof of 3.4: Let G be a finite ℓ-group, $R = O_F$ and $\mathbb{F}_q = R/\mathcal{P}$. One can easily show that $\mathcal{P}$ is retractible if and only if for every finite ℓ-group G, every representation of G over $\mathbb{F}_q$ lifts to an R-free representation over R. Taking G to be an ℓ-Sylow subgroup of $GL_n\mathbb{F}_q$, this means in particular we have a lift

$$(*) \qquad \begin{tikzcd} & BGLR^+ \arrow[d] \\ BG \arrow[r] \arrow[ur, dashed] & BGL\mathbb{F}_q^+ \end{tikzcd}$$

Since $BGLR^+$ is an infinite loop space, and we have localized at ℓ, a standard transfer argument shows we can replace G in $*$ by $GL_n\mathbb{F}_q$, and then by $GL\mathbb{F}_q$ by a standard limit argument. Thus we have

$$\begin{tikzcd} & BGLR^+ \arrow[d] \\ BGL\mathbb{F}_q \arrow[r] \arrow[ur, dashed, "f"] & BGL\mathbb{F}_q^+ \end{tikzcd}$$

By a universal property of the plus construction, f factors through $BGL\mathbb{F}_q^+$, yielding the desired section. This is essentially the argument of Harris-Segal, although their proof is more elementary in that it avoids explicitly using the infinite loop space structure on $BGLR^+$.

Browder's approach to 3.4 involves K-theory with coefficients, which we now describe. For any spectrum E we define $\pi_*(E;\mathbb{Z}/n) = \pi_*E \wedge M\mathbb{Z}/n$. We are mainly interested in the case $n = \ell^\nu$. The cofibre sequence $S^0 \xrightarrow{\ell^\nu} S^0 \to M\mathbb{Z}/\ell^\nu$ leads to short exact sequences

$$O \to \pi_k E/\ell^\nu \to \pi_k(E;\mathbb{Z}/\ell^\nu) \to_{\ell^\nu} \pi_{k-1}E \to 0. \tag{3.7}$$

In particular we define $K_*(X;\mathbb{Z}/\ell^\nu) = \pi_*(X;\mathbb{Z}/\ell^\nu)$, and this fits into an exact sequence as above. The spectrum $M\mathbb{Z}/\ell^\nu$ is a ring spectrum unless $\ell^\nu = 2$. It is associative and commutative unless $\ell^\nu = 3$, 4, or 8. We will generally ignore these exceptions for low ℓ^ν. Setting these aside, whenever E is a commutative associative ring spectrum the same is true of $E \wedge M\mathbb{Z}/\ell^\nu$, and hence $\pi_*(E;\mathbb{Z}/\ell^\nu)$ is a commutative ring. We also remark that the above short exact sequence splits unless $\ell^\nu = 2$. The trouble is that $M\mathbb{Z}/2$ has exponent 4 instead of 2. This exception, as well as the fact that $M\mathbb{Z}/2$ is not a ring spectrum, can be blamed on the generator η of $\pi_1 S^0 = \mathbb{Z}/2$.

K-theory with coefficients $\mathbb{Z}/\ell^\nu$ is often better behaved than the integral version. For example, take ℓ odd and consider $K\mathbb{F}_q$. Since $K_*\mathbb{F}_q$ is all in odd dimensions (except for $* = 0$), the ring structure is trivial. On the other hand $K_*(\mathbb{F}_q;\mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu[\beta, \partial\beta]/(\partial\beta)^2$. Here $|\partial\beta| = |\beta| - 1$ and $|\beta|$ is read off from 2.1–for example if $\ell^\nu|_{q-1}$ then $|\beta| = 2$. In particular $K_*(\mathbb{F}_q, \mathbb{Z}/\ell^\nu)$ is periodic, with the period increasing with ν. For example, suppose $K\mathbb{F}_p \cong j$ as in 3.3a. Then the period is $2(\ell-1)$ for $\nu = 1$ and $2(\ell-1)\ell^{\nu-1}$ in general, a phenomenon which is quite familiar to both homotopy theorists and number theorists.

In fact one can produce elements β of infinite height in $K_*(R;\mathbb{Z}/\ell^\nu)$ quite generally, provided $\ell^\nu \neq 2$, as follows: suppose first that R contains an ℓ^ν-th root of unity ξ; in fact, we may as well take $R = \mathbb{Z}[\xi_{\ell^\nu}]$. Then ξ_{ℓ^ν} is an element of ${}_{\ell^\nu}K_1 R$ and therefore lifts to an element β in $K_2(R;\mathbb{Z}/\ell^\nu)$. (This can be done canonically; in fact one should construct β in $\pi_2^s(B\mathbb{Z}/\ell^\nu_+;\mathbb{Z}/\ell^\nu)$). One can easily check that β maps to a generator of $\pi_2(bu;\mathbb{Z}/\ell^\nu)$. Hence β has infinite height and is called a *Bott element.* In general it is enough to consider the case $R = \mathbb{Z}$. When $\nu = 1$ we use $\mathbb{Z}[\xi_\ell]$ and the transfer to produce $\beta_1 \in K_{2\ell-2}(\mathbb{Z};\mathbb{Z}/\ell)$. By considering powers of β_1 and a Bockstein spectral sequence, we easily obtain Bott elements $\beta_\nu \in K_{2(\ell-1)\ell^{\nu-1}}(\mathbb{Z};\mathbb{Z}/\ell^\nu)$. These can be pushed into the K-theory of any ring or scheme. Modulo nilpotent elements, β_ν is essentially independent of the choices made. See the discussion in [Dwyer-Friedlander-Snaith-Thomason] for further details.

4. K-Theory of Algebraically Closed Fields and Hensel Local Rings

Up to this point, the only fields whose K-theory we can compute are the algebraic extensions of finite fields. What about algebraically closed fields? Quillen and Lichtenbaum conjectured early on that if $\mathbb{F}$ is algebraically closed and char $F \neq \ell$, $K_*/\ell^\nu F$ should be the same as ordinary topological K-theory ([Quillen 1974], [Lichtenbaum]). About ten years later, the conjectures were proved by Suslin in two spectacular papers ([Suslin 1983] and [Suslin 1984]). The first paper proves:

4.1. Theorem. *Let $i : F \subset E$ be an extension of algebraically closed fields. Then $i_* : K_*(F;\mathbb{Z}/n) \to K_*(E;\mathbb{Z}/n)$ is an isomorphism for all n.*

Hence it is enough to compute $K_*(F,\mathbb{Z}/n)$ for one algebraically closed F of each characteristic. In particular, by Quillen's work this settles the case char $F = p$. It remains to compute $K_*(F,\mathbb{Z}/n)$ for some F of characteristic zero. In the second paper Suslin shows:

4.2. Theorem. *The natural map $K_*(\mathbb{C},\mathbb{Z}/n) \to K_*^{\text{top}}(\mathbb{C},\mathbb{Z}/n)$ is an isomorphism for all n.*

Thus in terms of spectra we have:

4.3. Theorem. *Let F be any algebraically closed field, ℓ a prime $\neq$ char F. Then $KF\hat{} \cong bu\hat{}$.*

Remarks.

(a) Instead of completing at a fixed prime ℓ, we could of course use profinite completion away from char $\mathbb{F}$.
(b) Suslin in fact proves a stronger result: if char $F = 0$ then modulo uniquely divisible groups $K_nF \cong \mathbb{Q}/\mathbb{Z}$ if n is odd and $K_nF = 0$ if n even, $n > 0$; with a similar result for char $F = p$. We also have:

4.4. Theorem. *(Suslin) For any prime ℓ, the natural map $K\mathbb{R}\hat{} \to bo\hat{}$ is an equivalence.*

The above remarks apply to 4.4 as well.

The proofs of 4.1 and 4.2 are beautiful and ingenious ("diabolically clever", in the phrase of one highly placed source). An excellent outline of the proof of 4.1 can be found in [Grayson 1989], so we will give only a very brief sketch.

Consider the extension $F \subset E$ in 4.1. E is the direct limit of its finitely generated F-subalgebras. It follows at once from the Nullstellensatz that i_* is injective on any functor whatsoever that commutes with direct limits. In particular this is true for K-theory, with or without coefficients. For the surjectivity, let $A \subset E$ be a finitely-generated F-subalgebra. Fix a homomorphism $f : A \to F$. If $(if)_* = j_*$ on $K_*(\ \ , \mathbb{Z}/n)$, we are done. Both if and j extend to homomorphisms $A \otimes_F E \to E$, so it would be enough to show that any two such homomorphisms agree on $K_*(\ \ , \mathbb{Z}/n)$. In other words, translating this into algebraic geometry, we have reduced to the following *Rigidity Theorem*:

4.5. Theorem. *Let p, q be points on a connected algebraic variety X over an algebraically closed field E. Let i_p, i_q denote the inclusions. Then $i_p^* = i_q^* : K_*(X, \mathbb{Z}/n) \to K_*(E, \mathbb{Z}/n)$.*

The theorem is easily seen to be false without finite coefficients: consider, for example, K_1 and the variety $\mathbb{C} - \{0\}$. The proof first reduces to the case of a smooth projective curve X, and then uses a brilliant argument based on the divisibility of the Picard group. As noted in [Grayson 1989], the entire argument can be done axiomatically: all one needs is a contravariant functor schemes $\to$ abelian torsion groups that commutes appropriately with limits, has a suitable transfer, and satisfies a homotopy axiom.

Before discussing Theorem 4.2 we must digress to consider Hensel local rings, which will appear several times in later sections. Let A be a local ring with maximal ideal m and residue field k. Then A is *Hensel* if Hensel's lemma holds for the projection $A \to k$. A is a *strict* Hensel local ring (or "strictly local ring") if in addition k is separably closed.

Examples

(a) Any complete local ring is Hensel

(b) Let $W(\overline{\mathbb{F}}_p)$ denote the Witt ring of $\overline{\mathbb{F}}_p$-i.e., the completion of the ring of integers in the maximal unramified extension of $\mathbb{Q}_p$. Then $W(\mathbb{F}_p)$ is strict Hensel.

(c) The ring of germs of continuous $\mathbb{C}$-valued functions at a point p of a topological space is strict Hensel. More generally $\mathbb{C}$ could be replaced by a suitable topological field-cf. [Suslin 1984].

A less elementary but more enlightening definition of Hensel rings will be given in §5. If A is any local ring one can define its Henselization A^h and *strict Henselization* A^{sh}. For example, if A is contained in its m-adic completion $\hat{A}$, then A^h is roughly the smallest Hensel local ring in $\hat{A}$ containing A. If $A = \mathbb{Z}_p$, $A^{sh} = W(\mathbb{F}_p)$. Henselization $A \xrightarrow{f} A^h$ is characterized by the property: if $A \xrightarrow{g} B$ is a map of A to a Hensel local ring B, with $g^{-1}(m_B) = m_A$, there is a unique $\varphi : A^h \to B$ such that $\varphi^{-1}m_B = m_{A^h}$ and $\varphi f = g$.

4.6. Theorem ([Gillet-Thomason], (Gabber)). *Let F be a field and x a smooth rational point of a variety over F. Let $\mathcal{O}_x^h$ denote the Henselization of the local ring at x. Then if $\ell \neq$ char F, the natural map $K_*(\mathcal{O}_x^h, \mathbb{Z}/\ell) \to K_*(F, \mathbb{Z}/\ell)$ is an isomorphism.* □

The proof involves a generalization of Suslin's rigidity theorem, and can also be axiomatized [Grayson 1989]. According to Gabber the analogue of 4.6 for arbitrary Hensel local rings is valid. However we will need only the following, which will be proved below:

4.7. Theorem [Suslin 1984]. *Let A be a Hensel local ring which is either (a) an algebra over a field F or (b) a complete discrete valuation ring with residue field F. Then if $\ell \neq$ char F, $K_*(A, \mathbb{Z}/\ell) \underset{\cong}{\to} K_*(F, \mathbb{Z}/\ell)$.* □

In fact Suslin proves case (b) for more general valuation rings.

To apply these theorems we need the following extremely useful theorem of [Charney 1982] and [Suslin 1984]: Let I be a 2-sided ideal in a ring R, and let $GL_n(R, I) \subset GL_nR$ denote the normal "congruence subgroup" of matrices equal to the identity mod I-i.e., Ker $(GL_nR \to GL_n(R/I)$.

4.8. Theorem. *If ℓ is a unit in R/I, the conjugation action of GLR on $H_*(GL(R, I); \mathbb{Z}/\ell)$ is trivial.*

4.9. Corollary. *Let $\overline{GL}(R/I)$ denote the image of $GLR \to GL(R/I)$. Then*

(a) *The local coefficient system in mod ℓ homology of the fibration $BGL(R, I) \xrightarrow{i} BGLR \xrightarrow{\pi} B\overline{GL}(R/I)$ is trivial.*
(b) *π_* is an isomorphism on H_* ($, \mathbb{Z}/\ell)$ if and only if $H_*(BGL(R, I), \mathbb{Z}/\ell) = 0$.*
(c) *Let F denote the homotopy fibre of $BGLR^+ \to B\overline{GL}(R/I)^+$. Then the natural map $BGL(R, I) \to F$ is a mod ℓ homology isomorphism.*

4.10. Corollary. *Let A be Hensel as in 4.7, with maximal ideal m, $k = A/m$. If $\ell \neq$ char k, $\widetilde{H}_*(BGL(A, m); \mathbb{Z}/\ell) = 0$.*

Remarks.

(a) If R is local with maximal ideal I in 4.9, $\overline{GL}(R/I) = GL(R/I)$.

(b) In general the plus construction does not "commute with fibrations"; and in any case $BGL(R, I)$ need not have perfect commutator subgroup so this wouldn't even make sense in 4.9c. However 4.9c does show that F is ℓ-adically the group completion of $\coprod_n BGL_n(R, I)$.

This completes our digression on Hensel local rings. We are in fact going to give essentially complete proofs of 4.2 and 4.7, assuming 4.6. However the beautiful proof of 4.2 has a very quick intuitive explanation, and we will give this first.

Let G be a Lie group with finitely many components, and let G^δ denote G with the discrete topology. Clearly 4.2 holds if and only if $BGL\mathbb{C}^\delta \to BGL\mathbb{C}^{\text{ top}}$ is an isomorphism on $H_*(\ \ ;\mathbb{Z}/\ell)$. This raises an obvious question:

4.11. Isomorphism Conjecture (Milnor, Friedlander) $BG^\delta \xrightarrow{\varphi} BG^{\text{ top}}$ is an isomorphism on mod ℓ homology for all ℓ.

By [Milnor 1983] φ_* is surjective. Let $F \xrightarrow{i} BG^\delta$ be the homotopy fibre of φ. If G is connected, so the local coefficient system is trivial, it follows that φ_* is an isomorphism $\iff \widetilde{H}_*F = 0 \iff i_* : \widetilde{H}_*F \to \widetilde{H}_*BG^\delta$ is the zero map. Thus Suslin's theorem would follow from this last assertion for $G = GL(n, \mathbb{C})$. This is still unknown, but it's enough to prove the stable analogue:

4.12. Lemma. *Let F_n be the homotopy fibre of $BGL_n\mathbb{C}^\delta \to BGL_n\mathbb{C}^{\text{ top}}$. Then the composite map $j : F_n \to BGL_n\mathbb{C}^\delta \to BGL\mathbb{C}^\delta$ is zero on $H_*(\ \ ;\mathbb{Z}/\ell)$.*

Consider the fibre F of a general φ as above. It is intuitively plausible, and shown precisely by Suslin, that F is the realization of the "infinitesmal bar complex" BG_ε, which we will vaguely imagine as the subcomplex of the usual bar complex $\{G^p\}$ consisting of p-tuples $(g_1, \dots, g_p)$ with the g_i "arbitrarily close" to the identity. We now come to the crux of the proof: At level p the map j is in effect the *germ* at 1 of a map $j^p : GL_n^p\mathbb{C} \hookrightarrow GL^p\mathbb{C}$; namely, the inclusion. The set of all such continuous map germs is precisely $GL^p(\mathcal{O}_{n,p}^{\text{cont}})$, where $\mathcal{O}_{n,p}^{\text{cont}}$ is the ring of germs at 1 of continuous $\mathbb{C}$-valued functions on $GL_n^p\mathbb{C}$. The fact that $j^p(1) = 1$ says precisely that when we regard $j^p \in GL(\mathcal{O}_{n,p}^{\text{cont}})$, it in fact lies in the congruence subgroup $GL(\mathcal{O}_{n,p}^{\text{cont}}, m)$. But $\mathcal{O}_{n,p}^{\text{cont}}$ is a Hensel local ring. Hence by Corollary 4.10, there is no obstruction to inductively constructing a chain nullhomotopy of j!

A rigorous version of this argument will be given shortly. For the moment we just note the precise definition of BG_ε: Fix $\varepsilon > 0$ and let NG_ε denote the sub-simplicial set of the bar complex NG consisting of $(g_1, \dots, g_p)$:

$U_\varepsilon \cap g_1 U_\varepsilon \cap g_1 g_2 U_\varepsilon \cap \ldots \cap (g_1 \ldots g_p) U_\varepsilon \neq \emptyset$, where U_ε is a disc of radius ε (in some invariant metric) at 1. Then BG_ε is the realization $|NG_\varepsilon|$. Suslin shows that for all sufficiently small ε, $BG_\varepsilon \cong F$.

The rigorous proof of theorem 4.2, as well as 4.7, depends on Suslin's idea of a "universal nullhomotopy". One convenient way to describe this construction is as follows: Let $f_\bullet : X_\bullet \to Y_\bullet$ be a map between simplicial objects in a category $\mathcal{C}$ with null object. The sets $W_{p,q} = \operatorname{Hom}_{\mathcal{C}}(X_p, Y_q)$ form a cosimplicial simplicial pointed set and in particular we get a suitably augmented double complex $\mathbb{Z}/\ell \cdot W_{p,q}$. Let ∂^H, ∂^V denote the horizontal and vertical boundary maps in $\mathbb{Z}/\ell \cdot W_{p,q}$. Then one might define a *universal nullhomotopy* of f as a sequence of elements $s_p \in \mathbb{Z}/\ell W_{p,p+1}$ such that $\partial^V s_p + \partial^H s_{p-1} = f_p$. We have the trivial consequence:

4.13. Proposition. *Let A be an object of $\mathcal{C}$, $f_\#$ the induced map of simplicial sets $\operatorname{Hom}_{\mathcal{C}}(A, X_\bullet) \to \operatorname{Hom}_{\mathcal{C}}(A, Y_\bullet)$. Then a universal nullhomotopy of f induces a nullhomotopy of the chain map $\mathbb{Z}/\ell f_\#$.*

Equally trivial is:

4.14. Proposition. *Suppose the columns of $\mathbb{Z}/\ell W_{p,q}$ are acyclic. Then f has a universal nullhomotopy.*

The applications of the universal homotopy require a little bit of scheme-theoretic language. The reader who is unfamiliar with schemes need only accept the following: If A is a commutative ring, the corresponding *affine scheme* is Spec A, the set of prime ideals of A. This set is equipped with sufficient additional structure to make the contravariant correspondence $A \longleftrightarrow \operatorname{Spec} A$ an equivalence of categories (opposite of commutative rings) $\longleftrightarrow$ (affine schemes). Thus if F is a field an affine scheme over Spec F is the same thing as an F-algebra. We let $\mathcal{S}GL_nF$ denote the scheme over Spec F corresponding to the usual F-algebra $AGL_nF = F[a_{ij}, \det^{-1}(a_{ij})]$. Then if R is an F-algebra, and $\mathcal{C}^F$ is the category of affine schemes over Spec F, it is immediate that $GL_nR = \operatorname{Hom}_{\mathcal{C}^F}$ (Spec R, $\mathcal{S}GL_nF$). Now let $\mathcal{C}_0^F$ denote the category of pointed affine schemes over Spec F–i.e., the opposite of the category of augmented F-algebras. Here "augmented" means equipped with an F-algebra homomorphism to some extension field E of F. For example, we regard AGL_nF as augmented by evaluation at the identity and similarly for AGL_n^pF, where $GL_n^pF = GL_nF \times \cdots \times GL_nF$ (p times). Let $R \xrightarrow{\varepsilon} E$ be an augmented F-algebra, so $m \equiv \operatorname{Ker} \varepsilon$ is a prime ideal. Then $\operatorname{Hom}_{\mathcal{C}_0^F}$ (Spec R, $\mathcal{S}GL_nF$) is just the congruence subgroup $GL_n(R, m)$. As a mild abuse of notation we will allow the case $n = \infty$, and define

$$\operatorname{Hom}_{\mathcal{C}_0^F}(\operatorname{Spec} R, \mathcal{S}GLF) = \varinjlim_n \operatorname{Hom}_{\mathcal{C}_0^F}(\operatorname{Spec} R, \mathcal{S}GL_nF), \text{ etc.}$$

