Elliptic Cohomology I: Spectral Abelian Varieties

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Introduction

Elliptic cohomology studies a special class of cohomology theories which are "associated" to elliptic curves, in the following sense:

Definition 0.0.1. An *elliptic cohomology theory* is a triple (A, E, α), where A is an even periodic cohomology theory, E is an elliptic curve over the commutative ring $R = A^0(*)$, and α is an isomorphism of $\operatorname{Spf} A^0(\mathbf{CP}^\infty)$ with the formal completion $\widehat{E}$ of E (in the category of formal groups over R).

In [5], we proposed that many aspects of the theory of elliptic cohomology can be elucidated by considering a refinement of Definition 0.0.1, where the elliptic curve E and the isomorphism α are required to be defined over the cohomology theory A, rather than over the commutative ring R. This is the first in a series of papers whose ultimate aim is to carry out the details of the program outlined in [5]. The more modest goal of the present paper is to answer the following:

Question 0.0.2. What does it mean to give an elliptic curve (or, more generally, an abelian variety) over a cohomology theory A? To what extent do such objects behave like their counterparts in classical algebraic geometry?

Let us now outline the contents of this paper. We begin in §1 by addressing the first half of Question 0.0.2. For this, we will assume that the cohomology theory A is representable by an $\mathbb{E}_\infty$-ring (which, by abuse of notation, we will also denote by A). The language of spectral algebraic geometry (developed extensively in [8]) then allows us to "do" algebraic geometry over A, much as the language of schemes allows one to "do" algebraic geometry over an arbitrary commutative ring. Using this language, we introduce the notion of *abelian variety* over A (Definition 1.4.1) and the closely related notion of *strict abelian variety* (Definition 1.5.1). Specializing to the case of abelian varieties having relative dimension 1, we obtain a notion of *elliptic curve* (and *strict elliptic curve*) over A.

In the setting of classical algebraic geometry, elliptic curves themselves admit an algebro-geometric parametrization. More precisely, there exists a Deligne-Mumford stack $\mathcal{M}_{1,1}$ called the *moduli stack of elliptic curves* such that, for any commutative ring R, the groupoid of maps $\operatorname{Sp\acute{e}t} R \to \mathcal{M}_{1,1}$ can be identified with the groupoid of elliptic curves over R. In §2, we show that the same phenomenon occurs in spectral algebraic geometry: there exist spectral Deligne-Mumford stacks $\mathcal{M}$ and $\mathcal{M}^s$ which classify elliptic curves and strict elliptic curves (respectively) over arbitrary $\mathbb{E}_\infty$-rings

(Theorem 2.0.3). In contrast with the classical case, there is no obvious "hands-on" construction of these moduli stacks (via geometric invariant theory, for example); instead, we establish existence using a spectral version of Artin's representability theorem.

Let κ be a field and let X be an abelian variety over κ. One can then associate to X another abelian variety $\widehat{\mathsf{X}}$, called the *dual* of X, with the following features:

(i) The dual $\widehat{\mathsf{X}}$ can be identified with (or defined as) the identity component $\mathrm{Pic}^{\circ}(\mathsf{X})$ of the Picard variety $\mathrm{Pic}(\mathsf{X})$.

(ii) The construction $\mathsf{X} \mapsto \widehat{\mathsf{X}}$ is involutive: that is, we can identify X with the dual of $\widehat{\mathsf{X}}$.

(iii) The derived categories of coherent sheaves on X and $\widehat{\mathsf{X}}$ are canonically equivalent to one another by means of the *Fourier-Mukai transform* of [9].

Roughly half of this paper is devoted to generalizing this duality theory to abelian varieties over an arbitrary (connective) $\mathbb{E}_{\infty}$-ring A. In classical algebraic geometry, one can take (i) as the definition of the dual abelian variety $\widehat{\mathsf{X}}$, and (ii) and (iii) as theorems that can be deduced from this definition. In the setting of spectral algebraic geometry, assertion (i) fails: we can no longer identify the dual $\widehat{\mathsf{X}}$ of an abelian variety X with a connected component of $\mathrm{Pic}(\mathsf{X})$. We therefore adopt a different approach, using a categorified form of Cartier duality (developed in §3) to construct the ∞-category $\mathrm{QCoh}(\widehat{\mathsf{X}})$ of quasi-coherent sheaves on the dual $\widehat{\mathsf{X}}$, from which we can recover $\widehat{\mathsf{X}}$ using the formalism of Tannaka duality (as developed in [8]). From this perspective, it takes some work to show that the dual $\widehat{\mathsf{X}}$ actually exists (this is the subject of §5), but the counterparts of assertions (ii) and (iii) are more or less immediate from the definition (as we will see in §4).

Our final goal in this paper is to generalize the theory of p-divisible groups to the setting of spectral algebraic geometry. In §6, we define the notion of p-divisible group over an arbitrary $\mathbb{E}_{\infty}$-ring A (Definition 6.5.1). We show that every strict abelian variety X over A determines a p-divisible group $\mathsf{X}[p^{\infty}]$ (Proposition 6.7.1), and that this construction is compatible with duality (Proposition 6.8.2). In §7, we use these ideas to formulate and prove a "spectral" version of the classical Serre-Tate theorem: when working over $\mathbb{E}_{\infty}$-rings A which are p-complete, the deformation theory of a strict abelian variety X is equivalent to the deformation theory of its p-divisible group $\mathsf{X}[p^{\infty}]$ (Theorem 7.0.1).

Remark 0.0.3. To understand the construction of topological modular forms using the formalism of spectral algebraic geometry (as outlined in [5]), the duality theory of abelian varieties is not necessary: one only needs the ideas of §1, §2, §6, and §7 of the present paper. However, duality theory (at least for elliptic curves) plays a role in the construction of the string orientation of the theory of topological modular forms.

Remark 0.0.4. In many parts of this paper, we work in the setting of (spectral) algebraic geometry over an $\mathbb{E}_\infty$-ring A which might be nonconnective. However, this seeming generality is illusory: all of the geometric objects we study are required to be flat over A, so there is no difference between working over A or over its connective cover $\tau_{\geqslant 0} A$ (see Proposition 1.1.3). Nevertheless, it will be convenient to allow nonconnective ring spectra since we will later wish to consider *oriented* elliptic curves (see [5]), which exist only in the nonconnective setting.

Warning 0.0.5. The theory of topological modular forms can be understood as arising from a certain nonconnective spectral Deligne-Mumford stack $\mathcal{M}^{\mathrm{der}}$ which classifies *oriented* elliptic curves (see [5]). This object does *not* agree with the moduli stacks $\mathcal{M}$ and $\mathcal{M}^s$ constructed in §2, which parametrize elliptic curves without an orientation. However, the moduli stack $\mathcal{M}^s$ constructed here is a useful first approximation to the moduli stack $\mathcal{M}^{\mathrm{der}}$ of [5]. More details will be given in the sequel.

Notation and Terminology

Throughout this paper, we will assume that the reader is familiar with the language of higher category theory developed in [6] and [7], as well as the language of spectral algebraic geometry as developed in [8]. Since we will need to refer to these texts frequently, we adopt the following conventions:

(HTT) We will indicate references to [6] using the letters HTT.

(HA)   We will indicate references to [7] using the letters HA.

(SAG) We will indicate references to [8] using the letters SAG.

For example, Theorem HTT.6.1.0.6 refers to Theorem 6.1.0.6 of [6].

For the reader's convenience, we now review some cases in which the conventions of this this paper differ from those of the texts listed above, or from the established mathematical literature.

- We will generally not distinguish between a category $\mathcal{C}$ and its nerve $\mathrm{N}(\mathcal{C})$. In particular, we regard every category $\mathcal{C}$ as an ∞-category.

- We will generally abuse terminology by not distinguishing between an abelian group M and the associated Eilenberg-MacLane spectrum: that is, we view the ordinary category of abelian groups as a full subcategory of the ∞-category Sp of spectra. Similarly, we regard the ordinary category of commutative rings as a full subcategory of the ∞-category CAlg of $\mathbb{E}_\infty$-rings.

- Let A be an $\mathbb{E}_\infty$-ring. We will refer to A-module spectra simply as *A-modules*. The collection of A-modules can be organized into a stable ∞-category which we will denote by Mod_A and refer to as *the ∞-category of A-modules*. This convention has an unfortunate feature: when A is an ordinary commutative ring, it does not reduce to the usual notion of A-module. In this case, Mod_A is not the abelian category of A-modules but is closely related to it: the homotopy category hMod_A is equivalent to the derived category $D(A)$. *Unless otherwise specified, the term "A-module" will be used to refer to an object of Mod_A, even when A is an ordinary commutative ring.* When we wish to consider an A-module M in the usual sense, we will say that M is a *discrete $A*-module or an *ordinary A-module.

- Unless otherwise specified, all algebraic constructions we consider in this book should be understood in the "derived" sense. For example, if we are given discrete modules M and N over a commutative ring A, then the tensor product $M \otimes_A N$ denotes the *derived* tensor product $M \otimes_A^L N$. This may not be a discrete A-module: its homotopy groups are given by $\pi_n(M \otimes_A N) \simeq \mathrm{Tor}_n^A(M, N)$. When we wish to consider the usual tensor product of M with N over A, we will denote it by $\mathrm{Tor}_0^A(M, N)$ or by $\pi_0(M \otimes_A N)$.

- If M and N are spectra, we will denote the smash product of M with N by $M \otimes N$, rather than $M \wedge N$. More generally, if M and N are modules over an $\mathbb{E}_\infty$-ring A, then we will denote the smash product of M with N over A by $M \otimes_A N$, rather than $M \wedge_A N$. Note that when A is an ordinary commutative ring and the modules M and N are discrete, this agrees with the preceding convention.

- If $\mathcal{C}$ is an ∞-category, we let $\mathcal{C}^\simeq$ denote the largest Kan complex contained in $\mathcal{C}$: that is, the ∞-category obtained from $\mathcal{C}$ by discarding all non-invertible

morphisms.

- We will say that a functor $f : \mathcal{C} \to \mathcal{D}$ between ∞-categories is *left cofinal* if, for every object $D \in \mathcal{D}$, the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{D/}$ is weakly contractible (this differs from the convention of [6], which refers to a functor with this property simply as *cofinal*; see Theorem HTT.4.1.3.1). We will say that f is *right cofinal* if the induced map $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is left cofinal, so that f is right cofinal if and only if the ∞-category $\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/D}$ is weakly contractible for each $D \in \mathcal{D}$.

- If A is an $\mathbb{E}_\infty$-ring, we denote the associated affine nonconnective spectral Deligne-Mumford stack by $\mathrm{Sp\acute{e}t}\, A$. If A is an ordinary commutative ring, we let $\mathrm{Spec}\, A$ denote the associated affine scheme. Note that in this case, $\mathrm{Spec}\, A$ and $\mathrm{Sp\acute{e}t}\, A$ are essentially "the same" object, and we will often abuse notation by identifying them with one another.

Divergence with the Classical Theory

Roughly speaking, our objective in this paper is to define the notion of abelian variety over an arbitrary $\mathbb{E}_\infty$-ring A and to show that, in many respects, the theory of abelian varieties over A resembles the classical theory of abelian varieties. However, there are several respects in which the theory we develop here does *not* mirror its classical counterpart. For the reader's convenience, we collect here some of the most important:

- In the setting of classical algebraic geometry, the group structure on an abelian variety is essentially unique. If X is a smooth projective algebraic variety over a field κ equipped with a "zero-section" $e : \mathrm{Spec}\, \kappa \to \mathsf{X}$, then there is at most one multiplication map $m : \mathsf{X} \times_{\mathrm{Spec}\,\kappa} \mathsf{X} \to \mathsf{X}$ having e as a (two-sided) identity. Moreover, if such a multiplication exists, then it is automatically commutative and associative.

  In the setting of spectral algebraic geometry, the analogous statement fails: if X is an abelian variety over an $\mathbb{E}_\infty$-ring A, then we generally cannot recover the group structure on X from the underlying map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$, even if the identity section $e : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ is specified. Moreover, commutativity is not automatic (and different commutativity assumptions lead to slightly different theories: this is what distinguishes the *abelian varieties* of Definition 1.4.1 from the *strict abelian varieties* of Definition 1.5.1).

- In classical algebraic geometry, the theory of algebraic curves provides a rich supply of examples of abelian varieties: every smooth projective curve of genus g $\mathsf{X} \to \operatorname{Spec} \kappa$ admits a Jacobian $\operatorname{Pic}^{\circ}(\mathsf{X})$, which is an abelian variety of dimension g over κ. In the setting of spectral algebraic geometry over an $\mathbb{E}_\infty$-ring A, the analogous construction generally does *not* yield an abelian variety over A, unless A is assumed to be of characteristic zero (if A is connective, it yields an object which is differentially smooth over A, rather than fiber smooth).

- In classical algebraic geometry, the dual of an abelian variety X can be identified with a connected component of the Picard variety $\operatorname{Pic}(\mathsf{X})$. In the setting of spectral algebraic geometry, this is almost never true (except for elliptic curves in characteristic zero). However, it is still true that the dual of X can be identified with the moduli space of *multiplicative* line bundles on X (that is, extensions of X by the multiplicative group): see §5 for details.

- In classical algebraic geometry, the theory of line bundles is controlled by the *theorem of the cube*: if X, Y, Z are smooth projective varieties over a field κ equipped with base points $x \in \mathsf{X}(\kappa)$, $y \in \mathsf{Y}(\kappa)$, and $z \in \mathsf{Z}(\kappa)$, and $\mathscr{L}$ is a line bundle on the product $\mathsf{X} \times \mathsf{Y} \times \mathsf{Z}$ which is trivial when restricted to $\{x\} \times \mathsf{Y} \times \mathsf{Z}$, $\mathsf{X} \times \{y\} \times \mathsf{Z}$, and $\mathsf{X} \times \mathsf{Y} \times \{z\}$, then $\mathscr{L}$ itself is trivial. This result plays an essential role in the analysis of line bundles on an abelian variety X (and in the construction of the dual abelian variety $\widehat{\mathsf{X}}$). However, it has no counterpart in the setting of spectral algebraic geometry.

Acknowledgements

1 Abelian Varieties in Spectral Algebraic Geometry

Let κ be a field. Recall that an *abelian variety* over κ is a commutative group scheme X over κ for which the map $\mathsf{X} \to \kappa$ is smooth, proper, and geometrically connected. Our goal in this section is to introduce a generalization of this definition,

where we replace the field κ by an arbitrary $\mathbb{E}_\infty$-ring R. We will give the definition in two steps:

- We first introduce the notion of a *variety* over R, given by a morphism of (possibly nonconnective) spectral Deligne-Mumford stacks $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ satisfying a few requirements (Definition 1.1.1). The collection of varieties over R can be organized into an ∞-category $\mathrm{Var}(R)$ which admits finite products.

- Roughly speaking, we would like to define an *abelian variety* over R to be an object $\mathsf{X} \in \mathrm{Var}(R)$ equipped with a commutative group structure. However, in the setting of homotopy theory, commutativity comes in a variety of flavors. For the applications we have in mind, two of these will be relevant:

  (a) We could take X to be a *grouplike commutative monoid* object of $\mathrm{Var}(R)$ (see §1.3): in other words, we could require that for every $A \in \mathrm{CAlg}_R$, the space $\mathsf{X}(A)$ of A-valued points of X has the structure of an infinite loop space. This leads to the notion we call an *abelian variety over R*, which we will study in §1.4).

  (b) We could take X to be an *abelian group object* of $\mathrm{Var}(R)$ (see §1.2): in other words, we could require that for every $A \in \mathrm{CAlg}_R$, the space $\mathsf{X}(A)$ of A-valued points of X is homotopy equivalent to a topological abelian group. This leads to the notion we call a *strict abelian variety over R*, which we study in §1.5.

1.1 Varieties over $\mathbb{E}_\infty$-Rings

We begin by reviewing some terminology. We will use the following slight generalization of Definition SAG.19.4.5.3:

Definition 1.1.1. Let R be an $\mathbb{E}_\infty$-ring. A *variety over R* is a nonconnective spectral Deligne-Mumford stack $\mathsf{X} = (\mathcal{X}, \mathcal{O}_\mathsf{X})$ equipped with a flat map $f : \mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ having the following property: the underlying map of spectral Deligne-Mumford stacks $(\mathcal{X}, \tau_{\geqslant 0}\, \mathcal{O}_\mathsf{X}) \to \mathrm{Sp\acute{e}t}(\tau_{\geqslant 0} R)$ is proper, locally almost of finite presentation, geometrically reduced, and geometrically connected. We let $\mathrm{Var}(R)$ denote the full subcategory of $\mathrm{SpDM}^{\mathrm{nc}}_{/\mathrm{Sp\acute{e}t}\, R}$ spanned by the varieties over R.

Remark 1.1.2. Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks

$$\begin{array}{ccc} \mathsf{X}' & \longrightarrow & \mathsf{X} \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ \mathrm{Sp\acute{e}t}\, R' & \longrightarrow & \mathrm{Sp\acute{e}t}\, R. \end{array}$$

If f exhibits X as a variety over R, then f' exhibits X' as a variety over R'. Consequently, we can view the construction $R \mapsto \mathrm{Var}(R)$ as a functor $\mathrm{Var} : \mathrm{CAlg} \to \mathcal{S}$.

When the $\mathbb{E}_\infty$-ring R is connective, Definition 1.1.1 coincides with the definition of variety given in §SAG.19.4.5 . The extension of this definition to nonconnective $\mathbb{E}_\infty$-rings does not really provide any additional generality, by virtue of the following:

Proposition 1.1.3. *For every $\mathbb{E}_\infty$-ring R, the canonical map $\tau_{\geqslant 0} R \to R$ induces an equivalence of ∞-categories $\rho : \mathrm{Var}(\tau_{\geqslant 0} R) \to \mathrm{Var}(R)$.*

Proof. The construction $(\mathcal{X}, \mathscr{O}_\mathsf{X}) \mapsto (\mathcal{X}, \tau_{\geqslant 0}\, \mathscr{O}_\mathsf{X})$ determines a homotopy inverse to ρ. See Proposition SAG.2.8.2.10 . $\qquad\square$

Proposition 1.1.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism in $\mathrm{Var}(R)$. Suppose that, for every field κ and every morphism of $\mathbb{E}_\infty$-rings $R \to \kappa$, the induced map $\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$ is an equivalence. Then f is an equivalence.*

Proof. Apply Corollary SAG.6.1.4.12 . $\qquad\square$

Remark 1.1.5 (The Functor of Points)**.** Let R be an $\mathbb{E}_\infty$-ring. The construction $\mathsf{X} \mapsto \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}_{/\mathrm{Sp\acute{e}t}\, R}}(\mathrm{Sp\acute{e}t}\, \bullet, \mathsf{X})$ determines a fully faithful embedding of ∞-categories $\mathrm{Var}(R) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$. We will refer to the image of X under this functor as the *functor of points* of X.

If the $\mathbb{E}_\infty$-ring R is connective and n-truncated (for $0 \leqslant n \leqslant \infty$), then the same construction determines a fully faithful embedding $\mathrm{Var}(R) \hookrightarrow \mathrm{Fun}(\tau_{\leqslant n}\, \mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$, which we will also refer to as the *functor of points*.

1.2 Abelian Group Objects of ∞-Categories

We begin with some general remarks about abelian group objects of ∞-categories.

Definition 1.2.1. A *lattice* is a free abelian group of finite rank. We let Ab denote the category of abelian groups, and Lat denote the full subcategory of Ab spanned by the lattices. We will refer to Lat as the *category of lattices*.

Remark 1.2.2. The category of lattices Lat is essentially small (it is equivalent to the full subcategory of the category of abelian groups spanned by the objects $\{\mathbf{Z}^n\}_{n \geqslant 0}$).

Remark 1.2.3. The category of lattices Lat is additive and admits finite product and coproducts.

Definition 1.2.4. Let $\mathcal{C}$ be an ∞-category which admits finite products. An *abelian group object* of $\mathcal{C}$ is a functor $A : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}$ which commutes with finite products. We let $\mathrm{Ab}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}}, \mathcal{C})$ spanned by the abelian group objects of $\mathcal{C}$. We will refer to $\mathrm{Ab}(\mathcal{C})$ as the ∞-*category of abelian group objects of $\mathcal{C}$*.

Example 1.2.5. Let M be an abelian group. We let $h_M : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{S}\mathrm{et}$ denote the functor represented by M, given by $h_M(\Lambda) = \mathrm{Hom}_{\mathrm{Ab}}(\Lambda, M)$. The functor h_M commutes with finite products, and is therefore an abelian group object of $\mathcal{S}\mathrm{et}$ in the sense of Definition 1.2.4.

Remark 1.2.6 (Duality)**.** The category Lat of lattices is canonically equivalent to its opposite $\mathrm{Lat}^{\mathrm{op}}$, via the duality functor $\Lambda \mapsto \Lambda^{\vee} = \mathrm{Hom}(\Lambda, \mathbf{Z})$. Consequently, if $\mathcal{C}$ is an ∞-category and $X : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}$ is an abelian group object of $\mathcal{C}$, then the construction $\Lambda \mapsto X(\Lambda^{\vee})$ determines a functor $X^{\vee} : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. If the ∞-category $\mathcal{C}$ is semiadditive (see Definition SAG.C.4.1.6), then $X^{\vee}$ is an abelian group object of $\mathcal{C}^{\mathrm{op}}$. In this case, the construction $X \mapsto X^{\vee}$ induces an equivalence of ∞-categories $\mathrm{Ab}(\mathcal{C})^{\mathrm{op}} \simeq \mathrm{Ab}(\mathcal{C}^{\mathrm{op}})$.

Proposition 1.2.7. *The construction $M \mapsto h_M$ of Example 1.2.5 induces an equivalence of categories* $\mathrm{Ab} \to \mathrm{Ab}(\mathcal{S}\mathrm{et})$.

Proof. Let $A \in \mathrm{Ab}(\mathcal{S}\mathrm{et}) \subseteq \mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}}, \mathcal{S}\mathrm{et})$. Then the set $A(\mathbf{Z})$ has the structure of an abelian group, with multiplication induced by the map

$$A(\mathbf{Z}) \times A(\mathbf{Z}) \simeq A(\mathbf{Z} \times \mathbf{Z}) \xrightarrow{A(\delta)} A(\mathbf{Z})$$

where $\delta : \mathbf{Z} \to \mathbf{Z} \times \mathbf{Z}$ is the diagonal map. It is easy to check that the construction $A \mapsto A(\mathbf{Z})$ is a homotopy inverse to the functor $M \mapsto h_M$. $\square$

Remark 1.2.8. Proposition 1.2.7 is a formal consequence of the fact that lattices form compact generators for the category Ab.

Example 1.2.9. Let $\mathcal{S}$ denote the ∞-category of spaces. Then $\mathrm{Ab}(\mathcal{S})$ can be identified with the ∞-category $\mathcal{P}_{\Sigma}(\mathrm{Lat})$ (see §HTT.5.5.8), obtained by freely adjoining sifted colimits to the ∞-category Lat. Combining Propositions HTT.5.5.9.2 and 1.2.7, we deduce that $\mathrm{Ab}(\mathcal{S})$ is equivalent to the underlying ∞-category of the model category $\mathrm{Ab}^{\mathbf{\Delta}^{\mathrm{op}}}$ of simplicial abelian groups.

Remark 1.2.10. Let $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ denote the ∞-category of connective $\mathbf{Z}$-module spectra. We will abuse notation by identifying the category Ab of abelian groups with the full subcategory $\mathrm{Mod}_{\mathbf{Z}}^{\heartsuit} \subseteq \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ spanned by the discrete objects. In particular, we can identify Lat with a full subcategory of $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Invoking the universal property of Example 1.2.9, we see that the inclusion Lat $\hookrightarrow \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ admits an essentially unique extension to a functor $\rho : \mathrm{Ab}(\mathcal{S}) \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ which commutes with sifted colimits. By virtue of Proposition HA.7.1.1.15, the functor ρ is an equivalence of ∞-categories. That is, we can identify connective $\mathbf{Z}$-module spectra with abelian group objects of the ∞-category $\mathcal{S}$ of spaces.

Remark 1.2.11. Let $\mathcal{C}$ be an essentially small ∞-category which admits finite products, and let $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the Yoneda embedding. Then j induces a fully faithful embedding $\mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}(\mathcal{S})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$. Unwinding the definitions, we see that the essential image of this embedding consists of those functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ for which the composition $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^{\infty}} \mathcal{S}$ is representable by an object of $\mathcal{C}$.

Example 1.2.12. Let $\mathcal{C}$ be an ordinary category which admits finite products. If $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is any functor having the property that the composition $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \xrightarrow{\Omega^{\infty}} \mathcal{S}$ is representable, then F must take values in the full subcategory $\mathrm{Ab} \simeq \mathrm{Mod}_{\mathbf{Z}}^{\heartsuit} \subseteq \mathrm{Mod}_{\mathbf{Z}}$. Consequently, Remark 1.2.11 supplies an equivalence of $\mathrm{Ab}(\mathcal{C})$ with the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ spanned by those functors F for which the composition $\mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Ab} \to \mathcal{S}\mathrm{et}$ is representable. We can summarize the situation more informally as follows: an object of $\mathrm{Ab}(\mathcal{C})$ consists of an object $C \in \mathcal{C}$ together with an abelian group structure on the functor $\mathrm{Hom}_{\mathcal{C}}(\bullet, C)$ represented by C. This abelian group structure can be described more explicitly via a multiplication map $m : C \times C \to C$ satisfying suitable analogues of the axioms defining the notion of abelian group.

Example 1.2.13. Let $\mathcal{C}$ be a presentable ∞-category. Using the adjoint functor theorem (Corollary HTT.5.5.2.9 and Remark HTT.5.5.2.10), we see that that essential image of the fully faithful embedding $\mathrm{Ab}(\mathcal{C}) \hookrightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ consists of those functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ which preserve small limits. It follows that $\mathrm{Ab}(\mathcal{C})$ can be identified with the tensor product $\mathcal{C} \otimes \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$, formed in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories (see Proposition HA.4.8.1.17).

Example 1.2.14. Let $\mathcal{C}$ be an additive presentable ∞-category. Then we can regard $\mathcal{C}$ as tensored over the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ (see Theorem SAG.C.4.1.1). Combining

Example 1.2.13 with Theorem HA.4.8.4.6, we obtain an equivalence of ∞-categories $\mathrm{Ab}(\mathcal{C}) \simeq \mathrm{LMod}_{\mathbf{Z}}(\mathcal{C})$.

Remark 1.2.15. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite products and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite products. Then composition with F induces a functor $\mathrm{Ab}(\mathcal{C}) \to \mathrm{Ab}(\mathcal{D})$. This observation applies in particular when F is the tautological map from $\mathcal{C}$ to its homotopy category $\mathrm{h}\mathcal{C}$. Consequently, every abelian group object of $\mathcal{C}$ can be regarded (by neglect of structure) as an abelian group object of the homotopy category $\mathrm{h}\mathcal{C}$.

1.3 Commutative Monoid Objects of ∞-Categories

We now compare Definition 1.2.4 with the theory of commutative monoid objects introduced in §HA.2.4.2. We begin by recalling some definitions.

Notation 1.3.1. For each $n \geqslant 0$, we let $\langle n \rangle$ denote the pointed set $\{1, \ldots, n, *\}$. We let $\mathcal{F}\mathrm{in}_*$ denote the category whose objects are the sets $\langle n \rangle$ and whose morphisms are pointed maps. For $1 \leqslant i \leqslant n$, we let $\rho^i : \langle n \rangle \to \langle 1 \rangle = \{1, *\}$ denote the morphism in $\mathcal{F}\mathrm{in}_*$ described by the formula $\rho^i(j) = \begin{cases} 1 & \text{if } i = j \\ * & \text{otherwise.} \end{cases}$

Definition 1.3.2. Let $\mathcal{C}$ be an ∞-category which admits finite products. A *commutative monoid object of* $\mathcal{C}$ is a functor $M : \mathcal{F}\mathrm{in}_* \to \mathcal{C}$ with the following property: for each $n \geqslant 0$, the maps $\{M(\rho^i) : M(\langle n \rangle) \to M(\langle 1 \rangle)\}_{1 \leqslant i \leqslant n}$ determine an equivalence $M(\langle n \rangle) \to M(\langle 1 \rangle)^n$ in the ∞-category $\mathcal{C}$. We let $\mathrm{CMon}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$ spanned by the commutative monoid objects of $\mathcal{C}$.

In the special case where $\mathcal{C} = \mathcal{S}$ is the ∞-category of spaces, we will denote the ∞-category $\mathrm{CMon}(\mathcal{C})$ simply by CMon. We will sometimes refer to CMon as the *∞-category of $\mathbb{E}_\infty$-spaces*.

Example 1.3.3. Let $M : \mathcal{F}\mathrm{in}_* \to \mathcal{S}\mathrm{et}$ be a commutative monoid object of the category of sets. The unique map $u : \langle 2 \rangle \to \langle 1 \rangle$ satisfying $u^{-1}\{*\} = \{*\}$ determines a multiplication map $m : M(\langle 1 \rangle) \times M(\langle 1 \rangle) \simeq M(\langle 2 \rangle) \xrightarrow{M(u)} M(\langle 1 \rangle)$. The multiplication m is commutative, associative and unital: that is, it equips the set $M(\langle 1 \rangle)$ with the structure of a commutative monoid. The construction $M \mapsto M(\langle 1 \rangle)$ induces an equivalence from the category $\mathrm{CMon}(\mathcal{S}\mathrm{et})$ to the category of commutative monoids.

Example 1.3.4. Let $\mathcal{C}$ be a semiadditive ∞-category (Definition SAG.C.4.1.6). Then the forgetful functor $\mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence of ∞-categories: that is, every

object of $\mathcal{C}$ admits an essentially unique commutative monoid structure (see Remark SAG.C.1.5.3).

Remark 1.3.5. Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite products and let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite products. Then composition with F determines a functor $\mathrm{CMon}(\mathcal{C}) \to \mathrm{CMon}(\mathcal{D})$.

Remark 1.3.6. In the situation of Definition 1.3.2, we will often abuse notation by identifying a commutative monoid object $M \in \mathrm{CMon}(\mathcal{C})$ with the object $M(\langle 1 \rangle) \in \mathcal{C}$.

Example 1.3.7. Let $\mathcal{C}$ be an ∞-category which admits finite products. For each object $C \in \mathcal{C}$, the construction $D \mapsto \pi_0 \mathrm{Map}_{\mathcal{C}}(C, D)$ determines a functor $e_C : \mathcal{C} \to \mathcal{S}\mathrm{et}$ which commutes with finite products. It follows that if M is a commutative monoid object of $\mathcal{C}$, we can regard $e_C \circ M$ as a commutative monoid object of $\mathcal{S}\mathrm{et}$: that is, the set $\pi_0 \mathrm{Map}_{\mathcal{C}}(C, M) = \mathrm{Hom}_{\mathrm{h}\mathcal{C}}(C, M)$ inherits the structure of a commutative monoid.

Definition 1.3.8. Let $\mathcal{C}$ be an ∞-category which admits finite products and let M be a commutative monoid object of $\mathcal{C}$. We say that M is *grouplike* if, for every object $C \in \mathcal{C}$, the commutative monoid $\pi_0 \mathrm{Map}_{\mathcal{C}}(C, M)$ of Example 1.3.7 is an abelian group. We let $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{C})$ denote the full subcategory of $\mathrm{CMon}(\mathcal{C})$ spanned by the grouplike commutative monoid objects of $\mathcal{C}$.

Construction 1.3.9. Let I_* be a pointed set. We let $\mathrm{Hom}_*(I_*, \mathbf{Z})$ denote the set of pointed maps from I_* into $\mathbf{Z}$ (that is, functions $e : I_* \to \mathbf{Z}$ satisfying $e(*) = 0$). Note that $\mathrm{Hom}_*(I_*, \mathbf{Z})$ forms a group under pointwise addition. If the set I_* is finite, then $\mathrm{Hom}_*(I_*, \mathbf{Z})$ is a lattice. Consequently, the construction $\langle n \rangle \mapsto \mathrm{Hom}_*(\langle n \rangle, \mathbf{Z})$ determines a functor $\lambda : \mathcal{F}\mathrm{in}_* \to \mathrm{Lat}^{\mathrm{op}}$.

For any ∞-category $\mathcal{C}$, precomposition with λ induces a functor $\mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{C})$. If $\mathcal{C}$ admits finite products, then this functor carries abelian group objects of $\mathcal{C}$ (in the sense of Definition 1.2.4) to grouplike commutative monoid objects of $\mathcal{C}$ (in the sense of Definition 1.3.2). We therefore obtain a forgetful functor $\theta : \mathrm{Ab}(\mathcal{C}) \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{C})$.

Warning 1.3.10. If $\mathcal{C}$ is an ordinary category, then the forgetful functor $\theta : \mathrm{Ab}(\mathcal{C}) \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{C})$ of Construction 1.3.9 is an equivalence. That is, we can identify abelian group objects of $\mathcal{C}$ with grouplike commutative monoid objects $M \in \mathcal{C}$. In general, this does not hold. For example, if $\mathcal{C}$ is the ∞-category of spectra, then θ can be identified with the forgetful functor $\mathrm{Mod}_{\mathbf{Z}} = \mathrm{Mod}_{\mathbf{Z}}(\mathrm{Sp}) \to \mathrm{Sp}$, which is neither fully faithful nor essentially surjective.

1.4   Abelian Varieties

We are now ready to introduce the main objects of interest in this paper.

Definition 1.4.1. Let R be an $\mathbb{E}_\infty$-ring. An *abelian variety* over R is a commutative monoid object of the ∞-category $\mathrm{Var}(R)$. We let $\mathrm{AVar}(R)$ denote the ∞-category $\mathrm{CMon}(\mathrm{Var}(R))$ of abelian varieties over R.

Remark 1.4.2. If X is an abelian variety over R, we will generally abuse terminology by identifying X with its image under the forgetful functor $\mathrm{AVar}(R) \simeq \mathrm{CMon}(\mathrm{Var}(R)) \to \mathrm{Var}(R)$.

Remark 1.4.3. Let R be an $\mathbb{E}_\infty$-ring and let $\tau_{\geqslant 0} R$ denote the connective cover of R. Using Proposition 1.1.3, we deduce that extension of scalars along the tautological map $\tau_{\geqslant 0} R \to R$ induces an equivalence of ∞-categories $\mathrm{AVar}(\tau_{\geqslant 0} R) \to \mathrm{AVar}(R)$.

Proposition 1.4.4. *Let X be an abelian variety over an $\mathbb{E}_\infty$-ring R. Then X is a grouplike commutative monoid object of $\mathrm{Var}(R)$.*

Proof. By virtue of Proposition 1.1.3, we may assume that R is connective. Let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the multiplication map and let $p : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the projection onto the first factor. To show that X is grouplike, it will suffice to show that the "shearing" map $(p, m) : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is an equivalence. By virtue of Corollary SAG.6.1.4.12, it will suffice to show that for every field κ and every map $\eta : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{X}$, the induced map $\mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ is an equivalence. Without loss of generality, we may assume that κ is algebraically closed. Replacing R by κ, we can reduce to the case where $R = \kappa$ is a field and η is an element of the commutative monoid $\mathsf{X}(\kappa) = \mathrm{Map}_{\mathrm{Var}(\kappa)}(\mathrm{Sp\acute{e}t}\, \kappa, \mathsf{X})$ of κ-valued points of X. We will complete the proof by showing that η is an invertible element of $\mathsf{X}(\kappa)$.

Since X is proper over κ, the map $(p, m) : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X}$ has closed image. Let $U \subseteq |\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X}|$ be the complement of the image of (p, m). The projection map p is flat, so U has open image in $|\mathsf{X}|$. Let $K \subseteq |\mathsf{X}|$ be the complement of $p(U)$. Unwinding the definitions, we see that a κ-valued point $x \in \mathsf{X}(\kappa)$ factors through K if and only the multiplication map

$$m_x : \mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, \kappa \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X} \xrightarrow{x \times \mathrm{id}} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X} \xrightarrow{m} \mathsf{X}$$

is surjective. In particular, the unit element $e \in \mathsf{X}(\kappa)$ factors through K, so K is nonempty. We will complete the proof by showing that $K = |\mathsf{X}|$. It then follows that $m_\eta : \mathsf{X} \to \mathsf{X}$ is surjective. Using our assumption that κ is algebraically closed, it

follows that multiplication by η induces a surjective map from the set $\mathsf{X}(\kappa)$ to itself, so that η is an invertible element of $\mathsf{X}(\kappa)$ as desired.

Since the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ is geometrically connected, the topological space $|\mathsf{X}|$ is connected. Consequently, to show that $K = |\mathsf{X}|$, it will suffice to show that K is open. Let W be the interior of K in $|\mathsf{X}|$; we wish to show that $W = K$. Assume otherwise: then we can choose a closed point γ of K which does not belong to W. Let us identify γ with a κ-valued point of X. Our assumption that γ belongs to K guarantees that the translation map $m_\gamma : \mathsf{X} \to \mathsf{X}$ is surjective. Our assumption that κ is algebraically closed guarantees that multiplication by γ induces a surjection $\mathsf{X}(\kappa) \to \mathsf{X}(\kappa)$: that is, γ admits an inverse γ^{-1} in the commutative monoid $\mathsf{X}(\kappa)$. It follows that the map m_γ is an isomorphism, and is therefore flat. It follows that there exists an open subset $V \subseteq |\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X}|$ containing $|\{\gamma\} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X}|$ such that (p, m) is flat when restricted to V (Corollary SAG.6.1.4.6). Since the projection map p is closed, we can assume without loss of generality that V is the inverse image of some open set $V_0 \subseteq |\mathsf{X}|$ containing γ. Our assumption $\gamma \notin W$ guarantees that $V_0 \nsubseteq K$: that is, we can choose a point $\delta \in V_0$ which does not belong to K. Without loss of generality, we may assume that δ is closed in V_0 and therefore also in $|\mathsf{X}|$, so that we can identify δ with a κ-valued point of X. The assumption $\delta \in V_0$ guarantees that $m_\gamma : \mathsf{X} \to \mathsf{X}$ is flat. The induced map of topological spaces $|\mathsf{X}| \to |\mathsf{X}|$ is then open (since m_δ is flat) and closed (since X is proper over κ), and therefore surjective (since $|\mathsf{X}|$ is connected). This is a contradiction, since δ does not belong to K. $\qquad\square$

Remark 1.4.5 (The Functor of Points). Let R be an $\mathbb{E}_\infty$-ring. Then the functor of points construction $\mathrm{Var}(R) \to \mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})$ of Remark 1.1.5 induces a fully faithful embedding

$$
\begin{aligned}
\mathrm{AVar}(R) &= \mathrm{CMon}(\mathrm{Var}(R)) \\
&= \mathrm{CMon}^{\mathrm{gp}}(\mathrm{Var}(R)) \\
&\hookrightarrow \mathrm{CMon}^{\mathrm{gp}}(\mathrm{Fun}(\mathrm{CAlg}_R, \mathcal{S})) \\
&= \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})).
\end{aligned}
$$

The essential image of this embedding consists of those functors $X : \mathrm{CAlg}_R \to \mathrm{CMon}$ for which the underlying functor $\mathrm{CAlg}_R \xrightarrow{X} \mathrm{CMon} \to \mathcal{S}$ is representable by an object of $\mathrm{Var}(R)$.

Variant 1.4.6. Let R be an $\mathbb{E}_\infty$-ring which is connective and n-truncated for $0 \leqslant n \leqslant \infty$. Then the composite functor

$$
\mathrm{AVar}(R) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon}) \hookrightarrow \mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon})
$$

is also fully faithful. Moreover, this functor takes values in the full subcategory $\mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}, \tau_{\leqslant n}\,\mathrm{CMon})$. We therefore obtain a fully faithful embedding of ∞-categories $\mathrm{AVar}(R) \hookrightarrow \mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}, \tau_{\leqslant n}\,\mathrm{CMon})$. If X is an abelian variety over R, we will refer to its image under this embedding as the *functor of points* of X.

We now compare Definition 1.4.1 with the classical theory of abelian varieties.

Proposition 1.4.7 (Artin)**.** *Let κ be an algebraically closed field and let X be an abelian variety over κ. Then X is schematic and the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ is fiber smooth (see Definition SAG.11.2.5.5).*

Remark 1.4.8. It follows from Proposition 1.4.7 that in the special case where $R = \kappa$ is an algebraically closed field, the notion of abelian variety over R (in the sense of Definition 1.4.1) reduces to the usual notion of abelian variety over κ (in the sense of classical algebraic geometry): that is, to the notion of a group scheme over κ which is proper, smooth, and connected. More generally, if R is any commutative ring, then the theory of abelian varieties over R (in the sense of Definition 1.4.1) reduces to the classical theory of abelian schemes over R (see, for example, [10]). The nontrivial point (due to Raynaud) is that any abelian variety over R is automatically schematic. For a proof, we refer the reader to Theorem 1.9 of [2].

Corollary 1.4.9. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. Then the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\,R$ is fiber smooth.*

Proof. Using Proposition SAG.11.2.3.6, we can reduce to to the case where R is an algebraically closed field, in which case the desired result follows from Proposition 1.4.7. $\qquad\qquad\square$

Proof of Proposition 1.4.7. Let X be an abelian variety over an algebraically closed field κ. Then X is a nonempty separated spectral algebraic space. Using Corollary SAG.3.4.2.4, we can choose a nonempty open subspace $\mathsf{U}_0 \subseteq \mathsf{X}$ where $\mathsf{U}_0 \simeq \mathrm{Sp\acute{e}t}\,R$ is affine. Since X is flat and locally almost of finite presentation over κ, the $\mathbb{E}_\infty$-ring R is discrete and finitely generated as a κ-algebra. Moreover, the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\,\kappa$ is geometrically reduced, so that R is reduced. Replacing R by a localization, we can assume that it is smooth over κ (in the sense of commutative algebra), so that U_0 is fiber smooth over $\mathrm{Sp\acute{e}t}\,\kappa$ (in the sense of spectral algebraic geometry).

Let $\mathsf{U} \subseteq \mathsf{X}$ be the largest open subspace which is schematic and fiber smooth over κ. Let $\mathsf{X}(\kappa) = \mathrm{Map}_{\mathrm{Var}(\kappa)}(\mathrm{Sp\acute{e}t}\,\kappa, \mathsf{X})$ be the set of κ-valued points of X, which we

will identify with closed points of the topological space $|\mathsf{X}|$. For each $\eta \in \mathsf{X}(\kappa)$, let $m_\eta : \mathsf{X} \to \mathsf{X}$ be the map given by translation by η. It follows from Proposition 1.4.4 that m_η is an automorphism of X, and therefore carries U into itself.

Because U_0 is nonempty, we can choose a point $\gamma \in \mathsf{X}(\kappa)$ which factors through U_0. The map $m_{\gamma^{-1}}$ carries $\mathsf{U}_0 \subseteq \mathsf{U}$ into U, so that U contains the identity element of $\mathsf{X}(\kappa)$. Using the invariance of U under translations, we deduce that U contains all closed points of $|\mathsf{X}|$. We therefore have $\mathsf{U} \simeq \mathsf{X}$, so that X is schematic and fiber smooth over κ as desired. $\qquad\square$

1.5 Strict Abelian Varieties

Let κ be a field and let X be a group scheme over κ which is proper, smooth, and connected. The group structure on X is then automatically commutative (see [10]), so that X can be regarded as an abelian variety over κ. In the context of spectral algebraic geometry, the analogous statement is false: commutativity does not come for free. Moreover, in the ∞-categorical setting, one can consider many flavors of commutativity which are classically indistinguishable. We therefore introduce the following variant of Definition 1.4.1:

Definition 1.5.1. Let R be an $\mathbb{E}_\infty$-ring. A *strict abelian variety* over R is an abelian group object of the ∞-category $\mathrm{Var}(R)$ (see Definition 1.2.4). We let $\mathrm{AVar}^s(R) = \mathrm{Ab}(\mathrm{Var}(R))$ denote the ∞-category of abelian varieties over R.

Remark 1.5.2. Let R be an $\mathbb{E}_\infty$-ring. Then the ∞-category $\mathrm{AVar}^s(R)$ is semiadditive. Using Example 1.3.4, we deduce that the forgetful functor

$$
\begin{aligned}
\mathrm{Ab}(\mathrm{AVar}(R)) \quad &= \quad \mathrm{Ab}(\mathrm{CMon}(\mathrm{Var}(R)) \\
&\simeq \quad \mathrm{CMon}(\mathrm{Ab}(\mathrm{Var}(R)) \\
&\to \quad \mathrm{Ab}(\mathrm{Var}(R)) \\
&= \quad \mathrm{AVar}^s(R)
\end{aligned}
$$

is an equivalence of ∞-categories. In other words, we can identify $\mathrm{AVar}^s(R)$ with the ∞-category of abelian group objects of $\mathrm{AVar}(R)$, rather than $\mathrm{Var}(R)$.

Remark 1.5.3. Let R be an $\mathbb{E}_\infty$-ring and let $\tau_{\geqslant 0} R$ denote the connective cover of R. Using Proposition 1.1.3, we deduce that extension of scalars along the tautological map $\tau_{\geqslant 0} R \to R$ induces an equivalence of ∞-categories $\mathrm{AVar}^s(\tau_{\geqslant 0} R) \to \mathrm{AVar}^s(R)$.

Remark 1.5.4 (The Functor of Points). Let R be an $\mathbb{E}_\infty$-ring. Then the functor of points construction $\operatorname{Var}(R) \to \operatorname{Fun}(\operatorname{CAlg}_R, \mathcal{S})$ of Remark 1.1.5 induces a fully faithful embedding

$$
\begin{aligned}
\operatorname{AVar}^s(R) &= \operatorname{Ab}(\operatorname{Var}(R)) \\
&\hookrightarrow \operatorname{Ab}(\operatorname{Fun}(\operatorname{CAlg}_R, \mathcal{S})) \\
&= \operatorname{Fun}(\operatorname{CAlg}_R, \operatorname{Ab}(\mathcal{S})) \\
&\simeq \operatorname{Fun}(\operatorname{CAlg}_R, \operatorname{Mod}_{\mathbf{Z}}^{\mathrm{cn}}).
\end{aligned}
$$

The essential image of this embedding consists of those functors $X : \operatorname{CAlg}_R \to \operatorname{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ for which the 0th space $\Omega^\infty X : \operatorname{CAlg}_R \to \mathcal{S}$ is representable by an object of $\operatorname{Var}(R)$.

Remark 1.5.5. Applying Construction 1.3.9 to the ∞-category $\operatorname{Var}(R)$, we obtain a forgetful functor $\operatorname{AVar}^s(R) \to \operatorname{AVar}(R)$. In terms of the functors of points (Remark 1.4.6 and 1.5.4), this forgetful functor is obtained by composition with the forgetful functor $\operatorname{Ab}(\mathcal{S}) \simeq \operatorname{Mod}_{\mathbf{Z}}(\operatorname{Sp}^{\mathrm{cn}}) \to \operatorname{Sp}^{\mathrm{cn}} \simeq \operatorname{CMon}^{\mathrm{gp}}(\mathcal{S})$.

We will generally abuse notation by identifying a strict abelian variety $\mathsf{X} \in \operatorname{AVar}^s(R)$ with its image in $\operatorname{AVar}(R)$ (and, by extension, with its image under the forgetful functor $\operatorname{AVar}^s(R) \to \operatorname{AVar}(R) \to \operatorname{Var}(R)$).

Proposition 1.5.6. *For every commutative ring R, the forgetful functor $\operatorname{AVar}^s(R) \to \operatorname{AVar}(R)$ is an equivalence of categories.*

Proof. Since $\operatorname{Var}(R)$ is equivalent to an ordinary category, Warning 1.3.10 implies that the forgetful functor $\operatorname{AVar}^s(R) = \operatorname{Ab}(\operatorname{Var}(R)) \to \operatorname{CMon}(\operatorname{Var}(R)) \simeq \operatorname{AVar}(R)$ is a fully faithful embedding, whose essential image consists of the grouplike commutative monoid objects of $\operatorname{Var}(R)$. The desired result now follows from the fact that every abelian variety is grouplike (Proposition 1.4.4). $\qquad\square$

Warning 1.5.7. If R is a non-discrete $\mathbb{E}_\infty$-ring, then the forgetful functor $\operatorname{AVar}^s(R) \to \operatorname{AVar}(R)$ need not be an equivalence of ∞-categories. However, we will see later that it is an equivalence when $\pi_0 R$ is an algebra over the field $\mathbf{Q}$ of rational numbers (see Theorem 2.1.1).

Warning 1.5.8. If R is an ordinary commutative ring, then the forgetful functor $\operatorname{AVar}^s(R) \to \operatorname{Var}(R)$ is *almost* a fully faithful embedding: if X and Y are abelian varieties over R, then any morphism $f : \mathsf{X} \to \mathsf{Y}$ in $\operatorname{Var}(R)$ which preserves the identity sections is automatically a morphism of abelian varieties over R. The analogous

statement does not hold in the setting of spectral algebraic geometry: an abelian variety X over a non-discrete $\mathbb{E}_\infty$-ring R generally cannot be recovered from its underlying spectral algebraic space, even if the identity section is specified.

2 Moduli of Elliptic Curves

Let R be an $\mathbb{E}_\infty$-ring, let $X = (\mathcal{X}, \mathscr{O}_X)$ be an abelian variety over R, and let g be a nonnegative integer. We will say that X *has dimension g* if, for every field κ and every map $\pi_0 R \to \kappa$, the fiber product $\operatorname{Sp\acute{e}t} \kappa \times_{\operatorname{Sp\acute{e}t} \pi_0 R} (\mathcal{X}, \pi_0 \mathscr{O}_X)$ has Krull dimension g. We let $\operatorname{AVar}_g(R)$ denote the full subcategory of $\operatorname{AVar}(R)$ spanned by the abelian varieties of dimension g over R, and we let $\operatorname{AVar}_g^s(R)$ denote the full subcategory of $\operatorname{AVar}^s(R)$ spanned by the strict abelian varieties of dimension g over R.

Remark 2.0.1. Let R be an $\mathbb{E}_\infty$-ring and let $X = (\mathcal{X}, \mathscr{O}_X)$ be an abelian variety over R. Then the function

$$(x \in |\operatorname{Spec} R|) \mapsto \dim(\operatorname{Sp\acute{e}t} \kappa(x) \times_{\operatorname{Sp\acute{e}t} \pi_0 R} (\mathcal{X}, \pi_0 \mathscr{O}_X))$$

is locally constant on $\operatorname{Spec} R$. In particular, if $|\operatorname{Spec} R|$ is connected, then every abelian variety over R has dimension g for some uniquely determined nonnegative integer g.

Definition 2.0.2. Let R be an $\mathbb{E}_\infty$-ring. An *elliptic curve over R* is an abelian variety of dimension 1 over R. A *strict elliptic curve over R* is a strict abelian variety of dimension 1 over R. We let $\operatorname{Ell}^s(R) = \operatorname{AVar}_1^s(R)$ denote the ∞-category of strict elliptic curves over R, and $\operatorname{Ell}(R) = \operatorname{AVar}_1(R)$ the ∞-category of elliptic curves over R.

Our goal in this section is to prove the following result:

Theorem 2.0.3. *The functors $R \mapsto \operatorname{Ell}^s(R)^\simeq$ and $R \mapsto \operatorname{Ell}(R)^\simeq$ are representable by spectral Deligne-Mumford stacks. That is, there exists spectral Deligne-Mumford stacks $\mathcal{M}^s$ and $\mathcal{M}$ which are equipped with functorial homotopy equivalences*

$$\operatorname{Map}_{\operatorname{SpDM^{nc}}}(\operatorname{Sp\acute{e}t} R, \mathcal{M}^s) \simeq \operatorname{Ell}^s(R)^\simeq \qquad \operatorname{Map}_{\operatorname{SpDM^{nc}}}(\operatorname{Sp\acute{e}t} R, \mathcal{M}) \simeq \operatorname{Ell}(R)^\simeq.$$

We will refer to $\mathcal{M}$ as the moduli stack of elliptic curves, and $\mathcal{M}^s$ as the moduli stack of strict elliptic curves.

We will deduce Theorem 2.0.3 using the spectral version of Artin's representability theorem (Theorem SAG.18.3.0.1). To verify the hypotheses of Artin's criterion, we will need to study the deformation theory of the functors $R \mapsto \mathrm{Ell}^s(R)^{\simeq}$ and $R \mapsto \mathrm{Ell}(R)^{\simeq}$; this will occupy our attention throughout this section.

2.1 Comparing Abelian Varieties with Strict Abelian Varieties

We begin with a brief digression. In §1, we introduced the notions of *abelian variety* and *strict abelian variety* over an arbitrary $\mathbb{E}_{\infty}$-ring R. Over $\mathbb{E}_{\infty}$-rings of characteristic zero, these two notions coincide:

Theorem 2.1.1. *Let R be an $\mathbb{E}_{\infty}$-algebra over $\mathbf{Q}$. Then the forgetful functor*

$$\mathrm{AVar}^s(R) \to \mathrm{AVar}(R)$$

is an equivalence of ∞-categories.

Theorem 2.1.1 will not be needed elsewhere in this paper. However, our proof of Theorem 2.1.1 will showcase some of the ideas needed to establish Theorem 2.0.3. We begin with some preliminaries.

Proposition 2.1.2. *Let R be a connective $\mathbb{E}_{\infty}$-ring, let X be an abelian variety over R, let $e : \mathrm{Spec}\, R \to \mathsf{X}$ be the identity section, and set $\omega = e^* L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R} \in \mathrm{Mod}_R^{\mathrm{cn}}$. Let $X : \mathrm{CAlg}_R \to \mathrm{Sp}^{\mathrm{cn}}$ denote the functor of points of X (see Remark 1.4.5). For every R-module M, we have a canonical homotopy equivalence $X(R \oplus M) \simeq X(R) \oplus \tau_{\geqslant 0}\underline{\mathrm{Map}}_R(\omega, M)$.*

Proof. For every connective R-module M, we have canonical maps $X(R) \to X(R \oplus M) \to X(R)$ whose composition is the identity. We therefore obtain a direct sum decomposition $X(R \oplus M) \simeq X(R) \oplus F(M)$ for some functor $F : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$. Let $F' : \mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ denote the functor given by $M \mapsto \tau_{\geqslant 0}\underline{\mathrm{Map}}_R(\omega, M)$. We wish to show that the functors F and F' are equivalent.

