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The cyclic homology of an exact category

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Abstract

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We define Hochschild and cyclic homology groups for an exact category which generalize the usual definitions when one considers finitely generated projective modules. They satisfy additivity as well as many of the usual properties one expects from the homology groups of an algebra. The Dennis trace and its lift to negative homology are also (multiplicatively) generalized to this setting. We use the S construction of Waldhausen and a formal generalization of the usual cyclic complex from Hochschild homology for our definition.

1. Introduction

This paper arose from my desire to better understand the connections between algebraic K-theory and cyclic homology. I wanted a definition of Hochschild homology groups such that the Dennis trace map and its multiplicative properties were clear and that its extension (called here the Jones–Goodwillie Chern map) to negative homology was equally straightforward. To do this, it seemed that a definition of the Hochschild homology groups for an exact category with these properties would help to simplify the situation. The purpose of this paper is to construct such a theory and to re-derive the standard facts for cyclic homology groups in this generality.

Since Hochschild homology is defined for algebras over a commutative ground ring k, it is natural to introduce such a ground ring into our exact categories. We will call

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a category *k-linear* if its Hom sets are equipped with the structure of a k-module and composition is k-bi-linear. Every exact category is by definition an additive category and hence $\mathbb{Z}$-linear. For the purpose of this introduction we will simply state the results for the ground ring $\mathbb{Z}$. All statements hold for arbitrary ground ring k and are proved as such throughout the paper.

For $\mathscr{E}$ an exact category, we define Hochschild, cyclic, negative and periodic homology groups for $\mathscr{E}$ (denoted $\mathrm{HH}_*(\mathscr{E})$, $\mathrm{HC}_*(\mathscr{E})$, $\mathrm{HN}_*(\mathscr{E})$, and $\mathrm{HP}_*(\mathscr{E})$ respectively) with the following properties:

(1) *Agreement.* If $\mathscr{P}$ denotes the exact category of finitely generated projective modules over a (unital) algebra A, then $\mathrm{HH}_*(\mathscr{P})$ is naturally isomorphic to the usual Hochschild homology groups for A. Similar statements hold for cyclic, negative and periodic homology groups.

(2) *Exact sequence.* There is a natural commuting diagram with rows long exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{HN}_*(\mathscr{E}) \arrow[r] \arrow[d] & \mathrm{HP}_*(\mathscr{E}) \arrow[r] \arrow[d] & \mathrm{HC}_{*-2}(\mathscr{E}) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & \mathrm{HH}_*(\mathscr{E}) \arrow[r] & \mathrm{HC}_*(\mathscr{E}) \arrow[r] & \mathrm{HC}_{*-2}(\mathscr{E}) \arrow[r] & \cdots
\end{tikzcd}$$

(3) *Additivity.* If $0 \to F'' \to F \to F' \to 0$ is a short exact sequence of functors, then $\mathrm{HH}_*(F) = \mathrm{HH}_*(F'') + \mathrm{HH}_*(F')$. Again, identical statements hold for cyclic, negative and periodic homology groups.

(4) *Products.* If $F: \mathscr{C} \times \mathscr{D} \to \mathscr{E}$ is a bi-exact functor, then there is a natural product structure $\mathrm{HH}_p(\mathscr{C}) \otimes \mathrm{HH}_q(\mathscr{D}) \to \mathrm{HH}_{p+q}(\mathscr{E})$ induced by F. There is a similar product for negative and periodic homology groups. By the isomorphism of (1), one can recover all the products for the usual cyclic homology groups as found in [10] in this manner.

(5) *Trace maps.* There exists a natural transformation from the algebraic K-theory of an exact category to its Hochschild homology which recovers the Dennis trace map for rings by the isomorphism of (1). There is also an extension of this transformation to the negative homology which recovers the Jones–Goodwillie Chern map. These natural transformations are suitably multiplicative with respect to (4).

The construction

The construction for these groups arises by "twisting together" the Hochschild complex with the S construction of a category with cofibrations from [28]. This method was suggested to us by the idea of Tom Goodwillie in [8] which Jean-Louis Loday had provided to help with our investigations. These ideas were further encouraged by Daniel Grayson who showed us the importance of proving "additivity" ((3) above) for our model and by Christian Kassel who indicated how these techniques could be applied to bivariant cyclic cohomology (to appear elsewhere).

In order to "twist" these theories together, we define the *additive cyclic nerve* of a (small) linear category $\mathscr{C}$, denoted $\mathrm{CN}(\mathscr{C})$, to be the cyclic module defined by

$$\mathrm{CN}_n(\mathscr{C}) = \bigoplus \mathrm{Hom}_{\mathscr{C}}(C_1, C_0) \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(C_2, C_1) \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(C_0, C_n),$$

where the direct sum runs over all $(C_0, C_1, \ldots, C_n) \in \mathrm{Obj}(\mathscr{C})^{n+1}$ (the operators are like those for ordinary Hochschild homology). We discovered later that this complex is the same as one defined by Mitchell in [23]. If $\mathscr{E}$ is an exact category, (thought of as a category with cofibrations in the usual manner) then $S.\mathscr{E}$ is a simplicial additive category (see Section 3 for more details). Thus we can form the simplicial cyclic abelian group $\mathrm{CN}.S.\mathscr{E}$ and we define the Hochschild homology of $\mathscr{E}$ by

$$\mathrm{HH}_n(\mathscr{E}) = H_{n+1}(\mathrm{Tot}(\mathrm{CN}.S.\mathscr{E})).$$

The shift of one corresponds to the shift from looping the realization of $S.\mathscr{E}$ to obtain a model for algebraic K-theory. Applying the usual functors from cyclic modules to bi-complexes degree-wise to $\mathrm{CN}.S.\mathscr{E}$ we obtain tri-complexes which we use to define the cyclic, negative and periodic homology groups for $\mathscr{E}$.

The proof that these groups do agree with the usual ones in the case of finitely generated projectives is deduced from the following two facts. The first is "Morita invariance" for Hochschild homology and the second is modeled after Proposition 1.5.5 of [28] after one notes that for the category of projectives CN is suitably "additive."

We can consider any (unital) algebra A as a linear category having only one object $\star$ and morphisms the elements of A.

Proposition 2.4.3. *The natural inclusion from A into $\mathscr{P}_A$ which sends the object $\star$ to the object A induces a homotopy equivalence* $\mathrm{CN}(A) \to \mathrm{CN}(\mathscr{P}_A)$.

Theorem 3.3.3. *The sequence (where $PS.$ denotes the simplicial path space of $S.$)*

$$\mathrm{CN}.\mathscr{P} \to \mathrm{CN}.PS.\mathscr{P} \to \mathrm{CN}.S.\mathscr{P}$$

is a quasi-fibration (induces a long exact sequence of homotopy groups) and the center term is contractible.

One recovers the Dennis trace map in the following manner: There is a natural inclusion from $S.\mathscr{E}$ into $\mathrm{CN}_0 S.\mathscr{E}$ given by taking the identity endomorphism of an object. In Section 4 it is shown that the composite

$$S.\mathscr{P} \xrightarrow{\mathrm{id}} \mathrm{CN}_0 S.\mathscr{P} \to \mathrm{CN}.S.\mathscr{P}$$

recovers the Dennis trace map after looping and using the natural isomorphism arising from the two facts above. We note that this construction has the property of landing in the S^1-fixed point set of $\mathrm{CN}.S.\mathscr{P}$ and hence it is easy to construct a lift to negative homology. This fact also makes it easy to show that the lifting is suitably multiplicative.

So far we have been rather vague about the S^1-actions arising in all of this. We define a notion of "special homotopy" which is a sufficient condition for two maps to induce the same maps of Hochschild and its associated cyclic homology groups. Furthermore, a special homotopy has the property that it produces a *discrete homotopy*—a homotopy for the induced maps of fixed point sets for each *finite* subgroup of S^1 (though *not* necessarily on the S^1-fixed point sets). We will show throughout that all the necessary homotopies are discrete homotopies (or occasionally S^1-homotopies). Though our definition lands in the S^1-fixed point set of $\mathrm{CN}.S.\mathscr{P}$, the equivalence of this target with $\mathrm{CN}.A$ is only a discrete homotopy equivalence. Thus we recover (in a special case) the fact observed by Marcel Bökstedt that the Dennis trace lands in the homotopy inverse limit (given by divisibility) of the fixed point sets of the finite subgroups of S^1.

Organization

The paper is organized as follows. Section 2 introduces the additive cyclic nerve and the notion of a "special homotopy." We then derive several applications which we use later. Section 3 consists of the basic definition and the proof that this agrees with the usual case when considering the category of projectives. In Section 3.5 we prove additivity. The usual proofs of additivity for algebraic K-theory were not suitable for our setting and so we needed to devise another. Our proof, in a somewhat streamlined form, appears also in [22]. From additivity we derive the "delooping theorem" of [28] following the techniques found there.

We construct products for these groups in the fourth section. This is taken almost directly from [10] using the "cyclic Eilenberg–Zilber theorem." Our construction simply twists the description of products in algebraic K-theory as found in [28] using the delooping theorem with the constructions of [10]. We then define the Dennis trace map and its lifting to negative homology and prove that these maps are suitably multiplicative. Lastly we show that these maps recover the usual ones defined for a ring.

Notation and conventions

It seems traditional here to give one's own account of cyclic modules and cyclic homology. Since several very good ones already exist in the literature (see for example [7] or [18]), we will resist temptation and simply state which conventions we will be using and noting those which may be less familiar. There are a few non-standard things to first point out. First is that we use the notation $\mathrm{HN}_*(X)$ and $\mathrm{HP}_*(X)$ to denote the negative and periodic homology groups of a cyclic module X. These are usually written $\mathrm{HC}_*^-(X)$ and $\mathrm{HC}_*^{\mathrm{per}}(X)$. We denote the cyclic category of Connes by ΔC and not Λ as in [3]. We will let k be a fixed commutative ring and unlabeled tensor products are formed over k. By a *k-linear* category we mean a category whose Hom sets have the structure of a (unital) k-module and composition is k-bi-linear. We will need to use maps of cyclic modules which do not necessarily preserve degeneracies. We call such maps *semi-cyclic* maps and we refer the reader to the Appendix for facts about such maps which we use. Lastly, we will not differentiate in the notation between a simplicial (cyclic) k-module and the chain complex (mixed complex) naturally associated to it. Recall from [13] that a *mixed complex* is a chain complex with a differential operator of degree $+1$ which anti-commutes with the boundary operator.

Following the usual conventions, if X is a cyclic module, we let $B(X)$, $B^-(X)$ and $B^{\mathrm{per}}(X)$ denote the bi-complexes for cyclic, negative and periodic homology groups using the operator B. If Y is a cyclic $\times$ multi-simplicial module, then we consider its realization as an S^1-space by realizing the simplicial directions first and then giving the realization of the resulting cyclic space its usual circle action. Similarly, we define the Hochschild, cyclic, negative and periodic homology groups of Y as the homology of the multi-dimensional complexes obtained by applying the functors Identity, B, B^- or B^{per} to the cyclic module direction. We will always mean the total complex obtained by taking *products* when we want the homology of a multi-dimensional complex. Because of this, the result of [11] that a map of cyclic objects which produces an isomorphism of Hochschild homology groups does so on the related cyclic homology groups remains true for our convention about cyclic $\times$ multi-simplicial modules.

We will often be dealing with simplicial homotopies. A simplicial homotopy h can be described in terms of maps which raise the simplicial degree by one and satisfies certain relations with the face (and degeneracy) operators (see the Appendix (Section A.2) for more details). We will be using this combinatorial description since it is very convenient when dealing with semi-simplicial maps (not necessarily preserving degeneracies). We will give the definition of the homotopy but leave the straightforward details of checking the relations to the interested reader. In no case is this checking difficult but it can become technically tedious.

2. The additive cyclic nerve

Introduction

In this section we define the additive cyclic nerve of a (small) k-linear category and give a few examples. We use the additive cyclic nerve to define a notion of split homology groups of a k-linear category. We call these "split" to differentiate them from the homology groups of an exact category we will define in Section 3. We note that the split Hochschild homology we define is the same as that of Mitchell [23], a fact which we discovered later. We were encouraged to consider the additive cyclic nerve by Chase as a means of improving our proof of Morita invariance of cyclic homology as found in [21].

In Section 2.3, we define the notion of "special homotopy", which is a sufficient condition for two k-linear functors to induce the same map of the split Hochschild, cyclic, negative and periodic homology groups. Chase was the first to suggest to us that such a criterion might exist and the notion we define is a natural generalization of an argument of Karoubi [12, Theorem 2.12]. It has been pointed out to us by Bökstedt that our notion of special homotopy is related to work of his and we make some notes to this effect in the Appendix. We establish a criterion for two functors to be special homotopic and give several applications. These applications will be used later.

Recall that k is always assumed to be a commutative ring.

2.1. *Definition of the additive cyclic nerve*

Definition 2.1.1. Let $\mathscr{C}$ be a small k-linear category. We define the *additive cyclic nerve* of $\mathscr{C}$, denoted $\mathrm{CN}(\mathscr{C})$, to be the following cyclic k-module:

$$\mathrm{CN}_n(\mathscr{C}) = \bigoplus \mathrm{Hom}_{\mathscr{C}}(C_1, C_0) \otimes_k \mathrm{Hom}_{\mathscr{C}}(C_2, C_1) \otimes_k \cdots \otimes_k \mathrm{Hom}_{\mathscr{C}}(C_0, C_n),$$

where the direct sum runs over all $(C_0, C_1, \ldots, C_n) \in \mathrm{Obj}(\mathscr{C})^{n+1}$. The maps are defined by

$$d_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} (f_0 \otimes f_1 \otimes \cdots \otimes f_i \circ f_{i+1} \otimes \cdots \otimes f_n) & \text{if } 0 \le i \le n-1, \\ (f_n \circ f_0 \otimes f_1 \otimes \cdots \otimes f_n) & \text{if } i = n, \end{cases}$$

$$s_i(f_0 \otimes \cdots \otimes f_n) = \begin{cases} (f_0 \otimes f_1 \otimes \cdots \otimes f_i \otimes \mathrm{id}_{C_{i+1}} \otimes f_{i+1} \otimes \cdots \otimes f_n) \\ \quad \text{if } 0 \le i \le n-1, \\ (f_0 \otimes f_1 \otimes \cdots \otimes f_n \otimes \mathrm{id}_{C_0}) \\ \quad \text{if } i = n, \end{cases}$$

$$t(f_0 \otimes \cdots \otimes f_n) = (f_n \otimes f_0 \otimes \cdots \otimes f_{n-1}).$$

We note that the additive cyclic nerve is a covariant functor from the category of small k-linear categories to the category of cyclic k-modules.

Definition 2.1.2. For $\mathscr{C}$ a small k-linear category, we define the *split Hochschild homology groups of* $\mathscr{C}$ (with coefficients in k) as $\mathrm{HH}^{\mathrm{s}}_*(\mathscr{C}) = H_*(\mathrm{CN}(\mathscr{C}))$. We similarly define the split cyclic, negative and periodic homology groups of $\mathscr{C}$. We call these groups "split" to differentiate them from the homology groups of an *exact* k-linear category defined in Section 3.2. We will show in Corollary 3.3.4 that for a split exact category (every short exact sequence has a section) these two definitions agree. The functor $\mathrm{HH}^{\mathrm{s}}_*(\star)$ is sometimes called the Hochschild or Mitchell homology after [23].

2.2. *Some examples of the additive cyclic nerve*

Example 2.2.1. If A is a unital k-algebra, then we can think of it as a k-linear category with one object whose morphisms are the elements of A. The cyclic nerve of A is the standard bar resolution of Cartan and Eilenberg (written ZA in [7]). That is, $\mathrm{CN}(A)$ is the usual cyclic module one uses to construct the cyclic homology of A and so $\mathrm{HH}^{s}_{*}(A)$ is just the usual Hochschild homology for A.

Example 2.2.2. Following Quillen [24, Proposition 3], we let $\mathscr{J}$ be a small category which is filtering and $j \to \mathscr{C}_j$ a functor from $\mathscr{J}$ to the category of small k-linear categories. If $\mathscr{C}$ is the inductive limit of the $\mathscr{C}_j$ then $\mathscr{C}$ is a k-linear category. Since the filtered inductive limit also commutes with direct sums and tensors we have a natural isomorphism $\varinjlim \mathrm{CN}(\mathscr{C}_j) \cong \mathrm{CN}(\mathscr{C})$.

Example 2.2.3. Let X be a small category and let kX denote the k-linear category with $\mathrm{Obj}(kX) = \mathrm{Obj}(X)$ and $\mathrm{Hom}_{kX}(x, x')$ the free k-module on the set $\mathrm{Hom}_X(x, x')$. The functor $X \to kX$ is a faithful functor from the category of small categories to the category of small k-linear categories. We see that $\mathrm{CN}(kX)$ is the free cyclic k-module arising from the cyclic nerve $N^{\mathrm{cy}}(X)$ of the category X. By definition, $N^{\mathrm{cy}}(X)$ is the cyclic set which in degree n is the set of "cyclic" diagrams in X of the form

$$X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_n \leftarrow X_0.$$

The structure maps for $N^{\mathrm{cy}}(X)$ are like those for the additive cyclic nerve.

Example 2.2.4. Given a small category X, we can put an equivalence relation on its objects by declaring that $x \sim x'$ iff both $\mathrm{Hom}_X(x, x') \neq \emptyset$ and $\mathrm{Hom}_X(x', x) \neq \emptyset$. Let $[x]$ denote the full subcategory generated by the equivalence class of x and let $[X]$ be the set of equivalence classes, then

$$\mathrm{CN}(kX) \cong \bigoplus_{[x] \in [X]} \mathrm{CN}(k[x]).$$

Example 2.2.5. Given two small k-linear categories $\mathscr{C}$ and $\mathscr{D}$, we let $\mathscr{C} \cup \mathscr{D}$ denote the new k-linear category with objects the union of the objects of $\mathscr{C}$ and $\mathscr{D}$ and

$$\mathrm{Hom}_{\mathscr{C} \cup \mathscr{D}}(x, y) = \begin{cases} \mathrm{Hom}_{\mathscr{C}}(x, y) & \text{if } x, y \in \mathrm{Obj}(\mathscr{C}), \\ \mathrm{Hom}_{\mathscr{D}}(x, y) & \text{if } x, y \in \mathrm{Obj}(\mathscr{D}), \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\mathrm{CN}(\mathscr{C} \cup \mathscr{D}) \cong \mathrm{CN}(\mathscr{C}) \oplus \mathrm{CN}(\mathscr{D})$ and thus $\mathrm{HH}^{s}_{*}(\mathscr{C} \cup \mathscr{D}) \cong \mathrm{HH}^{s}_{*}(\mathscr{C}) \oplus \mathrm{HH}^{s}_{*}(\mathscr{D})$. Let 0 denote the trivial linear category with one object $+$ and morphisms the zero module. Then $\mathscr{C}_+ = \mathscr{C} \cup 0$ is a linear category with a zero object whose additive cyclic nerve is isomorphic to that of $\mathscr{C}$.

