# $A_\infty$ RING SPACES AND ALGEBRAIC K-THEORY

J.P. May

In [22], Waldhausen introduced a certain functor $A(X)$, which he thought of as the algebraic K-theory of spaces X, with a view towards applications to the study of the concordance groups of PL manifolds among other things. Actually, the most conceptual definition of $A(X)$, from which the proofs would presumably flow most smoothly, was not made rigorous in [22] on the grounds that the prerequisite theory of rings up to all higher coherence homotopies was not yet available.

As Waldhausen pointed out, my theory of $E_\infty$ ring spaces [12] gave a successful codification of the stronger notion of commutative ring up to all higher coherence homotopies. We begin this paper by pointing out that all of the details necessary for a comprehensive treatment of the weaker theory appropriate in the absence of commutativity are already implicit in [12]. Thus we define $A_\infty$ ring spaces in section 1, define $A_\infty$ ring spectra in section 2, and show how to pass back and forth between these structures in section 3. The reader is referred to [13] for an intuitive summary of the $E_\infty$ ring theory that the present $A_\infty$ ring theory will imitate.

The general theory does not immediately imply that Waldhausen's proposed definition of $A(X)$ can now be made rigorous. One must first analyze the structure present on the topological space $M_nX$ of $(n \times n)$-matrices with coefficients in an $A_\infty$ ring space X. There is no difficulty in giving $M_nX$ a suitable additive structure, but it is the multiplicative structure that is of interest and its analysis requires considerable work. We prove in section 4 that $M_nX$ is a multiplicative $A_\infty$ space and compare these $A_\infty$ structures as n varies in section 6. Technically, the freedom to use different $A_\infty$ operads is crucial to the definition of these $A_\infty$ structures, and a curious change of operad pairs trick is needed for their comparison as n varies. We study the relationship between the additive and multiplicative structures on $M_nX$ in section 5. It turns out that $M_nX$ is definitely not an $A_\infty$ ring space, although it may satisfy the requirements of an appropriate strong homotopy generalization of this notion.

With this theoretical background in place, we find ourselves in a position to develop a far more general theory than would be needed solely to obtain the algebraic K-theory of spaces. Thus we construct the algebraic K-theory of $A_\infty$ ring spaces X in section 7. The basic idea is simple enough. We take the homotopy groups of the plus construction KX on the telescope of the classifying spaces of the $A_\infty$ spaces of unit components $FM_nX$. The technical work here involves the construction of the relevant compatible classifying spaces and of a modified telescope necessary for functoriality. In section 8, we analyze the effect of restricting this chain of functors to the sub $A_\infty$ spaces $F_nX$ of monomial matrices. The resulting plus construction turns out to be equivalent to the zero component of $Q(BFX \amalg \{0\})$, where FX is the $A_\infty$ space of unit components of X and QY denotes colim $\Omega^n\Sigma^nY$. The proof involves the generalization of a standard consequence of the Barratt-Quillen theorem for wreath products of monoids to wreath products of $A_\infty$ spaces together with a comparison between the $A_\infty$ spaces $F_nX$ and $\Sigma_n \int FX$.

This completes the development of the technical machinery. Of course, the proofs in sections 3-8 (all of which are relegated to the ends of the sections) are necessarily addressed to those interested in a close look at the machinery. The consumer who wishes to inspect the finished product without taking the tour through the factory is invited to first read section 1 and skim sections 2 and 3 (up to the statement of Proposition 3.7), then skip to section 7 and read as far as Remarks 7.4, next read section 8 as far as Remarks 8.5, and finally turn to sections 9-12.

We begin the homotopical and homological analysis of our functors in section 9, giving general homotopy invariance properties and pointing out a spectral sequence converging to $H_*KX$ and some general formulas relevant to the computation of its $E_2$-term.

We discuss examples and display various natural maps and diagrams in section 10. On discrete rings, our theory reduces to Quillen's [18, 5]. For general $A_\infty$ ring spaces X, the discretization map $X \to \pi_0 X$ is a map of $A_\infty$ ring spaces. This establishes a natural map from the new algebraic K-groups of X to Quillen's algebraic K-groups of $\pi_0 X$. To illustrate the force of this assertion, we record the following trifling consequence of the diagram displayed in Theorem 10.7.

Corollary. The usual map from the $q^{\underline{th}}$ stable homotopy group of spheres to the $q^{\underline{th}}$ algebraic K-group of Z factors through the $q^{th}$ algebraic K-group of X for any $A_\infty$ ring space X such that $\pi_0 X = Z$.

On topological rings, our theory reduces to that of Waldhausen [22, § 1], or rather, to the topological analog of his simplicial theory. However, it should be pointed out that virtually all of the proofs in the earlier sections become completely trivial in this special case. The force of the theory is the translation of the obvious intuition, that much that is true for rings remains true for $A_\infty$ ring spaces, into rigorous mathematics. The crucial question is, how much? We have already observed that the matrix "ring" functor $M_n$ does not preserve $A_\infty$ ring spaces, and other examples of phenomena which do not directly generalize are discussed in Remarks 10.3 and 12.4. What I find truly remarkable is how very much does in fact generalize. The point is that there are vast numbers of interesting $A_\infty$ ring spaces which are far removed from our intuition of what a ring looks like.

Perhaps the most fascinating examples come from the fact that $E_\infty$ ring spaces are $A_\infty$ ring spaces by neglect of structure. For instance, the category of finitely generated projective modules (or free modules) over a commutative ring R gives rise to an $E_\infty$ ring space in which the addition comes from the direct sum of modules and the multiplication comes from the tensor product. The homotopy groups of this space are Quillen's algebraic K-groups of R, and the present theory gives rise to a second order algebraic K-theory based on its $A_\infty$ ring structure. I shall say a little bit more about such examples in section 10, but I should add at once that I have not yet had time even to contemplate the problem of making actual calculations.

We specialize to obtain the algebraic K-theory of topological spaces X in section 11, defining A(X) to be $KQ(\Omega X \amalg \{0\})$, as proposed by Waldhausen. The general theory gives an immediate calculation of the rational homotopy type of A(X), and various other properties claimed by Waldhausen also drop out by specialization. In particular, we discuss the algebraic K-theory of X with coefficients in a (discrete) commutative ring and give a complete account of the stabilization of the various algebraic K-theories of X to generalized homology theories, this being based on a general stabilization theorem given in the Appendix.

While the theory discussed above was inspired by Waldhausen's ideas, it is logically independent of his work and should be of independent interest. The connection with his theory is work in progress and is discussed very briefly in section 12. The basic point to be made is that, at this writing, there exist two algebraic K-theories of spaces, the one developed here and the one rigorously defined by Waldhausen, and a key remaining problem is to prove their equivalence.

I would like to thank Mel Rothenberg for insisting that I try to make Waldhausen's ideas rigorous and for many helpful discussions. Conversations with Bob Thomason have also been very useful. I am profoundly indebted to Waldhausen for envisioning the possibility of such a theory as that presented here.

Contents

## §1. $A_\infty$ ring spaces

We begin by recalling the definitional framework of [12]; details are in [12,VI §1 and §2] and a more leisurely discussion of some of the main ideas is in [10, §1-3].

The notion of an $E_\infty$ ring space is based on the notion of an operad pair $(\mathcal{C}, \mathcal{G})$, which consists of an "additive" operad $\mathcal{C}$, a "multiplicative" operad $\mathcal{G}$, and an action of $\mathcal{G}$ on $\mathcal{C}$. An operad $\mathcal{C}$ has associated to it a monad $(C, \mu, \eta)$ in $\mathcal{J}$, the category of (nice) based spaces. There is a notion of an action of $\mathcal{C}$ on a space $X$, and this is equivalent to the standard notion of an action of the monad $C$ on $X$. These notions apply equally well to $\mathcal{G}$. Actions by $\mathcal{G}$ are thought of as multiplicative, with basepoint $1$, and a $\mathcal{G}_0$-space is a $\mathcal{G}$-space with a second basepoint 0 which behaves as zero under the action. When $\mathcal{G}$ acts on $\mathcal{C}$, the monad $C$ restricts to a monad in $\mathcal{G}_0[\mathcal{J}]$, the category of $\mathcal{G}_0$-spaces. That is, $CX$ is a $\mathcal{G}_0$-space if $X$ is a $\mathcal{G}_0$-space and $\mu: CCX \to CX$ and $\eta: X \to CX$ are then maps of $\mathcal{G}_0$-spaces. An action of $(\mathcal{C}, \mathcal{G})$ on a space $X$ is an action of the monad $C$ in $\mathcal{G}_0[\mathcal{J}]$ on $X$. That is, a $(\mathcal{C}, \mathcal{G})$-space is both a $\mathcal{C}$-space and a $\mathcal{G}_0$-space such that the additive action $CX \to X$ is a map of $\mathcal{G}_0$-spaces. The last condition encodes distributivity homotopies in a simple conceptual way, and the multiplicative theory is to be thought of as obtained from the additive theory by a change of ground categories from spaces to $\mathcal{G}_0$-spaces.

$\mathcal{C}$ is an $E_\infty$ operad if its $j^{th}$ space $\mathcal{C}(j)$ is contractible and is acted on freely by the symmetric group $\Sigma_j$. An $E_\infty$ space is a space together with an action by any $E_\infty$ operad $\mathcal{C}$. Examples are the spaces CX for any space X. These are commutative monoids up to all coherence homotopies. $(\mathcal{C}, \mathcal{G})$ is an $E_\infty$ operad pair if $\mathcal{C}$ and $\mathcal{G}$ are both $E_\infty$ operads. An $E_\infty$ ring space is a space together with an action by any $E_\infty$ operad pair $(\mathcal{C}, \mathcal{G})$. Examples are the spaces CX for any $\mathcal{G}_0$-space X. These are commutative semi-rings up to all coherence homotopies (semi-ring because additive inverses are not built in).

A theory of $A_\infty$ spaces is developed in [10, § 3 and 13]. An operad $\mathcal{G}$ is an $A_\infty$ operad if $\pi_0 \mathcal{G}(j)$ is $\Sigma_j$ and each component of $\mathcal{G}(j)$ is contractible. An $A_\infty$ space is a space together with an action by any $A_\infty$ operad $\mathcal{G}$. These are monoids up to all coherence homotopies. As explained in [10, p. 134], $A_\infty$ spaces are equivalent to monoids, hence admit classifying spaces; there is also a direct delooping construction independent of the use of monoids.

Definition 1.1. An operad pair $(\mathcal{C}, \mathcal{G})$ is an $A_\infty$ operad pair if $\mathcal{C}$ is an $E_\infty$ operad and $\mathcal{G}$ is an $A_\infty$ operad. An $A_\infty$ ring space is a space together with an action by any $A_\infty$ operad pair $(\mathcal{C}, \mathcal{G})$. Examples are the spaces CX for any $\mathcal{G}_0$-space X.

This is the desired notion of a ring (or rather, semi-ring) up to all coherence homotopies. There is also a notion with both $\mathcal{C}$ and $\mathcal{G}$ $A_\infty$ operads, but it seems unprofitable to study rings up to homotopy for which not even addition is homotopy commutative.

Note that the product of an $E_\infty$ operad and an $A_\infty$ operad is an $A_\infty$ operad. Thus if $(\mathcal{C}, \mathcal{G})$ is an $E_\infty$ operad pair and $\mathcal{G}'$ is an $A_\infty$ operad, then $(\mathcal{C}, \mathcal{G}' \times \mathcal{G})$ is an $A_\infty$ operad pair, the action of $\mathcal{G}' \times \mathcal{G}$ on $\mathcal{C}$ being obtained by pullback from the action of $\mathcal{G}$ on $\mathcal{C}$. Since a $(\mathcal{C}, \mathcal{G})$-space is a $(\mathcal{C}, \mathcal{G}' \times \mathcal{G})$-space, again by pullback, $E_\infty$ ring spaces are $A_\infty$ ring spaces.

Some discussion of discrete operads may clarify ideas. There are operads $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}(j) = \Sigma_j$ and $\mathcal{N}(j)$ is a point. An $\mathcal{M}$-space is a monoid and an $\mathcal{N}$-space is a commutative monoid. Both $\mathcal{M}$ and $\mathcal{N}$ act on $\mathcal{N}$. An $(\mathcal{N}, \mathcal{M})$-space is a semi-ring and an $(\mathcal{N}, \mathcal{N})$-space is a commutative semi-ring. Say that a map of operads is an equivalence if the underlying map of $j^{th}$ spaces is a homotopy equivalence for each $j$. An $E_\infty$ operad $\mathcal{C}$ admits an evident equivalence $\mathcal{C} \to \mathcal{N}$. An operad $\mathcal{G}$ is an $A_\infty$ operad if and only if it admits an equivalence $\mathcal{G} \to \mathcal{M}$. Thus $A_\infty$ operad pairs and $E_\infty$ operad pairs map by equivalences onto the respective operad pairs $(\mathcal{N}, \mathcal{M})$ and $(\mathcal{N}, \mathcal{N})$.

Remark 1.2. I would like to correct an annoying misprint in the crucial definition, [12,VI. 1.6], of an action of $\mathcal{G}$ on $\mathcal{C}$. In (a') of the cited definition, the displayed formula is missing some symbols, $d_I$ being written for $\lambda(g, d_I)$, and should read

$$\gamma(\lambda(g; c_1, \ldots, c_k); \underset{I \in S(j_1, \ldots, j_k)}{\times} \lambda(g; d_I))\nu = \lambda(g; e_1, \ldots, e_k) .$$

## §2 $A_\infty$ ring spectra

An $E_\infty$ space determines a spectrum and thus a cohomology theory. The notion of $E_\infty$ ring spectrum encodes the additional multiplicative structure on the spectra derived from the underlying additive $E_\infty$ spaces of $E_\infty$ ring spaces. $A_\infty$ ring spaces also have underlying additive $E_\infty$ spaces, and we have an analogous notion of $A_\infty$ ring spectrum. Only the multiplicative operad $\mathcal{G}$ appears in these definitions. Let $\mathcal{L}$ denote the linear isometries $E_\infty$ operad of [12,I.1.2]. For good and sufficient reasons explained in [12,IV §1], we assume given a map of operads $\mathcal{G} \to \mathcal{L}$. In the cited section, $\mathcal{G}$ was assumed to be an $E_\infty$ operad. We may instead assume that $\mathcal{G}$ is an $A_\infty$ operad. For example, $\mathcal{G}$ might be the product of an $A_\infty$ operad and an $E_\infty$ operad which maps to $\mathcal{L}$. Now the assumption that $\mathcal{G}$ was an $E_\infty$ operad played no mathematical role whatever in the definition, [12,IV.1.1], of a $\mathcal{B}$-spectrum. This notion of an action by $\mathcal{B}$ only required the map $\mathcal{B} \to \mathcal{L}$ and is thus already on hand in our $A_\infty$ context.

Definition 2.1. An $A_\infty$ ring spectrum is a $\mathcal{B}$-spectrum over any $A_\infty$ operad $\mathcal{B}$ with a given morphism of operads $\mathcal{B} \to \mathcal{L}$.

As explained in [12, p. 68-70], an $A_\infty$ ring spectrum is a (not necessarily commutative) ring spectrum with additional structure. In particular, its zero$^{th}$ space is a $\mathcal{B}_0$-space. The formal lemmas [12, IV 1.4-1.9] apply verbatim to $\mathcal{B}$-spectra for any operad $\mathcal{B}$ which maps to $\mathcal{L}$. We summarize the conclusions they yield.

Recall that a (coordinate-free) spectrum $E$ consists of a space $EV$ for each finite-dimensional sub inner product space $V$ of $R^\infty$ together with an associative and unital system of homeomorphisms $EV \to \Omega^W E(V+W)$ for $V$ orthogonal to $W$; here $E_0 = E\{0\}$. The stabilization functor $Q_\infty$ from spaces to spectra is defined by

$$Q_\infty X = \{Q\Sigma^V X \mid V \subset R^\infty\}, \text{ where } QX = \operatorname{colim} \Omega^V \Sigma^V X;$$

here the loop and suspension functors $\Omega^V$ and $\Sigma^V$ are defined in terms of the sphere $tV$, the one-point compactification of $V$. The inclusion $\eta: X \to QX$ and colimit of loops on evaluation maps $\mu: QQX \to QX$ give a monad $(Q, \mu, \eta)$ in $\mathcal{T}$, and the analogous colimit map $\xi: QE_0 \to E_0$ gives an action of $Q$ on $E_0$. The sphere spectrum $S$ is defined to be $Q_\infty S^0$, and $S^0$ is a $\mathcal{B}_0$-space for any operad $\mathcal{B}$. Use of these notions is vital for rigor, but the reader may prefer to think of spectra in more classical terms, restricting attention to $V = R^i$ for $i \geq 0$.

Proposition 2.2. Let $X$ be a $\mathcal{B}_0$-space and $E$ a $\mathcal{B}$-spectrum.

(i) $Q_\infty X$ is a $\mathcal{B}$-spectrum and is the free $\mathcal{B}$-spectrum generated by $X$ in the sense that a map $f: X \to E_0$ of $\mathcal{B}_0$-spaces extends uniquely to a map $\tilde{f}: Q_\infty X \to E$ of $\mathcal{B}$-spectra such that $\tilde{f}_0 \eta = f$.

(ii) $S$ is a $\mathcal{J}$-spectrum and the unit $e: S \to E$ is a $\mathcal{J}$-map.

(iii) The monad $Q$ in $\mathcal{T}$ restricts to a monad in $\mathcal{J}_0[\mathcal{T}]$ and $\xi: QE_0 \to E$ is a $\mathcal{J}_0$-map, so that $E_0$ is a $Q$-algebra in $\mathcal{J}_0[\mathcal{T}]$.

The following analog of [12,IV. 1.10] is central to the definitions proposed by Waldhausen.

Example 2.3. For a $\mathcal{J}$-space $X$ without zero, construct a $\mathcal{J}$-space $X^+$ with zero by adjoining a disjoint basepoint $0$ to $X$ and extending the action in the evident way. $Q_\infty X^+$ is then a $\mathcal{J}$-spectrum and $\eta: X^+ \to QX^+$ is a map of $\mathcal{J}_0$-spaces. If $\mathcal{J}'$ is any $A_\infty$ operad and $\mathcal{J} = \mathcal{J}' \times \mathcal{L}$, then a $\mathcal{J}'$-space is a $\mathcal{J}$-space via the projection $\mathcal{J} \to \mathcal{J}'$, while the projection $\mathcal{J} \to \mathcal{L}$ allows $\mathcal{J}$ to be used in the present theory. Therefore $Q_\infty X^+$ is an $A_\infty$ ring spectrum for any $A_\infty$ space $X$.

Remarks 2.4. For what it is worth, we note that much of the discussion of orientation theory given in [12,IV §3] remains valid for $A_\infty$ ring spectra. One first checks that commutativity of the underlying ring spectra is not essential to the general theory in [12,III]. Independently of this, one finds that the assertion of [12,IV.3.1] is valid for $\mathcal{J}$-spectra $E$ for $A_\infty$ operads $\mathcal{J}$ as well as for $E_\infty$ operads $\mathcal{J}$. The cited result gives a certain commutative diagram of $\mathcal{J}$-spaces and $\mathcal{J}$-maps, the middle row of which yields a fibration sequence

$$G \xrightarrow{e} FE \xrightarrow{\tau} B(G;E) \xrightarrow{q} BG \xrightarrow{Be} BFE$$

after passage one step to the right by use of the classifying space functor on $\mathcal{J}$-spaces. Here $FE$ is the union of those components of $E_0$ which are units in the ring $\pi_0 E_0$, $G$ is the infinite group or monoid corresponding to some theory of bundles or fibrations, such as $O, U, Top,$ or $F$, and $B(G;E)$ is the classifying space for $E$-oriented $G$-bundles or fibrations. The map $q$ corresponds to neglect of orientation and the maps $e$ and $\tau$ are interpreted in [12,III. 2.5]. The point is that the notion of $A_\infty$ ring spectrum is just strong enough to yield $Be: BG \to BFE$, which is the universal obstruction to the E-orientability of G-bundles; compare [12, IV. 3.2].

## §3. The recognition principle

We first show that the zero$^{th}$ space of an $A_\infty$ ring spectrum is an $A_\infty$ ring space and then show that the spectrum determined by the additive $E_\infty$ structure of an $A_\infty$ ring space is an $A_\infty$ ring spectrum. We also obtain comparisons between the two evident composite functors and give an $A_\infty$ ring level version of the Barratt-Quillen theorem. All of this is in precise analogy with the corresponding development for $E_\infty$ ring spaces and spectra in [12, VII], and we need only point out the trivial changes of definition involved.

Let $(\mathcal{C}, \mathcal{G})$ be an $A_\infty$ operad pair and suppose given a map of operad pairs $(\pi, \rho): (\mathcal{C}, \mathcal{G}) \to (\mathcal{K}_\infty, \mathcal{L})$, where $\mathcal{K}_\infty$ is the infinite little convex bodies $E_\infty$ operad. Here $\mathcal{K}_\infty$ and its action by $\mathcal{L}$ are defined in [12, VII §1 and §2] (and we are suppressing technical problems handled there). $\mathcal{K}_\infty$ acts naturally on the zero$^{th}$ spaces of spectra, and there is a morphism $\alpha_\infty: K_\infty \to Q$ of monads in $\mathcal{T}$ Similarly, $\pi$ induces a morphism $C \to K_\infty$ of monads in $\mathcal{T}$. With these notations, the proof of [12, VII. 2.4] applies verbatim to prove the following result, in which the second part follows from the first via part (iii) of Proposition 2.2.

