A SPECTRUM-LEVEL SPLITTING OF THE $ku_{\mathbb{R}}$-COOPERATIONS ALGEBRA

GUCHUAN LI, SARAH PETERSEN, ELIZABETH TATUM

Abstract. In the 1980's, Mahowald and Kane used integral Brown–Gitler spectra to decompose $ku \wedge ku$ as a sum of finitely generated ku-module spectra. This splitting, along with an analogous decomposition of $ko \wedge ko$ led to a great deal of progress in stable homotopy computations and understanding of v_1-periodicity in the stable homotopy groups of spheres. In this paper, we construct a C_2-equivariant lift of Mahowald and Kane's splitting of $ku \wedge ku$. We also give a description of the resulting C_2-equivariant splitting in terms of C_2-equivariant Adams covers and record an analogous splitting for $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$. Similarly to the nonequivariant story, we expect the techniques of this paper to facilitate further C_2-equivariant stable homotopy computations and understanding of v_1-periodicity in C_2-equivariant stable stems.

Contents

1. Introduction 1
2. Underlying nonequivariant splittings 7
3. Equivariant preliminaries 11
4. C_2-equivariant homology results 14
5. Splitting $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ 20
6. Splitting $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ 25
References 39

1. Introduction

1.1. **Motivation.** In the 1970's Brown–Gitler and Cohen constructed Brown–Gitler spectra, a family of spectra realizing subcomodules of the dual Steenrod algebra $\mathcal{A}_*$ [BG73, Coh79]. Mahowald subsequently observed these spectra could be used to decompose the cooperations algebra for integral homology, $H\mathbb{Z} \wedge H\mathbb{Z}$, as a sum of finite $H\mathbb{Z}$-modules. Mahowald and Kane later used integral Brown–Gitler spectra, a family of spectra realizing subcomodules of $H_*H\mathbb{Z}$, to decompose $ku \wedge ku$ and $ko \wedge ko$ in an analogous way [Mah81, Kan81].

These splittings of $ku \wedge ku$ and $ko \wedge ko$ proved particularly useful in ku- or ko-Adams spectral sequences. As ku_*ku or ko_*ko is not flat over ku_* or ko_*, the corresponding E_2-page cannot be computed as an Ext group. One has to start the computation with the E_1-page. However, Mahowald and Kane's decompositions of $ku \wedge ku$ and $ko \wedge ko$ in terms of finitely generated ku- and ko-modules, respectively, made computation of the E_1-pages of these spectral sequences tractable by splitting the computation of the E_1-page into smaller pieces [Mah81, Dav87, DM89, BBB$^+$20]. Further, even the information on the E_2-pages of these spectral sequences provides a wealth of homotopical information. Both the ku- and ko-based Adams spectral sequences have vanishing lines which result in collapse at the E_2-page in a large range.

The ko-Adams spectral sequence in particular is a very effective tool for studying v_1-periodicity in the stable homotopy groups of the sphere, $\pi_* S_{(2)}$. For example, Mahowald famously used the ko-Adams spectral sequence to prove the Telescope Conjecture for height one at the prime 2 [Mah81]. The ko-based Adams spectral sequence is also a highly efficient way to compute $\pi_* S_{(2)}$ through the 40-stem [Dav87, DM89, BBB$^+$20]. Additionally, at odd primes, Gonzalez used the ku-Adams spectral sequence to study v_1-periodicity, as well as to classify stunted lens spaces [Gon00].

Given the range of successful applications of Mahowald and Kane's splittings in nonequivariant homotopy theory, one may ask whether analogous splittings exist in equivariant homotopy theory. In this paper, we answer this question affirmatively by constructing a C_2-equivariant splitting of $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$, in terms of finitely generated $ku_{\mathbb{R}}$-modules. Here, $ku_{\mathbb{R}}$ is the equivariant connective cover of $KU_{\mathbb{R}}$, the spectrum representing Atiyah real K-theory.

There are several reasons for working with the group of equivariance $G = C_2$, the cyclic group of order two. First, Mahowald and Kane's nonequivariant splittings involve (integral) Brown–Gitler spectral which realize subcomodules of the dual Steenrod algebra $\mathcal{A}_*$, or in the case of integral Brown–Gitler spectra, realize subcomodules of $H\mathbb{F}_{2*}H\mathbb{Z}$. Thus it is reasonable to begin with a group of equivariance where the dual Steenrod algebra has been computed and closely resembles the classical dual Steenrod algebra. This is the case when $G = C_2$ [HK01, Theorem 6.41] (see also [LSWX19, Theorem 2.14] for a description written in notation more obviously mirroring the classical dual Steenrod algebra), and in an earlier paper, we constructed C_2-equivariant integral Brown–Gitler spectra $\mathcal{B}_0(k)$ realizing certain subcomodules of $H\underline{\mathbb{F}}_{2\star}H\underline{\mathbb{Z}}$ [LPT23]. In this paper, these spectra play an integral role in our construction of a C_2-equivariant splitting of $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$.

For cyclic groups of prime order $p \neq 2$, the dual Steenrod algebra has been computed and is known not to be flat over the coefficients $H\underline{\mathbb{F}}_{p\star}$ [SW22, HKSZ23]. This suggests the construction of C_p-equivariant (integral) Brown–Gitler spectra, where p is an odd prime, may be more complicated, requiring techniques beyond those developed in [LPT23]. Additionally, odd primary analogues of $BP_{\mathbb{R}}$ are only beginning to be studied and also require the development of further computational techniques [HSW23].

Even in the case where $G = C_2$, constructing a splitting of $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ is significantly more complicated than in the nonequivariant case. This is largely due to the fact that the coefficients $H\underline{\mathbb{F}}_{2\star}$ form a bigraded ring (described in Section 3.1) rather than a single copy of $\mathbb{F}_2$ as in the nonequivariant case. In particular, many of the results in Section 4 deal with this technical difficulty. Further, much of the work in Sections 5 and 6, where we construct the $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ and $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ splittings, lies in keeping careful track of all the bigraded elements.

We now describe our main results. This is followed by a discussion of problems which are newly accessible with the methods developed in this paper.

1.2. **Main results.** The main result of this paper is a lift of Mahowald and Kane's splitting of $ku \wedge ku$ to the C_2-equivariant spectra:

Theorem (Theorem 6.1)**.** *Up to 2-completion, there is a splitting of $ku_{\mathbb{R}}$-modules*

$$ku_{\mathbb{R}} \wedge ku_{\mathbb{R}} \simeq ku_{\mathbb{R}} \wedge \Sigma^{\rho k}\mathcal{B}_0(k).$$

Our proof of this theorem requires the development of a number of C_2-equivariant homology results. Specifically, we give a Whitehead Theorem for Margolis homology in the C_2-equivariant setting (Proposition 4.16), compute the homology of C_2-equivariant mod 2 and integral Brown–Gitler spectra (Propositions 4.19 and 4.21), and record a computation of $H\underline{\mathbb{F}_2}_\star BP_{\mathbb{R}}\langle n\rangle$ which we first learned from Christian Carrick.

Just as in the nonequivariant setting, this splitting allows us to also describe $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ in terms of equivariant Adams covers $ku_{\mathbb{R}}^{\langle \nu_2(n!)\rangle}$ of $ku_{\mathbb{R}}$.

Theorem (Theorem 6.2)**.** *Up to 2-completion,*

$$ku_{\mathbb{R}} \wedge ku_{\mathbb{R}} \simeq \bigvee_{k=0}^{\infty} \Sigma^{2k} ku_{\mathbb{R}}^{\langle \nu_2(n!)\rangle} \vee V,$$

where V is a sum of suspensions of H.

Incidentally to our proof of the main theorem, we describe the cooperations algebra $ku_{\mathbb{R}\star}ku_{\mathbb{R}}$.

Theorem (Theorem 6.3)**.** *The $ku_{\mathbb{R}}$-cooperations algebra $ku_{\mathbb{R}\star}ku_{\mathbb{R}}$ splits as*

$$ku_{\mathbb{R}\star}ku_{\mathbb{R}} \cong \bigoplus_{k=0}^{\infty} ku_{\mathbb{R}\star - k\rho}\mathcal{B}_0(k),$$

where as a $ku_{\mathbb{R}\star}$-module

$$ku_{\mathbb{R}\star}\mathcal{B}_0(k) \cong \bigoplus_{k=0}^{\infty} H\underline{\mathbb{Z}}_\star\{x_0, x_1, \dots, x_{\nu_2(2k!)-1}\} \oplus ku_{\mathbb{R}\star}\{x_{\nu_2(2k!)}\} \oplus V_k,$$

with extensions $v_1 x_{i-1} = \rho x_i$, and where $|x_i| = \rho i$ and V_k is a sum of suspensions of $H_\star$.

We also describe the operations algebra $[ku_{\mathbb{R}}, ku_{\mathbb{R}}]$.

Theorem (Theorem 6.4)**.** *The cooperations algebra $[ku_{\mathbb{R}}, ku_{\mathbb{R}}]$ splits as*

$$[ku_{\mathbb{R}}, ku_{\mathbb{R}}] \cong \bigoplus_{k=0}^{\infty} [\Sigma^{\rho k}\mathcal{B}_0(k), ku_{\mathbb{R}}].$$

The Adams spectral sequence

$$Ext_{\mathcal{E}(1)_\star}(H_\star\mathcal{B}_0(k), H_\star ku_{\mathbb{R}}) \Longrightarrow [\Sigma^{\rho k}\mathcal{B}_0(k), ku_{\mathbb{R}}]$$

collapses at the E_2-page, and its E_2-page is described in Lemmas 6.9 and 6.10.

And, we also record an analogous splitting of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$.

Theorem (Theorem 5.1)**.** *Up to 2-completion there is a splitting*

$$H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{k=0}^{\infty} H\underline{\mathbb{Z}} \wedge \Sigma^{\rho k}\mathcal{B}_{-1}(k)$$

of $H\underline{\mathbb{Z}}$-modules.

Our proof of the splitting of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ in Section 5 largely serves as a warm-up to that of our main theorem.

We also show that, similarly to the nonequivariant setting, there is a splitting in the homology of $BP_{\mathbb{R}}\langle n\rangle \wedge BP_{\mathbb{R}}\langle n\rangle$ at all heights. Specifically,

Theorem (Theorem 4.1)**.** *Let $n \geqslant 0$. Then there exists a family of maps*

$$\{\theta_k : \Sigma^{\rho k} B_{n-1}(k) \to H_\star BP_{\mathbb{R}}\langle n\rangle \mid k \in \mathbb{N}\}$$

such that their sum

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star B_{n-1}(k) \to H_\star BP_{\mathbb{R}}\langle n\rangle$$

is an isomorphism of $\mathcal{E}(n)_\star$-comodules.

1.3. **New directions.** Similarly to the nonequivariant setting, we expect the splitting of $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ (Theorem 6.1) will make $ku_{\mathbb{R}}$-Adams spectral sequence computations newly accessible. Specifically, it would be interesting to investigate the extent to which this splitting allows one to compute C_2-equivariant stable stems and see if these techniques could extend Isaksen–Guillou's computations of C_2-equivariant stable stems [GI24]. Also, similarly to the nonequivariant setting, the $ku_{\mathbb{R}}$-Adams spectral sequence should have a vanishing line allowing one to identify v_1-periodicity in the C_2-equivariant stable stems.

One may further wonder whether ko_{C_2}-Adams spectral sequences are even more efficient for computing stable stems and detecting v_1-periodicity. Here, we use ko_{C_2} to denote the equivariant connective cover of KO_{C_2}, the spectrum representing the K-theory of C_2-equivariant real vector bundles. Thus one may also be interested in constructing a C_2-equivariant splitting for $ko_{C_2} \wedge ko_{C_2}$ in terms of finite ko_{C_2}-modules. The techniques and methods of this paper may be viewed as a stepping stone towards such a construction.

Additionally, computations with $ku_{\mathbb{R}}$-Adams spectral sequences are closely related to those of ko_{C_2}-Adams spectral sequences via the Wood cofiber sequence

$$\Sigma^\sigma ko_{C_2} \xrightarrow{\eta} ko_{C_2} \to ku_{\mathbb{R}}.$$

Here, η is the first C_2-equivariant Hopf map. Thus in a precise sense $ku_{\mathbb{R}}$-based computations already contain ko_{C_2} information.

In addition to computational applications of the splitting, the cooperations algebra $ku_{\mathbb{R}\star}ku_{\mathbb{R}}$ (Theorem 6.3) is of interest due to connections with number theory. In the underlying nonequivariant setting, ku_*ku can be described in terms of numerical polynomials. In particular, KU_*KU can be identified with the ring of finite Laurent series satisfying certain conditions [Ada74, Theorem 13.4]. This extends to a description of KU_*ku in terms of numerical polynomials [Ada74, Theorem 17.4], as well as to the torsion-free component of

ku_*ku [Ada74, p. 358]. Using this comparison, one can identify the summands of the splitting of ku_*ku in KU_*KU. A detailed discussion of this correspondence (and the analogous story for ko) can be found in Sections 3.2 and 3.3 of [BOSS19]. Our computation of $ku_{\mathbb{R}\star}ku_{\mathbb{R}}$ gives a starting point for developing a similar description in the C_2-equivariant setting.

Similarly to the splitting of $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$, the decomposition of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ into a sum of finite $H\underline{\mathbb{Z}}$-modules given by Theorem 5.1 and Corollary 5.9 provides a starting point for $H\underline{\mathbb{Z}}$-based Adams spectral sequence computations. For example, Burklund-Pstrągowksi use the nonequivariant analogue of Corollary 5.9 to give a cleaner and more concise rephrasing of Toda's original obstruction-theoretic approach to the construction of BP in terms of the $H\mathbb{Z}$-Adams spectral sequence [BP23, Thm 4.41]. It would be interesting to investigate such an argument in the C_2-equivariant case.

1.4. **Outline of the Paper.** In Section 2, we discuss various approaches to the nonequivariant splittings. We then outline the nonequivariant Adams spectral sequence argument underlying our equivariant arguments. In Section 3, we recall equivariant foundations necessary for our main computations. In Section 4 we introduce C_2-equivariant homology results which will later be used in our constructions of splittings. In particular, we describe the homology of $BP_{\mathbb{R}}\langle n\rangle$, state some C_2-equivariant Margolis homology results, and compute the homology of C_2-equivariant (integral) Brown–Gitler spectra in terms of C_2-equivariant lightning flash modules. We also prove a splitting in homology at all heights n. In Section 5, we construct a C_2-equivariant spectrum level splitting for $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$. In Section 6, we construct the splitting for $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$.

1.5. **Notation.** We will make use of the following notation:

(1) $H = H\underline{\mathbb{F}}_2$ denotes the Eilenberg–MacLane spectrum associated to the C_2-constant Mackey functor $\underline{\mathbb{F}}_2$.
(2) $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ is the motivic cohomology of $\mathbb{C}$ with $\mathbb{F}_2$ coefficients, where τ has bidegree $(0,1)$.
(3) $\mathbb{M}_2^{\mathbb{R}} = \mathbb{F}_2[\tau,\rho]$ is the motivic cohomology of $\mathbb{R}$ with $\mathbb{F}_2$ coefficients, where τ and ρ have bidegrees $(0,1)$ and $(1,1)$ respectively.
(4) $\mathbb{M}_2$ is the bigraded equivariant cohomology of a point with coefficients in the constant Mackey functor $\underline{\mathbb{F}}_2$. See Section 3.1 for a description.
(5) $\mathcal{A}^{cl}$, $\mathcal{A}^{\mathbb{C}}$, $\mathcal{A}^{\mathbb{R}}$ and $\mathcal{A}$ are the classical, $\mathbb{C}$-motivic, $\mathbb{R}$-motivic, and C_2-equivariant mod 2 Steenrod algebras.
(6) $\mathcal{E}(n)$ is the subalgebra of $\mathcal{A}$ generated by $Q_0 = \mathrm{Sq}^1$, $Q_1 = \mathrm{Sq}^1\mathrm{Sq}^2 + \mathrm{Sq}^2\mathrm{Sq}^1$, $\cdots, Q_n$. The analogously defined subalgebras of $\mathcal{A}^{\mathbb{R}}$ and $\mathcal{A}^{\mathbb{C}}$ are denoted $\mathcal{E}^{\mathbb{R}}(n)$ and $\mathcal{E}^{\mathbb{C}}(n)$, respectively.
(7) The square-zero extension $\mathbb{M}_2 \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$ induces a decomposition [GHIR20, p.8]
$$\mathrm{Ext}_{\mathcal{E}(n)_\star}(\mathbb{M}_2^{C_2}, \mathbb{M}_2^{C_2}) \cong \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \oplus \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(n)}(NC, \mathbb{M}_2^{\mathbb{R}}).$$
We will abuse notation by writing
$$\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(n)}(\mathbb{M}_2, \mathbb{M}_2) := \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$$
and
$$\mathrm{Ext}_{\mathcal{E}_\star^{NC}(n)}(\mathbb{M}_2, \mathbb{M}_2) := \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(n)}(NC, \mathbb{M}_2^{\mathbb{R}}).$$

Furthermore, we will inductively compute $\mathrm{Ext}_{\mathcal{E}(n)_\star}(M, N)$ for various modules M and N, which will split into a "positive cone summand" coming from $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(n)_\star}(\mathbb{M}_2, \mathbb{M}_2)$ and a "negative cone summand" coming from $\mathrm{Ext}_{\mathcal{E}_\star^{NC}(n)}(\mathbb{M}_2, \mathbb{M}_2)$. We will denote these summands respectively as $\mathrm{Ext}^{PC}_{\mathcal{E}(n)_\star}(M, N)$ and $\mathrm{Ext}^{NC}_{\mathcal{E}(n)_\star}(M, N)$.

