

# ENHANCED SIX OPERATIONS AND BASE CHANGE THEOREM FOR HIGHER ARTIN STACKS

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**ABSTRACT.** In this article, we develop a theory of Grothendieck’s six operations for derived categories in étale cohomology of Artin stacks. We prove several desired properties of the operations, including the base change theorem in derived categories. This extends all previous theories on this subject, including the recent one developed by Laszlo and Olsson, in which the operations are subject to more assumptions and the base change isomorphism is only constructed on the level of sheaves. Moreover, our theory works for higher Artin stacks as well.

Our method differs from all previous approaches, as we exploit the theory of stable  $\infty$ -categories developed by Lurie. We enhance derived categories, functors, and natural isomorphisms to the level of  $\infty$ -categories and introduce  $\infty$ -categorical (co)homological descent. To handle the “homotopy coherence”, we apply the results of our previous article [LZa] and develop several other  $\infty$ -categorical techniques.

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## INTRODUCTION

Derived categories in étale cohomology on Artin stacks and Grothendieck’s six operations between such categories have been developed by many authors including [Zhe15] (for Deligne–Mumford stacks), [LMB00], [Beh03], [Ols07] and [LO08]. These theories all have some restrictions. In the most recent and general one [LO08] by Laszlo and Olsson on Artin stacks, a technical condition was imposed on the base scheme which excludes, for example, the spectra of certain fields<sup>1</sup>. More importantly, the base change isomorphism was constructed only on the level of (usual) cohomology sheaves [LO08, §5]. The Base Change theorem is fundamental in many applications. In the Geometric Langlands Program for example, the theorem has already been used on the level of perverse cohomology. It is thus necessary to construct the Base Change isomorphism not just on the level of cohomology, but also in the derived category. Another limitation of most previous works is that they dealt only with constructible sheaves. When working with morphisms *locally* of finite type, it is desirable to have the six operations for more general lisse-étale sheaves.

In this article, we develop a theory that provides the desired extensions of previous works. Instead of the usual unbounded derived category, we work with its enhancement, which is a stable  $\infty$ -category in the sense of Lurie [HA, 1.1.1.9]. This makes our approach different from all previous ones. We construct functors and produce relations in the world of  $\infty$ -categories, which themselves form an  $\infty$ -category. We start by

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<sup>1</sup>For example, the field  $k(x_1, x_2, \dots)$  obtained by adjoining countably infinitely many variables to an algebraically closed field  $k$  in which  $\ell$  is invertible.

upgrading the known theory of six operations for (coproducts of) quasi-compact and separated schemes to  $\infty$ -categories. The coherence of the construction is carefully recorded. This enables us to apply  $\infty$ -categorical descent to carry over the theory of six operations, including the Base Change theorem, to algebraic spaces, higher Deligne–Mumford stacks and higher Artin stacks.

**0.1. Results.** In this section, we will state our results only in the classical setting of Artin stacks on the level of usual derived categories (which are homotopy categories of the derived  $\infty$ -categories), among other simplification. We refer the reader to Chapter 6 for a list of complete results for higher Deligne–Mumford stacks and higher Artin stacks, stated on the level of stable  $\infty$ -categories.

By an *algebraic space*, we mean a sheaf in the big fppf site satisfying the usual axioms [SP, 025Y]: its diagonal is representable (by schemes); and it admits an étale and surjective map from a scheme (in  $\mathrm{Sch}_{\mathcal{U}}$ ; see §0.5).

By an *Artin stack*, we mean an algebraic stack in the sense of [SP, 026O]: it is a stack in (1-)groupoids over  $(\mathrm{Sch}_{\mathcal{U}})_{\mathrm{fppf}}$ ; its diagonal is representable by algebraic spaces; and it admits a smooth and surjective map from a scheme. In particular, we do not assume that an Artin stack is quasi-separated. Our main results are the construction of the six operations for the derived categories of lisse-étale sheaves on Artin stacks and the expected relations among them. In what follows,  $\Lambda$  is a unital commutative ring, or more generally, a ringed diagram in Definition 2.2.5.

Let  $\mathcal{X}$  be an Artin stack. We denote by  $D(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  the unbounded derived category of  $(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ -modules, where  $\mathcal{X}_{\mathrm{lis-ét}}$  is the lisse-étale topos associated to  $\mathcal{X}$ . Recall that an  $(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ -module  $\mathcal{F}$  is equivalent to an assignment to each smooth morphism  $v: Y \rightarrow \mathcal{X}$  with  $Y$  an algebraic space a  $(Y_{\mathrm{ét}}, \Lambda)$ -module  $\mathcal{F}_v$  and to each 2-commutative triangle

$$\begin{array}{ccc} Y' & \xrightarrow{f} & Y \\ & \searrow v' & \swarrow v \\ & \mathcal{X} & \end{array}$$

with  $v, v'$  smooth and  $Y, Y'$  being algebraic spaces, a morphism  $\tau_\sigma: f^* \mathcal{F}_v \rightarrow \mathcal{F}_{v'}$  that is an isomorphism if  $f$  is étale, such that the collection  $\{\tau_\sigma\}$  satisfies a natural cocycle condition [LMB00, 12.2.1]. An  $(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ -module  $\mathcal{F}$  is *Cartesian* if in the above description, *all* morphisms  $\tau_\sigma$  are isomorphisms [LMB00, 12.3].

Let  $D_{\mathrm{cart}}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  be the full subcategory of  $D(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  spanned by complexes whose cohomology sheaves are all Cartesian. If  $\mathcal{X}$  is Deligne–Mumford, then we have an equivalence of categories  $D_{\mathrm{cart}}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda) \simeq D(\mathcal{X}_{\mathrm{ét}}, \Lambda)$ .

Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of Artin stacks. We define the following four operations in §6.2:

$$\begin{aligned} f^*: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda), \\ f_*: D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ - \otimes_{\mathcal{X}} -: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \times D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ \mathcal{H}\text{om}_{\mathcal{X}}: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

The pairs  $(f^*, f_*)$  and  $(- \otimes_{\mathcal{X}} K, \mathcal{H}\text{om}_{\mathcal{X}}(K, -))$  for every  $K \in D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  are pairs of adjoint functors.

To state the other two operations, we fix a nonempty set  $\square$  of rational primes. A ring is  $\square$ -torsion [SGA4, IX 1.1] if each element of it is killed by an integer that is a product of primes in  $\square$ . An Artin stack  $\mathcal{X}$  is  $\square$ -coprime if there exists a morphism  $\mathcal{X} \rightarrow \text{Spec } \mathbb{Z}[\square^{-1}]$ . If  $\mathcal{X}$  and  $\mathcal{Y}$  are  $\square$ -coprime (resp. Deligne–Mumford),  $f: \mathcal{Y} \rightarrow \mathcal{X}$  is locally of finite type, and  $\Lambda$  is  $\square$ -torsion (resp. torsion), then we have another pair of adjoint functors:

$$\begin{aligned} f_!: D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ f^!: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

Next we list some properties of the six operations. We refer the reader to §6.2 for a more complete list.

**Theorem 0.1.1** (Künneth Formula, Theorem 6.2.1). *Let  $\Lambda$  be a  $\square$ -torsion (resp. torsion) ring, and*

$$\begin{array}{ccccc} \mathcal{Y}_1 & \xleftarrow{q_1} & \mathcal{Y} & \xrightarrow{q_2} & \mathcal{Y}_2 \\ f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\ \mathcal{X}_1 & \xleftarrow{p_1} & \mathcal{X} & \xrightarrow{p_2} & \mathcal{X}_2 \end{array}$$

*a diagram of  $\square$ -coprime Artin stacks (resp. of arbitrary Deligne–Mumford stacks) that exhibits  $\mathcal{Y}$  as the limit  $\mathcal{Y}_1 \times_{\mathcal{X}_1} \mathcal{X} \times_{\mathcal{X}_2} \mathcal{Y}_2$ , where  $f_1$  and  $f_2$  are locally of finite type. Then we have a natural isomorphism of functors:*

$$\begin{aligned} f_!(q_1^* - \otimes_{\mathcal{Y}} q_2^* -) &\simeq (p_1^* f_{1!} -) \otimes_{\mathcal{X}} (p_2^* f_{2!} -): \\ D_{\text{cart}}(\mathcal{Y}_{1, \text{lis-ét}}, \Lambda) \times D_{\text{cart}}(\mathcal{Y}_{2, \text{lis-ét}}, \Lambda) &\rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

It has the following two corollaries.

**Corollary 0.1.2** (Base Change). *Let  $\Lambda$  be a  $\square$ -torsion (resp. a torsion) ring, and*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Z} \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

a Cartesian diagram of  $\square$ -coprime Artin stacks (resp. of arbitrary Deligne–Mumford stacks) where  $p$  is locally of finite type. Then we have a natural isomorphism of functors:

$$f^* \circ p_! \simeq q_! \circ g^*: D_{\text{cart}}(\mathcal{Z}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda).$$

**Corollary 0.1.3** (Projection Formula). *Let  $\Lambda$  be a  $\square$ -torsion (resp. torsion) ring, and  $f: \mathcal{Y} \rightarrow \mathcal{X}$  a morphism locally of finite type of  $\square$ -coprime Artin stacks (resp. of arbitrary Deligne–Mumford stacks). Then we have a natural isomorphism of functors:*

$$f_!(- \otimes_{\mathcal{Y}} f^* -) \simeq (f_! -) \otimes_{\mathcal{X}} -: D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \times D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda).$$

**Theorem 0.1.4** (Trace map and Poincaré duality, Theorem 6.2.9). *Let  $\Lambda$  be a  $\square$ -torsion ring, and  $f: \mathcal{Y} \rightarrow \mathcal{X}$  a flat morphism locally of finite presentation of  $\square$ -coprime Artin stacks. Then*

(1) *There is a functorial trace map*

$$\text{Tr}_f: \tau^{\geq 0} f_! \Lambda_{\mathcal{Y}} \langle d \rangle = \tau^{\geq 0} f_!(f^* \Lambda_{\mathcal{X}}) \langle d \rangle \rightarrow \Lambda_{\mathcal{X}},$$

*where  $d$  is an integer larger than or equal to the dimension of every geometric fiber of  $f$ ;  $\Lambda_{\mathcal{X}}$  and  $\Lambda_{\mathcal{Y}}$  denote the constant sheaves placed in degree 0; and  $\langle d \rangle = [2d](d)$  is the composition of the shift by  $2d$  and the  $d$ -th power of Tate’s twist.*

(2) *If  $f$  is moreover smooth, then the induced natural transformation*

$$u_f: f_! \circ f^* \langle \dim f \rangle \rightarrow \text{id}_{\mathcal{X}}$$

*is a counit transformation, where  $\text{id}_{\mathcal{X}}$  is the identity functor of  $D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ . In other words, we have a natural isomorphism of functors:*

$$f^* \langle \dim f \rangle \simeq f^!: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda).$$

**Corollary 0.1.5** (Smooth Base Change, Corollary 6.2.10). *Let  $\Lambda$  of a  $\square$ -torsion ring, and*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Z} \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

*a Cartesian diagram of  $\square$ -coprime Artin stacks where  $p$  is smooth. Then the natural transformation of functors*

$$p^* f_* \rightarrow g_* q^*: D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{Z}_{\text{lis-ét}}, \Lambda)$$

*is a natural isomorphism.*

**Theorem 0.1.6** (Descent, Corollary 6.2.14). *Let  $\Lambda$  be a ring,  $f: \mathcal{Y} \rightarrow \mathcal{X}$  a morphism of Artin stacks, and  $y: \mathcal{Y}_0^+ \rightarrow \mathcal{Y}$  a smooth surjective morphism. Let  $\mathcal{Y}_{\bullet}^+$  be the Čech nerve of  $y$  with the morphism  $y_n: \mathcal{Y}_n^+ \rightarrow \mathcal{Y}_{n-1}^+ = \mathcal{Y}$ . Put  $f_n = f \circ y_n: \mathcal{Y}_n^+ \rightarrow \mathcal{X}$ .*

(1) *For every complex  $K \in D^{\geq 0}(\mathcal{Y}, \Lambda)$ , we have a convergent spectral sequence*

$$E_1^{p,q} = H^q(f_{p*} y_p^* K) \Rightarrow H^{p+q} f_* K.$$

(2) If  $\mathcal{X}$  is  $\square$ -coprime;  $\Lambda$  is  $\square$ -torsion; and  $f$  is locally of finite type, then for every complex  $\mathbf{K} \in D^{\leq 0}(\mathcal{Y}, \Lambda)$ , we have a convergent spectral sequence

$$\tilde{E}_1^{p,q} = H^q(f_{-p!} y_{-p}^! \mathbf{K}) \Rightarrow H^{p+q} f_! \mathbf{K}.$$

*Remark 0.1.7.* Note that even in the case of schemes, Theorem 0.1.6 (2) seems to be a new result.

To state our results for constructible sheaves, we work over a  $\square$ -coprime base scheme  $\mathbb{S}$  that is either quasi-excellent finite-dimensional or regular of dimension  $\leq 1$ . We consider only Artin stacks  $\mathcal{X}$  that are locally of finite type over  $\mathbb{S}$ . Let  $\Lambda$  be a Noetherian  $\square$ -torsion ring. Recall that an  $(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ -module is *constructible* if it is Cartesian and its pullback to every scheme, finite type over  $\mathbb{S}$ , is constructible in the usual sense. Let  $D_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  be the full subcategory of  $D(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  spanned by complexes whose cohomology sheaves are constructible. Let  $D_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  (resp.  $D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$ ) be the full subcategory of  $D_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  spanned by complexes whose cohomology sheaves are locally bounded from below (resp. from above). The six operations mentioned previously restrict to the following refined ones as in §6.4 (see Propositions 6.4.4 and 6.4.5 for precise statements):

$$\begin{aligned} f^* &: D_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda), \\ f^! &: D_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda), \\ - \otimes_{\mathcal{X}} - &: D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \times D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ \mathcal{H}om_{\mathcal{X}} &: D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times D_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

If  $f$  is *quasi-compact and quasi-separated*, then we have

$$\begin{aligned} f_* &: D_{\text{cons}}^{(+)}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ f_! &: D_{\text{cons}}^{(-)}(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

We will also show that when the base scheme, the coefficient ring, and the morphism  $f$  are all in the range of [LO08], our operations for constructible complexes are compatible with those constructed by Laszlo and Olsson on the level of usual derived categories. In particular, Corollary 0.1.2 implies that their operations satisfy Base Change in derived categories, which was left open in [LO08].

In [LZb], we will develop an adic formalism and establish adic analogues of the above results. Let  $\lambda = (\Xi, \Lambda)$  be a partially ordered diagram of coefficient rings, that is,  $\Xi$  is a partially ordered set and  $\Lambda$  is a functor from  $\Xi^{op}$  to the category of commutative rings (with units). A typical example is the projective system

$$\cdots \rightarrow \mathbb{Z}/\ell^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/\ell^n\mathbb{Z} \rightarrow \cdots \rightarrow \mathbb{Z}/\ell\mathbb{Z},$$

where  $\ell$  is a fixed prime number and the transition maps are natural projections. We define the adic derived category  $D(X, \lambda)_a$  to be the “limit” of the diagram  $\xi \mapsto D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda(\xi))$  indexed by  $\Xi^{op}$ , where the limit is taken in certain  $\infty$ -categorical sense. We will show that  $D(X, \lambda)_a$  is canonically equivalent to a full subcategory of  $D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}^{\Xi}, \Lambda)$ .

**0.2. Why  $\infty$ -categories?** The  $\infty$ -categories in this article refer to the ones studied by A. Joyal [Joy02, Joy] (where they are called *quasi-categories*), J. Lurie [HTT], et al. Namely, an  $\infty$ -category is a simplicial set satisfying lifting properties of inner horn inclusions [HTT, 1.1.2.4]. In particular, they are models for  $(\infty, 1)$ -categories, that is, higher categories whose  $n$ -morphisms are invertible for  $n \geq 2$ . For readers who are not familiar with this language, we recommend [Gro] for a brief introduction of Lurie’s theory [HTT], [HA], etc. There are also other models for  $(\infty, 1)$ -categories such as topological categories, simplicial categories, complete Segal spaces, Segal categories, model categories, and, in a looser sense, differential graded (DG) categories and  $A_\infty$ -categories. We address two questions in this section. First, why do we need  $(\infty, 1)$ -categories instead of (usual) derived categories? Second, why do we choose this particular model of  $(\infty, 1)$ -categories?

To answer these questions, let us fix an Artin stack  $\mathcal{X}$  and an atlas  $u: X \rightarrow \mathcal{X}$ , that is, a smooth and surjective morphism with  $X$  an algebraic space. We denote by  $\mathrm{Mod}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  (resp.  $\mathrm{Mod}(X_{\mathrm{ét}}, \Lambda)$ ) the category of  $(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ -modules (resp.  $(X_{\mathrm{ét}}, \Lambda)$ -modules) which is a Grothendieck Abelian category. Let  $p_\alpha: X \times_{\mathcal{X}} X \rightarrow X$  ( $\alpha = 1, 2$ ) be the two projections. We know that if  $\mathcal{F} \in \mathrm{Mod}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  is Cartesian, then there is a natural isomorphism  $\sigma: p_1^* u^* \mathcal{F} \xrightarrow{\sim} p_2^* u^* \mathcal{F}$  satisfying a cocycle condition. Conversely, an object  $\mathcal{G} \in \mathrm{Mod}(X_{\mathrm{ét}}, \Lambda)$  such that there exists an isomorphism  $\sigma: p_1^* \mathcal{G} \xrightarrow{\sim} p_2^* \mathcal{G}$  satisfying the same cocycle condition is isomorphic to  $u^* \mathcal{F}$  for some  $\mathcal{F} \in \mathrm{Mod}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ . This descent property can be described in the following formal way. Let  $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  be the full subcategory of  $\mathrm{Mod}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$  spanned by Cartesian sheaves. Then it is the  $(2)$ -limit of the following diagram

$$\mathrm{Mod}(X_{\mathrm{ét}}, \Lambda) \xrightleftharpoons[p_2^*]{p_1^*} \mathrm{Mod}((X \times_{\mathcal{X}} X)_{\mathrm{ét}}, \Lambda) \rightrightarrows \mathrm{Mod}((X \times_{\mathcal{X}} X \times_{\mathcal{X}} X)_{\mathrm{ét}}, \Lambda)$$

in the  $(2, 1)$ -category of Abelian categories<sup>2</sup>. Therefore, to study  $\mathrm{Mod}_{\mathrm{cart}}(\mathcal{X}_{\mathrm{lis-ét}}, \Lambda)$ , we only need to study  $\mathrm{Mod}(X_{\mathrm{ét}}, \Lambda)$  for (all) algebraic spaces  $X$  in a “2-coherent way”, that is, we need to track down all the information of natural isomorphisms (2-cells). Such 2-coherence is not more complicated than the one in Grothendieck’s theory of descent [Gro95].

One may want to apply the same idea to derived categories. The problem is that the descent property mentioned previously, in its naïve sense, does not hold anymore, since otherwise the classifying stack  $\mathrm{BG}_m$  over an algebraically closed field will have finite cohomological dimension which is incorrect. In fact, when forming derived categories, we throw away too much information on the coherence of homotopy equivalences or quasi-isomorphisms, which causes the failure of such descent. A descent theory in a weaker sense, known as cohomological descent [SGA4, V bis] and due to Deligne, does exist partially on the level of objects. It is one of the main techniques used in Olsson [Ols07] and Laszlo–Olsson [LO08] for the definition of the six operations on Artin

<sup>2</sup>A  $(2, 1)$ -category is a 2-category in which all 2-cells are invertible.



stacks in certain cases. However, it has the following restrictions. First, Deligne’s cohomological descent is valid only for complexes bounded from below. Although a theory of cohomological descent for unbounded complexes was developed in [LO08], it comes at the price of imposing further finiteness conditions and restricting to constructible complexes when defining the remaining operators. Second, relevant spectral sequences suggest that cohomological descent cannot be used directly to define  $!$ -pushforward.

A more natural solution can be reached once the derived categories are “enhanced”. Roughly speaking (see Proposition 5.3.5 for the precise statement), if we write  $X_n = X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$  ( $(n+1)$ -fold), then  $\mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  is naturally equivalent to the limit of following cosimplicial diagram

$$\mathcal{D}(X_{0,\text{ét}}, \Lambda) \begin{array}{c} \xrightarrow{p_1^*} \\ \xRightarrow{p_2^*} \end{array} \mathcal{D}(X_{1,\text{ét}}, \Lambda) \begin{array}{c} \rightrightarrows \\ \rightrightarrows \end{array} \mathcal{D}(X_{2,\text{ét}}, \Lambda) \begin{array}{c} \rightrightarrows \\ \rightrightarrows \end{array} \cdots$$

in a suitable  $\infty$ -category of *closed symmetric monoidal presentable stable  $\infty$ -categories*. This is completely parallel to the descent property for module categories. Here  $\mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  (resp.  $\mathcal{D}(X_{n,\text{ét}}, \Lambda)$ ) is a closed symmetric monoidal presentable stable  $\infty$ -category which serves as the enhancement of  $\mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  (resp.  $\mathcal{D}(X_{n,\text{ét}}, \Lambda)$ ). Strictly speaking, the previous diagram is incomplete in the sense that we do not mark all the higher cells in the diagram, that is, all natural equivalences of functors, “equivalences between natural equivalences”, etc. In fact, there is an infinite hierarchy of (homotopy) equivalences hidden behind the limit of the previous diagram, not just the 2-level hierarchy in the classical case. To deal with such kind of “homotopy coherence” is the major difficulty of the work, that is, we need to find a way to encode all such hierarchy *simultaneously* in order to make the idea of descent work. In other words, we need to work in the *totality* of all  $\infty$ -categories of concern.

It is possible that such a descent theory (and other relevant higher-categorical techniques introduced below) can be realized by using other models for higher categories. We have chosen the theory developed by Lurie in [HTT], [HA] for its elegance and availability. Precisely, we will use the techniques of the (marked) straightening/unstraightening construction, Adjoint Functor Theorem, and the  $\infty$ -categorical Barr–Beck Theorem. Based on Lurie’s theory, we develop further  $\infty$ -categorical techniques to treat the homotopy-coherence problem mentioned as above. These techniques would enable us to, for example,

- take partial adjoints along given directions (§1.4);
- find a coherent way to decompose morphisms [LZa, §4];
- gluing data from Cartesian diagrams to general ones [LZa, §5];
- make a coherent choice of descent data (§4.2).

In the next section, we will have a chance to explain some of them.

We would also like to remark that Lurie’s theory has already been used, for example, in [BZFN10] to study quasi-coherent sheaves on certain (derived) stacks with many applications. This work, which studies lisse-étale sheaves, is another manifestation of the power of Lurie’s theory. Moreover, the  $\infty$ -categorical enhancement of six operations and its adic version, which is studied in the subsequent article [LZb], are necessary in



certain applications of geometric/categorical method to the Langlands program, as shown for example in the recent work of Bezrukavnikov, Kazhdan and Varshavsky [BKV15].

During the preparation of this article, Gaitsgory [Gai13] and Gaitsgory–Rozenblyum [GR] studied operations for ind-coherent sheaves on DG schemes and derived stacks in the framework of  $\infty$ -categories. Our work bears some similarity to his. We would like to point out that their approach uses  $(\infty, 2)$ -categories (see [GR, Chapter V]), while we stay in the world of  $(\infty, 1)$ -categories.

**0.3. What do we need to enhance?** In the previous section, we mention the enhancement of a single derived category. It is a stable  $\infty$ -category (which can be thought of as an  $\infty$ -categorical version of a triangulated category)  $\mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  (resp.  $\mathcal{D}(X_{\text{ét}}, \Lambda)$  for  $X$  an algebraic space) whose homotopy category (which is an ordinary category) is naturally equivalent to  $D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)$  (resp.  $D(X_{\text{ét}}, \Lambda)$ ). The enhancement of operations is understood in the similar way. For example, the enhancement of  $*$ -pullback for  $f: \mathcal{Y} \rightarrow \mathcal{X}$  should be an exact functor

$$(0.1) \quad f^*: \mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow \mathcal{D}_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$

such that the induced functor

$$hf^*: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}, \Lambda)$$

is the  $*$ -pullback functor of usual derived categories.

However, such enhancement is not enough for us to do descent. The reason is that we need to put all schemes and then algebraic spaces together. Let us denote by  $\text{Sch}^{\text{qc.sep}}$  the category of coproducts of quasi-compact and separated schemes. The enhancement of  $*$ -pullback for schemes in the strong sense is a functor:

$$(0.2) \quad {}_{\text{Sch}^{\text{qc.sep}}} \Lambda \text{EO}^*: N(\text{Sch}^{\text{qc.sep}})^{\text{op}} \rightarrow \mathcal{P}_{\text{r}_{\text{st}}}^{\text{L}}$$

where  $N$  denotes the nerve functor (see the definition preceding [HTT, 1.1.2.2]) and  $\mathcal{P}_{\text{r}_{\text{st}}}^{\text{L}}$  is certain  $\infty$ -category of presentable stable  $\infty$ -categories, which will be specified later. Then (0.1) is just the image of the edge  $f: \mathcal{Y} \rightarrow \mathcal{X}$  if  $f$  belongs to  $\text{Sch}^{\text{qc.sep}}$ . The construction of (0.2) (and its right adjoint which is the enhancement of  $*$ -pushforward) is not hard, with the help of the general construction in [HA]. The difficulty arises in the enhancement of  $!$ -pushforward. Namely, we need to construct a functor:

$${}_{\text{Sch}^{\text{qc.sep}}} \Lambda \text{EO}_!: N(\text{Sch}^{\text{qc.sep}})_F \rightarrow \mathcal{P}_{\text{r}_{\text{st}}}^{\text{L}},$$

where  $N(\text{Sch}^{\text{qc.sep}})_F$  is the subcategory of  $N(\text{Sch}^{\text{qc.sep}})$  only allowing morphisms that are locally of finite type. The basic idea is similar to the classical approach: using Nagata compactification theorem. The problem is the following: for a morphism  $f: Y \rightarrow X$  in  $\text{Sch}^{\text{qc.sep}}$ , locally of finite type, we need to choose (non-canonically!) a relative

compactification

$$\begin{array}{ccc} Y & \xrightarrow{i} & \overline{Y} \\ f \downarrow & & \downarrow \overline{f} \\ X & \xleftarrow{p} & \coprod_I X, \end{array}$$

that is,  $i$  is an open immersion and  $\overline{f}$  is proper, and define  $f_! = p_! \circ \overline{f}_* \circ i_!$  (in the derived sense). It turns out that the resulting functor of usual derived categories is independent of the choice, up to natural isomorphism. First, we need to upgrade such natural isomorphisms to natural equivalences between  $\infty$ -categories. Second and more importantly, we need to “remember” such natural equivalences for all different compactifications, and even “equivalences among natural equivalences”. We immediately find ourselves in the same scenario of an infinity hierarchy of homotopy equivalences again. For handling this kind of homotopy coherence, we use a technique called *multi-simplicial descent* in [LZa, §4], which can be viewed as an  $\infty$ -categorical generalization of [SGA4, XVII 3.3].

This is not the end of the story since our goal is to prove all expected relations among six operations. To use the same idea of descent, we need to “enhance” not just operations, but also relations as well. To simplify the discussion, let us temporarily ignore the two binary operations ( $\otimes$  and  $\mathcal{H}om$ ) and consider how to enhance the “Base Change theorem” which essentially involves  $*$ -pullback and  $!$ -pushforward. We define a simplicial set  $\delta_{2,\{2\}}^* \mathcal{N}(\mathrm{Sch}^{\mathrm{qc.sep}})_{F,\mathrm{all}}^{\mathrm{cart}}$  in the following way:

- The vertices are objects  $X$  of  $\mathrm{Sch}^{\mathrm{qc.sep}}$ .
- The edges are *Cartesian* diagrams

$$(0.3) \quad \begin{array}{ccc} X_{01} & \xrightarrow{g} & X_{00} \\ q \downarrow & & \downarrow p \\ X_{11} & \xrightarrow{f} & X_{10} \end{array}$$

with  $p$  locally of finite type, whose source is  $X_{00}$  and target is  $X_{11}$ .

- Simplices of higher dimensions are defined in a similar way.

Note that this is *not* an  $\infty$ -category. Assuming that  $\Lambda$  is torsion, the enhancement of the Base Change theorem (for  $\mathrm{Sch}^{\mathrm{qc.sep}}$ ) is a functor

$$\mathrm{Sch}^{\mathrm{qc.sep}} \xrightarrow{\Lambda} \mathrm{EO}_!^* : \delta_{2,\{2\}}^* \mathcal{N}(\mathrm{Sch}^{\mathrm{qc.sep}})_{F,\mathrm{all}}^{\mathrm{cart}} \rightarrow \mathcal{P}_{\mathrm{st}}^{\mathrm{L}}$$

such that it sends the edge

$$\begin{array}{ccc} X_{00} & \xrightarrow{\mathrm{id}} & X_{00} \\ \downarrow & & \downarrow p \\ X_{11} & \xrightarrow{\mathrm{id}} & X_{11} \end{array} \quad (\text{resp.} \quad \begin{array}{ccc} X_{11} & \xrightarrow{\quad} & X_{00} \\ \mathrm{id} \downarrow & & \downarrow \mathrm{id} \\ X_{11} & \xrightarrow{f} & X_{00} \end{array})$$

to  $p_! : \mathcal{D}(X_{00,\mathrm{ét}}, \Lambda) \rightarrow \mathcal{D}(X_{11,\mathrm{ét}}, \Lambda)$  (resp.  $f^* : \mathcal{D}(X_{11,\mathrm{ét}}, \Lambda) \rightarrow \mathcal{D}(X_{00,\mathrm{ét}}, \Lambda)$ ). The upshot is that the image of the edge (0.3) is a functor  $\mathcal{D}(X_{11,\mathrm{ét}}, \Lambda) \rightarrow \mathcal{D}(X_{00,\mathrm{ét}}, \Lambda)$  which

is naturally equivalent to both  $f^* \circ p_!$  and  $q_! \circ g^*$ . In other words, this functor has already encoded the Base Change theorem (for  $\mathcal{S}ch^{\text{qc.sep}}$ ) in a homotopy coherent way. This allows us to apply the descent method to construct the enhancement of the Base Change theorem for Artin stacks, which itself includes the enhancement of the four operations  $f^*$ ,  $f_*$ ,  $f_!$  and  $f^!$  by restriction and adjunction. To deal with the homotopy coherence involved in the construction of  ${}_{\mathcal{S}ch^{\text{qc.sep}}}^{\Lambda}\text{EO}_!^*$ , we use another technique called *Cartesian gluing* in [LZa, §5], which can be viewed as an  $\infty$ -categorical variant of [Zhe17, §6, §7].

In fact, we can even modify the map  ${}_{\mathcal{S}ch^{\text{qc.sep}}}^{\Lambda}\text{EO}_!^*$  such that its source is an  $\infty$ -category as well. We use the idea of *monoidal category of correspondence*. Let us continue the current setup, and define a new category<sup>3</sup>  $(\mathcal{S}ch^{\text{qc.sep}})_{\text{corr}: F, \text{all}}^{\otimes}$  as follows. An object of it is a sequence  $\{X_i\}_{1 \leq i \leq l}$  for  $l \geq 0$  (which is the empty sequence if  $l = 0$ ). A morphism  $\{X_i\}_{1 \leq i \leq l} \rightarrow \{Y_j\}_{1 \leq j \leq m}$  consists of the following data:

- a map  $\alpha: \{1, \dots, l\} \cup \{*\} \rightarrow \{1, \dots, m\} \cup \{*\}$  preserving  $*$ ,
- a morphism  $Y'_j \rightarrow X_i$  for every  $1 \leq j \leq m$  and every  $i$  with  $\alpha(i) = j$ ,
- a morphism  $Y'_j \rightarrow Y_j$  locally of finite type for every  $1 \leq j \leq m$ .

Given another morphism  $\{Y_j\}_{1 \leq j \leq m} \rightarrow \{Z_k\}_{1 \leq k \leq n}$  consisting of similar data  $(\beta, Z'_k \rightarrow Y_j, Z'_k \rightarrow Z_k)$ , its composition with the previous one is given by the data  $(\beta \circ \alpha, Z''_k \rightarrow X_i, Z''_k \rightarrow Z_k)$  where  $Z''_k$  is the fiber product as in the following square

$$\begin{array}{ccc} Z''_k & \longrightarrow & \prod_{\alpha(j)=k} Y'_j \\ \downarrow & & \downarrow \\ Z'_k & \longrightarrow & \prod_{\alpha(j)=k} Y_j. \end{array}$$

The category  $(\mathcal{S}ch^{\text{qc.sep}})_{\text{corr}: F, \text{all}}^{\otimes}$  has a canonical symmetric monoidal structure given by coproducts.

**Theorem 0.3.1** (see §6.1 for the precise statement). *We construct a “lax monoidal” functor*

$${}_{\mathcal{S}ch^{\text{qc.sep}}}^{\Lambda}\text{EO}_{\text{corr}}: \mathcal{N}((\mathcal{S}ch^{\text{qc.sep}})_{\text{corr}: F, \text{all}}^{\otimes}) \rightarrow \mathcal{C}at_{\infty}^{\times}$$

*between  $\infty$ -categories that encodes the four operations and Künneth Formula (hence Base Change and Projection Formula). Here,  $\mathcal{C}at_{\infty}^{\times}$  is the (canonical) Cartesian symmetric monoidal  $\infty$ -category of  $\mathcal{C}at_{\infty}$  [HA, 2.4.1].*

We hope the discussion so far explains the meaning of enhancement to some degree. The actual enhancement (3.8) constructed in the article is more complicated than the ones mentioned previously, since we need to include also the information of binary operations, the projection formula and extension of scalars.

<sup>3</sup>Strictly speaking, it is a  $(2, 1)$ -category.

**0.4. Structure of the article.** The main body of the article is divided into seven chapters.

Chapter 1 is a collection of preliminaries on  $\infty$ -categories, including the technique of partial adjoints (§1.4).

Chapter 2 is the starting point of the theory, where we construct enhanced operations for ringed topoi. The first two chapters do not involve algebraic geometry.

In Chapter 3, we construct the enhanced operation map for schemes in the category  $\mathrm{Sch}^{\mathrm{qc.sep}}$ . The enhanced operation map encodes even more information than the enhancement of the Base Change theorem we mentioned in §0.3. We also prove several properties of the map that are crucial for later constructions.

In Chapter 4, we develop an abstract program which we name DESCENT. The program allows us to extend the existing theory to a larger category. It will be run recursively from schemes to algebraic spaces, then to Artin stacks, and eventually to higher Artin or Deligne–Mumford stacks.

The detailed running process is described in Chapter 5. There, we also prove certain compatibility between our theory and existing ones.

In Chapter 6, we write down the resulting six operations for the most general situations and summarize their properties. We also develop a theory of constructible complexes, based on finiteness results of Deligne [SGA4d, Th. finitude] and Gabber [TGxiii]. Finally, we show that our theory is compatible with the work of Laszlo and Olsson [LO08].

For more detailed descriptions of the individual chapters, we refer to the beginning of these chapters.

We assume that the reader has some knowledge of Lurie’s theory of  $\infty$ -categories, especially Chapters 1 through 5 of [HTT], and Chapters 1 through 4 of [HA]. In particular, we assume that the reader is familiar with basic concepts of simplicial sets [HTT, A.2.7]. However, an effort has been made to provide precise references for notation, concepts, constructions, and results used in this article, (at least) at their first appearance.

## 0.5. Conventions and notation.

- All rings are assumed to be commutative with unity.

For *set-theoretical issues*:

- We fix two (Grothendieck) universes  $\mathcal{U}$  and  $\mathcal{V}$  such that  $\mathcal{U}$  belongs to  $\mathcal{V}$ . The adjective *small* means  $\mathcal{U}$ -small. In particular, Grothendieck Abelian categories and presentable  $\infty$ -categories are relative to  $\mathcal{U}$ . A topos means a  $\mathcal{U}$ -topos.
- All rings are assumed to be  $\mathcal{U}$ -small. We denote by  $\mathrm{Ring}$  the category of rings in  $\mathcal{U}$ . By the usual abuse of language, we call  $\mathrm{Ring}$  the category of  $\mathcal{U}$ -small rings.
- All schemes are assumed to be  $\mathcal{U}$ -small. We denote by  $\mathrm{Sch}$  the category of schemes belonging to  $\mathcal{U}$  and by  $\mathrm{Sch}^{\mathrm{aff}}$  the full subcategory consisting of affine schemes belonging to  $\mathcal{U}$ . We have an equivalence of categories  $\mathrm{Spec}: (\mathrm{Ring})^{\mathrm{op}} \rightarrow \mathrm{Sch}^{\mathrm{aff}}$ . The big fppf site on  $\mathrm{Sch}^{\mathrm{aff}}$  is not a  $\mathcal{U}$ -site, so that we need to consider

prestacks with values in  $\mathcal{V}$ . More precisely, for  $\mathcal{W} = \mathcal{U}$  or  $\mathcal{V}$ , let  $\mathcal{S}_{\mathcal{W}}$  [HTT, 1.2.16.1] be the  $\infty$ -category of spaces in  $\mathcal{W}$ . We define the  $\infty$ -category of prestacks to be  $\mathrm{Fun}(\mathrm{N}(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathcal{S}_{\mathcal{V}})$  [HTT, 1.2.7.2]. However, a (higher) Artin stack is assumed to be contained in the essential image of the full subcategory  $\mathrm{Fun}(\mathrm{N}(\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}}, \mathcal{S}_{\mathcal{U}})$ . See §5.4 for more details.

The (small) étale site of an algebraic scheme and the lisse-étale site of an Artin stack are  $\mathcal{U}$ -sites.

- For every  $\mathcal{V}$ -small set  $I$ , we denote by  $\mathrm{Set}_{I\Delta}$  the category of  $I$ -simplicial sets in  $\mathcal{V}$ . See also variants in §1.3. We denote by  $\mathrm{Cat}_{\infty}$  the (non  $\mathcal{V}$ -small)  $\infty$ -category of  $\infty$ -categories in  $\mathcal{V}$  [HTT, 3.0.0.1]<sup>4</sup>. (Multi)simplicial sets and  $\infty$ -categories are usually tacitly assumed to be  $\mathcal{V}$ -small.

For *lower categories*:

- Unless otherwise specified, a category will be understood as an ordinary category. A  $(2, 1)$ -category  $\mathcal{C}$  is a (strict) 2-category in which all 2-cells are invertible, or, equivalently, a category enriched in the category of groupoids. We regard  $\mathcal{C}$  as a simplicial category by taking  $\mathrm{N}(\mathrm{Map}_{\mathcal{C}}(X, Y))$  for all objects  $X$  and  $Y$  of  $\mathcal{C}$ .
- Let  $\mathcal{C}, \mathcal{D}$  be two categories. We denote by  $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$  the *category of functors* from  $\mathcal{C}$  to  $\mathcal{D}$ , whose objects are functors and morphisms are natural transformations.
- Let  $\mathcal{A}$  be an additive category. We denote by  $\mathrm{Ch}(\mathcal{A})$  the category of cochain complexes of  $\mathcal{A}$ .
- Recall that a *partially ordered set*  $P$  is an (ordinary) category such that there is at most one arrow (usual denoted as  $\leq$ ) between each pair of objects. For every element  $p \in P$ , we identify the overcategory  $P_{/p}$  (resp. undercategory  $P_{p/}$ ) with the full partially ordered subset of  $P$  consisting of elements  $\leq p$  (resp.  $\geq p$ ). For  $p, p' \in P$ , we identify  $P_{p//p'}$  with the full partially ordered subset of  $P$  consisting of elements both  $\geq p$  and  $\leq p'$ , which is empty unless  $p \leq p'$ .
- Let  $[n]$  be the ordered set  $\{0, \dots, n\}$  for  $n \geq 0$ , and put  $[-1] = \emptyset$ . Let us recall the *category of combinatorial simplices*  $\Delta$  (resp.  $\Delta^{\leq n}$ ,  $\Delta_+$ ,  $\Delta_+^{\leq n}$ ). Its objects are the linearly ordered sets  $[i]$  for  $i \geq 0$  (resp.  $0 \leq i \leq n$ ,  $i \geq -1$ ,  $-1 \leq i \leq n$ ) and its morphisms are given by (nonstrictly) order-preserving maps. In particular, for every  $n \geq 0$  and  $0 \leq k \leq n$ , we have the face map  $d_k^n: [n-1] \rightarrow [n]$  that is the unique injective map with  $k$  not in the image; and the degeneration map  $s_k^n: [n+1] \rightarrow [n]$  that is the unique surjective map such that  $s_k^n(k+1) = s_k^n(k)$ .

For *higher categories*:

- As we have mentioned, the word  $\infty$ -category refers to the one defined in [HTT, 1.1.2.4]. Throughout the article, an effort has been made to keep our notation consistent with those in [HTT] and [HA].

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<sup>4</sup>In [HTT],  $\mathrm{Cat}_{\infty}$  denotes the category of small  $\infty$ -categories. Thus our  $\mathrm{Cat}_{\infty}$  corresponds more closely to the notation  $\widehat{\mathrm{Cat}}_{\infty}$  in [HTT, 3.0.0.5], where the extension of universes is tacit.

- For  $\mathcal{C}$  a category, a  $(2, 1)$ -category, a simplicial category, or an  $\infty$ -category, we denote by  $\text{id}_{\mathcal{C}}$  the identity functor of  $\mathcal{C}$ . We denote by  $N(\mathcal{C})$  the (simplicial) nerve of a (simplicial) category  $\mathcal{C}$  [HTT, 1.1.5.5]. We identify  $\text{Ar}(\mathcal{C})$  (the set of arrows of  $\mathcal{C}$ ) with  $N(\mathcal{C})_1$  (the set of edges of  $N(\mathcal{C})$ ) if  $\mathcal{C}$  is a category. Usually, we will not distinguish between  $N(\mathcal{C}^{op})$  and  $N(\mathcal{C})^{op}$  for  $\mathcal{C}$  a category, a  $(2, 1)$ -category or a simplicial category.
- We denote the homotopy category [HTT, 1.1.3.2, 1.2.3.1] of an  $\infty$ -category  $\mathcal{C}$  by  $\text{h}\mathcal{C}$  and we view it as an ordinary category. In other words, we ignore the  $\mathcal{H}$ -enrichment of  $\text{h}\mathcal{C}$ .
- Let  $\mathcal{C}$  be an  $\infty$ -category, and  $c^\bullet: N(\Delta) \rightarrow \mathcal{C}$  (resp.  $c_\bullet: N(\Delta)^{op} \rightarrow \mathcal{C}$ ) a cosimplicial (resp. simplicial) object of  $\mathcal{C}$ . Then the limit [HTT, 1.2.13.4]  $\varprojlim(c^\bullet)$  (resp. colimit or geometric realization  $\varinjlim(c_\bullet)$ ), if it exists, is denoted by  $\varprojlim_{n \in \Delta} c^n$  (resp.  $\varinjlim_{n \in \Delta^{op}} c_n$ ). It is viewed as an object (up to equivalences parameterized by a contractible Kan complex) of  $\mathcal{C}$ .
- Let  $\mathcal{C}$  be an  $(\infty)$ -category, and  $\mathcal{C}' \subseteq \mathcal{C}$  a full subcategory. We say that a morphism  $f: y \rightarrow x$  in  $\mathcal{C}$  is *representable in  $\mathcal{C}'$*  if for every Cartesian diagram [HTT, 4.4.2]

$$\begin{array}{ccc} w & \xrightarrow{\quad} & z \\ \downarrow & & \downarrow \\ y & \xrightarrow{f} & x \end{array}$$

such that  $z$  is an object of  $\mathcal{C}'$ ,  $w$  is equivalent to an object of  $\mathcal{C}'$ .

- We refer the reader to the beginning of [HTT, 2.3.3] for the terminology *homotopic relative to  $A$  over  $S$* . We say that  $f$  and  $f'$  are *homotopic over  $S$*  (resp. *homotopic relative to  $A$* ) if  $A = \emptyset$  (resp.  $S = *$ ).
- Recall that  $\text{Cat}_\infty$  is the  $\infty$ -category of  $\mathcal{V}$ -small  $\infty$ -categories. In [HTT, 5.5.3.1], the subcategories  $\mathcal{P}\text{r}^L, \mathcal{P}\text{r}^R \subseteq \text{Cat}_\infty$  are defined<sup>5</sup>. We define subcategories  $\mathcal{P}\text{r}_{\text{st}}^L, \mathcal{P}\text{r}_{\text{st}}^R \subseteq \text{Cat}_\infty$  as follows:
  - The objects of both  $\mathcal{P}\text{r}_{\text{st}}^L$  and  $\mathcal{P}\text{r}_{\text{st}}^R$  are the  $\mathcal{U}$ -presentable stable  $\infty$ -categories in  $\mathcal{V}$  [HTT, 5.5.0.1], [HA, 1.1.1.9].
  - A functor  $F: \mathcal{C} \rightarrow \mathcal{D}$  of presentable stable  $\infty$ -categories is a morphism of  $\mathcal{P}\text{r}_{\text{st}}^L$  if and only if  $F$  preserves small colimits, or, equivalently,  $F$  is a left adjoint functor [HTT, 5.2.2.1, 5.5.2.9 (1)].
  - A functor  $G: \mathcal{C} \rightarrow \mathcal{D}$  of presentable stable  $\infty$ -categories is a morphism of  $\mathcal{P}\text{r}_{\text{st}}^R$  if and only if  $G$  is accessible and preserves small limits, or, equivalently,  $G$  is a right adjoint functor [HTT, 5.5.2.9(2)].

We adopt the notation of [HTT, 5.2.6.1]: for  $\infty$ -categories  $\mathcal{C}$  and  $\mathcal{D}$ , we denote by  $\text{Fun}^L(\mathcal{C}, \mathcal{D})$  (resp.  $\text{Fun}^R(\mathcal{C}, \mathcal{D})$ ) the full subcategory of  $\text{Fun}(\mathcal{C}, \mathcal{D})$  [HTT, 1.2.7.2] spanned by left (resp. right) adjoint functors. Small limits exist

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<sup>5</sup>Under our convention, the objects of  $\mathcal{P}\text{r}^L$  and  $\mathcal{P}\text{r}^R$  are the  $\mathcal{U}$ -presentable  $\infty$ -categories in  $\mathcal{V}$ .

in  $\mathcal{Cat}_\infty$ ,  $\mathcal{Pr}^L$ ,  $\mathcal{Pr}^R$ ,  $\mathcal{Pr}_{\text{st}}^L$  and  $\mathcal{Pr}_{\text{st}}^R$ . Such limits are preserved by the natural inclusions  $\mathcal{Pr}_{\text{st}}^L \subseteq \mathcal{Pr}^L \subseteq \mathcal{Cat}_\infty$  and  $\mathcal{Pr}_{\text{st}}^R \subseteq \mathcal{Pr}^R \subseteq \mathcal{Cat}_\infty$  by [HTT, 5.5.3.13, 5.5.3.18] and [HA, 1.1.4.4].

- For a simplicial model category  $\mathbf{A}$ , we denote by  $\mathbf{A}^\circ$  the subcategory spanned by fibrant-cofibrant objects.
- For the simplicial model category  $\mathcal{Set}_\Delta^+$  of marked simplicial sets in  $\mathcal{V}$  [HTT, 3.1.0.2] with respect to the Cartesian model structure [HTT, 3.1.3.7, 3.1.4.4], we fix a *fibrant replacement simplicial functor*

$$\text{Fibr}: \mathcal{Set}_\Delta^+ \rightarrow (\mathcal{Set}_\Delta^+)^\circ$$

via the Small Object Argument [HTT, A.1.2.5, A.1.2.6]. By construction, it commutes with finite products. If  $\mathcal{C}$  is a  $\mathcal{V}$ -small simplicial category [HTT, 1.1.4.1], we let  $\text{Fibr}^\mathcal{C}: (\mathcal{Set}_\Delta^+)^\mathcal{C} \rightarrow ((\mathcal{Set}_\Delta^+)^\circ)^\mathcal{C} \subseteq (\mathcal{Set}_\Delta^+)^\mathcal{C}$  be the induced fibrant replacement simplicial functor with respect to the projective model structure [HTT, A.3.3.1].

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## 1. PRELIMINARIES ON $\infty$ -CATEGORIES

This chapter is a collection of preliminaries on  $\infty$ -categories. In §1.1, we record some basic lemmas. In §1.2, we recall a key lemma and its variant established in [LZa], which will be subsequently used in this article. In §1.3, we recall the definitions of multisimplicial sets and multi-marked simplicial sets from [LZa]. In §1.4, we develop a method of taking partial adjoints, namely, taking adjoint functors along given directions. This will be used to construct the initial enhanced operation map for schemes. In §1.5, we collect some general facts and constructions related to symmetric monoidal  $\infty$ -categories.

**1.1. Elementary lemmas.** Let us start with the following lemma, which appears as [Lur, 2.4.6]. We include a proof for the convenience of the reader.

**Lemma 1.1.1.** *Let  $\mathcal{C}$  be a nonempty  $\infty$ -category that admits product of two objects. Then the geometric realization  $|\mathcal{C}|$  is contractible.*

*Proof.* Fix an object  $X$  of  $\mathcal{C}$  and a functor  $\mathcal{C} \rightarrow \mathcal{C}$  sending  $Y$  to  $X \times Y$ . The projections  $X \times Y \rightarrow X$  and  $X \times Y \rightarrow Y$  define functors  $h, h': \Delta^1 \times \mathcal{C} \rightarrow \mathcal{C}$  such that

- $h \mid \Delta^{\{0\}} \times \mathcal{C} = h' \mid \Delta^{\{0\}} \times \mathcal{C}$ ;



- $h|_{\Delta^{\{1\}} \times \mathcal{C}}$  is the constant functor of value  $X$ ;
- $h'|_{\Delta^{\{1\}} \times \mathcal{C}} = \text{id}_{\mathcal{C}}$ .

Then  $|h|$  and  $|h'|$  provide a homotopy between  $\text{id}_{|\mathcal{C}|}$  and the constant map of value  $X$ .  $\square$

The following is a variant of the Adjoint Functor Theorem [HTT, 5.5.2.9].

**Lemma 1.1.2.** *Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor between presentable  $\infty$ -categories. Let  $hF: h\mathcal{C} \rightarrow h\mathcal{D}$  be the functor of (unenriched) homotopy categories.*

- (1) *The functor  $F$  has a right adjoint if and only if it preserves pushouts and  $hF$  has a right adjoint.*
- (2) *The functor  $F$  has a left adjoint if and only if it is accessible and preserves pullbacks and  $hF$  has a left adjoint.*

*Proof.* The necessity follows from [HTT, 5.2.2.9]. The sufficiency in (1) follows from the fact that small colimits can be constructed out of pushouts and small coproducts [HTT, 4.4.2.7] and preservation of small coproducts can be tested on  $hF$ . The sufficiency in (2) follows from dual statements.  $\square$

We will apply the above lemma in the following form.

**Lemma 1.1.3.** *Let  $F: \mathcal{C} \rightarrow \mathcal{D}$  be a functor between presentable stable  $\infty$ -categories. Let  $hF: h\mathcal{C} \rightarrow h\mathcal{D}$  be the functor of (unenriched) homotopy categories. Then*

- (1) *The functor  $F$  admits a right adjoint if and only if  $hF$  is a triangulated functor and admits a right adjoint.*
- (2) *The functor  $F$  admits a left adjoint if  $F$  admits a right adjoint and  $hF$  admits a left adjoint.*

*Proof.* By [HA, 1.2.4.14], a functor  $G$  between stable  $\infty$ -categories is exact if and only if  $hG$  is triangulated. The lemma then follows from Lemma 1.1.2 and [HA, 1.1.4.1].  $\square$

**Lemma 1.1.4.** *Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be a left exact functor between Grothendieck Abelian categories that commutes with small coproducts. Assume that  $F$  has finite cohomological dimension. Then the right derived functor  $RF: \mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}(\mathcal{B})$  admits a right adjoint.*

*Proof.* By the previous lemma, it suffices to show that  $h(RF)$  commutes with small coproducts. This is standard. See [KS06, Proposition 14.3.4 (ii)].  $\square$

**1.2. Constructing functors via the category of simplices.** In this section we recall the technique in [LZa, §2] for constructing functors to  $\infty$ -categories. It is crucial for many constructions in both articles.

We start with some generalities on diagrams of simplicial sets. Let  $\mathcal{J}$  be a (small) ordinary category. We consider the injective model structure on the functor category  $(\text{Set}_{\Delta})^{\mathcal{J}} := \text{Fun}(\mathcal{J}, \text{Set}_{\Delta})$ . We say that a morphism  $i: \mathcal{N} \rightarrow \mathcal{M}$  in  $(\text{Set}_{\Delta})^{\mathcal{J}}$  is *anodyne* if  $i(\sigma): \mathcal{N}(\sigma) \rightarrow \mathcal{M}(\sigma)$  is anodyne for every object  $\sigma$  of  $\mathcal{J}$ . We say that a morphism  $\mathcal{R} \rightarrow \mathcal{R}'$  in  $(\text{Set}_{\Delta})^{\mathcal{J}}$  is an *injective fibration* if it has the right lifting property with respect to every anodyne morphism  $\mathcal{N} \rightarrow \mathcal{M}$  in  $(\text{Set}_{\Delta})^{\mathcal{J}}$ . We say that an object  $\mathcal{R}$  of  $(\text{Set}_{\Delta})^{\mathcal{J}}$  is *injectively fibrant* if the morphism from  $\mathcal{R}$  to the final object  $\Delta_{\mathcal{J}}^0$  is an

injective fibration. The right adjoint of the diagonal functor  $\text{Set}_\Delta \rightarrow (\text{Set}_\Delta)^{\mathcal{J}}$  is the global section functor

$$\Gamma: (\text{Set}_\Delta)^{\mathcal{J}} \rightarrow \text{Set}_\Delta$$

defined by the formula  $\Gamma(\mathcal{N})_q = \text{Hom}_{(\text{Set}_\Delta)^{\mathcal{J}}}(\Delta_j^q, \mathcal{N})$ , where  $\Delta_j^q: \mathcal{J} \rightarrow \text{Set}_\Delta$  is the constant functor of value  $\Delta^q$ .

**Notation 1.2.1.** Let  $\Phi: \mathcal{N} \rightarrow \mathcal{R}$  be a morphism of  $(\text{Set}_\Delta)^{\mathcal{J}}$ . We let  $\Gamma_\Phi(\mathcal{R}) \subseteq \Gamma(\mathcal{R})$  denote the simplicial subset, union of the images of  $\Gamma(\Psi): \Gamma(\mathcal{M}) \rightarrow \Gamma(\mathcal{R})$  for all factorizations

$$\mathcal{N} \xrightarrow{i} \mathcal{M} \xrightarrow{\Psi} \mathcal{R}$$

of  $\Phi$  such that  $i$  is anodyne.