Theorem 3.1. (i) The morphisms $\pi: C \to K_\infty$ and $\alpha_\infty: K_\infty \to Q$ of monads in $\mathcal{T}$ restrict to morphisms of monads in $\mathcal{G}_0[\mathcal{T}]$.

(ii) If $E$ is a $\mathcal{G}$-spectrum, then its zero$^{th}$ space $E_0$ is a $(\mathcal{C}, \mathcal{G})$-space by pullback of its Q-action $\xi: QE_0 \to E_0$ along $\alpha_\infty \pi$.

An $E_\infty$ space $X$ determines a spectrum $\underline{B}X$. Thanks to recent work by Thomason and myself [16], we now know that all infinite loop space machines yield equivalent spectra when applied to X, but it is essential to the present multiplicatively enriched theory that we use the construction presented in [12, VII §3]. We assume now that the $A_\infty$ operad pair $(\mathcal{C}, \mathcal{G})$ is $(\mathcal{C}' \times \mathcal{K}_\infty, \mathcal{G}' \times \mathcal{L})$ and that $(\pi, \rho)$ is given by the projections, where $(\mathcal{C}', \mathcal{G}')$ is an operad pair such that each $\mathcal{C}'(j)$ is contractible (but not necessarily $\Sigma_j$-free) and $\mathcal{G}'$ is an $A_\infty$ operad. For definiteness, one might think of the example $(\mathcal{C}', \mathcal{G}') = (\mathcal{N}, \mathcal{M})$. The proofs of [12, VII.4.1 and 4.2] apply verbatim to yield the following results.

<u>Theorem 3.2.</u> If $X$ is a $(\mathcal{C}, \mathcal{G})$-space, then $\underline{B}X$ (formed with respect to the $\mathcal{C}$-space structure) is a $\mathcal{G}$-spectrum.

The relationship between $X$ and the zero$^{\text{th}}$ space $B_0X$ is summarized by a natural diagram

$$\begin{tikzcd} X \arrow[r, leftarrow, "\simeq"] \arrow[rrr, dashed, bend right, "\iota"'] & B(C,C,X) \arrow[r] & B(Q,C,X) \arrow[r, "\simeq"] & B_0X. \end{tikzcd}$$

The first and third solid arrows are equivalences, and $\iota$ is obtained by use of a canonical homotopy inverse to the first arrow. The middle solid arrow, and therefore also $\iota$, is a group completion (see [12, p.168] or, for a full discussion, [11, §1]).

<u>Theorem 3.3.</u> The solid arrows in this diagram are maps of $(\mathcal{C}, \mathcal{G})$-spaces. The dotted arrow $\iota$ is a map of $\mathcal{G}_0$-spaces.

The canonical homotopy inverse, and $\iota$, are not $\mathcal{C}$-maps, but this is of little significance. The basic idea is that we have group completed the additive structure of $X$ while carrying along the multiplicative structure.

We have the following consistency statements in special cases, the proofs being identical to those in [12, p.191-192]. For a spectrum $E$, there is a natural map of spectra $\tilde{\omega}: \underline{B}E_0 \to E$, and $\tilde{\omega}$ is an equivalence if $E$ is connective (that is, if $\pi_i E = 0$ for $i < 0$).

Proposition 3.4. If $E$ is a $\mathcal{H}$-spectrum, then $\tilde{\omega}: \underline{B}E_0 \rightarrow E$ is a map of $\mathcal{H}$-spectra.

For a $\mathcal{H}_0$-space $Y$, $CY$ is a $(\mathcal{C}, \mathcal{H})$-space by Definition 1.1 and $QY$ is a $(\mathcal{C}, \mathcal{H})$-space by Proposition 2.2(i) and Theorem 3.1(ii). Moreover, $\alpha_\infty \pi: CY \rightarrow QY$ is a map of $(\mathcal{C}, \mathcal{H})$-spaces by Theorem 3.1(i).

Proposition 3.5. For a $\mathcal{H}_0$-space $Y$, the composite map of $\mathcal{H}$-spectra

$$\underline{B}CY \xrightarrow{\underline{B}(\alpha_\infty \pi)} \underline{B}QY \xrightarrow{\tilde{\omega}} Q_\infty Y$$

is a strong deformation retraction. Its inverse inclusion $\nu: Q_\infty Y \rightarrow \underline{B}CY$ is induced by the freeness of $Q_\infty Y$ from the $\mathcal{H}_0$-map $Y \xrightarrow{\eta} CY \xrightarrow{\iota} B_0 CY$ and is thus a map of $\mathcal{H}$-spectra.

When $Y = CS^0$, $CY = \coprod \mathcal{C}(j)/\Sigma_j = \coprod K(\Sigma_j, 1)$. Here the last result is a multiplicatively enriched form of the Barratt-Quillen theorem, the strongest form of which appears on the $E_\infty$ ring level. Interesting $A_\infty$ ring level applications come from $A_\infty$ spaces, such as monoids, via Example 2.3.

The previous result can be related to the Hurewicz homomorphism. The monad $N$ associated to $\mathcal{N}$ assigns to a space $Y$ its infinite symmetric product, or free commutative topological monoid, and any operad $\mathcal{H}$ acts on $\mathcal{N}$. Therefore $N$ restricts to a monad in $\mathcal{H}_0[\mathcal{T}]$. If $\varepsilon: \mathcal{C} \rightarrow \mathcal{N}$ is the augmentation, then $(\varepsilon, 1): (\mathcal{C}, \mathcal{H}) \rightarrow (\mathcal{N}, \mathcal{H})$ is a map of operad pairs. These observations imply the following result.

Lemma 3.6. For a $\mathcal{H}_0$-space $Y$, $\varepsilon: CY \rightarrow NY$ is a map of $(\mathcal{C}, \mathcal{H})$-spaces, hence $\underline{B}\varepsilon: \underline{B}CY \rightarrow \underline{B}NY$ is a map of $\mathcal{H}$-spectra.

Now forget all about the multiplicative structure on $Y$. By an oversight, the following result was omitted from my earlier works in this area.

Proposition 3.7. For a based space $Y$, $\pi_* \underline{B}NY$ is naturally isomorphic to $\tilde{H}_* Y$ and $h = \underline{B}\varepsilon \circ \nu : Q_\infty Y \to \underline{B}NY$ induces the stable Hurewicz homomorphism on passage to homotopy groups. In particular, $h$ is a rational equivalence.

Proof. The zero$^{\text{th}}$ map $h: QY \to B_0NY$ is obtained by passage to direct limits from the top composite in the commutative diagram

$$\begin{tikzcd}
\Omega^n\Sigma^n Y \arrow[r, "\Omega^n\Sigma^n\eta"] & \Omega^n\Sigma^n NY \arrow[r, "\Omega^n\Sigma^n\iota"] & \Omega^n\Sigma^n B_0NY \arrow[r, "\Omega^n\sigma_n"] & \Omega^n B_n NY \\
Y \arrow[u, "\eta"] \arrow[r, "\eta"] & NY \arrow[u, "\eta"] \arrow[r, "\iota"] & B_0NY \arrow[u, "\eta"] \arrow[r, equal] & B_0NY \arrow[u, equal]
\end{tikzcd}$$

where $\sigma_n$ is the iterated structure map of the spectrum $\underline{B}NY$ (and $\eta$ is written for the unit of both monads $N$ and $\Omega^n\Sigma^n$). Therefore $h$ will induce the stable Hurewicz homomorphism on homotopy groups if $\iota\eta: Y \to B_0NY$ induces the ordinary Hurewicz homomorphism. If $Y$ is connected, then $\iota$ is a natural equivalence while $\pi_*NY = \tilde{H}_*Y$ and $\eta$ induces the Hurewicz homomorphism on $\pi_*$ by results of Dold and Thom [3]. Thus the problem is to account for non-connected spaces. Let $\tilde{N}Y$ denote the free commutative topological group generated by $Y$ and let $\tilde{\eta}: Y \to \tilde{N}Y$ and $\lambda: NY \to \tilde{N}Y$ denote the natural inclusions, so that $\lambda\eta = \tilde{\eta}$. Dold and Thom give that $\pi_*\tilde{N}Y = \tilde{H}_*Y$ and $\tilde{\eta}$ induces the Hurewicz homomorphism on $\pi_*$. One could prove that $\lambda$ is a group completion by direct homological calculation and then deduce an equivalence $\tilde{N}Y \simeq B_0NY$, but we shall reverse this idea. Consider the following diagram.

$$\begin{tikzcd}
\tilde{N}Y \arrow[r, "\phi"] & \Omega\tilde{N}\Sigma Y & \Omega N\Sigma Y \arrow[l, "\Omega\lambda"'] \arrow[r, "\Omega\xi"] & \Omega BNY \arrow[r, "\psi"] & B_0NY \\
& & \Omega\Sigma Y \arrow[u, "\Omega\eta"'] \arrow[ul, "\Omega\tilde{\eta}"] & NY \arrow[u, "\zeta"'] \arrow[ur, "\iota"'] & \\
& & Y \arrow[u, "\eta"] \arrow[ur, "\eta"'] \arrow[uull, "\tilde{\eta}"] & &
\end{tikzcd}$$

Results of Dold and Thom imply that $\lambda$ is a weak equivalence (since $\Sigma Y$ is connected) and that there is a natural weak equivalence $\phi$ such that $\phi \circ \tilde{\eta} \simeq \Omega\tilde{\eta} \circ \eta$. By a result of Milgram [17, p. 245] (see also [9, 8.7 and 8.11]), there is a homeomorphism $\xi : N\Sigma Y \to BNY$ such that $\Omega\xi$ restricts on $Y$ to $\zeta\eta$, where $\zeta$ is the standard map [9, 8.7]. Finally, since the product on $NY$ is a map of monoids and thus of N-spaces, [12, 3.4] together with the proof of [11, 3.7 (p. 75)] give a natural weak equivalence $\psi$ such that $\psi\zeta \simeq \iota$ (modulo the use of weak equivalences with arrows going the wrong way). Since $\tilde{\eta}_*$ on the left is the Hurewicz homomorphism, so is $(\iota\eta)_*$ on the right. It is a standard consequence of the finiteness of the stable homotopy groups of spheres that, upon tensoring with the rationals, the stable Hurewicz homomorphism becomes an isomorphism of homology theories.

We record the following corollary of the proof and an elaboration of the diagram above which shows that $\phi\lambda$ is homotopic to $\Omega(\lambda\xi^{-1})\zeta : NY \to \Omega\tilde{N}\Sigma Y$.

<u>Corollary 3.8</u>. For any space $Y$, $\lambda : NY \to \tilde{N}Y$ is a group completion.

## §4. Matrices with entries in $A_\infty$ ring spaces

Let X be a $(\mathcal{C}, \mathcal{H})$-space, where $(\mathcal{C}, \mathcal{H})$ is any operad pair, and consider the set $M_nX$ of $n\times n$ matrices with entries in X. Clearly $M_nX$ is a $\mathcal{C}$-space, namely the $n^2$-fold Cartesion product of the $\mathcal{C}$-space X with itself. We wish to show that if $(\mathcal{C}, \mathcal{H})$ is an $A_\infty$ operad pair, then, while $M_nX$ is not an $A_\infty$ ring space, it is at least a (multiplicative) $A_\infty$ space. Even this much is non-trivial, since $M_nX$ is not actually a $\mathcal{H}$-space.

Since $M_nX$ is certainly not commutative, it is convenient to first eliminate the extraneous actions by symmetric groups on the spaces $\mathcal{H}(j)$; these serve only to handle commutativity homotopies in the general theory of operads. Thus assume now that $\mathcal{H}$ is a non-$\Sigma$ operad, in the sense of [10,3.12]. By [10,3.13], a typical $A_\infty$ operad has the form $\mathcal{H} \times \mathcal{M}$ (with $j^{th}$ space $\mathcal{H}(j) \times \Sigma_j$) for some non-$\Sigma$ operad $\mathcal{H}$ with each $\mathcal{H}(j)$ contractible. An action of $\mathcal{H}$ on the operad $\mathcal{C}$ is given by maps

$$\lambda : \mathcal{H}(k) \times \mathcal{C}(j_1) \times \dots \times \mathcal{C}(j_k) \to \mathcal{C}(j_1 \cdots j_k)$$

with the properties specified in [12,VI.1.6] (see Remark 1.2), except that its extraneous equivariance condition (c) must be deleted. A slight elaboration of [10,3.13] shows that the non-$\Sigma$ operad $\mathcal{H}$ acts on $\mathcal{C}$ if and only if the operad $\mathcal{H} \times \mathcal{M}$ acts on $\mathcal{C}$. A $(\mathcal{C}, \mathcal{H})$-space is defined to be a $\mathcal{C}$-space and $\mathcal{H}_0$-space X such that the additive action $CX \to X$ is a map of $\mathcal{H}_0$-spaces, and the notions of $(\mathcal{C}, \mathcal{H})$-space and $(\mathcal{C}, \mathcal{H} \times \mathcal{M})$-space are then equivalent. Therefore the theory of $A_\infty$ ring spaces may as well be developed in terms of $(\mathcal{C}, \mathcal{H})$-spaces for an $E_\infty$ operad $\mathcal{C}$ and a non-$\Sigma$ operad $\mathcal{H}$ with each $\mathcal{H}(j)$ contractible. However, the work in the rest of this section requires only that $\mathcal{C}$ be an operad acted upon by a non-$\Sigma$ operad $\mathcal{H}$ and that X be a $(\mathcal{C}, \mathcal{H})$-space. Let

$$\theta_j : \mathcal{C}(j) \times X^j \to X \quad \text{and} \quad \xi_j : \mathcal{H}(j) \times X^j \to X$$

denote the actions of $\mathcal{C}$ and of $\mathcal{H}$ on X.

We want to use matrix multiplication to define an $A_\infty$ space structure on $M_nX$. Obviously this entails use of both the multiplication and the addition on $X$. We are quite happy to use arbitrary j-fold products, that is, the products given by arbitrary elements of $\mathcal{G}(j)$. We are less happy to use arbitrary $n^{j-1}$-fold additions, but, since a canonical addition is present only in the trivial case of actual commutativity, with $\mathcal{C} = \mathcal{N}$, we have no choice. Thus define

$$\mathcal{H}_n(j) = \mathcal{C}(n^{j-1}) \times \mathcal{G}(j).$$

Let $T(j)$ denote the set of all sequences $U = (u_0, \ldots, u_j)$ with $1 \leq u_i \leq n$ and order $T(j)$ lexicographically. Let $T(r, s, j)$ denote the subset of those $U$ such that $u_0 = r$ and $u_j = s$. Define

$$\psi_j\colon \mathcal{H}_n(j) \times (M_nX)^j \to M_nX$$

by the following formula, where $x(r, s)$ denotes the $(r, s)^{\text{th}}$ entry of a matrix $x$.

$$(1) \quad \psi_j(c, g; x_1, \ldots, x_j)(r, s) = \theta_{n^{j-1}}(c; \mathop{\times}_{U \in T(r,s,j)} \xi_j(g; \mathop{\times}_{q=1}^{j} x_q(u_{q-1}, u_q))) .$$

All we have done is to write down ordinary iterated matrix multiplication, allowing for parametrized families of both multiplications and additions on the underlying "ring". The rest of this section will be devoted to the proof of the following result.

Theorem 4.1. The $\mathcal{H}_n(j)$ are the $j^{\text{th}}$ spaces of a non-$\Sigma$ operad $\mathcal{H}_n$, and the maps $\psi_j$ specify an action of $\mathcal{H}_n$ on $M_nX$.

Proof. By convention, $\mathcal{H}_n(0) = \{*\}$ and $\psi_0$ is the inclusion of the identity matrix $I_n$ in $M_nX$. Let $1 = (1, 1) \in \mathcal{C}(1) \times \mathcal{G}(1) = \mathcal{H}_n(1)$. Clearly $\psi_1(1)$ is the identity map of $M_nX$. We must specify maps

$$\gamma\colon \mathcal{H}_n(k) \times \mathcal{H}_n(j_1) \times \ldots \times \mathcal{H}_n(j_k) \to \mathcal{H}_n(j), \qquad j = j_1 + \ldots + j_k ,$$

with respect to which $\mathcal{H}$ is an operad and which make the following diagram commute, where $\mu$ is the evident shuffle homeomorphism:

$$
\begin{tikzcd}
\mathcal{H}_n(k)\times\mathcal{H}_n(j_1)\times\dots\times\mathcal{H}_n(j_k)\times(M_nX)^j \arrow[r, "\gamma\times 1"] \arrow[d, "1\times\mu"'] & \mathcal{H}_n(j)\times(M_nX)^j \arrow[r, "\psi_j"] & M_nX \\
\mathcal{H}_n(k)\times\mathcal{H}_n(j_1)\times(M_nX)^{j_1}\times\dots\times\mathcal{H}_n(j_k)\times(M_nX)^{j_k} \arrow[rr, "1\times\psi_{j_1}\times\dots\times\psi_{j_k}"] & & \mathcal{H}_n(k)\times(M_nX)^k \arrow[u, "\psi_k"']
\end{tikzcd}
\qquad (*)
$$

(Compare [10, 1.5].) We first chase the diagram to see how $\gamma$ must be defined and then verify that, with this definition, $\mathcal{H}_n$ is an operad. In principle, the details are perfectly straightforward: one does what one has to do and it works. However, since I omitted all such routine verifications from [12] and since this one is much less intuitively obvious than most, I will try to give some idea of the combinatorics involved.

We first calculate the composite around the bottom of the diagram (*). By (1), we have

(2) $$\psi_k(1\times\psi_{j_1}\times\dots\times\psi_{j_k})(1\times\mu)(c,g;c_1,g_1,\dots,c_k,g_k;x_1,\dots,x_j)(r,s)$$
$$= \theta_{n^{k-1}}(c;\ \underset{U\in T(r,s,k)}{\times}\ \xi_k(g;\ \underset{q=1}{\overset{k}{\times}}\ z_q(u_{q-1},u_q))),$$

where, with $e_q = j_1+\dots+j_q$,

(3) $$z_q = \psi_{j_q}(c_q,g_q;x_{e_{q-1}+1},x_{e_{q-1}+2},\dots,x_{e_q}).$$

If $j_q = 0$, then $z_q = I_n$. Since $X$ is a $\mathcal{G}_0$-space,

(4) $$\xi_k(g;\ \underset{q=1}{\overset{k}{\times}}\ z_q(u_{q-1},u_q)) = 0 \text{ if } j_q = 0 \text{ and } u_{q-1}\neq u_q \text{ for any } q.$$

Let $\{q_1,\dots,q_m\}$, $m\le k$, denote those $q$, in order, such that $j_q > 0$ and set

(5) $$i_p = j_{q_p},\quad d_p = e_{q_p},\quad b_p = c_{q_p},\quad f_p = g_{q_p},\ \text{and}\ y_p = z_{q_p}\quad \text{for } 1\le p\le m.$$

Let $s(j_1, \dots, j_k) \in \mathcal{M}(\varepsilon_1) \times \dots \times \mathcal{M}(\varepsilon_k)$, $\varepsilon_q = 0$ or $1$, have $q^{th}$ coordinate $1 \in \mathcal{M}(1)$ if $j_q > 0$ and $q^{th}$ coordinate $* \in \mathcal{M}(0)$ if $j_q = 0$. Set

(6) $$f = \gamma(g; s(j_1, \dots, j_k)) \in \mathcal{M}(m).$$

If $U \in T(k)$ satisfies $u_{q-1} = u_q$ whenever $j_q = 0$, define $V = (v_0, \dots, v_m) \in T(m)$ by deletion of the duplicated entries $u_q$. Then, since $1 \in X$ is the basepoint for the $\mathcal{M}$-action and $z_q(u_{q-1}, u_q) = 1$ if $j_q = 0$,

(7) $$\xi_k(g; \underset{q=1}{\overset{k}{\times}} z_q(u_{q-1}, u_q)) = \xi_m(f; \underset{p=1}{\overset{m}{\times}} y_p(v_{p-1}, v_p))$$

Note that each $V \in T(m)$ arises uniquely from such a $U \in T(k)$ and let $t(j_1, \dots, j_k) \in \mathcal{C}(\varepsilon_1) \times \dots \times \mathcal{C}(\varepsilon_{n^{m-1}})$, $\varepsilon_r = 0$ or $1$, have $r^{th}$ coordinate $1 \in \mathcal{C}(1)$ if $j_q = 0$ implies $u_{q-1} = u_q$ for the $r^{th}$ element $U \in T(k)$ and $r^{th}$ coordinate $* \in \mathcal{C}(0)$ otherwise. Set

(8) $$b = \gamma(c; t(j_1, \dots, j_k) \in \mathcal{C}(n^{m-1}).$$

Since $0 \in X$ is the basepoint for the $\mathcal{C}$-action, (4)-(8) imply

(9) $$\theta_{n^{k-1}}(c; \underset{U \in T(r,s,k)}{\times} \xi_k(g; \underset{q=1}{\overset{k}{\times}} z_q(u_{q-1}, u_q)))$$
$$= \theta_{n^{m-1}}(b; \underset{V \in T(r,s,m)}{\times} \xi_m(f; \underset{p=1}{\overset{m}{\times}} y_p(v_{p-1}, v_p))).$$

Evaluating $y_p$ by (1), (3), and (5) and then using the definition, [12, VI. 1.8 and VI. 1. 10], of a $(\mathcal{C}, \mathcal{M})$-space, we find

(10) $$\xi_m(f; \underset{p=1}{\overset{m}{\times}} y_p(v_{p-1}, v_p)) = \xi_m(f; \underset{p=1}{\overset{m}{\times}} \theta_{n^{i_p-1}}(b_p; \underset{W_p \in T(v_{p-1}, v_p, i_p)}{\times} y(W_p)))$$
$$= \theta_{n^{j-m}}(\lambda(f; b_1, \dots, b_m); \underset{H \in S(n^{i_1-1}, \dots, n^{i_m-1})}{\times} \xi_m(f; y_H(V))) .$$

where if $W_p = (w_{p,0}, \dots, w_{p,i_p}) \in T(i_p)$, then

(11) $$y(W_p) = \xi_{i_p}(f_p; \underset{t=1}{\overset{p}{\times}} x_{d_{p-1}+t}(w_{p,t-1}, w_{p,t}))$$

and if $H = (h_1, \ldots, h_m)$ with $1 \leq h_p \leq n^{i_p - 1}$, then

(12) $y_H(V) \in X^m$ has $p^{th}$ coordinate the $h_p^{th}$ coordinate of $\underset{W_p \in T(v_{p-1}, v_p, i_p)}{\times} y(W_p)$ .