Example 2.2.6. Let $\mathscr{C}$ be a small k-linear category. We define an associative k-algebra $\mathrm{Arr}_k \mathscr{C}$ as

$$\mathrm{Arr}_k \mathscr{C} = \bigoplus_{a, b \in \mathrm{Obj}(\mathscr{C})} \mathrm{Hom}_{\mathscr{C}}(a, b).$$

For $f \in \mathrm{Arr}_k\mathscr{C}$, we let $f_{b,a}$ denote the component of f in $\mathrm{Hom}_{\mathscr{C}}(a,b)$. Given f and g of $\mathrm{Arr}_k\mathscr{C}$ we define $f \cdot g$ by

$$(f \cdot g)_{c,a} = \sum_{b \in \mathrm{Obj}(\mathscr{C})} f_{c,b} \circ g_{b,a}.$$

One can show that $Z\mathrm{Arr}_k\mathscr{C}$ generates the same homology groups as the additive cyclic nerve of $\mathscr{C}$ (see for example [23]). We prefer the additive cyclic nerve for what follows because a k-linear functor $\mathscr{C} \to \mathscr{C}'$ does not always give rise to a k-algebra homomorphism $\mathrm{Arr}_k\mathscr{C} \to \mathrm{Arr}_k\mathscr{C}'$ (a functor may convert noncomposable pairs of arrows to ones which are composable). Thus Arr_k is at best a functor from the category of small k-linear categories and k-linear functors which are injective on the sets of objects to associative k-algebras and k-algebra homomorphisms.

Example 2.2.7. Let $\mathscr{C}$ be a small k-linear category and let $\mathscr{C}^{\mathrm{op}}$ denote its opposite category. We define $\gamma_n : \mathrm{CN}_n(\mathscr{C}) \to \mathrm{CN}_n(\mathscr{C}^{\mathrm{op}})$ to be the k-module map given by

$$\gamma_n(f_0 \otimes \cdots \otimes f_n) = (-1)^{n(n+1)/2}(f_0^{\mathrm{op}} \otimes f_n^{\mathrm{op}} \otimes \cdots \otimes f_1^{\mathrm{op}}).$$

One can show that $b \circ \gamma = \gamma \circ b$ and $B \circ \gamma = (-1)\gamma \circ B$. The proof is the same as Lemma 1.1 of [17] or II.6 of [15].

Example 2.2.8. For M and N cyclic k-modules, we let $M \otimes_k N$ denote the cyclic k-module $(M \otimes_k N)_n = M_n \otimes_k N_n$ (and letting the morphisms act diagonally). For $\mathscr{C}$ and $\mathscr{D}$ small k-linear categories, we define the k-linear category $\mathscr{C} \otimes_k \mathscr{D}$ by

$$\mathrm{Obj}(\mathscr{C} \otimes_k \mathscr{D}) = \mathrm{Obj}(\mathscr{C}) \times \mathrm{Obj}(\mathscr{D}),$$

$$\mathrm{Hom}_{\mathscr{C} \otimes_k \mathscr{D}}((C,D),(C',D')) = \mathrm{Hom}_{\mathscr{C}}(C,C') \otimes_k \mathrm{Hom}_{\mathscr{D}}(D,D').$$

It is easy to see that $\mathrm{CN}(\mathscr{C} \otimes_k \mathscr{D}) \cong \mathrm{CN}(\mathscr{C}) \otimes_k \mathrm{CN}(\mathscr{D})$.

2.3. *Special homotopies*

Let $\mathscr{I}$ be the trivial connected groupoid on two objects. That is, $\mathscr{I}$ is a category with two objects 0 and 1, and two non-identity morphisms $0 \to 1$ and $1 \to 0$ (necessarily inverse to each other). For $i = 0$ or 1, we let $\varepsilon_i : \mathrm{CN}(k) \to \mathrm{CN}(k\mathscr{I})$ be the cyclic k-module map $\varepsilon_i(k) = k \cdot (\mathrm{id}_i, \ldots, \mathrm{id}_i)$. In other words, ε_i is the map induced by the k-linear functor which sends the trivial category $*$ to i.

Lemma 2.3.1. *The maps ε_0 and ε_1 induce the* same *maps of split Hochschild, cyclic, negative and periodic homology groups.*

Proof. Let $\pi : \mathscr{I} \to *$ denote the unique functor. Naturally identifying $k*$ with k we have induced a k-linear functor from $k\mathscr{I}$ to k which we also denote by π. The cyclic map $\mathrm{CN}(\pi)$ is a homotopy inverse to ε_0 and ε_1 since $\pi \circ \varepsilon_i = \mathrm{id}$ for $i = 0,1$ and $\varepsilon_i \circ \pi \simeq \mathrm{id}$ by

$$h_j(t \cdot (g_0, \ldots, g_n)) = t \cdot (i \leftarrow \mathrm{Ran}(g_1), g_1, \ldots, g_j, \mathrm{Dom}(g_j) \leftarrow i, \overbrace{\mathrm{id}_i, \ldots, \mathrm{id}_i}^{n-j \text{ times}}).$$

Thus the ε_i induce isomorphisms of special homology groups with inverse induced by π. It follows that the induced maps are the same. □

We call a map f between two cyclic modules X and Y *semi-cyclic* if it is a map of graded modules that commutes with the face maps and the cyclic operators but not necessarily with the degeneracy operators. Various facts about such maps are collected in the Appendix. Such a map f naturally produces a map of the associated cyclic, negative and periodic homology groups but it does not necessarily produce an S^1-equivariant map of realizations. It does, however, give rise to natural maps of the associated C_r-fix point spaces for each *finite* subgroup C_r of S^1.

Definition 2.3.2. Given two semi-cyclic module maps $f,g: X \to Y$ of cyclic k-modules X and Y, we say they are *special homotopic* if there exists a semi-cyclic module map h from $X \otimes_k \mathrm{CN}(k\mathscr{I})$ to Y (called a *special homotopy*) such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id} \otimes \varepsilon_0"] \arrow[dr, "f"'] & X \otimes_k \mathrm{CN}(k\mathscr{I}) \arrow[d, "h"] & X \arrow[l, "\mathrm{id} \otimes \varepsilon_1"'] \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$

A semi-cyclic module map $f: X \to Y$ is a *special homotopy equivalence* if it has a special homotopy inverse, that is, a semi-cyclic module map $g: Y \to X$ such that $f \circ g$ and $g \circ f$ are both special homotopic to the identity. We note that the composition of two special homotopy equivalences is again a special homotopy equivalence.

Proposition 2.3.3. *If $f,g: X \to Y$ are semi-cyclic maps of cyclic k-modules which are special homotopic, then they induce the* same *maps of Hochschild, cyclic, negative and periodic homology groups.*

Proof. Returning to the notation of Lemma 2.3.1, we see that $\mathrm{id} \otimes_k \pi$ is a deformation retract for $\mathrm{id} \otimes_k \varepsilon$ by the homotopy $h_j = s_j \otimes_k (h_j)$ where s_j is the jth degeneracy of X. □

We would like to note that one needs X to be a *cyclic* k-module and not simply a semi-cyclic k-module for Proposition 2.3.3 to hold in general. Also, a special homotopy implies the two maps are homotopic but not all homotopic maps are special homotopic.

The statement of the following theorem is somewhat technical. It was devised to handle several arguments with a single tool. It is helpful to first consider the exceptional case when each object of $\mathscr{C}$ is a retract of an object of the full subcategory $\mathscr{D}$. The map $E(\alpha,\beta)$ we obtain in this case is a deformation retract of $\mathrm{CN}(\mathscr{C})$ to $\mathrm{CN}(\mathscr{D})$ as semi-cyclic objects.

Theorem 2.3.4 (Special homotopy criterion). *Let $\mathscr{C}$ be a small k-linear category and $\mathscr{D}$ a sub-k-linear category of $\mathscr{C}$. Suppose that for each $C \in \mathscr{C}$ there exists an $n_C \in \mathbb{N}$, and for each $i = 1, \dots, n_C$ there are*

$$D_C(i) \in \mathscr{D}, \quad \alpha_i(C) \in \mathrm{Hom}_{\mathscr{C}}(C, D_C(i)), \quad \beta_i(C) \in \mathrm{Hom}_{\mathscr{C}}(D_C(i), C)$$

such that $\sum_{i=1}^{n_C} \beta_i(C) \circ \alpha_i(C) = \mathrm{id}_C$.

We define the semi-cyclic k-module map $E(\alpha,\beta): \mathrm{CN}(\mathscr{C}) \to \mathrm{CN}(\mathscr{C})$ by

$$E(\alpha,\beta)(f_0 \otimes \cdots \otimes f_n) = \sum (\alpha_{i_0}(C_0) \circ f_0 \circ \beta_{i_1}(C_1) \otimes \cdots \otimes \alpha_{i_n}(C_n) \circ f_n \circ \beta_{i_0}(C_0)),$$

where the sum is over all $(i_0, \dots, i_n)$ such that $1 \le i_j \le n_{C_j}$. The semi-cyclic map $E(\alpha,\beta)$ is special homotopic to the identity map of $\mathrm{CN}(\mathscr{C})$. In particular, if $\mathscr{D}$ is also assumed to be a full subcategory of $\mathscr{C}$ then the inclusion functor is a special homotopy equivalence with inverse $E(\alpha,\beta)$.

Proof. For $f \in \mathrm{Hom}_{\mathscr{C}}(C, C')$, $g \in \mathrm{Mor}(\mathscr{I})$, $1 \le i \le n_{C'}$ and $1 \le j \le n_C$ we define $\mu_{i,j}(f,g) \in \mathrm{Mor}(\mathscr{C})$ as

$$\begin{array}{rl} f & \text{if } g = \mathrm{id}_0 \text{ and } i = j = 1, \\ \alpha_i(C') \circ f & \text{if } g = (0 \to 1) \text{ and } j = 1, \\ f \circ \beta_j(C) & \text{if } g = (1 \to 0) \text{ and } i = 1, \\ \alpha_i(C') \circ f \circ \beta_j(C) & \text{if } g = \mathrm{id}_1, \\ 0 & \text{otherwise.} \end{array}$$

We define the semi-cyclic k-module map $h: \mathrm{CN}(\mathscr{C}) \otimes_k \mathrm{CN}(k\mathscr{I}) \to \mathrm{CN}(\mathscr{C})$ by

$$h((f_0 \otimes \cdots \otimes f_n) \otimes (g_0 \otimes \cdots \otimes g_n)) = \sum (\mu_{i_0,i_1}(f_0, g_0) \otimes \cdots \otimes \mu_{i_n,i_0}(f_n, g_n)),$$

where the sum is over all $(i_0, \dots, i_n)$ such that $1 \le i_j \le n_{C_j}$. We see that h is a special homotopy of $E(\alpha,\beta)$ with the identity and we are done. □

2.4. Applications of special homotopies

Naturally isomorphic functions

Proposition 2.4.1. *If $F,G: \mathscr{C} \to \mathscr{D}$ are naturally isomorphic k-linear functors of small k-linear categories then $\mathrm{CN}(F)$ and $\mathrm{CN}(G)$ are special homotopic. Thus, two equivalent k-linear categories have the same split Hochschild, cyclic, negative and periodic homology groups.*

Proof. Let σ be a natural isomorphism from F to G. We define h from $\mathrm{CN}(\mathscr{C}) \otimes \mathrm{CN}(k\mathscr{I})$ to $\mathrm{CN}(\mathscr{D})$ by

$$h((f_0 \otimes \cdots \otimes f_n) \otimes (g_0 \otimes \cdots \otimes g_n)) = (\mu(f_0, g_0) \otimes \cdots \otimes \mu(f_n, g_n))$$

where we define $\mu(f, g)$ by

$$\begin{array}{rl} F(f) & \text{if } g = \mathrm{id}_0, \\ \sigma_{\mathrm{Ran}(f)} \circ F(f) & \text{if } g = 0 \to 1, \\ F(f) \circ \sigma_{\mathrm{Dom}(f)}^{-1} & \text{if } g = 1 \to 0, \\ G(f) = \sigma_{\mathrm{Ran}(f)} \circ F(f) \circ \sigma_{\mathrm{Dom}(f)}^{-1} & \text{if } g = \mathrm{id}_1. \quad \square \end{array}$$

If we have a k-linear category $\mathscr{B}$ which is naturally equivalent to a small k-linear category $\mathscr{B}'$, we define $\mathrm{CN}(\mathscr{B}) \equiv \mathrm{CN}(\mathscr{B}')$; which is well defined up to special homotopy. We now assume that all our categories are naturally equivalent to a small category.

Example. It follows from the above that conjugation by an invertible element of a k-algebra A induces the identity map on Hochschild, cyclic, negative and periodic homology groups. This is done in [4] and as Theorem 2.12 of [12].

Cofinality

Let $\mathscr{P}$ be a full (k-linear) sub-category of the additive k-linear category $\mathscr{M}$. We say $\mathscr{P}$ is *cofinal* in $\mathscr{M}$ if for any $M \in \mathrm{Obj}(\mathscr{M})$ there exists a $M' \in \mathrm{Obj}(\mathscr{M})$ such that $M \oplus M' \in \mathrm{Obj}(\mathscr{P})$.

Proposition 2.4.2. *Let $\mathscr{P}$ be cofinal in $\mathscr{M}$, then the inclusion functor $F: \mathscr{P} \to \mathscr{M}$ induces a special homotopy equivalence* CN.F.

Proof. For each $M \in \mathrm{Obj}(\mathscr{M})$, choose $M' \in \mathrm{Obj}(\mathscr{M})$ such that $M \oplus M' \in \mathrm{Obj}(\mathscr{P})$ and if $P \in \mathrm{Obj}(\mathscr{P})$ then $P' = 0$. Define $\alpha(M): M \to M \oplus M'$ by $(\mathrm{id}_M, 0)$ and $\beta(M): M \oplus M' \to M$ by the projection. Then $\beta(M) \circ \alpha(M) = \mathrm{id}_M$ and so by the special homotopy criterion we are done. $\square$

Replacing A by $\mathscr{P}_A$

Proposition 2.4.3. *If A is a unitary k-algebra and $\mathscr{P}_A$ is the k-linear category of finitely generated projective right A-modules, then the natural inclusion from A to $\mathscr{P}_A$ induces a special homotopy equivalence from* $\mathrm{CN}(A)$ *to* $\mathrm{CN}(\mathscr{P}_A)$.

Proof. Let $i: \mathrm{CN}(A) \to \mathrm{CN}(\mathscr{P}_A)$ be the cyclic k-module map induced by the inclusion functor. For each $P \in \mathrm{Obj}(\mathscr{P}_A)$, choose an $n_P \in \mathbb{N}$ and maps $\alpha_j(P)$ of $\mathrm{Hom}_A(P, A)$ and $\beta_j(P)$ of $\mathrm{Hom}_A(A, P)$ for $j = 1, \ldots, n_p$ such that $\sum \alpha_j(P) \circ \beta_j(P) = \mathrm{id}_p$ (these maps can be obtained by expressing P as a direct summand of a finitely generated free module). For $P = A$, we choose $n_A = 1$ and $\alpha_1(A) = \mathrm{id}$. The image of $E(\alpha, \beta)$ of the special homotopy criterion is equal to the image of i and hence induces a cyclic module map from $\mathrm{CN}(\mathscr{P}_A)$ to $\mathrm{CN}(A)$ such that $E(\alpha, \beta) \circ i = \mathrm{id}_{\mathrm{CN}(A)}$ and $i \circ E(\alpha, \beta) = E(\alpha, \beta)$ which is special homotopic to the identity. □

Corollary 2.4.4. *There are natural isomorphisms* $\mathrm{HH}_n(A) \cong \mathrm{HH}_n^s(\mathscr{P}_A)$ *for all* $n \geq 0$ *and similarly for split cyclic, negative and periodic homology groups.* □

Corollary 2.4.5. *Hochschild, cyclic, negative and periodic homology groups are Morita invariant.* □

We would like to note that the case for matrices was done as 1.7 of [18] as well as by others. The general case was done in [21] and also independently by Kassel in [14] as Corollary 2.3 and as Proposition IV.6.2. of [15].

Matrices

Definition 2.4.6. For $\mathscr{A}$ a k-linear category and $n \in \mathbb{N}$, we define the matrix category $M_n(\mathscr{A})$ by $\mathrm{Obj}(M_n(\mathscr{A})) = \mathrm{Obj}(\mathscr{A})^n$ and $\mathrm{Hom}_{M_n(\mathscr{A})}((A_1, \ldots, A_n)(B_1, \ldots, B_n))$ is the set of all $n \times n$ "matrices" of the form $(\alpha_{i,j})$ where $\alpha_{i,j} \in \mathrm{Hom}_{\mathscr{A}}(A_j, B_i)$ and composition is defined by

$$(\alpha \circ \beta)_{i,j} = \sum_{k=1}^{n} \alpha_{i,k} \circ \beta_{k,j}$$

(i.e. matrix multiplication).

Proposition 2.4.7. *Let* $\mathscr{A}$ *be a k-linear category,* $\xi \in \mathrm{Obj}(\mathscr{A})$, *and* $n \in \mathbb{N}$. *Let* I *be the k-linear functor from* $\mathscr{A}$ *to* $M_n(\mathscr{A})$ *given by taking an object* A *of* $\mathscr{A}$ *to* $(A, \xi, \ldots, \xi)$ *and a morphism* f *to* $I(f)$ *where* $(I(f))_{i,j} = f$ *if* $i = j = 1$ *and* 0 *otherwise. The induced map* $\mathrm{CN}(I)$ *is a special homotopy equivalence.*

Proof. Given $\vec{A} = (A_1, \ldots, A_n) \in \mathrm{Obj}(M_n(\mathscr{A}))$ and $1 \leq t \leq n$ we define $\beta_t(\vec{A}) : (A_t, \xi, \ldots, \xi) \to \vec{A}$ and $\alpha_t(\vec{A}) : \vec{A} \to (A_t, \xi, \ldots, \xi)$ by

$$[\beta_t(\vec{A})]_{i,j} = \begin{cases} \mathrm{id}: A_t \to A_t & \text{for } i = t \text{ and } j = 1, \\ 0: A_t \to A_i & \text{for } i \neq t \text{ and } j = 1, \\ 0: \xi \to A_i & \text{otherwise,} \end{cases}$$

$$[\alpha_t(\vec{A})]_{i,j} = \begin{cases} \mathrm{id}: A_t \to A_t & \text{for } i = 1 \text{ and } j = t, \\ 0: A_j \to A_t & \text{for } i = 1 \text{ and } j \neq t, \\ 0: A_j \to \xi & \text{otherwise.} \end{cases}$$

We see that $\sum \beta_t(\vec{A})\circ\alpha_t(\vec{A}) = \mathrm{id}_{\vec{A}}$. For $f\in \mathrm{Hom}_{M_n(\mathscr{A})}(\vec{A},\vec{B})$, we note that

$$[\alpha_s(\vec{B})\circ f\circ \beta_t(\vec{A})]_{i,j} = \begin{cases} f_{s,t}: A_t \to B_s & \text{for } i=j=1,\\ 0:\xi\to\xi & \text{otherwise.}\end{cases}$$

By the special homotopy criterion there is a special homotopy equivalence $E(\alpha,\beta)$ which implies the result since $E(\alpha,\beta)\circ \mathrm{CN}(I)$ is the identity on $\mathrm{CN}(\mathscr{A})$. □

Note. Proposition 2.4.7 is just a straightforward generalization of the Dennis trace map for Hochschild homology. See for example [5].