Using the definition of the cotangent complex $L_{\mathsf{X}/\mathrm{Sp\acute{e}t}\, R}$, we obtain a homotopy equivalence $\alpha : \Omega^{\infty}F(M) \simeq \mathrm{Map}_{\mathrm{Mod}_R}(\omega, M) \simeq \Omega^{\infty}F'(M)$, depending functorially on M. Let $\mathrm{Exc}_*(\mathrm{Mod}_R^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Mod}_R^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}})$ spanned by those functors which are reduced and excisive (see Definition HA.1.4.2.1), and define $\mathrm{Exc}_*(\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{S})$ similarly. Since the ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$ is prestable, the functor $M \mapsto \mathrm{Map}_{\mathrm{Mod}_R}(\omega, M)$ belongs to $\mathrm{Exc}_*(\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{S})$. It follows that the functors $\Omega^{\infty} \circ F$

and $\Omega^\infty \circ F'$ are reduced and excisive. The functor $\Omega^\infty : \mathrm{Sp}^{\mathrm{cn}} \to \mathcal{S}$ is conservative and left exact, so the functors F and F' are also reduced and excisive. Moreover, Ω^∞ induces an equivalence on stabilizations, so Proposition HA.1.4.2.22 guarantees that composition with Ω^∞ induces an equivalence of ∞-categories $\mathrm{Exc}_*(\mathrm{Mod}_R^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}}) \to \mathrm{Exc}_*(\mathrm{Mod}_R^{\mathrm{cn}}, \mathcal{S})$. Consequently, the equivalence α can be lifted (in an essentially unique way) to an equivalence of functors $F \simeq F$. $\qquad\square$

We now use Proposition 2.1.2 to establish a weak form of Theorem 2.1.1.

Proposition 2.1.3. *Let R be a connective $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$ and let X and Y be strict abelian varieties over R. Then the canonical map $\mathrm{Map}_{\mathrm{AVar}^s(R)}(\mathsf{X}, \mathsf{Y}) \to \mathrm{Map}_{\mathrm{AVar}(R)}(\mathsf{X}, \mathsf{Y})$ is a homotopy equivalence.*

Proof. Let $X, Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ be the functors represented by X and Y (see Remark 1.5.4). Let $\theta : \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ denote the forgetful functor. We wish to show that the canonical map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}})}(\theta X, \theta Y)$$

is a homotopy equivalence.

Let us say that a functor $Z : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is *good* if the canonical map $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(X, Z) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}})}(\theta X, \theta Z)$ is a homotopy equivalence. We wish to show that Y is good. We begin with some elementary observations:

(a) Let $Z : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ be a functor which factors through the full subcategory $\mathrm{Mod}_{\mathbf{Z}}^{\heartsuit} \subseteq \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Then Z is good. This follows from the fact that the forgetful functor $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ induces an equivalence $\mathrm{Mod}_{\mathbf{Z}}^{\heartsuit} \to \mathrm{Sp}^{\heartsuit}$.

(b) Let $Z : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ be a functor which factors through the forgetful functor $\mathrm{Mod}_{\mathbf{Q}}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Then Z is good (since the forgetful functors $\mathrm{Mod}_{\mathbf{Q}}^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ and $\mathrm{Mod}_{\mathbf{Q}}^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ are both fully faithful).

(c) The collection of good functors $Z : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is closed under limits.

For each $n \geqslant 0$, define $Y_n : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ by the formula $Y_n(A) = Y(\tau_{\leqslant n} A)$. Using Proposition SAG.17.3.2.3, we see that Y can be identified with the limit of the tower $\{Y_n\}_{n \geqslant 0}$. Consequently, to show that Y is good, it will suffice to show that each Y_n is good. The proof proceeds by induction on n. In the case $n = 0$, the functor Y_0 takes values in $\mathrm{Mod}_{\mathbf{Z}}^{\heartsuit}$, so the desired result follows from (a).

We now carry out the inductive step. Assume that $n > 0$ and that the functor Y_{n-1} is good. For every object $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, Theorem HA.7.4.1.26 supplies a canonical pullback diagram

$$\begin{array}{ccc} \tau_{\leqslant n}A & \longrightarrow & \tau_{\leqslant n-1}A \\ \downarrow & & \downarrow \\ \tau_{\leqslant n-1}A & \longrightarrow & (\tau_{\leqslant n-1}A) \oplus \Sigma^{n+1}(\pi_n A). \end{array}$$

It follows from Example SAG.17.3.1.2 that this diagram remains a pullback square after applying the functor Y. Writing $Y((\tau_{\leqslant n-1}A) \oplus \Sigma^{n+1}(\pi_n A)) = Y(\tau_{\leqslant n-1}A) \oplus K(A)$, we obtain a pullback square

$$\begin{array}{ccc} Y_n(A) & \longrightarrow & Y_{n-1}(A) \\ \downarrow & & \downarrow \\ Y_{n-1}(A) & \longrightarrow & Y_{n-1}(A) \oplus K(A) \end{array}$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$, depending functorially on A. By virtue of (c), to show that the functor Y_n is good, it will suffice to show that the functor $A \mapsto K(A)$ is good. Let $\omega \in \mathrm{Mod}_R$ be as in the statement of Proposition 2.1.2. Then Proposition 2.1.2 supplies canonical equivalences $\theta K(A) \simeq \tau_{\geqslant 0}\underline{\mathrm{Map}}_R(\omega, \Sigma^{n+1}\pi_n A)$. Since R is a $\mathbf{Q}$-algebra, the homotopy group $\pi_n A$ admits the structure of a vector space over $\mathbf{Q}$. It follows that the homotopy groups of $K(A)$ are also vector spaces over $\mathbf{Q}$, so that the functor $K : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ factors through $\mathrm{Mod}_{\mathbf{Q}}^{\mathrm{cn}}$. Applying (b), we deduce that the functor K is good, as desired. $\square$

Proposition 2.1.4. *The constructions $R \mapsto \mathrm{AVar}(R)$ and $R \mapsto \mathrm{AVar}^s(R)$ determine cohesive functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$, in the sense of Definition SAG.19.2.1.1. In other words, for every pullback diagram*

$$\begin{array}{ccc} R_{01} & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_1 & \longrightarrow & R \end{array}$$

of connective $\mathbb{E}_\infty$-rings for which the underlying ring homomorphisms $\pi_0 R_0 \to \pi_0 R \leftarrow \pi_0 R_1$ are surjective, the diagrams of ∞-categories

$$\begin{array}{ccc} \mathrm{AVar}(R_01) & \longrightarrow & \mathrm{AVar}(R_0) \\ \downarrow & & \downarrow \\ \mathrm{AVar}(R_1) & \longrightarrow & \mathrm{AVar}(R) \end{array} \qquad \begin{array}{ccc} \mathrm{AVar}^s(R_{01}) & \longrightarrow & \mathrm{AVar}^s(R_0) \\ \downarrow & & \downarrow \\ \mathrm{AVar}^s(R_1) & \longrightarrow & \mathrm{AVar}^s(R) \end{array}$$

are pullback squares.

Proof. It follows from Theorem SAG.19.4.0.2 and Proposition SAG.19.4.5.6 that the construction $R \mapsto \mathrm{Var}(R)$ is cohesive. The desired result now follows from the observation that the constructions $\mathcal{C} \mapsto \mathrm{CMon}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{Ab}(\mathcal{C})$ preserve limits. $\square$

Proposition 2.1.5. *The constructions $R \mapsto \mathrm{AVar}(R)$ and $R \mapsto \mathrm{AVar}^s(R)$ determine nilcomplete functors $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$, in the sense of Definition SAG.19.2.1.1. In other words, for every connective $\mathbb{E}_\infty$-ring R, the canonical maps*

$$\mathrm{AVar}(R) \to \varprojlim \mathrm{AVar}(\tau_{\leqslant n}R) \qquad \mathrm{AVar}^s(R) \to \varprojlim \mathrm{AVar}^s(\tau_{\leqslant n}R)$$

are equivalences of ∞-categories.

Proof. It follows from Theorem SAG.19.4.0.2 and Proposition SAG.19.4.5.6 that the construction $R \mapsto \mathrm{Var}(R)$ is nilcomplete. The desired result now follows from the observation that the constructions $\mathcal{C} \mapsto \mathrm{CMon}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{Ab}(\mathcal{C})$ preserve limits. $\square$

Proof of Theorem 2.1.1. Let R be an $\mathbb{E}_\infty$-algebra over $\mathbf{Q}$; we wish to show that the forgetful functor $\theta_R : \mathrm{AVar}^s(R) \to \mathrm{AVar}(R)$ is an equivalence of ∞-categories. Using Remarks 1.4.3 and 1.5.3, we can assume without loss of generality that R is connective. In this case, Proposition 2.1.5 guarantees that we can write θ as the limit of a tower of functors $\theta_{\tau_{\leqslant n}R} : \mathrm{AVar}^s(\tau_{\leqslant n}R) \to \mathrm{AVar}(\tau_{\leqslant n}R)$. It will therefore suffice to show that each θ_n is an equivalence. In oother words, we are reduced to proving Theorem 2.1.1 in the special case where R is connective and n-truncated for some $n \geqslant 0$.

The proof now proceeds by induction on n. In the case $n = 0$, the $\mathbb{E}_\infty$-ring R is discrete and the desired result follows from Proposition 1.5.6. To carry out the inductive step, suppose that R is n-truncated for $n > 0$ and set $R_0 = \tau_{\leqslant n-1}R$. Set $M = \pi_n R$, so that R is a square-zero extension of R_0 by $\Sigma^n M$ (Theorem HA.7.4.1.26). We therefore have a pullback diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} R & \longrightarrow & R_0 \\ \downarrow & & \downarrow \\ R_0 & \longrightarrow & R_0 \oplus \Sigma^{n+1}M. \end{array}$$

Applying Proposition 2.1.4, we deduce that the diagram

$$\begin{array}{ccc} \theta_R & \longrightarrow & \theta_{R_0} \\ \downarrow & & \downarrow \\ \theta_{R_0} & \longrightarrow & \theta_{R_0 \oplus \Sigma^{n+1}M} \end{array}$$

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is a pullback square in the ∞-category $\mathrm{Fun}(\Delta^1, \mathcal{C}\mathrm{at}_\infty)$. The functor θ_{R_0} is an equivalence by virtue of our inductive hypothesis and the functor $\theta_{R_0 \oplus \Sigma^{n+1} M}$ is fully faithful by virtue of Proposition 2.1.3, so that θ_R is also an equivalence of ∞-categories. $\square$

2.2 Deformation Theory of Abelian Varieties

We begin with an analysis of the functor $R \mapsto \mathrm{AVar}(R)$.

Proposition 2.2.1. *Let $n \geqslant 0$ be a nonnegative integer and let $F : \tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ be the functor given by $F(R) = \mathrm{AVar}(R)$. Then F commutes with filtered colimits. (More informally, the functor AVar is locally almost of finite presentation.)*

The proof of Proposition 2.2.1 will require some preliminaries.

Lemma 2.2.2. *Let $\{\mathcal{C}_\alpha\}$ be a filtered diagram of ∞-categories. Suppose that each $\mathcal{C}_\alpha$ admits finite products and that each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves finite products. Assume further that there exists an integer n such that each $\mathcal{C}_\alpha$ is equivalent to an n-category. Then the canonical map*

$$\theta : \varinjlim \mathrm{Mon}(\mathcal{C}_\alpha) \to \mathrm{Mon}(\varinjlim \mathcal{C}_\alpha)$$

is an equivalence of ∞-categories.

Proof. For every ∞-category $\mathcal{D}$ which admits finite products, let $U(\mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}, \mathcal{D})$ spanned by the category objects of $\mathcal{D}$. Note that if $\mathcal{D}$ is equivalent to an n-category, Theorem SAG.A.8.2.3 allows us to identify $U(\mathcal{D})$ with the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+2}, \mathcal{D})$ spanned by the $(n+2)$-skeletal category objects (see Definition SAG.A.8.2.2). Since the $(n+1)$-skeleton of the simplicial set $\mathrm{N}(\mathbf{\Delta}^{\mathrm{op}}_{\leqslant n+3})$ is finite, the canonical map $\varinjlim U(\mathcal{C}_\alpha) \to U(\varinjlim \mathcal{C}_\alpha)$ is an equivalence of ∞-categories. Moreover, if $\mathcal{D}$ admits finite products, then we can identify $\mathrm{Mon}(\mathcal{D})$ with the full subcategory of $U(\mathcal{D})$ spanned by those simplicial objects $D_\bullet$ for which D_0 is a final object of $\mathcal{D}$. It follows that we have a pullback diagram of ∞-categories

$$
\begin{array}{ccc}
\varinjlim \mathrm{Mon}(\mathcal{C}_\alpha) & \overset{\theta}{\longrightarrow} & \mathrm{Mon}(\varinjlim \mathcal{C}_\alpha) \\
\downarrow & & \downarrow \\
\varinjlim U(\mathcal{C}_\alpha) & \longrightarrow & U(\varinjlim \mathcal{C}_\alpha),
\end{array}
$$

which shows that θ is an equivalence of ∞-categories. $\square$

Lemma 2.2.3. *Let $\{\mathcal{C}_\alpha\}$ be a filtered diagram of ∞-categories. Suppose that each $\mathcal{C}_\alpha$ admits finite products and that each of the transition maps $\mathcal{C}_\alpha \to \mathcal{C}_\beta$ preserves finite products. Assume further that there exists an integer n such that each $\mathcal{C}_\alpha$ is equivalent to an n-category. Then the canonical map*

$$\theta : \varinjlim \mathrm{CMon}(\mathcal{C}_\alpha) \to \mathrm{CMon}(\varinjlim \mathcal{C}_\alpha)$$

is an equivalence of ∞-categories.

Proof. For each $k \geqslant 0$, we have a commutative diagram

$$
\begin{array}{ccc}
\varinjlim \mathrm{CMon}(\mathcal{C}_\alpha) & \xrightarrow{\ \theta\ } & \mathrm{CMon}(\varinjlim \mathcal{C}_\alpha) \\
\downarrow & & \downarrow \\
\varinjlim \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{C}_\alpha) & \xrightarrow{\ \theta_k\ } & \mathrm{Mon}_{\mathbb{E}_k}(\varinjlim \mathcal{C}_\alpha);
\end{array}
$$

here $\mathrm{Mon}_{\mathbb{E}_k}(\mathcal{D})$ denotes the ∞-category $\mathbb{E}_k$-monoid objects of $\mathcal{D}$ (see Definition HA.2.4.2.1). Our assumption that each $\mathcal{C}_\alpha$ is equivalent to an n-category guarantees that the vertical maps are equivalences of ∞-categories for $k > n$ (Example HA.5.1.2.3). It will therefore suffice to show that θ_k is an equivalence of ∞-categories for each $k \geqslant 1$. We proceed by induction on k. For $k > 1$, Theorem HA.5.1.2.2 allows us to factor θ_k as a composition

$$
\begin{aligned}
\varinjlim_\alpha \mathrm{Mon}_{\mathbb{E}_k}(\mathcal{C}_\alpha) \quad &\simeq \quad \varinjlim_\alpha \mathrm{Mon}(\mathrm{Mon}_{\mathbb{E}_{k-1}}(\mathcal{C}_\alpha)) \\
&\xrightarrow{\ \theta'\ } \quad \mathrm{Mon}(\varinjlim_\alpha \mathrm{Mon}_{\mathbb{E}_{k-1}}(\mathcal{C}_\alpha)) \\
&\xrightarrow{\ \theta''\ } \quad \mathrm{Mon}(\mathrm{Mon}_{\mathbb{E}_{k-1}}(\varinjlim \mathcal{C}_\alpha)) \\
&\simeq \quad \mathrm{Mon}_{\mathbb{E}_k}(\varinjlim \mathcal{C}_\alpha),
\end{aligned}
$$

where θ' is an equivalence by virtue of Lemma 2.2.2 and θ'' is an equivalence by our inductive hypothesis. $\qquad\square$

Remark 2.2.4. In the situation of Lemma 2.2.3, the functor θ restricts to an equivalence of full subcategories $\varinjlim \mathrm{CMon}^{\mathrm{gp}}(\mathcal{C}_\alpha) \to \mathrm{CMon}^{\mathrm{gp}}(\varinjlim \mathcal{C}_\alpha)$.

Proof of Proposition 2.2.1. According to Corollary SAG.19.4.5.7, the functor $R \mapsto \mathrm{Var}(R)$ commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}$. Moreover, if R is n-truncated, then $\mathrm{Var}(R)$ is equivalent to an $(n+1)$-category (Lemma SAG.1.6.8.8). Invoking Lemma 2.2.3, we conclude that the functor

$$R \mapsto \mathrm{AVar}(R) \simeq \mathrm{CMon}(\mathrm{Var}(R))$$

also commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}$. $\qquad\square$

Lemma 2.2.5. *Let K be a simplicial set and let $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula $F_K(R) = \mathrm{Fun}(K, \mathrm{AVar}(R))^{\simeq}$. Then the functor F_K admits a (-1)-connective cotangent complex.*

Proof. Define $G : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $G(R) = \mathrm{Fun}(\mathcal{F}\mathrm{in}_* \times K, \mathrm{Var}(R))^{\simeq}$. For every connective $\mathbb{E}_\infty$-ring R, we can identify $F_K(R)$ with the summand of $G(R)$ spanned by those functors $X : \mathcal{F}\mathrm{in}_* \times K \to \mathrm{Var}(R)$. which satisfy the following condition:

(∗) For every vertex $v \in K$, the restriction $X|_{\mathcal{F}\mathrm{in}_* \times \{v\}} : \mathcal{F}\mathrm{in}_* \to \mathrm{Var}(R)$ is a commutative monoid object of $\mathrm{Var}(R)$.

Theorem SAG.19.4.0.2 implies that the functor G admits a (-1)-connective cotangent complex $L_G \in \mathrm{QCoh}(G)$. We claim that the image L_G in the ∞-category $\mathrm{QCoh}(F_K)$ is a cotangent complex for F_K. To prove this, it will suffice to show that for every connective $\mathbb{E}_\infty$-ring R and every connective R-module M, the diagram

$$
\begin{array}{ccc}
F_K(R \oplus M) & \longrightarrow & F_K(R) \\
\downarrow & & \downarrow \\
G(R \oplus M) & \longrightarrow & G(R)
\end{array}
$$

is a pullback square. Without loss of generality, we may assume that $K = \Delta^0$; in this case, we must show that a diagram $\mathcal{F}\mathrm{in}_* \to \mathrm{Var}(R \oplus M)$ is a commutative monoid object of $\mathrm{Var}(R \oplus M)$ if and only if the composite map $\mathcal{F}\mathrm{in}_* \to \mathrm{Var}(R \oplus M) \to \mathrm{Var}(R)$ is a commutative monoid object of $\mathrm{Var}(R)$. This is clear, since the extension-of-scalars functor $\mathrm{Var}(R \oplus M) \to \mathrm{Var}(R)$ is conservative (Proposition 1.1.4). $\qquad\square$

Proposition 2.2.6. *Let K be a finite simplicial set, and let $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by $F_K(R) = \mathrm{Fun}(K, \mathrm{AVar}(R))^{\simeq}$. Then the functor F_K admits a cotangent complex which is connective and almost perfect.*

Remark 2.2.7. In the situation of Proposition 2.2.6, the connectivity of the cotangent complex L_{F_K} does not require the assumption that K is finite. In fact, the general case can be deduced from the case where K is finite, using Remark SAG.17.2.4.5.

Proof. Proposition 2.2.1 implies that the functor F_K is locally almost of finite presentation, and Lemma 2.2.5 guarantees that the cotangent complex $L_{F_K} \in \mathrm{QCoh}(F_K)$

exists and is (-1)-connective. Applying Corollary SAG.17.4.2.2, we deduce that L_{F_K} is almost perfect. We will complete the proof by showing that L_{F_K} is connective.

Let R be a connective $\mathbb{E}_\infty$-ring and choose a point in $\eta \in F_K(R)$. We wish to show that $M = \eta^* L_{F_K} \in \mathrm{Mod}_R$ is connective. Since M is (-1)-connective and almost perfect, the homotopy group $\pi_{-1} M$ is finitely generated as a module over the commutative ring $\pi_0 R$. We wish to show that $\pi_{-1} M$ vanishes. Using Nakayama's lemma, we are reduced to proving that the vector space

$$\pi_{-1}(\kappa \otimes_R M) \simeq \mathrm{Tor}_0^{\pi_0 R}(\kappa, \pi_{-1} M)$$

vanishes for every residue field κ of R. We may therefore replace R by κ and thereby reduce to the case where $R = \kappa$ is a field. Moreover, we may assume without loss of generality that κ is algebraically closed.

Let $A = \kappa[\epsilon]/(\epsilon^2)$ denote the ring of dual numbers over κ, and let $\eta_A \in F_K(A) = \mathrm{Fun}(K, \mathrm{AVar}(A))^\simeq$ denote the image of η. Unwinding the definitions, we see that the dual space $\mathrm{Hom}_\kappa(\pi_{-1} M, \kappa)$ can be identified with the set of automorphisms of η_A which restrict to the identity automorphism of η. We wish to prove that every such automorphism is trivial. To establish this, it suffices to treat the case where $K = \Delta^0$, in which case we are reduced to the following classical assertion:

$(*)$ Let X be an abelian variety over κ and let f be an automorphism of $\mathsf{X}_A = \mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{X}$ (in the category of abelian varieties over A). If f restricts to the identity on X, then $f = \mathrm{id}_{\mathsf{X}_A}$.

To prove $(*)$, let $g : \mathsf{X}_A \to \mathsf{X}_A$ denote the difference $f - \mathrm{id}_{\mathsf{X}_A}$ (computed with respect to the group structure on X). If f restricts to the identity on X, then g factors set-theoretically through the closed point of $|\mathsf{X}_A| \simeq |\mathsf{X}|$. Choose an affine open subspace $\mathsf{U} \subseteq \mathsf{X}$ containing the identity of X (which exists by virtue of Proposition 1.4.7), and write $\mathsf{U} = \mathrm{Sp\acute{e}t}\, R$ for some commutative κ-algebra R. The identity section of X then determines a κ-algebra map $e : R \to \kappa$. The map g factors as a composition $\mathsf{X}_A \to g_0 \mathrm{Sp\acute{e}t}\, A \times_{\mathrm{Sp\acute{e}t}\, \kappa} \mathsf{U} \hookrightarrow \mathsf{X}_A$, where g_0 is classified by a morphism of A-algebras $\phi : R[\epsilon]/(\epsilon^2) \to \pi_0 \Gamma(\mathsf{X}_A; \mathscr{O}_{\mathsf{X}_A})$.

Let $u : A \to \pi_0 \Gamma(\mathsf{X}_A; \mathscr{O}_{\mathsf{X}_A})$ be the unit map. Since the projection map $\mathsf{X}_A \to \mathrm{Sp\acute{e}t}\, A$ is proper, locally almost of finite presentation, geometrically reduced, and geometrically connected, the map u is an isomorphism (Proposition SAG.8.6.4.1). Let $\bar{e} : \pi_0 \Gamma(\mathsf{X}_A; \mathscr{O}_{\mathsf{X}_A}) \to A$ be the map given by evaluation at the identity of X_A. Then $\bar{e}$ is left inverse to u, and is therefore also an isomorphism. To show that $f = \mathrm{id}_{\mathsf{X}_A}$, we

must show that ϕ coincides with the composition

$$\psi : R[\epsilon]/(\epsilon^2) \xrightarrow{e} \kappa[\epsilon]/(\epsilon^2) \xrightarrow{u} \pi_0\Gamma(\mathsf{X}_A; \mathscr{O}_{\mathsf{X}_A}).$$

Equivalently, we must show that $\bar{e} \circ \phi = \bar{e} \circ \psi$. This follows from the observation that g vanishes along the identity section of X_A (since f is an automorphism of abelian varieties over X_A, and therefore preserves the identity sections). $\qquad\square$

2.3 Deformation Theory of Strict Abelian Varieties

We now establish analogues of the results of §2.2 for the theory of strict abelian varieties.

Proposition 2.3.1. *Let R be a connective $\mathbb{E}_\infty$-ring and suppose we are given strict abelian varieties $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}^s(R)$. For every morphism of connective $\mathbb{E}_\infty$-rings $R \to R'$, let $\mathsf{X}_{R'} = \mathrm{Sp\acute{e}t}\, R' \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ and $\mathsf{Y}_{R'} = \mathrm{Sp\acute{e}t}\, R' \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ be the images of X and Y in $\mathrm{AVar}^s(R)$, and set $F(R') = \mathrm{Map}_{\mathrm{AVar}^s(R')}(\mathsf{X}_{R'}, \mathsf{Y}_{R'})$. For each $n \geqslant 0$, the functor $F : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}$.*

Proof. Fix an integer $n \geqslant 0$; we will show that the functor $F|_{\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}}$ commutes with filtered colimits. Without loss of generality, we may replace R by $\tau_{\leqslant n}R$ and thereby reduce to the case where R is n-truncated. For every object $R' \in \tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}$, let us use Remark 1.4.6 identify $\mathrm{AVar}^s(R')$ with its essential image in $\mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$, and let $\mathsf{X}_{R'}$ and $\mathsf{Y}_{R'}$ denote the images of X and Y in $\mathrm{AVar}^s(R)$. If M is a connective $\mathbf{Z}$-$\mathbf{Z}$ bimodule spectrum, we define $F_M : \tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ by the formula $F_M(R') = \mathrm{Map}_{\mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(M \otimes_{\mathbf{Z}} \mathsf{X}_{R'}, \mathsf{Y}_{R'})$. We will say that M is *good* if the functor F_M commutes with filtered colimits. We now proceed in several steps:

(a) The construction $M \mapsto F_M$ carries colimits in the ∞-category $\mathrm{Mod}_{\mathbf{Z}\otimes\mathbf{Z}}^{\mathrm{cn}}$ to limits in the ∞-category $\mathrm{Fun}(\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$. Consequently, the collection of good objects $M \in \mathrm{Mod}_{\mathbf{Z}\otimes\mathbf{Z}}^{\mathrm{cn}}$ is closed under finite colimits.

(b) The module $M = \mathbf{Z} \otimes \mathbf{Z}$ is good. This follows from the calculation

$$
\begin{aligned}
F_M(R') &= \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}((\mathbf{Z} \otimes \mathbf{Z}) \otimes_{\mathbf{Z}} \mathsf{X}_{R'}, \mathsf{Y}_{R'}) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathrm{Sp}^{\mathrm{cn}})}(\mathsf{X}_{R'}, \mathsf{Y}_{R'}) \\
&\simeq \mathrm{Map}_{\mathrm{AVar}(R)}(\mathsf{X}_{R'}, \mathsf{Y}_{R'}),
\end{aligned}
$$

together with Proposition 2.2.1.

(c) Every connective perfect $(\mathbf{Z} \otimes \mathbf{Z})$-module is good; this follows from (a) and (b).

(d) If $f : M \to M'$ is a morphism of connective $(\mathbf{Z} \otimes \mathbf{Z})$-modules that induces an equivalence $\tau_{\leqslant n} M \to \tau_{\leqslant n} M'$, then the induced map $F_{M'} \to F_M$ is an equivalence of functors. Consequently, M is good if and only if M' is good.

(e) Every almost perfect $(\mathbf{Z} \otimes \mathbf{Z})$-module is good (this follows from (c) and (d), by virtue of Corollary SAG.2.7.2.2).

To complete the proof, we note that $F|_{\tau_{\leqslant n} \mathrm{CAlg}_R^{\mathrm{cn}}}$ can be identified with the functor $F_{\mathbf{Z}}$. By virtue of (e), we are reduced to showing that $\mathbf{Z}$ is almost perfect when regard as a $(\mathbf{Z} \otimes \mathbf{Z})$-module. This follows from the criterion of Proposition HA.7.2.4.17, since the $\mathbb{E}_\infty$-ring $\mathbf{Z} \otimes \mathbf{Z}$ is Noetherian (which follows from Theorem HA.7.2.4.31, since $\mathbf{Z} \otimes \mathbf{Z}$ is almost of finite presentation over $\mathbf{Z}$). $\qquad\square$

We need the following analogue of Lemma 2.2.5:

Lemma 2.3.2. *Let K be a simplicial set and let $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula $F_K(R) = \mathrm{Fun}(K, \mathrm{AVar}^s(R))^{\simeq}$. Then the functor F_K admits a (-1)-connective cotangent complex.*

Proof. We proceed as in the proof of Lemma 2.2.5, with some minor modifications. Define $G : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $G(R) = \mathrm{Fun}(\mathrm{Lat}^{\mathrm{op}} \times K, \mathrm{Var}(R))^{\simeq}$. For every connective $\mathbb{E}_\infty$-ring R, we can identify $F_K(R)$ with the summand of $G(R)$ spanned by those functors $X : \mathrm{Lat}^{\mathrm{op}} \times K \to \mathrm{Var}(R)$. which satisfy the following condition:

($*$) For every vertex $v \in K$, the restriction $X|_{\mathrm{Lat}^{\mathrm{op}} \times \{v\}} : \mathrm{Lat}^{\mathrm{op}} \to \mathrm{Var}(R)$ is an abelian group object of $\mathrm{Var}(R)$.

Theorem SAG.19.4.0.2 implies that the functor G admits a (-1)-connective cotangent complex $L_G \in \mathrm{QCoh}(G)$. We claim that the image L_G in the ∞-category $\mathrm{QCoh}(F_K)$ is a cotangent complex for F_K. To prove this, it will suffice to show that for every connective $\mathbb{E}_\infty$-ring R and every connective R-module M, the diagram

$$
\begin{array}{ccc}
F_K(R \oplus M) & \longrightarrow & F_K(R) \\
\downarrow & & \downarrow \\
G(R \oplus M) & \longrightarrow & G(R)
\end{array}
$$

is a pullback square. Without loss of generality, we may assume that $K = \Delta^0$; in this case, we must show that a diagram $\mathrm{Lat}^{\mathrm{op}} \to \mathrm{Var}(R \oplus M)$ commutes with finite

products if and only if the composite map $\mathrm{Lat}^{\mathrm{op}} \to \mathrm{Var}(R \oplus M) \to \mathrm{Var}(R)$ commutes with finite products $\mathrm{Var}(R)$. This is clear, since the extension-of-scalars functor $\mathrm{Var}(R \oplus M) \to \mathrm{Var}(R)$ is conservative (Proposition 1.1.4). $\qquad\square$

Proposition 2.3.3. *The construction $R \mapsto \mathrm{AVar}^s(R)$ is locally almost of finite presentation: that is, it commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$, for each $n \geqslant 0$.*

Proof. For every simplicial set K, define $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $F_K(R) = \mathrm{Fun}(K, \mathrm{AVar}^s(R))^{\simeq}$. To prove Proposition 2.3.3, it will suffice to show that whenever K is finite, the functor F_K is locally almost of finite presentation (that is, it commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$ for each $n \geqslant 0$). In fact, we prove more generally that for every inclusion of finite simplicial sets $K' \hookrightarrow K$, the restriction map $F_K \to F_{K'}$ is locally almost of finite presentation. Our proof proceeds by induction on the dimension k of K. Since the collection of morphisms which are locally almost of finite presentation is closed under composition, we can reduce to the case where K is obtained from K' by adjoining a single nondegenerate simplex: that is, there exists a pushout diagram

$$
\begin{array}{ccc}
\partial\,\Delta^d & \longrightarrow & \Delta^d \\
\downarrow & & \downarrow \\
K' & \longrightarrow & K
\end{array}
$$

for some $d \leqslant k$. We then have a pullback diagram

$$
\begin{array}{ccc}
F_K & \longrightarrow & F_{K'} \\
\downarrow & & \downarrow \\
F_{\Delta^d} & \longrightarrow & F_{\partial\,\Delta^d}.
\end{array}
$$

We may therefore replace K by Δ^d and K' by $\partial\,\Delta^d$. If $d \geqslant 2$, then we have a commutative diagram

$$
\begin{array}{ccc}
F_{\Delta^d} & \longrightarrow & F_{\partial\,\Delta^d} \\
& \searrow \quad \swarrow & \\
& F_{\Lambda_1^d} &
\end{array}
$$

where the left diagonal morphism is an equivalence (and therefore locally almost of finite presentation), and the right diagonal morphism is locally almost of finite

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presentation by our inductive hypothesis. It then follows that the horizontal arrow is a locally almost of finite presentation, as desired. If $d = 1$, then the assertion that that $F_{\Delta^d} \to F_{\partial \Delta^d}$ is locally almost of finite presentation is a reformulation of Proposition 2.3.1.

It will therefore suffice to treat the case $d = 0$. We have evident maps of simplicial sets $\partial \Delta^1 \hookrightarrow \Delta^1 \to \Delta^0$ which induce maps $F_{\Delta^0} \xrightarrow{\delta} F_{\Delta^1} \to F_{\partial \Delta^1}$, hence a fiber sequence $\delta^* L_{F_{\Delta^1}/F_{\partial \Delta^1}} \to L_{F_{\Delta^0}/F_{\partial \Delta^1}} \to L_{F_{\Delta^0}/F_{\Delta^1}}$ (where the relevant cotangent complexes are well-defined by virtue of Lemma 2.3.2). The first term in this fiber sequence is almost perfect (this follows from Proposition 2.3.1 and Corollary SAG.17.4.2.2) and the third term vanishes (since the forgetful functor $\mathrm{Var}(R \oplus M) \to \mathrm{Var}(R)$ is conservative for any connective $\mathbb{E}_\infty$-ring R and every connective R-module M, by virtue of Proposition 1.1.4). It follows that the cotangent complex $L_{F_{\Delta^0}/F_{\partial \Delta^1}}$ is almost perfect. We now observe that the identification $F_{\partial \Delta^1} \simeq F_{\Delta^0} \times F_{\Delta^0}$ induces an equivalence $L_{F_{\Delta^0}/F_{\partial \Delta^1}} \simeq \Sigma L_{F_{\Delta^0}}$, so that $L_{F_{\Delta^0}}$ is almost perfect. Using Propositions 1.5.6 and 2.2.1, we deduce that the functor F_{Δ^0} commutes with filtered colimits when restricted to $\mathrm{CAlg}^\heartsuit$. Applying Corollary SAG.17.4.2.2 (note that F_{Δ^0} is infinitesimally cohesive by Proposition 2.1.4), we conclude that F_{Δ^0} is locally almost of finite presentation, as desired. $\qquad\square$

Proposition 2.3.4. *Let K be a finite simplicial set, and let $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by $F_K(R) = \mathrm{Fun}(K, \mathrm{AVar}^s(R))^\simeq$. Then the functor F_K admits a cotangent complex which is connective and almost perfect.*

Proof. Proposition 2.3.3 implies that the functor F_K is locally almost of finite presentation, and Lemma 2.3.2 guarantees that the cotangent complex $L_{F_K} \in \mathrm{QCoh}(F_K)$ exists. Applying Corollary SAG.17.4.2.2, we deduce that L_{F_K} is almost perfect. To complete the proof, it will suffice to show that L_{F_K} is connective. Equivalently, we must show that if R is a discrete $\mathbb{E}_\infty$-ring and M is a discrete R-module, then the forgetful functor $\theta : \mathrm{Fun}(K, \mathrm{AVar}^s(R \oplus M))^\simeq \to \mathrm{Fun}(K, \mathrm{AVar}^s(R))^\simeq$ is injective on automorphism groups. Using Proposition 1.5.6, we can identify θ with the forgetful functor $\mathrm{Fun}(K, \mathrm{AVar}(R \oplus M))^\simeq \to \mathrm{Fun}(K, \mathrm{AVar}(R))^\simeq$, so that the desired result follows from Proposition 2.2.6. $\qquad\square$

2.4 The Moduli Stack of Elliptic Curves

We are now ready to give the proof of Theorem 2.0.3. In fact, we will establish the following slightly stronger assertion:

Theorem 2.4.1. *The functors $R \mapsto \mathrm{Ell}^s(R)^{\simeq}$ and $R \mapsto \mathrm{Ell}(R)^{\simeq}$ are representable by spectral Deligne-Mumford 1-stacks $\mathcal{M}^s$ and $\mathcal{M}$ which are locally almost of finite presentation over the sphere spectrum.*

Remark 2.4.2. Let $\mathcal{M}^c$ denote the classical moduli stack of elliptic curves, which we regard as a 0-truncated spectral Deligne-Mumford stack. Then $\mathcal{M}^c$ represents the functor $R \mapsto \mathrm{Ell}^s(R)^{\simeq} \simeq \mathrm{Ell}(R)^{\simeq}$ on the category $\mathrm{CAlg}^{\heartsuit}$ of commutative rings. It follows that we can identify $\mathcal{M}^c$ with the ordinary Deligne-Mumford stack underlying both $\mathcal{M}$ and $\mathcal{M}^w$. In particular, we have closed immersions of spectral Deligne-Mumford stacks $\mathcal{M}^c \to \mathcal{M}^s \to \mathcal{M}$ which are equivalences at the level of the underlying ∞-topoi. At the level of structure sheaves, one can show that both of these maps are rational equivalences (for the map $\mathcal{M}^s \to \mathcal{M}$, this follows from Theorem 2.1.1). However, their integral structures are quite different.

Proof of Theorem 2.4.1. We will give the proof for the functor $R \mapsto \mathrm{Ell}^s(R)^{\simeq}$; the proof in the other case is similar. Let $F : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by $F(R) = \mathrm{Ell}^s(R)^{\simeq}$. Note that the inclusion map $\mathrm{Ell}^s(\bullet)^{\simeq} \hookrightarrow \mathrm{AVar}^s(\bullet)^{\simeq}$ is an open immersion of functors (Remark 2.0.1). Combining Proposition SAG.19.2.4.3 with Propositions 2.1.4, 2.1.5, 2.3.3, and 2.3.4, we deduce that the functor F is cohesive, nilcomplete, locally almost of finite presentation, and admits a cotangent complex which is connective and almost perfect. It follows immediately from the definitions that the functor F satisfies descent for the étale topology and carries discrete $\mathbb{E}_\infty$-rings to 1-truncated spaces. By virtue of the spectral version of Artin's representability theorem (Theorem SAG.18.3.0.1), the functor F is representable by a spectral Deligne-Mumford 1-stack which is locally almost of finite presentation over the sphere spectrum if and only if it is integrable (Definition SAG.17.3.1.1). Using Proposition SAG.17.3.5.1, we can formulate the integrability of F as follows:

($*$) Let A be a complete local Noetherian commutative ring with maximal ideal $\mathfrak{m}$. Then the canonical map $F(A) \to \varprojlim_n F(\varprojlim A/\mathfrak{m}^n)$ is a homotopy equivalence.

In the situation of ($*$), Corollary SAG.8.5.3.3 implies that the canonical map $\rho : \mathrm{Var}(A) \to \varprojlim_n \mathrm{Var}(A/\mathfrak{m}^n)$ is a fully faithful embedding (which commutes with finite products). It follows that we have a pullback diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{AVar}^s(A) & \longrightarrow & \varprojlim_n \mathrm{AVar}^s(A/\mathfrak{m}^n) \\
\downarrow & & \downarrow \\
\mathrm{Var}(A) & \longrightarrow & \varprojlim_n \mathrm{Var}(A/\mathfrak{m}^n).
\end{array}
$$

Consequently, the upper horizontal map is a fully faithful embedding. We wish to show that its essential image contains every object of $\varprojlim_n \mathrm{Ell}^s(A/\mathfrak{m}^n)$, which is an immediate consequence of Proposition SAG.**??** .

Let $\mathcal{M}$ be a spectral Deligne-Mumford stack which represents the functor F and let $F^+ : \mathrm{CAlg} \to \mathcal{S}$ be the functor given by $F^+(R) = \mathrm{Map}_{\mathrm{SpDM}^{\mathrm{nc}}}(\mathrm{Sp\acute{e}t}\, R, \mathcal{M})$. Since the structure sheaf $\mathscr{O}_{\mathcal{M}}$ is connective, the functor F^+ is a left Kan extension of its restriction $F^+|_{\mathrm{CAlg}^{\mathrm{cn}}}$ to connective $\mathbb{E}_\infty$-rings: more concretely, the canonical map $F^+(\tau_{\geqslant 0}R) \to F^+(R)$ is an equivalence for every $\mathbb{E}_\infty$-ring R. To complete the proof, it will suffice to show that the functor $R \mapsto \mathrm{Ell}^s(R)^{\simeq}$ has the same property, which follows from Remark 1.5.3. $\qquad\square$

3 Cartier Duality

Let A be a commutative ring and let $\mathrm{FFG}(A)$ denote the category of commutative finite flat group schemes over A. The category $\mathrm{FFG}(A)$ is equipped with a canonical anti-involution $G \mapsto \mathbf{D}(G)$, given by *Cartier duality.* This operation admits several descriptions:

(a) Let G be a commutative finite flat group scheme over A. For every commutative A-algebra B, the abelian group $\mathbf{D}(G)(B)$ of B-valued points of $\mathbf{D}(G)$ can be identified with the set of morphisms $f : G_B \to \mathrm{GL}_1$ in the category of group schemes over B, where $G_B = \mathrm{Spec}\, B \times_{\mathrm{Spec}\, A} G$ denote the group scheme over B obtained from G by extension of scalars.

(b) Let G be a commutative finite flat group scheme over A. Then we can write $G = \mathrm{Spec}\, H$, where H is a (commutative and cocommutative) Hopf algebra over A which is projective of finite rank as an A-module. The A-linear dual $H^\vee$ is also a commutative and cocommutative Hopf algebra over A, whose spectrum $\mathrm{Spec}\, H^\vee$ can be identified with the Cartier dual $\mathbf{D}(G)$ of G.

In this section, we study a generalization of Cartier duality, where we replace category $\mathrm{Mod}_A^{\heartsuit}$ of (discrete) A-modules by an arbitrary symmetric monoidal ∞-category. To any symmetric monoidal ∞-category $\mathcal{C}$, we associate an ∞-category $\mathrm{bAlg}(\mathcal{C})$ of (commutative and cocommutative) *bialgebra objects* of $\mathcal{C}$ (Definition 3.3.1). The ∞-category $\mathrm{bAlg}(\mathcal{C})$ comes equipped with fully faithful "functor of points" $\mathrm{Spec}^{\mathcal{C}} : \mathrm{bAlg}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ (Variant 3.4.2). Our main result (Proposition 3.8.1) asserts that if H is a bialgebra object of $\mathcal{C}$ which is dualizable in $\mathcal{C}$, then the dual

$H^\vee$ can also be regarded as a bialgebra object of $\mathcal{C}$, so that the spectrum $\mathrm{Spec}^{\mathcal{C}} H^\vee$ classifies natural transformations $\mathrm{Spec}^{\mathcal{C}} H \to \mathbf{A}^1$, where $\mathbf{A}^1 : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ denotes the "affine line" (that is, the functor corepresented by the unit object $\mathbf{1} \in \mathcal{C}$; see Construction 3.5.1). More informally, this result asserts that the equivalence between (*a*) and (*b*) extends to the setting of an arbitrary symmetric monoidal ∞-category $\mathcal{C}$ (and to commutative monoids which are not necessarily grouplike; we specialize to the grouplike case in §3.9).

In this paper, we will be primarily interested in the following examples:

- When A is a commutative ring and we take $\mathcal{C}$ to be the category $\mathrm{Mod}_A^\heartsuit$ of discrete A-modules, then our theory reproduces the classical theory of Cartier duality (in a slightly more general form: it applies to all finite flat commutative monoid schemes over A, rather than merely to finite flat commutative group schemes).

- If A is an $\mathbb{E}_\infty$-ring and we take $\mathcal{C}$ to be the ∞-category Mod_A of all A-modules, then we obtain an analogue of Cartier duality in the setting of spectral algebraic geometry. We will return to this example in §6.3.

- Let A be a connective $\mathbb{E}_\infty$-ring and take $\mathcal{C} = \mathrm{LinCat}_A^{\mathrm{St}}$ to be the ∞-category of stable A-linear ∞-categories (see Definition SAG.D.1.4.1). The theory of Tannaka duality supplies a fully faithful embedding $\mathrm{Var}(A) \hookrightarrow \mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$, which extends to a fully faithful embedding $\mathrm{AVar}(A) \hookrightarrow \mathrm{bAlg}(\mathcal{C})^{\mathrm{op}}$ (see Proposition 4.4.1). In this case, the theory of Cartier duality developed in this section specializes to the duality theory of abelian varieties, which we will explain in detail in §4 and §5.

3.1 Coalgebra Objects of ∞-Categories

We begin with some general remarks. For every symmetric monoidal ∞-category $\mathcal{C}$, we let $\mathrm{CAlg}(\mathcal{C})$ denote the ∞-category of commutative algebra objects of $\mathcal{C}$ (Definition HA.2.1.3.1). Note that a symmetric monoidal structure on an ∞-category $\mathcal{C}$ determines a symmetric monoidal structure on the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$ (Remark HA.2.4.2.7), which motivates the following:

Definition 3.1.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *commutative coalgebra object of* $\mathcal{C}$ is a commutative algebra object of the opposite ∞-category $\mathcal{C}^{\mathrm{op}}$. We let $\mathrm{cCAlg}(\mathcal{C})$ denote the ∞-category $\mathrm{CAlg}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$; we will refer to $\mathrm{cCAlg}(\mathcal{C})$ as the ∞-*category of commutative coalgebra objects of* $\mathcal{C}$.

Proposition 3.1.2. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let K be a simplicial set. If $\mathcal{C}$ admits K-indexed colimits, then the ∞-category $\mathrm{cCAlg}(\mathcal{C})$ admits K-indexed colimits, and the forgetful functor $\mathrm{cCAlg}(\mathcal{C}) \to \mathcal{C}$ preserves K-indexed colimits.*

Proof. Apply Corollary HA.3.2.2.5 to the symmetric monoidal ∞-category $\mathcal{C}^{\mathrm{op}}$. $\square$

Proposition 3.1.3. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that $\mathcal{C}$ is accessible and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is accessible. Then the ∞-category $\mathrm{cCAlg}(\mathcal{C})$ of commutative coalgebra objects of $\mathcal{C}$ is accessible.*

Proof. Let us regard $\mathcal{C}$ as a commutative monoid object of the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of ∞-categories which are not necessarily small, given by a map $F : \mathcal{F}\mathrm{in}_* \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. The functor F is classified by a Cartesian fibration $q : \mathcal{E} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, whose fiber over an object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ can be identified with $\mathcal{C}^n$. Let $\mathcal{D} = \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}}(\mathcal{F}\mathrm{in}_*^{\mathrm{op}}, \mathcal{E})$ denote the ∞-category of sections of q. Unwinding the definitions, we can identify $\mathrm{cCAlg}(\mathcal{C})$ with the full subcategory of $\mathcal{D}$ spanned by those functors $F : \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \to \mathcal{E}$ which carry inert morphisms of $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ to q-Cartesian morphisms in $\mathcal{E}$. It follows from Corollary HTT.5.4.7.17 that the ∞-category $\mathcal{D}$ is accessible and that, for each object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, evaluation at $\langle n \rangle$ determines an accessible functor $\mathcal{E} \to \mathcal{C}^n$. Applying Proposition HTT.5.4.6.6, we deduce that $\mathrm{cCAlg}(\mathcal{C})$ is also accessible. $\square$

Corollary 3.1.4. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that the ∞-category $\mathcal{C}$ is presentable and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is accessible (this condition is satisfied, for example, if the functor $\otimes$ preserves small colimits separately in each variable). Then the ∞-category $\mathrm{cCAlg}(\mathcal{C})$ is presentable and the forgetful functor $\mathrm{cCAlg}(\mathcal{C}) \to \mathcal{C}$ preserves small colimits.*

Proof. Combine Propositions 3.1.2 and 3.1.3. $\square$

Corollary 3.1.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that the ∞-category $\mathcal{C}$ is presentable and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is accessible. Then the forgetful functor $\mathrm{cCAlg}(\mathcal{C}) \to \mathcal{C}$ admits a right adjoint $V : \mathcal{C} \to \mathrm{cCAlg}(\mathcal{C})$.*

Proof. Use Corollary 3.1.4 and the adjoint functor theorem (Corollary HTT.5.5.2.9). $\square$

Remark 3.1.6. In the situation of Corollary 3.1.5, we will refer to the functor $V : \mathcal{C} \to \mathrm{cCAlg}(\mathcal{C})$ as the *cofree coalgebra* functor. We do not know any explicit description of the functor V, except in special cases. On a similar note, Corollary 3.1.4 implies that the ∞-category $\mathrm{cCAlg}(\mathcal{C})$ admits small limits, which we do not know how to construct directly.

3.2 Duality between Algebra and Coalgebra Objects

Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. Recall that an object $C \in \mathcal{C}$ is said to be *dualizable* if there exists another object $C^{\vee} \in \mathcal{C}$, together with maps

$$e : C^{\vee} \otimes C \to \mathbf{1} \qquad c : \mathbf{1} \to C \otimes C^{\vee}$$

for which the composite maps

$$C \simeq \mathbf{1} \otimes C \xrightarrow{c \otimes \mathrm{id}} C \otimes C^{\vee} \otimes C \xrightarrow{\mathrm{id} \otimes e} C \otimes \mathbf{1} \simeq C$$

$$C^{\vee} \simeq C^{\vee} \otimes \mathbf{1} \xrightarrow{\mathrm{id} \otimes c} C^{\vee} \otimes C \otimes C^{\vee} \xrightarrow{e \otimes \mathrm{id}} \mathbf{1} \otimes C^{\vee} \simeq C^{\vee}$$

are homotopic to the identity (for more details, we refer the reader to §HA.4.6.1). We let $\mathcal{C}_{\mathrm{fd}}$ denote the full subcategory of $\mathcal{C}$ spanned by the dualizable objects. This choice of notation is motivated by the following example:

Example 3.2.1. Let κ be a field and let $\mathcal{C} = \mathrm{Mod}_{\kappa}^{\heartsuit}$ be the category of vector spaces over κ. Then $\mathcal{C}_{\mathrm{fd}}$ is the category of finite-dimensional vector spaces over κ. More generally, if $\mathcal{C} = \mathrm{Mod}_{A}^{\heartsuit}$ is the category of (discrete) modules over a commutative ring A, then $\mathcal{C}_{\mathrm{fd}}$ is the full subcategory of $\mathcal{C}$ spanned by those A-modules which are projective of finite rank.

Example 3.2.2. Let A be a connective $\mathbb{E}_{\infty}$-ring and let $\mathcal{C} = \mathrm{Mod}_{A}^{\mathrm{cn}}$ denote the ∞-category of connective A-modules. Then $\mathcal{C}_{\mathrm{fd}}$ is the full subcategory of $\mathcal{C}$ spanned by the projective A-modules of finite rank.

Example 3.2.3. Let A be an $\mathbb{E}_{\infty}$-ring and let $\mathcal{C} = \mathrm{Mod}_{A}$ denote the ∞-category of all A-modules. Then $\mathcal{C}_{\mathrm{fd}}$ is the full subcategory of $\mathcal{C}$ spanned by the perfect A-modules.

If C is a dualizable object of a symmetric monoidal ∞-category $\mathcal{C}$, then the dual $C^{\vee}$ depends functorially on C. In fact, we can be more precise:

Proposition 3.2.4. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the construction $C \mapsto C^{\vee}$ determines an equivalence of symmetric monoidal ∞-categories $\mathcal{C}_{\mathrm{fd}}^{\mathrm{op}} \to \mathcal{C}_{\mathrm{fd}}$.*

Proof. Replacing $\mathcal{C}$ by $\mathcal{C}_{\mathrm{fd}}$ if necessary, we may assume that every object of $\mathcal{C}$ is dualizable. Let $\mathcal{M}$ denote the fiber product $(\mathcal{C} \times \mathcal{C}) \times_{\mathcal{C}} \mathcal{C}_{/\mathbf{1}}$ whose objects are triples (C, D, e), where C and D are objects of $\mathcal{C}$ and $e : C \otimes D \to \mathbf{1}$ is a morphism in $\mathcal{C}$. Then projection onto the first factor determines a right fibration $\lambda : \mathcal{M} \to \mathcal{C} \times \mathcal{C}$. We regard λ as a pairing of ∞-categories, in the sense of Definition HA.5.2.1.8. Our

assumption that every object of $\mathcal{C}$ is dualizable guarantees that λ is left (and right) representable, and therefore determines a duality functor $\mathfrak{D}_\lambda : \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$, given on objects by $C \mapsto C^\vee$. Note that an object $(C, D, e) \in \mathcal{M}$ is left universal (in the sense of Definition HA.5.2.1.8) if and only if it is right universal: both conditions are equivalent to the requirement that e exhibits C and D as duals of one another. Applying Corollary HA.5.2.1.22, we deduce that $\mathfrak{D}_\lambda$ is an equivalence of ∞-categories.

Note that the symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal structure on $\mathcal{M}$ and on the functor λ. We may therefore regard λ as a pairing of symmetric monoidal ∞-categories, in the sense of Definition HA.5.2.2.20. It follows from Remark HA.5.2.2.25 that the functor $\mathfrak{D}_\lambda$ inherits the structure of a lax symmetric monoidal functor. To show that this functor is actually symmetric monoidal, it suffices to observe that the collection of left representable objects of $\mathcal{M}$ is closed under tensor products. $\qquad\square$

In the situation of Proposition 3.2.4, the construction $C \mapsto C^\vee$ exchanges commutative algebras with commutative coalgebras:

Corollary 3.2.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then there is a canonical equivalence of ∞-categories* $\mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \simeq \mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}})$, *given on objects by the construction $A \mapsto A^\vee$.*

Remark 3.2.6 (Functoriality). The equivalence of Corollary 3.2.5 depends functorially on $\mathcal{C}$, in the following precise sense: the constructions $\mathcal{C} \mapsto \mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}}$ and $\mathcal{C} \mapsto \mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}})$ are equivalent when regarded as functors $\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty) \to \mathcal{C}\mathrm{at}_\infty$. More informally: if $F : \mathcal{C} \to \mathcal{D}$ is a symmetric monoidal functor, then F carries dualizable objects of $\mathcal{C}$ to dualizable objects of $\mathcal{D}$, and we have a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} & \xrightarrow{\ F\ } & \mathrm{CAlg}(\mathcal{D}_{\mathrm{fd}})^{\mathrm{op}} \\
\downarrow & & \downarrow \\
\mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}}) & \xrightarrow{\ F\ } & \mathrm{cCAlg}(\mathcal{D}_{\mathrm{fd}}),
\end{array}
$$

where the vertical maps are the equivalences supplied by Corollary 3.2.5.