Let  $K$  be a simplicial set. The *category of simplices of  $K$* , which we denote by  $\Delta_{/K}$  following [HTT, Notation 6.1.2.5], plays a key role in our construction technique. Recall that  $\Delta_{/K}$  is the strict fiber product  $\Delta \times_{\text{Set}_\Delta} (\text{Set}_\Delta)_{/K}$ . An object of  $\Delta_{/K}$  is a pair  $(n, \sigma)$ , where  $n \geq 0$  is some integer and  $\sigma \in \text{Hom}_{\text{Set}_\Delta}(\Delta^n, K)$ . A morphism  $(n, \sigma) \rightarrow (n', \sigma')$  is a map  $d: \Delta^n \rightarrow \Delta^{n'}$  such that  $\sigma = \sigma' \circ d$ . In what follows, we sometimes simply write  $\sigma$  for an object of  $\Delta_{/K}$  if  $n$  is insensitive. Moreover, when  $\mathcal{J} = (\Delta_{/K})^{op}$ , we simply write  $\Delta_K^q$  for  $\Delta_j^q$ .

**Notation 1.2.2.** We define a functor  $\text{Map}[K, -]: \text{Set}_\Delta^+ \rightarrow (\text{Set}_\Delta)^{(\Delta_{/K})^{op}}$  as follows. For a marked simplicial set  $M$ , we define  $\text{Map}[K, M]$  by

$$\text{Map}[K, M](n, \sigma) = \text{Map}^\sharp((\Delta^n)^\flat, M),$$

for every object  $(n, \sigma)$  of  $\Delta_{/K}$ . A morphism  $d: (n, \sigma) \rightarrow (n', \sigma')$  in  $\Delta_{/K}$  goes to the natural restriction map  $\text{Res}^d: \text{Map}^\sharp((\Delta^{n'})^\flat, M) \rightarrow \text{Map}^\sharp((\Delta^n)^\flat, M)$ . For an  $\infty$ -category  $\mathcal{C}$ , we set  $\text{Map}[K, \mathcal{C}] = \text{Map}[K, \mathcal{C}^\flat]$ .

By [LZa, Remark 2.7], the map

$$\text{Map}^\sharp(K^\flat, M) \rightarrow \Gamma(\text{Map}[K, M])$$

induced by the restriction maps  $\text{Map}^\sharp(K^\flat, M) \rightarrow \text{Map}^\sharp((\Delta^n)^\flat, M)$  is an isomorphism of simplicial sets.

If  $g: K' \rightarrow K$  is a map, then composition with the functor  $\Delta_{/K'} \rightarrow \Delta_{/K}$  induced by  $g$  defines a functor  $g^*: (\text{Set}_\Delta)^{(\Delta_{/K})^{op}} \rightarrow (\text{Set}_\Delta)^{(\Delta_{/K'})^{op}}$ . We have  $g^* \text{Map}[K, M] = \text{Map}[K', M]$ .

**Lemma 1.2.3.** *Let  $f: Z \rightarrow T$  be a fibration in  $\text{Set}_\Delta^+$  with respect to the Cartesian model structure,  $K$  a simplicial set,  $a: K^\flat \rightarrow T$  a map, and  $\mathcal{N} \in (\text{Set}_\Delta)^{(\Delta_{/K})^{op}}$  such that  $\mathcal{N}(\sigma)$  is weakly contractible for all  $\sigma \in \Delta_{/K}$ . We let  $\text{Map}[K, f]_a$  denote the fiber of  $\text{Map}[K, f]: \text{Map}[K, Z] \rightarrow \text{Map}[K, T]$  at the section  $\Delta_K^0 \rightarrow \text{Map}[K, T]$  corresponding to  $a$ .*

- (1) *For every morphism  $\Phi: \mathcal{N} \rightarrow \text{Map}[K, f]_a$ , the simplicial set  $\Gamma_\Phi(\text{Map}[K, f]_a)$  is a (nonempty) connected component of  $\Gamma(\text{Map}[K, f]_a)$ .*

(2) For homotopic  $\Phi, \Phi': \mathcal{N} \rightarrow \text{Map}[K, f]_a$ , we have

$$\Gamma_\Phi(\text{Map}[K, f]_a) = \Gamma_{\Phi'}(\text{Map}[K, f]_a).$$

The condition in (2) means that there exists a morphism  $H: \Delta_K^1 \times \mathcal{N} \rightarrow \text{Map}[K, f]_a$  in  $(\text{Set}_\Delta)^{(\Delta/\kappa)^{op}}$  such that  $H|_{\Delta_K^{\{0\}} \times \mathcal{N}} = \Phi$ ,  $H|_{\Delta_K^{\{1\}} \times \mathcal{N}} = \Phi'$ .

*Proof.* This is [LZa, Lemma 2.4] applied to  $\mathcal{R} = \text{Map}[K, f]_a$ . The latter is injectively fibrant by [LZa, Proposition 2.8].  $\square$

**Lemma 1.2.4** ([LZa, Proposition 2.14]). *Let  $K$  be a simplicial set,  $\mathcal{C}$  an  $\infty$ -category, and  $i: A \hookrightarrow B$  a monomorphism of simplicial sets. Denote by  $f: \text{Fun}(B, \mathcal{C}) \rightarrow \text{Fun}(A, \mathcal{C})$  the map induced by  $i$ . Let  $\mathcal{N}$  be an object of  $(\text{Set}_\Delta)^{(\Delta/\kappa)^{op}}$  such that  $\mathcal{N}(\sigma)$  is weakly contractible for all  $\sigma \in \Delta_{/K}$ , and  $\Phi: \mathcal{N} \rightarrow \text{Map}[K, \text{Fun}(B, \mathcal{C})]$  a morphism such that  $\text{Map}[K, f] \circ \Phi: \mathcal{N} \rightarrow \text{Map}[K, \text{Fun}(A, \mathcal{C})]$  factorizes through  $\Delta_{(\Delta/\kappa)^{op}}^0$  to give a functor  $a: K \rightarrow \text{Fun}(A, \mathcal{C})$ . Then there exists  $b: K \rightarrow \text{Fun}(B, \mathcal{C})$  such that  $b \circ p = a$  and for every map  $g: K' \rightarrow K$  and every global section  $\nu \in \Gamma(g^*\mathcal{N})_0$ , the maps  $b \circ g$  and  $g^*\Phi \circ \nu: K' \rightarrow \text{Fun}(B, \mathcal{C})$  are homotopic over  $\text{Fun}(A, \mathcal{C})$ . Here,  $g^*\Phi$  denotes the induced map  $g^*\mathcal{N} \rightarrow g^*\text{Map}[K, \text{Fun}(B, \mathcal{C})] = \text{Map}[K', \text{Fun}(B, \mathcal{C})]$ .*

**1.3. Multisimplicial sets.** We recall the definitions of multisimplicial sets and multi-marked simplicial sets from [LZa, §3].

**Definition 1.3.1** (Multisimplicial set). Let  $I$  be a  $\mathcal{V}$ -small set. We define the category of  $I$ -simplicial sets to be  $\text{Set}_{I\Delta} := \text{Fun}((\Delta^I)^{op}, \text{Set})$ , where  $\Delta^I := \text{Fun}(I, \Delta)$ . For an integer  $k \geq 0$ , we define the category of  $k$ -simplicial sets to be  $\text{Set}_{k\Delta} := \text{Set}_{I\Delta}$ , where  $I = \{1, \dots, k\}$ . We identify  $\text{Set}_{1\Delta}$  with  $\text{Set}_\Delta$ .

We denote by  $\Delta^{n_i|i \in I}$  the  $I$ -simplicial set represented by the object  $([n_i])_{i \in I}$  of  $\Delta^I$ . For an  $I$ -simplicial set  $S$ , we denote by  $S_{n_i|i \in I}$  the value of  $S$  at the object  $([n_i])_{i \in I}$  of  $\Delta^I$ . An  $(n_i)_{i \in I}$ -cell of an  $I$ -simplicial set  $S$  is an element of  $S_{n_i|i \in I}$ . By Yoneda's lemma, there is a canonical bijection between the set  $S_{n_i|i \in I}$  and the set of maps from  $\Delta^{n_i|i \in I}$  to  $S$ .

For  $J \subseteq I$ , composition with the partial opposite functor  $\Delta^I \rightarrow \Delta^I$  sending  $(\dots, P_{j'}, \dots, P_j, \dots)$  to  $(\dots, P_{j'}, \dots, P_j^{op}, \dots)$  (taking  $op$  for  $P_j$  when  $j \in J$ ) defines a functor  $\text{op}_J^I: \text{Set}_{I\Delta} \rightarrow \text{Set}_{I\Delta}$ . We put  $\Delta_J^{n_i|i \in I} = \text{op}_J^I \Delta^{n_i|i \in I}$ . Although  $\Delta_J^{n_i|i \in I}$  is isomorphic to  $\Delta^{n_i|i \in I}$ , the notational distinction will be useful in specifying the variance of many constructions. When  $J = \emptyset$ ,  $\text{op}_\emptyset^I$  is the identity functor so that  $\Delta_\emptyset^{n_i|i \in I} = \Delta^{n_i|i \in I}$ .

**Definition 1.3.2.** Let  $I, J$  be two  $\mathcal{V}$ -small sets.

- (1) Let  $f: J \rightarrow I$  be a map of sets. Composition with  $f$  defines a functor  $\Delta^f: \Delta^I \rightarrow \Delta^J$ . Composition with  $(\Delta^f)^{op}$  induces a functor  $(\Delta^f)^*: \text{Set}_{J\Delta} \rightarrow \text{Set}_{I\Delta}$ , which has a right adjoint  $(\Delta^f)_*: \text{Set}_{I\Delta} \rightarrow \text{Set}_{J\Delta}$ . We will now look at two special cases.
- (2) Let  $f: J \rightarrow I$  be an injective map. The functor  $\Delta^f$  has a right adjoint  $c_f: \Delta^J \rightarrow \Delta^I$  given by  $c_f(F)_i = F_j$  if  $f(j) = i$  and  $c_f(F)_i = [0]$  if  $i$  is not in the image of  $f$ .

The functor  $(\Delta^f)_*$  can be identified with the functor  $\epsilon^f$  induced by composition with  $(c_f)^{op}$ . If  $J = \{1, \dots, k'\}$ , we write  $\epsilon_{f(1)\dots f(k')}^I$  for  $\epsilon^f$ .

- (3) Consider the map  $f: I \rightarrow \{1\}$ . Then  $\delta_I := \Delta^f: \Delta \rightarrow \Delta^I$  is the diagonal map, and composition with  $(\delta_I)^{op}$  induces the *diagonal functor*  $\delta_I^* = (\Delta^f)^*: \text{Set}_{I\Delta} \rightarrow \text{Set}_\Delta$ . We define

$$\Delta^{[n_i]_{i \in I}} := \delta_I^* \Delta^{n_i | i \in I} = \prod_{i \in I} \Delta^{n_i}.$$

We define the *multisimplicial nerve* functor to be the right adjoint  $\delta_*^I: \text{Set}_\Delta \rightarrow \text{Set}_{I\Delta}$  of  $\delta_I^*$ . An  $(n_i)_{i \in I}$ -cell of  $\delta_*^I X$  is given by a map  $\Delta^{[n_i]_{i \in I}} \rightarrow X$ .

- (4) For  $J \subseteq I$ , we define the *twisted diagonal functor*  $\delta_{I,J}^*$  as  $\delta_I^* \circ \text{op}_J^I: \text{Set}_{I\Delta} \rightarrow \text{Set}_\Delta$ . We define

$$\Delta_J^{[n_i]_{i \in I}} := \delta_{I,J}^* \Delta^{n_i | i \in I} = \delta_I^* \Delta_J^{n_i | i \in I} = \left( \prod_{i \in I-J} \Delta^{n_i} \right) \times \left( \prod_{j \in J} (\Delta^{n_j})^{op} \right).$$

When  $J = \emptyset$ , we have  $\delta_{I,\emptyset}^* = \delta_I^*$  and  $\Delta_\emptyset^{[n_i]_{i \in I}} = \Delta^{[n_i]_{i \in I}}$ .

When  $I = \{1, \dots, k\}$ , we write  $k$  instead of  $I$  in the previous notation. For example, in (2) we have  $(\epsilon_j^k K)_n = K_{0, \dots, n, \dots, 0}$ , where  $n$  is at the  $j$ -th position and all other indices are 0. In (3) we have  $\delta_k^*: \text{Set}_{k\Delta} \rightarrow \text{Set}_\Delta$  defined by  $(\delta_k^* X)_n = X_{n, \dots, n}$ .

**Definition 1.3.3** (Exterior product). Let  $I = \coprod_{j \in J} I_j$  be a partition. We define a functor

$$\boxtimes_{j \in J}: \prod_{j \in J} \text{Set}_{I_j \Delta} \rightarrow \text{Set}_{I\Delta}$$

by the formula  $\boxtimes_{j \in J} S^j = \prod_{j \in J} (\Delta^{\iota_j})^* S^j$ , where  $\iota_j: I_j \hookrightarrow I$  is the inclusion. For  $J = \{1, \dots, m\}$ ,  $I_j = \{1, \dots, k_j\}$ , we define

$$- \boxtimes \dots \boxtimes -: \text{Set}_{k_1 \Delta} \times \dots \times \text{Set}_{k_m \Delta} \rightarrow \text{Set}_{k \Delta}.$$

by  $(S^1 \boxtimes \dots \boxtimes S^m)_{n_1^1, \dots, n_{k_1}^1, \dots, n_1^m, \dots, n_{k_m}^m} = S_{n_1^1, \dots, n_{k_1}^1}^1 \times \dots \times S_{n_1^m, \dots, n_{k_m}^m}^m$ .

We have the isomorphisms  $\boxtimes_{i \in I} \Delta^{n_i} \simeq \Delta^{n_i | i \in I}$  and  $\delta_I^* \boxtimes_{j \in J} S^j \simeq \prod_{j \in J} \delta_{I_j}^* S^j$ .

*Remark 1.3.4.* For a map  $f: J \rightarrow I$ , we have  $(\Delta^f)^* \Delta^{n_i | i \in I} \simeq \boxtimes_{i \in I} \Delta^{[n_j]_{j \in f^{-1}(i)}}$ , so that an  $(n_j)_{j \in J}$ -cell of  $(\Delta^f)_* X$  is given by a map  $\boxtimes_{i \in I} \Delta^{[n_j]_{j \in f^{-1}(i)}} \rightarrow X$ .

**Definition 1.3.5** (Multi-marked simplicial set). An *I-marked simplicial set* (resp. *I-marked  $\infty$ -category*) is the data  $(X, \mathcal{E} = \{\mathcal{E}_i\}_{i \in I})$ , where  $X$  is a simplicial set (resp. an  $\infty$ -category) and, for all  $i \in I$ ,  $\mathcal{E}_i$  is a set of edges of  $X$  that contains every degenerate edge. A morphism  $f: (X, \{\mathcal{E}_i\}_{i \in I}) \rightarrow (X', \{\mathcal{E}'_i\}_{i \in I})$  of *I-marked simplicial sets* is a map  $f: X \rightarrow X'$  having the property that  $f(\mathcal{E}_i) \subseteq \mathcal{E}'_i$  for all  $i \in I$ . We denote the category of *I-marked simplicial sets* by  $\text{Set}_\Delta^{I+}$ . It is the strict fiber product of  $I$  copies of  $\text{Set}_\Delta^+$  over  $\text{Set}_\Delta$ .

**Definition 1.3.6** (Cartesian nerve). For an *I-marked  $\infty$ -category*  $(\mathcal{C}, \mathcal{E})$ , we denote by  $\mathcal{C}_\mathcal{E}^{\text{cart}} \subseteq \delta_*^I \mathcal{C}$  the *Cartesian I-simplicial nerve* of  $(\mathcal{C}, \mathcal{E})$  [LZa, Definition 3.16]. Roughly

speaking, its  $(n_i)_{i \in I}$ -simplices are functors  $\Delta^{[n_i]_{i \in I}} \rightarrow \mathcal{C}$  such that the image of a morphism in the  $i$ -th direction belongs to  $\mathcal{E}_i$  for  $i \in I$ , and the image of every “unit square” is a Cartesian diagram. For a marked  $\infty$ -category  $(\mathcal{C}, \mathcal{E})$ , we write  $\mathcal{C}_{\mathcal{E}}^{\text{cart}} \simeq \text{Map}^{\sharp}((\Delta^0)^b, (\mathcal{C}, \mathcal{E}))$ .

**1.4. Partial adjoints.** We first recall the notion of adjoints of squares.

**Definition 1.4.1.** Consider diagrams of  $\infty$ -categories

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{F} & \mathcal{D} \\ U \downarrow & \sigma & \downarrow V \\ \mathcal{C}' & \xleftarrow{F'} & \mathcal{D}' \end{array} \quad \begin{array}{ccc} \mathcal{C} & \xrightarrow{G} & \mathcal{D} \\ U \downarrow & \tau & \downarrow V \\ \mathcal{C}' & \xrightarrow{G'} & \mathcal{D}' \end{array}$$

that commute up to specified equivalences  $\alpha: F' \circ V \rightarrow U \circ F$  and  $\beta: V \circ G \rightarrow G' \circ U$ . We say that  $\sigma$  is a *left adjoint* to  $\tau$  and  $\tau$  is a *right adjoint* to  $\sigma$ , if  $F$  is a left adjoint of  $G$ ,  $F'$  is a left adjoint of  $G'$ , and  $\alpha$  is equivalent to the composite transformation

$$F' \circ V \rightarrow F' \circ V \circ G \circ F \xrightarrow{\beta} F' \circ G' \circ U \circ F \rightarrow U \circ F.$$

*Remark 1.4.2.* The diagram  $\tau$  has a left adjoint if and only if  $\tau$  is left adjointable in the sense of [HTT, 7.3.1.2] and [HA, 4.7.5.13]. If  $G$  and  $G'$  are equivalences, then  $\tau$  is left adjointable. We have analogous notions for ordinary categories. A square  $\tau$  of  $\infty$ -categories is left adjointable if and only if  $G$  and  $G'$  admit left adjoints and the square  $h\tau$  of homotopy categories is left adjointable. When visualizing a square  $\Delta^1 \times \Delta^1 \rightarrow \mathcal{C}$ , we adopt the convention that the first factor of  $\Delta^1 \times \Delta^1$  is vertical and the second factor is horizontal.

**Lemma 1.4.3.** *Consider a diagram of right Quillen functors*

$$\begin{array}{ccc} \mathbf{A} & \xrightarrow{G} & \mathbf{B} \\ U \downarrow & & \downarrow V \\ \mathbf{A}' & \xrightarrow{G'} & \mathbf{B}' \end{array}$$

*of model categories, that commutes up to a natural equivalence  $\beta: V \circ G \rightarrow G' \circ U$  and is endowed with Quillen equivalences  $(F, G)$  and  $(F', G')$ . Assume that  $U$  preserves weak equivalences and all objects of  $\mathbf{B}'$  are cofibrant. Let  $\alpha$  be the composite transformation*

$$F' \circ V \rightarrow F' \circ V \circ G \circ F \xrightarrow{\beta} F' \circ G' \circ U \circ F \rightarrow U \circ F.$$

*Then for every fibrant-cofibrant object  $Y$  of  $\mathbf{B}$ , the morphism  $\alpha(Y): (F' \circ V)(Y) \rightarrow (U \circ F)(Y)$  is a weak equivalence.*

*Proof.* The square  $R\beta$

$$\begin{array}{ccc} h\mathbf{A} & \xrightarrow{RG} & h\mathbf{B} \\ RU \downarrow & & \downarrow RV \\ h\mathbf{A}' & \xrightarrow{RG'} & h\mathbf{B}' \end{array}$$

of homotopy categories is left adjointable. Let  $\sigma: LF' \circ RV \rightarrow RU \circ LF$  be its left adjoint. For fibrant-cofibrant  $Y$ ,  $\alpha(Y)$  computes  $\sigma(Y)$ .  $\square$

We apply Lemma 1.4.3 to the straightening functor [HTT, 3.2.1]. Let  $p: S' \rightarrow S$  be a map of simplicial sets, and  $\pi: \mathcal{C}' \rightarrow \mathcal{C}$  a functor of simplicial categories fitting into a diagram

$$\begin{array}{ccc} \mathcal{C}[S'] & \xrightarrow{\phi'} & \mathcal{C}'^{op} \\ \mathcal{C}[p] \downarrow & & \downarrow \pi^{op} \\ \mathcal{C}[S] & \xrightarrow{\phi} & \mathcal{C}^{op} \end{array}$$

which is commutative up to a simplicial natural equivalence. By [HTT, 3.2.1.4], we have a diagram

$$\begin{array}{ccc} (\mathrm{Set}_{\Delta}^+)^{\mathcal{C}} & \xrightarrow{\mathrm{Un}_{\phi}^+} & (\mathrm{Set}_{\Delta}^+)_{/S} \\ \pi^* \downarrow & & \downarrow p^* \\ (\mathrm{Set}_{\Delta}^+)^{\mathcal{C}'} & \xrightarrow{\mathrm{Un}_{\phi'}^+} & (\mathrm{Set}_{\Delta}^+)_{/S'}, \end{array}$$

which satisfies the assumptions of Lemma 1.4.3 if  $\phi$  and  $\phi'$  are equivalences of simplicial categories. In this case, for every fibrant object  $f: X \rightarrow S$  of  $(\mathrm{Set}_{\Delta}^+)_{/S}$ , endowed with the Cartesian model structure, the morphism

$$(St_{\phi'}^+ \circ p^*)X \rightarrow (\pi^* \circ St_{\phi}^+)X$$

is a pointwise Cartesian equivalence.

Similarly, if  $g: \mathcal{C} \rightarrow \mathcal{D}$  is a functor of ( $\mathcal{V}$ -small) categories, then [HTT, 3.2.5.14] provides a diagram

$$\begin{array}{ccc} (\mathrm{Set}_{\Delta}^+)^{\mathcal{D}} & \xrightarrow{N_{\bullet}^+(\mathcal{D})} & (\mathrm{Set}_{\Delta}^+)_{/N(\mathcal{D})} \\ g^* \downarrow & & \downarrow N(g)^* \\ (\mathrm{Set}_{\Delta}^+)^{\mathcal{C}} & \xrightarrow{N_{\bullet}^+(\mathcal{C})} & (\mathrm{Set}_{\Delta}^+)_{/N(\mathcal{C})} \end{array}$$

satisfying the assumptions of Lemma 1.4.3. Thus for every fibrant object  $Y$  of  $(\mathrm{Set}_{\Delta}^+)_{/N(\mathcal{D})}$ , endowed with the coCartesian model structure, the morphism

$$\mathfrak{F}_{N(g)^*Y}^+(\mathcal{C}) \rightarrow g^* \mathfrak{F}_Y^+(\mathcal{D})$$

is a pointwise coCartesian equivalence.

**Proposition 1.4.4** (partial adjoint). *Consider quadruples  $(I, J, R, f)$  where  $I$  is a set,  $J \subseteq I$ ,  $R$  is an  $I$ -simplicial set and  $f: \delta_I^* R \rightarrow \mathrm{Cat}_{\infty}$  is a functor, satisfying the following conditions:*

- (1) *For every  $j \in J$  and every edge  $e$  of  $\epsilon_j^I R$ , the functor  $f(e)$  has a left adjoint.*
- (2) *For every  $i \in J^c := I \setminus J$ , every  $j \in J$  and every  $\tau \in (\epsilon_{i,j}^I R)_{1,1}$ , the square  $f(\tau): \Delta^1 \times \Delta^1 \rightarrow \mathrm{Cat}_{\infty}$  is left adjointable.*

There exists a way to associate, to every such quadruple, a functor  $f_J: \delta_{I,J}^* R \rightarrow \mathbf{Cat}_\infty$ , satisfying the following conclusions:

- (1)  $f_J \mid \delta_{J^c}^*(\Delta^\iota)_* R = f \mid \delta_{J^c}^*(\Delta^\iota)_* R$ , where  $\iota: J^c \rightarrow I$  is the inclusion.
- (2) For every  $j \in J$  and every edge  $e$  of  $\epsilon_j^I R$ , the functor  $f_J(e)$  is a left adjoint of  $f(e)$ .
- (3) For every  $i \in J^c$ , every  $j \in J$  and every  $\tau \in (\epsilon_{i,j}^I R)_{1,1}$ , the square  $f_J(\tau)$  is a left adjoint of  $f(\tau)$ .
- (4) For two quadruples  $(I, J, R, f)$ ,  $(I', J', R', f')$  and maps  $\mu: I' \rightarrow I$ ,  $u: (\Delta^\mu)^* R' \rightarrow R$  such that  $J' = \mu^{-1}(J)$  and  $f' = f \circ \delta_{I'}^* u$ , the functor  $f'_{J'}$  is equivalent to  $f_J \circ \delta_{I,J}^* u$ .

Note that in Conclusion (1),  $\delta_{J^c}^*(\Delta^\iota)_* R$  is naturally a simplicial subset of both  $\delta_I^* R$  and  $\delta_{I,J}^* R$ . When visualizing  $(1,1)$ -simplices of  $\epsilon_{i,j}^I R$ , we adopt the convention that direction  $i$  is vertical and direction  $j$  is horizontal. If  $J^c$  is nonempty, then Condition (2) implies Condition (1), and Conclusion (3) implies Conclusion (2).

*Proof.* Recall that we have fixed a fibrant replacement functor  $\mathbf{Fibr}: \mathbf{Set}_\Delta^+ \rightarrow \mathbf{Set}_\Delta^+$ .

Let  $\sigma \in (\delta_{I,J}^* R)_n$  be an object of  $\Delta_{/\delta_{I,J}^* R}$ , corresponding to  $\Delta_J^{[n_i]_{i \in I}} \rightarrow R$ , where  $n_i = n$ . It induces a functor  $f(\sigma): \mathbf{N}(D) \simeq \Delta_J^{[n_i]_{i \in I}} \rightarrow \mathbf{Cat}_\infty$ , where  $D$  is the partially ordered set  $S \times T^{op}$  with  $S = [n]^{J^c}$  and  $T = [n]^J$ . This corresponds to a projectively fibrant simplicial functor  $\mathcal{F}: \mathfrak{C}[\mathbf{N}(D)] \rightarrow \mathbf{Set}_\Delta^+$ . Let  $\phi_D: \mathfrak{C}[\mathbf{N}(D)] \rightarrow D$  be the canonical equivalence of simplicial categories and put  $\mathcal{F}' = (\mathbf{Fibr}^D \circ St_{\phi_D}^+ \circ \mathbf{Un}_{\mathbf{N}(D)^{op}}^+) \mathcal{F}: D \rightarrow \mathbf{Set}_\Delta^+$ . We have weak equivalences

$$\begin{aligned} \mathcal{F} &\leftarrow (St_{\mathbf{N}(D)^{op}}^+ \circ \mathbf{Un}_{\mathbf{N}(D)^{op}}^+) \mathcal{F} \rightarrow (\phi_D^* \circ \phi_{D!} \circ St_{\mathbf{N}(D)^{op}}^+ \circ \mathbf{Un}_{\mathbf{N}(D)^{op}}^+) \mathcal{F} \\ &\simeq (\phi_D^* \circ St_{\phi_D}^+ \circ \mathbf{Un}_{\mathbf{N}(D)^{op}}^+) \mathcal{F} \rightarrow \phi_D^*(\mathcal{F}'). \end{aligned}$$

Thus, for every  $\tau \in (\epsilon_{i,j}^I \mathbf{N}(D))_{1,1}$ , the square  $\mathcal{F}'(\tau)$  is equivalent to  $f(\tau)$ , both taking values in  $\mathbf{Cat}_\infty$ .

Let  $\mathcal{F}''$  be the composition

$$S \rightarrow (\mathbf{Set}_\Delta^+)^{T^{op}} \xrightarrow{\mathbf{Un}_{\phi_T}^+} (\mathbf{Set}_\Delta^+)_{/\mathbf{N}(T)},$$

where the first functor is induced by  $\mathcal{F}'$ . For every  $s \in S$ , the value  $\mathcal{F}''(s): X(s) \rightarrow \mathbf{N}(T)$  is a fibrant object of  $(\mathbf{Set}_\Delta^+)_{/\mathbf{N}(T)}$  with respect to the Cartesian model structure. In other words, there exists a Cartesian fibration  $p(s): Y(s) \rightarrow \mathbf{N}(T)$  and an isomorphism  $X(s) \simeq Y(s)^\natural$ . By Condition (1), for every morphism  $t \rightarrow t'$  of  $T$ , the induced functor  $Y(s)_{t'} \rightarrow Y(s)_t$  has a left adjoint. By [HTT, 5.2.2.5],  $p(s)$  is also a coCartesian fibration. We consider the object  $(p(s), \mathcal{E}(s))$  of  $(\mathbf{Set}_\Delta^+)_{/\mathbf{N}(T)}$ , where  $\mathcal{E}(s)$  is the set of  $p$ -coCartesian edges of  $Y(s)$ . By Condition (2), this construction is functorial in  $s$ , giving rise to a functor  $\mathcal{G}': S \rightarrow (\mathbf{Set}_\Delta^+)_{/\mathbf{N}(T)}$ .

The composition

$$S \xrightarrow{\mathcal{G}'} (\mathbf{Set}_\Delta^+)_{/\mathbf{N}(T)} \xrightarrow{\mathfrak{F}_\bullet^+(T)} (\mathbf{Set}_\Delta^+)^T \xrightarrow{\mathbf{Fibr}^T} (\mathbf{Set}_\Delta^+)^T$$



induces a projectively fibrant diagram

$$\mathcal{G}: S \times T \rightarrow \mathrm{Set}_\Delta^+.$$

We denote by  $\mathcal{G}_\sigma: [n] \rightarrow \mathrm{Set}_\Delta^+$  the composition

$$[n] \rightarrow S \times T \rightarrow \mathrm{Set}_\Delta^+,$$

where the first functor is the diagonal functor. The construction of  $\mathcal{G}_\sigma$  is not functorial in  $\sigma$  because the straightening functors do not commute with pullbacks, even up to natural equivalences. Nevertheless, for every morphism  $d: \sigma \rightarrow \tilde{\sigma}$  in  $\Delta_{/\delta_{I,J}^*R}$ , we have a canonical morphism  $\mathcal{G}_\sigma \rightarrow d^*\mathcal{G}_{\tilde{\sigma}}$  in  $(\mathrm{Set}_\Delta^+)^{[n]}$ , which is a weak equivalence by Lemma 1.4.3. The functor

$$(\Delta_{/\delta_{I,J}^*R})_{\sigma/} \rightarrow (\mathrm{Set}_\Delta^+)^{[n]}$$

sending  $d: \sigma \rightarrow \tilde{\sigma}$  to  $d^*\mathcal{G}_{\tilde{\sigma}}$  induces a map

$$\mathcal{N}(\sigma) := \mathcal{N}((\Delta_{/\delta_{I,J}^*R})_{\sigma/}) \rightarrow \mathrm{Map}^\sharp((\Delta^n)^b, (\mathrm{Cat}_\infty)^\natural),$$

which we denote by  $\Phi(\sigma)$ . Since the category  $(\Delta_{/\delta_{I,J}^*R})_{\sigma/}$  has an initial object, the simplicial set  $\mathcal{N}(\sigma)$  is weakly contractible. This construction is functorial in  $\sigma$  so that  $\Phi: \mathcal{N} \rightarrow \mathrm{Map}[\delta_{I,J}^*R, \mathrm{Cat}_\infty]$  is a morphism of  $(\mathrm{Set}_\Delta)^{(\Delta_{/\delta_{I,J}^*R})^{op}}$ . Applying Lemma 1.2.3 (1), we obtain a functor  $\widetilde{f}_J: \delta_{I,J}^*R \rightarrow \mathrm{Cat}_\infty$  satisfying Conclusions (2) and (3) up to homotopy.

Under the situation of Conclusion (4),  $\delta_{I,J}^*u: \delta_{I,J'}^*R' \rightarrow \delta_{I,J}^*R$  induces  $\varphi: \mathcal{N}' \rightarrow (\delta_{I,J}^*u)^*\mathcal{N}$ . By construction, there exists a homotopy between  $\Phi'$  and  $((\delta_{I,J}^*u)^*\Phi) \circ \varphi$ . By Lemma 1.2.3 (2), this implies that  $\widetilde{f}_{J'}$  and  $\widetilde{f}_J \circ \delta_{I,J'}^*u$  are homotopic.

By construction, there exists a homotopy between  $r^*\Phi$  and the composite map  $r^*\mathcal{N} \rightarrow \Delta_Q^0 \xrightarrow{f|Q} \mathrm{Map}[Q, \mathrm{Cat}_\infty]$ , where  $Q = \delta_{J^c}^*(\Delta^t)_*R$  and  $r: Q \rightarrow \delta_{I,J}^*R$  is the inclusion. By Lemma 1.2.3 (2), this implies that  $\widetilde{f}_J|Q$  and  $f|Q$  are homotopic. Since the inclusion

$$Q^\natural \times (\Delta^1)^\sharp \coprod_{Q^\natural \times (\Delta^{\{0\}})^\sharp} (\delta_{I,J}^*R)^\natural \times (\Delta^{\{0\}})^\sharp \rightarrow (\delta_{I,J}^*R)^\natural \times (\Delta^1)^\sharp$$

is marked anodyne, there exists  $f_J: \delta_{I,J}^*R \rightarrow \mathrm{Cat}_\infty$  homotopic to  $\widetilde{f}_J$  such that  $f_J|Q = f|Q$ .  $\square$

*Remark 1.4.5.* We have the following remarks concerning Proposition 1.4.4.

- (1) There is an obvious dual version of Proposition 1.4.4 for right adjoints.
- (2) Proposition 1.4.4 holds without the (implicit) convention that  $R$  is  $\mathcal{V}$ -small.

To see this, it suffices to apply the proposition to the composite map  $\delta_I^*R \xrightarrow{f} \mathrm{Cat}_\infty \rightarrow \mathrm{Cat}_\infty^\mathcal{W}$ , where  $\mathcal{W} \supseteq \mathcal{V}$  is a universe containing  $R$  and  $\mathrm{Cat}_\infty^\mathcal{W}$  is the  $\infty$ -category of  $\infty$ -categories in  $\mathcal{W}$ .

- (3) Consider the 2-tiled  $\infty$ -category  $(\mathrm{Cat}_\infty, \mathcal{T})$  where  $\mathcal{T}_1 = (\mathrm{Cat}_\infty)_1$ ,  $\mathcal{T}_2$  consists of all functors that admit a left adjoint, and  $\mathcal{T}_{12}$  consists of all squares that are left adjointable. Let

$$\phi: \delta_*^{2\Box}(\mathrm{Cat}_\infty, \mathcal{T}) \hookrightarrow \delta_2^*\delta_{2*}\mathrm{Cat}_\infty \rightarrow \mathrm{Cat}_\infty$$

be the natural functor induced by the counit map. Applying Proposition 1.4.4 (and Remark 1.4.5 (2)) to the quadruple  $(\{1, 2\}, \{2\}, \delta_*^{2\Box}(\mathcal{C}at_\infty, \mathcal{T}), \phi)$ , we get a functor

$$\phi_{\{2\}}: \delta_{2, \{2\}}^* \delta_*^{2\Box}(\mathcal{C}at_\infty, \mathcal{T}) \rightarrow \mathcal{C}at_\infty.$$

This functor is *universal* in the sense that for any quadruple  $(I, J, R, f)$  satisfying the conditions in Proposition 1.4.4, if we denote by  $\mu: I \rightarrow \{1, 2\}$  the map given by  $\mu^{-1}(2) = J$ , then  $f: \delta_2^*(\Delta^\mu)^* R \rightarrow \mathcal{C}at_\infty$  uniquely determines a map  $u: (\Delta^\mu)^* R \rightarrow \delta_{2*} \mathcal{C}at_\infty$  by adjunction which factorizes through  $\delta_*^{2\Box}(\mathcal{C}at_\infty, \mathcal{T})$  and  $f_J$  can be taken to be the composite functor

$$\delta_{I,J}^* R \simeq \delta_{2, \{2\}}^* (\Delta^\mu)^* R \xrightarrow{\delta_{2, \{2\}}^{*u}} \delta_{2, \{2\}}^* \delta_*^{2\Box}(\mathcal{C}at_\infty, \mathcal{T}) \xrightarrow{\phi_{\{2\}}} \mathcal{C}at_\infty.$$

- (4) For the quadruple  $(\{1\}, \{1\}, \mathcal{P}r^R, \phi)$  where  $\phi: \mathcal{P}r^R \rightarrow \mathcal{C}at_\infty$  is the natural inclusion, the functor  $\phi_{\{1\}}$  constructed in Proposition 1.4.4 induces an equivalence  $\phi_{\mathcal{P}r}: (\mathcal{P}r^R)^{op} \rightarrow \mathcal{P}r^L$ . This gives another proof of the second assertion of [HTT, 5.5.3.4]. By restriction, this equivalence induces an equivalence  $\phi_{\mathcal{P}r_{st}}: \mathcal{P}r_{st}^L \rightarrow (\mathcal{P}r_{st}^R)^{op}$  of  $\infty$ -categories.
- (5) For the quadruple  $(\{1, 2\}, \{1\}, S^{op} \boxtimes \text{Fun}^{LAd}(S^{op}, \mathcal{C}at_\infty), f)$  where

$$f: S^{op} \times \text{Fun}^{LAd}(S^{op}, \mathcal{C}at_\infty) \rightarrow \mathcal{C}at_\infty$$

is the evaluation map, the functor

$$f_{\{1\}}: S \times \text{Fun}^{LAd}(S^{op}, \mathcal{C}at_\infty) \rightarrow \mathcal{C}at_\infty$$

constructed in Proposition 1.4.4 induces an equivalence  $\text{Fun}^{LAd}(S^{op}, \mathcal{C}at_\infty) \rightarrow \text{Fun}^{RAd}(S, \mathcal{C}at_\infty)$ . This gives an alternative proof of [HA, 4.7.5.18 (3)].

**1.5. Symmetric monoidal  $\infty$ -categories.** Let  $\mathcal{F}in_*$  be the category of pointed finite sets defined in [HA, 2.0.0.2]. It is (equivalent to) the category whose objects are sets  $\langle n \rangle = \langle n \rangle^\circ \cup \{*\}$ , where  $\langle n \rangle^\circ = \{1, \dots, n\}$  ( $\langle 0 \rangle^\circ = \emptyset$ ) for  $n \geq 0$ , and morphisms are maps of sets that map  $*$  to  $*$ .

Let  $\mathcal{C}$  be an  $\infty$ -category that admits finite products. By [HA, 2.4.1.5], we have a symmetric monoidal  $\infty$ -category [HA, 2.0.0.7]  $\mathcal{C}^\times \rightarrow \mathcal{N}(\mathcal{F}in_*)$ , known as the *Cartesian symmetric monoidal  $\infty$ -category associated to  $\mathcal{C}$* . We put  $\mathcal{C}Alg(\mathcal{C}) = \mathcal{C}Alg(\mathcal{C}^\times)$  [HA, 2.1.3.1] as the  $\infty$ -category of commutative algebra objects in  $\mathcal{C}$ . We have the functor

$$(1.1) \quad G: \mathcal{C}Alg(\mathcal{C}) \rightarrow \mathcal{C}$$

by evaluating at  $\langle 1 \rangle$ .

*Remark 1.5.1.* In the above construction, if we put  $\mathcal{C} = \mathcal{C}at_\infty$ , then  $\mathcal{C}Alg(\mathcal{C}at_\infty)$  is canonically equivalent to  $\mathcal{C}at_\infty^\otimes$ , the  $\infty$ -category of symmetric monoidal  $\infty$ -categories [HA, 2.1.4.13]. The functor  $G$  restricts to a functor  $\mathcal{C}Alg(\mathcal{C}at_\infty) \rightarrow \mathcal{C}at_\infty$  sending  $\mathcal{C}^\otimes$  to its underlying  $\infty$ -category  $\mathcal{C}$ .

Recall that a symmetric monoidal  $\infty$ -category  $\mathcal{C}^\otimes$  is *closed* [HA, 4.1.1.17] if the functor  $- \otimes -: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , written as  $\mathcal{C} \rightarrow \text{Fun}(\mathcal{C}, \mathcal{C})$ , factorizes through  $\text{Fun}^L(\mathcal{C}, \mathcal{C})$ .

**Definition 1.5.2.** We define a subcategory  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr}}^L$  (resp.  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st}}^L$ ) of  $\mathrm{CAlg}(\mathrm{Cat}_\infty)$  as follows:

- An object that belongs to this subcategory is a symmetric monoidal  $\infty$ -categories  $\mathcal{C}^\otimes$  such that  $\mathcal{C} = G(\mathcal{C}^\otimes)$  is presentable (resp. and stable).
- A morphism that belongs to this subcategory is a symmetric monoidal functor  $F^\otimes: \mathcal{C}^\otimes \rightarrow \mathcal{D}^\otimes$  such that the underlying functor  $F = G(F^\otimes)$  is a left adjoint functor.

In particular, we have functors

$$G: \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr}}^L \rightarrow \mathrm{Pr}^L, \quad G: \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st}}^L \rightarrow \mathrm{Pr}_{\mathrm{st}}^L.$$

Moreover, we define  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{cl}} \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty)$ ,  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,cl}}^L \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr}}^L$  and  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st,cl}}^L \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st}}^L$  to be the full subcategories spanned by closed symmetric monoidal  $\infty$ -categories.

*Remark 1.5.3.* The  $\infty$ -categories  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,cl}}^L$  and  $\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st,cl}}^L$  admit small limits and such limits are preserved under the inclusions

$$\mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st,cl}}^L \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,cl}}^L \subseteq \mathrm{CAlg}(\mathrm{Cat}_\infty).$$

In fact, we only have to show that for a small simplicial set  $S$  and a diagram  $p^\otimes: S \rightarrow \mathrm{CAlg}(\mathrm{Pr}^L)$  such that  $p^\otimes(s) = \mathcal{C}_s^\otimes$  is closed for every vertex  $s$  of  $S$ , the limit  $\varprojlim(p^\otimes)$  is closed. Let  $p: S \rightarrow \mathrm{CAlg}(\mathrm{Pr}^L) \rightarrow \mathrm{Pr}^L$  (resp.  $p': S \rightarrow \mathrm{CAlg}(\mathrm{Pr}^L) \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ ) be the diagram induced by evaluating at the object  $\langle 1 \rangle$  (resp. unique active map  $\langle 2 \rangle \rightarrow \langle 1 \rangle$ ) of  $N(\mathcal{F}\mathrm{in}_*)$ . For every object  $c$  of  $\mathcal{C} = \varprojlim(p)$ , the diagram  $p'$  induces a diagram  $p'_c: S \rightarrow \mathrm{Fun}(\Delta^1, \mathrm{Pr}^L)$  such that  $p'_c(s)$  is the functor  $f_s^* c \otimes -: \mathcal{C}_s \rightarrow \mathcal{C}_s$  that admits right adjoints, where  $f_s^*: \mathcal{C} \rightarrow \mathcal{C}_s$  is the obvious functor. Since  $\mathrm{Pr}^L \subseteq \mathrm{Cat}_\infty$  is stable under small limits, the limit  $\varprojlim(p'_c)$  is an object of  $\mathrm{Fun}^L(\mathcal{C}, \mathcal{C})$ , which shows that the limit  $\varprojlim(p^\otimes)$  is closed.

A diagram  $p: S^\triangleleft \rightarrow \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st,cl}}^L$  is a limit diagram if and only if  $G \circ p: S^\triangleleft \rightarrow \mathrm{CAlg}(\mathrm{Cat}_\infty)_{\mathrm{pr,st,cl}}^L \xrightarrow{G} \mathrm{Cat}_\infty$  is a limit diagram, by the dual version of [HTT, 5.1.2.3].

Let  $\mathcal{C}$  be an  $\infty$ -category. Recall that by [HA, 2.4.3.1, 2.4.3.3], we have an  $\infty$ -operad  $p: \mathcal{C}^\mathrm{II} \rightarrow N(\mathcal{F}\mathrm{in}_*)$ . Suppose that  $\mathcal{C}$  is a fibrant simplicial category. We define  $\mathcal{C}^\mathrm{II}$  to be the fibrant simplicial category such that an object of  $\mathcal{C}^\mathrm{II}$  consists of an object  $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$  together with a sequence of objects  $(Y_1, \dots, Y_n)$  in  $\mathcal{C}$ , and

$$\mathrm{Map}_{\mathcal{C}^\mathrm{II}}((X_1, \dots, X_m), (Y_1, \dots, Y_n)) = \coprod_{\alpha} \prod_{i \in \alpha^{-1}\langle n \rangle^\circ} \mathrm{Map}_{\mathcal{C}}(X_i, Y_{\alpha(i)}),$$

where  $\alpha$  runs through all maps of pointed sets from  $\langle m \rangle$  to  $\langle n \rangle$ . By construction, we have a forgetful functor  $\mathcal{C}^\mathrm{II} \rightarrow \mathcal{F}\mathrm{in}_*$ , and its simplicial nerve  $N(\mathcal{C}^\mathrm{II}) \rightarrow N(\mathcal{F}\mathrm{in}_*)$  is canonically isomorphic to  $N(\mathcal{C})^\mathrm{II} \rightarrow N(\mathcal{F}\mathrm{in}_*)$ .

**Definition 1.5.4.** Let  $p: \mathcal{C} \rightarrow N(\mathcal{F}\mathrm{in}_*)$  be a functor of  $\infty$ -categories. We say that a diagram in  $\mathcal{C}$  is *p-static* (or simply *static* if  $p$  is clear) if its composition with  $p$  is constant.

**Lemma 1.5.5.** *Let  $\mathcal{C}$  be an  $\infty$ -category that admits finite colimits. Then a square*

$$(1.2) \quad \begin{array}{ccc} (X_1, \dots, X_m) & \longrightarrow & (Y_1, \dots, Y_n) \\ \downarrow & & \downarrow \\ (X'_1, \dots, X'_m) & \longrightarrow & (Y'_1, \dots, Y'_n) \end{array}$$

*in  $\mathcal{C}^\Pi$  with static vertical morphisms is a pushout square if and only if for every  $1 \leq j \leq n$ , the induced square*

$$(1.3) \quad \begin{array}{ccc} \coprod_{\alpha(i)=j} X_i & \longrightarrow & Y_j \\ \downarrow & & \downarrow \\ \coprod_{\alpha(i)=j} X'_i & \longrightarrow & Y'_j \end{array}$$

*in  $\mathcal{C}$  is a pushout square.*

*Proof.* It follows from the fact that for every pair of objects  $\{X_i\}_{1 \leq i \leq m}$ ,  $\{Y_j\}_{1 \leq j \leq n}$  of  $\mathcal{C}^\Pi$ , the mapping space  $\mathrm{Map}_{\mathcal{C}^\Pi}(\{X_i\}_{1 \leq i \leq m}, \{Y_j\}_{1 \leq j \leq n})$  is naturally equivalent to

$$\coprod_{\alpha \in \mathrm{Hom}_{\mathcal{F}\mathrm{in}_*}(\langle m \rangle, \langle n \rangle)} \coprod_{i \in \alpha^{-1}\langle n \rangle^\circ} \mathrm{Map}_{\mathcal{C}}(X_i, Y_{\alpha(i)}),$$

and the discussion in [HTT, 4.4.2]. □

*Remark 1.5.6.* Let  $\mathbf{T}: \mathcal{C}^\Pi \rightarrow \mathrm{Cat}_\infty$  be a functor that is a *weak Cartesian structure* [HA, 2.4.1.1]. Then we have an induced  $\infty$ -operad map  $\mathbf{T}^\otimes: \mathcal{C}^\Pi \rightarrow \mathrm{Cat}_\infty^\times$  [HA, 2.4.1.7], which is an object of  $\mathrm{Alg}_{\mathcal{C}^\Pi}(\mathrm{Cat}_\infty^\times)$ . The choice of such  $\mathbf{T}^\otimes$  is parameterized by a trivial Kan complex. Since the obvious map  $\mathrm{Alg}_{\mathcal{C}^\Pi}(\mathrm{Cat}_\infty^\times) \rightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{CAlg}(\mathrm{Cat}_\infty))$  is a trivial Kan fibration [HA, 2.4.3.18], in what follows, we will regard  $\mathbf{T}^\otimes$  as a functor  $\mathcal{C} \rightarrow \mathrm{CAlg}(\mathrm{Cat}_\infty)$ .

## 2. ENHANCED OPERATIONS FOR RINGED TOPOI

In this chapter, we construct a functor  $\mathbf{T}$  (2.1) and its induced functor  $\mathbf{T}^\otimes$  (2.2) that enhance the derived  $*$ -pullback and derived tensor product for ringed topoi. It also encodes the symmetric monoidal structures in a homotopy-coherent way. This serves as a starting point for the construction of the enhanced operation map.

The construction is based on the flat model structure. This marks a major difference with the study of quasi-coherent sheaves. For the latter one can simply start with the dual version of the model structure constructed in [HA, 1.3.5.3], because the category of quasi-coherent sheaves on affine schemes have enough projectives. The flat model structure for a ringed topological space has been constructed by [Gil06, Gil07]. In §2.1, we adapt the construction to every topos with enough points.

**2.1. The flat model structure.** Let  $(X, \mathcal{O}_X)$  be a ringed topos. In other words,  $X$  is a (Grothendieck) topos and  $\mathcal{O}_X$  is a sheaf of rings in  $X$ . An  $\mathcal{O}_X$ -module  $C$  is called *cotorsion* if  $\text{Ext}^1(F, C) = 0$  for every flat  $\mathcal{O}_X$ -module  $F$ . The following definition is a special case of [Gil07, 2.1].

**Definition 2.1.1.** Let  $K$  be a cochain complex of  $\mathcal{O}_X$ -modules.

- $K$  is called a *flat complex* if it is exact and  $Z^n K$  is flat for all  $n$ .
- $K$  is called a *cotorsion complex* if it is exact and  $Z^n K$  is cotorsion for all  $n$ .
- $K$  is called a *dg-flat complex* if  $K^n$  is flat for every  $n$ , and every cochain map  $K \rightarrow C$ , where  $C$  is a cotorsion complex, is homotopic to zero.
- $K$  is called a *dg-cotorsion complex* if  $K^n$  is cotorsion for every  $n$ , and every cochain map  $F \rightarrow K$ , where  $F$  is a flat complex, is homotopic to zero.

**Lemma 2.1.2.** Let  $(f, \gamma): (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$  be a morphism of ringed topoi. Then

- (1)  $(f, \gamma)^*$  preserves flat modules, flat complexes, and dg-flat complexes;
- (2)  $(f, \gamma)_*$  preserves cotorsion modules, cotorsion complexes, and dg-cotorsion complexes.

Recall that the functor  $(f, \gamma)^* = \mathcal{O}_Y \otimes_{f^* \mathcal{O}_X} f^* - : \text{Mod}(X, \mathcal{O}_X) \rightarrow \text{Mod}(Y, \mathcal{O}_Y)$  is a left adjoint of the functor  $(f, \gamma)_* : \text{Mod}(Y, \mathcal{O}_Y) \rightarrow \text{Mod}(X, \mathcal{O}_X)$ .

*Proof.* Let  $F \in \text{Mod}(X, \mathcal{O}_X)$  be flat, and  $C \in \text{Mod}(Y, \mathcal{O}_Y)$  cotorsion. We have a monomorphism  $\text{Ext}^1(F, (f, \gamma)_* C) \rightarrow \text{Ext}^1((f, \gamma)^* F, C) = 0$ . Thus  $(f, \gamma)_* C$  is cotorsion. Moreover, since short exact sequences of cotorsion  $\mathcal{O}_Y$ -modules are exact as sequences of presheaves,  $(f, \gamma)_*$  preserves short exact sequences of cotorsion modules, hence it preserves cotorsion complexes. It follows that  $(f, \gamma)^*$  preserves dg-flat complexes.

It is well known that  $(f, \gamma)^*$  preserves flat modules and short exact sequences of flat modules. It follows that  $(f, \gamma)^*$  preserves flat complexes and hence  $(f, \gamma)_*$  preserves dg-cotorsion complexes.  $\square$

The model structure in the following generalization of [Gil07, 7.8] is called the *flat model structure*.

**Proposition 2.1.3.** Assume that  $X$  has enough points. Then there exists a combinatorial model structure on  $\text{Ch}(\text{Mod}(X, \mathcal{O}_X))$  such that

- The cofibrations are the monomorphisms with dg-flat cokernels.
- The fibrations are the epimorphisms with dg-cotorsion kernels.
- The weak equivalences are quasi-isomorphisms.

Furthermore, this model structure is monoidal with respect to the usual tensor product of chain complexes.

For a morphism  $(f, \gamma): (Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)$  of ringed topoi with enough points, the pair of functors  $((f, \gamma)^*, (f, \gamma)_*)$  is a Quillen adjunction between the categories  $\text{Ch}(\text{Mod}(Y, \mathcal{O}_Y))$  and  $\text{Ch}(\text{Mod}(X, \mathcal{O}_X))$  endowed with the flat model structures.

*Remark 2.1.4.* We have the following remarks about different model structures.

- (1) The functor  $\text{id}: \text{Ch}(\text{Mod}(X, \mathcal{O}_X))^{\text{flat}} \rightarrow \text{Ch}(\text{Mod}(X, \mathcal{O}_X))^{\text{inj}}$  is a right Quillen equivalence. Here  $\text{Ch}(\text{Mod}(X, \mathcal{O}_X))^{\text{flat}}$  (resp.  $\text{Ch}(\text{Mod}(X, \mathcal{O}_X))^{\text{inj}}$ ) is the model category  $\text{Ch}(\text{Mod}(X, \mathcal{O}_X))$  endowed with the flat model structure (resp. the injective model structure [HA, 1.3.5.3]).
- (2) If  $X = *$  and  $\mathcal{O}_X = R$  is a (commutative) ring, then  $\text{id}: \text{Ch}(\text{Mod}(*, R))^{\text{proj}} \rightarrow \text{Ch}(\text{Mod}(*, R))^{\text{flat}}$  is a *symmetric monoidal* left Quillen equivalence between symmetric monoidal model categories. Here  $\text{Ch}(\text{Mod}(*, R))^{\text{proj}}$  is the model category  $\text{Ch}(\text{Mod}(*, R))$  endowed with the (symmetric monoidal) projective model structure [HA, 7.1.2.11].

To prove Proposition 2.1.3, we adapt the proof of [Gil07, 7.8]. Let  $S$  be a site, and  $G$  a small topologically generating family [SGA4, II 3.0.1] of  $S$ . For a presheaf  $F$  on  $S$ , we put  $|F|_G = \sup_{U \in G} \text{card}(F(U))$ .

