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L. G. Lewis, Jr.
J. P. May
M. Steinberger

with contributions by J. E. McClure

Equivariant Stable Homotopy Theory

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Preface

Our primary purpose in this volume is to establish the foundations of equivariant stable homotopy theory. To this end, we shall construct a stable homotopy category of G-spectra enjoying all of the good properties one might reasonably expect, where G is a compact Lie group. We shall use this category to study equivariant duality, equivariant transfer, the Burnside ring, and related topics in equivariant homology and cohomology theory.

This volume originated as a sequel to the volume "H_∞ ring spectra and their applications" in this series [20]. However, our goals changed as work progressed, and most of this volume is now wholly independent of [20]. In fact, we have two essentially disjoint motives for undertaking this study. On the one hand, we are interested in equivariant homotopy theory, the algebraic topology of spaces with group actions, as a fascinating subject of study in its own right. On the other hand, we are interested in equivariant homotopy theory as a tool for obtaining useful information in classical nonequivariant homotopy theory. This division of motivation is reflected in a division of material into two halves. The first half, chapters I-V, is primarily addressed to the reader interested in equivariant theory. The second half, chapters VI-X, is primarily addressed to the reader interested in nonequivariant applications. It gives the construction and analysis of extended powers of spectra that served as the starting point for [20]. It also gives a systematic study of generalized Thom spectra. With a very few minor and peripheral exceptions, the second half depends only on chapter I and the first four sections of chapter II from the first half. The reader is referred to [105] for a very brief guided tour of some of the high spots of the second half.

Chapter I gives the more elementary features of the equivariant stable category, such as the theory of G-CW spectra and a desuspension theorem allowing for desuspension of G-spectra by all representations of G in the given ambient "indexing universe". Chapter II gives the construction of smash products and function G-spectra. It also gives various change of universe and change of groups theorems. Chapter III gives a reasonably comprehensive treatment of equivariant duality theory, including Spanier-Whitehead, Atiyah, and Poincaré duality. Chapter IV studies transfer maps associated to equivariant bundles, with emphasis on their calculational behavior in cohomology. Chapter V studies the Burnside ring and its role in equivariant stable homotopy theory. It includes various related splitting theorems in equivariant homology and cohomology theory.

Although we have encountered quite a few new phenomena, our main goals in the first half have been the equivariant generalization of known nonequivariant results and the generalization and sharpening of known equivariant results. We therefore owe ideas and material to numerous other mathematicians. Our general debt to the work of Boardman [13,14] and Adams [1] in nonequivariant stable homotopy theory will be apparent throughout. The idea for a key proof in chapter I is due to Hauschild. The main change of groups theorems in chapter II are generalizations of results of Wirthmuller [144] and Adams [3], and the study of subquotient cohomology theories in II§9 is based on ideas of Costenoble.

Our debts are particularly large in chapters III, IV, and V. Our treatment of duality is largely based on ideas in the lovely paper [47] of Dold and Puppe and on (nonequivariant) details in the papers [63,64,65] of their students Henn and Hommel; equivariant duality was first studied by Wirthmuller [145]. Our treatment of transfer naturally owes much to the basic work of Becker and Gottlieb [10,11] and Dold [46], and transfer was first studied equivariantly by Nishida [117] and Waner [141]. Our IV§6 is a reexposition and equivariant generalization of Feshbach's work [53,54] on the double coset formula, and he cleared away our confusion on several points. While our initial definitions are a bit different, a good deal of chapter V is a reexposition in our context of tom Dieck's pioneering work [38-44] on the Burnside ring of a compact Lie group and the splitting of equivariant stable homotopy. This chapter also includes new proofs and generalizations of results of Araki [4].

A word about our level of generality is in order. We don't restrict to finite groups since, for the most part, relatively little simplification would result. We don't generalize beyond compact Lie groups because we believe that only the most formal and elementary portions of equivariant stable homotopy theory would then be available. The point is that, in all of our work, the depth and interest lies in the interplay between homotopy theory and representation theory. Technically, part of the point is that the cohomology theories represented by our G-spectra are RO(G)-graded and not just Z-graded. This implies huge amounts of algebraic structure which would be invisible in more formal and less specific homotopical contexts.

While a great deal of our work concerns equivariant cohomology theory, we have not given a systematic study here. Lewis, McClure, and I have used the equivariant stable category to invent "ordinary RO(G)-graded cohomology theories" [88], and the three of us and Waner are preparing a more thorough account [90]. (Hauschild, Waner, and I are also preparing an account of equivariant infinite loop space theory, which is less directly impinged upon by this volume.)

Chapters VI-VIII establish rigorous foundations for the earlier volume [20], which we shall refer to as $[H_\infty]$ here. That volume presupposed extended powers $D_jE = E\Sigma_j \ltimes_{\Sigma_j} E^{(j)}$ of spectra with various good properties. There E was a nonequivariant spectrum, but our construction will apply equally well to G-spectra E for any compact Lie group G.

In fact, extended powers result by specialization of what is probably the most fundamental construction in equivariant stable homotopy theory, namely the twisted

half-smash product $X \ltimes E$ of a G-space X and a G-spectrum E. (The "twisting" is encoded by changes of universe continuously parametrized by X.) This construction is presented in chapter VI, although various special cases will have been encountered earlier.