(8) $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}$ is the $\mathbb{F}_2[\tau]$-module $\mathrm{colim}_n\, \mathbb{F}_2[\tau]/\tau^n$. We write $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$ for the infinitely divisible $\mathbb{F}_2[\tau]$-module consisting of elements of the form $\frac{x}{\tau^k}$ for $k \geqslant 1$.
(9) $MU_{\mathbb{R}}$ is the spectrum MU with complex conjugation action.
(10) $BP_{\mathbb{R}}$ is the analogue of the nonequivariant Brown–Peterson spectrum BP with complex conjugation action.
(11) $BP_{\mathbb{R}}\langle n\rangle$ models the classical truncated Brown–Peterson spectrum $BP\langle n\rangle$ with C_2-action via complex conjugation.
(12) $ku_{\mathbb{R}}$ is the equivariant connective cover of $KU_{\mathbb{R}}$, the spectrum representing Atiyah real K-theory. See [GHIR20, p. 25] for the definition of
(13) ko_{C_2} is the equivariant connective cover of KO_{C_2}, the spectrum representing the K-theory of C_2-equivariant real vector bundles.
(14) Given a C_2-spectrum X, we use X^e to denote the underlying nonequivariant spectrum.

Grading conventions: Consider the real representation ring of C_2, $RO(C_2) \cong \mathbb{Z}[\sigma]/(\sigma^2 - 1)$. Here σ is the one-dimensional real sign representation. We express the equivariant degree $i + j\sigma$ according to the motivic convention as $(i + j, j)$ where $i + j$ is the total degree and j is the weight. We will also at times use representation spheres to denote the appropriate suspension. For example, instead of writing $\Sigma^{2,1}$, we will write Σ^ρ where ρ is the C_2-regular representation. Whether ρ is an element in the homology of the point or the C_2-regular representation will be clear from context.

We grade Ext groups in the form (s, f, w), where s is the stem, i.e., the total degree minus the homological degree; f is the Adams filtration, i.e., the homological degree; and w is the weight. We will also refer to the Milnor–Witt degree, which is $s - w$.

Acknowledgments. The authors would like to thank Mark Behrens, Christian Carrick, Mike Hill, and Vesna Stojanoska for enlightening conversations. The first and second authors would also like to thank the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support. The second and third authors would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme Equivariant homotopy theory in context, where work on this paper was undertaken. This work was supported by EPSRC grant EP/Z000580/1. The third author thanks the Hausdorff Institute in Bonn for its hospitality and financial support, as well as the Knut and Alice Wallenberg Foundation for financial support. All three authors are grateful to the Hausdorff Institute in Bonn for its hospitality during the Spectral Methods in Algebra, Geometry, and Topology trimester program in Fall of 2022, funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy- EXC-2047/1-390685813. This material is based upon work supported by the National Science Foundation under Grant No. DMS 2135884.

2. Underlying nonequivariant splittings

In this section, we give a summary of the nonequivariant splittings related to ku and ko found in the literature. We also outline a construction of the splitting of $ku \wedge ku$ using the same strategy as we use in Sections 5 and 6 where we construct C_2-equivariant splittings of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ and $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$.

2.1. **Nonequivariant Brown-Gitler spectra.** Let $\mathcal{A}^{cl}$ denote the classical Steenrod algebra, and let $\mathcal{A}^{cl}_* := H\mathbb{F}_{p*}H\mathbb{F}_p$ denote its dual. When $p = 2$,

$$\mathcal{A}^{cl}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots \xi_n],$$

where $|\xi_i| = 2^i - 1$. When p is odd,

$$\mathcal{A}^{cl}_* \cong \mathbb{F}_2[\xi_1, \xi_2, \dots \xi_n] \otimes E(\tau_0, \tau_1, \dots),$$

where $|\xi_i| = 2(p^i - 1)$ and $|\tau_i| = 2p^i - 1$.

The weight filtration on $\mathcal{A}^{cl}_*$ is defined by setting $wt(\bar{\xi}_k) = wt(\bar{\tau}_k) = p^k$ and $wt(xy) = wt(x) + wt(y)$.

Let $\mathcal{E}^{cl}(n)$ denote the subalgebra $\mathcal{E}^{cl}(n) = E(Q_0, Q_1, \dots Q_n)$. Then

$$H_*BP\langle n\rangle \cong \mathcal{A}^{cl}//\mathcal{E}^{cl}(n)_*$$

. In particular, $H_*H\mathbb{Z} \cong \mathcal{A}^{cl}//\mathcal{E}^{cl}(0)_*$. At each prime p, there is a family of integral Brown-Gitler spectra $\{B_0(k) |\ k \in \mathbb{N}\}$ such that $H_*B_0(k) \cong \mathbb{F}_p\{x \in H_*H\mathbb{Z}|\ wt(x) \leqslant k\}$ [GJM86].

2.2. Nonequivariant Splittings.

2.2.1. *Literature on nonequivariant splittings.* In [Mah81], Mahowald constructs the splitting

$$ko \wedge ko \simeq ko \wedge \Sigma^{4k} B_0(k).$$

Subsequently, Kane constructed a splitting

$$BP\langle 1\rangle \wedge BP\langle 1\rangle \simeq BP\langle 1\rangle \wedge \Sigma^{2k} B_0(k)$$

at odd primes. While the splitting for ku at the prime 2 does not appear in the literature, it can be deduced either from Mahowald's construction for ko, or by using the same arguments as Kane's odd primary splitting. In [Kli89], Klippenstein also constructs a decomposition of $BP\langle 1\rangle \wedge BP\langle n\rangle$ in terms of $BP\langle 1\rangle$-module spectra at odd primes, but notes that the same statements hold at the prime 2.

2.2.2. *Strategies for the nonequivariant splitting.* Following Mahowald's strategy for the construction of the ko-splitting, Kane used pairings $B_0(k) \wedge B_0(m) \to B_0(k+m)$ to inductively construct maps

$$\theta_k : \Sigma^{2k} B_0(k) \to BP\langle 1\rangle \wedge BP\langle 1\rangle$$

such that the sum of the composites

$$\bar{\theta}_k : \bigvee_{k=0}^{\infty} BP\langle 1\rangle \wedge \Sigma^{2k} B_0(k) \xrightarrow{1\wedge\theta_k} BP\langle 1\rangle^{\wedge 3} \xrightarrow{\mu\wedge 1} BP\langle 1\rangle^{\wedge 2}$$

is a homotopy equivalence. Towards constructing the splitting, assume inductively that appropriate maps $\theta_k : \Sigma^{2k} B_0(k) \to BP\langle 1\rangle \wedge BP\langle 1\rangle$ have been constructed. If n is not a power of p, take the p-adic decomposition $n = n_0 + n_1 p + \cdots + n_r p^r$, and consider the cofiber sequence

$$B_0(1)^{n_0} \wedge B_0(p)^{n_1} \wedge \cdots \wedge B_0(p^r)^{n_r} \to B_0(n) \to C(n),$$

where $C(n)$ is the cofiber of the composite of the pairing maps. Consider the composite

$$\begin{aligned} B_0(1)^{n_0} \wedge B_0(p)^{n_1} \wedge \cdots \wedge B_0(p^r)^{n_r} &\xrightarrow{\theta_0^{n_0} \wedge \theta_1^{n_1} \wedge \theta_r^{n_r}} BP\langle 1\rangle \wedge BP\langle 1\rangle \cdots BP\langle 1\rangle \\ &\xrightarrow{\qquad\quad} BP\langle 1\rangle \wedge BP\langle 1\rangle, \end{aligned}$$

where the second map is the standard multiplication map. The key step in the construction is to show that this composite factors through $B_0(n)$. One can then show that this factorization is the appropriate map $\theta_n : B_0(n) \to BP\langle 1\rangle \wedge BP\langle 1\rangle$.

However, the strategy for showing that the map factors through $B_0(n)$ does not lend itself well to the equivariant setting for two reasons: first, it relies in part on connectivity arguments that the negative cone prevents us from replicating. Second, extending this argument to the equivariant setting would require us to deduce that in homology, various complicated cofibers of free $\mathbb{M}_2$-modules are also free. While these obstacles may be surmountable, we have found that the alternative strategy of using an Adams spectral sequence to lift the isomorphism on homology is most readily adaptable to the equivariant setting. Since this exact argument does not appear in the literature, we present it here. This Adamas spectral sequence argument is most similar to Klippenstein's splitting construction. However, we use a ku-relative Adams spectral sequence as it makes the construction slightly cleaner. We also do not need his comparison with the Universal Coefficient Spectral Sequence as we only address the case $BP\langle 1\rangle \wedge BP\langle 1\rangle$, not $BP\langle 1\rangle \wedge BP\langle n\rangle$. For ease of comparison to the rest of this paper, we present the 2-primary version and use the notation ku. However, the same argument works for $BP\langle 1\rangle$ for all primes.

2.2.3. *Constructing the nonequivariant splitting.* Consider the ku-relative Adams spectral sequence

$$\mathrm{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}\left(\Sigma^{2k} H_* B_0(k), H_* ku\right) \Longrightarrow [ku \wedge \Sigma^{2k} B_0(k), ku \wedge ku]. \tag{2.1}$$

Consider $\theta_k : H_*\Sigma^{2k} B_0(k) \to H_* ku$ as a class in filtration $f = 0$. If we can show that this class survives the spectral sequence for each k, then we have constructed a family of maps realizing the $\mathcal{E}^{cl}(1)_\star$-comodule isomorphism $H_* ku \cong \oplus_{k=0}^{\infty} \Sigma^{2k} H_* B_0(k)$, and thus will have a ku-module splitting

$$ku \wedge ku \simeq \bigvee_{k=0}^{\infty} ku \wedge \Sigma^{2k} B_0(k).$$

The following identification of $H_* ku$ is useful for computing the E_2-page of this spectral sequence.

Proposition 2.2. *There is an isomorphism*

$$H_* ku \cong \bigoplus_{k=0}^{\infty} H_* \Sigma^{2k} B_0(k)$$

of $\mathcal{E}^{cl}(1)_*$-comodules.

Thus to compute the E_2-page of 2.1, it suffices to compute $\mathrm{Ext}_{\mathcal{E}^{cl}(1)_*}(H_*B_0(k), B_0(m))$ for all m. Towards this goal, it is helpful to use the following decomposition of the homology of the Brown-Gitler spectra in terms of homological lighting fash modules.

Definition 2.3. *The homological lightning flash module is given by*

$$L(k) = \mathcal{E}^{cl}(1)\{x_1, x_2, \cdots, x_k \mid x_{i+1}Q_1 = x_iQ_0 \,\forall 1 \leqslant i \leqslant k\}$$

where $|x_i| = 2i+1$. *Further define* $L(0) = \mathbb{F}_2$.

Proposition 2.4. *As an* $\mathcal{E}^{cl}(1)$*-module,*

$$H_*B_0(k) \cong L(\nu_2(k!)) \oplus F_k$$

where F_k *is a sum of suspensions of copies of* $\mathcal{E}^{cl}(1)_*$.

Thus we can rewrite the E_2-page of the Adams spectral sequence 2.1 as

$$\mathrm{Ext}_{\mathcal{E}^{cl}(1)_*}(H_*B_0(k), H_*ku) \cong \bigoplus_{m=0}^{\infty} \mathrm{Ext}_{\mathcal{E}^{cl}(1)_*}\big(L(\nu_2(k!)), \Sigma^{2m}L(\nu_2(m!))\big) \oplus W, \tag{2.5}$$

where W is a sum of suspensions of $\mathbb{F}_2$ in filtration $f = 0$.

The $\mathrm{Ext}_{\mathcal{E}^{cl}(1)_*}(L(k), L(m))$ terms have a fairly simple description as $\mathbb{F}_p[v_0, v_1]$-modules.

Proposition 2.6. *For* $k \leqslant m$,

$$\mathrm{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}(L(k), L(m)) \cong \mathbb{F}_2[v_0, v_1]\{x_0, x_1, \ldots x_{m-k} \mid v_1x_i = v_0x_{i+1}\},$$

where $|x_i| = (2i, 0)$.

For $k > m$,

$$\mathrm{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}(L(k), L(m)) \cong \mathbb{F}_p[v_0, v_1]\{x\} \oplus \mathbb{F}_p[v_0, v_1]\left\{ y_0, y_1, \ldots, y_{k-m} \;\middle|\; \begin{array}{l} v_1y_i = v_0y_{i+1}, \\ v_0y_0 = 0, \\ v_1y_{k-m} = 0 \end{array} \right\},$$

where $|x| = (0, k-m)$ *and* $|y_i| = (-1-2(k-m-i), 0)$.

Furthermore, one can lift the $\mathcal{E}^{cl}(1)_*$-comodule isomorphism of Proposition 2.4 to a splitting of ku-module spectra, using a ku-relative Adams spectral sequence.

Proposition 2.7. *There are ku-module splittings*

$$ku \wedge B_0(k) \simeq C_k \vee V_k$$

$$ku \wedge ku \simeq C \vee V,$$

where V_k *and* V *are sums of suspensions of* $H\mathbb{F}_2$, *and* C *is a* v_1*-torsion-free ku-module.*

Thus the splitting of the E_2-page (2.5) is in fact a splitting of spectral sequences, and we are ready to prove that the spectral sequence collapses.

Proposition 2.8. *For all* k, *the Adams spectral sequence 2.1 collapses.*

Proof. First, recall that no nontrivial differentials can go from v_1-torsion classes to v_1-torsion free classes. Next, note that all of the v_1-torsion free classes are concentrated in even stem s, and the Adams differential d_r has degree $(s, f) = (-1, r)$, so there is no possibility of differentials from the v_1-torsion free classes to other v_1-torsion free classes. Finally, we must consider the possibility of differentials from the v_1-torsion free classes to v_1-torsion classes. This is impossible for the following degree reasons.

First, note that $\Sigma^{2(m-k)} y_i$ only occur in stem $s \leqslant -3$. On the other hand, for $m' \geqslant k$, $\Sigma^{2(m'-k)} \operatorname{Ext}_{\mathcal{E}^{cl}(1)_*}(L(\nu_2(k!)), L(\nu_2(m'!)))$ is contained entirely in stem $s \geqslant 0$. So we are just left to show that there are no differentials from the summand

$$\Sigma^{2(m'-k)} \operatorname{Ext}_{\mathcal{E}^{cl}(1)_*}(L(\nu_2(k!)), L(\nu_2(m'!)))$$

to $\Sigma^{2(m-k)} v_0^i y_j$ for $m' < k$. Let $r \geqslant 0$. In stem $-2r$, any generators $\Sigma^{2(m'-k)} x$ must be in filtration at least $\nu_2(k!) - \nu_2((k-r)!)$. On the other hand, any class of the form $\Sigma^{k-m} v_0^j y_i$ in stem $-2r-1$ has filtration at most $\nu_2(k!) - \nu_2((k-r+1)!) + 1$. Thus no classes of the form $\Sigma^{2(m-k)} y_i$ are the target of differentials, and indeed the Adams spectral sequence collapses. □

So indeed the isomorphism of Proposition 2.4 lifts, and the splitting is an immediate consequence.

Theorem 2.1. *Up to 2-completion, there exists a splitting of ku-modules*

$$ku \wedge ku \simeq \bigvee_{k=0}^{\infty} ku \wedge \Sigma^{2k} B_0(k).$$

This splitting can also be reinterpreted in terms of Adams covers. Let $ku^{\langle n \rangle}$ denote the n-th Adams cover of ku, that is, the n^{th} term in a minimal Adams resolution of ku over $H\mathbb{F}_2$ (See [Lel84, p.2-3] and Proposition 6.18 of [Kan81] for discussion of the definition and the proof of the next proposition. Note that their phrasing is slightly different as they discuss cohomology and do not reference relative homology).

Proposition 2.9. *The ku-relative homology of the n-th Adams cover of ku is*

$$H_*^{ku} ku^{\langle n \rangle} \cong L(\nu_2(n)).$$

The Adams cover $ku^{\langle n \rangle}$ is uniquely determined up to homotopy by its homology, and the fact that it is a ku-module.

Theorem 2.2. *Up to 2-completion,*

$$ku \wedge B_0(n) \simeq ku^{\langle \nu_2(n!) \rangle} \vee V_n,$$

where V_n is a sum of suspensions of $H\mathbb{F}_2$.