3.3 Bialgebra Objects of ∞-Categories

We now study objects of symmetric monoidal ∞-categories which simultaneously admit the structure of a commutative algebra and a commutative coalgebra, in a compatible way.

Definition 3.3.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, so that the ∞-category $\mathrm{CAlg}(\mathcal{C})$ inherits a symmetric monoidal structure (see Construction HA.3.2.4.1). A *bialgebra object of* $\mathcal{C}$ is a commutative coalgebra object of $\mathrm{CAlg}(\mathcal{C})$. We let $\mathrm{bAlg}(\mathcal{C})$ denote the ∞-category $\mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ of bialgebra objects of $\mathcal{C}$.

Remark 3.3.2. The notion of bialgebra object introduced in Definition 3.3.1 might more properly be referred to as a *commutative and cocommutative bialgebra object.* We will omit mention of commutativity, since we have no need to consider bialgebras which are not commutative (and cocommutative) in this paper.

It is not immediately obvious from Definition 3.3.1 that the notion of bialgebra object is self-dual: that is, that the ∞-categories $\mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ and $\mathrm{CAlg}(\mathrm{cCAlg}(\mathcal{C}))$ are equivalent. To see that this is the case, it is convenient to characterize the ∞-category $\mathrm{bAlg}(\mathcal{C})$ by a universal property. Recall that an ∞-category $\mathcal{E}$ is said to be *semiadditive* if it admits both finite products and finite coproducts, and the Cartesian symmetric monoidal structure on $\mathcal{E}$ agrees with the coCartesian symmetric monoidal structure on $\mathcal{E}$ (see Definition SAG.C.4.1.6).

Proposition 3.3.3. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then:*

(a) *The ∞-category $\mathrm{bAlg}(\mathcal{C})$ of bialgebra objects of $\mathcal{C}$ is semiadditive (Definition SAG.C.4.1.6): that is, it admits a zero object, and the canonical map $C \amalg D \to C \times D$ is an equivalence for every pair of objects $C, D \in \mathcal{C}$.*

(b) *The symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal structure on $\mathrm{bAlg}(\mathcal{C})$ which is both Cartesian and coCartesian.*

(c) *Let $\mathcal{E}$ be any semiadditive ∞-category, and regard $\mathcal{E}$ as equipped with the Cartesian symmetric monoidal structure (or, equivalently, with the coCartesian symmetric monoidal structure). Then the forgetful functor $\mathrm{bAlg}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathrm{bAlg}(\mathcal{C})) \to \mathrm{Fun}^{\otimes}(\mathcal{E}, \mathcal{C})$. Here $\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathcal{C})$ denotes the ∞-category of symmetric monoidal functors from $\mathcal{E}$ to $\mathcal{C}$, and $\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathrm{bAlg}(\mathcal{C}))$ is defined similarly.*

We can summarize Proposition 3.3.3 more informally as follows: if $\mathcal{C}$ is a symmetric monoidal ∞-category, then $\mathrm{bAlg}(\mathcal{C})$ is universal among Cartesian and coCartesian symmetric monoidal ∞-categories $\mathcal{E}$ equipped with a symmetric monoidal functor $\mathcal{E} \to \mathcal{C}$. Note that this description is manifestly self-dual:

Corollary 3.3.4. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then there is a canonical (symmetric monoidal) equivalence* $\mathrm{bAlg}(\mathcal{C}^{\mathrm{op}}) \simeq \mathrm{bAlg}(\mathcal{C})^{\mathrm{op}}$. *In other words, the ∞-categories* $\mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ *and* $\mathrm{CAlg}(\mathrm{cCAlg}(\mathcal{C}))$ *are canonically equivalent.*

Proof of Proposition 3.3.3. We first prove (*b*). Observe that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is coCartesian (Proposition HA.3.2.4.7); in particular, the ∞-category $\mathrm{CAlg}(\mathcal{C})$ admits finite coproducts. Applying Corollary HA.3.2.2.5 (to the ∞-category $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$), we deduce that the ∞-category $\mathrm{bAlg}(\mathcal{C}) = \mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ also admits finite coproducts, and that the forgetful functor $\theta : \mathrm{bAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ preserves finite coproducts. Moreover, Proposition HA.3.2.4.7 implies that the symmetric monoidal structure on $\mathrm{bAlg}(\mathcal{C})$ is Cartesian (and in particular that $\mathrm{bAlg}(\mathcal{C})$ admits finite products). We will complete the proof of (*b*) by showing that the symmetric monoidal structure on $\mathrm{bAlg}(\mathcal{C})$ is also coCartesian: that is, it satisfies conditions (1) and (2) of Definition HA.2.4.0.1:

(1) Let $\mathbf{1}$ denote the unit object of $\mathrm{bAlg}(\mathcal{C})$; we wish to show that $\mathbf{1}$ is initial. Since the forgetful functor θ is conservative and preserves finite coproducts, it will suffice to show that $\theta(\mathbf{1})$ is an initial object of $\mathrm{CAlg}(\mathcal{C})$. Because θ is a symmetric monoidal functor, we can identify $\theta(\mathbf{1})$ with the unit object of $\mathrm{CAlg}(\mathcal{C})$. The desired result now follows from the fact that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is coCartesian (Proposition HA.3.2.4.7).

(2) Let C and D be objects of $\mathrm{bAlg}(\mathcal{C})$; we wish to show that the canonical maps

$$C \simeq C \otimes \mathbf{1} \xrightarrow{\mu} C \otimes D \xleftarrow{\nu} \mathbf{1} \otimes D \simeq D$$

exhibit $C \otimes D$ as a coproduct of C and D in the ∞-category $\mathrm{bAlg}(\mathcal{C})$. Because the functor θ is conservative and preserves finite coproducts, it will suffice to show that $\theta(\mu)$ and $\theta(\nu)$ exhibit $\theta(C \otimes D)$ as a coproduct of $\theta(C)$ and $\theta(D)$ in the ∞-category $\mathrm{CAlg}(\mathcal{C})$. This again follows from the fact that the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is coCartesian (Proposition HA.3.2.4.7).

Assertion (*a*) is a formal consequence of (*b*) (see Remark SAG.C.4.1.8). We now prove (*c*). Let $\mathcal{E}$ be a semiadditive ∞-category, and regard $\mathcal{E}$ as equipped with the Cartesian (or equivalently the coCartesian) symmetric monoidal structure. We wish to show that the composite functor $\mathrm{Fun}^{\otimes}(\mathcal{E}, \mathrm{bAlg}(\mathcal{C})) \xrightarrow{\rho} \mathrm{Fun}^{\otimes}(\mathcal{E}, \mathrm{CAlg}(\mathcal{C})) \xrightarrow{\rho'} \mathrm{Fun}^{\otimes}(\mathcal{E}, \mathcal{C})$ is an equivalence of ∞-categories. Since the symmetric monoidal structure on $\mathcal{E}$ is coCartesian, we can use Remark HA.3.2.4.9 to identify ρ' with the forgetful functor $\mathrm{Fun}(\mathcal{E}, \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C}))) \to \mathrm{Fun}(\mathcal{E}, \mathrm{CAlg}(\mathcal{C}))$, which is an equivalence by virtue of Example HA.3.2.4.5. A similar argument shows that ρ is an equivalence. $\square$

Corollary 3.3.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that the ∞-category $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Then the ∞-category $\mathrm{bAlg}(\mathcal{C})$ is presentable.*

Proof. Corollary HA.3.2.3.5 implies that the ∞-category $\mathrm{CAlg}(\mathcal{C})$ is presentable. Moreover, the tensor product functor $\otimes : \mathrm{CAlg}(\mathcal{C}) \times \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{C})$ preserves sifted colimits, and is therefore accessible. Applying Corollary 3.1.4, we deduce that the ∞-category $\mathrm{bAlg}(\mathcal{C}) = \mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}))$ is presentable. $\qquad\square$

Remark 3.3.6. Let $\mathcal{C}$ be as in Corollary 3.3.5. Using Corollaries 3.1.4 and 3.3.4, we deduce that the forgetful functor $\mathrm{bAlg}(\mathcal{C}) \simeq \mathrm{CAlg}(\mathrm{cCAlg}(\mathcal{C})) \to \mathrm{cCAlg}(\mathcal{C})$ admits a left adjoint, given by the construction $C \mapsto \mathrm{Sym}_{\mathcal{C}}^{*} C$. In other words, the commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{bAlg}(\mathcal{C}) & \longrightarrow & \mathrm{cCAlg}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{CAlg}(\mathcal{C}) & \longrightarrow & \mathcal{C}
\end{array}
$$

is left adjointable. More informally: if C is a commutative coalgebra object of $\mathcal{C}$, then the free commutative algebra $\mathrm{Sym}_{\mathcal{C}}^{*} C$ inherits the structure of a bialgebra object of $\mathcal{C}$. In particular, if $\mathbf{1}$ denotes the unit object of $\mathcal{C}$, then we can regard $\mathrm{Sym}_{\mathcal{C}}^{*} \mathbf{1}$ as a bialgebra object of $\mathcal{C}$.

3.4   The Spectrum of a Bialgebra

Let κ be a field and let H be a (commutative and cocommutative) Hopf algebra over κ. Then the comultiplication $\Delta : H \to H \otimes_{\kappa} H$ endows the affine scheme $\mathrm{Spec}\, H$ with the structure of a (commutative) group scheme over κ. In other words, the functor of points $(A \in \mathrm{CAlg}_{\kappa}^{\heartsuit}) \mapsto \mathrm{Map}_{\mathrm{CAlg}_{\kappa}^{\heartsuit}}(H, A)$ can be regarded as an abelian group in the functor category $\mathrm{Fun}(\mathrm{CAlg}_{\kappa}^{\heartsuit}, \mathcal{S}\mathrm{et})$. We now extend this observation to bialgebra objects of an arbitrary symmetric monoidal ∞-category.

Notation 3.4.1 (The Spectrum of a Commutative Algebra)**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category. For every commutative algebra object $A \in \mathrm{CAlg}(\mathcal{C})$, we let $\mathrm{Spec}^{\mathcal{C}}(A) : \mathrm{CAlg}(\mathcal{C}) \to \mathcal{S}$ denote the functor corepresented by A, given by the formula $\mathrm{Spec}^{\mathcal{C}}(A)(B) = \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(A, B)$. We will refer to $\mathrm{Spec}^{\mathcal{C}}(A)$ as the *spectrum of A*. We regard $\mathrm{Spec}^{\mathcal{C}}$ as a functor from the ∞-category $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ to the functor ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathcal{S})$. This functor is fully faithful (it is the Yoneda embedding for the ∞-category $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$).

Variant 3.4.2 (The Spectrum of a Bialgebra). Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then the symmetric monoidal structure on $\mathrm{CAlg}(\mathcal{C})$ is coCartesian (Proposition HA.3.2.4.7), so the induced monoidal structure on $\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$ is Cartesian. Applying Proposition HA.2.4.2.5, we obtain a canonical equivalence of ∞-categories $\mathrm{bAlg}(\mathcal{C})^{\mathrm{op}} = \mathrm{CAlg}(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}) \simeq \mathrm{CMon}(\mathrm{CAlg}(\mathcal{C})^{\mathrm{op}})$. Composing this equivalence with the Yoneda embedding $\mathrm{Spec}^{\mathcal{C}} : \mathrm{CAlg}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathcal{S})$ of Notation 3.4.1, we obtain a fully faithful functor

$$\mathrm{bAlg}(\mathcal{C})^{\mathrm{op}} \to \mathrm{CMon}(\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathcal{S})) = \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}).$$

We will abuse notation by denoting this functor also by $\mathrm{Spec}^{\mathcal{C}}$. By construction, we have a commutative diagram of ∞-categories

$$\begin{array}{ccc}
\mathrm{bAlg}(\mathcal{C})^{\mathrm{op}} & \longrightarrow & \mathrm{CAlg}(\mathcal{C})^{\mathrm{op}} \\
\downarrow{\scriptstyle \mathrm{Spec}^{\mathcal{C}}} & & \downarrow{\scriptstyle \mathrm{Spec}^{\mathcal{C}}} \\
\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\mathcal{S})) & \longrightarrow & \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathcal{S}),
\end{array}$$

where the horizontal maps are the evident forgetful functors. We can summarize the situation more informally as follows: if H is a bialgebra object of $\mathcal{C}$, then the spectrum $\mathrm{Spec}^{\mathcal{C}}(H)$ can be regarded as a CMon-valued functor on the ∞-category $\mathrm{CAlg}(\mathcal{C})$. In particular, if A is a commutative algebra object of $\mathcal{C}$, then the mapping space $\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(H, A)$ inherits the structure of an $\mathbb{E}_{\infty}$-space.

Remark 3.4.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ be a functor. The following conditions are equivalent:

(a) There exists an equivalence $X \simeq \mathrm{Spec}^{\mathcal{C}}(H)$ for some bialgebra object $H \in \mathrm{bAlg}(\mathcal{C})$.

(b) The composite functor $\mathrm{CAlg}(\mathcal{C}) \xrightarrow{X} \mathrm{CMon} \to \mathcal{S}$ is corepresentable.

In this case, we will say that the functor X is *corepresentable.*

3.5 The Affine Line

Let κ be a field and let $\mathbf{A}^1_{\kappa} = \mathrm{Spec}\,\kappa[t]$ denote the affine line over κ. For every commutative κ-algebra R, the set $\mathbf{A}^1_{\kappa}(R)$ of R-valued points of $\mathbf{A}^1_{\kappa}$ can be identified with R itself, and therefore inherits the structure of a commutative ring. We can

summarize the situation by saying that the affine line $\mathbf{A}^1_\kappa$ is a commutative ring object in the category of κ-schemes.

In this section, we will consider a generalization of the affine line, replacing the category $\mathrm{Mod}^\heartsuit_\kappa$ of κ-vector spaces by an arbitrary symmetric monoidal ∞-category $\mathcal{C}$. In this more general setting, one does not expect the affine line to admit an addition (the addition on $\mathbf{A}^1_\kappa$ is tied to the additive structure of the category $\mathrm{Mod}^\heartsuit_\kappa$). However, the multiplicative structure survives:

Construction 3.5.1 (The Affine Line)**.** Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $\mathbf{1}$ be the unit object of $\mathcal{C}$, and let $e : \mathcal{C} \to \mathcal{S}$ denote the functor given by $e(C) = \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, C)$. Then we can regard e as a lax symmetric monoidal functor (where we endow $\mathcal{S}$ with the Cartesian monoidal structure). Passing to commutative algebra objects, we obtain a functor $\mathbf{A}^1 : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CAlg}(\mathcal{S}) = \mathrm{CMon}$. We will refer to $\mathbf{A}^1$ as the *affine line*.

Example 3.5.2. Let R be a commutative ring and let $\mathcal{C} = \mathrm{Mod}^\heartsuit_R$ be the category of (discrete) R-modules, so that $\mathrm{CAlg}(\mathcal{C})$ is equivalent to the category of commutative R-algebras. Then the functor $\mathbf{A}^1 : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ assigns to each commutative R-algebra A its underlying set, regarded as a commutative monoid with respect to multiplication.

Example 3.5.3. Let A be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{C} = \mathrm{Mod}^{\mathrm{cn}}_A$ be the ∞-category of connective A-modules. Then $\mathrm{CAlg}(\mathcal{C})$ can be identified with the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}_A$ of connective $\mathbb{E}_\infty$-algebras over A. In this case, the affine line $\mathbf{A}^1 : \mathrm{CAlg}^{\mathrm{cn}}_A \to \mathrm{CMon}$ is given by the construction $B \mapsto \Omega^\infty B$, where we regard the 0th space $\Omega^\infty B$ as an $\mathbb{E}_\infty$-space using the multiplication on B.

Example 3.5.4. [Representability of $\mathbf{A}^1$] Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that $\mathcal{C}$ is presentable and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. Let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, and regard the symmetric algebra $\mathrm{Sym}^*_{\mathcal{C}} \mathbf{1}$ as a bialgebra object as in Remark 3.3.6 (so that, for every bialgebra object $H \in \mathrm{bAlg}(\mathcal{C})$, we have a canonical homotopy equivalence $\mathrm{Map}_{\mathrm{bAlg}(\mathcal{C})}(\mathrm{Sym}^*_{\mathcal{C}} \mathbf{1}, H) \simeq \mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C})}(\mathbf{1}, H))$. Unwinding the definitions, we see that the affine line $\mathbf{A}^1$ can be identified with the spectrum $\mathrm{Spec}^{\mathcal{C}}(\mathrm{Sym}^*_{\mathcal{C}} \mathbf{1})$ (see Variant 3.4.2).

Remark 3.5.5 (Functoriality)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞-categories and let $f : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. Suppose that f admits a

right adjoint g. Then g is a lax symmetric monoidal functor, and therefore induces a functor $G : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CAlg}(\mathcal{C})$. Let $\mathbf{A}_{\mathcal{C}}^1$ and $\mathbf{A}_{\mathcal{D}}^1$ be the functors obtained by applying Construction 3.5.1 to the ∞-categories $\mathcal{C}$ and $\mathcal{D}$, respectively. Unwinding the definitions, we see that $\mathbf{A}_{\mathcal{D}}^1$ is given by the composition $\mathrm{CAlg}(\mathcal{D}) \xrightarrow{G} \mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathbf{A}_{\mathcal{C}}^1} \mathrm{CMon}$.

Proposition 3.5.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let H be a bialgebra object of $\mathcal{C}$. Then there is a canonical homotopy equivalence*

$$\alpha : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(\mathrm{Spec}^{\mathcal{C}} H, \mathbf{A}^1) \simeq \mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C})}(\mathbf{1}, H).$$

The proof of Proposition 3.5.6 is based on the following:

Lemma 3.5.7. *Let $\mathcal{C}'$ be a symmetric monoidal ∞-category, let $\mathcal{C} \subseteq \mathcal{C}'$ be a full subcategory which contains the unit object and is closed under tensor products, and let H be a bialgebra object of $\mathcal{C}$ (so that we can also regard H as a bialgebra object of $\mathcal{C}'$). Then the functor $\mathrm{Spec}^{\mathcal{C}'}(H) : \mathrm{CAlg}(\mathcal{C}') \to \mathrm{CMon}$ is a left Kan extension of the functor $\mathrm{Spec}^{\mathcal{C}}(H) : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$.*

Proof. We prove a stronger claim: the functor $\mathrm{Spec}^{\mathcal{C}'}(H)$ is a left Kan extension of $\mathrm{Spec}^{\mathcal{C}} H$ when regarded as a functor from $\mathrm{CAlg}(\mathcal{C}')$ to $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{S})$. To prove this, it suffices to show that for each object $\langle k \rangle \in \mathcal{F}\mathrm{in}_*$, the functor $(\mathrm{Spec}^{\mathcal{C}'} H)(\langle k \rangle) : \mathrm{CAlg}(\mathcal{C}') \to \mathcal{S}$ is a left Kan extension of its restriction to $\mathrm{CAlg}(\mathcal{C})$. This is clear, since the functor $(\mathrm{Spec}^{\mathcal{C}'} H)(\langle k \rangle)$ is corepresented by an object of $\mathrm{CAlg}(\mathcal{C})$ (namely, the kth tensor power of H). $\square$

Proof of Proposition 3.5.6. Enlarging the universe if necessary, we may assume that the ∞-category $\mathcal{C}$ is small. Let $\mathcal{C}' = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ denote the ∞-category of $\mathcal{S}$-valued presheaves on $\mathcal{C}$, and let $j : \mathcal{C} \to \mathcal{C}'$ be the Yoneda embedding. We will abuse notation by identifying $\mathcal{C}$ with its essential image in $\mathcal{C}'$. Using Proposition HA.4.8.1.10, we deduce that $\mathcal{C}'$ admits an essentially unique symmetric monoidal structure (given by Day convolution) for which the functor j is symmetric monoidal and the tensor product $\otimes : \mathcal{C}' \times \mathcal{C}' \to \mathcal{C}'$ preserves small colimits separately in each variable. Let $\mathbf{A}_{\mathcal{C}}^1 : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ and $\mathbf{A}_{\mathcal{C}'}^1 : \mathrm{CAlg}(\mathcal{C}') \to \mathrm{CMon}$ be given by applying Construction 3.5.1 in the ∞-categories $\mathcal{C}$ and $\mathcal{C}'$, respectively. Using Example 3.5.4, we obtain canonical homotopy equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C})}(\mathbf{1}, H) &\simeq \mathrm{Map}_{\mathrm{cCAlg}(\mathcal{C}')}(\mathbf{1}, H) \\
&\simeq \mathrm{Map}_{\mathrm{bAlg}(\mathcal{C}')}(\mathrm{Sym}_{\mathcal{C}'}^* \mathbf{1}, H) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}'),\mathrm{CMon})}(\mathrm{Spec}^{\mathcal{C}'} H, \mathbf{A}_{\mathcal{C}'}^1).
\end{aligned}$$

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To complete the proof, it will suffice to show that the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}'),\mathrm{CMon})}(\mathrm{Spec}^{\mathcal{C}'} H, \mathbf{A}^1_{\mathcal{C}'}) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(\mathrm{Spec}^{\mathcal{C}} H, \mathbf{A}^1_{\mathcal{C}})$$

is a homotopy equivalence, which follows from Lemma 3.5.7. $\qquad\square$

3.6 Smash Products of $\mathbb{E}_\infty$-Spaces

Let X, Y, and Z be commutative monoids. We say that a map $b : X \times Y \to Z$ is *bilinear* if it satisfies the identities

$$b(x,0) = 0 = b(0,y) \quad b(x + x', y) = b(x,y) + b(x',y) \quad b(x, y + y') = b(x,y) + b(x,y').$$

For fixed X and Y, there exists a bilinear map $b_0 : X \times Y \to Z_0$ which is universal in the following sense: every bilinear map $b : X \times Y \to Z$ factors uniquely as a composition

$$X \times Y \xrightarrow{b_0} Z_0 \xrightarrow{\phi} Z,$$

where ϕ is a homomorphism of commutative monoids. In this case, we say that b_0 *exhibits Z_0 as the tensor product of X and Y* (in the category of commutative monoids). Note that when X and Y are abelian groups, this recovers the usual theory of tensor products in the category of abelian groups.

We now generalize this notion of tensor product to the setting of $\mathbb{E}_\infty$-spaces (that is, commutative monoid objects of the ∞-category $\mathcal{S}$ of spaces, rather than the ordinary category of sets). To avoid confusion with the various other notions of tensor product appearing in this paper, we will denote the relevant operation by $\wedge : \mathrm{CMon} \times \mathrm{CMon} \to \mathrm{CMon}$, which we refer to as the *smash product of $\mathbb{E}_\infty$-spaces* (this terminology is motivated by a close relationship with the usual smash product of spectra; see Remark 3.6.5 below).

Proposition 3.6.1 (The Smash Product)**.** *Let* CMon *denote the ∞-category of $\mathbb{E}_\infty$-spaces, and let* $\mathrm{Sym}^* : \mathcal{S} \to \mathrm{CMon}$ *denote a left adjoint to the forgetful functor. Then there exists an essentially unique symmetric monoidal structure on the ∞-category* CMon *with the following properties:*

(i) *The underlying bifunctor* $\mathrm{CMon} \times \mathrm{CMon} \to \mathrm{CMon}$ *preserves small colimits separately in each variable.*

(ii) *The functor* $\mathrm{Sym}^*_{\mathcal{S}} : \mathcal{S} \to \mathrm{CMon}$ *is symmetric monoidal, where we endow $\mathcal{S}$ with the Cartesian symmetric monoidal structure (in particular, for every pair of*

spaces X and Y, we have a canonical homotopy equivalence $\mathrm{Sym}^*_{\mathcal{S}}(X \times Y) \simeq$
$\mathrm{Sym}^*_{\mathcal{S}}(X) \wedge \mathrm{Sym}^*_{\mathcal{S}}(Y)$.

Notation 3.6.2. We will refer to the symmetric monoidal structure of Proposition 3.6.1 as the *smash product symmetric monoidal structure* on the ∞-category CMon, and will denote the underlying bifunctor by $\mathrm{CMon} \times \mathrm{CMon} \to \mathrm{CMon}$ by $(X, Y) \mapsto X \wedge Y$.

Proof of Proposition 3.6.1. Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are colimit-preserving functors (see Definition HTT.5.5.3.1). We will regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ as a symmetric monoidal ∞-category, whose tensor product $\otimes : \mathcal{P}\mathrm{r}^{\mathrm{L}} \times \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ can be characterized as follows: for every triple of objects $\mathcal{C}, \mathcal{D}, \mathcal{E} \in \mathcal{P}\mathrm{r}^{\mathrm{L}}$, we can identify colimit-preserving functors $F : \mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ with bifunctors $f : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserve colimits separately in each variable (see §HA.4.8.1). Let $L : \mathcal{P}\mathrm{r}^{\mathrm{L}} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ be the functor given by $\mathcal{C} \mapsto \mathcal{C} \otimes \mathrm{CMon}$. The functor $\mathrm{Sym}^*_{\mathcal{S}}$ determines a natural transformation $\mathrm{id}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}} \to L$. It follows from Proposition SAG.C.4.1.9 that L is a localization functor whose essential image consists of the presentable semiadditive ∞-categories. It follows that the functor $\mathrm{Sym}^*_{\mathcal{S}} : \mathcal{S} \to \mathrm{CMon}$ exhibits CMon as an idempotent object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$, in the sense of Definition HA.4.8.2.1). Applying Proposition HA.4.8.2.9 , we deduce that $\mathrm{Sym}^*_{\mathcal{S}}$ can be promoted (in an essentially unique way) to a morphism of commutative algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. $\qquad\square$

Remark 3.6.3. Let $\mathrm{Fin}^{\simeq}$ denote the category whose objects are finite sets and whose morphisms are bijections. Then the nerve $\mathrm{N}(\mathrm{Fin}^{\simeq})$ is a Kan complex which we can identify with the image of the one-point space $*$ under the functor $\mathrm{Sym}^*_{\mathcal{S}} : \mathcal{S} \to \mathrm{CMon}$. In particular, $\mathrm{N}(\mathrm{Fin}^{\simeq})$ admits the structure of an $\mathbb{E}_{\infty}$-space (whose multiplication is induced by the disjoint union functor $\amalg : \mathrm{Fin}^{\simeq} \times \mathrm{Fin}^{\simeq} \to \mathrm{Fin}^{\simeq}$). Moreover, since $\mathcal{S}$ is the unit object of the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ and a colimit-preserving functor $F : \mathcal{S} \to \mathrm{CMon}$ is determined by the object $F(*)$, we can replace condition (*ii*) of Proposition 3.6.1 with the following *a priori* weaker condition:

(*ii'*) The $\mathbb{E}_{\infty}$-space $\mathrm{N}(\mathrm{Fin}^{\simeq})$ is a unit object of CMon.

Remark 3.6.4 (Comparision with Tensor Products of Commutative Monoids)**.** Let $\mathcal{S}\mathrm{et}$ denote the category of sets. Then $\mathcal{S}\mathrm{et}$ can be regarded as an idempotent object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ (see Example HA.4.8.1.23). Note that the category $\mathrm{CMon}(\mathcal{S}\mathrm{et})$ of commutative monoids can be identified with the tensor product $\mathcal{S}\mathrm{et} \otimes \mathrm{CMon}$, and is therefore also an idempotent object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Applying Proposition

HA.4.8.2.9, we deduce that there is an essentially unique symmetric monoidal structure on the category $\mathrm{CMon}(\mathcal{S}\mathrm{et})$ with the following properties:

(i) The tensor product on $\mathrm{CMon}(\mathcal{S}\mathrm{et})$ preserves small colimits separately in each variable.

(ii) The unit object of $\mathrm{CMon}(\mathcal{S}\mathrm{et})$ is the commutative monoid $\mathbf{Z}_{\geqslant 0}$.

It follows from uniqueness that this symmetric monoidal structure must coincide (up to essentially unique equivalence) with the usual tensor product on commutative monoids, described in the introduction to this section.

Note that the functor $\pi_0 : \mathrm{CMon} \to \mathrm{CMon}(\mathcal{S}\mathrm{et})$ can be identified with the tensor product of the identity functor $\mathrm{id}_{\mathrm{CMon}}$ with the functor $\pi_0 : \mathcal{S} \to \mathcal{S}\mathrm{et}$, and can therefore be regarded as a morphism of commutative algebra objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$. In particular, if X and Y are $\mathbb{E}_\infty$-spaces, then the commutative monoid $\pi_0(X \wedge Y)$ can be identified with the tensor product of the commutative monoids $\pi_0 X$ and $\pi_0 Y$.

Remark 3.6.5 (Comparison with Smash Products of Spectra). Let $\mathrm{Sp}^{\mathrm{cn}}$ denote the ∞-category of connective spectra, and regard $\mathrm{Sp}^{\mathrm{cn}}$ as a symmetric monoidal ∞-category via the usual smash product of spectra. Since the ∞-category $\mathrm{Sp}^{\mathrm{cn}}$ is semiadditive, there is an essentially unique symmetric monoidal functor $F : \mathrm{CMon} \to \mathrm{Sp}^{\mathrm{cn}}$ which commutes with small colimits. Moreover, if we neglect symmetric monoidal structures, then F is characterized up to equivalence by the requirement that F carries the unit object $\mathrm{N}(\mathrm{Fin}^\simeq) \in \mathrm{CMon}$ to the sphere spectrum $S \in \mathrm{Sp}^{\mathrm{cn}}$. It follows that the composition of F with the equivalence $\mathrm{Sp}^{\mathrm{cn}} \simeq \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ can be identified with a left adjoint to the inclusion $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}) \hookrightarrow \mathrm{CMon}(\mathcal{S})$ (this left adjoint is given by the formation of *group completion* in the setting of $\mathbb{E}_\infty$-spaces).

Remark 3.6.6 (Comparison with Tensor Products of ∞-Operads). Let Op_∞ denote the ∞-category of ∞-operads (see Definition HA.2.1.4.1). Then we can identify the ∞-category CMon of $\mathbb{E}_\infty$-spaces with the full subcategory of Op_∞ spanned by those ∞-operads $q : \mathcal{O}^\otimes \to \mathcal{F}\mathrm{in}_*$ for which q is a left fibration. This observation determines a fully faithful embedding $\mathrm{CMon} \hookrightarrow \mathrm{Op}_\infty$, which admits a left adjoint $U : \mathrm{Op}_\infty \to \mathrm{CMon}$. One can show that this left adjoint is a symmetric monoidal functor, where we regard Op_∞ as equipped with the symmetric monoidal structure described in §HA.2.2.5, and CMon as equipped with the symmetric monoidal structure of Proposition 3.6.1. In particular, the functor U carries tensor products of ∞-operads to smash products of $\mathbb{E}_\infty$-spaces.

3.7 The Cartier Dual of a Functor

Let R be a commutative ring and let G be a finite flat commutative group scheme over R. Recall that the *Cartier dual* of G is another finite flat group scheme over R which parametrizes homomorphisms $G \to \mathbf{G}_m$. We now consider an analogous construction in a more general setting.

Construction 3.7.1 (Cartier Duality). Let $\mathcal{C}$ be a symmetric monoidal ∞-category. We regard the ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ as equipped with the symmetric monoidal structure given by pointwise smash product (see Proposition 3.6.1), whose underlying multiplication

$$\wedge : \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}) \times \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}) \to \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$$

is described by the formula $(X \wedge Y)(A) = X(A) \wedge Y(A)$. Note that this operation preserves small colimits separately in each variable.

Assume now that the ∞-category $\mathcal{C}$ is essentially small. Then the functor ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ is presentable (Proposition HTT.5.5.3.6). It follows that the smash product monoidal structure on $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ is closed (see Definition HA.4.1.1.15). In particular, for every object $X \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ there exists another object $\mathbf{D}(X) \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ equipped with a map $\alpha : X \wedge \mathbf{D}(X) \to \mathbf{A}^1$ with the following universal property: for every functor $Y \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(Y, \mathbf{D}(X)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(X \wedge Y, \mathbf{A}^1).$$

In this case, we will refer to $\mathbf{D}(X)$ as the *Cartier dual* of X, and we will say that α *exhibits* $\mathbf{D}(X)$ *as a Cartier dual of* X.

Variant 3.7.2. For some applications, it is inconvenient to assume that the ∞-category $\mathcal{C}$ appearing in Construction 3.7.1 is essentially small. Without this assumption, one cannot apply Construction 3.7.1 directly, because the symmetric monoidal ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\mathcal{C}))$ is generally not closed. However, one can correct this defect by passing to a larger universe which contains $\mathcal{C}$: if we then let $\widehat{\mathcal{S}}$ denote the ∞-category of spaces which are not necessarily small, then the symmetric monoidal structure on the ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\widehat{\mathcal{S}}))$ is closed, so that any functor $X \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}) \subseteq \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\widehat{\mathcal{S}}))$ admits a Cartier dual $\mathbf{D}(X) \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\widehat{\mathcal{S}}))$. In cases of interest to us, one can then check directly that $\mathbf{D}(X)$ belongs to (the essential image of) the full subcategory $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}) \subseteq \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}(\widehat{\mathcal{S}}))$.

Our next goal is to give a a more explicit description of the Cartier dual $\mathbf{D}(X)$ as a functor. We begin by evaluating $\mathbf{D}(X)$ at the initial object of $\mathrm{CAlg}(\mathcal{C})$.

Example 3.7.3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $\mathbf{1}$ denote the unit object of $\mathcal{C}$, which we regard as a commutative algebra object (namely, the initial object of $\mathrm{CAlg}(\mathcal{C})$). Let E denote the unit object of $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$, given by the constant functor taking the value $\mathrm{N}(\mathrm{Fin}^{\simeq}) \in \mathrm{CMon}$ (see Remark 3.6.3). Note that E is a left Kan extension of its restriction to the subcategory $\{\mathbf{1}\} \subseteq \mathrm{CAlg}(\mathcal{C})$. For any functor $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$, we obtain canonical homotopy equivalences

$$
\begin{aligned}
\mathbf{D}(X)(\mathbf{1}) &\simeq \mathrm{Map}_{\mathcal{S}}(*, \mathbf{D}(X)(\mathbf{1})) \\
&\simeq \mathrm{Map}_{\mathrm{CMon}}(\mathrm{N}(\mathrm{Fin}^{\simeq}), \mathbf{D}(X)(\mathbf{1})) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})}(E, \mathbf{D}(X)) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})}(X \wedge E, \mathbf{A}^1) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})}(X, \mathbf{A}^1).
\end{aligned}
$$

We now study the behavior of Construction 3.7.1 as the ∞-category $\mathcal{C}$ varies.

Notation 3.7.4. Let $f : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between essentially small symmetric monoidal ∞-categories, and suppose that f admits a right adjoint g. Then g inherits the structure of a lax symmetric monoidal functor, so that f and g determine an adjunction $\mathrm{CAlg}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{CAlg}(\mathcal{D})$. The functor G induces another adjunction

$$
\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}) \underset{G_!}{\overset{G^*}{\rightleftarrows}} \mathrm{Fun}(\mathrm{CAlg}(\mathcal{D}), \mathrm{CMon}),
$$

where G^* is given by pointwise composition with G and $G_!$ is given by left Kan extension along G. Note that G^* is a symmetric monoidal functor (with respect to formation of pointwise smash products). For every pair of functors $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ and $Y : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CMon}$, the composition

$$
G^*(X) \wedge Y \to G^*(X) \wedge G^* G_!(Y) \simeq G^*(X \wedge G_!(Y))
$$

classifies a map $\beta_{X,Y} : G_!(G^*(X) \wedge Y) \to X \wedge G_!(Y)$.

Lemma 3.7.5 (Projection Formula). *In the situation of Notation 3.7.4, suppose that the functor G is a coCartesian fibration of ∞-categories. Then, for every pair of functors $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ and $Y : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CMon}$, the map $\beta_{X,Y} : G_!(G^*(X) \wedge Y) \to X \wedge G_!(Y)$ is an equivalence.*

Proof. Fix an object $C \in \mathrm{CAlg}(\mathcal{C})$; we wish to show that $\beta_{X,Y}$ induces an equivalence when evaluated at C. Let $\mathrm{CAlg}(\mathcal{D})_{/C} = \mathrm{CAlg}(\mathcal{D}) \times_{\mathrm{CAlg}(\mathcal{C})} \mathrm{CAlg}(\mathcal{C})_{/C}$ denote the ∞-category whose objects are pairs (D, u), where $D \in \mathrm{CAlg}(\mathcal{D})$ and $u : G(D) \to C$ is a morphism in $\mathrm{CAlg}(\mathcal{C})$. Unwinding the definitions, we wish to show that the canonical map

$$\rho : \varinjlim_{(D,u) \in \mathrm{CAlg}(\mathcal{D})_{/C}} (X(G(D)) \wedge Y(D)) \to X(C) \wedge \varinjlim_{(D,u) \in \mathrm{CAlg}(\mathcal{D})_{/C}} Y(D)$$

is an equivalence in the ∞-category CMon. Let $\mathrm{CAlg}(\mathcal{D})_{/C}^{\circ}$ denote the full subcategory of $\mathrm{CAlg}(\mathcal{D})_{/C}$ spanned by those pairs (D, u) where u is an equivalence. Our assumption that G is a coCartesian fibration guarantees that the inclusion $\mathrm{CAlg}(\mathcal{D})_{/C}^{\circ} \hookrightarrow \mathrm{CAlg}(\mathcal{D})_{/C}$ admits a left adjoint, and is therefore left cofinal. Consequently, we can identify ρ with the canonical map

$$\varinjlim_{(D,u) \in \mathrm{CAlg}(\mathcal{D})_{/C}^{\circ}} (X(G(D)) \wedge Y(D)) \to X(C) \wedge \varinjlim_{(D,u) \in \mathrm{CAlg}(\mathcal{D})_{/C}^{\circ}} Y(D),$$

which is an equivalence by virtue of the fact that the smash product functor $\wedge : \mathrm{CMon} \times \mathrm{CMon} \to \mathrm{CMon}$ preserves small colimits separately in each variable. $\qquad \square$

Proposition 3.7.6. *In the situation of Notation 3.7.4, suppose that the functor G is a coCartesian fibration of ∞-categories. Then, for any functor $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$, there is a canonical equivalence $G^*(\mathbf{D}(X)) \simeq \mathbf{D}(G^*X)$.*

Proof. Let $\mathbf{A}_{\mathcal{C}}^1$ and $\mathbf{A}_{\mathcal{D}}^1$ be the functors obtained by applying Construction 3.5.1 to the ∞-categories $\mathcal{C}$ and $\mathcal{D}$, respectively. Remark 3.5.5 then supplies a canonical equivalence $\mathbf{A}_{\mathcal{D}}^1 \simeq G^* \mathbf{A}_{\mathcal{C}}^1$. Let $Y : \mathrm{CAlg}(\mathcal{D}) \to \mathrm{CMon}$ be any functor. Using Lemma 3.7.5, we obtain canonical homotopy equivalences

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{D}),\mathrm{CMon})}(Y, G^*(\mathbf{D}(X))) &\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(G_! Y, \mathbf{D}(X)) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(X \wedge G_! Y, \mathbf{A}_{\mathcal{C}}^1) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}),\mathrm{CMon})}(G_!((G^*X) \wedge Y), \mathbf{A}_{\mathcal{C}}^1) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{D}),\mathrm{CMon})}((G^*X) \wedge Y, G^* \mathbf{A}_{\mathcal{C}}^1) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{D}),\mathrm{CMon})}((G^*X) \wedge Y, \mathbf{A}_{\mathcal{D}}^1) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{D}),\mathrm{CMon})}(Y, \mathbf{D}(G^*X)).
\end{aligned}
$$

$\qquad \square$

Example 3.7.7. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let A be a commutative algebra object of $\mathcal{C}$. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects, so that the ∞-category of A-modules $\mathrm{Mod}_A(\mathcal{C})$ inherits a symmetric monoidal structure (given by the relative tensor product $\otimes_A$); see §HA.4.5.2. Extension and restriction of scalars then determine an adjunction $\mathcal{C} \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Mod}_A(\mathcal{C})$. Passing to commutative algebra objects, we obtain an adjunction $\mathrm{CAlg}(\mathcal{C}) \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{C}))$. Let us identify the ∞-category $\mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{C}))$ with the ∞-category $\mathrm{CAlg}(\mathcal{C})_{A/}$. Given any functor $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$, we let $X_A = G^*X$ denote the composite functor $\mathrm{CAlg}(\mathcal{C})_{A/} \xrightarrow{G} \mathrm{CAlg}(\mathcal{C}) \xrightarrow{X} \mathrm{CMon}$. Since the functor G is a left fibration of ∞-categories, Proposition 3.7.6 supplies a canonical equivalence $\mathbf{D}(X_A) \simeq \mathbf{D}(X)_A$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C})_{A/}, \mathrm{CMon})$. In other words, the formation of Cartier duality commutes with extension of scalars.

Proposition 3.7.8. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Suppose that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Then, for every functor $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$, we have canonical homotopy equivalences $\mathbf{D}(X)(A) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C})_{A/}, \mathrm{CMon})}(X_A, \mathbf{A}_A^1)$, where X_A and $\mathbf{A}_A^1$ are defined as in Example 3.7.7.*

Proof. Combine Examples 3.7.7 and 3.7.3. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.8 Duality for Bialgebras

We now specialize the Cartier duality construction of §3.7 to the case of corepresentable functors.

Proposition 3.8.1. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Assume that $\mathcal{C}$ admits geometric realizations of simplicial objects and that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves geometric realizations of simplicial objects. Let H be a bialgebra object of $\mathcal{C}$, and set $X = \mathrm{Spec}^{\mathcal{C}} H \in \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$. Assume that H is dualizable as an object of $\mathcal{C}$, and let $H^{\vee}$ denote the image of H under the forgetful functor*

$$\mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}}) \to \mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}}) \simeq \mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \subseteq \mathrm{CAlg}(\mathcal{C})^{\mathrm{op}}$$

(see Corollary 3.2.5). Then $H^{\vee}$ corepresents the functor $\mathrm{CAlg}(\mathcal{C}) \xrightarrow{\mathbf{D}(X)} \mathrm{CMon} \to \mathcal{S}$.

We can summarize the contents of Proposition 3.8.1 more informally as follows: if H is a bialgebra object of $\mathcal{C}$ which is dualizable as an object of $\mathcal{C}$, then the dual $H^\vee$ also admits the structure of a bialgebra object of $\mathcal{C}$. Moreover, we can identify $\mathrm{Spec}^{\mathcal{C}} H^\vee$ with the Cartier dual of $\mathrm{Spec}^{\mathcal{C}} H$.

Proof of Proposition 3.8.1. Let A be a commutative algebra object of $\mathcal{C}$. Combining Proposition 3.7.8, Proposition 3.5.6, and Corollary 3.2.5, we obtain homotopy equivalences

$$
\begin{aligned}
\mathbf{D}(X)(A) \quad &\simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{C})),\mathrm{CMon})}(X_A, \mathbf{A}^1_A) \\
&\simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{C})),\mathrm{CMon})}(\mathrm{Spec}^{\mathrm{Mod}_A(\mathcal{C})}(A \otimes H), \mathbf{A}^1_A) \\
&\simeq \quad \mathrm{Map}_{\mathrm{cCAlg}(\mathrm{Mod}_A(\mathcal{C}))}(A, A \otimes H) \\
&\simeq \quad \mathrm{Map}_{\mathrm{CAlg}(\mathrm{Mod}_A(\mathcal{C}))}(A \otimes H^\vee, A) \\
&\simeq \quad \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(H^\vee, A).
\end{aligned}
$$

Moreover, these homotopy equivalences depend functorially on A (see Remark 3.2.6). □

Remark 3.8.2. In the statement of Proposition 3.8.1, the assumption that $\mathcal{C}$ admits geometric realizations of simplicial objects is not necessary: it can be eliminated by embedding $\mathcal{C}$ into a larger symmetric monoidal ∞-category, as in the proof of Proposition 3.5.6. We leave the details to the reader.

Remark 3.8.3. Let $\mathcal{C}$ be as in Proposition 3.8.1. It follows from Proposition 3.8.1 that there exists a unique functor $D : \mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} \to \mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}})$ for which the diagram

$$
\begin{array}{ccc}
\mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} & \xrightarrow{\quad D \quad} & \mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}}) \\
\downarrow{\scriptstyle \mathrm{Spec}^{\mathcal{C}}} & & \downarrow{\scriptstyle \mathrm{Spec}^{\mathcal{C}}} \\
\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})^{\mathrm{op}} & \longrightarrow & \mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})
\end{array}
$$

commutes, where the bottom vertical map is given by Cartier duality. Moreover, Proposition 3.8.1 also establishes the commutativity of the diagram

$$
\begin{array}{ccc}
\mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} & \xrightarrow{\quad D \quad} & \mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}}) \\
\downarrow & & \downarrow \\
\mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} & \longrightarrow & \mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}}),
\end{array}
$$

where the vertical maps are the forgetful functors and the bottom horizontal map is the equivalence of Corollary 3.2.5.

Using more refined arguments, one can give a more direct description of the functor D: it is obtained by combining Proposition 3.2.4 with Corollary 3.3.4 to obtain equivalences

$$
\begin{aligned}
\mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}})^{\mathrm{op}} &= \mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}}))^{\mathrm{op}} \\
&\simeq \mathrm{CAlg}(\mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}}^{\mathrm{op}})) \\
&\simeq \mathrm{CAlg}(\mathrm{cCAlg}(\mathcal{C}_{\mathrm{fd}})) \\
&\simeq \mathrm{cCAlg}(\mathrm{CAlg}(\mathcal{C}_{\mathrm{fd}})) \\
&= \mathrm{bAlg}(\mathcal{C}_{\mathrm{fd}}).
\end{aligned}
$$

Remark 3.8.4. Let $\mathcal{C}$ be a symmetric monoidal ∞-category, let $H, H' \in \mathrm{bAlg}(\mathcal{C})$ be bialgebra objects of $\mathcal{C}$, and suppose that H is dualizable as an object of $\mathcal{C}$. Let us regard $R = H \otimes H'$ as a commutative algebra object of $\mathcal{C}$, so that we have tautologcial points $\eta \in (\mathrm{Spec}^{\mathcal{C}} H)(A)$ and $\eta' \in (\mathrm{Spec}^{\mathcal{C}} H')(R)$. Suppose that we are given a natural transformation $\mu : (\mathrm{Spec}^{\mathcal{C}} H) \wedge (\mathrm{Spec}^{\mathcal{C}} H') \to \mathbf{A}^1$. Then the induced map of $\mathcal{S}$-valued functors

$$
(\mathrm{Spec}^{\mathcal{C}} H) \times (\mathrm{Spec}^{\mathcal{C}} H') \to (\mathrm{Spec}^{\mathcal{C}} H) \wedge (\mathrm{Spec}^{\mathcal{C}} H') \xrightarrow{\mu} \mathbf{A}^1
$$

carries (η, η') to a point of the space $\mathbf{A}^1(R)$, which we can identify with a map $\mu_{\eta,\eta'} : \mathbf{1} \to R = H \otimes H'$ in the ∞-category $\mathcal{C}$.

It follows from Proposition 3.8.1 that the dual $H^\vee$ admits the structure of a bialgebra object of $\mathcal{C}$, and that μ is classified by a map of bialgebras $u : H^\vee \to H'$. Unwinding the definitions, we see that the image of u under the forgetful functor $\mathrm{bAlg}(\mathcal{C}) \to \mathcal{C}$ coincides with the composition

$$
H^\vee \xrightarrow{\mathrm{id} \otimes \mu_{\eta,\eta'}} H^\vee \otimes H \otimes H' \xrightarrow{e} \mathbf{1} \otimes H' \simeq H',
$$

where $e : H^\vee \otimes H \to \mathbf{1}$ is the evaluation map. In particular, u is an equivalence if and only if the map $\mu_{\eta,\eta'}$ exhibits H' as a dual of H in the symmetric monoidal ∞-category $\mathcal{C}$.

Suppose we are given a symmetric monoidal ∞-category $\mathcal{C}$ and a pair of functors $X, X' : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$. Every natural transformation $\mu : X \wedge X' \to \mathbf{A}^1$ induces maps

$$
\alpha : X \to \mathbf{D}(X') \qquad \beta : X' \to \mathbf{D}(X).
$$

We will say that μ *exhibits X as a Cartier dual of X'* if α is an equivalence, and that μ *exhibits X' as a Cartier dual of X* if β is an equivalence. Combining Proposition 3.8.1 with Remark 3.8.4, we obtain the following:

Proposition 3.8.5. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category containing bialgebra objects $H, H' \in \mathrm{bAlg}(\mathcal{C})$ representing functors $X = \mathrm{Spec}^{\mathcal{C}} H$ and $X' = \mathrm{Spec}^{\mathcal{C}} H'$, and suppose a morphism $\mu : X \wedge X' \to \mathbf{A}^1$. The following conditions are equivalent:*

(1) *The image of H in $\mathcal{C}$ is dualizable, and the map μ exhibits functor μ exhibits X' as a Cartier dual of X.*

(2) *The image of H' in $\mathcal{C}$ is dualizable, and the map μ exhibits functor μ exhibits X as a Cartier dual of X'.*

(3) *Let $\eta \in X(H)$ and $\eta' \in X'(H')$ be the tautological points, and let θ denote the image of (η, η') under the composite map*

$$
\begin{aligned}
X(H) \times X'(H') \quad &\to \quad X(H \otimes H') \times X'(H \otimes H') \\
&\to \quad (X \wedge X')(H \otimes H') \\
&\xrightarrow{\mu} \quad \mathbf{A}^1(H \otimes H') \\
&= \quad \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, H \otimes H').
\end{aligned}
$$

Then $\theta : \mathbf{1} \to H \otimes H'$ determines a duality between H and H' in the symmetric monoidal ∞-category $\mathcal{C}$.

3.9 Duality for Hopf Algebras

In the classical theory of Cartier duality, it is more common describe the Cartier dual of a finite flat commutative group scheme G as the scheme parametrizing group scheme homomorphisms from G into GL_1, rather than monoid scheme homomorphisms from G into $\mathbf{A}^1$. Of course, there is no difference: if G is a group scheme, then any monoid scheme homomorphism $G \to \mathbf{A}^1$ automatically factors through the open immersion $\mathrm{GL}_1 \hookrightarrow \mathbf{A}^1$. This observation extends to the more general setting of Construction 3.7.1.

Definition 3.9.1. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $X : \mathrm{CAlg}(\mathcal{C}) \to$ CMon be a functor. We will say that X is *grouplike* if, for every object $A \in \mathrm{CAlg}(\mathcal{C})$, the $\mathbb{E}_\infty$-space $X(A)$ is grouplike (that is, the commutative monoid $\pi_0 X(A)$ is an abelian group).

Remark 3.9.2. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $X, Y : \mathrm{CAlg}(\mathcal{C}) \to$ CMon be two functors. If either X or Y is grouplike, then the (pointwise) smash product $X \wedge Y$ is grouplike.

Remark 3.9.3. The inclusion functor $\iota : \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}) \hookrightarrow \mathrm{CMon}$ admits both a left adjoint ι^L (given by group completion) and a right adjoint ι^R. For any symmetric monoidal ∞-category $\mathcal{C}$, the inclusion functor $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})) \to$ $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$ also admits left and right adjoints, given by postcomposition with ι^L and ι^R respectively.

Construction 3.9.4 (The Functor GL_1). Let $\mathcal{C}$ be a symmetric monoidal ∞-category with unit object $\mathbf{1}$. For every commutative algebra object $R \in \mathrm{CAlg}(\mathcal{C})$, we let $\mathrm{GL}_1(R)$ denote the summand of $\mathrm{Map}_{\mathcal{C}}(\mathbf{1}, R)$ given by the union of those connected components which are invertible in the commutative monoid $\pi_0 \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, R)$ (where the monoid structure is induced by the multiplication on R). We regard GL_1 as a functor from the ∞-category $\mathrm{CAlg}(\mathcal{C})$ to the ∞-category $\mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ of grouplike $\mathbb{E}_\infty$-spaces.

Remark 3.9.5. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. Then we have a canonical inclusion of functors $\mathrm{GL}_1 \hookrightarrow \mathbf{A}^1$, which exhibits GL_1 as universal among grouplike functors $X : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ equipped with a map $X \to \mathbf{A}^1$ (see Remark 3.9.3).