We develop a theory of "operad ring G-spectra" and in particular construct free operad ring G-spectra in chapter VII. When G is finite, special cases give approximations of iterated loop G-spaces $\Omega^V\Sigma^V X$, and we obtain equivariant generalizations of Snaith's stable splittings of spaces $\Omega^n\Sigma^n X$.

We prove some homological properties of nonequivariant extended powers that were used in $[H_\infty]$ in chapter VIII.

Chapters IX and X give a careful treatment of the Thom spectra associated to maps into stable classifying spaces. These have been used extensively in recent years, and many people have felt a need for a detailed foundational study. In chapter IX, we work nonequivariantly and concentrate on technical problems arising in the context of spherical fibrations (as opposed to vector bundles). In chapter X, we work equivariantly but restrict ourselves to the context of G-vector bundles. There result two specializations to the context of nonequivariant vector bundles, the second of which is the more useful since it deals naturally with elements of KO(X) of arbitrary virtual dimension.

We must again acknowledge our debts to other mathematicians. We owe various details to Bruner, Elmendorf, and McClure. The paper of Tsuchiya [138] gave an early first approximation of our definitions of extended powers and H_∞ ring spectra. As explained at the end of VII§2, Robinson's A_∞ ring spectra [124] fit naturally into our context. The proof of the splitting theorem in VII§5 is that taught us by Ralph Cohen [34]. We owe the formulations of some of our results on Thom spectra to Boardman [12] and of others to Mahowald [93], whose work led to our detailed study of these objects.

Each chapter of this book has an introduction summarizing its main ideas and results. There is a preamble comparing our approach to the nonequivariant stable category with earlier ones, and there is an appendix giving some of the more esoteric proofs. References are generally by name (Lemma 5.4) when to results in the same chapter and by number (II.5.4) when to results in other chapters.

Finally, I should say a word about the genesis and authorship of this volume. Chapter VIII and part of chapter VI are based on Steinberger's thesis [133], and chapter VII started from unpublished 1978 notes of his. Chapter IX and the Appendix are based on Lewis' thesis [83], and the definition and axiomatization of the transfer in chapter IV are simplifications of his work in [85]. Chapter V incorporates material from unpublished 1980 notes of McClure. All of the rest of the equivariant material is later joint work of Lewis and myself.

The authorship of the several chapters is as follows.

Chapters I through IV: Lewis and May

Chapter V: Lewis, May, and McClure

Chapters VI and VII: Lewis, May, and Steinberger

Chapter VIII: May and Steinberger

Chapter IX: Lewis

Chapter X: Lewis and May

The Appendix and the indices were prepared by Lewis.

J. Peter May
June 20, 1985

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Contents

Preamble: A polemical introduction to the stable category

by J. P. May

Nonequivariantly, the virtues of having a good stable category are by now well understood. In such a category, the basic formal properties of homology and cohomology theories become trivialities. Many arguments that could be carried out ad hoc without a stable category became much cleaner with one. More important, many common arguments simply cannot be made rigorous without use of such a category.

Equivariantly, it is even more important to have a good stable category. Much basic equivariant algebra only arises in a fully stable context. For example, one already has that $[S^n,S^n]$ is Z for $n \geq 1$ and that $[S^n,X]$ is a Z-module for $n \geq 2$. The equivariant analog of Z is the Burnside ring $A(G)$, and, unless G is finite, there need be no representation V of G large enough that $[S^V,S^V]_G$ is $A(G)$ or that $[S^V,X]_G$ is an $A(G)$-module. Even when G is finite, and regardless of connectivity hypotheses, none of the ordinary homotopy groups $[S^n,X]_G$ of a G-space X need be $A(G)$-modules, whereas all of the homotopy groups of G-spectra are $A(G)$-modules. Much more evidence will appear as we proceed.

Our construction of the equivariant stable category is a generalization of my construction of the nonequivariant stable category. Since the latter is less familiar than the earlier constructions of Boardman and Adams, a comparison of the various approaches may be helpful to the informed reader. I can't resist quoting from Boardman's 1969 Historical Introduction [13, p.1]. "This introduction... is addressed without compromise to the experts. (The novice has the advantage of not having been misled by previous theories.)"

Boardman continues "In this advertisement we compare our category $\underline{S}$ of CW-spectra, <u>or rather its homotopy category</u> $\underline{S}_h$, with various competing products. We find the comparison quite conclusive, because the more good properties the competitors have, the closer they are to $\underline{S}_h$". All experts now accept this absolutely. Boardman's category $\underline{S}_h$ is definitively the right one, and any good stable category must be equivalent to it. It does not follow, however, that his category $\underline{S}$, <u>before passage to homotopy</u>, is the right one, and we are convinced that it is not.

Boardman's construction of his category $\underline{S}$ proceeds as follows. He begins with the category $\mathcal{F}$ of finite CW complexes. He constructs the category $\mathcal{F}_s$ of finite CW spectra by a purely categorical procedure of stabilization with respect to the suspension functor. He then constructs $\underline{S}$ from $\mathcal{F}_s$ by a much deeper purely categorical procedure of adjoining colimits of all directed diagrams of finite CW spectra and inclusions. The intuition is that, however CW spectra are defined, they ought to be the colimits of their finite subspectra, and finite CW spectra ought to be desuspensions of finite CW complexes. An advantage of this approach is that one can obtain conceptual proofs of theorems about $\underline{S}_h$ almost automatically by feeding information about finite CW complexes into a categorical black box. A disadvantage, to paraphrase Adams [1, p.123], is that the construction is inaccessible to those without a specialized knowledge of category theory.