Proof. Consider the ku-relative Adams spectral sequence

$$E_2^{s,f} \cong \operatorname{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}\left(H_*^{ku} C_n, H_*^{ku} ku^{\langle \nu_2(n!) \rangle}\right) \Longrightarrow [ku \wedge \Sigma^{2k} B_0(k), ku \wedge ku], \tag{2.10}$$

where the homological degree is denoted f, and s denotes the stem, ie the topological degree minus the homological degree. Note that

$$\operatorname{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}\left(H_*^{ku} C_n, H_*^{ku} ku^{\langle \nu_2(n!) \rangle}\right) \cong \operatorname{Ext}^{s,f}_{\mathcal{E}^{cl}(1)_*}\left(L(\nu_2(n!)), L(\nu_2(n!))\right).$$

It follows from the arguments of Proposition 2.8 that this spectral sequence collapses at the E_2-page, and we can lift the isomorphism $H^{ku}_* C_n \to H^{ku}_* ku^{\langle \nu_2(n!) \rangle}$ to the level of spectra. □

Corollary 2.11. *Up to 2-completion,*

$$ku \wedge ku \simeq \bigvee_{k=0}^{\infty} \Sigma^{2k} ku^{\langle \nu_2(k!) \rangle} \vee V,$$

where V is a sum of suspensions of $H\mathbb{F}_2$.

3. Equivariant preliminaries

3.1. **The C_2-equivariant homology of a point.** A detailed description of the coefficients $H_\star$ will be useful for our calculations. Our description closely follows that of [GHIR20, Section 2.1], which in turn is a reinterpretation of results in [HK01, Proposition 6.2]. Throughout, we frequently denote the coefficients $H_\star$ as $\mathbb{M}_2$.

Additively, $\mathbb{M}_2$ is

(1) $\mathbb{F}_2$ in degree (s, w) if $s \leqslant 0$ and $w \leqslant s$.
(2) $\mathbb{F}_2$ in degree (s, w) if $s \geqslant 0$ and $w \geqslant s + 2$.
(3) 0 otherwise.

This additive structure is represented by the dots in Figure 3.1. The non-zero element in degree $(0, -1)$ is called τ, and the non-zero element in degree $(-1, -1)$ is called ρ. Sometimes in the equivariant literature, the element τ is called u or u_σ, and ρ is called a or a_σ. We choose to use the names τ and ρ common in motivic literature so we can easily write $\mathbb{M}_2$ as a square-zero extension of $\mathbb{M}_2^{\mathbb{R}}$, the motivic homology of $\mathbb{R}$ with $\mathbb{F}_2$-coefficients.

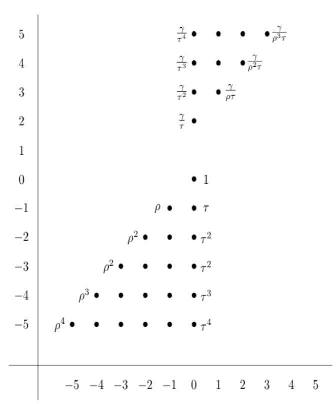

Figure 3.1. $\mathbb{M}_2$

The "positive cone" refers to the part of $\mathbb{M}_2$ in degrees (s, w) with $w \leqslant 0$. The positive cone is isomorphic to the $\mathbb{R}$-motivic homology ring $\mathbb{M}_2^{\mathbb{R}}$ of a point. Multiplicatively, the positive cone is a polynomial ring on two variables, ρ and τ.

The "negative cone" NC refers to the part of $\mathbb{M}_2$ in degrees (s, w) with $w \leqslant -2$. Multiplicatively, the product of any two elements of NC is zero, so $\mathbb{M}_2$ is a square-zero extension of $\mathbb{M}_2^{\mathbb{R}}$. Multiplications by ρ and τ are non-zero in NC whenever they make sense. Thus elements of NC are infinitely divisible by both ρ and τ. We use the notation $\frac{\gamma}{a^j \tau^k}$ for the non-zero element in degree $(j, 1+j+k)$, which is consistent with the described multiplicative properties. The symbol γ, which does not correspond to an actual element of $\mathbb{M}_2$, has degree $(0, 1)$.

The $\mathbb{F}_2[\tau]$-module structure on $\mathbb{M}_2$ is essential for calculations filtered by powers of ρ. Thus we describe the $\mathbb{F}_2[\tau]$-module structure on NC in further detail. We define $\mathbb{F}_2[\tau]/\tau^\infty$ to be the $\mathbb{F}_2[\tau]$-module $\operatorname{colim}_n \mathbb{F}_2[\tau]/\tau^k$. Following [GHIR20], we write $\frac{\mathbb{F}_2[\tau^k]}{\tau^\infty}\{x\}$ for the infinitely divisible $\mathbb{F}_2[\tau]$-module consisting of elements of the form $\frac{x}{\tau^k}$ for $k \geqslant 1$. Note that x itself is not an element of $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$. The idea is that x represents the infinitely many relations $\tau^k \cdot \frac{x}{\tau^k} = 0$ that define $\frac{\mathbb{F}_2[\tau]}{\tau^\infty}\{x\}$.

With this notation in place, $\mathbb{M}_2$ is equal to

$$\mathbb{M}_2^{\mathbb{R}} \oplus NC = \mathbb{M}_2^{\mathbb{R}} \oplus \bigoplus_{s \geqslant 0} \frac{\mathbb{F}_2[\tau]}{\tau^\infty} \left\{ \frac{\gamma}{\rho^s} \right\}$$

as an $\mathbb{F}_2[\tau]$-module.

3.2. **Comparison with nonequivariant homology.** Suppose X is a C_2-spectrum. The cofiber of τ,

$$S^0 \hookrightarrow S^\sigma \to C\tau,$$

is stably given by $C\tau \simeq \Sigma^{1-\sigma} C_{2+}$. Smashing $H \wedge X$ with the transfer map

$$S^{1-\sigma} \xrightarrow{\mathrm{tr}} \Sigma^{1-\sigma} C_{2+} \simeq C\tau$$

gives a map

$$\Sigma^{1-\sigma} H \wedge X \simeq H \wedge X \to H \wedge X \wedge C\tau.$$

where the equivalence is given by $\tau \in H_\star$.

Applying $\pi_V^{C_2}$ to this map gives a homomorphism

$$\Phi^e : H_V(X) \to H_{|V|}(X^e),$$

which can be used to compare the C_2-equivariant homology with the underlying nonequivariant homology of the nonequivariant spectrum underlying X.

Later, in Section 6, we will smash $ku_{\mathbb{R}} \wedge X$ with the transfer map and apply $\pi_V^{C_2}$ in order to compare the $ku_{\mathbb{R}\star}$-module structure of $ku_{\mathbb{R}\star}X$ structure to the ku_*-module structure of $ku_* X^e$ and deduce extensions.

3.3. **Equivariant connective covers.** Suppose X is a C_2-spectrum. The equivariant connective cover $X\langle 0 \rangle \xrightarrow{q} X$ is a C_2-spectrum such that:

(1) the restriction of q is the connective cover of the underlying spectrum X^e, and
(2) the categorical fixed points of q is the categorical fixed points of X.

See [GHIR20, p. 25] for a more detailed description of the restriction and categorical fixed points functors (cf. [Lew95, Proposition 3.3]).

3.4. **The C_2-equivariant Dual Steenrod Algebra.** In [HK01], Hu–Kriz computed the C_2-equivariant dual Steenrod algebra

$$\mathcal{A}_\star \cong H_\star[\xi_1, \xi_2, \cdots, \tau_0, \tau_1 \cdots]/({\tau_i}^2 = a\tau_{i+1} + (u + a\tau_0)\xi_{i+1}),$$

where $|\xi_i| = (2^i - 1)\rho$, $|\tau_i| = 2^i\rho - \sigma$, and

$$\psi(\xi_i) = \sum_{0\leqslant j\leqslant i} \xi_{i-j}^{2^j} \otimes \xi_j \qquad\qquad \psi(\tau_i) = \tau_i \otimes 1 + \sum_{0\leqslant j\leqslant i} \xi_{i-j}^{2^j} \otimes \tau_j.$$

We denote the images of the generators ξ_i and τ_i under the conjugation map

$$c : \mathcal{A}_\star^{C_2} \to \mathcal{A}_\star^{C_2}$$

by $\bar{\xi}_i$ and $\bar{\tau}_i$ respectively.

The coproduct formulas on the conjugates are

$$\psi(\bar{\xi}_i) = \sum_{0\leqslant j\leqslant i} \bar{\xi}_j \otimes \bar{\xi}_{i-j}^{2^j} \qquad\qquad \psi(\bar{\tau}_i) = 1 \otimes \bar{\tau}_j \otimes + \sum_{0\leqslant j\leqslant i} \bar{\xi}_{i-j}^{2^j}.$$

3.5. **Relative homology and Adams spectral sequences.** We will make use of relative homology and relative Adams spectral sequences in our computations. In general, if E is an R-algebra and M is an R-module in spectra, then E-homology in the category of R-modules is

$$E_\star^R(M) := \pi_\star(E \underset{R}{\wedge} M).$$

Note that

$$E_\star^R(R \wedge M) = \pi_\star(E \underset{R}{\wedge} R \wedge M) = E_\star M.$$

In [BL01, Prop 2.1], Baker–Lazarev introduce a relative Adams spectral sequence in the category of R-modules. The $RO(C_2)$-graded version exists by the same construction.

Proposition 3.1. *Let E be a C_2-equivariant R-algebra spectrum, and let X, Y be R-modules. If $E_\star^R X$ is projective as an $E_\star$-module, then there exists an E-based Adams spectral sequence in the category of R-modules*

$$E_2^{s,f,w} \cong \mathrm{Ext}^{s,f,w}_{E_\star^R E}(E_\star^R X, E_\star^R Y) \Longrightarrow [X, Y]^{R\ s,f,w}_{\hat{E}}$$

where $[X, Y]^R_{\hat{E}}$ denotes the E-nilpotent completion of R-module maps from X to Y, s denotes the stem, f denotes the homological degree, and w denotes the weight.

Proposition 3.2. *The $BP\langle 0\rangle$-relative dual Steenrod algebra is*

$$H_\star^{BP\langle 0\rangle} H \cong E(0)_\star$$

and the $BP\langle 1\rangle$-relative dual Steenrod algebras is

$$H_\star^{BP\langle 1\rangle} H \cong E(1)_\star.$$

Proof. We give an argument for the second statement and leave the first to the reader. Consider the cofiber sequence

$$BP_{\mathbb{R}}\langle 1\rangle \xrightarrow{v_1} BP_{\mathbb{R}}\langle 1\rangle \to H\underline{\mathbb{Z}}.$$

Smashing $H \underset{BP_{\mathbb{R}}\langle 1\rangle}{\wedge} -$ with the cofiber sequence yields a splitting

$$H \underset{BP_{\mathbb{R}}\langle 1\rangle}{\wedge} H\underline{\mathbb{Z}} \simeq H \wedge (S^0 \vee S^{\rho+1}),$$

which suffices to determine the additive structure of $H_\star^{BP_{\mathbb{R}}\langle 1\rangle} H\underline{\mathbb{Z}}$.

The multiplicative structure can be deduced from the algebra map

$$\pi_\star\big(H \wedge H\big) \to \pi_\star\big(H \underset{BP_{\mathbb{R}}\langle 1\rangle}{\wedge} H\big).$$

□

4. C_2-equivariant homology results

4.1. **The homology of $BP_{\mathbb{R}}\langle n\rangle$.** The authors first learned of a computation of the equivariant homology groups $H_\star BP_{\mathbb{R}}\langle n\rangle$ from Christian Carrick in the context of Carrick, Hill, and Ravenel's work on the homological slice spectral sequence in motivic and Real bordism [CHR24]. While Carrick–Hill–Ravenel did indeed compute the equivariant homology groups $H_\star BP_{\mathbb{R}}\langle n\rangle$, the result does not appear in the final version of their paper, so we include an argument due to Carrick here.

Recall that $\pi_{*\rho} BP_{\mathbb{R}} \cong \mathbb{Z}[\bar{v}_1, \bar{v}_2, \cdots]$. Consider the cofibre sequence

$$H \wedge \Sigma^{(2^i-1)\rho} BP_{\mathbb{R}} \xrightarrow{\cdot \bar{v}_i} H \wedge BP_{\mathbb{R}} \to H \wedge BP_{\mathbb{R}}/\bar{v}_i.$$

Note that $\bar{v}_i = 0 \in H_\star BP_{\mathbb{R}}$, so applying homotopy yields a short exact sequence

$$0 \to H_\star BP_{\mathbb{R}} \to H_\star(BP_{\mathbb{R}}/\bar{v}_i) \to H_{\star-((2^i-1)\rho+1)} BP_{\mathbb{R}} \to 0.$$

This sequence splits as a sum of $H_\star BP_{\mathbb{R}}$-modules, so

$$H_\star(BP_{\mathbb{R}}/\bar{v}_i) \cong H_\star BP_{\mathbb{R}} \oplus H_{\star-((2^i-1)\rho+1)} BP_{\mathbb{R}}.$$

In particular, $H_\star(BP_{\mathbb{R}}/\bar{v}_i)$ is a free $H_\star BP_{\mathbb{R}}$-module.

Proposition 4.1. *The map $H_\star(BP_{\mathbb{R}}/\bar{v}_i) \to H_\star H$ is injective, with image the free $H_\star BP_{\mathbb{R}}$-submodule generated by 1 and $\bar{\tau}_i$.*

Proof. The above splitting of $H_\star BP_{\mathbb{R}}$-modules is not split as $\mathcal{A}_\star$-comodules: if it were, one could run the H-based Adams spectral sequence for $BP_{\mathbb{R}}/\bar{v}_i$ and use this splitting to find that $\bar{v}_i \neq 0$ in $\pi_\star(BP_{\mathbb{R}}/\bar{v}_i)$, a contradiction. It follows that, under the splitting as $H_\star BP_{\mathbb{R}}$-modules, the map $H_\star(BP_{\mathbb{R}}/\bar{v}_i) \to A_\star^{C_2}$ cannot send the generator in degree $(2^i-1)\rho+1$ to zero, as the kernel of $H_\star(BP_{\mathbb{R}}/\bar{v}_i) \to \mathcal{A}_\star^{C_2}$ must be a sub $\mathcal{A}_\star$-comodule.

The generator of $H_{\star-((2^i-1)\rho+1)} BP_{\mathbb{R}}$ in degree $(2^i-1)\rho+1$ must be sent to an $\mathcal{A}_\star$-comodule primitive in $\mathcal{A}_\star/H_\star BP_{\mathbb{R}}$ and we have isomorphisms

$$\mathrm{Prim}(\mathcal{A}_\star^{C_2}/H_\star BP_{\mathbb{R}}) \cong \mathrm{Ext}^1_{\mathcal{A}_\star^{C_2}}(H_\star, H_\star BP_{\mathbb{R}}) \cong \mathrm{Ext}^1_{\mathcal{E}(n)_\star}(H_\star, H_\star).$$

by Proposition 3.2.

Extending the computation of [Hil11, Theorem 3.1] to a C_2-equivariant result (i.e., analyzing the negative cone) shows that $\mathrm{Ext}^1_{\mathcal{E}(n)_\star}(H_\star, H_\star)$ has only one nonzero class in this degree. We find that since $\bar{\tau}_i$ is a primitive in $\mathcal{A}^{C_2}_\star / H_\star BP_{\mathbb{R}}$, our generator must hit $\bar{\tau}_i$ mod $H_\star BP_{\mathbb{R}}$. It follows that the map $H_\star(BP_{\mathbb{R}}/\bar{v}_i) \to \mathcal{A}^{C_2}_\star$ is injective. □

By freeness, for any i, j, the Künneth spectral sequence

$$\mathrm{Tor}^{H_\star BP_{\mathbb{R}}}_{*,\star}(H_\star(BP_{\mathbb{R}}/\bar{v}_i), H_\star(BP_{\mathbb{R}}/\bar{v}_j)) \Longrightarrow H_\star(BP_{\mathbb{R}}/(\bar{v}_i, \bar{v}_j))$$

collapses to give an isomorphism

$$H_\star(BP_{\mathbb{R}}/(\bar{v}_i, \bar{v}_j)) \cong H_\star(BP_{\mathbb{R}}/\bar{v}_i) \otimes_{H_\star BP_{\mathbb{R}}} H_\star(BP_{\mathbb{R}}/\bar{v}_j).$$

We can iterate this process to compute $H_\star BP_{\mathbb{R}}/(\bar{v}_{n+1}, \cdots, \bar{v}_{n+m})$. Taking the colimit of $H_\star BP_{\mathbb{R}}/(\bar{v}_{n+1}, \cdots, \bar{v}_{n+m})$ over m yields the following proposition.