Terminology. In order to formulate the next several applications, it is convenient to introduce the following terminology. A k-linear *semi-functor* F between two k-linear categories is a k-linear "functor" which does not necessarily take identity morphisms to identity morphisms. By abuse of notation, we let $\mathrm{CN}(F)$ denote the map of *semi-cyclic* modules naturally induced by F.

Twisted products

Given a k-linear functor F from $\mathscr{A}$ to $\mathscr{B}$, we define the "twisted" product category $\mathscr{A}_F\mathscr{B}$ as follows. We set $\mathrm{Obj}(\mathscr{A}_F\mathscr{B})$ to be $\mathrm{Obj}(\mathscr{A})\times\mathrm{Obj}(\mathscr{B})$ and

$$\mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((A,B),(A',B')) = \mathrm{Hom}_{\mathscr{A}}(A,A')\oplus \mathrm{Hom}_{\mathscr{B}}(B,B')\oplus \mathrm{Hom}_{\mathscr{B}}(F(A),B')$$

with composition defined by $(f,g,h)\circ(f',g',h') = (f\circ f', g\circ g', h\circ F(f') + g\circ h)$.

Proposition 2.4.8. *Assume $\mathscr{A}$ contains at least one object. Then the natural cyclic k-module map from $\mathrm{CN}(\mathscr{A}_F\mathscr{B})$ to $\mathrm{CN}(\mathscr{A})\oplus\mathrm{CN}(\mathscr{B})\cong\mathrm{CN}(\mathscr{A}\cup\mathscr{B})$ (see Example 2.2.5) is a special homotopy equivalence.*

Proof. We let p denote the cyclic k-module map from $\mathrm{CN}(\mathscr{A}_F\mathscr{B})$ to $\mathrm{CN}(\mathscr{A})\oplus\mathrm{CN}(\mathscr{B})$ defined by

$$((f_0,h_0,h_0)\otimes\cdots\otimes(f_n,g_n,h_n)) \to (f_0\otimes\cdots\otimes f_n)\oplus(g_0\otimes\cdots\otimes g_n).$$

Choose some $a\in\mathrm{Obj}(\mathscr{A})$ and define the k-linear semi-functor F_a from $\mathscr{A}\cup\mathscr{B}$ to $\mathscr{A}_F\mathscr{B}$ by sending an object A of $\mathscr{A}$ to $(A,F(a))$ and a morphism f of $\mathscr{A}$ to $(f,0,0)$. Similarly define F_a on objects and morphisms of $\mathscr{B}$ using a instead of $F(a)$. We note that $p\circ\mathrm{CN}(F_a) = \mathrm{id}$. Given $(A,B)\in\mathrm{Obj}(\mathscr{A}_F\mathscr{B})$, we define

$$\alpha_1(A,B) = (\mathrm{id}_A,0,0)\in\mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((A,B),(A,F(a))),$$

$$\beta_1(A,B) = (\mathrm{id}_A,0,0)\in\mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((A,F(a)),(A,B)),$$

$$\alpha_2(A,B) = (0,\mathrm{id}_B,0)\in\mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((A,B),(a,B)),$$

$$\beta_2(A,B) = (0,\mathrm{id}_B,0)\in\mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((a,B),(A,B)).$$

Since $\sum \beta_i(A,B)\circ\alpha_i(A,B)\circ\alpha_i(A,B) = \mathrm{id}_{(A,B)}$ the special homotopy criterion produces a special homotopy equivalence $E(\alpha,\beta)$. For any $(f,g,h)\in \mathrm{Hom}_{\mathscr{A}_F\mathscr{B}}((A,B),(A',B'))$

$$\alpha_1(A,B)\circ(f,g,h)\circ\beta_1(A',B') = (f, F(a)\xrightarrow{0} F(a), F(A)\xrightarrow{0} F(a)),$$

$$\alpha_2(A,B)\circ(f,g,h)\circ\beta_2(A',B') = (a\xrightarrow{0} a, g, F(a)\xrightarrow{0} B'),$$

$$\alpha_2(A,B)\circ(f,g,h)\circ\beta_1(A',B') = 0.$$

Now we note that in the formula for $E(\alpha,\beta)$, if a summand contains an element of the form $\alpha_1(A,B)\circ(f,g,h)\circ\beta_2(A',B')$ then it must also contain an element of the form $\alpha_2(A,B)\circ(f,g,h)\circ\beta_1(A',B')$ and hence be zero. Thus, $E(\alpha,\beta) = \mathrm{CN}(F_a)\circ p$ and by the special homotopy criterion we are done. $\square$

Finite products

Proposition 2.4.9. *If $\mathscr{A}$ and $\mathscr{B}$ each contain at least one object, then the natural map of cyclic modules p from $\mathrm{CN}(\mathscr{A}\times\mathscr{B})$ to $\mathrm{CN}(\mathscr{A})\oplus\mathrm{CN}(\mathscr{B})$ is a special homotopy equivalence. This is similar to* I.4 *of* [15].

Proof. We let T denote the k-linear functor from $\mathscr{A}$ to $\mathscr{B}_+$ (see Example 2.2.5) defined by sending every object of $\mathscr{A}$ to "+". Since $\mathscr{A}_T(\mathscr{B}_+)$ is naturally isomorphic to $\mathscr{A}\times(\mathscr{B}_+)$ we see by Proposition 2.4.8 that we have a special homotopy equivalence $\mathrm{CN}(\mathscr{A}\times(\mathscr{B}_+)) \cong \mathrm{CN}(\mathscr{A}_T(\mathscr{B}_+)) \xrightarrow{\simeq} \mathrm{CN}(\mathscr{A}\cup\mathscr{B})$. Since $\mathscr{A}\times\mathscr{B}$ is naturally a full sub-category of $\mathscr{A}\times(\mathscr{B}_+)$ and every object of the latter category is a retract of an object of the former, we see by cofinality (Proposition 2.4.2) that the natural inclusion induces a special homotopy equivalence $\mathrm{CN}(\mathscr{A}\times\mathscr{B}) \xrightarrow{\simeq} \mathrm{CN}(\mathscr{A}\times(\mathscr{B}_+))$. Since the composition of special homotopy equivalences is again a special homotopy equivalence we are done.

Upper triangular matrices

Definition. We let $T_n(\mathscr{A})$ denote the k-linear sub-category of $M_n(\mathscr{A})$ (see Definition 2.4.6) which has the same objects but whose morphisms are the upper triangular matrices. That is, $\mathrm{Mor}(T_n(\mathscr{A}))\subset\mathrm{Mor}(M_n(\mathscr{A}))$ is the subset of "matrices" α with the property that $\alpha_{i,j}=0$ for $j<i$.

Proposition 2.4.10. *If $\mathscr{A}$ has at least one object, then the natural cyclic module map p from $\mathrm{CN}(T_n(\mathscr{A}))$ to $\mathrm{CN}(\mathscr{A})^{\oplus n}$ is a special homotopy equivalence.*

Proof. First assume $\mathscr{A}$ has a zero object "+". Let G denote the k-linear functor from $\mathscr{A}$ to $T_{n-1}(\mathscr{A})$ defined by sending an object A to $(A, +, \ldots, +)$. The categories $T_n(\mathscr{A})$ and $\mathscr{A}_G T_{n-1}(\mathscr{A})$ are naturally isomorphic and thus by Proposition 2.4.8 the natural cyclic module map $\mathrm{CN}(T_n(\mathscr{A})) \to \mathrm{CN}(\mathscr{A} \cup T_{n-1}(\mathscr{A}))$ is a special homotopy equivalence. Thus, if $\mathscr{A}$ has a zero object the result follows by induction.

If $\mathscr{A}$ does not have a zero object, then we note that $T_n(\mathscr{A})$ can be naturally identified with a full sub-category of $T_n(\mathscr{A}_+)$ and every object of the latter is a retract of an object of the former. Thus, by cofinality (Proposition 2.4.2) the natural inclusion produces a composite of special homotopy equivalences

$$\mathrm{CN}(T_n(\mathscr{A})) \xrightarrow{\simeq} \mathrm{CN}(T_n(\mathscr{A}_+)) \xrightarrow{\simeq} \mathrm{CN}(\mathscr{A}_+)^{\oplus n} \cong \mathrm{CN}(\mathscr{A})^{\oplus n}. \qquad \square$$

Alternate proof. One can show that there are natural isomorphisms of k-algebras $\mathrm{Arr}_k(M_n(\mathscr{A})) \cong M_n(\mathrm{Arr}_k(\mathscr{A}))$ and $\mathrm{Arr}_k(T_n(\mathscr{A})) \cong T_n(\mathrm{Arr}_k(\mathscr{A}))$ (see Example 2.2.6) and since these are k-algebras with local units, they are H-unital in the sense of Wodzicki and the Propositions 2.4.7 and 2.4.10 can then be obtained (essentially) from Corollaries 9.8 and 11.3 of [31].

Corollary 2.4.11. *Let ϕ be the k-linear functor from $T_n(\mathscr{A})$ to $\mathscr{A}^n$ defined by the identity on objects and sending an upper triangular matrix to its diagonal. Let ψ be the k-linear functor from $\mathscr{A}^n$ to $T_n(\mathscr{A})$ defined by the identity on objects and sending a morphism to the corresponding diagonal matrix. Thus, the composite $\psi \circ \phi$ is the endo-functor of $T_n(\mathscr{A})$ which is the identity on objects and sends a matrix to its diagonal. Since $p \circ \mathrm{CN}(\psi \circ \phi) = p$ we conclude that $\mathrm{CN}(\psi \circ \phi)$ is a special homotopy equivalence. Since $\phi \circ \psi = \mathrm{id}_{\mathscr{A}^n}$ we can conclude further that both $\mathrm{CN}(\psi)$ and $\mathrm{CN}(\phi)$ are special homotopy equivalences.*

Note. The composite $\mathrm{HH}^{\mathrm{s}}_*(\mathscr{A}) \cong \mathrm{HH}^{\mathrm{s}}_*(T_n(\mathscr{A})) \to \mathrm{HH}^{\mathrm{s}}_*(M_n(\mathscr{A})) \cong \mathrm{HH}^{\mathrm{s}}_*(\mathscr{A})$ obtained by Proposition 2.4.10 and the natural inclusion $T^{\mathrm{s}}_n(\mathscr{A}) \to M^{\mathrm{s}}_n(\mathscr{A})$ is simply addition. The same is true if we apply this composition to split cyclic, negative or periodic homologies.

3. Definition and first properties

Introduction

In this section we give our definition for the Hochschild (cyclic, negative, periodic) homology of an exact category. This is done by combining the S construction of Waldhausen [28] with the additive cyclic nerve of Section 2. We first review the S construction for a category with cofibrations. After defining our homology groups, we list a few immediate consequences as a series of lemmas. An important theorem of this section is that the homology groups of a split (i.e. semi-simple) exact category agrees with its split homology groups as defined in Section 2 (this is false in general, see Example 3.3.5). Thus, we can recover the Hochschild (cyclic, negative, periodic) homology groups of a unital k-algebra A by considering the homology groups of the exact category of finitely generated projective modules of A.

The additivity theorem for algebraic K-theory states that if $F'' \to F \to F'$ is a short exact sequence of exact functors, then $K(F)$ is naturally homotopic to $K(F') \oplus K(F'')$. This was proved for exact categories by Quillen in [24, Section 3] and for categories with cofibrations by Waldhausen in [28, Section 1.4]. Our second main goal in this section is to prove the analog of the additivity theorem for our Hochschild (cyclic, negative and periodic) homology groups (of a k-linear category with cofibrations). The "classical" proofs of [24] and [28] were not "additive" enough for our purposes and so we needed to design another (see [22]). The proof here uses a slight variant of Quillen's "theorem A" (which we provide in Section 3.4) and a two-stage homotopy construction whose first part was suggested to us by Grayson. We then follow the treatment of [28, Section 1.5] to deduce several useful corollaries from the additivity theorem. The proofs found in Section 3.6 are essentially transliterations of the corresponding statements found in [28].

3.1. The S construction

We recall some definitions and facts about Waldhausen's S construction as found in [28]. A category with cofibrations $\mathscr{C}$ is a category with a distinguished zero object together with a subcategory $co\mathscr{C}$ satisfying the axioms (Cof 1)–(Cof 3) below. The feathered arrows "$\rightarrowtail$" will be used to denote the morphisms in $co\mathscr{C}$ and will be called the *cofibrations* in $\mathscr{C}$.

(Cof 1) The isomorphisms in $\mathscr{C}$ are cofibrations
(Cof 2) For every $A \in \mathrm{Obj}(\mathscr{C})$, the arrow $0 \to A$ is a cofibration
(Cof 3) Cofibrations admit cobase change. This means the following two things. If $A \rightarrowtail B$ is a cofibration, and $A \to C$ any arrow, then firstly the pushout $C \amalg_A B$ exists in $\mathscr{C}$, and secondly the canonical arrow $C \to C \amalg_A B$ is a cofibration again.

If $A \rightarrowtail B$ is a cofibration, then B/A will denote any representative of $0 \amalg_A B$ and the arrows like "$\twoheadrightarrow$" are reserved to denote the quotient map $B \twoheadrightarrow B/A$. A cofibration sequence is a sequence $A \rightarrowtail B \twoheadrightarrow B/A$ where $B \twoheadrightarrow B/A$ is the quotient map associated to $A \rightarrowtail B$. A functor between categories with cofibrations is called *exact* if it preserves all the relevant structure: it takes 0 to 0, cofibrations to cofibrations, and it preserves the pushout diagrams which arise from axiom Cof 3.

The examples of categories with cofibrations which we are primarily concerned with here are those of an exact category in the sense of Quillen [24]. Any exact category can be considered as a category with cofibrations by choosing a zero object, and declaring the admissible monomorphisms to be cofibrations.

We will call a category $\mathscr{C}$ a *k-linear category with cofibrations* if it is a k-linear category and a category with cofibrations. It follows from Cof 3 that such a category has direct sums, and thus it is an additive category. Therefore, one can naturally associate an exact category to it by declaring the cofibration sequences to be the short exact sequences. We do *not* insist that $co\mathscr{C}$ be a k-linear subcategory.

Given a cofibered category $\mathscr{C}$, we form a simplicial category with cofibrations $S.\mathscr{C}$ as follows. Let $[n]$ denote the ordered set $(0 < 1 < \cdots < n)$ (which we think of as a category), let $\mathscr{A}r[n]$ denote the category of arrows in $[n]$, and let (j/i) denote the arrow from i to j in $[n]$, for $i \le j$. We consider the functors $A : \mathscr{A}r[n] \to \mathscr{C}$ which send $(i/j) \to A_{i,j}$ having the property that for every j, $A_{j,j} = 0$, and for every triple $i \le j \le k$, the map $A_{i,j} \to A_{i,k}$ is a cofibration, and the diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ 0 = A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout; in other words, $A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k}$ is a cofibration sequence. We write $S_n\mathscr{C}$ for the category of these functors and all their natural transformations. To give an object $A \in S_n\mathscr{C}$ is really the same as giving a sequence of cofibrations

$$0 = A_{0,0} \rightarrowtail A_{0,1} \rightarrowtail A_{0,2} \rightarrowtail \cdots \rightarrowtail A_{0,n}$$

together with a choice of subquotients $A_{i,j} = A_{0,j}/A_{0,i}$. The simplicial category $S.\mathscr{C}$ defined by $[n] \to S_n\mathscr{C}$ is naturally a simplicial category with cofibrations (see [28, 1.3]).

3.2. *Homology groups of a category with cofibrations*

For this section, $\mathscr{C}$ and $\mathscr{D}$ denote k-linear categories with cofibrations.

If $\mathscr{A}$ is a small k-linear category and $\mathscr{J}$ is a small category, then any sub-category of the category of functors from $\mathscr{J}$ to $\mathscr{A}$ (where the morphisms are natural transformations) is naturally a k-linear category by defining $(x\sigma)_C \equiv x(\sigma_C)$ for a natural transformation σ, $x \in k$ and $C \in \mathrm{Obj}(\mathscr{C})$. Thus, if $\mathscr{C}$ is a k-linear category with cofibrations, then $S.\mathscr{C}$ is a simplicial k-linear category.

Definition 3.2.1. *We define the* nth *Hochschild* (*cyclic, negative, periodic*) *homology of* $\mathscr{C}$ (with coefficients in k) to be the $(n+1)$st "Hochschild" (cyclic, negative, periodic) homology of thc simplicial $\times$ cyclic k-module $\mathrm{CN}.S.\mathscr{C}$. The shift of one dimension corresponds to the looping in algebraic K-theory.

Lemma 3.2.2. *There exists a natural diagram* (*with exact rows*)

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathrm{HN}_n(\mathscr{C}) & \longrightarrow & \mathrm{HP}_n(\mathscr{C}) & \longrightarrow & \mathrm{HC}_{n-2}(\mathscr{C}) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \| & & \\ \cdots & \longrightarrow & \mathrm{HH}_n(\mathscr{C}) & \longrightarrow & \mathrm{HC}_n(\mathscr{C}) & \longrightarrow & \mathrm{HC}_{n-2}(\mathscr{C}) & \longrightarrow & \cdots \end{array}$$

□

Lemma 3.2.3. *For an exact k-linear functor F from $\mathscr{C}$ to $\mathscr{D}$, the following are equivalent:*
(a) $\mathrm{HH}_n(F)$ *is an isomorphism for all* $n \in \mathbb{Z}$,
(b) $\mathrm{HC}_n(F)$ *is an isomorphism for all* $n \in \mathbb{Z}$,
(c) $\mathrm{HN}_n(F)$ *is an isomorphism for all* $n \in \mathbb{Z}$;
and if these conditions hold, then $\mathrm{HP}_n(F)$ *is an isomorphism for all* $n \in \mathbb{Z}$ *as well.* □

Let $\mathscr{J}$ be a small category which is filtering and $j \to \mathscr{C}_j$ a functor from $\mathscr{J}$ to small k-linear categories with cofibrations and exact k-linear functors. If $\mathscr{C}$ is the inductive limit of the $\mathscr{C}_j$, then $\mathscr{C}$ is made into a category with cofibrations by letting $co\mathscr{C}$ be $\varinjlim co\mathscr{C}_j$. We see that $S.\mathscr{C} = \varinjlim S.\mathscr{C}_j$ and so the following lemma follows from Example 2.2.2.

Lemma 3.2.4. *There exists a natural isomorphism* $\varinjlim \mathrm{CN}.S.\mathscr{C}_j \cong \mathrm{CN}.S.(\varinjlim \mathscr{C}_j)$. □

Terminology. Rather than invent artificial notions for maps of cyclic $\times$ simplicial sets, we resort to topology. Recall (say from [10]) that the topological group S^1 acts on the geometric realization $|X| \to |Y|$. We shall say that an S^1-equivariant map f is a *discrete* homotopy equivalence if the induced map of fix point sub-spaces for every finite subgroup C_r of S^1 is a homotopy equivalence. Any special homotopy equivalence (Definition 2.3.2) is a discrete homotopy equivalence; see Lemma A.5.2. Since a discrete homotopy equivalence is a homotopy equivalence (take $C_r = 1$), it induces isomorphisms on homotopy groups. Thus a discrete homotopy equivalence $\mathrm{CN}.S.\mathscr{C} \to \mathrm{CN}.S.\mathscr{D}$ will induce isomorphisms $\mathrm{HH}_*(\mathscr{C}) \xrightarrow{\cong} \mathrm{HH}_*(\mathscr{D})$ and therefore isomorphisms on the associated cyclic, negative and periodic homology groups as well.