Proof of 4.7a. Take $\mathcal{C} = \mathcal{C}_0^F$. We have simplicial objects $\mathcal{S}GL_n^\bullet F$ and $\mathcal{S}GL^\bullet F$ (bar construction) with a natural map $g : \mathcal{S}GL_n^\bullet F \to \mathcal{S}GL^\bullet F$. Let $\mathcal{O}_{n,p}^h$ be the Henselization of the local ring at 1 of $GL_n^p F$, and let $X_p = \operatorname{Spec}\mathcal{O}_{n,p}^h$. Then by naturality of Henselization the X_p assemble into a simplicial object in $\mathcal{C}_0^F$, and there is a map $h : X_\bullet \to \mathcal{S}GL_n^\bullet F$. Take $Y = \mathcal{S}GL^\bullet F$ and $f = gh$. The columns of $W_{p,q}$ are just the bar constructions for $GL(\mathcal{O}_{n,p}^h, m)$ and so are $\mathbb{Z}/\ell$-acyclic by 4.10. Hence f has a universal nullhomotopy s. Applying $\operatorname{Hom}_{\mathcal{C}_0^F}(\operatorname{Spec} R, -)$ we get a chain nullhomotopy $f_\#$. But the universal property of Henselization shows $\operatorname{Hom}_{\mathcal{C}_0^F}(\operatorname{Spec} R, X_\bullet) \underset{\cong}{\to} \operatorname{Hom}_{\mathcal{C}_0^F}(\operatorname{Spec} R, \mathcal{S}GL_n^\bullet F)$, so $g_\#$ also has a nullhomotopy. In otherwords, $GL_n(R,m) \to GL(R,m)$ is zero on $\widetilde{H}_*(\quad, \mathbb{Z}/\ell)$ for all n, and the theorem follows. □

Proof of 4.2. We keep the notation of the preceeding proof, with $F = \mathbb{C}$ (or $\mathbb{R}$!), except that $X_p = \operatorname{Spec}\mathcal{O}_{n,p}^{\text{cont}}$, the ring of germs at the identity of continuous functions on $GL_n^p\mathbb{C}$. Theorem 4.7a applied to $\mathcal{O}_{n,p}^{\text{cont}}$ leads to a universal nullhomotopy s for f, as before. Each s_p is a finite linear combination of map germs $GL_n^p\mathbb{C} \to GL_m^{p+1}\mathbb{C}$, $m >> 0$. If we fix $P >> 0$, the s_p for $p \le P$ are all defined on some $F_\bullet^\varepsilon$, ε fixed. In other words, through dimension P we have a universal nullhomotopy for the map of simplicial *spaces* $F_\bullet^\varepsilon \to GL^\bullet\mathbb{C}$. Applying 4.13 with A = point we get a chain nullhomotopy up to dimension P for the underlying map of simplicial sets. Since P was arbitrary, this completes the proof. □

Before sketching the proof of 4.7b, we first note that the theorem itself can be viewed as very much analogous to 4.2: Wagoner defined topological K-theory for complete discrete valuation rings and showed $K_i^{\text{top}} A \cong \varprojlim_n K_i(A/m^n)$, at least when the residue field $F = A/m$ is finite (see [Wagoner 1976]). On the other hand if $\ell \neq$ char F each projection $A/m^n \to A/m^{n-1}$ is an isomorphism on mod ℓ K-theory—in fact $GL_s(A/m^n) \xrightarrow{\pi} GL_s(A/m^{n-1})$ is an isomorphism on $H_*(\quad, \mathbb{Z}/\ell)$ for all s. (To see this, note $\operatorname{Ker}\pi$ is the additive group of $s \times s$ matrices over F, and hence is an F-vector space). Hence in 4.7b one can think of $K_*A \to K_*F$ as a map from algebraic to "topological" K-theory, as in 4.2.

Proof sketch of 4.7b. Let $H_* = H_*(\quad; \mathbb{Z}/\ell)$. We need to show that the natural map $BGLA \xrightarrow{\pi} BGLF$ is an isomorphism on H_*. Here we will show only that π_* is injective; the proof of surjectivity is in a similar spirit (see also the remark below). Let E denote the quotient field of A. Then GL_nA is an open subgroup of the topological group GL_nE, and in fact the subgroups $GL_nA \supset GL_n(A,m) \supset GL_n(A,m^2) \cdots$ form a neighborhood base at the identity. Thus the $GL_n^\bullet(A, m^b)$, b large, will play the role of the $F_{n,\epsilon}^\bullet$, ϵ

small, in 4.2. As before we obtain a universal nullhomotopy in a range for the map of simplicial spaces $GL_n^\bullet(A, m^b) \to GL^\bullet E$, b sufficiently large. However by taking b even larger, we can assume the nullhomotopy actually maps into $GL^\bullet A$, or even into a given $GL^\bullet(A, m^a)$. This leads easily to:

4.15. Fix k, n and a. Then for some $s \geq n$ and some $b \geq a$, $\widetilde{H}_* GL_n(A, m^b) \to \widetilde{H}_* GL_s(A, m^a)$ is zero through dimension k. Here s depends on k and n but not on a or b.

Now let $(H_* GL_n A)_{j,a}$ denote the jth Hochshild-Serre filtration associated to the extension

$$GL_n(A, m^a) \to GL_n A \to GL_n(A/m^a).$$

A short but delicate induction on j, using all of 4.15, yields:

4.16. Fix k, n, and $j < k$. Then for some $a \geq 0$, $(H_k GL_n A)_{j,a} \to H_k GLA$ is zero.

Note that the case $j = 0$ of 4.16 is contained in 4.15. On the other hand, taking $j = k - 1$ we see that the kernel of the natural map $H_k GL_n A \xrightarrow{\theta} H_k GL_n(A/m^a)$ stabilizes to zero. But A and A/m^a are local rings, and local rings satisfy homological stability [Wagoner 1976b]. Hence in the stable range $\operatorname{Ker}\theta$ is actually zero. It follows that $H_k GLA \to H_k GL(A/m^a)$ is injective, and since $H_* GL(A/m^a) \cong H_* GL(A/m)$ the proof is complete.

Remark. Let G be a finite group of order prime to char F. Then by lifting idempotents in the group ring FG, one can show that every representation of G over F lifts to a representation over A. Now suppose F is a subfield of $\overline{\mathbb{F}}_p$. Then $GL_n F$ is a direct limit of finite groups G_α. Letting G above range over the ℓ-Sylow subgroups of the G_α, we obtain an elementary proof that the maps $H_* GL_n A \to H_* GL_n F$ are surjective, $n \leq \infty$.

Remark. Suslin also deduces the following from 4.7b: Suppose F is algebraically closed of characteristic p, $W(F)$ is the ring of Witt vectors over F, and E is the quotient field of $W(F)$. Then if $\overline{E}$ denotes the algebraic closure, there is a canonical isomorphism $K_*(F, \mathbb{Z}/\ell) \cong K_*(\overline{E}, \mathbb{Z}/\ell)$. (Here $\ell \neq p$ as usual). Combining this with Quillen's calculation of $K_* \overline{\mathbb{F}}_p$, we obtain $K_*(\overline{\mathbb{Q}}_p, \mathbb{Z}/\ell)$ and hence $K_*(\mathbb{C}, \mathbb{Z}/\ell)$ by 4.1, independently of 4.2. Conversely if we start from 4.2, we get a new proof of Quillen's theorem on $K_* \overline{\mathbb{F}}_p$ (at least mod ℓ).

5. The Lichtenbaum-Quillen Conjectures

We now know the mod ℓ K-theory of any separably closed field F, char $F \neq \ell$. Given an arbitrary field F, we might hope to somehow recover

$K_*(F, \mathbb{Z}/\ell^\nu)$ from $K_*(\overline{F}; \mathbb{Z}/\ell^\nu)$ where $\overline{F}$ is the separable closure. Now if E/F is any Galois extension, $BGLF^+$ is exactly the fixed point set of $G = G(E/F) : BGLF^+ = (BGLE^+)^G$. Assume for a while what G is finite. A very naive hope would be that $K/\ell_*^\nu F = (K/\ell_*^\nu E)^G$. This is indeed true in the very special case when G is has order prime to ℓ; see 2.12. But in general it is clearly false; e.g. for $F = \mathbb{R}$, $E = \mathbb{C}$. A more reasonable but still optimistic hope would be that there is a *descent spectral sequence*:

$$E_2^{p,q} = H^p(GL(E/F); K/\ell_q^\nu E) \Longrightarrow K/\ell_{q-p}^\nu F.$$

Here the indexing is such that the differentials d_r have bidegree $(r, r-1)$. We will use this indexing throughout, but the reader is certainly free to re-index it however he or she prefers. A spectral sequence of this type does in fact arise, whenever a group G acts on a space X, as follows: filter EG by skeleta and apply the equivariant mapping space functor $F^G(-, X)$. We obtain a tower of fibrations and hence, applying homotopy, a spectral sequence. The E_2-term is easily identified as $H^p(G; \pi_q X)$, and under favorable circumstances, the spectral sequence converges to $\pi_*(F^G(EG, X))$. Hence the question of whether X admits a "descent" spectral sequence converging to the homotopy of the fixed point set X^G is transformed into the question of whether the natural map from X^G to the "homotopy fixed point set" $X^{hG} = F^G(EG, X)$ is an equivalence (at least ℓ-adically). This question has played a central role in homotopy theory over the last decade; for a discussion including K-theory, see [Carlsson 1987].

Example. $F = \mathbb{R}$. In view of Suslin's theorems 4.3 and 4.4, ℓ-adic descent for $\mathbb{R}$ is equivalent to descent for ordinary topological K-theory BO. But in fact the results of [Atiyah 1966] imply that BO satisfies descent globally. Since this often cited implication is not particularly obvious, we sketch the argument. Let $\mathbb{Z}/2$ act on BU by complex conjugation. Then the result to be shown is that the natural map $f : BO = BU^{\mathbb{Z}/2} \to BU^{h\mathbb{Z}/2} = F^{\mathbb{Z}/2}(S^\infty, BU)$ is a weak equivalence. The key results from Atiyah's paper are the following: Let X be a compact space with involution σ and let $KR(X)$ denote the Grothendieck group of "vector bundles with conjugation" over X; that is, complex vector bundles with a conjugate linear involution covering σ. Then $\widetilde{KR}(X)$ is the same as unbased $\mathbb{Z}/2$–equivariant homotopy classes of maps to $BU : \widetilde{KR}(X) = [X, BU]^{\mathbb{Z}/2}$. Now suppose the involution on X is trivial and let S^k have the antipodal involution. Then Corollary 3.8 of [Atiyah 1966] yields an exact sequence (for $k \geq 2$)

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{KO}^0 X \arrow[r] \arrow[d, equal] & KR^0(X \times S^k, * \times S^k) \arrow[r] \arrow[d, equal] & \widetilde{KO}^{k+2} X \arrow[r] & 0 \\
& {[X, BO]} \arrow[r, "j"'] & {[X, F^{\mathbb{Z}/2}(S^k, BU)]} & &
\end{tikzcd}$$

where j is induced by $BO \to BU^{h\mathbb{Z}/2} \to F^{\mathbb{Z}/2}(S^k, BU)$. Now take $X = S^n$. Then j is an isomorphism for infinitely many values of k. It is then obvious that the inverse system $\pi_n F^{\mathbb{Z}/2}(S^k, BU)$ is Mittag-Leffler and that $\pi_n BO \underset{\cong}{\to} \pi_n BU^{h\mathbb{Z}/2} \underset{\cong}{\to} \underset{\leftarrow}{\lim}\, \pi_n F^{\mathbb{Z}/2}(S^k, BU)$, as desired. As an amusing exercise, the reader can show that the descent spectral sequence collapses at E_4 (a fact I first learned from Bill Dwyer). This also provides an example of the "fringe effect" that arises in the spectral sequence of a tower of fibrations: there are copies of $\mathbb{Z}$ and $\mathbb{Z}/2$ along the line $p = q$ that are not eliminated by differentials and yet cannot represent anything in $\pi_* BO$. However this problem disappears, and the exercise is much easier, if one replaces BO, BU by KO, KU (equivalently, just formally invert the generator of $\pi_8 BO$).

Return now to the extension $\overline{F}/F$ and write G_F for $G(\overline{F}/F)$. Usually G_F is not finite, but it is always profinite. In that case the E_2-term of the conjectural spectral sequence should be interpreted as Galois cohomology, which we digress to explain. Let $G = \underset{\leftarrow}{\lim} G_\alpha$ be a profinite group. If M is a trivial G-module we define the continuous cohomology $H^*_{\text{cont}}(G, M)$ as $\underset{\underset{\alpha}{\rightarrow}}{\lim}\, H^*(G_\alpha, M)$. If M is nontrivial but at least is discrete in the sense that $M = \cup_\alpha M^{U_\alpha}$, where U_α is the kernel of the projection $G \to G_\alpha$, we define $H^*_{\text{cont}}(G, M) = \underset{\underset{\alpha}{\rightarrow}}{\lim}\, H^*(G_\alpha; M^{U_\alpha})$. To see how the limit works the reader should inspect the main example: $G = G_F$, $M = \overline{F}^*$. In this case we write H^*_{Gal} (or later $H^*_{\text{ét}}$) in place of H^*_{cont}. Of course $H^0 = M^G$, as usual. The basic results we need can be found in [Serre 1964]:

5.1. Theorem. *(Hilbert's Theorem 90) Let E/F be any Galois extension. Then $H^1_{\text{Gal}}(G(E/F); E^*) = 0$.*

5.2. Theorem. *$H^2_{\text{Gal}}(G_F; \overline{F}^*) \cong Br\, F$.*

If char $F \nmid n$ there is a short exact Kummer sequence of G_F-modules

$$O \to \mu_n(1) \to \overline{F}^* \xrightarrow{x^n} \overline{F}^* \to 0$$

where $\mu_n(1)$ is the group of nth roots of unity with its natural G_F-action. From the resulting long exact cohomology sequence we obtain:

5.3. Theorem. *(a) $H^1_{\text{Gal}}(G_F; \mu_n(1)) \cong F^*/n$.*
(b) $H^2_{\text{Gal}}(G_F, \mu_n(1)) \cong {}_nBr\, F$.

Let $\mu_n(i) = \otimes^i \mu_n(1)$, with the diagonal G_F-action. Note that if $\xi_n \in F$, $\mu_n(1)$ is the trivial module $\cong \mathbb{Z}/n$, and hence the same is true for $\mu_n(i)$.

This fact will be used repeatedly in the sequel. In general $\mu_n(i)$ can also be identified with the group μ_n, but with the "Tate twisted" G_F-action given by $\sigma \cdot a = \sigma^i(a)$ ($\sigma \in G_F, a \in \mu_n$). With either interpretation it is clear that if $n = \ell^\nu$ with ℓ odd, for example, $\mu_{\ell^\nu}(i)$ is the trivial module if i is divisible by $(\ell-1)\ell^{\nu-1}$. More generally $\mu_{\ell^\nu}(i) \cong \mu_{\ell^\nu}(i+(\ell-1)\ell^{\nu-1})$. Hence the cohomology groups $H^*_{\mathrm{Gal}}(G_F;\mu_{\ell^\nu}(i))$ are periodic in i, with period $(\ell-1)\ell^{\nu-1}$.

Let $\mathrm{cd}_\ell G$ denote the maximal value of n such that there is a discrete ℓ-torsion module M with $H^n_{\mathrm{cont}}(G,M) \neq 0$. For example, if $G = \hat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, $\mathrm{cd}_\ell G = 1$ for all ℓ (exercise).

5.4. Theorem. *Let F be a number field and suppose either ℓ is odd or $\sqrt{-1} \in F$. Then $\mathrm{cd}_\ell G_F = 2$.*

Remark. 5.4 is clearly false if $\ell = 2$ and $F = \mathbb{Q}$: for the existence of complex conjugation shows $\mathbb{Z}/2$ is a retract of $G_{\mathbb{Q}}$, and of course $\mathrm{cd}_2\mathbb{Z}/2 = \infty$.

Thus one can ask for a descent spectral sequence with $E_2^{p,q} = H^p_{\mathrm{Gal}}(G_F; K/\ell^\nu_q\overline{F})$. Again, a spectral sequence of this type does arise from the homotopy fixed point set $(BGL\overline{F}^+)^{hG_F}$. Here one needs to interpret X^{hG} for a profinite $G = \varprojlim G_\alpha$ in a suitable way. The precise definition is a bit technical and we will not give it here, although see §7. The reader should interpret the notation X^{hG}, G profinite, as standing for a suitable "pro" version of the homotopy fixed point set. The main point to keep in mind is that the definition is cooked up precisely so that the E_2-term of the descent spectral sequence involves *continuous* cohomology.

Let's imagine for a moment we had the ℓ-primary descent spectral sequence for F a number field. What would it look like? Recall that by Suslin's theorem $K/\ell^\nu_*\overline{F} = \mathbb{Z}/\ell^\nu[\beta]$. Then it is obvious from the definition of β that $K/\ell^\nu_{2i}(\overline{F}) = \mu_{\ell^\nu}(i)$ as G_F-module. Hence we would have $E_2^{p,q} = 0$ if q is odd and $E_2^{p,q} = H^p_{\mathrm{Gal}}(G_F;\mu_{\ell^\nu}(i))$ if $q = 2i$. The checkerboard pattern forces all even d_r's to be zero. Now suppose ℓ is odd or $\sqrt{-1} \in F$, so that 5.4 applies. Then since $d_2 = 0$ the spectral sequence collapses! Now suppose further, for convenience, that $\xi_{\ell^\nu} \in F$. Then $\mu_{\ell^\nu}(1)$ has trivial G_F-action and hence the same is true for $\mu_{\ell^\nu}(i)$. Hence the E_2-term is completely and explicitly computed by 5.1 and 5.2. Let's see what this would imply for $K/\ell^\nu_*\, F$. In degree zero ($p = q$) we expect to find $K/\ell^\nu_0 F = \mathbb{Z}/\ell^\nu$ and indeed this shows up as $E_2^{0,0}$, while $E_2^{1,1}$ is automatically zero. Unfortunately $E_2^{2,2} = {}_{\ell^\nu}Br\, F$, which is nonzero – indeed infinite. *Hence there is no descent for K_0.* Undeterred by this cruel twist of fate, we consider K_1. We would have only $E_2^{1,2} = H^1(G_F;\mathbb{Z}/\ell^\nu(1)) = F^*/\ell^\nu = K_1/\ell^\nu F$. Hence descent holds for K_1. Now consider K_2. We have $E_2^{0,2} = (K_2/\ell^\nu\overline{F})^{G_F} \cong \mathbb{Z}/\ell^\nu$, generated by β, and $E_2^{1,3} = 0$. Finally, $E_2^{2,4} = H^2_{\mathrm{Gal}}(G_F;\mu_{\ell^\nu}(2)) = {}_{\ell^\nu}Br\, F$

(since $\xi_{\ell^\nu} \in F$). Hence Tate's theorem 1.15 can be interpreted as verifying descent for K_2. This gives some hope that descent holds in positive dimensions.

5.5. Example. F is a finite field $\mathbb{F}_q$. In this case the Brauer group is zero, so there is no obvious obstruction to descent. In fact, Quillen's work on $K_*\mathbb{F}_q$ can be interpreted as verifying descent, as follows: As usual we fix a prime $\ell \neq p$, and suppose we have computed $K_*(\overline{\mathbb{F}}_q; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu[\beta]$, either by Quillen or by Suslin's work (see the remark in §4). Let $X = BGL\overline{\mathbb{F}}_q^+$. Then $BGL\mathbb{F}_q^+$ is the fixed point set $X^{\mathbf{Z}}$ of the $\mathbb{Z}$-action obtained from the Frobenius $\sigma : X \to X$. The homotopy fixed point set $X^{h\mathbf{Z}}$ is easily identified with the homotopy pullback E in the diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle \Delta} \\ X & \xrightarrow[(1,\sigma)]{} & X \times X \end{array}$$

This in turn is equivalent to the fibre of $\sigma - 1 : X \to X$, since X is an H-space. Since σ corresponds to ψ^q under the equivalence $X\hat{} \cong BU\hat{}$, we see that $(X^{h\mathbf{Z}})\hat{} \cong (F\psi^q)\hat{}$, and hence by Quillen $(BGL\mathbb{F}_q^+)\hat{} \cong (X^{h\mathbf{Z}})\hat{}$. The desired descent spectral sequence is then obtained from $X^{h\mathbf{Z}}$. Here the reader may object that the Galois group $G_{\mathbf{F}_q}$ is $\hat{\mathbb{Z}}$, not $\mathbb{Z}$, and that we have completely ignored the profinite topology on $\mathbb{Z}$. But the inclusion $\mathbb{Z} \to \hat{\mathbb{Z}}$ induces an isomorphism on $H^*(\quad; M)$ for any discrete torsion $\hat{\mathbb{Z}}$-module M, and hence $X^{h\mathbf{Z}} \to X^{h\hat{\mathbf{Z}}}$ is an equivalence after ℓ-adic completion. The details of this example provide a highly recommended exercise. Note $E_2^{p,q} = 0$ for $p > 1$, the spectral sequence collapses with no extensions, and E_2 is easily computed.