Proposition 3.9.6. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let X be a grouplike object of* $\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})$. *Then:*

(a) *The Cartier dual $\mathbf{D}(X) : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ is grouplike.*

(b) *The canonical map $X \wedge \mathbf{D}(X) \to \mathbf{A}^1$ factors through GL_1 (in an essentially unique way).*

(c) *The resulting map $\mu : X \wedge \mathbf{D}(X) \to \mathrm{GL}_1$ has the following universal property: for every functor $Y : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$, composition with μ induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})}(Y, \mathbf{D}(X)) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathcal{C}), \mathrm{CMon})}(X \wedge Y, \mathrm{GL}_1).$$

Proof. Assertions (b) and (c) follow immediately from Remarks 3.9.2 and 3.9.5. We will prove (a). Let $\iota^L : \mathrm{CMon} \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$ be as in Remark 3.9.3 and set $Y = \iota^L \circ \mathbf{D}(X)$. It follows from Remark 3.6.5 that the localization functor ι^L on CMon is compatible with the smash product monoidal structure. Consequently, the canonical

map $f : X \wedge \mathbf{D}(X) \to X \wedge Y$ induces an equivalence after group completion. Since $X \wedge \mathbf{D}(X)$ and $X \wedge Y$ are already grouplike (Remark 3.9.2), it follows that f is an equivalence. It follows that the canonical map $X \wedge \mathbf{D}(X) \to \mathbf{A}^1$ factors through f (in an essentially unique way). Invoking the definition of $\mathbf{D}(X)$, we see that the canonical map $\mathbf{D}(X) \to Y$ admits a left homotopy inverse, so that $\mathbf{D}(X)$ is a retract of Y and is therefore grouplike, as desired. $\square$

Definition 3.9.7. Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let $H \in \mathrm{bAlg}(\mathcal{C})$ be a bialgebra object of $\mathcal{C}$. We will say that H is a *Hopf algebra object of $\mathcal{C}$* if the functor $\mathrm{Spec}^{\mathcal{C}}(H) : \mathrm{CAlg}(\mathcal{C}) \to \mathrm{CMon}$ is grouplike, in the sense of Definition 3.9.1.

More concretely: a bialgebra object $H \in \mathrm{bAlg}(\mathcal{C})$ is a Hopf algebra if and only if every commutative algebra morphism $f : H \to A$ in $\mathcal{C}$ is invertible (with respect to the commutative monoid structure on $\pi_0 \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C})}(H, A)$ determined by the comultiplication on H).

Remark 3.9.8. The terminology of Definition 3.9.7 is potentially misleading: our notion of Hopf algebra object might more properly be referred to as a commutative, cocommutative Hopf algebra object (see Remark 3.3.2).

Proposition 3.9.9. *Let $\mathcal{C}$ be a symmetric monoidal ∞-category and let H be a Hopf algebra object of $\mathcal{C}$. Suppose that H is dualizable as an object of $\mathcal{C}$, so that the dual $H^{\vee}$ admits the structure of a bialgebra object of $\mathcal{C}$ (as in Proposition 3.8.5). The $H^{\vee}$ is also a Hopf algebra object of $\mathcal{C}$.*

Proof. Combine Propositions 3.8.5 and 3.9.6. $\square$

4 Biextensions and the Fourier-Mukai Transform

Let A be a connective $\mathbb{E}_{\infty}$-ring and let X be an abelian variety over A. Then $\mathrm{QCoh}(\mathsf{X})$ can be regarded as a symmetric monoidal ∞-category in two different ways:

- Via the usual tensor product of quasi-coherent sheaves $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \otimes \mathscr{G}$.

- Via the convolution product $(\mathscr{F}, \mathscr{G}) \mapsto \mathscr{F} \star \mathscr{G} = m_*(\mathscr{F} \boxtimes \mathscr{G})$, where $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denotes the addition law on X (see §4.6 for more details).

In this section, we will show that these two symmetric monoidal structures are compatible with one another in the following sense: they exhibit $\mathrm{QCoh}(\mathsf{X})$ as a

Hopf algebra in the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$ of stable R-linear ∞-categories (where the multiplication on $\mathrm{QCoh}(\mathsf{X})$ is given by the usual tensor product, and the tensor product is dual to the convolution product). Pairing this observation with the general version of Cartier developed in §3, we obtain a notion of "duality pairing" between abelian varieties. In §4.4, we use Tannaka duality to show that such a pairing can be identified with the classical notion of *biextension* (see Theorem 4.4.4). We will apply these ideas in §5 to extend the classical duality theory of abelian varieties to the setting of spectral algebraic geometry.

4.1 Line Bundles and Invertible Sheaves

We begin by reviewing some terminology (for more details, we refer the reader to §SAG.2.9.4).

Definition 4.1.1. Let X be a spectral Deligne-Mumford stack and let $\mathscr{L}$ be a quasi-coherent sheaf. We say that $\mathscr{L}$ is an *invertible sheaf* if it is an invertible object of $\mathrm{QCoh}(\mathsf{X})$: that is, if there exists an object $\mathscr{L}^{-1} \in \mathrm{QCoh}(\mathsf{X})$ and an equivalence $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{O}_{\mathsf{X}}$. We let $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$ denote the subcategory of $\mathrm{QCoh}(\mathsf{X})$ whose objects are invertible sheaves and whose morphisms are equivalences.

We say that a sheaf $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X})$ is a *line bundle* if there exists an étale surjection $\mathsf{U} \to \mathsf{X}$ and an equivalence $\mathscr{L}|_{\mathsf{U}} \simeq \mathscr{O}_{\mathsf{U}}$. We let $\mathscr{P}\mathrm{ic}(\mathsf{X})$ denote the subcategory of $\mathrm{QCoh}(\mathsf{X})$ whose objects are line bundles and whose morphisms are equivalences.

Remark 4.1.2. Let X be a spectral Deligne-Mumford stack. Every line bundle on X is an invertible sheaf on X: that is, we have an inclusion $\mathscr{P}\mathrm{ic}(\mathsf{X}) \subseteq \mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$.

Remark 4.1.3 (Tensor Products of Line Bundles)**.** Let X be a spectral Deligne-Mumford stack. The collection of line bundles on X contains the structure sheaf $\mathscr{O}_{\mathsf{X}}$ and is closed under tensor products. Consequently, the symmetric monoidal structure on $\mathrm{QCoh}(\mathsf{X})$ restricts to symmetric monoidal structures on the Kan complex $\mathscr{P}\mathrm{ic}(\mathsf{X})$. Consequently, the space $\mathscr{P}\mathrm{ic}(\mathsf{X})$ admits a commutative monoid structure, depending functorially on X. Moreover, this monoid structure is grouplike (note that if $\mathscr{L}$ is a line bundle on X, then the inverse $\mathscr{L}^{-1}$ is also a line bundle on X): that is, we can regard $\mathscr{P}\mathrm{ic}(\mathsf{X})$ as an infinite loop space. Similarly, the space $\mathscr{P}\mathrm{ic}^{\dagger}(\mathsf{X})$ also admits a grouplike commutative monoid structure, depending functorially on X.

Remark 4.1.4. Let X be a spectral Deligne-Mumford stack, and suppose that the underlying topological space X is connected. Then a sheaf $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X})$ is invertible

if and only if some suspension $\Sigma^n \mathscr{L}$ is a line bundle. A similar assertion holds when $|\mathsf{X}|$ is not connected, but in this case we must replace the integer n by a (locally constant) function on $|\mathsf{X}|$; see Corollary SAG.2.9.5.7.

Construction 4.1.5 (The Classifying Stack of Line Bundles)**.** We define functors $\mathrm{BGL}_1, \mathrm{BGL}_1^\dagger : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{CMon}$ by the formulae

$$\mathrm{BGL}_1(A) = \mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\, A) \qquad \mathrm{BGL}_1^\dagger(A) = \mathscr{P}\mathrm{ic}^\dagger(\mathrm{Sp\acute{e}t}\, A).$$

We will refer to BGL_1 as the *classifying stack of line bundles*, and $\mathrm{BGL}_1^\dagger$ as the *classifying stack of invertible sheaves.*

Remark 4.1.6. In more concrete terms: the functor $\mathrm{BGL}_1^\dagger$ assigns to each connective $\mathbb{E}_\infty$-ring A the Kan complex of projective A-modules of rank 1, while the functor BGL_1 assigns to each connective $\mathbb{E}_\infty$-ring A the Kan complex of invertible A-modules.

Remark 4.1.7. Let A be a connective $\mathbb{E}_\infty$-ring. Then $\pi_0 \mathrm{BGL}_1(A)$ can be identified with the set of isomorphism classes of locally free A-modules of rank 1. Moreover, we have canonical isomorphisms

$$\pi_n \mathrm{BGL}_1(A) \simeq \begin{cases} (\pi_0 A)^\times & \text{if } n = 1 \\ \pi_{n-1} A & \text{if } n > 1. \end{cases}$$

Applying Corollary HA.7.2.2.19, we see that the canonical map

$$\pi_i \mathrm{BGL}_1(A) \to \pi_i \mathrm{BGL}_1(\pi_0 A)$$

is an isomorphism for $i \in \{0, 1\}$.

Remark 4.1.8. Let X be a spectral Deligne-Mumford stack. Then the space $\mathscr{P}\mathrm{ic}(\mathsf{X})$ can be identified with the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})}(\mathsf{X}, \mathrm{BGL}_1)$; here we abuse notation by identifying X with its functor of points, and BGL_1 with the composite functor $\mathrm{CAlg} \xrightarrow{\mathrm{BGL}_1} \mathrm{CMon} \to \mathcal{S}$. Similarly, we have a canonical homotopy equivalence $\mathscr{P}\mathrm{ic}^\dagger(\mathsf{X}) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}, \mathcal{S})}(\mathsf{X}, \mathrm{BGL}_1^\dagger)$.

Variant 4.1.9. Let R be some fixed $\mathbb{E}_\infty$-ring. Then the composite functors

$$\mathrm{CAlg}_R \to \mathrm{CAlg} \xrightarrow{\mathrm{BGL}_1} \mathrm{CMon} \qquad \mathrm{CAlg}_R \to \mathrm{CAlg} \xrightarrow{\mathrm{BGL}_1^\dagger} \mathrm{CMon}$$

determine functors $\mathrm{CAlg}_R \to \mathrm{CMon}$. We will abuse notation by denoting these functors also by BGL_1 and $\mathrm{BGL}_1^\dagger$.

4.2 Biextensions of Abelian Varieties

We begin by adapting the theory of biextensions to the setting of spectral algebraic geometry.

Definition 4.2.1 (Biextensions)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let X and Y be abelian varieties over R, which we identify with their functors of points $\mathsf{X}, \mathsf{Y} : \mathrm{CAlg}_R \to \mathrm{CMon}$ (Remark 1.4.5). A *biextension of* (X, Y) is a morphism $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon})$. We let $\mathrm{BiExt}(\mathsf{X}, \mathsf{Y}) = \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R, \mathrm{CMon})}(\mathsf{X} \wedge \mathsf{Y}, \mathrm{BGL}_1)$ denote the space of biextensions of (X, Y).

Remark 4.2.2. Let X and Y be abelian varieties over an $\mathbb{E}_\infty$-ring R. Given a line bundle $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y})$, an R-algebra A, and a pair of A-valued points $x \in \mathsf{X}(A)$, $y \in \mathsf{Y}(A)$, we let $\mathscr{L}_{x,y}$ denote the A-module obtained by pulling $\mathscr{L}$ back along the map $\mathrm{Sp\acute{e}t}\,A \xrightarrow{(x,y)} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}$. Definition 4.2.1 can be phrased more informally as follows: a biextension of (X, Y) is a line bundle $\mathscr{L}$ on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}$ which is equipped with equivalences

$$\mathscr{L}_{x+x',y} \simeq \mathscr{L}_{x,y} \otimes \mathscr{L}_{x',y} \qquad \mathscr{L}_{x,y+y'} \simeq \mathscr{L}_{x,y} \otimes \mathscr{L}_{x,y'},$$

for $x, x' \in \mathsf{X}(A)$ and $y, y' \in \mathsf{Y}(A)$, which are coherently commutative and associative and depend functorially on A. For a more complete discussion from this point of view, we refer the reader to [3].

In the setting of Definition 4.2.1, it does not matter if we use line bundles or invertible sheaves:

Proposition 4.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring and let X and Y be abelian varieties over R, which we idenitfy with their functors of points. Then any map $\mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1^\dagger$ factors through the subfunctor $\mathrm{BGL}_1 \subseteq \mathrm{BGL}_1^\dagger$.*

Proof. Let A be a connective R-algebra; we wish to show that the canonical map $\mathsf{X}(A) \wedge \mathsf{Y}(A) \to \mathrm{BGL}_1^\dagger(A)$ factors through $\mathrm{BGL}_1(A)$. Note that we can identify $\pi_0(\mathsf{X}(A) \wedge \mathsf{Y}(A))$ with the tensor product of the abelian groups $\pi_0\,\mathsf{X}(A)$ and $\pi_0\,\mathsf{Y}(A)$, so that $\pi_0(\mathsf{X}(A) \wedge \mathsf{Y}(A))$ is generated as an abelian group by the image of the map $\pi_0(\mathsf{X}(A) \times \mathsf{Y}(A)) \to \pi_0(\mathsf{X}(A) \wedge \mathsf{Y}(A))$. It will therefore suffice to show that the composite map

$$\mathsf{X}(A) \times \mathsf{Y}(A) \to \mathsf{X}(A) \wedge \mathsf{Y}(A) \to \mathrm{BGL}_1^\dagger(A)$$

factors through $\mathrm{BGL}_1(A)$, for each $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$. Equivalently, we wish to show that the composite map

$$\rho : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1^{\dagger}$$

factors through BGL_1. Let us identify the map ρ with an invertible sheaf $\mathscr{L} \in \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y})$ (see Remark 4.1.8). The failure of $\mathscr{L}$ to be a line bundle is measured by some locally constant function $s : |\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}| \to \mathbf{Z}$ (Remark 4.1.4); we wish to show that s vanishes. Since the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, R$ is geometrically connected, the map s factors as a composition $|\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}| \to |\mathsf{Y}| \xrightarrow{s'} \mathbf{Z}$; it will therefore suffice to show that s' vanishes. Let $e : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$ be the identity section, so that e induces a map $e_{\mathsf{Y}} : \mathsf{Y} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$. Then s' can be identified with the composition $|\mathsf{Y}| \xrightarrow{e_{\mathsf{Y}}} |\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}| \xrightarrow{s} \mathbf{Z}$. We are therefore reduced to showing that the pullback $e_{\mathsf{Y}}^* \mathscr{L}$ is a line bundle on Y. In fact, we can say more: since the map ρ factors through the smash product $\mathsf{X} \wedge \mathsf{Y}$, there is a canonical equivalence $e_{\mathsf{Y}}^* \mathscr{L} \simeq \mathscr{O}_{\mathsf{Y}}$. $\qquad\square$

In the setting of classical algebraic geometry, biextensions exist in great abundance:

Proposition 4.2.4. *Let R be a commutative ring and let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$. Then there is a canonical fiber sequence*

$$\mathrm{BiExt}(\mathsf{X}, \mathsf{Y}) \to \mathscr{P}\mathrm{ic}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}) \xrightarrow{e} \mathscr{P}\mathrm{ic}(\mathsf{X}) \times_{\mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\, R)} \mathscr{P}\mathrm{ic}(\mathsf{Y}),$$

where e is the map defined by restriction along the zero sections of X and Y.

More informally: in the setting of classical algebraic geometry, any line bundle $\mathscr{L}$ on the product $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$ which is equipped with compatible trivializations along X and Y can be regarded as a biextension of (X, Y) in an essentially unique way.

Example 4.2.5. [The Biextension Associated to a Line Bundle] Let R be a commutative ring, let X be an abelian variety over R, and let $\mathscr{L}$ be a line bundle on X which is equipped with a trivialization α along the zero section $e : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}$. Let $\pi, \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the projection maps and let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the multiplication. Then $\mathscr{P} = m^* \mathscr{L} \otimes \pi^* \mathscr{L}^{-1} \otimes \pi'^* \mathscr{L}^{-1}$ is a line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$, and α determines compatible trivializations of $\mathscr{P}$ along $\mathsf{X} \times \{e\}$ and $\{e\} \times \mathsf{X}$. Using Proposition 4.2.4, we see that $\mathscr{P}$ is the underlying line bundle of a biextension $\mathsf{X} \wedge \mathsf{X} \to \mathrm{BGL}_1$.

Proof of Proposition 4.2.4. For the proof, it will be convenient to borrow some notations and results from §5.3 and 5.4. Let R be a commutative ring, let X and Y be abelian varieties over R with identity sections $e_{\mathsf{X}} \in \mathsf{X}(R)$ and $e_{\mathsf{Y}} \in \mathsf{Y}(R)$, and let $\mathscr{L}$

be a line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$ equipped with compatible trivializations α of $\mathscr{L}\,|_{\mathsf{X} \times \{e_\mathsf{Y}\}}$ and β of $\mathscr{L}\,|_{\{e_\mathsf{X}\} \times \mathsf{Y}}$.

Let $\mathscr{P}\mathrm{ic}^e_\mathsf{X}$ denote the spectral algebraic space over R which parametrizes line bundles on X equipped with a trivialization along the identity section (Notation 5.4.1) and let $\mathscr{P}\mathrm{ic}^m_\mathsf{X}$ be the spectral algebraic space parametrizing multiplicative line bundles on X (see Definition 5.3.1 and Proposition 5.5.1). The pair $(\mathscr{L}, \alpha)$ is classified by a map of spectral algebraic spaces $f : \mathsf{Y} \to \mathscr{P}\mathrm{ic}^e_\mathsf{X}$. We will complete the proof by showing that f admits an essentially unique factorization as a composition $\mathsf{Y} \xrightarrow{\bar{f}} \mathscr{P}\mathrm{ic}^m_\mathsf{X} \to \mathscr{P}\mathrm{ic}^e_\mathsf{X}$, and that $\bar{f}$ can be promoted to a morphism of commutative monoids in an essentially unique way. Let $\tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^m_\mathsf{X}$ and $\tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^e_\mathsf{X}$ denote the underlying classical algebraic spaces of $\mathscr{P}\mathrm{ic}^m_\mathsf{X}$ and $\mathscr{P}\mathrm{ic}^e_\mathsf{X}$, respectively. Using Proposition 5.4.9, we see that the canonical map $\tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^m_\mathsf{X} \to \tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^e_\mathsf{X}$ is a closed and open immersion. Set $\mathsf{U} = \mathsf{Y} \times_{\tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^e_\mathsf{X}} \tau_{\leqslant 0}\, \mathscr{P}\mathrm{ic}^m_\mathsf{X}$, so that we can regard U as a closed and open substack of Y. The existence of the trivialization β shows that the identity section e_Y factors through U. Since the projection map $\mathsf{Y} \to \mathrm{Sp\acute{e}t}\, R$ is geometrically connected, it follows that $\mathsf{U} = \mathsf{Y}$, which is equivalent to the existence (and uniqueness) of the map $\bar{f}$. To complete the proof, it will suffice to show that the map $\bar{f}$ can be promoted to a morphism of commutative monoids. Using Proposition 5.4.9, we see that this is equivalent to promoting f to a morphism of commutative monoids. Unwinding the definitions, we see that this is equivalent to showing that the map $g : \mathsf{X} \to \mathscr{P}\mathrm{ic}^e_\mathsf{Y}$ classifying $(\mathscr{L}, \beta)$ factors through $\mathscr{P}\mathrm{ic}^m_\mathsf{Y}$. This follows by repeating the preceding argument, with roles of X and Y exchanged. $\square$

4.3   Digression: Tannaka Duality

We now briefly review the theory of Tannaka duality for spectral algebraic spaces and introduce some terminology which we will need. For a more detailed discussion, we refer the reader to §SAG.9.6 .

Notation 4.3.1. Let R be an $\mathbb{E}_\infty$-ring and let Mod_R denote the ∞-category of R-module spectra. Then we can regard Mod_R as a commutative algebra object of the ∞-category $\mathcal{P}\mathrm{r}^\mathrm{L}$ of presentable ∞-categories. Recall that a *stable R-linear ∞-category* is a presentable ∞-category $\mathcal{C}$ which is tensored over Mod_R, for which the action $\otimes_R : \mathrm{Mod}_R \times \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable. We let $\mathrm{LinCat}^{\mathrm{St}}_R$ denote the ∞-category $\mathrm{Mod}_{\mathrm{Mod}_R}(\mathcal{P}\mathrm{r}^\mathrm{L})$. We will refer to $\mathrm{LinCat}^{\mathrm{St}}_R$ as *the ∞-category of stable R-linear ∞-categories* (the morphisms in $\mathrm{LinCat}^{\mathrm{St}}_R$ are colimit-preserving R-linear functors). For more details, we refer the reader to §SAG.D.1 .

Warning 4.3.2. The ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$ is not locally small: if $\mathcal{C}$ and $\mathcal{D}$ are stable R-linear ∞-categories, then the collection of equivalences classes of R-linear functors from $\mathcal{C}$ to $\mathcal{D}$ generally forms a proper class. For readers who prefer to avoid such constructs, this is easily remedied. For our applications, it is sufficient to restrict our attention to R-linear ∞-categories which are compactly generated and R-linear functors which preserve compact objects. These restrictions single out a symmetric monoidal subcategory $\mathcal{E} \subseteq \mathrm{LinCat}_R^{\mathrm{St}}$ which is locally small (in fact, even presentable), which can be used as a replacement for $\mathrm{LinCat}_R^{\mathrm{St}}$ in all the constructions which follow.

Example 4.3.3. Let R be a connective $\mathbb{E}_\infty$-ring and let X be a spectral Deligne-Mumford stack over R. Then $\mathrm{QCoh}(\mathsf{X})$ is a stable R-linear ∞-category, which we can regard as an object of $\mathrm{LinCat}_R^{\mathrm{St}}$. Moreover, the tensor product of quasi-coherent sheaves on X determines a symmetric monoidal structure on $\mathrm{QCoh}(\mathsf{X})$ (compatible with the symmetric monoidal structure on Mod_R), so that $\mathrm{QCoh}(\mathsf{X})$ can be regarded as a commutative algebra object of $\mathrm{LinCat}_R^{\mathrm{St}}$.

Theorem 4.3.4 (Tannaka Duality for Spectral Algebraic Spaces). *Let R be a connective $\mathbb{E}_\infty$-ring. Suppose that X and Y are spectral Deligne-Mumford stacks over R. If X is a quasi-compact, quasi-separated spectral algebraic space, then the canonical map*

$$\mathrm{Map}_{\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\,R}}(\mathsf{Y}, \mathsf{X}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}(\mathsf{X}), \mathrm{QCoh}(\mathsf{Y}))$$

is a homotopy equivalence.

Proof. This is an immediate consequence of Theorem SAG.9.6.0.1 . $\qquad\square$

Corollary 4.3.5. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the construction $A \mapsto \mathrm{Mod}_A$ induces a fully faithful embedding $\Theta : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$.*

Remark 4.3.6. Corollary 4.3.5 is much more elementary (and general) than Theorem 4.3.4; see §HA.4.8.5 .

Notation 4.3.7. Let R be a connective $\mathbb{E}_\infty$-ring. We let $\mathrm{AlgSpace}(R)$ denote the ∞-category of quasi-compact, quasi-separated spectral algebraic spaces over R (which we regard as a full subcategory of $\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\,R}$; here SpDM denotes the ∞-category of spectral Deligne-Mumford stacks). We regard $\mathrm{Var}(R)$ as a full subcategory of $\mathrm{AlgSpace}(R)$.

Corollary 4.3.8. *Let R be a connective $\mathbb{E}_\infty$-ring. Then the construction $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ determines a fully faithful embedding $\mathrm{AlgSpace}(R) \to \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$. Moreover, this embedding commutes with finite limits.*

Proof. The first assertion follows from Theorem 4.3.4 and the second is a special case of Remark SAG.10.2.6.4 . □

We would like to apply the Cartier duality formalism of §3 to study bialgebra objects of $\mathrm{LinCat}_R^{\mathrm{St}}$. To this end, let us consider some examples of functors whose domain is the ∞-category $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$.

Construction 4.3.9 (The Spectrum of a Symmetric Monoidal ∞-Category)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathcal{A}$ be a commutative algebra object of $\mathrm{LinCat}_R^{\mathrm{St}}$: that is, a presentable symmetric monoidal ∞-category, equipped with a symmetric monoidal functor $\mathrm{Mod}_R \to \mathcal{A}$. We let $\mathrm{Spec}^\star(\mathcal{A}) : \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}) \to \widehat{\mathcal{S}}$ denote the functor corepresented by $\mathcal{A}$ (see Notation 3.4.1); here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small (Warning 4.3.2).

In the special case where $\mathcal{A}$ is a bialgebra object of the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$, we regard $\mathrm{Spec}^\star(A)$ as a functor from the ∞-category $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$ to the ∞-category $\mathrm{CMon}(\widehat{\mathcal{S}})$ of (not necessarily small) $\mathbb{E}_\infty$-spaces (see Variant 3.4.2).

Remark 4.3.10. In the situation of Construction 4.3.9, suppose that the ∞-category $\mathcal{A}$ is compactly generated and that every compact object of $\mathcal{A}$ is dualizable (this condition is satisfied, for example, if $\mathcal{A} = \mathrm{QCoh}(\mathsf{X})$ where X is a quasi-compact, quasi-separated spectral algebraic space; see Proposition SAG.9.6.1.1). Then, for every commutative algebra object $\mathcal{B} \in \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$, the space $\mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathcal{A}, \mathcal{B})$ is essentially small. Consequently, we can regard $\mathrm{Spec}^\star(\mathcal{A})$ as a functor from $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$ to $\mathcal{S}$.

To prove this, we observe that there exists a regular cardinal κ such that the unit object of $\mathcal{B}$ is κ-compact. It follows that every dualizable object of $\mathcal{B}$ is κ-compact, so that every symmetric monoidal functor $F : \mathcal{A} \to \mathcal{B}$ carries compact objects of $\mathcal{A}$ to κ-compact objects of $\mathcal{B}$.

Construction 4.3.11 (The Functor of Invertible Objects)**.** Let R be a connective $\mathbb{E}_\infty$-ring. We let $\mathrm{GL}_1^\star : \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}) \to \mathrm{CMon}(\mathcal{S})$ denote the functor obtained by applying Construction 3.9.4 to the symmetric monoidal ∞-category $\mathcal{C} = \mathrm{LinCat}_R^{\mathrm{St}}$. More concretely, if $\mathcal{A}$ is a commutative algebra object of $\mathrm{LinCat}_R^{\mathrm{St}}$ (that is, an R-linear symmetric monoidal ∞-category), then $\mathrm{GL}_1^\star(\mathcal{A})$ can be identified with the full subcategory of $\mathcal{A}^\simeq$ spanned by the invertible objects.

Remark 4.3.12. In the situation of Construction 4.3.11, the space $\mathrm{GL}_1^\star(\mathcal{A})$ is always essentially small. To see this, we observe that if the unit object $\mathbf{1} \in \mathcal{A}$ is κ-compact for some regular cardinal κ, then every invertible object of $\mathcal{A}$ is κ-compact.

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Example 4.3.13. Let X be a spectral Deligne-Mumford stack over a connective $\mathbb{E}_\infty$-ring R. Then we have a canonical homotopy equivalence $\mathrm{GL}_1^\star(\mathrm{QCoh}(\mathsf{X})) \simeq \mathscr{P}\mathrm{ic}^\dagger(\mathsf{X})$.

Example 4.3.14. Let R be a connective $\mathbb{E}_\infty$-ring, let $\Theta : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$ denote the fully faithful embedding of Corollary 4.3.5, and let

$$\Theta^* : \mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}), \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$$

be the functor given by composition with Θ. Then we have a canonical equivalence $\Theta^*(\mathrm{GL}_1^\star) \simeq \mathrm{BGL}_1^\dagger$ (this is a restatement of Example 4.3.13). If X is a quasi-compact, quasi-separated spectral algebraic space over R, then Theorem 4.3.4 supplies a canonical equivalence $\Theta^*(\mathrm{Spec}^\star \mathrm{QCoh}(\mathsf{X})) \simeq \mathsf{X}$ (here we abuse notation by identifying X with its functor of points $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$).

4.4 Biextensions: Tannakian Perspective

We now apply the ideas of §4.3 to give a reformulation of Definition 4.2.1. Our starting point is the following observation:

Proposition 4.4.1 (Tannaka Duality for Abelian Varieties). *Let R be a connective* $\mathbb{E}_\infty$-*ring. Then the construction* $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ *determines a fully faithful embedding* $\mathrm{AVar}(R) \to \mathrm{Hopf}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$.

Proof. Combine Corollary 4.3.8 with Proposition 1.4.4. $\qquad\square$

Remark 4.4.2. If X is an abelian variety over R, then the algebra structure on $\mathrm{QCoh}(\mathsf{X})$ is given by the usual tensor product of quasi-coherent sheaves, while the coalgebra structure on $\mathrm{QCoh}(\mathsf{X})$ is related to the operation of *convolution* (with respect to the multiplication on X); we will explain this in more detail in §4.6.

Construction 4.4.3. Let R be a connective $\mathbb{E}_\infty$-ring and suppose we are given abelian varieties $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$, which we regard as functors $\mathsf{X}, \mathsf{Y} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$. Using Proposition 4.4.1, we can regard $\mathrm{QCoh}(\mathsf{X})$ and $\mathrm{QCoh}(\mathsf{Y})$ as Hopf algebra objects of the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$, which in turn determine functors

$$\mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{X})), \mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{Y})) : \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}) \to \mathrm{CMon}(\widehat{\mathcal{S}})$$

(see Construction 4.3.9). Set $\mathcal{E} = \mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}), \mathcal{S})$, and let $\Theta^* : \mathcal{E} \to \mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathcal{S})$ be as in Example 4.3.14. Combining Example 4.3.14 with Propo-

sition 4.2.3, we obtain a canonical map

$$\mathrm{Map}_{\mathcal{E}}(\mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{X})) \wedge \mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{Y})), \mathrm{GL}_1^{\star})$$

$$\downarrow \Theta^*$$

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon}(\mathcal{S}))}(\mathsf{X} \wedge \mathsf{Y}, \mathrm{BGL}_1^{\dagger})$$

$$\uparrow \sim$$

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon}(\mathcal{S}))}(\mathsf{X} \wedge \mathsf{Y}, \mathrm{BGL}_1)$$

$$\|$$

$$\mathrm{BiExt}(\mathsf{X}, \mathsf{Y}).$$

Theorem 4.4.4. *Let R be a connective $\mathbb{E}_{\infty}$-ring and let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$ be abelian varieties over R. Then Construction 4.4.3 determines a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}), \mathrm{CMon})}(\mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{X})) \wedge \mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{Y})), \mathrm{GL}_1^{\star}) \to \mathrm{BiExt}(\mathsf{X}, \mathsf{Y}).$$

Proof. Let $\mathcal{C}'$ denote the ∞-category $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$, let $\mathcal{C}$ denote the full subcategory of $\mathcal{C}'$ spanned by those ∞-categories of the form $\mathrm{QCoh}(\mathsf{Z})$, where Z is a quasi-compact, quasi-separated spectral algebraic space over R, and let $\mathcal{C}_0 \subseteq \mathcal{C}$ denote the full subcategory spanned by those ∞-categories of the form $\mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, A)$ for $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$. Define

$$F_{\mathsf{X}} = \mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{X})) \qquad F_{\mathsf{Y}} = \mathrm{Spec}^{\star}(\mathrm{QCoh}(\mathsf{Y})) \qquad G = \mathrm{GL}_1^{\star},$$

which we regard as functors from $\mathcal{C}'$ to the ∞-category $\mathrm{CMon}(\mathcal{S})$. Unwinding the definitions, we are reduced to showing that the composite map

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}', \mathrm{CMon}(\mathcal{S}))}(F_{\mathsf{X}} \wedge F_{\mathsf{Y}}, G)$$

$$\downarrow \phi$$

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}, \mathrm{CMon}(\mathcal{S}))}((F_{\mathsf{X}} \wedge F_{\mathsf{Y}})|_{\mathcal{C}}, G|_{\mathcal{C}})$$

$$\downarrow \psi$$

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_0, \mathrm{CMon}(\mathcal{S}))}((F_{\mathsf{X}} \wedge F_{\mathsf{Y}})|_{\mathcal{C}_0}, G|_{\mathcal{C}_0})$$

is a homotopy equivalence. We first observe that $G|_{\mathcal{C}}$ is a left Kan extension of $G|_{\mathcal{C}_0}$ (see Proposition SAG.6.2.4.1), so the map ψ is a homotopy equivalence. To complete

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the proof, it will suffice to show that the functor $F_{\mathsf{X}} \wedge F_{\mathsf{Y}}$ is a left Kan extension of its restriction to $\mathcal{C}$, so that ϕ is also a homotopy equivalence.

Fix an object $C \in \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}) = \mathcal{C}'$ and set $\mathcal{C}_{/C} = \mathcal{C} \times_{\mathcal{C}'} \mathcal{C}'_{/C}$. We then have a commutative diagram

$$
\begin{array}{ccc}
(\varinjlim_{B \in \mathcal{C}_{/C}} F_{\mathsf{X}}(B)) \wedge \varinjlim_{B \in \mathcal{C}_{/C}} (F_{\mathsf{Y}}(B)) & & \\
\Big\downarrow{\alpha} & \overset{\gamma}{\searrow} & \\
\varinjlim_{B \in \mathcal{C}_{/C}} (F_{\mathsf{X}}(B) \wedge F_{\mathsf{Y}}(B)) \overset{\beta}{\longrightarrow} & F_{\mathsf{X}}(C) \wedge F_{\mathsf{Y}}(C)
\end{array}
$$

in the ∞-category $\mathrm{CMon}(\widehat{\mathcal{S}})$, and we wish to show that β is an equivalence. Since the ∞-category $\mathcal{C}$ is closed under finite colimits in $\mathcal{C}'$ (Corollary 4.3.8), the ∞-category $\mathcal{C}_{/C}$ is filtered. The smash product functor $\wedge : \mathrm{CMon}(\widehat{\mathcal{S}}) \times \mathrm{CMon}(\widehat{\mathcal{S}}) \to \mathrm{CMon}(\widehat{\mathcal{S}})$ commutes with (not necessarily small) colimits separately in each variable, so the map α is an equivalence. We are therefore reduced to proving that γ is an equivalence. This is clear, since the functors $F|_{\mathsf{X}}$ and $F|_{\mathsf{Y}}$ are left Kan extensions of their restrictions to $\mathcal{C}$ (Lemma 3.5.7). $\qquad\square$

4.5 Categorical Digression

We now collect some general categorical remarks which will be useful for our study of convolution of quasi-coherent sheaves in §4.6. Recall that a stable monoidal ∞-category $\mathcal{C}$ is *locally rigid* if it is compactly generated, the tensor product $\otimes : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ preserves small colimits separately in each variable, and the compact objects of $\mathcal{C}$ are precisely the dualizable objects (see Definition SAG.D.7.4.1).

Proposition 4.5.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be locally rigid symmetric monoidal ∞-categories and let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor which preserves small colimits, so that the tensor product on $\mathcal{D}$ determines a (symmetric monoidal) functor $m : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \to \mathcal{D}$. Then F admits a $\mathcal{C}$-linear right adjoint $G : \mathcal{D} \to \mathcal{C}$ which preserves small colimits. Moreover, the composite functor*

$$
e : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \xrightarrow{m} \mathcal{D} \xrightarrow{G} \mathcal{C}
$$

exhibits $\mathcal{D}$ as a self-dual object of the ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$ of $\mathcal{C}$-linear presentable stable ∞-categories.

Proof. Let us regard $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ as a symmetric monoidal ∞-category, and let $\boxtimes :$ $\mathcal{D} \times \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ denote the canonical map. Note that if $D, D' \in \mathcal{D}$ are compact

objects, then the object $D \boxtimes D' \in \mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is also compact (see the proof of Lemma SAG.D.5.3.3). In particular, the unit object $\mathbf{1} \boxtimes \mathbf{1} \in \mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is compact.

Since $\mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is generated under small colimits by objects of the form $D \boxtimes D'$, it follows that $\mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is compactly generated. Moreover, the full subcategory of $\mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ spanned by the compact objects is the smallest stable subcategory which is closed under retracts and contains $D \boxtimes D'$, whenever $D, D' \in \mathcal{D}$ are compact. It follows that every compact object of $\mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is dualizable: that is, the symmetric monoidal ∞-category $\mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ is locally rigid (see Definition SAG.D.7.4.1).

It follows from Example SAG.D.7.4.5 that the functor $F : \mathcal{C} \to \mathcal{D}$ admits a $\mathcal{C}$-linear right adjoint $G : \mathcal{D} \to \mathcal{C}$ which preserves small colimits. Similarly, Example SAG.D.7.4.5 guarantees that the functor $m : \mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ admits a colimit-preserving right adjoint $\delta : \mathcal{D} \to \mathcal{D} \otimes_\mathcal{C} \mathcal{D}$, which can be regarded as a $(\mathcal{D} \otimes_\mathcal{C} \mathcal{D})$-linear functor (and, in particular, a $\mathcal{C}$-linear functor). Let $c : \mathcal{C} \to \mathcal{D} \otimes_\mathcal{C} \mathcal{D}$ denote the $\mathcal{C}$-linear functor given by the composition

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{\delta} \mathcal{D} \otimes_\mathcal{C} \mathcal{D}.$$

We will show that e and c are compatible evaluation and coevaluation maps for a duality datum in the symmetric monoidal ∞-category $\mathrm{Mod}_\mathcal{C}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. To prove this, it suffices to show that the composite maps

$$\mathcal{D} \xrightarrow{\mathrm{id} \otimes c} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{e \otimes \mathrm{id}} \mathcal{D}$$

$$\mathcal{D} \xrightarrow{c \otimes \mathrm{id}} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{\mathrm{id} \otimes e} \mathcal{D}$$

are homotopic to the identity. We consider the first of these composite maps; the proof in the other case is identical. Unwinding the definitions, we must show that the composite functor

$$\mathcal{D} \xrightarrow{\mathrm{id} \otimes F} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{\mathrm{id} \otimes \delta} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{m \otimes \mathrm{id}} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{G \otimes \mathrm{id}} \mathcal{D}$$

is homotopic to the identity (as a $\mathcal{C}$-linear functor from $\mathcal{D}$ to itself).

Note that the composite functors

$$\mathcal{D} \xrightarrow{\mathrm{id} \otimes F} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{m} \mathcal{D}$$

$$\mathcal{D} \xrightarrow{\delta} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{G \otimes \mathrm{id}} \mathcal{D}$$

are homotopic to the identity. Concatenating these, we deduce that the functor

$$\mathcal{D} \xrightarrow{\mathrm{id} \otimes F} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{m} \mathcal{D} \xrightarrow{\delta} \mathcal{D} \otimes_\mathcal{C} \mathcal{D} \xrightarrow{G \otimes \mathrm{id}} \mathcal{D}$$

is homotopic to the identity. To complete the proof, it will suffice to show that that the $\mathcal{C}$-linear functors

$$\delta \circ m, (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$$

are homotopic. We have a commutative diagram of $\mathcal{C}$-linear ∞-categories σ :

$$
\begin{array}{ccc}
\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\mathrm{id} \otimes m} & \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \\
\downarrow{\scriptstyle m \otimes \mathrm{id}} & & \downarrow{\scriptstyle m} \\
\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\quad m \quad} & \mathcal{D},
\end{array}
$$

which determines a $\mathcal{C}$-linear natural transformation $\beta : (m \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta) \to \delta \circ m$. To complete the proof, it will suffice to show that β is an equivalence. For this, we can ignore the $\mathcal{C}$-linearity: we must show that σ is right adjointable as a diagram of ∞-categories.

For $Z \in \mathcal{D}$, let $r_Z : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ be the functor given by the action of Z on the second tensor factor (so that $r_Z(X \boxtimes Y) \simeq X \boxtimes (Y \otimes Z)$). For every pair of objects $X, Y \in \mathcal{D}$, β induces a map

$$\beta_{X,Y} : r_Y(\delta X) \to \delta(X \otimes Y)$$

in the ∞-category $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$. We wish to show that each of the maps $\beta_{X,Y}$ is an equivalence. Since the construction $Y \mapsto \beta_{X,Y}$ commutes with filtered colimits, we may assume without loss of generality that $Y \in \mathcal{D}$ is compact and therefore admits a dual $Y^{\vee}$. Since $\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ is generated (under small colimits) by objects of the form $D \boxtimes D'$, it will suffice to show that $\beta_{X,Y}$ induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}}(D \boxtimes D', r_Y(\delta X)) \to \mathrm{Map}_{\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}}(D \boxtimes D', \delta(X \otimes Y))$$

for each $D, D' \in \mathcal{D}$. Unwinding the definitions, we see that θ is given by the composition of homotopy equivalences

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}}(D \boxtimes D', r_Y(\delta X)) & \simeq & \mathrm{Map}_{\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}}(D \boxtimes (D' \otimes Y^{\vee}), \delta X) \\
& \simeq & \mathrm{Map}_{\mathcal{D}}(D \otimes D' \otimes Y^{\vee}, X) \\
& \simeq & \mathrm{Map}_{\mathcal{D}}(D \otimes D', X \otimes Y) \\
& \simeq & \mathrm{Map}_{\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}}(D \boxtimes D', \delta(X \otimes Y)).
\end{aligned}
$$

$\square$

Proposition 4.5.2. *Suppose we are given a commutative diagram*

$$
\begin{array}{ccc}
 & \mathcal{C} & \\
{\scriptstyle f^*}\swarrow & & \searrow{\scriptstyle h^*} \\
\mathcal{D} & \xrightarrow{\;\;g^*\;\;} & \mathcal{E},
\end{array}
$$

where $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ are locally rigid symmetric monoidal ∞-categories and f^*, g^*, and h^* are colimit-preserving symmetric monoidal functors. Let us regard g^* as a $\mathcal{C}$-linear functor from $\mathcal{D}$ to $\mathcal{E}$, and let $F : \mathcal{E} \to \mathcal{D}$ denote the dual of g^* (where we use Proposition 4.5.1 to identify $\mathcal{D}$ and $\mathcal{E}$ with their own duals in the symmetric monoidal ∞-category $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L}})$). Then F can be identified with the right adjoint g_* of g^* (as a $\mathcal{C}$-linear functor; see Example SAG.D.7.4.5).

Proof. Let $e_{\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \to \mathcal{C}$ be the evaluation map appearing in the statement of Proposition 4.5.1, and $c_{\mathcal{D}} : \mathcal{C} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ be the coevaluation map appearing in the proof of Proposition 4.5.1. Unwinding the definitions, we see that F is given by the composition

$$
\mathcal{E} \xrightarrow{\;\mathrm{id}\otimes c_{\mathcal{D}}\;} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \xrightarrow{\;\mathrm{id}\otimes g^*\otimes\mathrm{id}\;} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \xrightarrow{\;e_{\mathcal{E}}\otimes\mathrm{id}\;} \mathcal{D}\,.
$$

Let $m_{\mathcal{D}} : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \to \mathcal{D}$ and $m_{\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \to \mathcal{E}$ be the multiplication maps, and denote their right adjoints by $\delta_{\mathcal{D}} : \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D}$ and $\delta_{\mathcal{E}} : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E}$. Then F is homotopic to the composition

$$
\begin{array}{rcl}
\mathcal{E} & \xrightarrow{\;\mathrm{id}\otimes f^*\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;\mathrm{id}\otimes\delta_{\mathcal{D}}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;\mathrm{id}\otimes g^*\otimes\mathrm{id}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;m_{\mathcal{E}}\otimes\mathrm{id}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;h_*\otimes\mathrm{id}\;} & \mathcal{D};
\end{array}
$$

here h_* denotes a right adjoint to h^*.

Let F_0 denote the composition

$$
\begin{array}{rcl}
\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\;\mathrm{id}\otimes\delta_{\mathcal{D}}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;\mathrm{id}\otimes g^*\otimes\mathrm{id}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \\[4pt]
 & \xrightarrow{\;m_{\mathcal{E}}\otimes\mathrm{id}\;} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D},
\end{array}
$$

so that $F \simeq (h_* \otimes \mathrm{id}) \circ F_0 \circ (\mathrm{id} \otimes f^*)$. Let Γ^* denote the composition

$$\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \xrightarrow{\mathrm{id} \otimes g^*} \mathcal{E} \otimes_{\mathcal{C}} \mathcal{E} \xrightarrow{m_{\mathcal{E}}} \mathcal{E}.$$

Then Γ^* admits a $\mathcal{C}$-linear right adjoint $\Gamma_* = \delta_{\mathcal{E}} \circ (\mathrm{id} \otimes g_*)$. We will prove:

(∗) There is a canonical equivalence of $\mathcal{C}$-linear functors $F_0 \simeq \Gamma_* \circ \Gamma^*$.

Assuming (∗), we deduce that F is homotopic to the composition $((h_* \otimes \mathrm{id}) \circ \Gamma_*) \circ (\Gamma^* \circ (\mathrm{id} \otimes f^*))$. Combining this with the evident equivalences

$$(h_* \otimes \mathrm{id}) \circ \Gamma_* \simeq g_* \qquad \Gamma^* \circ (\mathrm{id} \otimes f^*) \simeq \mathrm{id}$$

we conclude that F is homotopic to g_*, as desired.

It remains to prove (∗). We have a commutative diagram of $\mathcal{C}$-linear functors σ :

$$
\begin{array}{ccc}
\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\mathrm{id} \otimes m_{\mathcal{D}}} & \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} \\
\downarrow{\scriptstyle \Gamma^* \otimes \mathrm{id}} & & \downarrow{\scriptstyle \Gamma^*} \\
\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\Gamma^*} & \mathcal{E},
\end{array}
$$

which induces a $\mathcal{C}$-linear natural transformation

$$\beta : F_0 = (\Gamma^* \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta_{\mathcal{D}}) \to \Gamma^* \circ \Gamma_*.$$

In order to prove (∗), it suffices to show that β is an equivalence. Once again, this is a statement that can be checked at the level of the underlying functor: we can ignore the $\mathcal{C}$-linearity.

We now proceed as in the proof of Proposition 4.5.1. Given a pair of objects $X \in \mathcal{E}$, $Y \in \mathcal{D}$, we let $X \boxtimes Y$ denote the image of the pair (X, Y) in the ∞-category $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}$. If $Z \in \mathcal{E}$, we let $r_Z : \mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} \to \mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}$ be given by the action of Z on the first factor. For every pair of objects $X \in \mathcal{E}$, $Y \in \mathcal{D}$, β induces a map

$$\beta_{X,Y} : r_X(\delta_{\mathcal{D}} Y) \to \Gamma_* \Gamma^* (X \boxtimes Y)$$

in the ∞-category $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}$. We wish to show that each of the maps $\beta_{X,Y}$ is an equivalence. Since the construction $X \mapsto \beta_{X,Y}$ commutes with filtered colimits, we may assume without loss of generality that $X \in \mathcal{E}$ is compact and therefore admits a dual $X^{\vee}$. Note that $\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}$ is generated (under small colimits) by objects of the form

$E \boxtimes D$, where $E \in \mathcal{E}$ and $D \in \mathcal{D}$. Consequently, to prove that $\beta_{X,Y}$ is an equivalence, it will suffice to show that $\beta_{X,Y}$ induces a homotopy equivalence

$$\theta : \mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}(E \boxtimes D, r_X(\delta_{\mathcal{D}} Y)) \to \mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}(E \boxtimes D, \Gamma_* \Gamma^*(X \boxtimes Y)).$$

Unwinding the definitions, we see that θ is obtained by composing homotopy equivalences

$$
\begin{aligned}
\mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}(E \boxtimes D, m_X(\delta_{\mathcal{D}} Y)) \quad &\simeq \quad \mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}((X^{\vee} \otimes E) \boxtimes D, (g^* \otimes \mathrm{id})(\delta_{\mathcal{D}} Y)) \\
&\xrightarrow{\phi} \quad \mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}((X^{\vee} \otimes E) \boxtimes D, \Gamma_* g^* Y) \\
&\simeq \quad \mathrm{Map}_{\mathcal{E}}(X^{\vee} \otimes E \otimes g^* D, g^* Y) \\
&\simeq \quad \mathrm{Map}_{\mathcal{E}}(E \otimes g^* D, X \otimes g^* Y) \\
&\simeq \quad \mathrm{Map}_{\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D}}(E \boxtimes D, \Gamma_* \Gamma^*(X \boxtimes Y)),
\end{aligned}
$$

where ϕ is given by composition with the map $\alpha_Y : (g^* \otimes \mathrm{id}) \circ \Delta_*(Y) \to \Gamma_* g^* Y$ determined by the commutative diagram of ∞-categories τ :

$$
\begin{array}{ccc}
\mathcal{D} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{m_{\mathcal{D}}} & \mathcal{D} \\
\downarrow{\scriptstyle g^* \otimes \mathrm{id}} & & \downarrow{\scriptstyle g^*} \\
\mathcal{E} \otimes_{\mathcal{C}} \mathcal{D} & \xrightarrow{\Gamma^*} & \mathcal{E} .
\end{array}
$$

It now suffices to observe that the diagram τ is right adjointable, which follows from the assumption that $\mathcal{D}$ is locally rigid (see Remark SAG.D.7.4.4). $\qquad \square$

4.6 The Convolution Product

Let R be a connective $\mathbb{E}_{\infty}$-ring, let X be an abelian variety over R. Let $p, q : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the projection maps onto the first and second factor, and let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the addition map. Then the construction $(\mathscr{F}, \mathscr{G}) \mapsto m_*(p^* \mathscr{F} \otimes q^* \mathscr{G})$ determines a functor $\star : \mathrm{QCoh}(\mathsf{X}) \times \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})$, which we will refer to as the *convolution product*. In this section, we will apply the results of §4.5 to show that the convolution product endows $\mathrm{QCoh}(\mathsf{X})$ with the structure of a symmetric monoidal ∞-category (Example 4.6.6).

Construction 4.6.1. Let R be a connective $\mathbb{E}_{\infty}$-ring. According to Corollary 4.3.8, the construction $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ determines a functor $\mathrm{QCoh} : \mathrm{AlgSpace}(R) \to \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$ which commutes with finite products. Let us regard $\mathrm{AlgSpace}(R)$

as equipped with the Cartesian symmetric monoidal structure, so that we can regard QCoh as a symmetric monoidal functor (note that the symmetric monoidal structure on $\mathrm{LinCat}_R^{\mathrm{St}}$ induces a Cartesian symmetric monoidal structure on $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$; see Proposition HA.3.2.4.7). Composing with the (symmetric monoidal) forgetful functor $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}) \to \mathrm{LinCat}_R^{\mathrm{St}}$, we obtain a symmetric monoidal functor $\mathrm{AlgSpace}(R) \to (\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$, which we will also denote by QCoh. For each $\mathsf{X} \in \mathrm{AlgSpace}(R)$, the R-linear ∞-category $\mathrm{QCoh}(\mathsf{X})$ is compactly generated (Proposition SAG.9.6.1.1), and is therefore dualizable as an object of $\mathrm{LinCat}_R^{\mathrm{St}}$ (Theorem SAG.D.7.0.7). We denote the dual of $\mathrm{QCoh}(\mathsf{X})$ by $\mathrm{QCoh}^\vee(\mathsf{X})$. Using Proposition 3.2.4, we see that the construction $\mathsf{X} \mapsto \mathrm{QCoh}^\vee(\mathsf{X})$ can be regarded as a symmetric monoidal functor $\mathrm{QCoh}^\vee : \mathrm{AlgSpace}(R) \to \mathrm{LinCat}_R^{\mathrm{St}}$.

Remark 4.6.2 (Behavior of $\mathrm{QCoh}^\vee$ on Objects). Let $q : \mathsf{X} \to \mathsf{Y}$ be a morphism of quasi-compact, quasi-separated spectral algebraic spaces. Applying Proposition 4.5.1 to the pullback functor $q^* : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$, we deduce that the composite functor

$$\mathrm{QCoh}(\mathsf{X}) \otimes_{\mathrm{QCoh}(\mathsf{Y})} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{\otimes} \mathrm{QCoh}(\mathsf{X}) \xrightarrow{q_*} \mathrm{QCoh}(\mathsf{Y})$$

exhibits $\mathrm{QCoh}(\mathsf{X})$ as a self-dual object in the ∞-category $\mathrm{Mod}_{\mathrm{QCoh}(\mathsf{Y})}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$. In the special case where $\mathsf{Y} = \mathrm{Sp\acute{e}t}\, R$ is affine, we obtain a canonical R-linear equivalence $\mathrm{QCoh}^\vee(\mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{X})$. Nevertheless, it will be convenient to distinguish in notation between $\mathrm{QCoh}(\mathsf{X})$ and $\mathrm{QCoh}^\vee(\mathsf{X})$, because they have (a priori) different variance in X: the construction $\mathsf{X} \mapsto \mathrm{QCoh}(\mathsf{X})$ determines a contravariant functor from $\mathrm{AlgSpace}(R)$ to $\mathrm{LinCat}_R^{\mathrm{St}}$, while the construction $\mathsf{X} \mapsto \mathrm{QCoh}^\vee(\mathsf{X})$ determines a covariant functor from $\mathrm{AlgSpace}(R)$ to $\mathrm{LinCat}_R^{\mathrm{St}}$.

Remark 4.6.3 (Behavior of $\mathrm{QCoh}^\vee$ on Products). Let R be a connective $\mathbb{E}_\infty$-ring. The functor $\mathrm{QCoh}^\vee : \mathrm{AlgSpace}(R) \to \mathrm{LinCat}_R^{\mathrm{St}}$ is symmetric monoidal. Consequently, for every pair of objects $\mathsf{X}, \mathsf{Y} \in \mathrm{AlgSpace}(R)$, we have a canonical R-linear equivalence

$$\alpha : \mathrm{QCoh}^\vee(\mathsf{X}) \otimes_R \mathrm{QCoh}^\vee(\mathsf{Y}) \simeq \mathrm{QCoh}^\vee(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y})$$

Under the equivalence of Remark 4.6.2, we can identify α with the R-linear equivalence

$$\beta : \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{Y}) \simeq \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y})$$

determines by the symmetric monoidal structure on the functor QCoh. Concretely, β classifies the R-bilinear functor $\boxtimes : \mathrm{QCoh}(\mathsf{X}) \times \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y})$ given by $\mathscr{F} \boxtimes \mathscr{G} = p^* \mathscr{F} \otimes q^* \mathscr{G}$, where $p : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{X}$ and $q : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{Y}$ denote the projection maps.