Lemma 3.2.5. *There is a natural discrete homotopy equivalence* $\mathrm{CN}.S.(\mathscr{C} \times \mathscr{D}) \to \mathrm{CN}.S.\mathscr{C} \times \mathrm{CN}.S.\mathscr{D}$.

Proof. We first note that $S.(\mathscr{C} \times \mathscr{D}) \cong S.\mathscr{C} \times S.\mathscr{D}$ and that by Proposition 2.4.9 the natural cyclic map is degree-wise a special homotopy equivalence. The result now follows from the realization lemma (Lemma A.6.4). □

Lemma 3.2.6. *Naturally isomorphic exact k-linear functors induce the same maps of Hochschild, cyclic, negative and periodic homology groups. Thus, two equivalent k-linear categories with cofibrations have the same Hochschild, cyclic, negative and periodic homology groups.*

Proof. Let F and G be exact k-linear functors from $\mathscr{C}$ to $\mathscr{D}$ and $\sigma : F \to G$ a natural isomorphism. For each $n \in \mathbb{N}$, the functors $F_n, G_n : S_n\mathscr{C} \to S_n\mathscr{D}$ have an induced natural isomorphism σ_n. The proof of Proposition 2.4.1 produces special homotopies $h(n)$ from $\mathrm{CN}.S_nF$ to $\mathrm{CN}.S_nG$ for each $n \in \mathbb{N}$ which commute with the face and degeneracy operators of the S construction. The result now follows from Lemma A.6.2. □

Lemma 3.2.7 (Cofinality). *Let $\mathscr{D}$ be a k-linear exact sub-category of $\mathscr{C}$ such that $S_n\mathscr{D}$ is cofinal in $S_n\mathscr{C}$ for each $n \in \mathbb{N}$, then the natural inclusion $\mathrm{CN}.S.\mathscr{D} \to \mathrm{CN}.S.\mathscr{C}$ is a discrete homotopy equivalence. In particular, this holds if either*

(a) *$\mathscr{D}$ is strictly cofinal in $\mathscr{C}$ (that is, $\mathscr{D}$ is full and for every $D \in \mathrm{Obj}(\mathscr{D})$ there exists $C \in \mathscr{C}$ such that $D \oplus C \in \mathrm{Obj}(\mathscr{C})$), or*

(b) *$\mathscr{D}$ is a cofinal exact sub-category closed under exact sequences and extensions.*

Proof. The general statement follows from Proposition 2.4.3 and Lemma A.6.4. Part (a) is assertion 1 from Proposition 1.5.9 of [28] and part (b) follows from the argument found in Theorem 6.1 of [9]. □

3.3. Split exact categories

Definition 3.3.1. We will call an exact category *split* exact or *semi-simple* if all the short exact sequences split.

Theorem 3.3.2. *Let $\mathscr{M}$ be a split exact k-linear category. Let $\Phi : S_n\mathscr{M} \to \mathscr{M}^n$ be the exact k-linear functor*

$$(0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \to M_n) \xrightarrow{\Phi} (M_1, M_2/M_1, \ldots, M_n/M_{n-1}).$$

The induced map $\mathrm{CN}.(\Phi) : \mathrm{CN}.S_n\mathscr{M} \to \mathrm{CN}.\mathscr{M}^n$ is a special homotopy equivalence.

Proof. Define the exact functor $\Psi : \mathscr{M}^n \to S_n\mathscr{M}$ by

$$(M_1, M_2, \ldots, M_n) \xrightarrow{\Psi} (M_1 \to M_1 \oplus M_2 \to \cdots \to M_1 \oplus M_2 \oplus \cdots \oplus M_n)$$

with the obvious auxiliary data. Thus $\Phi \circ \Psi = \mathrm{id}$ and we want to show that $\Psi \circ \Phi$ is special homotopic to the identity.

Reduction step. Let $S_n'\mathscr{M}$ be the *full* subcategory generated by the image of Ψ. Then $S_n'\mathscr{M}$ is naturally equivalent to $S_n\mathscr{M}$ since for every element of $S_n\mathscr{M}$ we have an isomorphism

$$\begin{array}{ccccccccc}
0 = M_0 & \rightarrowtail & M_2 & \rightarrowtail & M_2 & \rightarrowtail & \cdots & \rightarrowtail & M_n \\
\downarrow\cong & & \downarrow\cong & & \downarrow\cong & & & & \downarrow\cong \\
0 = M_0 & \rightarrowtail & M_1 & \rightarrowtail & M_1 \oplus M_2/M_1 & \rightarrowtail & \cdots & \rightarrowtail & M_1 \oplus \cdots \oplus M_n/M_{n-1}
\end{array}$$

Using Proposition 2.4.1, $\mathrm{CN}.S_n\mathscr{M}$ "special" deformation retracts to $\mathrm{CN}.S_n'\mathscr{M}$ and so it suffices to show $\Psi \circ \Phi$ is a special homotopy equivalence on $S_n'\mathscr{M}$.

We now note that $S_n'\mathscr{M}$ is naturally isomorphic to $T_n(\mathscr{M})$ (Proposition 2.4.10) by sending

$$(M_1 \to M_1 \oplus M_2 \to \cdots \to M_1 \oplus M_2 \oplus \cdots \otimes M_n) \quad \text{to} \quad (M_1, M_2, \ldots, M_n)$$

(this is well defined since we are now using the category $S'_n\mathcal{M}$). By this isomorphism, $\Psi \circ \Phi$ is identified to $\psi \circ \phi$ of Corollary 2.4.11 (i.e. the map which replaces the off-diagonal matrix elements with 0) and hence is special homotopic to the identity. □

Given a pointed simplicial set X, we let PX denote the simplicial path space of X. Therefore, $PX_n = X_{n+1}$, the ith face map of PX is d_{i+1} of X and the ith degeneracy map of PX is s_{i+1} of X. We note that PX is contractible to the trivial simplicial set $[n] \to X_0$: the contraction is constructed from the unused degeneracy s_0. There is a simplicial map from PX to X which applies the unused face map d_0. There is also a map of the trivial simplicial set $[n] \to X_1$ to PX since $PX_0 = X_1$. Thus, given a cofibered category $\mathscr{C}$, we obtain a sequence of simplicial maps $\mathscr{C} \to PS.\mathscr{C} \to S.\mathscr{C}$. Now assume that $\mathscr{C}$ is also a k-linear category, then $PS.\mathscr{C}$ is a simplicial k-linear category and since d_0 is a k-linear functor, we obtain a sequence $\mathrm{CN}.\mathscr{C} \to \mathrm{CN}.PS.\mathscr{C} \to \mathrm{CN}.S.\mathscr{C}$ of bi-simplicial $\times$ cyclic k-modules. Since the contraction of $PS.\mathscr{C}$ is constructed by using s_0, which is a k-linear functor, we see that $\mathrm{CN}_m PS.\mathscr{C}$ is contractible for all m and hence $\mathrm{HH}_n(\mathrm{CN}.PS.\mathscr{C}) = 0$ for all $n \in \mathbb{N}$ (similarly for cyclic, negative and periodic homology groups).

Terminology. We will call a sequence of pointed (base point is fixed by the S^1-action) S^1 maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ a *discrete* quasi-fibration if $g \circ f = *$ and the natural map from the homotopy fiber to X (which is naturally an S^1-map since the base-point is fixed by the action) is a discrete homotopy equivalence. We note that a sequence of simplicial $\times$ cyclic k-modules which is a discrete quasi-fibration naturally produces long exact sequences of Hochschild, cyclic, negative and periodic homology groups.

Theorem 3.3.3. *If $\mathcal{M}$ is a split exact k-linear category, then the sequence*

$$\mathrm{CN}.\mathcal{M} \xrightarrow{F} \mathrm{CN}.PS.\mathcal{M} \xrightarrow{G} \mathrm{CN}.S.\mathcal{M}$$

is a discrete quasi-fibration.

Proof. (This proof is modeled after Proposition 1.5.5 of [28].)

We will show that the sequence $\mathrm{CN}.\mathcal{M} \xrightarrow{F} \mathrm{CN}.PS_m\mathcal{M} \xrightarrow{G} \mathrm{CN}.S_m\mathcal{M}$ is a discrete quasi-function for all m, and this will imply the result by the fibration Lemma A.6.5. We record the maps F and G explicitly as

$$F(M) = (0 \rightarrowtail M = M = \cdots = M),$$

$$G(0 = M_0 \rightarrowtail M_1 \rightarrowtail \cdots \rightarrowtail M_m) = (0 = M_1/M_1 \rightarrowtail M_2/M_1 \rightarrowtail \cdots \to M_m/M_1)$$

and note that $G \circ F = 0$. Using the map Φ defined in Theorem 3.3.2 composed with the natural map $\mathrm{CN}.(\mathcal{M}^m) \to (\mathrm{CN}.\mathcal{M})^m$ of Proposition 2.4.9, we obtain the following commutative diagram:

$$\begin{tikzcd}
\mathrm{CN}.\mathcal{M} \arrow[r, "F"] \arrow[d, equal] & \mathrm{CN}.PS_m\mathcal{M} \arrow[r, "G"] \arrow[d, "\Phi"] & \mathrm{CN}.S_m\mathcal{M} \arrow[d, "\Phi"] \\
\mathrm{CN}.\mathcal{M} \arrow[r, "F'"] & (\mathrm{CN}.\mathcal{M})^{m+1} \arrow[r, "G'"] & (\mathrm{CN}.\mathcal{M})^m
\end{tikzcd}$$

where F' imbeds into the first coordinate (putting 0 in the other positions) and G' forgets the first coordinate. The bottom row is an S^1-fibration sequence, and since $\mathcal{M}$ is split, Theorem 3.3.2 (and Proposition 2.4.9) tells us the two vertical maps Φ are special homotopy equivalences and therefore the top row is a discrete quasi-fibration. □

Corollary 3.3.4. *If $\mathcal{M}$ is a split exact k-linear category, then there exists natural isomorphisms $\mathrm{HH}^{\mathrm{s}}_*(\mathcal{M}) \cong \mathrm{HH}_*(\mathcal{M})$ (similarly for cyclic, negative and periodic homology groups). In particular, for A a unital k-algebra, Corollary 2.4.5 implies there are natural isomorphisms $\mathrm{HH}_*(A) \cong \mathrm{HH}_*(\mathcal{P}_A)$ (and similarly for cyclic, negative and periodic homology groups).* □

Example 3.3.5. This example is to demonstrate that Corollary 3.3.4 is not necessarily true if one drops the hypothesis that $\mathcal{M}$ is a split category. Let $\mathcal{Ab}_{\mathrm{f}}$ be the $\mathbb{Z}$-linear category of finitely generated $\mathbb{Z}$-modules (i.e. the category of finitely generated abelian groups). Let $\mathcal{F}$ be the sub-category of free modules. For $p \in \mathbb{N}$ prime, let $\mathcal{F}_p$ be the sub-$\mathbb{Z}$-linear-category of $\mathcal{Ab}_{\mathrm{f}}$ generated by the groups killed by p. We have a $\mathbb{Z}$-linear (but *not* exact) functor F_p from $\mathcal{Ab}_{\mathrm{f}}$ to $\mathcal{F}_p$ given by $F_p(A) = \ker(A \xrightarrow{p} A)$. Since the inclusion of $\mathcal{F}_p$ is a section to F_p, we see that $\mathrm{HH}^{\mathrm{s}}_*(\mathcal{Ab}_{\mathrm{f}}) \cong \mathrm{HH}^{\mathrm{s}}_*(\mathcal{F}_p) \oplus$ (something).

The category $\mathcal{F}_p$ is isomorphic to the category of finitely generated $\mathbb{Z}/p\mathbb{Z}$-modules. By Corollary 2.4.5 we see that $\mathrm{HH}^{\mathrm{s}}_*(\mathcal{F}_p) \cong \mathrm{HH}_*(\mathbb{Z}/p\mathbb{Z})$ which is $\mathbb{Z}/p\mathbb{Z}$ in degree 0 and 0 otherwise. We also see that $\mathrm{HH}^{\mathrm{s}}_*(\mathcal{F}) \cong \mathrm{HH}_*(\mathbb{Z})$ which is $\mathbb{Z}$ in degree 0 and 0 otherwise.

Claim. There exists a natural isomorphism $\mathrm{HH}_0(\mathcal{F}) \cong HH_0(\mathcal{Ab}_{\mathrm{f}})$ induced by the inclusion.

Assuming the claim, Theorem 3.3.3 tells us that

$$\mathbb{Z} \cong \mathrm{HH}^{\mathrm{s}}_0(\mathcal{F}) \cong \mathrm{HH}_0(\mathcal{F}) \cong \mathrm{HH}_0(\mathcal{Ab}_{\mathrm{f}}) \not\cong \mathrm{HH}^{\mathrm{s}}_0(\mathcal{Ab}_{\mathrm{f}}) \cong \mathbb{Z}_p \oplus (\text{something})$$

and therefore Corollary 3.3.4 does not hold for the exact $\mathbb{Z}$-linear category $\mathcal{Ab}_{\mathrm{f}}$.

Proof of claim. Choose $[A \xrightarrow{\alpha} A] \in \mathrm{HH}_0(\mathcal{Ab}_{\mathrm{f}})$. Let $F' \rightarrowtail F \twoheadrightarrow A$ be a resolution of A by free modules. We obtain an induced commutative diagram:

$$\begin{tikzcd}
F' \arrow[r, tail] \arrow[d, "\gamma"] & F \arrow[r, two heads] \arrow[d, "\beta"] & A \arrow[d, "\alpha"] \\
F' \arrow[r, tail] & F \arrow[r, two heads] & A
\end{tikzcd}$$

By standard K-theory arguments, we see that $[A \xrightarrow{\alpha} A] = [F \xrightarrow{\beta} F] - [F \xrightarrow{\gamma} F']$ and that this is independent of the choice of resolution *and* extensions of α. Thus we have a well-defined inverse map and the result follows.

3.4. Theorem A

In this section, we essentially redo "theorem A" of [24] but keeping track of certain additive conditions we will want in applying it to the additive cyclic nerve. Our treatment here follows the methods of [6] and we liberally adopt their notation.

Notation. If X is a simplicial set, we let XR and XL denote the bisimplicial sets $XR([m],[n]) = X([n])$ and $XL([m],[n]) = X([m])$ (with trivial simplicial maps in the first and second variables respectively). If X is a simplicial k-linear category, then the natural maps of bi-simplicial $\times$ cyclic k-modules $\mathrm{CN}.X \to \mathrm{CN}.XL$ ($\mathrm{CN}.XR$) are S^1-homotopy equivalences.

Definition 3.4.1. We will call two (semi-) simplicial k-linear category maps $F,G : X \to Y$ *k-linearly homotopic* if they are (semi-) simplicially homotopic by a homotopy $H = \{H_i\}$ such that the maps $H_i : X_n \to Y_{n+1}$ (see Section A.2) are all functors of k-linear categories. If F and G are k-linearly homotopic, then $\mathrm{CN}.F$ and $\mathrm{CN}.G$ are S^1 homotopic by Lemma A.6.1.

Let $\mathscr{C}$ and $\mathscr{D}$ be k-linear categories with cofibrations, $F : \mathscr{C} \to \mathscr{D}$ an exact (k-linear) functor, and let $S.F : S.\mathscr{C} \to S.\mathscr{D}$ denote the simplicial functor induced by F. For $m,n \in \mathbb{N}$, we let the diagram

$$(*) \qquad \left(\frac{0 = C_0 \rightarrowtail \cdots \rightarrowtail C_m}{0 = D_0 \rightarrowtail \cdots \rightarrowtail D_m} \rightarrowtail E_0 \rightarrowtail \cdots \rightarrowtail E_n \right)$$

denote the following information (suppressing the chosen quotients),

$$(0 = D_0 \rightarrowtail D_1 \rightarrowtail \cdots \to D_m \rightarrowtail E_0 \rightarrowtail \cdots \rightarrowtail E_n) \in S_{m+n+1}\mathscr{D},$$

$$(0 = C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_m) \in S_m\mathscr{C},$$

plus the identity in $S_m\mathscr{D}$,

$$\begin{pmatrix} 0 = & F(C_0) & \rightarrowtail & F(C_1) & \rightarrowtail \cdots \rightarrowtail & F(C_m) \\ & \| & & \| & & \| \\ 0 = & D_0 & \rightarrowtail & D_1 & \rightarrowtail \cdots \rightarrowtail & D_m \end{pmatrix}$$

Definition 3.4.2. Following [6], we let $S.F|\mathscr{D}$ denote the following bisimplicial k-linear category:

$$(S.F|\mathscr{D})([m],[n]) = \{\text{diagrams of the type } (*) \text{ above}\}$$

(where a morphism is a commuting diagram and a cofibration is a morphism consisting of cofibrations).

Lemma 3.4.3. *The natural projection of bisimplicial k-linear categories* $\pi: S.F|\mathscr{D} \to (S.\mathscr{C})L$ *produces an* S^1*-equivalence* CN.π.

Proof. The map $\pi([m], -)$ is split by the simplicial k-linear category map ι defined by setting $\iota(0 = C_0 \rightarrowtail \cdots \rightarrowtail C_m)$ to be

$$\left(\frac{0 = C_0 \rightarrowtail \cdots \rightarrowtail C_m}{0 = (C_0) \rightarrowtail \cdots \rightarrowtail F(C_m) = F(C_m) = \cdots = F(C_m)}\right)$$

and $\iota \circ \pi([m], -)$ is k-linearly homotopic to the identity by defining h_i of diagram $(*)$ to be (suppressing some notation)

$$\left(\frac{\cdots \rightarrowtail C_m}{\cdots \rightarrowtail D_m \;=\; D_m \underbrace{= \cdots =}_{i} D_m \rightarrowtail E_i \rightarrowtail \cdots}\right)$$

The result now follows from Lemma A.6.1. □

We let ρ denote the natural projection of bisimplicial k-linear categories $\rho: S.F|\mathscr{D} \to (S.\mathscr{D})R$ defined by taking diagram $(*)$ to $(0 = E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \cdots \rightarrowtail E_n/E_0)$.

Lemma 3.4.4. *The map* $\tilde{\rho}$ *produces an* S^1*-homotopy equivalence* CN.ρ.

Proof. The map $\tilde{\rho}(-, [n])$ is split by the simplicial k-linear functor ν defined by

$$\nu = (0 = F_0 \rightarrowtail \cdots \rightarrowtail F_n) = \left(\frac{0 = 0 = \cdots = 0}{0 = 0 = \cdots = 0 = F_0 \rightarrowtail \cdots \rightarrowtail F_n}\right)$$

and $\nu \circ \tilde{\rho}(-, [n])$ is k-linearly homotopic to the identity by defining h_i of diagram $(*)$ to be

$$\left(\frac{\cdots \rightarrowtail D_i \rightarrowtail E_0 = \cdots = E_0}{\cdots \rightarrowtail D_i \rightarrowtail E_0 \underbrace{= \cdots =}_{m+1-i} E_0 = E_0 \rightarrowtail E_1 \cdots}\right)$$

The result now follows from Lemma A.6.1. □

The following is essentially Theorem A′ of [6] which is one possible reformulation of Theorem A of [24] in the setting of simplicial sets.