Proposition 4.2. *The map $H_\star BP_{\mathbb{R}}\langle n\rangle \to H_\star H$ is injective with image the sub-$H_\star$-algebra*

$$H_\star BP_{\mathbb{R}}\langle n\rangle \cong H_\star[\bar{\xi}_1, \bar{\xi}_2, \bar{\xi}_3, \cdots, \bar{\tau}_{n+1}, \bar{\tau}_{n+2}, \bar{\tau}_{n+3}, \cdots]/(\bar{\tau}_i^2 = a\bar{\tau}_{i+1} + u\bar{\xi}_{i+1})$$

In particular,

$$H_\star BP_{\mathbb{R}}\langle n\rangle \cong \mathcal{A}^{C_2}_\star \underset{\mathcal{E}(n)_\star}{\square} H_\star. \tag{4.3}$$

Remark 4.4. *The description of $H_\star BP_{\mathbb{R}}\langle n\rangle$ given in Equation* (4.3) *is analogous to the nonequivariant isomorphism $H_* BP\langle n\rangle \cong A//E(n)_*$. Likewise, the motivic homology of the 2-complete algebraic Johnson-Wilson spectra $BPGL\langle n\rangle$ over p-adic fields* [Orm11, Theorem 3.9] *has the same form, with the appropriate motivic analogues substituted for A and $E(n)$.*

The left $\mathcal{E}(i)_\star$-coaction on $\mathcal{A}^{C_2}_\star \underset{\mathcal{E}(n)_\star}{\square} H_\star$ is given by

$$\alpha : \mathcal{A}^{C_2}_\star \underset{\mathcal{E}(n)_\star}{\square} H_\star \xrightarrow{\psi\otimes 1} \mathcal{A}^{C_2}_\star \otimes (\mathcal{A}^{C_2}_\star \square_{\mathcal{E}(n)} H_\star) \xrightarrow{\pi\otimes 1} \mathcal{E}(n) \otimes (\mathcal{A}^{C_2}_\star \square_{\mathcal{E}(n)} H_\star), \tag{4.5}$$

which on generators $\bar{\xi}_k$ and $\bar{\tau}_k$ is

$$\begin{aligned} \alpha(\bar{\xi}_k) &= 1 \otimes \bar{\xi}_k \\ \alpha(\bar{\tau}_{n+k}) &= 1 \otimes \bar{\tau}_{n+k} + \sum_{0\leqslant i\leqslant n} \bar{\tau}_i \otimes \bar{\xi}^{2^i}_{n+k-i}, \qquad \text{for all } k. \end{aligned}$$

Note that $\mathcal{E}(i)_\star$ is the dual of the exterior algebra $\mathcal{E}(i)$, and that $\mathcal{E}(i) \cong E[Q_0, Q_1, \ldots, Q_n]$, where Q_i is defined to be the dual of τ_i.

Let M be a left $\mathcal{E}(i)_\star$-comodule. Then there is an induced right $\mathcal{E}(i)$-module action $\lambda : M \otimes \mathcal{E}(i) \to M$, defined by

$$\lambda(x, \theta) = (\theta \otimes Id_M) \circ \alpha(x)$$

where $\alpha(x) = \Sigma_i \theta_i \otimes x_i$.

Note that $\bar{\tau}_i$ is the dual of Q_i, and so the right $\mathcal{E}(1)$-module action on $H_\star BP_{\mathbb{R}}\langle 1\rangle$ is

$$\begin{aligned} \bar{\xi}_j Q_i &= 0 \\ \bar{\tau}_k Q_0 &= \bar{\xi}_k \\ \bar{\tau}_k Q_1 &= \bar{\xi}^2_{(k-1)}. \end{aligned}$$

Note that $\mathcal{E}(i)_\star$ is finitely generated and projective over $\mathbb{M}_2^{C_2}$ [Ric15], so the following proposition is standard algebra (see for example [BW03, Theorem 4.7]).

Proposition 4.6. *There is an equivalence of categories between left $\mathcal{E}(n)_\star$-comodules and right $\mathcal{E}(n)$-modules.*

4.2. **Mahowald weight and Brown–Gitler Subcomodules.** Define a weight filtration on $A_\star^{C_2}$ by

$$wt(\bar{\xi}_k) = 2^k = wt(\bar{\tau}_k) \qquad \text{and} \qquad wt(xy) = wt(x) + wt(y).$$

For $i \geqslant 0$, the j^{th} Brown–Gitler comodule $N_i(j)$ is the subspace of $\mathcal{A}_\star^{C_2} \square_{\mathcal{E}(i)_\star} H_\star$ spanned by monomials of weight less than or equal to $2j$. For $i = -1$, let $N_{-1}(j)$ denote the subspace of $\mathcal{A}_\star^{C_2} \square_{\mathcal{E}(-1)_\star} H_\star$ spanned by monomials of weight less than or equal to j.

4.3. **A Splitting in Homology.** Define $M_i(j)$ to be the $\mathbb{M}_2$-submodule of $\mathcal{A}_\star^{C_2} \square_{\mathcal{E}(i)_\star} H_\star$ spanned by monomials of weight exactly $2j$. Observe from the coaction on the generators $\bar{\tau}_j$ and $\bar{\xi}_j$ (4.5) that the $\mathcal{E}(i)_\star$-coaction on $A//\mathcal{E}(n)_\star$ preserves Mahowald weight, so $M_i(j)$ is an $\mathcal{E}(i)_\star$-subcomodule of $A//\mathcal{E}(n)_\star$. The next proposition follows immediately, and is analogous to the underlying nonequivariant statement [Cul19, Proposition 3.3].

Proposition 4.7. *There is a natural isomorphism*

$$\mathcal{A}_\star^{C_2} \underset{\mathcal{E}(i)_\star}{\square} H_\star \cong \bigoplus_{k \geqslant 0} M_i(k)$$

of left $\mathcal{E}(i)_\star$-comodules.

Just as in the nonequivariant case [Cul19, Lemma 3.4],[Cul20, Lemma 4.10], we also have the following $\mathcal{E}(i)_\star$-comodule isomorphism.

Proposition 4.8. *For $n \geqslant 0$, there is an isomorphism of $\mathcal{E}(n)_\star$-comodules*

$$\Sigma^{\rho k} N_{n-1}(k) \xrightarrow{\cong} M_n(k)$$

for all k, where

$$\bar{\xi}_1^{k_1} \bar{\xi}_2^{k_2} \cdots \bar{\xi}_{i+1}^{k_1} \bar{\tau}_{i+1}^{\epsilon_{i+1}} \bar{\xi}_{i+2}^{k_{i+2}} \bar{\tau}_{i+2}^{\epsilon_{i+2}} \cdots \quad \mapsto \quad \bar{\xi}_1^{a} \bar{\xi}_2^{k_1} \cdots \bar{\xi}_{i+2}^{k_1} \bar{\tau}_{i+2}^{\epsilon_{i+1}} \bar{\xi}_{i+3}^{k_{i+2}} \bar{\tau}_{i+3}^{\epsilon_{i+2}} \cdots$$

and $a = j - \frac{1}{2} wt(\bar{\xi}_2^{k_1} \cdots \bar{\xi}_{i+2}^{k_1} \bar{\tau}_{i+2}^{\epsilon_{i+1}} \bar{\xi}_{i+3}^{k_{i+2}} \bar{\tau}_{3+2}^{\epsilon_{i+2}})$.

Thus the following analogue of [Cul20, Rmk 4.12] is immediate.

Theorem 4.1. *Let $n \geqslant 0$. Then there is a family of maps*

$$\{\theta_k : \Sigma^{\rho k} B_{n-1}(k) \to H_\star BP_{\mathbb{R}}\langle n\rangle \mid k \in \mathbb{N}\}$$

such that their sum

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star B_{n-1}(k) \to H_\star BP_{\mathbb{R}}\langle n\rangle$$

is an isomorphism of $\mathcal{E}(n)_\star$-comodules.

The following $\mathcal{A}_\star$-isomorphism is an immediate consequence.

Corollary 4.9. *For $n \geqslant 0$, there are $\mathcal{A}_\star$-comodule isomorphisms*

$$H_\star(BP_{\mathbb{R}}\langle n\rangle \wedge BP_{\mathbb{R}}\langle n\rangle) \cong H_\star(BP_{\mathbb{R}}\langle n\rangle \otimes \Sigma^{\rho k} N_{n-1}(k)).$$

This naturally leads one to ask for which heights n these isomorphisms can be realized. The first hurdle is to construct spectra realizing the Brown-Gitler comodules $N_i(k)$.

4.4. C_2-equivariant Brown–Gitler spectra.

In [BW18], Behrens–Wilson give an equivalence

$$(\Omega^\rho S^{\rho+1})^\mu \simeq H$$

of C_2-spectra. In [HW20], Hahn–Wilson observe that the left hand side of this equivalence caries a natural filtration, which produces a filtration of H by spectra. This filtration is analogous to the May-Milgram filtration of $H\mathbb{F}_2$, which Mahowald observed and Cohen proved could be used to construct Brown–Gitler spectra [Coh85, Mah77]. Hahn–Wilson point out that one can simply define equivariant Brown–Gitler spectra using this filtration. In [LPT23, Proposition 3.4], we confirm that these spectra, which we will denote $\mathcal{B}_{-1}(k)$, indeed have the property that

$$H_\star\mathcal{B}_{-1}(k) \cong \mathbb{M}_2\{x \in \mathcal{A}_\star^{C_2} \mid wt(x) \leqslant k\}.$$

In [CDGM88], Cohen-Davis-Goerss-Mahowald give a construction of integral Brown-Gitler spectra. In [LPT23, Thm 6.1], we construct C_2-equivariant lifts of these spectra, that is finite spectra $\{\mathcal{B}_0(k) |\ k \in \mathbb{N}\}$ such that

$$H_\star\mathcal{B}_0(k) \cong \mathbb{M}_2\{x \in H_\star H\mathbb{Z} |\ wt(x) \leqslant k\}.$$

Remark 4.10. *There is a small difference between our notation for the integral Brown-Gitler spectra here and in* [LPT23]. *Here we use $\mathcal{B}_0(k)$ to denote the Brown-Gitler spectrum realizing the subcomodule $H_\star H\mathbb{Z}$ having weight at most* **k**. *In* [LPT23], *we used $\mathcal{B}_0(k)$ to denote the Brown-Gitler spectrum realizing the subcomodule $H_\star H\mathbb{Z}$ having weight at most* **2k**. *Since every element of $H_\star H\mathbb{Z}$ has weight divisible by 2, these are indeed the same family of spectra. The convention in this paper makes the results of Section 4.3 easier to state, and generally makes the formulas in Section 6 cleaner and more readable.*

4.5. Margolis homology and free $\mathcal{E}(n)$-modules.

The goal of this subsection is to give a Whitehead Theorem for Margolis homology in the C_2-equivariant setting. To this end, we first establish some C_2-equivariant freeness criteria. These criteria are C_2-equivariant analogues of the $\mathbb{R}$-motivic freeness criteria studied in [BGL22, Section 2], and we follow the techniques developed there closely.

Proposition 4.11. *A finitely generated $\mathcal{E}(n)$-module M is free if and only if*

1. *M is free as an $\mathbb{M}_2$-module and*
2. *$\mathbb{F}_2 \otimes_{\mathbb{M}_2} M$ is free as an $\mathcal{E}(n)$-module.*

Note that

$$\mathbb{F}_2 \otimes \mathbb{M}_2 \otimes \mathcal{E}(n) := \mathcal{E}(n) \Big/ (a, u) \cong \mathcal{E}^{\mathbb{C}}(n) \Big/ (\tau), \tag{4.12}$$

where $\mathcal{E}^{\mathbb{C}}(n)$ is the analogously defined subalgebra of the $\mathbb{C}$-motivic Steenrod algebra, and $\tau \in \mathbb{M}_2^{\mathbb{C}}$ with $|\tau| = (0, 1)$. We will make use of the following $\mathbb{C}$-motivic lemma, which is an immediate consequence of [HK18, Theorem B(i)].

Lemma 4.13. *Let M be a $\mathcal{E}^{\mathbb{C}}(1)$-module that is free and of finite type over $\mathbb{M}_2^{\mathbb{C}}$. If $\mathcal{M}_*(M/\tau, Q_0)$ and $\mathcal{M}_*(M/\tau, Q_1)$ are zero, then M is $\mathcal{E}^{\mathbb{C}}(1)$-free.*

Combining the isomorphism of Equation (4.12) with Lemma 4.13 gives

Corollary 4.14. *Let M be a finitely generated left $\mathcal{E}(n)$-module and let*

$$M \Big/ (a, u) := M \otimes_{\mathbb{M}_2} \mathbb{F}_2.$$

Then M is a free $\mathcal{E}(n)$-module if and only if

(1) *M is free over $\mathbb{M}_2$*
(2) $\mathcal{M}_* \left(M \Big/ (a, u), Q_i \right) = 0$ *for $0 \leqslant i \leqslant n$.*

To finish our proof of Proposition 4.11, we need a C_2-equivariant analogue of the $\mathbb{R}$-motivic statement given in [BGL22, Lemma 2.1].

Lemma 4.15 ([BGL22] Lemma 2.1)**.** *A finitely generated $\mathcal{E}(n)$-module M is free if and only if*

1. *M is a free $\mathbb{M}_2$-module*
2. *$M/(a)$ is a free $\mathcal{E}(n)/(a)$-module*

Proof. If a finitely generated $\mathcal{E}(n)$-module M is free, the two conditions are immediately satisfied. Thus we consider M where the two conditions are satisfied and show M is a free $\mathcal{E}(n)$-module.

Choose a basis $\beta = \{b_1, \cdots, b_n\}$ of $M/(a)$ and let $\bar{b}_i \in M$ be any lift of b_i. Let F denote the free $\mathcal{E}(n)$-module generated by β and consider the map

$$f : F \to M$$

defined by $f(b_i) = \bar{b}_i$. We show f is an isomorphism by inductively proving f induces an isomorphism $F/(a^n) \cong M/(a^n)$ for all $n \geqslant 1$. The case $n = 1$ is true by assumption. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & F/(a^{n-1}) \arrow[r] \arrow[d, "f_{n-1}"] & F/(a^n) \arrow[r] \arrow[d, "f_n"] & F/(a) \arrow[r] \arrow[d, "f_0"] & 0 \arrow[d, equal] \\
0 \arrow[r] & M/(a^{n-1}) \arrow[r] & M/(a^n) \arrow[r] & M/(a) \arrow[r] & 0
\end{tikzcd}$$

Since a is in the center, that is $\eta_R(a) = a$ (see [HK01, Theorem 6.41] and [LSWX19, Theorems 2.6 and 2.14]), this is a diagram of $\mathcal{E}(n)$-modules. By assumption f_0 is an isomorphism and by induction, f_{n-1} is an isomorphism, so applying the five lemma gives that f_n is an isomorphism. $\square$

The proof of Proposition 4.11 then follows from Lemma 4.15 combined with Lemma 4.13 and the fact that

$$\mathcal{E}^{\mathbb{C}}(n) = \mathcal{E}(n)/(a).$$

Moreover, since the second condition of Proposition 4.11, that is that

$$\mathbb{F}_2 \otimes_{\mathbb{M}_2} \mathcal{E}(n) := \mathcal{E}(n)/(a,u) \cong \mathcal{E}^{\mathbb{C}}(n)/(\tau)$$

is free over $\mathcal{E}(1)$, can be calculated using Lemma 4.15, we get a C_2-equivariant Whitehead Theorem as a corollary.

Proposition 4.16 (Whitehead Theorem)**.** *Let M and N be finitely generated $\mathcal{E}(1)$-modules that are $\mathbb{M}_2$-free, and let $f : M \to N$ be an $\mathcal{E}(1)$-module map. Then f is a stable equivalence if and only if $f/(a,u) : M/(a,u) \to N/(a,u)$ induces an isomorphism in Margolis homologies with respect to Q_0 and Q_1.*

Remark 4.17. *Proposition 4.16 is the C_2-equivariant analogue of the $\mathbb{C}$-motivic result of* [GIR18, Corollary 4.10].

4.6. **The homology of mod-2 Brown–Gitler spectra.** Recall that $H_\star B_{-1}(0) \cong \mathbb{M}_2$. We now show that $H_\star B_{-1}(k)$ is a free and injective $\mathcal{E}\star(0)$-comodule for all $k > 0$. To do so, we need a few propositions

Proposition 4.18. *If M is a free $\mathcal{E}(n)_\star$-comodule, then M is an injective $\mathcal{E}(n)_\star$-comodule.*

Proof. First, recall from Proposition 4.6 that we can instead work with $\mathcal{E}(n)$-modules. Since every free $\mathcal{E}(n)$-module is a sum of copies of $\mathcal{E}(n)$, it will suffice to show that $\mathcal{E}(n)$ is self-injective. This follows from modifying May's proof that $\mathbb{M}_2$ is self-injective [May20, Appendix]. In particular, the graded ideals of $\mathcal{E}(n)$ are just those of $\mathbb{M}_2$, with the addition of various Q_i. □

Proposition 4.19. *For all $k > 0$, $H_\star B_{-1}(k)$ is a free and injective $\mathcal{E}(0)_\star$-comodule.*

Proof. We use Proposition 4.11 to show that $H_\star B_{-1}(k)$ is a free $\mathcal{E}(0)$-module for all $k > 0$. First, note that $H_\star B_{-1}(k)$ is a free $\mathbb{M}_2$-module. Then observe that

$$\mathbb{F}_2 \otimes_{\mathbb{M}_2} H_\star B_{-1}(k) \cong \mathbb{F}_2\{x = \bar{\xi}_1^{i_1}\bar{\xi}_2^{i_2}\cdots\bar{\xi}_r^{i_r}\bar{\tau}_0^{\epsilon_0}\bar{\tau}_1^{\epsilon_1}\cdots\bar{\tau}_s^{\epsilon_s} \mid wt(x) \leqslant n\},$$

which is exactly the homology of the classical Brown–Gitler spectrum of weight k. The homology of the classical Brown–Gitler spectrum of weight k is a free $E(0)$-module. Thus $H_\star B_{-1}(k)$ is a free $\mathcal{E}(0)$-module. So by Proposition 4.18, we know that $H_\star B_{-1}(k)$ is a free and injective $\mathcal{E}(0)_\star$-comodule. □

4.7. **The homology of integral Brown–Gitler spectra.**

Definition 4.20. *The homological lightning flash module is given by*

$$L(k) = \mathcal{E}(1)\{x_1, x_2, \cdots, x_k \mid x_{i+1}Q_1 = x_iQ_0 \,\forall 1 \leqslant i \leqslant k\}$$

where $|x_i| = i\rho + 1$. Further define $L(0) = \mathbb{M}_2$.