**Lemma 2.1.5.** *Let  $\beta \geq \text{card}(G)$  be an infinite cardinal such that  $\beta \geq \text{card}(\text{Hom}(U, V))$  for all  $U$  and  $V$  in  $G$ , and  $\kappa$  a cardinal  $\geq 2^\beta$ . Let  $F$  be a presheaf on  $S$  such that  $|F|_G \leq \kappa$ , and  $F^+$  the sheaf associated to  $F$ . Then  $|F^+|_G \leq \kappa$ .*

*Proof.* By the construction in [SGA4, II 3.5], we have  $F^+ = L LF$ , where

$$(LF)(U) = \varinjlim_{R \in J(U)} \text{Hom}_{\hat{S}}(R, F)$$

for  $U \in S$  in which  $J(U)$  is the set of sieves covering  $U$  and  $\hat{S}$  is the category of presheaves on  $S$ . By [SGA4, II 3.0.4] and its proof,  $|LF|_G \leq \beta^2 \kappa^{\beta^2} = \kappa$ .  $\square$

Let  $\mathcal{O}_S$  be a sheaf of rings on  $S$ . For an element  $U \in S$ , we denote by  $j_{U!}$  the left adjoint of the restriction functor  $\text{Mod}(S, \mathcal{O}_S) \rightarrow \text{Mod}(U, \mathcal{O}_U)$ . Using the fact that  $(j_{U!}\mathcal{O}_U)_{U \in G}$  is a family of flat generators of  $\text{Mod}(S, \mathcal{O}_S)$ , we have the following analogue of [Gil07, 7.7] with essentially the same proof.

**Lemma 2.1.6.** *Let  $\beta \geq \text{card}(G)$  be an infinite cardinal such that  $\beta \geq \text{card}(\text{Hom}(U, V))$  for all  $U$  and  $V$  in  $G$ . Let  $\kappa \geq \max\{2^\beta, |\mathcal{O}_S|_G\}$  be a cardinal such that  $j_{U!}\mathcal{O}_U$  is  $\kappa$ -generated for every  $U$  in  $G$ . Then the following conditions are equivalent for an  $\mathcal{O}_S$ -module  $F$ :*

- (1)  $|F|_G \leq \kappa$ ;
- (2)  $F$  is  $\kappa$ -generated;
- (3)  $F$  is  $\kappa$ -presentable.

Let  $F$  be an  $\mathcal{O}_S$ -premodule. We say that an  $\mathcal{O}_S$ -subpremodule  $E \subseteq F$  is  $G$ -pure if  $E(U) \subseteq F(U)$  is pure for every  $U$  in  $G$ . This implies that  $E^+ \subseteq F^+$  is pure. As in [EO02, 2.4], one proves the following.

**Lemma 2.1.7.** *Let  $\beta \geq \text{card}(G)$  be an infinite cardinal such that  $\beta \geq \text{card}(\text{Hom}(U, V))$  for all  $U$  and  $V$  in  $G$ . Let  $\kappa \geq \max\{2^\beta, |\mathcal{O}_S|_G\}$  be a cardinal, and let  $E \subseteq F$  be  $\mathcal{O}_S$ -premodules such that  $|E|_G \leq \kappa$ . Then there exists a  $G$ -pure  $\mathcal{O}_S$ -subpremodule  $E'$  of  $F$  containing  $E$  such that  $|E'|_G \leq \kappa$ .*

*Proof of Proposition 2.1.3.* We choose a site  $S$  of  $X$ , and a small topologically generating family  $G$ , and a cardinal  $\kappa$  satisfying the assumptions of Lemma 2.1.6. Using the previous lemmas, one shows as in the proof of [Gil07, 7.8] that the conditions of [Gil07, 4.12, 5.1] are satisfied for  $\kappa$ , which finishes the proof.  $\square$

*Remark 2.1.8.* Using the sheaves  $i_*(\mathbb{Q}/\mathbb{Z})$ , where  $i$  runs through points  $P \rightarrow X$  of  $X$ , one can show as in [Gil06, 5.6] that a complex  $K$  of  $\mathcal{O}_X$ -modules is dg-flat if and only if  $K^n$  is flat for each  $n$  and  $K \otimes_{\mathcal{O}_X} L$  is exact for each exact sequence  $L$  of  $\mathcal{O}_X$ -modules.

**2.2. Enhanced operations.** Let us start by recalling the category of ringed topoi.

**Definition 2.2.1.** Let  $\mathbf{RingedPTopos}$  be the  $(2, 1)$ -category of ringed  $\mathcal{U}$ -topoi in  $\mathcal{V}$  with enough points:

- An object of  $\mathbf{RingedPTopos}$  is a ringed topoi  $(X, \mathcal{O}_X)$  such that  $X$  has enough points.
- A morphism  $(X, \mathcal{O}_X) \rightarrow (X', \mathcal{O}_{X'})$  in  $\mathbf{RingedPTopos}$  is a morphism of ringed topoi in the sense of [SGA4, IV 13.3], namely a pair  $(f, \gamma)$ , where  $f: X \rightarrow X'$  is a morphism of topoi and  $\gamma: f^*\mathcal{O}_{X'} \rightarrow \mathcal{O}_X$ .
- A 2-morphism  $(f_1, \gamma_1) \rightarrow (f_2, \gamma_2)$  in  $\mathbf{RingedPTopos}$  is an equivalence  $\epsilon: f_1 \rightarrow f_2$  such that  $\gamma_2$  equals the composition  $f_2^*\mathcal{O}_{X'} \xrightarrow{\epsilon^*} f_1^*\mathcal{O}_{X'} \xrightarrow{\gamma_1} \mathcal{O}_X$ .
- Composition of morphisms and 2-morphisms are defined in the obvious way.

We sometimes simply write  $X$  for an object of  $\mathbf{RingedPTopos}$  if the structure sheaf is insensitive.

Our goal in this section is to construct a functor

$$(2.1) \quad \mathbf{T}: \mathbf{N}(\mathbf{RingedPTopos}^{op})^{\mathbf{H}} \rightarrow \mathbf{Cat}_{\infty}$$

that is a weak Cartesian structure such that the induced functor  $\mathbf{T}^{\otimes}$  (see Remark 1.5.6) factorizes through  $\mathbf{CAlg}(\mathbf{Cat}_{\infty})_{\text{pr, st, cl}}^L \subseteq \mathbf{CAlg}(\mathbf{Cat}_{\infty})$ . In other words, we have the induced functor

$$(2.2) \quad \mathbf{T}^{\otimes}: \mathbf{N}(\mathbf{RingedPTopos}^{op}) \rightarrow \mathbf{CAlg}(\mathbf{Cat}_{\infty})_{\text{pr, st, cl}}^L,$$

where  $\mathbf{CAlg}(\mathbf{Cat}_{\infty})_{\text{pr, st, cl}}^L$  is defined in Definition 1.5.2.

Let  $\mathbf{Cat}_1^+$  be the  $(2, 1)$ -category of *marked categories*, namely pairs  $(\mathcal{C}, \mathcal{E})$  consisting of an (ordinary) category  $\mathcal{C}$  and a set of arrows  $\mathcal{E}$  containing all identity arrows. We have a simplicial functor  $\mathbf{Cat}_1^+ \rightarrow \mathbf{Set}_{\Delta}^+$  sending  $(\mathcal{C}, \mathcal{E})$  to  $(\mathbf{N}(\mathcal{C}), \mathcal{E})$ . We start by constructing a pseudofunctor

$$\mathbf{T}: (\mathbf{RingedPTopos}^{op})^{\mathbf{H}} \rightarrow \mathbf{Cat}_1^+.$$

Recall that to every object  $X \in \mathbf{RingedPTopos}$ , we can associate a marked simplicial set

$$(\mathbf{N}(\mathbf{Ch}(\mathbf{Mod}(X))_{\text{dg-flat}}), W(X)),$$

where  $\mathbf{Ch}(\mathbf{Mod}(X))_{\text{dg-flat}} \subseteq \mathbf{Ch}(\mathbf{Mod}(X))$  is the full subcategory spanned by the dg-flat complexes, and  $W(X)$  is the set of quasi-isomorphisms. We define the image of an



object  $(X_1, \dots, X_m)$  under  $\mathbf{T}$  to be

$$\prod_{i=1}^m (\mathrm{Ch}(\mathrm{Mod}(X_i))_{\mathrm{dg-flat}}, W(X_i)).$$

By definition, a (1-)morphism  $f: (X_1, \dots, X_m) \rightarrow (Y_1, \dots, Y_n)$  in  $(\mathrm{RingedPTopos})^{\mathrm{op}}$  consists of a map  $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  and a morphism  $f_i: Y_{\alpha(i)} \rightarrow X_i$  in  $\mathrm{RingedPTopos}$  for every  $i \in \alpha^{-1}\langle n \rangle^\circ$ . Now we define the image of  $f$  under  $\mathbf{T}$  to be the functor

$$\prod_{i=1}^m (\mathrm{Ch}(\mathrm{Mod}(X_i))_{\mathrm{dg-flat}}, W(X_i)) \rightarrow \prod_{j=1}^n (\mathrm{Ch}(\mathrm{Mod}(Y_j))_{\mathrm{dg-flat}}, W(Y_j))$$

$$\{K_i\}_{1 \leq i \leq m} \mapsto \left\{ \bigotimes_{\alpha(i)=j} f_i^* K_i \right\}_{1 \leq j \leq n},$$

where we take the unit object as the tensor product over an empty set. The image of 2-morphisms are defined in the obvious way. Composing with the simplicial functor  $\mathrm{Cat}_1^+ \rightarrow \mathrm{Set}_\Delta^+ \xrightarrow{\mathrm{Fibr}} (\mathrm{Set}_\Delta^+)^{\circ}$  and taking nerves, we obtain the desired functor  $\mathbf{T}$  (2.1).

**Lemma 2.2.2.** *We have that*

- (1) *the functor  $\mathbf{T}$  is a weak Cartesian structure [HA, 2.4.1.1];*
- (2) *the functor  $\mathbf{T}^\otimes$  factorizes through  $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{L}}_{\mathrm{pr,st,cl}})$ ; and*
- (3) *the functor  $\mathbf{T}^\otimes$  sends small coproducts to products.*

*Proof.* Part (1) is clear from the construction.

For (2), we note that for an object  $X$  of  $\mathrm{RingedPTopos}$ , its image under  $\mathbf{T}$ , denoted by  $\mathcal{D}(X)$ , is the fibrant replacement of  $(\mathrm{N}(\mathrm{Ch}(\mathrm{Mod}(X))_{\mathrm{dg-flat}}), W(X))$ . In particular, by Remark 2.1.4 (1) and [HA, 1.3.4.16, 1.3.5.15],  $\mathcal{D}(X)$  is equivalent to the derived  $\infty$ -category of  $\mathrm{Mod}(X)$  defined in [HA, 1.3.5.8]. It is a presentable stable  $\infty$ -category by [HA, 1.3.5.9, 1.3.5.21 (1)]. Combining this with Lemma 1.1.3, we deduce that the image of  $\mathbf{T}^\otimes$  is actually contained in  $\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{L}}_{\mathrm{pr,st,cl}})$ . This proves part (2).

Part (3) follows from the construction and Remark 1.5.3.  $\square$

**Notation 2.2.3.** For an object  $X$  of  $\mathrm{RingedPTopos}$ , we denote the image of  $X$  under  $\mathbf{T}^\otimes$  by  $\mathcal{D}(X)^\otimes$ , which is a symmetric monoidal  $\infty$ -category, whose underlying  $\infty$ -category is denoted by  $\mathcal{D}(X)$  as in the proof of the previous lemma.

*Remark 2.2.4.* We have the following remarks.

- (1) The  $\infty$ -category  $\mathbf{T}((X_1, \dots, X_m))$  is equivalent to  $\prod_{i=1}^m \mathcal{D}(X_i)$ .
- (2) By Remark 2.1.4 (2) and [HA, 4.1.3.5], for every (commutative) ring  $R$ ,  $\mathcal{D}(*, R)^\otimes$  is equivalent to the symmetric monoidal  $\infty$ -category  $\mathcal{D}(\mathrm{Ch}(R))^\otimes$  defined in [HA, 7.1.2.12].
- (3) Let  $f: X \rightarrow X'$  be a morphism of  $\mathrm{RingedPTopos}$ . It follows from Remark 2.1.8 and [KS06, 14.4.1, 18.6.4] that the functors  $f^*: \mathcal{D}(X') \rightarrow \mathcal{D}(X)$  and  $- \otimes_X -: \mathcal{D}(X) \times \mathcal{D}(X) \rightarrow \mathcal{D}(X)$  induced by  $\mathbf{T}^\otimes$  are equivalent to the respective functors constructed in [KS06, 18.6], where  $\mathcal{D}(X) = \mathrm{h}\mathcal{D}(X)$  and  $\mathcal{D}(X') = \mathrm{h}\mathcal{D}(X')$ .

Let  $\mathbf{Ring}$  be the category of (small commutative) rings. To deal with torsion and adic coefficients simultaneously. We introduce the category  $\mathcal{R}\mathbf{ind}$  of ringed diagrams as follows.

**Definition 2.2.5** (Ringed diagram). We define a category  $\mathcal{R}\mathbf{ind}$  as follows:

- An object of  $\mathcal{R}\mathbf{ind}$  is a pair  $(\Xi, \Lambda)$ , called a *ringed diagram*, where  $\Xi$  is a small partially ordered set and  $\Lambda: \Xi^{op} \rightarrow \mathbf{Ring}$  is a functor. We identify  $(\Xi, \Lambda)$  with the topos of presheaves on  $\Xi$ , ringed by  $\Lambda$ . A typical example is  $(\mathbb{N}, n \mapsto \mathbb{Z}/\ell^{n+1}\mathbb{Z})$  with transition maps given by projections.
- A morphism of ringed diagrams  $(\Xi', \Lambda') \rightarrow (\Xi, \Lambda)$  is a pair  $(\Gamma, \gamma)$  where  $\Gamma: \Xi' \rightarrow \Xi$  is a functor (that is, an order-preserving map) and  $\gamma: \Gamma^*\Lambda := \Lambda \circ \Gamma^{op} \rightarrow \Lambda'$  is a morphism of  $\mathbf{Ring}^{\Xi'^{op}}$ .

For an object  $(\Xi, \Lambda)$  of  $\mathcal{R}\mathbf{ind}$  and an object  $\xi$  of  $\Xi$ , we define the *over ringed diagram*  $(\Xi, \Lambda)_{/\xi}$  to be the ringed diagram whose underlying category is  $\Xi_{/\xi}$  and the corresponding functor is  $\Lambda_{/\xi} := \Lambda|_{\Xi_{/\xi}}$ .

For a topos  $X$  and a small partially ordered set  $\Xi$ , we denote by  $X^\Xi$  the topos  $\mathbf{Fun}(\Xi^{op}, X)$ . If  $(\Xi, \Lambda)$  is a ringed diagram, then  $\Lambda$  defines a sheaf of rings on  $X^\Xi$ , which we still denote by  $\Lambda$ . We thus obtain a pseudofunctor

$$(2.3) \quad \mathcal{PTopos} \times \mathcal{R}\mathbf{ind} \rightarrow \mathcal{R}\mathbf{ingedPTopos}$$

carrying  $(X, (\Xi, \Lambda))$  to  $(X^\Xi, \Lambda)$ , where  $\mathcal{PTopos}$  is the  $(2, 1)$ -category of ringed topoi with enough points. Composing the nerve of (2.3) with  $\mathbf{T}$  (2.1), we obtain a functor

$$(2.4) \quad {}_{\mathcal{PTopos}}\mathbf{EO}^I: (\mathbf{N}(\mathcal{PTopos})^{op} \times \mathbf{N}(\mathcal{R}\mathbf{ind})^{op})^{\mathbf{II}} \rightarrow \mathbf{Cat}_\infty$$

that is a weak Cartesian structure.

**Notation 2.2.6.** By abuse of notation, we denote by  $\mathcal{D}(X, \lambda)^\otimes$  the image of an object  $(X, \lambda)$  of  $\mathcal{PTopos} \times \mathcal{R}\mathbf{ind}$  under the induced functor

$${}_{\mathcal{PTopos}}\mathbf{EO}^\otimes := ({}_{\mathcal{PTopos}}\mathbf{EO}^I)^\otimes: \mathbf{N}(\mathcal{PTopos})^{op} \times \mathbf{N}(\mathcal{R}\mathbf{ind})^{op} \rightarrow \mathbf{CAlg}(\mathbf{Cat}_\infty)_{\mathbf{pr}, \mathbf{st}, \mathbf{cl}}^L,$$

whose underlying  $\infty$ -category is denoted by  $\mathcal{D}(X, \lambda)$  which is (equivalent to) the image of  $(X, \lambda, \langle 1 \rangle, \{1\})$  under the functor  ${}_{\mathcal{PTopos}}\mathbf{EO}^I$ .

**Definition 2.2.7.** A morphism  $(\Gamma, \gamma): (\Xi', \Lambda') \rightarrow (\Xi, \Lambda)$  of  $\mathcal{R}\mathbf{ind}$  is said to be *perfect* if for every  $\xi \in \Xi'$ ,  $\Lambda'(\xi)$  is a perfect complex in the derived category of  $\Lambda(\Gamma(\xi))$ -modules.

**Lemma 2.2.8.** Let  $f: Y \rightarrow X$  be a morphism of  $\mathcal{PTopos}$ , and  $\pi: \lambda' \rightarrow \lambda$  a perfect morphism of  $\mathcal{R}\mathbf{ind}$ . Then the square

$$(2.5) \quad \begin{array}{ccc} \mathcal{D}(Y, \lambda') & \xleftarrow{f^*} & \mathcal{D}(X, \lambda') \\ \pi^* \uparrow & & \uparrow \pi^* \\ \mathcal{D}(Y, \lambda) & \xleftarrow{f^*} & \mathcal{D}(X, \lambda) \end{array}$$

is right adjointable and its transpose is left adjointable.

*Proof.* Write  $\lambda = (\Xi, \Lambda)$  and  $\lambda' = (\Xi', \Lambda')$ . We denote by  $e_\xi$  the natural morphism  $(\{\xi\}, \Lambda'(\xi)) \rightarrow (\Xi', \Lambda')$ . We show that (2.5) is right adjointable and  $\pi^*$  preserves small limits. As the family of functors  $(e_\xi^*)_{\xi \in \Xi'}$  is conservative, it suffices to show these assertions with  $\pi$  replaced by  $e_\xi$  and by  $\pi \circ e_\xi$ . In other words, we may assume  $\Xi' = \{*\}$ . We decompose  $\pi$  as

$$(\{*\}, \Lambda') \xrightarrow{t} (\{\zeta\}, \Lambda(\zeta)) \xrightarrow{s} (\Xi, \Lambda)_{/\zeta} \xrightarrow{i} (\Xi, \Lambda).$$

We show that the assertions hold with  $\pi^*$  replaced by  $i^*$ , by  $s^*$ , and by  $t^*$ . The assertions for  $i^*$  follow from Lemma 2.2.9 below. The assertions for  $s^*$  are trivial as  $s^* \simeq p_*$ , where  $p: (\Xi, \Lambda)_{/\zeta} \rightarrow (\{\zeta\}, \Lambda(\zeta))$ . As  $t_*$  is conservative, the assertions for  $t^*$  follow from the assertions for  $t_*$  and  $t_* t^* \simeq \mathcal{H}om_{\Lambda(\zeta)}(\Lambda'^\vee, -)$ , which are trivial. Here we used the fact that for any perfect complex  $M$  in the derived category of  $\Lambda(\zeta)$ -modules, the natural transformation  $M \otimes_{\Lambda(\zeta)} - \rightarrow \mathcal{H}om_{\Lambda(\zeta)}(M^\vee, -)$  is a natural equivalence, where  $M^\vee = \mathcal{H}om_{\Lambda(\zeta)}(M, \Lambda(\zeta))$ . This applies to  $M = \Lambda'$  by the assumption that  $\pi$  is perfect.  $\square$

**Lemma 2.2.9.** *Let  $f: (X', \Lambda') \rightarrow (X, \Lambda)$  be a morphism of ringed topoi, and  $j: V \rightarrow U$  a morphism of  $X$ . Put  $j' := f^{-1}(j): V' = f^{-1}(V) \rightarrow f^{-1}(U) = U'$ . Then the square*

$$\begin{array}{ccc} \mathcal{D}(X_{/U}, \Lambda \times U) & \xrightarrow{j^*} & \mathcal{D}(X_{/V}, \Lambda \times V) \\ f_{/U}^* \downarrow & & \downarrow f_{/V}^* \\ \mathcal{D}(X'_{/U'}, \Lambda' \times U') & \xrightarrow{j'^*} & \mathcal{D}(X'_{/V'}, \Lambda' \times V') \end{array}$$

*is left adjointable and its transpose is right adjointable.*

*Proof.* The functor  $j_!: \text{Mod}(X_{/V}, \Lambda \times V) \rightarrow \text{Mod}(X_{/U}, \Lambda \times U)$  is exact and induces a functor  $\mathcal{D}(X_{/V}, \Lambda \times V) \rightarrow \mathcal{D}(X_{/U}, \Lambda \times U)$ , left adjoint of  $j^*$ . The same holds for  $j'_!$ . The first assertion of the lemma follows from the existence of these left adjoints and the second assertion. The second assertion follows from the fact that  $j'^*$  preserves fibrant objects in  $\text{Ch}(\text{Mod}(-))^{\text{inj}}$ .  $\square$

### 3. ENHANCED OPERATIONS FOR SCHEMES

In this chapter, we construct the enhanced operation map for the category of co-products of quasi-compact and separated schemes, and establish several properties of the map. In §3.1, we introduce an abstract notion of (universal) descent and collect some basic properties. In §3.2, we construct the enhanced operation maps (3.3) and (3.8) based on the techniques developed in the previous two chapters. In §3.3, we establish some properties of the maps constructed in the previous sections, including an enhanced version of (co)homological descent for smooth coverings. This property is crucial for the extension of the enhanced operation map to algebraic spaces and stacks in Chapter 5.

**3.1. Abstract descent properties.** We start from the definition of morphisms with descent properties.

**Definition 3.1.1** (*F*-descent). Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks,  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  a functor of  $\infty$ -categories, and  $f: X_0^+ \rightarrow X_{-1}^+$  a morphism of  $\mathcal{C}$ . We say that  $f$  is of *F*-descent if  $F \circ (X_\bullet^+)^{op}: N(\Delta_+) \rightarrow \mathcal{D}$  is a limit diagram in  $\mathcal{D}$ , where  $X_\bullet^+: N(\Delta_+)^{op} \rightarrow \mathcal{C}$  is a Čech nerve of  $f$  (see the definition after [HTT, 6.1.2.11]). We say that  $f$  is of *universal F*-descent if every pullback of  $f$  in  $\mathcal{C}$  is of *F*-descent. Dually, for a functor  $G: \mathcal{C} \rightarrow \mathcal{D}$ , we say that  $f$  is of *G*-codescent (resp. of *universal G*-codescent) if it is of  $G^{op}$ -descent (resp. of universal  $G^{op}$ -descent).

We say that a morphism  $f$  of an  $\infty$ -category  $\mathcal{C}$  is a *retraction* if it is a retraction in the homotopy category  $h\mathcal{C}$ . Equivalently,  $f$  is a retraction if it can be completed into a *weak retraction diagram* [HTT, 4.4.5.4]  $\text{Ret} \rightarrow \mathcal{C}$  of  $\mathcal{C}$ , corresponding to a 2-cell of  $\mathcal{C}$  of the form

$$\begin{array}{ccc} & Y & \\ s \nearrow & & \searrow f \\ X & \xrightarrow{\text{id}_X} & X. \end{array}$$

The following is an  $\infty$ -categorical version of [Gir64, 10.10, 10.11] (for ordinary descent) and [SGA4, Vbis 3.3.1] (for cohomological descent). See also [TGxii, Proposition 1.5, Corollary 1.6, Remark 2.4].

**Lemma 3.1.2.** *Let  $\mathcal{C}$  be an  $\infty$ -category admitting pullbacks, and  $F: \mathcal{C}^{op} \rightarrow \mathcal{D}$  a functor of  $\infty$ -categories. Then*

- (1) *Every retraction  $f$  in  $\mathcal{C}$  is of universal  $F$ -descent.*
- (2) *Let*

$$(3.1) \quad \begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

*be a pullback diagram in  $\mathcal{C}$  such that the base change of  $f$  to  $(Z/X)^i$  is of  $F$ -descent for  $i \geq 0$  and the base change of  $p$  to  $(Y/X)^j$  is of  $F$ -descent for  $j \geq 1$ . Then  $p$  is of  $F$ -descent.*

- (3) *Let*

$$\begin{array}{ccc} & Y & \\ g \nearrow & & \searrow f \\ Z & \xrightarrow{h} & X \end{array}$$

*be a 2-cell of  $\mathcal{C}$  such that  $h$  is of universal  $F$ -descent. Then  $f$  is of universal  $F$ -descent.*

(4) Let

$$\begin{array}{ccc} & Y & \\ g \nearrow & & \searrow f \\ Z & \xrightarrow{h} & X \end{array}$$

be a 2-cell of  $\mathcal{C}$  such that  $f$  is of  $F$ -descent and  $g$  is of universal  $F$ -descent. Then  $h$  is of  $F$ -descent.

The assumptions on  $f$  and  $p$  in (2) are satisfied if  $f$  is of  $F$ -descent and  $g$  and  $q$  are of universal  $F$ -descent.

*Proof.* For (1), it suffices to show that  $f$  is of  $F$ -descent. Consider the map  $N(\Delta_+)^{op} \times \text{Ret} \rightarrow \mathcal{C}$ , right Kan extension along the inclusion

$$K = \{[-1]\} \times \text{Ret} \coprod_{\{[-1]\} \times \{\emptyset\}} N(\Delta_+^{\leq 0})^{op} \times \{\emptyset\} \subseteq N(\Delta_+)^{op} \times \text{Ret}$$

of the map  $K \rightarrow \mathcal{C}$  corresponding to the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\text{id}_Y} & Y \\ f \downarrow & & \downarrow f \\ & Y & \\ s \nearrow & & \searrow f \\ X & \xrightarrow{\text{id}_X} & X \end{array}$$

Then by [HA, 4.7.3.9], the Čech nerve of  $f$  is split. Therefore, the assertion follows from the dual version of [HTT, 6.1.3.16].

For (2), let  $X_{\bullet\bullet}^+ : N(\Delta_+)^{op} \times N(\Delta_+)^{op} \rightarrow \mathcal{C}$  be an augmented bisimplicial object of  $\mathcal{C}$  such that  $X_{\bullet\bullet}^+$  is a right Kan extension of (3.1), considered as a diagram  $N(\Delta_+^{\leq 0})^{op} \times N(\Delta_+^{\leq 0})^{op} \rightarrow \mathcal{C}$ . By assumption,  $F \circ (X_{i\bullet}^+)^{op}$  is a limit diagram in  $\mathcal{D}$  for  $i \geq -1$  and  $F \circ (X_{\bullet j}^+)^{op}$  is a limit diagram in  $\mathcal{D}$  for  $j \geq 0$ . By the dual version of [HTT, 5.5.2.3],  $F \circ (X_{\bullet-1}^+)^{op}$  is a limit diagram in  $\mathcal{D}$ , which proves (2) since  $X_{\bullet-1}^+$  is a Čech nerve of  $p$ .

For (3), it suffices to show that  $f$  is of  $F$ -descent. Consider the diagram

(3.2)

$$\begin{array}{ccccc} Z & & \xrightarrow{\text{id}_Z} & & Z \\ & \searrow & & \searrow & \\ & Y \times_X Z & \xrightarrow{\text{pr}_Z} & & Z \\ & \downarrow \text{pr}_Y & & \downarrow h & \\ & Y & \xrightarrow{f} & & X \end{array}$$

in  $\mathcal{C}$ . Since  $\text{pr}_Z$  is a retraction, it is of universal  $F$ -descent by (1). It then suffices to apply (2).

For (4), consider the diagram (3.2). By (3),  $\text{pr}_Y$  is of universal  $F$ -descent. It then suffices to apply (2).  $\square$

Next, we prove a descent lemma for general topoi. Let  $X$  be a topos that has enough points, with a fixed final object  $e$ . Let  $u_0: U_0 \rightarrow e$  be a covering, which induces a hypercovering  $u_\bullet: U_\bullet \rightarrow e$  by taking the Čech nerve. Let  $\Lambda$  be a sheaf of rings in  $X$ , and put  $\Lambda_n = \Lambda \times U_n$ . In particular, we obtain an augmented simplicial ringed topoi  $(X/U_\bullet, \Lambda_\bullet)$ , where  $U_{-1} = e$  and  $\Lambda_{-1} = \Lambda$ . Suppose that for every  $n \geq -1$ , we are given a strictly full subcategory  $\mathcal{C}_n$  ( $\mathcal{C} = \mathcal{C}_{-1}$ ) of  $\text{Mod}(X/U_n, \Lambda_n)$  such that for every morphism  $\alpha: [m] \rightarrow [n]$  of  $\Delta_+$ ,  $u_\alpha^*: \text{Mod}(X/U_n, \Lambda_n) \rightarrow \text{Mod}(X/U_m, \Lambda_m)$  sends  $\mathcal{C}_n$  to  $\mathcal{C}_m$ . Then, applying the functor  $G \circ \mathbf{T}^\otimes$  (2.2), we obtain an augmented cosimplicial  $\infty$ -category  $\mathcal{D}_\bullet(X/U_\bullet, \Lambda_\bullet)$ , where  $\mathcal{D}_{\mathcal{C}_n}(X/U_n, \Lambda_n)$  is the full subcategory of  $\mathcal{D}(X/U_n, \Lambda_n)$  spanned by complexes whose cohomology sheaves belong to  $\mathcal{C}_n$ .

**Lemma 3.1.3.** *Assume that for every object  $\mathcal{F}$  of  $\text{Mod}(X, \Lambda)$  such that  $u_{d_0}^* \mathcal{F}$  belongs to  $\mathcal{C}_0$ , we have  $\mathcal{F} \in \mathcal{C}$ . Then the natural map*

$$\mathcal{D}_\mathcal{C}(X, \Lambda) \rightarrow \varprojlim_{n \in \Delta} \mathcal{D}_{\mathcal{C}_n}(X/U_n, \Lambda_n)$$

*is an equivalence of  $\infty$ -categories.*

*Proof.* We first consider the case where  $\mathcal{C}_n = \text{Mod}(X/U_n, \Lambda_n)$  for  $n \geq -1$ . We apply [HA, 4.7.6.3]. Assumption (1) follows from the fact that  $u_{d_0}^*: \mathcal{D}(X, \Lambda) \rightarrow \mathcal{D}(X/U_0, \Lambda_0)$  is a morphism of  $\text{Pr}_{\text{st}}^{\text{L}}$ . Moreover, the functor  $u_{d_0}^*$  is conservative since  $u_0$  is a covering. Therefore, we only need to check Assumption (2) of [HA, 4.7.6.3], that is, the left adjointability of the diagram

$$\begin{array}{ccc} \mathcal{D}(X/U_m, \Lambda_m) & \xrightarrow{u_{d_0}^{*m+1}} & \mathcal{D}(X/U_{m+1}, \Lambda_{m+1}) \\ u_\alpha^* \downarrow & & \downarrow u_{\alpha'}^* \\ \mathcal{D}(X/U_n, \Lambda_n) & \xrightarrow{u_{d_0}^{*n+1}} & \mathcal{D}(X/U_{n+1}, \Lambda_{n+1}) \end{array}$$

for every morphism  $\alpha: [m] \rightarrow [n]$  of  $\Delta_+$ , where  $\alpha': [m+1] \rightarrow [n+1]$  is the induced morphism. This is a special case of Lemma 2.2.9.

Now the general case follows from Lemma 3.1.4 below and the fact that  $u_{d_0}^*$  is exact.  $\square$

**Lemma 3.1.4.** *Let  $p: K^\triangleleft \rightarrow \text{Cat}_\infty$  be a limit diagram. Suppose that for each vertex  $k$  of  $K^\triangleleft$ , we are given a strictly full subcategory  $\mathcal{D}_k \subseteq \mathcal{C}_k = p(k)$  such that*

- (1) *For every morphism  $f: k \rightarrow k'$ , the induced functor  $p(f)$  sends  $\mathcal{D}_k$  to  $\mathcal{D}_{k'}$ .*
- (2) *An object  $c$  of  $\mathcal{C}_\infty$  belongs to  $\mathcal{D}_\infty$  if and only if for every vertex  $k$  of  $K$ ,  $p(f_k)(c)$  belongs to  $\mathcal{D}_k$ , where  $\infty$  denotes the cone point of  $K^\triangleleft$ ,  $f_k: \infty \rightarrow k$  is the unique edge.*

*Then the induced diagram  $q: K^\triangleleft \rightarrow \text{Cat}_\infty$  sending  $k$  to  $\mathcal{D}_k$  is also a limit diagram.*

*Proof.* Let  $\tilde{p}: X \rightarrow (K^{op})^\triangleright$  be a Cartesian fibration classified by  $p$  [HTT, 3.3.2.2]. Let  $Y \subseteq X$  be the simplicial subset spanned by vertices in each fiber  $X_k$  that are in the essential image of  $\mathcal{D}_k$  for all vertices  $k$  of  $K^\triangleleft$ . The map  $\tilde{q} = \tilde{p}|_Y: Y \rightarrow (K^{op})^\triangleright$  has

the property that if  $f: x \rightarrow y$  is  $\tilde{p}$ -Cartesian and  $y$  belongs to  $Y$ , then  $x$  also belongs to  $Y$  by assumption (1), and  $f$  is  $\tilde{q}$ -Cartesian by the dual version of [HTT, 2.4.1.8]. It follows that  $\tilde{q}$  is a Cartesian fibration, which is in fact classified by  $q$ . By assumption (2) and [HTT, 3.3.3.2],  $q$  is a limit diagram.  $\square$

### 3.2. Enhanced operation map.

**Notation 3.2.1.** For a property (P) in the category  $\mathcal{R}\text{ing}$ , we say that a ringed diagram  $(\Gamma, \Lambda)$  (Definition 2.2.5) has the property (P) if for every object  $\xi$  of  $\Xi$ , the ring  $\Lambda(\xi)$  has the property (P). We denote by  $\mathcal{R}\text{ind}_{\text{tor}}$  the full subcategory of  $\mathcal{R}\text{ind}$  consisting of torsion ringed diagrams.

Let  $\mathcal{S}\text{ch}^{\text{qc.sep}} \subseteq \mathcal{S}\text{ch}$  be the full subcategory spanned by (small) coproducts of quasi-compact and separated schemes. For each object  $X$  of  $\mathcal{S}\text{ch}$  (resp.  $\mathcal{S}\text{ch}^{\text{qc.sep}}$ ), we denote by  $\acute{\text{E}}\text{t}(X) \subseteq \mathcal{S}\text{ch}/X$  (resp.  $\acute{\text{E}}\text{t}^{\text{qc.sep}}(X) \subseteq \mathcal{S}\text{ch}/X^{\text{qc.sep}}$ ) the full subcategory spanned by the étale morphisms, which is naturally a site. We denote by  $X_{\acute{\text{E}}\text{t}}$  (resp.  $X_{\text{qc.sep.}\acute{\text{E}}\text{t}}$ ) the associated topos, namely the category of sheaves on  $\acute{\text{E}}\text{t}(X)$  (resp.  $\acute{\text{E}}\text{t}^{\text{qc.sep}}(X)$ ). In [SGA4, VII 1.2],  $\acute{\text{E}}\text{t}(X)$  is called the étale site of  $X$  and  $X_{\acute{\text{E}}\text{t}}$  is called the étale topos of  $X$ . The inclusion  $\acute{\text{E}}\text{t}^{\text{qc.sep}}(X) \subseteq \acute{\text{E}}\text{t}(X)$  induces an equivalence of topoi  $X_{\acute{\text{E}}\text{t}} \rightarrow X_{\text{qc.sep.}\acute{\text{E}}\text{t}}$ . In this chapter, we will not distinguish between  $X_{\acute{\text{E}}\text{t}}$  and  $X_{\text{qc.sep.}\acute{\text{E}}\text{t}}$ .

**Definition 3.2.2.** In what follows, we will often deal with  $\infty$ -categories of the form

$$(\mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}} := ((\mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}})^{\text{op}}$$

where  $\mathcal{C}$  is an  $\infty$ -category and  $\mathcal{D}$  is a subcategory of  $\mathcal{N}(\mathcal{R}\text{ind})$ . Suppose that  $\mathcal{E}$  is a subset of edges of  $\mathcal{C}$  that contains every isomorphism.

We say that an edge  $f: (\{(X'_i, Y'_i)\}_{1 \leq i \leq m}) \rightarrow (\{(X_i, Y_i)\}_{1 \leq i \leq m})$  of  $(\mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}}$  *statically belongs to*  $\mathcal{E}$  if  $f^{\text{op}}$  is static (Definition 1.5.4) and the corresponding edge  $X'_i \rightarrow X_i$  (resp.  $Y'_i \rightarrow Y_i$ ) of  $\mathcal{C}$  (resp.  $\mathcal{D}$ ) belongs to  $\mathcal{E}$  (resp. is an isomorphism). By abuse of notation, we will denote again by  $\mathcal{E}$  the subset of edges of  $(\mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}}$  that statically belong to  $\mathcal{E}$ . Moreover, if sometimes  $\mathcal{E}$  is defined by a property  $P$ , then edges that statically belong to  $\mathcal{E}$  are said to *statically have the property*  $P$ . We also denote by “all” the set of all edges of  $(\mathcal{C}^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}}$ .

For  $\mathcal{C} = \mathcal{N}(\mathcal{S}\text{ch}^{\text{qc.sep}})$ , we denote by

- $F$  the set of morphisms of  $\mathcal{C}$  locally of finite type;
- $P \subseteq F$  the subset consisting of proper morphisms;
- $I \subseteq F$  the subset consisting of local isomorphisms.

**Lemma 3.2.3.** *Let  $\mathcal{D}$  be a subcategory of  $\mathcal{N}(\mathcal{R}\text{ind})$ . The natural map*

$$\delta_{3, \{3\}}^* ((\mathcal{N}(\mathcal{S}\text{ch}^{\text{qc.sep}})^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}})^{\text{cart}}_{P, I, \text{all}} \rightarrow \delta_{2, \{2\}}^* ((\mathcal{N}(\mathcal{S}\text{ch}^{\text{qc.sep}})^{\text{op}} \times \mathcal{D}^{\text{op}})^{\text{II}, \text{op}})^{\text{cart}}_{F, \text{all}}$$

*is a categorical equivalence.*

*Proof.* The proof is similar to [LZa, Corollary 0.4]. Let  $F_{\text{ft}} \subseteq F$  be the set consisting of morphisms of finite type, and put  $I_{\text{ft}} = I \cap F_{\text{ft}}$ . Consider the following commutative



diagram

$$\begin{array}{ccc}
 \delta_{4,\{4\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II},op})_{P,I_{\mathrm{ft}},I,\mathrm{all}}^{\mathrm{cart}} & \longrightarrow & \delta_{3,\{3\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II},op})_{F_{\mathrm{ft}},I,\mathrm{all}}^{\mathrm{cart}} \\
 \downarrow & & \downarrow \\
 \delta_{3,\{3\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II},op})_{P,I,\mathrm{all}}^{\mathrm{cart}} & \longrightarrow & \delta_{2,\{2\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II},op})_{F,\mathrm{all}}^{\mathrm{cart}}.
 \end{array}$$

To show that the lower horizontal map is a categorical equivalence, it suffices to show that the other three maps are categorical equivalences.

In [LZa, Theorem 0.1], we put  $k = 4$ ,  $\mathcal{C} = (N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II},op}$ ,  $\mathcal{E}_0 = F_{\mathrm{ft}}$ ,  $\mathcal{E}_1 = P$ ,  $\mathcal{E}_2 = I_{\mathrm{ft}}$ ,  $\mathcal{E}_3 = I$ , and  $\mathcal{E}_4 = \mathrm{all}$ . Note that we have a canonical isomorphism

$$(N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times \mathcal{D}^{op})^{\mathrm{II}} \simeq (N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op})^{\mathrm{II}} \times_{N(\mathcal{F}\mathrm{in}_*)} (\mathcal{D}^{op})^{\mathrm{II}}.$$

By Nagata compactification theorem [Con07, 4.1], condition (2) of [LZa, Theorem 0.1] is satisfied. The other conditions are also satisfied by Lemma 1.5.5. It follows that the map in the upper horizontal arrow is a categorical equivalence. Similarly, using [LZa, Theorem 0.1], one proves that the vertical arrows are also categorical equivalences.  $\square$

*Remark 3.2.4.* The same proof shows that the lemma also holds with  $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}$  replaced by the category of disjoint unions of quasi-compact *quasi-separated* schemes and  $F$  replaced by the set of *separated* morphisms locally of finite type.

Our goal is to construct a map (3.8) which encodes  $f^*$ ,  $f_!$  and the monoidal structure given by tensor product.

We start by encoding  $f^*$  and the monoidal structure. Composing the nerve of the pseudofunctor  $\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}} \rightarrow \mathcal{P}\mathcal{T}\mathrm{opos}$  carrying  $X$  to  $X_{\mathrm{et}}$  with  ${}_{\mathcal{P}\mathcal{T}\mathrm{opos}}\mathrm{EO}^{\mathrm{I}}$  (2.4), we obtain a functor

$$(3.3) \quad {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{\mathrm{I}}: (N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times N(\mathcal{R}\mathrm{ind})^{op})^{\mathrm{II}} \rightarrow \mathcal{C}\mathrm{at}_{\infty}$$

that is a weak Cartesian structure, which induces a functor (Notation 2.2.3)

$$(3.4) \quad {}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{\otimes} := ({}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{\mathrm{I}})^{\otimes}: N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times N(\mathcal{R}\mathrm{ind})^{op} \rightarrow \mathcal{C}\mathrm{Alg}(\mathcal{C}\mathrm{at}_{\infty})_{\mathrm{pr,st,cl}}^{\mathrm{L}}$$

by Lemma 2.2.2.

To encode  $f_!$ , we resort to the technique of taking partial adjoints. Consider the composite map

$$\begin{aligned}
 (3.5) \quad \delta_{3,\{1,2,3\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times N(\mathcal{R}\mathrm{ind})^{op})^{\mathrm{II},op})_{P,I,\mathrm{all}}^{\mathrm{cart}} \\
 \rightarrow (N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times N(\mathcal{R}\mathrm{ind})^{op})^{\mathrm{II}} \xrightarrow{{}_{\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}}}\mathrm{EO}^{\mathrm{I}}(3.3)} \mathcal{C}\mathrm{at}_{\infty}.
 \end{aligned}$$

First, we apply the dual version of Proposition 1.4.4 to (3.5) for direction 1 to construct the partial right adjoint

$$(3.6) \quad \delta_{3,\{2,3\}}^* ((N(\mathcal{S}\mathrm{ch}^{\mathrm{qc.sep}})^{op} \times N(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{op})^{\mathrm{II},op})_{P,I,\mathrm{all}}^{\mathrm{cart}} \rightarrow \mathcal{C}\mathrm{at}_{\infty}.$$

The adjointability condition for direction (1, 2) is a special case of that for direction (1, 3). We check the latter as follows.

**Lemma 3.2.5.** *Let  $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  be a morphism of  $\mathbf{Fin}_*$ . Let  $f_i: X'_i \rightarrow X_i$  be proper morphisms of schemes in  $\mathbf{Sch}^{\text{qc.sep}}$  and take  $\lambda_i \in \mathcal{R}\text{ind}_{\text{tor}}$  for  $1 \leq i \leq m$ . For pullback squares*

$$\begin{array}{ccc} Y'_j & \xrightarrow{\quad} & Y_j \\ \downarrow & & \downarrow \\ \prod_{\alpha(i)=j} X'_i & \xrightarrow{\prod f_i} & \prod_{\alpha(i)=j} X_i \end{array}$$

*of schemes in  $\mathbf{Sch}^{\text{qc.sep}}$  and morphisms  $\mu_j \rightarrow \prod_{\alpha(i)=j} \lambda_i$  in  $\mathcal{R}\text{ind}_{\text{tor}}$  for  $1 \leq j \leq n$ , the square*

$$\begin{array}{ccc} \prod_{j \in T} \mathcal{D}(Y'_j, \mu_j) & \longleftarrow & \prod_{j \in T} \mathcal{D}(Y_j, \mu_j) \\ \uparrow & & \uparrow \\ \prod_{i \in S} \mathcal{D}(X'_i, \lambda_i) & \longleftarrow & \prod_{i \in S} \mathcal{D}(X_i, \lambda_i) \end{array}$$

*given by pullback and tensor product is right adjointable.*

Note that the right adjoints of the horizontal arrows admit right adjoints. Indeed, for the lower arrow we may assume  $X_i$  quasi-compact and apply Lemma 1.1.4.

*Proof.* Decomposing the product categories with respect to  $\langle n \rangle$ , we are reduced to two cases: (a)  $n = 0$ ; (b)  $n = 1$  and  $\alpha(\langle m \rangle^\circ) \subseteq \{1\}$ . Case (a) is trivial. For case (b), writing  $(f_i)_{1 \leq i \leq m}$  as a composition, we may further assume that at most one  $f_i$  is not the identity. Changing notation, we are reduced to showing that for every pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & Y \\ g' \downarrow & & \downarrow g \\ X' & \xrightarrow{f} & X \end{array}$$

of schemes in  $\mathbf{Sch}^{\text{qc.sep}}$  with  $f$  proper and every morphism  $\pi: \mu \rightarrow \lambda$  in  $\mathcal{R}\text{ind}_{\text{tor}}$ , the diagram

$$\begin{array}{ccc} \mathcal{D}(Y', \mu) & \xleftarrow{f'^*} & \mathcal{D}(Y, \mu) \\ (g', \pi)^* - \otimes f'^* \mathbf{K} \uparrow & & \uparrow (g, \pi)^* - \otimes \mathbf{K} \\ \mathcal{D}(X', \lambda) & \xleftarrow{f^*} & \mathcal{D}(X, \lambda) \end{array}$$

is right adjointable for every  $\mathbf{K} \in \mathcal{D}(Y, \mu)$ . As in the proof of Lemma 2.2.8, we easily reduce to the case with  $\lambda = (\{*\}, \Lambda)$  and  $\mu = (\{*\}, M)$ . This case is the combination of proper base change and projection formula. See [SGA4, XVII 4.3.1] for a proof in  $\mathbf{D}^-$ . Finally, the right completeness of unbounded derived categories [HA, 1.3.5.21] implies that every object  $\mathbf{L}$  of  $\mathcal{D}(X, \lambda)$  is the sequential colimit of  $\tau^{\leq n} \mathbf{L}$ . The unbounded case follow since the vertical arrows and the right adjoints of the horizontal arrows preserve sequential colimits.  $\square$

Second, we apply Proposition 1.4.4 to (3.6) for direction 2 to construct a map

$$(3.7) \quad \delta_{3,\{3\}}^* ((N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind_{\text{tor}})^{op})^{\text{II},op})_{P,I,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}at_{\infty}.$$

The adjointability condition for direction (2,1) follows from the fact that, for every separated étale morphism  $f$  of finite type between quasi-separated and quasi-compact schemes, the functor  $f_!$  constructed in [SGA4, XVII 5.1.8] is a left adjoint of  $f^*$  [SGA4, XVII 6.2.11]. The adjointability condition for direction (2,3) follows from étale base change and a trivial projection formula [KS06, 18.2.5].

Third, we compose (3.7) with (a quasi-inverse) of the categorical equivalence in Lemma 3.2.3 to construct a map

$$(3.8) \quad {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^* ((N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind_{\text{tor}})^{op})^{\text{II},op})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}at_{\infty}.$$

Now we explain how to encode  $f_*$  and  $f^!$  via adjunction. Note that we have a natural map from  $\delta_{2,\{2\}}^* ((N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind_{\text{tor}})^{op})^{\text{II},op})_{F,\text{all}}^{\text{cart}}$  to  $N(\mathcal{F}in_*)$ , whose fiber over  $\langle 1 \rangle$  is isomorphic to  $\delta_{2,\{2\}}^* N(\mathcal{S}ch^{\text{qc.sep}})_{F,\text{all}}^{\text{cart}} \times N(\mathcal{R}ind_{\text{tor}})^{op}$ . Denote by  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!^*$  the restriction of  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  to the above fiber. By construction, we see that the image of  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!^*$  actually factorizes through the subcategory  $\mathcal{P}r_{\text{st}}^{\text{L}} \subseteq \mathcal{C}at_{\infty}$ . In other words, (3.8) induces a map

$$(3.9) \quad {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!^*: \delta_{2,\{2\}}^* N(\mathcal{S}ch^{\text{qc.sep}})_{F,\text{all}}^{\text{cart}} \times N(\mathcal{R}ind_{\text{tor}})^{op} \rightarrow \mathcal{P}r_{\text{st}}^{\text{L}}.$$

Evaluating (3.4) at the object  $\langle 1 \rangle \in \mathcal{F}in_*$ , we obtain the map

$$(3.10) \quad {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^*: N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind)^{op} \rightarrow \mathcal{P}r_{\text{st}}^{\text{L}}.$$

Note that this is equivalent to the map by restricting (3.9) to the second direction, on  $N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind_{\text{tor}})^{op}$ . Composing the equivalence  $\phi_{\mathcal{P}r_{\text{st}}}$  in Remark 1.4.5 with  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^*$ , we obtain the map

$${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_*: N(\mathcal{S}ch^{\text{qc.sep}}) \times N(\mathcal{R}ind) \rightarrow \mathcal{P}r_{\text{st}}^{\text{R}}.$$

Restricting (3.9) to the first direction, we obtain the map

$$(3.11) \quad {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!: N(\mathcal{S}ch^{\text{qc.sep}})_F \times N(\mathcal{R}ind_{\text{tor}})^{op} \rightarrow \mathcal{P}r_{\text{st}}^{\text{L}}.$$

Composing the equivalence  $\phi_{\mathcal{P}r_{\text{st}}}$  in Remark 1.4.5 with  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!$ , we obtain the map

$$(3.12) \quad {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^!: N(\mathcal{S}ch^{\text{qc.sep}})_F^{op} \times N(\mathcal{R}ind_{\text{tor}}) \rightarrow \mathcal{P}r_{\text{st}}^{\text{R}}.$$

**Variant 3.2.6.** Let  $Q(\subseteq F) \subseteq \text{Ar}(\mathcal{S}ch^{\text{qc.sep}})$  be the set of locally quasi-finite morphisms [SP, 01TD]. Recall that base change for an integral morphism [SGA4, VIII 5.6] holds for all Abelian sheaves. Replacing proper base change by finite base change in the construction of (3.8), we obtain

$${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^* ((N(\mathcal{S}ch^{\text{qc.sep}})^{op} \times N(\mathcal{R}ind)^{op})^{\text{II},op})_{Q,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}at_{\infty}.$$

When restricted to their common domain of definition, this map is equivalent to  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  (3.8).

**Notation 3.2.7.** We introduce the following notation.

- (1) For an object  $(X, \lambda)$  of  $\mathrm{Sch}^{\mathrm{qc.sep}} \times \mathcal{R}\mathrm{ind}$ , we denote its image under  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\otimes}$  by  $\mathcal{D}(X, \lambda)^{\otimes}$ , with the underlying  $\infty$ -category  $\mathcal{D}(X, \lambda)$ . In other words, we have  $\mathcal{D}(X, \lambda)^{\otimes} = \mathcal{D}(X_{\mathrm{\acute{e}t}}, \lambda)^{\otimes}$  and  $\mathcal{D}(X, \lambda) = \mathcal{D}(X_{\mathrm{\acute{e}t}}, \lambda)$ . By construction and Remark 2.2.4 (2),  $\mathcal{D}(X, \lambda)$  is equivalent to the derived  $\infty$ -category of  $\mathrm{Mod}(X_{\mathrm{\acute{e}t}}^{\Xi}, \Lambda)$  if  $\lambda = (\Xi, \Lambda)$ , and the monoidal structure on  $\mathcal{D}(X, \lambda)^{\otimes}$  is an  $\infty$ -categorical enhancement of the usual (derived) tensor product in the classical derived category.
- (2) For a morphism  $f: (X', \lambda') \rightarrow (X, \lambda)$  of  $\mathrm{Sch}^{\mathrm{qc.sep}} \times \mathcal{R}\mathrm{ind}$ , we denote its image under  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\otimes}$  by

$$f^{*\otimes}: \mathcal{D}(X, \lambda)^{\otimes} \rightarrow \mathcal{D}(X', \lambda')^{\otimes},$$

with the underlying functor  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(X', \lambda')$ . Note that  $f^*$  is an  $\infty$ -categorical enhancement of the usual (derived) pullback functor in the classical derived category, which is monoidal. If  $\lambda' \rightarrow \lambda$  is the identity, we denote the image of  $f$  under  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}_*$  by

$$f_*: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(X, \lambda),$$

which is an  $\infty$ -categorical enhancement of the usual (derived) pushforward functor.

- (3) For a morphism  $f: Y \rightarrow X$  locally of finite type of  $\mathrm{Sch}^{\mathrm{qc.sep}}$  and an object  $\lambda$  of  $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$ , we denote its image under  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}_!$  and  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^!$  by

$$f_!: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(X, \lambda), \quad f^!: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$$

which are  $\infty$ -categorical enhancement of the usual  $f_!$  and  $f^!$  in the classical derived category, respectively.

*Remark 3.2.8.* In the previous discussion, we have constructed two maps

$${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\mathrm{I}}, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\mathrm{II}}$$

from which we deduce the other six maps

$${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\otimes}, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}_!^*, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^*, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}_*, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}_!, \quad {}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^!.$$

Moreover, maps  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\mathrm{I}}$  and  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\mathrm{II}}$  are equivalent on their common part of domain, which is  $(\mathrm{N}(\mathrm{Sch}^{\mathrm{qc.sep}})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II}}$ .