Remark. [Carlsson 1987] proves a sort of descent for finite Galois groups but with BGL replaced by BGL_n, n finite. Unfortunately, the argument fails for $n = \infty$.

We turn now to the general descent question for a scheme X. The reader who is unfamiliar with schemes should systematically translate "scheme" as either "algebraic variety" or "commutative ring", bearing in mind that the correspondence between affine schemes and commutative rings is just like the correspondence between affine varieties over k and k-algebras, and so in particular is contravariant. In order to remain flexible and to avoid tedious technicalities, we will assume X is "sufficiently nice", without specifying what that means. Various hypotheses on X will be discussed as they arise.

However "sufficiently nice" should always be taken to include smooth varieties over an algebraically closed or finite field of characteristic not equal to ℓ, and number fields and their rings of S-integers.

The first thing we need to discuss is étale cohomology. This certainly isn't the place to attempt an introduction to the subject – see [Milne], and [Deligne]; topologists will want to consult Sullivan's marvelous "Intuitive discussion of the étale homotopy type", which appears in Chapter 5 of [Sullivan]. However, for the benefit of the reader who may know even less about étale cohomology than the author, we will include some brief remarks.

Consider the scheme X with its usual Zariski topology. For example when $X = \operatorname{Spec} R$, R a Dedekind domain, the proper closed subsets are just the finite sets of nonzero prime ideals. A presheaf on X is just a contravariant functor from the category of open sets and inclusions to abelian groups. Presheafs can be sheafified, there are enough injectives, and sheaf cohomology is defined in terms of right derived functors of the global sections functor. This sort of ordinary sheaf cohomology will be written $H^*_{\text{Zar}}(X, \mathcal{F})$, for a given sheaf $\mathcal{F}$. Étale cohomology $H^*_{\text{ét}}(X, -)$ is defined in roughly the same way, except that the category of open inclusions $U \subset X$ is replaced by the category of étale covers $U' \to U$, U open in X. The technical definition of étale is "flat and unramified". Geometrically one should picture a smooth unramified covering; algebraically, a finite extension of Dedekind domains which is unramified in the sense of number theory. A crucial new feature of the étale setting is that there are nontrivial *automorphisms* of the objects $U' \to U$-i.e. maps $U' \to U'$ covering the projection. For example, suppose $X = \operatorname{Spec} F$, F a field. As a space X consists of a single point, so Zariski sheaves aren't very interesting. On the other hand a connected étale cover of X is the same thing as a finite separable extension E of F; thus $\operatorname{Spec} E \to \operatorname{Spec} F$ is an étale open, and if E is Galois its automorphism group is just $G(E/F)$. Hence an étale sheaf $\mathcal{F}$ on $\operatorname{Spec} F$ would in particular assign to each finite Galois extension E some $G(E/F)$-module, in a compatible way. From this one can construct a discrete G_F-module $M_{\mathcal{F}}$, and it is an exercise to show ([Milne], p. 53):

5.6. Theorem. *The category of étale sheaves on* $\operatorname{Spec} F$ *is equivalent to the category of discrete G_F-modules, and* $H^*_{\text{ét}}(\operatorname{Spec} F; \mathcal{F}) = H^*_{\text{Gal}}(G_F; M_{\mathcal{F}})$.

At the opposite extreme, we have ([Milne], p.117, see also [Sullivan], loc. cit.):

5.7. Theorem. *Let X be a smooth complex algebraic variety. Then* $H^*_{\text{ét}}(X, \mathbb{Z}/n) \cong H^*(X, \mathbb{Z}/n)$.

Here $\mathbb{Z}/n$ is regarded as the constant étale sheaf on the left, and on the right we have ordinary singular homology. Thus étale cohomology with finite coefficients can be viewed, as a first approximation, as a concatenation of Galois cohomology and singular cohomology. In the sequel we will refer somewhat vaguely to the "étale site" or "étale topology". Roughly this just means X together with the étale covers of its open subsets – again, see [Milne] for a precise definition. Similarly the "Zariski site" refers to X together with its Zariski open subsets. Sometimes we write $X_{\text{ét}}$ or X_{Zar} to indicate which site is being considered.

The main examples of étale sheaves that we need are the following: Let $\mathbb{G}_m$ denote the functor assigning to any scheme U the group $\Gamma(U, \mathcal{O}_U)^*$ of invertible regular functions on U. If $U = \operatorname{Spec} R$, this is just R^*. Then $\mathbb{G}_m$ defines a sheaf on both the Zariski and étale sites of any scheme X. If $X = \operatorname{Spec} F$, F a field, the corresponding discrete G_F-module as in 5.6 is just $\overline{F}^*$. Using a "change of site" spectral sequence one shows ([Milne], p.124):

5.8. Theorem. *("Hilbert's Theorem 90").* $H^1_{\text{ét}}(X; \mathbb{G}_m) = H^1_{\text{Zar}}(X, \mathbb{G}_m)$.

Note that when $X = \operatorname{Spec} F$, $H^k_{\text{Zar}}(X, \mathcal{F}) = 0$ (trivially) for $k > 0$ and any sheaf $\mathcal{F}$. Hence we indeed recover the classical Hilbert's theorem 90 as a special case. Recall also that in general $H^1_{\text{Zar}}(X, \mathbb{G}_m) = \operatorname{Pic} X$, the group of line bundles on X. Generalizing 5.2, we have:

5.9. Theorem. *Let R be a commutative ring. Then $H^2_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m) \cong Br\, R$.*

For a discussion of Brauer groups of schemes, see [Milne], Ch.IV. Now let $\mu_n(1)$ denote the étale sheaf assigning to each étale open U the group of nth roots of unity in $\Gamma(U, \varphi_U)^*$. One would like to have an exact Kummer sequence

$$O \to \mu_n(1) \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

as we did for fields. The only possible problem is with surjectivity of the nth power map. Consider for example the case $n = \ell$, $X = \operatorname{Spec} \mathcal{O}_F$. If $a \in \mathcal{O}_F^*$ and the extension ring $S = \mathcal{O}_F[T]/(T^\ell - a)$ is étale over $\mathcal{O}_F$, then "a has an ℓth root locally in the étale topology", which is exactly what we need. Unfortunately this is never the case since S is ramified over ℓ. However it is ramified *only* over ℓ, so if we replace $\mathcal{O}_F$ by $\mathcal{O}_F[\frac{1}{\ell}]$ (cutting out the primes over ℓ) the problem disappears. This is the main reason for insisting on inverting ℓ in the sequel. In general, write $\frac{1}{\ell} \in X$ as shorthand for "the residue field characteristics of X are all prime to ℓ". Then (see [Milne], p. 66):

5.10. Theorem. *If $\frac{1}{\ell} \in X$ the Kummer sequence*

$$O \to \mu_{\ell^\nu}(1) \to \mathbb{G}_m \to \mathbb{G}_m \to 0$$

is exact on the étale site. In particular this is true for $X = \operatorname{Spec} \mathcal{O}_F[\frac{1}{\ell}]$.

As in the case of Galois cohomology we can define $\mu_{\ell^\nu}(i) = \otimes^i \mu_{\ell^\nu}(1)$. If, for example, $X = \operatorname{Spec} R$ and $\xi_{\ell^\nu} \in R$, then $\mu_{\ell^\nu}(i)$ is isomorphic to the constant sheaf $\mathbb{Z}/\ell^\nu$. A similar remark applies to the general case.

We conclude our discussion of étale cohomology with some miscellaneous remarks that may be helpful to the reader.

5.11. Remark. In §4 we said that a local ring A is Hensel if it satisfies Hensel's lemma. An equivalent definition (cf. [Milne], §4) is that A has no nontrivial finite étale extensions with trivial residue field extension. A is strict Hensel if it has no nontrivial étale extensions whatsoever. Here "trivial étale extension" means a product A^n. In fact if A is Hensel with residue field k, the map $\operatorname{Spec} k \to \operatorname{Spec} A$ is in a suitable sense an "étale homotopy equivalence". We also note that the strict Henselization $A^{\,\mathrm{sh}}$ can be viewed as the direct limit of all connected finite étale extensions of A, with a similar definition of A^h. This means that the strict Hensel local rings are precisely the local rings for the étale topology. In a bit more detail, note that a "point" in the étale world is a scheme of the form $\operatorname{Spec} F$, F separably closed. Thus a point $\overline{x}$ of a scheme X is Spec of a separably closed field F containing the residue field $k(x)$, together with the induced map $\overline{x} \xrightarrow{\varepsilon} X$. An étale neighbourhood of $\overline{x}$ is an étale open $U \to X$ with a factorization of ε through U. Hence the stalk $\mathcal{F}_{\overline{x}}$ of an étale presheaf is $\varinjlim \mathcal{F}(U)$, where U runs over all such neighborhoods of $\overline{x}$. In particular the stalk of the structure sheaf is the strict Henselization $\mathcal{O}_x^{\,\mathrm{sh}}$ of the (ordinary) local ring at x. This also means that the Gabber/Gillet-Thomason/Suslin theorems (§4) can be rephrased as follows: Let $\mathcal{K}/\ell^\nu_*$ denote the sheafification of the étale presheaf $U \mapsto K_*(U; \mathbb{Z}/\ell^\nu)$. Then $\mathcal{K}/\ell^\nu_q$ is isomorphic to $\mathbb{Z}/\ell^\nu(i)$ if $q = 2i$ and is zero for q odd. This should be kept in mind while pondering 5.12 below.

We are now ready to state our first version of the Lichtenbaum–Quillen conjectures. Here "sufficiently nice" should be taken to include at least the following: (a) $\frac{1}{\ell} \in X$. (b) X is regular (c) $\operatorname{cd}_\ell X < \infty$.

5.12. Lichtenbaum-Quillen Conjectures: First Version (LQCI). If X is a sufficiently nice scheme, then there is a descent spectral sequence with

$$\begin{aligned} E_2^{p,q} &= H^p_{\text{ét}}(X, \mu_{\ell^\nu}(i)) \ (q = 2i) \\ &= 0 \ (q \, \textit{odd}) \end{aligned}$$

converging to $K_{q-p}/\ell^\nu X$ if $q - p$ is sufficiently large. Here $q - p \geq 1$ should suffice for $X = \operatorname{Spec} \mathcal{O}_F[\frac{1}{\ell}]$.

Remarks.

(a) As we have seen, even for $X = \operatorname{Spec} F$ there is no such spectral sequence converging precisely to $K_*/\ell^\nu X$. As another example take X to be a smooth projective variety over $\mathbb{C}$. In that case the descent spectral sequence would surely have to be the Atiyah-Hirzebruch spectral sequence for ordinary topological K-theory mod ℓ^ν. Hence if it converged on the nose the topological and algebraic K-theory of X would be equal. But this is well-known to be false even for K_0: there are topological vector bundles with no algebraic structure, and nonisomorphic algebraic vector bundles which are isomorphic topologically.

(b) The conjecture as stated is very awkward. What one wants of course is an auxiliary space or spectrum EX that does have descent, with a map $KX \xrightarrow{f} EX$. Then LQC can be rephrased as "f_* is an isomorphism on $\pi_n(\quad; \mathbb{Z}/\ell^\nu)$ for $n > N$". In other words, EX would be analogous to the homotopy fixed point set discussed earlier. This will be the subject of §7. There are conjectures concerning N–see [Quillen 1974] and [Thomason 1986].

(c) Again the checkerboard pattern would force all even d_r's to be zero. Hence the spectral sequence would collapse if $E_2^{p,q} = 0$ for p odd – e.g. X is a flag variety over an algebraically closed field – or if $\operatorname{cd}_\ell X \leq 2$.

(d) The assumption $\operatorname{cd}_\ell X < \infty$ ensures that for $p \gg 0$, $E_2^{p,q} = 0$ for all q, so we have finite convergence. Unfortunately this excludes many interesting X when $\ell = 2$ – e.g. $\operatorname{Spec} Q$ or $\operatorname{Spec} \mathbb{Z}$. However the reformulation of LQCI in terms of étale K-theory (§7) will cover these cases as well.

6. The Conjectures for a Ring of Algebraic Integers

Throughout this section, $R = \mathcal{O}_F[\frac{1}{\ell}]$, where $\mathcal{O}_F$ is the ring of integers in a number field F. Our goal is to give some explicit examples of the Lichtenbaum-Quillen conjectures for R. The first problem is of course to compute the relevant étale cohomology groups. Combining the Kummer sequence 5.10, Hilbert's theorem 90 in the form 5.8, and the Brauer group theorem 5.9, we have:

6.1. Theorem. *There are natural short exact sequences*

$$0 \to R^*/\ell^\nu \to H^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/\ell^\nu(1)) \to {}_{\ell^\nu}C\ell R \to 0,$$

$$0 \to (C\ell R)/\ell^\nu \to H^2_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/\ell^\nu(1)) \to {}_{\ell^\nu}BrR \to 0.$$

Alternatively (see [Milne]) one can obtain 6.1 from the Leray spectral sequence of the map $\operatorname{Spec} F \to \operatorname{Spec} R$. This method also computes the higher cohomology groups $H^*_{\text{ét}}(\ \ , \mathbb{Z}/\ell^\nu(1))$. However, if $\mathbb{Z}/\ell^\nu(1)$ is replaced by $\mathbb{Z}/\ell^\nu(i)$, the computation becomes difficult; compare 6.20 below. Hence, we will be content with the following:

6.2. Theorem. *Suppose $\xi_{\ell^\nu} \in R$, and that either ℓ is odd or $\nu \geq 2$. Then $H^k_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/\ell^\nu(i))$ is zero for $k > 2$, and is given by 6.1 for $k = 1, 2$.*

6.3. Remarks.

(a) Recall that if $\xi_{\ell^\nu} \in R$, $\mathbb{Z}/\ell^\nu(i) \cong \mathbb{Z}/\ell^\nu$ for all i.

(b) Let E denote the maximal extension of F which is unramified away from ℓ. Then one can show that for any F,

$$H^*_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/\ell^\nu(i)) \cong H^*{}_{\text{Gal}}(G_{E/F}; \mathbb{Z}/\ell^\nu(i))$$

for all ν, i.

(c) Let $\ell = 2$. As noted in §5, the field F can have infinite cohomological dimension at 2. The same is true for R. For example, $H^*_{\text{ét}}(\operatorname{Spec} \mathbb{Z}[\frac{1}{2}], \mathbb{Z}/2) \cong \mathbb{Z}/2[x, e]/(xe, e^2)$, where $x, e \in H^1$-compare §7.

Suppose now that we are in the situation of 6.2, and that the Lichtenbaum-Quillen conjectures hold for R. Then the descent spectral sequence collapses. Furthermore the E_2-term is periodic, in positive dimension with period 2. This periodicity would appear in $K_*(R, \mathbb{Z}/\ell^\nu)$ as "Bott periodicity". That is, let $\beta \in K_2(R; \mathbb{Z}/\ell^\nu)$ denote the Bott class (§3). Then the conjectural computation of $K_*(R, \mathbb{Z}/\ell^\nu)$ is most conveniently expressed as follows:

6.4. Theorem. *Suppose $\xi_{\ell^\nu} \in R$, and either ℓ is odd or $\nu \geq 2$. Assume LQCI 5.12 holds for R. Then there are split short exact sequences*

$$0 \to R^*/\ell^\nu \to K_1(R, \mathbb{Z}/\ell^\nu) \to {}_{\ell^\nu}(C\ell R) \to 0,$$

$$0 \to \langle\beta\rangle \oplus C\ell R/\ell^\nu \to K_2(R, \mathbb{Z}/\ell^\nu) \to {}_{\ell^\nu}BrR \to 0.$$

Furthermore, as $\mathbb{Z}/\ell^\nu[\beta]$-module

$$K_*(R; \mathbb{Z}/\ell^\nu) \cong \mathbb{Z}/\ell^\nu[\beta] \otimes_{\mathbb{Z}/\ell^\nu} (K_0 R/\ell^\nu \oplus K_1(R; \mathbb{Z}/\ell^\nu) \oplus_{\ell^\nu} BrR). \qquad \square$$

6.5. Remarks.

(a) From the short exact sequences we see that the Bass-Milnor-Serre theorem 1.5 and Tate's theorem 1.18 can be viewed as verifying descent for K_1 and K_2, respectively.

(b) Recall (§1) that $R^*/\ell^\nu \cong (\mathbb{Z}/\ell^\nu)^{r_2+1}$ and ${}_{\ell^\nu}BrR \cong (\mathbb{Z}/\ell^\nu)^{k-1}$, where k is the number of primes over ℓ. Hence 6.4 gives, conjecturally, a completely explicit computation modulo determination of the class group.

(c) Comparing 6.4 and Borel's theorem 2.4, it is natural to ask whether the Borel classes can be constructed ℓ-adically in such a way that they reduce mod ℓ^ν to the $\mathbb{Z}/\ell^\nu[\beta]$-module generated by the units. This was shown by [Soulé 1980]. A beautiful homotopy-theoretic reformulation of Soulé's construction, due to [Bökstedt-Hsiang-Madsen], allows one to realize these classes by actual maps $(\Sigma\mathbb{C}P^\infty_+)\hat{} \to (BGLR^+)\hat{}$.

(d) Note that when $\nu = 1$, the last part of 6.4 says that $K_*\ (R;\mathbb{Z}/\ell)$ is actually a *free* module over $\mathbb{Z}/\ell[\beta]$. In general the only relations are those arising from the orders of the cyclic summands of $C\ell R/\ell^\nu$.

6.6. Example. Suppose ℓ is odd and $R = \mathbb{Z}[\xi_{\ell^\nu}, \frac{1}{\ell}]$. Then ${}_{\ell^\nu}BrR = 0$ 1.19 and hence $K_*(R, \mathbb{Z}/\ell^\nu)$ is determined, conjecturally, by the class group and the unit group. Now suppose further that ℓ is regular; i.e, $\ell \nmid |C\ell\mathbb{Z}[\xi_\ell]|$. Then the class group terms in 6.1 and 6.4 also vanish (see 1.19). Thus Spec R has mod ℓ étale cohomological dimension *one*, and $K_*(R;\mathbb{Z}/\ell^\nu)$ is, conjecturally, just the free $\mathbb{Z}/\ell^\nu[\beta]$-module generated by $1 \in K_0R$ and the units! This suggests a conjecture on the nature of the spectrum KR, which will be considered in §7.

Now suppose $R = \mathbb{Z}[\frac{1}{\ell}]$, ℓ odd. By Example 2.13, we have $K_*(R;\mathbb{Z}/\ell^\nu) = (K_*(S;\mathbb{Z}/\ell^\nu)^G$, where $G = G_{\mathbb{Q}(\xi_\ell)/\mathbb{Q}}$ and $S = \mathbb{Z}[\xi_\ell, \frac{1}{\ell}]$. Hence, taking $\nu = 1$, a conjectural calculation of $K_*(R, \mathbb{Z}/\ell)$ can be obtained from 6.6-provided we can determine the action of G on $K_*(S, \mathbb{Z}/\ell)$. Recall that G is cyclic of order $\ell - 1$. In particular every representation of G over $\mathbb{Z}/\ell$ is completely reducible. Let λ denote the natural representation of G on the ℓ-th roots of unity $\mu_\ell \subseteq S^*$. Then the irreducible representations of G over $\mathbb{Z}/\ell$ are precisely λ^0, $\lambda, \ldots, \lambda^{\ell-2}$. Note that G has a unique element c of order two, namely, complex conjugation. Call a representation of G *even* if c acts trivially and *odd* if c acts as multiplication by (-1). For example λ^k is even if k is even and odd otherwise. Now by definition the submodule $\langle\beta\rangle \in K_2(S, \mathbb{Z}/\ell)$ is isomorphic to λ. Thus β^k is fixed by G if and only if $k = 0 \bmod \ell - 1$. In particular $K_*(R, \mathbb{Z}/\ell)$ is a module over $\mathbb{Z}/\ell[\beta^{\ell-1}]$. What about the units? The torsion subgroup μ_ℓ is a copy of λ. On the other hand inspection of the proof of Dirichlet's theorem 1.6 easily yields the general result:

6.7. Theorem. *Let F be a finite Galois extension of $\mathbb{Q}$ with group G. Then $(\mathcal{O}_F^* \otimes_{\mathbb{Z}} \mathbb{R})$ is isomorphic to the reduced regular representation of G if F is real, and is isomorphic to the reduced form of the induced representation*

$\mathbb{R}G \otimes_{\mathbb{R}\mathbb{Z}/2} \mathbb{R}$ if F is imaginary. Here $\mathbb{Z}/2 \subseteq G$ is generated by complex conjugation.