By (10) and the definition of a $\mathcal{C}$-space [10, 1.3 or 12, VI. 1.3],

(13) $$\theta_{n^{m-1}}(b; \underset{V \in T(r,s,m)}{\times} \xi_m(f; \underset{p=1}{\overset{m}{\times}} y_p(v_{p-1}, v_p)))$$

$$= \theta_{n^{j-1}}(\gamma(b; \lambda(f; b_1, \ldots, b_m)^{n^{m-1}}); \underset{V \in T(r,s,m)}{\times} \underset{H \in S(n^{i_1 - 1}, \ldots, n^{i_m - 1})}{\times} \xi_m(f; y_H(V))).$$

Here $\lambda(f; b_1, \ldots, b_m) \in \mathcal{C}(n^{j-m})$, since $i_1 + \ldots + i_m = j$, hence application of $\gamma(b; -)$ to its $n^{m-1}$ st power yields an element of $\mathcal{C}(n^{j-1})$. Further, by (11), (12), and the definition of a $\mathcal{B}$-space,

(14) $$\xi_m(f; y_H(V)) = \xi_j(\gamma(f; f_1, \ldots, f_m); \underset{q=1}{\overset{j}{\times}} x_q(u_{q-1}, u_q)),$$

where $U_H(V) = (u_0, \ldots, u_j) \in T(j)$ is the sequence

$$(w_{1,0}, \ldots, w_{1,i_1}, w_{2,1}, \ldots, w_{2,i_2}, \ldots, w_{m,1}, \ldots, w_{m,i_m})$$

obtained by splicing together the $h_p$ th elements $W_p$ of the ordered sets $T(v_{p-1}, v_p, i_p)$ for $1 \leq p \leq m$. As $V$ runs through $T(r, s, m)$ and $H$ runs through $S(n^{i_1 - 1}, \ldots, n^{i_m - 1})$, $U_H(V)$ runs through $T(r, s, j)$. Let $\zeta \in \Sigma_{n^{j-1}}$ be that permutation which changes the given lexicographic ordering of $T(r, s, j)$ to the ordering specified by $U_H(V) < U_{H'}(V')$ if $V < V'$ or if $V = V'$ and $H < H'$ (in the lexicographic ordering; see [12, VI. 1.4]). Substituting (14) into (13), (13) into (9), and (9) into (2) and using the evident equivariance identification (to rearrange "addends"), we arrive at the formula

$$(15)\quad \psi_k(1\times\psi_{j_1}\times\ldots\times\psi_{j_k})(1\times\mu)(c,g;c_1,g_1,\ldots,c_k,g_k;x_1,\ldots,x_j)(r,s)$$
$$=\theta_{n^{j-1}}(\gamma(b;\lambda(f;b_1,\ldots,b_m)^{n^{m-1}})\zeta;\ \underset{U\in T(r,s,j)}{\times}\ \xi_j(\gamma(f;f_1,\ldots,f_m);\ \overset{j}{\underset{q=1}{\times}} x_q(u_{q-1},u_q))).$$

Comparing (1) and (15), we see that the diagram (*) will commute provided that we define

$$(16)\quad \gamma(c,g;c_1,g_1,\ldots,c_k,g_k)=(\gamma(b;\lambda(f;b_1,\ldots,b_m)^{n^{m-1}})\zeta,\gamma(f;f_1,\ldots,f_m))\,.$$

Here, when $j_q=0$, $(c_q,g_q)=*$ and we may think of $g_q$ as $*\in\mathcal{G}(0)$; (5),(6), and the definition of an operad then imply

$$(17)\qquad \gamma(f;f_1,\ldots,f_m)\ =\ \gamma(g;g_1,\ldots,g_k).$$

No such reinterpretation of the first factor of (16) is possible (as we would have to interpret $c_q$ as an element of $\mathcal{C}(n^{-1})$ to make the numbers work out).

We must show that, with this definition, $\mathcal{H}_n$ is a non-$\Sigma$ operad. Certainly $\gamma(1;c,g)\ =\ (c,g)$ and $\gamma(c,g;1^k)=(c,g)$ for $(c,g)\in\mathcal{H}_n(k)$, by [12,VI. 1.6(b) and (b')]. It remains to check the associativity formula [12,VI. 1.2(a)] for iteration of the maps $\gamma$, and the reader who has followed the combinatorics so far should not have too much trouble carrying out the requisite verification for himself. The details involve use of the corresponding associativity formulas for $\mathcal{C}$ and $\mathcal{G}$, the equivariance formulas [12,VI. 1.2(c) and VI. 1.6(c')], the interaction formulas [12,VI. 1.6(a) and (a')] as corrected in Remark 1.2, and a rather horrendous check that the permutations come out right.

## § 5. Strong homotopy $(\mathcal{C}, \mathcal{G})$-spaces and matrix rings

In this rather speculative section (which will play little part in our later work), we make an initial definition in the direction of an up to homotopy elaboration of the theories of $A_\infty$ and $E_\infty$ ring spaces and explain its likely relevance to the matrix "rings" studied in the previous section.

Lada, in [2, V], has developed an up to homotopy generalization of the theory of $\mathcal{C}$-spaces. (See [13,§6] for a sketch.) His theory is based on use of the associated monad $C$, and the essential starting point of the analogous up to homotopy generalization of the theory of $(\mathcal{C}, \mathcal{G})$-spaces surely must be the fact that $C$ restricts to a monad in the category of $\mathcal{G}_0$-spaces. The following definition should be appropriate.

Definition 5.1. Let $(\mathcal{C}, \mathcal{G})$ be an operad pair (where $\mathcal{G}$ might be a non-$\Sigma$ operad). A strong homotopy, or sh, $(\mathcal{C}, \mathcal{G})$-space $X$ is a $\mathcal{C}$-space $(X, \theta)$ with basepoint $0$ and a $\mathcal{G}_0$-space $(X, \xi)$ with basepoint $1$ such that $\theta: CX \to X$ is an sh $G$-map.

The notion of an sh $G$-map is defined and discussed in [2, V § 3]. It is required that the basic distributivity diagram

$$\begin{array}{ccc} GCX & \xrightarrow{G\theta} & GX \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \xi} \\ CX & \xrightarrow{\theta} & X \end{array},$$

the commutativity of which is the defining property of a $(\mathcal{C}, \mathcal{G})$-space, should homotopy commute and that this homotopy should be the first of an infinite sequence of higher coherence homotopies.

More general notions of sh $(\mathcal{C}, \mathcal{G})$-spaces, with $X$ only an sh $\mathcal{C}$-space or only an sh $\mathcal{G}$-space, surely also exist but would be much more complicated.

Unfortunately, the most general notion would presumably be essential to a fully homotopy invariant theory.

I would hope that if $X$ is an sh $(\mathcal{C},\mathcal{G})$-space, then there is an actual $(\mathcal{C},\mathcal{G})$-space $UX$ equivalent to $X$ in an appropriately structured way, so that the passage from $A_\infty$ ring spaces to $A_\infty$ ring spectra and from $E_\infty$ ring spaces to $E_\infty$ ring spectra directly generalizes to sh $(\mathcal{C},\mathcal{G})$-spaces for suitable pairs $(\mathcal{C},\mathcal{G})$. This would be in analogy to Lada's one operad theory, and his cubical bar construction $UX = \widetilde{B}(C,C,X)$ of [2, V §2] would be the obvious candidate for $UX$. However, I have not attempted to pursue these ideas.

We return to consideration of $M_nX$ for a $(\mathcal{C},\mathcal{G})$-space $X$, with $\mathcal{G}$ being a non-$\Sigma$ operad for convenience. As formulae (4.16) and (4.17) make clear, projection on the second factor gives a morphism of non-$\Sigma$ operads $\mathcal{H}_n \to \mathcal{G}$. Therefore $\mathcal{H}_n$ acts on $\mathcal{C}$ by pullback of the action of $\mathcal{G}$ on $\mathcal{C}$. However, $M_nX$ is not a $(\mathcal{C},\mathcal{H}_n)$-space because the distributivity diagram fails to commute (for this or any other action of $\mathcal{H}_n$ on $\mathcal{C}$). As explained in [12, VI §1 and 2, p. 77], the diagram in question results by passage to disjoint unions and then to quotients from the following diagrams:

$$(*)\qquad \begin{tikzcd}
\mathcal{H}_n(k)\times\mathcal{C}(j_1)\times(M_nX)^{j_1}\times\cdots\times\mathcal{C}(j_k)\times(M_nX)^{j_k} \arrow[r, "1\times\theta_{j_1}\times\cdots\times\theta_{j_k}"] \arrow[d, "1\times\mu"'] & \mathcal{H}_n(k)\times(M_nX)^k \arrow[dd, "\psi_k"] \\
\mathcal{H}_n(k)\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\times(M_nX)^{j_1}\times\cdots\times(M_nX)^{j_k} \arrow[d, "\Delta\times 1\times\delta"'] & \\
\mathcal{H}_n(k)^{j+1}\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\times((M_nX)^k)^j \arrow[d, "\mu"'] & M_nX \\
\mathcal{H}_n(k)\times\mathcal{C}(j_1)\times\cdots\times\mathcal{C}(j_k)\times(\mathcal{H}_n(k)\times(M_nX)^k)^j \arrow[r, "\lambda\times(\psi_k)^j"] & \mathcal{C}(j)\times(M_nX)^j \arrow[u, "\theta_j"']
\end{tikzcd}$$

Here $j = j_1 \cdots j_k$, the $\mu$ are shuffle homeomorphisms, $\Delta$ is the iterated diagonal, $\psi_k$ is as defined in formula (4.1), the $\theta_j$ give the additive action on $(M_n X) = X^{n^2}$, and

$$\delta : (M_n X)^{j_1} \times \ldots \times (M_n X)^{j_k} \to ((M_n X)^k)^j$$

is specified by

$$\delta(y_1, \ldots, y_k) = \mathop{\times}_{I \in S(j_1, \ldots, j_k)} y_I \,, \quad \text{with}$$

$$y_I = (x_{1i_1}, \ldots, x_{ki_k}) \quad \text{if} \quad y_q = (x_{q1}, \ldots, x_{qj_q}) \,.$$

The following result shows that, as far as the relationships between any homological and homotopical invariants of the maps $\theta_j$ and $\psi_k$ go, any results valid for $A_\infty$ ring spaces are also valid for matrix rings with entries in $A_\infty$ ring spaces. (Compare the analysis of the homology of $E_\infty$ ring spaces in [2, II]; we shall return to this point in section 9.)

<u>Proposition 5.2.</u> If X is a $(\mathcal{C}, \mathcal{H})$-space, where $\mathcal{C}$ is an $E_\infty$ operad and $\mathcal{H}$ is any non-$\Sigma$ operad which acts on $\mathcal{C}$, then the diagram (*) is $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$-equivariantly homotopy commutative.

<u>Proof.</u> On the one hand, the corresponding diagram for the $(\mathcal{C}, \mathcal{H})$-space X, formula (4.1), and the diagram which expresses that X is a $\mathcal{C}$-space [10, 1.5] imply

$$(1) \quad \psi_k(1 \times \theta_{j_1} \times \ldots \times \theta_{j_k})(c, g; c_1, y_1, \ldots, c_k, y_k)$$

$$= \theta_{jn^{k-1}}(\gamma(c; \lambda(g; c_1, \ldots, c_k)^{n^{k-1}}); \mathop{\times}_{U \in T(r,s,k)} \mathop{\times}_{I \in S(j_1, \ldots, j_k)} \xi_k(g; y_I(U))),$$

where $y_I(U) = (x_{1i_1}(u_0, u_1), \ldots, x_{ki_k}(u_{k-1}, u_k))$. On the other hand, formula (4.1) and the fact that X is a $\mathcal{C}$-space imply

$$(2) \quad \theta_j(\lambda \times \psi_k^j)\mu(\Delta \times 1 \times \delta)(1 \times \mu)(c, g; c_1, y_1, \ldots, c_k, y_k)$$

$$= \theta_{jn^{k-1}}(\gamma(\lambda(c, g; c_1, \ldots, c_k); c^j); \mathop{\times}_{I \in S(j_1, \ldots, j_k)} \mathop{\times}_{U \in T(r,s,k)} \xi_k(g; y_I(U))).$$

No definition of $\lambda(c, g; c_1, \ldots, c_k)$ will make the right sides of (1) and (2) agree, and we take the pullback definition $\lambda(g; c_1, \ldots, c_k)$. In view of the difference in order of appearance of the indexing sets $S(j_1, \ldots, j_k)$ and $T(r, s, k)$, the addends in (1) and (2) differ by a permutation $\nu \in \Sigma_{jn^{k-1}}$. The maps

$$f, g: \mathcal{H}_n(k) \times \mathcal{C}(j_1) \times \ldots \times \mathcal{C}(j_k) \to \mathcal{C}(jn^{k-1})$$

specified by

$$f(c, g; c_1, \ldots, c_k) = \gamma(c; \lambda(g; c_1, \ldots, c_k)^{n^{k-1}})$$

and

$$g(c, g; c_1, \ldots, c_k) = \gamma(\lambda(g; c_1, \ldots, c_k); c^j)\nu$$

are $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$-equivariant, where the action of this group on $\mathcal{C}(jn^{k-1})$ is determined by its tensorial embedding in $\Sigma_j$ [10,VI.1.4], the diagonal embedding of $\Sigma_j$ in $(\Sigma_j)^{n^{k-1}}$, and the block sum embedding of the latter in $\Sigma_{jn^{k-1}}$. Since $\mathcal{C}$ is an $E_\infty$ operad, the domain and codomain of f and g are $\Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$-free and contractible, hence f and g are equivariantly homotopic. The conclusion follows.

Of course, if we had chosen to work with permutations in our multiplicative operads, then the diagram (*) would be $\Sigma_k \times \Sigma_{j_1} \times \ldots \times \Sigma_{j_k}$-equivariantly homotopy commutative.

If the homotopies of the proposition can be chosen with suitable compatibility as k and the $j_q$ vary, they will together yield the first of the infinite sequence of homotopies needed to verify the following assertion.

Conjecture 5.3. If X is a $(\mathcal{C}, \mathcal{H})$-space, where $\mathcal{C}$ is an $E_\infty$ operad and $\mathcal{H}$ is any non-$\Sigma$ operad which acts on $\mathcal{C}$, then $M_n X$ is an sh $(\mathcal{C}, \mathcal{H}_n)$-space.

With $\mathcal{C}$ an $E_\infty$ operad and each $\mathcal{H}(j)$ contractible, the notion of an sh $(\mathcal{C}, \mathcal{H})$-space is an up to homotopy generalization of the notion of an $A_\infty$ ring space.

The conjecture gives the appropriate sense in which it might be true that $M_nX$ is an $A_\infty$ ring space if $X$ is an $A_\infty$ ring space.

The proof of Proposition 5.2 is precisely analogous to that of [10, 1.9(ii)], which gives a similar result about the product $\phi$ on a $\mathcal{C}$-space when $\mathcal{C}$ is an $E_\infty$ operad. Lada [6] has studied the passage from that result to the assertion that $\phi$ is an sh C-map, and the problems he encountered there illustrate what would be involved in a proof of Conjecture 5.3.

## §6. The comparison between $M_nX$ and $M_{n+1}X$

As in section 4, let $X$ be a $(\mathcal{C}, \mathcal{H})$-space, where $\mathcal{C}$ is any operad and $\mathcal{H}$ is any non-$\Sigma$ operad which acts on $X$. We have exhibited a $\mathcal{C}$-space structure and an $\mathcal{H}_n$-space structure on $M_nX$, where $\mathcal{H}_n(j) = \mathcal{C}(n^{j-1}) \times \mathcal{H}(j)$, and have studied the relationship between these actions. We here study the relationship between $M_nX$ and $M_{n+1}X$. We adopt the notations of section 4, but with an identifying subscript $n$ where necessary for clarity. Let $\nu_n : M_nX \to M_{n+1}X$ denote the natural inclusion. First consider the diagram

$$\begin{array}{ccc} \mathcal{C}(j) \times (M_nX)^j & \xrightarrow{\theta_{nj}} & M_nX \\ \Big\downarrow{\scriptstyle 1 \times \nu_n^j} & & \Big\downarrow{\scriptstyle \nu_n} \\ \mathcal{C}(j) \times (M_{n+1}X)^j & \xrightarrow{\theta_{n+1,j}} & M_{n+1}X \end{array}$$

Here $\theta_n$ and $\theta_{n+1}$ are determined entrywise from the action of $\mathcal{C}$ on $X$, and this diagram certainly commutes on the $(r,s)^{th}$ matrix entries for $r \le n$ and $s \le n$. Similarly, both composites always give $(r,s)^{th}$ matrix entry 0 if either but not both of $r$ and $s$ is $n+1$. However, for $c \in \mathcal{C}(j)$ and $x_i \in M_nX$, we have

$$\nu_n\theta_{nj}(c, x_1, \ldots, x_j)(n+1, n+1) = 1$$

but

$$\theta_{n+1,j}(1 \times \nu_n^j)(c, x_1, \ldots, x_j)(n+1, n+1) = \theta_j(c; 1^j) \quad .$$

Thus $\nu_n$ fails to be a $\mathcal{C}$-map. Indeed, as for discrete rings, 1 and $\theta_j(c; 1^j)$ lie in different components in all non-trivial cases since, by [12, p.140], X is contractible if 0 and 1 lie in the same component. For the multiplicative structures, we have the following result. Its hypothesis $\mathcal{C}(1) = \{1\}$ will be discussed after the proof.

Theorem 6.1. Assume that $\mathcal{C}(1)$ is the point 1. For each $n \geq 1$, there is a map $\tau_n: \mathcal{H}_{n+1} \to \mathcal{H}_n$ of non-$\Sigma$ operads such that $\nu_n: M_n X \to M_{n+1} X$ is an $\mathcal{H}_{n+1}$-map, where $M_n X$ is an $\mathcal{H}_{n+1}$-space by pullback along $\nu_n$.