Remark 4.6.4 (Behavior of QCoh$^\vee$ on Morphisms)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let $g : \mathsf{X} \to \mathsf{Y}$ be a morphism between quasi-compact, quasi-separated spectral algebraic spaces over R. Then g induces an R-linear functor QCoh$^\vee(g)$: QCoh$^\vee(\mathsf{X}) \to$ QCoh$^\vee(\mathsf{Y})$. Applying Proposition 4.5.2 to the diagram

$$
\begin{array}{ccc}
 & \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\, R) & \\
 \swarrow & & \searrow \\
 \mathrm{QCoh}(\mathsf{Y}) \xrightarrow{\hspace{2em} g^* \hspace{2em}} & & \mathrm{QCoh}(\mathsf{X}),
\end{array}
$$

we deduce that the functor QCoh$^\vee(g)$ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{QCoh}^\vee(\mathsf{X}) & \xrightarrow{\mathrm{QCoh}^\vee(g)} & \mathrm{QCoh}^\vee(\mathsf{Y}) \\
\downarrow & & \downarrow \\
\mathrm{QCoh}(\mathsf{X}) & \xrightarrow{\hspace{1em} g_* \hspace{1em}} & \mathrm{QCoh}(\mathsf{Y}),
\end{array}
$$

where the vertical maps are the equivalences of Remark 4.6.2.

Remark 4.6.5 (Functoriality)**.** With more effort, one can show that the identification of functors $g_* \simeq \mathrm{QCoh}^\vee(g)$ supplied by Remark 4.6.4 is coherently associative (with respect to composition in $\mathrm{AlgSpace}(R)$). More precisely, one can show that the composite functor $\mathrm{AlgSpace}(R) \xrightarrow{\mathrm{QCoh}^\vee} \mathrm{LinCat}_R^{\mathrm{St}} \to \mathcal{C}\mathrm{at}_\infty$ classifies a Cartesian fibration $\mathcal{C} \to \mathrm{AlgSpace}(R)^{\mathrm{op}}$, which is also a coCartesian fibration classified by the composite functor $\mathrm{AlgSpace}(R)^{\mathrm{op}} \xrightarrow{\mathrm{QCoh}} \mathrm{LinCat}_R^{\mathrm{St}} \to \mathcal{C}\mathrm{at}_\infty$.

Example 4.6.6 (The Convolution Product)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R, which we can identify with a commutative monoid object of $\mathrm{AlgSpace}(R)$. Applying the symmetric monoidal functor QCoh$^\vee$: $\mathrm{AlgSpace}(R) \to$ $\mathrm{LinCat}_R^{\mathrm{St}}$, we see that the ∞-category QCoh$^\vee(\mathsf{X})$ can be identified with a commutative algebra object of $\mathrm{LinCat}_R^{\mathrm{St}}$. In other words, there is a symmetric monoidal structure on the ∞-category QCoh$^\vee(\mathsf{X})$, and the action of R on QCoh$^\vee(\mathsf{X})$ is given by a symmetric monoidal functor $\mathrm{Mod}_R \to$ QCoh$^\vee(\mathsf{X})$.

Using Remark 4.6.2, we obtain an R-linear equivalence of ∞-categories QCoh$^\vee(\mathsf{X}) \simeq$ QCoh(X). Transporting the symmetric monoidal structure on QCoh$^\vee(\mathsf{X})$ along this equivalence, we obtain a new symmetric monoidal structure on the ∞-category QCoh(X). Using Remarks 4.6.3 and 4.6.4, we see that this symmetric monoidal structure agrees with the convolution product

$$
\star : \mathrm{QCoh}(\mathsf{X}) \times \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X}) \qquad \mathscr{F} \star \mathscr{G} = m_*(p^* \mathscr{F} \otimes \mathscr{G})
$$

defined at the beginning of this section. Here $p, q : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ denote the projection maps onto the first and second factor, while $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ denotes the addition map on X.

We can summarize the situation more informally as follows: if X is an abelian variety over R, then the convolution product $\star : \mathrm{QCoh}(\mathsf{X}) \times \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{X})$ endows the ∞-category $\mathrm{QCoh}(\mathsf{X})$ with the structure of a symmetric monoidal ∞-category. That is, the convolution product is commutative and associative up to coherent homotopy.

4.7   The Fourier-Mukai Transform

We now show that every biextension of abelian varieties $\mu \in \mathrm{BiExt}(\mathsf{X}, \mathsf{Y})$ gives rise to a functor $\mathrm{FMuk}_\mu : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$, which we will refer to as the *Fourier-Mukai transform*.

Construction 4.7.1 ($\mathrm{QCoh}^\vee(\mathsf{X})$ as a Hopf algebra). Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. Then we can regard the ∞-category $\mathrm{QCoh}(\mathsf{X})$ as a Hopf algebra object of the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$. Note that $\mathrm{QCoh}(\mathsf{X})$ is dualizable as an object of $\mathrm{LinCat}_R^{\mathrm{St}}$ (see Construction 4.6.1). Applying Proposition 3.9.9, we see that the dual $\mathrm{QCoh}^\vee(\mathsf{X})$ inherits the structure of a Hopf algebra object of X, which is characterized (up to equivalence) by the requirement that $\mathrm{Spec}^\star(\mathrm{QCoh}^\vee(\mathsf{X}))$ is the Cartier dual of $\mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{X}))$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}), \mathrm{CMon})$.

Remark 4.7.2. Let X be as in Construction 4.7.1. Then the Hopf algebra structure on $\mathrm{QCoh}^\vee(\mathsf{X})$ determines a commutative algebra structure on $\mathrm{QCoh}^\vee(\mathsf{X})$: that is, it exhibits $\mathrm{QCoh}^\vee(\mathsf{X})$ as a symmetric monoidal ∞-category (equipped with a symmetric monoidal functor $\mathrm{Mod}_R \to \mathrm{QCoh}^\vee(\mathsf{X})$). Using Proposition 3.8.1, we see that this commutative algebra structure coincides with the structure described in Example 4.6.6. In other words, under the equivalence $\mathrm{QCoh}^\vee(\mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{X})$ supplied by Remark 4.6.2, the multiplication on $\mathrm{QCoh}^\vee(\mathsf{X})$ is given by convolution of quasi-coherent sheaves.

Construction 4.7.3 (Fourier-Mukai Transform Associated to a Biextension). Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$ be abelian varieties over R, and let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of (X, Y). Using Theorem 4.4.4, we can identify μ with a map

$$\overline{\mu} : \mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{X})) \wedge \mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{Y})) \to \mathrm{GL}_1^\star$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}}), \mathrm{CMon})$, which we can identify with a map

$$\mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{X})) \to \mathbf{D}(\mathrm{Spec}^\star(\mathrm{QCoh}(\mathsf{Y}))) \simeq \mathrm{Spec}^\star(\mathrm{QCoh}^\vee(\mathsf{Y}))$$

or with a map $\mathrm{FMuk}_\mu : \mathrm{QCoh}^\vee(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ in the ∞-category $\mathrm{Hopf}(\mathrm{LinCat}_R^{\mathrm{St}})$. We will refer to FMuk_μ as the *Fourier-Mukai transform associated to μ.*

Remark 4.7.4. In the situation of Construction 4.7.3, we can identify μ with a map $\mathsf{Y} \wedge \mathsf{X} \to \mathrm{BGL}_1$, which determines another Fourier-Mukai transform $\mathrm{FMuk}'_\mu :$ $\mathrm{QCoh}^\vee(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$.

Remark 4.7.5. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$ be abelian varieties over R. The passage from a biextension $\mu \in \mathrm{BiExt}(\mathsf{X}, \mathsf{Y})$ to the associated Fourier-Mukai transform FMuk_μ does not lose any information. More precisely, the construction $\mu \mapsto \mathrm{FMuk}_\mu$ determines a homotopy equivalence $\mathrm{BiExt}(\mathsf{X}, \mathsf{Y}) \simeq \mathrm{Map}_{\mathrm{Hopf}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}^\vee(\mathsf{X}), \mathrm{QCoh}(\mathsf{Y}))$ (this is the essential content of Theorem 4.4.4).

Remark 4.7.6 (The Underlying R-Linear Functor of FMuk_μ). Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R, and let $\mathscr{L}_\mu$ denote the underlying line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}$, which we can identify with an R-linear functor $\mathrm{Mod}_R \to \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}) \simeq \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{Y})$.

Let $\mathrm{FMuk}_\mu : \mathrm{QCoh}^\vee(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ denote the Fourier-Mukai transform associated to μ, regarded as an R-linear functor (that is, we ignore the Hopf algebra structures on $\mathrm{QCoh}^\vee(\mathsf{X})$ and $\mathrm{QCoh}(\mathsf{Y})$). Using Remark 3.8.4, we can identify the functor FMuk_μ with the composition

$$\mathrm{QCoh}^\vee(\mathsf{X}) \xrightarrow{\mathrm{id} \otimes \mathscr{L}_\mu} \mathrm{QCoh}^\vee(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{Y}) \xrightarrow{e \otimes \mathrm{id}} \mathrm{QCoh}(\mathsf{Y}),$$

where $e : \mathrm{QCoh}^\vee(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \to \mathrm{Mod}_R$ is the evaluation map.

Let us identify $\mathrm{QCoh}^\vee(\mathsf{X})$ with $\mathrm{QCoh}(\mathsf{X})$ using Remark 4.6.2, so that the evaluation functor e is given by the composition

$$\begin{aligned}
\mathrm{QCoh}(\mathsf{X}) \otimes_R \mathrm{QCoh}(\mathsf{X}) \quad &\simeq \quad \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X}) \\
&\xrightarrow{\delta^*} \quad \mathrm{QCoh}(\mathsf{X}) \\
&\xrightarrow{\Gamma} \quad \mathrm{Mod}_R,
\end{aligned}$$

where $\delta : \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X}$ denotes the diagonal map. Under this identification, we see that the Fourier-Mukai transform FMuk_μ corresponds to the functor $\mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ given by $\mathscr{F} \mapsto q_*(\mathscr{L}_\mu \otimes p^* \mathscr{F})$, where

$$p : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \to \mathsf{X} \qquad q : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \to \mathsf{Y}$$

denote the projection maps.

Remark 4.7.7. Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R. The Fourier-Mukai transform FMuk_μ is a morphism of Hopf algebra objects of $\mathrm{LinCat}_R^{\mathrm{St}}$. In particular, it is a morphism of commutative algebra objects of $\mathrm{LinCat}_R^{\mathrm{St}}$. It follows that the functor

$$\mathrm{FMuk}_\mu : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y}) \qquad \mathscr{F} \mapsto q_*(\mathscr{L}_\mu \otimes p^* \mathscr{F})$$

of Remark 4.7.6 is symmetric monoidal, where we regard $\mathrm{QCoh}(\mathsf{X})$ as equipped with the symmetric monoidal structure given by convolution of quasi-coherent sheaves, and $\mathrm{QCoh}(\mathsf{Y})$ with the symmetric monoidal structure given by tensor products of quasi-coherent sheaves.

5 Duality Theory for Abelian Varieties

Let κ be a field and let X be an abelian variety over κ. We let $\mathrm{Pic}(\mathsf{X})$ denote the Picard scheme of X (whose A-valued points are given by line bundles $\mathscr{L}$ on the product $\mathsf{X}_A = \mathsf{X} \times_{\mathrm{Spec}\,\kappa} \mathrm{Spec}\,A$, together with a trivialization of $\mathscr{L}$ along the identity section of X_A). Then the identity component $\mathrm{Pic}^\circ(\mathsf{X}) \subseteq \mathrm{Pic}(\mathsf{X})$ is an abelian variety, called the *dual abelian variety* of X. Moreover, the relationship between X and $\mathrm{Pic}^\circ(\mathsf{X})$ is symmetric: we can recover the original abelian variety X (up to canonical isomorphism) as the dual of $\mathrm{Pic}^\circ(\mathsf{X})$.

In the setting of spectral algebraic geometry, the situation is more subtle. Suppose that R is a connective $\mathbb{E}_\infty$-ring and that X is an abelian variety over R. We can again consider line bundles on X which are trivialized along the identity section $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$: these are parametrized by a spectral algebraic space $\mathscr{P}\mathrm{ic}_\mathsf{X}^e$ (see Notation 5.4.1). Inside $\mathscr{P}\mathrm{ic}_\mathsf{X}^e$, we can consider the open subspace given by the unions of the identity components of each fiber of the projection map $\mathscr{P}\mathrm{ic}_\mathsf{X}^e \to \mathrm{Sp\acute{e}t}\,R$. However, this open subspace is usually *not* an abelian variety over R, because it need not be flat over R.

To remedy the situation, let us return for a moment to the setting of classical algebraic geometry and consider an alternative description of the dual abelian variety $\mathrm{Pic}^\circ(\mathsf{X})$: it parametrizes line bundles $\mathscr{L}$ on X which are *multiplicative* in the sense that the associated $\mathbf{G}_m$-torsor on X determines an extension $0 \to \mathbf{G}_m \to \widetilde{\mathsf{X}} \to \mathsf{X} \to 0$ in the category of commutative group schemes. From this description, it follows immediately that the universal multiplicative line bundle $\mathscr{P}$ on the product $\mathsf{X} \times_{\mathrm{Spec}\,\kappa} \mathrm{Pic}^\circ(\mathsf{X})$ can be regarded as a biextension of $(\mathsf{X}, \mathrm{Pic}^\circ(\mathsf{X}))$ (in the sense of Definition 4.2.1). Moreover, this biextension is *universal* in the following (closely related) senses:

(*a*) For any abelian variety Y over κ, there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{AVar}^s(\kappa)}(\mathsf{Y}, \mathrm{Pic}^{\circ}(\mathsf{X})) \simeq \mathrm{BiExt}(\mathsf{X}, \mathsf{Y}).$$

(*b*) The Fourier-Mukai transform associated to $\mathscr{P}$ (Construction 4.7.3) induces an equivalence of ∞-categories $\mathrm{QCoh}(\mathsf{X}) \simeq \mathrm{QCoh}(\mathsf{Y})$.

In the setting of spectral algebraic geometry, we will *define* the dual of an abelian variety X to be another abelian variety $\widehat{\mathsf{X}}$ for which there is a biextension $\mu \in \mathrm{BiExt}(\mathsf{X}, \widehat{\mathsf{X}})$ having the property (*b*) (Definition 5.1.1). From this perspective, it follows immediately that the dual $\widehat{\mathsf{X}}$ is uniquely determined by X up to equivalence, and that the relation of duality is symmetric in X and $\widehat{\mathsf{X}}$ (Proposition 5.1.3). However, it is not obvious that such a universal biextension should exist: proving this is the main goal of this section. Our strategy can be outlined as follows:

(*i*) Given a connective $\mathbb{E}_{\infty}$-ring R and an abelian variety $\mathsf{X} \in \mathrm{AVar}(R)$, we introduce the notion of a *multiplicative extension* of X. The classification of multiplicative extensions determines a functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$. Essentially by definition, there is a biextension $\mu : \mathsf{X} \wedge \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m \to \mathrm{BGL}_1$ which is universal in the sense (*a*) above.

(*ii*) Using Artin's representability theorem, we show that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is representable by a spectral algebraic space over R (Proposition 5.5.1), which we will also denote by $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$.

(*iii*) In the special case where $R = \kappa$ is an algebraically closed field, we show that the identity component Y of the reduced subspace $(\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m)_{\mathrm{red}}$ is an abelian variety over κ, and that the biextension $\mathsf{X} \wedge \mathsf{Y} \to \mathsf{X} \wedge \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m \xrightarrow{\mu} \mathrm{BGL}_1$ satisfies condition (*b*) (see the proof of Lemma 5.6.3). It follows formally that the same biextension also satisfies (*a*), so that $\mathsf{Y} = \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ and therefore $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is also an abelian variety over κ.

(*iv*) Returning to the case where R is an arbitrary connective $\mathbb{E}_{\infty}$-ring, we use (*iii*) to show that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is an abelian variety over R and that the biextension μ satisfies the universal property (*b*) (Theorem 5.6.4), and therefore exhibits $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ as a dual of X.

Warning 5.0.1. In the setting of classical algebraic geometry, there is direct approach to step (*ii*) which avoids the use of Artin's representability theorem. Any abelian

variety X over a field κ is a projective variety over κ, so we can choose an ample line bundle $\mathscr{L}$ on X. The construction $(x \in X) \mapsto m_x^* \mathscr{L} \otimes \mathscr{L}^{-1}$ then determines a map $f : X \to \mathrm{Pic}^\circ(X)$ (where we regard $\mathrm{Pic}^\circ(X)$ as a functor, not *a priori* assumed to be representable). Using the assumption that $\mathscr{L}$ is ample, one can then argue that $K = \ker(f)$ is representable by a finite group scheme over κ and that f induces an isomorphism $X/K \simeq \mathrm{Pic}^\circ(X)$, thereby proving that $\mathrm{Pic}^\circ(X)$ is representable by an abelian variety (for more details, we refer the reader to [10]).

There are several obstacles to adapting this style of argument to the spectral setting:

- In general, one does not expect an abelian variety X over an $\mathbb{E}_\infty$-ring R to be "projective" in any useful sense. Consequently, there is no obvious candidate for the line bundle $\mathscr{L}$ on X.

- Given a line bundle $\mathscr{L}$ on X, there is no reason to expect line bundles of the form $m_x^* \mathscr{L} \otimes \mathscr{L}^{-1}$ to be multiplicative (in classical algebraic geometry, this depends on the "theorem of the cube", which does not extend to spectral algebraic geometry).

In general, it does not seem reasonable to expect that the dual $\widehat{X} \simeq \mathscr{P}\mathrm{ic}_X^m$ can be constructed as a quotient of X.

5.1 Perfect Biextensions

We begin by introducing the notion of a *perfect biextension* of abelian varieties.

Definition 5.1.1. Let R be a connective $\mathbb{E}_\infty$-ring, let X and Y be abelian varieties over R, and let $\mu : X \wedge Y \to \mathrm{BGL}_1$ be a biextension of (X, Y).

We will say that μ is *perfect* if the functor

$$\mathrm{Mod}_R \xrightarrow{\otimes_R \mathscr{L}_\mu} \mathrm{QCoh}(X \times_{\mathrm{Sp\acute{e}t}\, R} Y) \simeq \mathrm{QCoh}(X) \otimes_R \mathrm{QCoh}(Y)$$

exhibits $\mathrm{QCoh}(X)$ as a dual of $\mathrm{QCoh}(Y)$ in the ∞-category $\mathrm{LinCat}_R^{\mathrm{St}}$ of stable R-linear ∞-categories; here $\mathscr{L}_\mu$ denotes the underlying line bundle of μ. In this case, we will also say that μ *exhibits* Y *as a dual of* X, or that μ *exhibits* X *as a dual of* Y.

Remark 5.1.2. Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. If there exists an abelian variety $Y \in \mathrm{AVar}(R)$ and a biextension $\mu \in \mathrm{BiExt}(X, Y)$ which exhibits Y as a dual of X, then the abelian variety Y and the

biextension μ are determined up to equivalence (in fact, up to a contractible space of choices). However, it is not immediately clear from the definitions that the dual Y exists: to prove this, we must show that $\mathrm{QCoh}^\vee(\mathsf{X}) \in \mathrm{Hopf}(\mathrm{LinCat}_R^{\mathrm{St}})$ belongs to the essential image of the embedding $\mathrm{AVar}(R) \hookrightarrow \mathrm{Hopf}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$ of Proposition 4.4.1. Note that this is equivalent to the *a priori* weaker assertion that, as an object of $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$, the ∞-category $\mathrm{QCoh}^\vee(\mathsf{X})$ to the essential image of the embedding $\mathrm{Var}(R) \hookrightarrow \mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})^{\mathrm{op}}$ provided by Corollary 4.3.8.

Specializing Proposition 3.8.5 to the present situation, we obtain the following:

Proposition 5.1.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$ be abelian varieties over R, and let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension. The following conditions are equivalent:*

(1) *The Fourier-Mukai transform $\mathrm{FMuk}_\mu : \mathrm{QCoh}^\vee(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ is an equivalence of ∞-categories.*

(2) *The Fourier-Mukai transform $\mathrm{FMuk}_\mu' : \mathrm{QCoh}^\vee(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ (see Remark 4.7.4) is an equivalence of ∞-categories.*

(3) *The biextension μ is perfect.*

We now supply a more concrete criterion for a biextension to be perfect:

Proposition 5.1.4. *Let R be a connective $\mathbb{E}_\infty$-ring, let $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$ be abelian varieties over R, let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension, and let $\mathscr{L}_\mu$ denote the underlying line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$. Then μ is perfect if and only if it satisfies both of the following conditions:*

(a) *Let $\pi : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{X}$ denote the projection map. Then $\pi_* \mathscr{L}_\mu$ is invertible with respect to the convolution product on $\mathrm{QCoh}(\mathsf{X})$.*

(b) *Let $\pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{Y}$ denote the projection map. Then $\pi'_* \mathscr{L}_\mu$ is invertible with respect to the convolution product on $\mathrm{QCoh}(\mathsf{Y})$.*

Proof. In what follows, let us abuse notation by using Remark 4.6.2 to identify $\mathrm{QCoh}^\vee(\mathsf{X})$ with $\mathrm{QCoh}(\mathsf{X})$ and $\mathrm{QCoh}^\vee(\mathsf{Y})$ with $\mathrm{QCoh}(\mathsf{Y})$. Note first that if μ is perfect, then the Fourier-Mukai functors

$$\mathrm{FMuk}_\mu : \mathrm{QCoh}(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y}) \qquad \mathrm{FMuk}_\mu' : \mathrm{QCoh}(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$$

are equivalences of ∞-categories (Proposition 5.1.3). In this case, it follows that the composite functors

$$(\text{FMuk}'_\mu \circ \text{FMuk}_\mu) : \text{QCoh}(\mathsf{X}) \to \text{QCoh}(\mathsf{X})$$

$$(\text{FMuk}_\mu \circ \text{FMuk}'_\mu) : \text{QCoh}(\mathsf{Y}) \to \text{QCoh}(\mathsf{Y})$$

are also equivalences of ∞-categories. Conversely, if the functors $(\text{FMuk}'_\mu \circ \text{FMuk}_\mu)$ and $(\text{FMuk}_\mu \circ \text{FMuk}'_\mu)$ are both equivalences, then FMuk_μ admits a left homotopy inverse (given by $(\text{FMuk}'_\mu \circ \text{FMuk}_\mu)^{-1} \circ \text{FMuk}'_\mu$) and a right homotopy inverse (given by $\text{FMuk}'_\mu \circ (\text{FMuk}_\mu \circ \text{FMuk}'_\mu)^{-1}$), and is therefore an equivalence of ∞-categories. Using Proposition 5.1.3 again, we see that μ is perfect if and only if the following conditions are satisfied:

(a') The functor $(\text{FMuk}'_\mu \circ \text{FMuk}_\mu) : \text{QCoh}(\mathsf{X}) \to \text{QCoh}(\mathsf{X})$ is an equivalence of ∞-categories.

(b') The functor $(\text{FMuk}_\mu \circ \text{FMuk}'_\mu) : \text{QCoh}(\mathsf{Y}) \to \text{QCoh}(\mathsf{Y})$ is an equivalence of ∞-categories.

To complete the proof, it will suffice to show that $(a) \Leftrightarrow (a')$ and $(b) \Leftrightarrow (b')$. We will show that $(a) \Leftrightarrow (a')$; the other case follows by symmetry. For this, we need to give an explicit description of the composite functor $\text{FMuk}'_\mu \circ \text{FMuk}_\mu$.

Consider the diagram

Using the description of the Fourier-Mukai transform supplied by Remark 4.7.6, we can identify the composition $\text{FMuk}'_\mu \circ \text{FMuk}_\mu$ with the functor

$$
\begin{aligned}
\mathscr{F} &\mapsto q''_*(\mathscr{L}_\mu \otimes p''^* q'_*(\mathscr{L}_\mu \otimes p'^* \mathscr{F})) \\
&\simeq q''_*(\mathscr{L}_\mu \otimes q_* p^*(\mathscr{L}_\mu \otimes p'^* \mathscr{F})) \\
&\simeq q''_* q_*(q^* \mathscr{L}_\mu \otimes p^* \mathscr{L}_\mu \otimes p^* p'^* \mathscr{F}).
\end{aligned}
$$

Let $r_0, r_1 : \mathsf{X} \times_{\text{Spét } R} \mathsf{X} \to \mathsf{X}$ denote the projection onto the first and second factor, respectively, and let $u : \mathsf{X} \times_{\text{Spét } R} \mathsf{Y} \times_{\text{Spét } R} \mathsf{X} \to \mathsf{X} \times_{\text{Spét } R} \mathsf{X}$ denote the projection onto

the outer factors. Then $q'' \circ q \simeq r_1 \circ u$ and $p' \circ p \simeq r_0 \circ u$, so that $\mathrm{FMuk}'_\mu \circ \mathrm{FMuk}_\mu$ is given by

$$
\begin{aligned}
\mathscr{F} &\mapsto r_{1*} u_* (q^* \mathscr{L}_\mu \otimes p^* \mathscr{L}_\mu \otimes u^* r_0^* \mathscr{F}) \\
&\simeq r_{1*} (u_* (q^* \mathscr{L}_\mu \otimes p^* \mathscr{L}_\mu) \otimes r_0^* \mathscr{F}) \\
&= r_{1*} (\mathscr{E} \otimes r_0^* \mathscr{F}),
\end{aligned}
$$

where $\mathscr{E} = u_* (q^* \mathscr{L}_\mu \otimes p^* \mathscr{L}_\mu)$.

Let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X}$ denote the multiplication on X, so that m induces a map $m_\mathsf{Y} : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$. Since $\mathscr{L}_\mu$ is the line bundle associated to a biextension μ, we have a canonical equivalence $q^* \mathscr{L}_\mu \otimes p^* \mathscr{L}_\mu \simeq m_\mathsf{Y}^* \mathscr{L}_\mu$. The pullback diagram

$$
\begin{array}{ccc}
\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} & \xrightarrow{\ m_\mathsf{Y}\ } & \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle \pi} \\
\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X} & \xrightarrow{\ \ \ m\ \ \ } & \mathsf{X}
\end{array}
$$

determines an equivalence of functors $m^* \pi_* \to u_* m_\mathsf{Y}^*$, and therefore an equivalence $\mathscr{E} \simeq m^* \pi_* \mathscr{L}_\mu$.

Let $s : \mathsf{X} \to \mathsf{X}$ denote the map given by multiplication by (-1), and let us regard the pair $(m, s \circ r_0)$ as an equivalence of $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{X}$ with itself, having homotopy inverse $(m, s)^{-1}$. We then compute

$$
\begin{aligned}
(\mathrm{FMuk}'_\mu \circ \mathrm{FMuk}_\mu)(\mathscr{F}) &\simeq r_{1*} (m^* \pi_* \mathscr{L}_\mu \otimes r_0^* \mathscr{F}) \\
&\simeq r_{1*} (m, s)^* (r_0^* \pi_* \mathscr{L}_\mu \otimes r_1^* s^* \mathscr{F}) \\
&\simeq r_{1*} (m, s)_*^{-1} (r_0^* \pi_* \mathscr{L}_\mu \otimes r_1^* s^* \mathscr{F}) \\
&\simeq m_* (r_0^* \pi_* \mathscr{L}_\mu \otimes r_1^* s^* \mathscr{F}) \\
&= (\pi_* \mathscr{L}_\mu) \star (s^* \mathscr{F}),
\end{aligned}
$$

where $\star$ denotes the convolution product of §4.6. Since the pullback functor s^* is an equivalence, it follows that $\mathrm{FMuk}'_\mu \circ \mathrm{FMuk}_\mu$ is an equivalence if and only if the object $\pi_* \mathscr{L}_\mu \in \mathrm{QCoh}(\mathsf{X})$ is invertible with respect to convolution. $\qquad\square$

5.2 Dualizing Sheaves

Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R, let $\mathscr{L}_\mu$ be the underlying line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}$, and let

$$
\pi : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{X} \qquad \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y} \to \mathsf{Y}
$$

denote the projection maps. If μ is perfect, then $\pi_* \mathscr{L}_\mu \in \mathrm{QCoh}(\mathsf{X})$ and $\pi'_* \mathscr{L}_\mu \in \mathrm{QCoh}(\mathsf{Y})$ are invertible objects with respect to the convolution product on $\mathrm{QCoh}(\mathsf{X})$ and $\mathrm{QCoh}(\mathsf{Y})$ respectively. In this case, we can explicitly identify the objects $\pi_* \mathscr{L}_\mu$ and $\pi'_* \mathscr{L}_\mu$ using the language of Grothendieck duality (see §SAG.6). We begin with some general remarks.

Notation 5.2.1 (The Dualizing Sheaf). Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. We let $\omega_{\mathsf{X}} = \omega_{\mathsf{X}/\mathrm{Sp\acute{e}t}\,R}$ denote the relative dualizing sheaf of the projection map $\mathsf{X} \to \mathrm{Sp\acute{e}t}\,R$ (Definition SAG.6.4.5.3). We let $\omega_{\mathsf{X}}^\circ$ denote the pullback $e^* \omega_{\mathsf{X}} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R) \simeq \mathrm{Mod}_R$, where $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$ denotes the identity section.

Remark 5.2.2. Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. Since X is fiber smooth (Corollary 1.4.9), the sheaf ω_{X} is an invertible object of $\mathrm{QCoh}(\mathsf{X})$ (with respect to the tensor product). If X is an abelian variety of dimension g over R, then $\Sigma^{-g}\omega_{\mathsf{X}}$ is a line bundle on X.

Proposition 5.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be an abelian variety over R, and let $q : \mathsf{X} \to \mathrm{Sp\acute{e}t}\,R$ denote the projection map. Then there is a canonical equivalence $\omega_{\mathsf{X}} \simeq q^* \omega_{\mathsf{X}}^\circ$ in the ∞-category $\mathrm{QCoh}(\mathsf{X})$.*

Proof. Let $\pi, \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ denote the projection onto the first and second factor, respectively, and let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ be the addition map. Then we have a diagram of pullback squares

$$
\begin{array}{ccccc}
\mathsf{X} & \xleftarrow{\;\pi\;} & \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} & \xrightarrow{\;\pi'\;} & \mathsf{X} \\
{\scriptstyle q}\downarrow & & {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle q} \\
\mathrm{Sp\acute{e}t}\,R & \xleftarrow{\;q\;} & \mathsf{X} & \xrightarrow{\;q\;} & \mathrm{Sp\acute{e}t}\,R.
\end{array}
$$

Let $\omega_m \in \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X})$ denote the relative dualizing sheaf of m. Applying Remark SAG.6.4.2.6, we obtain equivalences $\pi^* \omega_{\mathsf{X}} \simeq \omega_m \simeq \pi'^* \omega_{\mathsf{X}}$. The desired result now follows by applying the functor f^* to this equivalence, where $f : \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X}$ is the map given by $(\mathrm{id}, 0)$. $\square$

Remark 5.2.4 (The Adjoint Fourier-Mukai Transform). Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R and let $\mathrm{FMuk}_\mu : \mathrm{QCoh}^\vee(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ be the Fourier-Mukai transform associated to μ. According

to Remark 4.7.6, the functor FMuk_μ is given by the composition of functors

$$
\begin{array}{rcl}
\mathrm{QCoh}^\vee(\mathsf{X}) & \simeq & \mathrm{QCoh}(\mathsf{X}) \\
& \xrightarrow{\pi^*} & \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}) \\
& \xrightarrow{\otimes \mathscr{L}_\mu} & \mathrm{QCoh}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}) \\
& \xrightarrow{\pi'_*} & \mathrm{QCoh}(\mathsf{Y}),
\end{array}
$$

where $\pi : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \to \mathsf{X}$ and $\pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \to \mathsf{Y}$ denote the projection maps and $\mathscr{L}_\mu$ is the underlying line bundle of μ. It follows that the functor FMuk_μ admits a right adjoint, given by the construction

$$
\mathscr{F} \mapsto \pi_*(\mathscr{L}_\mu^{-1} \otimes \pi^* \omega_\mathsf{X} \otimes \pi'^* \mathscr{F}) \simeq \omega_\mathsf{X} \otimes \pi_*(\mathscr{L}_\mu^{-1} \otimes \pi'^* \mathscr{F}).
$$

Remark 5.2.5. Let X be an abelian variety over a connective $\mathbb{E}_\infty$-ring R and let $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$ be the zero section. The morphism e is proper, locally almost of finite presentation, and of finite Tor-amplitude, and therefore admits a relative dualizing sheaf $\omega_e = \omega_{\mathrm{Sp\acute{e}t}\,R/\mathsf{X}} \in \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R)$. Applying Corollary SAG.6.4.2.8 to the diagram

we obtain a canonical equivalence $\omega_\mathsf{X}^\circ \otimes \omega_e \simeq \omega_{\mathrm{Sp\acute{e}t}\,R/\mathrm{Sp\acute{e}t}\,R} \simeq \mathscr{O}_{\mathrm{Sp\acute{e}t}\,R}$. It follows that the sheaf ω_e can be identified with $(\omega_\mathsf{X}^\circ)^{-1}$. Applying Corollary SAG.6.4.2.7, we deduce that the direct image functor $e_* : \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R) \to \mathrm{QCoh}(\mathsf{X})$ admits a right adjoint, given by $\mathscr{F} \mapsto (\omega_\mathsf{X}^\circ)^{-1} \otimes e^* \mathscr{F}$.

Construction 5.2.6 (The Roots of a Biextension). Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R and let $\mathscr{L}_\mu$ be the underlying line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}$. Let $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$ denote the identity section and form a pullback square

$$
\begin{array}{ccc}
\mathsf{Y} & \xrightarrow{e_\mathsf{Y}} & \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \\
\downarrow{\scriptstyle \overline{\pi}} & & \downarrow{\scriptstyle \pi} \\
\mathrm{Sp\acute{e}t}\,R & \xrightarrow{e} & \mathsf{X}.
\end{array}
$$

The multiplicativity of $\mathscr{L}_\mu$ supplies a trivialization of the pullback $e_\mathsf{Y}^* \mathscr{L}_\mu$, which determines an equivalence

$$
\alpha : \overline{\pi}_* \mathscr{O}_\mathsf{Y} \simeq \overline{\pi}_* e_\mathsf{Y}^* \mathscr{L}_\mu \simeq e^* \pi_* \mathscr{L}_\mu.
$$

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Using Remark 5.2.5, we obtain a canonical homotopy equivalence

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(\mathscr{O}_{\mathsf{Y}}, \mathscr{O}_{\mathsf{Y}}) \;&\simeq\; \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R)}(\mathscr{O}_{\mathrm{Sp\acute{e}t}\,R}, \overline{\pi}_{*}\,\mathscr{O}_{\mathsf{Y}}) \\
&\xrightarrow{\alpha}\; \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R)}(\mathscr{O}_{\mathrm{Sp\acute{e}t}\,R}, e^{*}\pi_{*}\,\mathscr{L}_{\mu}) \\
&\simeq\; \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(e_{*}(\omega_{\mathsf{X}}^{\circ})^{-1}, \pi_{*}\,\mathscr{L}_{\mu}).
\end{aligned}
$$

We let $\rho_{\mu} : e_{*}(\omega_{\mathsf{X}}^{\circ})^{-1} \to \pi_{*}\,\mathscr{L}_{\mu}$ denote the image of the identity map $\mathrm{id}_{\mathscr{O}_{\mathsf{Y}}}$ under this homotopy equivalence.

Applying the same construction with the roles of X and Y reversed, we obtain a morphism $\rho'_{\mu} : e'_{*}(\omega_{\mathsf{Y}}^{\circ})^{-1} \to \pi'_{*}\,\mathscr{L}_{\mu}$, where $e' : \mathrm{Sp\acute{e}t}\,R \to \mathsf{Y}$ and $\pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \to \mathsf{Y}$ denote the identity section and projection map, respectively. We will refer to ρ_{μ} and ρ'_{μ} as the *roots* of the biextension μ.

Remark 5.2.7. Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_{\infty}$-ring R and let $\rho_{\mu} : e_{*}(\omega_{\mathsf{X}}^{\circ})^{-1} \to \pi_{*}\,\mathscr{L}_{\mu}$ be as in Construction 5.2.6. If $R \not\simeq 0$, then ρ_{μ} is not nullhomotopic (by construction, ρ_{μ} is nullhomotopic if and only if the identity map $\mathrm{id} : \mathscr{O}_{\mathsf{Y}} \to \mathscr{O}_{\mathsf{Y}}$ is nullhomotopic).

We have the following refinement of Proposition 5.1.4:

Proposition 5.2.8. *Let* $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ *be a biextension of abelian varieties over a connective* $\mathbb{E}_{\infty}$-*ring* R. *Then* μ *is perfect if and only if the roots*

$$
\rho_{\mu} : e_{*}(\omega_{\mathsf{X}}^{\circ})^{-1} \to \pi_{*}\,\mathscr{L}_{\mu} \qquad \rho'_{\mu} : e'_{*}(\omega_{\mathsf{Y}}^{\circ})^{-1} \to \pi'_{*}\,\mathscr{L}_{\mu}
$$

of Construction 5.2.6 are equivalences in the ∞-*categories* $\mathrm{QCoh}(\mathsf{X})$ *and* $\mathrm{QCoh}(\mathsf{Y})$, *respectively.*

Proof. We will employ the notation of Construction 5.2.6. Note that $(\omega_{\mathsf{X}}^{\circ})^{-1}$ is an invertible object of the ∞-category $\mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R) \simeq \mathrm{Mod}_R$. Moreover, the direct image functor $e_{*} : \mathrm{QCoh}(\mathrm{Sp\acute{e}t}\,R) \to \mathrm{QCoh}(\mathsf{X})$ is symmetric monoidal, if we regard $\mathrm{QCoh}(\mathsf{X})$ as equipped with the convolution product of §4.6. Consequently, if ρ_{μ} is an equivalence, then $\pi_{*}\,\mathscr{L}_{\mu} \in \mathrm{QCoh}(\mathsf{X})$ is invertible with respect to the convolution product. Similarly, if ρ'_{μ} is an equivalence, then $\pi'_{*}\,\mathscr{L}_{\mu}$ is invertible with respect to the convolution product on $\mathrm{QCoh}(\mathsf{Y})$. Applying Proposition 5.1.4, we see that if ρ_{μ} and ρ'_{μ} are both equivalences, then μ is perfect.

We now prove the converse. Suppose that μ is a perfect biextension; we will show that ρ_{μ} and ρ'_{μ} are equivalences. Working locally with respect to the Zariski topology on R, we may assume without loss of generality that $R \neq 0$ and that X and Y have

fixed dimensions g and g', respectively. Localizing further if necessary, we may assume that $\omega_{\mathsf{X}}^{\circ} \simeq \Sigma^g \mathscr{O}_{\mathrm{Sp\acute{e}t}\,R}$ and $\omega_{\mathsf{Y}}^{\circ} \simeq \Sigma^{g'} \mathscr{O}_{\mathrm{Sp\acute{e}t}\,R}$ (see Remark 5.2.2). Moreover, we may assume without loss of generality that $g \leqslant g'$.

Let $\mathrm{FMuk}'_\mu : \mathrm{QCoh}^\vee(\mathsf{Y}) \to \mathrm{QCoh}(\mathsf{X})$ denote the Fourier-Mukai transform associated to μ, so that FMuk'_μ is an equivalence of ∞-categories (Proposition 5.1.3). Let G denote a homotopy inverse of the functor FMuk'_μ. Note that $\mathrm{FMuk}'_\mu(\mathscr{O}_{\mathsf{Y}}) \simeq \pi_* \mathscr{L}_\mu$ (see Remark 4.7.6), so there exists an equivalence $G(\pi_* \mathscr{L}_\mu) \simeq \mathscr{O}_{\mathsf{Y}}$. The functor G is also right adjoint to FMuk'_μ, so Remark 5.2.4 supplies a calculation

$$
\begin{aligned}
G((e^* \omega_{\mathsf{X}}^{\circ})^{-1}) &\simeq \omega_{\mathsf{Y}} \otimes \pi'_*(\mathscr{L}_\mu^{-1} \otimes \pi^* e_*(\omega_{\mathsf{X}}^{\circ})^{-1}) \\
&\simeq \omega_{\mathsf{Y}} \otimes \pi'_*(\mathscr{L}_\mu^{-1} \otimes e_{\mathsf{Y}*} \overline{\pi}^*(\omega_{\mathsf{X}}^{\circ})^{-1}) \\
&\simeq \omega_{\mathsf{Y}} \otimes \pi'_* e_{\mathsf{Y}*}(e_{\mathsf{Y}}^* \mathscr{L}_\mu^{-1} \otimes \overline{\pi}^*(\omega_{\mathsf{X}}^{\circ})^{-1}) \\
&\simeq \omega_{\mathsf{Y}} \otimes \pi'_* e_{\mathsf{Y}*}(\overline{\pi}^*(\omega_{\mathsf{X}}^{\circ})^{-1}) \\
&\simeq \omega_{\mathsf{Y}} \otimes \overline{\pi}^*(\omega_{\mathsf{X}}^{\circ})^{-1} \\
&\simeq \Sigma^{g'-g} \mathscr{O}_{\mathsf{Y}}.
\end{aligned}
$$

It follows that we can identify $G(\rho_\mu)$ with a morphism $\gamma : \Sigma^{g'-g} \mathscr{O}_{\mathsf{Y}} \to \mathscr{O}_{\mathsf{Y}}$ in $\mathrm{QCoh}(\mathsf{Y})$. We claim that γ is an equivalence. To prove this, we may assume without loss of generality that $R = \kappa$ is a field (see Corollary SAG.2.7.4.4). Our assumption that Y is geometrically reduced and geometrically connected then supplies isomorphisms

$$
\pi_* \Gamma(\mathsf{Y}; \mathscr{O}_{\mathsf{Y}}) \simeq \begin{cases} \kappa & \text{if } * = 0 \\ 0 & \text{if } * > 0. \end{cases}
$$

Since the map γ does not vanish (Remark 5.2.7), we conclude that $g = g'$ and that γ is an equivalence. Since G is an equivalence of ∞-categories, we conclude that ρ_μ is an equivalence. Applying the same argument (with the roles of X and Y reversed; note that the above argument shows that $g' \leqslant g$), we conclude that ρ'_μ is also an equivalence. $\qquad\square$

Corollary 5.2.9. *Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R. Suppose that, for every morphism $R \to \kappa$ where κ is an algebraically closed field, the induced map $\mu_\kappa : \mathsf{X}_\kappa \wedge \mathsf{Y}_\kappa \to \mathrm{BGL}_1$ is a perfect biextension of abelian varieties over κ. Then μ is perfect.*

Proof. Combine Proposition 5.2.8 with Corollary SAG.2.7.4.4. $\qquad\square$

Remark 5.2.10. Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R with underlying line bundle $\mathscr{L}_\mu$. If μ is perfect, then Proposition 5.1.4 supplies canonical equivalences

$$(\omega_\mathsf{X}^\circ)^{-1} \simeq \Gamma(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, R} \mathsf{Y}; \mathscr{L}_\mu) \simeq (\omega_\mathsf{Y}^\circ)^{-1}.$$

In particular, ω_X° and ω_Y° are canonically equivalent (as invertible objects of Mod_R).

5.3   Multiplicative Line Bundles

Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. If X admits a dual $\widehat{\mathsf{X}}$ (in the sense of Definition 5.1.1), then $\widehat{\mathsf{X}}$ is determined by X up to equivalence (Remark 5.1.2). We now make this observation more explicit by describing $\widehat{\mathsf{X}}$ in terms of its functor of points.

Definition 5.3.1 (Multiplicative Line Bundles)**.** Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R, which we identify with its functor of points $\mathsf{X} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$, and let $\mathrm{BGL}_1 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$ be as in Construction 4.1.5. A *multiplicative line bundle* on X is a morphism $\mathscr{L} : \mathsf{X} \to \mathrm{BGL}_1$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon})$.

For every connective R-algebra A, we let $\mathscr{P}\mathrm{ic}_\mathsf{X}^m(A)$ denote the mapping space $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon})}(\mathsf{X}_A, \mathrm{BGL}_1)$ of multiplicative line bundles on $\mathsf{X}_A = \mathsf{X}|_{\mathrm{CAlg}_A^{\mathrm{cn}}}$.

Remark 5.3.2. In the situation of Definition 5.3.1, we can regard the construction $A \mapsto \mathscr{P}\mathrm{ic}_\mathsf{X}^m(A)$ as a functor from the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ to the ∞-category CMon of $\mathbb{E}_\infty$ spaces. This functor is equipped with a map $e : \mathsf{X} \wedge \mathscr{P}\mathrm{ic}_\mathsf{X}^m \to \mathrm{BGL}_1$ with the following universal property: for every functor $Y : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$, composition with e induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon}(\mathcal{S}))}(Y, \mathscr{P}\mathrm{ic}_\mathsf{X}^m) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon})}(\mathsf{X} \wedge Y, \mathrm{BGL}_1).$$

This follows from a slight variant of the arguments given in §3.7.

Example 5.3.3. Let R be a connective $\mathbb{E}_\infty$-ring and suppose we are given abelian varieties $\mathsf{X}, \mathsf{Y} \in \mathrm{AVar}(R)$. Then we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_R^{\mathrm{cn}}, \mathrm{CMon})}(\mathsf{Y}, \mathscr{P}\mathrm{ic}_\mathsf{X}^m) \simeq \mathrm{BiExt}(\mathsf{X}, \mathsf{Y}).$$

Combining Example 5.3.3 with Theorem 4.4.4, we obtain the following:

Proposition 5.3.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a perfect biextension of abelian varieties over R. Then the induced map $\alpha : \mathsf{Y} \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ is an equivalence.*

Corollary 5.3.5. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. If X admits a dual $\widehat{\mathsf{X}}$, then there is a canonical equivalence $\widehat{\mathsf{X}} \simeq \mathscr{P}\mathrm{ic}_\mathsf{X}^m$. In particular, $\mathscr{P}\mathrm{ic}_\mathsf{X}^m$ is (representable by) an abelian variety over R.*

Warning 5.3.6. Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over a connective $\mathbb{E}_\infty$-ring R. If X admits a dual $\widehat{\mathsf{X}}$, then the converse of Proposition 5.3.4 holds: if $\alpha : \mathsf{Y} \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m \simeq \widehat{\mathsf{X}}$ is an equivalence, then the biextension μ is perfect. However, if we do not yet know that X admits a dual, then the converse of Proposition 5.3.4 not obvious.

To understand the issue, let $\mathrm{FMuk}_\mu : \mathrm{QCoh}^\vee(\mathsf{X}) \to \mathrm{QCoh}(\mathsf{Y})$ be the Fourier-Mukai transform associated to μ. For every connective R-algebra A, we can identify $\alpha(A) : \mathsf{Y}(A) \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m(A)$ with the map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}(\mathsf{Y}), \mathrm{Mod}_A) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}^\vee(\mathsf{X}), \mathrm{Mod}_A)$$

given by precomposition with FMuk_μ. Using Proposition 5.1.3, we obtain the following:

(*a*) The biextension μ is perfect if and only if, for every commutative algebra object $\mathcal{C}$ of $\mathrm{LinCat}_R^{\mathrm{St}}$, the Fourier-Mukai transform FMuk_μ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}(\mathsf{Y}), \mathcal{C}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}^\vee(\mathsf{X}), \mathcal{C}).$$

(*b*) The map α is an equivalence if and only if, for every commutative algebra object $\mathcal{C}$ of $\mathrm{LinCat}_R^{\mathrm{St}}$ having the form Mod_A for some $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the Fourier-Mukai transform FMuk_μ induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}(\mathsf{Y}), \mathcal{C}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})}(\mathrm{QCoh}^\vee(\mathsf{X}), \mathcal{C}).$$

The implication (*a*) $\Rightarrow$ (*b*) is the contents of Proposition 5.3.4. To prove the converse, it would suffice to show that $\mathrm{QCoh}^\vee(\mathsf{X})$ can be obtained as an inverse limit (in the ∞-category $\mathrm{CAlg}(\mathrm{LinCat}_R^{\mathrm{St}})$) of ∞-categories of the form Mod_A, for $A \in \mathrm{CAlg}_R^{\mathrm{cn}}$. This condition is satisfied if X admits a dual (we then have $\mathrm{QCoh}^\vee(\mathsf{X}) \simeq \mathrm{QCoh}(\widehat{\mathsf{X}})$), but is not obvious from the definitions.

We now study the deformation-theoretic properties of the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$.

Proposition 5.3.7. *Let R be a connective $\mathbb{E}_\infty$-ring, let X be an abelian variety over R, and let us regard $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ as a functor from the ∞-category $\mathrm{CAlg}_R^{\mathrm{cn}}$ to the ∞-category $\mathcal{S}$ of spaces. Then:*

(a) *For every object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the space $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m(R')$ is essentially small.*

(b) *The functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is nilcomplete.*

(c) *The functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is cohesive.*

(d) *The functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ admits a (-1)-connective cotangent complex (relative to R).*

Proof. For every object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, let us regard the commutative monoids $\mathsf{X}_{R'}$ and BGL_1 as defining a map of simplicial sets $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \partial\,\Delta^1 \to \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathcal{S})$. For every map of simplicial sets $f : T \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$, let T_∂ denote the inverse image of $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \partial\,\Delta^1$ in T, and let $F_T(R')$ denote the fiber of the categorical fibration

$$\mathrm{Fun}(T, \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathcal{S})) \to \mathrm{Fun}(T_\partial, \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathcal{S}))$$

over the point determined by the pair $(\mathsf{X}_{R'}, \mathrm{BGL}_1)$. Then we can regard F_T as defining a functor $\mathrm{CAlg}_R^{\mathrm{cn}} \to \widehat{\mathcal{S}}$.

Let T' be a simplicial subset of T containing T_∂. We will prove the following:

(a') *For every $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the map of spaces $F_T(R') \to F_{T'}(R')$ has essentially small homotopy fibers.*

(b') *The natural transformation $F_T \to F_{T'}$ is nilcomplete.*

(c') *The natural transformation $F_T \to F_{T'}$ is cohesive.*

(d') *The natural transformation $F_T \to F_{T'}$ admits a (-1)-connective cotangent complex.*

Taking $T = \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ and $T' = T_\partial$, we will obtain the desired result.

We proceed by induction on the (possibly infinite) dimension of T. Choose a transfinite sequence of simplicial subsets $T' = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_\alpha = T$ satisfying the following conditions:

(i) *If γ is a nonzero limit ordinal, then $T_\gamma = \bigcup_{\beta < \gamma} T_\beta$.*

(ii) If $\gamma = \beta + 1$ is a successor ordinal, then T_γ is obtained from T_β by adjoining a single nondegenerate simplex (whose boundary is already contained in T_β).

We will prove, using induction on $\gamma \leqslant \alpha$, that assertions (a') through (d') hold for the inclusion $T' \subseteq T_\gamma$. If $\gamma = 0$, then $T' = T_\gamma$ and there is nothing to prove. If γ is a nonzero limit ordinal, then $F_{T_\gamma} \simeq \varprojlim_{\beta < \gamma} F_{T_\beta}$, so that the desired result follows from the inductive hypothesis (see Remark SAG.17.2.4.5 in case (d')). It therefore suffices to treat the case where $\gamma = \beta + 1$ is a successor ordinal. Using the inductive hypothesis (and Proposition SAG.17.3.9.1 in case (d')), we are reduced to proving that the inclusion $T_\beta \subseteq T_\gamma$ satisfies the analogues of (a') through (d'). We have a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial\Delta^n & \longrightarrow & \Delta^n \\
\downarrow & & \downarrow \\
T_\beta & \longrightarrow & T_\gamma,
\end{array}
$$

hence a pullback diagram of functors

$$
\begin{array}{ccc}
F_{T_\gamma} & \longrightarrow & F_{T_\beta} \\
\downarrow & & \downarrow \\
F_{\Delta^n} & \longrightarrow & F_{\partial\Delta^n}.
\end{array}
$$

We may therefore replace T by Δ^n and T' by $\partial\Delta^n$.

Note that the map $f : T \to \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ cannot factor through $\mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \Delta^1$ (since $T_\partial \subseteq \partial\Delta^n$). It follows that there exists $1 \leqslant i \leqslant n$ such that $f(i-1) \in \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \{0\}$ and $f(i) \in \mathrm{N}(\mathcal{F}\mathrm{in}_*) \times \{1\}$.

Suppose now that $i < n$, and let $T'' \subseteq T = \Delta^n$ denote the union of the simplices $\Delta^{\{0,\dots,i\}}$ and $\Delta^{\{i,\dots,n\}}$. We have a commutative diagram of functors

$$
\begin{array}{ccc}
F_T & \xrightarrow{\ \psi\ } & F_{T'} \\
 & \underset{\phi}{\searrow} \quad \underset{\psi'}{\swarrow} & \\
 & F_{T''}. &
\end{array}
$$

Since the inclusion $T'' \subseteq T$ is inner anodyne, the map ϕ is an equivalence. Consequently, to verify that assertions (a') through (d') hold for the morphism ψ, it suffices to verify that (a') through (d') hold for the morphism ψ'. This follows from the inductive hypothesis, since T' has dimension smaller than that of T.

If $i > 1$, then we define T'' to be the union of the simplices $\Delta^{\{0,\dots,i-1\}}$ and $\Delta^{\{i-1,\dots,n\}}$ and proceed as above. We are therefore reduced to the case where $i = n = 1$. Then $F_{T'}$ is contractible, and f determines a morphism $(\langle m \rangle, 0) \to (\langle m' \rangle, 1)$ in $N(\mathcal{F}\mathrm{in}_*) \times \Delta^1$. Unwinding the definitions, we see that F_T is the functor which classifies maps X^m to $\mathrm{BGL}_1^{m'}$; here X^m denotes the mth power of X in the ∞-category of spectral algebraic spaces over R. Assertions (a') through (d') now follow from Proposition SAG.19.2.4.7. $\qquad\square$

5.4 The Functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{\circ}$

In the setting of classical algebraic geometry, the theory of multiplicative line bundles simplifies: if X is an abelian variety over a field κ, then a line bundle $\mathscr{L}$ is multiplicative if and only if there exists an isomorphism $m^* \mathscr{L} \simeq \mathscr{L} \boxtimes \mathscr{L}$, where $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to \mathsf{X}$ is the addition law on X (see Proposition 5.4.9 below).