Here the reduced form of a permutation representation V is the kernel of the natural map $V \to \mathbb{R}$. Thus if ℓ is a rational prime which is either inert or totally ramified in the extension $F/\mathbb{Q}$, we can replace $\mathcal{O}_F$ by $\mathcal{O}_F[\frac{1}{\ell}]$ and eliminate the word "reduced" from 6.7. If $\ell \nmid |G|$ and $\mathbb{Z}/\ell$ is a splitting field for G, we can replace $\mathcal{O}_F^* \otimes_{\mathbf{Z}} \mathbb{R}$ by $(\mathcal{O}_F^*/\mathit{torsion}) \otimes_{\mathbf{Z}} \mathbb{Z}/\ell$ in this discussion. Thus, returning to our situation:

6.8. Corollary. *Let $M = \mathbb{Z}[\xi_\ell, \frac{1}{\ell}]^*/\mathit{torsion}$. Then $M/\ell M$ is isomorphic as a G-module to $\lambda^0 \oplus \lambda^2 \oplus \ldots \oplus \lambda^{\ell-3}$. In particular $M/\ell M$ is even.*

It is convenient to rewrite the above decomposition as $\lambda^0 \oplus \lambda^{-2} \oplus \ldots \oplus \lambda^{-(\ell-3)}$. Now observe that if $x_k \in M/\ell M$ generates the eigenspace λ^{-2k}, $y_k = \beta^{2k} x_k$ is fixed by G. Thus each y_k, $0 \leq k \leq (\ell-3)/2$, generates a free $\mathbb{Z}/\ell[\beta^{\ell-1}]$-submodule of $K_*(S, \mathbb{Z}/\ell)$. Note $\dim y_k = 1$ mod. Hence if ℓ is regular, our discussion so far can be summarized as follows:

6.9. Theorem. *Suppose ℓ is an odd regular prime and LQCI 5.12 holds for $\mathbb{Z}[\frac{1}{\ell}]$. Then $K_*(\mathbb{Z}[\frac{1}{\ell}], \mathbb{Z}/\ell)$ is a free $\mathbb{Z}/\ell[\beta^{\ell-1}]$-module of rank $(\ell+3)/2$. The generators are the elements $y_k \in K_{2k-1}$ described above, the identity in K_0, and $\beta^{\ell-2}[\xi_\ell] \in K_{2\ell-3}$.*

6.10. Remark. The last two summands in 6.9 correspond to $K_*(\mathbb{F}_p, \mathbb{Z}/\ell)$, where p is a retractible prime as in §3. Thus if F is the fibre of the reduction map $K\mathbb{Z}[\frac{1}{\ell}] \to K\mathbb{F}_p$, we have conjecturally that $\pi_n(F; \mathbb{Z}/\ell)$ is $\mathbb{Z}/\ell$ if $n = 1$ mod 4 and zero otherwise. In fact, since F has finite type, it follows from Borel's theorem 2.4 that modulo torsion prime to ℓ, $\pi_n F$ is $\mathbb{Z}$ if $n = 1$ mod 4 and zero otherwise, assuming LQCI. As in Example 6.6, this suggests a conjecture about F and $K\mathbb{Z}[\frac{1}{\ell}]$, that will be considered in §7.

Now suppose that ℓ is an irregular prime. Let A denote the ℓ-primary part of the class group of $\mathbb{Z}[\xi_\ell]$. Then A splits into eigenspaces for the G-action. $A \cong A_0 \oplus \ldots \oplus A_{\ell-2}$. Here A_i/ℓ is a direct sum of copies of λ^{-i}. (N.B. Our A_i is the $A_{\ell-1-i}$ of [Washington]). Note that $A_0 = 0$ since $\mathbb{Z}[\frac{1}{\ell}]$ has trivial class group. It is also known, for example, that $A_{\ell-2} = 0$ and $A_i = 0$ for $i \leq 5$ [Washington, p.102]. That $A_1 = 0$ is already reflected in K-theory, since $K_2(\mathbb{Z}; \mathbb{Z}/\ell) = 0$ for ℓ odd and $\beta : K_0(S; \mathbb{Z}/\ell) \to K_2(S; \mathbb{Z}/\ell)$ is injective. In general, each A_i determines a free $\mathbb{Z}/\ell[\beta^{\ell-1}]$-module in $K_*(\mathbb{Z}[\frac{1}{\ell}]; \mathbb{Z}/\ell)$, generated by $\beta^i \cdot (A_i/\ell)$. In order to be more specific, we will assume a famous conjecture from number theory ([Washington], p. 159).

6.11. Vandiver's Conjecture. $A_k = 0$ for k even. In other words, the natural representation of $G = G_{\mathbb{Q}(\xi_\ell)/\mathbb{Q}}$ on $(C\ell(\mathbb{Z}(\xi_\ell)))/\ell$ is odd.

The conjecture is usually stated in the form $\ell \nmid h^+(\mathbb{Q}(\xi_\ell))$, where h^+ denotes the order of the class group of the maximal real subfield. But the ℓ-part of h^+ is just $|A^{\mathbb{Z}/2}|$, where $\mathbb{Z}/2$ acts via complex conjugation, so this is equivalent to 6.11. Vandiver's conjecture is known to be true for all $\ell < 125,000$, but according to the number theorists this is no reason to believe it. It has the following consequence (see [Washington], Theorem 10.9).

6.12. Theorem. *If Vandiver's conjecture holds for ℓ, then A_i is cyclic for all i.*

Assuming this, we arrive at the following conjectural calculation of $K_*(\mathbb{Z}[\frac{1}{\ell}], \mathbb{Z}/\ell)$:

6.13. Theorem. *Let ℓ be odd. Assume Vandiver's conjecture for ℓ and LQCI 5.12 for $\mathbb{Z}[\frac{1}{\ell}]$. Then $K_*(\mathbb{Z}[\frac{1}{\ell}], \mathbb{Z}/\ell)$ is a free $\mathbb{Z}/\ell[\beta^{\ell-1}]$-module on r generators, where $(\ell+3)/2 \leq r \leq \ell$. The first $(\ell+3)/2$ generators are as in 6.9. Let $A_{i}, \dots, A_{i_m}$ $(0 \leq m \leq \frac{\ell-3}{2}$, i_k odd) denote the nontrivial eigenspaces of A, and let x_{i_k} generate A_{i_k}/ℓ. Then the remaining generators are the elements $\beta^{i_k} X_{i_k}$, $1 \leq k \leq m$.* □

6.14. Remark. In fact very few of the A_i are nontrivial, at least for $\ell \leq 4001$; see [Washington] p. 350-51 and Remark 6.16 below.

We next give a conjectural global calculation of $K_*\mathbb{Z}$. This calculation depends on §7 below for the 2-primary information. Let $\frac{B_n}{n} = c_n/d_n$ in lowest terms, where B_n is the nth Bernoulli number (in the notation of [Milnor-Stasheff, Appendix B], so that all B_n are nonzero).

6.15. Theorem. *Assume the Lichtenbaum-Quillen conjectures in the form LQCI and for $\ell = 2$ in the form LQCII 7.3. Assume also Vandiver's conjecture 6.11. Then for $n \geq 2$, $K_n\mathbb{Z}$ is given by:*

n mod 8	$K_n\mathbb{Z}$	
0	0	
1	$\mathbb{Z} \oplus \mathbb{Z}/2$	
2	$\mathbb{Z}/c_k \oplus \mathbb{Z}/2$	$(n = 4k-2)$
3	$\mathbb{Z}/8d_k$	$(n = 4k-1)$
4	0	
5	$\mathbb{Z}$	
6	$\mathbb{Z}/c_k$	$(n = 4k-2)$
7	$\mathbb{Z}/4d_k$	$(n = 4k-1)$

6.16. Remark. With our notation, the theorems of Herbrand and Ribet ([Washington], p.102) can be combined to read: If $1 \le k \le (\ell-1)/2$, $A_{2k-1} \neq 0$ if and only if ℓ divides the numerator of B_k/k. In the fantasy world of Theorem 6.15, we can append the condition "if and only if ℓ divides $|K_{4k-2}\mathbb{Z}|$". In fact the implication $A_{2k-1} \neq 0 \implies \ell$ divides $|K_{4k-2}\mathbb{Z}|$ is a theorem of [Soulé 1979]; see also 7.5 below. Note that the order in which irregular primes appear in 6.15 is the order in which they appear as divisors of the B_k/k. For example, the first nontrivial numerator is $c_6 = 691$, which is a prime. Hence 691 appears in $K_{22}\mathbb{Z}$ by Soulé's theorem; indeed conjecturally $K_{22}\mathbb{Z} = \mathbb{Z}/691$. (According to the tables in [Washington], $K_*(\mathbb{Z}, \mathbb{Z}/691)$ would have exactly one more generator, as $\mathbb{Z}/691[\beta^{690}]$-module, in dimension 398). On the other hand the smallest irregular prime, 37, first appears as a divisor of c_{16}, and so should not appear in $K_n\mathbb{Z}$ until $n = 62$.

We conclude with a brief discussion of the original zeta-function conjectures of [Lichtenbaum]. The zeta function of a number field F is defined by $\zeta_F(s) = \Sigma_I |A/I|^{-s}$, where the sum is over all proper nonzero ideals of $\mathcal{O}_F$. Note this is the classical Riemann zeta function when $F = \mathbb{Q}$. The zeta function extends to a function analytic on the whole plane, except for a simple pole at $s = 1$. When F is totally real, it is known to take nonzero rational values on the odd negative integers.

6.17. Conjecture [Lichtenbaum] Suppose F is totally real. Then up to powers of 2, $\zeta_F(1-2n) = |K_{4n-2}\mathcal{O}_F|/|K_{4n-1}\mathcal{O}_F|$.

Note the righthand side makes sense since both groups are finite by Quillen's theorem 2.3 and Borel's theorem 2.4. Note also this agrees with 6.15 when $F = \mathbb{Q}$. Now Lichtenbaum also conjectured a relation between values of zeta functions and étale cohomology. Define $H^p_{\text{ét}}(-;\mathbb{Z}_\ell(i)) = \varprojlim_\nu H^p_{\text{ét}}(-;\mathbb{Z}/\ell^\nu(i))$–and beware the misleading notation; the inverse limit must be taken on the outside as shown. Most of the results and conjectures of this paper can be formulated ℓ-adically in this way, rather than working modulo ℓ^ν. In particular this is true of Conjecture 5.12. Now suppose ℓ is odd, and let $X = \operatorname{Spec}\mathcal{O}_F[\frac{1}{\ell}]$. It turns out that because of the way the inverse limit works, $H^0_{\text{ét}}(X, \mathbb{Z}_\ell(n)) = 0$ for all $n > 0$ (the groups $H^0_{\text{ét}}(X, \mathbb{Z}/\ell^\nu(n))$ in effect are shifted to $H^1_{\text{ét}}(X, \mathbb{Z}_\ell(n))$. Hence 5.12 predicts, at ℓ, that $K_{4n-1}\mathcal{O}_F \cong H^1_{\text{ét}}(X, \mathbb{Z}_\ell(2n))$ and $K_{4n-2}\mathcal{O}_F \cong H^2_{\text{ét}}(X, \mathbb{Z}_\ell(2n))$. The resulting refomulation of 6.17 is now a theorem:

6.18. Theorem. [Wiles] *Suppose ℓ is odd and F is totally real. Then*

$$\zeta_F(1-2n) = |H^2_{\text{ét}}(\operatorname{Spec}\mathcal{O}_F[\frac{1}{\ell}]; \mathbb{Z}_\ell(2n)|/|H^1_{\text{ét}}(\operatorname{Spec}\mathcal{O}_F[\frac{1}{\ell}]; \mathbb{Z}_\ell(2n))$$

up to an ℓ-adic unit.

Of course if follows that 6.17 would be an immediate corollary of 5.12.

7. ÉTALE AND BOTT-PERIODIC K-THEORY

We first discuss the étale K-theory of [Dwyer-Friedlander 1985]. This theory was inspired by the work of [Soulé 1979]. The authors define, for any connected scheme X over $\operatorname{Spec}\mathbb{Z}[\frac{1}{\ell}]$, an ℓ-adic étale K-theory spectrum $K^{\text{ét}}X$ (which they denote $\hat{K}^{\text{ét}}X$, but we will omit the "$\wedge$"). Define $K_n^{\text{ét}}X = \pi_n K^{\text{ét}}X$. It has the following basic properties:

7.1. Suppose $cd_\ell X < \infty$. Then there is a strongly convergent spectral sequence

$$E_2^{p,q} = \left\{ \begin{matrix} H^p_{\text{ét}}(X, \mathbb{Z}/\ell^v(i)) & (q = 2i) \\ 0 & (q \text{ odd}) \end{matrix} \right\} \Longrightarrow K^{\text{ét}}_{q-p}(X; \mathbb{Z}/\ell^v)$$

(differentials as in §5, $q - p \geq 0$).

7.2. There is a good map $KX \xrightarrow{\varphi} K^{\text{ét}}X$.

Here "good" implies naturality and that φ is a map of ring spectra, at least when $X = \operatorname{Spec}R$, R a Noetherian $\mathbb{Z}[\frac{1}{\ell}]$-algebra; see [Dwyer-Friedlander 1975], Proposition 4.4. It also justifies a reformulation of the Lichtenbaum-Quillen conjectures:

7.3. Conjecture (*LQCII*). If X is a sufficiently nice scheme, the map $\varphi : KX \to K^{\text{ét}}X$ induces an isomorphism $K_n(X; \mathbb{Z}/\ell^v) \to K_n^{\text{ét}}(X; \mathbb{Z}/\ell^v)$ for all $n \gg 0$. Here $n \geq 1$ should suffice for $X = \operatorname{Spec}\mathcal{O}_F[\frac{1}{\ell}]$.

Remarks.

(a) Here "sufficiently nice" has the same vague meaning as in §5, except that we do not assume $cd_\ell X < \infty$. We wish to include examples like $X = \operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ with $\ell = 2$, where 7.3 is a viable conjecture even though $cd_\ell X = \infty$ (see below).

(b) Note that if φ is an isomorphism on $\pi_n(\ ; \mathbb{Z}/\ell)$ for $n \geq N$, φ is an isomorphism on $\pi_n(\ ; \mathbb{Z}/\ell^v)$ for $n \geq N$ and all v.

Let $E(X) = \Omega_0^\infty K^{\text{ét}}X$: the étale K-theory *space*. If $X = \operatorname{Spec}R$ we also write $E(R)$ in place of $E(X)$. The precise definition of $K^{\text{ét}}X$ or $E(X)$ is complicated and technical, as the reader can discover by inspecting the references cited. However one can give a quick intuitive description of $E(X)$ as follows (needless to say, none of this should be taken literally): One can associate to X its *étale homotopy* type $X^{\text{ét}}$. This is really an inverse system of homotopy types. For example, when $X = \operatorname{Spec} F$, F a field, $X^{\text{ét}}$ is essentially the inverse system $\{BG(L/F)\}$, where L ranges over all finite Galois extensions of F. However we will simplify matters by speaking of $X^{\text{ét}}$ as though it was an honest homotopy type. As another example, take X to be a smooth variety over $\mathbb{C}$. Then $X^{\text{ét}}$ is essentially the profinite completion of the ordinary homotopy type of X (compare 5.7). One can think of $E(X)$ as the space of sections of a certain bundle over $X^{\text{ét}}$ with fibre $BU\hat{}$ ($\wedge$ = ℓ-adic completion). For example when X is a smooth variety over $\mathbb{C}$, the bundle in question is the trivial bundle, and $E(X)$ is just the function space $F(X, BU\hat{})$. Hence the étale K-groups $\pi_n E(X)$ are just the ordinary (ℓ-adic) topological K-groups, and the spectral sequence 7.1 is a truncated form of the Atiyah-Hirzebruch spectral sequence. If $X = \operatorname{Spec} F$, let G_F denote the Galois group $G(\overline{F}/F)$, where $\overline{F}$ is the separable closure. The action of G_F on the ℓ-power roots of unity yields a homomorphism $G_F \to \mathbb{Z}_\ell^*$, and since the latter group acts on $BU\hat{}$ via ψ^k operations we obtain an action of G_F on $BU\hat{}$. The bundle defining $E(F)$ is $EG_F \times_{G_F} BU\hat{}$ (we continue to ignore the profinite topology on G_F). Now in general the space of sections of a bundle of the form $EG \times_G X \to BG$ is precisely the homotopy fixed point set X^{hG}. Here we could even use Suslin's theorem to replace $BU\hat{}$ by $(BGL\overline{F}^+)\hat{}$. This shows that the map $BGLF^+ \to E(F)$ is exactly the map $(BGL\overline{F}^+)^{G_F} \to (BGL\overline{F}^+)^{hG_F}$ discussed in §5.

Now suppose $X = \operatorname{Spec} R$, where $R = \mathcal{O}_F[\frac{1}{\ell}]$. Let $L(F)$ denote the maximal algebraic extension of F which is unramified away from ℓ, and let $\Gamma_F = G(L(F)/F)$. Note that $L(F)$ contains $F(\xi_{\ell^\infty})$. Then Γ_F is the fundamental group of $(\operatorname{Spec} R)^{\text{ét}}$, and $E(R)$ is the space of sections of a flat $BU\hat{}$-bundle over $(\operatorname{Spec} R)^{\text{ét}}$ arising from a homomorphism $\Gamma_F \to \mathbb{Z}_\ell^*$ as above. Recall, however, that $H^*_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}/\ell^\nu(i)) = H^*{}_{\text{Gal}}(\Gamma_F, \mathbb{Z}/\ell^\nu(i))$ 6.3b. From this it is more or less clear that $E(R)$ is just the homotopy fixed-point set $(BU\hat{})^{h\Gamma_F}$.

Having subjected the reader to an assortment of lies and half-truths, we now state two honest theorems from [Dwyer-Friedlander 1985].

7.4. Theorem. *Let X be a connected scheme over $\operatorname{Spec}\mathbb{Z}[\xi_{\ell^\nu}, \frac{1}{\ell}]$, with $cd_\ell X < \infty$. If $\ell = 2$, assume $\nu \geq 2$. Then for $i \geq 0$, multiplication by the Bott class β_ν is an isomorphism $K_i^{\text{ét}}(X, \mathbb{Z}/\ell)^\nu \underset{\cong}{\to} K_{i+2}^{\text{ét}}(X, \mathbb{Z}/\ell^\nu)$.*

For example, X could be $\operatorname{Spec}\mathcal{O}_F[\frac{1}{\ell}]$, where ℓ is odd and $\xi_{\ell^\nu} \in F$. The Bott class β_ν comes form $K_2(X, \mathbb{Z}/\ell^\nu)$ via the map φ of 7.2. Theorem 7.4 follows easily from the spectral sequence 7.1 and the corresponding periodicity in étale cohomology.

7.5. Theorem. *Let F be a number field. Suppose either ℓ is odd or $\nu \geq 2$ and $\sqrt{-1} \in F$. Let $\varphi_* : K_i(\mathcal{O}_F[\frac{1}{\ell}]; \mathbb{Z}/\ell^\nu) \to K_i^{\text{ét}}(\mathcal{O}_F[\frac{1}{\ell}]; \mathbb{Z}/\ell^\nu)$ denote the map induced by 7.2. Then*

(a) φ_* *is an isomorphism for* $i = 1, 2$.
(b) φ_* *is surjective for* $i \geq 1$.

Part (a) is a reformulation of descent for K_1 (Bass-Milnor-Serre) and K_2 (Tate). If $\xi_{\ell^\nu} \in F$, part (b) is then immediate from 7.4, and in fact one obtains a naturally split surjection. If ℓ is odd and $\nu = 1$ one can then use an obvious transfer argument. The complete proof is more difficult, and makes use of a "secondary transfer." We remark that when $i = 0$, φ_* is injective, but is surjective only when ${}_\ell Br\mathcal{O}_F[\frac{1}{\ell}] = 0$.