Proof. Let $t_{nj} \in \mathcal{C}(\varepsilon_1) \times \ldots \times \mathcal{C}(\varepsilon_{(n+1)j-1})$, $\varepsilon_i = 0$ or $1$, have $i^{th}$ coordinate $* \in \mathcal{C}(0)$ if the $i^{th}$ element $U \in T_{n+1}(j-2)$ has any $u_q = n+1$ and have $i^{th}$ coordinate $1 \in \mathcal{C}(1)$ otherwise. Observe that if elements of $T_{n+1}(j-2)$ are written in the form $U = (u_1, \ldots, u_{j-1})$, then $U \longleftrightarrow (r, u_1, \ldots, u_{j-1}, s)$ gives a bijective correspondence between the ordered sets $T_{n+1}(j-2)$ and $T_{n+1}(r, s, j)$ for each r and s between 1 and n+1. Define

$$\tau_{nj}: \mathcal{H}_{n+1}(j) = \mathcal{C}((n+1)^{j-1}) \times \mathcal{G}(j) \to \mathcal{C}(n^{j-1}) \times \mathcal{G}(j) = \mathcal{H}_n(j)$$

by

$$\tau_{nj}(c, g) = (\gamma(c; t_{nj}), g).$$

By convention, $\tau_{n0}(*) = *$ and $t_{n1} = 1 \in \mathcal{C}(1)$ so that $\tau_{n1}$ is the identity map. Another laborious combinatorial argument, which uses formula (4.16), the associativity and equivariance formulas for the operad $\mathcal{C}$ [12, VI. 1.2(a) and (c)], the interaction, unit, and equivariance formulas [12, VI. 1.6(a'), (b'), (c')] for the action of $\mathcal{G}$ on $\mathcal{C}$ (see Remark 1.2), and a careful consideration of permutations based on the description of $\zeta$ given after formula (4.14) shows that the following diagrams commute.

$$\begin{tikzcd}
\mathcal{H}_{n+1}(k) \times \mathcal{H}_{n+1}(j_1) \times \ldots \times \mathcal{H}_{n+1}(j_k) \arrow[r, "\gamma"] \arrow[d, "\tau_{nk} \times \tau_{nj_1} \times \ldots \times \tau_{nj_k}"'] & \mathcal{H}_{n+1}(j_1 + \ldots + j_k) \arrow[d, "\tau_{n, j_1 + \ldots + j_k}"] \\
\mathcal{H}_n(k) \times \mathcal{H}_n(j_1) \times \ldots \times \mathcal{H}_n(j_k) \arrow[r, "\gamma"] & \mathcal{H}_n(j_1 + \ldots + j_k)
\end{tikzcd}$$

Thus $\tau_n$ is a map of non-$\Sigma$ operads. Now consider the following diagrams.

$$\begin{array}{ccccc}
\mathcal{H}_{n+1}(j)\times(M_nX)^j & \xrightarrow{\tau_{nj}\times 1} & \mathcal{H}_n(j)\times(M_nX)^j & \xrightarrow{\psi_{nj}} & M_nX \\
\downarrow{\scriptstyle 1\times\nu_n^j} & & & & \downarrow{\scriptstyle \nu_n} \\
\mathcal{H}_{n+1}(j)\times(M_{n+1}X)^j & & \xrightarrow{\psi_{n+1,j}} & & M_{n+1}X
\end{array}$$

Our claim is that these diagrams commute, and it is for this that we require $\mathcal{C}(1)$ to be a point. Consider

$$\psi_{n+1,j}(c,g;y_1,\ldots,y_j)(r,s)$$

$$= \theta_{(n+1)^{j-1}}(c;\ \underset{U\in T_{n+1}(r,s,j)}{\times}\ \xi_j(g;\ \overset{j}{\underset{q=1}{\times}}\ y_q(u_{q-1},u_q)))\ ,$$

where $c\in\mathcal{C}((n+1)^{j-1})$, $g\in\mathcal{B}(j)$, and $y_i=\nu_n(x_i)$. The $U^{th}$ factor on the right is $0\in X$ if either but not both of $u_{q-1}$ and $u_q$ is n+1 for any q. If $r\le n$ and $s\le n$, it follows that the right side is equal to

$$\theta_{n^{j-1}}(\gamma(c;t_{nj});\ \underset{V\in T_n(r,s,j)}{\times}\ \xi_j(g;\ \overset{j}{\underset{q=1}{\times}}\ x_q(u_{q-1},u_q)))$$

$$= \psi_j(\tau_{nj}(c,g);x_1,\ldots,x_j)(r,s)\ .$$

If either but not both of r and s is n+1, then, for any $U\in T_{n+1}(r,s,j)$, there exists q such that either but not both of $u_{q-1}$ and $u_q$ is n+1, hence $\psi_j(c,g;y_1,\ldots,y_j)(r,s)=0$. We therefore have

$$\nu_n\psi_{nj}(\tau_{nj}\times 1)(c,g;x_1,\ldots,x_j)(r,s)=\psi_{n+1,j}(1\times\nu_n^j)(c,g;x_1,\ldots,x_j)(r,s)\ ,$$

as desired, unless $r=s=n+1$. Here we find that the left side is $1\in X$ whereas the right side reduces to

$$\theta_{(n+1)^{j-1}}(c;0,\ldots,0,1)\ =\ \theta_1(\gamma(c;s_{nj}),1)\in X\ ,$$

where

$$s_{nj}=\ (*,\ldots,*,1)\in\mathcal{C}(0)^{(n+1)^{j-1}-1}\times\mathcal{C}(1)\ .$$

Indeed, all elements of $T_{n+1}(n+1, n+1, j)$ except the last have either but not both of $u_{q-1}$ and $u_q$ equal to n+1 for some q, whereas the last U is $(n+1, \ldots, n+1)$ for which $y_q(n+1, n+1) = 1$ and $\xi_j(g; \times_{q=1}^{j} 1) = 1$. The assumption $\mathcal{C}(1) = \{1\}$ ensures that $\gamma(c; s_{nj}) = 1$ and therefore $\theta_1(\gamma(c; s_{nj}); 1) = 1$.

Unfortunately, it is not in general the case that $\mathcal{C}(1) = \{1\}$; for example, this fails for the canonical $E_\infty$ operad $\mathcal{K}_\infty$ used in section 3. We could avoid this assumption by appealing to Lada's theory. The contractibility of $\mathcal{C}(1)$ can be used to prove that $\nu_n$ is an sh $H_{n+1}$-map. However, this solution (which I worked out in detail in an earlier draft) leads to further complications in later sections. Our preferred solution is to prove that $A_\infty$ ring spaces can be functorially replaced by equivalent $A_\infty$ ring spaces with respect to a different $A_\infty$ operad pair for which $\mathcal{C}(1)$ is a point. This replacement process works equally well in the $E_\infty$ ring context.

We exploit the fact that the particular $E_\infty$ operad $\mathcal{D}$ of [11, §4] has $\mathcal{D}(1) = \{1\}$. Moreover, as explained in [12, VI §2 and 4], $\mathcal{D}$ acts on itself and thus $(\mathcal{D}, \mathcal{D})$ is an $E_\infty$ operad pair. Let $(\mathcal{C}, \mathcal{G})$ be any operad pair such that $\mathcal{C}$ is an $E_\infty$ operad. $\mathcal{G}$ might be either an $A_\infty$ or an $E_\infty$ operad. By use of products and projections, we then have operad pairs, and maps thereof,

$$(\mathcal{C}, \mathcal{G}) \xleftarrow{(\pi_1, \pi_1)} (\mathcal{C} \times \mathcal{D}, \mathcal{G} \times \mathcal{D}) \xrightarrow{(\pi_2, 1)} (\mathcal{D}, \mathcal{G} \times \mathcal{D}).$$

Therefore $(\mathcal{C}, \mathcal{G})$-spaces are $(\mathcal{C} \times \mathcal{D}, \mathcal{G} \times \mathcal{D})$-spaces by pullback, while both D and $C \times D$ are monads in the category of $(\mathcal{G} \times \mathcal{D})_0$-spaces and $\pi_2 : C \times D \to D$ is a morphism of monads in this category.

We proceed to construct a functor W from $(\mathcal{C}, \mathcal{G})$-spaces to $(\mathcal{D}, \mathcal{G} \times \mathcal{D})$-spaces. As explained in [12, VI. 2.7(iii)], there is a functor W from $\mathcal{C}$-spaces to $\mathcal{D}$-spaces specified in terms of the two-sided bar construction of [10, §9] by $WX = B(D, C \times D, X)$ and there is a natural diagram of $(\mathcal{C} \times \mathcal{D})$-spaces

$$X \xleftarrow{\varepsilon} B(C \times D, C \times D, X) \xrightarrow{B\pi_2} B(D, C \times D, X) = WX .$$

Here $\varepsilon$ is a homotopy equivalence with a natural homotopy inverse and $B\pi_2$ is also an equivalence. Technically, we should assume or arrange (without loss of structure by an elaboration of the arguments in [10,A.8 and A.11 ]) that $1 \in \mathcal{C}(1)$ and $0 \in X$ are non-degenerate basepoints, so that the simplicial spaces used in our constructions are proper [10,11.2 and 11,A.5]. We have the following result.

Proposition 6.2. If $X$ is a $(\mathcal{C}, \mathcal{G})$-space, then $WX$ is a $(\mathcal{Q}, \mathcal{G} \times \mathcal{Q})$-space, $B(C \times D, C \times D, X)$ is a $(\mathcal{C} \times \mathcal{Q}, \mathcal{G} \times \mathcal{Q})$-space, and $\varepsilon$ and $B\pi_2$ are maps of $(\mathcal{C} \times \mathcal{Q}, \mathcal{G} \times \mathcal{Q})$-spaces.

Proof. By formal verifications from [10,9.6 and 9. 9], the action $DWX \to WX$ of $D$ on $WX$, the action of $C \times D$ on $B(C \times D, C \times D, X)$, and $\varepsilon$ and $B\pi_2$ are all geometric realizations of maps of simplicial $(\mathcal{G} \times \mathcal{Q})_0$-spaces and are therefore maps of $(\mathcal{G} \times \mathcal{Q})_0$-spaces by [10,12.2].

Clearly, we may as well start our analysis of matrix rings of $A_\infty$ ring spaces by first replacing $X$ by $WX$. In particular, our assumption that $\mathcal{C}(1) = \{1\}$ results in no real loss of generality. This construction also handles a different technical problem, one that we have heretofore ignored.

Remarks 6.3. As explained in [12,VII § 1 and 2], $\mathcal{K}_\infty$ and the product operads $\mathcal{C} = \mathcal{C}' \times \mathcal{K}_\infty$ necessary to the proof of the recognition principle in section 3 are only partial operads and their associated monads are only partial monads. However, it is not hard to see that our replacement argument above works perfectly well for such $\mathcal{C}$. Thus, by use of the functor $W$, we may assume without loss of generality that all operads in sight are honest operads in the development of the present theory since $\mathcal{Q}$ and all multiplicative operads (see [12,p.178]) are honest.

Remarks 6.4. By the method of proof of Theorem 6.1, one can construct maps $\tau_{pq}: \mathcal{H}_{p+q} \to \mathcal{H}_p$ and $\tau'_{pq}: \mathcal{H}_{p+q} \to \mathcal{H}_q$ of non-$\Sigma$ operads such that the usual block sum of matrices $\oplus: M_pX \times M_qX \to M_{p+q}X$ specifies an $\mathcal{H}_{p+q}$-map, where $M_pX$ and $M_qX$ are regarded as $\mathcal{H}_{p+q}$-spaces by pullback along $\tau_{pq}$ and $\tau'_{pq}$ respectively and where $M_pX \times M_qX$ is given the product $\mathcal{H}_{p+q}$-structure [10,1.7]. From this point of view, the problem with $\mathscr{C}(1)$ above simply reflects the fact that the inclusion $\{1\} \to X = M_1X$ is not an $\mathcal{H}_1$-map unless $\mathscr{C}(1) = \{1\}$. In order for these sum maps to be useful, one would have to understand their stabilization, that is, to analyze the diagrams

$$\begin{tikzcd}
M_pX \times M_qX \arrow[r, "\nu_p \times \nu_q"] \arrow[d, "\oplus"'] & M_{p+1}X \times M_{q+1}X \arrow[d, "\oplus"] \\
M_{p+q}X \arrow[r, "\nu_{p+q+1} \circ \nu_{p+q}"] & M_{p+q+2}X
\end{tikzcd}$$

If $X$ is a ring, these composites obviously differ only by conjugation by a permutation matrix. In general, the definition of such a conjugation entails an arbitrary choice of product and yields only a homotopy commutative diagram. A full analysis of the situation would presumably entail application of Lada's theory of strong homotopy maps.

## §7. The Algebraic K-theory of $A_\infty$ ring spaces

Let $(\mathcal{C}, \mathcal{G})$ be an $A_\infty$ operad pair. A $(\mathcal{C}, \mathcal{G})$-space X will be said to be grouplike if $\pi_0 X$ is a group under addition and therefore a ring rather than just a semi-ring. Up to weak equivalence then, X is a grouplike $A_\infty$ ring space if and only if it is the zero$^{th}$ space of an $A_\infty$ ring spectrum.

For a grouplike $(\mathcal{C}, \mathcal{G})$-space X, define $FM_nX$ to be the pullback in the following diagram, where d denotes the discretization map.

$$\begin{tikzcd} FM_nX \arrow[r] \arrow[d] & M_nX \arrow[d, "d"] \\ GL(n, \pi_0X) \arrow[r, "\subset"] & M_n(\pi_0X) \end{tikzcd}$$

That is, $FM_nX$ is the space of unit components in $M_nX$. The notation is chosen in analogy with that in Remarks 2.4; Waldhausen would write $\widehat{GL}_n(X)$ for $FM_nX$. If X is a discrete ring then $FM_nX = GL(n, X)$; for general topological rings, $FM_nX$ is larger than $GL(n, X)$. We reiterate that 0 and 1 must be in different components for non-triviality. Clearly $\nu_n$ maps $FM_nX$ into $FM_{n+1}X$. If each $\nu_n$ is a cofibration, we let $FM_\infty X$ denote the union of the $FM_nX$; otherwise we let $FM_\infty X$ be their telescope. It is the purpose of this section to prove that $FM_\infty X$ has a functorial delooping.

Theorem 7.1. There is a functor T from grouplike $(\mathcal{C}, \mathcal{G})$-spaces to connected based spaces together with a natural weak equivalence between $\Omega TX$ and $FM_\infty X$.

In particular, $\pi_1 TX = \pi_0 FM_\infty X = GL(\infty, \pi_0 X)$ has a perfect commutator subgroup. Replacing TX by a naturally weakly equivalent CW-complex if necessary, we can take its plus construction in the sense of Quillen (see e.g. [21, §1]). We have the following definition of the algebraic K-theory of $A_\infty$ ring spaces.

Definition 7.2. Let $i: TX \to KX$ be the plus construction of TX, so that i induces an isomorphism on homology and KX is a simple space. KX is called the connected algebraic K-space of X. For $q > 0$, let $K_q X = \pi_q(KX)$. $K_q X$ is called the $q^{th}$ algebraic K-group of X.

Remarks 7.3. As a space, $FM_n X$ is just the union of some of the components of $M_n X = X^{n^2}$ and each of these components is equivalent to the component $SFM_n X$ of the identity matrix. Indeed, $FM_n X$ is equivalent, although not in general as an H-space, to $SFM_n X \times GL(n, \pi_0 X)$ [2, I.4.6]. Further, $SFM_n X$ is equivalent to the component of the zero matrix, and the latter is just $M_n X_0$ where $X_0$ is the component of zero in X (and $M_n X$ and $M_n X_0$ are additive infinite loop spaces). It follows that, for $q > 0$, $\pi_q FM_\infty X$ is the direct sum of infinitely many copies of $\pi_q X$; of course, $i_*$ maps this group naturally to $K_{q+1} X$.

Remarks 7.4. By restriction to the spaces $SFM_n X$, the proof of Theorem 7.1 will yield a functor UT from grouplike $(\mathcal{C}, \mathcal{G})$-spaces to simply connected based spaces together with a natural weak equivalence between $\Omega UTX$ and $SFM_\infty X = \mathrm{Tel}\, SFM_n X$ and a natural map $UTX \to TX$ compatible with the weak equivalences. Thus, homotopically, UTX will be the universal cover of TX.

The rest of this section will be devoted to the proof of Theorem 7.1. We begin by reviewing the basic theory of classifying spaces of $A_\infty$ spaces. Let $\mathcal{G}$ be an $A_\infty$ operad. There is a functor V from $\mathcal{G}$-spaces to topological monoids specified in terms of the two-sided bar construction of [10, §9] by $VX = B(M, G, X)$. Here M denotes the free monoid, or James construction, monad. The augmentation $\delta: \mathcal{G} \to \mathcal{M}$ induces a map of monads $\delta: G \to M$, and we obtain a natural diagram of $\mathcal{G}$-maps

$$(1) \qquad X \xleftarrow{\varepsilon} B(G, G, X) \xrightarrow{B\delta} B(M, G, X) = VX .$$

Here $\varepsilon$ is a homotopy equivalence with a natural homotopy inverse. The map $B\delta$ is also a homotopy equivalence (by [11, A.2(ii)], in which the connectivity assumption of [10, 13.5] is removed). If X is itself a monoid considered as a $\mathcal{G}$-space by pullback along $\delta$, there is a natural composite

$$(2) \qquad VX = B(M,G,X) \xrightarrow{B\delta} B(M,M,X) \xrightarrow{\varepsilon} X$$

which is both a map of monoids and a homotopy equivalence. If $X = GY$ for a based space Y, there is a natural equivalence of monoids $VX \to MY$. Moreover, all of this is natural with respect to maps of $A_\infty$ operads. See [10, 13.5] for details.

We can therefore deloop $\mathcal{G}$-spaces by applying the standard product-preserving classifying space functor B of [17 or 9, §7-8] to VX. For our purposes, the crucial property of B is that there is a natural map $\zeta : X \to \Omega BX$ for monoids X such that $\zeta$ is a weak equivalence if $\pi_0 X$ is a group (e.g. [9, 8.7]).

Turning to the proof of Theorem 7.1, fix an $A_\infty$ operad pair $(\mathcal{C}, \mathcal{G})$ and construct $A_\infty$ operads $\mathcal{H}_n$ as in section 4 (either crossing with $\mathcal{M}$ to obtain actual operads, with permutations, or rephrasing the arguments above in terms of non-$\Sigma$ operads). By use of the argument at the end of the previous section, replacing X by WX if necessary, we may assume without loss of generality that $\mathcal{C}(1) = \{1\}$.

Clearly Theorems 4.1 and 6.1 imply that $FM_nX$ is a sub $\mathcal{H}_n$-space of $M_nX$ and that $\nu_n$ restricts to an $\mathcal{H}_{n+1}$-map $FM_nX \to FM_{n+1}X$, where $FM_nX$ is an $\mathcal{H}_{n+1}$-space by pullback along $\tau_n : \mathcal{H}_{n+1} \to \mathcal{H}_n$. Write $V_n$ for the functor V above defined with respect to the operad $\mathcal{H}_n$. Consider the following maps, where the notation on the left abbreviates that on the right.

$$\begin{array}{ccc}
V_n FM_n X & = & B(M, H_n, FM_n X) \\
\uparrow{\scriptstyle \tau_n} & & \uparrow{\scriptstyle B(1,\tau_n,1)} \\
V_{n+1} FM_n X & = & B(M, H_{n+1}, FM_n X) \\
\downarrow{\scriptstyle V_{n+1}\nu_n} & & \downarrow{\scriptstyle B(1,1,\nu_n)} \\
V_{n+1} FM_{n+1} X & = & B(M, H_{n+1}, FM_{n+1} X)
\end{array}$$

Again, in order to ensure that all simplicial spaces involved in our constructions are proper [10, 11.2 and 11, A.5], we should assume or arrange that $I_n \epsilon FM_n X$ and $1 \epsilon \mathcal{H}_n(1)$ are nondegenerate basepoints before applying the functors $V_n$ and B; see [10, p. 127 and 167-171]. With this precaution, we have the following result.

Lemma 7.5. $\tau_n: V_{n+1}FM_nX \to V_nFM_nX$ is a homotopy equivalence. Therefore $B\tau_n$ is also a homotopy equivalence.

Proof. By [11, A.2(ii)], $\tau_n: H_{n+1}Y \to H_nY$ is a homotopy equivalence for any $\mathcal{H}_n$-space Y. By an argument just like the proof of [11, A.4 (see 16, 5.5 and 5.6)], $Mf: MX \to MX'$ is a homotopy equivalence for any homotopy equivalence $f: X \to X'$. Therefore $B(1, \tau_n, 1)$ is a homotopy equivalence by [11, A.4(ii)], and the conclusion for $B\tau_n$ follows from the same result; see [9, p. 32].

At this point, we could choose a homotopy inverse $(B\tau_n)^{-1}$ to $B\tau_n$ and let TX be the telescope of the spaces $BV_nFM_nX$. However, we would then run into a naturality problem. Certainly $(B\tau_n)^{-1}$ is natural up to homotopy since $B\tau_n$ is natural (by a trivial formal argument), but functoriality up to homotopy of the telescope would require $(B\tau_n)^{-1}$ to be natural up to natural homotopy. In fact, tracing very carefully through the proofs cited above, one may check that $(B\tau_n)^{-1}$ can be so chosen, but there is a much simpler and more precise solution to the problem.

Definition 7.6. Construct TX as follows. Let $T_nX$ be the reduced double mapping cylinder

$$BV_nFM_nX \bigcup_{B\tau_n} (BV_{n+1}FM_nX \wedge I^+) \bigcup_{BV_{n+1}\nu_n} BV_{n+1}FM_{n+1}X$$

and let TX be obtained from the disjoint union of the $T_nX$ by identifying the top, $BV_nFM_nX$, of $T_{n-1}X$ with the bottom, $BV_nFM_nX$, of $T_nX$ for $n > 1$. Clearly T is then the object function of a functor from $(\mathcal{C}, \mathcal{G})$-spaces to connected based spaces.

Visibly, TX is homotopy equivalent to the telescope of the $BV_nFM_nX$ with respect to composites $(BV_{n+1}\nu_n)(B\tau_n)^{-1}$, any questions of naturality being thrown irrelevantly onto the choice of equivalence. The properties of T stated in Theorem 7.1 are immediate from the definition and the general theory discussed above.

Remarks 7.7. One would like to construct a product $TX \times TX \to TX$ (not an H-space structure of course) by use of block sum of matrices so as to be able to obtain an H-space structure on KX by mimicry of Wagoner's proof [21, §1] in the case when X is a discrete ring. The main obstruction is explained in Remarks 6.4.