Now we explain how K nneth Formula is encoded in the map  ${}_{\mathrm{Sch}^{\mathrm{qc.sep}}} \mathrm{EO}^{\mathrm{II}}$ . In particular, as special cases, Base Change and Projection Formula are also encoded. Suppose that we have a diagram

$$\begin{array}{ccccc} Y_1 & \xleftarrow{q_1} & Y & \xrightarrow{q_2} & Y_2 \\ f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\ X_1 & \xleftarrow{p_1} & X & \xrightarrow{p_2} & X_2, \end{array}$$

which exhibits  $Y$  as the limit  $Y_1 \times_{X_1} \times X \times_{X_2} Y_2$  and such that  $f_1$  and  $f_2$  (hence  $f$ ) are locally of finite type. Fix an object  $\lambda$  of  $\mathcal{R}\text{ind}_{\text{tor}}$ . They together induce an edge

$$\begin{array}{ccc} ((X_1, \lambda), (X_2, \lambda)) & \longrightarrow & (X, \lambda) \\ \downarrow & & \downarrow \\ ((Y_1, \lambda), (Y_2, \lambda)) & \longrightarrow & (Y, \lambda) \end{array}$$

of  $\delta_{2,\{2\}}^*((N(\text{Sch}^{\text{qc.sep}})^{\text{op}} \times N(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}})^{\text{II},\text{op}})^{\text{cart}}_{F,\text{all}}$  above the unique active map  $\langle 2 \rangle \rightarrow \langle 1 \rangle$  of  $\mathcal{F}\text{in}_*$ . Applying the map  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  and by adjunction, we obtain the following square

$$\begin{array}{ccc} \mathcal{D}(Y_1, \lambda) \times \mathcal{D}(Y_2, \lambda) & \xrightarrow{q_1^* - \otimes_Y q_2^*} & \mathcal{D}(Y, \lambda) \\ f_{1!} \times f_{2!} \downarrow & & \downarrow f_! \\ \mathcal{D}(X_1, \lambda) \times \mathcal{D}(X_2, \lambda) & \xrightarrow{p_1^* - \otimes_X p_2^*} & \mathcal{D}(X, \lambda) \end{array}$$

in  $\text{Cat}_{\infty}$ . At the level of homotopy categories, this recovers the classical Künneth Formula.

We end this section by the following adjointability result.

**Lemma 3.2.9.** *Let  $f: Y \rightarrow X$  be a morphism locally of finite type of  $\text{Sch}^{\text{qc.sep}}$ , and  $\pi: \lambda' \rightarrow \lambda$  a perfect morphism of  $\mathcal{R}\text{ind}_{\text{tor}}$  (Definition 2.2.7). Then the square*

$$\begin{array}{ccc} \mathcal{D}(Y, \lambda') & \xrightarrow{f_!} & \mathcal{D}(X, \lambda') \\ \pi^* \uparrow & & \uparrow \pi^* \\ \mathcal{D}(Y, \lambda) & \xrightarrow{f_!} & \mathcal{D}(X, \lambda), \end{array}$$

*is right adjointable and its transpose is left adjointable.*

*Proof.* The assertion being trivial for  $f$  in  $I$ , we may assume  $f$  in  $P$ . As in the proof of Lemma 2.2.8, we are reduced to the case where  $\pi^*$  is replaced  $e_{\zeta}^*$  and  $t_* \circ t^*$ , respectively.

Here, we have maps  $(\{*\}, \Lambda') \xrightarrow{t} (\{\zeta\}, \Lambda(\zeta)) \xrightarrow{e_{\zeta}} (\Xi, \Lambda)$ .

The assertion for  $t_* \circ t^*$  is trivial, since a left adjoint of  $t_* \circ t^*$  is  $- \otimes_{\Lambda(\zeta)} \Lambda'^{\vee} \simeq \mathcal{H}\text{om}_{\Lambda(\zeta)}(\Lambda', -)$ , where  $\Lambda'^{\vee} = \mathcal{H}\text{om}_{\Lambda(\zeta)}(\Lambda', \Lambda(\zeta))$ . We denote by  $e_{\zeta}^*$  a left adjoint of  $e_{\zeta}$ . For  $\xi \in \Xi$ , since  $e_{\xi}^*$  commutes with  $f_*$  by Lemma 2.2.8, it suffices to check that  $e_{\xi}^* \circ e_{\zeta}^*$  commutes with  $f_*$ . Here  $e_{\xi}: (\{\xi\}, \Lambda(\xi)) \rightarrow (\Xi, \Lambda)$  is the obvious morphism. For  $\xi \leq \zeta$ , we have  $e_{\xi}^* \circ e_{\zeta}^* \simeq - \otimes_{\Lambda(\zeta)} \Lambda(\xi)$  and the assertion follows from projection formula. For other  $\xi \in \Xi$ , the map  $e_{\xi}^* \circ e_{\zeta}^*$  is zero.  $\square$

**3.3. Poincaré duality and (co)homological descent.** For an object  $X$  of  $\text{Sch}^{\text{qc.sep}}$  and an object  $\lambda = (\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}$ , we have a t-structure  $(\mathcal{D}^{\leq 0}(X, \lambda), \mathcal{D}^{\geq 0}(X, \lambda))^6$  on

<sup>6</sup>We use a *cohomological* indexing convention, which is different from [HA, 1.2.1.4].

$\mathcal{D}(X, \lambda)$ , which induces the usual t-structure on its homotopy category  $D(X_{\text{ét}}^{\Xi}, \Lambda)$ . We denote by  $\tau^{\leq 0}$  and  $\tau^{\geq 0}$  the corresponding truncation functors. The heart

$$\mathcal{D}^{\heartsuit}(X, \lambda) := \mathcal{D}^{\leq 0}(X, \lambda) \cap \mathcal{D}^{\geq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$$

is canonically equivalent to (the nerve of) the Abelian category

$$\text{Mod}(X, \lambda) := \text{Mod}(X_{\text{ét}}^{\Xi}, \Lambda).$$

The constant sheaf  $\lambda_X$  on  $X^{\Xi}$  of value  $\Lambda$  is an object of  $\mathcal{D}^{\heartsuit}(X, \lambda)$ .

We fix a nonempty set  $\square$  of rational primes. Recall that a ring  $R$  is a  $\square$ -torsion ring if each element is killed by an integer that is a product of primes in  $\square$ . In particular, a  $\square$ -torsion ring is a torsion ring. We denote by  $\mathcal{R}\text{ind}_{\square\text{-tor}} \subseteq \mathcal{R}\text{ind}_{\text{tor}}$  the full subcategory spanned by  $\square$ -torsion ringed diagrams. Recall that a scheme  $X$  is  $\square$ -coprime if  $\square$  does not contain any residue characteristic of  $X$ . Let  $\text{Sch}_{\square}^{\text{qc.sep}}$  be the full subcategory of  $\text{Sch}^{\text{qc.sep}}$  spanned by  $\square$ -coprime schemes. In particular,  $\text{Spec } \mathbb{Z}[\square^{-1}]$  is a final object of  $\text{Sch}_{\square}^{\text{qc.sep}}$ . By abuse of notation, we still use  $A$  and  $F$  to denote  $A \cap \text{Ar}(\text{Sch}_{\square}^{\text{qc.sep}})$  and  $F \cap \text{Ar}(\text{Sch}_{\square}^{\text{qc.sep}})$ , respectively. Moreover, let  $L \subseteq F$  be the set of smooth morphisms.

**Definition 3.3.1** (Tate twist). We define a functor

$$\text{tw}: (\mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op})^{\triangleleft} \rightarrow \mathcal{C}\text{at}_{\infty}$$

such that

- (1) the restriction of  $\text{tw}$  to  $\mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op}$  coincides with the restriction of the functor  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^*$  (3.10) to  $\{\text{Spec } \mathbb{Z}[\square^{-1}]\} \times \mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op}$ ;
- (2)  $\text{tw}(-\infty)$  equals  $\Delta^0$ ;
- (3) for every object  $\lambda$  of  $\mathcal{R}\text{ind}_{\square\text{-tor}}$ , the image of 0 under the functor  $\text{tw}(-\infty \rightarrow \lambda)$  is the Tate twisted sheaf, denoted by  $\lambda_{\square}(1)$ , is dualizable in the symmetric monoidal  $\infty$ -category  $\mathcal{D}(\text{Spec } \mathbb{Z}[\square^{-1}], \lambda)^{\otimes}$ .

Let  $(X, \lambda)$  be an object of  $\text{Sch}_{\square}^{\text{qc.sep}} \times \mathcal{R}\text{ind}_{\square\text{-tor}}$ . We define the following functor

$$-\langle 1 \rangle := (- \otimes s_X^* \lambda_{\square}(1))[2]: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(X, \lambda),$$

where  $s_X: X \rightarrow \text{Spec } \mathbb{Z}[\square^{-1}]$  is the structure morphism. We know that  $-\langle 1 \rangle$  is an auto-equivalence since  $\lambda_{\square}(1)$  is dualizable and  $s_X^*$  is monoidal. In general, for  $d \in \mathbb{Z}$ , we define  $-\langle d \rangle$  to be the (inverse of the, if  $d < 0$ )  $|d|$ -th iteration of  $-\langle 1 \rangle$ .

We adapt the classical theory of trace maps and the Poincaré duality to the  $\infty$ -categorical setting, as follows. Let  $f: Y \rightarrow X$  be a flat morphism in  $\text{Sch}_{\square}^{\text{qc.sep}}$ , locally of finite presentation, and such that every geometric fiber has dimension  $\leq d$ . Let  $\lambda$  be an object of  $\mathcal{R}\text{ind}_{\square\text{-tor}}$ . In [SGA4, XVIII 2.9], Deligne constructed the trace map

$$(3.13) \quad \text{Tr}_f = \text{Tr}_{f, \lambda}: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X,$$

which turns out to be a morphism of  $\mathcal{D}^{\heartsuit}(X, \lambda)$ . The construction satisfies the following functorial properties.

**Lemma 3.3.2** (Functoriality of trace maps, [SGA4, XVIII §2]). *The trace maps  $\text{Tr}_f$  for all such  $f$  and  $\lambda$  are functorial in the following sense:*

(1) For every morphism  $\lambda \rightarrow \lambda'$  of  $\mathcal{R}\text{ind}_{\square\text{-tor}}$ , the diagram

$$\begin{array}{ccc} & \tau^{\geq 0} f_! \lambda_Y \langle d \rangle & \\ \sim \nearrow & & \searrow \text{Tr}_{f, \lambda} \\ \tau^{\geq 0} ((\tau^{\geq 0} f_! \lambda'_Y \langle d \rangle) \otimes_{\lambda'_X} \lambda_X) & \xrightarrow{\tau^{\geq 0} (\text{Tr}_{f, \lambda'} \otimes_{\lambda'_X} \lambda_X)} & \lambda_X \end{array}$$

commutes.

(2) For every Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

of  $\mathcal{S}\text{ch}_{\square}^{\text{qc.sep}}$ , the diagram

$$\begin{array}{ccc} u^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle & \xrightarrow{u^* \text{Tr}_f} & u^* \lambda_X \\ \simeq \downarrow & & \downarrow \simeq \\ \tau^{\geq 0} f'_! \lambda_{Y'} \langle d \rangle & \xrightarrow{\text{Tr}_{f'}} & \lambda_{X'} \end{array}$$

commutes.

(3) Consider a 2-cell

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ & \searrow g & \nearrow f \\ & Y & \end{array}$$

of  $\mathcal{N}(\mathcal{S}\text{ch}_{\square}^{\text{qc.sep}})$  with  $f$  (resp.  $g$ ) flat, locally of finite presentation, and such that every geometric fiber has dimension  $\leq d$  (resp.  $\leq e$ ). Then  $h$  is flat, locally of finite presentation, and such that every geometric fiber has dimension  $\leq d + e$ , and the diagram

$$\begin{array}{ccc} \tau^{\geq 0} f_! (\tau^{\geq 0} g_! \lambda_Z \langle e \rangle) \langle d \rangle & \xrightarrow{\tau^{\geq 0} f_! \text{Tr}_g \langle d \rangle} & \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \\ \simeq \downarrow & & \downarrow \text{Tr}_f \\ \tau^{\geq 0} h_! \lambda_Z \langle d + e \rangle & \xrightarrow{\text{Tr}_h} & \lambda_X \end{array}$$

commutes.



Let  $f: Y \rightarrow X$  be as above. We have the following 2-cell

$$\begin{array}{ccc} & \mathcal{D}(Y, \lambda) & \\ f^* \nearrow & \downarrow f_! & \\ \mathcal{D}(X, \lambda) & & \mathcal{D}(X, \lambda) \\ f_! \lambda_Y \otimes - \searrow & & \end{array}$$

of  $\mathcal{C}at_\infty$ . If we abuse of notation by writing  $f^*\langle d \rangle$  for  $-\langle d \rangle \circ f^*$ , then the composition

$$(3.14) \quad u_f: f_! \circ f^*\langle d \rangle \xrightarrow{\sim} f_! \lambda_Y \langle d \rangle \otimes - \rightarrow \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \otimes - \xrightarrow{\text{Tr}_f \otimes -} \lambda_X \otimes - \xrightarrow{\sim} \text{id}_X$$

is a natural transformation, where  $\text{id}_X$  is the identity functor of  $\mathcal{D}(X, \lambda)$ .

**Lemma 3.3.3.** *If  $f: Y \rightarrow X$  is smooth and of pure relative dimension  $d$ , then  $u_f$  is a counit transformation. In particular, the functors  $f^*\langle d \rangle$  and  $f^!$  are equivalent.*

*Proof.* It follows from [SGA4, XVIII 3.2.5] and the fact that  $f^!$  is right adjoint to  $f_!$ .  $\square$

*Remark 3.3.4.* Let  $f: Y \rightarrow X$  be a morphism in  $\text{Sch}^{\text{qc.sep}}$  that is flat, locally quasi-finite, and locally of finite presentation. Let  $\lambda$  be an object of  $\mathcal{R}\text{ind}$  (see Variant 3.2.6 for the definition of the enhanced operation map in this setting). In [SGA4, XVII 6.2.3], Deligne constructed the trace map

$$\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \rightarrow \lambda_X,$$

which is a morphism of  $\mathcal{D}^\heartsuit(X, \lambda)$ . It coincides with the trace map (3.13) when both are defined, and satisfies similar functorial properties. Moreover, by [SGA4, XVII 6.2.11], the map  $u_f: f_! \circ f^* \rightarrow \text{id}_X$  constructed similarly as (3.14) is a counit transform when  $f$  is étale. Thus, the functors  $f^!$  and  $f^*$  are equivalent in this case.

The following proposition will be used in the construction of the enhanced operation map for quasi-separated schemes.

**Proposition 3.3.5** ((Co)homological descent). *Let  $f: X_0^+ \rightarrow X_{-1}^+$  be a smooth and surjective morphism of  $\text{Sch}^{\text{qc.sep}}$ . Then*

- (1)  $(f, \text{id}_\lambda)$  is of universal  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}^\otimes$ -descent (3.4), where  $\lambda$  is an arbitrary object of  $\mathcal{R}\text{ind}$ ;
- (2)  $(f, \text{id}_\lambda)$  is of universal  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}_!$ -codescent (3.11), where  $\lambda$  is an arbitrary object of  $\mathcal{R}\text{ind}_{\text{tor}}$ .

See Definition 3.1.1 for the definition of universal (co)descent.

*Proof.* We restrict both functors to a fixed object  $\lambda$  of  $\mathcal{R}\text{ind}$  or  $\mathcal{R}\text{ind}_{\text{tor}}$ .

We first prove the case where  $f$  is étale. For (1), let  $X_\bullet^+$  be a Čech nerve of  $f$ , and put  $(\mathcal{D}^{\otimes*})_+^\bullet = {}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^\otimes \circ (X_\bullet^+)^{op}$ . By Remark 1.5.3, we only need to check that  $(\mathcal{D}^*)_+^\bullet = G \circ (\mathcal{D}^{\otimes*})_+^\bullet$  is a limit diagram, where  $G$  is the functor (1.1). This is a special case of Lemma 3.1.3 by letting  $U_\bullet$  be the sheaf represented by  $X_\bullet^+$ , and

$\mathcal{C}_\bullet$  be the whole category. For (2), by [Gaia, 1.3.3]<sup>7</sup>, we only need to prove that  $(\mathcal{D}^!)^\bullet_+ := \phi \circ_{\mathcal{S}\text{ch}^{\text{qc.sep}}} \text{EO}^! \circ (X_\bullet^+)^{op}$  is a limit diagram. Here  $\phi: \mathcal{P}\text{r}_{\text{st}}^{\text{R}} \rightarrow \text{Cat}_\infty$  is the natural inclusion, and the functor  $_{\mathcal{S}\text{ch}^{\text{qc.sep}}} \text{EO}^!$  is the one in (3.12). We apply Lemma 3.3.6 below. Assumption (1) follows from the fact that  $(\mathcal{D}^!)^\bullet_+$  admits small limits and such limits are preserved by  $f^!$ . Assumption (2) follows from the Poincaré duality for étale morphisms recalled in Remark 3.3.4. Moreover,  $f^!$  is conservative since it is equivalent to  $f^*$ .

The general case where  $u$  is smooth follows from the above case by Lemma 3.1.2 (3) (and its dual version), and the fact that there exists an étale surjective morphism  $g: Y \rightarrow X$  of  $\mathcal{S}\text{ch}^{\text{qc.sep}}$  that factorizes through  $f$  [EGAIV, 17.16.3 (ii)].  $\square$

**Lemma 3.3.6.** *Let  $\mathcal{C}^\bullet: \mathbf{N}(\Delta_+) \rightarrow \text{Cat}_\infty$  be an augmented cosimplicial  $\infty$ -category, and put  $\mathcal{C} = \mathcal{C}^{-1}$ . Let  $G: \mathcal{C} \rightarrow \mathcal{C}^0$  be the evident functor (1.1). Assume that:*

- (1) *The  $\infty$ -category  $\mathcal{C}^{-1}$  admits limits of  $G$ -split cosimplicial objects [HA, 4.7.3.2], and those limits are preserved by  $G$ .*
- (2) *For every morphism  $\alpha: [m] \rightarrow [n]$  of  $\Delta_+$ , the diagram*

$$\begin{array}{ccc} \mathcal{C}^m & \xrightarrow{d^0} & \mathcal{C}^{m+1} \\ \downarrow & & \downarrow \\ \mathcal{C}^n & \xrightarrow{d^0} & \mathcal{C}^{n+1} \end{array}$$

*is right adjointable.*

- (3)  *$G$  is conservative.*

*Then the canonical map  $\theta: \mathcal{C} \rightarrow \varprojlim_{n \in \Delta} \mathcal{C}^n$  is an equivalence.*

*Proof.* We only need to apply [HA, 4.7.6.3] to the augmented cosimplicial  $\infty$ -category  $\mathbf{N}(\Delta_+) \rightarrow \text{Cat}_\infty \xrightarrow{R} \text{Cat}_\infty$ , where  $R$  is the equivalence that associates to every  $\infty$ -category its opposite [HA, 2.4.2.7].  $\square$

#### 4. THE PROGRAM DESCENT

From Remark 3.2.8, we know that all useful information of six operations for  $\mathcal{S}\text{ch}^{\text{qc.sep}}$  is encoded in the maps  $_{\mathcal{S}\text{ch}^{\text{qc.sep}}} \text{EO}^{\text{I}}$  (3.3) and  $_{\mathcal{S}\text{ch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  (3.8) constructed in §3.2. In this chapter, we develop a program called DESCENT, which is an abstract categorical procedure to extend the above two maps to larger categories. The extended maps satisfy similar properties as the original ones. This program will be run in the next chapter to extend our theory successively to quasi-separated schemes, to algebraic spaces, to Artin stacks, and eventually to higher Deligne–Mumford and higher Artin stacks.

In §4.1, we describe the program by formalizing the data for  $\mathcal{S}\text{ch}^{\text{qc.sep}}$ . In §4.2, we construct the extension of the maps. In §4.3, we prove the required properties of the extended maps.

<sup>7</sup>Although in [Gaia] the author works with the  $\infty$ -category  $\text{DGCat}_{\text{cont}}$ , the proof for Lemma 1.3.3 works for  $\mathcal{P}\text{r}_{\text{st}}^{\text{L}}$  as well.

**4.1. Description.** In §3.2, we constructed two maps  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{I}}$  (3.3) and  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  (3.8). They satisfy certain properties such as descent for smooth morphisms (Proposition 3.3.5). We would like to extend these maps to maps defined on the  $\infty$ -category of higher Deligne–Mumford or higher Artin stacks, satisfying similar properties. We will achieve this in many steps, by first extending the maps to quasi-separated schemes, and then to algebraic spaces, and then to Artin stacks, and so on. All the steps are similar to each other. The output of one step provides the input for the next step. We will think of this as recursively running a program, which we name DESCENT. In this section, we axiomatize the input and output of this program in an abstract setting.

Let us start with a toy model.

**Proposition 4.1.1.** *Let  $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}})$  be a marked  $\infty$ -category such that  $\tilde{\mathcal{C}}$  admits pullbacks and  $\tilde{\mathcal{E}}$  is stable under composition and pullback. Let  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$  be a full subcategory stable under pullback such that for every object  $X$  of  $\tilde{\mathcal{C}}$ , there exists a morphism  $Y \rightarrow X$  in  $\tilde{\mathcal{E}}$  representable in  $\mathcal{C}$  with  $Y$  in  $\mathcal{C}$ . Let  $\mathcal{D}$  be an  $\infty$ -category such that  $\mathcal{D}^{\text{op}}$  admits geometric realizations. Let  $\text{Fun}^{\mathcal{E}}(\mathcal{C}^{\text{op}}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}^{\text{op}}, \mathcal{D})$  (resp.  $\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{\text{op}}, \mathcal{D}) \subseteq \text{Fun}(\tilde{\mathcal{C}}^{\text{op}}, \mathcal{D})$ ) be the full subcategory spanned by functors  $F$  such that every edge in  $\mathcal{E} = \tilde{\mathcal{E}} \cap \mathcal{C}_1$  (resp. in  $\tilde{\mathcal{E}}$ ) is of  $F$ -descent. Then the restriction map*

$$\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{\text{op}}, \mathcal{D}) \rightarrow \text{Fun}^{\mathcal{E}}(\mathcal{C}^{\text{op}}, \mathcal{D})$$

*is a trivial fibration.*

The proof will be given at the end of §4.2.

*Example 4.1.2.* Let  $\text{Sch}^{\text{qs}} \subseteq \text{Sch}$  be the full subcategory spanned by quasi-separated schemes. It contains  $\text{Sch}^{\text{qc.sep}}$  as a full subcategory. By Proposition 3.3.5 (1), we may apply Proposition 4.1.1 to

- $\tilde{\mathcal{C}} = (\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}, \text{op}}$ ,
- $\mathcal{C} = (\text{N}(\text{Sch}^{\text{qc.sep}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}, \text{op}}$ ,
- $\mathcal{D} = \mathcal{C}\text{at}_{\infty}$ ,
- and the set  $\tilde{\mathcal{E}}$  consists of edges  $f$  that are statically smooth surjective (Definition 3.2.2).

Then we obtain an extension of the map  $_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{I}}$  with larger source  $(\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$ .

Now we describe the program in full. We begin by summarizing the categorical properties we need on the geometric side into the following definition.

**Definition 4.1.3.** An  $\infty$ -category  $\mathcal{C}$  is *geometric* if it admits small coproducts and pullbacks such that

- (1) *Coproducts are disjoint:* every coCartesian diagram

$$\begin{array}{ccc} \emptyset & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \amalg Y \end{array}$$

is also Cartesian, where  $\emptyset$  denotes an initial object of  $\mathcal{C}$ .

- (2) *Coproducts are universal*: For a small collection of Cartesian diagrams

$$\begin{array}{ccc} Y_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X_i & \longrightarrow & X, \end{array}$$

$i \in I$ , the diagram

$$\begin{array}{ccc} \coprod_{i \in I} Y_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \coprod_{i \in I} X_i & \longrightarrow & X, \end{array}$$

is also Cartesian.

*Remark 4.1.4.* We have the following remarks about geometric categories.

- (1) Let  $\mathcal{C}$  be geometric. Then a small coproduct of Cartesian diagrams of  $\mathcal{C}$  is again Cartesian.
- (2) The  $\infty$ -categories  $N(\text{Sch}^{\text{qc.sep}})$ ,  $N(\text{Sch}^{\text{qs}})$ ,  $N(\mathcal{E}\text{sp})$ ,  $N(\mathcal{C}\text{hp})$ ,  $\mathcal{C}\text{hp}^{k\text{-Ar}}$  and  $\mathcal{C}\text{hp}^{k\text{-DM}}$  ( $k \geq 0$ ) appearing in this article are all geometric.

We now describe the input and the output of the program. The input has three parts: 0, I, and II. The output has two parts: I and II. We refer the reader to Example 4.1.12 for a typical example.

**Input 0.** We are given

- A 5-marked  $\infty$ -category  $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}}_s, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'', \tilde{\mathcal{E}}_t, \tilde{\mathcal{F}})$ , a full subcategory  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$ , and a morphism  $s'' \rightarrow s'$  of  $(-1)$ -truncated objects of  $\mathcal{C}$  [HTT, 5.5.6.1].
- For each  $d \in \mathbb{Z} \cup \{-\infty\}$ , a subset  $\tilde{\mathcal{E}}''_d$  of  $\tilde{\mathcal{E}}''$ .
- A sequence of inclusions of  $\infty$ -categories  $\mathcal{L}'' \subseteq \mathcal{L}' \subseteq \mathcal{L}$ .
- A function  $\dim^+ : \tilde{\mathcal{F}} \rightarrow \mathbb{Z} \cup \{-\infty, +\infty\}$ .

Put  $\mathcal{E}_s = \tilde{\mathcal{E}}_s \cap \mathcal{C}_1$ ,  $\mathcal{E}' = \tilde{\mathcal{E}}' \cap \mathcal{C}_1$ ,  $\mathcal{E}'' = \tilde{\mathcal{E}}'' \cap \mathcal{C}_1$ ,  $\mathcal{E}''_d = \tilde{\mathcal{E}}''_d \cap \mathcal{C}_1$  ( $d \in \mathbb{Z} \cup \{-\infty\}$ ),  $\mathcal{E}_t = \tilde{\mathcal{E}}_t \cap \mathcal{C}_1$  and  $\mathcal{F} = \tilde{\mathcal{F}} \cap \mathcal{C}_1$ . Let  $\mathcal{C}'$  (resp.  $\tilde{\mathcal{C}}'$ ,  $\mathcal{C}''$ , and  $\tilde{\mathcal{C}}''$ ) be the full subcategory of  $\mathcal{C}$  (resp.  $\tilde{\mathcal{C}}$ ,  $\mathcal{C}$ , and  $\tilde{\mathcal{C}}$ ) spanned by those objects that admit morphisms to  $s'$  (resp.  $s'$ ,  $s''$ , and  $s''$ ). Put  $\mathcal{F}' = \mathcal{F} \cap \mathcal{C}'_1$  and  $\tilde{\mathcal{F}}' = \tilde{\mathcal{F}} \cap \tilde{\mathcal{C}}'_1$ . They satisfy

- (1)  $\tilde{\mathcal{C}}$  is geometric, and the inclusion  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$  is stable under finite limits. Moreover, for every small coproduct  $X = \coprod_{i \in I} X_i$  in  $\tilde{\mathcal{C}}$ ,  $X$  belongs to  $\mathcal{C}$  if and only if  $X_i$  belongs to  $\mathcal{C}$  for all  $i \in I$ .
- (2)  $\mathcal{L}'' \subseteq \mathcal{L}'$  and  $\mathcal{L}' \subseteq \mathcal{L}$  are full subcategories.
- (3)  $\tilde{\mathcal{E}}_s, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'', \tilde{\mathcal{E}}_t, \tilde{\mathcal{F}}$  are stable under composition, pullback and small coproducts; and  $\tilde{\mathcal{E}}' \subseteq \tilde{\mathcal{E}}'' \subseteq \tilde{\mathcal{E}}_t \subseteq \tilde{\mathcal{F}}$ .
- (4) For every object  $X$  of  $\tilde{\mathcal{C}}$ , there exists an edge  $f : Y \rightarrow X$  in  $\tilde{\mathcal{E}}_s \cap \tilde{\mathcal{E}}'$  with  $Y$  in  $\mathcal{C}$ . Such an edge  $f$  is called an *atlas* for  $X$ .

- (5) For every object  $X$  of  $\tilde{\mathcal{C}}$ , the diagonal morphism  $X \rightarrow X \times X$  is representable in  $\mathcal{C}$ .
- (6) For every edge  $f: Y \rightarrow X$  in  $\tilde{\mathcal{E}}''$ , there exist 2-simplices

(4.1)

$$\begin{array}{ccc} & Y & \\ i_d \nearrow & & \searrow f \\ Y_d & \xrightarrow{f_d} & X \end{array}$$

of  $\tilde{\mathcal{C}}$  with  $f_d$  in  $\tilde{\mathcal{E}}''_d$  for  $d \in \mathbb{Z}$ , such that the edges  $i_d$  exhibit  $Y$  as the coproduct  $\coprod_{d \in \mathbb{Z}} Y_d$ .

- (7) For every  $d \in \mathbb{Z} \cup \{-\infty\}$ , we have  $\tilde{\mathcal{E}}''_d \subseteq \tilde{\mathcal{E}}''$ , that  $\tilde{\mathcal{E}}''_d$  is stable under pullback and small coproducts, and that  $\tilde{\mathcal{E}}''_{-\infty}$  is the set of edges whose source is an initial object. For distinct integers  $d$  and  $e$ , we have  $\tilde{\mathcal{E}}''_d \cap \tilde{\mathcal{E}}''_e = \tilde{\mathcal{E}}''_{-\infty}$ .
- (8) For every small set  $I$  and every pair of objects  $X$  and  $Y$  of  $\tilde{\mathcal{C}}$ , the morphisms  $X \rightarrow X \coprod Y$  and  $\coprod_I X \rightarrow X$  are in  $\tilde{\mathcal{E}}''_0$ . For every 2-cell

(4.2)

$$\begin{array}{ccc} & Y & \\ g \nearrow & & \searrow f \\ Z & \xrightarrow{h} & X \end{array}$$

of  $\tilde{\mathcal{C}}$  with  $f$  in  $\tilde{\mathcal{E}}''_d$  and  $g$  in  $\tilde{\mathcal{E}}''_e$ , where  $d$  and  $e$  are integers,  $h$  is in  $\tilde{\mathcal{E}}''_{d+e}$ .

- (9) The function  $\dim^+$  satisfies the following conditions.
- (a)  $\dim^+(f) = -\infty$  if and only if  $f$  is in  $\tilde{\mathcal{E}}''_{-\infty}$ .
  - (b) The restriction of  $\dim^+$  to  $\tilde{\mathcal{E}}''_d - \tilde{\mathcal{E}}''_{-\infty}$  is of constant value  $d$ .
  - (c) For every 2-cell (4.2) in  $\tilde{\mathcal{C}}$  with edges in  $\tilde{\mathcal{F}}$ , we have  $\dim^+(h) \leq \dim^+(f) + \dim^+(g)$ , and that the equality holds when  $g$  belongs to  $\tilde{\mathcal{E}}_s \cap \tilde{\mathcal{E}}''$ .
  - (d) For every Cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

in  $\tilde{\mathcal{C}}$  with  $f$  (and hence  $g$ ) in  $\tilde{\mathcal{F}}$ , we have  $\dim^+(g) \leq \dim^+(f)$ , and equality holds when  $p$  belongs to  $\tilde{\mathcal{E}}_s$ .

- (e) For every edge  $f: Y \rightarrow X$  in  $\tilde{\mathcal{F}}$  and every small collection

$$\begin{array}{ccc} & Y & \\ g_i \nearrow & & \searrow f \\ Z_i & \xrightarrow{h_i} & X \end{array}$$

of 2-simplices with  $g_i$  in  $\tilde{\mathcal{E}}''_{d_i}$  such that the morphism  $\coprod_{i \in I} Z_i \rightarrow Y$  is in  $\tilde{\mathcal{E}}_s$ , we have  $\dim^+(f) = \sup_{i \in I} \{\dim^+(h_i) - d_i\}$ .

- (10) We have  $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''_0$ .

*Remark 4.1.5.* By (8) and (9c,d,e), for every small collection  $\{Y_i \xrightarrow{f_i} X_i\}_{i \in I}$  of edges in  $\tilde{\mathcal{F}}$ , we have  $\dim^+(\coprod_{i \in I} f_i) = \sup_{i \in I} \{\dim^+(f_i)\}$ .

**Input I.** Input I consists of two maps as follows.

- The *first abstract operation map*:

$${}_{\mathcal{C}}\mathrm{EO}^{\mathrm{I}}: (\mathcal{C}^{op} \times \mathcal{L}^{op})^{\mathrm{II}} \rightarrow \mathrm{Cat}_{\infty}.$$

- The *second abstract operation map*:

$${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}: \delta_{2,\{2\}}^* ((\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op})_{\mathcal{F}',\mathrm{all}}^{\mathrm{cart}} \rightarrow \mathrm{Cat}_{\infty}.$$

Input I is subject to the following properties:

- P0: Monoidal symmetry.** The functor  ${}_{\mathcal{C}}\mathrm{EO}^{\mathrm{I}}$  is a weak Cartesian structure, and the induced functor  ${}_{\mathcal{C}}\mathrm{EO}^{\otimes} := ({}_{\mathcal{C}}\mathrm{EO}^{\mathrm{I}})^{\otimes}$  factorizes through  $\mathrm{CAlg}(\mathrm{Cat}_{\infty})_{\mathrm{pr,st,cl}}^{\mathrm{L}}$  (see Remark 1.5.6).
- P1: Disjointness.** The map  ${}_{\mathcal{C}}\mathrm{EO}^{\otimes}$  sends small coproducts to products.
- P2: Compatibility.** The restrictions of  ${}_{\mathcal{C}}\mathrm{EO}^{\mathrm{I}}$  and  ${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}$  to  $(\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II}}$  are equivalent functors.

Before stating the remaining properties, we have to fix some notation. Similar to the construction of (3.9), we obtain a map

$${}_{\mathcal{C}'}\mathrm{EO}_!^*: \delta_{2,\{2\}}^* \mathcal{C}_{\mathcal{F}',\mathcal{C}'_1}^{\mathrm{cart}} \times \mathcal{L}'^{op} \rightarrow \mathcal{P}_{\mathrm{st}}^{\mathrm{L}}.$$

from  ${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}$ . Similar to the construction of (3.10) and (3.11), we obtain maps

$${}_{\mathcal{C}}\mathrm{EO}^*: \mathcal{C}^{op} \times \mathcal{L}^{op} \rightarrow \mathcal{P}_{\mathrm{st}}^{\mathrm{L}}, \quad {}_{\mathcal{C}'}\mathrm{EO}_!: \mathcal{C}'_{\mathcal{F}'} \times \mathcal{L}'^{op} \rightarrow \mathcal{P}_{\mathrm{st}}^{\mathrm{L}}.$$

Moreover, we will use similar notation as in Notation 3.2.7 for the image of 0 and 1-cells under above maps, after replacing  $\mathrm{Sch}^{\mathrm{qc,sep}}$  (resp.  $\mathcal{R}\mathrm{ind}$ ) by  $\mathcal{C}$  (resp.  $\mathcal{L}$ ). Now we are ready to state the remaining properties.

- P3: Conservativeness.** If  $f: Y \rightarrow X$  belongs to  $\mathcal{E}_s$ , then  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is conservative for every object  $\lambda$  of  $\mathcal{L}$ .
- P4: Descent.** Let  $f$  be a morphism of  $\mathcal{C}$  (resp.  $\mathcal{C}'$ ) and  $\lambda$  an object of  $\mathcal{L}$  (resp.  $\mathcal{L}'$ ). If  $f$  belongs to  $\mathcal{E}_s \cap \mathcal{E}''$  (resp.  $\mathcal{E}_s \cap \mathcal{E}'' \cap \mathcal{C}'_1$ ), then  $(f, \mathrm{id}_{\lambda})$  is of universal  ${}_{\mathcal{C}}\mathrm{EO}^{\otimes}$ -descent (resp.  ${}_{\mathcal{C}'}\mathrm{EO}_!$ -codescent).
- P5: Adjointability for  $\mathcal{E}'$ .** Let

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

be a Cartesian diagram of  $\mathcal{C}'$  with  $f$  in  $\mathcal{E}'$ , and  $\lambda$  an object of  $\mathcal{L}'$ . Then

(1) The square

$$\begin{array}{ccc} \mathcal{D}(Z, \lambda) & \xleftarrow{p^*} & \mathcal{D}(X, \lambda) \\ g^* \downarrow & & \downarrow f^* \\ \mathcal{D}(W, \lambda) & \xleftarrow{q^*} & \mathcal{D}(Y, \lambda) \end{array}$$

has a right adjoint which is a square of  $\mathcal{P}\mathcal{R}_{\text{st}}^{\text{R}}$ .

(2) If  $p$  is also in  $\mathcal{E}'$ , then the square

$$\begin{array}{ccc} \mathcal{D}(X, \lambda) & \xleftarrow{f_!} & \mathcal{D}(Y, \lambda) \\ p^* \downarrow & & \downarrow q^* \\ \mathcal{D}(Z, \lambda) & \xleftarrow{g_!} & \mathcal{D}(W, \lambda) \end{array}$$

is right adjointable.

**P5<sup>bis</sup>:** *Adjointability for  $\mathcal{E}''$ .* We have the same statement as in (P5) after replacing  $\mathcal{C}'$  by  $\mathcal{C}''$ ,  $\mathcal{E}'$  by  $\mathcal{E}''$ , and  $\mathcal{L}'$  by  $\mathcal{L}''$ .

**Input II.** Input II consists of the following data.

- A functor  $\text{tw}: (\mathcal{L}''^{\text{op}})^{\triangleleft} \rightarrow \mathcal{C}\text{at}_{\infty}$  satisfying that
  - the restriction of  $\text{tw}$  to  $\mathcal{L}''^{\text{op}}$  coincides with the restriction of  ${}_{\mathcal{C}}\text{EO}^*$  to  $\{s''\} \times \mathcal{L}''^{\text{op}}$ ;
  - $\text{tw}(-\infty)$  equals  $\Delta^0$ ;
  - for every object  $\lambda$  of  $\mathcal{R}\text{ind}_{\square\text{-tor}}$ , if we denote the image of 0 under the functor  $\text{tw}(-\infty \rightarrow \lambda): \Delta^0 \rightarrow \mathcal{D}(s'', \lambda)$  by  $\lambda(1)$ , then it is dualizable in the symmetric monoidal  $\infty$ -category  $\mathcal{D}(s'', \lambda)^{\otimes}$ .
- A t-structure on  $\mathcal{D}(X, \lambda)$  for every object  $X$  of  $\mathcal{C}$  and every object  $\lambda$  of  $\mathcal{L}$ .
- (*Trace map for  $\mathcal{E}_t$* ) A map  $\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$  for every edge  $f: Y \rightarrow X$  in  $\mathcal{E}_t \cap \mathcal{C}_1''$ , every integer  $d \geq \dim^+(f)$ , and every object  $\lambda$  of  $\mathcal{L}''$ . Here,  $\lambda_X$  is a unit object of the monoidal  $\infty$ -category  $\mathcal{D}(X, \lambda)$  and similarly for  $\lambda_Y$ ;  $-\langle d \rangle$  is defined in the same way as in Definition 3.3.1.
- (*Trace map for  $\mathcal{E}'$* ) A map  $\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \rightarrow \lambda_X$  for every edge  $f: Y \rightarrow X$  in  $\mathcal{E}' \cap \mathcal{C}_1'$  and every object  $\lambda$  of  $\mathcal{L}'$ , which coincides with the one above for  $f \in \mathcal{E}' \cap \mathcal{C}_1''$ .

Input II is subject to the following properties.

**P6:** *t-structure.* Let  $\lambda$  be an arbitrary object of  $\mathcal{L}$ . We have

- (1) For every object  $X$  of  $\mathcal{C}$ , we have  $\lambda_X \in \mathcal{D}^{\heartsuit}(X, \lambda)$ .
- (2) If  $\lambda$  belongs to  $\mathcal{L}''$  and  $X$  is an object of  $\mathcal{C}''$ , then the auto-equivalence  $-\otimes s_X^* \lambda(1)$  of  $\mathcal{D}(X, \lambda)$  is t-exact.
- (3) For every object  $X$  of  $\mathcal{C}$ , the t-structure on  $\mathcal{D}(X, \lambda)$  is accessible, right complete, and  $\mathcal{D}^{\leq -\infty}(X, \lambda) := \bigcap_n \mathcal{D}^{\leq -n}(X, \lambda)$  consists of zero objects.
- (4) For every morphism  $f: Y \rightarrow X$  of  $\mathcal{C}$ , the functor  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is t-exact.

**P7:** *Poincaré duality for  $\mathcal{E}''$ .* We have



- (1) For every  $f$  in  $\mathcal{E}_t \cap \mathcal{C}_1''$ , every integer  $d \geq \dim^+(f)$ , and every object  $\lambda$  of  $\mathcal{L}''$ , the source of the trace map  $\mathrm{Tr}_f$  belongs to the heart  $\mathcal{D}^\heartsuit(X, \lambda)$ . Moreover,  $\mathrm{Tr}_f$  is functorial in the way as in Lemma 3.3.2. See Remark 4.1.6 below for more details.
- (2) For every  $f$  in  $\mathcal{E}_d'' \cap \mathcal{C}_1''$ , and every object  $\lambda$  of  $\mathcal{L}''$ , the map  $u_f: f_! \circ f^* \langle d \rangle \rightarrow \mathrm{id}_X$ , induced by the trace map  $\mathrm{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$  similarly as (3.14), is a counit transformation. Here  $\mathrm{id}_X$  is the identity functor of  $\mathcal{D}(X, \lambda)$ .
- P7<sup>bis</sup>:** *Poincaré duality for  $\mathcal{E}'$ .* We have the same statement as in (P7) after letting  $d = 0$ , and replacing  $\mathcal{C}''$  by  $\mathcal{C}'$ ,  $\mathcal{E}_t$  by  $\mathcal{E}'$ , and  $\mathcal{L}''$  by  $\mathcal{L}'$ .

*Remark 4.1.6.* In (P7)(1) above, the trace maps  $\mathrm{Tr}_f$  for all such  $f$  and  $\lambda$  are functorial in the following sense:

- (1) For every morphism  $\lambda \rightarrow \lambda'$  of  $\mathcal{L}''$ , the diagram

$$\begin{array}{ccc} & \tau^{\geq 0} f_! \lambda_Y \langle d \rangle & \\ \nearrow \sim & & \searrow \mathrm{Tr}_{f, \lambda} \\ \tau^{\geq 0} ((\tau^{\geq 0} f_! \lambda'_Y \langle d \rangle) \otimes_{\lambda'_X} \lambda_X) & \xrightarrow{\tau^{\geq 0} (\mathrm{Tr}_{f, \lambda'} \otimes_{\lambda'_X} \lambda_X)} & \lambda_X \end{array}$$

commutes.

- (2) For every Cartesian diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

of  $\mathcal{C}''$ , the diagram

$$\begin{array}{ccc} u^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle & \xrightarrow{u^* \mathrm{Tr}_f} & u^* \lambda_X \\ \simeq \downarrow & & \downarrow \simeq \\ \tau^{\geq 0} f'_! \lambda_{Y'} \langle d \rangle & \xrightarrow{\mathrm{Tr}_{f'}} & \lambda_{X'} \end{array}$$

commutes.

- (3) Consider a 2-cell

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \\ & \searrow g & \nearrow f \\ & Y & \end{array}$$

of  $\mathcal{C}''$  with  $f, g \in \mathcal{E}_t \cap \mathcal{C}_1''$  such that  $\dim^+(f) \leq d$  and  $\dim^+(g) \leq e$ . In particular, we have  $h \in \mathcal{E}_t \cap \mathcal{C}_1''$  and  $\dim^+(h) \leq d + e$ . Then the diagram

$$\begin{array}{ccc} \tau^{\geq 0} f_! (\tau^{\geq 0} g_! \lambda_Z \langle e \rangle) \langle d \rangle & \xrightarrow{\tau^{\geq 0} f_! \text{Tr}_g \langle d \rangle} & \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \\ \simeq \downarrow & & \downarrow \text{Tr}_f \\ \tau^{\geq 0} h_! \lambda_Z \langle d + e \rangle & \xrightarrow{\text{Tr}_h} & \lambda_X \end{array}$$

commutes.

*Remark 4.1.7.* We have the following remarks concerning input.

- (1) (P0) and (P4) imply the following: If  $f$  is an edge of  $(\mathcal{C}^{op} \times \mathcal{L}^{op})^{\text{II}, op}$  that statically belongs to  $\mathcal{E}_s \cap \mathcal{E}''$ , then it is of universal  ${}_{\mathcal{C}}\text{EO}^{\text{I}}$ -descent.
- (2) (P4) implies that (P3) holds for  $f \in \mathcal{E}_s \cap \mathcal{E}''$ .
- (3) If  $d > \dim^+(f)$ , then the trace map  $\text{Tr}_f$  is not interesting because its source  $\tau^{\geq 0} f_! \lambda_Y \langle d \rangle$  is a zero object. We have included such maps in the data in order to state the functoriality as in Remark 4.1.6 more conveniently.
- (4) We extend the trace map to morphisms  $f: Y \rightarrow X$  in  $\mathcal{E}_t \cap \mathcal{C}_1''$  endowed with 2-simplices (4.1) satisfying  $\dim^+(f_d) \leq d$  and such that the morphisms  $i_d$  exhibit  $Y$  as  $\coprod_{d \in \mathbb{Z}} Y_d$ . For every object  $\lambda$  of  $\mathcal{L}''$ , the map

$$\mathcal{D}(Y, \lambda) \rightarrow \prod_{d \in \mathbb{Z}} \mathcal{D}(Y_d, \lambda),$$

induced by  $i_d$  is an equivalence by (P1). We write  $-\langle \dim^+ \rangle: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  for the product of  $(-\langle d \rangle: \mathcal{D}(Y_d, \lambda) \rightarrow \mathcal{D}(Y_d, \lambda))_{d \in \mathbb{Z}}$ . Since  $\lambda_Y \simeq \bigoplus_{d \in \mathbb{Z}} i_d^* \lambda_{Y_d}$ , the maps  $\text{Tr}_{f_d}$  induce a map  $\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle \dim^+ \rangle \rightarrow \lambda_X$ . Moreover, the trace map is functorial in the sense that an analogue of Remark 4.1.6 holds.

- (5) (P7) (2) still holds for morphisms  $f: Y \rightarrow X$  in  $\mathcal{E}'' \cap \mathcal{C}_1''$ . For such morphisms, the 2-simplices in Input 0 (6) are unique up to equivalence by Input 0 (7). We write  $-\langle \dim f \rangle: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  for the product of  $(-\langle d \rangle: \mathcal{D}(Y_d, \lambda) \rightarrow \mathcal{D}(Y_d, \lambda))_{d \in \mathbb{Z}}$ . Then, (P7) (2) for the morphisms  $f_d$  implies that the map  $u_f: f_! \circ f^* \langle \dim f \rangle \rightarrow \text{id}_X$  induced by the trace map  $\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$  is a counit transformation.

The output has two parts: I & II.

**Output I.** Output I consists of two maps as follows.

- The *first abstract operation map*:

$${}_{\mathcal{C}}\text{EO}^{\text{I}}: (\tilde{\mathcal{C}}^{op} \times \mathcal{L}^{op})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

extending  ${}_{\mathcal{C}}\text{EO}^{\text{I}}$ .

- The *second abstract operation map*:

$$\tilde{{}_{\mathcal{C}}}\text{EO}^{\text{II}}: \delta_{2, \{2\}}^* ((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II}, op})_{\tilde{\mathcal{F}}', \text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

extending  ${}_{\mathcal{C}'}\text{EO}^{\text{II}}$ .

**Output II.** Output II consists of the following data, all extending the existed data in Input II.

- A functor  $\mathrm{tw}: (\mathcal{L}''^{\mathrm{op}})^{\triangleleft} \rightarrow \mathcal{C}\mathrm{at}_{\infty}$  same as in Input II.
- A t-structure on  $\mathcal{D}(X, \lambda)$  for every object  $X$  of  $\tilde{\mathcal{C}}$  and every object  $\lambda$  of  $\mathcal{L}$ .
- (*Trace map* for  $\tilde{\mathcal{E}}_t$ ) A map  $\mathrm{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$  for every edge  $f: Y \rightarrow X$  in  $\tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$ , every integer  $d \geq \dim^+(f)$ , and every object  $\lambda$  of  $\mathcal{L}''$ .
- (*Trace map* for  $\tilde{\mathcal{E}}'$ ) A map  $\mathrm{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \rightarrow \lambda_X$  for every edge  $f: Y \rightarrow X$  in  $\tilde{\mathcal{E}}' \cap \tilde{\mathcal{C}}_1'$  and every object  $\lambda$  of  $\mathcal{L}'$ , which coincides with the one above for  $f \in \tilde{\mathcal{E}}' \cap \tilde{\mathcal{C}}_1''$ .

We introduce properties (P0) through (P7<sup>bis</sup>) for Output I and II by replacing  $\mathcal{C}'$ ,  $\mathcal{C}''$  and  $(\mathcal{C}, \mathcal{E}_s, \mathcal{E}', \mathcal{E}'', \mathcal{E}_t, \mathcal{F})$  by  $\tilde{\mathcal{C}}'$ ,  $\tilde{\mathcal{C}}''$  and  $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}}_s, \tilde{\mathcal{E}}', \tilde{\mathcal{E}}'', \tilde{\mathcal{E}}_t, \tilde{\mathcal{F}})$ , respectively. The following theorem shows how our program works.

**Theorem 4.1.8.** *Fix an Input 0. Then*

- (1) *Every Input I satisfying (P0) through (P5<sup>bis</sup>) can be extended to an Output I satisfying (P0) through (P5<sup>bis</sup>).*
- (2) *For given Input I, II satisfying (P0) through (P7<sup>bis</sup>) and given Output I extending Input I and satisfying (P0) through (P5<sup>bis</sup>), there exists an Output II extending Input II and satisfying (P6), (P7), (P7<sup>bis</sup>).*

Output I will be accomplished in §4.2. Output II and the proof of properties (P1) through (P7<sup>bis</sup>) will be accomplished in §4.3.

**Variant 4.1.9.** Let us introduce a variant of DESCENT. In Input 0, we let  $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''$ ,  $s' \rightarrow s''$  be a degenerate edge,  $\mathcal{L}' = \mathcal{L}''$ , and *ignore* (10). In Input II (resp. Output II), we also *ignore* the trace map for  $\mathcal{E}'$  (resp.  $\tilde{\mathcal{E}}'$ ) and property (P7<sup>bis</sup>). In particular, (P5) and (P5<sup>bis</sup>) coincide. Theorem 4.1.8 for this variant still holds and will be applied to (higher) Artin stacks.

**Remark 4.1.10.** We have the following remarks concerning Theorem 4.1.8.

- (1) If the only goal is to extend the first and second operation maps, the statement of Theorem 4.1.8 (1) can be made more compact: every Input I satisfying properties (P0), (P2), (P4), and (P5) can be extended to an Output I satisfying (P0), (P2), (P4), and (P5). This will follow from our proof of Theorem 4.1.8 in this chapter.
- (2) The Output I in Theorem 4.1.8 (1) is unique up to equivalence. More precisely, we can define a simplicial set  $K$  classifying those Input I that satisfy (P2) and (P4). The vertices of  $K$  are triples  $({}_c\mathrm{EO}^I, {}_c\mathrm{EO}^{\mathrm{II}}, h)$ , where  $h$  is the equivalence in (P2). Similarly, let  $\tilde{K}$  be the simplicial set classifying those Output II that satisfy (P2) and (P4). Then the restriction map  $\tilde{K} \rightarrow K$  satisfies the right lifting property with respect to  $\partial\Delta^n \subseteq \Delta^n$  for all  $n \geq 1$ . One can show this by adapting our proof of Theorem 4.1.8. Moreover, in all the above,  $h$  can be taken to be the identity without loss of generality.

- (3) The Output II in Theorem 4.1.8 (2) is also unique up to equivalence. More precisely, let us fix an Output I extending Input I and satisfying (P2) and (P4). Note that the functor  $\mathrm{tw}$  remains the same. Fix an assignment of t-structures for the Input satisfying (P6). Then there exists a unique extension to the Output satisfying (P6). Moreover, for every assignment of traces for the Input satisfying (P7) (resp. (P7<sup>bis</sup>)), there exists a unique extension to the Output satisfying (P7) (resp. (P7<sup>bis</sup>)). Note that the trace map is defined in the heart, so that no homotopy issue arises.

**Definition 4.1.11.** For a morphism  $f: Y \rightarrow X$  locally of finite type between algebraic spaces, we define the *upper relative dimension* of  $f$  to be

$$\sup\{\dim(Y \times_X \mathrm{Spec} \Omega)\} \in \mathbb{Z} \cup \{-\infty, +\infty\}$$

[SP, 04N6], where the supremum is taken over all geometric points  $\mathrm{Spec} \Omega \rightarrow X$ . We adopt the convention that the empty scheme has dimension  $-\infty$ .

*Example 4.1.12.* The initial input for DESCENT is the following:

- $\tilde{\mathcal{C}} = \mathrm{N}(\mathrm{Sch}^{\mathrm{qs}})$ , where  $\mathrm{Sch}^{\mathrm{qs}} \subseteq \mathrm{Sch}$  is the full subcategory spanned by quasi-separated schemes as in Example 4.1.2. It is geometric and admits  $\mathrm{Spec} \mathbb{Z}$  as a final object.
- $\mathcal{C} = \mathrm{N}(\mathrm{Sch}^{\mathrm{qc.sep}})$ , and  $\mathfrak{s}'' \rightarrow \mathfrak{s}'$  is the unique morphism  $\mathrm{Spec} \mathbb{Z}[\square^{-1}] \rightarrow \mathrm{Spec} \mathbb{Z}$ . In particular,  $\mathcal{C}' = \mathcal{C}$  and  $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}$ .
- $\tilde{\mathcal{E}}_{\mathrm{s}}$  is the set of *surjective* morphisms.
- $\tilde{\mathcal{E}}'$  is the set of *étale* morphisms.
- $\tilde{\mathcal{E}}''$  is the set of *smooth* morphisms.
- $\tilde{\mathcal{E}}''_d$  is the set of *smooth* morphisms of pure relative dimension  $d$ .
- $\tilde{\mathcal{E}}_{\mathrm{t}}$  is the set of morphisms that are *flat and locally of finite presentation*.
- $\tilde{\mathcal{F}}$  is the set of morphisms *locally of finite type*.
- $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind})^{\mathrm{op}}$ ,  $\mathcal{L}' = \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}}$ , and  $\mathcal{L}'' = \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square\text{-tor}})^{\mathrm{op}}$ .
- $\dim^+$  is the (function of) upper relative dimension (Definition 4.1.11).
- ${}_{\mathcal{C}}\mathrm{EO}^{\mathrm{I}}$  is (3.3), and  ${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}$  is (3.8).
- $\mathrm{tw}$  is defined in Definition 3.3.1.
- $\mathcal{D}(X, \lambda)$  is endowed with its usual t-structure recalled at the beginning of §3.3.
- The trace maps are the classical ones (3.13); see also Remark 3.3.4.