In fact, in his Historical Introduction [13, p.4], Boardman pointed out an alternative description of a category equivalent to his, and he gave details of the comparison in [14,§10]. Define a CW prespectrum D to be a sequence of CW complexes D_n and cellular inclusions $\Sigma D_n \to D_{n+1}$. Define a map $f: D \to D'$ to be a family of based maps $f_n: D_n \to D'_n$ strictly compatible with the given inclusions. Let $\mathcal{P}(D,D')$ denote the set of maps $D \to D'$. Say that a subprespectrum C is dense (or cofinal) in D if for any finite subcomplex X of D_n, $\Sigma^k X$ is contained in C_{n+k} for some k. Then [14,10.3] implies that $\underline{S}$ is equivalent to the category of CW-prespectra D and morphisms

$$\underline{S}(D,D') = \coprod_{(C,C')} \mathcal{P}(C,C')/(\approx) = \coprod_{C} \mathcal{P}(C,D')/(\approx),$$

where C and C' run through the dense subprespectra of D and D' and where $f: C \to C'$ is equivalent to $\bar{f}: \bar{C} \to \bar{C}'$ if and only if the composites

$$C \cap \bar{C} \xrightarrow{f} C' \subset D' \quad \text{and} \quad C \cap \bar{C} \xrightarrow{\bar{f}} \bar{C}' \subset D'$$

are equal.

Adams [1] turned this result into a definition and proceeded from there. (He called a map $D \to D'$ a "function", an element of $\underline{S}(D,D')$ a "map", and a homotopy class of "maps" a "morphism"; he also called a CW prespectrum a CW spectrum.) A similarly explicit starting point was taken by Puppe [122]. An advantage of this approach (to some people!) is that it is blessedly free of category theory. A disadvantage is that many proofs, for example in the theory of smash products, become unpleasantly ad hoc. To quote Boardman again [14,p.52], "The complication will show why we do not adopt this as definition".

It seems reasonable to seek an alternative construction with all of the advantages and none of the disadvantages. Staring at the definition, we see that $\underline{S}$ is constructed from the category of CW prespectra and maps by applying a kind of limit procedure to morphisms while leaving the objects strictly alone. This is the meaning of Adams' slogan [1, p.142] "cells now - maps later".

From our point of view, this is precisely analogous to developing sheaf theory without ever introducing sheaves or sheafification. There is a perfectly sensible way to "spectrify" so as to force elements of $\underline{S}(D,D')$ to be on the same concrete level as maps $D \to D'$. Define a spectrum E to be a sequence of based spaces E_n

and based homeomorphisms $E_n \to \Omega E_{n+1}$. Define a map $f: E \to E'$ to be a sequence of based maps $f_n: E_n \to E'_n$ strictly compatible with the homeomorphisms. Let $\mathcal{S}(E,E')$ denote the set of maps $E \to E'$. Define the spectrum LD associated to a CW prespectrum D by

$$(LD)_n = \operatorname*{colim}_{k \geq 0} \Omega^k D_{n+k},$$

where the colimit is taken with respect to iterated loops on adjoint inclusions $D_i \to \Omega D_{i+1}$. Since Ω commutes with colimits, there are evident homeomorphisms $\Omega(LD)_{n+1} \cong (LD)_n$. One finds by a laborious inspection of definitions that

$$\underline{S}(D,D') \cong \mathcal{S}(LD,LD').$$

Of course, only the expert seeking concordance with earlier definitions need worry about the verification: we shall take the category $\mathcal{S}$ as our starting point.

Obviously the spaces $(LD)_n$ are no longer CW complexes (although they do have the homotopy types of CW complexes), and we have imposed no CW requirement in our definition of spectra. It should now be apparent that, despite their rigid structure, spectra are considerably more general objects than CW prespectra. Working in a stable world in which the only spectra are those coming from CW prespectra is precisely analogous to working in an unstable world in which the only spaces are the CW complexes. Just as any space has the weak homotopy type of a CW complex, so any spectrum has the weak homotopy type of one coming from a CW prespectrum. (Verification of the last assertion requires only elementary constructions with space level CW approximations and mapping cylinders and was already implicit in my 1969 paper [95].)

The extra generality allowed by our definition of spectra is vital to our theory. Throughout this volume, we shall be making concrete spectrum-level constructions which simply don't exist in the world of CW prespectra.