Proposition 3.4.5. *The following are equivalent:*

(a) *The simplicial map* $S.F: S.\mathscr{C} \to S.\mathscr{D}$ *induces an* S^1*-homotopy equivalence* CN.$S.F$.

(b) *The bisimplicial map* $\rho: (S.F|\mathscr{D}) \to (S.\mathscr{D}R)$ *induces an* S^1*-homotopy equivalence* CN.ρ.

Proof. There is a commutative diagram of bisimplicial k-linear categories:

$$\begin{tikzcd} S.\mathscr{D}R \arrow[d, equal] & S.F|\mathscr{D} \arrow[l] \arrow[r, "\simeq"] \arrow[d, "F"] & S.\mathscr{C}L \arrow[d, "F"] \\ S.\mathscr{D}R & S.\mathrm{id}_{\mathscr{D}}|\mathscr{D} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & S.\mathscr{D}L \end{tikzcd}$$

By Lemmas 3.4.3 and 3.4.4, the arrows marked $\simeq$ are S^1-homotopy equivalences when we apply CN.(–) and we are done. □

For each $n \in \mathbb{N}$, we define the simplicial map E_n from $S.F|\mathscr{D}(-,[n])$ to itself by sending diagram $(*)$ to

$$\left(\frac{0 = 0 = \cdots = 0}{0 = 0 = \cdots = 0 = E_0/E_0 \rightarrowtail E_1/E_0 \rightarrowtail \cdots \rightarrowtail E_n/E_0} \right)$$

Corollary 3.4.6. *If the simplicial maps E_n are k-linearly homotopic to the identity for all $n \in \mathbb{N}$ then* CN.*S.F. is an S^1-homotopy equivalence.*

Proof. For $n \in \mathbb{N}$, the simplicial map $\rho(-,[n])$ from $S.F|\mathscr{D}(-,[n])$ to $S.\mathscr{D}R(-,[n])$ as split by a simplicial map I_n defined by

$$I_n(0 = F_0 \rightarrowtail \cdots \rightarrowtail F_n) = \left(\frac{0 = 0 = \cdots = 0}{0 = 0 = \cdots = 0 = F_0 \rightarrowtail \cdots \rightarrowtail F_n} \right)$$

Since $I_n \circ \rho(-,[n]) = E_n$, if E_n is k-linearly homotopic to the identity for all n then CN.ρ is an S^1-homotopy equivalence by Lemma A.6.1 and we are done by Proposition 3.4.5. □

3.5. The additivity theorem

For $\mathscr{C}$ a k-linear category with cofibrations, we let $E(\mathscr{C})$ denote the category with objects the cofibration sequences $A \rightarrowtail C \twoheadrightarrow B$ in $\mathscr{C}$. This is naturally a k-linear category with cofibrations which is equivalent to $S_2\mathscr{C}$ (see Proposition 1.3.2 of [28]).

Theorem 3.5.1 (Additivity theorem). ([24, Section 3] and [28, Section 1.4]) *The exact (k-linear) functor $F: E(\mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ defined by sending $(A \rightarrowtail C \twoheadrightarrow B)$ to (A, B) induces a homotopy equivalence $S.F: S.E(\mathscr{C}) \to S.(\mathscr{C}) \times S.(\mathscr{C})$ such that* CN.*S.F is an S^1-homotopy equivalence. Thus, by Lemma* 3.2.5, *the natural map* CN.*S.E*$(\mathscr{C}) \to$ (CN.*S.*$\mathscr{C})^2$ *is a discrete homotopy equivalence.*

Proof. We will show that in this situation the map E_n above is k-linearly homotopic to the identity for all $n \in \mathbb{N}$. The result will then follow from Corollary 3.4.6. Define the simplicial map Γ from $S.F|\mathscr{C}^2(-,[n])$ to itself by taking an arbitrary simplex $e \in S.F|\mathscr{C}^2([m],[n])$ like

$$\left(\begin{array}{cccccccccl}
0 = & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_m & & \\
& \downarrow & & \downarrow & & & & \downarrow & & \\
0 = & C_0 & \rightarrowtail & C_1 & \rightarrowtail & \cdots & \rightarrowtail & C_m & & \\
& \downarrow & & \downarrow & & & & \downarrow & & \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_m & & \\
\hline
0 = & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_m & \rightarrowtail & S_0 \rightarrowtail S_1 \rightarrowtail \cdots \rightarrowtail S_n \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_m & \rightarrowtail & T_0 \rightarrowtail T_1 \rightarrowtail \cdots \rightarrowtail T_n
\end{array}\right)$$

and setting $\Gamma(e)$ to be (suppressing notation)

$$\left(\begin{array}{cccccccl}
0 = & 0 & = & \cdots & = & 0 & & \\
& \downarrow & & & & \downarrow & & \\
0 = & B_0 & \rightarrowtail & \cdots & \rightarrowtail & B_m & & \\
& \| & & & & \| & & \\
0 = & B_0 & \rightarrowtail & \cdots & \rightarrowtail & B_m & & \\
\hline
0 = & 0 & = & \cdots & = & 0 & = & S_0/S_0 \rightarrowtail S_1/S_0 \rightarrowtail \cdots \\
0 = & B_0 & \rightarrowtail & \cdots & \rightarrowtail & B_m & \rightarrowtail & T_0 \rightarrowtail T_1 \rightarrowtail \cdots
\end{array}\right)$$

Let X be the subspace of $S.F|C^2(-,[n])$ determined by elements e such that all the A_is are 0. Thus, Γ is a retraction of $S.F|\mathscr{C}^2(-,[n])$ to X. The result will follow from:

(1) Γ is k-linearly homotopic to the identity,

(2) $E_n|_X$ is k-linearly homotopic to the identity of X.

The homotopy for (1) is defined by taking $e \in S.F|\mathscr{C}^2([m],[n])$, setting $X_j = C_j \amalg_{A_j} S_0$, and letting $h_i(e)$ be (suppressing notation)

$$\left(\begin{array}{ccccccccccccccc}
0 = & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = & S_0 & = \cdots = & S_0 \\
& \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = & C_0 & \rightarrowtail & C_1 & \rightarrowtail & \cdots & \rightarrowtail & C_i & \rightarrowtail & X_i & \rightarrowtail & X_{i+1} & \rightarrowtail \cdots \rightarrowtail & X_m \\
& \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & = & B_i & = & B_{i+1} & \rightarrowtail \cdots \rightarrowtail & B_m \\
\hline
0 = & A_0 & \rightarrowtail & A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_i & \rightarrowtail & S_0 & = & S_0 & = \cdots = & S_0 \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & = & B_i & \rightarrowtail & B_{i+1} & \rightarrowtail \cdots \rightarrowtail & B_m
\end{array}\right)$$

For $i = 0, 1, \ldots, m$ the crucial row of the diagram (the A_i row) is given by

$$\begin{array}{ll}
h_0\colon & 0 = A_0 \rightarrowtail S_0 = S_0 = \cdots \\
h_1\colon & 0 = A_0 \rightarrowtail A_1 \rightarrowtail S_0 = \cdots \\
\vdots & \vdots \\
h_m\colon & 0 = A_0 \rightarrowtail A_1 \rightarrowtail \cdots \rightarrowtail A_m \rightarrowtail S_0
\end{array}$$

The homotopy for (2) is given by h' defined by taking $e \in SF|\mathscr{C}^2([m],[n])$, and letting $h_i'(e)$ be

$$\left(\begin{array}{cccccccccccccccc}
0 = & 0 & = & 0 & = & \cdots & = & 0 & = & 0 & = & 0 & = & \cdots & = & 0 \\
 & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & \rightarrowtail & T_0 & = & T_0 & = & \cdots & = & T_0 \\
 & \| & & \| & & & & \| & & \| & & \| & & & & \| \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & \rightarrowtail & T_0 & = & T_0 & = & \cdots & = & T_0 \\
\hline
0 = & 0 & = & 0 & = & \cdots & = & 0 & = & 0 & = & 0 & = & \cdots & = & 0 \\
0 = & B_0 & \rightarrowtail & B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_i & \rightarrowtail & T_0 & = & T_0 & = & \cdots & = & T_0
\end{array}\right)$$

Definition 3.5.2. Following [28], we define a *cofibration sequence of exact k-linear functors* from $\mathscr{C}'$ to $\mathscr{C}$ to be a sequence of natural transformations $F' \to F \to F''$ between exact k-linear functors having the following two properties:

(a) For every $A \in \mathscr{C}'$, the sequence $F'(A) \to F(A) \to F''(A)$ is a cofibration sequence.

(b) For every cofibration $A' \rightarrowtail A$ in $\mathscr{C}'$, the square of cofibrations

$$\begin{array}{ccc}
F'(A') & \rightarrowtail & F'(A) \\
\downarrow & & \downarrow \\
F(A') & \rightarrowtail & F(A)
\end{array}$$

is *admissible* in the sense that the unique pushout map $F(A') \amalg_{F'(A')} F'(A) \to F(A)$ is also a cofibration (in $\mathscr{C}$).

Terminology. We say that two S^1-equivariant maps are *discretely homotopic* if the induced maps of fixed subspaces for every finite sub-group of S^1 are homotopic.

Corollary 3.5.3. *If $F' \to F \to F''$ is a cofibration sequence of exact k-linear functors from $\mathscr{C}'$ to $\mathscr{C}$, then* $\mathrm{CN}.S.F$ *and* $\mathrm{CN}.S.F' \oplus \mathrm{CN}.S.F'' = \mathrm{CN}.S.(F' \oplus F'')$ *are discretely homotopic. In particular,* $\mathrm{HH}_*(F) = \mathrm{HH}_*(F') + \mathrm{HH}_*(F'')$ *and similarly for cyclic, negative and periodic homology groups.*

Proof. This follows from a direct transliteration of Proposition 1.3.2 of [28] using Theorem 3.5.1. In fact, all four conditions found there hold but we do not need all of them. $\square$

Example 3.5.4. This is an example to show that the additivity theorem does not hold in general for the split homology groups defined in Section 2. Returning to the notation of Example 3.3.5, let s,t,q be the exact $\mathbb{Z}$-linear functors from $E(\mathscr{A}\ell_{\mathrm{f}})$ to $\mathscr{A}\ell_{\mathrm{f}}$ defined by $(A \rightarrowtail C \twoheadrightarrow B) \to A,B,C$ respectively. By Corollary 3.5.3 we see that

$$\mathrm{HH}_*(t) = \mathrm{HH}_*(s \oplus q) = \mathrm{HH}_*(s) \oplus \mathrm{HH}_*(q).$$

We show that $H_0^{\mathrm{s}}(t)$ and $H_0^{\mathrm{s}}(s \oplus q)$ from $H_0^{\mathrm{s}}(E(\mathscr{A}b_{\mathrm{f}}))$ to $H_0^{\mathrm{s}}(\mathscr{A}b_{\mathrm{f}})$ are not equal. As in Example 3.3.5, we let F_p denote the $\mathbb{Z}$-linear (but *not* exact) functor from $\mathscr{A}b_{\mathrm{f}}$ to $\mathscr{F}_p$ (the p-torsion groups of $\mathscr{A}b_{\mathrm{f}}$). We see that

$$F_p \circ t((\mathbb{Z} \xrightarrow{p(-)} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z})) = 0,$$

$$F_p \circ (s \oplus q)((\mathbb{Z} \xrightarrow{p(-)} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z})) = \mathbb{Z}/p\mathbb{Z}.$$

Since $\mathbb{Z}/p\mathbb{Z}$ is the generator of $\mathrm{HH}_0^{\mathrm{s}}(\mathscr{F}_p) \cong \mathbb{Z}/p\mathbb{Z}$ it is not a boundary and we see that

$$\mathrm{HH}_0^{\mathrm{s}}(F_p) \circ \mathrm{HH}_0^{\mathrm{s}}(t) \neq \mathrm{HH}_0^{\mathrm{s}}(F_p) \circ \mathrm{HH}_0^{\mathrm{s}}(s \oplus q).$$

3.6. The de-looping theorem

Definition 3.6.1. Let $F: \mathscr{C} \to \mathscr{D}$ be an exact k-linear functor of k-linear categories with cofibrations. Following [28], we define the simplicial k-linear category with cofibrations $S.(F: \mathscr{C} \to \mathscr{D})$ as the pull-back of the following diagram:

$$\begin{tikzcd} & PS.\mathscr{D} \arrow[d, "d_0"] \\ S.\mathscr{C} \arrow[r, "SF"] & S.\mathscr{D} \end{tikzcd}$$

For each $n \in \mathbb{N}$, an arbitrary element $e \in S_n(F: \mathscr{C} \to \mathscr{D})$ can be represented by

$$\left(\frac{0 = \quad C_0 \rightarrowtail C_1 \rightarrowtail \cdots \rightarrowtail C_n}{0 = D_0 \rightarrowtail D_1 \rightarrowtail D_2 \rightarrowtail \cdots \rightarrowtail D_{n+1}} \right)$$

where

$$\begin{array}{ccccccc} 0 = & F(C_0) & \rightarrowtail & F(C_1) & \rightarrowtail \cdots \rightarrowtail & F(C_n) \\ & \cong & & \cong & & \cong \\ 0 = & D_1/D_1 & \rightarrowtail & D_2/D_1 & \rightarrowtail \cdots \rightarrowtail & D_{n+1}/D_1 \end{array}$$

Proposition 3.6.2. *Thinking of $\mathscr{D}$ as a trivial simplicial object we obtain a simplicial exact k-linear inclusion $\mathscr{D} \to S.(F: \mathscr{C} \to \mathscr{D})$ and the following sequence is a discrete quasi-fibration*

$$\mathrm{CN}.S.\mathscr{D} \to \mathrm{CN}.S.S.(F: \mathscr{C} \to \mathscr{D}) \to \mathrm{CN}.S.S.\mathscr{C}.$$

Proof. It is clear from the definitions that the composition is trivial. By the fibration Lemma A.6.5 it suffices to show that for each $n \in \mathbb{N}$, the induced sequence

$$\mathrm{CN}.S.\mathscr{D} \to \mathrm{CN}.S.S_n(F: \mathscr{C} \to \mathscr{D}) \to \mathrm{CN}.S.S_n\mathscr{C}$$

is a discrete quasi-fibration. We let $\Psi : \mathscr{D} \times S_n\mathscr{C} \to S_n(F : \mathscr{C} \to \mathscr{D})$ be the exact (k-linear) functor defined by sending $D \times (0 = C_0 \rightarrowtail \cdots \rightarrowtail C_n)$ to:

$$\left(\frac{0 = C_0 \rightarrowtail \quad C_1 \quad \rightarrowtail \cdots \rightarrowtail \quad C_n}{0 = 0 \rightarrowtail D \rightarrowtail D \oplus F(C_1) \rightarrowtail \cdots \rightarrowtail D \oplus F(C_n)} \right)$$

By the commutative diagram

$$\begin{array}{ccccc} S.\mathscr{D} & \longrightarrow & S.(\mathscr{D} \times S_n\mathscr{C}) & \longrightarrow & S.S_n\mathscr{C} \\ \| & & \downarrow{\scriptstyle S\Psi} & & \| \\ S.\mathscr{D} & \longrightarrow & S.S_n(F : \mathscr{C} \to \mathscr{D}) & \longrightarrow & S.S_n\mathscr{C} \end{array}$$

and the fact that $\mathrm{CN}.S.(\mathscr{D} \times S_n\mathscr{C})$ is discretely homotopy equivalent to $\mathrm{CN}.S.\mathscr{D} \times \mathrm{CN}.S.S_n\mathscr{C}$ (Lemma 3.2.5) it suffices to show $S.\Psi$ is a discrete homotopy equivalence. Define the exact k-linear endofunctors F' and F'' of $S_n(F : \mathscr{C} \to \mathscr{D})$ by

$$F'(e) = \left(\frac{0 = \ 0 \ = \ 0 \ = \cdots \rightarrowtail \ 0}{0 = D_0 \rightarrowtail D_1 \rightarrowtail D_2 \rightarrowtail \cdots \rightarrowtail D_{n+1}} \right)$$

$$F''(e) = \left(\frac{0 = \quad C_0 \ \rightarrowtail \ C_1 \ \rightarrowtail \cdots \rightarrowtail \ C_n}{0 = 0 = D_1/D_1 \rightarrowtail D_2/D_1 \rightarrowtail \cdots \rightarrowtail D_{n+1}/D_1} \right)$$

There is a cofibration sequence of endofunctors $F' \rightarrowtail \mathrm{id} \twoheadrightarrow F''$, and so by the additivity theorem (Theorem 3.5.3) $F' \oplus F''$ is a discrete homotopy equivalence which implies $S.\Psi$ is also. □

Corollary 3.6.3 (Delooping theorem). *For any k-linear category with cofibrations, the following sequence is a discrete quasi-fibration,*

$$\mathrm{CN}.S.\mathscr{C} \to \mathrm{CN}.PS.S.\mathscr{C} \to \mathrm{CN}.S.S.\mathscr{C}$$

and thus there exists natural isomorphisms $\mathrm{HH}_*(\mathscr{C}) \cong \mathrm{HH}_{*+1}(S.\mathscr{C}) \cong \mathrm{HH}_{*+2}(S.S.\mathscr{C})$ *and similarly for cyclic, negative and periodic homology groups.*

Proof. This follows from noting that $S.(\mathrm{id} : \mathscr{C} \to \mathscr{C}) = PS.\mathscr{C}$ and applying Proposition 3.6.2. □

4. Products and Chern maps

Introduction

The first goal of this section is to construct products for our various homology groups of an exact category analogous to those constructed by Hood and Jones in [10] and by Kassel in [13]. Our method is to combine the techniques found in [10] with the description for products in algebraic K-theory as done by Waldhausen in [28] using bi-exact functors and the delooping theorem. We first introduce some notation and recall the cyclic Eilenberg–Zilber theorem of [10]. Next we recall the construction of products in algebraic K-theory using the delooping theorem and introduce some definitions. The third part consists of the construction of our product structures.

The second goal of this chapter is to generalize the Dennis trace map from algebraic K-theory to Hochschild homology and its extension to negative homology (which we call the Jones–Goodwillie Chern map) for unital k-algebras A to k-linear categories with cofibrations. As for products, one recovers the original constructions when considering the finitely generated projective modules of A. We construct these maps in Section 4 and show that they are suitably multiplicative. The last section shows in detail that our constructions do agree with the Jones–Goodwillie construction. An interesting difference occurring in our treatment is that we begin with a model for algebraic K-theory which is not a cyclic space but which maps to the S^1-fixed points of $\mathrm{CN}.S.\mathscr{C}$. The Jones–Goodwillie construction uses a map of cyclic spaces whose domain is an "epi-cyclic" space and thus produces a natural map to the homotopy fixed point space. We choose to use our particular model here for its ease in showing the Chern map is suitably multiplicative. We also point out a method to recover the result of Bökstedt that the Dennis trace maps into the homotopy inverse limit of the fixed point sets for *finite* subgroups of S^1 if $\mathscr{M}$ is a split category with cofibrations.