Properties (P0) through (P7<sup>bis</sup>) are satisfied as follows:

- (P0) This is Lemma 2.2.2 (1,2).
- (P1) This is Lemma 2.2.2 (3).
- (P2) This follows from our construction. In fact, the two maps are equal in this case.
- (P3) This is obvious.
- (P4) This is Proposition 3.3.5.
- (P5) This follows from Lemma 4.1.13 below. Part (1) of (P5), namely the étale base change, is trivial.
- (P5<sup>bis</sup>) This follows from Lemma 4.1.13 below. Part (1) of (P5<sup>bis</sup>) is the smooth base change.

(P6) Part (3) follows from [HA, 1.3.5.21]. The rest follows from construction.

(P7) This has been recalled in Lemmas 3.3.2 and 3.3.3.

(P7<sup>bis</sup>) This has been recalled in Remark 3.3.4.

**Lemma 4.1.13.** *Assume (P7). Then (P5) holds. In fact, we have the stronger result that part (2) of (P5) holds without the assumption that  $p$  is also in  $\mathcal{E}'$ . The similar statements hold concerning (P7<sup>bis</sup>) and (P5<sup>bis</sup>).*

*Proof.* We denote by  $p_*$  (resp.  $q_*$ ) a right adjoint of  $p^*$  (resp.  $q^*$ ) and by  $f^!$  (resp.  $g^!$ ) a right adjoint of  $f_!$  (resp.  $g_!$ ).

By (P7) or (P7<sup>bis</sup>),  $f^*$  and  $g^*$  have left adjoints. Moreover, the diagram

$$(4.3) \quad \begin{array}{ccccccc} f^*p_*\langle \dim f \rangle & \longrightarrow & q_*g^*\langle \dim f \rangle & \xlongequal{\quad} & q_*g^*\langle \dim f \rangle & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \simeq \left( f^!f_!f^*p_*\langle \dim f \rangle \longrightarrow f^!f_!q_*g^*\langle \dim f \rangle \longrightarrow f^!p_*g_!g^*\langle \dim f \rangle \xrightarrow{\sim} q_*g^!g_!g^*\langle \dim f \rangle \right) & & & & & & \simeq \\ \downarrow \text{Tr}_f & & \downarrow \text{Tr}_g & & \downarrow \text{Tr}_g & & \\ f^!p_* & \xlongequal{\quad} & f^!p_* & \xrightarrow{\sim} & q_*g^! & & \end{array}$$

is commutative up to homotopy. It follows that the top horizontal arrow is an equivalence.

Since the diagram

$$\begin{array}{ccccccc} q^*f^*\langle \dim f \rangle & \xlongequal{\quad} & q^*f^*\langle \dim f \rangle & \xrightarrow{\sim} & g^*p^*\langle \dim f \rangle & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \simeq \left( q^*f^!f_!f^*\langle \dim f \rangle \longrightarrow g^!p^*f_!f^*\langle \dim f \rangle \xrightarrow{\sim} g^!g_!q^*f^*\langle \dim f \rangle \xrightarrow{\sim} g^!g_!g^*p^*\langle \dim f \rangle \right) & & & & & & \simeq \\ \downarrow \text{Tr}_f & & \downarrow \text{Tr}_f & & \downarrow \text{Tr}_g & & \\ q^*f^! & \longrightarrow & g^!p^* & \xlongequal{\quad} & g^!p^* & & \end{array}$$

is commutative up to homotopy, the bottom horizontal arrow is an equivalence.  $\square$

**4.2. Construction.** The goal of this subsection is to construct the maps  $\tilde{\mathcal{C}}\text{EO}^{\text{I}}$  and  $\tilde{\mathcal{C}}'\text{EO}^{\text{II}}$  in Output I in §4.1. We will construct Output II and check the properties (P0) – (P7<sup>bis</sup>) in the next section.

Let us start from the construction of second abstract operation map  $\tilde{\mathcal{C}}'\text{EO}^{\text{II}}$ . The first one  $\tilde{\mathcal{C}}\text{EO}^{\text{I}}$  will be constructed at the end of this section, after the proof of Proposition 4.1.1.

Let  $\mathcal{R} \subseteq \tilde{\mathcal{F}}'$  be the subset of morphisms that are representable in  $\mathcal{C}'$ . We have successive inclusions

$$\delta_{2,\{2\}}^*((\mathcal{C}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II},\text{op}})_{\mathcal{F}',\text{all}}^{\text{cart}} \subseteq \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II},\text{op}})_{\mathcal{R},\text{all}}^{\text{cart}} \subseteq \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II},\text{op}})_{\tilde{\mathcal{F}}',\text{all}}^{\text{cart}}.$$

We proceed in two steps.

**Step 1.** We first extend  ${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}$  to the map  ${}_{\mathcal{C}'}^{\mathcal{R}}\mathrm{EO}^{\mathrm{II}}$  with the new source

$$\delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op})_{\mathcal{R},\mathrm{all}}^{\mathrm{cart}}.$$

An  $n$ -cell of the above source is given by a functor

$$\sigma: \Delta^n \times (\Delta^n)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op}$$

We define  $\mathrm{Cov}(\sigma)$  to be the full subcategory of

$$\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\Delta_+)^{op}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op}) \times_{\mathrm{Fun}(\Delta^n \times (\Delta^n)^{op} \times \{[-1]\}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op})} \{\sigma\}$$

spanned by functors  $\sigma^0: \Delta^n \times (\Delta^n)^{op} \times \mathrm{N}(\Delta_+)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op}$  such that

- for every  $0 \leq j \leq n$ , the restriction  $\sigma^0|_{\Delta^{(n,j)} \times \mathrm{N}(\Delta_+^{\leq 0})^{op}}$ , regarded as an edge of  $(\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op}$ , is statically an atlas (see Definition 3.2.2 and Input 0 (4));
- $\sigma^0$  is a right Kan extension of  $\sigma^0|_{\Delta^{\{n\}} \times (\Delta^n)^{op} \times \mathrm{N}(\Delta_+^{\leq 0})^{op} \cup \Delta^n \times (\Delta^n)^{op} \times \{[-1]\}}$  along the obvious inclusion.

In particular, for every object  $(i, j)$  of  $\Delta^n \times (\Delta^n)^{op}$ , the restriction  $\sigma^0|_{\Delta^{(i,j)} \times \mathrm{N}(\Delta_+)^{op}}$  is a Čech nerve of the restriction  $\sigma^0|_{\Delta^{(i,j)} \times \mathrm{N}(\Delta_+^{\leq 0})^{op}}$ .

The  $\infty$ -category  $\mathrm{Cov}(\sigma)$  is nonempty by Input 0 (4) and (5), and admits product of two objects. Indeed, for every pair of objects  $\sigma_1^0$  and  $\sigma_2^0$  of  $\mathrm{Cov}(\sigma)$ , the assignment

$$(i, j, [k]) \mapsto \sigma_1^0(i, j, [k]) \times_{\sigma(i,j)} \sigma_2^0(i, j, [k])$$

induces a product of  $\sigma_1^0$  and  $\sigma_2^0$  by Lemma 1.5.5. Therefore, by Lemma 1.1.1,  $\mathrm{Cov}(\sigma)$  is a weakly contractible Kan complex.

The restriction functor

$$\mathrm{Cov}(\sigma) \rightarrow \mathrm{Fun}(\mathrm{N}(\Delta)^{op} \times \Delta^n \times (\Delta^n)^{op}, (\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op})$$

induces a map

$$\mathrm{Cov}(\sigma)^{op} \rightarrow \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Fun}(\Delta^n, \delta_{2,\{2\}}^*((\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\mathrm{II},op})_{\mathcal{F},\mathrm{all}}^{\mathrm{cart}})).$$

Composing with the map  ${}_{\mathcal{C}'}\mathrm{EO}^{\mathrm{II}}$ , we obtain a functor

$$\phi(\sigma): \mathrm{Cov}(\sigma)^{op} \rightarrow \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty})).$$

Let  $\mathcal{K} \subseteq \mathrm{Fun}(\mathrm{N}(\Delta_+), \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty}))$  be the full subcategory spanned by those functors  $F: \mathrm{N}(\Delta_+) \rightarrow \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty})$  that are right Kan extensions of  $F|_{\mathrm{N}(\Delta)}$ . Consider the following diagram

$$\begin{array}{ccc} \mathrm{N}(\sigma) & \xrightarrow{\quad} & \mathrm{Cov}(\sigma)^{op} \\ \downarrow \mathrm{res}_1^* \phi(\sigma) & & \downarrow \phi(\sigma) \\ \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty}) & \xleftarrow{\mathrm{res}_2} \mathcal{K} \xrightarrow{\mathrm{res}_1} & \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty})) \end{array}$$

in which the right square is Cartesian, and  $\mathrm{res}_2$  is the restriction to  $\{[-1]\}$ . Put

$$\Phi(\sigma) = \mathrm{res}_2 \circ \mathrm{res}_1^* \phi(\sigma): \mathrm{N}(\sigma) \rightarrow \mathrm{Fun}(\Delta^n, \mathcal{C}\mathrm{at}_{\infty}).$$

It is easy to see that the above process is functorial so that the collection of  $\Phi(\sigma)$  defines a morphism  $\Phi$  of the category

$$(\text{Set}_\Delta)^{(\Delta_{2,\{2\}}^* ((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})_{\mathcal{R},\text{all}}^{\text{cart}})^{op}}.$$

See §1.2 for more backgrounds.

**Lemma 4.2.1.** *The map  $\Phi(\sigma)$  takes values in  $\text{Map}^\#((\Delta^n)^b, \text{Cat}_\infty^b)$ .*

*Proof.* Let  $X_{-1}$  be an object of  $(\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}$ , and  $\text{Cov}(X_{-1})$  the full subcategory of

$$\text{Fun}(\text{N}(\Delta_+)^{op}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}) \times_{\text{Fun}(\{-1\}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})} \{X_{-1}\}$$

spanned by functors  $X_\bullet$  such that the edge  $X_0 \rightarrow X_{-1}$  is statically an atlas and  $X_\bullet$  is a Čech nerve of  $X_0 \rightarrow X_{-1}$ . By (P2), it suffices to show that for every morphism  $f$  of  $\text{Cov}(X_{-1})$ , considered as a functor  $f: \Delta^1 \times \text{N}(\Delta_+)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}$ , and every right Kan extension  $F$  of  ${}_{\mathcal{C}}\text{EO}^I \circ (f| \Delta^1 \times \text{N}(\Delta_+)^{op})^{op}$ , the morphism  $F|(\Delta^1 \times \{-1\})^{op}$  is an equivalence in  $\text{Cat}_\infty$ .

In fact, let  $f: X_\bullet^0 \rightarrow X_\bullet^1$  be a morphism of  $\text{Cov}(X_{-1})$ . Let  $X_\bullet^2$  be an object of  $\text{Cov}(X_{-1})$ . Then we have a diagram

$$\begin{array}{ccccc} & & X_\bullet^0 \times X_\bullet^2 & \xrightarrow{\text{pr}} & X_\bullet^0 \\ & \swarrow \text{pr} & \downarrow f \times X_\bullet^2 & & \downarrow f \\ X_\bullet^2 & \xleftarrow{\text{pr}} & X_\bullet^1 \times X_\bullet^2 & \xrightarrow{\text{pr}} & X_\bullet^1 \end{array}$$

Here products are taken in  $\text{Cov}(X_{-1})$ . Thus it suffices to show the assertion for the projection  $X_\bullet \times X'_\bullet \rightarrow X'_\bullet$ , where  $X_\bullet$  and  $X'_\bullet$  are objects of  $\text{Cov}(X_{-1})$ .

Let  $Y_{\bullet\bullet}: \text{N}(\Delta_+)^{op} \times \text{N}(\Delta_+)^{op} \rightarrow \tilde{\mathcal{C}}'$  be an augmented bisimplicial object of  $\tilde{\mathcal{C}}'$  such that

- $Y_{-1\bullet} = X'_\bullet$ ,  $Y_{\bullet-1} = X_\bullet$ .
- $Y_{\bullet\bullet}$  is a right Kan extension of  $Y_{-1\bullet} \cup Y_{\bullet-1}$ .

Let  $\delta: [1] \times \Delta_+^{op} \rightarrow \Delta_+^{op} \times \Delta_+^{op}$  be the functor sending  $(0, [n])$  to  $([n], [n])$  and  $(1, [n])$  to  $([-1], [n])$ . It suffices to show the assertion for  $Y_{\bullet\bullet} \circ \text{N}(\delta)$ , regarded as a morphism of  $\text{Cov}(X_{-1})$ . This follows from Lemma 4.2.2 below by taking  $p$  to be  $\text{Cat}_\infty \rightarrow *$  and  $c^{\bullet\bullet}$  to be a right Kan extension of  ${}_{\mathcal{C}}\text{EO}^I \circ (Y_{\bullet\bullet}| \text{N}(\Delta_{++})^{op})^{op}$ . Here,  $\Delta_{++} \subseteq \Delta_+ \times \Delta_+$  is the full subcategory spanned by all objects except the initial one. Assumptions (2) and (3) of Lemma 4.2.2 are satisfied thanks to (P0) and (P4); see Remark 4.1.7 (1).  $\square$

**Lemma 4.2.2.** *Let  $p: \mathcal{C} \rightarrow \mathcal{D}$  be a categorical fibration of  $\infty$ -categories. Let  $c^{\bullet\bullet}: \text{N}(\Delta_+) \times \text{N}(\Delta_+) \rightarrow \mathcal{C}$  be an augmented bicosimplicial object of  $\mathcal{C}$ . For  $n \geq -1$ , put  $c^{n\bullet} := c^{\bullet\bullet}| \{[n]\} \times \text{N}(\Delta_+)$  and  $c^{\bullet n} := c^{\bullet\bullet}| \text{N}(\Delta_+) \times \{[n]\}$ , respectively. Assume that*

- (a)  $c^{\bullet\bullet}$  is a  $p$ -limit [HTT, 4.3.1.1] of  $c^{\bullet\bullet}| \text{N}(\Delta_{++})$ , where  $\Delta_{++} \subseteq \Delta_+ \times \Delta_+$  is the full subcategory spanned by all objects except the initial one.
- (b) For every  $n \geq 0$ ,  $c^{n\bullet}$  is a  $p$ -limit of  $c^{n\bullet}| \text{N}(\Delta)$ .
- (c) For every  $n \geq 0$ ,  $c^{\bullet n}$  is a  $p$ -limit of  $c^{\bullet n}| \text{N}(\Delta)$ .

Then



- (1)  $c^{-1\bullet}$  is a  $p$ -limit of  $c^{-1\bullet} \mid \{[-1]\} \times N(\Delta)$ .
- (2)  $c^{\bullet-1}$  is a  $p$ -limit of  $c^{\bullet-1} \mid N(\Delta) \times \{[-1]\}$ .
- (3)  $c^{\bullet\bullet} \mid N(\Delta_+)_{\text{diag}}$  is a  $p$ -limit of  $c^{\bullet\bullet} \mid N(\Delta)_{\text{diag}}$ , where  $N(\Delta_+)_{\text{diag}} \subseteq N(\Delta_+) \times N(\Delta_+)$  is the image of the diagonal inclusion  $\text{diag}: N(\Delta_+) \rightarrow N(\Delta_+) \times N(\Delta_+)$  and  $N(\Delta)_{\text{diag}}$  is defined similarly.

*Proof.* For (1), we apply (the dual version of) [HTT, 4.3.2.8] to  $p$  and  $N(\Delta_+ \times \Delta) \subseteq N(\Delta_{++}) \subseteq N(\Delta_+ \times \Delta_+)$ . By (the dual version of) [HTT, 4.3.2.9] and assumption (b), the restriction  $c^{\bullet\bullet} \mid N(\Delta \times \Delta_+)$  is a  $p$ -right Kan extension of the restriction  $c^{\bullet\bullet} \mid N(\Delta \times \Delta)$  [HTT, 4.3.2.2]. It follows that  $c^{\bullet\bullet} \mid N(\Delta_{++})$  is a  $p$ -right Kan extension of  $c^{\bullet\bullet} \mid N(\Delta_+ \times \Delta)$ . By assumption (a),  $c^{\bullet\bullet}$  is a  $p$ -right Kan extension of  $c^{\bullet\bullet} \mid N(\Delta_{++})$ . Therefore,  $c^{\bullet\bullet}$  is a  $p$ -right Kan extension of  $c^{\bullet\bullet} \mid N(\Delta_+ \times \Delta)$ . By [HTT, 4.3.2.9] again,  $c^{-1\bullet}$  is a  $p$ -limit of  $c^{-1\bullet} \mid \{[-1]\} \times N(\Delta)$ .

For (2), it follows from conclusion (1) by symmetry.

For (3), we view  $(\Delta \times \Delta)^\triangleleft$  as a full subcategory of  $\Delta_+ \times \Delta_+$  by sending the cone point to the initial object. By [HTT, 4.3.2.7], we find that  $c^{\bullet\bullet} \mid (\Delta \times \Delta)^\triangleleft$  is a  $p$ -limit diagram. By [HTT, 5.5.8.4], the simplicial set  $N(\Delta)^{op}$  is *sifted* [HTT, 5.5.8.1], that is, the diagonal map  $N(\Delta)^{op} \rightarrow N(\Delta)^{op} \times N(\Delta)^{op}$  is cofinal. Therefore,  $c^{\bullet\bullet} \mid N(\Delta_+)_{\text{diag}}$  is a  $p$ -limit of  $c^{\bullet\bullet} \mid N(\Delta)_{\text{diag}}$ .  $\square$

Since  $\text{res}_1$  is a trivial fibration by [HTT, 4.3.2.15], the simplicial set  $N(\sigma)$  is weakly contractible. By Lemma 4.2.1, we can apply Lemma 1.2.4 to

$$K = \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})_{\mathcal{R},\text{all}}^{\text{cart}}, \quad K' = \delta_{2,\{2\}}^*((\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})_{\mathcal{F}',\text{all}}^{\text{cart}}, \quad g: K' \hookrightarrow K,$$

and the section  $\nu$  given by  ${}_{\mathcal{C}'}\text{EO}^{\text{II}}$ . This extends  ${}_{\mathcal{C}'}\text{EO}^{\text{II}}$  to a map

$${}_{\tilde{\mathcal{C}'}}\text{EO}^{\text{II}}: \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})_{\mathcal{R},\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_\infty.$$

**Step 2.** Now we are going to extend  ${}_{\tilde{\mathcal{C}'}}\text{EO}^{\text{II}}$  to the map  ${}_{\tilde{\mathcal{C}'}}\text{EO}^{\text{II}}$  with the new source

$$\delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})_{\mathcal{F}',\text{all}}^{\text{cart}}.$$

An  $n$ -cell of the above source is given by a functor

$$\varsigma: \Delta^n \times (\Delta^n)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}$$

We define  $\text{Kov}(\varsigma)$  to be the full subcategory of

$$\text{Fun}(\Delta^n \times (\Delta^n)^{op} \times N(\Delta_+)^{op}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}) \times_{\text{Fun}(\Delta^n \times (\Delta^n)^{op} \times \{[-1]\}, (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op})} \{\varsigma\}$$

spanned by functors  $\varsigma^0: \Delta^n \times (\Delta^n)^{op} \times N(\Delta_+)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}$  such that

- for every  $0 \leq i \leq n$ , the restriction  $\varsigma^0 \mid \Delta^{(i,0)} \times N(\Delta_+^{\leq 0})^{op}$ , regarded as an edge of  $(\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}$ , statically belongs to  $\tilde{\mathcal{E}}_s \cap \tilde{\mathcal{E}}' \cap \mathcal{R}$ ;
- $\varsigma^0$  is a right Kan extension of  $\varsigma^0 \mid \Delta^n \times (\Delta^{\{0\}})^{op} \times N(\Delta_+^{\leq 0})^{op} \cup \Delta^n \times (\Delta^n)^{op} \times \{[-1]\}$  along the obvious inclusion;
- the restriction  $\varsigma^0 \mid \Delta^n \times (\Delta^{\{0\}})^{op} \times \{[0]\}$  corresponds to an  $n$ -cell of  $(\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\text{II},op}_{\mathcal{R}}$ .

In particular, for every object  $(i, j)$  of  $\Delta^n \times (\Delta^n)^{op}$ , the restriction  $\varsigma^0 \mid \Delta^{(i,j)} \times N(\Delta_+)^{op}$  is a Čech nerve of the restriction  $\varsigma^0 \mid \Delta^{(i,j)} \times N(\Delta_+^{\leq 0})^{op}$ . Moreover, the restriction  $\varsigma^0 \mid \Delta^n \times (\Delta^n)^{op} \times \{[0]\}$  corresponds to an  $n$ -cell of  $\delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\mathcal{R},\text{all}}^{\text{cart}}$ .

Similar to  $\text{Cov}(\sigma)$ , the  $\infty$ -category  $\text{Kov}(\varsigma)$  is nonempty and admits product of two objects. Therefore, by Lemma 1.1.1,  $\text{Kov}(\varsigma)$  is a weakly contractible Kan complex.

The restriction functor

$$\text{Kov}(\varsigma) \rightarrow \text{Fun}(N(\Delta)^{op} \times \Delta^n \times (\Delta^n)^{op}, (\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})$$

induces a map

$$\text{Kov}(\varsigma) \rightarrow \text{Fun}(N(\Delta)^{op}, \text{Fun}(\Delta^n, \delta_{2,\{2\}}^*((\mathcal{C}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\mathcal{F}',\text{all}}^{\text{cart}})).$$

Composing with the map  $\mathbb{R}_{\tilde{\mathcal{C}}'}\text{EO}^{\Pi}$ , we obtain a functor

$$\phi(\varsigma): \text{Kov}(\sigma) \rightarrow \text{Fun}(N(\Delta)^{op}, \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty})).$$

Let  $\mathcal{K}' \subseteq \text{Fun}(N(\Delta_+)^{op}, \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty}))$  be the full subcategory spanned by those functors  $F: N(\Delta_+)^{op} \rightarrow \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty})$  that are right Kan extensions of  $F \mid N(\Delta)^{op}$ . Consider the following diagram

$$\begin{array}{ccc} N(\varsigma) & \xrightarrow{\quad} & \text{Kov}(\varsigma) \\ \downarrow \text{res}_1^* \phi(\varsigma) & & \downarrow \phi(\varsigma) \\ \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty}) & \xleftarrow{\text{res}_2} \mathcal{K}' \xrightarrow{\text{res}_1} & \text{Fun}(N(\Delta)^{op}, \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty})) \end{array}$$

in which the right square is Cartesian, and  $\text{res}_2$  is the restriction to  $\{[-1]\}$ . Put

$$\Phi(\varsigma) = \text{res}_2 \circ \text{res}_1^* \phi(\varsigma): N(\varsigma) \rightarrow \text{Fun}(\Delta^n, \mathcal{Cat}_{\infty}).$$

It is easy to see that the above process is functorial so that the collection of  $\Phi(\varsigma)$  defines a morphism  $\Phi$  of the category

$$(\text{Set}_{\Delta})^{(\Delta_{\delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\mathcal{F}',\text{all}}^{\text{cart}}})^{op}}.$$

**Lemma 4.2.3.** *The map  $\Phi(\varsigma)$  takes values in  $\text{Map}^{\sharp}((\Delta^n)^b, \mathcal{Cat}_{\infty}^{\sharp})$ .*

*Proof.* Let  $X_{\bullet}: N(\Delta_+)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op}$  be an augmented simplicial object that is a Čech nerve of  $f: X_0 \rightarrow X_{-1}$  such that  $f$  statically belongs to  $\tilde{\mathcal{E}}_s \cap \tilde{\mathcal{E}}' \cap \mathcal{R}$ . By the construction of  $\Phi(\varsigma)$ , it suffices to show that  $R \circ X_{\bullet}$  is a left Kan extension of  $R \circ X_{\bullet} \mid N(\Delta)^{op}$ , where  $R = \mathbb{R}_{\tilde{\mathcal{C}}'}\text{EO}^{\Pi} \mid ((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\mathcal{R}}$  is the restriction along direction 1.

Choose an object  $X'_{-1}$  of  $\text{Cov}(X_{-1})$  and form a bisimplicial object  $Y_{\bullet\bullet}: N(\Delta_+)^{op} \times N(\Delta_+)^{op} \rightarrow (\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op}$  as in the proof of Lemma 4.2.1, which is static. Applying  $\mathbb{R}_{\tilde{\mathcal{C}}'}\text{EO}^{\Pi}$  to  $Y_{\bullet\bullet}$  and by adjunction, we obtain a diagram  $\chi_{\bullet}^{\bullet}: N(\Delta_+)^{op} \times N(\Delta_+)^{op} \rightarrow \mathcal{Cat}_{\infty}$ . By the construction of  $\mathbb{R}_{\tilde{\mathcal{C}}'}\text{EO}^{\Pi}$ , we have that  $\chi_n^{\bullet}$  is a limit diagram for  $n \geq -1$ . By (P4),  $\chi_n^{\bullet}$  is a colimit diagram for  $n \geq 0$ . Therefore, by (P5) (2) and [HA, 4.7.5.19] applied to the restriction  $\chi_{\bullet}^{\bullet} \mid N(\Delta_{s,+})^{op} \times N(\Delta_{s,+})$ , we have that  $R \circ X_{\bullet} = \chi_{\bullet}^{-1}$  is a colimit diagram. In the last sentence, we used [HTT, 6.5.3.7] twice.  $\square$

Since  $\text{res}_1$  is a trivial fibration by [HTT, 4.3.2.15], the simplicial set  $\mathcal{N}(\varsigma)$  is weakly contractible. By Lemma 4.2.3, we can apply Lemma 1.2.4 to

$$K = \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\tilde{\mathcal{F}}',\text{all}}^{\text{cart}}, \quad K' = \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\mathcal{R},\text{all}}^{\text{cart}}, \quad g: K' \hookrightarrow K,$$

and the section  $\nu$  given by  $\mathbb{R}\text{EO}^{\Pi}$ . This extends  $\mathbb{R}\text{EO}^{\Pi}$  to a map

$$\tilde{\mathcal{C}}'\text{EO}^{\Pi}: \delta_{2,\{2\}}^*((\tilde{\mathcal{C}}'^{op} \times \mathcal{L}'^{op})^{\Pi,op})_{\tilde{\mathcal{F}}',\text{all}}^{\text{cart}} \rightarrow \text{Cat}_{\infty},$$

as demanded.

Now we prove Proposition 4.1.1, which will be applied to construct the first abstract operation map  $\tilde{\mathcal{C}}\text{EO}^{\text{I}}$  in Output I.

*Proof of Proposition 4.1.1.* The proof is similar to Step 1 above. Consider the diagram

$$\begin{array}{ccc} \partial\Delta^n & \xrightarrow{G} & \text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D}) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow{F} & \text{Fun}^{\mathcal{C}}(\mathcal{C}^{op}, \mathcal{D}). \end{array}$$

Let  $\sigma: (\Delta^m)^{op} \rightarrow \mathcal{C}$  be an  $m$ -cell of  $\tilde{\mathcal{C}}^{op}$ . We denote by  $\text{Cov}(\sigma)$  the full subcategory of

$$\text{Fun}((\Delta^m)^{op} \times \mathcal{N}(\Delta_+)^{op}, \tilde{\mathcal{C}}) \times_{\text{Fun}((\Delta^m)^{op} \times \{-1\}, \tilde{\mathcal{C}})} \{\sigma\}$$

spanned by Čech nerves  $\sigma^0: (\Delta^m)^{op} \times \mathcal{N}(\Delta_+)^{op} \rightarrow \tilde{\mathcal{C}}$  such that  $\sigma^0|_{(\Delta^m)^{op} \times \mathcal{N}(\Delta)^{op}}$  factorizes through  $\mathcal{C}$ , and that  $\sigma^0|_{\Delta^{\{j\}} \times \mathcal{N}(\Delta_+^{\leq 0})^{op}}$  belongs to  $\tilde{\mathcal{E}}$  and is representable in  $\mathcal{C}$  for all  $0 \leq j \leq m$ . Since  $\text{Cov}(\sigma)$  admits product of two objects, it is a contractible Kan complex by Lemma 1.1.1.

Let  $\mathcal{K} \subseteq \text{Fun}(\mathcal{N}(\Delta_+), \text{Fun}(\Delta^m, \mathcal{D}))$  be the full subcategories spanned by augmented cosimplicial objects  $X_{\bullet}^+$  that are right Kan extensions of  $X_{\bullet}^+|_{\mathcal{N}(\Delta)}$ . By [HTT, 4.3.2.15], the restriction map  $\mathcal{K} \rightarrow \text{Fun}(\mathcal{N}(\Delta), \text{Fun}(\Delta^m, \mathcal{D}))$  is a trivial fibration. We have a diagram

$$\begin{array}{ccccc} \text{Cov}(\sigma)^{op} & \xrightarrow{\alpha} & & & \\ \downarrow \phi & & \mathcal{K}' & \xrightarrow{\quad} & \text{Fun}(\Delta^n, \text{Fun}(\mathcal{N}(\Delta) \times \Delta^m, \mathcal{D})) \\ \downarrow \beta & & \downarrow & & \downarrow \\ \text{Fun}(\partial\Delta^n, \mathcal{K}) & \xrightarrow{\quad} & \text{Fun}(\partial\Delta^n, \text{Fun}(\mathcal{N}(\Delta) \times \Delta^m, \mathcal{D})) & & \end{array}$$

where the square is Cartesian,  $\alpha$  is induced by  $F$ , and  $\beta$  is induced by  $G$ . Consider the diagram

$$\begin{array}{ccccc} & & \mathcal{N}(\sigma) & \xrightarrow{\quad} & \text{Cov}(\sigma)^{op} \\ & & \downarrow \text{res}_1^* \phi & & \downarrow \phi \\ \text{Fun}(\Delta^n, \text{Fun}(\Delta^m, \mathcal{D})) & \xleftarrow{\text{res}_2} & \text{Fun}(\Delta^n, \mathcal{K}) & \xrightarrow{\text{res}_1} & \mathcal{K}', \end{array}$$

where the square is Cartesian and  $\text{res}_2$  is the restriction to  $\{[-1]\}$ . Since  $\text{res}_1$  is a trivial fibration,  $\mathcal{N}(\sigma)$  is a contractible Kan complex.

Put  $\Phi(\sigma) = \text{res}_2 \circ \text{res}_1^* \phi$ . The construction is functorial in  $\sigma$  in the sense that it defines a morphism  $\Phi$  of the category  $(\text{Set}_\Delta)^{(\Delta^{\tilde{\mathcal{C}}^{op}})^{op}}$ . Moreover,  $\Phi(\sigma)$  takes values in  $\text{Map}^\sharp((\Delta^m)^\flat, \text{Fun}(\Delta^n, \mathcal{D})^\sharp)$ . In fact, this is trivial for  $n > 0$  and the proof of Lemma 4.2.1 can be easily adapted to treat the case  $n = 0$ . Applying Lemma 1.2.3 to  $\Phi$  and  $a = G$ , we obtain a lifting  $\tilde{F}: \Delta^n \rightarrow \text{Fun}(\tilde{\mathcal{C}}^{op}, \mathcal{D})$  of  $F$  extending  $G$ .

It remains to show that  $\tilde{F}$  factorizes through  $\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}^{op}, \mathcal{D})$ . This is trivial for  $n > 0$ . For  $n = 0$ , we need to show that every morphism  $f: Y \rightarrow X$  in  $\tilde{\mathcal{E}}$  is of  $\tilde{F}$ -descent, where we regard  $\tilde{F}$  as a functor  $\mathcal{C}^{op} \rightarrow \mathcal{D}$ . Let  $u: X' \rightarrow X$  be a morphism in  $\tilde{\mathcal{E}}$  with  $X'$  in  $\mathcal{C}$ , and  $v$  the composite morphism  $Y' \xrightarrow{w} Y \times_X X' \rightarrow Y$  of the pullback of  $u$  and a morphism  $w$  in  $\mathcal{E}$  with  $Y'$  in  $\mathcal{C}$ . This provides a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

where  $u$  and  $v$  are in  $\tilde{\mathcal{E}}$  and  $f'$  belongs to  $\mathcal{E}$ . Then  $f'$  and  $u$  are of  $\tilde{F}$ -descent by construction. It follows that  $f$  is of  $F$ -descent by Lemma 3.1.2 (3), (4).  $\square$

Thanks to (P0) and (P4) (see Remark 4.1.7 (1)), we may apply Proposition 4.1.1 to

- $\tilde{\mathcal{C}} = (\tilde{\mathcal{C}}^{op} \times \mathcal{L}^{op})^{\text{II}, op}$ ,
- $\mathcal{C} = (\mathcal{C}^{op} \times \mathcal{L}^{op})^{\text{II}, op}$ ,
- $\mathcal{D} = \text{Cat}_\infty$ ,
- and the set  $\tilde{\mathcal{E}}$  consists of edges  $f$  that statically belong to  $\tilde{\mathcal{E}}_s \cap \tilde{\mathcal{E}}''$ ,

and obtain an extension of the functor  ${}_{\mathcal{C}}\text{EO}^{\text{I}}$  to a functor

$${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}}: (\tilde{\mathcal{C}}^{op} \times \mathcal{L}^{op})^{\text{II}} \rightarrow \text{Cat}_\infty$$

as demanded.

**4.3. Properties.** We construct Output II and prove that Output I and Output II satisfy all required properties.

**Lemma 4.3.1** (P0). *The functor  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}}$  is a weak Cartesian structure, and the induced functor  ${}_{\tilde{\mathcal{C}}}\text{EO}^\otimes := ({}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}})^\otimes$  factorizes through  $\text{CAlg}(\text{Cat}_\infty)_{\text{pr, st, cl}}^{\text{L}}$ .*

*Proof.* This follows from the construction of  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}}$  as the properties in (P0) are preserved under limits.  $\square$

**Lemma 4.3.2** (P1). *The map  ${}_{\tilde{\mathcal{C}}}\text{EO}^\otimes$  sends small coproducts to products.*

*Proof.* Since  $\tilde{\mathcal{C}}$  is geometric (Definition 4.1.3), small coproducts commute with pullbacks. Therefore, forming Čech nerves commutes with the such coproducts. Then the lemma follows from the construction of  ${}_{\tilde{\mathcal{C}}}\text{EO}^\otimes$  and the property (P1) for  ${}_{\mathcal{C}}\text{EO}^\otimes$ .  $\square$

**Lemma 4.3.3** (P2). *The restrictions of  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}}$  and  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{II}}$  to  $(\tilde{\mathcal{C}}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II}}$  are equivalent functors.*

*Proof.* By Proposition 4.1.1 and the original (P2), it suffices to show that the restriction  $F := {}_{\tilde{\mathcal{C}}'}\text{EO}^{\text{II}} \mid (\tilde{\mathcal{C}}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II}}$  belongs to  $\text{Fun}^{\tilde{\mathcal{E}}}((\tilde{\mathcal{C}}'^{\text{op}} \times \mathcal{L}'^{\text{op}})^{\text{II}}, \text{Cat}_{\infty})$  where set  $\tilde{\mathcal{E}}$  consists of edges  $f$  of that statically belong to  $\tilde{\mathcal{E}}_{\text{s}} \cap \tilde{\mathcal{E}}'' \cap \tilde{\mathcal{C}}'_1$ . In other words, it suffices to show that  $f$  is of  $F$ -descent.

By construction, the assertions are true if  $f$  is statically an atlas. Moreover, by the original (P4), the assertions are also true if  $f$  is a morphism of  $\mathcal{C}'$ . In the general case, consider a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

where  $u$  is an atlas and  $f'$  belongs to  $\mathcal{E}_{\text{s}} \cap \mathcal{E}''$ . For example, we can take  $v$  to be an atlas of  $Y \times_X X'$ . The proposition then follows from Lemma 3.1.2 (3), (4).  $\square$

**Lemma 4.3.4** (P3). *If  $f: Y \rightarrow X$  belongs to  $\tilde{\mathcal{E}}_{\text{s}}$ , then  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is conservative for every object  $\lambda$  of  $\mathcal{L}$ .*

*Proof.* We may put  $f$  into the following diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

where  $u$  is an atlas,  $Y$  belongs to  $\mathcal{C}$  and  $f'$  belongs to  $\mathcal{E}_{\text{s}}$ . Then we only need to show that  $v^* \circ f^*$ , which is equivalent to  $f'^* \circ u^*$ , is conservative. By [HA, 4.7.6.2 (3)],  $u^*$  is conservative, and  $f'^*$  is also conservative by the original (P3). Therefore,  $f^*$  is conservative.  $\square$

**Proposition 4.3.5** (P4). *Let  $f$  be a morphism of  $\tilde{\mathcal{C}}^{\text{op}}$  (resp.  $\tilde{\mathcal{C}}'$ ).*

- (1) *If  $f$  belongs to  $\tilde{\mathcal{E}}_{\text{s}} \cap \tilde{\mathcal{E}}''$ , then  $(f, \text{id}_{\lambda})$  is of universal  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\otimes}$ -descent for every object  $\lambda$  of  $\mathcal{L}$ .*
- (2) *If  $f$  belongs to  $\tilde{\mathcal{E}}_{\text{s}} \cap \tilde{\mathcal{E}}'' \cap \tilde{\mathcal{C}}'_1$ , then  $(f, \text{id}_{\lambda})$  is of universal  ${}_{\tilde{\mathcal{C}}'}\text{EO}_1$ -codescent for every object  $\lambda$  of  $\mathcal{L}'$ .*

*Proof.* Part (1) follows from the construction of  ${}_{\tilde{\mathcal{C}}}\text{EO}^{\text{I}}$ . Part (2) follows from the same argument as in Lemma 4.3.3.  $\square$

We will only check (P5), and (P5<sup>bis</sup>) follows in the same way.

**Proposition 4.3.6** (P5). *Let*

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ q \downarrow & & \downarrow p \\ Y & \xrightarrow{f} & X \end{array}$$

be a Cartesian diagram of  $\tilde{\mathcal{C}}'$  with  $f$  in  $\tilde{\mathcal{E}}'$ , and  $\lambda$  an object of  $\mathcal{L}'$ . Then

(1) The square

$$(4.4) \quad \begin{array}{ccc} \mathcal{D}(Z, \lambda) & \xleftarrow{p^*} & \mathcal{D}(X, \lambda) \\ g^* \downarrow & & \downarrow f^* \\ \mathcal{D}(W, \lambda) & \xleftarrow{q^*} & \mathcal{D}(Y, \lambda) \end{array}$$

has a right adjoint which is a square of  $\mathcal{P}\mathcal{r}_{\text{st}}^{\text{R}}$ .

(2) If  $p$  is also in  $\tilde{\mathcal{E}}'$ , the square

$$(4.5) \quad \begin{array}{ccc} \mathcal{D}(X, \lambda) & \xleftarrow{f!} & \mathcal{D}(Y, \lambda) \\ p^* \downarrow & & \downarrow q^* \\ \mathcal{D}(Z, \lambda) & \xleftarrow{g!} & \mathcal{D}(W, \lambda) \end{array}$$

is right adjointable.

We first prove a technical lemma.

**Lemma 4.3.7.** *Let  $K$  be a simplicial set, and  $p: K \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}\text{at}_{\infty})$  a diagram of squares of  $\infty$ -categories. We view  $p$  as a functor  $K \times \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}\text{at}_{\infty}$ . If for every edge  $\sigma: \Delta^1 \rightarrow K \times \Delta^1$ , the induced square  $p \circ (\sigma \times \text{id}_{\Delta^1}): \Delta^1 \times \Delta^1 \rightarrow \mathcal{C}\text{at}_{\infty}$  is right adjointable (resp. left adjointable), then the limit square  $\varprojlim(p)$  is right adjointable (resp. left adjointable).*

Recall from the remark following Proposition 1.4.4 that when visualizing squares, we adopt the convention that direction 1 is vertical and direction 2 is horizontal.

*Proof.* Let us prove the right adjointable case, the proof of the other case being essentially the same. The assumption allows us to view  $p$  as a functor

$$p': K \rightarrow \text{Fun}(\Delta^1, \text{Fun}^{\text{RAd}}(\Delta^1, \mathcal{C}\text{at}_{\infty}))$$

[HA, 4.7.5.16]. By [HA, 4.7.5.18] and (the dual version of) [HTT, 5.1.2.3], the  $\infty$ -category  $\text{Fun}(\Delta^1, \text{Fun}^{\text{RAd}}(\Delta^1, \mathcal{C}\text{at}_{\infty}))$  admits all limits and these limits are preserved by the inclusion

$$\text{Fun}(\Delta^1, \text{Fun}^{\text{RAd}}(\Delta^1, \mathcal{C}\text{at}_{\infty})) \subseteq \text{Fun}(\Delta^1, \text{Fun}(\Delta^1, \mathcal{C}\text{at}_{\infty})).$$

Therefore, the limit square  $\varprojlim(p)$  is equivalent to  $\varprojlim(p')$  which is right adjointable.  $\square$

*Proof of Proposition 4.3.6.* For (1), it is clear from the construction and the original (P5) (1) that both  $f^*$  and  $g^*$  admit left adjoints. Therefore, we only need to show that (4.4) is right adjointable. By Lemma 4.3.7, we may assume that  $f$  belongs to  $\mathcal{E}'$ . Then it reduces to show that the transpose of (4.4) is left adjointable, which allows us to assume that  $p$  is a morphism of  $\mathcal{C}'$ , again by Lemma 4.3.7. Then it follows from the original (P5) (1).

For (2), by Lemma 4.3.7, we may assume that  $p$  belongs to  $\mathcal{E}'$ . Then  $p^*$  and  $q^*$  admit left adjoints. Therefore, we only need to prove that the transpose of (4.5) is left adjointable, which allows us to assume that  $f$  is also in  $\mathcal{E}'$ , again by Lemma 4.3.7. Then it follows from the original (P5) (2).  $\square$

Next we define the t-structure. Let  $X$  be an object of  $\tilde{\mathcal{C}}$  and let  $\lambda$  be an object of  $\mathcal{L}$ . For an atlas  $f: X_0 \rightarrow X$ , we denote by  $\mathcal{D}_f^{\leq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$  (resp.  $\mathcal{D}_f^{\geq 0}(X, \lambda) \subseteq \mathcal{D}(X, \lambda)$ ) the full subcategory spanned by complexes  $\mathbf{K}$  such that  $f^*\mathbf{K}$  belongs to  $\mathcal{D}^{\leq 0}(X_0, \lambda)$  (resp.  $\mathcal{D}^{\geq 0}(X_0, \lambda)$ ).

**Lemma 4.3.8.** *We have*

- (1) *The pair of subcategories  $(\mathcal{D}_f^{\leq 0}(X, \lambda), \mathcal{D}_f^{\geq 0}(X, \lambda))$  determine a t-structure on  $\mathcal{D}(X, \lambda)$ .*
- (2) *The pair of subcategories  $(\mathcal{D}_f^{\leq 0}(X, \lambda), \mathcal{D}_f^{\geq 0}(X, \lambda))$  do not depend on the choice of  $f$ .*

In what follows, we will write  $(\mathcal{D}^{\leq 0}(X, \lambda), \mathcal{D}^{\geq 0}(X, \lambda))$  for  $(\mathcal{D}_f^{\leq 0}(X, \lambda), \mathcal{D}_f^{\geq 0}(X, \lambda))$  for an arbitrary atlas  $f$ . Moreover, if  $X$  is an object of  $\mathcal{C}$ , then the new t-structure coincides with the old one since  $\text{id}_X: X \rightarrow X$  is an atlas.

*Proof.* For (1), let  $f_\bullet: X_\bullet \rightarrow X$  be a Čech nerve of  $f_0 = f$ . We need to check the axioms of [HA, 1.2.1.1]. To check axiom (1), let  $\mathbf{K}$  be an object of  $\mathcal{D}_f^{\leq 0}(X, \lambda)$  and  $\mathbf{L}$  an object of  $\mathcal{D}_f^{\geq 1}(X, \lambda)$ . By (P6) for the input and Proposition 4.3.5 (1),  $\text{Map}(\mathbf{K}, \mathbf{L})$  is a homotopy limit of  $\text{Map}(f_n^*\mathbf{K}, f_n^*\mathbf{L})$  by [HTT, Theorem 4.2.4.1, Corollary A.3.2.28] and is thus a weakly contractible Kan complex. Axiom (2) is trivial. By (P6) for the input, we have a cosimplicial diagram  $p: N(\Delta) \rightarrow \text{Fun}(\Delta^1, \text{Cat}_\infty)$  sending  $[n]$  to the functor  $\mathcal{D}(X_n, \lambda) \rightarrow \text{Fun}(\Delta^1 \times \Delta^1, \mathcal{D}(X_n, \lambda))$  that corresponds to the following Cartesian diagram of functors:

$$\begin{array}{ccc} \tau_n^{\leq 0} & \longrightarrow & \text{id}_{X_n} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \tau_n^{\geq 1}, \end{array}$$

where  $\tau_n^{\leq 0}$  and  $\tau_n^{\geq 1}$  (resp.  $\text{id}_{X_n}$ ) are the truncation functors (resp. is the identity functor) of  $\mathcal{D}(X_n, \lambda)$ . Axiom (3) follows from the fact that  $\varprojlim(p)$  provides a similar Cartesian diagram of endofunctors of  $\mathcal{D}(X, \lambda)$ .

For (2), by (1) it suffices to show that for every other atlas  $f': X'_0 \rightarrow X$ , we have  $\mathcal{D}_f^{\leq 0}(X, \lambda) = \mathcal{D}_{f'}^{\leq 0}(X, \lambda)$ . Let  $\mathbf{K}$  be an object of  $\mathcal{D}_f^{\leq 0}(X, \lambda)$  and form a Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & X'_0 \\ g' \downarrow & & \downarrow f' \\ X_0 & \xrightarrow{f} & X. \end{array}$$



By (P6) for the input, the functors  $g^*$  and  $g'^*$  are t-exact, so that

$$g^* \tau^{\geq 1} f'^* K \simeq \tau^{\geq 1} g^* f'^* K \simeq \tau^{\geq 1} g'^* f^* K \simeq g'^* \tau^{\geq 1} f^* K = 0.$$

As  $g^*$  is conservative by (P3) for the input, we have  $\tau^{\geq 1} f'^* K = 0$ . In other words,  $f'^* K$  belongs to  $\mathcal{D}^{\leq 0}(X'_0, \lambda)$ . Therefore, we have  $\mathcal{D}_{\tilde{f}}^{\leq 0}(X, \lambda) \subseteq \mathcal{D}_{\tilde{f}'}^{\leq 0}(X, \lambda)$ . By symmetry, we have  $\mathcal{D}_{\tilde{f}}^{\leq 0}(X, \lambda) \supseteq \mathcal{D}_{\tilde{f}'}^{\leq 0}(X, \lambda)$ . It follows that  $\mathcal{D}_{\tilde{f}}^{\leq 0}(X, \lambda) = \mathcal{D}_{\tilde{f}'}^{\leq 0}(X, \lambda)$ .  $\square$

**Lemma 4.3.9** (P6). *Let  $\lambda$  be an arbitrary object of  $\mathcal{L}$ . We have*

- (1) *For every object  $X$  of  $\tilde{\mathcal{C}}$ , we have  $\lambda_X \in \mathcal{D}^\heartsuit(X, \lambda)$ .*
- (2) *If  $\lambda$  belongs to  $\mathcal{L}''$  and  $X$  is an object of  $\tilde{\mathcal{C}}''$ , then the auto-equivalence  $-\otimes_X^* \lambda(1)$  of  $\mathcal{D}(X, \lambda)$  is t-exact.*
- (3) *For every object  $X$  of  $\tilde{\mathcal{C}}$ , the t-structure on  $\mathcal{D}(X, \lambda)$  is accessible, right complete, and  $\mathcal{D}^{\leq -\infty}(X, \lambda) := \bigcap_n \mathcal{D}^{\leq -n}(X, \lambda)$  consists of zero objects.*
- (4) *For every morphism  $f: Y \rightarrow X$  of  $\tilde{\mathcal{C}}$ , the functor  $f^*: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  is t-exact.*

*Proof.* We choose an atlas  $f: X_0 \rightarrow X$ . Then (1) and (2) follows from (4), the definition of the t-structure, and that  $f^* \lambda_X \simeq \lambda_{X_0}$ . Moreover, (3) follows from the construction, the conservativeness of  $f^*$ , and the corresponding properties for  $X_0$ . Therefore, it remains to show (4).

However, we may put  $f: Y \rightarrow X$  into a diagram

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ v \downarrow & & \downarrow u \\ Y & \xrightarrow{f} & X \end{array}$$

where  $u$  and  $v$  are both atlases. Then the assertion follows from the definition of the t-structure and the fact that  $f'^*$  is t-exact.  $\square$

Finally we construct the trace maps. We will construct the trace maps for  $\tilde{\mathcal{E}}_t$  and check (P7). Construction of the trace maps for  $\tilde{\mathcal{E}}'$  and verification of (P7<sup>bis</sup>) are similar and in fact easier.

Same as before, we have two steps. We first construct the trace maps for  $\mathcal{R} \cap \tilde{\mathcal{E}}_t$ .

**Lemma 4.3.10.** *There exists a unique way to define the trace map*

$$\mathrm{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X,$$

*for morphisms  $f: Y \rightarrow X$  in  $\mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$  and integers  $d \geq \dim^+(f)$ , satisfying (P7) (1) and extending the input. In particular, for such a morphism  $f$ , we have  $f_! \lambda_Y \langle d \rangle \in \mathcal{D}^{\leq 0}(X, \lambda)$ .*

*Proof.* Let

$$(4.6) \quad \begin{array}{ccc} Y_0 & \xrightarrow{f_0} & X_0 \\ y_0 \downarrow & & \downarrow x_0 \\ Y & \xrightarrow{f} & X \end{array}$$

be a Cartesian diagram in  $\tilde{\mathcal{C}}''$ , where  $x_0$  and hence  $y_0$  are atlases. Let  $N(\Delta_+)^{op} \times \Delta^1 \rightarrow \tilde{\mathcal{C}}''$  be a Čech nerve, as shown in the following diagram

$$(4.7) \quad \begin{array}{ccc} Y_\bullet & \xrightarrow{f_\bullet} & X_\bullet \\ y_\bullet \downarrow & & \downarrow x_\bullet \\ Y & \xrightarrow{f} & X \end{array}$$

We call such a diagram a *simplicial Cartesian atlas* of  $f$ . We have  $\dim^+(f_n) = \dim^+(f)$  for every  $n \geq 0$ . By Base Change which is encoded in  $\tilde{\mathcal{C}}'EO^{\text{II}}$  and the definition of  $-\langle d \rangle$ , we have

$$x_0^* f_! \lambda_Y \langle d \rangle \simeq f_0! y_0^* \lambda_Y \langle d \rangle \simeq f_0! \lambda_{Y_0} \langle d \rangle \in \mathcal{D}^{\leq 0}(X_0, \lambda),$$

which implies that  $f_! \lambda_Y \langle d \rangle$  belongs to  $\mathcal{D}^{\leq 0}(X, \lambda)$  by the definition of the t-structure. The uniqueness of the trace map follows from condition (2) of Remark 4.1.6 applied to the diagram (4.6) and (P3) applied to  $x_0$ .

For  $n \geq 0$ , we have trace maps  $\text{Tr}_{f_n} : \tau^{\geq 0} f_n! \lambda_{Y_n} \langle d \rangle \rightarrow \lambda_{X_n}$ . By condition (2) of Remark 4.1.6 applied to the squares induced by  $f_\bullet$ , we know that  $\tau^{\leq 0} x_{\bullet*} \text{Tr}_{f_\bullet}$  is a morphism of cosimplicial objects of  $\mathcal{D}^\heartsuit(X, \lambda)$ . Taking limit, we obtain a map

$$\varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} \text{Tr}_{f_n} : \varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} \tau^{\geq 0} f_n! \lambda_{Y_n} \langle d \rangle \rightarrow \varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} \lambda_{X_n} \simeq \lambda_X.$$

However, the left-hand side is isomorphic to

$$\begin{aligned} \varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} \tau^{\geq 0} f_n! y_n^* \lambda_Y \langle d \rangle &\simeq \varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} \tau^{\geq 0} x_n^* f_! \lambda_Y \langle d \rangle \\ &\simeq \varprojlim_{n \in \Delta} \tau^{\leq 0} x_{n*} x_n^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \simeq \tau^{\geq 0} f_! \lambda_Y \langle d \rangle. \end{aligned}$$

Therefore, we obtain a map  $\text{Tr}_{f_\bullet} : \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$ .

This extends the trace map of the input. In fact, for  $f$  in  $\mathcal{C}_1''$ , by condition (2) of Remark 4.1.6 applied to (4.7),  $\text{Tr}_{f_\bullet}$  can be identified with  $\varprojlim_{n \in \Delta} x_{n*} x_n^* \text{Tr}_f$ . Moreover, condition (2) of Remark 4.1.6 holds in general if one interprets  $\text{Tr}_f$  as  $\text{Tr}_{f_\bullet}$  and  $\text{Tr}_{f'}$  as  $\text{Tr}_{f'_\bullet}$ , where  $f'_\bullet$  is a simplicial Cartesian atlas of  $f'$ , compatible with  $f_\bullet$ . In fact, by

condition (2) of Remark 4.1.6 for the input, the bottom square of the diagram

$$\begin{array}{ccccc}
 u^* \tau^{\geq 0} f_! \lambda_Y \langle d \rangle & \xrightarrow{u^* \text{Tr}_{f_\bullet}} & u^* \lambda_X & & \\
 \downarrow \simeq & \searrow \simeq & \downarrow \text{Tr}_{f'_\bullet} & \searrow \simeq & \\
 & \tau^{\geq 0} f'_! \lambda_{Y'} \langle d \rangle & & & \lambda_{X'} \\
 \downarrow \simeq & \downarrow \simeq & \downarrow \simeq & & \downarrow \simeq \\
 \varprojlim \tau^{\leq 0} x'_{n*} u_n^* \tau^{\geq 0} f_{n!} \lambda_{Y_n} \langle d \rangle & \xrightarrow{\varprojlim \tau^{\leq 0} x'_{n*} u_n^* \text{Tr}_{f_n}} & \varprojlim \tau^{\leq 0} x'_{n*} u_n^* \lambda_{X_n} & & \\
 \downarrow \simeq & \downarrow \simeq & \downarrow \simeq & & \downarrow \simeq \\
 \varprojlim \tau^{\leq 0} x'_{n*} \tau^{\geq 0} f'_{n!} \lambda_{Y'_n} & \xrightarrow{\varprojlim \tau^{\leq 0} x'_{n*} \text{Tr}_{f'_n}} & \varprojlim \tau^{\leq 0} x'_{n*} \lambda_{X'_n} & & 
 \end{array}$$

is commutative, where all the limits are taken over  $n \in \Delta$ . Since the vertical squares are commutative, it follows that the top square is commutative as well. The case of condition (2) of Remark 4.1.6 where  $u$  is an atlas then implies that  $\text{Tr}_{f_\bullet}$  does not depend on the choice of  $f_\bullet$ . We may therefore denote it by  $\text{Tr}_f$ .