These lightning flash modules can be easily visualized as in Figure 4.1, which depicts $L(4)$ using the motivic grading convention that the total topological degree (number of sign plus trivial real C_2-representations) is plotted along the horizontal axis and the number of sign representations is plotted along the vertical axis. Here, a point denotes a copy of $\mathbb{M}_2$, straight arrows indicate the (non-trivial) operation of Q_0, and curved arrows indicate the (non-trivial) operation of Q_1.

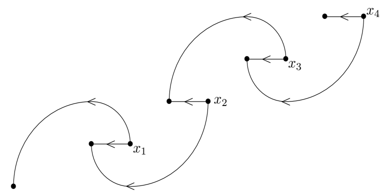

FIGURE 4.1. Homological lightning flash module: $L(4)$

Proposition 4.21. *Let $k \geqslant 0$. Then*

$$H_\star B_0\,(k) \cong L(\nu_2(k!)) \oplus W_k,$$

where W_k is a sum of suspensions of $\mathcal{E}(1)_\star$.

Proof. First, observe that $\mathcal{M}_*\big(L(k), Q_0\big) \cong \mathbb{F}_2\{1\}$ and $\mathcal{M}_*\big(L(\nu_2(k!)), Q_1\big) \cong \mathbb{F}_2\{Q_0 x_{\nu_2(k!)}\}$. Next, observe

$$\mathcal{M}_*\left(H_\star \mathcal{B}_0\,(k)\,/(a,u),\, Q_0\right) \cong \mathbb{F}_2\{1\}$$
$$\mathcal{M}_*\left(H_\star \mathcal{B}_0\,(k)\,/(a,u),\, Q_1\right) \cong \mathbb{F}_2\{\bar{\xi}_{i_1}\bar{\xi}_{i_2}\dots\bar{\xi}_{i_n}\},$$

where $i_1 < i_2 < \cdots < i_n$ and $2^{i_1} + 2^{i_2} + \cdots + 2^{i_n}$ is the 2-adic expansion of k. (This is best seen by comparison with the underlying case.) Note that $|x_{\nu_2(k!)}| = |\bar{\xi}_{i_1}\bar{\xi}_{i_2}\cdots\bar{\xi}_{i_n}|$. So if we construct an $\mathcal{E}(1)$-module map $L(\nu_p(k!)) \to H_\star \mathcal{B}_0\left(\left\lfloor \frac{k}{2}\right\rfloor\right)$ realizing the isomorphism, then we can apply Proposition 4.16 to conclude the proof. Let $S(k)$ denote the submodule of $L(k)$

$$S(k) = \mathcal{E}(1)\{\bar{\xi}_{i_1}\bar{\xi}_{i_2}\dots\bar{\xi}_{i_n}\bar{\tau}_j | j \leqslant i_1 < i_2 < \cdots < i_n\}.$$

Observe that $S(k)$ is naturally isomorphic to $L(k)$, and so the inclusion $S(k) \to H_\star B_0\left(\left\lfloor \frac{k}{2}\right\rfloor\right)$ is exactly the map we are looking for. □

5. SPLITTING $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$

In this section, we construct a family of $H\underline{\mathbb{Z}}$-module maps

$$f_k : \Sigma^{\rho k} H\underline{\mathbb{Z}} \wedge \mathcal{B}_{-1}(k) \to H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$$

such that

$$\bigvee_{k=0}^{\infty} f_k : \bigvee_{k=0}^{\infty} \Sigma^{\rho k} H\underline{\mathbb{Z}} \wedge \mathcal{B}_{-1}(k) \xrightarrow{\simeq} H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$$

is an equivalence (up to p-completion).

5.1. **Strategy.** Theorem 4.1 gives a family of maps

$$\{\theta_k : \Sigma^{\rho k}\mathcal{B}_{-1}(k) \to H_\star H\underline{\mathbb{Z}} \mid k \in \mathbb{N}\}$$

such that their sum

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star \mathcal{B}_{-1}(k) \to H_\star H\underline{\mathbb{Z}}$$

is an isomorphism of $\mathcal{E}(0)_\star$-comodules.

Using $H\underline{\mathbb{Z}}$-relative homology (discussed in Section 3.5), we can think of the family of $\mathcal{E}(0)_\star$-comodule maps

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star B_{-1}(k) \to H_\star H\underline{\mathbb{Z}}$$

as a family of maps

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star^{H\underline{\mathbb{Z}}}(H\underline{\mathbb{Z}}_\star B_{-1}(k)) \to H_\star^{H\underline{\mathbb{Z}}}(H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}})$$

and consider the $H\underline{\mathbb{Z}}$-relative Adams spectral sequence

$$E_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\Sigma^{\rho k} H_\star B_{-1}(k), H_\star H\underline{\mathbb{Z}}) \implies [H\underline{\mathbb{Z}} \wedge \Sigma^{\rho k} B_{-1}(k), H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}]^{H\underline{\mathbb{Z}}}. \tag{5.1}$$

We grade Ext-groups in the form (s, f, w), where s is the stem, that is, the total degree minus the homological degree, f is the Adams filtration, that is the homological degree, and w is the weight. So $|\theta_k| = (0, 0, 0)$. The Adams differential d_r decreases stem by 1, increases filtration by r, and preserves motivic weight.

Since $E_2^{s,f,w}$ is finite in each degree, the spectral sequence converges [Boa99, Thm 15.6, Thm 7.1]. Thus constructing maps $f_k : H\underline{\mathbb{Z}} \wedge \Sigma^{\rho k} B_{-1}(k) \to H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$ for all $n \in \mathbb{N}$ is the same as showing the θ_k survive the spectral sequence. In fact, we will show that the Adams spectral sequence collapses at the E_2-page.

5.2. **Analyzing the E_2-page.**

5.2.1. *Starting the analysis.*

Proposition 5.2. *The E_2-page of the Adams spectral sequence has the form*

$$E_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\Sigma^{\rho k} H_\star B_{-1}(k), H_\star H\underline{\mathbb{Z}}) \cong \mathrm{Ext}^{s,f,w}_{E(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V,$$

where V is an $\mathbb{M}_2$-vector space concentrated in Adams filtration $f = 0$.

Proof. Using the isomorphism

$$H_\star H\underline{\mathbb{Z}} \cong \bigoplus_{m=0}^{\infty} \theta_m : \Sigma^{\rho m} H_\star B_{-1}(m)$$

of $\mathcal{E}(0)_\star$-comodules given by Theorem 4.1 yields,

$$\begin{aligned} E_2^{s,f,w} &\cong Ext^{s,f,w}_{\mathcal{E}(0)_\star}(\Sigma^{\rho k} H_\star B_{-1}(k), H_\star H\underline{\mathbb{Z}}) \\ &\cong \bigoplus_{m=0}^{\infty} Ext^{s,f,w}_{\mathcal{E}(0)_\star}\big(\Sigma^{\rho k} H_\star B_{-1}(k), \Sigma^{\rho m} H_\star B_{-1}(m)\big). \end{aligned}$$

Since $H_\star B_{-1}(k)$ is a free and injective $\mathcal{E}(0)_\star$-comodule when $k > 0$ by Proposition 4.19, any summand $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}\left(\Sigma^{\rho k} H_\star B_{-1}(k), \Sigma^{\rho m} H_\star B_{-1}(m)\right)$ with k or m nonzero must be concentrated on the ($f = 0$)-line. Then since $H_\star B_{-1}(0) \cong \mathbb{M}_2$ by definition,

$$E_2^{s,f,w} \cong \mathrm{Ext}^{s,f,w}_{E(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V,$$

where V is an $\mathbb{M}_2$-vector space concentrated in filtration $f = 0$. □

We now compute $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$. Our computation closely follows that of $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$ in [GHIR20], which uses simpler $\mathbb{C}$-motivic and $\mathbb{R}$-motivic calculations as stepping stones. In that vein, we view C_2-equivariant coefficients as tensored up from $\mathbb{R}$-motivic coefficients. In particular,

$$\mathcal{E}(n)_\star \cong \mathbb{M}_2 \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{E}^{\mathbb{R}}_\star(n)$$

(see [GHIR20, Equation 2.4] for more details).

Let NC denote the negative cone, so that $\mathbb{M}_2 \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$ as an $\mathbb{F}_2[\tau]$-module. Then the square-zero extension $\mathbb{M}_2 \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$ induces a decomposition [GHIR20, page 8]

$$\mathrm{Ext}_{\mathcal{E}(n)_\star}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}_\star(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \oplus \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}_\star(n)}(NC, \mathbb{M}_2^{\mathbb{R}})$$

and one can use the ρ-Bockstein spectral sequence to analyze each of these summands.

Proposition 5.3 ([GHIR20] Proposition 3.1)**.** *There is a ρ-Bockstein spectral sequence*

$$E_1 = \mathrm{Ext}_{\mathrm{gr}_\rho\, \mathcal{E}(n)_\star}(\mathrm{gr}_\rho \mathbb{M}_2, \mathrm{gr}_\rho \mathbb{M}_2) \Longrightarrow \mathrm{Ext}_{\mathcal{E}(n)_\star}(\mathbb{M}_2, \mathbb{M}_2),$$

such that the differential d_r sends a class in degree (s, f, w) to a class of degree $(s-1, f+1, w)$. Furthermore, the spectral sequence decomposes into the following two pieces:

$$E_1^+ = \mathrm{Ext}_{\mathbb{C}}[\rho] \Longrightarrow Ext_{\mathbb{R}}$$

and

$$E_1^- \cong \bigoplus_{s=0}^{\infty} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \underset{\mathbb{M}_2^{\mathbb{C}}}{\otimes} Ext_{E^{\mathbb{C}}(n)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}}) \Longrightarrow Ext_{NC}.$$

5.2.2. *Analyzing E_1^+.* In [Hil11], Hill gives a complete calculation of $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(n)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$. We are only working over $\mathcal{E}(0)$ right now, so we state that portion of the result here.

Proposition 5.4 ([Hil11] Thm 3.2)**.** *There exists a ρ-Bockstein spectral sequence*

$$E_1 = \mathrm{Ext}_{\mathcal{E}^{\mathbb{C}}(0)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})[\rho] = \mathbb{F}_2[\tau, v_0, \rho] \Longrightarrow Ext_{\mathcal{E}^{\mathbb{R}}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}),$$

with differential $d_1(\tau) = \rho v_0$.

The values of $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ follow immediately.

Proposition 5.5 ([Hil11] Thm 3.1)**.**

$$\mathrm{Ext}_{E^{\mathbb{R}}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) = \mathbb{F}_2[\rho, \tau^2, v_0]/(\rho v_0),$$

where $|v_0| = (0, 1, 0)$, $|\tau^2| = (0, 0, -2)$ and $|\rho| = (-1, 0, -1)$.

5.2.3. *Analyzing E_1^-.* It remains to calculate the E_1^- summand in the ρ-Bockstein spectral sequence. Since $\mathrm{Ext}^{\mathbb{C}}_{E(n)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}})$ is free over $\mathbb{M}_2^{\mathbb{C}}$, we can write E_1^- in the following form.

Proposition 5.6 ([GHIR20, Proposition 3.1])**.**

$$E_1^- \cong \bigoplus_{s=0}^{\infty} \frac{\mathbb{M}_2^{\mathbb{C}}}{\tau^\infty}\left\{\frac{\gamma}{\rho^s}\right\} \otimes_{\mathbb{M}_2^{\mathbb{C}}} Ext_{E^{\mathbb{C}}(n)}(\mathbb{M}_2^{\mathbb{C}}, \mathbb{M}_2^{\mathbb{C}}) \Longrightarrow Ext_{\mathbb{M}_2^{\mathbb{R}}}(NC, \mathbb{M}_2^{\mathbb{R}}).$$

To determine the differentials in the negative cone, we use the strategy described on pages 17-18 of [GHIR20]. Note that E^- is an E^+-module. The E_1^- page is generated over E_1^+ by the elements $\frac{\gamma}{\rho^j \tau^k}$. The differentials in E^- are infinitely divisible by ρ, meaning that if $d_r(x) = y$, then $d_r(\frac{x}{\rho^j}) = \frac{y}{\rho^j}$ for all $j \geqslant 0$. So all differentials in the E^--summand of the ρ-Bockstein spectral sequence are determined by the differential

$$d_1\left(\frac{\gamma}{\rho\tau^{2k+1}}\right) = \frac{\gamma}{\tau^{2k+2}} v_0,$$

in combination with the Leibniz rule, E^+-module structure, and infinite-ρ-divisibility.

Thus using the ρ-Bockstein spectral sequence, we conclude

Proposition 5.7. *The summand $\mathrm{Ext}_{\mathcal{E}_\star^{NC}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ consists of two components:*

1. *elements of the form $\frac{\gamma}{\tau^{2n}}\left[\frac{1}{\rho}\right]$, which are v_0-torsion and concentrated on the $(f = 0)$-line, and*
2. *v_0-towers, of the form $\mathbb{F}_2[v_0]\{\frac{\gamma}{\tau^{2n+1}}\}$*

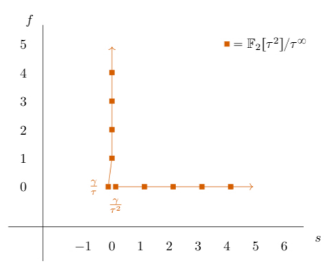

FIGURE 5.1. $\mathrm{Ext}_{\mathcal{E}_\star^{NC}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$

Now that we have both the positive cone summand $\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ and negative cone summand $\mathrm{Ext}_{\mathcal{E}_\star^{NC}(0)}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ of $\mathrm{Ext}_{\mathcal{E}_\star(n)}(\mathbb{M}_2, \mathbb{M}_2)$, we are ready to show that the Adams spectral sequence (Equation (5.1)) collapses.

5.2.4. *Running the Adams spectral sequence.*

Proposition 5.8. *The Adams spectral sequence*

$$E_2^{s,f,w} = \mathrm{Ext}^{s,t}_{\mathcal{E}(0)_\star}(\Sigma^{\rho k} H_\star \mathcal{B}_{-1}(k), H_\star H\underline{\mathbb{Z}}) \implies [H\underline{\mathbb{Z}} \wedge \mathcal{B}_{-1}(k), H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}]^{H\underline{\mathbb{Z}}}$$

collapses at the E_2-page.

Proof. We showed in Proposition 5.2 that the E_2-page of the Adams spectral sequence has the form

$$\begin{aligned} E_2^{s,f,w} &\cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\Sigma^{\rho k} H_\star \mathcal{B}_{-1}(k), H_\star H\underline{\mathbb{Z}}) \\ &\cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V, \end{aligned}$$

where V is an $\mathbb{M}_2$-vector space concentrated in Adams filtration $f = 0$.

We will first show that elements in V do not support Adams differentials. Suppose towards a contradiction that there exists $x \in V$ such that $d_r(x) = y$ is nonzero and r is the smallest natural number for which such a nonzero differential exits. Then y must be an element of $\mathrm{Ext}^{s,f,w}_{E(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V$. As calculated in Propositions 5.5 and 5.7, the only classes in $\mathrm{Ext}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$ with filtration $f > 0$ are contained in v_0-towers. Thus, by v_0-linearity, $d_r(v_0 x) = v_0 y$ must be nonzero. However, this is a contradiction, as $v_0 x = 0$ since V is concentrated in Adams filtration $f = 0$.

Therefore the only possible nonzero differentials are those from $\mathrm{Ext}^{s,f,w}_{E(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V$ to itself. However, the filtration greater than zero portions of $\mathrm{Ext}^{s,f,w}_{E(0)_\star}(\mathbb{M}_2, \mathbb{M}_2) \oplus V$ are concentrated in even Milnor-Witt degree, where Milnor-Witt degree is defined to be $s - w$, that is stem minus motivic weight. Since the Adams differential has Milnor-Witt degree -1, this rules out any nontrivial differentials. $\square$

Thus we can lift the maps $\theta_k : \Sigma^{\rho k} H_\star B_{-1}(k) \to H_\star H\underline{\mathbb{Z}}$ to maps

$$f_k : H\underline{\mathbb{Z}} \wedge \Sigma^{\rho k} B_{-1}(k) \to H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$$

for all $n \in \mathbb{N}$, and we have proved a C_2-equivariant analogue of Mahowald's splitting of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$.

Theorem 5.1. *Up to 2-completion there is a splitting*

$$H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}} \simeq \bigvee_{k=0}^{\infty} H\underline{\mathbb{Z}} \wedge \Sigma^{\rho k} \mathcal{B}_{-1}(k)$$

of $H\underline{\mathbb{Z}}$-modules.