In another beautiful paper [Dwyer-Friedlander 1986], the authors explicitly identify the space $E(R)$, and even the étale K-theory spectrum, in many cases of interest. The idea is quite simple. Think of $E(R)$ as the space of sections of a bundle $BU\hat{} \to D \to (\operatorname{Spec} R)^{\text{ét}}$, as above. When $R = \mathcal{O}_F[\frac{1}{\ell}]$, complete determination of the homotopy type $(\operatorname{Spec} R)^{\text{ét}}$ is too difficult, since no one even knows how to explicitly describe the fundamental group Γ_F. However the étale cohomology is very well understood (§6). The idea is then to produce a known space Y and a map $Y \xrightarrow{f} (\operatorname{Spec} R)^{\text{ét}}$ inducing an isomorphism on $H^*(\ \ , \mathbb{Z}/\ell^\nu(i))$. If D' is the pullback of D along f, and E' is its space of sections, the induced map $E(R) \to E'$ will be an ℓ-adic equivalence. For example, let $JK(\mathbb{Z})$ denote the homotopy pullback of the diagram

$$\begin{array}{ccc} & & BO \\ & & \big\downarrow \\ BGL\mathbb{F}_p^+ & \xrightarrow[\theta]{} & BU \end{array}$$

Here $p = 3$ if $\ell = 2$ and p generates $(\mathbb{Z}/\ell^2)^*$ if ℓ is odd; θ is the Brauer lift. If ℓ is odd, $JK(\mathbb{Z}) \cong BGL\mathbb{F}_p^+ \times U/0$.

7.6. Theorem. [Dwyer-Friedlander 1986, 1991] *Suppose ℓ is a regular prime. Then* $E(\mathbb{Z}[\frac{1}{\ell}]) \cong JK(\mathbb{Z})\hat{}$.

In this example the space Y is $\mathbb{R}P^\infty \vee S^1$. A quick explanation can be given as follows, thinking of $E(\mathbb{Z}[\frac{1}{\ell}])$ as $(BU\hat{})^{h\Gamma_{\mathbb{Q}}}$ as above. Note that $\mathbb{R}P^\infty \vee S^1$ is the classifying space of the free product $\mathbb{Z} * \mathbb{Z}/2$. In spite of the complicated nature of $\Gamma_{\mathbb{Q}}$, for regular ℓ there is a homomorphism $\mathbb{Z} * \mathbb{Z}/2 \xrightarrow[\eta]{} \Gamma_{\mathbb{Q}}$ inducing an isomorphism on ℓ-torsion cohomology. To define η, let K denote the maximal abelian ℓ-extension of $\mathbb{Q}(\xi_\ell)$ which is unramified away from ℓ, and note $\mathbb{Q}(\xi_{\ell^\infty}) \subseteq K$. One can show that there is an element $T \in G(K/\mathbb{Q})$ such that T and complex conjugation c topologically generate $G(K/\mathbb{Q})$, and such that (taking ℓ odd for simplicity) T projects to a topological generator of $\mathbb{Z}_\ell^*$ under the natural map. Note $K \subseteq L_{\mathbb{Q}}$ and choose any $\widetilde{T} \in \Gamma_{\mathbb{Q}}$ projecting to T. The elements $\widetilde{T}, c$ define η. The choices are such that $(BU\hat{})^{h\mathbb{Z}/2} = BO\hat{}$ and $(BU\hat{})^{h\mathbb{Z}} = (BGL\mathbb{F}_p^+)\hat{}$ (compare §5), and since function spaces convert wedge sums to homotopy pullbacks we have $(BU\hat{})^{h(\mathbb{Z}*\mathbb{Z}/2)} = JK(\mathbb{Z})\hat{}$. Since η is a cohomology isomorphism, the induced map $(BU\hat{})^{h\Gamma_{\mathbb{Q}}} \to (BU\hat{})^{h(\mathbb{Z}*\mathbb{Z}/2)}$ is an equivalence.

7.7. Remarks.

(a) The notation $JK(\mathbb{Z})$ is due to [Bokstedt], who earlier constructed a map $h : BGL\mathbb{Z}[\frac{1}{2}]^+ \to JK(\mathbb{Z})$ when $\ell = 2$, and proved the striking theorem: Ωh is a homotopy retraction. See below for a discussion of the map.

(b) Note 7.6 is consistent with 6.9. In particular when $\ell = 2$, 7.6–or Bokstedt's work cited above–"explains", from a homotopy-theoretic viewpoint, the "extra" factor of 2 in $K_3\mathbb{Z} \cong \mathbb{Z}/48$. "Extra" factors of two commonly arise in topological K-theory, merely bacause the natural maps $BO \to BU$ and $BU \to BO$ induce isomorphisms half the time and multiplication by 2 the rest of the time in degrees $\equiv 0 \mod 4$. That is exactly what happens here. $JK(\mathbb{Z})$ (for $\ell = 2$) is the fibre of the composite $BO \to BU \xrightarrow{\psi^3-1} BU$, and hence $\pi_{8k+3}JK(\mathbb{Z}) \cong \mathbb{Z}/16$.

(c) 7.6 holds for the associated spectra as well. Thus if we define $jk(\mathbb{Z})$ by the homotopy fibre square

$$\begin{tikzcd} jk(\mathbb{Z}) \arrow[r] \arrow[d] & bo \arrow[d] \\ K\mathbb{F}_p \arrow[r] & bu \end{tikzcd}$$

so that $\Omega_0^\infty jk(\mathbb{Z}) = JK(\mathbb{Z})$, we have $K^{\text{ét}}\,\mathbb{Z}[\frac{1}{\ell}]\hat{} \cong jk(\mathbb{Z})\hat{}$ for ℓ regular. Note that for ℓ odd, $jk(\mathbb{Z}) \cong K\mathbb{F}_p \vee \Sigma bo$, and for all ℓ, $jk(\mathbb{Z})$ is the fibre of $\psi^p - 1 : bo \to \Sigma^2 bu$.

7.8. Corollary. *There is a map $K\mathbb{Z}[\frac{1}{\ell}]\hat{} \xrightarrow{h} jk(\mathbb{Z})\hat{}$, such that for ℓ regular the Lichtenbaum-Quillen conjecture for $\mathbb{Z}[\frac{1}{\ell}]$ holds if and only if h is an equivalence.*

7.9. Remark. Maps from $K\mathbb{Z}[\frac{1}{\ell}]$ to $jk(\mathbb{Z})\hat{}$ can be constructed without using étale K-theory, as follows: choose an embedding $i : \mathbb{Z}_p \hookrightarrow \mathbb{C}$. This yields a commutative diagram of rings

$$\begin{tikzcd} \mathbb{Z}[\frac{1}{\ell}] \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathbb{Z}_p \arrow[r] & \mathbb{C} \end{tikzcd}$$

and hence a *strictly* commutative diagram of spectra

$$\begin{tikzcd} KZ[\frac{1}{\ell}] \arrow[r] \arrow[d] & bo \arrow[d] \\ K\mathbb{Z}_p \arrow[r] & bu \end{tikzcd}$$

By Suslin's theorem 4.7b, $K\mathbb{Z}_p\hat{}$ is canonically equivalent to $K\mathbb{F}_p\hat{}$. This yields a map $K\mathbb{Z}[\frac{1}{\ell}]\hat{} \to jk(\mathbb{Z})\hat{}$. However the map depends on the choice of embedding i. When $\ell = 2$ the choice is not important, but for ℓ odd it is essential to choose i carefully to get an interesting map–in particular, to get 7.8 (Dwyer-Friedlander). We should also remark that the construction above doesn't really avoid the étale theory; the latter is merely hidden in Suslin's theorem. Bokstedt's original construction also depended on étale homotopy theory.

Let us now contemplate Conjecture 7.3 (LQCII) in the light of Theorem 7.4. In that situation, LQCII is evidently equivalent to the assertion that $K_*(X;\mathbb{Z}/\ell^\nu)$ has Bott periodicity as in 7.4–at least in sufficiently high degrees. This suggests studying the localized theory defined by formally inverting the Bott element. Now recall from §3 that for any X, with or without roots of unity, we can define Bott elements $\beta_\nu \in K_*(X, \mathbb{Z}/\ell^\nu)$ (if $\ell = 2, \nu \geq 2$). Choose such a β and form the mapping telescope $\beta^{-1}KX \wedge M\mathbb{Z}/\ell^\nu$. This spectrum is independent of the choice of β. Its homotopy groups are precisely $\beta^{-1}K_*(X, \mathbb{Z}/\ell^\nu)$. In particular it is, of course, non-connective. The following remarkable theorem plays a crucial role in later sections.

7.10. Theorem. [Thomason 1985] *Let X be a nice scheme. If $\ell = 2$, assume $\sqrt{-1} \in X$. Then the Lichtenbaum-Quillen conjectures hold for $\beta^{-1}KX \wedge M\mathbb{Z}/\ell^\nu$. That is, there is a descent spectral sequence as in 7.1, converging to $\beta^{-1}K_*(X, \mathbb{Z}/\ell^\nu)$.*

For the precise list of hypotheses summarized by the work "nice" in 7.10, see Theorem 4.1, p.516 of [Thomason 1985]. In particular X should be Noetherian, regular, and of finite Krull dimension, and as usual $\frac{1}{\ell} \in X$. There are further technical hypotheses, but these are satisfied by any scheme the reader is likely to think of. The only serious restriction is the usual one: $\sqrt{-1} \in X$ when $\ell = 2$. This is annoying since for $\ell = 2$ it eliminates $\mathbb{Z}[\frac{1}{2}]$, $\mathbb{Q}$, totally real number fields, etc. One hopes that this assumption will some day be removed.

7.11. Remarks.

(a) In view of 7.4, 7.10 can be neatly summarized by the assertion that $\beta^{-1}KX \wedge M\mathbb{Z}/\ell^\nu \to \beta^{-1}K^{\text{ét}}X \wedge M\mathbb{Z}/\ell^\nu$ is an equivalence.

(b) Note that in 7.10 the spectral sequence occupies an entire half-plane.

(c) Localizations of the sort considered in 7.10–i.e., formed by inverting an element in a ring spectrum–were studied extensively in [Snaith 1983], and applied to algebraic K-theory. For example, Snaith showed that if $\beta \in \pi_2^s \mathbb{C}P_+^\infty$ is a generator, $\beta^{-1}\mathbb{C}P_+^\infty \cong KU$, and used this to deduce a similar result for BGL_1R_+, where R is a suitable strict Hensel ring. For an application pre-dating 7.10, see [Dwyer-Friedlander-Snaith-Thomason]. The equivalence $\beta^{-1}\mathbb{C}P_+^\infty \cong KU$ also illustrates the fact that such localizations usually drastically alter the homotopy type of the given ring spectrum (for another example, see §8.8).

The proof of 7.10 is, unfortunately, rather difficult to summarize. The first step is to reduce descent for a scheme to descent for all of its local rings. Then one shows that descent for a local ring follows from descent for its residue field. These two steps are carried out in a very general setting, and do not require inverting the Bott element. Inversion of β is necessary only at the final step: descent for fields. The argument uses, among other things, the Kummer isomorphism 5.3a and a homotopy-theoretic realization of a spectral sequence of Tate (see [Serre]). The reader should consult Thomason's survey article [Thomason 1989] for further enlightenment; the truly daring can also attempt the one hundred and sixteen pages of [Thomason 1985].

We are going to show (§11), that inverting the Bott element amounts to forming the "Bousfield localization with respect to complex K-theory," denoted $L_{KU}KX$. This reformulation is extremely convenient. It eliminates the choice of β, the integer ν and even, if desired, the prime ℓ from the definitions; these can be replaced by a global integral funtor L_{KU} defined on all spectra. More importantly, it brings algebraic K-theory squarely into contact with state-of-the-art stable homotopy theory. This is the subject of the next four sections.

8. Complex Cobordism, Brown-Peterson Cohomology and the Morava K-Theories

In this section we give a brief survey of some relevent aspects of stable homotopy theory. For further details and references we suggest [Ravenel 1986], especially Chapter 4 and Appendix 2.

Our story begins with complex cobordism. The complex cobordism ring MU_* is the ring of cobordism classes of weakly complex manifolds-that is, smooth manifolds with a complex structure on the stable normal bundle. Let $MU(k)$ denote the Thom space of the universal bundle over $BU(k)$. Then the fundamental work of Thom shows that $MU_n \cong \pi_{n+2k}MU(k)$, $k \gg 0$. In other words, if MU is the spectrum whose $2k$-th space is $MU(k)$, $MU_n \cong \pi_n MU$. The spectrum MU is a commutative associative ring spectrum and so defines a multiplicative generalized cohomology theory. This theory has a geometric interpretation, and has applications to the topology of complex manifolds. However it can also be viewed purely as a cohomology theory, to be used as a tool for attacking homotopy-theoretic problems. In this respect complex cobordism turns out to be an extremely powerful theory. It has an amazingly rich and deep structure, and has played a central role in stable homotopy theory over the last twenty years. The first problem is of course to compute the coefficient ring.

8.1. Theorem. (Milnor) *$\pi_* MU \cong \mathbb{Z}[x_1, x_2, \dots]$, where $|x_i| = 2i$.*

Remark. The generators x_i can in fact be taken to be complex projective varieties. However there is no good canonical choice of the x_i's, which makes calculations cumbersome.

Over the next few years there were a number of applications of MU^* to homotopy-theoretic problems. Then the subject was revolutionized by the work of Quillen on complex cobordism and formal group laws. The mere fact that formal groups arise is not deep or surprising, as we now explain. Let E be a commutative ring spectrum. A *complex orientation* on E is a class $z^E \in E^2\mathbb{C}P^\infty$ such that z^E restricts to the identity element of E^2S^2. For example ordinary cohomology $H\mathbb{Z}$, complex K-theory KU, and MU all have natural complex orientations. Complex-oriented theories behave in many respects like ordinary cohomology. For example, $E^*\mathbb{C}P^\infty_+ \cong E^*[[z^E]]$, there are Chern classes c_n^E satisfying the usual axioms, Thom isomorphisms for complex vector bundles, etc. The big difference comes when we ask for a formula for the first Chern class of a tensor product of line bundles. In ordinary cohomology we of course have $c_1(\lambda_1 \otimes \lambda_2) = c_1(\lambda_1) + c_1(\lambda_2)$. For a general E we have $c_1^E(\lambda_1 \otimes \lambda_2) = F(c_1^E(\lambda_1), c_1^E(\lambda_2))$, where $F(x,y)$ is a formal power series in $E^*[[x,y]] = E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$. Since tensor product of line bundles is associative, commutative and has the trivial bundle as identity, $F(x,y)$ is a (commutative, one-dimensional) formal group law.

Now it is trivial to show that there is a universal formal group law. That is, there is a commutative ring L-the Lazard ring and a formal group law F^L over L with the following universal property:

8.2. Given any formal group F over a commutative ring R, there is a unique ring homomorphism $L \xrightarrow{\varphi} R$ such that $\varphi_*(F^L) = F$.

The ring L can be given a natural grading. But what sort of ring is it?

8.3. Theorem. (Lazard) *$L \cong \mathbb{Z}[y_1, y_2, \dots]$, where $|y_i| = 2i$.*

Now MU also has a universal property:

8.4. Theorem. *Let E be a complex oriented ring spectrum, with orientation class z^E. Then there is a unique map of ring spectra $MU \to E$ carrying z^{MU} to z^E.*

Comparing 8.1, 8.2, 8.3, and 8.4, it requires only a leap of faith to believe the amazing:

8.5. Theorem. (Quillen) *Let $\varphi : L \to \pi_* MU$ denote the unique ring homomorphism induced by F^{MU}. Then φ is an isomorphism.*

Theorem 8.5 and its consequences have dominated the subject ever since. The main point is that the highly developed theory of formal groups can now be systematically applied. We next explain how it applies to Brown-Peterson cohomology and Morava K-theories.

First, an analogy: Thom's computation of the unoriented cobordism ring $\pi_* MO$ includes as a key step the computation of $H^*(MO, \mathbb{Z}/2)$ as a module over the mod 2 Steenrod algebra. He showed it is a free module, which implies that the spectrum MO is a wedge of Eilenberg-Maclane spectra $H\mathbb{Z}/2$. From our present point of view this is something of a disappointment: it shows that the cohomology theory MO contains the same information as mod 2 cohomology, but in a grossly redundant form. Now fix an arbitrary prime p and let A denote the mod p Steenrod algebra. Let $\beta \in A^1$ denote the Bockstein. Then $H^*(MU; \mathbb{Z}/p)$ is a free module over $A/A\beta A$. This suggests that after localizing at p, MU might split as a wedge of copies of a ring spectrum X with $H^*(X, \mathbb{Z}/p) = A/A\beta A$. It isn't at all obvious that such a spectrum even exists, but nevertheless X, now known as BP, was constructed by Brown and Peterson. The homotopy of BP is a polynomial algebra $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$, where $|v_i| = 2(p^i - 1)$. Thus BP is much "smaller" then MU, and yet carries the same p-primary information, and so should be a more efficient theory. And in contrast to the unoriented case cited above, BP is nothing like ordinary cohomology; it is a new and extraordinarily powerful theory.

Now although BP has a smaller coefficient ring than MU, the description of $\pi_* BP$ given above still suffers from the lack of a canonical choice of the generators v_i - indeed the Brown-Peterson construction itself was non-canonical. A beautiful, canonical construction of BP is obtained from the theory of formal groups as follows: Let R be a torsion-free ring. Any formal group F over R is isomorphic over $R \otimes \mathbb{Q}$ to the additive group $G_a(x, y) = x + y$. The isomorphism is denoted $\log_F$; it is a power series with coefficients in $R \otimes \mathbb{Q}$. Then F is *p-typical* for a prime p if $\log_F(x) = \sum_{i \geq 0} m_i x^{p^i}$. This notion can be extended to arbitrary R, and by a theorem of Cartier every formal group over R is canonically isomorphic to a p-typical formal group. Combining this theorem with Quillen's theorem leads to a canonical idempotent map of ring spectra $MU_{(p)} \xrightarrow{e} MU_{(p)}$, and hence to a splitting

$MU_{(p)} = eMU_{(p)} \vee (1-e)MU_{(p)}$. The spectrum $eMU_{(p)}$ is BP. In fact this is an independent construction of BP that is now usually taken as the definition. Thus we have a canonical map of ring spectra $MU_{(p)} \to BP$, and $\pi_* BP$ is the p-typical Lazard ring $\mathbb{Z}_{(p)}[v_1, v_2, \ldots]$. Here the v_i's are the *Araki generators*, namely, the coefficients of the formal sum expansion $[p]_F(x) = \sum_{n\geq 0}^F v_n x^{p^n}$, where F is the universal p-typical formal group and $v_0 = p$. This canonical choice of the generators v_n leads to good, explicit formulae that have had many applications in stable homotopy theory; see [Ravenel 1986].

Suppose now R is a ring of characteristic p. Then any formal group law has $[p]_F(x) = a_n x^{p^n} +$ higher terms for some n, $1 \leq n \leq \infty$ ($n = \infty$ is the case $[p]_F(x) = 0$). The integer n is obviously an isomorphism invariant and is called the *height* of F: $htF = n$. For example, the additive law has infinite height and the multiplicative law $x + y + xy$ has height one. If R is a separably closed field of characteristic p, a theorem of Lazard states that the height actually classifies formal groups over R up to isomorphism. In any case one can ask whether there are complex-oriented cohomology theories that realize various "height n Lazard rings." For example, the ring $\mathbb{F}_p[v_n, v_n^{-1}]$ is obviously universal for formal groups F with $[p]_F(x) = ax^{p^n}$, where a is a unit. This ring can indeed be realized: for each fixed prime p and each n, $1 \leq n < \infty$, there is a ring spectrum $K(n)$, the *nth Morava K-theory*, with $\pi_* K(n) = \mathbb{F}_p[v_n, v_n^{-1}]$. Furthermore there is a map of ring spectra $BP \to K(n)$ inducing the obvious map on homotopy. $K(n)$ is a commutative ring spectrum except when $p = 2$. However the noncommutativity when $p = 2$ rarely causes any significant problems, and will generally be ignored in the sequel. We define $K(0) = H\mathbb{Q}$ (rational cohomology) and $K(\infty) = H\mathbb{Z}/p$. The $K(n)$ have a number of pleasant properties:

8.6. Theorem. *Fix p. Then for all n, $0 \leq n \leq \infty$:*

(a) *$\pi_* K(n)$ is a graded field – i.e. every nonzero homogeneous element is invertible, and every graded module is free.*

(b) *If X is any spectrum, $K(n) \wedge X$ is a wedge of suspensions of $K(n)$.*

(c) *$K(n)_*$ satisfies the Künneth theorem:*

$$K(n)_* X \otimes_{K(n)_*} K(n)_* Y \cong K(n)_*(X \wedge Y)$$

(d) *If $m \neq n$, $K(m) \wedge K(n)$ is contractible.*

Here (b) and (c) follow easily from (a). Part (d) follows from the easy fact that for $n < m$, v_n is in the kernel of the $K(m)$-Hurewicz map $\pi_* BP \to K(m)_* BP$.