Notation 5.4.1. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathsf{X} \in \mathrm{Var}(R)$ be a variety over R equipped with an R-valued point $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$. We let $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{e} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula

$$\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{e}(A) = \mathrm{fib}(\mathscr{P}\mathrm{ic}(\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathrm{Sp\acute{e}t}\,A) \xrightarrow{e^*} \mathscr{P}\mathrm{ic}(\mathrm{Sp\acute{e}t}\,A)).$$

By virtue of Theorem SAG.19.2.0.5, the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{e}$ is representable by a spectral algebraic space which is quasi-separated and locally of finite presentation over R (which we will also denote by $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{e}$).

Construction 5.4.2. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathsf{X}, \mathsf{Y} \in \mathrm{Var}(R)$ be varieties over R, equipped with R-valued points $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{X}$ and $e' : \mathrm{Sp\acute{e}t}\,R \to \mathsf{Y}$. Then e and e' determine maps

$$\mathsf{Y} \xrightarrow{e_{\mathsf{Y}}} \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y} \xleftarrow{e_{\mathsf{X}}^*} \mathsf{X}.$$

We let $\mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$ denote the fiber of the pullback map

$$\mathscr{P}\mathrm{ic}_{\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}}^{(e,e')} \xrightarrow{e_{\mathsf{X}}^* \times e_{\mathsf{Y}}^*} \mathscr{P}\mathrm{ic}_{\mathsf{X}}^{e} \times_{\mathrm{Sp\acute{e}t}\,R} \mathscr{P}\mathrm{ic}_{\mathsf{Y}}^{e'}.$$

More informally, $\mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$ parametrizes line bundles $\mathscr{L}$ on the product $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}$ which are equipped with (compatible) trivializations along X and Y.

Proposition 5.4.3. *In the situation of Construction 5.4.2, the projection map $\pi : \mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta} \to \mathrm{Sp\acute{e}t}\,R$ admits a cotangent complex which is perfect and 1-connective.*

Proof. For each variety Z over R, let $\mathscr{P}\mathrm{ic}_{\mathsf{Z}}$ denote the classifying stack of line bundles on Z (not equipped with any trivialization), and let $q_{\mathsf{Z}} : \mathscr{P}\mathrm{ic}_{\mathsf{Z}} \to \mathrm{Spét}\, R$ be the projection map. Using Remark SAG.19.2.4.8, we deduce that q_{Z} admits a relative cotangent complex, given by $\Sigma^{-1} q_{\mathsf{Z}}^{*} \Gamma(\mathsf{Z}; \mathscr{O}_{\mathsf{Z}})^{\vee}$. Unwinding the definitions, we see that $\mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$ can be described as the total fiber of the commutative diagram

$$\begin{array}{ccc} \mathscr{P}\mathrm{ic}_{\mathsf{X} \times_{\mathrm{Spét}\,R} \mathsf{Y}} & \longrightarrow & \mathscr{P}\mathrm{ic}_{\mathsf{X}} \\ \downarrow & & \downarrow \\ \mathscr{P}\mathrm{ic}_{\mathsf{Y}} & \longrightarrow & \mathscr{P}\mathrm{ic}_{\mathrm{Spét}\,R} . \end{array}$$

It follows that the relative cotangent complex of π can be obtained by applying the functor $\Sigma^{-1}\pi^{*}$ to the total cofiber of the diagram σ:

$$\begin{array}{ccc} \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})^{\vee} \otimes_{R} \Gamma(\mathsf{Y}; \mathscr{O}_{\mathsf{Y}})^{\vee} & \longleftarrow & \Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}})^{\vee} \\ \uparrow & & \uparrow \\ \Gamma(\mathsf{Y}; \mathscr{O}_{\mathsf{Y}})^{\vee} & \longleftarrow & R \end{array}$$

It will therefore suffice to show that the total cofiber of σ is perfect and 2-connective. This is clear, since the cofiber is dual to the tensor product

$$\mathrm{fib}(\Gamma(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \to R) \otimes_{R} \mathrm{fib}(\Gamma(\mathsf{Y}; \mathscr{O}_{\mathsf{Y}}) \to R),$$

where each factor is perfect of Tor-amplitude $\leqslant -1$ by virtue of our assumption that X and Y are varieties over R (see Proposition SAG.8.6.4.1). $\qquad\square$

Corollary 5.4.4. *Let R be a commutative ring, let $\mathsf{X}, \mathsf{Y} \in \mathrm{Var}(R)$ be varieties over R equipped with points $e : \mathrm{Spét}\, R \to \mathsf{X}$ and $e' : \mathrm{Spét}\, R \to \mathsf{Y}$, and let $\tau_{\leqslant 0} \mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$ denote the underlying 0-truncated spectral algebraic space of $\mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$. Then the structure sheaf $\mathscr{O}_{\mathsf{X} \times_{\mathrm{Spét}\,R} \mathsf{Y}}$ is classified by an étale morphism $u : \mathrm{Spét}\, R \to \tau_{\leqslant 0} \mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$.*

Proof. Set $\mathsf{Z} = \mathscr{P}\mathrm{ic}_{\mathsf{X},\mathsf{Y}}^{\Delta}$ and $\mathsf{Z}_0 = \tau_{\leqslant 0} \mathsf{Z}$, so that we have a tautological closed immersion $\iota : \mathsf{Z}_0 \to \mathsf{Z}$. Using the fiber sequence $u^{*} L_{\mathsf{Z}/\mathrm{Spét}\,R} \to L_{\mathrm{Spét}\,R/\mathrm{Spét}\,R} \to L_{\mathrm{Spét}\,R/\mathsf{Z}}$ together with Proposition 5.4.3, we deduce that $L_{\mathrm{Spét}\,R/\mathsf{Z}}$ is 2-connective. Applying Corollary SAG.17.1.4.3, we deduce that the relative cotangent complex $L_{\mathsf{Z}_0/\mathsf{Z}}$ is also 2-connective. Using the fiber sequence $\iota^{*} L_{\mathsf{Z}_0/\mathsf{Z}} \to L_{\mathrm{Spét}\,R/\mathsf{Z}} \to L_{\mathrm{Spét}\,R/\mathsf{Z}_0}$, we deduce that $L_{\mathrm{Spét}\,R/\mathsf{Z}_0}$ is also 2-connective. Using Proposition SAG.11.2.4.1, we deduce that the morphism u is fiber smooth. The map u admits a left homotopy inverse (given by the composition $\mathsf{Z}_0 \xrightarrow{\iota} \mathsf{Z} \to \mathrm{Spét}\, R$) and therefore locally quasi-finite. It follows that u is étale, as desired. $\qquad\square$

Remark 5.4.5. In the situation of Corollary 5.4.4, suppose that X and Y are fiber smooth over R. Applying Proposition SAG.19.2.5.3, we deduce that the spectral algebraic spaces $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}}$, $\mathscr{P}\mathrm{ic}^{e'}_{\mathsf{Y}}$, and $\mathscr{P}\mathrm{ic}^{(e,e')}_{\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{Y}}$ are separated. It follows that $\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{Y}}$ is also separated, so that the morphism $u : \mathrm{Sp\acute{e}t}\,R \to \tau_{\leqslant 0}\,\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{Y}}$ is a closed immersion. It follows that u is a clopen immersion (see Definition SAG.3.1.7.2): that is, u exhibits $\mathrm{Sp\acute{e}t}\,R$ as a summand of $\tau_{\geqslant 0}\,\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{Y}}$.

Construction 5.4.6. Let R be a connective $\mathbb{E}_\infty$-ring, let X be an abelian variety over R, and let $u : \mathrm{Sp\acute{e}t}\,\pi_0 R \to \tau_{\leqslant 0}\,\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}}$ be the closed and open immersion of Corollary 5.4.4 (and Remark 5.4.5). Then there is a unique (up to equivalence) closed and open subspace $\mathsf{U} \subseteq \mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}}$ which fits into a pullback square

$$
\begin{array}{ccc}
\mathrm{Sp\acute{e}t}\,\pi_0 R & \xrightarrow{\;u\;} & \tau_{\leqslant 0}\,\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}} \\
\downarrow & & \downarrow \\
\mathsf{U} & \longrightarrow & \mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}}\,.
\end{array}
$$

Let $\pi, \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ denote the projection maps and $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X} \to \mathsf{X}$ is the addition map on X. The construction $\mathscr{L} \mapsto m^* \mathscr{L} \otimes \pi^* \mathscr{L}^{-1} \otimes \pi'^* \mathscr{L}^{-1}$ determines a map $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}} \to \mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}}$. We let $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$ denote the fiber product $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}} \times_{\mathscr{P}\mathrm{ic}^\Delta_{\mathsf{X},\mathsf{X}}} \mathsf{U}$, which we regard as a closed and open subspace of $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}}$.

Remark 5.4.7. More informally, the spectral algebraic space $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$ of Construction 5.4.6 parametrizes line bundles $\mathscr{L}$ on X (equipped with a trivialization along the identity section) for which the line bundles $m^* \mathscr{L}$ and $\mathscr{L} \boxtimes \mathscr{L}$ become equivalent when restricted to the 0-truncation of $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,R} \mathsf{X}$. Note that this condition is automatically satisfied if $\mathscr{L}$ is a multiplicative line bundle on X. Consequently, the forgetful map $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}} \to \mathscr{P}\mathrm{ic}^e_{\mathsf{X}}$ factors through $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$.

Warning 5.4.8. Let X be as in Construction 5.4.6. By construction, $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$ is an open and closed subspace of $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}}$. In fact, we can say more: $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$ is the "identity component" of $\mathscr{P}\mathrm{ic}^e_{\mathsf{X}}$, so that the fibers of the map $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}} \to \mathrm{Sp\acute{e}t}\,R$ are connected. This is not *a priori* obvious from the definition. However, it follows from Theorem 5.6.4 (which shows that $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}}$ is an abelian variety over R) together with Proposition 5.4.9 (which shows that $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}}$ and $\mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}$ have the same underlying classical algebraic space).

Proposition 5.4.9. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. For every discrete R-algebra R', the map $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}}(R') \to \mathscr{P}\mathrm{ic}^\circ_{\mathsf{X}}(R')$ is a homotopy equivalence. In particular, the space $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}}(R')$ is discrete.*

Proof. Without loss of generality, we may replace R by R' and thereby reduce to the case where R is discrete. In what follows, we work in the category of algebraic spaces over R; to simplify the notation, we will denote fiber products over $\text{Spét}\,R$ simply as Cartesian products. Let $e : \text{Spét}\,R \to \mathsf{X}$ denote the identity section of X, let $\pi, \pi' : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ denote the projection maps, and let $m : \mathsf{X} \times \mathsf{X} \to \mathsf{X}$ denote the addition map.

Unwinding the definitions, we see that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m(R)$ can be identified with the classifying space of the groupoid of extensions of X by the multiplicative group $\mathbb{G}_m$. To prove that this classifying space is discrete, it suffices to show that there are do not exist any nontrivial group homomorphisms $\phi : \mathsf{X} \to \mathbb{G}_m$. As a map of algebraic spaces, ϕ is determined by an (invertible) global section $f \in \mathrm{H}^0(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$. Since the base point e induces an isomorphism $\mathrm{H}^0(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \to R$, ϕ is determined by the composite map $\text{Spét}\,R \xrightarrow{e} \mathsf{X} \xrightarrow{\phi} \mathbb{G}_m$, which is necessarily trivial whenever ϕ is a group homomorphism.

Now suppose we are given an element of $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^e(R)$, classifying a line bundle $\mathscr{L}$ on X together with a trivialization of $e^* \mathscr{L}$. Let $\widetilde{\mathsf{X}}$ be the associated principal $\mathbb{G}_m$-bundle over X, so that the trivialization of $e^* \mathscr{L}$ determines a point $\widetilde{e} : \text{Spec}\,R \to \widetilde{\mathsf{X}}$. To promote $\mathscr{L}$ to a point of $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m(R)$, we need to supply a compatible abelian group structure on $\widetilde{X}$ having $\widetilde{e}$ as the identity section. Such an abelian group structure is determined by a multiplication map $\widetilde{m} : \widetilde{\mathsf{X}} \times \widetilde{\mathsf{X}} \to \widetilde{\mathsf{X}}$. Such a map is necessarily $\mathbf{G}_m \times \mathbf{G}_m$-equivariant and therefore determined by an isomorphism of line bundles $\alpha : m^* \mathscr{L} \simeq \pi^* \mathscr{L} \otimes \pi'^* \mathscr{L}$, which exists if and only if $\mathscr{L}$ is classified by an R-valued point of $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^\circ(R)$ (and, in this case, the isomorphism α is uniquely determined by the requirement that it is compatible with the trivialization of $e^* \mathscr{L}$). To complete the proof, it will suffice to show that for any such isomorphism α, the induced multiplication $\widetilde{m} : \widetilde{\mathsf{X}}^2 \to \widetilde{\mathsf{X}}$ is commutative and associative, has $\widetilde{e}$ as a unit, and admits inverses. We verify each of these conditions in turn:

(a) The point $\widetilde{e} : \text{Spét}\,R \to \widetilde{\mathsf{X}}$ is a left unit for the multiplication $\widetilde{m}$. To prove this, we must show that the composite map

$$\{\widetilde{e}\} \times \widetilde{\mathsf{X}} \hookrightarrow \widetilde{\mathsf{X}} \times \widetilde{\mathsf{X}} \to \widetilde{\mathsf{X}}$$

is the identity. Unwinding the definitions, this amounts to the assertion that the map of line bundles

$$\beta : \mathscr{L} \simeq (e \times \mathrm{id})^* m^* \mathscr{L} \xrightarrow{\alpha} (e \times \mathrm{id})^* (\pi^* \mathscr{L} \otimes \pi'^* \mathscr{L}) \simeq 0^* \mathscr{L} \otimes \mathscr{L} \simeq \mathscr{L}$$

is the identity map, where $0 : \mathsf{X} \to \mathsf{X}$ denotes the zero map. This map is given by multiplication by an invertible element of $\mathrm{H}^0(\mathsf{X}; \mathscr{O}_{\mathsf{X}})$. Since evaluation at e

induces an isomorphism $\mathrm{H}^0(\mathsf{X}; \mathscr{O}_\mathsf{X}) \to R$, it suffices to check that β restricts to the identity map from $e^* \mathscr{L}$ to itself, which is evident.

(b) The multiplication $\widetilde{m}$ is commutative. Let $\widetilde{m}'$ denote the opposite multiplication on $\widetilde{X}$ (obtained by composing $\widetilde{m}$ with the automorphism of $\widetilde{X} \times \widetilde{X}$ which exchanges the factors). Then $\widetilde{m}'$ is determined by another isomorphism of line bundles $\alpha' : m^* \mathscr{L} \simeq \pi^* \mathscr{L} \otimes \pi'^* \mathscr{L}$. Note that α and α' differ by multiplication by an invertible element of $\mathrm{H}^0(\mathsf{X}^2; \mathscr{O}_{\mathsf{X}^2})$. The map $(e, e) : \mathrm{Sp\acute{e}t}\, R \to \mathsf{X}^2$ induces an isomorphism $\mathrm{H}^0(\mathsf{X}^2; \mathscr{O}_{\mathsf{X}^2}) \to R$. Consequently, to prove that $\alpha = \alpha'$, it suffices to show that they induce the same isomorphism after restricting the base point of X^2, which is evident.

(c) The multiplication $\widetilde{m}$ is associative. Let $f, g : \widetilde{\mathsf{X}}^3 \to \widetilde{\mathsf{X}}$ denote the maps given by the compositions

$$\widetilde{\mathsf{X}}^3 \xrightarrow{\widetilde{m} \times \mathrm{id}} \widetilde{\mathsf{X}}^2 \xrightarrow{\widetilde{m}} \widetilde{\mathsf{X}}$$

$$\widetilde{\mathsf{X}}^3 \xrightarrow{\mathrm{id} \times \widetilde{m}} \widetilde{\mathsf{X}}^2 \xrightarrow{\widetilde{m}} \widetilde{\mathsf{X}}.$$

Let $m_3 : \mathsf{X}^3 \to \mathsf{X}$ denote the multiplication map, and let $p_0, p_1, p_2 : \mathsf{X}^3 \to \mathsf{X}$ be the projections onto the three factors. Then f and g are determined by isomorphisms of line bundles

$$\beta, \beta' : m_3^* \mathscr{L} \to p_0^* \mathscr{L} \otimes p_1^* \mathscr{L} \otimes p_2^* \mathscr{L} \, .$$

We wish to prove that $\beta = \beta'$. Note that β and β' differ by multiplication by an invertible element of $\mathrm{H}^0(\mathsf{X}^3, \mathscr{O}_{\mathsf{X}^3})$. Since the base point $(e, e, e) : \mathrm{Spec}\, R \to \mathsf{X}^3$ induces an isomorphism $\mathrm{H}^0(\mathsf{X}^3, \mathscr{O}_{\mathsf{X}^3}) \to R$, we are reduced to proving that β and β' agree after restriction the base point of X^3, which is obvious.

(d) The commutative monoid structure on $\widetilde{\mathsf{X}}$ admits inverses. Suppose we are given a commutative algebra A over R and an A-valued point $x : \mathrm{Sp\acute{e}t}\, A \to \widetilde{\mathsf{X}}$. We wish to prove that there exists another point $y : \mathrm{Sp\acute{e}t}\, A \to \widetilde{\mathsf{X}}$ such that the composite map

$$\mathrm{Sp\acute{e}t}\, A \xrightarrow{(x,y)} \widetilde{\mathsf{X}} \times \widetilde{\mathsf{X}} \xrightarrow{\widetilde{m}} \widetilde{\mathsf{X}}$$

factors through e. Since y is uniquely determined (if it exists), the existence of y can be tested locally with respect to the Zariski topology on A. Let $y_0 : \mathrm{Sp\acute{e}t}\, A \to \mathsf{X}$ be an inverse to the composite map $\mathrm{Sp\acute{e}t}\, A \xrightarrow{x} \widetilde{\mathsf{X}} \to \mathsf{X}$. Passing

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to a localization of A if necessary, we may suppose that $y_0^* \mathscr{L}$ is trivial so that y_0 lifts to a point $y : \operatorname{Sp\acute{e}t} A \to \widetilde{\mathsf{X}}$. Then the composite map

$$\gamma : \operatorname{Sp\acute{e}t} A \overset{(x,y)}{\to} \widetilde{\mathsf{X}}^2 \overset{\widetilde{m}}{\to} \widetilde{\mathsf{X}}$$

factors through $\ker(\widetilde{\mathsf{X}} \to \mathsf{X}) \simeq \mathbb{G}_m$, and therefore admits an inverse in $\mathbb{G}_m$. Modifying y if necessary, we may assume that γ factors through e, as desired.

$\square$

5.5 Representability of the Functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$

We have now assembled most of the ingredients needed for the proof of the following result:

Proposition 5.5.1. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. Then the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ is representable by a separated spectral algebraic space which is locally almost of finite presentation over R. Moreover, if $i : \mathsf{Z} \to \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is a closed immersion of spectral algebraic spaces and Y is quasi-compact, then Z is proper over R.*

Proof. Let $e : \operatorname{Sp\acute{e}t} R \to \mathsf{X}$ denote the identity section of X. Using Proposition SAG.19.2.0.5 and Corollary SAG.19.2.5.3, we see that the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^\circ$ is representable by a separated spectral algebraic space. By virtue of Proposition 5.4.9, the functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^\circ$ and $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ agree when restricted to the category $\mathrm{CAlg}_R^\heartsuit \subseteq \mathrm{CAlg}_R^{\mathrm{cn}}$ of discrete R-algebras. Since the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is cohesive, nilcomplete, and admits a cotangent complex (Proposition 5.3.7), Theorem SAG.18.1.0.2 guarantees that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is representable by a spectral Deligne-Mumford stack over R. Since $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^\circ$ and $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ agree on discrete R-algebras, we conclude that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is also a separated spectral algebraic space. Note that if $i : \mathsf{Z} \to \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is a closed immersion, then the composite map $\mathsf{Z} \overset{i}{\to} \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m \to \mathscr{P}\mathrm{ic}_{\mathsf{X}}^e$ is also a closed immersion. If Z is quasi-compact, then Corollary SAG.19.2.5.3 guarantees that Z is proper over R.

We now complete the proof by showing that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ is locally almost of finite presentation over R. By virtue of Proposition SAG.17.4.3.1, we are reduced to proving the following:

($*$) For each integer $n \geqslant 0$, the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m$ commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}_R^{\mathrm{cn}}$.

Let us now regard n as fixed. For each n-truncated object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, we let $\mathcal{C}_{R'}$ denote the full subcategory of $\mathrm{CAlg}_{R'}^{\mathrm{cn}}$ spanned by the n-truncated objects, and regard $\mathcal{C}_{R'}$ as equipped with the étale topology. Let $\widehat{\mathcal{S}\mathrm{hv}}^{\,\text{ét}}(R')$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ which are sheaves with respect to the étale topology. Consider the following condition on a functor F:

($*'$) Let $F' : \mathrm{CAlg}_{R'}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of $F|_{\mathcal{C}_{R'}}$. Then the canonical map $F' \to F$ exhibits F as a sheafification of F' with respect to the étale topology.

Note that the collection of functors F which satisfy $(*)$ is closed under colimits in $\widehat{\mathcal{S}\mathrm{hv}}^{\,\text{ét}}(R')$. Moreover, any functor which is corepresentable by an n-truncated object of $\mathrm{CAlg}_{R'}^{\mathrm{cn}}$ satisfies $(*')$. It follows that if a functor F is representable by an n-truncated spectral Deligne-Mumford stack over R', then F satisfies $(*')$.

Suppose we are given a simplicial set K and a pair of diagrams $Y, Z : K \to \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Assume that for every vertex $v \in K$, the functor $Z(v) : \mathrm{CAlg}_{R'}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a sheaf for the étale topology, and the functor $Y(v) : \mathrm{CAlg}_{R'}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ satisfies $(*')$. Then the restriction map

$$\mathrm{Map}_{\mathrm{Fun}(K, \mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, Z) \to \mathrm{Map}_{\mathrm{Fun}(K, \mathrm{Fun}(\mathcal{C}_{R'}, \widehat{\mathcal{S}}))}(Y|_{\mathcal{C}_{R'}}, Z|_{\mathcal{C}_{R'}})$$

is a homotopy equivalence. In particular, we obtain a homotopy equivalence

$$\begin{aligned} \mathscr{P}\mathrm{ic}_{\mathsf{X}}^m(R') &\simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Fun}(\mathrm{CAlg}_{R'}^{\mathrm{cn}}, \mathcal{S}))}(\mathsf{X}|_{\mathrm{CAlg}_{R'}^{\mathrm{cn}}}, \mathrm{BGL}_1|_{\mathrm{CAlg}_{R'}^{\mathrm{cn}}}) \\ &\simeq \mathrm{Map}_{\mathrm{CMon}(\mathrm{Fun}(\mathcal{C}_{R'}, \mathcal{S}))}(\mathsf{X}|_{\mathcal{C}_{R'}}, \mathrm{BGL}_1|_{\mathcal{C}_{R'}}). \end{aligned}$$

Note that $\mathsf{X}|_{\mathcal{C}_{R'}}$ is an n-truncated object of $\mathrm{Fun}(\mathcal{C}_{R'}, \mathcal{S})$ and that $\mathrm{BGL}_1|_{\mathcal{C}_R}$ is an $(n+1)$-truncated object of $\mathrm{Fun}(\mathcal{C}_{R'}, \mathcal{S})$.

For every integer $k \geqslant 0$, let $\mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S}))$ denote the ∞-category of $\mathbb{E}_k$-monoid objects of $\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S})$ (see §HA.5.1). Let $\mathsf{X}|_{R'}^{(k)}$ denote the image of $\mathsf{X}|_{\mathcal{C}_{R'}}$ in $\mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S}))$, define $\mathrm{BGL}_{1R'}^{(k)}$ similarly, and let $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k)}(R')$ denote the mapping space $\mathrm{Map}_{\mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S}))}(\mathsf{X}_{R'}^{(k)}, \mathrm{BGL}_{1R'}^{(k)})$. The construction $R' \mapsto \mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k)}(R')$ determines a functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k)} : \tau_{\leqslant n}\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$. Since $\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S})$ is equivalent to an $(n+2)$-category, Example HA.5.1.2.3 implies that the restriction functor $\mathrm{CMon}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S})) \to \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}, \mathcal{S}))$ is an equivalence of ∞-categories for $k \geqslant n+3$. It follows that the restriction map $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^m(R') \to \mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k)}(R')$ is a homotopy equivalence for $k \geqslant n+3$, for any n-truncated object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$. To complete the proof, it will suffice to show that the functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k)}$ commute with filtered colimits for $k \geqslant 0$.

We now proceed by induction on k. In the case $k = 0$, the desired result is an immediate consequence of Proposition SAG.19.2.4.7. To carry out the inductive step, let us suppose that $k \geqslant 0$ and the functor $\mathscr{P}\mathrm{ic}_{\mathsf{Y}}^{(k)}$ commutes with filtered colimits for *every* abelian variety Y over R; we will show that $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k+1)}$ commutes with filtered colimits. For every n-truncated object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, let $\mathcal{E}_{R'} = \mathrm{Mon}_{\mathbb{E}_k}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S}))$. Using Theorem HA.5.1.2.2, we obtain an equivalence of ∞-categories $\mathrm{Mon}_{\mathbb{E}_{k+1}}(\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S})) \simeq \mathrm{Mon}(\mathcal{E}_{R'})$. In particular, $\mathsf{X}_{R'}^{(k+1)}$ and $\mathrm{BGL}_{1R'}^{(k+1)}$ determine monoid objects $E_{R'}, E'_{R'} \in \mathrm{Mon}(\mathcal{E}) \subseteq \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{E}_{R'})$, and the functor $\mathscr{P}\mathrm{ic}_{\mathsf{X}}^{(k+1)}$ is given by $R' \mapsto \mathrm{Map}_{\mathrm{Mon}(\mathcal{E}_{R'})}(E_{R'}, E'_{R'})$. For every map of simplicial sets $K \to \mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}$, let $F_K : \tau_{\leqslant n}\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathcal{S}$ be the functor given by the formula

$$R' \mapsto \mathrm{Map}_{\mathrm{Fun}(K, \mathcal{E}_{R'})}(E_{R'}|_K, E'_{R'}|_K).$$

We wish to prove that the functor $F_{\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}}$ commutes with filtered colimits.

For any n-truncated object $R' \in \mathrm{CAlg}_R^{\mathrm{cn}}$, the ∞-category $\mathrm{Fun}(\mathcal{C}_{R'}, \tau_{\leqslant n+1}\mathcal{S})$ is equivalent to an $(n + 2)$-category. It follows that $\mathcal{E}_{R'}$ is also equivalent to an $(n + 2)$-category. Applying Theorem SAG.A.8.2.3, we deduce that the inclusion $\boldsymbol{\Delta}_{\leqslant n+3}^{\mathrm{op}} \hookrightarrow \boldsymbol{\Delta}^{\mathrm{op}}$ induces an equivalence $F_{\mathrm{N}(\boldsymbol{\Delta})^{\mathrm{op}}} \to F_{\mathrm{N}(\boldsymbol{\Delta}_{\leqslant n+3}^{\mathrm{op}})}$. It will therefore suffice to prove that the functor F_K commutes with filtered colimits for every finite simplicial set K equipped with a map $K \to \mathrm{N}(\boldsymbol{\Delta}^{\mathrm{op}})$.

We now proceed by induction on the dimension p of K and on the number of nondegenerate p-simplices of K. If $K = \varnothing$ there is nothing to prove. Otherwise, we can choose a pushout diagram of simplicial sets

$$
\begin{array}{ccc}
\partial \Delta^p & \longrightarrow & \Delta^p \\
\downarrow & & \downarrow \\
K' & \longrightarrow & K,
\end{array}
$$

hence a pullback diagram of functors

$$
\begin{array}{ccc}
F_K & \longrightarrow & F_{K'} \\
\downarrow & & \downarrow \\
F_{\Delta^p} & \longrightarrow & F_{\partial \Delta^p}.
\end{array}
$$

The inductive hypothesis implies that the functors $F_{\partial \Delta^p}$ and $F_{K'}$ commute with filtered colimits. We are therefore reduced to proving that the functor F_{Δ^p} commutes with filtered colimits. If $p \geqslant 2$, then there exists an inner anodyne inclusion $\Lambda_1^p \hookrightarrow \Delta^p$ which

induces an equivalence $F_{\Delta^p} \to F_{\Lambda_1^p}$, and the desired result follows from the inductive hypothesis. We may therefore reduce to the case where $K = \Delta^p$ and $p \leqslant 1$.

Let us now consider the case where $p = 0$. In this case, the map $K \to \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ determines an object $[a] \in \mathbf{\Delta}$, and the functor F_K is given by the formula

$$F_K(R') = \mathrm{Map}_{\mathcal{E}_{R'}}(E_{R'}([a]), E'_{R'}([a])) \simeq \mathrm{Map}_{\mathcal{E}_{R'}}(E_{R'}([a]), E'_{R'}([1]))^a \simeq \mathscr{P}\mathrm{ic}_{\mathsf{X}^a}^{(k)}(R')^a,$$

and therefore commutes with filtered colimits by the inductive hypothesis (applied to the abelian variety X^a over R).

We conclude by treating the case where $p = 1$, so that the map $K \to \mathrm{N}(\mathbf{\Delta})^{\mathrm{op}}$ classifies a map of finite linearly ordered sets $[a] \to [b]$. Unwinding the definitions, we obtain a pullback square (depending functorially on R'):

$$
\begin{array}{ccc}
F_K(R') & \longrightarrow & \mathrm{Map}_{\mathcal{E}_{R'}}(E_{R'}([a]), E'_{R'}([a])) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathcal{E}_{R'}}(E_{R'}([b]), E'_{R'}([b])) & \longrightarrow & \mathrm{Map}_{\mathcal{E}_{R'}}(E_{R'}([b]), E'_{R'}([a])),
\end{array}
$$

which gives a pullback square of functors

$$
\begin{array}{ccc}
F_K & \longrightarrow & (\mathscr{P}\mathrm{ic}_{\mathsf{X}^a}^{(k)})^a \\
\downarrow & & \downarrow \\
(\mathscr{P}\mathrm{ic}_{\mathsf{X}^b}^{(k)})^b & \longrightarrow & (\mathscr{P}\mathrm{ic}_{\mathsf{X}^b}^{(k)})^a.
\end{array}
$$

Since the functors $\mathscr{P}\mathrm{ic}_{\mathsf{X}^a}^{(k)}$ and $\mathscr{P}\mathrm{ic}_{\mathsf{X}^b}^{(k)}$ commute with filtered colimits by the inductive hypothesis, we deduce that the functor F_K commutes with filtered colimits. $\qquad\square$

5.6 Existence of the Dual Abelian Variety

Our goal in this section is to show that every abelian variety X over a connective $\mathbb{E}_\infty$-ring R admits a dual (Theorem 5.6.4). The strategy of proof is to first treat the case where R is an algebraically closed field (in which case the result is classical, but we include a proof here for completeness). We begin with some preliminary observations.

Lemma 5.6.1. *Let κ be a field, let X be an abelian variety over κ, and let $\mathscr{L}$ be a multiplicative line bundle on X. If $\mathscr{L}$ is nontrivial, then the cohomology $\mathrm{H}^*(\mathsf{X}; \mathscr{L})$ vanishes.*

Proof. Let $\pi, \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to \mathsf{X}$ denote the two projection maps and let $m :$ $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to \mathsf{X}$ denote the addition map. Then $(\pi, m) : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X}$ is an automorphism. Moreover, the multiplicativity of $\mathscr{L}$ supplies an an isomorphism $(\pi, m)^*(\pi^* \mathscr{O}_{\mathsf{X}} \otimes \pi'^* \mathscr{L}) \simeq \pi^* \mathscr{L} \otimes \pi'^* \mathscr{L}$. Passing to cohomology, we obtain an isomorphism of graded vector spaces

$$\alpha : \mathrm{H}^*(\mathsf{X}; \mathscr{O}_{\mathsf{X}}) \otimes_\kappa \mathrm{H}^*(\mathsf{X}; \mathscr{L}) \simeq \mathrm{H}^*(\mathsf{X}; \mathscr{L}) \otimes_\kappa \mathrm{H}^*(\mathsf{X}; \mathscr{L}).$$

Assume that $\mathrm{H}^*(\mathsf{X}; \mathscr{L})$ is nonzero. Then there exists some smallest integer $n \geqslant 0$ such that $\mathrm{H}^n(\mathsf{X}; \mathscr{L})$ is nonzero. Note that the domain of α is nonzero in (cohomological) degree n, and the codomain of α vanishes in (cohomological) degrees $< 2n$. Since α is an isomorphism, we conclude that $n = 0$: that is, there exists a nonzero global section u of $\mathscr{L}$. Note that $\mathscr{L}^{-1}$ is isomorphic to the pullback of $\mathscr{L}$ along the map $(-1) : \mathsf{X} \to \mathsf{X}$, and therefore also has nonzero cohomology. Applying the same argument to $\mathscr{L}^{-1}$, we deduce that there exists a nonzero global section v of $\mathscr{L}^{-1}$. Since X is irreducible, the product uv is a nonzero global section of $\mathscr{L} \otimes \mathscr{L}^{-1} \simeq \mathscr{O}_{\mathsf{X}}$. It follows that uv is nowhere vanishing, so that u determines an isomorphism $\mathscr{O}_{\mathsf{X}} \simeq \mathscr{L}$. $\qquad\square$

Lemma 5.6.2. *Let κ be an algebraically closed field and let X be an abelian variety of dimension g over κ. Then there exists another abelian variety Y of dimension g over κ and a biextension $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ with the following property:*

$(*)$ *Let $\mathscr{L}_\mu$ be the underlying line bundle of μ. Then there are only finitely many κ-valued points $x \in \mathsf{X}(\kappa)$ such that $\mathscr{L}_\mu|_{\{x\} \times \mathsf{Y}}$ is trivial, and only finitely many κ-valued points $y \in \mathsf{Y}(\kappa)$ such that $\mathscr{L}_\mu|_{\mathsf{X} \times \{y\}}$ is trivial.*

Proof. Choose a nonempty affine open subset $U \subseteq |\mathsf{X}|$ (such an open set exists by virtue of Proposition 1.4.7 or Corollary SAG.3.4.2.4). Shrinking U if necessary, we may assume that the complement of U is a Cartier divisor $D \subseteq |\mathsf{X}|$. Let $\mathscr{O}_{\mathsf{X}}(-D)$ denote the ideal sheaf of D and let $\mathscr{O}_{\mathsf{X}}(D) = \mathscr{O}_{\mathsf{X}}(-D)^{-1}$ denote its inverse. Without loss of generality, we may assume that U contains the identity section of X, so that $\mathscr{O}_{\mathsf{X}}(D)$ is canonically trivialized at the identity of X. Let $\pi, \pi' : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to$ X denote the projection maps and let $m : \mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X} \to \mathsf{X}$ denote the addition map, and let $\mathscr{L}_\mu$ denote the line bundle on $\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{X}$ given by the tensor product $m^* \mathscr{O}_{\mathsf{X}}(D) \otimes \pi^* \mathscr{O}_{\mathsf{X}}(-D) \otimes \pi'^* \mathscr{O}_{\mathsf{X}}(-D)$. Then $\mathscr{L}_\mu$ is the underlying line bundle of a biextension $\mu : \mathsf{X} \wedge \mathsf{X} \to \mathrm{BGL}_1$ (see Example 4.2.5). We claim that μ satisfies condition $(*)$. To prove this, let us identify μ with a map $f : \mathsf{X} \to \mathscr{P}\mathrm{ic}^m_{\mathsf{X}}$, and let $K \subseteq |\mathsf{X}|$ be the inverse image of the identity element of $\mathscr{P}\mathrm{ic}^m_{\mathsf{X}}(\kappa)$. We wish to show that K is

finite. To prove this, we let X' denote the reduced closed subspace of X corresponding to the connected component of the identity in K. Then X' is also an abelian variety over κ, and the biextension μ is trivial on $\mathsf{X}' \wedge \mathsf{X}'$ (even on $\mathsf{X}' \wedge \mathsf{X}$). We wish to show that $\mathsf{X}' \simeq \mathrm{Sp\acute{e}t}\,\kappa$. Replacing X by X', we are reduced to the case where the map f is nullhomotopic, so that the line bundle $\mathscr{L}_\mu$ is trivial. Applying Proposition 5.4.9, we deduce that $\mathscr{O}_\mathsf{X}(D)$ admits the structure of a multiplicative line bundle on X. Since $\mathrm{H}^0(\mathsf{X}; \mathscr{O}_\mathsf{X}(D))$ is nonzero, Lemma 5.6.1 implies that the line bundle $\mathscr{O}_\mathsf{X}(D)$ is trivial. It follows that the structure sheaf $\mathscr{O}_\mathsf{X}$ admits a section s which vanishes exactly on the Cartier divisor D. Since X is connected and proper, any nonvanishing section of $\mathscr{O}_\mathsf{X}$ is nowhere vanishing. It follows that $D = \varnothing$, so that $U = |\mathsf{X}|$. Then the abelian variety X is proper and affine over κ, hence zero-dimensional as desired. $\qquad\square$

Lemma 5.6.3. *Let κ be an algebraically closed field and let X be an abelian variety over κ. Then X admits a dual.*

Proof. Let $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$ be a biextension of abelian varieties over κ which satisfies condition $(*)$ of Lemma 5.6.2. Then μ determines an additive map $f : \mathsf{Y} \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ of spectral algebraic spaces over κ. Let Y' denote the schematic image of f (see Construction SAG.3.1.5.1). Then the canonical map $\mathsf{Y}' \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ is a closed immersion. Since $\mathscr{P}\mathrm{ic}_\mathsf{X}^m$ is of finite type over κ (Proposition 5.5.1), the algebraic space Y' is of finite type over κ. Because Y is connected, reduced, and proper over κ, we conclude that Y' is also connected, reduced, and proper over κ. Using the additivity of f, we see that the composite map

$$\mathsf{Y}' \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathsf{Y}' \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m \times_{\mathrm{Sp\acute{e}t}\,\kappa} \mathscr{P}\mathrm{ic}_\mathsf{X}^m \xrightarrow{\otimes} \mathscr{P}\mathrm{ic}_\mathsf{X}^m$$

factors (in an essentially unique way) through Y'. It follows that Y' inherits the structure of an abelian variety over κ and that the closed immersion $\mathsf{Y}' \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ can be regarded as a morphism of commutative monoids. This morphism determines a biextension $\mu' : \mathsf{X} \wedge \mathsf{Y}' \to \mathrm{BGL}_1$ which also satisfies condition $(*)$ of Lemma 5.6.2 (note that condition $(*)$ implies that the kernel of the map $\mathsf{Y} \to \mathsf{Y}'$ is finite, so that Y and Y' have the same dimension). Replacing μ by μ', we may reduce to the case where $f : \mathsf{Y} \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ is a closed immersion.

We will complete the proof by showing that the biextension μ is perfect. Since f is a closed immersion, Proposition 5.4.9 implies the following:

(i) The map $\mathsf{Y}(R) \to \mathscr{P}\mathrm{ic}_\mathsf{X}^m(R) \to \mathscr{P}\mathrm{ic}_\mathsf{X}^e(R)$ is a monomorphism whenever R is a discrete κ-algebra.

In particular, if $y \in \mathsf{Y}(\kappa)$ is a κ-valued point for which $\mathscr{L}_\mu |_{\mathsf{X} \times \{y\}}$ is trivial, then y must be the identity element $y_0 \in \mathsf{Y}(\kappa)$. Let $\pi : \mathsf{X} \times_{\mathrm{Spét}\,\kappa} \mathsf{Y} \to \mathsf{X}$ and $\pi' : \mathsf{X} \times_{\mathrm{Spét}\,\kappa} \mathsf{Y} \to \mathsf{Y}$ denote the projection maps. Using Lemma 5.6.1, we deduce that the direct image $\mathscr{F} = \pi'_* \mathscr{L}_\mu$ is supported at the point y_0.

Let $\mathscr{E}_0 = y_{0*} \mathscr{O}_{\mathrm{Spét}\,\kappa} \in \mathrm{QCoh}(\mathsf{Y})$ denote the skyscraper sheaf at the point y_0. Since the map π' is proper and flat, the sheaf $\mathscr{F}$ is perfect (Theorem SAG.6.1.3.2). Moreover, the map π' has relative dimension g (where g denotes the dimension of the abelian variety X), so Corollary SAG.9.6.1.4 guarantees that $\mathscr{F}$ is $(-g)$-connective. It follows that $\mathscr{F}$ can be written as a finite extension of sheaves of the form $\Sigma^a \mathscr{E}_0$ for $a \geqslant -g$. By assumption, Y is also an abelian variety of dimension g over κ, so the local ring of Y at the point y_0 is regular of dimension g. It follows that there exists an equivalence $\mathscr{E}_0^\vee \simeq \Sigma^{-g} \mathscr{E}_0$, so that the dual $\mathscr{F}^\vee$ can be written as a finite extension of sheaves of the form $\Sigma^{a-g} \mathscr{E}_0$ for $a \geqslant -g$, and therefore belongs to $\mathrm{QCoh}(\mathsf{Y})_{\leqslant 0}$. On the other hand, the direct image $\mathscr{F} = \pi'_* \mathscr{L}$ has Tor-amplitude $\leqslant 0$ (Proposition SAG.6.1.3.1), so that $\mathscr{F}^\vee$ is connective. It follows that $\mathscr{F}^\vee$ belongs to $\mathrm{QCoh}(\mathsf{Y})^\heartsuit$.

Note that we have canonical equivalences

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(\mathsf{Y})}(\mathscr{F}^\vee, \mathscr{E}_0) &\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Spét}\,\kappa)}(y_0^* \mathscr{F}^\vee, \mathscr{O}_{\mathrm{Spét}\,\kappa}) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Spét}\,\kappa)}(\mathscr{O}_{\mathrm{Spét}\,\kappa}, y_0^* \mathscr{F}) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathrm{Spét}\,\kappa)}(\mathscr{O}_{\mathrm{Spét}\,\kappa}, y_0^* \pi'_* \mathscr{L}_\mu) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{O}_{\mathsf{X}}, \mathscr{L}_\mu |_{\mathsf{X} \times \{y_0\}} \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(\mathsf{X})}(\mathscr{O}_{\mathsf{X}}, \mathscr{O}_{\mathsf{X}}) \\
&\simeq \kappa.
\end{aligned}
$$

In particular, there exists a nonzero map $u : \mathscr{F}^\vee \to \mathscr{E}_0$, which is unique up to scalar multiplication. Since $\mathscr{F}^\vee$ is set-theoretically supported at the point y, we can choose a map $v : \mathscr{O}_{\mathsf{Y}} \to \mathscr{F}^\vee$ such that the composition $\mathscr{O}_{\mathsf{Y}} \xrightarrow{v} \mathscr{F}^\vee \xrightarrow{u} \mathscr{E}_0$ is an epimorphism (in the abelian category $\mathrm{QCoh}(\mathsf{Y})^\heartsuit$). Using Nakayama's lemma, we deduce that v is an epimorphism: that is, it induces an isomorphism $\overline{v} : \mathscr{O}_{\mathsf{Y}} / \mathcal{I} \simeq \mathscr{F}^\vee$, where $\mathcal{I} \subseteq \mathscr{O}_{\mathsf{Y}}$ is some quasi-coherent ideal sheaf. Let $\iota : \mathsf{Y}_0 \to \mathsf{Y}$ be the closed immersion determined by the ideal sheaf $\mathcal{I}$. Then we can identify the inverse isomorphism $\overline{v}^{-1}$ with a nonzero global section of the pullback $\iota^* \mathscr{F}$, or equivalently with a nonzero global section of $\mathscr{L}_\mu |_{\mathsf{X} \times_{\mathrm{Spét}\,\kappa} \mathsf{Y}_0}$. This global section is nowhere vanishing (since this can be checked after replacing Y_0 by the closed point y_0), and therefore determines a trivialization of the restriction $\mathscr{L}_\mu |_{\mathsf{X} \times_{\mathrm{Spét}\,\kappa} \mathsf{Y}_0}$. Applying (i), we deduce that the map ι factors through y_0, so that $\mathscr{O}_{\mathsf{Y}} / \mathcal{I}$ is equivalent to $\mathscr{E}_0$. Since $\overline{v}$ is an isomorphism, we conclude that

$\mathscr{F}^\vee$ is equivalent to $\mathscr{E}_0$, so that $\mathscr{F} \simeq \mathscr{E}_0^\vee \simeq \Sigma^{-g}\mathscr{E}_0$. It follows that $\mathscr{F}$ is invertible with respect to the convolution product on $\mathrm{QCoh}(\mathsf{Y})$.

Set $\mathscr{G} = \pi_* \mathscr{L}_\mu \in \mathrm{QCoh}(\mathsf{X})$. Note that if $x \in \mathsf{X}(\kappa)$ is a κ-valued point for which the restriction $\mathscr{L}_\mu|_{\{x\}\times\mathsf{Y}}$ is nontrivial, then $x^*\mathscr{G} \simeq 0$ (Lemma 5.6.1). This condition is satisfied for all but finitely many points $x \in \mathsf{X}(\kappa)$ (by virtue of our assumption that μ satisfies condition $(*)$ of Lemma 5.6.2), so that the quasi-coherent sheaf $\mathscr{G}$ is supported on a finite subset of X. Moreover, we have a canonical equivalence

$$
\begin{aligned}
\Gamma(\mathsf{X};\mathscr{G}) &\simeq \Gamma(\mathsf{X}\times_{\mathrm{Sp\acute{e}t}\,\kappa}\mathsf{Y},\mathscr{L}_\mu) \\
&\simeq \Gamma(\mathsf{Y};\mathscr{F}) \\
&\simeq \Gamma(\mathsf{Y};\Sigma^{-g}\mathscr{E}_0) \\
&\simeq \Sigma^{-g}\kappa.
\end{aligned}
$$

It follows that $\mathscr{G}$ is equivalent to the suspension of a skyscraper sheaf at some point $x \in \mathsf{X}(\kappa)$, and is therefore invertible with respect to the convolution product on $\mathrm{QCoh}(\mathsf{X})$. Applying the criterion of Proposition 5.1.4, we deduce that the biextension μ is perfect, as desired. $\qquad\square$

We now come to our main result.

Theorem 5.6.4. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be an abelian variety over R. Then X admits a dual.*

Proof. Let $\mathsf{Y} = \mathscr{P}\mathrm{ic}_\mathsf{X}^m$ be as in Definition 5.3.1, so that Y is representable by a spectral algebraic space over R (Proposition 5.5.1). Let $m : \mathsf{Y}\times_{\mathrm{Sp\acute{e}t}\,R}\mathsf{Y} \to \mathsf{Y}$ denote the multiplication map and let $e : \mathrm{Sp\acute{e}t}\,R \to \mathsf{Y}$ be the zero section. Since $\mathrm{Sp\acute{e}t}\,R$ is quasi-compact, we can choose a quasi-compact open subspace $\mathsf{U} \subseteq \mathsf{Y}$ which contains the image of e. We first prove the following:

$(*)$ The composite map $m_\mathsf{U} : \mathsf{U}\times_{\mathrm{Sp\acute{e}t}\,R}\mathsf{U} \hookrightarrow \mathsf{Y}\times_{\mathrm{Sp\acute{e}t}\,R}\mathsf{Y} \xrightarrow{m} \mathsf{Y}$ is surjective.

To prove $(*)$, we can assume without loss of generality that $R = \kappa$ is an algebraically closed field. In this case, Lemma 5.6.3 guarantees that X admits a dual, so that Corollary 5.3.5 implies that Y is dual to X. In particular, Y is an abelian variety over R and therefore connected. Fix a κ-valued point $y \in |\mathsf{Y}|$, and let $\mathsf{U}' \subseteq \mathsf{Y}$ denote the image of U under the map $y' \mapsto y - y'$. Then U and U' are nonempty open subspaces of Y. Since Y is irreducible, the intersection $\mathsf{U}\cap\mathsf{U}'$ is nonempty and therefore contains a κ-valued point $y' \in \mathsf{Y}(\kappa)$. In this case, the map m_U carries the point $(y', y - y')$ to y. Allowing y to vary, we deduce that m_U is surjective, as desired.

We now return to the general case. Using $(*)$, we deduce that Y is quasi-compact. It then follows from Proposition 5.5.1 that Y is proper and locally almost of finite presentation over R. We claim that Y is also an abelian variety over R. To prove this, we must show that the projection map $\mathsf{Y} \to \mathrm{Sp\acute{e}t}\, R$ is flat, geometrically reduced, and geometrically connected. By virtue of Corollary SAG.6.1.4.8, it suffices to check these conditions after replacing R by an algebraically closed field. In this case, the desired result follows again by combining Lemma 5.6.3 with Corollary 5.3.5.

By construction, there is a tautological biextension of abelian varieties $\mu : \mathsf{X} \wedge \mathsf{Y} \to \mathrm{BGL}_1$. To complete the proof, it will suffice to show that μ is perfect. Using Corollary 5.2.9, we can again reduce to the case where κ is an algebraically closed field, in which case the desired result follows from Proposition 5.3.4. $\qquad\square$

6 p-Divisible Groups

Let X be an abelian variety of dimension g over a commutative ring R. For every integer $n > 0$, let $[n] : \mathsf{X} \to \mathsf{X}$ denote the map given by multiplication by n (with respect to the group structure on X, and let $\mathsf{X}[n]$ denote the kernel $\ker([n])$ (formed in the category of group schemes over R). A foundational result in the theory of abelian varieties guarantees that $\mathsf{X}[n]$ is a finite flat group scheme of degree n^{2g} over R (see Proposition 6.7.3). In [11], Tate observed that for each prime number p, the collection of group schemes $\{\mathsf{X}[p^k]\}_{k \geq 0}$ could be organized into a mathematical object called a *p-divisible group* (also known as a Barsotti-Tate module), and can be regarded as a useful replacement for the Tate module of X in the case where p is not invertible in R.

Our goal in this section is to generalize the theory of p-divisible groups, replacing the commutative ring R by an arbitrary $\mathbb{E}_\infty$-ring A. With an eye to later applications, we consider more generally the notion of S-divisible group, where S is a set of prime numbers (this is essentially just for terminological convenience: specifying an S-divisible group is equivalent to specifying a p-divisible group for each $p \in$ S; see Remark 6.5.6). We begin in §6.1 by introducing the notion of a *(commutative) finite flat group scheme* over A (Definition 6.1.2). Moreover, we show that the theory of finite flat group schemes comes equipped with good notions of monomorphism, epimorphism, exact sequence, and Cartier duality (see §6.2 and §6.3).

In §6.4, we introduce the notion of a S-*torsion object* of an arbitrary ∞-category $\mathcal{C}$ which admits finite limits. Moreover, we show that there is a close relationship between S-torsion objects of $\mathcal{C}$ and abelian group objects of $\mathcal{C}$, given by a functor $X \mapsto X[\mathrm{S}^\infty]$ which extracts the "S-torsion part" of an abelian group object X. In §6.5

we apply these ideas to define the notion of S-divisible group over an arbitrary $\mathbb{E}_\infty$-ring A (Definition 6.5.1), and in §6.7 we show that the construction $X \mapsto X[S^\infty]$ takes strict abelian varieties to S-divisible groups (Proposition 6.7.1). Finally, we show that there is a good theory of Cartier duality for S-divisible groups (Proposition 6.6.3), and that the construction $X \mapsto X[S^\infty]$ intertwines the duality theory of abelian varieties (studied in §5) with the Cartier duality of S-divisible groups (Proposition 6.8.2).

6.1 Finite Flat Group Schemes

We begin by introducing some terminology.

Definition 6.1.1. Let A be an $\mathbb{E}_\infty$-ring and let M be an A-module. We will say that M is *finite flat* if the following conditions are satisfied:

(*i*) The abelian group $\pi_0 M$ is a projective module of finite rank over the commutative ring $\pi_0 A$.

(*ii*) For each integer n, the canonical map $(\pi_0 M) \otimes_{\pi_0 A} (\pi_n A) \to \pi_n M$ is an isomorphism.

We will say that f is *finite flat of rank d* if, in addition, the module $\pi_0 M$ has rank d over $\pi_0 A$.

For every $\mathbb{E}_\infty$-ring A, we let $\mathrm{Mod}_A^{\mathrm{ff}}$ denote the full subcategory of Mod_A spanned by those A-modules which are finite flat.

Definition 6.1.2. Let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of nonconnective spectral Deligne-Mumford stacks. We will say that f is *finite flat (of degree d)* if, for every map $\mathrm{Sp\acute{e}t}\, A \to \mathsf{Y}$, the fiber product $\mathsf{X} \times_\mathsf{Y} \mathrm{Sp\acute{e}t}\, A$ has the form $\mathrm{Sp\acute{e}t}\, B$, where B is finite flat (of rank d) when regarded as an A-module.

For each $\mathbb{E}_\infty$-ring A, we let $\mathrm{FF}(A)$ denote the full subcategory of $\mathrm{SpDM}_A^{\mathrm{nc}}$ spanned by the finite flat morphisms $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, A$.

Remark 6.1.3. In the setting of spectral Deligne-Mumford stacks (with connective structure sheaves), Definition 6.1.2 reduces to Definition SAG.5.2.3.1 . More precisely, if Y is a spectral Deligne-Mumford stack and $f : \mathsf{X} \to \mathsf{Y}$ is a morphism of nonconnective spectral Deligne-Mumford stacks, then f is finite flat (in the sense of Definition 6.1.2) if and only if the structure sheaf $\mathscr{O}_\mathsf{X}$ is connective and f is finite flat as a morphism of spectral Deligne-Mumford stacks (in the sense of Definition SAG.5.2.3.1).

Moreover, the notion of finite flat morphism in the nonconnective setting immediately reduces to the connective one: a morphism $f : (\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \to (\mathcal{Y}, \mathscr{O}_{\mathsf{Y}})$ is finite flat if and only if f is flat and the underlying map of spectral Deligne-Mumford stacks $(\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathsf{X}}) \to (\mathcal{Y}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathsf{Y}})$ is finite flat.