4.1. The cyclic Eilenberg–Zilber theorem

Notation. If X and Y are simplicial (cyclic) modules, we let $X\Delta Y$ denote the simplicial (cyclic) module with $(X\Delta Y)_n = X_n \otimes_k Y_n$ and operators given by the diagonal action. By abuse of notation, we will let X denote both the simplicial (cyclic) k-module and the chain (mixed) complex associated to it. If C and D are chain complexes, we let $C \otimes_k D$ denote the bi-complex of k-modules with bi-degree (m, n) defined by $C_m \otimes_k D_n$.

Recall that if X and Y are simplicial k-modules, then by the Eilenberg–Zilber theorem (see [19, VIII.8]) there are natural chain equivalences

$$\mathrm{Tot}(X \otimes_k Y) \underset{g_0}{\overset{f_0}{\rightleftarrows}} X\Delta Y.$$

We can define g_0 by the Alexander–Whitney map and f_0 by the Eilenberg–Mac Lane shuffle map, both defined in terms of the fundamental operators (i.e. face and degeneracy maps).

We now introduce some notation from [10]. Consider the graded ring $k[u]$, where u has degree -2. For K and L graded k-modules, we define the graded k-module $K \hat{\otimes} L$ to be the product $(K \hat{\otimes} L)_n = \prod K_i \otimes L_{n-i}$. So an element of degree n of $k[u] \hat{\otimes} L$ is given by an infinite sum of the form $\sum u^i \otimes_k l_i$ where $l_i \in L_{n+2i}$. Now suppose L is a mixed complex. We introduce the boundary operator ∂^- in $k[u] \hat{\otimes} L$ by

$$\partial^-(u^t \otimes e) = u^t \otimes b(e) + u^{t+1} \otimes B(e).$$

We similarly define boundary operators $\hat{\partial}$ of $k[u,u^{-1}]\,\hat{\otimes}\,L$ and ∂ of $k[u,u^{-1}]/uk[u]\,\hat{\otimes}\,L$. For any cyclic k-module X we see that X can be interpreted as a mixed complex and there exist natural isomorphisms (the functors B, B^- and B^{per} are like those in [7])

$$\mathrm{Tot}(B^-(X)) \cong k[u]\,\hat{\otimes}\,X,$$

$$\mathrm{Tot}(B^{\mathrm{per}}(X)) \cong k[u,u^{-1}]\,\hat{\otimes}\,X,$$

$$\mathrm{Tot}(B(X)) \cong k[u,u^{-1}]/uk[u]\,\hat{\otimes}\,X.$$

Note. By definition, $B^-(X)$ is a bi-complex of k-modules and thus $\mathrm{Tot}(B^-(X))$ is a chain complex of k-modules. By the above isomorphisms, we see that $\mathrm{Tot}(B^-(X))$ can also be considered as a chain complex of $k[u]$-modules which is how we will be working with it below. Similar remarks apply to $\mathrm{Tot}(B(X))$ and $\mathrm{Tot}(B^{\mathrm{per}}(X))$. If X and Y are cyclic k-modules, we abuse notation and set

$$\mathrm{Tot}(B^-(X))\,\hat{\otimes}_{k[u]}\,\mathrm{Tot}(B^-(Y)) = k[u]\,\hat{\otimes}\,X\,\hat{\otimes}\,Y \cong k[u]\,\hat{\otimes}\,\mathrm{Tot}(X\otimes_k Y).$$

Thus an element of degree n is an infinite sum of the form $\Sigma u^t \otimes x_i \otimes y_j$ with $n = i + j - 2t$. This is again a chain complex of $k[u]$-modules.

If X and Y are cyclic k-modules and

$$f\colon \mathrm{Tot}(B^-(X))\,\hat{\otimes}_{k[u]}\,\mathrm{Tot}(B^-(Y)) \to \mathrm{Tot}(B^-(X\Delta Y))$$

is a $k[u]$-module map, there is an induced map $f_0\colon \mathrm{Tot}(X\otimes_k Y) \to X\Delta Y$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Tot}(B^-(X))\,\hat{\otimes}_{k[u]}\,\mathrm{Tot}(B^-(Y)) & \xrightarrow{f} & \mathrm{Tot}(B^-(X\Delta Y)) \\
\downarrow & & \downarrow \\
\mathrm{Tot}(X\otimes_k Y) & \xrightarrow{f_0} & X\Delta Y
\end{array}$$

The vertical arrows above are induced by the natural projections of complexes, that is, by $k[u]\,\hat{\otimes}\,L \to k\,\hat{\otimes}\,L \cong L$. One calls the $k[u]$-module map f a *coextension* of f_0. Of course there is a similar notion for $k[u]$-module maps $g\colon \mathrm{Tot}(B^-(X\Delta Y)) \to \mathrm{Tot}(B^-(X))\,\hat{\otimes}_{k[u]}\,\mathrm{Tot}(B^-(Y))$.

Theorem 4.1.1 (The cyclic Eilenberg–Zilber theorem [10, Theorem 2.3]). *Let X and Y be cyclic k-modules. Let $f_0\colon \mathrm{Tot}(X\otimes_k Y) \to X\Delta Y$ be a natural chain equivalence which is the identity in degree zero.*

(a) *There exists a coextension f of f_0.*

(b) *Any such coextension f is a chain equivalence.*

(c) *Let f_0 and f_0' be natural chain equivalences from $\mathrm{Tot}(X\otimes_k Y)$ to $X\Delta Y$ which both give the identity in degree zero. Let f and f' be coextensions of f_0 and f_0'. Then there is a natural chain homotopy between f and f'.*

There are similar statements for a chain map $g_0\colon X\Delta Y \to \mathrm{Tot}(X\otimes_k Y)$ which is the identity in degree zero. $\square$

Notes on the proof. The method used in [10] to prove this theorem (which is similar to the technique found in [13]) actually yields more information than is contained in its statement. The first step to note is that one can pass to the normalized complexes since the operator B is closed on the sub-module of degeneracy elements. Letting $N_* X$ denote the normalized complex of X one can see that the natural projection $\mathrm{Tot}(B^- X) \to k[u] \hat{\otimes} N_*(X)$ is a natural chain equivalence. There is a natural isomorphism of chain complexes of $k[u]$-modules

$$(k[u] \hat{\otimes} N_* X \hat{\otimes} N_* Y) \cong k[u] \hat{\otimes} \mathrm{Tot}(N_* X \otimes_k N_* Y)$$

and one shows that $f_0 : \mathrm{Tot}(N_* X \otimes N_* Y) \to N_*(X \Delta Y)$ can be lifted to a $k[u]$-module map $k[u] \hat{\otimes} \mathrm{Tot}(N_* X \otimes_k N_* Y) \to k[u] \hat{\otimes} N_*(X \Delta Y)$. This lift is constructed by finding k-linear maps

$$f_t : \mathrm{Tot}(N_* X \otimes_k N_* Y) \to N_*(X \Delta Y) \quad t \geq 0$$

of degree $2t$ satisfying the formulas

$$b \circ f_t = f_t \circ b - B \circ f_{t-1} + f_{t-1} \circ B \quad (f_{-1} = 0).$$

The k-linear map $\Sigma u^t f_t : \mathrm{Tot}(N_* X \otimes_k N_* Y) \to k[u] \hat{\otimes} N_*(X \Delta Y)$ extends to a $k[u]$-linear map $k[u] \hat{\otimes} \mathrm{Tot}(N_* X \otimes_k N_* Y) \to k[u] \hat{\otimes} N_*(X \Delta Y)$.

The proof uses a variant of acyclic models, where the models are the cyclic sets $\mathrm{Hom}_{\Delta C}(-, [n])$. These are not acyclic spaces, since they have homology in degrees zero and one, but Hood and Jones show how to explicitly handle these low-dimensional cases in terms of the face, degeneracy and cyclic operators. Thus, if f_0 and g_0 are given in terms of face and degeneracy maps, then the associated co-extensions can also be given in terms of the face, degeneracy and cyclic operators. In addition, given any two coextensions of the same map given in terms of the fundamental operators, the natural chain homotopy between them can be chosen in terms of the fundamental operators also.

4.2. Products in algebraic K-theory

Definition 4.2.1. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be categories with cofibrations. A *bi-exact functor* of categories with cofibrations F from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$ is a functor having the following properties [28, p. 342]:

(a) For all $A \in \mathrm{Obj}(\mathscr{A})$ and $B \in \mathrm{Obj}(\mathscr{B})$, the induced functors $F(A, -)$ and $F(-, B)$ are exact.

(b) For every pair of cofibrations $A \rightarrowtail A'$ and $B \rightarrowtail B'$ in $\mathscr{A}$ and $\mathscr{B}$ respectively, the map from $F(A', B) \amalg_{F(A,B)} F(A, B')$ to $F(A', B')$ is a cofibration in $\mathscr{C}$.

A bi-exact functor induces a map of bisimplicial categories from $S.\mathscr{A} \times S.\mathscr{B}$ to $S.S.\mathscr{C}$ by sending

$$(0 = A_0 \rightarrowtail \cdots \rightarrowtail A_m) \times (0 = B_0 \rightarrowtail \cdots \rightarrowtail B_n) \in S_m\mathscr{A} \times S_n\mathscr{B}$$

to the element of $S_m S_n \mathscr{C}$ represented by

$$
\begin{array}{ccccccc}
F(A_0,B_0) & \rightarrowtail & F(A_0,B_1) & \rightarrowtail \cdots \rightarrowtail & F(A_0,B_n) \\
\downarrow & & \downarrow & & \downarrow \\
F(A_1,B_0) & \rightarrowtail & F(A_1,B_1) & \rightarrowtail \cdots \rightarrowtail & F(A_1,B_n) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & \ddots & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
F(A_m,B_0) & \rightarrowtail & F(A_m,B_1) & \rightarrowtail \cdots \rightarrowtail & F(A_m,B_n)
\end{array}
$$

with associated quotients (for $0 \le i < j \le m$ and $0 \le k < l \le n$) given by

$$F(A_j, B_l)/F(A_i, B_k) = F(A_j/A_i, B_l/B_k).$$

Condition (b) of the definition of a bi-exact functor ensures that this map is well defined.

Taking the geometric realization, we have produced a map $|S.\mathscr{A}| \wedge |S.\mathscr{B}| \to |S.S.\mathscr{C}|$. Passing to homotopy groups and using the delooping theorem (like Corollary 3.6.3), we obtain the K-theory product $K_i(\mathscr{A}) \otimes_{\mathbb{Z}} K_j(\mathscr{B}) \to K_{i+j}(\mathscr{C})$. This product agrees with that defined by Loday in [16] (which uses the "plus" construction). For a proof of this agreement, see for example [29].

4.3. *Products of cyclic homology groups*

Recall that if $\mathscr{A}$ and $\mathscr{B}$ are k-linear categories, we defined the k-linear category $\mathscr{A} \otimes_k \mathscr{B}$ by

$$\mathrm{Obj}(\mathscr{A} \otimes_k \mathscr{B}) = \mathrm{Obj}(\mathscr{A}) \times \mathrm{Obj}(\mathscr{B}),$$

$$\mathrm{Hom}_{\mathscr{A} \otimes_k \mathscr{B}}((A, B), (A', B')) = \mathrm{Hom}_{\mathscr{A}}(A, A') \otimes_k \mathrm{Hom}_{\mathscr{B}}(B, B').$$

Definition 4.3.1. Let $\mathscr{C}$ and $\mathscr{D}$ be k-linear categories with cofibrations. We define the bi-simplicial k-linear category $S.\mathscr{C} \otimes_k S.\mathscr{D}$ by

$$(S.\mathscr{C} \otimes_k S.\mathscr{D})[m, n] = S_m\mathscr{C} \otimes_k S_n\mathscr{D}.$$

We can form the cyclic $\times$ bi-simplicial k-module $\mathrm{CN}.(S.\mathscr{C} \otimes_k S.\mathscr{D})$ and we define

$$\mathrm{HH}_*(\mathscr{C} \otimes_k \mathscr{D}) \equiv \mathrm{H}_{*+2}(\mathrm{Tot}(\mathrm{CN}.(S.\mathscr{C} \otimes_k S.\mathscr{D}))).$$

The shift of two comes from the use of two $S.$ in our construction. We define the cyclic, negative and periodic homology groups of the symbol $\mathscr{C} \otimes_k \mathscr{D}$ similarly.

Definition 4.3.2. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be k-linear categories with cofibrations. A *bi-exact k-linear functor* F is a bi-exact functor from $\mathscr{A} \times \mathscr{B}$ to $\mathscr{C}$ such that:

(a′) For all $A \in \mathrm{Obj}(\mathscr{A})$ and $B \in \mathrm{Obj}(\mathscr{B})$, the induced functors $F(A, -)$ and $F(-, B)$ are exact *k-linear* functors.

Given a bi-exact k-linear functor F, the map of bi-simplicial categories $S.\mathscr{A} \times S.\mathscr{B} \xrightarrow{F} S.S.\mathscr{C}$ of Definition 4.2.1 induces a map of bi-simplicial k-linear categories $S.\mathscr{A} \otimes_k S.\mathscr{B} \xrightarrow{F} S.S.\mathscr{C}$ and thus a map of cyclic $\times$ bi-simplicial k-modules

$$\mathrm{CN}.(S.\mathscr{A} \otimes_k S.\mathscr{B}) \xrightarrow{F} \mathrm{CN}.(S.S.\mathscr{C}).$$

By the delooping theorem (Corollary 3.6.3) we see that F naturally produces a map from $\mathrm{HH}_*(\mathscr{A} \otimes_k \mathscr{B})$ to $\mathrm{HH}_*(\mathscr{C})$, and similarly for cyclic, negative and periodic homology groups.

Lemma 4.3.3. *For $\mathscr{C}$ and $\mathscr{D}$ k-linear categories with cofibrations, there exist natural chain equivalences*

$$\mathrm{Tot}[\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C}) \otimes_k \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D})] \underset{g_0}{\overset{f_0}{\rightleftarrows}} \mathrm{Tot}(\mathrm{CN}.(S.\mathscr{C} \otimes_k S.\mathscr{D}))$$

which are the identity in degree zero and given in terms of face and degeneracy maps. There exist coextensions to natural $k[u]$-module chain equivalences given in terms of the fundamental operators

$$\mathrm{Tot}(B^- S.\mathscr{C}) \hat{\otimes}_{k[u]} \mathrm{Tot}(B^- S.\mathscr{D}) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Tot}(B^-(S.\mathscr{C} \otimes_k S.\mathscr{D}))$$

These extensions f and g are unique up to natural chain homotopies which can be given in terms of the fundamental operators.

Proof. This is essentially an exercise in homological algebra using the naturality of the cyclic Eilenberg–Zilber theorem. The technical but straightforward details are left to the interested reader. □

Theorem 4.3.4. (After Theorems 2.4, 2.5 and 2.6 of [10].) *Given $\mathscr{C}$ and $\mathscr{D}$ k-linear categories with cofibrations there exist well-defined natural external product operations as follows:*

(0) $\mathrm{HH}_*(\mathscr{C}) \otimes_k \mathrm{HH}_*(\mathscr{D}) \to \mathrm{HH}_*(\mathscr{C} \otimes_k \mathscr{D})$ *which is associative and graded commutative.*

(1) $\mathrm{HN}_*(\mathscr{C}) \otimes_{k[u]} \mathrm{HN}_*(\mathscr{D}) \to \mathrm{HN}_*(\mathscr{C} \otimes_k \mathscr{D})$ *which is associative and graded commutative.*

(2) $\mathrm{HP}_*(\mathscr{C}) \otimes_{k[u,u^{-1}]} \mathrm{HP}_*(\mathscr{D}) \to \mathrm{HP}_*(\mathscr{C} \otimes_k \mathscr{D})$ *which is associative and graded commutative.*

(3) $\mathrm{HN}_*(\mathscr{C}) \otimes_{k[u]} \mathrm{HC}_*(\mathscr{D}) \to \mathrm{HC}_*(\mathscr{C} \otimes_k \mathscr{D})$ *which is associative.*

(4) $\mathrm{HC}_*(\mathscr{C}) \otimes_{k[u]} \mathrm{HC}_*(\mathscr{D}) \to \mathrm{HC}_{*+1}(\mathscr{C} \otimes_k \mathscr{D})$ *which is associative and graded commutative.*

(5) *The natural maps relating negative, Hochschild and periodic homology groups preserve these product operations.*

Proof. The product of (0) is constructed as follows:

$$\begin{aligned}\mathrm{HH}_m(\mathscr{C}) \otimes_k \mathrm{HH}_n(\mathscr{D}) &\equiv H_{m+1}(\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C})) \otimes_k H_{n+1}(\mathrm{Tot}(\mathrm{CN}.S.\mathscr{D}))\\ &\to H_{m+n+2}(\mathrm{Tot}[\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C}) \otimes_k \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D})])\\ &\xrightarrow{\simeq} H_{m+n+2}(\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C} \otimes_k S.\mathscr{D}))) \equiv \mathrm{HH}_{m+n}(\mathscr{C} \otimes_k \mathscr{D}).\end{aligned}$$

The first map is the external homology product and the second is the chain equivalence given by Lemma 4.3.3. Associativity holds because each of the above maps are (the proofs of associativity can be done using homotopies which are expressed in terms of the fundamental operators). The product is graded commutative because the last map is.

The product of (1) is constructed as follows:

$$\begin{aligned}\mathrm{HN}_m(\mathscr{C}) \otimes_k \mathrm{HN}_n(\mathscr{D}) &\equiv H_{m+1}(\mathrm{Tot}(B^- S.\mathscr{C}) \otimes_{k[u]} H_{n+1}(\mathrm{Tot}(B^- S.\mathscr{D}))\\ &\to H_{m+n+2}(\mathrm{Tot}(B^- S.\mathscr{C}) \hat{\otimes}_{k[u]} \mathrm{Tot}(B^- S.\mathscr{D}))\\ &\xrightarrow{\simeq} H_{m+n+2}(\mathrm{Tot}(B^-(S.\mathscr{C} \otimes_k S.\mathscr{D}))) \equiv \mathrm{HH}_{m+n}(\mathscr{C} \otimes_k \mathscr{D}),\end{aligned}$$

where the first map is the external homology product and the second is the chain equivalence induced by Lemma 4.3.3. Associativity and graded commutativity follows as in the above case and similarly for the cases below.