It remains to check conditions (1) and (3) of Remark 4.1.6. Similarly to the situation of condition (2), these follow from the input by taking limits.  $\square$

**Lemma 4.3.11.** *If  $f: Y \rightarrow X$  belongs to  $\mathcal{R} \cap \tilde{\mathcal{E}}''_d \cap \tilde{\mathcal{C}}''_1$ , then the induced natural transformation*

$$f^* \langle d \rangle = \text{id}_Y \circ f^* \langle d \rangle \rightarrow f^! \circ f_! \circ f^* \langle d \rangle \xrightarrow{f^! \circ u_f} f^!$$

is an equivalence, where the first arrow is given by the unit transformation and  $u_f$  is defined similarly as (3.14).

*Proof.* Consider diagram (4.7). We need to show that for every object  $K$  of  $\mathcal{D}(X, \lambda)$ , the natural map  $f^* K \langle d \rangle \rightarrow f^! K$  is an equivalence. By Proposition 4.3.5 (1), the map  $K \rightarrow \varprojlim_{n \in \Delta} u_{n*} u_n^* K$  is an equivalence. Moreover,  $f^!$  preserves small limits, and, by (P5<sup>bis</sup>) (1), so does  $f^*$ , since  $f$  belongs to  $\tilde{\mathcal{E}}''$ . Therefore, we may assume  $K = x_{n*} L$ , where  $L \in \mathcal{D}(X_n, \lambda)$ . Similarly to (4.3), the diagram

$$\begin{array}{ccc}
 f^* x_{n*} L \langle d \rangle & \longrightarrow & y_{n*} f_n^* L \langle d \rangle \\
 \downarrow & & \downarrow \\
 f^! x_{n*} L & \longrightarrow & y_{n*} f_n^! L
 \end{array}$$

is commutative up to homotopy. The upper horizontal arrow is an equivalence by (P5<sup>bis</sup>) (1), the lower horizontal arrow is an equivalence by  $\tilde{\mathcal{C}}'' \text{EO}_!^*$ , and the right vertical arrow is an equivalence by (P6) for the input. It follows that the left vertical arrow is an equivalence.  $\square$

**Proposition 4.3.12** (P7 (1)). *There exists a unique way to define the trace map*

$$\text{Tr}_f: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X,$$

for morphisms  $f: Y \rightarrow X$  in  $\tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$  and integers  $d \geq \dim^+(f)$ , satisfying (P7) (1) and extending the input. In particular, for such a morphism  $f$ , we have  $f_! \lambda_Y \langle d \rangle \in \mathcal{D}^{\leq 0}(X, \lambda)$ .

*Proof.* Let  $Y_\bullet: \mathbf{N}(\Delta_+)^{op} \rightarrow \tilde{\mathcal{C}}'$  be a Čech nerve of an atlas  $y_0: Y_0 \rightarrow Y$ , and form a triangle

$$(4.8) \quad \begin{array}{ccc} & Y & \\ y_\bullet \nearrow & & \searrow f \\ Y_\bullet & \xrightarrow{f_\bullet} & X. \end{array}$$

For  $n \geq 0$ , we have  $f_n \in \mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$ . By Proposition 4.3.5 (2), we have equivalences

$$\varinjlim_{n \in \Delta^{op}} f_n! y_n^! \lambda_Y \simeq \varinjlim_{n \in \Delta^{op}} f_n! y_n! y_n^! \lambda_Y \xrightarrow{\sim} f_! \varinjlim_{n \in \Delta^{op}} y_n! y_n^! \lambda_Y \xrightarrow{\sim} f_! \lambda_Y.$$

Since  $y_n$  belongs to  $\mathcal{R} \cap \tilde{\mathcal{E}}'' \cap \tilde{\mathcal{C}}_1''$ , by Lemmas 4.3.11 and Remark 4.1.7 (5), we have equivalences

$$\varinjlim_{n \in \Delta^{op}} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle \simeq \varinjlim_{n \in \Delta^{op}} f_n! y_n^* \lambda_Y \langle d + \dim y_n \rangle \xrightarrow{\sim} \varinjlim_{n \in \Delta^{op}} f_n! y_n^! \lambda_Y \langle d \rangle.$$

Combining the above ones, we obtain an equivalence

$$\varinjlim_{n \in \Delta^{op}} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle \xrightarrow{\sim} f_! \lambda_Y \langle d \rangle.$$

By Lemma 4.3.10,  $f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle$  belongs to  $\mathcal{D}^{\leq 0}(X, \lambda)$  for every  $n \geq 0$ . It follows that the colimit is as well by [HA, 1.2.1.6]. Moreover, the composite map

$$\begin{aligned} \tau^{\geq 0} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle &\rightarrow \varinjlim_{n \in \Delta^{op}} \tau^{\geq 0} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle \\ &\xrightarrow{\sim} \tau^{\geq 0} \varinjlim_{n \in \Delta^{op}} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle \xrightarrow{\sim} \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \end{aligned}$$

is induced by  $\mathrm{Tr}_{f_n}$ . The uniqueness of  $\mathrm{Tr}_f$  then follows from condition (3) of Remark 4.1.6 applied to the triangle (4.8).

Condition (3) of Remark 4.1.6 applied to the triangles induced by  $f_\bullet$  implies the compatibility of

$$\mathrm{Tr}_{f_n}: \tau^{\geq 0} f_n! \lambda_{Y_n} \langle d + \dim y_n \rangle \rightarrow \lambda_X$$

with the transition maps, so that we obtain a map  $\mathrm{Tr}_{f_\bullet}: \tau^{\geq 0} f_! \lambda_Y \langle d \rangle \rightarrow \lambda_X$ . This extends the trace map of Lemma 4.3.10, by condition (3) of Remark 4.1.6 applied to (4.8) for  $f \in \mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$ . Moreover, condition (3) of Remark 4.1.6 holds for  $g \in \mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$ , if we interpret  $\mathrm{Tr}_f$  as  $\mathrm{Tr}_{f_\bullet}$  and  $\mathrm{Tr}_h$  as  $\mathrm{Tr}_{h_\bullet}$ , where  $h_\bullet: Y_\bullet \times_Y Z \rightarrow X$ .

In fact, by condition (3) of Remark 4.1.6 for morphisms in  $\mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$ , the diagram

$$\begin{array}{ccccc}
 \varinjlim \tau^{\geq 0} f_{n!}(\tau^{\geq 0} g_{!} \lambda_Z \langle e \rangle) \langle d + \dim y_n \rangle & \xrightarrow{\varinjlim \tau^{\geq 0} f_{n!} \operatorname{Tr}_g \langle d + \dim y_n \rangle} & \varinjlim \tau^{\geq 0} f_{!} \lambda_Y \langle d \rangle & & \\
 \downarrow & \searrow \simeq & \searrow \simeq & & \\
 \varinjlim \tau^{\geq 0} h_{n!} \lambda_Z \langle d + e + \dim y_n \rangle & \xrightarrow{\tau^{\geq 0} f_{!}(\tau^{\geq 0} g_{!} \lambda_Z \langle e \rangle) \langle d \rangle} & \tau^{\geq 0} f_{!} \operatorname{Tr}_g \langle d \rangle & \xrightarrow{\tau^{\geq 0} f_{!} \operatorname{Tr}_g \langle d \rangle} & \tau^{\geq 0} f_{!} \lambda_Y \langle d \rangle \\
 & \searrow \simeq & \downarrow \simeq & & \downarrow \operatorname{Tr}_{f_{\bullet}} \\
 & \xrightarrow{\tau^{\geq 0} h_{!} \lambda_Z \langle d + e \rangle} & \tau^{\geq 0} h_{!} \lambda_Z \langle d + e \rangle & \xrightarrow{\operatorname{Tr}_{h_{\bullet}}} & \lambda_X
 \end{array}$$

commutes, where all the colimits are taken over  $n \in \Delta^{op}$ . It follows that  $\operatorname{Tr}_{f_{\bullet}}$  does not depend on the choice of  $f_{\bullet}$ . We may therefore denote it by  $\operatorname{Tr}_f$ .

It remains to check the functoriality of the trace map. Similarly to the above special case of condition (2) of Remark 4.1.6, this follows from the functoriality of the trace map for morphisms in  $\mathcal{R} \cap \tilde{\mathcal{E}}_t \cap \tilde{\mathcal{C}}_1''$  by taking colimits.  $\square$

**Proposition 4.3.13** (P7 (2)). *If  $f: Y \rightarrow X$  belongs to  $\tilde{\mathcal{E}}_d'' \cap \tilde{\mathcal{C}}_1''$ , the induced natural transformation*

$$f^* \langle d \rangle = \operatorname{id}_Y \circ f^* \langle d \rangle \rightarrow f^! \circ f_{!} \circ f^* \langle d \rangle \xrightarrow{f^! \circ u_f} f^!$$

is an equivalence, where the first arrow is given by the unit transformation and  $u_f$  is defined similarly as (3.14).

*Proof.* We need to show that  $f^* K \langle d \rangle \rightarrow f^! K$  is an equivalence of every object  $K$  of  $\mathcal{D}(X, \lambda)$ . Let  $y_0: Y_0 \rightarrow Y$  be an atlas. Since  $v_0^*$  is conservative by Lemma 4.3.4, we only need to show that the composite map

$$y_0^* K \langle \dim f_0 \rangle \xrightarrow{\sim} y_0^* f^* K \langle d + \dim y_0 \rangle \rightarrow y_0^* f^! K \langle \dim y_0 \rangle \xrightarrow{\sim} y_0^! f^! K \xrightarrow{\sim} f_0^! K$$

is an equivalence, where  $f_0: Y_0 \rightarrow X$  is a composite of  $f$  and  $y_0$ . However, this follows from Lemma 4.3.11 applied to  $f_0$ .  $\square$

## 5. RUNNING DESCENT

In this chapter, we run the program DESCENT recursively to construct the theory of six operations of quasi-separated schemes in §5.1, algebraic spaces in §5.2, (classical) Artin stacks in §5.3, and eventually higher Artin stacks in §5.4. Moreover, we start from algebraic spaces to construct the theory for higher Deligne–Mumford (DM) stacks as well in §5.5. We would like to point out that although higher DM stacks are special cases of higher Artin stacks, we have less restrictions on the coefficient rings for the former.

Throughout this chapter, we fix a nonempty set  $\square$  of rational primes. See Remark 5.5.5 for the relevance on  $\square$ .

**5.1. Quasi-separated schemes.** Recall from Example 4.1.12 that  $\text{Sch}^{\text{qs}}$  is the full subcategory of  $\text{Sch}$  spanned by quasi-separated schemes, which contains  $\text{Sch}^{\text{qc.sep}}$  as a full subcategory. We run the program DESCENT with the input data in Example 4.1.12. Then the output consists of the following two maps: a functor

$$(5.1) \quad {}_{\text{Sch}^{\text{qs}}} \text{EO}^{\text{I}}: (\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$$(5.2) \quad {}_{\text{Sch}^{\text{qs}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^* ((\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}})^{\text{II},\text{op}})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty},$$

and Output II. Here we recall that  $F$  denotes the set of morphisms locally of finite type of quasi-separated schemes.

For each object  $X$  of  $\text{Sch}^{\text{qs}}$ , we denote by  $\dot{\text{E}}\text{t}^{\text{qs}}(X)$  the quasi-separated étale site of  $X$ . Its underlying category is the full subcategory of  $\text{Sch}_{/X}^{\text{qs}}$  spanned by étale morphisms. We denote by  $X_{\text{qs.ét}}$  the associated topos, namely the category of sheaves on  $\dot{\text{E}}\text{t}^{\text{qs}}(X)$ . For every object  $X$  of  $\text{Sch}^{\text{qc.sep}}$ , the inclusions  $\dot{\text{E}}\text{t}^{\text{qc.sep}}(X) \subseteq \dot{\text{E}}\text{t}^{\text{qs}}(X) \subseteq \dot{\text{E}}\text{t}(X)$  induce equivalences of topoi  $X_{\text{qc.sep.ét}} \rightarrow X_{\text{qs.ét}} \rightarrow X_{\text{ét}}$ .

The pseudofunctor  $\text{Sch}^{\text{qs}} \times \mathcal{R}\text{ind} \rightarrow \mathcal{R}\text{ingedPTopos}$  sending  $(X, (\Xi, \Lambda))$  to  $(X_{\text{qs.ét}}^{\Xi}, \Lambda)$  induces a map  $\text{N}(\text{Sch}^{\text{qs}}) \times \text{N}(\mathcal{R}\text{ind}) \rightarrow \text{N}(\mathcal{R}\text{ingedPTopos})$ . Composing with **T** (2.1), we obtain a functor

$$(5.3) \quad {}_{\text{Sch}^{\text{qs}}}^{\text{qs.ét}} \text{EO}^{\text{I}}: (\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure. It is clear that the restriction of  ${}_{\text{Sch}^{\text{qs}}}^{\text{qs.ét}} \text{EO}^{\text{I}}$  to  $(\text{N}(\text{Sch}^{\text{qc.sep}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$  is equivalent to  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{I}}$ . By the same proof of Proposition 3.3.5 (1), we have the following.

**Proposition 5.1.1** (Cohomological descent for étale topoi). *Let  $f$  be an edge of  $(\text{N}(\text{Sch}^{\text{qs}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$  that is statically a smooth surjective morphism of quasi-separated schemes. Then  $f$  is of universal  ${}_{\text{Sch}^{\text{qs}}}^{\text{qs.ét}} \text{EO}^{\text{I}}$ -descent.*

From the above proposition and Proposition 4.1.1, we obtain the following compatibility result.

**Proposition 5.1.2.** *The two functors  ${}_{\text{Sch}^{\text{qs}}} \text{EO}^{\text{I}}$  (5.1) and  ${}_{\text{Sch}^{\text{qs}}}^{\text{qs.ét}} \text{EO}^{\text{I}}$  (5.3) are equivalent.*

*Remark 5.1.3.* Let  $X$  be object of  $\text{Sch}^{\text{qs}}$ , and  $\lambda = (\Xi, \Lambda)$  an object of  $\mathcal{R}\text{ind}$ . Then it is easy to see that the usual t-structure on  $\mathcal{D}(X_{\text{qs.ét}}^{\Xi}, \Lambda)$  coincides with the one on  $\mathcal{D}(X, \lambda)$  obtained in Output II of the program DESCENT.

**5.2. Algebraic spaces.** Let  $\mathcal{E}\text{sp}$  be the category of algebraic spaces (§0.1). It contains  $\text{Sch}^{\text{qs}}$  as a full subcategory. We run the program DESCENT with the following input:

- $\tilde{\mathcal{C}} = \text{N}(\mathcal{E}\text{sp})$ . It is geometric.
- $\mathcal{C} = \text{N}(\text{Sch}^{\text{qs}})$ , and  $\mathbf{s}'' \rightarrow \mathbf{s}'$  is the unique morphism  $\text{Spec } \mathbb{Z}[\square^{-1}] \rightarrow \text{Spec } \mathbb{Z}$ . In particular,  $\mathcal{C}' = \mathcal{C}$  and  $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}$ .
- $\tilde{\mathcal{E}}_{\text{s}}$  is the set of *surjective* morphisms of algebraic spaces.
- $\tilde{\mathcal{E}}'$  is the set of *étale* morphisms of algebraic spaces.

- $\tilde{\mathcal{E}}''$  is the set of *smooth* morphisms of algebraic spaces.
- $\tilde{\mathcal{E}}_d''$  is the set of *smooth* morphisms of algebraic spaces of pure relative dimension  $d$ . In particular,  $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}_0''$ .
- $\tilde{\mathcal{E}}_t$  is the set of *flat* morphisms *locally of finite presentation* of algebraic spaces.
- $\tilde{\mathcal{F}} = F$  is the set of morphisms *locally of finite type* of algebraic spaces.
- $\mathcal{L} = N(\mathcal{R}\text{ind})^{op}$ ,  $\mathcal{L}' = N(\mathcal{R}\text{ind}_{\text{tor}})^{op}$ , and  $\mathcal{L}'' = N(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op}$ .
- $\dim^+$  is the upper relative dimension (Definition 4.1.11).
- Input I and II are the output of §5.1. In particular,  ${}_c\text{EO}^I$  is (5.1), and  ${}_{c'}\text{EO}^{II}$  is (5.2).

Then the output consists of the following two maps: a functor

$$(5.4) \quad {}_{\mathcal{E}\text{sp}}\text{EO}^I: (N(\mathcal{E}\text{sp})^{op} \times N(\mathcal{R}\text{ind})^{op})^{II} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$$(5.5) \quad {}_{\mathcal{E}\text{sp}}\text{EO}^{II}: \delta_{2,\{2\}}^*((N(\mathcal{E}\text{sp})^{op} \times N(\mathcal{R}\text{ind}_{\text{tor}})^{op})^{II,op})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty},$$

and Output II.

For each object  $X$  of  $\mathcal{E}\text{sp}$ , we denote by  $\dot{\text{E}}\text{t}^{\text{esp}}(X)$  the spatial étale site of  $X$ . Its underlying category is the full subcategory of  $\mathcal{E}\text{sp}/X$  spanned by étale morphisms. We denote by  $X_{\text{esp.ét}}$  the associated topos, namely the category of sheaves on  $\dot{\text{E}}\text{t}^{\text{esp}}(X)$ . For every object  $X$  of  $\text{Sch}^{\text{qs}}$ , the inclusion of the original étale site  $\dot{\text{E}}\text{t}^{\text{qs}}(X)$  of  $X$  into  $\dot{\text{E}}\text{t}^{\text{esp}}(X)$  induces an equivalence of topoi  $X_{\text{esp.ét}} \rightarrow X_{\text{qs.ét}}$ .

As in §5.1, we obtain a functor

$$(5.6) \quad {}_{\mathcal{E}\text{sp}}^{\text{esp.ét}}\text{EO}^I: (N(\mathcal{E}\text{sp})^{op} \times N(\mathcal{R}\text{ind})^{op})^{II} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure. It is clear that the restriction  ${}_{\mathcal{E}\text{sp}}^{\text{esp.ét}}\text{EO}^I|_{(N(\text{Sch}^{\text{qs}})^{op} \times N(\mathcal{R}\text{ind})^{op})^{II}}$  is equivalent to  ${}_{\text{Sch}^{\text{qs}}}\text{EO}^I$ . By the same proof of Proposition 3.3.5 (1), we have the following.

**Proposition 5.2.1** (Cohomological descent for étale topoi). *Let  $f$  be an edge of  $(N(\mathcal{E}\text{sp})^{op} \times N(\mathcal{R}\text{ind})^{op})^{II}$  that is statically a smooth surjective morphism of algebraic spaces. Then  $f$  is of universal  ${}_{\mathcal{E}\text{sp}}^{\text{esp.ét}}\text{EO}^I$ -descent.*

From the above proposition and Proposition 4.1.1, we obtain the following compatibility result.

**Proposition 5.2.2.** *The two functors  ${}_{\mathcal{E}\text{sp}}\text{EO}^I$  (5.4) and  ${}_{\mathcal{E}\text{sp}}^{\text{esp.ét}}\text{EO}^I$  (5.6) are equivalent.*

*Remark 5.2.3.* Let  $X$  be object of  $\mathcal{E}\text{sp}$ , and  $\lambda = (\Xi, \Lambda)$  an object of  $\mathcal{R}\text{ind}$ . Then it is easy to see that the usual t-structure on  $\mathcal{D}(X_{\text{qs.ét}}^{\Xi}, \Lambda)$  coincides with the one on  $\mathcal{D}(X, \lambda)$  obtained in Output II of the program DESCENT.

*Remark 5.2.4.* In our construction of the map (3.8) in §3.2, the essential facts we used from algebraic geometry are Nagata's compactification and proper base change. Nagata's compactification has been extended to separated morphisms of finite type



between quasi-compact and quasi-separated algebraic spaces [CLO12, 1.2.1]. Proper base change for algebraic spaces follows from the case of schemes by cohomological descent and Chow's lemma for algebraic spaces [RG71, I 5.7.13] or the existence theorem of a finite cover by a scheme. The latter is a special case of [Ryd15, Theorem B] and also follows from the Noetherian case [LMB00, 16.6] by Noetherian approximation of algebraic spaces [CLO12, 1.2.2].

Therefore, if we denote by  $\mathcal{E}\mathrm{sp}^{\mathrm{qc},\mathrm{sep}}$  the full subcategory of  $\mathcal{E}\mathrm{sp}$  spanned by (small) coproducts of quasi-compact and separated algebraic spaces (hence contains  $\mathcal{S}\mathrm{ch}^{\mathrm{qc},\mathrm{sep}}$  as a full subcategory), and repeat the process in §3.2, then we obtain a map

$$\varepsilon_{\mathcal{S}\mathrm{p}^{\mathrm{qc},\mathrm{sep}}}^{\mathrm{var}} \mathrm{EO}^{\mathrm{II}}: \delta_{2,\{2\}}^* ((\mathrm{N}(\mathcal{E}\mathrm{sp}^{\mathrm{qc},\mathrm{sep}})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II},\mathrm{op}})^{\mathrm{cart}}_{F,\mathrm{all}} \rightarrow \mathcal{C}\mathrm{at}_{\infty},$$

whose restriction to  $\delta_{2,\{2\}}^* ((\mathrm{N}(\mathcal{S}\mathrm{ch}^{\mathrm{qc},\mathrm{sep}})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II},\mathrm{op}})^{\mathrm{cart}}_{F,\mathrm{all}}$  is equivalent to the map  $\varepsilon_{\mathcal{S}\mathrm{ch}^{\mathrm{qc},\mathrm{sep}}} \mathrm{EO}^{\mathrm{II}}$ .

Moreover, the restriction  $\varepsilon_{\mathcal{S}\mathrm{p}} \mathrm{EO}^{\mathrm{II}} | \delta_{2,\{2\}}^* ((\mathrm{N}(\mathcal{E}\mathrm{sp}^{\mathrm{qc},\mathrm{sep}})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II},\mathrm{op}})^{\mathrm{cart}}_{F,\mathrm{all}}$  is equivalent to the map  $\varepsilon_{\mathcal{S}\mathrm{p}^{\mathrm{qc},\mathrm{sep}}}^{\mathrm{var}} \mathrm{EO}^{\mathrm{II}}$ . In fact, by Remark 4.1.10 (2), it suffices to prove that  $\varepsilon_{\mathcal{S}\mathrm{p}^{\mathrm{qc},\mathrm{sep}}}^{\mathrm{var}} \mathrm{EO}^{\mathrm{II}}$  satisfies (P4). For this, we can repeat the proof of Proposition 3.3.5. The analogue of Remark 3.3.4 holds for algebraic spaces because the definition of trace maps is local for the étale topology on the target.

**5.3. Artin stacks.** Let  $\mathcal{C}\mathrm{hp}$  be the  $(2,1)$ -category of Artin stacks (§0.1). It contains  $\mathcal{E}\mathrm{sp}$  as a full subcategory. We run the *simplified* DESCENT (see Variant 4.1.9) with the following input:

- $\tilde{\mathcal{C}} = \mathrm{N}(\mathcal{C}\mathrm{hp})$ . It is geometric.
- $\mathcal{C} = \mathrm{N}(\mathcal{E}\mathrm{sp})$ , and  $\mathbf{s}'' \rightarrow \mathbf{s}'$  is the identity morphism of  $\mathrm{Spec} \mathbb{Z}[\square^{-1}]$ . In particular,  $\mathcal{C}' = \mathcal{C}'' = \mathrm{N}(\mathcal{E}\mathrm{sp}_{\square})$  (resp.  $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}'' = \mathrm{N}(\mathcal{C}\mathrm{hp}_{\square})$ ), where  $\mathcal{E}\mathrm{sp}_{\square}$  (resp.  $\mathcal{C}\mathrm{hp}_{\square}$ ) is the category of  $\square$ -coprime algebraic spaces (resp. Artin stacks).
- $\tilde{\mathcal{E}}_{\mathrm{s}}$  is the set of *surjective* morphisms of Artin stacks.
- $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''$  is the set of *smooth* morphisms of Artin stacks.
- $\tilde{\mathcal{E}}''_d$  is the set of *smooth* morphisms of Artin stacks of pure relative dimension  $d$ .
- $\tilde{\mathcal{E}}_{\mathrm{t}}$  is the set of *flat* morphisms *locally of finite presentation* of Artin stacks.
- $\tilde{\mathcal{F}} = F$  is the set of morphisms *locally of finite type* of Artin stacks.
- $\mathcal{L} = \mathrm{N}(\mathcal{R}\mathrm{ind})^{\mathrm{op}}$ , and  $\mathcal{L}' = \mathcal{L}'' = \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square-\mathrm{tor}})^{\mathrm{op}}$ .
- $\dim^+$  is upper relative dimension, which is defined as a special case in Definition 5.4.4 later.
- Input I and II are given by the output of §5.2. In particular,  ${}_{\mathrm{e}}\mathrm{EO}^{\mathrm{I}}$  is (5.5), and  ${}_{\mathrm{e}}\mathrm{EO}^{\mathrm{II}} = \varepsilon_{\mathcal{S}\mathrm{p}_{\square}} \mathrm{EO}^{\mathrm{II}}$  is defined as the restriction of  $\varepsilon_{\mathcal{S}\mathrm{p}} \mathrm{EO}^{\mathrm{I}}$  (5.4) to

$$\delta_{2,\{2\}}^* ((\mathrm{N}(\mathcal{E}\mathrm{sp}_{\square})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square-\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II},\mathrm{op}})^{\mathrm{cart}}_{F,\mathrm{all}}.$$

Then the output consists of the following two maps: a functor

$$(5.7) \quad {}_{\mathrm{chp}}\mathrm{EO}^{\mathrm{I}}: (\mathrm{N}(\mathcal{C}\mathrm{hp})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind})^{\mathrm{op}})^{\mathrm{II}} \rightarrow \mathcal{C}\mathrm{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$${}_{\mathrm{chp}_{\square}}\mathrm{EO}^{\mathrm{II}}: \delta_{2,\{2\}}^* ((\mathrm{N}(\mathcal{C}\mathrm{hp}_{\square})^{\mathrm{op}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square-\mathrm{tor}})^{\mathrm{op}})^{\mathrm{II},\mathrm{op}})^{\mathrm{cart}}_{F,\mathrm{all}} \rightarrow \mathcal{C}\mathrm{at}_{\infty},$$

and Output II.

Now we study the values of objects under the above two maps. Let us recall the lisse-étale site  $\text{Lis-ét}(X)$  of an Artin stack  $X$ . Its underlying category, the full subcategory (which is in fact an ordinary category) of  $\text{Chp}/_X$  spanned by smooth morphisms whose sources are algebraic spaces, is equivalent to a  $\mathcal{U}$ -small category. In particular,  $\text{Lis-ét}(X)$  endowed with the étale topology is a  $\mathcal{U}$ -site. We denote by  $X_{\text{lis-ét}}$  the associated topos. Let  $M \subseteq \text{Ar}(\text{Chp})$  be the set of smooth representable morphisms of Artin stacks. The lisse-étale topos has enough points by [LMB00, 12.2.2], and is functorial with respect to  $M$ , so that we obtain a functor  $\text{Chp}_M \times \mathcal{R}\text{ind} \rightarrow \mathcal{R}\text{ingedPTopos}$ . Composing with **T** (2.1), we obtain a functor

$$(5.8) \quad (\text{N}(\text{Chp})_M^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure.

To simplify the notation, for an algebraic space  $U$ , we will write  $U_{\text{ét}}$  instead of  $U_{\text{esp.ét}}$  in what follows. Let  $\lambda = (\Xi, \Lambda)$  be an object of  $\mathcal{R}\text{ind}$ . We denote by

$$\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda) \subseteq \mathcal{D}(X_{\text{lis-ét}}, \lambda)$$

(Notation 2.2.6) the full subcategory consisting of complexes whose cohomology sheaves are all Cartesian (§0.1), or, equivalently, complexes  $\mathbf{K}$  such that for every morphism  $f: Y' \rightarrow Y$  of  $\text{Lis-ét}(X)$ , the map  $f^*(\mathbf{K}|_{Y_{\text{ét}}}) \rightarrow (\mathbf{K}|_{Y'_{\text{ét}}})$  is an equivalence. This full subcategory is *functorial under T* in the sense that (5.8) restricts to a new functor

$$(5.9) \quad {}^{\text{lis-ét}}_{\text{Chp}}\text{EO}^{\text{I}}: (\text{N}(\text{Chp})_M^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, whose value at  $(X, \lambda)$  is  $\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)$ . It is clear that the restrictions of  ${}^{\text{lis-ét}}_{\text{Chp}}\text{EO}^{\text{I}}$  and  ${}_{\text{Esp}}\text{EO}^{\text{I}}$  (5.4) to  $(\text{N}(\mathcal{E}\text{sp})_{M'}^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$  are equivalent, where  $M' = M \cap \text{Ar}(\mathcal{E}\text{sp})$ . In order to compare  ${}^{\text{lis-ét}}_{\text{Chp}}\text{EO}^{\text{I}}$  and  ${}_{\text{Chp}}\text{EO}^{\text{I}}$  more generally, we start from the following lemma, which is a variant of Proposition 4.1.1. Its proof is similar to Proposition 4.1.1, and we leave the details to the reader.

**Lemma 5.3.1.** *Let  $(\tilde{\mathcal{C}}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}})$  be a 2-marked  $\infty$ -category such that  $\tilde{\mathcal{C}}$  admits pullbacks and  $\tilde{\mathcal{E}} \subseteq \tilde{\mathcal{F}}$  are stable under composition and pullback. Let  $\mathcal{C} \subseteq \tilde{\mathcal{C}}$  be a full subcategory stable under pullback such that every edge in  $\tilde{\mathcal{F}}$  is representable in  $\mathcal{C}$  and for every object  $X$  of  $\tilde{\mathcal{C}}$ , there exists a morphism  $Y \rightarrow X$  in  $\tilde{\mathcal{E}}$  with  $Y$  in  $\mathcal{C}$ . Let  $\mathcal{D}$  be an  $\infty$ -category such that  $\mathcal{D}^{\text{op}}$  admits geometric realizations. Put  $\mathcal{E} = \tilde{\mathcal{E}} \cap \mathcal{C}_1$ ,  $\mathcal{F} = \tilde{\mathcal{F}} \cap \mathcal{C}_1$ . Let  $\text{Fun}^{\tilde{\mathcal{E}}}(\mathcal{C}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{D}) \subseteq \text{Fun}(\mathcal{C}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{D})$  (resp.  $\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{D}) \subseteq \text{Fun}(\tilde{\mathcal{C}}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{D})$ ) be the full subcategory spanned by functors  $F$  such that for every edge  $f: X_0^+ \rightarrow X_{-1}^+$  in  $\mathcal{E}$  (resp. in  $\tilde{\mathcal{E}}$ ),  $F \circ (X_{\bullet}^{s,+})^{\text{op}}: \text{N}(\Delta_{s,+}) \rightarrow \mathcal{D}$  is a limit diagram, where  $X_{\bullet}^{s,+}$  is a semisimplicial Čech nerve of  $f$  in  $\mathcal{C}$  (resp.  $\tilde{\mathcal{C}}$ ) [HTT, 6.5.3.6]. Then the restriction map*

$$\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{D}) \rightarrow \text{Fun}^{\mathcal{E}}(\mathcal{C}_{\mathcal{F}}^{\text{op}}, \mathcal{D})$$

*is a trivial fibration.*

For an object  $V \rightarrow X$  of  $\text{Lis-ét}(X)$ , we denote by  $\tilde{V}$  the sheaf in  $X_{\text{lis-ét}}$  represented by  $V$ . The overcategory  $(X_{\text{lis-ét}})_{/\tilde{V}}$  is equivalent to the topos defined by the site  $\text{Lis-ét}(X)_{/V}$  endowed with the étale topology [SGA4, III 5.4]. A morphism  $f: U \rightarrow U'$  of  $\text{Lis-ét}(X)_{/V}$  induces a 2-commutative diagram

$$\begin{array}{ccccc} & & (X_{\text{lis-ét}})_{/\tilde{U}} & \xrightarrow{\epsilon_{U*}} & U_{\text{ét}} \\ & \swarrow u_* & \downarrow f_* & & \downarrow f_{\text{ét}*} \\ (X_{\text{lis-ét}})_{/\tilde{V}} & \xleftarrow{u'^*} & (X_{\text{lis-ét}})_{/\tilde{U}'} & \xrightarrow{\epsilon_{U'*}} & U'_{\text{ét}} \end{array}$$

of topoi [SGA4, IV 5.5].

For an object  $\lambda = (\Xi, \Lambda)$  of  $\mathcal{R}\text{ind}$ , let  $\mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\tilde{V}}, \lambda)^{\otimes} \subseteq \mathcal{D}((X_{\text{lis-ét}})_{/\tilde{V}}, \lambda)^{\otimes}$  be the full (monoidal) subcategory [HTT, 2.2.1] spanned by complexes on which the natural transformation  $f^* \circ \epsilon_{U'*} \circ u'^* \rightarrow \epsilon_{U*} \circ u^*$  is an isomorphism for all  $f$ . We have a functor

$$[1] \times \text{Lis-ét}(X) \times \mathcal{R}\text{ind} \rightarrow \text{RingedPTopos}$$

sending  $[1] \times \{f: U \rightarrow V\} \times \{\lambda\}$  to the square

$$\begin{array}{ccc} ((X_{\text{lis-ét}})_{/\tilde{U}}^{\Xi}, \Lambda) & \xrightarrow{\epsilon_{U*}} & (U_{\text{ét}}^{\Xi}, \Lambda) \\ f_* \downarrow & & \downarrow f_{\text{ét}*} \\ ((X_{\text{lis-ét}})_{/\tilde{V}}^{\Xi}, \Lambda) & \xrightarrow{\epsilon_{V*}} & (V_{\text{ét}}^{\Xi}, \Lambda). \end{array}$$

Composing with the functor  $\mathbf{T}^{\otimes}$  (2.2), we obtain a functor

$$F: (\Delta^1)^{\text{op}} \times \text{N}(\text{Lis-ét}(X))^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_{\infty})_{\text{pr, st, cl}}^{\text{L}}.$$

By construction,  $F(0, V, \lambda) = \mathcal{D}((X_{\text{lis-ét}})_{/\tilde{V}}, \lambda)^{\otimes}$ . Replacing  $F(0, V, \lambda)$  by the full subcategory  $\mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\tilde{V}}, \lambda)^{\otimes}$ , we obtain a new functor

$$F': (\Delta^1)^{\text{op}} \times \text{N}(\text{Lis-ét}(X))^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}} \rightarrow \text{CAlg}(\text{Cat}_{\infty})$$

sending  $(\Delta^1)^{\text{op}} \times \{f: U \rightarrow V\} \times \{\lambda\}$  to the square

$$\begin{array}{ccc} \mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\tilde{U}}, \lambda)^{\otimes} & \xleftarrow{\epsilon_U^*} & \mathcal{D}(U_{\text{ét}}, \lambda)^{\otimes} \\ f^* \uparrow & & \uparrow f_{\text{ét}}^* \\ \mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\tilde{V}}, \lambda)^{\otimes} & \xleftarrow{\epsilon_V^*} & \mathcal{D}(V_{\text{ét}}, \lambda)^{\otimes}. \end{array}$$

We have the following two lemmas.

**Lemma 5.3.2.** *The functor  $F'$ , viewed as an edge of*

$$\text{Fun}(\text{N}(\text{Lis-ét}(X))^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}}, \text{CAlg}(\text{Cat}_{\infty})),$$

*is an equivalence. In particular, the functor  $F'$  factorizes through  $\text{CAlg}(\text{Cat}_{\infty})_{\text{pr, st, cl}}^{\text{L}}$ .*

*Proof.* We only need to prove that for every object  $V$  of  $\text{Lis-ét}(X)$ , the functor

$$\epsilon_V^*: \mathcal{D}(V_{\text{ét}}, \lambda) \rightarrow \mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}}, \lambda)$$

is an equivalence. This follows from the fact that

$$\epsilon_V^*: \text{Mod}(V_{\text{ét}}, \lambda) \rightarrow \text{Mod}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}}, \lambda)$$

is an equivalence of categories and that the functor

$$\epsilon_{V*}: \text{Mod}((X_{\text{lis-ét}})_{/\widetilde{V}}, \lambda) \rightarrow (V_{\text{ét}}, \lambda)$$

is exact, by the following lemma.  $\square$

**Lemma 5.3.3.** *Let  $F: \mathcal{A} \rightarrow \mathcal{B}$  be an exact fully faithful functor between Grothendieck Abelian categories that admit an exact right adjoint  $G$ . Then  $F$  induces an equivalence of  $\infty$ -categories  $\mathcal{D}(\mathcal{A}) \rightarrow \mathcal{D}_{\mathcal{A}}(\mathcal{B})$ , where  $\mathcal{D}_{\mathcal{A}}(\mathcal{B})$  denotes the full subcategory of  $\mathcal{D}(\mathcal{B})$  spanned by complexes with cohomology in the essential image of  $\epsilon$ .*

*Proof.* This is standard. The pair  $(F, G)$  induce a pair of t-exact adjoint between  $\mathcal{D}(\mathcal{A})$  and  $\mathcal{D}_{\mathcal{A}}(\mathcal{B})$ . To check that the unit and counit are natural equivalences, we may reduce to objects in the Abelian categories, for which the assertion follows from the assumptions.  $\square$

**Lemma 5.3.4.** *Let  $v: V \rightarrow X$  be an object of  $\text{Lis-ét}(X)$ , viewed as a morphism of  $\text{Chp}$ . Assume that  $v$  is surjective. Then a complex  $\mathbf{K} \in \mathcal{D}(X_{\text{lis-ét}}, \lambda)$  belongs to  $\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)$  if and only if  $v^*\mathbf{K}$  belongs to  $\mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}}, \lambda)$ .*

*Proof.* The necessity is trivial. Assume that  $v^*\mathbf{K}$  belongs to  $\mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}}, \lambda)$ . We need to show that for every morphism  $f: Y' \rightarrow Y$  of  $\text{Lis-ét}(X)$ , the map  $f^*(\mathbf{K}|_{Y_{\text{ét}}}) \rightarrow (\mathbf{K}|_{Y'_{\text{ét}}})$  is an equivalence. The problem is local for the étale topology on  $Y$ . However, locally for the étale topology on  $Y$ , the morphism  $Y \rightarrow X$  factorizes through  $v$  [EGAIV, 17.16.3 (ii)]. The assertions thus follows from the assumption.  $\square$

Now let  $V_{\bullet}: \mathbf{N}(\Delta_+)^{\text{op}} \rightarrow \mathbf{N}(\text{Chp})$  be a Čech nerve of  $v$  where  $v: V \rightarrow X$  be an object of  $\text{Lis-ét}(X)$ , which can be viewed as a simplicial object of  $\text{Lis-ét}(X)$ . By Lemma 5.3.4, we can apply Lemma 3.1.3 to  $U_{\bullet} = \widetilde{V}_{\bullet}^{\Xi}$  and  $\mathcal{C}_{\bullet} = \text{Mod}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}_{\bullet}}^{\Xi}, \Lambda)$ . We obtain a natural equivalence of symmetric monoidal  $\infty$ -categories

$$\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)^{\otimes} \xrightarrow{\sim} \varprojlim_{n \in \Delta} \mathcal{D}_{\text{cart}}((X_{\text{lis-ét}})_{/\widetilde{V}_n}, \lambda)^{\otimes}.$$

Combining this with a quasi-inverse of the equivalence in Lemma 5.3.2, we obtain the following result.

**Proposition 5.3.5** (Cohomological descent for lisse-étale topoi). *Let  $X$  be an Artin stack,  $V$  an algebraic space, and  $v: V \rightarrow X$  a surjective smooth morphism. Then there is an equivalence in  $\text{Fun}(\mathbf{N}(\mathbf{Rind})^{\text{op}}, \text{CAlg}(\text{Cat}_{\infty})_{\text{pr, st, cl}}^{\text{L}})$  sending  $\lambda$  to the equivalence*

$$\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)^{\otimes} \xrightarrow{\sim} \varprojlim_{n \in \Delta} \mathcal{D}(V_n, \text{ét}, \lambda)^{\otimes},$$

where  $V_{\bullet}$  is a Čech nerve of  $v$ .

The proposition has the following corollaries.

**Corollary 5.3.6.** *Let  $f: Y \rightarrow X$  be a smooth surjective representable morphism of Artin stacks,  $\lambda$  an object of  $\mathcal{R}\text{ind}$ , and  $Y_\bullet$  a Čech nerve of  $f$ . Then the functor*

$$\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)^\otimes \xrightarrow{\sim} \varprojlim_{n \in \Delta_s} \mathcal{D}_{\text{cart}}(Y_n, \text{lis-ét}, \lambda)^\otimes$$

*is an equivalence.*

**Corollary 5.3.7.** *The functor  ${}^{\text{lis-ét}}_{\text{Chp}}\text{EO}^{\text{I}}$  (5.9) belongs to  $\text{Fun}^{\tilde{\mathcal{E}}}(\tilde{\mathcal{C}}_{\tilde{\mathcal{F}}}^{\text{op}}, \mathcal{C}\text{at}_\infty)$  with the notation in Lemma 5.3.1, where*

- $\tilde{\mathcal{C}} = (\text{N}(\text{Chp})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}, \text{op}}$ ;
- $\tilde{\mathcal{F}}$  consists of edges of that statically belong to  $M$ ; and
- $\tilde{\mathcal{E}} \subseteq \tilde{\mathcal{F}}$  consists of edges that are also statically surjective.

**Corollary 5.3.8.** *The functor  ${}^{\text{lis-ét}}_{\text{Chp}}\text{EO}^{\text{I}}$  (5.9) is equivalent to the restriction of the functor  ${}_{\text{Chp}}\text{EO}^{\text{I}}$  (5.7) to  $(\text{N}(\text{Chp})_M^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$ . In particular, for every Artin stack  $X$  and every object  $\lambda$  of  $\mathcal{R}\text{ind}$ , we have an equivalence*

$$\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)^\otimes \simeq \mathcal{D}(X, \lambda)^\otimes$$

*of symmetric monoidal  $\infty$ -categories. Consequently,  $\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)^\otimes$  is a closed presentable stable symmetric monoidal  $\infty$ -category. Here we recall that  $\mathcal{D}(X, \lambda)^\otimes$  is the value of  $(X, \lambda, \langle 1 \rangle, \{1\})$  under the functor  ${}_{\text{Chp}}\text{EO}^{\text{I}}$ .*

**Corollary 5.3.9.** *Let  $X$  be an Artin stack, and  $\lambda$  an object of  $\mathcal{R}\text{ind}$ . Under the equivalence in Corollary 5.3.8, the usual  $t$ -structure on  $\mathcal{D}_{\text{cart}}(X_{\text{lis-ét}}, \lambda)$  coincides with the  $t$ -structure on  $\mathcal{D}(X, \lambda)$  obtained in Output II. In particular, the heart of  $\mathcal{D}(X, \lambda)$  is equivalent to (the nerve of)  $\text{Mod}_{\text{cart}}(X_{\text{lis-ét}}^\Xi, \Lambda)$ , the Abelian category of Cartesian  $(X_{\text{lis-ét}}^\Xi, \Lambda)$ -modules.*

*Remark 5.3.10* (de Jong). The  $*$ -pullback encoded by  ${}_{\text{Chp}}\text{EO}^{\text{I}}$  can be described more directly using big étale topoi of Artin stacks. For any Artin stack  $X$ , we consider the full subcategories  $\mathcal{E}\text{sp}_{\text{lfp}/X} \subseteq \text{Chp}_{\text{rep.lfp}/X}$  of  $\text{Chp}_{/X}$  spanned by morphisms locally of finite presentation whose sources are algebraic spaces and by representable morphisms locally of finite presentation<sup>8</sup>, respectively. They are ordinary categories and we endow them with the étale topology. The corresponding topoi are equivalent, and we denote them by  $X_{\text{big.ét}}$ . The construction of  $X_{\text{big.ét}}$  is functorial in  $X$ , so that we obtain a functor  $\text{Chp} \times \mathcal{R}\text{ind} \rightarrow \text{RingedPTopos}$ . Composing with  $\mathbf{T}$ , we obtain a functor

$$(\text{N}(\text{Chp})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_\infty$$

that is a weak Cartesian structure, sending  $(X, \lambda)$  to  $\mathcal{D}(X_{\text{big.ét}}, \lambda)$ . Replacing the latter by the full subcategory  $\mathcal{D}_{\text{cart}}(X_{\text{big.ét}}, \lambda)$  consisting of complexes  $\mathbf{K}$  such that  $f^*(\mathbf{K} | Y'_{\text{ét}}) \rightarrow (\mathbf{K} | Y_{\text{ét}})$  is an equivalence for every morphism  $f: Y \rightarrow Y'$  of  $\mathcal{E}\text{sp}_{/X}$ , we obtain a new functor

$${}^{\text{big}}_{\text{Chp}}\text{EO}^{\text{I}}: (\text{N}(\text{Chp})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_\infty$$

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<sup>8</sup>We impose the “locally of finite presentation” condition here to avoid set-theoretic issues.

that is a weak Cartesian structure. Using similar arguments as in this section, with Lemma 5.3.1 replaced by Proposition 4.1.1, one shows that  ${}^{\text{big}}_{\text{chp}}\text{EO}^I$  and  ${}_{\text{chp}}\text{EO}^I$  are equivalent.

**5.4. Higher Artin stacks.** We begin by recalling the definition of higher Artin stacks. We will use the fppf topology instead of the étale topology adopted in [Toë10]. The two definitions are equivalent [Toë11]. Let  $\text{Sch}^{\text{aff}} \subseteq \text{Sch}$  be the full subcategory spanned by affine schemes. Recall that  $\mathcal{S}_{\mathcal{W}}$  is the  $\infty$ -category of spaces in  $\mathcal{W} \in \{\mathcal{U}, \mathcal{V}\}$ <sup>9</sup>.

**Definition 5.4.1** (Prestack and stack). We defined the  $\infty$ -category of  $(\mathcal{V}\text{-})\text{prestacks}$  to be  $\text{Chp}^{\text{pre}} := \text{Fun}(\text{N}(\text{Sch}^{\text{aff}})^{\text{op}}, \mathcal{S}_{\mathcal{V}})$ . We endow  $\text{N}(\text{Sch}^{\text{aff}})$  with the fppf topology. We define the  $\infty$ -category of (small) stacks  $\text{Chp}^{\text{fppf}}$  to be the essential image of the following inclusion

$$\text{Shv}(\text{N}(\text{Sch}^{\text{aff}})_{\text{fppf}}) \cap \text{Fun}(\text{N}(\text{Sch}^{\text{aff}})^{\text{op}}, \mathcal{S}_{\mathcal{U}}) \subseteq \text{Chp}^{\text{pre}},$$

where  $\text{Shv}(\text{N}(\text{Sch}^{\text{aff}})_{\text{fppf}}) \subseteq \text{Fun}(\text{N}(\text{Sch}^{\text{aff}})^{\text{op}}, \mathcal{S}_{\mathcal{V}})$  is the full subcategory spanned by fppf sheaves [HTT, 6.2.2.6]. A prestack  $F$  is *k-truncated* [HTT, 5.5.6.1] for an integer  $k \geq -1$ , if  $\pi_i(F(A)) = 0$  for every object  $A$  of  $\text{Sch}^{\text{aff}}$  and every integer  $i > k$ .

The Yoneda embedding  $\text{N}(\text{Sch}^{\text{aff}}) \rightarrow \text{Chp}^{\text{pre}}$  extends to a fully faithful functor  $\text{N}(\mathcal{E}\text{sp}) \rightarrow \text{Chp}^{\text{pre}}$  sending  $X$  to the discrete Kan complex  $\text{Hom}_{\mathcal{E}\text{sp}}(\text{Spec } A, X)$ . The image of this functor is contained in  $\text{Chp}^{\text{fppf}}$ . We will generally not distinguish between  $\text{N}(\mathcal{E}\text{sp})$  and its essential image in  $\text{Chp}^{\text{fppf}}$ . A stack  $X$  belongs to (the essential image of)  $\text{N}(\mathcal{E}\text{sp})$  if and only if it satisfies the following conditions.

- It is 0-truncated.
- The diagonal morphism  $X \rightarrow X \times X$  is schematic, that is, for every morphism  $Z \rightarrow X \times X$  with  $Z$  a scheme, the fiber product  $X \times_{X \times X} Z$  is a scheme.
- There exists a scheme  $Y$  and an (automatically schematic) morphism  $f: Y \rightarrow X$  that is smooth (resp. étale) and surjective. In other words, for every morphism  $Z \rightarrow X$  with  $Z$  a scheme, the induced morphism  $Y \times_X Z \rightarrow Z$  is smooth (resp. étale) and surjective. The morphism  $f$  is called an *atlas* (resp. *étale atlas*) for  $X$ .

**Definition 5.4.2** (Higher Artin stack; see [Toë10, Gaib]). We define *k*-Artin stacks inductively for  $k \geq 0$ .

- A stack  $X$  is a *0-Artin stack* if it belongs to (the essential image of)  $\text{N}(\mathcal{E}\text{sp})$ .

For  $k \geq 0$ , assume that we have defined *k*-Artin stacks. We define:

- A morphism  $F' \rightarrow F$  of prestacks is *k-Artin* if for every morphism  $Z \rightarrow F$  where  $Z$  is a *k*-Artin stack, the fiber product  $F' \times_F Z$  is a *k*-Artin stack.
- A *k*-Artin morphism  $F' \rightarrow F$  is flat (resp. locally of finite type, resp. locally of finite presentation, resp. smooth, resp. surjective) if for every morphism  $Z \rightarrow F$  and every atlas  $f: Y \rightarrow F' \times_F Z$  where  $Y$  and  $Z$  are schemes, the composite morphism  $Y \rightarrow F' \times_F Z \rightarrow Z$  is a flat (resp. locally of finite type, resp. locally of finite presentation, resp. smooth, resp. surjective) morphism of schemes.

<sup>9</sup>We refer to §0.5 for conventions on set-theoretical issues.



- A stack  $X$  is a  $(k+1)$ -Artin stack if the diagonal morphism  $X \rightarrow X \times X$  is  $k$ -Artin, and there exists a scheme  $Y$  together with an (automatically  $k$ -Artin) morphism  $f: Y \rightarrow X$  that is smooth and surjective. The morphism  $f$  is called an *atlas* for  $X$ .

We denote by  $\mathcal{C}hp^{k\text{-Ar}} \subseteq \mathcal{C}hp^{\text{fppf}}$  the full subcategory spanned by  $k$ -Artin stacks. We define *higher Artin stacks* to be objects of  $\mathcal{C}hp^{\text{Ar}} := \bigcup_{k \geq 0} \mathcal{C}hp^{k\text{-Ar}}$ . A morphism  $F' \rightarrow F$  of prestacks is *higher Artin* if for every morphism  $Z \rightarrow F$  where  $Z$  is a higher Artin stack, the fiber product  $F' \times_F Z$  is a higher Artin stack.

To simplify the notation, we put  $\mathcal{C}hp^{(-1)\text{-Ar}} = \mathcal{N}(\text{Sch}^{\text{qs}})$  and  $\mathcal{C}hp^{(-2)\text{-Ar}} = \mathcal{N}(\text{Sch}^{\text{qc.sep}})$ , and we call their objects  $(-1)$ -Artin stacks and  $(-2)$ -Artin stacks, respectively.

By definition,  $\mathcal{C}hp^{0\text{-Ar}}$  and  $\mathcal{C}hp^{1\text{-Ar}}$  are equivalent to  $\mathcal{N}(\mathcal{E}sp)$  and  $\mathcal{N}(\mathcal{C}hp)$ , respectively. For  $k \geq 0$ ,  $k$ -Artin stacks are  $k$ -truncated prestacks. Higher Artin stacks are *hypercomplete* sheaves [HTT, 6.5.2.9]. Every flat surjective morphism locally of finite presentation of higher Artin stacks is an *effective epimorphism* in the  $\infty$ -topos  $\text{Shv}(\mathcal{N}(\text{Sch}^{\text{aff}})_{\text{fppf}})$  in the sense after [HTT, 6.2.3.5]. A higher Artin morphism of prestacks is  $k$ -Artin for some  $k \geq 0$ .

**Definition 5.4.3.** We have the following notion of quasi-compactness.

- A higher Artin stack  $X$  is *quasi-compact* if there exists an atlas  $f: Y \rightarrow X$  such that  $Y$  is a quasi-compact scheme.
- A higher Artin morphism  $F' \rightarrow F$  of prestacks is *quasi-compact* if for every morphism  $Z \rightarrow F$  where  $Z$  is a quasi-compact scheme, the fiber product  $F' \times_F Z$  is a quasi-compact higher Artin stack.

We define quasi-separated higher Artin morphisms of prestacks by induction as follows.

- A 0-Artin morphism of prestacks  $F' \rightarrow F$  is *quasi-separated* if the diagonal morphism  $F' \rightarrow F' \times_F F'$ , which is automatically schematic, is quasi-compact.
- For  $k \geq 0$ , a  $(k+1)$ -Artin morphism of prestacks  $F' \rightarrow F$  is *quasi-separated* if the diagonal morphism  $F' \rightarrow F' \times_F F'$ , which is automatically  $k$ -Artin, is quasi-separated and quasi-compact.

We say that a morphism of higher Artin stacks is *of finite presentation* if it is quasi-compact, quasi-separated, and locally of finite presentation.

We say that a higher Artin stack  $X$  is  $\square$ -coprime if there exists a morphism  $X \rightarrow \text{Spec } \mathbb{Z}[\square^{-1}]$ . This is equivalent to the existence of a  $\square$ -coprime atlas. We denote by  $\mathcal{C}hp_{\square}^{\text{Ar}} \subseteq \mathcal{C}hp^{\text{Ar}}$  the full subcategory spanned by  $\square$ -coprime higher Artin stacks. We put  $\mathcal{C}hp_{\square}^{k\text{-Ar}} = \mathcal{C}hp^{k\text{-Ar}} \cap \mathcal{C}hp_{\square}^{\text{Ar}}$ .

**Definition 5.4.4** (Relative dimension). We define by induction the class of smooth morphisms *of pure relative dimension*  $d$  of  $k$ -Artin stacks for  $d \in \mathbb{Z} \cup \{-\infty\}$  and the *upper relative dimension*  $\dim^+(f)$  for every morphism  $f$  locally of finite type of  $k$ -Artin stacks. If in Input 0 of §4.1, we let  $\tilde{\mathcal{F}}$  (resp.  $\tilde{\mathcal{E}}''$ ,  $\tilde{\mathcal{E}}''_d$ ) be the set of morphisms locally of finite type (resp. smooth morphisms, smooth morphisms of pure relative dimension  $d$ ) of  $k$ -Artin stacks, then such definitions should satisfy conditions (6) through (9) of Input 0.