The construction of $K(n)$ uses the Baas-Sullivan method of manifolds with singularities. The idea is that one can modify complex cobordism by allowing specified manifolds with cone-like singularities. If we allow the cone on M, obviously M is as dead as a doornail in the modified cobordism ring (if it *is* a ring-but we are ignoring such technicalities). In this way, we can kill off arbitrary generators of $\pi_* MU$. In particular we can kill off everything (including p) except v_n, yielding a connective ring spectrum $k(n)$ with $\pi_* k(n) = \mathbb{F}_p[v_n]$. Multiplication by v_n yields a map $S^{2(p^n-1)} \wedge k(n) \xrightarrow{f} k(n)$. The mapping telescope of f (i.e., the direct limit of $k(n) \xrightarrow{f} k(n) \xrightarrow{f} k(n) \xrightarrow{f} \dots$) is $K(n)$.

8.7. Remarks.

(a) The generator v_n in $K(n)_*$ can be thought of as the Fermat hypersurface of degree p in $\mathbb{C}P^{p^n}$. This fact is primarily useful for creating the illusion that one is doing geometry.

(b) By a theorem of Adams, p-local complex K-theory splits as a wedge of $(p-1)$ copies of a spectrum $E(1)$ with $\pi_* E(1) = \mathbb{Z}_{(p)}[v_1]$, where $|v_1| = 2(p-1)$. $K(1)$ is just the mod p reduction of $E(1)$, and so in particular is a wedge summand of mod p complex K-theory. We also remark that Adams' proof involved p-typifying the formal group law $x + y + \beta xy$ of KU, and apparently was the inspiration for Quillen's construction of BP.

(c) Although it is irrelevant for the purposes of this paper, it would be criminal not to mention the beautiful insight of Morava's that has made his K-theories so powerful: Briefly, the ring E^*E of cohomology operations of a complex-oriented theory E tends to be related to the automorphism group Γ of the associated formal group law F^E. (This is even true for $E = H\mathbb{Z}/p$, where E^*E is the Steenrod algebra and F^E is the additive law; see e.g. [Ravenel 1986], p.378.) When $E = K(n)$ and we extend scalars to $\mathbb{F}_{p^n}$, the group Γ is essentially the p-adic Lie group of units in the maximal order of the division algebra over $\mathbb{Q}_p$ with Hasse invariant $1/n$. Hence the group cohomology of Γ is related to suitable Ext groups of $K(n)^*K(n)$, which in term feed into the chromatic spectral sequence [Miller-Ravenel-Wilson], which converges to the E_2-term of the Adams-Novikov spectral sequence for the stable homotopy groups of spheres. For more details of this remarkable story, again see [Ravenel 1986].

Brown-Peterson cohomology and the Morava K-theories have been used to detect periodic families of elements in the stable homotopy groups of spheres. We conclude this section by explaining roughly how this works. In particular we discuss the Adams map, which plays a crucial role in later sections.

Adams showed there is a map $\Sigma^d M\mathbb{Z}/p \xrightarrow{A} M\mathbb{Z}/p$ inducing an isomorphism on $K(1)_*$, where $d = 2p-2$ for p odd and $d = 8$ for $p = 2$. In fact the induced map on $BP_*M\mathbb{Z}/p = BP_*/p$ is multiplication by v_1 (p odd) or $v_1^4 (p = 2)$. Note A is necessarily nonnilpotent. When p is odd, A can be defined as follows: let α_1 be a fixed generator of $\pi_{2p-3}S^0 \cong \mathbb{Z}/p$, and let $r : M\mathbb{Z}/p \to S^1$ denote the pinch map to the top cell. Then there is a unique *Adams element* $\widetilde{A} \in \pi_{2p-2}M\mathbb{Z}/p$ such that $r\widetilde{A} = \Sigma\alpha_1$. Now since p is odd, $M\mathbb{Z}/p$ is a commutative ring spectrum, associative if $p > 3$. (This is a trivial exercise. The fact that $M\mathbb{Z}/3$ is nonassociative is much more subtle, but rarely causes any significant problems in our context.) We then define $A =$ multiplication by $\widetilde{A}$. Note that by definition, the composite

$$S^{2p-2} \hookrightarrow S^{2p-2} \wedge M\mathbb{Z}/p \xrightarrow{A} M\mathbb{Z}/p \xrightarrow{r} S^1$$

is α_1. More generally, the composite

$$S^{k(2p-2)} \hookrightarrow S^{k(2p-2)} \wedge M\mathbb{Z}/p \xrightarrow{A^k} M\mathbb{Z}/p \xrightarrow{r} S^1$$

is α_k, a generator of the elements of order p in the image of J. Furthermore, the localized groups $A^{-1}\pi_*M\mathbb{Z}/p$ $(= \widetilde{A}^{-1}\pi_*M\mathbb{Z}/p$ for p odd) have been explicitly computed by Mahowald ($p = 2$) and H. Miller (p odd). For example, when p is odd we have

8.8. Theorem. (Miller) $A^{-1}\pi_*M\mathbb{Z}/p \cong \mathbb{F}_p[\widetilde{A}, \widetilde{A}^{-1}]\langle\partial\widetilde{A}\rangle$.

One can hope to generalize the preceeding constructions as follows. Suppose there is a finite spectrum $V(n)$ with $BP_*V(n) = BP_*/(v_0, \dots, v_{n-1})$. Such a spectrum would in particular have the properties (i) $K(n)_*V(n) = 0$ for $m < n$, and (ii) $K(n)_*V(n) \neq 0$. Suppose further that $V(n)$ admits a selfmap (analogous to the Adams map) $f : \Sigma^d V(n) \to V(n)$ inducing multiplication by v_n on BP_* and hence inducing an isomorphism on $K(n)_*$. Note every iterate of f is essential. One could then construct "v_n-periodic" families in π_*S^0 by using the composites

$$S^{kd} \to \Sigma^{kd}V(n) \xrightarrow{f^k} V(n) \to S^m,$$

where the first and last maps are inclusion of the bottom cell and pinch to the top cell, respectively. Of course there is no guarantee that the composite is essential, but the chromatic spectral sequence machinery mentioned in Remark 8.7c is designed precisely for this kind of detection problem. Thus one may hope to sort the stable homotopy groups into such v_n-periodic families, with the image of J corresponding to $n = 1$. One could even hope to generalize the Mahowald-Miller theorems by calculating $f^{-1}\pi_*V(n)$. Unfortunately, $V(n)$ does not exist in general. For example if $n = 1$ and p is odd, we can take $V(1) =$ cofibre of the Adams map, but there is no $V(1)$ for $p = 2$. If $V(1)$ has a self-map f as above, we can of course take $V(2) =$ cofibre of f, and so on, but the problem is a very difficult one. At present the evidence for small n suggests that for each n, $V(n)$ exists provided p is sufficiently large. However one could ask instead for a finite spectrum with the weaker properties (i) and (ii) above, and require only that the selfmap f induce an isomorphism on $K(n)_*$. This will be considered in the next section.

9. The Prime Spectrum of the Stable Homotopy Category

This section is for the most part just a summary of portions of Mike Hopkins' beautiful paper "Global methods in homotopy theory" [Hopkins]. The reader should certainly consult the original source as well, and should blame only the present author for any misleading assertions in what follows.

9.1. Theorem. [Devinatz-Hopkins-J. Smith] *Let E be an arbitrary ring spectrum, not necessarily associative or commutative. If $\alpha \in \pi_*E$ and $MU_*\alpha = 0$, α is nilpotent.*

9.2. Corollary. (Ravenel's Nilpotence Conjecture). *Let X be a finite spectrum, $f : \Sigma^d X \to X$ a self-map. If $MU_*f = 0$, f is nilpotent–i.e. some iterate of f is nullhomotopic.*

The corollary follows by applying 9.1 to the ring spectrum $X \wedge DX$, where X is the Spanier-Whitehead dual of X. If we work with p-local spectra, MU can be replaced by BP in 9.1 and 9.2. There is also the following variant of 9.1:

9.3. Theorem. [Hopkins-J. Smith] *Let E be a p-local ring spectrum. If $\alpha \in \pi_*E$ and $K(n)_*\alpha = 0$ for all n, $0 \le n \le \infty$, α is nilpotent.*

One of the most remarkable consequences of the nilpotence theorem is the complete determination of the "prime ideal spectrum" of the stable homotopy category. What is a prime in the stable homotopy category? A quick, if rather crude, explanation is to note that the prime ideals of a commutative ring R are in bijective correspondence with the "prime fields" over R – that is, the residue fields k_p of R. For example if $R = \mathbb{Z}$ we mean prime fields in the usual sense: $\mathbb{Z}/p$ or $\mathbb{Q}$. Note that a commutative $\mathbb{Z}$-algebra A is a field if and only if for every abelian group M, $A \otimes_{\mathbb{Z}} M$ is a direct sum (possibly empty) of copies of A. Such a field is a prime field if and only if it is indecomposable as an abelian group. Hence:

9.4. Definition. (Hopkins) Let E be a ring spectrum. E is a *field* if for every spectrum X, $E \wedge X$ is a wedge (possibly empty) of copies of suspensions of E. A field E is a *prime field* if E is indecomposable as a spectrum – i.e. $E \cong Y \vee Z$ implies Y or Z trivial.

Note. E is not assumed associative or commutative in 9.4. However we will see shortly that the prime fields are always associative, and always commutative if p is odd.

Examples.

(a) $K(n)$ is a prime field, $0 \leq n \leq \infty$. (By 8.6b, one only has to check that $K(n)$ is indecomposable.) Note also that 8.6d is what one would expect for prime fields.
(b) $KU \wedge M\mathbb{Z}/p$ is a field, which is not prime if p is odd.

9.5. Theorem. (Hopkins-J. Smith) *Let E be a prime field. Then $E \cong K(n)$ for some n, $0 \leq n \leq \infty$.*

The idea is that $E \wedge K(n)$ is nontrivial for some n by 9.3, since the identity in $\pi_0 E$ is nonnilpotent. Hence $E \wedge K(n)$ is simultaneously a wedge of copies of E and a wedge of copies of $K(n)$, which at least makes 9.5 plausible.

Thus if we write $\mathcal{S}$ for the stable homotopy category, we are at least morally justified in writing $\operatorname{Spec} \mathcal{S}$ for the set $\{(p, n)$: p an ordinary prime, $0 \leq n \leq \infty\}$ modulo the identifications $(p, 0) \sim (q, 0)$ for all p, q. Here the "integer" n, $1 \leq n \leq \infty$, ultimately corresponds to the height invariant of formal group laws in characteristic p (§8). Writing $\mathcal{S}_p$ for the p-local stable homotopy category, we may similarly write $\operatorname{Spec} \mathcal{S}_p = \{n : 0 \leq n \leq \infty\}$. However, this approach is not entirely convincing; for example, it doesn't reveal the Zariski topology. Recall the following characterization of the closed sets in $\operatorname{Spec} A$, A a commutative ring. If M is an A-module, the *support* of M is the set $\operatorname{Supp} M = \{p \in \operatorname{Spec} A : M_{(p)} \neq 0\}$. If M is finitely generated this is the same as $\{p \in \operatorname{Spec} A : M \otimes_A k_p \neq 0\}$. The closed sets are precisely the sets of the form $\operatorname{Supp} M$, M finitely-generated. The analogous definition in our setting is now obvious: if X is a finite spectrum (p-local, as usual), we let $\operatorname{Supp} X = \{n : K(n) \wedge X \ncong *, 0 \leq n \leq \infty\}$, declare closed the subsets of $\{0, 1, \dots, \infty\}$ of the form $\operatorname{Supp} X$, and take the topology this generates on $\operatorname{Spec} \mathcal{S}$.

9.6. Theorem. *If X is a finite spectrum, $K(n)_* X \neq 0$ for some $n < \infty$. Furthermore, if $K(n)_* X \neq 0$ then $K(m)_* X \neq 0$ for all $m > n$.*

Remark. It follows trivially from the Atiyah-Hirzebruch spectral sequence that $K(n)_*X \neq 0$ for all sufficiently large n. The second assertion of 9.6. is more difficult, see [Ravenel 1984], Theorem 2.11.

Thus the nonempty closed sets all have the form $[n, \infty]$ for some n. However this still leaves the question: which intervals $[n, \infty]$ actually occur? Let us say that a finite spectrum X has *type n* if $K(n)_*X \neq 0$ and $K(m)_*X = 0$ for $m < n$. For example, S^0, $M\mathbb{Z}/p$, and the cofibre of the Adams map have type $0, 1, 2$, respectively. The question is whether such complexes exist in general. The hypothetical spectra $V(n)$ discussed in §8 would have type n, but do not exist, in general.

9.7. Theorem. [Mitchell 1985] *Let p be any prime. Then for all n, $0 \leq n < \infty$, there exists a finite spectrum X_n of type n.*

The construction of X_n can be briefly described as follows, taking p odd. Embed $(\mathbb{Z}/p)^n$ in the unitary group $U(p^n)$ via the regular representation. This extends to an embedding of the affine group $GL_n\mathbb{F}_p \tilde{\times} (\mathbb{Z}/p)^n$ and hence $GL_n\mathbb{F}_p$ acts on the homogeneous space $Y_n = U(p^n)/(\mathbb{Z}/p)^n$. Hence the group ring $\mathbb{Z}_{(p)}GL_n\mathbb{F}_p$ acts on the p-localized suspension spectrum of Y_n, and idempotents in this group ring can be used to split Y_n into various wedge summands. A twisted form of the famous Steinberg idempotent yields a wedge summand X_{n-1}.

This completes our description of the prime spectrum of the stable homotopy category: it can be visualized as an infinite "comb", with teeth corresponding to the ordinary primes and linked by the zero ideal. Each tooth supports an infinite sequence of primes (p, n) with "residue field" $K(n)$. Lurking beyond the end of each tooth is a "prime at infinity", $H\mathbb{Z}/p$. The closed subsets of a tooth are the intervals $[n, \infty]$.

Remark. The preceeding discussion is not meant to be in historical order. In particular, it was only after the fact that the author was informed, by Jack Morava via Haynes Miller, that he had proved "Euclid's theorem for stable homotopy".

9.8. Remark. Another construction of spectra of type n was discovered by Jeff Smith. The construction is similar in spirit to 9.7, but involves representations of the symmetric groups rather than the general linear groups. Smith's construction has the advantage that it can be easily modified to produce complexes that admit a "v_n-map"; see below.

9.9. Remark. There is an even more elegant classification of "primes" in terms of categories of finite spectra that are "closed under cofibrations and retracts"; see [Hopkins].

Let X be a finite spectrum. A v_n-map ($n \geq 1$) is a map $f : \Sigma^d X \to X$ such that $K(n)_* f$ is an isomorphism and $K(m)_* f$ is nilpotent for $m \neq n$. It is clear that some iterate f^k is zero on $K(m)_*$ for all $m \neq n$, and we may assume v_n-maps already have this form.

9.10. Theorem. [Hopkins-J. Smith], [Hopkins]. *Let X be a finite spectrum of type n. Then X admits a v_n-map f. Furthermore (a) f^k is central in $[X, X]_*$ for some k and (b) any two such maps are isogenous: if f, g are v_n-maps of X, $f^i = g^j$ for some i, j.*

9.11. Remark. Let R be a finite ring spectrum of type n. A *v_n-element* $\alpha \in \pi_* R$ is a class such that $K(n)_* \alpha$ is a unit and $K(m)_* \alpha$ is nilpotent for $m \neq n$. These exist for all n: just choose (X, f) as in 9.10, take $R = X \wedge DX$, and use the isomorphism $\pi_* X \wedge DX \cong [X, X]_*$.

9.12. Remark. The proof of 9.10 relies heavily on the nilpotence theorem. However the first step is to construct just one example (X, f) for each n. This depends on Smith's construction but not on the nilpotence theorem.

We will call a pair (X, f) as in 9.10 a v_n-complex. We may then define *v_n-periodic homotopy with coefficients in X* as $f^{-1}[X, E]_*$. This is independent of the choice of f by 9.10b. We conclude by recording some trivial properties of this construction that will be used frequently in §11, 12:

9.13. Proposition.

(a) *If (X, f) is a v_n-complex, so is (DX, Df), and $f^{-1}[X, E]_* = (1 \wedge Df)^{-1} \pi_*(E \wedge DX)$.*

(b) *Suppose $f^{-1}[X, E]_* = 0$ for some v_n-complex (X, f). Then $K(n)_* E = 0$.*

(c) *If E is bounded above–i.e. $\exists N$ such that $\pi_n E = 0$ for $n \geq N$-then $f^{-1}[X, E]_* = 0$.*

Here (b) follows from (a) and the Künneth theorem, while (c) is a triviality valid for any connective spectrum X with selfmap $f : \Sigma^d X \to X$, $d > 0$.

10. Bousfield Localization

Much of the material in this section is taken from [Ravenel 1984]. In the stable world, localization in the classical sense is easy. For example, let $S = \{p_1, p_2, \dots\}$ be a set of primes. For any abelian group A, $S^{-1}A$ is the direct limit of the sequence $A \xrightarrow{p_1} A \xrightarrow{p_1 p_2} A \xrightarrow{p_1 p_2 p_3} A \to \dots$. If X is a spectrum, $S^{-1}X$ can be defined in exactly the same way. It is immediate that $\pi_*(S^{-1}X) = S^{-1}\pi_*X$. In particular we can define the rationalization $X_\mathbb{Q}$ and the localization $X_{(p)}$, which we've been using throughout this paper, in this way. The spectrum $X_{(p)}$ retains exactly the information about X which is visible to p-local homotopy. Now let E be another spectrum. The *Bousfield localization of X with respect to E*, denoted $L_E X$, is a spectrum that retains exactly the information about X which is visible to E. More precisely, call a spectrum W *E-acyclic* if $E_*W = 0$. Then a spectrum X is said to be *E-local* if $[W, X] = 0$ for every E-acyclic W, and we have:

10.1. Theorem. [Bousfield 1979] *Let E and X be arbitrary spectra. Then there exists a spectrum $L_E X$ and a map $X \xrightarrow{j} L_E X$, natural in X, such that*

(a) *j is an isomorphism on E_*.*
(b) *$L_E X$ is E-local.*

10.2. Remark. L_E is an idempotent functor on the stable homotopy category, and annihilates E-acyclic spectra. The existence of such a localization functor was considered earlier by Adams; see [Adams 1974]. Unfortunately, Adams' approach runs into set-theoretic difficulties. Bousfield's approach runs into the same difficulties, but these are overcome by a series of ingenius arguments.

10.3. Examples.

(a) Take $E = S_{(p)}$ or $S_\mathbb{Q}(= H\mathbb{Q})$. Then $L_E X$ is $X_{(p)}$ or $X_\mathbb{Q}$, respectively.
(b) $E = M\mathbb{Z}/p$. Then $L_E X$ is $X\hat{}$, the p-completion of X. In fact for us, this is the definition of $X\hat{}$. An alternative definition is $X\hat{} = \underleftarrow{holim}\, X \wedge M\mathbb{Z}/p^n$. It is an easy exercise to show this X satisfies 10.2a, b. It also follows that if X has finitely-generated homotopy groups, $\pi_*X \to \pi_*X\hat{}$ is just p-completion. $X\hat{}$ can also be described as the "function spectrum" $F(M\mathbb{Z}/p^\infty, \Sigma X)$. Here $M\mathbb{Z}/p^\infty$ is the cofibre of $S^0 \to S^0_\mathbb{Q}$ (as usual, S^0 means $S^0_{(p)}$). This leads to a functorial short exact sequence $0 \to \operatorname{Ext}(\mathbb{Z}/p^\infty, \pi_n X) \to \pi_n X\hat{} \to \operatorname{Hom}(\mathbb{Z}/p^\infty, \pi_{n-1}X) \to 0$ for all X. Note this means that $\mathbb{Q}/\mathbb{Z}$'s in $\pi_n X$ disappear from $\pi_n X\hat{}$ but show up as copies of the p-adic integers in $\pi_{n+1}X\hat{}$. This happens frequently in algebraic K-theory; cf. remark (b) following 4.3. Finally, note the $M\mathbb{Z}/p$-acyclic spectra are precisely the spectra with uniquely p-divisible homotopy groups.

Bousfield showed that if E is connective, $L_E X$ is essentially an ordinary arithmetic localization or completion functor as in the preceeding examples. When E is nonconnective, however, fascinating new phenomena arise. For example, consider $E = KU$: K-theoretic localization. A great deal is now known about this case; the following results are the most important for us:

10.4. Theorem. [Bousfield 1979] $L_{KU}M\mathbb{Z}/p = A^{-1}M\mathbb{Z}/p$.

Here A is the Adams map (§8). For a description of $L_{KU}S^0$, see [Ravenel 1984].