The product for (2) is constructed by noting that there is a natural isomorphism of chain complexes of $k[u, u^{-1}]$-modules

$$\begin{aligned}&\mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{C}) \hat{\otimes}_{k[u,u^{-1}]} \mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{D})\\ &\quad = k[u, u^{-1}] \hat{\otimes} \mathrm{Tot}(\mathrm{CN}.S.\mathscr{C}) \hat{\otimes} \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D})\\ &\quad \cong k[u, u^{-1}] \hat{\otimes} \mathrm{Tot}[\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C}) \otimes_k \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D})]\end{aligned}$$

and we can extend Lemma 4.3.3 to this complex. Thus we define the product by

$$\begin{aligned}\mathrm{HP}_m(\mathscr{C}) \otimes_k \mathrm{HP}_n(\mathscr{D}) &\equiv H_{m+1}(\mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{C})) \otimes_{k[u,u^{-1}]} H_{n+1}(\mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{D}))\\ &\to H_{m+n+2}(\mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{C}) \hat{\otimes}_{k[u,u^{-1}]} \mathrm{Tot}(B^{\mathrm{per}}S.\mathscr{D}))\\ &\xrightarrow{\simeq} H_{m+n+2}(\mathrm{Tot}(B^{\mathrm{per}}(S.\mathscr{C} \otimes_k S.\mathscr{D}))) \equiv \mathrm{HP}_{m+n}(\mathscr{C} \otimes_k \mathscr{D}).\end{aligned}$$

The product in (3) is constructed by noting that there is a natural isomorphism of chain complexes of $k[u, u^{-1}]/uk[u]$-modules

$$\begin{aligned}&(k[u] \hat{\otimes} \mathrm{Tot}(\mathrm{CN}.S.\mathscr{C})) \hat{\otimes}_{k[u]} (k[u, u^{-1}]/uk[u] \hat{\otimes} \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D}))\\ &\quad \cong k[u, u^{-1}]/uk[u] \hat{\otimes} \mathrm{Tot}[\mathrm{Tot}(\mathrm{CN}.S.\mathscr{C}) \otimes_k \mathrm{Tot}(\mathrm{CN}.S.\mathscr{D})]\end{aligned}$$

and we can extend Lemma 4.3.3 to this complex. The product is defined by

$$\mathrm{HN}_m(\mathscr{C}) \otimes_k \mathrm{HC}_n(\mathscr{D}) \equiv H_{m+1}(\mathrm{Tot}(B^- S.\mathscr{C})) \otimes_{k[u]} H_{n+1}(\mathrm{Tot}(B.S.\mathscr{D}))$$

$$\to H_{m+n+2}(\mathrm{Tot}(B^- S.\mathscr{C}) \mathbin{\hat{\otimes}_{k[u]}} \mathrm{Tot}(B.S.\mathscr{D}))$$

$$\xrightarrow{\simeq} H_{m+n+2}(\mathrm{Tot}(B.(S.\mathscr{C} \otimes_k S.\mathscr{D}))) \equiv \mathrm{HC}_{m+n}(\mathscr{C} \otimes_k \mathscr{D}).$$

The product in (4) is defined as follows:

$$\mathrm{HC}_m(\mathscr{C}) \otimes_k \mathrm{HC}_n(\mathscr{D}) \to \mathrm{HN}_{m+n+1}(\mathscr{C}) \otimes_k \mathrm{HC}_n(\mathscr{D}) \to \mathrm{HC}_{m+n+1}(\mathscr{C} \otimes_k \mathscr{D}),$$

where the first arrow is induced by the boundary map in the long exact sequence relating cyclic, negative and periodic homology groups and the second is the product defined in (3). □

Example 4.3.5. We let A and B be k-algebras, $\mathscr{P}_A$, $\mathscr{P}_B$ and $\mathscr{P}_{\mathscr{A} \otimes_k \mathscr{B}}$ be the exact k-linear categories of finitely generated projective modules of A, B and $A \otimes_k B$ respectively. We let $F : \mathscr{P}_A \times \mathscr{P}_B \to \mathscr{P}_{A \otimes_k B}$ be the bi-exact k-linear functor which sends (P, Q) to $P \otimes_k Q$. Combining the products of Theorem 4.3.4 with the natural map induced by F in Definition 4.3.2 we obtain natural products like $\mathrm{HN}_*(\mathscr{P}_A) \otimes_k \mathrm{HN}_*(\mathscr{P}_B) \to \mathrm{HN}_*(\mathscr{P}_{A \otimes_k B})$ for all statements (0)–(5) of Theorem 4.3.4. It can be checked that the products of [10] and the ones we have just described are interwined by the natural isomorphisms produced by Corollaries 3.3.4 and 3.6.3. That is, by the natural discrete homotopic maps

$$\mathrm{CN}.\mathscr{P}_A \xrightarrow{\simeq} \Omega\mathrm{CN}.S.\mathscr{P}_A \xrightarrow{\simeq} \Omega\Omega\mathrm{CN}.S.S.\mathscr{P}_A.$$

4.4. *The Jones–Goodwillie Chern map*

In this section $\mathscr{C}$ will denote a k-linear category with cofibrations. To define the algebraic K-theory of $\mathscr{C}$ we shall always let weak equivalences be the isomorphisms in $\mathscr{C}$. By [28], the simplicial set $\mathrm{obj}(S.\mathscr{C})$ is homotopy equivalent to the bisimplicial set $\mathrm{iN}.S.\mathscr{C}$, so we may consider the algebraic K-theory of $\mathscr{C}$ as being the space $K(\mathscr{C}) = \Omega|\mathrm{obj}(S.\mathscr{C})|$. When $\mathscr{C}$ is an exact k-linear category we know by [28], that this space is homeomorphic to the space $\Omega|BQ\mathscr{C}|$ giving the K-theory in the sense of Quillen [24].

Definition 4.4.1. We define the *Dennis trace map* (of a k-linear category with cofibrations $\mathscr{C}$) to be the natural map from the algebraic K-theory of $\mathscr{C}$ to the Hochschild homology of $\mathscr{C}$ induced by the map

$$\mathrm{obj}\, S.\mathscr{C} \xrightarrow{\mathrm{id}} \mathrm{CN}_0 S.\mathscr{C} \xrightarrow{\mathrm{incl}} \mathrm{CN}.S.\mathscr{C}.$$

The map of loop spaces will be written as $D: K(\mathscr{C}) \to \mathrm{HH}(\mathscr{C})$ where we have written $\mathrm{HH}(\mathscr{C})$ for $\Omega\,|\mathrm{CN}.S.\mathscr{C}|$. We will show in Section 4.5 that this map agrees with the usual definition of the Dennis trace when $\mathscr{C}$ is $\mathscr{P}_A$ for some k-algebra A.

For X a cyclic set, the S^1-fixed points of $|X|$ are the vertices of $x \in X_0$ such that $t \cdot s_0(x) = s_0(x)$. Thus, the Dennis trace map takes $|\mathrm{obj}\, S.\mathscr{C}|$ to the S^1-fixed point set of $|\mathrm{CN}.S.\mathscr{C}|$. Given a unital k-algebra A, the elements of the form $k \cdot \mathrm{id}_A$ in $\mathrm{CN}_0(A)$ are contained in the S^1-fixed points of $|\mathrm{CN}.A|$. There is a natural lift from the submodule of these elements to the zero cycles of B^-A defined by

$$\prod_{t=0}^{\infty} (-1)^t \frac{(2t)!}{t!} k \cdot (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \cdots \otimes \mathrm{id}_A) \in Z_0(B^-(A)).$$

We note that on the normalized complex $N_*^-(A)$ associated to B^-A this lift is simply $k \cdot \mathrm{id}_A \times 0 \times 0 \times \cdots \in Z_0(N_*^-(A))$.

By abuse of notation, and in order to make our diagrams legible, we shall write $B^-(S.\mathscr{C})$ for the simplicial double complex $B^-(\mathrm{CN}.S.\mathscr{C})$, and $Z_0 B^-(\mathscr{C})$ for the simplicial k-module of cycles in $\mathrm{Tot}_0 B^-(S.\mathscr{C})$. By naturality, it is easy to see that we can lift the Dennis trace map to a natural map $\mathrm{obj}\, S.\mathscr{C} \xrightarrow{\alpha} Z_0(B^- S.\mathscr{C})$ (of a simplicial set to a simplicial k-module).

Definition 4.4.2. We define the *Jones–Goodwillie Chern map*, denoted J–G, to be the natural map $K(\mathscr{C}) \to \mathrm{HN}(\mathscr{C})$ obtained by the composition

$$\mathrm{obj}\, S.\mathscr{C} \xrightarrow{\alpha} Z_0(B^- S.\mathscr{C}) \xrightarrow{\mathrm{incl}} B^- S.\mathscr{C}.$$

It is clear from the definitions that there is a commuting diagram

$$\begin{tikzcd} & \mathrm{HN}(\mathscr{C}) \arrow[d] \\ K(\mathscr{C}) \arrow[ur, "\text{J–G}"] \arrow[r, "D"] & \mathrm{HH}(\mathscr{C}) \end{tikzcd}$$

Proposition 4.4.3. *The Jones–Goodwillie Chern map is suitably multiplicative. That is, given a bi-exact k-linear functor $F: \mathscr{A} \times \mathscr{B} \to \mathscr{C}$, then the induced products of algebraic K-theory and those of negative homology commute with the map* J–G. *A similar statement is true for the Dennis trace map D.*

Proof. We first note the following commutative diagram:

$$\begin{tikzcd}
\pi_m(|\mathrm{obj}\, S.\mathscr{A}|) \otimes_{\mathbb{Z}} \pi_n(|\mathrm{obj}\, S.\mathscr{B}|) \arrow[r, "\mu_1"] \arrow[d, "\alpha \,\otimes_k\, \alpha"'] & \pi_{m+n}(|\mathrm{obj}\, S.\mathscr{A}| \wedge |\mathrm{obj}\, S.\mathscr{B}|) \arrow[d, "\alpha \otimes_k \alpha"] \\
H_m(Z_0(B^-(S.\mathscr{A}))) \otimes_k H_n(Z_0(B^-(S.\mathscr{B}))) \arrow[r, "\mu_2"] \arrow[d, "\mathrm{incl} \,\otimes_{k[u]}\, \mathrm{incl}"'] & H_{m+n}(Z_0 B^-(S.\mathscr{A}) \otimes_k Z_0 B^-(S.\mathscr{B})) \arrow[d, "\mathrm{incl}"] \\
\mathrm{HN}_{m-1}(\mathscr{A}) \otimes_{k[u]} \mathrm{HN}_{n-1}(\mathscr{B}) \arrow[r, "\mu_3"] & H_{m+n}(B^-(S.\mathscr{A}) \hat{\otimes}_{k[u]} B^-(S.\mathscr{B})) \arrow[d, "\mathrm{lemma}"] \\
 & H_{m+n}(B^-(S.\mathscr{A} \otimes_k S.\mathscr{B}))
\end{tikzcd}$$

Where μ_1 is the usual homotopy product, μ_2 and μ_3 are the product defined by the exterior homotopy product. The map "lemma" is a coextension of the Eilenberg–Mac Lane shuffle map (Theorem 4.1.1) as given by the cyclic Eilenberg–Zilber theorem and extended by naturality as in Lemma 4.3.3. The upper square commutes by standard simplicial techniques (see for example [30, Section 3]).

We next consider the diagram

$$\begin{tikzcd}
\pi_t(|\mathrm{obj}\, S.\mathscr{A}| \wedge |\mathrm{obj}\, S.\mathscr{B}|) \arrow[r, "F"] \arrow[d] & \pi_t(|\mathrm{obj}\, S.S.\mathscr{C}|) \arrow[d, "\alpha"] \\
H_t(B^-(S.\mathscr{A} \otimes_k S.\mathscr{B})) \arrow[r, "F"] & H_t(B^-(S.S.\mathscr{C}))
\end{tikzcd}$$

which commutes since the Eilenberg–Mac Lane shuffle map is the identity in dimension zero. This proves the proposition since the two delooping theorems for algebraic K-theory and negative homology certainly agree via the Jones–Goodwillie Chern map. □

4.5. *Agreement for $\mathscr{P}_A$*

We now want to show that our definitions of the Dennis trace and the Jones–Goodwillie Chern map agree with the usual ones via the natural isomorphisms established in Corollary 3.3.4. Let $\mathscr{P}$ denote the category of finitely generated projective modules over a unital algebra A. We recall that $\mathscr{P}$ is a Γ category in the sense of [26] (using direct sums). There is a natural map from the classifying space $\mathbb{B}\mathscr{P}$ to $\mathrm{iN}.S.\mathscr{P}$ which is a homotopy equivalence since $\mathbb{B}_n\mathscr{P}$ is equivalent to $S_n\mathscr{P}$ for all $n \in \mathbb{N}$. Also, by Section 1.4 of [28], the natural map $S.\mathscr{P} = \mathrm{iN}_0 S.\mathscr{P} \to \mathrm{iN}.S.\mathscr{P}$ is a homotopy equivalence.

For $\mathscr{C}$ a category, let $\mathrm{iN}.\mathscr{C}$ denote the nerve of the sub-category of $\mathscr{C}$ determined by its isomorphisms. We consider this as a cyclic set by setting

$$\tau(\alpha_1, \ldots, \alpha_n) = ((\alpha_1 \circ \cdots \circ \alpha_n)^{-1}, \alpha_2, \ldots, \alpha_{n-1}).$$

For any k-linear category $\mathscr{C}$, there is a natural map of cyclic sets, which we write as $\tau(\mathscr{C})$, from $\mathrm{iN}.\mathscr{C}$ to $\mathrm{CN}.\mathscr{C}$ defined by

$$(\alpha_1, \ldots, \alpha_n) \to ((\alpha_1 \circ \cdots \circ \alpha_n)^{-1} \otimes \alpha_1 \otimes \cdots \otimes \alpha_n).$$

At this point we would like to appeal to the universal property of "group completion" (see, for example, 4.1 of [26]) to say that the natural map $|\mathrm{iN}.\mathscr{P}| \to |\mathrm{CN}.\mathscr{P}|$ so obtained lifts uniquely (up to homotopy) to a map from $\Omega|\mathbb{B}\mathscr{P}|$. In order to do this we need to note that the map in question is a *monoidal* map. There are several equivalent ways to see this. Consider the diagram

$$\begin{tikzcd} \mathrm{iN}.\mathscr{P} \times \mathrm{iN}.\mathscr{P} \arrow[r, "\tau"] \arrow[d, "\oplus"] & \mathrm{CN}(\mathscr{P} \times \mathscr{P}) \arrow[r, "\simeq"] \arrow[d, "\oplus"] & \mathrm{CN}(\mathscr{P}) \oplus \mathrm{CN}(\mathscr{P}) \arrow[dl, "+"] \\ \mathrm{iN}.(\mathscr{P}) \arrow[r, "\tau"] & \mathrm{CN}(\mathscr{P}) & \end{tikzcd}$$

where $\oplus$ denotes the operation of direct sum and $+$ denotes the operation of addition component-wise. The square commutes but the triangle only commutes up to homotopy. To see this, use the explicit homotopy inverse given by Proposition 2.4.9 for finite products with the choice of some object given by the zero object (we have allowed ourselves the liberty of a *strict* monoidal unit here, otherwise we need to appeal to more homotopies). Thus the map $\tau(\mathscr{P})$ takes (up to homotopy) the monoidal action of direct sum to the group action of addition. Applying the "plus" construction to $\tau(\mathscr{P})$ gives the usual Dennis trace map.

Now we can appeal to the universal property of group completion and from the commutative diagram below we see that the resulting lift is the same (up to homotopy) as the Dennis trace we have defined. Thus our Dennis trace map, which is the right vertical map in the following diagram, agrees with the usual Dennis trace defined via the "plus" construction.

$$\begin{tikzcd} & & \Omega|S.\mathscr{P}| \arrow[d, "\wr"] \\ |\mathrm{iN}.\mathscr{P}| \arrow[r] \arrow[d, "\tau(\mathscr{P})"] & \Omega|\mathbb{B}\mathscr{P}| \arrow[r, "\simeq"] & \Omega|\mathrm{iN}.S.\mathscr{P}| \arrow[d] \\ |\mathrm{CN}.\mathscr{P}| \arrow[rr, "\simeq"] & & \Omega|\mathrm{CN}.S.\mathscr{P}| \end{tikzcd}$$

For X a cyclic set, we let $\mathbb{Z}[X]$ denote the $\mathbb{Z}$-module mixed complex obtained by taking the free cyclic object generated by X and denote the associated homologies simply by $\mathrm{HH}_*(X)$ and so on. Following [10], we note that $\mathrm{HN}_*(\mathrm{iN}.\mathscr{C})$ is isomorphic to $\mathbb{Z}[u] \hat{\otimes}_{\mathbb{Z}} H_*(iN\mathscr{C})$ since $iN.\mathscr{C}$ is equivalent to a union of $\mathrm{iN}.G_\alpha$ for groups G_α and this is true for groups. We conclude that $\mathrm{HN}_*(\mathrm{iN}.S.\mathscr{P})$ is isomorphic to $\mathbb{Z}[u] \hat{\otimes}_{\mathbb{Z}} H_*(\mathrm{iN}.S.\mathscr{P})$. Let β be the natural map of mixed complexes from $\mathbb{Z}[\mathrm{iN}.\mathscr{P}]$ to $\mathbb{Z}[\mathrm{iN}.S._{-1}]$ induced by the identity $\mathscr{P} = S_1\mathscr{P}$. It is now clear that we can construct the following commutative diagram:

$$\begin{tikzcd} H_*(\mathrm{iN}.\mathscr{P}) \arrow[r] \arrow[d, "\beta"] & \mathbb{Z}[u] \hat{\otimes}_{\mathbb{Z}} H_*(\mathrm{iN}.\mathscr{P}) \arrow[r, "\simeq"] \arrow[d, "\mathrm{id} \otimes \beta"] & \mathrm{HN}_*(\mathrm{iN}.\mathscr{P}) \arrow[d, "\beta"] \\ H_*(\mathrm{iN}.S._{-1}\mathscr{P}) \arrow[r] & \mathbb{Z}[u] \hat{\otimes}_{\mathbb{Z}} H_*(\mathrm{iN}.S._{-1}\mathscr{P}) \arrow[r, "\simeq"] & \mathrm{HN}_*(\mathrm{iN}.S._{-1}\mathscr{P}) \end{tikzcd}$$

We now want to consider the following commutative diagram:

$$\begin{array}{ccccccc}
 & & H_*(\mathrm{iN}.\mathscr{P}) & \longrightarrow & \mathrm{HN}_*(\mathrm{iN}.\mathscr{P}) & \xrightarrow{\tau(\mathscr{P})} & \mathrm{HN}^s_*(\mathscr{P}) \\
 & & \downarrow \beta & & \downarrow \beta & & \downarrow \simeq \\
H_*(S._{-1}\mathscr{P}) & \xrightarrow{\simeq} & H_*(\mathrm{iN}.S._{-1}\mathscr{P}) & \longrightarrow & \mathrm{HN}_*(\mathrm{iN}.S._{-1}\mathscr{P}) & \xrightarrow{\tau(\mathscr{P})} & \mathrm{HN}_*(\mathscr{P})
\end{array}$$

One construction of the usual Jones–Goodwillie Chern map arises from the composite of the top row. The vertical map on the right is the isomorphism from Corollary 3.3.4 and thus we will be done if we show the lower composition agrees with the Jones–Goodwillie map we defined in Definition 4.4.1. This follows by noting that one can describe the map from $H_0(\mathrm{iN}.S_n\mathscr{P})$ to $\mathrm{HN}_0(\mathrm{iN}.S_n\mathscr{P})$ (n is fixed) on generators by

$$Q \to \prod_{t=0}^{\infty} (-1)^t \frac{(2t)!}{t!} (Q = Q = \cdots = Q).$$

(see for example [10, Section 5]).