Corollary 5.9.

$$H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}} \simeq H\underline{\mathbb{Z}} \vee V,$$

where V is a sum of suspensions of $H\underline{\mathbb{F}}_2$.

Proof. By definition $\mathcal{B}_{-1}(0) \simeq S^0$, so $H\underline{\mathbb{Z}} \wedge \mathcal{B}_{-1}(0) \simeq H\underline{\mathbb{Z}}$, and for $k > 0$, $H_\star \mathcal{B}_{-1}(k)$ is a sum of suspensions of $\mathcal{E}(0)_\star$. Thus for $k > 0$,

$$H_\star(H\underline{\mathbb{Z}} \wedge \mathcal{B}_{-1}(k)) \cong \bigoplus H_\star V.$$

$\square$

6. Splitting $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$

Let $B_0(k)$ denote the C_2-equivariant integral Brown–Gitler spectrum defined in [LPT23]. In this section we construct a family of $ku_{\mathbb{R}}$-module maps

$$f_k : \Sigma^{\rho k} ku_{\mathbb{R}} \wedge B_0(k) \to ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$$

such that

$$\bigvee_{k=0}^{\infty} f_k : \bigvee_{k=0}^{\infty} \Sigma^{\rho k} ku_{\mathbb{R}} \wedge B_0(k) \xrightarrow{\simeq} ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$$

is an equivalence (up to 2-completion).

6.1. **Strategy.** Our strategy is similar to that of Section 5.1 where we constructed a splitting of $H\underline{\mathbb{Z}} \wedge H\underline{\mathbb{Z}}$. We begin by considering the family of $\mathcal{E}(1)_\star$-comodule isomorphisms

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{k\rho} H_\star B_0(k) \to H_\star ku_{\mathbb{R}}$$

given by Theorem 4.1 in the case where $n = 1$. We write these as a family of maps

$$\bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{k\rho} H_\star^{ku_{\mathbb{R}}} (ku_{\mathbb{R}} \wedge B_0(k)) \to H_\star^{ku_{\mathbb{R}}}(ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}).$$

and then study the Adams spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)_\star} \left(H_\star \Sigma^{\rho k} B_0(k), H_\star ku_{\mathbb{R}} \right) \implies [ku_{\mathbb{R}} \wedge \Sigma^{\rho k} B_0(k), ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}]^{ku_{\mathbb{R}}}. \tag{6.1}$$

Since $E_2^{s,f,w}$ is finite in each degree, the spectral sequence converges [Boa99, Thm 15.6, Thm 7.1]. Moreover, each θ_k is in $E_2^{0,0}$, so constructing maps $f_k : ku_{\mathbb{R}} \wedge \Sigma^{k\rho} B_0(k) \to ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$ for all $k \in \mathbb{N}$ is the same as showing that the θ_k survive the spectral sequence.

On our way to showing the θ_k survive the Adams spectral sequence, it will be helpful to first record a simpler splitting of $ku_{\mathbb{R}} \wedge \mathcal{B}_0(k)$ and $ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}$. This will allow us to decompose the Adams spectral sequence 6.1 into a sum of four separate spectral sequences.

Proposition 6.2. *There are $ku_{\mathbb{R}}$-module splittings*

$$ku_{\mathbb{R}} \wedge \mathcal{B}_0(k) \simeq C_k \vee V_k,$$
$$ku_{\mathbb{R}} \wedge ku_{\mathbb{R}} \simeq C \vee V,$$

where V_k and V are sums of suspensions of H, and C_k and C contain no H-summands.

Proof. In Proposition 4.21 we showed that

$$H_\star B_0(k) \cong L(\nu_p(k!)) \oplus W_k,$$

where W_k is a finite sum of suspensions of $\mathcal{E}(1)_\star$. In Proposition 3.2, we also showed that $H_\star^{ku_{\mathbb{R}}} H \cong \mathcal{E}(1)_\star$. Take V_k to be a sum of suspensions of H such that $H_\star^{ku_{\mathbb{R}}} V_k \cong W_k$. We can use the Adams spectral sequences

$$E_2^{s,f,w} \cong \mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star^{ku_{\mathbb{R}}} V_k, H_\star \mathcal{B}_0(k) \Longrightarrow [V_k, ku_{\mathbb{R}} \wedge \mathcal{B}_0(k)]$$
$$E_2^{s,f,w} \cong \mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \mathcal{B}_0(k), H_\star^{ku_{\mathbb{R}}} V_k) \Longrightarrow [ku_{\mathbb{R}} \wedge \mathcal{B}_0(k), V_k]$$

to lift the homology splitting $H_\star B_0(k) \cong L(\nu_2(k!)) \oplus W_k$ to a splitting of $ku_{\mathbb{R}}$-module spectra by viewing the inclusion $i : H_\star^{ku_{\mathbb{R}}} V_k \cong W_k \hookrightarrow H_\star \mathcal{B}_0(k)$ as a class in filtration zero in the first E_2-page, and the projection $j : H_\star \mathcal{B}_0(k) \to W_k \cong H_\star^{ku_{\mathbb{R}}} V_k$ as a class in filtration zero of the second E_2-page. Since $\mathcal{E}(1)_\star$ is a free and injective module (see [Ric15] and also Proposition 4.18), both spectral sequences are entirely concentrated in filtration $f = 0$. Therefore, there are no differentials and both the class of the inclusion and the class of the projection lift to maps of $ku_\mathbb{R}$-module spectra

$$V_k \to ku_\mathbb{R} \wedge B_0(k) \to V_k$$

and we have a splitting $ku_\mathbb{R} \wedge B_0(k) \simeq V_k \vee C_k$.

To prove $ku_\mathbb{R} \wedge ku_\mathbb{R} \simeq C \wedge V$, consider the $\mathcal{E}(1)_\star$-comodule isomorphism

$$H_\star ku_\mathbb{R} \cong \bigoplus_{k=0}^{\infty} \theta_k : \Sigma^{\rho k} H_\star B_0(k)$$

of Theorem 4.1. Composing this isomorphism with the isomorphism $H_\star B_0\,(k) \cong L(\nu_2(k!)) \oplus W_k$ of Proposition 4.21 yields

$$H_\star ku_\mathbb{R} \cong \bigoplus_{k=0}^{\infty} \Sigma^{\rho k} L(\nu_p(k!)) \oplus H_\star V,$$

where V is a finite-type sum of copies of H. By the same spectral sequence argument as the first splitting,

$$ku_\mathbb{R} \wedge ku_\mathbb{R} \simeq ku_\mathbb{R} \wedge V.$$

□

As an immediate consequence, we get a decomposition of the Adams spectral sequence (6.1).

Proposition 6.3. *The Adams spectral sequence*

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{E}(1)_\star}\left(H_\star \Sigma^{\rho k} B_0(k), H_\star ku_\mathbb{R}\right) \implies [ku_\mathbb{R} \wedge \Sigma^{\rho k} B_0(k), ku_\mathbb{R} \wedge ku_\mathbb{R}]^{ku_\mathbb{R}}$$

decomposes into a sum of four separate spectral sequences, listed below. All but the first is concentrated on the $(f = 0)$-line.

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star C) \Longrightarrow [\Sigma^{\rho k} C_k, C],$$
$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star V) \Longrightarrow [\Sigma^{\rho k} C_k, V],$$
$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} V_k, H_\star C) \Longrightarrow [\Sigma^{\rho k} V_k, C],$$
$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} V_k, H_\star V) \Longrightarrow [\Sigma^{\rho k} V_k, V].$$

Since each summand containing V or V_k consists solely of copies of suspensions of $\mathbb{M}_2$ on the filtration $(f = 0)$-line, we will focus on the spectral sequence

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star C) \Longrightarrow [\Sigma^{\rho k} C_k, C].$$

By construction, $H_\star^{ku_\mathbb{R}} C_k \cong L(\nu_2(k!))$ and $H_\star^{ku_\mathbb{R}} C \cong \oplus_{m=0}^{\infty} L(\nu_2(m!))$. Thus in order to calculate $\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star C)$ in the proof of Proposition 6.11. we inductively compute

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(\nu_2(k!)), L(\nu_2(m!)))$$

using the long exact sequences induced by $\mathrm{Ext}_{\mathcal{E}(1)_\star}(—, L(m))$ and $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(m), —)$ applied to the short exact sequence

$$0 \to \Sigma^\rho L(k-1) \to L(k) \to \mathcal{E}(1)//\mathcal{E}(0)_\star \to 0. \tag{6.4}$$

We will phrase our long exact sequence computations in terms of the spectral sequence associated to each long exact sequence.

Since $L(0) \simeq \mathbb{M}_2$, the base case for this inductive computation is $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$. This Ext term is also the E_2-page of the Adams spectral sequence for $\pi_\star ku_{\mathbb{R}}$ and is computed in [GHIR20]. In order to build inductively on their result, we describe $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$.

Similarly to the height zero case, the square-zero extension $\mathbb{M}_2 \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$ again induces a decomposition

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}((\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \oplus \mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}((NC, \mathbb{M}_2^{\mathbb{R}})$$

[GHIR20, Proposition 2.2].

In [Hil11, Theorem 3.1], Hill computed $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}(\mathbb{M}_2, \mathbb{M}_2)$. In the notation of [GHIR20, Proposition 6.3],

$$\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}(\mathbb{M}_2, \mathbb{M}_2) \cong \mathbb{F}_2[\rho, \tau^4, v_0, \tau^2 v_0, v_1]/(\rho v_0, \rho^3 v_1, (\tau^2 v_0)^2 + \tau^4 v_0^2),$$

where the Milnor-Witt weight, stem s, filtration f, and motivic weight w are given in Table 1.

Milnor-Witt	$(s,\ f,\ w)$	$x \in \mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}(1)}$
0	$(-1, 0, -1)$	ρ
0	$(0, 1, 0)$	v_0
1	$(2, 1, 1)$	v_1
2	$(0, 1, -2)$	$\tau^2 v_0$
4	$(0, 0, -4)$	τ^4

TABLE 1. Generators for $\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}(\mathbb{M}_2, \mathbb{M}_2)$

In charts, $\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}(\mathbb{M}_2, \mathbb{M}_2)$ is given in Figure 6.1. The horizontal axis is the stem (s), and the vertical axis is the Adams filtration (f). Note that the motivic weight (w) is suppressed in this depiction. We will use this grading in each of the following charts. One can also refere to the charts in [GHIR20, Section 12].

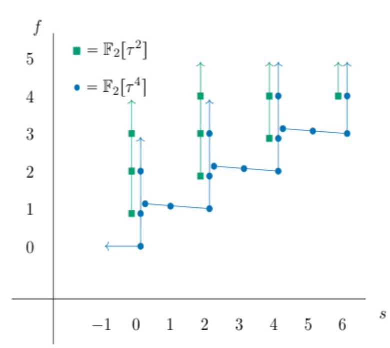

FIGURE 6.1. $\mathrm{Ext}_{\mathcal{E}_\star^{\mathbb{R}}(1)}(\mathbb{M}_2, \mathbb{M}_2)$

The summand $\mathrm{Ext}_{\mathcal{E}^{NC}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$, is a module over $\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$. As such, it is best understood by reading the charts in Figure 6.2. In particular, $\mathrm{Ext}_{\mathcal{E}^{NC}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$ is a direct sum of parts A and B in Figure 6.2. For full algebraic details see [GHIR20, §7 - 9].

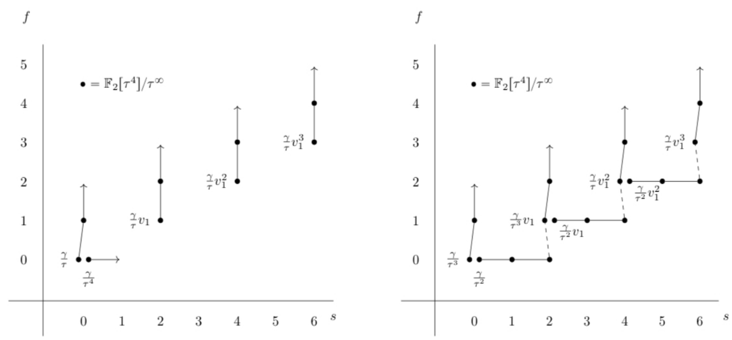

FIGURE 6.2. $\mathrm{Ext}_{\mathcal{E}^{NC}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$

Computing inductively, we find the module $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m))$ has the structure that the reader familiar with the non-equivariant case might hope for: up to v_1-extensions, it consists of a shifted copy of $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m))$, along with a sum of copies of $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$. The equivariant v_1-extensions also parallel the nonequivariant case.

Proposition 6.5. *As an* $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$*-module,* $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m))$ *is generated by* $\{x_i \mid 0 \leqslant i \leqslant m\}$*, where* $|x_i| = (2i, 0, i)$*. The generator* x_m *carries a copy of* $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$*, and for each* $0 \leqslant i < m$*,* x_i *carries a copy of* $\mathrm{Ext}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$*. There are extensions* $v_1 x_i = v_0 x_{i+1}$ *for each* i*.*

Proof. Suppose that for $m' < m$, the Ext group $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m'))$ is generated by $\{x_i \mid 0 \leqslant i \leqslant m'\}$, where $|x_i| = (2i, 0, i)$. The generator $x_{m'}$ carries a copy of $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$, while for each $0 \leqslant i < m'$, x_i carries a copy of $Ext_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$. There are extensions $v_1 x_i = v_0 x_{i+1}$ for each i. We will show that $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m))$ has the desired description.

Consider the long exact sequence

$$\begin{aligned} \cdots \to Ext^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m)) &\to Ext^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathcal{E}(1)//\mathcal{E}(0)_\star) \\ &\xrightarrow{d} Ext^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \Sigma^\rho L(m-1)) \to \cdots . \end{aligned}$$

induced by the short exact sequence

$$0 \to \Sigma^\rho L(k-1) \to L(k) \to \mathcal{E}(1)//\mathcal{E}(0)_\star \to 0$$

of Equation (6.4),

First, observe that we can use change-of-rings to write

$$Ext^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathcal{E}(1)//\mathcal{E}(0)_\star) \cong Ext^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2).$$

The induction hypothesis together with the description of $Ext^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$ in Proposition 5.5 and Proposition 5.7 imply the differential

$$d : \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathcal{E}(0)_\star) \to \mathrm{Ext}^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \Sigma^\rho L(m-1))$$

must be zero for degree reasons. For example, this can be observed in Figure 6.3. Note that in Figure 6.3, the classes have been relabeled as described at the end of the proof.

Therefore,

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m)) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathcal{E}(0)_\star) \oplus \mathrm{Ext}^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \Sigma^\rho L(m-1)),$$

$$\begin{aligned}\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m)) &\cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathcal{E}(0)_\star) \oplus \mathrm{Ext}^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \Sigma^\rho L(m-1)) \\ &\cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathcal{E}(0)_\star) \oplus \Sigma^\rho \,\mathrm{Ext}^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m-1)),\end{aligned}$$

up to extensions.

Let x denote the generator of $\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, \mathbb{M}_2)$. By comparison with the underlying calculation, there must be an extension $v_1 x = v_0(\Sigma^\rho x_0)$. There are no other additional new multiplicative extensions for degree reasons.

Relabeling x by x_0 and $\Sigma^\rho x_i$ by x_{i+1} finishes the proof. $\square$

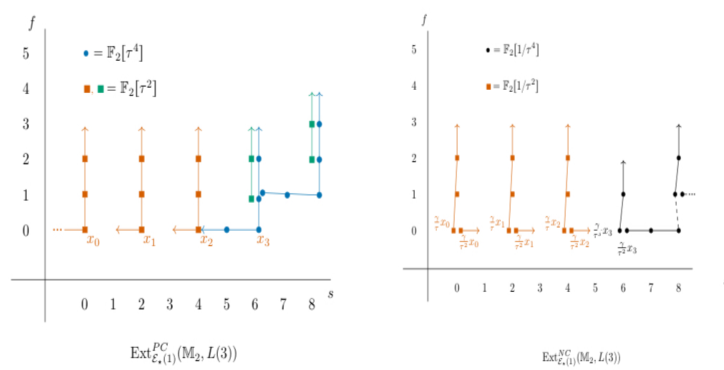

FIGURE 6.3. $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(3))$

We now move on to compute $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m)$ for $k > 0$. We will make use the following two lemmas.

Lemma 6.6. *There is a 'wrong-side' change-of-rings isomorphism*

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathcal{E}(1)//\mathcal{E}(0)_\star, -) \cong \Sigma^{-\rho-1}\,\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, -).$$

Proof. By the equivalence of categories between left $\mathcal{E}(n)_\star$-comodules and right $\mathcal{E}(n)$-modules (Proposition 4.6),

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\mathcal{E}(1)//\mathcal{E}(0)_\star, -) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)}(\mathcal{E}(1)//\mathcal{E}(0)_\star, -).$$

As an $\mathcal{E}(1)$-module, $\mathcal{E}(1)//\mathcal{E}(0)_\star \cong \Sigma^{\rho+1}\mathcal{E}(1)//\mathcal{E}(0)$. By ordinary change-of-rings,

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)}(\Sigma^{\rho+1}\mathcal{E}(1)//\mathcal{E}(0), -\,) \cong \Sigma^{-\rho-1}\,\mathrm{Ext}^{s,f,w}_{\mathcal{E}(0)}(\mathbb{M}_2, -\,).$$

Applying the equivalence of categories again concludes the proof. □

Lemma 6.7. *Let $m > k$. For $s < 0$ and even,* $\mathrm{Hom}^{s,*}_{\mathcal{E}(1)_\star}(L(k), L(m)) = 0$.