When  $k = 0$ , we use the usual definitions for algebraic spaces, with the upper relative dimension given in Definition 4.1.11. For  $k \geq 0$ , assuming that these notions are defined for  $k$ -Artin stacks. We first extend these definitions to  $k$ -representable morphisms locally of finite type of  $(k+1)$ -Artin stacks. Let  $f: Y \rightarrow X$  be such a morphism, and  $X_0 \xrightarrow{u} X$  an atlas of  $X$ . Let  $f_0: Y_0 \rightarrow X_0$  be the base change of  $f$  by  $u$ . Then  $f_0$  is a morphism locally of finite type of  $k$ -Artin stacks. We define  $\dim^+(f) = \dim^+(f_0)$ . It is easy to see that this is independent of the atlas we choose, by assumption (9d) of Input 0. We say that  $f$  is smooth of pure relative dimension  $d$  if  $f_0$  is. This is independent of the atlas we choose by assumption (7) of Input 0. We need to check (6) through (9) of Input 0. (7) through (9) are easy and (6) can be argued as follows. Since  $f_0$  is a smooth morphism of  $k$ -Artin stacks, there is a decomposition  $f_0: Y_0 \simeq \coprod_{d \in \mathbb{Z}} Y_{0,d} \xrightarrow{(f_0,d)} X_0$ . Let  $X_\bullet \rightarrow X$  be a Čech nerve of  $u$ , and put  $Y_{\bullet,d} := Y_{0,d} \times_{X_0} X_\bullet$ . Then  $\coprod_{d \in \mathbb{Z}} Y_{\bullet,d} \rightarrow Y$  is a Čech nerve of  $v: Y_0 \rightarrow Y$ . Put  $Y_d = \varinjlim_{n \in \Delta^{op}} Y_{n,d}$ . Then  $Y \simeq \coprod_{d \in \mathbb{Z}} Y_d$  is the desired decomposition.

Next we extend these definitions to all morphisms locally of finite type of  $(k+1)$ -Artin stacks. Let  $f: Y \rightarrow X$  be such a morphism, and  $v_0: Y_0 = \coprod_{d \in \mathbb{Z}} Y_{0,d} \xrightarrow{(v_0,d)} Y$  an atlas of  $Y$  such that  $v_{0,d}$  is smooth of pure relative dimension  $d$ . We define

$$\dim^+(f) = \sup_{d \in \mathbb{Z}} \{\dim^+(f \circ v_{0,d}) - d\}.$$

We say that  $f$  is smooth of pure relative dimension  $d$  if for every  $e \in \mathbb{Z}$ , the morphism  $f \circ v_{0,e}$  is smooth of pure relative dimension  $d + e$ . We leave it to the reader to check that these definitions are independent of the atlas we choose, and satisfy (7) through (9) of Input 0. We sketch the proof for (6). Since  $f \circ v_{0,e}$  is smooth and  $k$ -representable, it can be decomposed as  $Y_{0,e} \simeq \coprod_{e' \in \mathbb{Z}} Y_{0,e,e'} \xrightarrow{(f_{e,e'},e')} X$  such that  $f_{e,e'}$  is of pure relative dimension  $e'$ . We let  $Y_d$  be the colimit of the underlying groupoid object of the Čech nerve of  $\coprod_{e' - e = d} Y_{0,e,e'} \rightarrow X$ . Then  $Y \simeq \coprod_{d \in \mathbb{Z}} Y_d \rightarrow X$  is the desired decomposition.

Let  $F$  be the set of morphisms locally of finite type of higher Artin stacks. For every  $k$ , we are going to construct a functor

$$\mathcal{C}_{\text{hp}}^{k-\text{Ar}} \text{EO}^I: ((\mathcal{C}_{\text{hp}}^{k-\text{Ar}})^{op} \times \mathcal{N}(\mathcal{R}\text{ind})^{op})^{\text{II}} \rightarrow \mathcal{C}\text{at}_\infty$$

that is a weak Cartesian structure, and a map

$$\mathcal{C}_{\text{hp}}^{k-\text{Ar}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^* (((\mathcal{C}_{\text{hp}}^{k-\text{Ar}})^{op} \times \mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op})^{\text{II},op})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_\infty,$$

such that their restrictions to  $(k-1)$ -Artin stacks coincide with those for the latter.

We construct by induction. When  $k = -2, -1, 0, 1$ , they have been constructed in §3.2, §5.1, §5.2, and §5.3, respectively. Assume that they have been extended to  $k$ -Artin stacks. We run the version of DESCENT in Variant 4.1.9 with the following input:

- $\tilde{\mathcal{C}} = \mathcal{C}_{\text{hp}}^{(k+1)-\text{Ar}}$ . It is geometric.
- $\mathcal{C} = \mathcal{C}_{\text{hp}}^{k-\text{Ar}}$ ,  $s'' \rightarrow s'$  is the identity morphism of  $\text{Spec } \mathbb{Z}[\square^{-1}]$ . In particular,  $\mathcal{C}' = \mathcal{C}'' = \mathcal{C}_{\text{hp}}^{k-\text{Ar}}$ , and  $\tilde{\mathcal{C}}' = \tilde{\mathcal{C}}'' = \mathcal{C}_{\text{hp}}^{(k+1)-\text{Ar}}$ .

- $\tilde{\mathcal{E}}_s$  is the set of *surjective* morphisms of  $(k+1)$ -Artin stacks.
- $\tilde{\mathcal{E}}' = \tilde{\mathcal{E}}''$  is the set of *smooth* morphisms of  $(k+1)$ -Artin stacks.
- $\tilde{\mathcal{E}}_d''$  is the set of *smooth* morphisms of  $(k+1)$ -Artin stacks of pure relative dimension  $d$ .
- $\tilde{\mathcal{E}}_t$  is the set of *flat* morphisms *locally of finite presentation* of  $(k+1)$ -Artin stacks.
- $\tilde{\mathcal{F}} = F$  is the set of morphisms *locally of finite type* of  $(k+1)$ -Artin stacks.
- $\mathcal{L} = N(\mathcal{R}\text{ind})^{op}$ , and  $\mathcal{L}' = \mathcal{L}'' = N(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op}$ .
- $\dim^+$  is the upper relative dimension in Definition 5.4.4.
- Input I and II is given by induction hypothesis. In particular, we take

$${}_e\text{EO}^I = {}_{\text{Chp}^{k\text{-Ar}}}\text{EO}^I, \quad {}_{e'}\text{EO}^{II} = {}_{\text{Chp}_{\square}^{k\text{-Ar}}}\text{EO}^{II}.$$

Then the output consists of desired two maps  ${}_{\text{Chp}^{k+1\text{-Ar}}}\text{EO}^I$ ,  ${}_{\text{Chp}_{\square}^{k+1\text{-Ar}}}\text{EO}^{II}$  and Output II, satisfying (P0) – (P7<sup>bis</sup>). Taking union of all  $k \geq 0$ , we obtain the following two maps: a functor

$$(5.10) \quad {}_{\text{Chp}^{\text{Ar}}}\text{EO}^I: ((\text{Chp}^{\text{Ar}})^{op} \times N(\mathcal{R}\text{ind})^{op})^{II} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$$(5.11) \quad {}_{\text{Chp}_{\square}}\text{EO}^{II}: \delta_{2,\{2\}}^*(((\text{Chp}_{\square}^{\text{Ar}})^{op} \times N(\mathcal{R}\text{ind}_{\square\text{-tor}})^{op})^{II,op})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty}.$$

**5.5. Higher Deligne–Mumford stacks.** The definition of higher Deligne–Mumford (DM) stacks is similar to that of higher Artin stacks (Definition 5.4.2).

**Definition 5.5.1** (Higher DM stack).

- A stack  $X$  is a *0-DM stack* if it belongs to (the essential image of)  $N(\mathcal{E}\text{sp})$ .

For  $k \geq 0$ , assume that we have defined  $k$ -DM stacks. We define:

- A morphism  $F' \rightarrow F$  of prestacks is  *$k$ -DM* if for every morphism  $Z \rightarrow F$  where  $Z$  is a  $k$ -DM stack, the fiber product  $F' \times_F Z$  is a  $k$ -DM stack.
- A  $k$ -DM morphism  $F' \rightarrow F$  of prestacks is *étale* (resp. *locally quasi-finite*) if for every morphism  $Z \rightarrow F$  and every étale atlas  $f: Y \rightarrow F' \times_F Z$  where  $Y$  and  $Z$  are schemes, the composite morphism  $Y \rightarrow F' \times_F Z \rightarrow Z$  is an étale (resp. locally quasi-finite) morphism of schemes.
- A stack  $X$  is a  $(k+1)$ -DM stack if the diagonal morphism  $X \rightarrow X \times X$  is  $k$ -DM, and there exists a scheme  $Y$  together with an (automatically  $k$ -DM) morphism  $f: Y \rightarrow X$  that is étale and surjective. The morphism  $f$  is called an *étale atlas* for  $X$ .

We denote by  $\text{Chp}^{k\text{-DM}} \subseteq \text{Chp}^{\text{fppf}}$  the full subcategory spanned by  $k$ -DM stacks. We define *higher DM stacks* to be objects of  $\text{Chp}^{\text{DM}} := \bigcup_{k \geq 0} \text{Chp}^{k\text{-DM}}$ . We put  $\text{Chp}_{\square}^{\text{DM}} = \text{Chp}^{\text{DM}} \cap \text{Chp}_{\square}^{\text{Ar}}$ , and  $\text{Chp}_{\square}^{k\text{-DM}} = \text{Chp}^{k\text{-DM}} \cap \text{Chp}_{\square}^{\text{DM}}$ .

A morphism of higher DM stacks is étale if and only if it is smooth of pure relative dimension 0.

Let  $F$  be the set of morphisms locally of finite type of higher DM stacks. For every  $k$ , we are going to construct a functor

$${}_{\text{Chp}^{k\text{-DM}}} \text{EO}^{\text{I}}: ((\text{Chp}^{k\text{-DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$${}_{\text{Chp}^{k\text{-DM}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^*(((\text{Chp}^{k\text{-DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}})^{\text{II},\text{op}})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty},$$

such that their restrictions to  $(k-1)$ -DM stacks coincide with those for the latter. Note that the first functor has already been constructed in §5.4, after restriction. However for induction, we construct it again, which in fact coincides with the previous one.

We construct by induction. When  $k = 0$ , they have been constructed in §5.2. Assuming that they have been extended to  $k$ -DM stacks. We run the program DESCENT with the following input:

- $\tilde{\mathcal{C}} = \text{Chp}^{(k+1)\text{-DM}}$ . It is geometric.
- $\mathcal{C} = \text{Chp}^{k\text{-DM}}$ ,  $\mathbf{s}'' \rightarrow \mathbf{s}'$  is the morphism  $\text{Spec } \mathbb{Z}[\square^{-1}] \rightarrow \text{Spec } \mathbb{Z}$ .
- $\tilde{\mathcal{E}}_{\text{s}}$  is the set of *surjective* morphisms of  $(k+1)$ -DM stacks.
- $\tilde{\mathcal{E}}'$  is the set of *étale* morphisms of  $(k+1)$ -DM stacks.
- $\tilde{\mathcal{E}}''$  is the set of *smooth* morphisms of  $(k+1)$ -DM stacks.
- $\tilde{\mathcal{E}}_d''$  is the set of *smooth* morphisms of  $(k+1)$ -DM stacks of pure relative dimension  $d$ .
- $\tilde{\mathcal{E}}_{\text{t}}$  is the set of *flat* morphisms *locally of finite presentation* of  $(k+1)$ -DM stacks.
- $\tilde{\mathcal{F}} = F$  is the set of morphisms *locally of finite type* of  $(k+1)$ -DM stacks.
- $\mathcal{L} = \text{N}(\mathcal{R}\text{ind})^{\text{op}}$ ,  $\mathcal{L}' = \text{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}}$ , and  $\mathcal{L}'' = \text{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{\text{op}}$ .
- $\dim^+$  is the upper relative dimension.
- Input I and II is given by induction hypothesis. In particular, we take

$${}_{\mathcal{C}} \text{EO}^{\text{I}} = {}_{\text{Chp}^{k\text{-DM}}} \text{EO}^{\text{I}}, \quad {}_{\mathcal{C}'} \text{EO}^{\text{II}} = {}_{\text{Chp}^{k\text{-DM}}} \text{EO}^{\text{II}}.$$

Then the output consists of desired two maps  ${}_{\text{Chp}^{k+1\text{-DM}}} \text{EO}^{\text{I}}$ ,  ${}_{\text{Chp}^{k+1\text{-DM}}} \text{EO}^{\text{II}}$  and Output II, satisfying (P0) – (P7<sup>bis</sup>). Taking union of all  $k \geq 0$ , we obtain a functor

$$(5.12) \quad {}_{\text{Chp}^{\text{DM}}} \text{EO}^{\text{I}}: ((\text{Chp}^{\text{DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}} \rightarrow \mathcal{C}\text{at}_{\infty}$$

that is a weak Cartesian structure, and a map

$$(5.13) \quad {}_{\text{Chp}^{\text{DM}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^*(((\text{Chp}^{\text{DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind}_{\text{tor}})^{\text{op}})^{\text{II},\text{op}})_{F,\text{all}}^{\text{cart}} \rightarrow \mathcal{C}\text{at}_{\infty}.$$

*Remark 5.5.2.* We have the following compatibility properties:

- The restriction of  ${}_{\text{Chp}^{\text{Ar}}} \text{EO}^{\text{I}}$  to  $((\text{Chp}^{\text{DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II}}$  is equivalent to  ${}_{\text{Chp}^{\text{DM}}} \text{EO}^{\otimes}$ .
- The restrictions of  ${}_{\text{Chp}^{\text{DM}}} \text{EO}^{\text{II}}$  and  ${}_{\text{Chp}^{\square\text{-tor}}} \text{EO}^{\text{II}}$  to the common domain

$$\delta_{2,\{2\}}^*(((\text{Chp}_{\square}^{\text{DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{\text{op}})^{\text{II},\text{op}})_{F,\text{all}}^{\text{cart}}$$

are equivalent.

**Variante 5.5.3.** We denote by  $Q \subseteq F$  the set of locally quasi-finite morphisms. Applying DESCENT to the map  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{II}}$  constructed in Variant 3.2.6 (and  ${}_{\text{Sch}^{\text{qc.sep}}} \text{EO}^{\text{I}}$ ), we obtain a map

$$(5.14) \quad {}_{\text{Chp}^{\text{DM}}} \text{EO}^{\text{II}}: \delta_{2,\{2\}}^* (((\text{Chp}^{\text{DM}})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind})^{\text{op}})^{\text{II},\text{op}})^{\text{cart}}_{Q,\text{all}} \rightarrow \text{Cat}_{\infty}.$$

This map and  ${}_{\text{Chp}^{\text{DM}}} \text{EO}^{\text{II}}$  are equivalent when restricted to their common domain.

*Remark 5.5.4.* The  $\infty$ -category  $\text{Chp}^{\text{DM}}$  can be identified with a full subcategory of the  $\infty$ -category  $\text{Sch}(\mathcal{G}_{\text{ét}}(\mathbb{Z}))$  of  $\mathcal{G}_{\text{ét}}(\mathbb{Z})$ -schemes in the sense of [Lur, 2.3.9, 2.6.11]. The constructions of this section can be extended to  $\text{Sch}(\mathcal{G}_{\text{ét}}(\mathbb{Z}))$  by hyperdescent. We will provide more details in [LZb].

*Remark 5.5.5.* Note that in this chapter, we have fixed a non-empty set  $\square$  of rational primes. In fact, our constructions are compatible for different  $\square$  in the obvious sense. For example, if we are given  $\square_1 \subseteq \square_2$ , then the maps  ${}_{\text{Chp}^{\square_1}} \text{EO}^{\text{II}}$  and  ${}_{\text{Chp}^{\square_2}} \text{EO}^{\text{II}}$  are equivalent when restricted to their common domain, which is

$$\delta_{2,\{2\}}^* (((\text{Chp}^{\square_2})^{\text{op}} \times \text{N}(\mathcal{R}\text{ind}_{\square_1\text{-tor}})^{\text{op}})^{\text{II},\text{op}})^{\text{cart}}_{F,\text{all}}.$$

We also have obvious compatibility properties for Output II under different  $\square$ .

## 6. SUMMARY AND COMPLEMENTS

In this chapter we summarize the construction in the previous chapter and presents several complements. In §6.1, we study the relation of our construction with category of correspondences. In §6.2, we write down the resulting six operations for the most general situations and summarize their properties. In §6.3, we prove some additional adjointness properties in the finite-dimensional Noetherian case. In §6.4, we develop a theory of constructible complexes, based on finiteness results of Deligne [SGA4d, Th. finitude] and Gabber [TGxiii]. In §6.5, we show that our results for constructible complexes are compatible with those of Laszlo–Olsson [LO08].

We remark that §6.1 is independent to the later sections, so readers may skip the first section if they are not interested in the relation with category of correspondences.

Once again, we fix a nonempty set  $\square$  of rational primes.

**6.1. Monoidal category of correspondences.** The  $\infty$ -category of correspondences was first introduced by Gaitsgory [Gai13]. We start by recalling the construction of the simplicial set of correspondences from [LZa, Example 4.30].

For  $n \geq 0$ , we define  $\mathcal{C}(\Delta^n)$  to be the full subcategory of  $\Delta^n \times (\Delta^n)^{\text{op}}$  spanned by  $(i, j)$  with  $i \leq j$ . An edge of  $\mathcal{C}(\Delta^n)$  is *vertical* (resp. *horizontal*) if its projection to the second (resp. first) factor is degenerate. A square of  $\mathcal{C}(\Delta^n)$  is *exact* if it is both a pushout square and a pullback square. We extend the above construction to a colimit preserving functor  $\mathcal{C}: \text{Set}_{\Delta} \rightarrow \text{Set}_{\Delta}$ . Then  $\mathcal{C}$  also preserves finite products. The right adjoint functor is denoted by  $\text{Corr}$ . In particular, we have  $\text{Corr}(K)_n = \text{Hom}(\mathcal{C}(\Delta^n), K)$  for a simplicial set  $K$ .

**Definition 6.1.1.** Let  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  be a 2-marked  $\infty$ -category. We define a simplicial subset  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$  of  $\text{Corr}(\mathcal{C})$ , called the *simplicial set of correspondences*, such that its  $n$ -cells are given by maps  $\mathcal{C}(\Delta^n) \rightarrow \mathcal{C}$  that send vertical (resp. horizontal) edges into  $\mathcal{E}_1$  (resp.  $\mathcal{E}_2$ ), and exact squares to pullback squares.

By construction, there is an obvious map

$$\delta_{2, \{2\}}^* \mathcal{C}_{\mathcal{E}_1, \mathcal{E}_2}^{\text{cart}} \rightarrow \mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2},$$

which is a categorical equivalence by [LZa, Example 4.30].

The following lemma shows that under certain mild conditions,  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$  is an  $\infty$ -category.

**Lemma 6.1.2.** *Let  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  be a 2-marked  $\infty$ -category such that*

- (1) *both  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are stable under composition;*
- (2) *pullbacks of  $\mathcal{E}_1$  by  $\mathcal{E}_2$  exist and remain in  $\mathcal{E}_1$ ;*
- (3) *pullbacks of  $\mathcal{E}_2$  by  $\mathcal{E}_1$  exist and remain in  $\mathcal{E}_2$ .*

*Then  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$  is an  $\infty$ -category.*

*Proof.* We check that  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2} \rightarrow *$  has the right lifting property with respect to the collection  $A_2$  in [HTT, 2.3.2.1]. Since  $\mathcal{C}$  preserves colimits and finite products, to give a map

$$f: (\Delta^m \times \Lambda_1^2) \coprod_{\partial \Delta^m \times \Lambda_1^2} (\partial \Delta^m \times \Delta^2) \rightarrow \text{Corr}(\mathcal{C})$$

is equivalent to give a map

$$f^\sharp: (\mathcal{C}(\Delta^m) \times \mathcal{C}(\Lambda_1^2)) \coprod_{\mathcal{C}(\partial \Delta^m) \times \mathcal{C}(\Lambda_1^2)} (\mathcal{C}(\partial \Delta^m) \times \mathcal{C}(\Delta^2)) \rightarrow \mathcal{C}.$$

Let  $\mathcal{K}$  and  $\mathcal{K}'$  be defined as in the dual version of [HTT, 4.2.3.15] with  $\mathcal{C} = \mathcal{C}(\Delta^2)$ ,  $\mathcal{C}^0 = \mathcal{C}(\Lambda_1^2)$ , and  $\mathcal{D} = \mathcal{C}$  (in our setup). If  $f$  factorizes through  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$ , then  $f^\sharp$  induces a commutative square

$$\begin{array}{ccc} \mathcal{C}(\partial \Delta^m) & \longrightarrow & \mathcal{K} \\ \downarrow & \nearrow & \downarrow \\ \mathcal{C}(\Delta^m) & \longrightarrow & \mathcal{K}' \end{array}$$

by assumption (2) or (3). Since the restriction map  $\mathcal{K} \rightarrow \mathcal{K}'$  is a trivial fibration by the dual of [HTT, 4.2.3.15], there exists a dotted arrow  $g^\sharp: \mathcal{C}(\Delta^m) \rightarrow \mathcal{K}$  as indicated above. We regard  $g^\sharp$  as a map  $\mathcal{C}(\Delta^m \times \Delta^2) \simeq \mathcal{C}(\Delta^m) \times \mathcal{C}(\Delta^2) \rightarrow \mathcal{C}$ , thus induces a map  $g: \Delta^m \times \Delta^2 \rightarrow \text{Corr}(\mathcal{C})$ . Since all exact squares of  $\mathcal{C}(\Delta^m \times \Delta^2)$  can be obtained by composition from exact squares either contained in the source of  $f^\sharp$  or being constant under the projection to  $\mathcal{C}(\Delta^m)$ , the three assumptions ensure that if  $f$  factorizes through  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$ , then so does  $g$ .  $\square$

Now we study certain natural coCartesian symmetric monoidal structure on the  $\infty$ -category  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$ . Let  $(\mathcal{C}, \mathcal{E})$  be a marked  $\infty$ -category. We construct a 2-marked

$\infty$ -categories  $((\mathcal{C}^{op})^{\Pi, op}, \mathcal{E}^-, \mathcal{E}^+)$  as follows: We write an edge  $f$  of  $(\mathcal{C}^{op})^{\Pi, op}$  in the form  $\{Y_j\}_{1 \leq j \leq n} \rightarrow \{X_i\}_{1 \leq i \leq m}$  lying over an edge  $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  of  $N(\mathcal{F}in_*)$ . Then  $\mathcal{E}^+$  consists of  $f$  such that the induced edge  $Y_{\alpha(i)} \rightarrow X_i$  belongs to  $\mathcal{E}$  for every  $i \in \alpha^{-1}\langle n \rangle^\circ$ . Define  $\mathcal{E}^-$  to be the subset of  $\mathcal{E}^+$  such that the edge  $\alpha$  is degenerate.

**Proposition 6.1.3.** *Let  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  be a 2-marked  $\infty$ -category satisfying the assumptions in Lemma 6.1.2 and such that  $\mathcal{C}_{\mathcal{E}_2}$  admits finite products. Then*

$$(6.1) \quad p: ((\mathcal{C}^{op})^{\Pi, op})_{\text{corr}: \mathcal{E}_1^-, \mathcal{E}_2^+} \rightarrow N(\mathcal{F}in_*)$$

is a coCartesian symmetric monoidal  $\infty$ -category [HA, 2.4.0.1], whose underlying  $\infty$ -category is  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$ .

*Proof.* Put  $\mathcal{O}^\otimes := ((\mathcal{C}^{op})^{\Pi, op})_{\text{corr}: \mathcal{E}_1^-, \mathcal{E}_2^+}$  for simplicity. If  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  satisfies the assumptions in Lemma 6.1.2, then so does  $((\mathcal{C}^{op})^{\Pi, op}, \mathcal{E}_1^-, \mathcal{E}_2^+)$ . Therefore, by Lemma 6.1.2,  $\mathcal{O}^\otimes$  is an  $\infty$ -category hence (6.1) is an inner fibration by [HTT, 2.3.1.5]. By Lemma 6.1.4 below, we know that  $p$  is a coCartesian fibration since  $\mathcal{C}_{\mathcal{E}_2}$  admits finite products. Moreover, we have the obvious isomorphism  $\mathcal{O}_{\langle n \rangle}^\otimes \simeq \prod_{1 \leq i \leq n} \mathcal{O}_{\langle 1 \rangle}^\otimes$  induced by  $\rho_i^!: \mathcal{O}_{\langle n \rangle}^\otimes \rightarrow \mathcal{O}_{\langle 1 \rangle}^\otimes$ . By [HA, 2.0.0.7], (6.1) is a symmetric monoidal  $\infty$ -category. The remaining assertions are obvious from definition and construction.  $\square$

**Lemma 6.1.4.** *Suppose that  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  satisfies the assumptions in Lemma 6.1.2. If we write an edge  $f$  of  $((\mathcal{C}^{op})^{\Pi, op})_{\text{corr}: \mathcal{E}_1^-, \mathcal{E}_2^+}$  in the form*

$$\begin{array}{ccc} \{Z_j\}_{1 \leq j \leq n} & \longrightarrow & \{X_i\}_{1 \leq i \leq m} \\ \downarrow & & \\ \{Y_j\}_{1 \leq j \leq n} & & \end{array}$$

lying over an edge  $\alpha: \langle m \rangle \rightarrow \langle n \rangle$  of  $N(\mathcal{F}in_*)$  under (6.1), then  $f$  is  $p$ -coCartesian [HTT, 2.4.2.1] if and only if

- (1) for every  $1 \leq j \leq n$ , the induced morphism  $Z_j \rightarrow Y_j$  is an isomorphism; and
- (2) for every  $1 \leq j \leq n$ , the induced morphisms  $Z_j \rightarrow X_i$  with  $\alpha(i) = j$  exhibit  $Z_j$  as the product of  $\{X_i\}_{\alpha(i)=j}$  in  $\mathcal{C}_{\mathcal{E}_2}$ .

*Proof.* The *only if* part: Suppose that  $f$  is a  $p$ -coCartesian edge.

We first show (1). Without loss of generality, we may assume that  $\alpha$  is the degenerate edge at  $\langle 1 \rangle$ . In particular, the edge  $f$  we consider has the form

$$\begin{array}{ccc} z & \longrightarrow & x. \\ \downarrow & & \\ y & & \end{array}$$

Assume that  $f$  is  $p$ -coCartesian. In terms of the dual version of [HTT, 2.4.1.4], we are going to construct a diagram of the form

$$(6.2) \quad \begin{array}{ccc} \Delta^{\{0,1\}} & \xrightarrow{f} & ((\mathcal{C}^{op})^{\Pi, op})_{\text{corr}: \varepsilon_1^-, \varepsilon_2^+} \\ \downarrow & \searrow g & \downarrow p \\ \Lambda_0^n & \xrightarrow{g} & N(\mathcal{F}in_*) \\ \downarrow & \nearrow & \\ \Delta^n & \xrightarrow{\quad} & \end{array}$$

in which  $n = 3$  and the bottom map is constant with value  $\langle 1 \rangle$ . We may construct a map  $g$  in (6.2) such that its image of  $\mathcal{C}(\Delta^{\{0,1,2\}})$ ,  $\mathcal{C}(\Delta^{\{0,1,3\}})$ ,  $\mathcal{C}(\Delta^{\{0,2,3\}})$  are

$$\begin{array}{ccc} \begin{array}{c} z \longrightarrow z \longrightarrow x, \\ \downarrow \quad \downarrow \\ y' \longrightarrow y \\ \downarrow \\ z \end{array} & \begin{array}{c} z \xlongequal{\quad} z \longrightarrow x, \\ \downarrow \quad \downarrow \\ y \xlongequal{\quad} y \\ \downarrow \\ y \end{array} & \begin{array}{c} z \xlongequal{\quad} z \longrightarrow x, \\ \downarrow \quad \downarrow \\ z \xlongequal{\quad} z \\ \downarrow \\ y \end{array} \end{array}$$

respectively, in which

- all squares are Cartesian diagrams;
- all edges  $z \rightarrow x$  are same as the one in the presentation of  $f$ ;
- all vertical edges  $z \rightarrow y$  are same as the one in the presentation of  $f$ ;
- in the second and third diagrams, all 2-cells are degenerate.

Note that the existence of the first diagram is due to the lifting property for  $n = 2$ . Now we lift  $g$  to a dotted arrow as in (6.2). The image of the unique nondegenerate exact square in  $\mathcal{C}(\Delta^{\{1,2,3\}})$  provides a pullback square

$$\begin{array}{ccc} y & \longrightarrow & y' \\ \downarrow & & \downarrow \\ z & \xlongequal{\quad} & z. \end{array}$$

Therefore, the edge  $y \rightarrow y'$  is an isomorphism, and it is easy to check that the left vertical edge  $y \rightarrow z$  is an inverse of the edge  $z \rightarrow y$  in the presentation of  $f$ .

Next we show (2). Without loss of generality, we may assume that  $\alpha$  is the unique active map from  $\langle m \rangle$  to  $\langle 1 \rangle$  [HA, 2.1.2.1]; and the edge  $f$  has the form

$$\begin{array}{ccc} y & \longrightarrow & \{x_i\}_{1 \leq i \leq m} \\ \parallel & & \\ y & & \end{array}$$



We construct a diagram (6.2) as follows. The bottom map  $\Delta^n \rightarrow N(\mathcal{F}in_*)$  is given by the sequence of morphisms

$$\langle m \rangle \xrightarrow{\alpha} \langle 1 \rangle \xrightarrow{\text{id}} \dots \xrightarrow{\text{id}} \langle 1 \rangle.$$

Note that we have a projection map  $\pi: \mathbb{C}(\Delta^n) \rightarrow (\Delta^n)^{op}$  to the second factor. Denote by  $\mathbb{C}(\Delta^n)_0$  the preimage of  $(\Delta^{\{1, \dots, n\}})^{op}$  under  $\pi$ , and  $\mathbb{C}(\Delta^n)_{00}$  the preimage of  $(\partial\Delta^{\{1, \dots, n\}})^{op}$  under  $\pi$ . It is clear that  $\mathbb{C}(\Lambda_0^n) \cap \mathbb{C}(\Delta^n)_0 \subseteq \mathbb{C}(\Delta^n)_{00}$ . Suppose that we are given a map

$$\alpha: (\partial\Delta^{\{1, \dots, n\}})^{op} \rightarrow (\mathcal{C}_{\mathcal{E}_2})_{/\{x_i\}_{1 \leq i \leq m}}$$

such that  $\alpha|_{\Delta^{\{0\}}}$  is represented by  $y \rightarrow \{x_i\}_{1 \leq i \leq m}$  as in the edge  $f$ . We regard  $\alpha$  as a map  $\alpha': (\partial\Delta^{\{1, \dots, n\}})^{op} \star \langle m \rangle^\circ \rightarrow \mathcal{C}_{\mathcal{E}_2}$ . Note that  $\pi$  induces a projection map

$$\pi': (\mathbb{C}(\Lambda_0^n) \cap \mathbb{C}(\Delta^n)_0) \star \langle m \rangle^\circ \rightarrow (\partial\Delta^{\{1, \dots, n\}})^{op} \star \langle m \rangle^\circ.$$

We then have a map  $g_\alpha := \alpha' \circ \pi': (\mathbb{C}(\Lambda_0^n) \cap \mathbb{C}(\Delta^n)_0) \star \langle m \rangle^\circ \rightarrow \mathcal{C}_{\mathcal{E}_2}$ , which induces a map  $g$  as in (6.2). The existence of the dotted arrow in (6.2) will provide a filling of  $\alpha$  to  $(\Delta^{\{1, \dots, n\}})^{op}$ . This implies that  $y \rightarrow \{x_i\}_{1 \leq i \leq m}$  is a final object of  $(\mathcal{C}_{\mathcal{E}_2})_{/\{x_i\}_{1 \leq i \leq m}}$ .

The *if* part: Let  $f$  be an edge satisfying (1) and (2). To show that  $f$  is  $p$ -coCartesian, we again consider the diagram (6.2). Define  $\mathbb{C}(\Delta^n)'$  to be the  $\infty$ -category by adding one more object  $(0, 0)'$  emitting from  $(0, 0)$  in  $\mathbb{C}(\Delta^n)$ , which can be depicted as in the following diagram

$$\begin{array}{ccccccc} \dots & (0, 2) & \longrightarrow & (0, 1) & \longrightarrow & (0, 0) & \longrightarrow & (0, 0)' \\ & \downarrow & & \downarrow & & & & \\ \dots & (1, 2) & \longrightarrow & (1, 1) & & & & \\ & \downarrow & & & & & & \\ \dots & (2, 2) & & & & & & \\ & & & & & & & \\ \dots & & & & & & & \end{array}$$

We have maps  $\mathbb{C}(\Delta^n) \xrightarrow{\iota} \mathbb{C}(\Delta^n)' \xrightarrow{\gamma} \mathbb{C}(\Delta^n)$ , in which  $\iota$  is the obvious inclusion, and  $\gamma$  collapse the edge  $(0, 1) \rightarrow (0, 0)$  to the single object  $(0, 1)$  and sends  $(0, 0)'$  to  $(0, 0)$ . Let  $K \subseteq \mathbb{C}(\Delta^n)$  be the simplicial subset that is the union of  $\mathbb{C}(\Lambda_0^n)$  and the top row of  $\mathbb{C}(\Delta^n)$ . Define  $K'$  to be the inverse image of  $K$  under  $\gamma$ . Then  $\iota$  sends  $\mathbb{C}(\Lambda_0^n)$  into  $K'$ . We have one more inclusion  $\iota': \mathbb{C}(\Delta^n) \rightarrow \mathbb{C}(\Delta^n)'$  that sends  $(0, 0)$  to  $(0, 0)'$  and keeps the other objects.

A map  $g$  as in (6.2) gives rise to a map  $g^\sharp: \mathbb{C}(\Lambda_0^n) \rightarrow (\mathcal{C}^{op})^{\text{II}, op}$ . By (2) and [HA, 2.4.3.4], we may extend  $g^\sharp$  to  $K$ . Consider the new map  $g^\sharp \circ \gamma \circ \iota: \mathbb{C}(\Lambda_0^n) \rightarrow (\mathcal{C}^{op})^{\text{II}, op}$ , which gives rise to a map  $g'$  as in (6.2) however with the restriction  $g'|_{\Delta^{\{0, 1\}}}$  being an equivalence in the  $\infty$ -category  $((\mathcal{C}^{op})^{\text{II}, op})_{\text{corr}: \varepsilon_1^-, \varepsilon_2^+}$  by (1). Therefore, we may lift  $g'$  to an edge  $\tilde{g}'$  as the dotted arrow in (6.2) by [HTT, 2.4.1.5]. Now  $\tilde{g}'$  induces a map  $\tilde{g}^\sharp: \mathbb{C}(\Delta^n) \rightarrow (\mathcal{C}^{op})^{\text{II}, op}$ . To find a lifting of  $g$  as the dotted arrow in (6.2), it suffices

to extend  $\tilde{g}^\#$  to  $\mathcal{C}(\Delta^n)'$  under the inclusion  $\iota$  such that its restriction to  $\mathcal{C}(\Lambda_0^n)$  with respect to the other inclusion  $\iota'$  coincides with  $g^\#$ . However, this lifting problem only involves the top row of  $\mathcal{C}(\Delta^n)'$ , which can be solved because of (2).  $\square$

**Definition 6.1.5** (symmetric monoidal  $\infty$ -category of correspondences). Let  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  be a 2-marked  $\infty$ -category satisfying the assumptions in Proposition 6.1.3 (3). We call (6.1) the *symmetric monoidal  $\infty$ -category of correspondences* associated to  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$ , denoted by  $p: \mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}^\otimes \rightarrow \mathcal{N}(\mathcal{F}\text{in}_*)$  or simply  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}^\otimes$ . It is a reasonable abuse of notation since its underlying  $\infty$ -category is  $\mathcal{C}_{\text{corr}: \mathcal{E}_1, \mathcal{E}_2}$ .

We apply the above construction to the source of the map  ${}_{\text{Chp}}^{\text{Ar}}\text{EO}^\Pi$  (5.11). Take  $\mathcal{C} = \text{Chp}_\square^{\text{Ar}} \times \mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})$ ,  $\mathcal{E}_1 := \mathcal{E}_F$  to be the set of edges of the form  $(f, g)$  where  $f$  belongs to  $F$  and  $g$  is an isomorphism, and  $\mathcal{E}_2 := \text{all}$  to be the set of all edges. Note that  $(\mathcal{C}, \mathcal{E}_1, \mathcal{E}_2)$  satisfies the assumptions in Proposition 6.1.3 (2) hence defines a symmetric monoidal  $\infty$ -category  $\mathcal{C}_{\text{corr}: \mathcal{E}_F, \text{all}}^\otimes$ .

By definition, we have the identity

$$\delta_{2, \{2\}}^* (((\text{Chp}_\square^{\text{Ar}})^{\text{op}} \times \mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}})^{\text{op}})^{\Pi, \text{op}})^{\text{cart}}_{F, \text{all}} = \delta_{2, \{2\}}^* ((\mathcal{C}^{\text{op}})^{\Pi, \text{op}})^{\text{cart}}_{\mathcal{E}_1^-, \mathcal{E}_2^+}.$$

Since the map

$$\delta_{2, \{2\}}^* ((\mathcal{C}^{\text{op}})^{\Pi, \text{op}})^{\text{cart}}_{\mathcal{E}_1^-, \mathcal{E}_2^+} \rightarrow ((\mathcal{C}^{\text{op}})^{\Pi, \text{op}})_{\text{corr}: \mathcal{E}_1^-, \mathcal{E}_2^+} = \mathcal{C}_{\text{corr}: \mathcal{E}_F, \text{all}}^\otimes$$

is a categorical equivalence, by Proposition 6.1.3 (1), the map (5.11) induces a map

$$(6.3) \quad \mathcal{C}_{\text{corr}: \mathcal{E}_F, \text{all}}^\otimes \rightarrow \mathcal{C}\text{at}_\infty.$$

**Lemma 6.1.6.** *The functor (6.3) is a weak Cartesian structure.*

*Proof.* It follows from the fact that (5.10) is a weak Cartesian structure, the construction of (5.11), and Lemma 6.1.4.  $\square$

From the above lemma, we know that (6.3) induces an  $\infty$ -operad map

$$(6.4) \quad {}_{\text{Chp}}^{\text{Ar}}\text{EO}_{\text{corr}}: (\text{Chp}_\square^{\text{Ar}} \times \mathcal{N}(\mathcal{R}\text{ind}_{\square\text{-tor}}))_{\text{corr}: \mathcal{E}_F, \text{all}}^\otimes \rightarrow \mathcal{C}\text{at}_\infty^\times$$

between symmetric monoidal  $\infty$ -categories. Similarly, we have two more  $\infty$ -operad maps

$$(6.5) \quad {}_{\text{Chp}}^{\text{DM}}\text{EO}_{\text{corr}}: (\text{Chp}^{\text{DM}} \times \mathcal{N}(\mathcal{R}\text{ind}_{\text{tor}}))_{\text{corr}: \mathcal{E}_F, \text{all}}^\otimes \rightarrow \mathcal{C}\text{at}_\infty^\times,$$

and

$$(6.6) \quad {}_{\text{Chp}}^{\text{lqf}}\text{EO}_{\text{corr}}: (\text{Chp}^{\text{DM}} \times \mathcal{N}(\mathcal{R}\text{ind}))_{\text{corr}: \mathcal{E}_Q, \text{all}}^\otimes \rightarrow \mathcal{C}\text{at}_\infty^\times,$$

induced from (5.13) and (5.14), respectively.

*Remark 6.1.7.* By all the constructions and (P2) of DESCENT, we obtain the following square

$$\begin{array}{ccc} ((\mathrm{Chp}_{\square}^{\mathrm{Ar}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square\text{-tor}}))^{\mathrm{op}})^{\mathrm{II}} & \xrightarrow{\quad} & ((\mathrm{Chp}^{\mathrm{Ar}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}))^{\mathrm{op}})^{\mathrm{II}} \\ \downarrow & & \downarrow \\ (\mathrm{Chp}_{\square}^{\mathrm{Ar}} \times \mathrm{N}(\mathcal{R}\mathrm{ind}_{\square\text{-tor}}))^{\otimes}_{\mathrm{corr}: \mathcal{E}_{F, \mathrm{all}}} & \xrightarrow{\mathrm{Chp}_{\square}^{\mathrm{Ar}} \mathrm{EO}_{\mathrm{corr}} (6.4)} & \mathrm{Cat}_{\infty}^{\times} \end{array}$$

in the  $\infty$ -category of symmetric monoidal  $\infty$ -categories with  $\infty$ -operad maps, where the right vertical map is induced from  ${}_{\mathrm{Chp}^{\mathrm{Ar}}} \mathrm{EO}^{\mathrm{I}}$  (5.10).

The new functor  ${}_{\mathrm{Chp}_{\square}^{\mathrm{Ar}}} \mathrm{EO}_{\mathrm{corr}}$  loses no information from the original one  ${}_{\mathrm{Chp}_{\square}^{\mathrm{Ar}}} \mathrm{EO}^{\mathrm{II}}$ . However, the new one has the advantage that its source is an  $\infty$ -category as well.

The above remarks can be applied to the other two cases as well.

**6.2. The six operations.** Now we can summarize our construction of Grothendieck's six operations. Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathrm{Chp}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}^{\mathrm{DM}}$ , resp.  $\mathrm{Chp}^{\mathrm{DM}}$ ), and  $\lambda$  an object of  $\mathcal{R}\mathrm{ind}$ . From  ${}_{\mathrm{Chp}^{\mathrm{Ar}}} \mathrm{EO}^{\mathrm{I}}$  (5.10) (resp.  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \mathrm{EO}^{\mathrm{I}}$  (5.12), resp.  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \mathrm{EO}^{\mathrm{I}}$ ) and  ${}_{\mathrm{Chp}_{\square}^{\mathrm{Ar}}} \mathrm{EO}_{\mathrm{corr}}$  (6.4) (resp.  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \mathrm{EO}_{\mathrm{corr}}$  (6.5), resp.  ${}_{\mathrm{Chp}^{\mathrm{DM}}} \mathrm{EO}_{\mathrm{corr}}^{\mathrm{lqf}}$  (6.6)), we directly obtain three operations:

**1L:**  $f^*: \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)$ , which underlies a monoidal functor

$$f^{*\otimes}: \mathcal{D}(\mathcal{X}, \lambda)^{\otimes} \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)^{\otimes};$$

**2L:**  $f_!: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$  if  $f$  is locally of finite type,  $\lambda$  belongs to  $\mathcal{R}\mathrm{ind}_{\square\text{-tor}}$  and  $\mathcal{X}$  is  $\square$ -coprime (resp.  $f$  is locally of finite type and  $\lambda$  belongs to  $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$ , resp.  $f$  is locally quasi-finite and  $\lambda$  is arbitrary);

**3L:**  $- \otimes - = - \otimes_{\mathcal{X}} -: \mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ .

If  $\mathcal{X}$  is a 1-Artin stack (resp. 1-DM stack), then  $\mathcal{D}(\mathcal{X}, \lambda)^{\otimes}$  is equivalent to  $\mathcal{D}_{\mathrm{cart}}(\mathcal{X}_{\mathrm{lis\text{-}\acute{e}t}}, \lambda)^{\otimes}$  (resp.  $\mathcal{D}(\mathcal{X}_{\acute{e}t}, \lambda)^{\otimes}$ ) as symmetric monoidal  $\infty$ -categories.

Taking right adjoints for (1L) and (2L), respectively, we obtain:

**1R:**  $f_*: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ ;

**2R:**  $f^!: \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)$  under the same condition as (2L).

For (3L), moving the first factor of the source  $\mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda)$  to the target side, we can write the functor  $- \otimes -$  in the form  $\mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ , since the tensor product on  $\mathcal{D}(\mathcal{X}, \lambda)$  is closed. Taking opposites and applying [HTT, 5.2.6.2], we obtain a functor  $\mathcal{D}(\mathcal{X}, \lambda)^{\mathrm{op}} \rightarrow \mathrm{Fun}^{\mathrm{R}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ , which can be written as

**3R:**  $\mathcal{H}\mathrm{om}(-, -) = \mathcal{H}\mathrm{om}_{\mathcal{X}}(-, -): \mathcal{D}(\mathcal{X}, \lambda)^{\mathrm{op}} \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ .

Besides these six operations, for every morphism  $\pi: \lambda' \rightarrow \lambda$  of  $\mathcal{R}\mathrm{ind}$ , we have the following functor of *extension of scalars*:

**4L:**  $\pi^*: \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda')$ , which underlies a monoidal functor

$$\pi^{*\otimes}: \mathcal{D}(\mathcal{X}, \lambda)^{\otimes} \rightarrow \mathcal{D}(\mathcal{X}, \lambda')^{\otimes}.$$

The right adjoint of the functor  $\pi^*$  is the functor of *restriction of scalars*:

**4R:**  $\pi_*: \mathcal{D}(\mathcal{X}, \lambda') \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ .

The following theorem is a consequence of existence of the map  ${}_{\text{Chp}\square^{\text{Ar}}}\text{EO}_{\text{corr}}$  (6.4) (resp.  ${}_{\text{Chp}^{\text{DM}}}\text{EO}_{\text{corr}}$  (6.5), resp.  ${}_{\text{Chp}^{\text{DM}}}\text{EO}_{\text{corr}}^{\text{lqf}}$  (6.6)).

**Theorem 6.2.1** (Künneth Formula). *Let  $f_i: \mathcal{Y}_i \rightarrow \mathcal{X}_i$  ( $i = 1, \dots, n$ ) be finitely many morphisms of  $\text{Chp}\square^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ , resp.  $\text{Chp}^{\text{DM}}$ ) that are locally of finite type (resp. locally of finite type, resp. locally quasi-finite). Given a pullbacks square*

$$\begin{array}{ccc} \mathcal{Y} & \xrightarrow{(q_1, \dots, q_n)} & \mathcal{Y}_1 \times \dots \times \mathcal{Y}_n \\ f \downarrow & & \downarrow f_1 \times \dots \times f_n \\ \mathcal{X} & \xrightarrow{(p_1, \dots, p_n)} & \mathcal{X}_1 \times \dots \times \mathcal{X}_n \end{array}$$

*of  $\text{Chp}\square^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ , resp.  $\text{Chp}^{\text{DM}}$ ), then for every object  $\lambda$  of  $\text{Rind}_{\square\text{-tor}}$  (resp.  $\text{Rind}_{\text{tor}}$ , resp.  $\text{Rind}$ ), the following square*

$$\begin{array}{ccc} \mathcal{D}(\mathcal{Y}_1, \lambda) \times \dots \times \mathcal{D}(\mathcal{Y}_n, \lambda) & \xrightarrow{q_1^* - \otimes_{\mathcal{Y}} \dots \otimes_{\mathcal{Y}} q_n^* -} & \mathcal{D}(\mathcal{Y}, \lambda) \\ f_1! \times \dots \times f_n! \downarrow & & \downarrow f! \\ \mathcal{D}(\mathcal{X}_1, \lambda) \times \dots \times \mathcal{D}(\mathcal{X}_n, \lambda) & \xrightarrow{p_1^* - \otimes_{\mathcal{X}} \dots \otimes_{\mathcal{X}} p_n^* -} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

*is commutative up to equivalence.*

It has the following two corollaries.

**Corollary 6.2.2** (Base Change). *Let*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Z} \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

*be a Cartesian diagram in  $\text{Chp}\square^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ , resp.  $\text{Chp}^{\text{DM}}$ ) where  $p$  is locally of finite type (resp. locally of finite type, resp. locally quasi-finite). Then for every object  $\lambda$  of  $\text{Rind}_{\square\text{-tor}}$  (resp.  $\text{Rind}_{\text{tor}}$ , resp.  $\text{Rind}$ ), the following square*

$$\begin{array}{ccc} \mathcal{D}(\mathcal{W}, \lambda) & \xleftarrow{g^*} & \mathcal{D}(\mathcal{Z}, \lambda) \\ q! \downarrow & & \downarrow p! \\ \mathcal{D}(\mathcal{Y}, \lambda) & \xleftarrow{f^*} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

*is commutative up to equivalence.*

**Corollary 6.2.3** (Projection Formula). *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\text{Chp}\square^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ , resp.  $\text{Chp}^{\text{DM}}$ ) that is locally of finite type (resp. locally of finite type, resp.*

locally quasi-finite). Then the following square

$$\begin{array}{ccc} \mathcal{D}(\mathcal{Y}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) & \xrightarrow{- \otimes_{\mathcal{Y}} f^* -} & \mathcal{D}(\mathcal{Y}, \lambda) \\ f_! \times \text{id} \downarrow & & \downarrow f_! \\ \mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) & \xrightarrow{- \otimes_{\mathcal{X}} -} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

is commutative up to equivalence.

**Proposition 6.2.4.** *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\text{Chp}^{\text{Ar}}$ , and  $\lambda$  an object of  $\mathcal{R}\text{ind}$ . Then*

- (1) *The functors  $f^*(- \otimes_{\mathcal{X}} -)$  and  $(f^* -) \otimes_{\mathcal{Y}} (f^* -)$  are equivalent.*
- (2) *The functors  $\mathcal{H}\text{om}_{\mathcal{X}}(-, f_* -)$  and  $f_* \mathcal{H}\text{om}_{\mathcal{Y}}(f^* -, -)$  are equivalent.*
- (3) *If  $f$  is a morphism of  $\text{Chp}_{\square}^{\text{Ar}}$  (resp.  $\text{Chp}^{\text{DM}}$ , resp.  $\text{Chp}^{\text{DM}}$ ) that is locally finite type (resp. locally of finite type, resp. locally quasi-finite), and  $\lambda$  belongs to  $\mathcal{R}\text{ind}_{\square\text{-tor}}$  (resp.  $\mathcal{R}\text{ind}_{\text{tor}}$ , resp.  $\mathcal{R}\text{ind}$ ), then the functors  $f^! \mathcal{H}\text{om}_{\mathcal{X}}(-, -)$  and  $\mathcal{H}\text{om}_{\mathcal{Y}}(f^* -, f^! -)$  are equivalent.*
- (4) *Under the same assumptions as in (3), the functors  $f_* \mathcal{H}\text{om}_{\mathcal{Y}}(-, f^! -)$  and  $\mathcal{H}\text{om}_{\mathcal{X}}(f_! -, -)$  are equivalent.*

*Proof.* For (1), it follows from the fact that  $f^*$  is a symmetric monoidal functor.

For (2), the functor  $\mathcal{H}\text{om}(-, f_* -): \mathcal{D}(\mathcal{X}, \lambda)^{\text{op}} \times \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$  induces a functor  $\mathcal{D}(\mathcal{X}, \lambda)^{\text{op}} \rightarrow \text{Fun}^{\text{R}}(\mathcal{D}(\mathcal{Y}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ . Taking opposite, we obtain a functor  $\mathcal{D}(\mathcal{X}, \lambda) \rightarrow \text{Fun}^{\text{L}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{Y}, \lambda))$ , which induces a functor  $\mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)$ . By construction, the latter is equivalent to the functor  $f^*(- \otimes_{\mathcal{X}} -)$ . Repeating the same process for  $f_* \mathcal{H}\text{om}(f^* -, -)$ , we obtain  $(f^* -) \otimes_{\mathcal{Y}} (f^* -)$ . Therefore, by (1), the functors  $\mathcal{H}\text{om}(-, f_* -)$  and  $f_* \mathcal{H}\text{om}(f^* -, -)$  are equivalent.

For (3), the functor  $f^! \mathcal{H}\text{om}(-, -): \mathcal{D}(\mathcal{X}, \lambda)^{\text{op}} \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)$  induces a functor  $\mathcal{D}(\mathcal{X}, \lambda)^{\text{op}} \rightarrow \text{Fun}^{\text{R}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{Y}, \lambda))$ . Taking opposite, we obtain a functor  $\mathcal{D}(\mathcal{X}, \lambda) \rightarrow \text{Fun}^{\text{L}}(\mathcal{D}(\mathcal{Y}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ , which induces a functor  $\mathcal{D}(\mathcal{X}, \lambda) \times \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ . By construction, the latter is equivalent to the functor  $- \otimes_{\mathcal{X}} (f_! -)$ . Repeating the same process for  $\mathcal{H}\text{om}(f^* -, f^! -)$ , we obtain  $f_!((f^* -) \otimes_{\mathcal{Y}} -)$ . Therefore, by Corollary 6.2.3, the functors  $f^! \mathcal{H}\text{om}(-, -)$  and  $\mathcal{H}\text{om}(f^* -, f^! -)$  are equivalent.