Dropping CW conditions in our definition of spectra clearly entails dropping CW conditions in our definition of prespectra. For us, a prespectrum is a sequence of spaces D_n and maps $\Sigma D_n \to D_{n+1}$. Maps of prespectra are defined as above. By our analogy with sheaf theory, we are morally bound to extend the construction L above to a spectrification functor $L: \mathcal{P} \to \mathcal{S}$ left adjoint to the obvious forgetful functor from spectra to prespectra. When the adjoints $D_i \to \Omega D_{i+1}$ are not inclusions, LD is slightly mysterious and its construction is due to Lewis [83], who will give details in the Appendix. Starting from D one constructs a prespectrum D' and map $D \to D'$ by letting D'_i be the image of D_i in ΩD_{i+1}. The resulting maps $D'_i \to \Omega D'_{i+1}$ are a bit closer to being inclusions. Iterating this construction (transfinitely many times!) one arrives at a prespectrum $\bar{D}$ and map $D \to \bar{D}$ such that the maps $\bar{D}_i \to \Omega\bar{D}_{i+1}$ are inclusions. One defines LD by applying the elementary colimit construction above to $\bar{D}$; one has a composite natural map $D \to LD$ which is the unit of the adjunction. Actually, the explicit construction is of little importance. The essential point is that, by standard and elementary category theory, L obviously exists and is obviously unique.

We now see that our category of spectra has arbitrary limits and colimits. Indeed, the category of prespectra obviously has all limits and colimits since these can be constructed spacewise. All such limit constructions preserve spectra. Colimit constructions do not, and colimits of spectra are obtained by applying L to prespectrum level colimits. Thus, and this will take some getting used to by the experts, limits for us are simpler constructions than colimits. In fact, right adjoints in general are simpler constructions than left adjoints. For example, it is trivial for us to write down explicit products and pullbacks of spectra and explicit function spectra. These don't exist in the world of CW-prespectra.

Moreover, we shall often prove non-obvious facts about left adjoints simply by quoting obvious facts about right adjoints. This might seem altogether too categorical, but in fact the opposite procedure has long been standard practice. Function spectra in $\underline{S}_h$ (not $\underline{S}$!) are usually obtained by quoting Brown's representability theorem - something at least as sophisticated as any category theory we use - and then proving things about these right adjoints by quoting known facts about the left adjoint smash product functors.

Of course, one does want a theory of CW spectra, but there is no longer the slightest reason to retreat to the space or prespectrum level to develop it. We have a good category of spectra, with cones, pushouts, and colimits. To define CW spectra, we need only define sphere spectra and proceed exactly as on the space level, using spectrum level attaching maps. The resulting CW spectra are all homotopy equivalent to spectra coming from CW prespectra; conversely, any spectrum coming from a CW-prespectrum is homotopy equivalent to a CW-spectrum. A CW-spectrum is the colimit of its finite subspectra, and a finite CW spectrum is a desuspension of a finite CW complex (that is, of its associated suspension spectrum). Our stable category is constructed from the homotopy category of spectra by adjoining formal inverses to the weak equivalences. It is equivalent to the homotopy category of CW spectra. By the discussion above, it is also equivalent to Boardman's category $\underline{S}_h$. Without exception, everything in the literature done in Boardman's category can just as well be interpreted as having been done in our category.

In one respect I have lied a bit above. We don't usually index prespectra and spectra on integers but rather on finite dimensional inner product spaces. When one thinks of D_n, one thinks of S^n and thus of $\mathbb{R}^n$. Implicitly, one is thinking

of $\mathbb{R}^\infty$ with its standard basis. Even nonequivariantly, a coordinate free approach has considerable advantages. For example, it leads to an extremely simple conceptual treatment of smash products and, as Quinn, Ray, and I realized in 1973 [99], it is vital to the theory of structured ring spectra. In the equivariant context, one must deal with all representations, and coordinate-free indexing is obviously called for, as tom Dieck realized even earlier.

Modulo the appropriate indexing, virtually everything said above about my approach to the nonequivariant stable category applies verbatim in the equivariant context. The few exceptions are relevant to possible generalizations of the earlier constructions. Our G-CW spectra are built up from "G-sphere spectra" $G/H^+ \wedge S^n$, and any G-CW spectrum is the colimit of its finite subspectra. However, it is not true that a finite G-CW spectrum is isomorphic (as opposed to homotopy equivalent) to a desuspension of a finite G-CW complex unless one redefines the latter by allowing G-spheres $G/H^+ \wedge S^V$ associated to G-representations V as domains of attaching maps. This loses the cellular approximation theorem and would presumably cause difficulties in a Boardman style approach to the G-stable category. Since G-spheres S^V are not known to have canonical G-CW structures, the appropriate notion of a G-CW prespectrum is not immediately apparent. We shall give a definition which is related to our notions exactly as described above in the nonequivariant case. However, a full treatment, including smash and twisted half-smash products, would be inordinately lengthy and complicated. In any case, right adjoints, such as fixed point functors, are even more important equivariantly than nonequivariantly, and a treatment lacking such constructions on the spectrum level would be most unnatural.

I should say that there is also a semisimplicial construction of the stable category due to Kan (and Whitehead) [69,70] and elaborated by Burghelea and Deleanu [21]. Except perhaps when G is finite, it is ill-adapted to equivariant generalization, and it is also inconvenient for the study of structured ring spectra.

One last point addressed to the experts. We shall not introduce graded morphisms here. Regardless of what approach one takes, graded morphisms are really nothing more than a notational device. The device can aid in keeping track of the signs which arise in the study of cohomology theories, but it can in principle add nothing substantive to the mathematics. In the equivariant context, the grading would have to be over RO(G) and its introduction would serve only to obscure the exposition.