This is a straightforward computation and is best checked by drawing the relevant lightning flash modules.

We are now ready to compute $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ for $k > 0$. We proceed by fixing $m \geqslant 0$ and inducting on k. Since the resulting Ext groups have a different form when $k > m$, we first describe $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ for $k \leqslant m$ before moving on to the case where $k > m$.

Proposition 6.8. *Let $k \leqslant m$. Then*

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m)) \cong \mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m-k)) \oplus V,$$

where V consists of v_0 and v_1-torsion concentrated on the $(f = 0)$-line, specifically:

1. *infinitely ρ-divisible towers, and*
2. *ρ-towers in odd stem s and filtration degree $(f = 0)$.*

Proof. The base case of the induction, when $k = 0$, holds by Proposition 6.5. Suppose that the claim holds for $k - 1$ and consider the long exact sequence

$$\cdots \to \mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(L(k), L(m)) \to \mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\Sigma^\rho L(k-1), L(m))$$
$$\xrightarrow{d} \mathrm{Ext}^{s+1,f,w}_{\mathcal{E}(1)_\star}(\mathcal{E}(1)//\mathcal{E}(0)_\star, L(m)) \to \cdots .$$

induced by the short exact sequence

$$0 \to \Sigma^\rho L(k-1) \to L(k) \to \mathcal{E}(1)//\mathcal{E}(0)_\star \to 0.$$

By the induction assumption,

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(\Sigma^\rho L(k-1), L(m)) \cong Ext_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m-k+1)) \oplus V.$$

The ‘wrong-side‘ change of rings isomorphism (Lemma 6.6) gives

$$Ext^{s,f,w}_{\mathcal{E}(1)_\star}(\mathcal{E}(1)//\mathcal{E}(0)_\star, L(m)) \cong \Sigma^{-\rho-1} Ext^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, L(m)).$$

So

$$\Sigma^{-\rho-1} Ext^{s,f,w}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, L(m)) \oplus Ext^{s,f,w}_{\mathcal{E}(1)_\star}(\Sigma^\rho L(k-1), L(m)) \implies Ext_{\mathcal{E}(1)_\star}(L(k), L(m)).$$

Let $\Sigma^{-\rho-1}y$ denote the generator of the v_0-tower in $\Sigma^{-\rho-1}Ext^{*,*,*}_{\mathcal{E}(0)_\star}(\mathbb{M}_2, L(m))$ as in Figure 6.4. There is a potential nonzero differential from $\Sigma^{-\rho}x_0$ to $\Sigma^{-\rho-1}v_0 y$. Lemma 6.7 implies that $Ext^{-2,0,*}(L(k), L(m))$ must be zero, so indeed $d(\Sigma^{-\rho}x_0) = \Sigma^{-\rho-1}v_0 y$.

The differential d is $\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, \mathbb{M}_2)$-linear, so this suffices to determine all other differentials. There is no room for extensions. Thus indeed

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m) \cong \mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(m-k)) \oplus V,$$

as illustrated in Figures 6.4 and 6.5 which depict the case where $m = 2$ and one inducts from $k = 0$ to $k = 1$. □

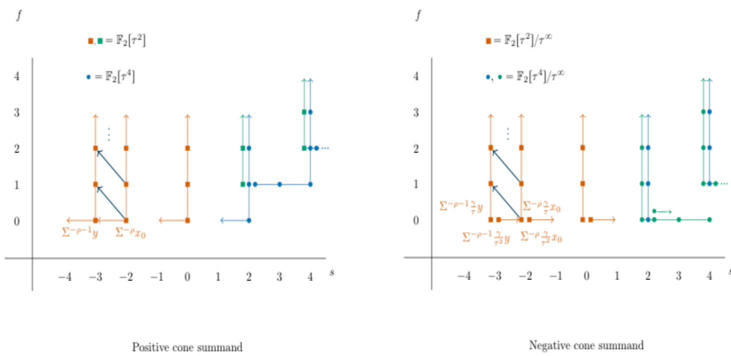

FIGURE 6.4. E_1-page computing $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), L(2))$

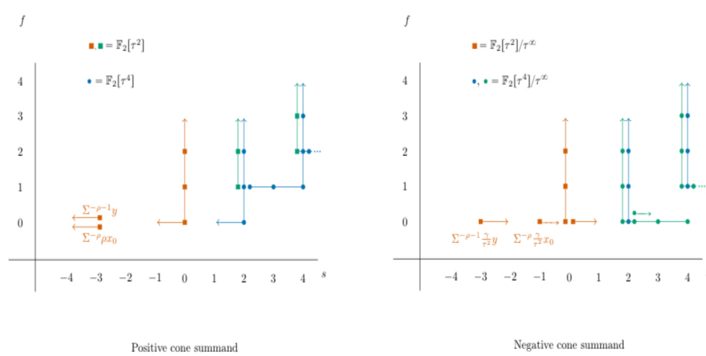

FIGURE 6.5. $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), L(2))$

Having computed $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ for fixed $m \geqslant 0$ by induction on $k \leqslant m$, we are now ready to continue the induction for $k > m$.

Lemma 6.9. *The positive cone* $\mathrm{Ext}^{*,*,*}_{E^{\mathbb{R}}_\star(1)}(L(k), L(m))$ *for* $k > m$ *consists of:*

(1) *a triangle formation:*

$$\mathbb{F}_2[\rho, \tau^2, v_0, v_1]/\rho v_0 \left\{ y_0, \ldots, y_{m-k-1} \;\middle|\; \begin{array}{ll} v_1 y_i = v_0 y_{i+1}, & v_0^2 y_0 = 0, \\ v_1 y_{m-k-1} = 0, & v_0^{i+1} y_i \tau^{4i+2} = 0 \;\; \forall i \end{array} \right\},$$

where $|y_i| = \big(-1 - 2(k-m-i), 0, -(k-m-i)\big)$,

(2) *infinite* ρ*-towers generated by* b *where* $|b| = (2(m-k)-1,\, 0,\, m-k-1)$,

(3) *a copy of* $v_1 \mathrm{Ext}_{E^{\mathbb{R}}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$*, with generator denoted* x *and* $|x| = (2(m-k), m-k, m-k)$,

(4) *ρ-pairs:*
$$\mathbb{F}_2[\rho, v_1]\left\{b \,|\, v_1^{m-k-1}b = \rho x,\ \rho^2 b = 0\right\}.$$

Lemma 6.10. *The negative cone* $\mathrm{Ext}^{*,*,*}_{\mathcal{E}(1)^-}(L(k), L(m))$ *for* $k > m$ *consists of:*

(1) *a triangle formation:*
$$\frac{\mathbb{F}_2[\tau^2]}{\tau^\infty, \rho^\infty}\left\{\ \frac{\gamma}{\tau^2}y_0, \cdots, \frac{\gamma}{\tau^2}y_{m-k-1}\ \middle|\ \begin{array}{ll} v_1\frac{\gamma}{\tau^2}y_i = v_0\frac{\gamma}{\tau^2}y_{i+1}, & v_0\frac{\gamma}{\tau^2}y_0 = 0, \\ v_1\frac{\gamma}{\tau^2}y_{m-k-1} = 0, & v_0^{i+1}\frac{\gamma}{\tau^2}y_i\tau^{4j+2} = 0\ \ \forall i \end{array}\right\},$$
(2) *infinite ρ-divisible towers in filtration $f = 0$,*
(3) *a copy of* $v_1\,\mathrm{Ext}_{\mathcal{E}(1)^{NC}_\star}(\mathbb{M}_2, \mathbb{M}_2)$*, with generators denoted* $\frac{\gamma}{\tau^i}x$ *and* $|x| = (2(m-k), m-k, m-k)$,
$$\frac{\mathbb{F}_2[v_0, v_1, \tau^2]}{\tau^\infty,\ \rho^\infty}\{x, c\}/(\rho^{-3}x,\ v_0\rho x,\ v_0\rho^{-2}x,\ \tau^{4k+2}x,\ \rho^{-i}y,\ \rho^{-2}v_0\tau^{4k} - \tau^{4k}c),$$
(4) *ρ-divisible triples:*
$$\frac{\mathbb{F}_2[\tau^4]}{\tau^\infty\rho^\infty}\{c\}/(\rho^{-3}c,\ v_1^n c - x).$$

Before proving Lemmas 6.9 and 6.10, we introduce some charts illustrating $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ in the cases where $k = 1$ and $m = 0$ (Figure 6.7) and where $k = 2$ and $m = 0$ (Figure 6.9). In both figures, one can see the triangle formation begin to emerge in stems $s \leqslant -3$. The shape of the triangle formation becomes more apparent as the difference between k and m grows.

Examples of infinite ρ-towers can be found starting in filtration $f = 0$ and stem $s = -3$ in the positive cone summands. As multiplication by ρ is represented by a horizontal line to the left, these ρ towers appear as arrows on the zero line pointing to the left. Examples of infinite ρ-divisible towers can be found starting in filtration $f = 0$ and stem $s = -3$ in the negative cone summands. As divisibility by ρ is represented by a horizontal line to the right, these ρ-divisible towers appear as arrows on the zero line pointing to the right.

The copies of $v_1\,\mathrm{Ext}_{\mathcal{E}^{\mathbb{R}}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$ in the positive cone summands and $v_1\,\mathrm{Ext}_{\mathcal{E}^{NC}_\star(1)}(\mathbb{M}_2, \mathbb{M}_2)$ are best identified by comparison with the charts in Figure 6.1 and Figure 6.2, respectivly.

The ρ-pairs and ρ-divisible triples only appear when $k - m \geqslant 2$. As such, a ρ-pair can be seen in the positive cone summand of Figure 6.9. Specifically, the ρ-pair there is the pair of two blue dots in filtration $f = 1$ and stems $s = -3$ and $s = -4$ linked by a ρ-multiplication. A ρ-divisible triple can be seen in the negative cone summand of Figure 6.9. Specically, the ρ-divisible triple is the three green dots in filtration $f = 1$ and stems $s = -2$, $s = -1$, and $s = 0$ linked by divisibility by ρ.

While proceeding through the inductive proof of Lemmas 6.9 and 6.10, it is helpful to keep similar charts of the Ext-groups $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ in mind.

Proof of Lemmas 6.9 and 6.10. Suppose the above description holds for all $k' < k$. For degree reasons, the only possible nontrivial differentials originate from the v_0-tower on $\Sigma^{-\rho}x$ in the positive cone, and the v_0-tower on $\Sigma^{-\rho}\frac{x}{\tau^2}$ in the negative one. This is illustrated in the case where $m = 0$ and we are inducting from $k = 0$ to $k = 1$ in **??** and in the case where $m = 0$ and we are inducting from $k = 1$ to $k = 2$ in Figure 6.8.

In Proposition 6.8 we computed $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ for $k = m$, by fixing m and inducting on k. Alternatively, we could fix k, and induct on m to compute $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ for $k = m$. We describe the base case of this alternate induction when $m = 0$, computing $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), \mathbb{M}_2)$, in detail. The cases where $m \neq 0$ are similar and left to the reader.

The E_1-page of the spectral sequence computing $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), \mathbb{M}_2)$ from the short exact sequence

$$0 \to \Sigma^\rho L(k-1) \to L(k) \to \mathcal{E}(1)//\mathcal{E}(0)_\star \to 0$$

is displayed in **??**. By comparison with the underlying calculation, there must be a differential

$$d_1(\Sigma^{-\rho}1) = \Sigma^{-\rho-1}v_0.$$

By linearity in τ and v_0, this implies there are also differentials

$$d_1(\Sigma^{-\rho}v_0^i\tau^{4j}) = \Sigma^{-\rho-1}v_0^{i+1}\tau^{4i}, \qquad i,j \geqslant 0$$
$$d_1(\Sigma^{-\rho}v_0^i\tau^{4j-2}) = \Sigma^{-\rho-1}v_0^{i+1}\tau^{4j-2} \qquad i \geqslant 0, j \geqslant 1$$

in the positive cone summand and differentials

$$d_1\left(\Sigma^{-\rho}v_0^i\frac{\gamma}{\tau^{1+4j}}\right) = \Sigma^{-\rho-1}v_0^{i+1}\frac{\gamma}{\tau^{1+4j}} \qquad i \geqslant 0, j \geqslant 1$$
$$d_1\left(\Sigma^{-\rho}v_0^i\frac{\gamma}{\tau^{3+4j}}\right) = \Sigma^{-\rho-1}v_0^{i+1}\frac{\gamma}{\tau^{3+4j}} \qquad i \geqslant 0, j \geqslant 1.$$

in the negative cone summand. Here it may be helpful to visualize the calculation using the chart in **??** and recall that $\mathbb{F}_2[\tau^2]/\tau^\infty\left\{\Sigma^{-\rho-1}\frac{\gamma}{\tau}\right\}$ consists of all classes $\Sigma^{-\rho-1}\frac{\gamma}{\tau^{1+2k}}$ where $k \geqslant 1$.

The only remaining classes on the E_1-page that could potentially support a differential are $\mathbb{F}_2[\tau^4]/\tau^\infty\left\{\Sigma^{-\rho}\frac{\gamma}{\tau^2}\right\}$ and $\mathbb{F}_2[\tau^4]/\tau^\infty\left\{\Sigma^{-\rho}\frac{\gamma}{\tau^4}\right\}$. However, none of these can support differentials due to ρ-linearity. Specifically, suppose some $\Sigma^{-\rho}\frac{\gamma}{\tau^{2+4j}}$ (or $\Sigma^{-\rho}\frac{\gamma}{\tau^{4+4j}}$) supports a differential. Then by ρ-linearity, $\rho^2\Sigma^{-\rho}\frac{\gamma}{\rho^2\tau^{2+4j}}$ (or $\rho^2\Sigma^{-\rho}\frac{\gamma}{\rho^2\tau^{4+4j}}$) must also support a differential, but this is not possible for degree reasons. Thus $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), \mathbb{M}_2)$ has the form described in Lemmas 6.9 and 6.10 and illustrated in Figure 6.7. This completes the base case calculation when $k = 1$.

We now return to the general case where we have fixed k and compute $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ where $k = m$ by induction on m. Comparing the result of this induction with that of Proposition 6.8 (where the same $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$ with $k = m$ is computed by fixing m and inducting on k), we see that the differential $d(\Sigma^{-\rho}x)$ must be nontrivial in order for the two methods to agree. Specifically, if, when we fix m and induct on k, we assume $d(\Sigma^{-\rho}x) = 0$, then we would get an infinite v_0-tower in odd stem in $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(m))$. If we then fixed k and inducted on m to compute $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(k))$, we would get the same infinite v_0-tower in odd stem in $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(k))$. But we already know from Proposition 6.8 that $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(k), L(k))$ has no infinite v_0 towers in odd stems. Thus $d(\Sigma^{-\rho}x) = v_0^{k-m}\Sigma^{-\rho-1}y$. Likewise, comparison with Proposition 6.8 tells us that $d(\Sigma^{-\rho}\frac{\gamma}{\tau^2}x) = v_0^{k-m}\Sigma^{-\rho-1}\frac{\gamma}{\tau^2 y}$. Linearity in τ and v_0 give the remaining differentials in the v_0-towers supported by these classes. $\square$

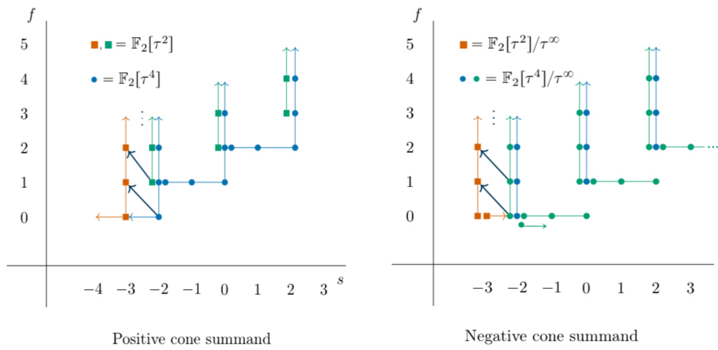

FIGURE 6.6. E_1-page computing $\mathrm{Ext}_{\mathcal{E}(1)_\star}(L(1), \mathbb{M}_2))$