## §8. Monomial matrices and $Q_0(BFX \amalg \{0\})$

As before, let $(\mathcal{C}, \mathcal{G})$ be an $A_\infty$ operad pair and let X be a grouplike $(\mathcal{C}, \mathcal{G})$-space. Let $FX = FM_1X$ be the space of unit components of X and let SFX be the component of $1 \in X$. Then FX and SFX are sub $\mathcal{G}$-spaces of X. Let V be the functor of the previous section, defined with respect to $\mathcal{G}$, and abbreviate $BFX = BVFX$ and similarly for SFX. If FX is a topological monoid regarded as a $\mathcal{G}$-space by pullback, then this classifying space agrees up to natural equivalence with the standard one. Write $Z \amalg \{0\}$ or $Z^+$ interchangeably for the union of a space $Z$ and a disjoint basepoint $0$. (We agree never to use the + notation for the plus construction.) Let $Q_0Y$ denote the component of zero in the space $QY$. The purpose of this section is to construct a natural map

$$\mu: Q_0(BFX \amalg \{0\}) \to KX$$

and thus a natural transformation from the stable homotopy groups of $BFX^+$ to the algebraic K-groups of X. Of course, $\pi_q^s(BFX^+)$ is the direct sum of $\pi_q^s BFX$ and the stable stem $\pi_q^s = \pi_q^s S^0$. When $X = QS^0$, such a map was asserted to exist by Waldhausen [22, § 2].

The construction is based on the use of monomial matrices.

Definition 8.1. Let $F_nX$ denote the subspace of $FM_nX$ which consists of the monomial matrices with entries in FX, namely those matrices with precisely one non-zero entry in each row and column and all non-zero entries in FX. Let $F_\infty X$ denote the telescope of the $F_nX$ with respect to the restrictions of the maps $\nu_n$. Similarly, let $SF_nX$ denote the space of monomial matrices with entries in SFX and let $SF_\infty X = \mathrm{Tel}\, SF_nX$.

As will become clear below, it is immediate from formula (4.1) that $F_nX$ and $SF_nX$ are sub $\mathcal{H}_n$-spaces of $FM_nX$. The arguments of the previous section can be carried out word for word with $FM_nX$ replaced by $F_nX$. There results the following analog of Theorem 7.1.

Theorem 8.2. There is a functor P from grouplike $(\mathcal{C}, \mathcal{H})$-spaces to connected based spaces together with a natural weak equivalence between $\Omega PX$ and $F_\infty X$. Moreover, there is a natural map $PX \to TX$ the loop of which agrees under the weak equivalences with the inclusion $F_\infty X \to FM_\infty X$.

In particular, $\pi_1 PX = \pi_0 F_\infty X = \operatorname{colim} \pi_0 F_n X$. It will soon become apparent that $\pi_0 F_n X$ is the wreath product $\Sigma_n \int \pi_0 FX$. It follows easily that $\pi_1 PX$ has a perfect commutator subgroup (see [7, 1.2]), and this will also drop out of our arguments below since they will give a homology isomorphism from PX to the simple space $Q_0(BFX \amalg \{0\})$. This homology isomorphism will immediately imply the following theorem.

Theorem 8.3. The plus construction on PX is naturally equivalent to $Q_0(BFX \amalg \{0\})$, hence the map $PX \to TX$ induces a natural (up to homotopy) map $\mu : Q_0(BFX \amalg \{0\}) \to KX$.

Remarks 8.4. Replacing $F_n X$ by $SF_n X$ throughout, we obtain SPX, a natural weak equivalence between $\Omega SPX$ and $SF_\infty X$, and a natural map $SPX \to PX$ compatible with the weak equivalences. Moreover, the plus construction on SPX is naturally equivalent to $Q_0(BSFX \amalg \{0\})$ and the resulting map from this space to KX agrees with the restriction of $\mu$.

Remarks 8.5. There is a natural inclusion $\Sigma FX \to BFX$ (adjoint to $\zeta : FX \to \Omega BFX$), hence $\mu$ restricts to a natural map $Q_0(\Sigma FX \amalg \{0\}) \to KX$. Via the basepoint of $\Sigma FX$, there is a further natural restriction $QS^0 \to KX$. Similar remarks hold with F replaced by SF.

The rest of this section will be devoted to the proof of Theorem 8.3. We begin with a well-known observation about the classifying spaces of wreath products and an equally well-known consequence of the Barratt-Quillen theorem.

Lemma 8.6. For a topological monoid X, $B(\Sigma_n \int X)$ is naturally homeomorphic to $E\Sigma_n \times_{\Sigma_n} (BX)^n$.

Proof. Our conventions on wreath products are in [2, p.51]. B and E are obtained as geometric realizations of certain simplicial spaces [9, p. 31]. We define a $\Sigma_n \int X$-equivariant homeomorphism of simplicial spaces

$$\psi_* : E_*(\Sigma_n \int X) \to E_*\Sigma_n \times (E_*X)^n$$

by the formula

$$\psi_q[(\sigma_1, y_1), \ldots, (\sigma_q, y_q)](\sigma_{q+1}, y_{q+1}) = ([\sigma_1, \ldots, \sigma_q]\sigma_{q+1}, z),$$

where $\sigma_i \in \Sigma_n$, $y_i = (x_{i1}, \ldots, x_{in}) \in X^n$, and $z \in (E_qX)^n$ has $i^{th}$ coordinate $[x_{1,\sigma_2 \cdots \sigma_{q+1}(i)}, \ldots, x_{q,\sigma_{q+1}(i)}]x_{q+1,i}$; $\psi_*^{-1}$ is given by

$$\psi_q^{-1}([\sigma_1, \ldots, \sigma_q]\sigma_{q+1}, [x_{11}, \ldots, x_{q1}]x_{q+1,1}, \ldots, [x_{1n}, \ldots, x_{qn}]x_{q+1,n})$$

$$= [(\sigma_1, y_1), \ldots, (\sigma_q, y_q)](\sigma_{q+1}, y_{q+1}),$$

where $y_i = (x_{i,\sigma_{q+1}^{-1}\sigma_q^{-1}\cdots\sigma_{i+1}^{-1}(1)}, \ldots, x_{i,\sigma_{q+1}^{-1}\sigma_q^{-1}\cdots\sigma_{i+1}^{-1}(n)})$. The conclusion follows since realization commutes with products.

For a based space Y, the inclusion of $\Sigma_n$ in $\Sigma_{n+1}$ as the subgroup fixing the last letter and the inclusion of $Y^n$ in $Y^{n+1}$ as $Y^n \times \{*\}$ induce an inclusion $E\Sigma_n \times_{\Sigma_n} Y^n \to E\Sigma_{n+1} \times_{\Sigma_{n+1}} Y^{n+1}$.

Proposition 8.7. For connected spaces Y, there is a natural homology isomorphism $\mathrm{Tel}\, E\Sigma_n \times_{\Sigma_n} Y^n \to Q_0(Y \amalg \{0\})$.

Proof. For the $E_\infty$ operad $\mathcal{Q}$ of [10, §4 and 12, §2 and 4], the space $DY^+$ is precisely $\coprod_{n \geq 0} E\Sigma_n \times_{\Sigma_n} Y^n$. Let $\mathcal{C} = \mathcal{Q} \times \mathcal{K}_\infty$ (or $\mathcal{Q} \times \mathcal{C}_\infty$ if one prefers to avoid partial operads) as in section 3, and note that the projection $CY^+ \to DY^+$ is a homotopy equivalence by [11, A.2(ii)]. By Proposition 3.5, with multiplicative structure ignored, $QY^+$ is naturally a group completion of $CY^+$. The conclusion follows as in [2, I. 5. 10].

We shall reduce Theorem 8.3 to an application of the previous two results. For this purpose, we require an understanding of $\Sigma_n \int X$ when $X$ is a $\mathcal{C}$-space rather than a monoid.

Definition 8.8. For a $\mathcal{C}$-space X, define a $\mathcal{C}$-space $\Sigma_n \int X$ as follows. As a space, $\Sigma_n \int X = \Sigma_n \times X^n$. The action $\xi_n$ of $\mathcal{C}$ on $\Sigma_n \int X$ is given by the maps

$$\xi_{nj} : \mathcal{C}(j) \times (\Sigma_n \times X^n)^j \rightarrow \Sigma_n \times X^n$$

specified for $g \in \mathcal{C}(j)$, $\sigma_q \in \Sigma_n$, and $x_q = (x_{q,1}, \ldots, x_{q,n}) \in X^n$ by

$$\xi_{nj}(g; \sigma_1, x_1, \ldots, \sigma_j, x_j) = (\sigma_1 \cdots \sigma_j, \underset{i=1}{\overset{n}{\times}} \xi_j(g; \underset{q=1}{\overset{j}{\times}} x_{q, \sigma_{q+1} \cdots \sigma_j(i)})).$$

(Technically, this formula is appropriate when $\mathcal{C}$ is taken as a non-$\Sigma$ operad; compare section 4.) This is just the ordinary iterated wreath product, but with a parametrized family of multiplications on X. Let $\nu_n : \Sigma_n \int X \rightarrow \Sigma_{n+1} \int X$ denote the natural inclusion, and observe that $\nu_n$ is a map of $\mathcal{C}$-spaces.

We wish to commute the functor V past wreath products. The following rather elaborate formal argument based on the maps displayed in (7.1) and (7.2) suffices.

Lemma 8.9. For $\mathcal{C}$-spaces X, the horizontal arrows are homotopy equivalences in the following commutative diagram of $\mathcal{C}$-spaces and $\mathcal{C}$-maps, where $UX = B(G, G, X)$.

$$\begin{tikzcd}
V(\Sigma_n \int X) \arrow[d, "V\nu_n"] & U(\Sigma_n \int X) \arrow[l, "B\delta"'] \arrow[r, "\varepsilon"] \arrow[d, "W\nu_n"] & \Sigma_n \int X \arrow[d, "\nu_n"] & \Sigma_n \int UX \arrow[l, "\Sigma_n \int \varepsilon"'] \arrow[r, "\Sigma_n \int B\delta"] \arrow[d, "\nu_n"] & \Sigma_n \int VX \arrow[d, "\nu_n"] \\
V(\Sigma_{n+1} \int X) & U(\Sigma_{n+1} \int X) \arrow[l, "B\delta"'] \arrow[r, "\varepsilon"] & \Sigma_{n+1} \int X & \Sigma_{n+1} \int UX \arrow[l, "\Sigma_{n+1} \int \varepsilon"'] \arrow[r, "\Sigma_{n+1} \int B\delta"] & \Sigma_{n+1} \int VX
\end{tikzcd}$$

Application of the functor V to this diagram gives a commutative diagram which can be extended to the left and right by the commutative diagrams

$$\begin{tikzcd} V(\Sigma_n\int X) \arrow[d, "V\nu_n"'] & VV(\Sigma_n\int X) \arrow[l, "\varepsilon\circ B\delta"'] \arrow[d, "VV\nu_n"] \\ V(\Sigma_{n+1}\int X) & VV(\Sigma_{n+1}\int X) \arrow[l, "\varepsilon\circ B\delta"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} V(\Sigma_n\int VX) \arrow[r, "\varepsilon\circ B\delta"] \arrow[d, "V\nu_n"'] & \Sigma_n\int VX \arrow[d, "\nu_n"] \\ V(\Sigma_{n+1}\int VX) \arrow[r, "\varepsilon\circ B\delta"] & \Sigma_{n+1}\int VX \end{tikzcd}$$

The resulting composite diagram is a commutative diagram of maps of topological monoids in which all horizontal arrows are homotopy equivalences.

Via the homotopy invariance properties of the classifying space functor B [9, 7.3(ii)] and the telescope, the previous result implies a chain of natural equivalences of telescopes which, together with the first two results above, leads to the following conclusion.

Theorem 8.10. For $\mathcal{H}$-spaces X, there is a natural homotopy equivalence

$$\mathrm{Tel}\ BV(\Sigma_n\int X) \simeq \mathrm{Tel}\ B(\Sigma_n\int VX) \cong \mathrm{Tel}\ E\Sigma_n \times_{\Sigma_n} (BVX)^n .$$

Therefore there is a natural homology isomorphism

$$\mathrm{Tel}\ BV(\Sigma_n\int X) \to Q_0(BVX \amalg \{0\})$$

We can now prove Theorem 8.3. By application of the previous theorem to the $\mathcal{H}$-space FX, with BFX = BVFX by notational convention, it suffices to prove the following result. The proof again makes strong use of the assumption $\mathcal{C}(1) = \{1\}$ justified at the end of section 6.

Theorem 8.11. For grouplike $(\mathcal{C}, \mathcal{H})$-spaces X, there is a natural homotopy equivalence $PX \to \mathrm{Tel}\ BV(\Sigma_n\int FX)$.

Proof. Define a homeomorphism $\alpha_n: F_nX \to \Sigma_n\int FX$ by sending a monomial matrix x to $(\sigma, x(1), \ldots, x(n))$, where $\sigma \in \Sigma_n$ is specified by $\sigma(i) = j$ if $x(i,j) \neq 0$ and where $x(j) = x(i,j)$ for this i. Let $\pi_n: \mathcal{H}_n \to \mathcal{H}$ be the evident projection of operads and regard $\Sigma_n\int FX$ as an $\mathcal{H}_n$-space by pullback along $\pi_n$. We claim first that $\alpha_n$ is an $\mathcal{H}_n$-map or, equivalently, that the actions

$\alpha_n^{-1}\xi_n\pi_n H_n\alpha_n$ and $\psi_n$ coincide on $F_nX$. Recall formula (4.1). Given $x_i \in F_nX$ for $1 \leq i \leq j$, there is for each $r$ between 1 and n a unique $s$ between 1 and n and a unique $U \in T_n(r,s,j)$ such that $x_q(u_{q-1}, u_q) \neq 0$. If this $U$ is the $k^{th}$ element of $T_n(r,s,j)$ and if

$$s_{n,j,k} = (*, \ldots, *, 1, *, \ldots, *) \in \mathcal{C}(0)^{k-1} \times \mathcal{C}(1) \times \mathcal{C}(0)^{n^{j-1}-k},$$

then, for $c \in \mathcal{C}(n^{j-1})$ and $g \in \mathcal{G}(j)$,

$$\begin{aligned}\psi_{n,j}(c, g; x_1, \ldots, x_j)(r,s) &= \theta_1(\gamma(c; s_{n,j,k}); \xi_j(g; \times_{q=1}^{j} x_q(u_{q-1}, u_q))) \\ &= \xi_j(g; \times_{q=1}^{j} x_q(u_{q-1}, u_q)),\end{aligned}$$

the last equality holding since $\gamma(c; s_{n,j,k}) = 1 \in \mathcal{C}(1)$ by assumption. The claim follows by comparison with Definition 8.8. Now consider the following diagram.

$$\begin{tikzcd}
BV_nF_nX \arrow[r, "BV_n\alpha_n"] & BV_n(\Sigma_n\int FX) \arrow[r, "B\pi_n"] & BV(\Sigma_n\int FX) \arrow[d, equal] \\
BV_{n+1}F_nX \arrow[u, "B\tau_n"] \arrow[r, "BV_{n+1}\alpha_n"] \arrow[d, "BV_{n+1}\nu_n"'] & BV_{n+1}(\Sigma_n\int FX) \arrow[u, "B\tau_n"'] \arrow[r, "B\pi_{n+1}"] \arrow[d, "BV_{n+1}\nu_n"] & BV(\Sigma_n\int FX) \arrow[d, "BV\nu_n"] \\
BV_{n+1}F_{n+1}X \arrow[r, "BV_{n+1}\alpha_{n+1}"] & BV_{n+1}(\Sigma_{n+1}\int FX) \arrow[r, "B\pi_{n+1}"] & BV(\Sigma_{n+1}\int X)
\end{tikzcd}$$

The left horizontal arrows are homeomorphisms and the right horizontal arrows are homotopy equivalences by [11, A.2(ii) and A.4(ii)]. The diagram commutes by naturality and the facts that $\pi_n\tau_n = \pi_{n+1}$ and $\nu_n\alpha_n = \alpha_{n+1}\nu_n$. The required natural equivalence $PX \to \mathrm{Tel}\; BV(\Sigma_n\int FX)$ follows by passage to reduced double mapping cylinders and then to unions as in Definition 7.6.

## §9. Some homotopical and homological properties of KX.

Again, let $(\mathcal{C}, \mathcal{H})$ be an $A_\infty$ operad pair and let $X$ be a grouplike $(\mathcal{C}, \mathcal{H})$-space. From the point of view of analysis of its invariants, the sophisticated functors $V_n$ (and W) which entered into the construction of KX are of no significance. They simply replace a given structured space by a homotopy equivalent space with different structure. Thus, up to homotopy, only the classifying space functor, the telescope, and the plus construction are involved. These facts and Remarks 7.3 imply that the analysis of KX is considerably less refractory than the complicated theory necessary for its construction would suggest.

We begin with two elementary homotopy invariance properties, which will be seen later to be simultaneous generalizations of Waldhausen's assertions [22, 1.1 and 2.3] and [22, 1.3 and 2.4].

Recall that a map $f: X \to Y$ is said to be an n-equivalence if $\pi_i f$ is an isomorphism for $i < n$ and an epimorphism for $i = n$ for all choices of basepoint in X (and analogously for maps of pairs and for n-homology equivalences).

Proposition 9.1. If $f: X \to Y$ is a map of grouplike $(\mathcal{C}, \mathcal{H})$-spaces and an n-equivalence, then $Kf: KX \to KY$ is an (n+1)-equivalence.

Proof. By Remarks 7.3, $FM_\infty f: FM_\infty X \to FM_\infty Y$ is an n-equivalence. By Theorem 7.1, $Tf: TX \to TY$ is thus an (n+1)-equivalence. Therefore Tf and thus also Kf are (n+1)-homology equivalences. Since KX and KY are simple spaces, the conclusion follows by the Whitehead theorem.

We next want the relative version of this result, and we need some preliminaries in order to take account of the non-existence of an unstable relative Whitehead theorem and to handle some technical points ubiquitously ignored in the literature. Consider a homotopy commutative diagram of spaces

(*)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ Z \arrow[r, "f'"] & W \end{tikzcd}$$

Definitions 9.2. The diagram (*) is said to be an $(m, n)$-equivalence if $f$ is an $m$-equivalence and $g$ is an $n$-equivalence. It is said to be $q$-homotopy Cartesion if there exists a map of triads

$$\overline{f} = (\overline{f}; f', f): (Mg; Z, X) \to (Mg'; W, Y)$$

such that the map of pairs $\overline{f}: (Mg, X) \to (Mg', Y)$ is a $q$-equivalence, where $Mg$ and $Mg'$ denote the mapping cylinders of $g$ and $g'$. If (*) commutes, with no homotopy required, we insist that this condition be satisfied with $\overline{f}(x,t) = (fx, t)$ on the cylinder, and it is then equivalent to require that the natural map $Fg \to Fg'$ of homotopy fibres be a $(q+1)$-equivalence for each choice of basepoint in $X$ (by the standard verification that the two definitions of the relative homotopy groups of a map agree).

Remarks 9.3. In the general case, with based spaces and maps, the map of triads $\overline{f}$ induces a map

$$\tilde{f}: Fg = X \times_g PZ \to Y \times_{g'} PW = Fg'$$

via

$$\tilde{f}(x, \zeta) = (fx, \omega), \quad \text{where } \omega(t) = \begin{cases} f'\zeta(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ r\overline{f}(x, 2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

with $r: Mg' \to W$ being the canonical retraction. If $\overline{f}$ is a $q$-equivalence of pairs then $\tilde{f}$ is a $(q+1)$-equivalence. A converse construction is not immediately obvious to me, and the definition has been given in the form we wish to use. Clearly a homotopy $h: f'g \simeq g'f$ induces a map of triads of the sort specified, via

$$\bar{f}(x,t) = \begin{cases} h(x,2t) & \text{if } 0 \le t \le \frac{1}{2} \\ (fx, 2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{cases},$$

but whether or not $\bar{f}$ is a q-equivalence of pairs really does depend on the choice of homotopy.

Lemma 9.4. Assume that (*) is an $(m,n)$-equivalence, where $m \ge 0$ and $n \ge 1$. For $q \le m+n$, (*) is q-homotopy Cartesian if and only if there exists a q-equivalence $\phi: M(g,f) \to W$ such that $\phi k \simeq f'$ and $\phi j \simeq g'$, where $M(g,f)$ is the double mapping cylinder of $g$ and $f$ and $k: Z \to M(g,f)$ and $j: Y \to M(g,f)$ are the natural inclusions. If (*) commutes, $\phi$ must be the natural map factoring through the quotient map to the pushout of $f$ and $g$.

Proof. The last statement will be a consequence of the conventions in Definitions 9.2. By the homotopy excision theorem, the natural map $(Mg, X) \to (M(g,f), Y)$ is an $(m+n)$-equivalence. (The range is misstated in [22].) Clearly maps $\bar{f}: Mg \to Mg'$ as in Definitions 9.2 factor uniquely through maps

$$\psi = (\psi; f', 1): (M(g,f); Z, Y) \to (Mg'; W, Y).$$

If $r: Mg' \to W$ is the retraction and $r\psi = \phi$, then $\phi k = f'$ and $\phi j = g$. Conversely, given $\phi$ as in the statement, let $\psi$ be the composite of $\phi$ and the inclusion $i: W \to Mg'$. Since $Y \amalg Z \to Mg'$ is a cofibration, $\psi$ is homotopic to a map of triads $\psi$ as displayed. By the five lemma, $\psi: (M(g,f), Y) \to (Mg', Y)$ is a q-equivalence if and only if $\psi: M(g,f) \to Mg'$ is a q-equivalence. The conclusion follows.