For (4), the functor  $f_* \mathcal{H}\text{om}(-, f^! -): \mathcal{D}(\mathcal{Y}, \lambda)^{\text{op}} \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$  induces a functor  $\mathcal{D}(\mathcal{Y}, \lambda)^{\text{op}} \rightarrow \text{Fun}^{\text{R}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ . Taking opposite, we obtain a functor  $\mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \text{Fun}^{\text{L}}(\mathcal{D}(\mathcal{X}, \lambda), \mathcal{D}(\mathcal{X}, \lambda))$ , which induces a functor  $\mathcal{D}(\mathcal{Y}, \lambda) \times \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ . By construction, the latter is equivalent to the functor  $f_!(- \otimes_{\mathcal{Y}} (f^* -))$ . Repeating the same process for  $\mathcal{H}\text{om}(f_! -, -)$ , we obtain  $(f_! -) \otimes_{\mathcal{X}} -$ . Therefore, by Corollary 6.2.3, the functors  $f_* \mathcal{H}\text{om}(-, f^! -)$  and  $\mathcal{H}\text{om}(f_! -, -)$  are equivalent.  $\square$

**Proposition 6.2.5.** *Let  $\mathcal{X}$  be an object of  $\text{Chp}^{\text{Ar}}$ , and  $\pi: \lambda' \rightarrow \lambda$  a morphism of  $\mathcal{R}\text{ind}$ . Then*

- (1) *The functors  $\pi^*(- \otimes_{\lambda} -)$  and  $(\pi^* -) \otimes_{\lambda'} (\pi^* -)$  are equivalent.*
- (2) *The functors  $\mathcal{H}\text{om}_{\lambda}(-, \pi_* -)$  and  $\pi_* \mathcal{H}\text{om}_{\lambda'}(\pi^* -, -)$  are equivalent.*

*Proof.* The proof is similar to Proposition 6.2.4.  $\square$

**Proposition 6.2.6.** *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathbf{Chp}^{\text{Ar}}$ , and  $\pi: \lambda' \rightarrow \lambda$  a perfect morphism of  $\mathbf{Rind}$ . Then the square*

$$(6.7) \quad \begin{array}{ccc} \mathcal{D}(\mathcal{Y}, \lambda') & \xleftarrow{f^*} & \mathcal{D}(\mathcal{X}, \lambda') \\ \pi^* \uparrow & & \uparrow \pi^* \\ \mathcal{D}(\mathcal{Y}, \lambda) & \xleftarrow{f^*} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

*is right adjointable and its transpose is left adjointable.*

In particular, if  $\mathcal{X}$  is an object of  $\mathbf{Chp}^{\text{Ar}}$  and  $\pi: \lambda' \rightarrow \lambda$  is a perfect morphism of  $\mathbf{Rind}$ , then  $\pi^*$  admits a left adjoint

$$\pi_!: \mathcal{D}(\mathcal{X}, \lambda') \rightarrow \mathcal{D}(\mathcal{X}, \lambda).$$

*Proof.* The first assertion follows from the second one. To show the second assertion, by Lemma 4.3.7, we may assume that  $f$  is a morphism of  $\mathbf{Sch}^{\text{qc.sep}}$ . In this case the proposition reduces to Lemma 2.2.8.  $\square$

**Proposition 6.2.7.** *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ , resp.  $\mathbf{Chp}^{\text{DM}}$ ) that is locally of finite type (resp. locally of finite type, resp. locally quasi-finite), and  $\pi: \lambda' \rightarrow \lambda$  a perfect morphism of  $\mathbf{Rind}_{\square\text{-tor}}$  (resp.  $\mathbf{Rind}_{\text{tor}}$ , resp.  $\mathbf{Rind}$ ). Then the square*

$$(6.8) \quad \begin{array}{ccc} \mathcal{D}(\mathcal{Y}, \lambda') & \xrightarrow{f_!} & \mathcal{D}(\mathcal{X}, \lambda') \\ \pi^* \uparrow & & \uparrow \pi^* \\ \mathcal{D}(\mathcal{Y}, \lambda) & \xrightarrow{f_!} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

*is right adjointable and its transpose is left adjointable.*

*Proof.* It follows from Lemmas 4.3.7 and 3.2.9.  $\square$

**Proposition 6.2.8.** *Let  $\mathcal{X}$  be an object of  $\mathbf{Chp}^{\text{Ar}}$ ,  $\lambda = (\Xi, \Lambda)$  an object of  $\mathbf{Rind}$ , and  $\xi$  an object of  $\Xi$ . Consider the obvious morphism  $\pi: \lambda' := (\Xi/\xi, \Lambda \mid \Xi/\xi) \rightarrow \lambda$ . Then*

- (1) *The natural transformation  $\pi_!(-\otimes_{\lambda'} \pi^* -) \rightarrow (\pi_! -) \otimes_{\lambda} -$  is a natural equivalence.*
- (2) *The natural transformation  $\pi^* \mathcal{H}om_{\lambda}(-, -) \rightarrow \mathcal{H}om_{\lambda'}(\pi^* -, \pi^* -)$  is a natural equivalence.*
- (3) *The natural transformation  $\mathcal{H}om_{\lambda}(\pi_! -, -) \rightarrow \pi_* \mathcal{H}om_{\lambda'}(-, \pi^* -)$  is a natural equivalence.*

*Proof.* Similarly to the proof of Proposition 6.2.4 (3), (4), one shows that the three assertions are equivalent (for every given  $\mathcal{X}$ ). For assertion (1), we may assume that  $\mathcal{X}$  is an object of  $\mathbf{Sch}^{\text{qc.sep}}$ . In this case, assertion (2) follows from the fact that  $\pi^*$  preserves fibrant objects in  $\mathbf{Ch}(\mathbf{Mod}(-))^{\text{inj}}$ .  $\square$

Let  $\mathcal{X}$  be an object of  $\mathbf{Chp}^{\text{Ar}}$ , and  $\lambda = (\Xi, \Lambda)$  an object of  $\mathbf{Rind}$ . There is a t-structure on  $\mathcal{D}(\mathcal{X}, \lambda)$ , such that if  $\mathcal{X}$  is a 1-Artin stack (resp. 1-DM stack), then it induces the usual t-structure on its homotopy category  $\mathbf{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}^{\Xi}, \Lambda)$  (resp.  $\mathbf{D}(\mathcal{X}_{\text{ét}}^{\Xi}, \Lambda)$ ). For an

object  $s_{\mathcal{X}}: \mathcal{X} \rightarrow \mathrm{Spec} \mathbb{Z}$  of  $\mathrm{Chp}_{\square}^{\mathrm{Ar}}$ , we put  $\lambda_{\mathcal{X}} := s_{\mathcal{X}}^* \lambda_{\mathrm{Spec} \mathbb{Z}}$ , which is a monoidal unit of  $\mathcal{D}(\mathcal{X}, \lambda)^{\otimes}$  and also an object of  $\mathcal{D}^{\heartsuit}(\mathcal{X}, \lambda)$ . We have the following theorem of *Poincaré duality* from (P7) of DESCENT.

**Theorem 6.2.9** (Poincaré duality). *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathrm{Chp}_{\square}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}^{\mathrm{DM}}$ ) that is flat (resp. flat and locally quasi-finite) and locally of finite presentation. Let  $\lambda$  be an object of  $\mathcal{R}\mathrm{ind}_{\square\text{-tor}}$  (resp.  $\mathcal{R}\mathrm{ind}$ ). Then*

(1) *There is a trace map*

$$\mathrm{Tr}_f: \tau^{\geq 0} f_! \lambda_{\mathcal{Y}} \langle d \rangle = \tau^{\geq 0} f_! (f^* \lambda_{\mathcal{X}}) \langle d \rangle \rightarrow \lambda_{\mathcal{X}}$$

*for every integer  $d \geq \dim^+(f)$ , which is functorial in the sense of Remark 4.1.6.*

(2) *If  $f$  is moreover smooth, the induced natural transformation*

$$u_f: f_! \circ f^* \langle \dim f \rangle \rightarrow \mathrm{id}_{\mathcal{X}}$$

*is a counit transformation, so that the induced map*

$$f^* \langle \dim f \rangle \rightarrow f^!: \mathcal{D}(\mathcal{X}, \lambda) \rightarrow \mathcal{D}(\mathcal{Y}, \lambda)$$

*is a natural equivalence of functors.*

**Corollary 6.2.10** (Smooth (resp. Étale) Base Change). *Let*

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{g} & \mathcal{Z} \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

*be a Cartesian diagram in  $\mathrm{Chp}_{\square}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}^{\mathrm{DM}}$ ) where  $p$  is smooth (resp. étale). Then for every object  $\lambda$  of  $\mathcal{R}\mathrm{ind}_{\square\text{-tor}}$  (resp.  $\mathcal{R}\mathrm{ind}$ ), the following square*

$$\begin{array}{ccc} \mathcal{D}(\mathcal{W}, \lambda) & \xleftarrow{g^*} & \mathcal{D}(\mathcal{Z}, \lambda) \\ q^* \uparrow & & \uparrow p^* \\ \mathcal{D}(\mathcal{Y}, \lambda) & \xleftarrow{f^*} & \mathcal{D}(\mathcal{X}, \lambda) \end{array}$$

*is right adjointable.*

*Proof.* This is part (2) of (P5<sup>bis</sup>). It also follows from Corollary 6.2.2 and Theorem 6.2.9 (2) as in Lemma 4.1.13.  $\square$

**Proposition 6.2.11.** *Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathrm{Chp}_{\square}^{\mathrm{Ar}}$  (resp.  $\mathrm{Chp}^{\mathrm{DM}}$ ), and  $\lambda$  an object of  $\mathcal{R}\mathrm{ind}_{\square\text{-tor}}$  (resp.  $\mathcal{R}\mathrm{ind}_{\mathrm{tor}}$ ). Assume that for every morphism  $X \rightarrow \mathcal{X}$  from an algebraic space  $X$ , the base change  $\mathcal{Y} \times_{\mathcal{X}} X \rightarrow X$  is a proper morphism of algebraic spaces; in particular,  $f$  is locally of finite type. Then*

$$f_*, f_!: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$$

*are equivalent functors.*

*Proof.* We only prove the proposition for  $\mathbf{Chp}_{\square}^{\text{Ar}}$  and leave the other case to readers. For simplicity, we call such morphism  $f$  in the proposition as *proper*. For every integer  $k \geq 0$ , denote by  $\mathcal{C}^k$  the subcategory of  $\text{Fun}(\Delta^1, \mathbf{Chp}_{\square}^{k-\text{Ar}})$  spanned by objects of the form  $f: \mathcal{Y} \rightarrow \mathcal{X}$  that is proper and edges of the form

$$(6.9) \quad \begin{array}{ccc} \mathcal{Y}' & \xrightarrow{f'} & \mathcal{X}' \\ q \downarrow & & \downarrow p \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \end{array}$$

that is a Cartesian diagram in which  $p$  hence  $q$  are smooth. In addition, we let  $\mathcal{C}^{-1}$  be the subcategory of  $\mathcal{C}^0$  spanned by  $f: \mathcal{Y} \rightarrow \mathcal{X}$  such that  $\mathcal{X}$  hence  $\mathcal{Y}$  are quasi-compact separated algebraic spaces. For  $k \geq -1$ , denote by  $\mathcal{E}^k$  the subset of  $(\mathcal{C}^k)_1$  consists of (6.9) in which  $p$  hence  $q$  are moreover surjective. We have  $\mathcal{E}^k \cap (\mathcal{C}^{k-1})_1 = \mathcal{E}^{k-1}$  for  $k \geq 0$ .

By Corollary 6.2.10 and the map  ${}_{\mathbf{Chp}_{\square}^{k-\text{Ar}}} \text{EO}^*$  (obtained from  ${}_{\mathbf{Chp}_{\square}^{k-\text{Ar}}} \text{EO}^{\text{II}}$  as in (3.9)), for every  $k \geq -1$ , we have two functors

$$F_*^k, F_!^k: (\mathcal{C}^k)^{op} \rightarrow \text{Fun}(\Delta^1, \mathcal{C}\text{at}_{\infty})$$

in which the first (resp. second) one sends  $f: \mathcal{Y} \rightarrow \mathcal{X}$  to  $f_*: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$  (resp.  $f_!: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$ ), and an edge (6.9) to

$$\begin{array}{ccc} \mathcal{D}(\mathcal{Y}', \lambda) & \xrightarrow{f'_* \text{ (resp. } f'_!)} & \mathcal{D}(\mathcal{X}', \lambda) \\ q^* \uparrow & & \uparrow p^* \\ \mathcal{D}(\mathcal{Y}, \lambda) & \xrightarrow{f_* \text{ (resp. } f_!)} & \mathcal{D}(\mathcal{X}, \lambda). \end{array}$$

By Remark 5.2.4,  $F_*^{-1}$  and  $F_!^{-1}$  are equivalence. Applying Proposition 4.1.1 successively to marked  $\infty$ -categories  $(\mathcal{C}^k, \mathcal{E}^k)$ , we conclude that  $F_*^k$  and  $F_!^k$  are equivalence for every  $k \geq 0$ . The proposition follows.  $\square$

*Remark 6.2.12.* Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ) that is locally of finite type and representable by DM stacks, and  $\lambda$  an object of  $\mathcal{R}\text{ind}_{\square-\text{tor}}$  (resp.  $\mathcal{R}\text{ind}_{\text{tor}}$ ). We can always construct a natural transformation

$$f_! \rightarrow f_*: \mathcal{D}(\mathcal{Y}, \lambda) \rightarrow \mathcal{D}(\mathcal{X}, \lambda)$$

of functors, which specializes to the equivalence in Proposition 6.2.11 if  $f$  satisfies the property there.

**Theorem 6.2.13** ((Co)homological descent). *Let  $f: X_0^+ \rightarrow X_{-1}^+$  be a smooth surjective morphism of  $\mathbf{Chp}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ), and  $X_{\bullet}^+$  a Čech nerve of  $f$ .*

(1) *For every object  $\lambda$  of  $\mathcal{R}\text{ind}$ , the functor*

$$\mathcal{D}(X_{-1}^+, \lambda) \rightarrow \varprojlim_{n \in \Delta} \mathcal{D}(X_n^+, \lambda)$$

*is an equivalence, where the transition maps in the limit are provided by  $*$ -pullback.*



- (2) Suppose that  $f$  is a morphism of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ). For every object  $\lambda$  of  $\mathbf{Rind}_{\square\text{-tor}}$  (resp.  $\mathbf{Rind}_{\text{tor}}$ ), the functor

$$\mathcal{D}(X_{-1}^+, \lambda) \rightarrow \varprojlim_{n \in \Delta} \mathcal{D}(X_n^+, \lambda)$$

is an equivalence, where the transition maps in the limit are provided by  $!$ -pullback.

*Proof.* It follows from (P4) of DESCENT.  $\square$

**Corollary 6.2.14.** *Let  $f: Y \rightarrow X$  be a morphism of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ) and let  $y: Y_0^+ \rightarrow Y$  be a smooth surjective morphism of  $\mathbf{Chp}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ). Denote  $Y_{\bullet}^+$  the Čech nerve of  $y$  with the morphism  $y_n: Y_n^+ \rightarrow Y_{-1}^+ = Y$ . Put  $f_n = f \circ y_n: Y_n^+ \rightarrow X$ .*

- (1) *For every object  $\lambda$  of  $\mathbf{Rind}$  and every object  $K \in \mathcal{D}^{\geq 0}(Y, \Lambda)$ , we have a convergent spectral sequence*

$$E_1^{p,q} = H^q(f_{p*} y_p^* K) \Rightarrow H^{p+q} f_* K.$$

- (2) *Suppose that  $f$  is a morphism of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ). For every object  $\lambda$  of  $\mathbf{Rind}_{\square\text{-tor}}$  (resp.  $\mathbf{Rind}_{\text{tor}}$ ) and every object  $K \in \mathcal{D}^{\leq 0}(Y, \Lambda)$ , we have a convergent spectral sequence*

$$\tilde{E}_1^{p,q} = H^q(f_{-p!} y_{-p}^! K) \Rightarrow H^{p+q} f_! K.$$

*Proof.* It essentially follows from Theorem 6.2.13 and [HA, 1.2.4.5 & 1.2.4.9].

For (1), we obtain a cosimplicial object  $N(\Delta) \rightarrow \mathcal{D}^{\geq 0}(Y, \Lambda)$  whose value at  $[n]$  is  $y_n y_n^* K$ , such that  $K$  is its limit by Theorem 6.2.13 (1); in other words, we have  $K \xrightarrow{\sim} \varprojlim_{n \in \Delta} y_n y_n^* K$ . Applying the functor  $f_*$ , we obtain another cosimplicial object  $N(\Delta) \rightarrow \mathcal{D}^{\geq 0}(X, \Lambda)$  whose value at  $[n]$  is  $f_n y_n^* K$ , such that  $f_* K$  is its limit. Put  $\mathcal{C} = \mathcal{D}(X, \Lambda)^{op}$  and let  $\mathcal{C}_{\geq 0} := \mathcal{D}^{\geq 0}(X, \Lambda)^{op}$ ,  $\mathcal{C}_{\leq 0} := \mathcal{D}^{\leq 0}(X, \Lambda)^{op}$  be the induced (homological) t-structure. Then we obtain a simplicial object  $N(\Delta)^{op} \rightarrow \mathcal{C}_{\geq 0}$  whose value at  $[n]$  is  $f_n y_n^* K$ , with  $f_* K$  its geometric realization. By [HA, 1.2.4.5 & 1.2.4.9], we obtain a spectral sequence  $\{E_r^{p,q}\}_{r \geq 1}$  abutting to  $H^{p+q} f_* K$ , with  $E_1^{p,q} = H^q(f_{p*} y_p^* K)$ .

For (2), by Theorem 6.2.13 (2), the functor  $\mathcal{D}(Y, \lambda)^{op} \rightarrow \varprojlim_{n \in \Delta} \mathcal{D}(Y_n^+, \lambda)^{op}$  is an equivalence, where the transition maps in the limit are provided by  $!$ -pullback. Similar to (1), we obtain a cosimplicial object  $N(\Delta) \rightarrow \mathcal{D}^{\leq 0}(Y, \Lambda)^{op}$  whose value at  $[n]$  is  $y_n y_n^! K$ , such that  $K$  is its limit. Applying the functor  $f_!$ , we obtain another cosimplicial object  $N(\Delta) \rightarrow \mathcal{D}^{\leq 0}(Y, \Lambda)^{op}$  whose value at  $[n]$  is  $f_n y_n^! K$ , such that  $f_! K$  is its limit. Put  $\mathcal{C} = \mathcal{D}(X, \Lambda)$  and let  $\mathcal{C}_{\geq 0} := \mathcal{D}^{\leq 0}(X, \Lambda)$ ,  $\mathcal{C}_{\leq 0} := \mathcal{D}^{\geq 0}(X, \Lambda)$  be the induced (homological) t-structure. Then we obtain a simplicial object  $N(\Delta)^{op} \rightarrow \mathcal{C}_{\geq 0}$  whose value at  $[n]$  is  $f_n y_n^! K$ , with  $f_! K$  its geometric realization. By [HA, 1.2.4.5 & 1.2.4.9], we obtain a spectral sequence  $\{\tilde{E}_r^{p,q}\}_{r \geq 1}$  abutting to  $H^{p+q} f_! K$ , with  $\tilde{E}_1^{p,q} = H^q(f_{-p!} y_{-p}^! K)$ .  $\square$

The following lemma will be used in §6.4.

**Lemma 6.2.15.** *Let  $f: Y \rightarrow X$  be a morphism locally of finite type of  $\mathbf{Chp}_{\square}^{\text{Ar}}$  (resp.  $\mathbf{Chp}^{\text{DM}}$ ), and  $\lambda$  an object of  $\mathbf{Rind}_{\square\text{-tor}}$  (resp.  $\mathbf{Rind}_{\text{tor}}$ ). Then  $f_!$  restricts to a functor*

$\mathcal{D}^{\leq 0}(Y, \lambda) \rightarrow \mathcal{D}^{\leq 2d}(X, \lambda)$ , where  $d = \dim^+(f)$ . Moreover, if  $f$  is smooth (resp. étale), then  $f_! \circ f^!$  restricts to a functor  $\mathcal{D}^{\leq 0}(X, \lambda) \rightarrow \mathcal{D}^{\leq 0}(X, \lambda)$ .

*Proof.* We may assume that  $X$  is the spectrum of a separably closed field.

We prove the first assertion by induction on  $k$  when  $Y$  is a  $k$ -Artin stack. Take an object  $K \in \mathcal{D}^{\leq 0}(Y, \lambda)$ . For  $k = -2$ ,  $Y$  is the coproduct of a family  $(Y_i)_{i \in I}$  of morphisms of schemes separated and of finite type over  $X$ , so that

$$f_! K = \bigoplus_{i \in I} f_{i!}(K|Y_i) \in \mathcal{D}^{\leq 2d}(X, \lambda),$$

where  $f_i$  is the composite morphism  $Y_i \rightarrow Y \xrightarrow{f} X$ . Assume the assertion proved for some  $k \geq -2$ , and let  $Y$  be a  $(k+1)$ -Artin stack. Let  $Y_\bullet$  be a Čech nerve of an atlas (resp. étale atlas)  $y_0: Y_0 \rightarrow Y$  and form a triangle

$$\begin{array}{ccc} & Y & \\ y_\bullet \nearrow & & \searrow f \\ Y_\bullet & \xrightarrow{f_\bullet} & X. \end{array}$$

Then, by Theorem 6.2.13 (2), we have  $f_! K \simeq \varinjlim_{n \in \Delta^{op}} f_{n!} y_n^! K$ . Thus it suffices to show that for every smooth (resp. étale) morphism  $g: Z \rightarrow X$  where  $Z$  is a  $k$ -Artin stack,  $(f \circ g)_! g^! K$  belongs to  $\mathcal{D}^{\leq 2d}(X, \lambda)$ . For this, we may assume that  $g$  is of pure dimension  $e$  (resp. 0). The assertion then follows from Theorem 6.2.9 and induction hypothesis.

For the second assertion, we may assume that  $f$  is of pure dimension  $d$  (resp. 0). It then follows from Theorem 6.2.9 (2) and the first assertion.  $\square$

*Remark 6.2.16.* Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a smooth morphism of (1-)Artin stacks, and  $\pi: \Lambda' \rightarrow \Lambda$  a ring homomorphism. Standard functors for the lisse-étale topoi induce

$$\begin{aligned} Lf_{\text{lisse-ét}}^* &: D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{Y}_{\text{lisse-ét}}, \Lambda), \\ - \overset{L}{\otimes}_{\mathcal{X}} - &: D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda) \times D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda), \\ L\pi^* &: D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda'). \end{aligned}$$

By Corollary 5.3.8, we have an equivalence of categories

$$(6.10) \quad h\mathcal{D}(\mathcal{X}, \Lambda) \simeq D_{\text{cart}}(\mathcal{X}_{\text{lisse-ét}}, \Lambda),$$

and isomorphisms of functors

$$hf^* \simeq Lf_{\text{lisse-ét}}^*, \quad h(- \otimes_{\mathcal{X}} -) \simeq (- \overset{L}{\otimes}_{\mathcal{X}} -), \quad h\pi^* \simeq L\pi^*,$$

compatible with (6.10).

Let  $f: \mathcal{Y} \rightarrow \mathcal{X}$  be a morphism of Artin stacks. Using the methods of [Ols07, (9.16.2)], one can define a functor

$$L^+ f^*: D_{\text{cart}}^+(\mathcal{X}_{\text{lisse-ét}}, \Lambda) \rightarrow D_{\text{cart}}^+(\mathcal{Y}_{\text{lisse-ét}}, \Lambda).$$

Similarly to Proposition 6.5.2 in §6.5, there is an isomorphism between  $hf^{*+} \simeq L^+ f_{\text{lisse-ét}}^*$ , compatible with (6.10), where  $f^{*+}$  denotes the obvious restriction of  $f^*$ .

Assume that there exists a nonempty set  $\square$  of rational primes such that  $\Lambda$  is  $\square$ -torsion and  $\mathcal{X}$  is  $\square$ -coprime. Then the functors  $R^+f_{\text{lis-ét}*}$  and  $R\mathcal{H}\text{om}_{\mathcal{X}}$  for the lisse-étale topoi induce

$$\begin{aligned} R^+f_{\text{lis-ét}*} &: D_{\text{cart}}^+(\mathcal{Y}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}^+(\mathcal{X}_{\text{lis-ét}}, \Lambda), \\ R\mathcal{H}\text{om}_{\mathcal{X}} &: D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda)^{op} \times D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda) \rightarrow D_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}, \Lambda). \end{aligned}$$

Indeed, the statement for  $Rf_{\text{lis-ét}*}$ , similar to [Ols07, 9.9], follows from smooth base change; and the statement for  $R\mathcal{H}\text{om}_{\mathcal{X}}$ , similar to [LO08, 4.2.2], follows from the fact that the map  $g^*R\mathcal{H}\text{om}_X(-, -) \rightarrow R\mathcal{H}\text{om}_Y(g^*- , g^*-)$  is an equivalence for every smooth morphism  $g: Y \rightarrow X$  of  $\square$ -coprime schemes, which in turn follows from the Poincaré duality. By adjunction, we obtain isomorphisms of functors  $\mathcal{H}\text{om}_{\mathcal{X}} \simeq R\mathcal{H}\text{om}_{\mathcal{X}}$  and  $\mathcal{H}f_*^+ \simeq R^+f_{\text{lis-ét}*}$ , compatible with (6.10).

**6.3. More adjointness in the finite-dimensional Noetherian case.** Recall the following result of Gabber: for every morphism  $f: Y \rightarrow X$  of finite type between finite-dimensional Noetherian schemes, and every prime number  $\ell$  invertible on  $X$ , the  $\ell$ -cohomological dimension of  $f_*$  is finite [TGxviii, 1.4]. In particular,  $f_*: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(X, \lambda)$  preserves small colimits and thus admits a right adjoint.

We say that a higher Artin stack  $X$  is locally Noetherian (resp. locally finite-dimensional) if  $X$  admitting an atlas  $Y \rightarrow X$  where  $Y$  is a coproduct of Noetherian (resp. finite-dimensional) schemes.

**Proposition 6.3.1.** *Let  $f: Y \rightarrow X$  be a morphism locally of finite type of  $\text{Chp}_{\square}^{\text{Ar}}$ , and  $\pi: \lambda' \rightarrow \lambda$  an arbitrary morphism of  $\text{Rind}_{\square\text{-tor}}$ . Assume that  $X$  is locally Noetherian and locally finite-dimensional. Then  $f^!: \mathcal{D}(X, \lambda) \rightarrow \mathcal{D}(Y, \lambda)$  admits a right adjoint; the squares (6.7) and (6.8) are right adjointable. Moreover, if  $f$  is 0-Artin, quasi-compact and quasi-separated, then  $f_*: \mathcal{D}(Y, \lambda) \rightarrow \mathcal{D}(X, \lambda)$  also admits a right adjoint.*

*Proof.* Let  $g: \coprod Z_i = Z \rightarrow Y$  be an atlas of  $Y$ . By the Poincaré duality,  $g^!$  is conservative, and  $h_i^!$  exhibits  $\mathcal{D}(Z, \lambda)$  as the product of  $\mathcal{D}(Z_i, \lambda)$ , where  $h_i: Z_i \rightarrow Z$ . Therefore, to show that  $f^!$  preserves small colimits, it suffices to show that, for every  $i$ ,  $(f \circ g_i)^!$  preserves small colimits, where  $g_i: Z_i \rightarrow Y$ . We may thus assume that  $X$  and  $Y$  are both affine schemes. Let  $i$  be a closed embedding of  $Y$  into an affine space over  $X$ . It then suffices to show that  $i^!$  preserves small colimits, which follows from the finiteness of cohomological dimension of  $j_*$ , where  $j$  is the complementary open immersion.

To show that (6.7) and (6.8) are right adjointable, we reduce by Lemma 4.3.7 to the case of affine schemes. By the factorization above and the Poincaré duality, the assertion for  $f^!$  reduces to the assertion for  $f_*$ . We may further assume that  $\Xi' = \Xi = \{*\}$  where  $\lambda = (\Xi, \Lambda)$  and  $\lambda' = (\Xi', \Lambda')$ . In this case, it suffices to take a resolution of  $\Lambda'$  by free  $\Lambda$ -modules.

For the second assertion, by smooth base change, we may assume that  $X$  is an affine Noetherian scheme. By alternating Čech resolution, we may assume that  $Y$  is a scheme. The assertion in this case has been recalled above.  $\square$

**6.4. Constructible complexes.** We study constructible complexes on higher Artin stacks and their behavior under the six operations. Let  $\lambda = (\Xi, \Lambda)$  be a Noetherian

ringed diagram. For every object  $\xi$  of  $\Xi$ , we denote by  $e_\xi$  the morphism  $(\{\xi\}, \Lambda(\xi)) \rightarrow (\Xi, \Lambda)$ .

We start from the case of schemes. Let  $X$  be a scheme. Recall from [SGA4, IX 2.3] that for a Noetherian ring  $R$ , a sheaf  $\mathcal{F}$  of  $R$ -modules on  $X$  is said to be *constructible* if the stalks of  $\mathcal{F}$  are finitely-generated  $R$ -modules and every affine open subset of  $X$  is the disjoint union of finitely many constructible subschemes  $U_i$  such that the restriction of  $\mathcal{F}$  to each  $U_i$  is locally constant.

**Definition 6.4.1.** We say that an object  $K$  of  $\mathcal{D}(X, \lambda)$  is a *constructible complex* or simply *constructible* if for every object  $\xi$  of  $\Xi$  and every  $q \in \mathbb{Z}$ , the sheaf  $H^q e_\xi^* K \in \text{Mod}(X, \Lambda(\xi))$  is constructible. We say that an object  $K$  of  $\mathcal{D}(X, \lambda)$  is *locally bounded from below* (resp. *locally bounded from above*) if for every object  $\xi$  of  $\Xi$  and every quasi-compact open subscheme  $U$  of  $X$ ,  $e_\xi^* K|_U$  is bounded from below (resp. bounded from above).

Note that we do not require constructible complexes to be bounded in either direction. Note that  $K \in \mathcal{D}(X, \lambda)$  is locally bounded from below (resp. from above) if and only if there exists a Zariski open covering  $(U_i)_{i \in I}$  of  $X$  such that  $K|_{U_i}$  is bounded from below (resp. from above).

**Lemma 6.4.2.** *Let  $f: Y \rightarrow X$  be a morphism of schemes. Let  $K$  be an object of  $\mathcal{D}(X, \lambda)$ . If  $K$  is constructible (resp. locally bounded from below, resp. locally bounded from above), then  $f^* K$  satisfies the same property. The converse holds when  $f$  is surjective and locally of finite presentation.*

*Proof.* The constructible case follows from [SGA4, IX 2.4 (iii), 2.8]. For the locally bounded case we use the characterization by open coverings. The first assertion is then clear. For the second assertion, by [SGA4, IX 2.8.1] we may assume  $f$  flat, hence open. In this case the image of an open covering of  $Y$  is an open covering of  $X$ .  $\square$

The lemma implies that Definition 6.4.1 is compatible with the following.

**Definition 6.4.3** (Constructible complex). Let  $X$  be a higher Artin stack. We say that an object  $K$  of  $\mathcal{D}(X, \lambda)$  is a *constructible complex* or simply *constructible* (resp. *locally bounded from below*, resp. *locally bounded from above*) if there exists an atlas  $f: Y \rightarrow X$  with  $Y$  a scheme,  $f^* K$  is constructible (resp. locally bounded from below, resp. locally bounded from above).

We denote by  $\mathcal{D}_{\text{cons}}(X, \lambda)$  (resp.  $\mathcal{D}^{(+)}(X, \lambda)$ ,  $\mathcal{D}^{(-)}(X, \lambda)$  or  $\mathcal{D}^{(b)}(X, \lambda)$ ) the full subcategory of  $\mathcal{D}(X, \lambda)$  spanned by objects that are constructible (resp. locally bounded from below, locally bounded from above, or locally bounded from both sides). Moreover, we put

$$\begin{aligned} \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda) &= \mathcal{D}_{\text{cons}}(X, \lambda) \cap \mathcal{D}^{(+)}(X, \lambda); \\ \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda) &= \mathcal{D}_{\text{cons}}(X, \lambda) \cap \mathcal{D}^{(-)}(X, \lambda); \\ \mathcal{D}_{\text{cons}}^{(b)}(X, \lambda) &= \mathcal{D}_{\text{cons}}(X, \lambda) \cap \mathcal{D}^{(b)}(X, \lambda). \end{aligned}$$

**Proposition 6.4.4.** *Let  $f: Y \rightarrow X$  be a morphism of higher Artin stacks.*

- (1) *Let  $K$  be an object of  $\mathcal{D}(X, \lambda)$ . If  $K$  is constructible (resp. locally bounded from below, resp. locally bounded from above), then  $f^*K$  satisfies the same property. The converse holds when  $f$  is surjective and locally of finite presentation. In particular,  $f^*$  restricts to a functor*  
 $\mathbf{1L'}: f^*: \mathcal{D}_{\text{cons}}(X, \lambda) \rightarrow \mathcal{D}_{\text{cons}}(Y, \lambda).$
- (2) *Suppose that  $X$  and  $Y$  are  $\square$ -coprime higher Artin stacks (resp. higher DM stacks), and  $f$  is of finite presentation (Definition 5.4.3). Let  $\lambda$  be a  $\square$ -torsion (resp. torsion) Noetherian ringed diagram. Then  $f_!$  restricts to*  
 $\mathbf{2L'}: f_!: \mathcal{D}_{\text{cons}}^{(-)}(Y, \lambda) \rightarrow \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda),$  and if  $f$  is 0-Artin (resp. 0-DM),  
 $f_!: \mathcal{D}_{\text{cons}}(Y, \lambda) \rightarrow \mathcal{D}_{\text{cons}}(X, \lambda).$
- (3) *The functor  $- \otimes_X -$  restricts to a functor*  
 $\mathbf{3L'}: - \otimes_X -: \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda) \times \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda) \rightarrow \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda).$   
*In particular,  $\mathcal{D}_{\text{cons}}^{(-)}(X, \lambda)^{\otimes}$  is a symmetric monoidal subcategory [HTT, 2.2.1].*

*Proof.* For (1), we reduce by taking atlases to the case of schemes, which is Lemma 6.4.2. The reduction for the second assertion is clear. The reduction for the first assertion uses the second assertion.

For (2), we may assume  $\Xi = \{*\}$ . We prove by induction on  $k$  that the assertion holds when  $f$  is a morphism of  $k$ -Artin (resp.  $k$ -DM) stacks. The case  $k = -2$  is [SGA4, XVII 5.3.6]. Now assume that the assertions hold for some  $k \geq -2$  and let  $f$  be a morphism of  $(k+1)$ -Artin (resp.  $(k+1)$ -DM) stacks. By smooth base change (Corollary 6.2.10), we may assume that  $X$  is an affine scheme. Then  $Y$  is a  $(k+1)$ -Artin (resp.  $(k+1)$ -DM) stack, of finite presentation over  $X$ . It suffices to show that for every object  $K$  of  $\mathcal{D}_{\text{cons}}^{\leq 0}(Y, \lambda)$ ,  $f_!K$  belongs to  $\mathcal{D}_{\text{cons}}^{\leq 2d}(Y, \lambda)$ , where  $d = \dim^+(f)$ . Let  $Y_{\bullet}$  be a Čech nerve of an atlas  $y_0: Y_0 \rightarrow Y$ , where  $Y_0$  is an affine scheme, and form a triangle

$$\begin{array}{ccc} & Y & \\ y_{\bullet} \nearrow & & \searrow f \\ Y_{\bullet} & \xrightarrow{f_{\bullet}} & X. \end{array}$$

Then for  $n \geq 0$ ,  $f_n$  is a quasi-compact and quasi-separated morphism of  $k$ -Artin (resp.  $k$ -DM) stacks. By Theorem 6.2.13 and the dual version of [HA, 1.2.4.9], we have a convergent spectral sequence

$$E_1^{p,q} = H^q(f_{-p!} y_{-p}^! K) \Rightarrow H^{p+q} f_! K.$$

By induction hypothesis and the Poincaré duality (Theorem 6.2.9 (2)),  $E_1^{p,q}$  is constructible for all  $p$  and  $q$ . Moreover,  $E_1^{p,q}$  vanishes for  $p > 0$  or  $q > 2d$  by Lemma 6.2.15. Therefore,  $f_!K$  belongs to  $\mathcal{D}_{\text{cons}}^{\leq 2d}(X, \lambda)$ .

For (3), we may assume  $X$  is an affine scheme. The assertion is then trivial.  $\square$

To state the results for the other operations, we work in a relative setting. Let  $\mathbb{S}$  be a  $\square$ -coprime higher Artin stack. Assume that there exists an atlas  $S \rightarrow \mathbb{S}$ , where  $S$  is

a coproduct of Noetherian quasi-excellent<sup>10</sup> schemes and regular schemes of dimension  $\leq 1$ . We denote by  $\mathbf{Chp}_{\text{lft}/\mathbb{S}}^{\text{Ar}} \subseteq \mathbf{Chp}_{/\mathbb{S}}^{\text{Ar}}$  the full subcategory spanned by morphisms  $X \rightarrow \mathbb{S}$  locally of finite type.

**Proposition 6.4.5.** *Let  $f: Y \rightarrow X$  be a morphism of  $\mathbf{Chp}_{\text{lft}/\mathbb{S}}^{\text{Ar}}$ , and  $\lambda$  a  $\square$ -torsion Noetherian ringed diagram. Then the operations introduced in §6.2 restrict to the following*

- 1R'**:  $f_*: \mathcal{D}_{\text{cons}}^{(+)}(Y, \lambda) \rightarrow \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda)$  if  $f$  is quasi-compact and quasi-separated, and  $f_*: \mathcal{D}_{\text{cons}}(Y, \lambda) \rightarrow \mathcal{D}_{\text{cons}}(X, \lambda)$  if  $\mathbb{S}$  is locally finite-dimensional and  $f$  is quasi-compact and quasi-separated and 0-Artin;
- 2R'**:  $f^!: \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda) \rightarrow \mathcal{D}_{\text{cons}}^{(+)}(Y, \lambda)$ , and, if  $\mathbb{S}$  is locally finite-dimensional,  $f^!: \mathcal{D}_{\text{cons}}(X, \lambda) \rightarrow \mathcal{D}_{\text{cons}}(Y, \lambda)$ ;
- 3R'**:  $\mathcal{H}\text{om}_X(-, -): \mathcal{D}_{\text{cons}}^{(-)}(X, \lambda)^{\text{op}} \times \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda) \rightarrow \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda)$  if  $\Xi_{/\xi}$  is finite for all  $\xi \in \Xi$ .

*Proof.* Suppose  $\lambda = (\Xi, \Lambda)$ . We first reduce to the case  $\Xi = \{*\}$ . The reduction follows from Propositions 6.2.6 and 6.2.7 for (1R') and (2R'). For (3R'), by Proposition 6.2.8 (2) and the assumption on  $\Xi_{/\xi}$ , we may assume  $\Xi$  finite. In this case, by Proposition 6.2.5 (2), it suffices to prove that every  $K \in \mathcal{D}_{\text{cons}}^{(+)}(X, \lambda)$  is a successive extension of  $e_{\xi*}L_{\xi}$ , where  $L_{\xi} \in \mathcal{D}_{\text{cons}}^{(+)}(X_{\xi}, \Lambda(\xi))$  for every object  $\xi \in \Xi$ . This being trivial for  $\Xi = \emptyset$ , we proceed by induction on the cardinality of  $\Xi$ . Let  $\Xi' \subseteq \Xi$  be the partially ordered subset spanned by the minimal elements of  $\Xi$ , and let  $\Xi''$  be the complement of  $\Xi'$ . Then we have a fibre sequence  $i_*L \rightarrow K \rightarrow \prod_{\xi \in \Xi'} e_{\xi*}e_{\xi}^*K$ , where  $i: (\Xi'', \Lambda | \Xi'') \rightarrow \lambda$  and  $L \in \mathcal{D}_{\text{cons}}^{(+)}(\Xi'', \Lambda | \Xi'')$ . Since  $\Xi'$  is nonempty, it then suffices to apply the induction hypothesis to  $L$ .

We then prove by induction on  $k$  that the assertions for  $\Xi = \{*\}$  hold when  $f$  is a morphism of  $k$ -Artin stacks. The case  $k = -2$  is due to Deligne [SGA4d, Th. Finitude 1.5, 1.6] if  $S$  is regular of dimension  $\leq 1$  and to Gabber [TGxiii] if  $S$  is quasi-excellent. In fact, in the latter case, by arguments similar to [SGA4d, Th. Finitude 2.2], we may assume  $\lambda = (*, \mathbb{Z}/n\mathbb{Z})$ . In the finite-dimensional case we also need the finiteness of cohomological dimension recalled at the beginning of §6.3. Now assume that the assertions hold for some  $k \geq -2$  and let  $f$  be a morphism of  $(k+1)$ -Artin stacks. Then (2R') follows from induction hypothesis, Theorem 6.2.9 (2) and (1L'); (3R') follows from induction hypothesis, Proposition 6.2.4 (3), Theorem 6.2.9 (2) and (1L'), (2R'). The proof of (1R') is similar to the proof of Proposition 6.4.4. Indeed, to show that for every object  $K$  of  $\mathcal{D}_{\text{cons}}^{\geq 0}(Y, \lambda)$ ,  $f_*K$  belongs to  $\mathcal{D}_{\text{cons}}^{\geq 0}(X, \lambda)$ , it suffices to apply the convergent spectral sequence

$$E_1^{p,q} = H^q(f_{p*}y_p^*K) \Rightarrow H^{p+q}f_*K$$

and induction hypothesis. □

<sup>10</sup>Recall from [TGi, 2.10] that a ring is *quasi-excellent* if it is Noetherian and satisfies conditions (2), (3) of [EGAIV, 7.8.2]. A Noetherian scheme is *quasi-excellent* if it admits a Zariski open cover by spectra of quasi-excellent rings.



**6.5. Compatibility with the work of Laszlo and Olsson.** In this section we establish the compatibility between our theory and the work of Laszlo and Olsson [LO08], under the (more restrictive) assumptions of the latter.

We fix  $\square = \{\ell\}$  and a Gorenstein local ring  $\Lambda$  of dimension 0 and residual characteristic  $\ell$ . We will suppress  $\Lambda$  from the notation when no confusion arises. Let  $\mathbb{S}$  be a  $\square$ -coprime scheme, endowed with a global dimension function, satisfying the following conditions.

- (1)  $\mathbb{S}$  is affine excellent and finite-dimensional;
- (2) For every  $\mathbb{S}$ -scheme  $X$  of finite type, there exists an étale cover  $X' \rightarrow X$  such that, for every scheme  $Y$  étale and of finite type over  $X'$ ,  $\text{cd}_\ell(Y) < \infty$ ;

*Remark 6.5.1.* In [LO08], the authors did not explicitly include the existence of a global dimension function in their assumptions. However, their method relies on pinned dualizing complexes (see below), which makes use of the dimension function. Note that assumption (2) above is slightly weaker than the assumption on cohomological dimension in [LO08]; for example, (2) allows the case  $\mathbb{S} = \text{Spec } \mathbb{R}$  and  $\ell = 2$  while the assumption in [LO08] does not. Nevertheless, assumption (2) implies that the right derived functor of the countable product functor on  $\text{Mod}(X_{\text{ét}}, \Lambda)$  has finite cohomological dimension, which is in fact sufficient for the construction in [LO08].

Let  $\text{Chp}_{\text{ift}/\mathbb{S}}^{\text{LMB}}$  be the full subcategory of  $\text{Chp}_{\text{ift}/\mathbb{S}}^{\text{Ar}}$  spanned by (1-)Artin stacks locally of finite type over  $\mathbb{S}$ , with quasi-compact and separated diagonal. Stacks with such diagonal are called algebraic stacks in [LMB00] and [LO08]. We adopt the notation  $\text{D}_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}) \subseteq \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}})$  from §0.1. For a morphism  $f: \mathcal{Y} \rightarrow \mathcal{X}$  of finite type (of  $\text{Chp}_{\text{ift}/\mathbb{S}}^{\text{LMB}}$ ), Laszlo–Olsson defined functors

$$\begin{aligned} \text{R}f_* &: \text{D}_{\text{cons}}^{(+)}(\mathcal{Y}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}), \\ \text{R}f_! &: \text{D}_{\text{cons}}^{(-)}(\mathcal{Y}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}), \\ \text{L}f^* &: \text{D}_{\text{cons}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cons}}(\mathcal{Y}_{\text{lis-ét}}), \\ \text{R}f^! &: \text{D}_{\text{cons}}(\mathcal{X}) \rightarrow \text{D}_{\text{cons}}(\mathcal{Y}_{\text{lis-ét}}), \\ \text{R}\mathcal{H}\text{om}_{\mathcal{X}} &: \text{D}_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}})^{\text{op}} \times \text{D}_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cons}}^{(+)}(\mathcal{X}_{\text{lis-ét}}), \\ - \overset{\text{L}}{\otimes}_{\mathcal{X}} - &: \text{D}_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}) \times \text{D}_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cons}}^{(-)}(\mathcal{X}_{\text{lis-ét}}). \end{aligned}$$

Three of the six functors,  $\text{R}f_*$ ,  $\text{R}\mathcal{H}\text{om}_{\mathcal{X}}$ , and  $- \overset{\text{L}}{\otimes}_{\mathcal{X}} -$ , are standard functors for the lisse-étale topoi and can be extended to  $\text{D}_{\text{cart}}$  (see Remarks 6.2.16 and 5.3.10):

$$\begin{aligned} \text{R}f_* &: \text{D}_{\text{cart}}^{(+)}(\mathcal{Y}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cart}}^{(+)}(\mathcal{X}_{\text{lis-ét}}), \\ \text{R}\mathcal{H}\text{om}_{\mathcal{X}} &: \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}})^{\text{op}} \times \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}), \\ - \overset{\text{L}}{\otimes}_{\mathcal{X}} - &: \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) \times \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}). \end{aligned}$$

Moreover, the construction of  $\text{L}f^*$  in [LO08, 4.3] can also be extended to  $\text{D}_{\text{cart}}$ :

$$\text{L}f^*: \text{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \text{D}_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}).$$



In fact, it suffices to apply [LO08, 2.2.3] to  $\mathcal{D}_{\text{cart}}$ . The six operations satisfy all the usual adjointness properties (cf. [LO08, 4.3.1, 4.4.2]). On the other hand, restricting our constructions in the two previous sections, we have

$$\begin{aligned} f_* &: \mathcal{D}^{(+)}(\mathcal{Y}) \rightarrow \mathcal{D}^{(+)}(\mathcal{X}), \\ f_! &: \mathcal{D}_{\text{cons}}^{(-)}(\mathcal{Y}) \rightarrow \mathcal{D}_{\text{cons}}^{(-)}(\mathcal{X}), \\ f^* &: \mathcal{D}(\mathcal{X}) \rightarrow \mathcal{D}(\mathcal{Y}), \\ f^! &: \mathcal{D}_{\text{cons}}(\mathcal{X}) \rightarrow \mathcal{D}_{\text{cons}}(\mathcal{Y}), \\ \mathcal{H}\text{om}_{\mathcal{X}} &: \mathcal{D}(\mathcal{X})^{op} \times \mathcal{D}(\mathcal{X}) \rightarrow \mathcal{D}(\mathcal{X}), \\ - \otimes_{\mathcal{X}} - &: \mathcal{D}(\mathcal{X}) \times \mathcal{D}(\mathcal{X}) \rightarrow \mathcal{D}(\mathcal{X}). \end{aligned}$$

The equivalence of categories  $\mathbf{h}\mathcal{D}(\mathcal{X}) \simeq \mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}})$  (6.10) restricts to an equivalence  $\mathbf{h}\mathcal{D}_{\text{cons}}(\mathcal{X}) \simeq \mathcal{D}_{\text{cons}}(\mathcal{X}_{\text{lis-ét}})$ . The main result of this section is the following.

**Proposition 6.5.2.** *We have equivalences of functors*

$$\begin{aligned} \mathbf{h}f_* &\simeq \mathbf{R}f_*, \quad \mathbf{h}f_! \simeq \mathbf{R}f_!, \quad \mathbf{h}f^* \simeq \mathbf{L}f^*, \quad \mathbf{h}f^! \simeq \mathbf{R}f^!, \\ \mathbf{h}\mathcal{H}\text{om}_{\mathcal{X}} &\simeq \mathbf{R}\mathcal{H}\text{om}_{\mathcal{X}}, \quad \mathbf{h}(- \otimes_{\mathcal{X}} -) \simeq (- \overset{\mathbf{L}}{\otimes}_{\mathcal{X}} -), \end{aligned}$$

compatible with (6.10).

*Proof.* The assertions for  $- \otimes_{\mathcal{X}} -$  and  $\mathcal{H}\text{om}_{\mathcal{X}}$  are special cases of Remark 6.2.16. Moreover, by adjunction, the assertion for  $f_*$  (resp.  $f_!$ ) will follow from the one for  $f^*$  (resp.  $f^!$ ).

Let us first prove that  $\mathbf{h}f^* \simeq \mathbf{L}f^*: \mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) \rightarrow \mathcal{D}_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}})$ . We choose a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{X} \end{array}$$

where the vertical morphisms are atlases. It induces a 2-commutative diagram

$$\begin{array}{ccc} Y_{\bullet} & \xrightarrow{f_{\bullet}} & X_{\bullet} \\ \eta_Y \downarrow & & \downarrow \eta_X \\ \mathcal{Y} & \xrightarrow{f} & \mathcal{X}. \end{array}$$

Using arguments similar to §5.4, we get the following diagram

$$\begin{array}{ccccc}
 \mathcal{D}_{\text{cart}}(\text{Mod}(Y_{\bullet, \text{ét}})) & \xleftarrow{f_{\bullet, \text{ét}}^*} & \mathcal{D}_{\text{cart}}(\text{Mod}(X_{\bullet, \text{ét}})) & & \\
 \uparrow \eta_{Y, \text{cart}}^* & \searrow & \uparrow & \searrow & \\
 \varprojlim_{n \in \Delta} \mathcal{D}(Y_{n, \text{ét}}) & \xleftarrow{\varprojlim_{n \in \Delta} f_{n, \text{ét}}^*} & \varprojlim_{n \in \Delta} \mathcal{D}(X_{n, \text{ét}}) & & \\
 \nearrow \sim & & \nearrow \sim & & \\
 \mathcal{D}_{\text{cart}}(\mathcal{Y}_{\text{lis-ét}}) & \xleftarrow{f^*} & \mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}}) & & 
 \end{array}$$

By [LO08, 2.2.3],  $\eta_{X, \text{cart}}^*$  and  $\eta_{Y, \text{cart}}^*$  are equivalences. By the construction of  $Lf^*$ ,  $Lf^*$  fits into a homotopy version of the rectangle in the above diagram. Therefore, we have an equivalence  $\text{hf}^* \simeq Lf^*$ .

Let  $\Omega_{\mathbb{S}} \in \mathcal{D}(\mathbb{S})$  be a potential dualizing complex (with respect to the fixed dimension function) in the sense of [TGxvii, 2.1.2], which is unique up to isomorphism by [TGxvii, 5.1.1] (see Remark 6.5.3). For every object  $\mathcal{X}$  of  $\text{Chp}_{\text{ift}/\mathbb{S}}^{\text{LMB}}$ , with structure morphism  $a: \mathcal{X} \rightarrow \mathbb{S}$ , we put  $\Omega_{\mathcal{X}} := a^! \Omega_{\mathbb{S}}$ . Let  $u: U \rightarrow \mathcal{X}$  be an object of  $\text{Lis-ét}(\mathcal{X})$ . Then  $u^* \Omega_{\mathcal{X}} \simeq \Omega_U \langle -d \rangle$  by the Poincaré duality (Theorem 6.2.9 (2)), where  $d = \dim u$ . Consider the morphism of topoi  $(\epsilon_*, \epsilon^*): (\mathcal{X}_{\text{lis-ét}})_{/\tilde{U}} \rightarrow U_{\text{ét}}$ . Applying Lemma 5.3.2, we get an equivalence  $\Omega_{\mathcal{X}} | (\mathcal{X}_{\text{lis-ét}})_{/\tilde{U}} \simeq \epsilon^* \Omega_U \langle -d \rangle$ , where we regard  $\Omega_{\mathcal{X}}$  as an object of  $\mathcal{D}_{\text{cart}}(\mathcal{X}_{\text{lis-ét}})$  and  $\Omega_U$  as an object of  $\mathcal{D}(U_{\text{ét}})$ . The equivalence is compatible with restriction by morphisms of  $\text{Lis-ét}(\mathcal{X})$ , so that  $\Omega_{\mathcal{X}}$  is a dualizing complex of  $\mathcal{X}$  in the sense of [LO08, 3.4.5], which is unique up to isomorphism by [LO08, 3.4.3, 3.4.4]. Put  $\mathcal{D}_{\mathcal{X}} = \mathcal{H}\text{om}_{\mathcal{X}}(-, \Omega_{\mathcal{X}})$  and  $\text{D}_{\mathcal{X}} = \text{R}\mathcal{H}\text{om}_{\mathcal{X}}(-, \Omega_{\mathcal{X}}) \simeq \text{h}\mathcal{D}_{\mathcal{X}}$ . By [LO08, 3.5.7], the biduality functor  $\text{id} \rightarrow \text{D}_{\mathcal{X}} \circ \text{D}_{\mathcal{X}}$  is a natural isomorphism of endofunctors of  $\text{D}_{\text{cons}}(\mathcal{X}_{\text{lis-ét}})$ . Therefore, the natural transformation  $\text{hf}^! \rightarrow \text{hf}^! \circ \text{D}_{\mathcal{X}} \circ \text{D}_{\mathcal{X}}$  is a natural equivalence when restricted to  $\text{D}_{\text{cons}}(\mathcal{X}_{\text{lis-ét}})$ . By Proposition 6.2.4 (3), we have

$$\begin{aligned}
 f^! \circ \text{D}_{\mathcal{X}} \circ \text{D}_{\mathcal{X}} &\simeq f^! \mathcal{H}\text{om}_{\mathcal{X}}(\text{D}_{\mathcal{X}} -, \Omega_{\mathcal{X}}) \simeq \mathcal{H}\text{om}_{\mathcal{Y}}(f^* \text{D}_{\mathcal{X}} -, f^! \Omega_{\mathcal{X}}) \\
 &\simeq \mathcal{H}\text{om}_{\mathcal{Y}}(f^* \text{D}_{\mathcal{X}} -, \Omega_{\mathcal{Y}}) = \text{D}_{\mathcal{Y}} \circ f^* \circ \text{D}_{\mathcal{X}}.
 \end{aligned}$$

Since  $\text{hf}^* \simeq Lf^*$ , this shows

$$\text{hf}^! \simeq \text{D}_{\mathcal{Y}} \circ Lf^* \circ \text{D}_{\mathcal{X}} = \text{R}f^!,$$

where the last identity is the definition of  $\text{R}f^!$  in [LO08, 4.4.1].  $\square$

*Remark 6.5.3.* As Joël Riou observed (private communication), although the definition, existence and uniqueness of potential dualizing complexes are only stated for the coefficient ring  $R = \mathbb{Z}/n\mathbb{Z}$  in [TGxvii, 2.1.2, 5.1.1], they can be extended to any Noetherian ring  $R'$  over  $R$ . In fact, if  $\delta$  is a dimension function of an excellent  $\mathbb{Z}[1/n]$ -scheme  $X$  and  $K_R$  is a potential dualizing complex for  $(X, \delta)$  relative to  $R$ , then  $K_{R'} = K_R \overset{\text{L}}{\otimes}_R R'$  is a potential dualizing complex for  $(X, \delta)$  relative to  $R'$  by the projection formula  $\text{R}\Gamma_x(K_R) \overset{\text{L}}{\otimes}_R R' \simeq \text{R}\Gamma_x(K_R \overset{\text{L}}{\otimes}_R R')$ , where  $x$  is a geometric point of  $X$ . The formula

follows from the fact that the punctured strict localization of  $X$  at  $x$  has finite cohomological dimension [TGxviii, 1.4]. Moreover, by the theorem of local biduality [TGxvii, 6.1.1, 7.1.2],  $K_{R'}$  is a dualizing complex for  $D_{\text{cons}}^b(X_{\text{ét}}, R')$  in the sense of [TGxvii, 7.1.1] as long as  $R'$  is Gorenstein of dimension 0.

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