Remark 6.1.4. Let $f : \mathsf{X} \to \mathsf{Z}$ and $g : \mathsf{Y} \to \mathsf{Z}$ be morphisms of nonconnective spectral Deligne-Mumford stacks. Suppose that f is a finite flat surjection and that $\mathsf{X} \times_{\mathsf{Z}} \mathsf{Y} \to \mathsf{Z}$ is finite flat. Then g is finite flat.

Remark 6.1.5. Let A be an $\mathbb{E}_\infty$-ring. Then the construction $B \mapsto \mathrm{Sp\acute{e}t}\, B$ induces an equivalence of ∞-categories $\mathrm{CAlg}(\mathrm{Mod}_A^{\mathrm{ff}})^{\mathrm{op}} \simeq \mathrm{FF}(A)$.

Remark 6.1.6. The functor $(A \in \mathrm{CAlg}^{\mathrm{cn}}) \mapsto \mathrm{FF}(A)$ is locally almost of finite presentation: that is, it commutes with filtered colimits when restricted to $\tau_{\leqslant n}\,\mathrm{CAlg}^{\mathrm{cn}}$, for each $n \geqslant 0$. This is a special case of Corollary SAG.19.4.5.7 .

Remark 6.1.7. Let A be an $\mathbb{E}_\infty$-ring. Then the ∞-category $\mathrm{FF}(A)$ depends only on the connective cover $\tau_{\geqslant 0}A$. More precisely, extension of scalars along the canonical map $\tau_{\geqslant 0}A \to A$ induces an equivalence of ∞-categories $\mathrm{FF}(\tau_{\geqslant 0}A) \to \mathrm{FF}(A)$; a homotopy inverse to this equivalence is given by the construction $(\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \mapsto (\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathsf{X}})$.

Definition 6.1.8. Let A be an $\mathbb{E}_\infty$-ring. A *commutative finite flat group scheme* over A is a grouplike commutative monoid object of the ∞-category $\mathrm{FF}(A)$. We let $\mathrm{FFG}(A) = \mathrm{CMon}^{\mathrm{gp}}(\mathrm{FF}(A))$ denote the ∞-category of commutative finite flat group schemes over A.

Remark 6.1.9. Let A be an $\mathbb{E}_\infty$-ring. Using Remark 6.1.5, we see that the construction $B \mapsto \mathrm{Sp\acute{e}t}\, B$ induces an equivalence of ∞-categories $\mathrm{Hopf}(\mathrm{Mod}_A^{\mathrm{ff}})^{\mathrm{op}} \simeq \mathrm{FFG}(A)$. In other words, we can identify commutative finite flat group schemes over A with Hopf algebras that are finite flat when regarded as A-modules.

Remark 6.1.10 (The Functor of Points)**.** Let A be an $\mathbb{E}_\infty$-ring. Then the functor of points $\mathrm{SpDM}_A^{\mathrm{nc}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$ restricts to a fully faithful embedding $\mathrm{FF}(A) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$. Passing to grouplike commutative monoid objects, we obtain a fully faithful embedding $\mathrm{FFG}(A) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}))$. We will refer to this embedding as the *functor of points*. In practice, we will generally abuse terminology by not distinguishing between a commutative finite flat group scheme $G \in \mathrm{FFG}(A)$ and its functor of points $\mathrm{CAlg}_A \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$.

Remark 6.1.11. The functor $(A \in \mathrm{CAlg}^{\mathrm{cn}}) \mapsto \mathrm{FFG}(A)$ is locally almost of finite presentation: that is, it commutes with filtered colimits when restricted to $\tau_{\leqslant n} \mathrm{CAlg}^{\mathrm{cn}}$, for each $n \geqslant 0$. This follows by combining Remark 6.1.6 with Lemma 2.2.3 (and Remark 2.2.4).

For future use, we note the following variant of Lemma 2.2.5 for finite flat group schemes:

Proposition 6.1.12. *Let K be a simplicial set and let $F_K : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ denote the functor given by the formula $F_K(R) = \mathrm{Fun}(K, \mathrm{FFG}(R))^{\simeq}$. Then the functor F_K admits a (-1)-connective cotangent complex.*

Proof. Define $G : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $G(R) = \mathrm{Fun}(\mathcal{F}\mathrm{in}_* \times K, \mathrm{FF}(R))^{\simeq}$. For every connective $\mathbb{E}_\infty$-ring R, we can identify $F_K(R)$ with the summand of $G(R)$ spanned by those functors $X : \mathcal{F}\mathrm{in}_* \times K \to \mathrm{FF}(R)$. which satisfy the following condition:

($*$) For every vertex $v \in K$, the restriction $X|_{\mathcal{F}\mathrm{in}_* \times \{v\}} : \mathcal{F}\mathrm{in}_* \to \mathrm{FF}(R)$ is a grouplike commutative monoid object of $\mathrm{FF}(R)$.

Theorem SAG.19.4.0.2 implies that the functor G admits a (-1)-connective cotangent complex $L_G \in \mathrm{QCoh}(G)$. We claim that the image L_G in the ∞-category $\mathrm{QCoh}(F_K)$ is a cotangent complex for F_K. To prove this, it will suffice to show that for every connective $\mathbb{E}_\infty$-ring R and every connective R-module M, the diagram

$$
\begin{array}{ccc}
F_K(R \oplus M) & \longrightarrow & F_K(R) \\
\downarrow & & \downarrow \\
G(R \oplus M) & \longrightarrow & G(R)
\end{array}
$$

is a pullback square. Without loss of generality, we may assume that $K = \Delta^0$; in this case, we must show that a diagram $\mathcal{F}\mathrm{in}_* \to \mathrm{FF}(R \oplus M)$ is a grouplike commutative monoid object of $\mathrm{FF}(R \oplus M)$ if and only if the composite map $\mathcal{F}\mathrm{in}_* \to \mathrm{FF}(R \oplus M) \to \mathrm{FF}(R)$ is a grouplike commutative monoid object of $\mathrm{FF}(R)$. This is clear, since the extension-of-scalars functor $\mathrm{FF}(R \oplus M) \to \mathrm{FF}(R)$ is conservative. $\qquad\square$

Remark 6.1.13. In the situation of Proposition 6.1.12, if K is a finite simplicial set, then the cotangent complex L_{F_K} is almost perfect. This follows by combining Remark 6.1.11 with Corollary SAG.17.4.2.2 .

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6.2 Epimorphisms and Monomorphisms

We now introduce the notion of *epimorphism* between commutative finite flat group schemes.

Proposition 6.2.1. *Let A be an $\mathbb{E}_\infty$-ring and let $f : G \to H$ be a morphism of commutative finite flat group schemes over A. The following conditions are equivalent:*

(1) *The morphism f is finite flat and surjective (when regarded as a map of nonconnective spectral Deligne-Mumford stacks).*

(2) *The morphism f is an effective epimorphism of sheaves with respect to the finite flat topology: that is, for every B-valued point $\eta \in H(B)$, there exists a finite flat surjection $\mathrm{Sp\acute{e}t}\, \widetilde{B} \to \mathrm{Sp\acute{e}t}\, B$ such that the fiber product the fiber product $G(\widetilde{B}) \times_{H(\widetilde{B})} \{\eta\}$ is nonempty.*

Proof. We first show that $(1) \Rightarrow (2)$. Choose any point $\eta \in H(B)$, and let X denote the fiber product $G \times_H \mathrm{Sp\acute{e}t}\, B$ (formed in the ∞-category of spectral Deligne-Mumford stacks). Using assumption (1), we see that the projection $\mathsf{X} \to \mathrm{Sp\acute{e}t}\, B$ is a finite flat surjection. Writing $\mathsf{X} \simeq \mathrm{Sp\acute{e}t}\, \widetilde{B}$, we observe that the fiber product $G(\widetilde{B}) \times_{H(\widetilde{B})} \{\eta\} \simeq \mathsf{X}(\widetilde{B})$ has a canonical point.

Now suppose that (2) is satisfied; we wish to show that f is finite flat and surjective. Write $H = \mathrm{Sp\acute{e}t}\, B$, where B is a (finite flat) Hopf algebra over A. Let $\widetilde{B}$ be as in (2). Set $K = \mathrm{fib}(f)$, where the fiber is formed in the ∞-category $\mathrm{SpDM}_A^{\mathrm{nc}}$. Using the group structure on G, we obtain an equivalence $\mathrm{Sp\acute{e}t}\, \widetilde{B} \times_H G \simeq \mathrm{Sp\acute{e}t}\, \widetilde{B} \times_{\mathrm{Sp\acute{e}t}\, A} K$. In particular, we conclude that $\mathrm{Sp\acute{e}t}\, \widetilde{B} \times_{\mathrm{Sp\acute{e}t}\, A} K$ is finite flat over A. Since the projection map $\mathrm{Sp\acute{e}t}\, \widetilde{B} \to \mathrm{Sp\acute{e}t}\, A$ is a finite flat surjection, Remark 6.1.4 implies that K is finite flat over A. It follows that the projection map $\mathrm{Sp\acute{e}t}\, \widetilde{B} \times_H G \to \mathrm{Sp\acute{e}t}\, \widetilde{B}$ is finite flat, so that G is finite flat over H. Moreover, f is surjective (since the composite map $\mathrm{Sp\acute{e}t}\, \widetilde{B} \times_H G \to G \xrightarrow{f} H$ is surjective). $\qquad\square$

Definition 6.2.2. Let A be an $\mathbb{E}_\infty$-ring and let $f : G \to H$ be a morphism of commutative finite flat group schemes over A. We will say that f is a *epimorphism* if it satisfies the equivalent conditions of Proposition 6.2.1. We will say that f is an *monomorphism* if the induced map $f^* : \pi_0\Gamma(H; \mathscr{O}_H) \to \pi_0\Gamma(G; \mathscr{O}_G)$ is surjective.

Warning 6.2.3. The terminology of Definition 6.2.2 is potentially misleading: if $g : G \to H$ is an epimorphism (monomorphism) of commutative finite flat group schemes over an $\mathbb{E}_\infty$-ring A, then f need not be a *categorical* epimorphism (monomorphism):

that is, a morphism $h : H \to H'$ $(f : G' \to G)$ need not be determined (even up to homotopy) by the composition $h \circ g$ $(g \circ f)$.

Remark 6.2.4. Let A be a connective $\mathbb{E}_\infty$-ring and let $f : G \to H$ be a morphism of commutative finite flat group schemes over A. Then f is a monomorphism if and only if it is a closed immersion, in the sense of Definition SAG.3.1.0.1 .

Remark 6.2.5. Let $f : G \to H$ be a morphism of commutative finite flat group schemes over an $\mathbb{E}_\infty$-ring A. Using Remark 6.1.7, we see that f can be obtained (in an essentially unique way) from a morphism $f_0 : G_0 \to H_0$ of commutative finite flat group schemes over the connective cover $\tau_{\geqslant 0} A$. Then f is an epimorphism (monomorphism) if and only if the map f_0 is an epimorphism (monomorphism).

Remark 6.2.6. Let $f : G \to H$ be a morphism of commutative finite flat group schemes over a connective $\mathbb{E}_\infty$-ring A. For every residue field κ of A, let $f_\kappa : G_\kappa \to H_\kappa$ denote the induced map of commutative finite flat group schemes over κ. Then f is an epimorphism (monomorphism) if and only if each f_κ is an epimorphism (monomorphism). For monomorphisms, this follows from Nakayama's lemma; for epimorphisms, it follows from Corollary SAG.6.1.4.10 .

Remark 6.2.7. Let κ be a field and let $f : G \to H$ be a morphism of commutative finite flat group schemes over κ. Then we can write $G = \mathrm{Sp\acute{e}t}\, A$ and $H = \mathrm{Sp\acute{e}t}\, B$, where A and B are finite-dimensional Hopf algebras over κ, and f determines a map of Hopf algebras $\phi : B \to A$. Then:

(a) The map f is an epimorphism if and only if ϕ is injective.

(b) The map f is a monomorphism if and only if ϕ is surjective.

Assertion (b) follows immediately from the definitions (and does not require the assumption that κ is a field). The "only if" direction of (a) is also immediate (note that f is an epimorphism if and only if ϕ is faithfully flat). To prove the converse, let K denote the kernel of f (formed in the ordinary category of finite flat group schemes over κ), so that f factors as a composition $G \xrightarrow{f'} G/K \xrightarrow{f''} H$ where f' is faithfully flat. We may then replace f by $f'' : G/K \to H$ and thereby reduce to the case where $K \simeq 0$. In this case, f is a monomorphism in the ordinary category of schemes, so the unit map $A \to \pi_0(A \otimes_B A)$ (induced by the inclusion of either tensor factor) is surjective (in fact, an isomorphism). The assumption that ϕ is injective guarantees that the map of Zariski spectra $|\mathrm{Spec}\, A| \to |\mathrm{Spec}\, B|$ is surjective. Applying Nakayama's lemma,

we deduce that that the map ϕ is surjective. It follows that ϕ is an isomorphism, so that f is an equivalence (and, in particular, an epimorphism).

Remark 6.2.8. Let $f : G \to H$ be a morphism of commutative finite flat group schemes over a connective $\mathbb{E}_\infty$-ring R, which we identify with their functors of points $G, H : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$. For every discrete R-algebra A, the spectra $G(A)$ and $H(A)$ are also discrete, and can therefore be viewed as abelian groups. The map f is a monomorphism if and only if, for every discrete R-algebra A, the induced map $G(A) \to H(A)$ is a monomorphism in the category of abelian groups.

Proposition 6.2.9. *Let A be an $\mathbb{E}_\infty$-ring and let $f : G \to H$ be a morphism of commutative finite flat group schemes over A. Suppose that f is an epimorphism. Then:*

(1) *The fiber $\mathrm{fib}(f)$ (formed in the ∞-category $\mathrm{CMon}^{\mathrm{gp}}(\mathrm{SpDM}_A^{\mathrm{nc}})$) is also a commutative finite flat group scheme over A.*

(2) *The canonical map $\mathrm{fib}(f) \to G$ is a monomorphism of commutative finite flat group schemes over A.*

(3) *The tautological fiber sequence $\mathrm{fib}(f) \to G \to H$ is also a cofiber sequence (of commutative finite flat group schemes over A).*

Proof. Assertion (1) follows from the assumption that f is finite flat. To prove (2), we can assume that A is connective (Remark 6.2.5). In this case, we wish to show that the projection map $G \times_H \mathrm{Sp\acute{e}t}\, A \to G$ is a closed immersion (Remark 6.2.4), which follows from the separatedness of H.

We now prove (3). Note that the ∞-category CAlg_A can be equipped with a Grothendieck topology, where the collection of coverings is generated by those finite collections of maps $\{\phi_i : B \to B_i\}_{i \in I}$ which induce a finite flat surjection $\sqcup \mathrm{Sp\acute{e}t}\, B_i \to \mathrm{Sp\acute{e}t}\, B$; we will refer to this Grothendieck topology as the *finite flat topology* (see Proposition SAG.A.3.2.1). Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S}))$ spanned by those functors which are sheaves with respect to the finite flat topology, so that the functor of points of Remark 6.1.10 induces a fully faithful embedding $j : \mathrm{FFG}(A) \to \mathcal{C}$. To show that the fiber sequence $\mathrm{fib}(f) \to G \to H$ is also a cofiber sequence, it suffices to show that its image under the functor j is a cofiber sequence in $\mathcal{C}$. Unwinding the definitions, we are reduced to proving that f induces an effective epimorphism of sheaves with respect to the finite flat topology, which is equivalent to our assumption that f is an epimorphism (Proposition 6.2.1). $\qquad\square$

6.3 Cartier Duality for Finite Flat Group Schemes

We now apply the categorical formalism of §3 to the setting of finite flat groups schemes.

Construction 6.3.1 (The Cartier Dual). Let A be an $\mathbb{E}_\infty$-ring and let G be a commutative finite flat group scheme over A, which we identify with its functor of points $\mathrm{CAlg}_A \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$. We let $\mathbf{D}(G)$ denote the Cartier dual of G, in the sense of Construction 3.7.1.

Proposition 6.3.2. *Let A be an $\mathbb{E}_\infty$-ring and let G be a commutative finite flat group scheme over A, which we identify with its functor of points $\mathrm{CAlg}_A \to \mathrm{CMon}^{\mathrm{gp}}(\mathcal{S})$. Then the Cartier dual $\mathbf{D}(G)$ is also a commutative finite flat group scheme over A.*

Proof. Since G is grouplike, the Cartier dual $\mathbf{D}(G)$ is also grouplike (Proposition 3.9.6). Write $G = \mathrm{Sp\acute{e}t}\, H$, where H is a Hopf algebra which is finite flat when regarded as an A-module. Then H is dualizable as an object of of Mod_A. Applying Proposition 3.8.1, we deduce that the dual $H^\vee$ admits the structure of a bialgebra object of Mod_A for which there is an equivalence $\mathbf{D}(G) \simeq \mathrm{Sp\acute{e}t}\, H^\vee$. Since H is a finite flat A-module, the dual $H^\vee$ is also a finite flat A-module. $\square$

Corollary 6.3.3. *Let A be an $\mathbb{E}_\infty$-ring. Then the Cartier duality construction $G \mapsto \mathbf{D}(G)$ induces an (involutive) equivalence of ∞-categories $\mathrm{FFG}(A) \simeq \mathrm{FFG}(A)^{\mathrm{op}}$.*

Proposition 6.3.4. *Let $f : G \to H$ be a morphism of finite flat group schemes over an $\mathbb{E}_\infty$-ring A. The following conditions are equivalent:*

(1) *The morphism f is an epimorphism (in the sense of Definition 6.2.2).*

(2) *The Cartier dual map $\mathbf{D}(f) : \mathbf{D}(H) \to \mathbf{D}(G)$ is a monomorphism (in the sense of Definition 6.2.2).*

Proof. Using Remarks 6.2.5 and 6.2.6, we can reduce to the case where $A = \kappa$ is a field. In this case, the equivalence of (1) and (2) follows immediately from the criterion of Remark 6.2.7. $\square$

Corollary 6.3.5. *Let A be an $\mathbb{E}_\infty$-ring and let $f : G \to H$ be a morphism of commutative finite flat group schemes over A. Suppose that f is a monomorphism. Then:*

(1) *The map f admits a cofiber $\mathrm{cofib}(f)$ in the ∞-category $\mathrm{FFG}(A)$.*

(2) *The canonical map $H \to \mathrm{cofib}(f)$ is an epimorphism of commutative finite flat group schemes over A.*

(3) *The tautological cofiber sequence $G \xrightarrow{f} H \to \mathrm{cofib}(f)$ is also a fiber sequence (in the ∞-category $\mathrm{FFG}(A)$).*

Proof. Using Cartier duality (Corollary 6.3.3) and Proposition 6.3.4, we see that assertions (1), (2) and (3) follow their counterparts in Proposition 6.2.9. $\square$

Corollary 6.3.6. *Let A be an $\mathbb{E}_\infty$-ring and suppose we are given a commutative diagram σ :*

$$
\begin{array}{ccc}
G' & \xrightarrow{f} & G \\
\downarrow & & \downarrow{\scriptstyle g} \\
\mathrm{Sp\acute{e}t}\, A & \longrightarrow & G''
\end{array}
$$

of commutative finite flat group schemes over A. The following conditions are equivalent:

(a) *The map f is an monomorphism and σ is a pushout square: that is, it determines a cofiber sequence $G' \to G \to G''$.*

(b) *The map g is an epimorphism and σ is a pullback square: that is, it determines a fiber sequence $G' \to G \to G''$.*

Proof. Combine Proposition 6.2.9 with Corollary 6.3.5. $\square$

Definition 6.3.7. Let A be an $\mathbb{E}_\infty$-ring. An *exact sequence* of commutative finite flat group schemes over A is a commutative diagram

$$
\begin{array}{ccc}
G' & \xrightarrow{f} & G \\
\downarrow & & \downarrow{\scriptstyle g} \\
\mathrm{Sp\acute{e}t}\, A & \longrightarrow & G''
\end{array}
$$

which satisfies the equivalent conditions of Corollary 6.3.6.

Remark 6.3.8. In the situation of Definition 6.3.7, we will abuse terminology by saying that the sequence $G' \xrightarrow{f} G \xrightarrow{g} G''$ is *exact*. In this case, we are implicitly assuming that some nullhomotopy of $g \circ f$ has been specified (which determines an extension of the pair (f, g) to a commutative diagram such as the one appearing in Definition 6.3.7 and Corollary 6.3.6).

Remark 6.3.9. The notion of exact sequence introduced in Definition 6.3.7 is Cartier self-dual: for every exact sequence $G' \xrightarrow{f} G \xrightarrow{g} G''$ of commutative finite flat group schemes over A, the dual sequence $\mathbf{D}(G'') \xrightarrow{\mathbf{D}(g)} \mathbf{D}(G) \xrightarrow{\mathbf{D}(f)} \mathbf{D}(G')$ is also exact.

6.4 S-Torsion Objects of ∞-Categories

We begin by introducing some terminology.

Notation 6.4.1. For every finite group G, we let $|G|$ denote the order of G. If S is a set of prime numbers, we will say that G is a S-*group* if every prime factor of $|G|$ belongs to S. We let $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ denote the category whose objects are finite abelian S-groups and whose morphisms are group homomorphisms. In the special case where S is the set of *all* prime numbers, we will denote $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ by $\mathrm{Ab}_{\mathrm{fin}}$ (so that $\mathrm{Ab}_{\mathrm{fin}}$ is the category of finite abelian groups). In the special case where S $= \{p\}$ for some prime number p, we will denote $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ by $\mathrm{Ab}_{\mathrm{fin}}^{p}$ (so that $\mathrm{Ab}_{\mathrm{fin}}^{p}$ is the category of finite abelian p-groups).

Definition 6.4.2. Let S be a set of prime numbers and let $\mathcal{C}$ be an ∞-category which admits finite limits. A S-*torsion object* of $\mathcal{C}$ is a functor $X : (\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \to \mathcal{C}$ satisfying the following pair of conditions:

(a) The functor X commutes with finite products.

(b) For every exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian S-groups, the diagram

$$
\begin{array}{ccc}
X(M'') & \longrightarrow & X(M) \\
\downarrow & & \downarrow \\
X(0) & \longrightarrow & X(M')
\end{array}
$$

is a pullback square (more informally, the functor X carries exact sequences to fiber sequences in $\mathcal{C}$).

In the special case where S is the set of all prime numbers, we will refer to an S-torsion object of $\mathcal{C}$ as a *torsion object* of $\mathcal{C}$. In the special case where S $= \{p\}$ for some prime number p, we will refer to a S-torsion object of $\mathcal{C}$ as a *p-torsion object* of $\mathcal{C}$.

We let $\mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}}, \mathcal{C})$ spanned by the S-torsion objects of $\mathcal{C}$. In the special case where S is the set of all prime numbers, we denote $\mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ by $\mathcal{T}\mathrm{ors}(\mathcal{C})$; in the special case where S $= \{p\}$ for some prime number p, we denote $\mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ by $\mathcal{T}\mathrm{ors}_{p}(\mathcal{C})$.

Remark 6.4.3 (Functoriality). Let $\mathcal{C}$ and $\mathcal{D}$ be ∞-categories which admit finite limits, let $f : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite limits, and let S be a set of prime numbers. Then composition with f induces a functor $F : \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{D})$ which also preserves finite limits. Moreover, if f is fully faithful, then F is also fully faithful.

In the situation of Definition 6.4.2, very little is lost by restricting our attention to the case where the set S consists of a single prime number p:

Proposition 6.4.4. *Let* S *be a set of prime numbers and let* $\mathcal{C}$ *be an* ∞-*category which admits finite limits. Then:*

(1) *For each prime number* $p \in$ S, *composition with the inclusion functor* $\mathrm{Ab}_{\mathrm{fin}}^{p} \hookrightarrow \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ *induces a forgetful functor* $\theta_p : \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_p(\mathcal{C})$.

(2) *The product map* $\theta : \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C}) \to \prod_{p \in \mathrm{S}} \mathcal{T}\mathrm{ors}_p(\mathcal{C})$ *is an equivalence of* ∞-*categories.*

Proof. Assertion (1) follows immediately from the definitions, since the inclusion functor $\mathrm{Ab}_{\mathrm{fin}}^{p} \hookrightarrow \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ preserves finite products and exact sequences. We now prove (2). Using a direct limit argument, we can reduce to the case where the set S is finite. For each finite abelian S-group M and each prime number $p \in$ S, let $M_{(p)}$ denote the localization of M at the prime p, so that $M_{(p)}$ is a finite abelian p-group. The construction $M \mapsto M_{(p)}$ determines a functor $L : \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}} \to \mathrm{Ab}_{\mathrm{fin}}^{p}$, which is right adjoint (and also left adjoint) to the inclusion $\mathrm{Ab}_{\mathrm{fin}}^{p} \hookrightarrow \mathrm{Ab}_{\mathrm{fin}}^{p}$. Since L also preserves finite products and short exact sequences, precomposition with L determines a functor $\rho_p : \mathcal{T}\mathrm{ors}_p(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ which is right adjoint to θ_p. Because the ∞-category $\mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ admits finite products, the functor θ admits a right adjoint $\rho : \prod_{p \in \mathrm{S}} \mathcal{T}\mathrm{ors}_p(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$, given by the formula $\rho(\{X_p\}_{p \in \mathrm{S}}) = \prod_{p \in \mathrm{S}} \rho_p(X_p)$. For each object $X \in \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$ and each finite abelian S-group M, the unit map

$$X(M) \to (\rho \circ \theta)(X)(M) = \prod_{p \in \mathrm{S}}(\rho_p \circ \theta_p)(X)(M) = \prod_{p \in \mathrm{S}} X(M_{(p)})$$

is an equivalence by virtue of the fact that X preserves finite products. To complete the proof, it will suffice to show that for every collection $\{X_p \in \mathcal{T}\mathrm{ors}_p(\mathcal{C})\}_{p \in \mathrm{S}}$, the counit map $v : (\theta \circ \rho)\{X_p\}_{p \in \mathrm{S}} \to \{X_p\}_{p \in \mathrm{S}}$ is an equivalence in the ∞-category $\prod_{p \in \mathrm{S}} \mathcal{T}\mathrm{ors}_p(\mathcal{C})$. Equivalently, we must show that v induces equivalences $v_p : (\theta_p \circ \rho)\{X_p\}_{p \in \mathrm{S}} \to X_p$ for each $p \in$ S. Taking N to be a finite abelian p-group, we are reduced to showing that the projection map

$$\prod_{p' \in \mathrm{S}} X_{p'}(N_{(p')}) \to X_p(N_{(p)}) \simeq X_p(N)$$

is an equivalence. This is clear, since the localization $N_{(p')}$ vanishes for $p' \neq p$ (so that $X_{p'}(N_{(p')})$ is a final object of $\mathcal{C}$). $\qquad\square$

Our next goal is to compare Definition 6.4.2 with the theory of abelian group objects introduced in §1.2.

Notation 6.4.5. Let S be a set of prime numbers and let Ab be the category of abelian groups. We let Lat(S) denote the full subcategory of Ab spanned by those abelian groups which are either lattices or finite abelian S-groups. We regard Lat and $\mathrm{Ab}^{\mathrm{S}}_{\mathrm{fin}}$ as full subcategories of Lat(S).

Proposition 6.4.6. *Let* S *be a set of prime numbers and let* $\mathcal{C}$ *be an* ∞-*category which admits finite limits. For every abelian group object* $A_0 : \mathrm{Lat}^{\mathrm{op}} \to \mathcal{C}$, *there exists a functor* $A : \mathrm{Lat(S)}^{\mathrm{op}} \to \mathcal{C}$ *which is a right Kan extension of* A_0. *Moreover, the restriction* $A|_{\mathrm{Ab}^{\mathrm{S}}_{\mathrm{fin}}}$ *is an* S-*torsion object of* $\mathcal{C}$.

Construction 6.4.7. Let $\mathcal{C}$ be an ∞-category which admits finite limits, let S be a set of prime numbers, and let $\mathcal{E}$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Lat(S)})^{\mathrm{op}}, \mathcal{C})$ spanned by those functors A which satisfy the following pair of conditions:

(i) The restriction $A_0 = A|_{\mathrm{Lat}^{\mathrm{op}}}$ is an abelian group object of $\mathcal{C}$.

(ii) The functor A is a right Kan extension of A_0.

Using Propositions 6.4.6 and HTT.4.3.2.15, we obtain restriction functors $\mathrm{Ab}(\mathcal{C}) \leftarrow^{\phi} \mathcal{E} \xrightarrow{\psi} \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$, where the functor ϕ is a trivial Kan fibration. Choosing a section s of ϕ, we obtain a functor $\mathrm{Ab}(\mathcal{C}) \xrightarrow{s} \mathcal{E} \xrightarrow{\psi} \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathcal{C})$. We will denote this functor by $X \mapsto X[\mathrm{S}^{\infty}]$. In the special case where $\mathrm{S} = \{p\}$ for some prime number p, we will denote $X[\mathrm{S}^{\infty}]$ by $X[p^{\infty}]$.

Proof of Proposition 6.4.6. Let A_0 be an abelian group object of $\mathcal{C}$. Our first goal is to show that A_0 admits a right Kan extension $A : \mathrm{Lat(S)}^{\mathrm{op}} \to \mathcal{C}$. By virtue of Lemma HTT.4.3.2.13, it will suffice to show that for every object $M \in \mathrm{Lat(S)}$, the induced map $(\mathrm{Lat}_{/M})^{\mathrm{op}} \to \mathrm{Lat}^{\mathrm{op}} \xrightarrow{A_0} \mathcal{C}$ admits a limit, where we let $\mathrm{Lat}_{/M}$ denote the category $\mathrm{Lat} \times_{\mathrm{Lat(S)}} \mathrm{Lat(S)}_{/M}$ whose objects are lattices Λ equipped with a map $\Lambda \to M$. Since the abelian group M is finitely generated, we can choose a surjection $u : M_0 \to M$, where M_0 is a lattice. Let $M_{\bullet}$ denote the Čech nerve of u (formed in the category of abelian groups). Then each M_n is isomorphic to a subgroup of a product of copies of M_0, hence a free abelian group of finite rank. The construction $[n] \mapsto M_n$ determines

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a functor $\mathbf{\Delta}^{\mathrm{op}} \to (\mathrm{Lat})_{/M}$. Using Theorem HTT.4.1.3.1, we see that this functor is right cofinal. We are therefore reduced to showing that the cosimplicial object $A_0(M_\bullet)$ admits a totalization in the ∞-category $\mathcal{C}$. Let $\mathbf{\Delta}_{s,\leqslant 1}$ denote the subcategory of $\mathbf{\Delta}$ whose objects are $[0]$ and $[1]$ and whose morphisms are injective maps of linearly ordered sets. Using our assumption that A_0 commutes with finite products, we deduce that the functor $[n] \mapsto A_0(M_n)$ is a right Kan extension of its restriction to $\mathbf{\Delta}_{s,\leqslant 1}$. Using Lemma HTT.4.3.2.7, we are reduced to proving that the diagram

$$\mathbf{\Delta}_{s,\leqslant 1} \hookrightarrow \mathbf{\Delta} \xrightarrow{M_\bullet} \mathrm{Lat}^{\mathrm{op}} \xrightarrow{A_0} \mathcal{C}$$

admits a limit. This is clear, since $\mathcal{C}$ admits finite limits and the simplicial set $\mathrm{N}(\mathbf{\Delta}_{s,\leqslant 1})$ is finite. This completes the proof of the existence of A. Moreover, the proof yields the following:

($*$) Let $A \in \mathrm{Fun}(\mathrm{Lat}(\mathrm{S})^{\mathrm{op}}, \mathcal{C})$ be a functor such that $A_0 = A|_{\mathrm{Lat}^{\mathrm{op}}}$ belongs to $\mathrm{Ab}(\mathcal{C})$. Then A is a right Kan extension of A_0 if and only if, for every object $M \in \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ and every surjective map $\Lambda \to M$ where Λ is a lattice, the diagram

$$
\begin{array}{ccc}
A(M) & \longrightarrow & A(\Lambda) \\
\downarrow & & \downarrow \\
A(\Lambda) & \longrightarrow & A(\Lambda \times_M \Lambda)
\end{array}
$$

is a pullback square in $\mathcal{C}$.

In the situation of ($*$), let Λ' denote the kernel of the surjection $\Lambda \to M$. We have a split exact sequence $0 \to \Lambda' \to \Lambda \times_M \Lambda \to \Lambda \to 0$, so the assumption that A_0 is an abelian group object of $\mathcal{C}$ guarantees that the right square in the diagram

$$
\begin{array}{ccccc}
A(M) & \longrightarrow & A(\Lambda) & \longrightarrow & A(0) \\
\downarrow & & \downarrow & & \downarrow \\
A(\Lambda) & \longrightarrow & A(\Lambda \times_M \Lambda) & \longrightarrow & A(\Lambda')
\end{array}
$$

is a pullback. It follows that the left square is a pullback if and only if the outer rectangle is a pullback. We may therefore reformulate the criterion of ($*$) as follows:

($*'$) Let $A \in \mathrm{Fun}(\mathrm{Lat}(\mathrm{S})^{\mathrm{op}}, \mathcal{C})$ be a functor such that $A_0 = A|_{\mathrm{Lat}^{\mathrm{op}}}$ belongs to $\mathrm{Ab}(\mathcal{C})$. Then A is a right Kan extension of A_0 if and only if, for every inclusion of lattices

$\Lambda' \subseteq \Lambda$ for which the quotient Λ/Λ' is a finite S-group, the associated diagram

$$
\begin{array}{ccc}
A(\Lambda/\Lambda') & \longrightarrow & A(\Lambda) \\
\downarrow & & \downarrow \\
A(0) & \longrightarrow & A(\Lambda)
\end{array}
$$

is a pullback square in $\mathcal{C}$.

To complete the proof, it will suffice to show that if A is a functor satisfying the criterion of $(*')$, then the restriction $X = A|_{(\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}}}$ is an S-torsion object of $\mathcal{C}$. We first show that X commutes with finite products. Fix a finite collection of objects $\{M_i\}_{i \in I}$ in the category $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$. For each $i \in I$, choose an exact sequence

$$
0 \to \Lambda_i' \to \Lambda_i \to M_i \to 0,
$$

where Λ_i is a lattice. Using our assumption on A, we obtain a commutative diagram of fiber sequences

$$
\begin{array}{ccc}
A(\bigoplus_{i \in I} M_i) & \longrightarrow A(\bigoplus_{i \in I} \Lambda_i) \longrightarrow & A(\bigoplus_{i \in I} \Lambda_i') \\
\downarrow & \downarrow & \downarrow \\
\prod_{i \in I} A(M_i) & \longrightarrow \prod_{i \in I} A(\Lambda_i) \longrightarrow & \prod_{i \in I} A(\Lambda_i').
\end{array}
$$

Our assumption that A_0 commutes with finite products guarantees that the right and center vertical maps are equivalences, so that the left vertical map is an equivalence as well.

We now complete the proof by verifying that the functor X satisfies condition (b) of Definition 6.4.2. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finite abelian S-groups; we wish to show that the diagram σ :

$$
\begin{array}{ccc}
A(M'') & \longrightarrow & A(M) \\
\downarrow & & \downarrow \\
A(0) & \longrightarrow & A(M')
\end{array}
$$

is a pullback square. Choose an epimorphism $\Lambda \to M$, where Λ is a lattice, and set $\Lambda' = \Lambda \times_M M'$. We can then extend σ to a commutative diagram $\overline{\sigma}$:

$$
\begin{array}{ccccc}
A(M'') & \longrightarrow A(M) & \longrightarrow & A(\Lambda) \\
\downarrow & \downarrow & & \downarrow \\
A(0) & \longrightarrow A(M') & \longrightarrow & A(\Lambda').
\end{array}
$$

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Our assumption on A guarantees that the right square and the outer rectangle are pullbacks, so that the left square is a pullback as well. $\square$

Remark 6.4.8. Let $\mathcal{C}$ be an ∞-category which admits finite limits and let A be an abelian group object of $\mathcal{C}$. The proof of Proposition 6.4.6 yields an explicit description of the S-torsion object $A[S^\infty]$: its value on any finite abelian S-group M is given by the fiber of the map $A(\Lambda) \to A(\Lambda')$, where $0 \to \Lambda' \to \Lambda \to M \to 0$ is any resolution of M by finitely generated free abelian groups. In particular, for each prime number $p \in S$, the value of $A[S^\infty]$ on the finite abelian group $\mathbf{Z}/p^n\mathbf{Z}$ can be identified with the fiber of the map $A(\mathbf{Z}) \xrightarrow{p^n} A(\mathbf{Z})$.

Proposition 6.4.9. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and sequential colimits, and suppose that the formation of sequential colimits in $\mathcal{C}$ is left exact. Let S be a set of prime numbers. Then the forgetful functor $[S^\infty] : \mathrm{Ab}(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_S(\mathcal{C})$ of Construction 6.4.7 admits a fully faithful left adjoint.*

Proof. It will suffice to show that the restriction functor $\mathcal{E} \to \mathcal{T}\mathrm{ors}_S(\mathcal{C})$ admits a fully faithful left adjoint, where $\mathcal{E}$ is defined as in Construction 6.4.7. By virtue of Proposition HTT.4.3.2.15, we are reduced to proving the following:

- For every S-torsion object $X : (\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \to \mathcal{C}$, there exists a functor $A : \mathrm{Lat}(S)^{\mathrm{op}} \to \mathcal{C}$ which is a left Kan extension of X. Moreover, the functor A belongs to $\mathcal{E}$: that is, $A_0 = A|_{\mathrm{Lat}^{\mathrm{op}}}$ is an abelian group object of $\mathcal{C}$, and A is a right Kan extension of A_0.

Let us regard $X \in \mathcal{T}\mathrm{ors}_S(\mathcal{C})$ as fixed. To show that X admits a left Kan extension $A : \mathrm{Lat}(S)^{\mathrm{op}} \to \mathcal{C}$, it will suffice to show that for every lattice Λ, the diagram

$$\rho : ((\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})_{\Lambda/})^{\mathrm{op}} \to (\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \xrightarrow{X} \mathcal{C}$$

admits a colimit in $\mathcal{C}$ (Lemma HTT.4.3.2.13); here $(\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})_{\Lambda/}$ denotes the fiber product $\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}} \times_{\mathrm{Lat}(S)} (\mathrm{Lat}(S))_{\Lambda/}$ (that is, the category of finite abelian S-groups M equipped with a map $\Lambda \to M$).

Choose a sequence of positive integers $\{N_i\}_{i \geqslant 0}$ with the following properties:

- Every prime divisor of each N_i belongs to S.

- Each N_i divides N_{i+1}.

- Every finite abelian S-group M is annihilated by N_i for $i \gg 0$.

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For example, we can take N_i to be the product $\prod_p p^i$, where p ranges over all prime numbers $p \leqslant i$ such that $p \in S$. There is an evident functor $\mathbf{Z}_{\geqslant 0}^{\mathrm{op}} \to (\mathrm{Ab}_{\mathrm{fin}}^p)_{\Lambda/}$, given by the tower of finite abelian p-groups

$$\cdots \to \Lambda/N_2\Lambda \to \Lambda/N_1\Lambda \to \Lambda/N_0\Lambda \to 0.$$

Using Theorem HTT.4.1.3.1, we see that this functor is left cofinal. Consequently, we can identify colimits of ρ with colimits of the diagram

$$X(\Lambda/N_0\Lambda) \to X(\Lambda/N_1\Lambda) \to X(\Lambda/N_2\Lambda) \to \cdots,$$

which exist by virtue of our assumption that $\mathcal{C}$ admits sequential colimits. This proves the existence of the functor A.

Set $A_0 = A|_{\mathrm{Lat}^{\mathrm{op}}}$. The preceding argument shows that the functor A_0 is given concretely by $A_0(\Lambda) \simeq \varinjlim X(\Lambda/N_i\Lambda)$. From this formula (and our assumption that sequential colimits in $\mathcal{C}$ commute with finite limits), we immediately deduce that A_0 commutes with finite products: that is, it is an abelian group object of $\mathcal{C}$.

We now complete the proof by showing that A is a right Kan extension of A_0. For this, it will suffice to show that the functor A satisfies the criterion of $(*')$ in the proof of Proposition 6.4.6. Suppose we are given an exact sequence $0 \to \Lambda' \to \Lambda \to M \to 0$, where Λ is a lattice and M is a finite abelian S-group; we wish to show that the diagram σ :

$$
\begin{array}{ccc}
A(M) & \longrightarrow & A(\Lambda) \\
\downarrow & & \downarrow \\
A(0) & \longrightarrow & A(\Lambda')
\end{array}
$$

is a pullback square. This follows from our assumption that the collection of pullback squares in $\mathcal{C}$ is closed under sequential colimits, since σ can be obtained as a filtered colimit of diagrams σ_n:

$$
\begin{array}{ccc}
X(M) & \longrightarrow & X(\Lambda/N_i\Lambda) \\
\downarrow & & \downarrow \\
X(0) & \longrightarrow & X(\Lambda'/(N_i\Lambda \cap \Lambda')),
\end{array}
$$

where i ranges over those positive integers for which M is annihilated by N_i; here each σ_n is a pullback square by virtue of our assumption that X is an S-torsion object of $\mathcal{C}$. $\qquad\square$

Remark 6.4.10. Let S be a set of prime numbers and let $\mathcal{C}$ be an ∞-category which satisfies the hypotheses of Proposition 6.4.9, and let $\{N_i\}_{i \geqslant 0}$ be as in the proof of Proposition 6.4.9. Then we can identify $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$ with the full subcategory of $\mathrm{Ab}^\mathrm{S}(\mathcal{C})$ spanned by those abelian group objects A which satisfy the following condition:

($*$) For every lattice Λ, the canonical map $\theta : \varinjlim A[\mathrm{S}^\infty](\Lambda/N_i\Lambda) \to A(\Lambda)$ is an equivalence.

Since sequential colimits in $\mathcal{C}$ commute with finite limits, the map θ fits into a pullback diagram

$$
\begin{array}{ccc}
\varinjlim A[p^\infty](\Lambda/N_i\Lambda) & \xrightarrow{\ \theta\ } & A(\Lambda) \\
\downarrow & & \downarrow \\
A(0) & \longrightarrow & \varinjlim A(N_i\Lambda).
\end{array}
$$

Consequently, it is sufficient (but not necessary) to satisfy the following stronger condition:

($*'$) The colimit $\varinjlim A(N_i\Lambda)$ is a final object of $\mathcal{C}$.

Moreover, in verifying conditions ($*$) or ($*'$), we are free to assume that $\Lambda = \mathbf{Z}$.

Example 6.4.11. According to Remark 1.2.10, we have a canonical equivalence of ∞-categories $\mathrm{Ab}(\mathcal{S}) \simeq \mathrm{Mod}_\mathbf{Z}^\mathrm{cn}$, where $\mathbf{Z}$ denotes the ring of integers. Consequently, Proposition 6.4.9 determines a fully faithful embedding $\rho : \mathcal{T}\mathrm{ors}_\mathrm{S}\,\mathcal{S} \to \mathrm{Mod}_\mathbf{Z}^\mathrm{cn}$. The essential image of ρ consists of those connective $\mathbf{Z}$-modules M having the following property:

- For every element $x \in \pi_* M$, there exists a positive integer N such that $Nx = 0$ and every prime factor of N belongs to S.

For example, if $\mathrm{S} = \{p\}$ for some prime number p, then the essential image consists of those connective $\mathbf{Z}$-modules which are p-nilpotent, in the sense of Definition SAG.7.1.1.1: that is, those modules M for which the localization $M[p^{-1}]$ vanishes.

Proposition 6.4.12. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and let S be a set of prime numbers. Then the ∞-category $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$ is additive (see Definition SAG.C.1.5.1).*

Proof. It follows from Example 6.4.11 that we can identify $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{S})$ with an additive (or even prestable) subcategory of the ∞-category $\mathrm{Mod}_\mathbf{Z}$. It follows that the ∞-category $\mathrm{Fun}(\mathcal{C}^\mathrm{op}, \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{S})) \simeq \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{Fun}(\mathcal{C}^\mathrm{op}, \mathcal{S}))$ is also additive. Using Remark 6.4.3, we see that the Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^\mathrm{op}, \mathcal{S})$ determines a fully faithful embedding $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C}) \to \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{Fun}(\mathcal{C}^\mathrm{op}, \mathcal{S}))$. In particular, we can identify $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$ with a full subcategory of $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{Fun}(\mathcal{C}^\mathrm{op}, \mathcal{S}))$ which is closed under finite products, so that $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$ is also additive (Example SAG.C.1.5.4). $\qquad\square$

Corollary 6.4.13. *Let $\mathcal{C}$ be an ∞-category which admits finite limits and let S be a set of prime numbers. Then the forgetful functor $\mathrm{CMon}^\mathrm{gp}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence $\theta : \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{CMon}^\mathrm{gp}(\mathcal{C})) \to \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$.*

Proof. Unwinding the definitions, we can identify θ with the forgetful functor

$$\mathrm{CMon}^\mathrm{gp}(\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})) \to \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C}).$$

Since the ∞-category $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{C})$ is additive (Proposition 6.4.12), this functor is an equivalence (see Proposition HA.2.4.2.5 and Proposition HA.2.4.3.9). $\qquad\square$

6.5   S-Divisible Groups

We now adapt the theory of p-divisible groups to the setting of spectral algebraic geometry.

Definition 6.5.1. Let A be an $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. A S-*divisible group* over A is a functor $X : (\mathrm{Ab}_\mathrm{fin}^\mathrm{S})^\mathrm{op} \to \mathrm{FFG}(A)$ with the following property:

(i) The commutative finite flat group scheme $X(0)$ is trivial (that is, it is equivalent to $\mathrm{Sp\acute{e}t}\, A$).

(ii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ of finite abelian S-groups, the induced diagram

$$
\begin{array}{ccc}
X(M'') & \longrightarrow & X(M) \\
\downarrow & & \downarrow \\
X(0) & \longrightarrow & X(M')
\end{array}
$$

is an exact sequence of commutative finite flat group schemes over A (Definition 6.3.7).

We will say that an S-divisible group X has *height h* (where h is some nonnegative integer) if it satisfies the following additional condition:

(iii) For every finite abelian S-group M, the commutative finite flat group scheme $X(M)$ has degree $|M|^h$ over A (where $|M|$ denotes the cardinality of M).

In the case where S consists of a single prime number p, we will refer to an S-divisible group over A as a *p-divisible group over A*.

We let $\mathrm{BT}^{\mathrm{S}}(A)$ denote the full subcategory of $\mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}}, \mathrm{FFG}(A))$ spanned by the S-divisible groups over A. For each $h \geqslant 0$, we let $\mathrm{BT}_h^{\mathrm{S}}(A)$ denote the full subcategory of $\mathrm{BT}^{\mathrm{S}}(A)$ spanned by those p-divisible groups having height h. In the special case where $\mathrm{S} = \{p\}$ for some prime number p, we write $\mathrm{BT}^p(A)$ and $\mathrm{BT}_h^p(A)$ in place of $\mathrm{BT}^{\mathrm{S}}(A)$ and $\mathrm{BT}_h^{\mathrm{S}}(A)$, respectively.

Remark 6.5.2. In the situation of Definition 6.5.1, it suffices to verify condition (iii) in the special case $M = \mathbf{Z}/p\mathbf{Z}$ for $p \in \mathrm{S}$ (the general case then follows from iterated application of (ii)).

Remark 6.5.3. Let A be an $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. Then extension of scalars determines an equivalence of ∞-categories $\mathrm{BT}^{\mathrm{S}}(\tau_{\leqslant 0}A) \to \mathrm{BT}^{\mathrm{S}}(A)$ (see Remark 6.1.7).

Remark 6.5.4 (Functoriality)**.** Let $\phi : A \to B$ be a morphism of $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. Then the extension-of-scalars map $\mathrm{FFG}(A) \to \mathrm{FFG}(B)$ determines a functor $\mathrm{BT}^{\mathrm{S}}(A) \to \mathrm{BT}^{\mathrm{S}}(B)$, which carries S-divisible groups of height h over A to S-divisible groups of height h over B. If the map ϕ induces an isomorphism $\pi_n(A) \to \pi_n(B)$ for $n \geqslant 0$, then the functor $\mathrm{BT}^{\mathrm{S}}(A) \to \mathrm{BT}^{\mathrm{S}}(B)$ is an equivalence of ∞-categories (this follows from Remark 6.1.7).

In the situation of Definition 6.5.1, it is not necessary to specify the group structures on the finite flat A-schemes $X(M)$: they are already encoded (in an essentially unique way) by the functoriality of X. More precisely, we have the following:

Proposition 6.5.5. *Let A be an $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. Then the forgetful functor $\mathrm{FFG}(A) \to \mathrm{SpDM}_A^{\mathrm{nc}}$ induces a fully faithful embedding $\mathrm{BT}^{\mathrm{S}}(A) \to \mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}}, \mathrm{SpDM}_A^{\mathrm{nc}})$, whose essential image is spanned by those functors $X : (\mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \to \mathrm{SpDM}_A^{\mathrm{nc}}$ which satisfy the following conditions:*

(a) *The functor X is an S-torsion object of $\mathrm{SpDM}_A^{\mathrm{nc}}$, in the sense of Definition 6.4.2.*

(b) *For every prime number $p \in \mathrm{S}$ and every positive integer n, the canonical map $X(\mathbf{Z}/p^n\,\mathbf{Z}) \to X(p\,\mathbf{Z}/p^n\,\mathbf{Z})$ is a finite flat surjection.*

Proof. Let us identify $\mathrm{FFG}(A)$ with a full subcategory of $\mathrm{CMon}^{\mathrm{gp}}(\mathrm{SpDM}_A^{\mathrm{nc}})$. It follows immediately from the definitions that every S-divisible group over A (in the sense of Definition 6.5.1) is an S-torsion object of $\mathrm{CMon}^{\mathrm{gp}}(\mathrm{SpDM}_A^{\mathrm{nc}})$ (in the sense of Definition 6.4.2). Applying Corollary 6.4.13, we deduce that the forgetful functor

$$\theta : \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathrm{CMon}^{\mathrm{gp}}(\mathrm{SpDM}_A^{\mathrm{nc}})) \to \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathrm{SpDM}_A^{\mathrm{nc}})$$

is an equivalence of ∞-categories. To complete the proof, it will suffice to show that an object $X \in \mathcal{T}\mathrm{ors}_{\mathrm{S}}(\mathrm{CMon}^{\mathrm{gp}}(\mathrm{SpDM}_A^{\mathrm{nc}}))$ is an S-divisible group if and only if $\theta(X)$ satisfies conditions (a) and (b). The "only if" direction follows immediately from the definitions. For the converse, suppose that $\theta(X)$ satisfies conditions (a) and (b). It follows from (a) that $X(0) \simeq \mathrm{Sp\acute{e}t}\, A$. Applying (b) repeatedly, we deduce that $X(\mathbf{Z}/p^n\,\mathbf{Z})$ is finite flat over A for every prime number $p \in \mathrm{S}$. Since every finite abelian S-group can be written as a finite direct sum of cyclic S-groups, it follows from (a) that $X(M)$ is a commutative finite flat group scheme over A for each $M \in \mathrm{Ab}_{\mathrm{fin}}^{\mathrm{S}}$.

Given an exact sequence of finite abelian S-groups $0 \to M' \to M \to M'' \to 0$, condition (a) implies that the diagram σ :

$$\begin{array}{ccc} X(M'') & \longrightarrow & X(M) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ X(0) & \longrightarrow & X(M') \end{array}$$

is a pullback square in the ∞-category $\mathrm{SpDM}_A^{\mathrm{nc}}$, hence also in the ∞-category $\mathrm{FFG}(A)$. We wish to show that σ is exact: that is, that the map g is a finite flat surjection (see Proposition 6.2.1). Proceeding by induction on the index of M' in M, we can reduce to the case where there exists a pushout diagram of finite abelian groups

$$\begin{array}{ccc} p\,\mathbf{Z}/p^n\,\mathbf{Z} & \xrightarrow{\ f_0\ } & \mathbf{Z}/p^n\,\mathbf{Z} \\ \downarrow & & \downarrow \\ M' & \longrightarrow & M \end{array}$$

for some prime number $p \in \mathrm{S}$. Using condition (a), we see that $X(f)$ is a pullback of $X(f_0)$, which is a finite flat surjection by virtue of assumption (b). $\qquad\square$

Remark 6.5.6 (Dependence on S). Let A be an $\mathbb{E}_\infty$-ring, let S be a set of prime numbers, and let X be an S-torsion object of $\mathrm{SpDM}_A^{\mathrm{nc}}$. For each prime number $p \in$ S, let X_p be the p-torsion object of $\mathrm{SpDM}_A^{\mathrm{nc}}$ determined by X. Note that X satisfies condition (b) of Proposition 6.5.5 if and only if each X_p satisfies condition (b) of Proposition 6.5.5. It follows that the equivalence $\mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{SpDM}_A^{\mathrm{nc}}) \simeq \prod_{p \in \mathrm{S}} \mathcal{T}\mathrm{ors}_p(\mathrm{SpDM}_A^{\mathrm{nc}})$ of Proposition 6.4.4 induces an equivalence of ∞-categories $\mathrm{BT}^\mathrm{S}(A) \simeq \prod_{p \in \mathrm{S}} \mathrm{BT}^p(A)$. More informally: the datum of a S-divisible group over A is equivalent to the datum of a collection of p-divisible groups over A, where p ranges over all prime numbers which belong to S.

Construction 6.5.7 (The Functor of Points). Let A be an $\mathbb{E}_\infty$-ring and let S be a set of prime numbers. Using the functor of points $j : \mathrm{SpDM}_A^{\mathrm{nc}} \to \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$, Proposition 6.5.5, and Example 6.4.11, we fully faithful embeddings

$$
\begin{aligned}
\mathrm{BT}^\mathrm{S}(A) &\to \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{SpDM}^{\mathrm{nc}}) \\
&\to \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})) \\
&\simeq \mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{T}\mathrm{ors}_\mathrm{S}(\mathcal{S})) \\
&\to \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_\mathbf{Z}^{\mathrm{cn}}).
\end{aligned}
$$

If X is an S-divisible group over A, we will refer to the image of X under this map as the *functor of points* of X.