I. THE EQUIVARIANT STABLE CATEGORY

by L. G. Lewis, Jr. and J. P. May

We gave a preliminary definition of spectra in $[H_\infty, I\S 1]$ as sequences of spaces E_i and homeomorphisms $E_i \cong \Omega E_{i+1}$ This "coordinatized" notion is wholly inadequate for the study of either structured ring spectra or equivariant stable homotopy theory. While our main concern in $[H_\infty]$ was with the first of these subjects, we are here most interested in the second. Because of the role played by permutation groups in the construction of extended powers, we need a fair amount of equivariant stable homotopy theory to make rigorous the constructions used in $[H_\infty]$ in any case. While this motivates us only as far as the study of G-spectra for finite groups G, it turns out that a complete treatment of the foundations of equivariant stable homotopy theory in the proper generality of compact Lie groups is obtainable with very little extra effort.

Thus, throughout the first five chapters, G will be a compact Lie group. Considerations special to permutation groups will not appear until late in Chapter VI. We shall construct a good "stable category" of G-spectra, where "stability" is to be interpreted as allowing for desuspensions by arbitrary finite dimensional real representations of G.

After some recollections about equivariant homotopy theory in section 1, we begin work in section 2 by setting up categories of G-prespectra and G-spectra and discussing various adjoint functors relating them to each other and to G-spaces. We give both coordinate-free and coordinatized notions of G-spectra and show that these give rise to equivalent categories. The freedom to pass back and forth between the two is vital to the theory.

In section 3, we introduce the smash products of G-spaces and G-spectra and the associated adjoint function G-spectra. The analogous constructions between G-spectra and G-spectra are deeper and will be presented in the next chapter. The simpler constructions suffice for development of most of the basic machinery of homotopy theory. We also introduce orbit spectra and fixed point spectra.

In section 4, we introduce left adjoints "$\Lambda^Z\Sigma^\infty$" to the $Z^{\underline{th}}$ space functors from G-spectra to G-spaces, where Z runs through the relevant indexing representations. These functors play a basic role in the passage back and forth between space level and spectrum level information. In particular, we use instances of these functors to define sphere G-spectra $S^n_H = G/H^+ \wedge S^n$ for integers n and closed subgroups H of G. (The term subgroup shall mean closed subgroup henceforward.) We then define homotopy groups in terms of these sphere spectra and define weak equivalences in terms of the resulting homotopy groups.

In section 5, we introduce G-CW spectra. We follow a general approach, developed in more detail in [107], in which such basic results as the cellular approximation theorem, Whitehead's theorem, and the Brown representability theorem are almost formal trivialities. With these results, we see that arbitrary G-spectra are weakly equivalent to G-CW spectra. This allows us to construct the equivariant stable category by formally inverting the weak equivalences in the homotopy category of G-spectra. The result is equivalent to the homotopy category of G-CW spectra, both points of view being essential to a fully satisfactory theory.

In section 6, we summarize the basic properties of the stable category, the most important being the equivariant desuspension theorem. This asserts that Ω^V and Σ^V are adjoint self equivalences of the stable category for any G-representation V. We then indicate briefly how to define represented equivariant cohomology theories. The natural representing objects for cohomology theories on G-spaces are cruder than our G-spectra, and we make use of an elementary iterated mapping cylinder construction on the G-prespectrum level to obtain a precise comparison. This cylinder construction has various other applications. On the G-spectrum level, it turns out to admit a simple description as a telescope, and this leads to a $\lim^1$ exact sequence for the calculation of the cohomology of G-spectra in terms of the cohomologies of their component G-spaces.

In section 7, we give a number of deferred proofs based on use of a shift desuspension functor Λ^Z (in terms of which the earlier functor $\Lambda^Z\Sigma^\infty$ is a composite). In particular, we prove the equivariant desuspension theorem. This depends on the assertion that a map of G-spectra is a weak equivalence if and only if each of its component maps of G-spaces is a weak equivalence. This is the only place in the chapter where equivariance plays a really major role in a proof, the corresponding nonequivariant assertion being utterly trivial. We learned the basic line of argument from Henning Hauschild, although the full strength of the result depends on our definitional framework.

In section 8, we give various results concerning special kinds of G-prespectra and G-spectra. In particular, we show that, up to homotopy, our G-CW spectra come from G-CW prespectra of a suitably naive sort.

We shall defer some details of proof to the Appendix, on the grounds that the arguments in question would unduly interrupt the exposition.

We remind the reader that G is always a compact Lie group and that everything in sight is G-equivariant. Once the definitions are in place, we generally omit the G from the notations, writing spectra for G-spectra, etc.

§1. Recollections about equivariant homotopy theory

Since the basic definitions of equivariant homotopy theory are not as widely known as they ought to be, we give a brief summary before turning to G-spectra.