Proposition 9.5. If (*) is a (strictly) commutative diagram of grouplike $(\mathcal{C}, \mathcal{J})$-spaces which is a (q-1)-homotopy Cartesian $(m-1, n-1)$-equivalence with $m \ge 2$, $n \ge 2$, and $q \le m+n$, then

$$\begin{tikzcd}
KX \arrow[r, "Kf"] \arrow[d, "Kg"'] & KY \arrow[d, "Kg'"] \\
KZ \arrow[r, "Kf'"] & KW
\end{tikzcd}$$

is a q-homotopy Cartesian (m, n)-equivalence.

Proof. By Remarks 7.3, application of $FM_\infty$ to (*) gives a (q-1)-homotopy Cartesian (m-1, n-1)-equivalence. By Theorem 7.1 and a little standard argument with homotopy fibres, the (strictly) commutative diagram

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "Tg"'] & TY \arrow[d, "Tg'"] \\
TZ \arrow[r, "Tf'"] & TW
\end{tikzcd}$$

is a q-homotopy Cartesian (m, n)-equivalence. Consider the following diagram

$$\begin{tikzcd}
TX \arrow[rr, "Tf"'] \arrow[dd, "Tg"'] \arrow[rrr, "i_x"] & & TY \arrow[dl, "j"'] \arrow[dd, "Tg'"] \arrow[rrr, "i_y"] & KX \arrow[rr, "Kf"] \arrow[dd, "Kg"] & & KY \arrow[dl, "j"'] \arrow[dd, "Kg'"] \\
& M(Tg, Tf) \arrow[dr, "\psi"] \arrow[rrr, "i"] & & & M(Kg, Kf) \arrow[dr, dashed, "\phi"] & \\
TZ \arrow[ur, "k"] \arrow[rr, "Tf'"'] \arrow[rrr, "i_z"] & & TW \arrow[rrr, "i_w"'] & KZ \arrow[ur, "k"] \arrow[rr, "Kf'"] & & KW
\end{tikzcd}$$

Breaking the cylinder of $M(Tg, Tf)$ into three parts, mapping the middle third $TX \times [1/3, 2/3]$ to $KX \times [0, 1]$ via $i_x$ and expansion, and using homotopies $i_z \circ Tg \simeq Kg \circ i_x$ and $Kf \circ i_x \simeq i_y \circ Tf$ on $TX \times [0, 1/3]$ and $TX \times [2/3, 1]$, we obtain a map $i: M(Tg, Tf) \to M(Kg, Kf)$ such that $ij = ji_y$ and $ik = ki_z$ on the bases. By the van Kampen theorem and the fact that $Tf, Tg, Kf$, and $Kg$ induce isomorphisms on $\pi_1$ since $m \geq 2$ and $n \geq 2$, $\pi_1 M(Tg, Tf) = \pi_1 TX$, $\pi_1 M(Kg, Kf) = H_1 TX$, and $\pi_1 i$ is Abelianization. By the Mayer-Vietoris sequence and the five lemma, i is a homology isomorphism. Therefore $i$ is equivalent to the plus construction on $M(Tg, Tf)$. In particular, $M(Kg, Kf)$ is a simple space. By the universal property of the plus construction, there is a map $\phi: M(Kg, Kf) \to KW$, unique up to homotopy, such that $\phi i \simeq i_W \psi$, where $\psi: M(Tg, Tf) \to TW$ is the natural map. Since

$$\phi j i_Y = \phi i j \simeq i_W \psi j = i_W Tg' \simeq Kg' \circ i_Y ,$$

$\phi j \simeq Kg'$ by the universal property. Similarly $\phi k \simeq kf'$. Since $\psi$ is a q-equivalence (by the lemma), it is a q-homology equivalence. Therefore $\phi$ is a q-homology equivalence and thus a q-equivalence by the Whitehead theorem. The conclusion follows from the lemma.

The proofs above have the following useful consequence.

Lemma 9.6. If (*) is a commutative diagram of grouplike $(\mathcal{C}, \mathcal{J})$-spaces, then there is a canonical homotopy class of maps $\tilde{f}: FKg \to FKg'$ which makes the following diagram homotopy commutative and which is functorial up to homotopy when (*) is regarded as a morphism $(f, f'): g \to g'$ in the category of maps of grouplike $(\mathcal{C}, \mathcal{J})$-spaces and is natural up to homotopy with respect to morphisms of such diagrams (*).

$$\begin{tikzcd} \Omega KZ \arrow[r] \arrow[d, "\Omega Kf'"'] & FKg \arrow[r] \arrow[d, "\tilde{f}"] & KX \arrow[r, "Kg"] \arrow[d, "Kf"] & KZ \arrow[d, "Kf'"] \\ \Omega KW \arrow[r] & FKg' \arrow[r] & KY \arrow[r, "Kg'"] & KW \end{tikzcd}$$

Here the unlabeled arrows are the natural maps of the displayed fibration sequences.

Proof. Of course, Barratt-Puppe sequence arguments give a map $\tilde{f}$, not uniquely determined up to homotopy. We ignore this. Construct $\phi: M(Kg, Kf) \to KW$ as in the previous proof, deform $i\phi$, $i: KW \to Mg'$ to a map of triads

$$\psi = (\psi; Kf', 1): (M(Kg, Kf); KZ, KY) \to (Mg'; KW, KY)$$

as in the proof of Lemma 9.4, let $\bar{f}$ be the composite of $\psi$ and the natural map $MKg \to M(Kg, Kf)$, and construct $\tilde{f}$ as in Remarks 9.3. It is simple to check (by standard cofibration arguments for the passage from $\phi$ to $\psi$) that $\tilde{f}$ is a well-defined homotopy class which makes the displayed diagram homotopy commute. Its functoriality and naturality are then easily verified by the same sorts of homotopical arguments as those above.

Turning to homology, we first record the form taken in our context by the standard spectral sequence for the calculation of the homology of classifying spaces of topological monoids.

Since $\nu_n : FM_nX \to FM_{n+1}X$ only commutes up to homotopy with the multiplications, there is a slight ambiguity in giving $FM_\infty X$ an H-space structure. There is no ambiguity in its Pontryagin product, however, and the spectral sequences of the filtered spaces $BV_nFM_nX$ pass to limits to give the following result. (See e.g. [9, 13.10].)

Proposition 9.7. Take homology with coefficients in a field k. There is then a natural spectral sequence $\{E^rX\}$ of differential coalgebras which converges from $E^2X = \mathrm{Tor}^{H_*FM_\infty X}(k, k)$ to $H_*KX$.

Field coefficients serve only to yield a conceptual description of $E^2X$. Since $i: TX \to KX$ induces an isomorphism on $k_*$ for any connective homology theory $k_*$ (by the Atiyah-Hirzebruch spectral sequence), we obtain a spectral sequence $\{E^rX\}$ converging to $k_*KX$ with $E^1_{pq}X = k_q((FM_\infty X)^p)$ for any such $k_*$, $d^1$ being induced by application of $k_*$ to the alternating sum of the standard bar construction face maps. (See [10, 11.14] for details.)

The proposition focuses attention on the problem of computing the Pontryagin algebras $H_*FM_nX$. It is clear from previous experience what procedures one should adopt: one should work in all of $H_*M_nX$ and exploit the diagram of Proposition 5.2. Since $M_nX = X^{n^2}$ is an (additive) infinite loop space, all of the machinery of homology operations explained in [2, I] is available. Assuming that $H_*M_nX$ is understood additively (as a Hopf algebra with product $*$, coproduct $\psi$, conjugation $\chi$, augmentation $\varepsilon$, homology operations $Q^s$ and Steenrod operations $P^r_*$, with $P^r_* = Sq^r_*$ at the prime 2), we can study its products by use of the following two results, the proofs of which are exactly the same as those in [2, p. 79-81].

<u>Proposition 9.8.</u> Take homology with coefficients in any field. Let $x, y, z \in H_*M_nX$, $n \geq 1$, and let $[0], [1]$, and $[-1]$ be the classes of the zero matrix, the identity matrix $I_n$, and any matrix in the component additively inverse to that of $I_n$.

(i) $[0]x = (\varepsilon x)[0]$, $[1]x = x$, and $[-1]x = \chi x$.

(ii) $(x * y)z = \sum (-1)^{\deg y \deg z'} xz' * yz''$, where $\psi z = \sum z' \otimes z''$.

<u>Proposition 9.9.</u> Take homology with mod p coefficients, where p is any prime. Let $x, y \in H_*M_nX$, $n \geq 1$. Then

$$(Q^s x)y = \sum_i Q^{s+i}(xP^i_* y) \quad \text{and, if } p > 2,$$
$$(\beta Q^s x)y = \sum_i \beta Q^{s+i}(xP^i_* y) - \sum_i (-1)^{\deg x} Q^{s+i}(xP^i_* \beta y).$$

<u>Remarks 9.10.</u> Applied to the $A_\infty$ ring spaces $QX^+$ of Examples 2.3, where X is an $A_\infty$ space, the propositions above completely determine the Pontryagin algebra $H_*FQX^+$ in terms of the additive structure of $H_*QX^+$. Compare [2, II §4], where the analogous assertion for the $E_\infty$ ring space $QX^+$ obtained from an $E_\infty$ space X is explained in detail. To determine $H_*FM_nQX^+$ for $n > 1$, one would

require additional formulas to explain the effect on homology of translations by more general matrices than in Proposition 9.8(i).

## §10. Examples, natural maps, and formal properties of KX

Recall from section 1 that any $A_\infty$ operad pair $(\mathcal{C}, \mathcal{G})$ admits an augmentation $(\mathcal{C}, \mathcal{G}) \to (\mathcal{N}, \mathcal{M})$. An $(\mathcal{N}, \mathcal{M})$-space is precisely a topological semi-ring and a grouplike $(\mathcal{N}, \mathcal{M})$-space is precisely a topological ring. Therefore a topological ring R is a grouplike $(\mathcal{C}, \mathcal{G})$-space by pullback. If R is discrete, then $FM_n R = GL(n, R)$, and the following example is immediate from the constructions of section 7.

Example 10.1. If R is a discrete ring, then KR is naturally equivalent to the plus construction on $BGL(\infty, R)$. Therefore $K_q R$, $q > 0$, is Quillen's $q^{th}$ algebraic K-group of R [5, 18].

For general topological rings, our theory reduces to the topological version of Waldhuasen's [22, §1].

Example 10.2. For simplicial rings $R_*$, Waldhausen defined a certain functor $KR_*$. It would be immediate from the definitions that his $KR_*$ is naturally equivalent to our $K|R_*|$, where $|R_*|$ is the geometric realization of $R_*$, were it not that he has chosen to throw in a discrete factor Z and we have not, so that his $KR_*$ is our $K|R_*| \times Z$.

Remarks 10.3. There is one vital distinction to be made between KR for a topological ring R and KX for an arbitrary grouplike $A_\infty$ ring space X. As is well-known in the discrete case [20; 21; 12] and will be proven in general at the end of this section, KR is an infinite loop space. In contrast, I see little reason to think that KX is an infinite loop space and have not yet been able to prove that it is even an H-space, although I believe this to be the case (compare Remarks 7.7).

It is immediate from the definitions that, for any $(\mathcal{C}, \mathcal{H})$-space $X$, the discretization map $d: X \to \pi_0 X$ is a map of $(\mathcal{C}, \mathcal{H})$-spaces. When X is grouplike, this has the following important consequence.

Proposition 10.4. For grouplike $(\mathcal{C}, \mathcal{H})$-spaces $X$, there is a natural augmentation $Kd: KX \to K\pi_0 X$.

This suggests the following reduced variant of $KX$.

Definition 10.5. Define $K'X$ to be the homotopy fibre of $Kd$. By Lemma 9.6, $K'$ is then a functor of $X$ such that the following diagram is homotopy commutative for any map $f: X \to Y$ of grouplike $(\mathcal{C}, \mathcal{H})$-spaces.

$$\begin{tikzcd}
\Omega K\pi_0 X \arrow[r] \arrow[d] & K'X \arrow[r] \arrow[d, "K'f"] & KX \arrow[r, "Kd"] \arrow[d, "Kf"] & K\pi_0 X \arrow[d, "K\pi_0 f"] \\
\Omega K\pi_0 Y \arrow[r] & K'Y \arrow[r] & KY \arrow[r, "Kd"] & K\pi_0 Y
\end{tikzcd}$$

Of course, $K'X$ is trivial if $X$ is discrete.

We can mimic this construction after restriction to monomial matrices. The resulting functor may be described, up to equivalence, as follows.

Definition 10.6. Define $Q_0'(BFX^+)$, $BFX^+ = BFX \amalg \{0\}$, to be the homotopy fibre of $Q_0(Bd^+): Q_0(BFX^+) \to Q_0(BF\pi_0 X^+)$, where $F\pi_0 X$ is the group of units in the ring $\pi_0 X$. As in the previous definition, Lemma 9.6 (or rather its monomial matrix analog) implies that this is the object function of a functor of $X$. Since $SF\pi_0 X = \{1\}$, the corresponding functor obtained with $FX$ replaced by $SFX$ is equivalent to $Q_0 BSFX$.

At the risk of belaboring the obvious, we combine the definitions above with Theorem 8.3, Remarks 8.5, and further applications of Lemma 9.6 (and analogs thereof) in the following theorem.

Theorem 10.7. For grouplike $(\mathscr{C}, \mathscr{J})$-spaces $X$, the following is a homotopy commutative diagram and is natural up to homotopy.

$$\begin{tikzcd}
Q_0\Sigma SFX \arrow[dr] \arrow[dd] & & & & \\
& Q_0BSFX \arrow[rr, "\nu"] \arrow[dd] & & Q_0'(BFX^+) \arrow[r, "\mu'"] \arrow[dd] & K'X \arrow[dd] \\
Q_0(\Sigma SFX^+) \arrow[rr] \arrow[dr] \arrow[dd, shift left] & & Q_0(\Sigma FX^+) \arrow[dr] & & \\
& Q_0(BSFX^+) \arrow[rr] \arrow[dd, shift left, "Q_0(Bd^+)"] & & Q_0(BFX^+) \arrow[r, "\mu"] \arrow[dd, "Q_0(Bd^+)"] & KX \arrow[dd, "Kd"] \\
Q_0S^0 \arrow[uu, dashed, shift left] \arrow[dr, equal] & & & & \\
& Q_0S^0 \arrow[uu, dashed, shift left] \arrow[rr, shift left] & & Q_0(BF\pi_0X^+) \arrow[ll, dashed, shift left] \arrow[r, "\mu"] & K(\pi_0X)
\end{tikzcd}$$

The dotted arrows denote the presence of evident sections, and the columns are fibrations; $\mu'$ and $\nu$ are obtained as in Lemma 9.6 and the remaining horizontal arrows are inclusions.

The diagram could be expanded further (by use of $Q_0(\Sigma F\pi_0X^+)$, etc.), but we desist. In view of its importance in the applications, we call special attention to the case $X = QS^0$, where $FX$ and $SFX$ are generally denoted $F$ and $SF$ (or $G$ and $SG$) and where $\pi_0X = Z$ and thus $F\pi_0X = Z_2$. (Up to notation, the resulting diagram is an elaboration of one claimed by Waldhausen [22, above 2.3].) Here the bottom composite

$$Q_0S^0 \longrightarrow Q_0(BZ_2^+) \xrightarrow{\mu} KZ$$

is induced by the inclusions $\Sigma_n \to GL(n, Z)$ of permutation matrices and is thus the standard map studied by Quillen [19] (see also [12, VI §5 and VIII. 3.6]); the monomial matrix map $\mu$ has been studied in [12, VI. 5.9, VII. 4.6 and VIII. 3.6] and also in Loday [7]. As the following remarks make clear, this example is universal:

the diagram for $QS^0$ maps naturally to that for $X$ for any grouplike $A_\infty$ ring space $X$.

Remarks 10.8. By Proposition 2.2 and Theorem 3.1, the zero$^{th}$ map of the unit of any $A_\infty$ ring spectrum E is a map $e: QS^0 \to E_0$ of grouplike $A_\infty$ ring spaces. Let us write $\Gamma X$ for the zero$^{th}$ space of the $A_\infty$ ring spectrum obtained as in section 3 from an $A_\infty$ ring space $X$. If $X$ is grouplike, we have a natural weak equivalence of $A_\infty$ ring spaces between $X$ and $\Gamma X$ by Theorem 3.3. By a slight abuse, we may thus regard $e$ as a unit map $QS^0 \to X$.

Proposition 2.2 and Theorem 3.1 also show that $QY$ is a grouplike $A_\infty$ ring space for any $A_\infty$ space with zero $Y$. We shall specialize this example to obtain Waldhausen's algebraic K-theory of spaces in the next section. We note here that the following generalized version of Waldhausen's assertion [22, 2.2] is an immediate consequence of Propositions 3.5 and 3.7 and Lemma 3.6 together with Theorem 7.1, Remarks 7.3, and Proposition 9.7.

Proposition 10.9. For any $A_\infty$ space with zero $Y$, the Hurewicz map $h: QY \to \Gamma NY$ is a map of grouplike $A_\infty$ ring spaces and a rational equivalence. Therefore $Kh: KQY \to K\Gamma NY$ is also a rational equivalence.

Remarks 10.10. $NS^0$ is precisely the additive monoid of non-negative integers, hence $\pi_0\Gamma NS^0 = Z$ and $d: \Gamma NS^0 \to Z$ and therefore also $Kd: K\Gamma NS^0 \to KZ$ are equivalences. Thus we may view the Hurewicz map of $KQS^0$ as having target $KZ$.

The deepest source of examples is the theory of $E_\infty$ ring spaces and $E_\infty$ ring spectra. By neglect of structure, these are $A_\infty$ ring spaces and $A_\infty$ ring spectra. For example, the zero$^{th}$ spaces of all of the various Thom spectra are $A_\infty$ ring spaces [12, IV §2]. Of greater interest are the examples coming out of the chain of functors constructed in [12, VI and VII]:

$$\begin{array}{c}
\text{symmetric bimonoidal categories} \\
\Big\Downarrow \text{ blow up} \\
\text{bipermutative categories} \\
\Big\Downarrow \text{ classifying space} \\
E_\infty \text{ ring spaces} \\
\Big\Downarrow \text{ recognition principle} \\
E_\infty \text{ ring spectra} \\
\Big\Downarrow \text{ zero}^{\text{th}} \text{ space} \\
\text{grouplike } E_\infty \text{ ring spaces}
\end{array}$$

For instance, starting with the symmetric bimonoidal category $\mathscr{P}R$ of finitely generated projective modules over a commutative topological ring R, we arrive at an associated grouplike $E_\infty$ ring space, denoted $\Gamma B\mathscr{P}R$, in which addition comes from the direct sum and multiplication comes from the tensor product. We interject the following note (compare [22, §1]).

Remarks 10.11. We are here faced with a conflict of definitions and notations. For a topological ring R (not necessarily commutative), I wrote $K_qR$ for $\pi_q\Gamma B\mathscr{P}R$ in [10, VIII §1]. For discrete R, this agrees with the present $K_qR$ for $q > 0$. For general R, it is quite different. I suggest writing $K^tR$ for the plus construction on BGLR and writing $K^t_qR = \pi_qK^tR$, thinking of this as a topological K-theory (which it is when R is the topologized complex numbers for example). In practice, the functors $K_*R$ and $K^t_*R$ tend to be of interest for different choices of R.

In fact, the theory sketched above applies equally well to both generalizations of Quillen's theory from discrete rings to topological rings. To see this, let $\mathscr{GL}R$ denote the permutative category of finitely generated free R-modules, as described explicitly in [12, VI. 5. 2]. Define $\mathscr{F}R$ to be the category defined in precisely the same way, except that the space of morphisms $n \to n$ is the topological monoid $FM_nR$. Then, with structure as specified in the cited definition, $\mathcal{F}R$ is a permutative category and is a bipermutative category if $R$ is commutative. Moreover, the inclusion $\mathcal{GL}R \to \mathcal{F}R$ preserves all structure in sight. The argument used to prove that $\Gamma_0 B\mathcal{GL}R$ is equivalent to the plus construction on BGLR in [12, VIII §1] applies equally well to prove that $\Gamma_0 B\mathcal{F}R$ is equivalent to the plus construction on $BFM_\infty R$ (and of course $\Gamma_0 B\mathcal{GL}R$ is equivalent to $\Gamma_0 B\mathcal{P}R$ [11, p. 85]). We have proven the following result.

Proposition 10.12. For a topological ring $R$, $K^tR$ and $KR$ are the zero components of infinite loop spaces $\Gamma B\mathcal{GL}R$ and $\Gamma B\mathcal{F}R$, and there is a natural infinite loop map $i: \Gamma B\mathcal{GL}R \to \Gamma B\mathcal{F}R$. If $R$ is commutative, then $\Gamma B\mathcal{GL}R$ and $\Gamma B\mathcal{F}R$ are $E_\infty$ ring spaces and $i$ is an $E_\infty$ ring map.

Here the additive infinite loop space structures associated to permutative categories are uniquely determined by the axioms in [14]; in particular, Segal's machine [20] and mine give equivalent spectra.