10.5. Theorem. [Ravenel 1984] *For any spectrum X, $L_{KU}X = X \wedge L_{KU}S^0$. Hence $L_{KU}\ (X \wedge M\mathbb{Z}/p) = X \wedge (A^{-1}M\mathbb{Z}/p)$.*

10.6. Remark. Spectra E and F are *Bousfield equivalent* if for all spectra X, $E \wedge X \cong * \iff F \wedge X \cong *$. Clearly, E and F are Bousfield equivalent if and only if the localization functors L_E and L_F coincide. For example, it is easy to show KU is Bousfield equivalent to $K(0) \vee K(1)$.

10.7. Remark. For any spectrum E, $L_{E\wedge M\mathbb{Z}/p}X = (L_EX)\hat{}$. For example, $(L_{KU}X)\hat{} = L_{K(1)}X$. Since our applications to algebraic K-theory almost invariably involve completion, we will usually work with $L_{K(1)}X$ rather than $L_{KU}\ X$.

10.8. Theorem. [Bousfield 1987] *There is a functor T: (spaces) $\to$ (spectra) such that $L_{K(1)} = T \circ \Omega^\infty$.*

10.9. Remark. The significance of this result is that it reduces spectrum level questions about $K(1)$-local spectra to space level questions. For example, suppose X, Y $K(1)$-local and $f, g : X \to Y$. Then if $\Omega^\infty f$ and $\Omega^\infty g$ are homotopic maps of spaces, f and g are homotopic maps of spectra. We also note, for future reference, that 10.8 can often be applied to a spectrum which is only a connective cover of its $K(1)$-localization. For example, in the application just cited the assumption "X, Y $K(1)$-local" can be replaced "$X \to L_{K(1)}X$ is a (-1)-connected cover, and similarly for Y", or even by a slightly weaker assumption.

Remark. Theorem 10.8 holds for all $L_{K(n)}$, $0 < n < \infty$ [Kuhn].

Bousfield's theorem provides a systematic way of analyzing the stable homotopy category "locally". The most natural thing to do is of course to localize at the "primes." That is, we should consider $L_{K(n)}$ or in view of 9.6 we might consider $L_n \equiv L_{E\langle n\rangle}$, where $E\langle n\rangle = K(0) \vee K(1) \vee \ldots \vee K(n)$. The functors L_n assemble into the *chromatic* or *harmonic tower*:

$$
\begin{tikzcd}
 & \vdots \arrow[d] \\
 & L_2X \arrow[d] \\
 & L_1X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & L_0X
\end{tikzcd}
$$

Recall here that $L_0X = X_{\mathbb{Q}}$ and $L_1X = L_{KU}X$. The fibre F_nX of L_nX is "monochromatic": $K(m)_*F_nX = 0$ for $m \neq n$ and $K(n)_*F_nX = K(n)_*X$. In picturesque terms, one may say that the harmonic tower attempts to resolve the homotopy-type of X into various "wavelengths"; the nth-stage corresponds in some sense to the v_n-periodic homotopy discussed in §8, 9. How much information is lost? Let $E\langle\infty\rangle = \bigvee_{0\leq n<\infty} K(n)$ and let $L_\infty X = L_{E\langle\infty\rangle}X$. Call X *harmonic* if X is $E\langle\infty\rangle$-local and *dissonant* if X is $E\langle\infty\rangle$-acyclic. Thus X is dissonant if and only if its harmonic tower is trivial: $L_nX \sim *$ for all n, and hence X is completely invisible to the chromatic theory. At the opposite extreme, it is still unknown whether or not the natural map $f : X \to \operatorname{holim}_n L_nX$ is an equivalence when X is harmonic. Certainly $\operatorname{holim}_n L_nX$ is harmonic; the problem is to show f_* is an isomorphism on $K(n)_*$ for all $n < \infty$. This is true for X finite (Hopkins-Ravenel).

10.10. Examples of harmonic spectra

(a) [Ravenel 1984] BP is harmonic. Any finite spectrum is harmonic. Any connective spectrum X with $H_*(X,\mathbb{Z})$ torsion-free and of finite type over $\mathbb{Z}_{(p)}$ is harmonic. More generally any connective spectrum of finite type with $\operatorname{hom}\dim_{BP_*}(BP_*X) < \infty$ is harmonic.

(b) [Hopkins-Ravenel] Any suspension spectrum is harmonic.

(c) any $E\langle n\rangle$-local spectrum–e.g., a KU-local spectrum-is automatically harmonic.

10.11. Examples of dissonant spectra

(a) [Ravenel 1984] any spectrum X such that π_*X is torsion and bounded above is dissonant. This is true for $X = H\mathbb{Z}/p$ by 8.6d. Hence it is true for finite p-torsion Postnikov towers, and the claim follows by a limit argument. Note also that if $0 < n < \infty$, $K(n)_*X = 0$ for any spectrum X which is bounded above, and hence $K(n)_*$ is invariant under passage to connective covers $(0 < n < \infty)$.

(b) (Hopkins) Here is an example very different from (a). Let X_2 denote the cofibre of a v_1-selfmap of the Moore spectrum. By 9.10, X_2 has a v_2-selfmap f_2. Let X_3 denote the cofibre. Continuing in this way, we obtain a sequence of finite spectra $X_2 \subseteq X_3 \subseteq X_4 \ldots$ with $K(m)_*X_n = 0$ for $m < n$. Then $X = \varinjlim X_n$ is a dissonant spectrum, and is nontrivial since $H_*(X;\mathbb{Z}/p) \neq 0$.

As these examples indicate, the class of harmonic spectra is very broad and includes many familiar spectra. The class of dissonant spectra seems to be somewhat more restricted. We conclude with one more example that is helpful for understanding §11, 12.

10.12. Example. Consider the natural map $bu \to KU$. Any ring spectrum is local with respect to itself [Ravenel 1984]. Hence $KU = L_1 KU$ and KU is harmonic by 10.10d. However bu is certainly not harmonic. For if it was, it would follow from 10.11a that $bu\hat{} \to KU\hat{}$ is an equivalence, which is absurd. (If a spectrum is both harmonic and bounded above, it must be a wedge of rational Eilenberg-Maclane spectra). Instead, bu is "semi-harmonic": any map from a sufficiently highly-connected $E\langle\infty\rangle$-acyclic spectrum W into bu is trivial ((-3)-connected will do in this case).

11. Reformulation of the Lichtenbaum-Quillen Conjectures, and Some Consequences

In this section we explain an observation of [Waldhausen 1984] that allows one to reformulate the Lichtenbaum-Quillen conjectures in terms of Bousfield localization. We then explore some of the consequences of this conjecture for algebraic K-theory spectra. We continue to assume all spectra are localized at a fixed prime ℓ.

Let ℓ be an odd prime and let $\beta \in K_2(\mathbb{Z}[\xi_\ell]; \mathbb{Z}/\ell)$ denote the Bott class. Recall that $\beta^{\ell-1}$ is an element of $K_{2\ell-2}(\mathbb{Z}; \mathbb{Z}/\ell)$, also referred to as a "Bott class". We also have an "Adams class" $\widetilde{A} \in K_{2\ell-2}(\mathbb{Z}; \mathbb{Z}/\ell)$, obtained by smashing the unit map $S^0 \to K\mathbb{Z}$ with $M\mathbb{Z}/\ell$ and pushing forward the Adams class $\widetilde{A} \in \pi_{2\ell-2} M\mathbb{Z}/\ell$ (§8).

11.1. Theorem. [Snaith 1984] *$\beta^{\ell-1} = c\widetilde{A}$, where $c \in \mathbb{Z}/\ell$ is nonzero (ℓ odd).*

Proof sketch. The element β arises from a map of ring spectra $\Sigma^\infty B\mathbb{Z}/\ell_+ \xrightarrow[j]{} K\mathbb{Z}[\xi_\ell]$. This map fits into a commutative diagram

$$\begin{tikzcd} \Sigma^\infty B\mathbb{Z}/\ell_+ \arrow[r, "j"] \arrow[d, "t-\epsilon"'] & K\mathbb{Z}[\xi_\ell] \arrow[d, "\tau"'] \\ S^0 \arrow[r, "i"'] & K\mathbb{Z} \end{tikzcd}$$

where i is the unit map, t is the classical transfer, ε is the augmentation and τ is the K-theory transfer (§2). The theorem follows easily by applying $\pi_*(\ \ , \mathbb{Z}/\ell)$ to the diagram. The diagram itself is not hard to establish; see [Snaith 1984] or the slight reworking of Snaith's proof in [Mitchell 1990b]. $\square$

11.2. Remark. The relation $\beta^{\ell-1} = c\widetilde{A}$ does not hold in $\pi_* B\mathbb{Z}\ell_+ \wedge M\mathbb{Z}/\ell$. One has only that $\beta^\ell = c\beta\widetilde{A}$.

11.3. Remark. The case $\ell = 2$ has been worked out by Zaldivar (unpublished).

It follows that Bott-periodic K-theory, $\beta^{-1}K_*(\ \ ,\mathbb{Z}/\ell^\nu)$, is the same as Adams-periodic K-theory $A^{-1}K_*(\ \ ;\mathbb{Z}/\ell^\nu)$. In view of Theorem 10.4, Thomason's theorem can then be restated in terms of the functor L_1: Bousfield localization with respect to ordinary complex K-theory.

11.4. Theorem. *Let X be a nice scheme. If $\ell = 2$ assume $\sqrt{-1} \in X$. Then LQCI holds for L_1KX: there is a descent spectral sequences as in 5.12, converging to $\pi_n(L_1KX \wedge M\mathbb{Z}/\ell^\nu)$ for $n \gg 0$.*

Hence, following [Waldhausen 1984] we may reformulate the Lichtenbaum-Quillen conjectures as follows:

11.5. Conjecture. (LQCIII) If X is a nice scheme, the map $KX \to L_1KX$ is an isomorphism on π_n for $n \gg 0$. ($n \geq 1$ if $X = \operatorname{Spec} \mathcal{O}_F$).

11.6. Remark. Recall that KX is a connective spectrum. On the other hand a KU-local spectrum can't be connective unless it is a wedge of rational Eilenberg-Maclane spectra; this follows from 10.4. Hence we must at least take $n \geq 0$ in 11.5. In general, however, $n \geq 0$ is not enough, as explained in §5.

11.7. Remark. Note we have not assumed $\sqrt{-1} \in X$ when $\ell = 2$ in 11.5. Thomason's theorem requires this assumption at present, but one hopes it could be eliminated.

Conjecture 11.5 has some interesting stable homotopy-theoretic corollaries. In order to clearly separate fact from fantasy, we will label these as "consequences" to indicate that we have assumed 11.5. However we will see in §12 that a few of these consequences are now theorems.

11.8. Consequence. Let $i : S^0 \to K\mathbb{Z}$ and $i' : S^0 \to j$ denote the unit maps. Then there is a unique factorization φ:

$$\begin{tikzcd} S^0 \arrow[r, "i"] \arrow[d, "i'"'] & K\mathbb{Z} \\ j \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

Furthermore, φ is a map of ring spectra.

Proof. The fibre of i' is KU-acyclic and connective, so the existence and uniqueness of φ follows at once from 11.5. Now consider the diagram

$$\begin{tikzcd} j \wedge j \arrow[r, "\varphi \wedge \varphi"] \arrow[d] & K\mathbb{Z} \wedge K\mathbb{Z} \arrow[d] \\ j \arrow[r, "\varphi"'] & K\mathbb{Z} \end{tikzcd}$$

Clearly it commutes after precomposition with $i' \wedge i' : S^0 \wedge S^0 \to j \wedge j$. But the fibre of $i' \wedge i'$ is again KU-acyclic and connective, so the diagram commutes by 11.5 and φ is a map of ring spectra.

Since any algebraic K-theory spectrum is a module spectrum over $K\mathbb{Z}$, we conclude:

11.9. Consequence. Let X be an arbitrary ring or scheme. Then KX is a module spectrum over j.

11.10. Consequence. Let X be an arbitrary ring or scheme. Then the higher Morava K-theories of KX vanish: $K(n)_*KX = 0$, $n \geq 2$. In fact, the higher v_n-periodic homotopy groups vanish: if Y is any finite spectrum of type n (§9), $n \geq 2$, with v_n-selfmap f, $f^{-1}[Y, KX]_* = 0$.

Proof. The second statement trivially implies the first. Since $f^{-1}[Y, j] = 0$ for Y as in 11.10, it follows from 11.9 that $f^{-1}[Y, KX] = 0$. Alternatively both statements can be deduced directly from 11.5. □

There are a number of further consequences along these lines. We conclude with two more examples:

11.11. Consequence. Let $\mathcal{P} \in \operatorname{Spec} \mathcal{O}_F$ be a retractible prime (§3). Then $K\hat{\mathcal{O}_F} \to K(\mathcal{O}_F/\mathcal{P})\hat{}$ is a retraction and hence $K(\mathcal{O}_F/\mathcal{P})\hat{}$ is a wedge summand of $K\hat{\mathcal{O}_F}$

Proof. This is true on the space level (i.e. after applying Ω^∞) by the Harris-Segal theorem 3.4. Assuming 11.5, we can apply Bousfield's theorem (10.8 and Remark 10.9) to deduce that it is true on the spectrum level. □

11.12. Consequence. Let A be a finite abelian group. Then for $n \geq 2$, $[K(A, n), BGL\mathcal{O}_F^+] = 0$. □

This follows from the fact that $K(1)_*K(A, n) = 0$, $n \geq 2$. Of course 11.12 is false for $n = 1$–consider a retractible prime. However if we consider infinite loop maps, 11.12 holds for $n \geq 1$ by 10.11a. The case $n = 1$ will be discussed further in §12.

12. Some Recent Results

We have seen (§11) that if the Lichtenbaum-Quillen conjectures are true, the unit map $S^0 \xrightarrow{i} K\mathbb{Z}$ factors through j, the "image of J" spectrum. The author has recently shown that the zero-th space analogue of this assertion holds:

12.1. Theorem. [Mitchell 1990a] *There is a homotopy commutative diagram*

$$\begin{tikzcd} Q_0S^0 \arrow[r, "\Omega_0^\infty i"] \arrow[d, "\Omega_0^\infty i'"'] & BGL\mathbb{Z}^+ \\ \operatorname{Im} J \arrow[ur, "g"'] & \end{tikzcd}$$

12.2. Remark. If our fixed prime ℓ is odd, we can take $\operatorname{Im} J = BGL\mathbb{F}_p^+$, where p generates $\mathbb{Z}_\ell^*$ as usual. When $\ell = 2$, we can take $\operatorname{Im} J = BNO\mathbb{F}_3^+$ (see [Fiedorowicz-Priddy]). Here $NO_n\mathbb{F}_3$ is the subgroup of the orthogonal group $O_n\mathbb{F}_3$ consisting of matrices A such that $\det A =$ spinor norm A, as every school child knows. We remind the reader that at 2, $\pi_* \operatorname{Im} J$ consists of the classical "Image of J" *and* the Adams μ-family. Note also that $\operatorname{Im} J$ cannot be replaced with $BGL\mathbb{F}_3^+$ in 12.1, as this would contradict 3.1b.

Write "coker J" for the kernel of the map $(\Omega_0^\infty i')_* : \pi_*^s \to \pi_* \operatorname{Im} J$.

12.3. Corollary. *$i_* : \pi_*^s \to K_*\mathbb{Z}$ annihilates coker J.*

Combined with Theorem 3.1, this completely determines the map i_*.

The proof of 12.1 will be discussed later. First we show how to deduce a version of "Consequence 11.10."

12.4. Theorem. [Mitchell 1990a]. *Let X be an arbitrary ring or scheme. Then for $n \geq 2$, $K(n)_*KX = 0$.*

The proof proceeds as follows: since KX is a module spectrum over $K\mathbb{Z}$, we reduce at once to the case $KX = K\mathbb{Z}$. Fix $n \geq 2$. It is sufficient to show that there is a finite spectrum X of type n such that $f^{-1}[X, K\mathbb{Z}]_*$, the v_n-periodic homotopy of $K\mathbb{Z}$ with coefficients in X, vanishes (see §9). If we take X to be a ring spectrum as in Remark 9.11, it is in fact sufficient to show that the unit map $S^0 \to K\mathbb{Z}$ induces the zero map on $f^{-1}[X, -]_*$. Now it is easy to see that the functor $f^{-1}[X, -]_*$ is defined on spaces as well as on spectra, and that for any spectrum E, $f^{-1}[X, E] = f^{-1}[X, \Omega^\infty E]$. Since the spectrum j is the connective cover of its KU-localization $L_1 j = J$, we have $f^{-1}[X, \operatorname{Im} J] = 0$ and the theorem follows from 12.1.

Remarks.

(1) It is not hard to show that for all primes and all n, $K(n)_*$ of the *space* $BGL\mathbb{Z}^+$ is nonzero-indeed, infinite dimensional over $K(n)_*$. In general the relationship between the value of a homology theory on a spectrum and on the associated infinite loop space is quite complicated. The Hopkins-Smith complexes therefore play a crucial role, allowing one to replace homology by homotopy.

(2) The proof shows that the higher v_n-periodic homotopy vanishes, at least for some choice of the spectrum KX. At one time it was thought that this was equivalent to the vanishing of the corresponding Morava K-theories. However Ravenel has recently announced a counterexample to his "telescope conjecture." This means that the vanishing of v_n-periodic homotopy is in fact a stronger condition than 12.4.

12.5. Corollary. *For all* $n \geq 1$, $L_1KX \cong L_nKX \cong L_\infty KX$.

Proof. For any spectrum E, L_nE is harmonic and $K(n)_*L_1E = 0$ for $n \geq 2$. Hence the natural maps $L_\infty KX \to L_nKX \to L_1KX$ are isomorphisms on $K(n)_*$ for *all* $n < \infty$, and the result follows. □

12.6. Corollary. *Let X be a nice scheme, and if $\ell = 2$ assume $\sqrt{-1} \in X$. Then the Lichtenbaum-Quillen conjectures hold for the harmonic localization $L_\infty KX$: there is a descent spectral sequence 5.12 converging to $\pi_{q-p}(L_\infty KX; \mathbb{Z}/\ell^\nu)$, $q - p \gg 0$.*

Proof. This is immediate from 12.5 and Thomason's theorem 7.10. (If the assumption for $\ell = 2$ could be eliminated from Thomason's theorem, it could be eliminated here too.)

Remark. This does *not* reduce the Lichtenbaum-Quillen conjectures to showing that KX is harmonic. On the contrary, 12.5 shows that KX is definitely *not* harmonic: any connective harmonic spectrum satisfying 12.5 is a wedge of rational Eilenberg-Maclane spectra; compare example 10.12.

The significance of 12.6 is best appreciated by considering the fibre F of the map $KX \to L_1KX$. Note that F is a torsion spectrum. Clearly $LQCIII$ 11.5 is equivalent to the assertion that F is bounded above. This would imply that F is dissonant (§9). Since $K(n)_*F$ is automatically zero for $n = 0, 1$, and $K(n)_*L_1KX$ is zero for $n \geq 2$, Theorem 12.4 says precisely that F *is* dissonant. However this does not, alas, imply conversely that F is bounded above. For example, all the results (but not the conjectures!) of this paper are consistent with the possibility that a spectrum of the form $\bigvee \Sigma^n H\mathbb{Z}/\ell$ is a retract of F. Still, Theorem 12.4 places severe restrictions on the nature of F. Further interpretation of this theorem will be left to the reader. True believers will interpret it as strong evidence for the Lichtenbaum-Quillen conjectures; heretics may view it as suggesting the presence of Eilenberg-Maclane spectra in the fibre F; wild-eyed fanatics may even entertain the possibility that the exotic spectra of Example 10.11b have a role to play in algebraic K-theory.

Combining the results above with the theorem of Dwyer and Friedlander 7.6 allows one to explicitly identify the completed harmonic localization $L_\infty^{\wedge} KR \equiv (L_\infty KR)^{\wedge}$ for certain R. For example:

12.7. Theorem. [Mitchell 1990b] *Let ℓ be an odd regular prime. Then*

$$L_\infty^{\wedge} K\mathbb{Z}[\frac{1}{\ell}] \cong J^{\wedge} \vee \Sigma KO^{\wedge}.$$

As an amusing corollary we have that for any harmonic spectrum E, $E^{\wedge}(K\mathbb{Z}[\frac{1}{\ell}]) \cong E^{\wedge}(J \vee \Sigma KO)$. In particular we could take $E = BP$. However we should point out, lest any of our younger readers be misled into fantasies involving the Adams-Novikov spectral sequence, that the BP-cohomology of a spectrum does not determine its BP-homology. For example, $BP^*H\mathbb{Z}/\ell \equiv 0$ (since BP is harmonic and $H\mathbb{Z}/\ell$ is dissonant), but $BP_*H\mathbb{Z}/\ell = \mathbb{Z}/\ell[t_1, t_2, \dots]$.