Let $G\mathcal{U}$ denote the category of compactly generated weak Hausdorff left G-spaces. (The weak Hausdorff condition asserts that the diagonal is closed in the compactly generated product; it is the most natural separation axiom to adopt for compactly generated spaces; see [111,83].) Let $G\mathcal{J}$ denote the category of based left G-spaces, with G acting trivially on basepoints. These categories are closed under such standard constructions as (compactly generated) products and function spaces, G acting diagonally on products and by conjugation on function spaces. The usual adjunction homeomorphisms hold and are G-equivariant. For unbased G-spaces X,Y, and Z we have

$$Z^{X \times Y} \cong (Z^Y)^X.$$

We write F(X,Y) for the function space of based maps $X \to Y$; for based G-spaces X,Y, and Z we have

$$F(X \wedge Y, Z) \cong F(X, F(Y,Z)).$$

The usual machinery of homotopy theory is available in the categories $G\mathcal{U}$ and $G\mathcal{J}$, homotopies being maps $X \times I \to Y$ in $G\mathcal{U}$ or $X \wedge I^+ \to Y$ in $G\mathcal{J}$. Cofibrations are defined in either category by the homotopy extension property (and are automatically closed inclusions). Similarly, fibrations are defined by the covering homotopy property. We shall use standard results without further comment; see e.g. [17,143, or 107]. Write $hG\mathcal{U}$ and $hG\mathcal{J}$ for the respective homotopy categories and write $\pi(X,Y)_G$ for the set of homotopy classes of based maps $X \to Y$.

Turning to homotopy groups, let $S^n = I^n/\partial I^n$ with trivial G-action and with $S^0 = \{0,1\}$. For $H \subset G$, define a G-space S^n_H by

$$S^n_H = (G/H)^+ \wedge S^n = (G/H) \times S^n/(G/H) \times \{*\}.$$

We think of S^n_H as a generalized sphere. It is well understood that the homotopy groups of a based G-space X should be taken to be the collection of homotopy groups

$$\pi^H_n X = \pi_n X^H \cong \pi(S^n,X)_H \cong \pi(S^n_H,X)_G \ .$$

Here the last isomorphism comes from the adjunction

$$H\mathcal{J}(Y,X) \cong G\mathcal{J}(G^+ \wedge_H Y, X) \quad \text{for } X \in G\mathcal{J} \text{ and } Y \in H\mathcal{J}.$$

X is said to be n-connected (or G-n-connected) if $\pi^H_q X = 0$ for all $q \leq n$ and $H \subset G$.

A G-map $f:X \to Y$ is said to be a weak equivalence (or weak G-equivalence) if each fixed point map $f^H:X^H \to Y^H$ is a weak equivalence in the usual nonequivariant sense. This means that $f^H_*:\pi_*(X^H,x) \to \pi_*(Y^H,fx)$ is an isomorphism for all possible choices of basepoint x. In our applications, the based G-spaces X and Y will always be double loop spaces. For such an X, each X^H is homotopy equivalent as a Hopf space to the product of the basepoint component $(X^H)_0$ and the discrete group $\pi_0 X^H$ [29,I.4.6]. Therefore f will be a weak equivalence if and only if $f_*:\pi_* X^H \to \pi_* Y^H$ is an isomorphism, where the homotopy groups are taken with respect to the given fixed basepoints.

G-CW complexes have been studied by Bredon [17] for finite G and by Matumoto [94], Illman [67], and Waner [140] for compact Lie groups G. Waner makes use of the general abstract pattern for CW-theory developed in [104 and 107]. There are two variants, one adapted to $G\mathcal{U}$ and the other to $G\mathcal{J}$. The basic definitions go as follows.

An ordinary G-CW complex is a G-space $X \in G\mathcal{U}$ which is the union of an expanding sequence of sub G-spaces X^n such that X^0 is a disjoint union of orbits G/H and X^{n+1} is obtained from X^n by attaching cells $(G/H) \times e^{n+1}$ by means of attaching G-maps $(G/H) \times S^n \to X^n$. When X is a based G-space, the basepoint is required to be a vertex and then X^0 may be described as a wedge of 0-spheres $S^0_H = (G/H)^+$. The successive quotients X^n/X^{n-1} for $n \geq 1$ are clearly wedges of n-spheres S^n_H, one for each attaching map.

A based G-CW complex is a G-space $X \in G\mathcal{J}$ which is the union of an expanding sequence of sub G-spaces X^n such that X^0 is the basepoint and X^{n+1} is the cofibre of a based G-map from a wedge of n-spheres S^n_H to X^n. The essential difference from the previous notion is the use of based attaching maps, and their use will lead to a closer connection with G-CW spectra.

Evidently based G-CW complexes are necessarily G-connected ordinary G-CW complexes. Conversely, it is easily seen that if X is a G-connected ordinary G-CW complex with base vertex, then X is equivalent to a based G-CW complex. In particular, this applies to the (reduced) suspension ΣX of an ordinary G-CW complex with base vertex. Indeed, with a little care one can give such a suspension a canonical structure of based G-CW complex.

In the unbased context, the Whitehead theorem asserts that if X is a G-CW complex and $f:Y \to Z$ is a weak equivalence, then

$$f_*:hG\mathcal{U}(X,Y) \to hG\mathcal{U}(X,Z)$$

is a bijection. In particular, it follows that f is an equivalence if Y and Z are G-CW complexes. Moreover, for any G-space X there is a G-CW complex ΓX and a weak equivalence $\gamma: \Gamma X \to X$. (See Seymour [128].) It is formal that any choices for the ΓX yield a functor $\Gamma: hG\mathcal{U} \to hG\mathcal{U}$ such that γ is natural. Using this, we can construct a category $\overline{hG\mathcal{U}}$ by formally inverting the weak equivalences of $hG\mathcal{U}$. The functor Γ induces an equivalence from $\overline{hG\mathcal{U}}$ to the homotopy cateory of G-CW complexes. A more categorical discussion of these ideas is given in section 5. By restriction to based G-maps of based G-spaces, we obtain $\overline{hG\mathcal{T}}$ analogously. Of course, to avoid restricting to G-connected G-spaces, we must use ordinary and not based G-CW complexes in the construction of $\Gamma: hG\mathcal{T} \to hG\mathcal{T}$.