At the moment, nothing is known about the resulting "second order" algebraic K-theories $K_*\Gamma B\mathcal{GL}R$ and $K_*\Gamma B\mathcal{F}R$ of commutative topological rings. They do not appear to be very closely related to the "first order" theories $K_*R$ or $K_*^tR$. Since $\pi_0\Gamma B\mathcal{GL}R = \pi_0\Gamma B\mathcal{F}R = Z$, Proposition 10.4 and naturality give maps

$$K\Gamma B\mathcal{GL}R \to K\Gamma B\mathcal{F}R \to KZ \to K^tR \to KR.$$

If we had followed Waldhausen and crossed everything with $Z$, we could consistently write

$$KK^tR = K\Gamma B\mathcal{GL}R \quad \text{and} \quad KKR = K\Gamma B\mathcal{F}R,$$

hence our view of these as second order theories.

## §11. The algebraic K-theory of spaces

Finally, we specialize our theory to make rigorous the algebraic K-theory of spaces suggested by Waldhausen [22]. In this section, X will denote a non-degenerately based CW-space with basepoint 1. There are at least three ways that $\Omega X$ can be interpreted as an $A_\infty$-space.

(1) The ordinary loop space of X is a $\mathcal{C}_1$-space, where $\mathcal{C}_1$ is the little 1-cubes $A_\infty$ operad of [10, §4].

(2) The Moore loop space of X is a topological monoid, or $\mathcal{M}$-space.

(3) $|GSX|$, the geometric realization of the Kan loop group of the total singular complex of X is also an $\mathcal{M}$-space.

We lean towards the first choice, but the theory works equally well with any choice. As in Example 2.3, we construct from $\Omega X$ an $A_\infty$ space with zero $\Omega X^+ = \Omega X \amalg \{0\}$ and thus a grouplike $A_\infty$ ring space $Q(\Omega X^+)$. For any based space Y,

$$\pi_0 QY = \operatorname{colim} \pi_n \Omega^n \Sigma^n Y = \operatorname{colim} \tilde{H}_n \Sigma^n Y = \tilde{H}_0 Y,$$

this isomorphism being realized by $h\colon QY \to \Gamma NY$ on passage to components. It follows easily that, as a ring, $\pi_0 Q(\Omega X^+) = \pi_0 \Gamma N(\Omega X^+)$ is the integral group ring $Z[\pi_1 X]$. In reading the following definitions, it will be useful to keep in mind the following commutative diagram of $A_\infty$ ring spaces (see Propositions 3.7, 10.4, and 10.9 and Remarks 10.10).

$$\begin{tikzcd}
QS^0 \arrow[r, "\eta"] \arrow[d, "h"'] & Q(\Omega X^+) \arrow[r, "\varepsilon"] \arrow[d, "h"] & QS^0 \arrow[d, "h"] \\
\Gamma NS^0 \arrow[r, "\eta"] \arrow[d, "d"', "\simeq"] & \Gamma N(\Omega X^+) \arrow[r, "\varepsilon"] \arrow[d, "d"] & \Gamma NS^0 \arrow[d, "\simeq"', "d"] \\
Z \arrow[r, "\eta"] & Z[\pi_1 X] \arrow[r, "\varepsilon"] & Z
\end{tikzcd}$$

For uniformity of notation, we write $\eta$ for maps induced by inclusions of basepoints of spaces or trivial subgroups of groups and write $\varepsilon$ for the corresponding projections, so that $\varepsilon\eta = 1$. The vertical composites in the diagram are again discretization maps d. We shall continue to write $\eta, \varepsilon$, d, and h upon application of the functor K.

Definitions 11.1. Define AX, the connected algebraic K-space of X, to be $KQ(\Omega X^+)$. Further, define the following reduced variants.

$$\tilde{A}X = \text{fibre}\,(\varepsilon : AX \to A\{*\}), \quad A\{*\} = KQS^0$$

$$A'X = \text{fibre}\,(d : AX \to KZ[\pi_1 X]) = K'Q(\Omega X^+)$$

$$\tilde{A}'X = \text{fibre}\,(\varepsilon' : A'X \to A'\{*\}), \quad A'\{*\} = K'QS^0$$

For $q > 0$, define $A_q X$, the $q^{th}$ algebraic K-group of X, to be $\pi_q AX$, and introduce similar notation for the reduced variants.

We have an analogous algebraic K-theory of X with coefficients in Z.

Definition 11.2. Define $A(X; Z) = K\Gamma N(\Omega X^+)$ and define the reduced variants

$$\tilde{A}(X; Z) = \text{fibre}\,(\varepsilon : A(X; Z) \to A(*; Z)), \quad A(*; Z) = KZ$$

$$A'(X; Z) = \text{fibre}\,(\varepsilon' : A(X; Z) \to KZ[\pi_1 X]) = K'\Gamma N(\Omega X^+).$$

Here $A'(*; Z) \simeq \{*\}$, hence we set $\tilde{A}'(X; Z) = A'(X; Z)$. Define $A_q(X; Z) = \pi_q A(X; Z)$ and similarly for the reduced variants.

For a (discrete) group $\pi$ and commutative ring R, write $\tilde{K}R[\pi] = \text{fibre}\,(\varepsilon : KR[\pi] \to KR)$.

By Lemma 9.6, these are all well-defined functors of their variables such that the various canonical maps in sight are natural. We summarize the relationships between these functors in the following result.

Theorem 11.3. The rows and columns are fibration sequences in the following natural homotopy commutative diagram.

$$\begin{tikzcd}
\tilde{A}'X \arrow[r, "\pi"] \arrow[d] & \tilde{A}X \arrow[r, "\tilde{d}"] \arrow[d] & \tilde{K}Z[\pi_1 X] \arrow[d] \\
A'X \arrow[r] \arrow[d, shift right, "\varepsilon'"'] & AX \arrow[r, "d"] \arrow[d, shift right, "\varepsilon"'] & KZ[\pi_1 X] \arrow[d, shift right, "\varepsilon"'] \\
A'\{*\} \arrow[r] \arrow[u, dotted, shift right, "\eta'"'] & A\{*\} \arrow[r, "d"] \arrow[u, dotted, shift right, "\eta"'] & KZ \arrow[u, dotted, shift right, "\eta"']
\end{tikzcd}$$

The dotted arrows denote the presence of sections. The Hurewicz map sends this diagram naturally, via a rational equivalence, to the corresponding natural $3 \times 3$ diagram of fibration sequences with $AX$ and its variants replaced by $A(X;Z)$ and its variants.

Proof. In view of our earlier results, only the construction of $\pi$ and the verification that it is equivalent to the fibre of $\tilde{d}$ are needed. This would be obvious enough if we knew that $AX$ were an H-space. In the absence of this, a technical argument with Barratt-Puppe sequences, which we defer to the appendix, is required.

Of course, Theorem 10.7 applies naturally to all $A_\infty$ ring spaces in sight and produces a maze of commutative diagrams. In particular, we have the maps

$$Q_0 X_1^+ \simeq Q_0(B\Omega X^+) \xrightarrow{Q_0(B\eta^+)} Q_0(BFQ(\Omega X^+)^+) \xrightarrow{\mu} AX ,$$

where $X_1$ is the component of the basepoint of $X$, $X_1 \simeq B\Omega X$ by [9,14.3 or 15.4], $\eta: \Omega X \to FQ(\Omega X^+)$ is the natural $A_\infty$ map of Example 2.3, and $\mu$ is the monomial matrix map of section 8. This agrees with assertions of Waldhausen [22, §2].

It is desirable to have algebraic K-theories of spaces with coefficients in arbitrary commutative rings R. For many rings, this can be obtained topologically. (For $Z_n$, compare Dold and Thom [3] to the material at the end of section 3.) However it is most natural to follow Waldhausen [22,§1] and do this simplicially.

Definitions 11.4. Define $A(X;R) = K|R[GSX]|$, where the group ring of a simplicial group is formed degreewise. Clearly

$$\pi_0|R[GSX]| = \pi_0 R[GSX] = R[\pi_0 GSX] = R[\pi_1 X].$$

Define reduced variants and algebraic K-groups as in Definitions 11.2.

We now have two definitions of $A(X;Z)$ in sight. With interpretation (3) above for $\Omega X$, the following result implies that these definitions agree up to natural equivalence.

Proposition 11.5. For simplicial groups $G$, there is a natural weak equivalence of grouplike $A_\infty$ ring spaces between $\Gamma N|G|^+$ and $|Z[G]|$.

Proof. $N|G|^+$ is the free topological Abelian monoid generated by $|G|$ (the disjoint basepoint being identified to zero). The inclusion $G \to Z[G]$ induces $|G| \to |Z[G]|$ and thus, by freeness, $N|G|^+ \to |Z[G]|$. Moreover, this map clearly extends over the free Abelian group $\tilde{N}|G|^+$ generated by $|G|$, and it is easy to verify from the fact that realization commutes with products that $\tilde{N}|G|^+ \to |Z[G]|$ is actually a homeomorphism. The natural map

$$\lambda : N|G|^+ \to \tilde{N}|G|^+ \cong |Z[G]|$$

is clearly a map of semi-rings, hence of $(\mathcal{C}, \mathcal{H})$-spaces for any $A_\infty$ operad pair $(\mathcal{C}, \mathcal{H})$. By Corollary 3.8 (recall that $\Gamma = B_0$), $\Gamma\lambda$ is a weak equivalence. By Theorem 3.3 and the fact that $|Z[G]|$ is grouplike, $\Gamma|Z[G]|$ is naturally weakly equivalent as an $A_\infty$ ring space to $|Z[G]|$.

In view of the unit map $Z \to R$, all of our diagrams remain present with $Z$ replaced by $R$, and the resulting diagrams are natural in $R$ as well as $X$.

The simplicial approach has the advantage of giving us infinite loop space structures on our algebraic K-spaces, by Proposition 10.12 applied to $K|R[G]|$ for a simplicial group $G$. When $G$ is the trivial simplicial group, $|R[G]| = R$. We thus have $\varepsilon : K|R[G]| \to KR$ and $\eta : KR \to K|R[G]|$. Write $\tilde{K}|R[G]| =$ fibre $\varepsilon$

and give it the induced infinite loop structure. The following result is immediate by a glance at homotopy groups.

Proposition 11.6. For simplicial groups $G$ and commutative rings $R$, the composite

$$\widetilde{K}|R[G]| \times KR \xrightarrow{p \times \eta} K|R[G]| \times K|R[G]| \xrightarrow{\phi} K|R[G]|,$$

where $p$ is the canonical map and $\phi$ is the product, is a natural equivalence of infinite loop spaces.

The following is a special case.

Corollary 11.7. $A(X; R)$ is naturally equivalent as an infinite loop space to $\widetilde{A}(X; R) \times KR$.

All of these algebraic K-theories on spaces admit stabilizations to generalized homology theories.

Theorem 11.8. There are natural homomorphisms $\widetilde{A}_n X \to \widetilde{A}_{n+1} \Sigma X$ such that if $\widetilde{A}^s_q X = \operatorname{colim} \widetilde{A}_{q+r} \Sigma^r X$, then $\widetilde{A}^s_*$ is a reduced homology theory. There are analogous theories $\widetilde{A}^s_*(X; R)$ defined and natural on $R$, and there is a Hurewicz homomorphism $\widetilde{h}^s_* : \widetilde{A}^s_* X \to \widetilde{A}^s_*(X; Z)$ which is a rational equivalence.

We need only construct the theories. The rest will follow from evident naturality arguments. Of course, $\widetilde{A}_* X$ maps naturally to $\widetilde{A}^s_* X$, and similarly and compatibly for $\widetilde{A}_*(X; R)$. The following consequence is immediate (see Corollary A.3).

Corollary 11.9. For unbased spaces $X$, the definitions $A^s_* X = \widetilde{A}^s_*(X^+)$ and $A^s_*(X; R) = \widetilde{A}^s_*(X^+; R)$ specify unreduced homology theories.

Note here that $\Omega(X^+)$ is a point, hence $\widetilde{A}_*(X^+) = \widetilde{A}_*(S^0)$. The corollary was asserted by Waldhausen [22, 1.4 and 2.8] (but with a rather misleading sketch proof). We shall prove a general result, Theorem A.2, on the stabilization of functors to homology theories. Theorem 11.8 will be an immediate special case in view of the following two results.

Proposition 11.10. The functors $AX$ and $A(X;R)$ preserve n-equivalences and q-homotopy Cartesian (m, n)-equivalences with $m \geq 2$, $n \geq 2$, and $q \leq m+n$ provided that, for the latter, the domain square is strictly commutative.

Proof. We lose a dimension upon appliction of $\Omega$ and gain it back upon passage to $K$ and quotation of Propositions 9.1 and 9.5. We need only verify that the intervening functors $Q$, $\Gamma N$, or $R[?]$ have the appropriate preservation properties, where $R[C]$ for a simplicial set $C$ is the free simplicial R-module generated by C. Since $\pi_* \Gamma N(X^+) = H_* X$ and, by [8, § 22], $\pi_* R[C] = H_*(C; R)$, the conclusions are obvious in these cases. Since $\pi_* QX^+ = \pi_*^s(X^+)$ is the unreduced homology theory associated to stable homotopy, the conclusion here follows by use of the Atiyah-Hirzebruch spectral sequence.

Lemma 11.11. The functors $AX$ and $A(X;R)$ from based CW-spaces to the homotopy category of based CW-spaces are homotopy preserving.

Proof. This is not obvious. Recall that a functor $T$ from based spaces to based spaces is said to be continuous if the function $T: F(X, Y) \to F(TX, TY)$ on function spaces is continuous. Since a homotopy between maps $X \to Y$ is a map $I \to F(X, Y)$, continuous functors are homotopy preserving. The various monads, bar constructions, and telescopes which entered into the construction of the functor $T$ on $(\mathcal{C}, \mathfrak{H})$-spaces are all continuous. (In particular, this uses the fact that geometric realization of simplicial spaces is a continuous functor [12, p. 21].) We pass from $T$ to $K$ by first converting to CW-complexes by applying geometric realization on the total singular complex, this composite being homotopy preserving although not continuous, and then applying the plus construction (which is a homotopy functor by definition). The functors $Q$, $\Gamma N$, and also $\Omega$ when interpreted as in (1) or (2) are continuous, and the functors $R[GSX]$ are homotopy preserving by standard facts on the relationship between simplicial and topological homotopy theory (e.g. [8, §16 and §26]).

## §12. Notes on Waldhausen's work

Since this work started with an attempt to understand Waldhausen's, a rundown of those things in [22, §1 and 2] not considered above may not be taken amiss.

There are two calculational results in [22, §1] concerning simplicial (or topological) rings. Our theory adds nothing new to the foundations here except for the infinite loop space structure on KR of Proposition 10.12 and the concomitant splittings of Proposition 11.6. Since the H-space level of these additions provides some clarification of Waldhausen's arguments, I shall run through the details (modulo the relevant algebra; these details are included at Rothenberg's request).

Proposition 12.1 ([22, 1.2]). Let $f: R \to R'$ be an (n-1)-equivalence of topological rings, where $n \geq 2$. Let F and F' be the homotopy fibres in the following diagram and choose $F \to F'$ which makes the top square homotopy commute.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & F' \arrow[d] \\ BFM_\infty R \arrow[r] \arrow[d, "BFM_\infty f"'] & KR \arrow[d, "Kf"] \\ BFM_\infty R' \arrow[r] & KR' \end{tikzcd}$$

For any Abelian group A, the diagram induces an isomorphism

$$H_0(GL(\pi_0 R'); M_\infty(\pi_{n-1}) \otimes A) \cong H_0(BFM_\infty R'; H_n(F; A)) \to H_n(F'; A),$$

where $\pi_{n-1}$ is the $(n-1)^{st}$ homotopy group of the fibre of f.

Proof. Waldhausen asserts further that the left side is clearly isomorphic, via the trace, to the Hochschild homology group $H_0(\pi_0 R'; \pi_{n-1} \otimes A)$; I haven't checked the algebra. By the Hurewicz theorem, Remarks 7.3, and universal coefficients, $H_n(F; A) \cong M_\infty(\pi_{n-1}) \otimes A$. The first isomorphism follows from the definition of local coefficients, the fact that $\pi_1 BFM_\infty R' \cong GL(\pi_0 R')$, and a check that the action on $H_n(F; A)$ agrees with the natural action on $M_\infty(\pi_{n-1}) \otimes A$. Let $\{E^r\} \to \{'E^r\}$ be the map of Serre spectral sequences (with coefficients in A) induced by the diagram. The essential topological fact is that, since Kf is a map of connected H-spaces which induces an epimorphism on fundamental groups, it has trivial local coefficients. Our assertion is thus that $E^2_{0n} \to {'E}^2_{0n}$ is an isomorphism. Since

$$E^2_{*0} = H_*(BGLR'; A) \to H_*(KR'; A) = {'E}^2_{*0} \quad \text{and} \quad H_*(BGLR; A) \to H_*(KR; A)$$

are isomorphisms and $E^r_{*q} = {'E}^r_{*q} = 0$ for $0 < q < n$, $E^2_{n0} = E^\infty_{n,0}$ and ${'E}^2_{n0} = {'E}^\infty_{n0}$, hence the five lemma gives that $E^\infty_{0n} \to {'E}^\infty_{0n}$ is an isomorphism. A diagram chase from the edge homomorphism gives that $E^\infty_{*0} \to {'E}^\infty_{*0}$ is an isomorphism, and another five lemma argument (involving the transgression $d^{n+1}: E^2_{n+1,0} \to E^2_{0n}$) gives the conclusion.

Proposition 12.2 ([22, 1.5])

$$A^s_i(\{*\}; Z) \otimes Q \cong \begin{cases} Q & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases}$$

Proof. Specialize the diagram of the previous result to

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & F' = \tilde{A}(S^n; Z) \arrow[d] \\
BFM_\infty R_n \arrow[r] \arrow[d, "BFM_\infty d"'] & KR_n = A(S^n; Z) \arrow[d] \\
BGLZ \arrow[r] & KZ = A(\{*\}; Z)
\end{tikzcd},$$

where $R_n = |Z[G S\Sigma^n S^0]|$ and $d: R_n \to Z$ is the discretization. $F'$ is $(n-1)$-connected. We claim that $H_n F' = Q$ and $H_q F' = 0$ for $n < q \leq 2n-2$ ($Q$ coefficients understood). By the Whitehead theorem applied to a $\pi_n$-isomorphism $F_0' \to K(Q,n)$, where $F_0'$ is the rationalization of $F'$, it will follow that $\pi_n F' \otimes Q = Q$ and $\pi_q F' \otimes Q = 0$ for $n < q \leq 2n-2$, hence the conclusion. By Remarks 7.3 and [8, § 22], for $q \geq 2$ and $n \geq 1$ we have

$$\begin{aligned}
\pi_q F &= \pi_q BFM_\infty R_n = \pi_{q-1} FM_\infty R_n = \pi_{q-1} SFM_\infty R_n \\
&\cong \pi_{q-1} M_\infty R_n = M_\infty \pi_{q-1} R_n = M_\infty H_{q-1}(\Omega S^n; Z),
\end{aligned}$$

which is zero for $q \leq 2n-2$ and $q \neq n$. By the Whitehead theorem applied to a $\pi_n$-isomorphism $F \to K(M_\infty Z, n)$, $H_n(F; Z) = M_\infty Z$ and $H_q(F; Z) = 0$ otherwise, $0 < q \leq 2n-2$, hence similarly with $Z$ replaced by $Q$. The key algebraic fact, due to Farrell and Hsiang [4] and based on work of Borel [1], is that $H_*(BGLZ; M_\infty Q)$ is isomorphic to $H_*(BGLZ; Q)$; the definition of the isomorphism, via the trace, is irrelevant to the argument here. Consider the rational homology Serre spectral sequences. Certainly $E^2$ is finite-dimensional in each degree $\leq 2n-2$ (say by Borel's calculations of $H_*(BGLZ)$ [1]) and, in this range,

$$E^2 = E^2_{*0} \oplus E^2_{*n} \quad (E^2_{*0} = H_* BGLZ, E^2_{*n} \cong H_* BGLZ).$$

By the previous result, $Q \cong E^2_{0n} \to {}'E^2_{0n}$ is an isomorphism. By Corollary 11.7, $A(S^n; Z) \simeq A(S^n; Z) \times KZ$ and therefore

$${}'E^2 = {}'E^\infty = H_* KZ \otimes H_* F' \quad ({}'E^2_{*0} = H_* KZ,\ {}'E^2_{*n} \cong H_* KZ).$$

Since $\{E^r\}$ and $\{'E^r\}$ converge to isomorphic homologies, $H_q F' = 0$ for $n < q \leq 2n-2$ by a trivial comparison of dimensions.

The following is an immediate consequence, by Theorem 11.8.

Corollary 12.3

$$A_i^s\{*\} \otimes Q = \begin{cases} Q & \text{if } i = 0 \\ 0 & \text{if } i > 0 \end{cases} .$$

The results claimed about Postnikov systems in [22, §2] seem much more problematical (and are fortunately much less essential to the overall program).