Before discussing the proof of 12.1 we digress to consider a general question concerning the algebraic K-theory of classifying spaces of finite groups. To motivate the question, we recall the classical theorem of Atiyah:

12.8. Theorem. [Atiyah 1961] *Let G be a finite group. Then there is a natural ring isomorphism $(R_{\mathbb{C}}G)^{\wedge} \underset{\cong}{\xrightarrow{\theta^{\wedge}}} KU^0(BG_+)$*

Here $(R_{\mathbb{C}}G)^{\wedge}$ is the completion of the complex representation ring at the augmentation ideal I. Recall the definition of θ: a representation of dimension n determines a conjugacy class of homomorphisms $G \to U(n)$ and hence a homotopy class of maps $BG \to BU(n)$. This leads easily to a ring homomorphism $R_{\mathbb{C}}G \xrightarrow{\theta} [BG_+, BU \times \mathbb{Z}] = KU^0BG_+$, which factors through the I-adic completion for general reasons.

We wish to consider an analogous construction with topological K-theory KU replaced by algebraic K-theory $K\Lambda$, Λ a commutative ring. The representation ring $R_\Lambda G$ is the Grothendieck group of finitely-generated Λ-projective ΛG-modules. A representation of rank n determines a conjugacy class of homomorphisms $G \to \operatorname{Aut}_\Lambda P$ and hence a homotopy class of maps $BG \to B\operatorname{Aut} P$. Again, this leads to a ring homomorphism $R_\Lambda G \xrightarrow{\theta} [BG_+, BGL\Lambda^+ \times K_0\Lambda] = K\Lambda^0 BG_+$. However in this setting the construction of θ is not so easy; the problem is that short exact sequences of Λ-projective ΛG-modules need not split. Nevertheless the map θ exists by a theorem of Quillen (see [Hiller]). We also note that if the order of G is a unit in Λ, as it will be in our applications, the construction of θ is an easy exercise (use the group completion method, §2). Again, θ factors through the I-adic completion, for the same general reasons. Here I is the kernel of the natural map $R_\Lambda G \to K_0\Lambda$. Thus we have:

12.9. Problem. Study the map $\theta\hat{} : (R_\Lambda G)\hat{} \to K\Lambda^0 BG_+$.

For example, when Λ is a finite field, $\theta\hat{}$ is an isomorphism by a theorem of [Rector]. In general, this is too much to expect, even for rings of S-integers. However one can formulate various conjectures. Here we will confine ourselves to a special case:

12.10. Conjecture. Suppose G is a finite ℓ-group and $\Lambda = \mathbb{Z}[\frac{1}{\ell}]$. Then the map $\theta\hat{}$ of 12.9 is an isomorphism.

An étale K-theory argument, due to Bill Dwyer, shows $\theta\hat{}$ is split injective. The same argument shows that 12.10 follows from the Lichtenbaum-Quillen conjectures. Combining all this with 12.6, we conclude that 12.10 holds after harmonic localization, at least if ℓ is odd.

What is the representation ring in 12.10? In general, if Λ is a Dedekind domain, define the *class group* $C\ell(\Lambda G)$ to be the kernel of the natural map $R_\Lambda G \xrightarrow{\varphi} R_F G$, where F is the quotient field.

12.11. Theorem. (cf. [Curtis-Reiner])

(a) *The map φ is surjective.*
(b) *If $\Lambda = S^{-1}\mathcal{O}_F$, $C\ell\Lambda$ is finite.*

Here it is important to note that even when G is a finite ℓ-group and $\Lambda = \mathbb{Z}[\frac{1}{\ell}]$, the group $C\ell\Lambda$ can have nontrivial ℓ-torsion. In order to give the reader a better feeling for these groups, we mention the following result, in which we take ℓ odd for simplicity:

12.12. Theorem. *Let G be a finite ℓ-group, ℓ odd, and let $V_1, \dots, V_m$ denote the simple $\mathbb{Q}G$-modules. Let $D_i = \mathrm{End}_{\mathbb{Q}G} V_i$ denote the corresponding division algebras. Then*

(a) *Each D_i is in fact a cyclotomic field $\mathbb{Q}(\xi_{\ell^k})$ for some k (depending on i).*
(b) *The group ring $\mathbb{Z}[\frac{1}{\ell}]G$ is isomorphic to $\Pi_{i=1}^m M_{n_i}\mathcal{O}_i$, where $n_i = \dim_{D_i} V_i$ and $\mathcal{O}_i$ is the ring of integers in D_i.*
(c) $C\ell(\mathbb{Z}[\frac{1}{\ell}]G) \cong \bigoplus_{i=1}^{m} C\ell\mathcal{O}_i$.

Note (c) follows from (b) and Morita equivalence. A discussion of the proof can be found in [Mitchell 1990a].

12.13. Remark. By a theorem of Iwasawa, the ℓ-component of $C\ell(\mathbb{Z}[\xi_{\ell^k}])$ is trivial if and only if the ℓ-component of $C\ell[\mathbb{Z}[\xi_\ell])$ is trivial. Hence, by 12.12c, the ℓ-component of $C\ell(\mathbb{Z}[\frac{1}{\ell}]G)$ is trivial if and only if ℓ is regular.

We conclude our digression by recalling another analog of Atiyah's theorem 12.8: the Segal conjecture. Instead of considering representations of G we consider finite G-sets. Let AG denote the Burnside ring–that is, the "Grothendieck group" of finite G-sets, with addition and multiplication given by disjoint union and cartesian product, respectively. A G-set S of cardinality n determines a conjugacy class of homomorphisms $G \to \Sigma_n$ and hence a homotopy class of maps $BG \to B\Sigma_n$. As before, this leads to a ring homomorphism $AG \xrightarrow{\psi} [BG_+, B\Sigma_\infty^+ \times \mathbb{Z}]$ (compare the discussion of group completion in §2). By the Barratt-Priddy-Quillen theorem [Priddy], $B\Sigma_\infty^+ \times \mathbb{Z} = QS^0$. Hence the target of ψ is actually the stable cohomotopy ring $\pi_s^0(BG_+)$. Again the map ψ factors through completion at the augmentation ideal I, and the Segal conjecture asserts that $\psi\hat{}: (AG)\hat{} \to \pi_s^0(BG_+)$ is an isomorphism. This conjecture is now a theorem, due to [Carlsson 1984].

We now turn to the proof of 12.1. Thus we need to study the diagram (localized at ℓ)

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r, "f"] \arrow[d, "r"'] & BGL\mathbb{Z}^+ \\ BGL\mathbb{F}_p^+ \arrow[ur, dashed, "g"'] & \end{tikzcd} \tag{12.14}$$

Here f and r are induced by inclusion of Σ_n in GL_n, and correspond to $\Omega_0^\infty i$ and $\Omega_0^\infty i'$ via the Barratt-Priddy-Quillen theorem and Remark 12.2, respectively. We will use the following device, which is well-known to those who know it well:

12.15. Lemma. *Let G_n denote the ℓ-Sylow subgroup of Σ_n. Let Y be any infinite loop space. Then a map h from $B\Sigma_\infty^+$ to Y is uniquely determined by the composites $BG_n \to B\Sigma_n \to B\Sigma_\infty^+ \xrightarrow{h} Y$. A similar result holds with $B\Sigma_\infty^+$ replaced by $BGL\mathbb{F}_q^+$ (q prime to ℓ).*

Think of G_n as an arbitrary ℓ-group G, with $BG_n \to B\Sigma_n$ induced by some G-set. The lemma allows one to translate algebraic results about Burnside rings and representation rings into results about maps of spaces. For example, consider the following algebraic fact:

12.16. Theorem. [Segal 1972] *Let G be a finite ℓ-group. Then the natural map $AG \to R_{\mathbb{Q}}G$ is surjective.*

12.17a. Corollary. *If ℓ is odd and p generates $(\mathbb{Z}/\ell^2)^*$, the natural map $AG \to R_{\mathbb{F}_p}G$ is surjective.*

The point of 12.17a is that for such ℓ, p, the rational and mod p representation rings of G coincide. The case $\ell = 2$ is more complicated, as usual, and hence *from now on we will assume*, for simplicity, that *ℓ is odd.* Then 12.17a and a trick with inverse limits yields:

12.17b. Theorem. (May-Tornehave; see [May 1977]) *If ℓ is odd and p generates $(\mathbb{Z}/\ell^2)^*$, the map $r : B\Sigma_\infty^+ \to BGL\mathbb{F}_p^+$ is a homotopy retraction. Thus* $\operatorname{Im} J$ *is a retract of Q_0S^0.*

Remark. If $\ell = 2$ the first statement is false for all p, but $\operatorname{Im} J$ is still a retract of Q_0S^0 (May-Tornehave, loc.cit.).

Now consider the diagram 12.14. As a first step one can prove the algebraic analogue:

12.18. Theorem. *Let ℓ, p be as in 12.17, G a finite ℓ-group. Then there is a factorization g' in the diagram*

(*i*)
$$\begin{tikzcd} AG \arrow[r, "f'"] \arrow[d, "r'"'] & R_{\mathbb{Z}[\frac{1}{\ell}]}G \\ R_{\mathbb{F}_p}G \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

In fact for any prime ℓ there is an factorization

(*ii*)
$$\begin{tikzcd} AG \arrow[r, "f'"] \arrow[d] & R_{Z[\frac{1}{\ell}]}G \\ R_{\mathbb{Q}}G \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Here diagram (i) follows from (ii) since $R_{\mathbb{F}_p}G = R_{\mathbb{Q}}G$. Note (ii) says precisely that if a virtual G-set is zero as a virtual permutation module over $\mathbb{Q}$, it must already be zero as a virtual permutation module over $\mathbb{Z}[\frac{1}{\ell}]$. If the map $R_{\mathbb{Z}[\frac{1}{\ell}]}G \xrightarrow{\varphi} R_{\mathbb{Q}}G$ is an isomorphism, this is trivially true. But we have seen that Ker $\varphi \equiv C\ell(\mathbb{Z}[\frac{1}{\ell}]G)$ can be nonzero, even after localization at ℓ. Hence one must show that the image of f' does not intersect the class group. This is done using an explicit construction of integral lattices in the simple $\mathbb{Q}G$-modules; the construction is based on the fact that the simple modules are all induced modules of a very special kind.

The proof of 12.1 now proceeds as follows. First one observes that there the analogous diagram with $BGL\mathbb{Z}^+$ replaced by BU does exist:

$$\begin{tikzcd} B\Sigma_\infty^+ \arrow[r] \arrow[d, "r"'] & BU \\ BGL\mathbb{F}_p^+ \arrow[ur, dashed, "h"'] & \end{tikzcd} \tag{12.19}$$

Hence h is in fact the Brauer lift (§2). (Alternatively one can cite a theorem of Snaith which asserts that r^* is an isomorphism on K-theory). Next, note that 12.17 implies that g is unique if it exists: let s be a right inverse of r; we must take $g = fs$ and show $f = fsr$. Using the principle of 12.15 we reduce to showing that if G is a finite ℓ-group and $\alpha : G \to \Sigma_n$ a homomorphism, $f\overline{\alpha} = fsr\overline{\alpha}$, where $\overline{\alpha}$ is the induced map $BG \to B\Sigma_\infty^+$. Ignore the completions and think of $\overline{\alpha}$ and $sr\overline{\alpha}$ as elements of the Burnside ring AG, and think of $f\overline{\alpha}$ and $fsr\overline{\alpha}$ as elements of $R_{\mathbb{Z}[\frac{1}{\ell}]}G$. By 12.19 $\overline{\alpha}$ and $sr\overline{\alpha}$ become equal in $R_{\mathbb{C}}G$. Hence they are equal in $R_{\mathbb{Q}}G$, since $R_{\mathbb{Q}}G \to R_{\mathbb{C}}G$ is injective. By 12.18(ii) they are equal in $R_{\mathbb{Z}[\frac{1}{\ell}]}G$-i.e. $f\overline{\alpha} = fsr\overline{\alpha}$, and the proof is complete.

Theorem 12.1 admits a natural generalization.

12.20. Theorem. [Dwyer-Friedlander-Mitchell] *Let F be a number field, μ the group of ℓ-power roots of unity in F. Let $\mathbb{F}_q = \mathcal{O}_F/\mathcal{P}$ be a retractible residue field of $\mathcal{O}_F$ (§3). Then there is a factorization g:*

$$\begin{tikzcd} Q_0B\mu \arrow[r] \arrow[d] & BGL\mathcal{O}_F^+ \\ BGL\mathbb{F}_q^+ \arrow[ur, dashed, "g"'] & \end{tikzcd}$$

Remarks.

(a) As with 12.1, one can show (with more difficulty in this case) that the Lichtenbaum-Quillen conjectures imply the spectrum-level version of 12.20.

(b) In particular, the natural map $\pi_*^s(B\mu_+) \to K_*\mathcal{O}_F$ factors through $K_*\mathbb{F}_q$.

(c) $Q_0B\mu$ is equivalent to $B(\Sigma_\infty \int \mu)^+$ ($\int$ = wreath product). The proof is then very similar to that of 12.1, with AG replaced by the "two-sided" Burnside ring $A(G, \mu)$ of μ-free $(G \times \mu)$-sets.

13. CONCLUDING REMARKS

The results of §12 have one very striking feature, which the reader has probably noticed: the proofs use very little in the way of explicit information about the space $BGL\mathbb{Z}^+$ or the spectrum $K\mathbb{Z}$. Indeed this feature was essential, since very little is known. Clearly some deeper analysis will be required to work back from §12 to the Lichtenbaum-Quillen conjecture itself. Somehow, one has to come to grips with the actual construction of the K-theory spaces. There is only one immediately apparent way to get a concrete hold on $BGLR^+$: via the cohomology of GLR. Virtually all of the *explicit* calculations (there aren't many) of higher K-theory to date ultimately involve computing or at least analyzing H^*GLR-Quillen's work on finite fields, Borel's rational computations, the Lee-Szczarba computation of $K_3\mathbb{Z}$, and Suslin's theorems on $\mathbb{C}$ and $\mathbb{R}$ are the prime examples.

From this point of view, then, the main problem in our setting is to compute the cohomology of $GL_n\mathcal{O}_F$ with $\mathbb{Z}/\ell$ coefficients. Two remarks should be made at once. The first is that the case $n = \infty$, which is the case we really need, is probably easier. For example, [Quillen 1971] shows that the map $H^*(GL\mathcal{O}_F; \mathbb{Z}/\ell) \to H^*(GL_n\mathcal{O}_F; \mathbb{Z}/\ell)$ need not be onto. Counterexamples arise whenever the class group of $\mathcal{O}_F$ has torsion prime to ℓ-e.g., $\mathbb{Z}[\xi_{23}]$. (Recall that there is nevertheless a "stable range;" see the proof of theorem 2.3). There are also counterexamples for $\mathcal{O}_F = \mathbb{Z}$. And even the rational cohomology is still unknown for $n < \infty$. The second remark is that it is once again advisable to invert ℓ in the ring $\mathcal{O}_F$. For example, Quillen (loc.cit.) showed that the Krull dimension of $H^*\Gamma$, Γ an S-arithmetic group, is the rank of a maximal elementary abelian ℓ-subgroup (ℓ-torus for short), and that the minimal primes of $H^*\Gamma$ correspond to conjugacy classes of maximal ℓ-tori. As Quillen notes, it is much easier to analyze ℓ-tori in $GL_n\mathcal{O}_F[\frac{1}{\ell}]$ than in $GL_n\mathcal{O}_F$. A more dramatic illustration of the same principle arises from a conjecture of Dwyer and Friedlander. Here we will state only a special case:

13.1. Conjecture. $H^*(GL_n\mathbb{Z}[\frac{1}{2}], \mathbb{Z}/2)$ is detected on the diagonal matrices, in the stable range.

This is actually equivalent to $LQCII$ 7.3 for $K\mathbb{Z}[\frac{1}{2}]$. Note that nothing like 13.1 can be true for $GL_n\mathbb{Z}$ itself: for if so it would follow that the natural map $BGL\mathbb{Z} \to BO$ is an isomorphism on mod 2 cohomology! Note that the diagonal subgroup of $GL_n\mathbb{Z}[\frac{1}{2}]$ is $(\mathbb{Z}/2)^n \times \mathbb{Z}^n$.

When studying the cohomology of $GL_n\mathcal{O}_F[\frac{1}{\ell}]$, it is natural to start by eliminating the "easy" part coming from the cohomology of a residue field. Thus let $\mathcal{P}$ denote a retractible prime of $\mathcal{O}_F[\frac{1}{\ell}]$ as in §3. We also include the case $\mathcal{O}_F = \mathbb{Z}$, $\ell = 2$, $\mathcal{P} = (3)$, even though there are no retractible primes for $\mathbb{Z}[\frac{1}{2}]$. Let $\Gamma_n = \Gamma_{n,\mathcal{P}}$ denote the congruence subgroup of $GL_n\mathcal{O}_F[\frac{1}{\ell}]$ consisting of matrices which are congruent to the identity mod $\mathcal{P}$. One can show that Γ_n is ℓ-torsion-free and has finite mod ℓ cohomological dimension. Furthermore, one can deduce the following from the Charney-Suslin theorem 4.9c: Let F denote the fibre of the reduction map $BGL\mathcal{O}_F[\frac{1}{\ell}]^+ \to BGL\mathcal{O}_F/\mathcal{P}^+$. Then the natural map $B\Gamma_\infty \to F$ is an isomorphism on $H_*(\ \ ,\mathbb{Z}/\ell)$. Hence we will refer to F as the "congruence fibre." By the Harris-Segal theorem 3.4, $H^*(GL\mathcal{O}_F[\frac{1}{\ell}]) \cong H^*(GL\mathcal{O}_F/\mathcal{P}) \otimes H^*F$. (This is true even in the case of $\mathbb{Z}[\frac{1}{2}]$; see [Arlettaz]). Thus, for example, $LQCII$ for $\mathbb{Z}[\frac{1}{\ell}]$, ℓ regular, is equivalent to the following:

13.2. Conjecture. (compare 7.6) Let $\mathcal{O}_F = \mathbb{Z}$ and suppose ℓ is regular. Let Γ_n denote a congruence subgroup of $GL_n\mathbb{Z}[\frac{1}{\ell}]$ as above. Then $H^*B\Gamma_\infty \cong H^*U/O$. (Note: $H^*U/O \cong \mathbb{Z}/\ell\langle x_1, x_5, x_9, \dots\rangle$ if ℓ is odd; $H^*U/O \cong \mathbb{Z}/\ell\langle w_1, w_2, w_3, \dots\rangle$ if $\ell = 2$).

This approach also leads to an interesting reformulation of Conjecture 12.10.

13.3. Conjecture. Let G be a finite ℓ-group and let F denote the congruence fibre for $\mathbb{Z}[\frac{1}{\ell}]$. Then $[BG_+, F]$ is isomorphic to the ℓ-torsion subgroup of the class group of $\mathbb{Z}[\frac{1}{\ell}]G$. In particular $[BG_+, F] = 0$ if ℓ is regular.

Note that in the regular case this conjecture follows at once from conjecture 7.3 in the form 7.8. In general Dwyer's argument (see 12.10) shows that $(C\ell(\mathbb{Z}[\frac{1}{\ell}]G))_{(\ell)}$ injects into $[BG_+, F]$. In particular the latter group is nonzero whenever ℓ is irregular. This is interesting since F is the group completion of $\coprod B\Gamma_n$, and there are no nontrivial homomorphisms $G \to \Gamma_n$. It also follows, using work of Lannes, that $H^*B\Gamma$ must contain elements of infinite height when ℓ is irregular–this should be contrasted with 13.2. However in some sense this phenomenon is already "explained" by [Quillen 1971], which detects "exotic" infinite height classes in $H^*GL_n(\mathbb{Z}[\xi_\ell, \frac{1}{\ell}])$ arising from "exotic" maximal ℓ-tori: i.e., ℓ-tori arising from a splitting of $(\mathbb{Z}[\xi_\ell, \frac{1}{\ell}])^n$ into nontrivial rank one projective modules. It would be interesting to make this explanation more explicit.

We conclude with a very speculative remark. We believe that a proof of 13.3 for elementary abelian ℓ-groups would lead to a proof of the Lichtenbaum-Quillen conjecture for $\mathbb{Z}[\frac{1}{\ell}]$. Furthermore the proof of 13.3 should not require an explicit cohomology calculation; a qualitative analysis should suffice. For example, if ℓ is regular, it is enough to show that H^*F is nil.

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