On the space level, one can usually get away with restricting attention to CW-homotopy types because they are preserved under most common constructions. This is much less true on the spectrum level, where formal inversion of weak equivalences plays a correspondingly more essential role. The difference comes from the absence of spectrum level analogs of Milnor's basic theorems [112] on spaces of the homotopy type of CW complexes. The equivariant version of Milnor's theorems were proven by Araki and Murayama [5] for finite G and by Waner [140] in the full generality of compact Lie groups G. In particular, Waner proved the following result.

Theorem 1.1. (i) Let $H \to G$ be any homomorphism of compact Lie groups. Then any G-CW complex has the H-homotopy type of an H-CW complex.
(ii) If X is a compact G-space and Y has the homotopy type of a G-CW complex, then so do Y^X and, if X and Y are based, $F(X,Y)$.

We should also record the following result (although we shall only need its easy up to homotopy type version). For finite G, it was proven by Illman [66]. As noted by Matumoto [94] and Illman [66,67], the general case is readily proven once one knows that the orbit space of a smooth action on a compact manifold is triangulable, and a correct proof of this fact has been supplied by Verona [139].

Theorem 1.2. Any smooth compact G-manifold is triangulable as a finite G-CW complex.

Remarks 1.3. While the preceding result is very important for the more geometric parts of equivariant theory, it is not helpful to us because it fails to provide canonical triangulations of even such simple G-spaces as $G/H \times G/K$ (where double coset choices would enter in the finite case) and spheres associated to representations of G. In particular, products of G-CW complexes and suspensions of G-CW complexes by representations fail to have canonical G-CW structures.

§2. Categories of G-prespectra and G-spectra

We need some preliminaries before we can give our basic definitions.

If V is a finite dimensional real inner product space, we let S^V denote its one-point compactification with basepoint at ∞. If G acts through isometries on V, then S^V is a based G-space. (We shall usually, but not invariably, use small letter superscripts for typographical reasons.) We write $V \oplus W$ for external direct sums and $V + W$ for internal direct sums of orthogonal subspaces of some ambient inner product space; we write $V \perp W$ to indicate that V is orthogonal to W. If $V \subset W$, we agree to write $W - V$ for the orthogonal complement of V in W. For any based G-space X and G-space S^V as above, we define

$$\Sigma^V X = X \wedge S^V \quad \text{and} \quad \Omega^V X = F(S^V, X).$$

Of course, these "suspensions" and "loop spaces" are based G-spaces.

In order to allow desuspension by general representations, it is essential to index spectra on representations. The proper way to do this is to start with an ambient real inner product space U of countably infinite dimension such that G acts on U through isometries and U is the direct sum of its finite dimensional G-invariant sub inner product spaces. For later purposes, U is to be topologized as the colimit of these finite dimensional subspaces (but no use will be made of the topology in this chapter).

We say that U is a "G-universe" if it contains a trivial representation and contains each of its finite dimensional subrepresentations infinitely often. The most interesting case occurs when U contains all irreducible representations, and we then refer to U as a complete G-universe. If G is finite, the sum of countably many copies of the regular representation gives a canonical complete G-universe. For the definition of sphere spectra and homotopy groups, it is convenient to insist that U contain a canonical infinite dimensional trivial G-representation, denoted R^∞.

An "indexing space" in a G-universe U is a finite dimensional G-invariant sub inner product space. An "indexing sequence" in U is an expanding sequence $A = \{A_i | i \geq 0\}$ of indexing spaces such that $A_0 = \{0\}$ and U is the union of the A_i. An "indexing set" in U is a set $\mathcal{A}$ of indexing spaces which contains some indexing sequence. Of course, indexing sequences are examples of indexing sets. The "standard indexing set" in U is the set of all indexing spaces in U.

With these conventions, we can now define G-prespectra and G-spectra.

Definition 2.1. Let $\mathcal{A}$ be an indexing set in a G-universe U. A G-prespectrum (D,σ) indexed on $\mathcal{A}$ consists of based G-spaces DV for $V \in \mathcal{A}$ and based G-maps

$$\sigma : \Sigma^{W-V} DV \to DW$$

for $V \subset W$ in $\mathcal{A}$ such that the following conditions hold.

(i) $\sigma : DV = \Sigma^0 DV \to DV$ is the identity map.