Remarks 12.4. The $n^{th}$ term $R_*^{(n)}$ of the natural Postnikov system of a simplicial set $R_*$ is $R_*/(\underset{\sim}{n})$, where $x \overset{n}{\sim} y$ for q-simplices $x$ and $y$ if all of their iterated faces of dimension $\leq n$ are equal [8, §8]. Visibly each $R_*^{(n)}$ is a simplicial ring if $R_*$ is so, and the natural maps $R_*^{(n)} \to R_*^{(m)}$ for $n > m$ are maps of simplicial rings. As Waldhausen states [22, §1], there results a spectral sequence the $E^2$-term of which is given by the homotopy groups of the fibres of the maps $K|R_*^{(n)}| \to K|R_*^{(n-1)}|$ and which converges to $K_*|R_*|$. He asserts further [22, 2.5 and sequel] that the same conclusions hold with $R_*$ replaced by an arbitrary ring up to homotopy, that is, in our terminology, by an arbitrary grouplike $A_\infty$ ring space $X$. If true, anything like this would be enormously difficult to prove. Certainly, the coskeleta $X^{(n)}$ of $X$ could at best be strong homotopy $A_\infty$ ring spaces of some sort (with more homotopies in sight than in Definition 5.1; see the discussion following that notion). He also asserts [22, 2.6 and sequel] that the coskeleta of $QS^0$ give rise to a spectral sequence the $E^2$-term of which is given by the homotopy groups of certain fibres and which converges to $A_*X$ for any space $X$. Here he thinks of $QS^0$ as the "coefficient ring" of AX, in analogy with the role of R in Definitions 11.4. Since this is at best only a metaphor, rigor seems still further away. The infinite loop space splitting $Q(\Omega X^+) \simeq QS^0 \times Q\Omega X$ does not seem relevant. Even if they do exist, there seems to be little reason to believe that such spectral sequences would help much with explicit calculations.

Of course, it is conceivable that there is a simplicial analog of our theory for which this difficulty disappears, but I am skeptical (and certain that other technical difficulties would appear in any such approach).

It is time to discuss the main issue. Waldhausen proposed our AX as a nice description of what he wanted, if it were to exist, but he gave an alternative definition in terms of which the proofs were all to proceed. We write WX for Waldhausen's functor (or rather its connected version). If GSX is the simplicial group of (11.3), then WX is the plus construction on the classifying space of the colimit over n and k of certain categories $(h\mathcal{S}GSX)^n_k$ with objects simplicial GSX-sets suitably related to the wedge of X and k copies of $S^n$. In the absence of any indications of proof, I for one find it hard to see how analogs for WX of some of the results above for AX are to be made rigorous from this definition. The technical details, for example of the rational equivalence required for Corollary 12.3, must surely be considerable. It would seem preferable to compare AX and WX. Waldhausen asserts (without proof, [22, 2.1]) that the loop of the classifying space of the colimit over n of the categories $(h\mathcal{S}GSX)^n_k$ is equivalent to $FM_kQ(|GSX|^+)$. While this certainly seems plausible, his further claim that the equivalence is one of H-spaces seems much more difficult, and this in turn is nowhere near strong enough to prove the following assertion.

Conjecture 12.5. AX and WX are naturally equivalent.

Except that the definitions of AX and WX seem farther apart, one might view this as analogous to the equivalence between his two definitions that was the pivotal result in Quillen's development of algebraic K-theory [5]. The point is that it is AX which is most naturally connected with Quillen's algebraic K-theory, but it is $WX$ and its various equivalents in [22] which Waldhausen's arguments relate to the Whitehead groups for stable PL concordance.

## Appendix. Stabilizations of functors to homology theories

We first give the technical lemma needed to complete the proof of Theorem 11.3 and then give a very general theorem (presumably part of the folklore) on the stabilization of homotopy functors to generalized homology theories.

We work in the category $\mathcal{V}$ of nondegenerately based spaces of the homotopy type of a CW-complex and in its homotopy category $h\mathcal{V}$. The proofs below use well-known facts about fibration sequences but, annoyingly, I know of no published source which contains everything we need; such details will appear in [15, I §1].

Lemma A.1. Consider the following diagram in $\mathcal{V}$, in which i and p are written generically for the canonical maps of fibration sequences, the solid arrow parts of the diagram homotopy commute, the bottom squares with solid vertical arrows erased also homotopy commute, and the dotted arrows $\zeta$, $\eta$, and $\theta$ are homotopy sections ($\varepsilon\theta \simeq 1$, etc.).

$$\begin{tikzcd}
\Omega F\varepsilon \arrow[rr, dashed, "\iota"] \arrow[dd, shift right, "\Omega p"'] \arrow[drrr, dashed, "i"] & & F\gamma \arrow[rr, dashed, "\pi"] \arrow[dr, dashed, "\alpha"] \arrow[dd, "p"'] & & F\delta \arrow[rr, dashed, "e"] \arrow[dd, "p"] & & F\varepsilon \arrow[dd, "p"] \\
& & & Fe \arrow[ur, dashed, "p"'] \arrow[dl, dashed, "\beta"] & & & \\
\Omega Y \arrow[rr, "i"] \arrow[uu, dashed, shift right, "\psi"'] \arrow[d, shift right, "\Omega\varepsilon"'] & & Ff \arrow[rr, "p"] \arrow[d, shift right, "\gamma"'] & & X \arrow[rr, "f"] \arrow[d, shift right, "\delta"'] & & Y \arrow[d, shift right, "\varepsilon"'] \\
\Omega W \arrow[rr, "i"] \arrow[u, dashed, shift right, "\Omega\theta"'] & & Fg \arrow[rr, "p"] \arrow[u, dashed, shift right, "\zeta"'] & & Z \arrow[rr, "g"] \arrow[u, dashed, shift right, "\eta"'] & & W \arrow[u, dashed, shift right, "\theta"']
\end{tikzcd}$$

There exist maps $e$, $\pi$, and $\iota$, unique up to homotopy, such that the top three squares homotopy commute and there exists an equivalence $\alpha : F\gamma \to Fe$ such that $p\alpha \simeq \pi$.

Proof. The homotopy commutativity of the lower three squares implies the existence of $e$, $\pi$ and $\iota$. Here $e$ is unique since two such maps differ by the action of a map $F\delta \to \Omega W$ and the action of $[F\delta, \Omega W]$ on $[F\delta, F\varepsilon]$ is trivial since

$i: \Omega W \to F\mathcal{E}$ is null homotopic (because of the section). Similarly $\pi$ and $\iota$ are unique. Since $pe\pi$ is null homotopic by the diagram, $pe$ is null homotopic (by the exact sequence of the right column). Thus there exists $\alpha': F\gamma \to Fe$ such that $p\alpha' \simeq \pi$. The top row clearly induces a long exact sequence of homotopy groups mapping onto a direct summand of that of the middle row, and the desired conclusion that $\alpha'$ is an equivalence would be immediate from the five lemma if $\alpha'\iota$ were homotopic to $i$. However, there seems to be no reason to suppose that $\alpha'$ can be so chosen. Choose a map $\beta: Fe \to Ff$ such that $\beta i \simeq i\Omega p$ and $pp \simeq p\beta$. Again, $\alpha'$ would be an equivalence if $\beta\alpha'$ were homotopic to $p: F\gamma \to Ff$. Here we have more room for maneuver. Since $\Omega Y$ is an H-space, the sum of $\Omega p$ and $\Omega\theta$ is an equivalence $\Omega F\mathcal{E} \times \Omega W \to \Omega Y$. Let $\psi: \Omega Y \to \Omega F\mathcal{E}$ be an inverse projection. Certainly $\beta \circ \alpha' = p\mu$ for some $\mu: F\gamma \to \Omega Y$, where $p\mu$ is given by the right action of $[F\gamma, \Omega Y]$ on $[F\gamma, Ff]$ coming from $Ff \times \Omega Y \to Ff$. Let $\alpha = \alpha'\psi_*(-\mu)$. Then

$$\beta \circ \alpha = \beta \circ (\alpha'\psi_*(-\mu)) = (\beta \circ \alpha')(\Omega p)_*\psi_*(-\mu) = p(\mu(\Omega p)_*\psi_*(-\mu)) .$$

Since $\psi_*(\mu(\Omega p)_*\psi_*(-\mu)) = \psi_*(\mu) - \psi_*(\mu) = 0$, we have

$$\mu(\Omega p)_*\psi_*(-\mu) = (\Omega\theta)_*(\nu)$$

for some $\nu \in [F\gamma, \Omega W]$ and thus $\beta \circ \alpha = p(\Omega\theta)_*(\nu)$. It follows that $\beta_* \circ \alpha_*$ and $p_*: \pi_* F\gamma \to \pi_* Ff$ become equal when one passes from the long exact homotopy sequence of the middle row to its quotient by the long exact homotopy sequence of the bottom row. Here $p_*$ and $\beta_*$ are isomorphisms, and we conclude that $\alpha_*$ is an isomorphism and thus that $\alpha$ is an equivalence.

The naturality of $\pi$ in Theorem 11.3 follows from the argument used to prove the uniqueness of $e$ in the lemma.

Turning to the desired construction of homology theories, we define a stability sequence $\{a_n\}$ to be a strictly increasing sequence of positive integers $a_n$ such that $a_n - n$ tends to infinity.

Theorem A.2. Let $k: h\mathcal{V} \to h\mathcal{V}$ be a functor with the following properties.

(1) Application of k to an n-equivalence yields a $b_n$-equivalence, where $\{2b_n\}$ is a stability sequence.

(2) Application of k to a strictly commutative 2n-homotopy Cartesian (n, n)-equivalence yields an $a_n$-homotopy Cartesian square, where $\{a_n\}$ is a stability sequence.

Let $\varepsilon: kX \to k\{*\}$ be induced by $X \to \{*\}$ and let $\tilde{k}X$ be the fibre of $\varepsilon$. Then there exist natural maps $\sigma: \tilde{k}X \to \Omega\tilde{k}\Sigma X$ such that if $\tilde{k}^s X$ is defined to be the telescope of the spaces $\Omega^n\tilde{k}\Sigma^n X$ with respect to the maps $\Omega^n\sigma$, $\sigma: \tilde{k}\Sigma^n X \to \Omega\tilde{k}\Sigma^{n+1}X$, and if $\tilde{k}^s_q X$ is defined to be $\pi_q\tilde{k}^s X$, then $\tilde{k}^s_*$ is a reduced homology theory which satisfies the wedge axiom.

The following is a standard consequence.

Corollary A.3. On unbased spaces X, define $k^s_* X = \tilde{k}^s_*(X^+)$. On unbased pairs (X, A), define $k^s_*(X, A) = \tilde{k}^s_*((X \cup CA)^+)$ where CA is the (unreduced) cone on A. Then $k^s_*$ is a generalized homology theory in the classical sense.

Returning to based spaces, we first discuss the statement of the theorem. It will turn out that property (1) is only needed for the wedge axiom, and then only for maps $X \to \{*\}$, hence may be omitted in obtaining a homology theory on finite complexes. Property (2) will also only be needed for a few simple types of diagrams, to be displayed in the proof. Since $\varepsilon$ is only given as a homotopy class of maps, we must choose a representative before constructing $\tilde{k}X$. The first part of the proof of Lemma A.1 gives the following result.

Lemma A.4. For $f: X \to Y$, there is a unique homotopy class $\tilde{k}f: \tilde{k}X \to \tilde{k}Y$ such that the following diagram commutes in $h\mathcal{V}$.

$$\begin{tikzcd} \tilde{k}X \arrow[r, "\tilde{k}f"] \arrow[d] & \tilde{k}Y \arrow[d] \\ kX \arrow[r, "kf"] & kY \end{tikzcd} .$$

It follows that $\tilde{k}$ is a well-defined functor $h\mathcal{V} \to h\mathcal{V}$ such that $\tilde{k} \to k$ is natural. We also need the following analog.

Lemma A.5. Let $\eta: k\{*\} \to kX$ be induced by $\{*\} \to X$ and let $\hat{k}X$ be the fibre of $\eta$. For $f: X \to Y$, there is a unique homotopy class $\hat{k}f: \hat{k}X \to \hat{k}Y$ such that the following diagram commutes in $h\mathcal{V}$.

$$\begin{tikzcd} \Omega kX \arrow[r, "\Omega kf"] \arrow[d] & \Omega kY \arrow[d] \\ \hat{k}X \arrow[r, "\hat{k}f"] & \hat{k}Y \end{tikzcd} .$$

Proof. The map $\Omega kX \to \Omega k\{*\} \times \hat{k}X$ with first coordinate $\Omega\varepsilon$ and second coordinate the canonical map is an equivalence.

It follows that $\hat{k}$ is a functor and $\Omega k \to \hat{k}$ is natural.

Lemma A.6. The composite $\Omega\tilde{k}X \to \Omega kX \to \hat{k}X$ is a natural equivalence.

Proof. The map $\Omega k\{*\} \times \Omega\tilde{k}X \to \Omega kX$ given by the sum of $\Omega\eta$ and the canonical map is also an equivalence.

These observations suffice for the construction of $\sigma$.

Lemma A.7. There is a natural map $\sigma: \tilde{k}X \to \Omega\tilde{k}\Sigma X$ such that $\sigma$ is an $(a_{n+1} - 1)$-equivalence if X is n-connected.

Proof. We define $\sigma$ to be the top composite in the diagram

$$\begin{tikzcd} \tilde{k}X \arrow[r, dashed] \arrow[d] & \hat{k}\Sigma X \arrow[r, "\simeq"] \arrow[d] & \Omega\tilde{k}\Sigma X \\ kX \arrow[r, "\varepsilon"] \arrow[d, "\varepsilon"'] & k\{*\} \arrow[d, "\eta"] & \\ k\{*\} \arrow[r, "\eta"] & k\Sigma X & \end{tikzcd}$$

Here we have the tautological strict equality $\eta\varepsilon = \eta\varepsilon$, and the dotted arrow is canonical; its naturality up to homotopy is easily checked by direct inspection. If X is n-connected, then the commutative square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \Sigma X \end{tikzcd}$$

is a (2n+2)-homotopy Cartesian (n+1, n+1)-equivalence by the homotopy excision theorem (compare Definition 9.2). By a mild interpretation, property (2) implies the result.

The spaces $\tilde{k}^s X$ of Theorem A.2 are now defined. Since $\Omega$ commutes with telescopes, there is an evident homotopy equivalence

$$\tau: \tilde{k}^s X = \operatorname*{Tel}_{n\geq 0} \Omega^n \tilde{k} \Sigma^n X \rightarrow \operatorname*{Tel}_{n\geq 0} \Omega^{n+1} \tilde{k} \Sigma^{n+1} X = \Omega \tilde{k}^s \Sigma X .$$

While $\tilde{k}^s$ need not be a functor and $\tau$ need not be natural, since $\lim^1$ terms might well be present, they induce functors $\tilde{k}^s_q X$ and natural isomorphisms $\tau_q : \tilde{k}^s_q X \rightarrow \tilde{k}^s_{q+1} \Sigma X$ on passage to homotopy groups. Alternatively, with $\tilde{k}_q X = \pi_q \tilde{k} X$, we could equally well define

$$\tilde{k}^s_q X = \operatorname{colim} \tilde{k}_{q+n} \Sigma^n X$$

and not bother with the telescopes, the isomorphisms $\tau_q$ then being evident.

For reduced homology theories, excision reduces to the suspension axiom just verified on trivial formal grounds, without use of properties (1) and (2). The things to be proven are exactness and the wedge axiom. The following lemma verifies the appropriate exactness axiom.

<u>Lemma A.8.</u> If $\iota: A \rightarrow X$ is a cofibration, then the sequence

$$k^s_q A \xrightarrow{k^s_q \iota} k^s_q X \xrightarrow{k^s_q \pi} k^s_q (X/A)$$

is exact for all q, where $\pi$ is the quotient map.

<u>Proof.</u> Since the functors $\Sigma^n$ commute with cofibration sequences, a glance at the relevant colimit systems shows that it suffices to prove

$$\tilde{k}_q A \xrightarrow{\tilde{k}_q \iota} \tilde{k}_q X \xrightarrow{\tilde{k}_q \pi} \tilde{k}_q (X/A)$$

to be exact in a suitable range when $A, X$, and $X/A$ are n-connected. By the homotopy excision theorem again, the square

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d] & X \arrow[d, "\pi"] \\ \{*\} \arrow[r] & X/A \end{tikzcd}$$

is a 2n-homotopy Cartesian $(n, n)$-equivalence. Consider the following diagram

$$\begin{tikzcd} \tilde{k}A \arrow[r, dashed] \arrow[d, "kA"'] & Fk\pi \arrow[r, dashed, "\simeq"] \arrow[d] & F\tilde{k}\pi \arrow[d] \\ kA \arrow[r, "k\iota"] \arrow[d, "\varepsilon"'] & kX \arrow[d, "k\pi"] & \tilde{k}X \arrow[l] \arrow[d, "\tilde{k}\pi"] \\ k\{*\} \arrow[r, "\eta"] & k(X/A) & \tilde{k}(X/A) \arrow[l] \end{tikzcd}$$

Verdier's axiom for fibration sequences applied to the triangle

$$\begin{tikzcd} kX \arrow[rr, "\varepsilon"] \arrow[dr, "k\pi"'] & & k\{*\} \\ & k(X/A) \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

gives an equivalence $Fk\pi \to F\tilde{k}\pi$ such that the upper right square homotopy commutes. Property (2) and Remarks 9.3 give an $(a_n+1)$-equivalence $\tilde{k}A \to Fk\pi$ such that the upper left square homotopy commutes. By Lemma A.4, the composite $\tilde{k}A \to \tilde{k}X$ in the diagram is $\tilde{k}\iota$. The conclusion follows from the long exact sequence of homotopy groups of the right column.

It remains only to verify the wedge axiom.

Lemma A.9. For any set of spaces $\{X_i\}$, the natural map

$$\bigoplus_i \tilde{k}^s_q X_i \to \tilde{k}^s_q \left(\bigvee_i X_i\right)$$

is an isomorphism for all $q$.

Proof. Since the functors $\Sigma^n$ commute with wedges, a glance at the relevant colimit systems shows that it suffices to prove

$$\bigoplus_i \widetilde{k}_q(X_i) \to \widetilde{k}_q(\bigvee_i X_i)$$

to be an isomorphism in a suitable range when each $X_i$ is n-connected. If $X$ is n-connected, then $X \to \{*\}$ is an $(n+1)$-equivalence, hence $\varepsilon: kX \to k\{*\}$ is a $b_{n+1}$-equivalence by property (1), hence $\widetilde{k}X$ is $(b_{n+1}-1)$-connected. Therefore the inclusion of $\bigvee_i \widetilde{k}X_i$ in the weak direct product of the $kX_i$ (all but finitely many coordinates at the basepoint) is a $(2b_{n+1}-1)$-equivalence, and the conclusion follows.

## Bibliography


1. A. Borel. Stable real cohomology of arithmetic groups. Ann. Sci. Ecole Normale Sup. $4^e$ serie t. 7 (1974), 235-272.

2. F. Cohen, T. Lada, and J. P. May. The Homology of Iterated Loop Spaces. Springer Lecture Notes in Mathematics, Vol. 533, 1976.

3. A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische produkte. Annals of Math. 67 (1958), 239-281.

4. F. T. Farrell and W. C. Hsiang. Proc. Amer. Math. Soc. Summer Institute. Stanford, 1976. To appear.

5. D. Grayson. Higher algebraic K-theory: II (after D. Quillen). Springer Lecture Notes in Mathematics. Vol. 551, 216-240, 1976.

6. T. Lada. An operad action on infinite loop space multiplication. Canadian J. Math. To appear.

7. J. L. Loday. Les matrices monomiales et le groupe de Whitehead $Wh_2$. Springer Lecture Notes in Mathematics, Vol. 551, 155-163, 1976.

8. J. P. May. Simplicial Objects in Algebraic Topology. D. van Nostrand, 1967.

9. J. P. May. Classifying Spaces and Fibrations. Memoirs Amer. Math. Soc. 155, 1975.

10. J. P. May. The Geometry of Iterated Loop Spaces. Springer Lecture Notes in Mathematics, Vol. 271, 1972.

11. J. P. May. $E_\infty$ spaces, group completions, and permutative categories. London Math. Soc. Lecture Note Series 11, 61-94, 1974.

12. J. P. May (with contributions by N. Ray, F. Quinn, and J. Tornehave). $E_\infty$ Ring Spaces and $E_\infty$ Ring Spectra. Springer Lecture Notes in Mathematics, Vol. 577, 1977.

13. J. P. May. Infinite loop space theory. Bull. Amer. Math. Soc. 83(1977), 456-494.

14. J. P. May. The spectra associated to permutative categories. To appear.

15. J. P. May. The Homotopical Foundations of Algebraic Topology. Academic Press. In preparation.

16. J.P. May and R. Thomason. The uniqueness of infinite loop space machines. To appear.

17. R.J. Milgram. The bar construction and Abelian H-spaces. Illinois J. Math. 11(1957), 242-250.

18. D. Quillen. Higher algebraic K-theory I. Springer Lecture Notes in Mathematics, Vol. 341, 85-147, 1973.

19. D. Quillen. Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i 0 \xrightarrow{J} \pi_i^s \longrightarrow K_i Z)$. Springer Lecture Notes in Mathematics, Vol. 551, 182-188, 1976.

20. G. Segal. Categories and cohomology theories. Topology 13(1974), 293-312.

21. J.B. Wagoner. Delooping classifying spaces in algebraic K-theory. Topology 11(1972), 349-370.

22. F. Waldhausen. Algebraic K-theory of topological spaces, I. Proc. Amer. Math. Soc. Summer Institute. Stanford, 1976. To appear.