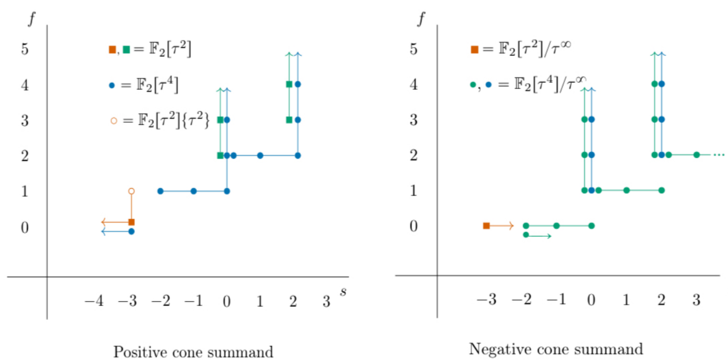

FIGURE 6.7. $\mathrm{Ext}_{E(1)_\star}(L(1), \mathbb{M}_2)$

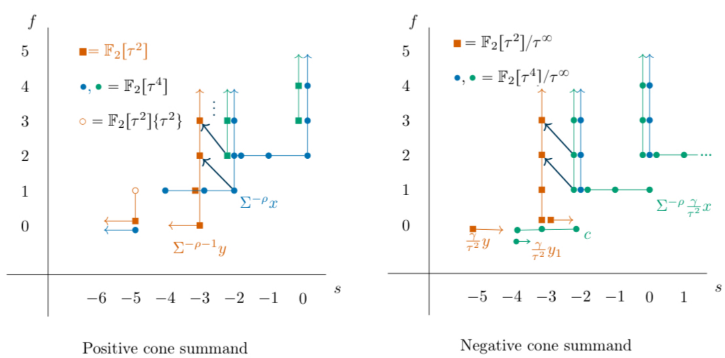

FIGURE 6.8. E_1-page computing $\mathrm{Ext}_{E(1)_\star}(L(2), \mathbb{M}_2)$

6.2. **Analyzing Adams differentials.** The goal of this section is to finish proving the Adams spectral sequence (6.1) collapses. Specifically we show,

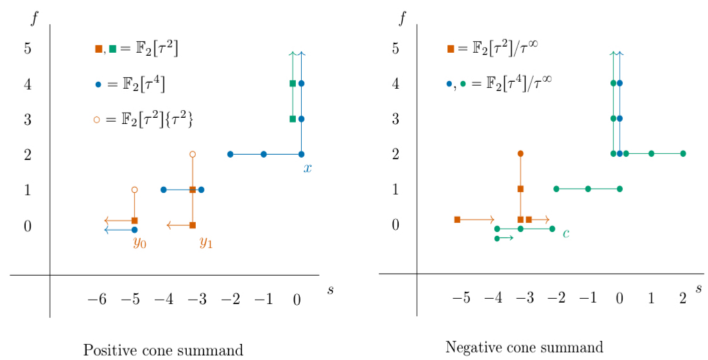

FIGURE 6.9. $\mathrm{Ext}_{E(1)_\star}(L(2), \mathbb{M}_2)$

Proposition 6.11. *For all $k \geqslant 0$, the Adams spectral sequence*

$$E_2^{s,t} = Ext_{\mathcal{E}(1)_\star}\left(H_\star \Sigma^{\rho k} B_0(k), H_\star ku_{\mathbb{R}}\right) \implies \left[ku_{\mathbb{R}} \wedge \Sigma^{\rho k} B_0(k), ku_{\mathbb{R}} \wedge ku_{\mathbb{R}}\right]^{ku_{\mathbb{R}}}$$

collapses.

Our main theorem (Theorem 6.1) will follow immediately from this proposition.

Proof. In Proposition 6.3, we showed that the Adams spectral sequence splits into a sum of four different Adams spectral sequences, and that all of the summands collapse at the E_2-page (and are concentrated on the $(f = 0)$-line) except

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star C) \Longrightarrow [\Sigma^{\rho k} C_k, C].$$

We also observed

$$\mathrm{Ext}^{s,f,w}_{\mathcal{E}(1)_\star}(H_\star \Sigma^{\rho k} C_k, H_\star C) \cong \bigoplus_{m=0}^{\infty} \Sigma^{\rho(m-k)} \mathrm{Ext}_{\mathcal{E}(1)_\star}(L(\nu_2(k!)), L(\nu_2(m!))).$$

To show the Adams spectral sequence collapses, we will begin by considering Adams differentials with source a v_1-torsion class. First, observe that by v_1-linearity, the only potential targets of such differentials are other v_1-torsion classes. There are two types of v_1-torsion classes in the positive cone. The first has the form $\Sigma^{\rho(m-k)} v_0^i y_j$ and consists of v_0-towers in the triangle formation (Lemma 6.9 (1). All other v_1-torsion classes in the positive cone summands are infinite ρ-towers based in odd stem and filtration $f = 0$ (Lemma 6.9 (2). In the negative cone summands, the v_1-torsion classes are $\Sigma^{\rho(m-k)} v_0^i \frac{\gamma}{\tau^2} y_j$, the v_0-towers in the triangle formation (Lemma 6.10 1), and the infinite ρ-divisible towers in filtration $f = 0$ (Lemma 6.10 (2)). By ρ-linearity, the ρ-towers in the positive cone summands cannot support differentials. Similarly, all the classes in the ρ-divisible towers in the negative cone summands cannot support differentials. Suppose that $d_r(x) \neq 0$ for some ρ^n-divisible x. Then since $d_r(x) = \rho^n d_r(x/\rho^n)$, $d_r(x/\rho^n)$ must also be nonzero. Thus the infinitely ρ-divisible towers cannot support differentials either.

Further, since the v_0-towers in the triangle formation are all in odd stem, the classes $\Sigma^{\rho(m-k)}v_0^i y_j$ and $\Sigma^{\rho(m-k)}v_0^i \frac{\gamma}{\tau^2} y_j$ cannot support differentials for degree reasons.

We will now consider the possibility of v_1-torsion free classes supporting differentials with target a v_1-torsion class. In particular, we will show all such potential differentials must be zero.

In positive cone summands, the possible sources for such differentials are of the form

(1) $\Sigma^{\rho(m-k)}x_i$, which are found in the summands $\Sigma^{\rho(m-k)}\operatorname{Ext}_{\mathcal{E}(1)}(L(\nu(k!), L(\nu(m!))$ when $\nu_2(k!) \leqslant \nu_2(m!)$.
(2) $\Sigma^{\rho(m-k)}x$, which are found in the summands $\Sigma^{\rho(m-k)}\operatorname{Ext}_{\mathcal{E}(1)}(L(\nu(k!), L(\nu(m!))$ when $\nu_2(k!) > \nu_2(m!)$.

The (stem, filtration, weight) degrees of these classes are $|\Sigma^{\rho(m-k)}x_i| = \big(2(i+m-k), 0, i+m-k\big)$ and $|\Sigma^{\rho(m-k)}x| = \big(2(m-k), k-m, m-k\big)$. More succinctly, they are all of the form $(2n, *, n)$ for some n. Likewise, in the negative summand the generators are of the form

(1) $\Sigma^{\rho(m-k)}\frac{\gamma}{\tau^{4j+1}}\,x_i$
(2) $\Sigma^{\rho(m-k)}\frac{\gamma}{\rho^2\tau^{4j+2}}\,x_{\nu_2(m!)-\nu_2(k!)}$
(3) $\Sigma^{\rho(m-k)}\frac{\gamma}{\tau^{4j+3}}\,x_{\nu_2(m!)-\nu_2(k!)}$
(4) $\Sigma^{\rho(m-k)}\frac{\gamma}{\rho^2\tau^{4j+2}}\,x$.

Observe that generators of form (1) are all in stem $2n$ with motivic weight congruent to $n+2$ mod 4 for some n, while generators of forms (2)-(4) the classes are in stem $2n$ with motivic weight congruent to n mod 4.

First we will show that no differentials can hit the v_1-torsion classes $\Sigma^{\rho(m-k)}v_0^i y_j$ and $\Sigma^{\rho(m-k)}v_0^i \frac{\gamma}{\tau^2} y_j$ for degree reasons. First, note that $\Sigma^{\rho(m-k)}v_0^i y_j$ and $\Sigma^{\rho(m-k)}v_0^i \frac{\gamma}{\tau^2} y_j$ only occur in stem $s \leqslant -3$. On the other hand, for $m' \geqslant k$, $\Sigma^{\rho(m'-k)}\operatorname{Ext}_{\mathcal{E}(1)}(L(\nu_2(k!)), L(\nu_2(m'!)))$ is contained entirely in stem $s \geqslant 0$. So we are just left to show that there are no differentials from

$$\Sigma^{\rho(m'-k)}\operatorname{Ext}_{\mathcal{E}(1)}(L(\nu_2(k!)), L(\nu_2(m'!)))$$

to $\Sigma^{\rho(m-k)}v_0^i y_j$ and $\Sigma^{\rho(m-k)}v_0^i \frac{\gamma}{\tau^2} y_j$ for $m' < k$. Let $r \geqslant 0$. In stem $-2r$, any generators $\Sigma^{\rho(m'-k)}x$ and $\Sigma^{\rho(m'-k)}\frac{\gamma}{\rho^2\tau^{4j+2}}x$ must be in filtration at least $\nu_2(k!) - \nu_2((k-r)!)$. On the other hand, any class of the form $\Sigma^{k-m}v_0^j y_i$ or $\Sigma^{k-m}v_0^j \frac{\gamma}{\tau^2} y_i$ in stem $-2r-1$ has filtration at most $\nu_2(k!) - \nu_2((k-r+1)!) + 1$. Thus no classes of the form $\Sigma^{\rho(m-k)}y_i$ and $\Sigma^{\rho(m-k)}\frac{\gamma}{\tau^2} y_i$ are the target of differentials.

Finally, we will show that there are no nontrivial differentials from v_1-torsion-free classes to other v_1-torsion-free classes. Note that the generators of the v_1-torsion-free classes are all in even stem, so any potential target will be in odd stem. The v_1-torsion-free classes in odd stem are all of the form

(1) $\Sigma^{(m'-k)\rho}\rho x_{\nu_2(m'!)-\nu_2(k!)}[v_1, \tau^4]$
(2) $\Sigma^{(m'-k)\rho} b\ [v_1, \tau^4]$

(3) $\Sigma^{(m'-k)\rho}\frac{\gamma}{\rho\tau^{4i+2}}v_1^j x_{\nu_2(m'!)-\nu_2(k!)}$, $i,j\in\mathbb{N}$
(4) $\Sigma^{(m'-k)\rho}\frac{\gamma}{\rho\tau^{4i+2}}v_1^j c$, $i,j\in\mathbb{N}$

Note that obstructions of type (1) and (2) are all in stem $2n-1$ for some n and weight congruent to $(n-1)$ mod 4, while those of type (3) and (4) are all in stem $2n-1$ for some n and weight congruent to $(n+3)$ mod 4. Recall that the Adams differential decreases stem by 1 and preserves motivic weight, so indeed no nontrivial differentials are possible here. □

Thus the class $\theta_k : H_\star\Sigma^{\rho k}B_0(k) \to H_\star\Sigma^{\rho k}ku_{\mathbb{R}}$ survives the Adams spectral sequence, and we have proved our main theorem.

Theorem 6.1. *Up to 2-completion, there exists a splitting of $ku_{\mathbb{R}}$-modules*

$$ku_{\mathbb{R}}\wedge ku_{\mathbb{R}} \simeq ku_{\mathbb{R}}\wedge \Sigma^{\rho k}\mathcal{B}_0(k).$$

6.3. **Description in terms of C_2-equivariant Adams covers.** We now give a description of the splitting of Theorem 6.1 in terms of C_2-equivariant Adams covers. Let $ku_{\mathbb{R}}^{\langle n\rangle}$ denote the n-th Adams cover of $ku_{\mathbb{R}}$, that is, the n^{th} term in a minimal Adams resolution of $ku_{\mathbb{R}}$ over H (see [Lel84, p.2-3] for discussion). Note that $H_\star^{ku_{\mathbb{R}}}ku_{\mathbb{R}}^{\langle n\rangle}\cong L(\nu_2(n!))$, and that $ku_{\mathbb{R}}^{\langle n\rangle}$ must be a $ku_{\mathbb{R}}$-module by construction. Up to homotopy, the Adams cover $ku_{\mathbb{R}}^{\langle n\rangle}$ is uniquely determined by its homology and the fact that it is a $ku_{\mathbb{R}}$-module. This can be seen from the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star^{ku_{\mathbb{R}}}ku_{\mathbb{R}}^{\langle n\rangle}, H_\star^{ku_{\mathbb{R}}}ku_{\mathbb{R}}^{\langle n\rangle}) \cong \mathrm{Ext}_{\mathcal{E}(1)_\star}(L(\nu_2(n!)), L(\nu_2(n!)) \Longrightarrow [ku_{\mathbb{R}}^{\langle n\rangle}, ku_{\mathbb{R}}^{\langle n\rangle}]^{ku_{\mathbb{R}}},$$

which the proof of our main theorem (Theorem 6.1) shows collapses at the E_2-page.

We prove

Theorem 6.2. *Up to 2-completion,*

$$ku_{\mathbb{R}}\wedge ku_{\mathbb{R}} \simeq \bigvee_{k=0}^{\infty}\Sigma^{\rho k}ku_{\mathbb{R}}^{\langle \nu_2(k!)\rangle}\vee V,$$

where V is a sum of suspensions of H.

Proof. Proposition 6.2 shows $ku_{\mathbb{R}}\wedge\mathcal{B}_0(k)\cong C_k\vee V_k$, where V_k is a sum of suspensions of H and C_k contains no H-summands. Specifically, $H_\star^{ku_{\mathbb{R}}}C_k\cong L(\nu_2(n!))$. Thus we can use the Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{E}(1)}(H_\star^{ku_{\mathbb{R}}}ku_{\mathbb{R}}^{\langle n\rangle}, H_\star C_k) \Longrightarrow [ku_{\mathbb{R}}^{\langle n\rangle}, C_k]^{ku_{\mathbb{R}}}$$

to lift the isomorphism to an equivalence of C_2-spectra. Since the E_2-page of this spectral sequence is simply a summand of the E_2-page of the Adams spectral sequence (6.1) which appeared in the proof of Theorem 6.1, the same arguments imply this spectral sequence also collapses. □

6.4. **$ku_{\mathbb{R}}$-operations and cooperations.**

6.4.1. *The $ku_{\mathbb{R}}$-cooperations algebra.* Another consequence of our proof of Theorem 6.1 is a computation of the $ku_\star$-cooperations algebra.

Theorem 6.3. *The $ku_{\mathbb{R}}$-cooperations algebra $ku_{\mathbb{R}\star}ku_{\mathbb{R}}$ splits as*

$$ku_{\mathbb{R}\star}ku_{\mathbb{R}} \cong \bigoplus_{k=0}^{\infty} ku_{\mathbb{R}\star-k\rho}\mathcal{B}_0(k),$$

where as a $ku_{\mathbb{R}\star}$-module

$$\begin{aligned} ku_{\mathbb{R}\star}\mathcal{B}_0(k) &\cong \bigoplus_{k=0}^{\infty} \mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, L(\nu_2(2k!)) \\ &\cong \bigoplus_{k=0}^{\infty} H\underline{\mathbb{Z}}_\star\{x_0, x_1, \ldots, x_{\nu_2(2k!)-1}\} \oplus ku_{\mathbb{R}\star}\{x_{\nu_2(2k!)}\} \oplus V_k, \end{aligned}$$

with extensions $v_1x_{i-1} = \rho x_i$, and where $|x_i| = \rho i$ and V_k is a sum of suspensions of $H_\star$.

While we do not specify the degree of the summands appearing in V_k in the statement of Theorem 6.3 due to the amount of book keeping that would be involved, our computational methods do allow one to precisely deduce them.

Proof. Consider the $ku_{\mathbb{R}}$-Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(\mathbb{M}_2, H_\star B_0(k)) \Longrightarrow \pi_\star\big(ku_{\mathbb{R}} \wedge \mathcal{B}_0(k)\big).$$

This is a summand of the spectral sequence of Equation (6.1) so this spectral sequence collapses. Further, Proposition 6.5 gives a complete description of the E_2-page. Observe there are no hidden extensions for degree reasons. □

6.4.2. *$ku_{\mathbb{R}}$-Operations.* Our computational methods further yield an inductive description of the operations algebra for $ku_{\mathbb{R}}$, that is $[ku_{\mathbb{R}}, ku_{\mathbb{R}}]$.

Theorem 6.4. *The cooperations algbera $[ku_{\mathbb{R}}, ku_{\mathbb{R}}]$ splits as*

$$[ku_{\mathbb{R}}, ku_{\mathbb{R}}] \cong \bigoplus_{k=0}^{\infty} [\Sigma^{\rho k}\mathcal{B}_0(k), ku_{\mathbb{R}}].$$

The Adams spectral sequence

$$Ext_{\mathcal{E}(1)_\star}(H_\star\mathcal{B}_0(k), H_\star ku_{\mathbb{R}}) \Longrightarrow [\Sigma^{\rho k}\mathcal{B}_0(k), ku_{\mathbb{R}}]$$

collapses at the E_2-page, and its E_2-page is described in Lemmas 6.9 and 6.10.

Proof. Consider the $ku_{\mathbb{R}}$-Adams spectral sequence

$$\mathrm{Ext}_{\mathcal{E}(1)_\star}(H_\star\mathcal{B}_0(k), \mathbb{M}_2) \Longrightarrow [\mathcal{B}_0(k), ku_{\mathbb{R}}]^{ku_{\mathbb{R}}}.$$

Lemma 6.9 gives a description of $Ext_{\mathcal{E}(1)_\star}(L(\nu(k!), \mathbb{M}_2)$ for all $k \geqslant 0$. By the same arguments as the proof of the main theorem (Theorem 6.1), no differentials are possible. □

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