A note about fixed point sets

Marcel Bökstedt has shown that the Dennis trace maps into the homotopy inverse limit of the fixed point sets of each finite subset of S^1. We would like mention how this phenomenon appears in our setting and would like to thank Marcel for explaining it to us. The Dennis trace we have defined maps into the S^1 fixed point set of $\Omega|\mathrm{CN}.S.\mathscr{C}|$ for any exact category $\mathscr{C}$. If $\mathscr{C}$ is split (or semi-simple), then by the proof of Theorem 3.3.3 and by Section A.6 we see that the natural map from $\mathrm{CN}.\mathscr{C}$ to $\Omega|\mathrm{CN}.S.\mathscr{C}|$ is an S^1 map and a discrete homotopy equivalence. This implies that the map produces a homotopy equivalence on the fixed point sets for each *finite* sub-group C_r of S^1 and hence we have a diagram:

$$\mathrm{holim}_{r|s} |CN.\mathscr{C}|^{C_r} \xrightarrow{\simeq} \mathrm{holim}_{r|s} \Omega|\mathrm{CN}.S.\mathscr{C}|^{C_r} \longleftarrow \Omega|\mathrm{CN}.S.\mathscr{C}|^{S^1} \xleftarrow{D} K(\mathscr{C}).$$

If $\mathscr{C}$ is the category of finite projective modules of the algebra A, then by Proposition 2.4.3 the natural inclusion of the usual Hochschild homology of A is also a discrete homotopy equivalence so we can replace $\mathrm{CN}.\mathscr{C}$ by $\mathrm{CN}.A$ in this case.

Appendix. Semi-cyclic objects

Several times throughout this paper we have used maps of cyclic objects which did not preserve degeneracies. Since such maps do not necessarily produce S^1 equivariant maps of realizations, we collect here a few basic observations about them which we have found useful.

A.1. Semi-simplicial sets

The observations of these next three paragraphs can be found for example in [25]. We let Δ denote the category of finite ordered sets and order preserving set maps with one object $[n] = \{0 < \cdots < n\}$ for each cardinality and we let Δ_m denote the subcategory of Δ generated by the injective (monomorphic) maps. We call a functor from (Δ_m) Δ to the category of sets a (semi-) simplicial set. We let $|X.|$ denote the realization of a simplicial set $X.$ and $|Y.|_m$ denote the realization of a semi-simplicial set $Y.$. That is,

$$|Y.|_m = \coprod_{n \in \mathbb{N}} Y_n \times \Delta^n / \sim, \qquad (f^* y, t) \sim (y, f_* t) \quad \text{for } f \in \Delta_m,$$

where Δ^n is the standard n-simplex.

Let $X.$ be a simplicial set. We can consider this a semi-simplicial set by forgetting structure. We let $q_X : |X.|_m \to |X.|$ denote the quotient map which is a homotopy equivalence. We define a section to q_X, denoted x_X, which is functorial with respect to simplicial maps and is "adjoint" to q_X. Let $x \times t \in X_n \times \Delta^n$; then by the Eilenberg–Zilber lemma x can be uniquely represented as $\sigma^*(\tilde{x})$ where $\tilde{x}$ is a non-degenerate element of X_m ($m \leq n$) and $\sigma \in \Delta$ is degenerate. We define s_X of the class $[x, t]$ to be $[\tilde{x}, \sigma_* t]$.

Let G be a group and suppose that $X.$ is a simplicial G-set. Then $|X.|_m$ and $|X.|$ are naturally G-spaces, q_X and s_X are G-maps and q_X (hence also $\mathbf{s}_X$) is a G-homotopy equivalence (strong sense).

A.2. Semi-simplicial homotopies

Let $\Delta(1)$ denote the simplicial set $\mathrm{Hom}_\Delta(\star, [1])$. We recall that a simplicial homotopy h between the two simplicial maps f and g is a simplicial map $h : X. \times \Delta(1) \to Y.$ such that $h(x \times \eta_0) = f$ and $h(x \times \eta_1) = g$ where $\eta_i \in \mathrm{Hom}_\Delta([1], [1])$ denotes the constant map to i. Since realizations commute with products, a simplicial homotopy induces a homotopy of realizations. One can also describe a simplicial homotopy as a set of maps $h_i(q) \in \mathrm{Hom}(X_q, Y_{q+1})$ $(0 \leq i \leq q)$ which satisfy the following relations (see [20]):

(a) $d_0 \circ h_0 = f, \qquad d_{q+1} \circ h_q = g,$

(b) $d_i \circ h_j = \begin{cases} h_{j-1} \circ d_i & \text{if } i < j, \\ d_{j+1} \circ h_{j+1} & \text{if } i = j+1, \\ h_j \circ d_{i-1} & \text{if } i > j+1, \end{cases}$

(c) $s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & \text{if } i \leq j, \\ h_j \circ s_{i-1} & \text{if } i < j. \end{cases}$

Suppose we are now given two semi-simplicial maps. We define a *semi-simplicial* homotopy h to be a collection of maps $h_i(q) \in \mathrm{Hom}(X_q, Y_{q+1})$ which satisfy conditions (a) and (b) above. A semi-simplicial homotopy is the same as giving a semi-simplicial map h from $X. \times \Delta(1)$ to $Y.$. Note that if $X.$ and $Y.$ are simplicial sets then a semi-simplicial homotopy produces a homotopy of realizations.

A.3. Semi-simplicial subdivision

The observations and notation of this paragraph come from [1, Section 1]. We let sd_r denote the functor from Δ to Δ defined by sending $[n]$ to $[n] \amalg \cdots \amalg [n]$ (r copies of concatenation of the ordered set $[n]$). Thus, $\mathrm{sd}_r([n-1]) = [rn-1]$ and $\mathrm{sd}_r(f)\ (am+b) = an + f(b)$ when $f: [m-1] \to [n-1]$ and $0 \leq b < m$. Given a simplicial set $X.$, we let $\mathrm{sd}_r(X.)$ denote the simplicial set $X \circ \mathrm{sd}_r$. The standard simplex Δ^{rn-1} is the r-fold join of Δ^{n-1} with itself and we let $d_r: \Delta^{n-1} \to \Delta^{rn-1}$ denote the diagonal map $d_r(u) = u/r \oplus \cdots \oplus u/r$. The map $D_r: |\mathrm{sd}_r(X.)| \to |X.|$ of realizations defined by $1 \times d_r: X_{rn-1} \times \Delta^{n-1} \to X_{rn-1} \times \Delta^{rn-1}$ is a homeomorphism. Furthermore, if $Y.$ is a cyclic set, then $|\mathrm{sd}_r(Y.)|$ has a natural structure of an $\mathbb{R}/r\mathbb{Z}$-space and D_r is an S^1-homeomorphism if we identify $\mathbb{R}/r\mathbb{Z}$ with $S^1 = \mathbb{R}/\mathbb{Z}$ in the usual manner ($t \to t/r$). Let C_r denote the cyclic group with r elements. The simplicial set $\mathrm{sd}_r(Y.)$ is naturally a C_r-simplicial set and D_r gives a homeomorphism of the C_r fixed point sets $|\mathrm{sd}_r(Y.)|^{C_r} \cong |Y.|^{C_r}$ (where the action on $|Y.|$ is by restriction $C_r \subset S^1$).

Now we can consider *semi-simplicial* subdivision. Unfortunately, the map $D_r: |\mathrm{sd}_r(X.)|_m \to |X.|_m$ is not a homeomorphism but only a homotopy equivalence. Given a semi-simplicial map f of simplicial sets, we can consider the various maps of realizations $f_r = D_r \circ q_{\mathrm{sd}_r Y} \circ |\mathrm{sd}_r(f)|_m \circ s_{\mathrm{sd}_r X} \circ D_r^{-1}$. That is,

$$\begin{array}{ccccc} & |\mathrm{sd}_r(X)|_m & \xrightarrow{\mathrm{sd}_r(f)} & |\mathrm{sd}_r(Y)|_m & \\ & \uparrow{\scriptstyle s_{\mathrm{sd}_r(X)}} & & \downarrow{\scriptstyle q_{\mathrm{sd}_r(Y)}} & \\ |X.| \cong & |\mathrm{sd}_r(X)| & & |\mathrm{sd}_r(Y)| & \cong |Y.| \end{array}$$

We will call a map f of cyclic sets a *semi-cyclic* map if it is a semi-simplicial map which preserves the cyclic operators. If f is a semi-cyclic map of cyclic spaces, then the maps f_r are C_r equivariant for all $r \in \mathbb{N}$ and hence produce natural maps $f^{[r]}$ of the associated C_r fixed point spaces $f^{[r]}: |X.|^{C_r} \to |Y.|^{C_r}$. (we set $f^{[0]}$ to be the natural map mentioned earlier).

Since $s_X \circ q_X$ is homotopic to the identity, we see that $f_r \circ g_r \simeq (f \circ g)_r$ but these are not necessarily equal unless g was a map of simplicial sets. Similarly, if f is only a semi-cyclic map and for $r|s$ we do not obtain either that f_r is the restriction of f_s or that $f^{[s]}$ is the restriction of $f^{[r]}$. We only obtain that these commute up to homotopy, one reason for these failures lies in the fact that $D_r \circ s_{\mathrm{sd}_r X} \neq s_X \circ D_r$. If f is a cyclic map then these conditions do hold.

A.4. The cyclic set $N^{\mathrm{cy}}(\mathscr{I})$

We let $\mathscr{I}$ denote the groupoid on two objects. That is, $\mathscr{I}$ is a category with two objects 0 and 1 and two (necessarily inverse) non-trivial morphisms $0 \to 1$ and $1 \to 0$. For any (small) category $\mathscr{C}$, we let $N^{\mathrm{cy}}(\mathscr{C})$ denote its cyclic nerve (see Example 2.2.3). One can check that the non-degenerate simplices of $N^{\mathrm{cy}}_{2n}(\mathscr{I})$ are:

$$x_n \equiv (0 = 0 \leftarrow 1 \leftarrow 0 \leftarrow \cdots \leftarrow 1 \leftarrow 0),$$

$$y_n \equiv (1 = 1 \leftarrow 0 \leftarrow 1 \leftarrow \cdots \leftarrow 0 \leftarrow 1)$$

and that the non-degenerate simplicies $N^{\mathrm{cy}}_{2n-1}(\mathscr{I})$ are simply $d_0(x_n)$ and $d_0(y_n)$. We let $N^{\mathrm{cy}}(\mathscr{I})[2n]$ denote the $2n$-skeleton of $N^{\mathrm{cy}}(\mathscr{I})$ which is not only a simplicial subset but also a cyclic subset. It is not too difficult to see that the realization of $N^{\mathrm{cy}}(\mathscr{I})[2n]$ is S^{2n} and that the inclusion of cyclic sets $N^{\mathrm{cy}}(\mathscr{I})[2n] \subset N^{\mathrm{cy}}(\mathscr{I})[2n+2]$ corresponds to the double suspension map. Thus, the realization of $N^{\mathrm{cy}}(\mathscr{I})$ is homeomorphic to the direct limit of $S^0 \to S^2 \to \cdots$ with structure maps double suspension. What we have not yet indicated is the S^1 action on $|N^{\mathrm{cy}}(\mathscr{I})|$ we obtain since it is the realization of a cyclic set.

Proposition A.4.1. *Considering S^{2n} as the subset of $\mathbb{R} \times \mathbb{C}^n$ determined by*

$$\{(r, z_1, \ldots, z_n) \,|\, |r| + \textstyle\sum \|z_i\| = 1\},$$

we can describe the action of $\lambda \in \mathbb{C}^*$ on $|N^{\mathrm{cy}}(\mathscr{I})|$ as

$$\lambda * (r, z_1, \ldots, z_n) = (r, \lambda z_1, \lambda^2 z_2, \ldots, \lambda^n z_n).$$

Proof. The cases of $n = 0$ and $n = 1$ are clear. We assume the proposition is true through case $n - 1$ and proceed by induction. We consider the C_n-fixed point set which by induction must consist of elements of the form $(r, 0, \ldots, 0, z_n)$. A straightforward calculation shows that the simplicial map from $N^{\mathrm{cy}}(\mathscr{I})[2]$ to $(\mathrm{sd}_n N^{\mathrm{cy}}(\mathscr{I})[2n])^{C_n}$ (the sub-simplicial set which is degree-wise fixed by the C_n action) defined by "concatenating" a cyclic diagram n times with itself is an isomorphism of simplicial sets. Furthermore, it is also straightforward to show that $(\mathrm{sd}_{tn} N^{\mathrm{cy}}(\mathscr{I})[2n])^{C_{tn}}$ consists of exactly two non-degenerate points if $t > 1$. The result follows. □

Corollary A.4.2. *Giving $|N^{\mathrm{cy}}(\mathscr{I})|$ the circle action it obtains as the realization of a cyclic set, then it is a contractible S^1-space with the property that the S^1-fixed points are simply two points (corresponding to the simplicies $0 = 0$ and $1 = 1$) but with the C_n-fixed point sets contractible for all $n \in \mathbb{N}$.* □

A.5. Special homotopies

Definition A.5.1. We will call two semi-cyclic maps f and g *special homotopic* if there exists a semi-cyclic map $h : X_\cdot \times N^{\mathrm{cy}}(\mathscr{I}) \to Y_\cdot$ such that the following diagram commutes,

$$\begin{tikzcd} X. \arrow[r, "\varepsilon_0"] \arrow[dr, "f"'] & X.\times N^{\mathrm{cy}}(\mathscr{I}) \arrow[d, "h"] & X. \arrow[l, "\varepsilon_1"'] \arrow[dl, "g"] \\ & Y. & \end{tikzcd}$$

where $\varepsilon_i(x) = (x \times i = i = \cdots = i)$. The natural simplicial embedding $\Delta(1) \to N^{\mathrm{cy}}(\mathscr{I})$ shows "special homotopic implies semi-homotopic" but the converse is in general false. We say that a semi-cyclic map f is a *special homotopy equivalence* if it has a special homotopy inverse. That is, a semi-cyclic map g such that $f \circ g$ and $g \circ f$ are both special homotopic to the identity.

Lemma A.5.2. *Special homotopic maps f and g have the property that $f^{[r]}$ is homotopic to $g^{[r]}$ for all $r \in \mathbb{N}$.*

Proof. One can see that the C_r fixed sub-simplicial set of $\mathrm{sd}_r(N^{\mathrm{cy}}(\mathscr{I}))$ is isomorphic to $N^{\mathrm{cy}}(\mathscr{I})$ by the simplicial isomorphism which sends a cyclic diagram to itself "concatenated r times." That is, $N^{\mathrm{cy}}(\mathscr{I})$ is an *epi-cyclic* space in the sense of [8]. The result follows because the semi-simplicial realization of $X. \times N^{\mathrm{cy}}(\mathscr{I})$ is naturally homotopic to the product of the semi-simplicial realizations. $\square$

Note. If f is a *cyclic* set map and g is a special homotopy inverse, then $|f|^{C_r}$ (the induced map of C_r fixed point sets) is a homotopy equivalence for all $r \in \mathbb{N}$. If $|f|^{S^1}$ is also a homotopy equivalence then f would be an S^1-homotopy equivalence by the equivariant Whitehead theorem. The following example is to show that this is not the case in general.

Example. For an algebra A, let $Z.A$ denote the usual cyclic $\mathbb{Z}$-module used to compute Hochschild homology (see Example 2.2.1). Then $Z.(\mathbb{Z} \times \mathbb{Z})$ is special homotopic to the cyclic module (operators act diagonally) $Z.(\mathbb{Z}) \oplus Z.(\mathbb{Z})$ but the S^1 fixed point set of the first is isomorphic to $\mathbb{Z}$ and for the second it is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ (the natural map of cyclic sets takes the first to the second as the diagonal map).

A.6. Cyclic × simplicial sets

Let $X..$ be a cyclic × simplicial set. As a bi-simplicial set, we know that the three natural realizations are homeomorphic. Realizing first the simplicial direction we obtain a cyclic space whose realization is naturally an S^1-space. We will give the other two possible realizations the S^1 structure they obtain via the natural homeomorphisms.

The following two lemmas are simple and left to the reader. They follow from the fact that the homotopies in question can naturally be assembled to give a homotopy of the total realizations.

Lemma A.6.1. *Let f and g be maps of cyclic $\times$ simplicial sets. If for each n, $f_{n,\star}$ is homotopic to $g_{n,\star}$ by $h(n)$ and for each p, q and $\alpha \in \mathrm{Hom}_{\Delta C}([p],[q])$, $\alpha^* \circ h(q) = h(p) \circ \alpha^*$, then $|f|$ is S^1-homotopic to $|g|$.* □

Lemma A.6.2. *Let f and g be maps of cyclic $\times$ simplicial sets. If for each m, $f_{\star,m}$ is special homotopic to $g_{\star,m}$ by $h(m)$ and for each p, q and $\beta \in \mathrm{Hom}_{\Delta}([p],[q])$, $\beta^* \circ h(q) = h(p) \circ \beta^*$, then $|f|$ is special homotopic to $|g|$.* □

Definition A.6.3. We will say that an S^1 map f is a *discrete* homotopy equivalence if $|f|^{C_r}$ is a homotopy equivalence for all $r \in \mathbb{N}$. Similarly we will call a sequence of (pointed, $* \in X^{S^1}$) S^1 maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ a *discrete* quasi-fibration if $g \circ f = *$ and the natural map from the homotopy fiber to X (which is naturally an S^1-map since the base-point is fixed under the action) is a discrete homotopy equivalence.

We note that the difference between a discrete homotopy equivalence and a special homotopy equivalence is that a special homotopy equivalence has the property that the homotopy equivalences for the various fixed point spaces can be assembled in a compatible fashion.

Lemma A.6.4 (Realization lemma). *Let f be a cyclic $\times$ simplicial set map from $X_{..}$ to $Y_{..}$ such that $f_{\star,n}$ is a discrete homotopy equivalence for all $n \in \mathbb{N}$. Then the map of realizations $|f|$ is a discrete homotopy equivalence.*

Proof. This is immediate from the usual realization lemma (see Lemma 5.1 of [27]) after one notices that subdividing the cyclic direction generates a bi-simplicial set with a natural C_r action whose realization is naturally S^1 homeomorphic to the realization of the original bi-simplicial set. □

Lemma A.6.5 (Fibration lemma). *Let $X_{..} \xrightarrow{f} Y_{..} \xrightarrow{g} Z_{..}$ be a sequence of (pointed) simplicial cyclic groups. Assume that $g \circ f = *$ and that the sub-sequences $f_{\star,n} \circ g_{\star,n}$ are all discrete quasi-fibrations. Then the sequence of total realizations is a discrete quasi-fibration.*

Proof. First note that if $X_{..}$ is a pointed cyclic $\times$ simplicial set then the realization has the property that the basepoint is invariant under the induced S^1-action. The sequence is a quasi-fibration by Theorem B.4 of [2]. When we subdivide the cyclic direction, we obtain new sequences which are again fibration sequences since they are homeomorphic to the original sequence. Considering the subsequences of these which are fixed by the group actions we obtain sequences which by assumption are again quasi-fibration sequences degree-wise and hence (again by [2]) they produce quasi-fibrations of their total realizations. □

Corollary A.6.6. *Suppose we are in the situation of the fibration lemma above and suppose further that we have an S^1-equivariant contraction for $Y.$, then the natural homotopy equivalence $X \xrightarrow{\simeq} \Omega Z$ is a discrete homotopy equivalence.*

Proof. The natural map from X to ΩZ in this setting is an S^1 map (we are using the fact that we have chosen an S^1-equivariant contraction here) and the result follows. □

Remark. In the situation of the corollary, we obtain the diagram

$$\mathrm{holim}_{r|s} X^{C_r} \xrightarrow{\simeq} \mathrm{holim}_{r|s} \Omega Z^{C_r} \longleftarrow \Omega Z^{S^1}.$$

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