We can use Construction 6.5.7 to give an alternate description of the ∞-category $\mathrm{BT}^\mathrm{S}(A)$:

Proposition 6.5.8. *Let A be an $\mathbb{E}_\infty$-ring, let S be a set of prime numbers, and let $Y : \mathrm{CAlg}_A \to \mathrm{Mod}_\mathbf{Z}^{\mathrm{cn}}$ be a functor. Then Y is representable by an S-divisible group over A (that is, it lies in the essential image of the embedding $\mathrm{BT}^\mathrm{S}(A) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Mod}_\mathbf{Z}^{\mathrm{cn}})$) if and only if it satisfies the following conditions:*

(1) *For each $B \in \mathrm{CAlg}_A$, the $\mathbf{Z}$-module $Y(B)$ is S-torsion: that is, every element of $\pi_* Y(B)$ is annihilated by some positive integer whose prime factors belong to S.*

(2) *For every finite abelian S-group M, the functor $B \mapsto \mathrm{Map}_{\mathrm{Mod}_\mathbf{Z}}(M, Y(B))$ is corepresentable by an $\mathbb{E}_\infty$-algebra which is finite flat over A.*

(3) *For each prime number $p \in$ S, the "multiplication by p" map $p : Y \to Y$ is an epimorphism locally with respect to the finite flat topology (see the proof of Proposition 6.2.9).*

Remark 6.5.9. In the situation of Proposition 6.5.8, if condition (3) is satisfied, then it suffices to verify condition (2) in the special case $M = \mathbf{Z}/p\,\mathbf{Z}$ where $p \in \mathrm{S}$.

Proof of Proposition 6.5.8. We will show that conditions (1), (2), and (3) are sufficient; the proof of necessity is similar. Using (1), we can identify Y with an S-torsion object of the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$. Using (2), we see that each $Y(M)$ is representable by a nonconnective spectral Deligne-Mumford stack $\overline{Y}(M) \in \mathrm{FF}(A)$. We wish to show that the functor $M \mapsto \overline{Y}(M)$ satisfies conditions (a) and (b) of Proposition 6.5.5. Condition (a) is automatic (since Y is S-torsion object of $\mathrm{Fun}(\mathrm{CAlg}_A, \mathcal{S})$. Using the criterion of Proposition 6.2.1, we see that condition (b) is equivalent to the requirement that the natural transformation $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(\mathbf{Z}/p^n\,\mathbf{Z}, Y(\bullet)) \to \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(p\,\mathbf{Z}/p^n\,\mathbf{Z}, Y(\bullet))$ is an effective epimorphism of sheaves with respect to the finite flat topology for each $n > 0$. This follows from assumption (3). $\qquad\square$

Warning 6.5.10 (Descent)**.** Let p be a prime number. We will say that an object $M \in \mathrm{Mod}_{\mathbf{Z}}$ is *p-nilpotent* if the localiztion $M[p^{-1}]$ vanishes: that is, if the action of p is locally nilpotent on $\pi_*(M)$. Let $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn,Nil}(p)}$ denote the full subcategory of $\mathrm{Mod}_{\mathbf{Z}}$ spanned by those $\mathbf{Z}$-module spectra which are connective and p-nilpotent. Let R be a connective $\mathbb{E}_\infty$-ring and let $X : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ be a functor which is (representable by) a p-divisible group over R. Then X takes values in the full subcategory $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn,Nil}(p)} \subseteq \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Moreover, if we regard X as a functor taking values in $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn,Nil}(p)}$, then it is a sheaf with respect to the fpqc topology on $\mathrm{CAlg}_R^{\mathrm{cn}}$. To see this, we observe that the ∞-category $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn,Nil}(p)}$ is generated (under small colimits) by finite abelian p-groups. It will therefore suffice to show that, for every finite abelian p-group M, the functor $A \mapsto \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, X(A))$ is a sheaf with respect to the fpqc topology. This is clear, since the functor $\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, X(\bullet))$ is corepresentable by a finite flat R-algebra.

Beware that it is not *a priori* clear that the functor X is a $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$-valued sheaf with respect to the fpqc topology (or even the finite flat topology), because the inclusion functor $\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn\,Nil}(p)} \hookrightarrow \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ does not preserve limits. However, it *is* true that the composite functor

$$\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}} \hookrightarrow \mathrm{CAlg}_R^{\mathrm{cn}} \xrightarrow{X} \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$$

is a sheaf with respect to the fpqc topology, for each fixed $n \geqslant 0$: this follows from the fact that X can be written as a filtered colimit of the functors $X_k = \mathrm{fib}(p^k : X \to X)$, each satisfying flat descent and taking n-truncated values on the subcategory $\tau_{\leqslant n}\,\mathrm{CAlg}_R^{\mathrm{cn}}$.

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6.6 Cartier Duality for S-Divisible Groups

We now extend the Cartier duality construction of §6.3 to the setting of S-divisible groups.

Notation 6.6.1. Let M be a finite abelian group. We let M^* denote the Pontryagin dual of M: that is, the group $\operatorname{Hom}(M, \mathbf{Q}/\mathbf{Z})$. For every set S of prime numbers, the construction $M \mapsto M^*$ determines an equivalence of the category $\operatorname{Ab}_{\mathrm{fin}}^{\mathrm{S}}$ with its opposite $(\operatorname{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}}$.

Construction 6.6.2 (The Cartier Dual of an S-Divisible Group)**.** Let A be an $\mathbb{E}_\infty$-ring, let S be a set of prime numbers, and let $X : (\operatorname{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \to \operatorname{FFG}(A)$ be an S-divisible group over A. We define a functor $\mathbf{D}(X) : (\operatorname{Ab}_{\mathrm{fin}}^{\mathrm{S}})^{\mathrm{op}} \to \operatorname{FFG}(A)$ by the formula $\mathbf{D}(X)(M) = \mathbf{D}(X(M^*))$. We will refer to $\mathbf{D}(X)$ as the *Cartier dual* of X.

Proposition 6.6.3. *Let A be an $\mathbb{E}_\infty$-ring, let S be a set of prime numbers, and let X be an S-divisible group over A. Then the Cartier dual $\mathbf{D}(X)$ is also an S-divisible group over A.*

Proof. Suppose we are given a short exact sequence of finite abelian S-groups $0 \to M' \to M \to M'' \to 0$; we wish to show that the induced sequence $\mathbf{D}(X)(M'') \to \mathbf{D}(X)(M) \to \mathbf{D}(X)(M')$ is exact. This follows by applying Remark 6.3.9 to the sequence $X(M'^*) \to X(M^*) \to X(M''^*)$ (which is exact by virtue of our assumption that X is an S-divisible group). $\qquad\square$

Remark 6.6.4. The Cartier duality operation of Construction 6.6.2 is involutive: for every S-divisible group X over A, there is a canonical equivalence $\mathbf{D}(\mathbf{D}(X)) \simeq X$. It follows that the construction $X \mapsto \mathbf{D}(X)$ determines an equivalence of ∞-categories $\operatorname{BT}^{\mathrm{S}}(A) \simeq \operatorname{BT}^{\mathrm{S}}(A)^{\mathrm{op}}$.

Remark 6.6.5. Let X be an S-divisible group over an $\mathbb{E}_\infty$-ring A and let $h \geqslant 0$ be an integer. Then X has height h if and only if $\mathbf{D}(X)$ has height h.

Remark 6.6.6. The formation of Cartier duals is compatible with the equivalence $\operatorname{BT}^{\mathrm{S}}(A) \simeq \prod_{p \in \mathrm{S}} \operatorname{BT}^p(A)$ of Remark 6.5.6. More precisely, if X is an S-divisible group over A corresponding to a collection $\{X_p\}_{p \in \mathrm{S}}$, where each X_p is a p-divisible group over A, then the Cartier dual $\mathbf{D}(X)$ corresponds (under the same equivalence) to the collection of p-divisible groups $\{\mathbf{D}(X_p)\}_{p \in \mathrm{S}}$.

6.7 The S-Divisible Group of a Strict Abelian Variety

As in classical algebraic geometry, the theory of abelian varieties provides a rich supply of p-divisible groups.

Proposition 6.7.1. *Let R be an $\mathbb{E}_\infty$-ring, let S be a set of prime numbers, and let X be a strict abelian variety of dimension g over R, which we regard as an abelian group object of the ∞-category $\mathrm{SpDM}_R^{\mathrm{nc}}$, and let $X[S^\infty]$ be defined as in Construction 6.4.7. Then $X[S^\infty]$ is an S-divisible group of height $2g$ over R (that is, it lies in the essential image of the functor $\mathrm{BT}_{2g}^S(R) \to \mathcal{T}\mathrm{ors}_S(\mathrm{SpDM}_R^{\mathrm{nc}})$ of Proposition 6.5.5).*

Remark 6.7.2. Let X be as in Proposition 6.7.1. For each prime number p, we let $X[p^\infty]$ denote the associated p-torsion object $\mathrm{SpDM}_R^{\mathrm{nc}}$. Proposition 6.7.1 is equivalent to the assertion that each $X[p^\infty]$ is a p-divisible group of height $2g$.

We will deduce Proposition 6.7.1 from the following:

Proposition 6.7.3. *Let R be an $\mathbb{E}_\infty$-ring and let X be an abelian variety of dimension g over R. For every integer $n > 0$, the map $[n] : X \to X$ is finite flat of degree n^{2g}.*

The statement of Proposition 6.7.3 can be easily reduced to the "classical" case, where $R = \kappa$ is a field. We refer the reader to [10] for two proofs of this statement. For the sake of completeness, we include a slightly different argument here.

Notation 6.7.4. Let κ be a field and let X be a variety over κ. If $\mathscr{L}$ is a line bundle on X, we define the *Euler characteristic* of $\mathscr{L}$ to be the sum $\chi(\mathscr{L}) = \sum_{n \geqslant 0}(-1)^n \dim_\kappa H^n(X; \mathscr{L})$.

Lemma 6.7.5. *Let X be an abelian variety of dimension g over a field κ and let $\mathscr{L}$ be a line bundle on X. For every integer n, we have an equality $\chi(\mathscr{L}^{\otimes n}) = n^g \chi(\mathscr{L})$.*

Proof. Since X is a smooth group scheme over κ, the tangent bundle of X is trivial. Applying the Hirzebruch-Riemann-Roch theorem, we obtain an equality $\chi(\mathscr{L}) = \frac{1}{g!}c_1(\mathscr{L})^g[X]$. Applying the same formula to $\mathscr{L}^{\otimes n}$, we compute

$$
\begin{aligned}
\chi(\mathscr{L}^n) &= \frac{1}{g!}c_1(\mathscr{L}^{\otimes n})^g[X] \\
&= \frac{n^g}{g!}c_1(\mathscr{L})^g[X] \\
&= n^g \chi(\mathscr{L}).
\end{aligned}
$$

$\square$

Lemma 6.7.6. *Let κ be a field and let $f : \mathsf{X} \to \mathsf{Y}$ be a morphism of abelian varieties having the same dimension g over κ, classified by a biextension $\mu : \mathsf{X} \wedge \widehat{\mathsf{Y}} \to \mathrm{BGL}_1$ with underlying line bundle $\mathscr{L}_\mu$. Then f is finite flat if and only if the Euler characteristic $\chi(\mathscr{L}_\mu)$ does not vanish. In this case, the degree of f is equal to $(-1)^g \chi(\mathscr{L}_\mu)$.*

Proof. Set $\mathscr{F} = f_* \mathscr{O}_\mathsf{X} \in \mathrm{QCoh}(\mathsf{Y})$. For each κ-valued point $y \in \mathsf{Y}(\kappa)$, let us identify the (derived) fiber $\mathscr{F}_y = y^* \mathscr{F}$ with an object of Mod_κ, and set $d_y = \sum_n (-1)^n \dim_\kappa \pi_n \mathscr{F}_y$. Since $\mathscr{F}$ is perfect, the function $y \mapsto d_y$ is locally constant and therefore constant (since Y is connected); let us denote the constant value by d. We now consider two cases:

(a) If the map f is finite flat of degree r, then $f_* \mathscr{O}_\mathsf{X}$ is a vector bundle of rank r over Y, so we have $d = r$.

(b) Suppose that the map f is not finite flat. Since X and Y are proper smooth κ-schemes of the same dimension, some fiber of f is not finite. Because f is a morphism of group schemes, it follows that the kernel of f is positive-dimensional, so that the image of f has Krull dimension $< g$. It follows that there exists a point $y \in \mathsf{Y}(\kappa)$ which does not belong to the image of f. We then have $\mathscr{F}_y \simeq 0$, so that $d = d_y = 0$.

To complete the proof of Lemma 6.7.6, it will suffice to show that $d = (-1)^g \chi(\mathscr{L}_\mu)$. Let $e : \mathrm{Sp\acute{e}t}\, \kappa \to \mathsf{Y}$ denote the zero section and let $\mu_0 : \mathsf{Y} \wedge \widehat{\mathsf{Y}} \to \mathrm{BGL}_1$ be the tautological (perfect) biextension and let $\mathscr{L}_{\mu_0}$ denote the underlying line bundle on $\mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \widehat{\mathsf{Y}}$, Unwinding the definitions, we see that μ is given by the composition

$$\mathsf{X} \wedge \widehat{\mathsf{Y}} \xrightarrow{f \wedge \mathrm{id}} \mathsf{Y} \wedge \widehat{\mathsf{Y}} \xrightarrow{\mu_0} \mathrm{BGL}_1,$$

so that $\mathscr{L}_\mu$ is the pullback of $\mathscr{L}_{\mu_0}$ along the map $f \times \mathrm{id}$. Using the pullback diagram

$$
\begin{array}{ccc}
\mathsf{X} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \widehat{\mathsf{Y}} & \xrightarrow{\ \pi\ } & \mathsf{X} \\
{\scriptstyle f \times \mathrm{id}} \big\downarrow & & \big\downarrow {\scriptstyle f} \\
\mathsf{Y} \times_{\mathrm{Sp\acute{e}t}\, \kappa} \widehat{\mathsf{Y}} & \xrightarrow{\ \pi'\ } & \mathsf{Y},
\end{array}
$$

together with Proposition 5.2.8, we compute

$$
\begin{aligned}
\Gamma(\mathsf{X} \times_{\operatorname{Sp\acute{e}t}\kappa} \mathsf{Y}; \mathscr{L}_\mu) & \simeq \Gamma(\mathsf{X}; \pi_* \mathscr{L}_\mu) \\
& \simeq \Gamma(\mathsf{X}; \pi_*(f \times \operatorname{id})^* \mathscr{L}_{\mu_0}) \\
& \simeq \Gamma(\mathsf{X}; f^* \pi_*' \mathscr{L}_{\mu_0}) \\
& \simeq \Gamma(\mathsf{X}; f^* e_*(\omega_{\mathsf{Y}}^\circ)^{-1}) \\
& \simeq \Gamma(\mathsf{Y}; f_* f^* e_*(\omega_{\mathsf{Y}}^\circ)^{-1}) \\
& \simeq \Gamma(\mathsf{Y}; \mathscr{F} \otimes (e_* \omega_{\mathsf{Y}}^\circ)^{-1}) \\
& \simeq \Gamma(\mathsf{Y}; e_*(e^* \mathscr{F}) \otimes (\omega_{\mathsf{Y}}^\circ)^{-1}) \\
& \simeq \mathscr{F}_e \otimes (\omega_{\mathsf{Y}}^\circ)^{-1}.
\end{aligned}
$$

Note that $\omega_{\mathsf{Y}}^\circ$ is (noncanonically) equivalent to the g-fold suspension of κ, this identification supplies isomorphisms $\mathrm{H}^n(\mathscr{L}_\mu) \simeq \pi_{g-n} \mathscr{F}_e$. Taking the dimensions of both sides and forming an (alternating) sum over n, we obtain the desired equality $d = d_e = (-1)^g \chi(\mathscr{L}_\mu)$. $\qquad \square$

Lemma 6.7.7. *Let κ be a field, let X be an abelian variety of dimension g over κ, and let $[n] : \mathsf{X} \to \mathsf{X}$ be the map given by multiplication by n for some $n > 0$. Then $[n]$ is a finite flat map of degree n^{2g}.*

Proof. Let $\mu : \mathsf{X} \wedge \widehat{\mathsf{X}} \to \mathrm{BGL}_1$ be a perfect biextension of abelian varieties over κ, and let $\mathscr{L}_\mu$ denote the underlying line bundle of μ. Applying Lemma 6.7.6 to the identity map $\operatorname{id} : \mathsf{X} \to \mathsf{X}$, we obtain an equality $\chi(\mathscr{L}_\mu) = (-1)^g$. The map $[n] : \mathsf{X} \to \mathsf{X}$ is classified by a biextenison whose underlying line bundle can be identified with $\mathscr{L}_\mu^{\otimes n}$. Since the product $\mathsf{X} \times_{\operatorname{Sp\acute{e}t}\kappa} \widehat{\mathsf{X}}$ is an abelian variety of dimension $2g$ over κ, Lemma 6.7.5 supplies an equality $\chi(\mathscr{L}_\mu^{\otimes n}) = n^{2g} \chi(\mathscr{L}_\mu) = (-1)^g n^{2g}$. Applying Lemma 6.7.6 again, we deduce that $[n] : \mathsf{X} \to \mathsf{X}$ is a finite flat map of degree n^{2g}. $\qquad \square$

Proof of Proposition 6.7.3. Let X be an abelian variety of dimension g over an arbitrary $\mathbb{E}_\infty$-ring R, let $n > 0$, and let $[n] : \mathsf{X} \to \mathsf{X}$ denote the map given by multiplication by n. We wish to show that the map $[n]$ is finite flat of degree n^{2g}. Using Remark 1.4.3, we can reduce to the case where R is connective. Using Corollary SAG.6.1.4.10, we can further reduce to the case where κ is a field. In this case, the desired result follows from Lemma 6.7.7. $\qquad \square$

Proof of Proposition 6.7.1. Let X be a strict abelian variety of dimension g over an $\mathbb{E}_\infty$-ring R and let S be a set of prime numbers; $\mathsf{X}[\mathsf{S}^\infty]$ satisfies conditions (a) and

(b) of Proposition 6.5.5, and that the associated S-divisible group has height $2g$. For each integer m, let $\mathsf{X}[m]$ denote the fiber of the map $[m] : \mathsf{X} \to \mathsf{X}$. Condition ($a$) is automatic. To verify (b) (and to show that the S-divisible group $\mathsf{X}[\mathrm{S}^\infty]$ has height $2g$), it will suffice to show that the natural map $\rho : \mathsf{X}[p^n] \to \mathsf{X}[p^{n-1}]$ is finite flat of degree p^{2g}, for each $n > 0$ and each $p \in \mathrm{S}$. Note that ρ fits into a commutative diagram of pullback squares

$$
\begin{array}{ccccc}
\mathsf{X}[p^n] & \longrightarrow & \mathsf{X}[p^{n-1}] & \longrightarrow & \mathrm{Sp\acute{e}t}\, R \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{X} & \xrightarrow{\;[p]\;} & \mathsf{X} & \xrightarrow{\;[p^{n-1}]\;} & \mathsf{X}.
\end{array}
$$

We are therefore reduced to showing that the map $[p] : \mathsf{X} \to \mathsf{X}$ is finite flat of degree p^{2g}, which is a special case of Proposition 6.7.3. $\qquad\square$

Remark 6.7.8. Let S be a set of prime numbers, lt R be an $\mathbb{E}_\infty$-ring, and let X be a strict abelian variety over R, which we identify with its functor of points $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Unwinding the definitions, we see that the S-divisible group $\mathsf{X}[\mathrm{S}^\infty]$ can be described by the formula

$$
\begin{aligned}
\mathsf{X}[\mathrm{S}^\infty](M)(A) & = \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \mathsf{X}(A)) \\
& \simeq \Omega^\infty(M^\vee \otimes_{\mathbf{Z}} \mathsf{X}(A)) \\
& \simeq \Omega^\infty(\Sigma^{-1} M^* \otimes_{\mathbf{Z}} \mathsf{X}(A)) \\
& \simeq \Omega^{\infty+1}(M^* \otimes_{\mathbf{Z}} \mathsf{X}(A)).
\end{aligned}
$$

Here $M^\vee$ denotes the dual of M in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$, and $M^* = \mathrm{Hom}(M, \mathbf{Q}/\mathbf{Z})$ denotes the Pontryagin dual of M (so that $M^\vee \simeq \Sigma^{-1} M^*$).

6.8 Comparison of Duality Theories

Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathrm{AVar}(R)$ denote the ∞-category of abelian varieties over R. Using Theorem 5.6.4, we see that the duality construction $\mathsf{X} \mapsto \widehat{\mathsf{X}}$ induces an equivalence of ∞-categories $\mathrm{AVar}(R) \to \mathrm{AVar}(R)^{\mathrm{op}}$. Combining this observation with Remark 1.2.6, we obtain an equivalence of ∞-categories

$$
D : \mathrm{Ab}(\mathrm{AVar}(R)) \simeq \mathrm{Ab}(\mathrm{AVar}(R)^{\mathrm{op}}) \simeq \mathrm{Ab}(\mathrm{AVar}(R))^{\mathrm{op}},
$$

given by the formula $(DX)(\Lambda) = \widehat{X(\Lambda^\vee)}$. Using the equivalence $\mathrm{AVar}^s(R) \simeq \mathrm{Ab}(\mathrm{AVar}(R))$ of Remark 1.5.2, we can identify D with an equivalence $\mathrm{AVar}^s(R) \to \mathrm{AVar}^s(R)^{\mathrm{op}}$. We can summarize the situation more informally as follows: if X is a

strict abelian variety over R, then the dual $\widehat{\mathsf{X}}$ inherits the structure of a strict abelian variety over R.

Remark 6.8.1. Let R be a connective $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R, so that the dual $\widehat{\mathsf{X}}$ also has the structure of a strict abelian variety over R. Let us identify X and $\widehat{\mathsf{X}}$ with their functors of points $\mathsf{X}, \widehat{\mathsf{X}} : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$. Unwinding the definitions, we see that the functor $\widehat{\mathsf{X}}$ is characterized by the formula

$$\mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \widehat{\mathsf{X}}(A)) = \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{CMon})}(\Omega^\infty(M \otimes_{\mathbf{Z}} \mathsf{X} \, |_{\mathrm{CAlg}_A^{\mathrm{cn}}}), \mathrm{BGL}_1 \, |_{\mathrm{CAlg}_A^{\mathrm{cn}}}).$$

Proposition 6.8.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let X be a strict abelian variety over R, so that the dual $\widehat{\mathsf{X}}$ inherits the structure of a strict abelian variety over R. For every set of prime numbers* S, *there is a canonical equivalence* $\widehat{\mathsf{X}}[\mathrm{S}^\infty] \simeq \mathbf{D}(\mathsf{X}[\mathrm{S}^\infty])$ *of* S-*divisible groups over R.*

Proof. Let us identify X and $\widehat{\mathsf{X}}$ with their functors of points $\mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}$. Let A be a connective $\mathbb{E}_\infty$-algebra over R, let X_A and $\widehat{\mathsf{X}}_A$ be the associated strict abelian varieties over A, and let M be a finite abelian S-group. To simplify the notation, let us not distinguish between the functor $\mathrm{BGL}_1 : \mathrm{CAlg}_R^{\mathrm{cn}} \to \mathrm{CMon}$ and its restriction to $\mathrm{CAlg}_A^{\mathrm{cn}}$. Using Remarks 6.8.1 and 6.7.8, we obtain a canonical map

$$
\begin{aligned}
\widehat{\mathsf{X}}[\mathrm{S}^\infty](M)(A) \quad &\simeq \quad \mathrm{Map}_{\mathrm{Mod}_{\mathbf{Z}}}(M, \widehat{\mathsf{X}}(A) \\
&\simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{CMon})}(\Omega^\infty(M \otimes_{\mathbf{Z}} \mathsf{X}_A), \mathrm{BGL}_1) \\
&\xrightarrow{\gamma} \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{CMon})}(\Omega^{\infty+1}(M \otimes_{\mathbf{Z}} \mathsf{X}_A), \mathrm{GL}_1) \\
&\simeq \quad \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{CMon})}(\mathsf{X}_A[p^\infty](M^*), \mathrm{GL}_1) \\
&\simeq \quad \mathbf{D}(\mathsf{X}[\mathrm{S}^\infty](M^*))(A) \\
&\simeq \quad \mathbf{D}(\mathsf{X}[\mathrm{S}^\infty])(M)(A)
\end{aligned}
$$

depending functorially on M and A. To complete the proof, it will suffice to show that the map γ is a homotopy equivalence. Since the functor BGL_1 satisfies descent for the finite flat topology, it will suffice to show that the the functor $M \otimes_{\mathbf{Z}} \mathsf{X}_A \in \mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}}, \mathrm{Mod}_{\mathbf{Z}})$ becomes 1-connective after sheafification with respect to the finite flat topology. Splitting M as a direct sum, we may assume that $M \simeq \mathbf{Z}/p^n\mathbf{Z}$ for some $n \geqslant 0$ and some prime number $p \in \mathrm{S}$. In this case, the desired result follows from the observation that the map $[p^n] : \mathsf{X}_A \to \mathsf{X}_A$ is a finite flat surjection (Proposition 6.7.3). $\qquad\square$

7 The Serre-Tate Theorem

Let p be a prime number, which we regard as fixed throughout this section. Let R be an $\mathbb{E}_\infty$-ring and let X be a strict abelian variety of dimension g over R. It follows from Proposition 6.7.1 that the associated p-torsion object $\mathsf{X}[p^\infty]$ is a p-divisible group of height $2g$ over R. The construction $\mathsf{X} \mapsto \mathsf{X}[p^\infty]$ determines a functor $\mathrm{AVar}_g^s(R) \to \mathrm{BT}_{2g}^p(R)$, depending functorially on R. Our goal in this section is to prove the following result:

Theorem 7.0.1 (Serre-Tate). *Let R be a connective $\mathbb{E}_\infty$-ring, let M be a connective R-module which is p-complete (Definition SAG.7.3.1.1), and let $\overline{R}$ be a square-zero extension of R by M (Definition HA.7.4.1.6). For every integer $g \geqslant 0$, the diagram of ∞-categories*

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(\overline{R}) & \longrightarrow & \mathrm{AVar}_g^s(R) \\
\downarrow & & \downarrow \\
\mathrm{BT}_{2g}^p(\overline{R}) & \longrightarrow & \mathrm{BT}_{2g}^p(R)
\end{array}
$$

is a pullback square.

Remark 7.0.2. In the special case where R and M are discrete and p is nilpotent in R (which guarantees that M is p-complete), Theorem 7.0.1 reduces to the classical Serre-Tate theorem.

We will prove Theorem 7.0.1 using an argument of Drinfeld (for an exposition in a more classical setting, see [4]). Roughly speaking, the idea is to use formal arguments to reduce to the case where $\overline{R}$ is a *trivial* square-zero extension of R by M, which we can analyze using an elaboration of Proposition 2.1.2.

7.1 Deformation Theory of the Functor $R \mapsto \mathrm{BT}^p(R)$

We begin by adapting some of the results of §2 to the setting of p-divisible groups. First, we need a bit of notation.

Notation 7.1.1. Let R be an $\mathbb{E}_\infty$-ring. We let $\mathrm{Var}^+(R)$ denote the full subcategory of $\mathrm{SpDM}_{/\mathrm{Sp\acute{e}t}\,R}^{\mathrm{nc}}$ spanned by those flat maps $f : \mathsf{X} = (\mathcal{X}, \mathscr{O}_{\mathsf{X}}) \to \mathrm{Sp\acute{e}t}\,R$ for which the underlying map of spectral Deligne-Mumford stacks $(\mathcal{X}, \tau_{\geqslant 0}\,\mathscr{O}_{\mathsf{X}}) \to \mathrm{Sp\acute{e}t}(\tau_{\geqslant 0} R)$ is proper and locally almost of finite presentation.

Note that any $\mathbb{E}_\infty$-ring R, we can identify the ∞-category $\mathrm{BT}^p(R)$ of p-divisible groups over R with a full subcategory of $\mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}, \mathrm{Var}^+(R))$.

Proposition 7.1.2. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X : (\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}} \to \mathrm{Var}^+(R)$ be a functor. The following conditions are equivalent:*

(a) *The functor X is a p-divisible group over R.*

(b) *For every residue field κ of R, the composite functor $(\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}} \xrightarrow{X} \mathrm{Var}^+(R) \to \mathrm{Var}^+(\kappa)$ is a p-divisible group over κ.*

Proof. The implication $(a) \Rightarrow (b)$ is obvious. Conversely, suppose that (b) is satisfied. Using Corollary SAG.6.1.4.12, we deduce that X is a p-torsion object of $\mathrm{SpDM}_R^{\mathrm{nc}}$. Let M be a finite abelian p-group and let $M' \subseteq M$ be a finite subgroup of index p^k. Combining (b) with Corollary SAG.6.1.4.10, we deduce that the map $\rho : X(M) \to X(M')$ is finite flat. $\qquad\square$

Remark 7.1.3. In the situation of Proposition 7.1.2, the p-divisible group X has height h if and only if each of the p-divisible groups $X_\kappa : (\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}} \xrightarrow{X} \mathrm{Var}^+(R) \to \mathrm{Var}^+(\kappa)$ has height h.

Proposition 7.1.4. *The construction $R \mapsto \mathrm{BT}^p(R)$ determines a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}\mathrm{at}_\infty$ which is nilcomplete and cohesive. Similarly, for each $h \geqslant 0$, the functor $R \mapsto \mathrm{BT}_h^p(R)$ is nilcomplete and cohesive.*

Proof. We will show that the functor $R \mapsto \mathrm{BT}^p(R)$ is cohesive; the proofs in the other cases are similar. Suppose we are given a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$
\begin{array}{ccc}
R & \longrightarrow & R_0 \\
\downarrow & & \downarrow \\
R_1 & \longrightarrow & R_{01}
\end{array}
$$

which induces surjections $\pi_0 R_0 \to \pi_0 R_{01} \leftarrow \pi_0 R_1$. Applying Theorem SAG.19.4.0.2, we deduce that the diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}, \mathrm{Var}^+(R)) & \longrightarrow & \mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}, \mathrm{Var}^+(R_0)) \\
\downarrow & & \downarrow \\
\mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}, \mathrm{Var}^+(R_1)) & \longrightarrow & \mathrm{Fun}((\mathrm{Ab}_{\mathrm{fin}}^p)^{\mathrm{op}}, \mathrm{Var}^+(R_{01}))
\end{array}
$$

is a pullback square. To complete the proof, it will suffice to show that if $X : (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \to \mathrm{Var}^+(R)$ is a functor having the property that the composite functors

$$X_0 : (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \xrightarrow{X} \mathrm{Var}^+(R) \to \mathrm{Var}^+(R_0) \qquad X_1 : (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \xrightarrow{X} \mathrm{Var}^+(R) \to \mathrm{Var}^+(R_1)$$

are p-divisible groups over R_0 and R_1, respectively, then X is a p-divisible group over R. By virtue of Proposition 7.1.2, it will suffice to show that for each residue field κ of R, the composite functor $(\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}} \xrightarrow{X} \mathrm{Var}^+(R) \to \mathrm{Var}^+(\kappa)$ is a p-divisible group over κ. This is clear, since our hypotheses guarantee that the map $R \to \kappa$ factors through either R_0 or R_1. $\qquad\square$

Proposition 7.1.5. *Let $h \geqslant 0$ be an integer and let K be a simplicial set. Define $F : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formulas $F(R) = \mathrm{Fun}(K, \mathrm{BT}^p(R))^{\simeq}$. Then the functor F has a (-1)-connective cotangent complex.*

Proof. Define $G : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ by the formula $G(R) = \mathrm{Fun}(K \times (\mathrm{Ab}^p_{\mathrm{fin}})^{\mathrm{op}}, \mathrm{Var}^+(R))^{\simeq}$. Note that for every connective $\mathbb{E}_\infty$-ring R, we can identify $F(R)$ with a summand of $G(R)$. Moreover, a point $\eta \in G(R)$ belongs to $F(R)$ if and only if, for every residue field κ of R, the image of η in $G(\kappa)$ belongs to $F(\kappa)$ (Proposition 7.1.2). It follows that if R is a connective $\mathbb{E}_\infty$-ring and M is a connective R-module, then the diagram

$$\begin{array}{ccc} F(R \oplus M) & \longrightarrow & F(R) \\ \downarrow & & \downarrow \\ G(R \oplus M) & \longrightarrow & G(R) \end{array}$$

is a pullback square. Consequently, if G admits a cotangent complex $L_G \in \mathrm{QCoh}(G)$, then the restriction $L_G|_F \in \mathrm{QCoh}(F)$ is a cotangent complex for F. We are therefore reduced to proving that the functor G admits a (-1)-connective cotangent complex, which is a special case of Theorem SAG.19.4.0.2 . $\qquad\square$

7.2 The Case of a Trivial Square-Zero Extension

We will deduce Theorem 7.0.1 from the following *a priori* weaker assertion:

Proposition 7.2.1. *Let R be a connective $\mathbb{E}_\infty$-ring, let M be a connective R-module which is p-complete, and let $g \geqslant 0$ be an integer. Then the canonical map*

$$\mathrm{AVar}^s_g(R) \to \mathrm{BT}^p_{2g}(R) \times_{\mathrm{BT}^p_{2g}(R \oplus M)} \mathrm{AVar}^s_g(R \oplus M)$$

is a fully faithful embedding of ∞-categories.

The proof of Proposition 7.2.1 will require a few preliminaries.

Notation 7.2.2. For every connective $\mathbb{E}_\infty$-ring A and every ∞-category $\mathcal{C}$, we let $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}_A^{\mathrm{cn}},\mathcal{C})$ spanned by those functors which are sheaves with respect to the fpqc topology. If X is a strict abelian variety over A and $X : \mathrm{CAlg}_A^{\mathrm{cn}} \to \mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}$ is its functor of points (see Remark 1.5.4), then X can be regarded as an object of the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$. In what follows, we will abuse terminology by identifying the ∞-category $\mathrm{AVar}^s(A)$ with its essential image under the fully faithful embedding

$$\mathrm{AVar}^s(A) \hookrightarrow \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}}) \qquad \mathsf{X} \mapsto X.$$

Lemma 7.2.3. *Let R be a connective $\mathbb{E}_\infty$-ring and let $X \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ be a strict abelian variety over R. Let M be a connective R-module. If M is p-complete, then the fiber of the projection map $X(R \oplus M) \to X(R)$ is p-complete.*

Proof. Using Proposition 2.1.2, we obtain an equivalence of spectra (not necessarily $\mathbf{Z}$-linear)

$$\mathrm{fib}(X(R \oplus M) \to X(R)) \simeq \tau_{\geqslant 0}\underline{\mathrm{Map}}_R(\omega, M)$$

for a certain connective A-module spectrum ω. If M is p-complete, then $\underline{\mathrm{Map}}_R(\omega, M)$ is also p-complete, so that the truncation $\tau_{\geqslant 0}\underline{\mathrm{Map}}_A(\omega, R)$ is likewise p-complete (see Corollary SAG.7.3.4.3). $\qquad\qquad\square$

The proof of Proposition 6.4.9 yields the following:

Lemma 7.2.4. *Let A be a connective $\mathbb{E}_\infty$-ring, let $X, Y \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ be strict abelian varieties of dimensions g over A and let $X[p^\infty], Y[p^\infty]$ denote their p-divisible groups. Then there is a canonical homotopy equivalence*

$$\mathrm{Map}_{\mathrm{BT}_{2g}^p(A)}(X[p^\infty], Y[p^\infty]) \simeq \varprojlim_n \mathrm{Map}_{\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(A;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})}(\mathrm{fib}(X \xrightarrow{p^n} X, Y).$$

Proof of Proposition 7.2.1. Let $X, Y \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ be strict abelian varieties of dimension g over R and let $X_M, Y_M \in \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R \oplus M;\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{cn}})$ denote their images in $\mathrm{AVar}_g^s(R \oplus M)$. We wish to show that the diagram of spaces σ_M :

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathrm{AVar}_g^s(R)}(X, Y) & \longrightarrow & \mathrm{Map}_{\mathrm{AVar}_g^s(R\oplus M)}(X_M, Y_M) \\
\downarrow & & \downarrow \\
\mathrm{Map}_{\mathrm{BT}_{2g}^p(R)}(X[p^\infty], Y[p^\infty]) & \longrightarrow & \mathrm{Map}_{\mathrm{BT}_{2g}^p(R\oplus M)}(X_M[p^\infty], Y_M[p^\infty])
\end{array}
$$

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is a pullback square.

It follows from Propositions 2.3.2 and 7.1.5 that the construction $M \mapsto \sigma_M$ commutes with filtered limits. For each $n \geqslant 0$, let M_n denote the cofiber of the map $p^n : M \to M$. Since M is p-complete, it is the limit of the tower $\{M_n\}$. It will therefore suffice to show that each σ_{M_n} is a pullback square. Note that there is an equivalence of spectra $M_n \simeq M \otimes (S/p^n)$, where S/p^n denotes the Moore spectrum (given by the cofiber of the map $p^n : S \to S$). A simple calculation shows that the multiplication map $p^k : S/p^n \to S/p^n$ is nullhomotopic for $k \gg 0$, so that $p^k : M_n \to M_n$ is nullhomotopic for $k \gg 0$. Replacing M by M_n, we may reduce to the case where $p^k : M \to M$ is nullhomotopic for $k \gg 0$.

Let $\widehat{\mathcal{S}}$ denote the ∞-category of spaces which are not necessarily small, and let us regard $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R; \widehat{\mathcal{S}})$ and $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R \oplus M; \widehat{\mathcal{S}})$ as ∞-topoi after a change of universe. The projection map $R \oplus M \to R$ induces a geometric morphism $q_* : \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R \oplus M; \widehat{\mathcal{S}}) \to \mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R; \widehat{\mathcal{S}})$, with a left adjoint q^*. Let $\mathcal{C}$ denote the ∞-category of abelian group objects of $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R; \widehat{\mathcal{S}})$ and $\mathcal{C}'$ the ∞-category of abelian group objects of $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R \oplus M; \widehat{\mathcal{S}})$. The functors q^* and q_* are both left exact, and therefore determine adjoint functors $\mathcal{C} \underset{q_*}{\overset{q^*}{\rightleftarrows}} \mathcal{C}'$. Unwinding the definitions, we have equivalences $X_M \simeq q^*X$ and $Y_M \simeq q^*Y$, hence canonical homotopy equivalences

$$\mathrm{Map}_{\mathrm{AVar}^s(R \oplus M)}(X_M, Y_M) \simeq \mathrm{Map}_{\mathcal{C}}(X, q_* q^* Y)$$

$$\mathrm{Map}_{\mathcal{C}'}(\mathrm{fib}(X_M \xrightarrow{p^n} X_M), Y_M) \simeq \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(X \xrightarrow{p^n} X), q_* q^* Y).$$

Using Lemma 7.2.4, we can identify σ_M with the diagram

$$
\begin{array}{ccc}
\mathrm{Map}_{\mathcal{C}}(X, Y) & \longrightarrow & \mathrm{Map}_{\mathcal{C}}(X, q_* q^* Y) \\
\downarrow & & \downarrow \\
\varprojlim_n \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(X \xrightarrow{p^n} X), Y) & \longrightarrow & \varprojlim_n \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(X \xrightarrow{p^n} X), q_* q^* Y).
\end{array}
$$

For every connective R-module N, define $E_N \in \mathcal{C}$ by the formula $E_N(A) = \mathrm{fib}(Y(A \oplus (A \otimes_R N)) \to Y(A))$. Note that if N has the property that $p^k : N \to N$ is nullhomotopic for $k \gg 0$, then $A \otimes_R N$ has the same property, and is therefore p-complete. It then follows from Lemma 7.2.3 that $E_N(A)$ is also p-complete (when viewed as an object of $\mathrm{Mod}_{\mathbf{Z}}$). Since the projection $R \oplus M \to R$ admits a section, the unit map $Y \to q_* q^* Y$ admits a left homotopy inverse, giving a splitting $q_* q^* Y \simeq Y \oplus E_M$. Consequently, to show that σ_M is a pullback square, it will suffice to show

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that the canonical map

$$\phi : \mathrm{Map}_{\mathcal{C}}(X, E_M) \to \varprojlim_n \mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(X \xrightarrow{p^n} X), E_M)$$

is a homotopy equivalence.

Let $\mathcal{E} = \mathrm{Sp}(\mathcal{C})$ denote the ∞-category of spectrum objects of $\mathcal{C}$, which we can identify with the ∞-category $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}(R, \widehat{\mathrm{Mod}}_{\mathbf{Z}})$, where $\widehat{\mathrm{Mod}}_{\mathbf{Z}}$ denotes the ∞-category of (not necessarily small) $\mathbf{Z}$-module spectra. Then $\mathcal{E}$ is a stable ∞-category equipped with a t-structure $(\mathcal{E}_{\geqslant 0}, \mathcal{E}_{\leqslant 0})$, and there is a fully faithful embedding $\iota : \mathcal{C} \to \mathcal{E}$ whose essential image is $\mathcal{E}_{\geqslant 0}$. Using Proposition 2.1.2, we deduce that the canonical map $E_M \to \Omega E_{\Sigma M}$ is an equivalence, and therefore induces an equivalence $\iota(E_M) \simeq \tau_{\geqslant 0}\Omega(\iota E_{\Sigma M})$ in the ∞-category $\mathcal{E}$. We may therefore identify ϕ with the canonical map

$$\mathrm{Map}_{\mathcal{E}}(\iota X, \Omega(\iota E_{\Sigma M})) \to \mathrm{Map}_{\mathcal{E}}(\varinjlim_n \iota \, \mathrm{fib}(X \xrightarrow{p^n} X), \Omega(\iota E_{\Sigma M})).$$

Since X is a strict abelian variety over R, multiplication by p^n induces a faithfully flat map from X to itself, and is therefore an effective epimorphism with respect to the flat topology (Proposition 6.7.3). It follows that the canonical map

$$\iota \, \mathrm{fib}(X \xrightarrow{p^n} X) \to \mathrm{fib}(\iota X \xrightarrow{p^n} \iota X)$$

is an equivalence for every integer n. Let $X[\frac{1}{p}]$ denote the colimit of the sequence

$$X \xrightarrow{p} X \xrightarrow{p} \cdots$$

in $\mathcal{C}$, so that we have a fiber sequence $\varinjlim_n \mathrm{fib}(\iota X \xrightarrow{p^n} \iota X) \to \iota X \to \iota X[p^{-1}]$ in the stable ∞-category $\mathcal{E}$ and therefore a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{E}}(\iota X, \Omega(\iota E_{\Sigma M})) \xrightarrow{\phi} \mathrm{Map}_{\mathcal{E}}(\varinjlim_n \mathrm{fib}(\iota X \xrightarrow{p^n} \iota X), \Omega(\iota E_{\Sigma M}) \to \mathrm{Map}_{\mathcal{E}}(\iota X[\frac{1}{p}], \iota E_{\Sigma M}).$$

To complete the proof, it will suffice to show that the space

$$K = \mathrm{Map}_{\mathcal{E}}(\iota X[p^{-1}], \iota E_{\Sigma M}) \simeq \mathrm{Map}_{\mathcal{C}}(X[p^{-1}], E_{\Sigma M}).$$

This is clear, since $X[p^{-1}]$ is p-local and $E_{\Sigma M}$ is p-complete. $\qquad \square$

7.3 Proof of the Serre-Tate Theorem

We now explain how to deduce Theorem 7.0.1 from Proposition 7.2.1. Let $\overline{R}$ be a square-zero extension of R by a connective R-module M, so that we have a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} \overline{R} & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & R \oplus \Sigma M, \end{array}$$

where the bottom horizontal map is the tautological map from R into the trivial square-zero extension $R \oplus \Sigma M$. We wish to show that the upper square appearing in the diagram

$$\begin{array}{ccc} \mathrm{AVar}_g^s(\overline{R}) & \longrightarrow & \mathrm{AVar}_g^s(R) \\ \downarrow & & \downarrow \\ \mathrm{BT}_{2g}^p(\overline{R}) & \longrightarrow & \mathrm{BT}_{2g}^p(R) \\ \downarrow & & \downarrow \\ \mathrm{BT}_{2g}^p(R) & \longrightarrow & \mathrm{BT}_{2g}^p(R \oplus \Sigma M) \end{array}$$

is a pullback. Using Proposition 7.1.4, we deduce that the lower square is a pullback; it will therefore suffice to show that the outer rectangle is a pullback. Consider the diagram

$$\begin{array}{ccc} \mathrm{AVar}_g^s(\overline{R}) & \longrightarrow & \mathrm{AVar}_g^s(R) \\ \downarrow & & \downarrow \\ \mathrm{AVar}_g^s(R) & \longrightarrow & \mathrm{AVar}_g^s(R \oplus \Sigma M) \\ \downarrow & & \downarrow \\ \mathrm{BT}_{2g}^p(R) & \longrightarrow & \mathrm{BT}_{2g}^p(R \oplus \Sigma M). \end{array}$$

Since the upper square is a pullback by Proposition 2.1.4, we are reduced to proving that the lower square is a pullback. In other words, we wish to show that the canonical functor

$$\phi : \mathrm{AVar}_g^s(R) \to \mathrm{AVar}^s(R \oplus \Sigma M) \times_{\mathrm{BT}_{2g}^p(R \oplus \Sigma M)} \mathrm{BT}_{2g}^p(R)$$

is an equivalence of ∞-categories.

Let $\mathcal{C}$ denote the fiber product $\mathrm{AVar}_g^s(R) \times_{\mathrm{BT}_{2g}^p(R)} \mathrm{BT}_{2g}^p(R \oplus \Sigma M)$, so that we have a commutative diagram

$$
\begin{array}{ccccc}
\mathrm{AVar}^s(R) & \longrightarrow & \mathcal{C} & \longrightarrow & \mathrm{AVar}^s(R) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BT}^p(R) & \longrightarrow & \mathrm{BT}^p(R \oplus \Sigma M) & \longrightarrow & \mathrm{BT}^p(R).
\end{array}
$$

The outer rectangle in this diagram is a pullback square (since the horizontal composite maps are equivalences), and the right square is a pullback by construction. It follows that the left square is a pullback, so that the composite functor

$$
\mathrm{AVar}^s(R) \xrightarrow{\phi} \mathrm{AVar}^s(R \oplus \Sigma M) \times_{\mathrm{BT}^p(R \oplus \Sigma M)} \mathrm{BT}^p(R) \xrightarrow{\psi} \mathcal{C} \times_{\mathrm{BT}^p(R \oplus \Sigma M)} \mathrm{BT}^p(R)
$$

is an equivalence of ∞-categories. To prove that ϕ is an equivalence of ∞-categories, it will suffice to show that ψ is fully faithful. The functor ψ is a pullback of the natural map $\psi_0 : \mathrm{AVar}^s(R \oplus \Sigma M) \to \mathcal{C}$. We are therefore reduced to proving that ψ_0 is fully faithful.

Note that $R \oplus \Sigma M$ is a square-zero extension of R by $\Sigma^2 M$, so we have a pullback diagram of connective $\mathbb{E}_\infty$-rings

$$
\begin{array}{ccc}
R \oplus \Sigma M & \longrightarrow & R \\
\downarrow & & \downarrow \\
R & \longrightarrow & R \oplus \Sigma^2 M,
\end{array}
$$

and therefore a commutative diagram of ∞-categories

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(R \oplus \Sigma M) & \longrightarrow & \mathrm{AVar}_g^s(R) \\
\downarrow & & \downarrow \\
\mathrm{BT}_{2g}^p(R \oplus \Sigma M) & \longrightarrow & \mathrm{BT}_{2g}^p(R) \\
\downarrow & & \downarrow \\
\mathrm{BT}_{2g}^p(R) & \longrightarrow & \mathrm{BT}_{2g}^p(R \oplus \Sigma^2 M).
\end{array}
$$

Proposition 7.1.4 implies that the lower square in this diagram is a pullback, so that we can identify ψ_0 with the functor $\mathrm{AVar}^s(R \oplus \Sigma M) \to \mathrm{BT}^p(R) \times_{\mathrm{BT}^p(R \oplus \Sigma^2 M)} \mathrm{AVar}^s(R)$ determined by the outer rectangle. This outer rectangle appears also in the

commutative diagram

$$\begin{array}{ccc}
\mathrm{AVar}^s(R \oplus \Sigma M) & \longrightarrow & \mathrm{AVar}^s(R) \\
\downarrow & & \downarrow \\
\mathrm{AVar}^s(R) & \longrightarrow & \mathrm{AVar}^s(R \oplus \Sigma^2 M) \\
\downarrow & & \downarrow \\
\mathrm{BT}^p(R) & \longrightarrow & \mathrm{BT}^p(R \oplus \Sigma^2 M)
\end{array}$$

where the upper square is a pullback (Proposition 2.1.4). It follows that ψ_0 is a pullback of the functor $\psi_1 : \mathrm{AVar}^s(R) \to \mathrm{BT}^p(R) \times_{\mathrm{BT}^p(R \oplus \Sigma^2 M)} \mathrm{AVar}^s(R \oplus \Sigma^2 M)$ determined by the lower square. We are therefore reduced to showing that ψ_1 is fully faithful, which is a special case of Proposition 7.2.1.

7.4 Application: Lifting Abelian Varieties from Classical to Spectral Algebraic Geometry

We close this section by noting the following consequence of Theorem 7.0.1:

Proposition 7.4.1. *Let $f : R \to R'$ be a map of $\mathbb{E}_\infty$-rings and p a prime number. Assume that f induces an isomorphism of commutative rings $\pi_0 R \to \pi_0 R'$, and that the abelian groups $\pi_i R$ and $\pi_i R'$ are p-complete for each $i > 0$. Then, for every integer $g \geq 0$, the diagram of ∞-categories σ :*

$$\begin{array}{ccc}
\mathrm{AVar}_g^s(R) & \longrightarrow & \mathrm{AVar}_g^s(R') \\
\downarrow & & \downarrow \\
\mathrm{BT}_{2g}^p(R) & \longrightarrow & \mathrm{BT}_{2g}^p(R')
\end{array}$$

is a pullback square.

Proof. We have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{AVar}_g^s(\tau_{\geq 0} R) & \longrightarrow & \mathrm{AVar}_g^s(R) & \longrightarrow & \mathrm{AVar}_g^s(R') \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BT}_{2g}^p(\tau_{\geq 0} R) & \longrightarrow & \mathrm{BT}_{2g}^p(R) & \longrightarrow & \mathrm{BT}_{2g}^p(R')
\end{array}$$

where the horizontal maps on the left are equivalences (Remarks 1.5.3 and 6.5.3). We may therefore replace R by $\tau_{\geq 0} R$ and thereby reduce to the case where R is connective.

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In this case, f factors through the connective cover of R', so we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(R) \longrightarrow \mathrm{AVar}_g^s(\tau_{\geqslant 0} R') \longrightarrow \mathrm{AVar}_g^s(R') \\
\downarrow \qquad\qquad \downarrow \qquad\qquad\quad \downarrow \\
\mathrm{BT}_{2g}^p(R) \longrightarrow \mathrm{BT}_{2g}^p(\tau_{\geqslant 0} R') \longrightarrow \mathrm{BT}_{2g}^p(R').
\end{array}
$$

Remarks 1.5.3 and 6.5.3 now imply that the right horizontal maps are equivalences, so we may replace R' by $\tau_{\geqslant 0} R'$ and thereby reduce to the case where R' is also connective. Let R'' denote the discrete commutative ring $\pi_0 R'$, so that we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(R) \longrightarrow \mathrm{AVar}_g^s(R') \longrightarrow \mathrm{AVar}_g^s(R'') \\
\downarrow \qquad\qquad \downarrow \qquad\qquad\quad \downarrow \\
\mathrm{BT}_{2g}^p(R) \longrightarrow \mathrm{BT}_{2g}^p(R') \longrightarrow \mathrm{BT}_{2g}^p(R'').
\end{array}
$$

To prove that the left square is a pullback, it will suffice to show that the right square is a pullback and that the outer rectangle is a pullback. It will therefore suffice to prove Proposition 7.4.1 under the additional assumption that R' is discrete (so that f exhibits R' as the 0-truncation of R).

Using Propositions 2.1.5 and 7.1.4, we deduce deduce that σ is the limit of a tower of diagrams σ_n :

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(\tau_{\leqslant n} R) \longrightarrow \mathrm{AVar}_g^s(R') \\
\downarrow \qquad\qquad\quad \downarrow \\
\mathrm{BT}_{2g}^p(\tau_{\leqslant n} R) \longrightarrow \mathrm{BT}_{2g}^p(R').
\end{array}
$$

It will therefore suffice to show that each σ_n is a pullback square. The proof proceeds by induction on n. When $n = 0$, this is trivial (the horizontal maps are equivalences, by virtue of our assumption that f induces an isomorphism $\pi_0 R \to \pi_0 R'$). If $n > 0$, we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{AVar}_g^s(\tau_{\leqslant n} R) \longrightarrow \mathrm{AVar}_g^s(\tau_{\leqslant n-1} R) \longrightarrow \mathrm{AVar}_g^s(R') \\
\downarrow \qquad\qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \\
\mathrm{BT}_{2g}^p(\tau_{\leqslant n} R) \longrightarrow \mathrm{BT}_{2g}^p(\tau_{\leqslant n-1} R) \longrightarrow \mathrm{BT}_{2g}^p(R')
\end{array}
$$

where the right square is a pullback by the inductive hypothesis. It will therefore suffice to prove that the left square is a pullback. This is a special case of Theorem

7.0.1, since $\tau_{\leqslant n}R$ is a square-zero extension of $\tau_{\leqslant n-1}R$ by the p-complete R-module $\Sigma^n(\pi_n R)$ (see Theorem HA.7.4.1.26). $\qquad\square$

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