(ii) For $V \subset W \subset Z$ in $\mathcal{A}$, the following diagram commutes:

$$\begin{tikzcd}
\Sigma^{Z-W}\Sigma^{W-V}DV \arrow[r, "\Sigma^{Z-W}\sigma"] \arrow[d, phantom, "\cong"] & \Sigma^{Z-W}DW \arrow[d, "\sigma"] \\
\Sigma^{Z-V}DV \arrow[r, "\sigma"] & DZ
\end{tikzcd}.$$

The G-map $DV \to \Omega^{W-V}DW$ adjoint to σ is denoted $\tilde{\sigma}$. A G-prespectrum (D,σ) is said to be an inclusion G-prespectrum if each $\tilde{\sigma}$ is an inclusion; (D,σ) is said to be a G-spectrum if each $\tilde{\sigma}$ is a homeomorphism. A map $f : D \to D'$ of G-prespectra is a system of based G-maps $fV : DV \to D'V$ such that the following diagram commutes for $V \subset W$ in $\mathcal{A}$:

$$\begin{tikzcd}
\Sigma^{W-V}DV \arrow[r, "\Sigma^{W-V}fV"] \arrow[d, "\sigma"'] & \Sigma^{W-V}D'V \arrow[d, "\sigma'"] \\
DW \arrow[r, "fW"] & D'W
\end{tikzcd}$$

We refer to f as a "spacewise" inclusion, surjection, weak equivalence, etc., if each fV is an inclusion, surjection, weak equivalence, etc. (We generally abbreviate fV to f when there is no danger of confusion.) We denote by

$$G\mathcal{P}\mathcal{A} \supset G\mathcal{I}\mathcal{A} \supset G\mathcal{S}\mathcal{A}$$

the category of G-prespectra indexed on $\mathcal{A}$ and its full subcategories of inclusion G-prespectra and G-spectra respectively. We write $G\mathcal{S}U$ for $G\mathcal{S}\mathcal{A}$ when $\mathcal{A}$ is the standard indexing set in U, and similarly for our categories of G-prespectra.

It is the category $G\mathcal{S}\mathcal{A}$ of G-spectra that is of primary interest. The category $G\mathcal{P}\mathcal{A}$ is needed because such basic constructions on G-spectra as colimits and smash products are obtained by carrying out the construction on the level of G-prespectra and then applying a suitable functor from G-prespectra to G-spectra. This functor is actually the composite of functors $G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ and $G\mathcal{I}\mathcal{A} \to G\mathcal{S}\mathcal{A}$, each of these being left adjoint to the evident forgetful functor the other way.

The functor $G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ can be obtained either by Freyd's adjoint functor theorem [55], or by a direct (and fairly unilluminating) point-set topological construction. Details will be given in the Appendix. This functor is needed because colimits and smash products of inclusion prespectra need not be inclusion prespectra. (May overlooked this point in [99,II] and, with some motivation, gave

inclusion prespectra an unnecessarily complicated definition; compare [99,II.1.10]. Also, the closed condition placed on inclusions in [99] is unnecessary and counterproductive.)

The functor $G\mathcal{I}\mathcal{A} \to G\mathcal{S}\mathcal{A}$ is completely elementary (and was introduced in [99,II.1.4]); it assigns to an inclusion G-prespectrum D the G-spectrum E with

$$EV = \operatorname*{colim}_{W \supset V} \Omega^{W-V}DW,$$

the colimit being taken over those $W \in \mathcal{A}$ which contain V. Summarizing, we have the following result.

<u>Theorem 2.2.</u> There is a left adjoint

$$L : G\mathcal{P}\mathcal{A} \to G\mathcal{S}\mathcal{A}$$

to the forgetful inclusion functor

$$\ell : G\mathcal{S}\mathcal{A} \to G\mathcal{P}\mathcal{A}.$$

That is,

$$G\mathcal{P}\mathcal{A}(D, \ell E) \cong G\mathcal{S}\mathcal{A}(LD, E) \quad \text{for } D \in G\mathcal{P}\mathcal{A} \text{ and } E \in G\mathcal{S}\mathcal{A}.$$

Let $\eta : D \to \ell LD$ and $\varepsilon : L\ell E \to E$ be the unit and counit of the adjunction. Then ε is an isomorphism for each G-spectrum E, hence η is an isomorphism if $D = \ell E$.

Except when making categorical assertions, we shall generally omit forgetful functors such as ℓ from the notations, writing $\eta : D \to LD$ and $\varepsilon : LE \to E$ for example.

It is essential to any really good stable category that it be derived from a category of spectra which has all colimits (wedges, pushouts, coequalizers, etc) and all limits (products, pullbacks, equalizers, etc). These constructions all exist in $G\mathcal{T}$. They also exist in $G\mathcal{P}\mathcal{A}$, where they are given by the evident spacewise constructions. It is easy to check that the limit in $G\mathcal{P}\mathcal{A}$ of a diagram of G-spectra is again a G-spectrum. Thus $G\mathcal{S}\mathcal{A}$ has all limits. Colimits in $G\mathcal{P}\mathcal{A}$ of diagrams of G-spectra are hardly ever G-spectra, but application of L to these colimits yields colimits in $G\mathcal{S}\mathcal{A}$. Thus $G\mathcal{S}\mathcal{A}$ also has all colimits. We remind the reader that functors which are left adjoints preserve colimits while functors which are right adjoints preserve limits [92].

We must point out one unpleasant fact of nature. Already in $G\mathcal{T}$, colimits need not be given by the obvious constructions. Pushouts, for example, must be formed as usual in the category of all G-spaces and then made weak Hausdorff if they are not already so. In practice, one gets around this by making appropriate point-set topological assumptions. This solution is less practicable for spectra, and here one must simply accept colimits as they come. This has the effect of retaining formal properties while losing control of homotopical information and, except when restricted to $G\mathcal{I}\mathcal{A}$, the functor L itself suffers from the same defect. This forces a certain amount of technical care, a discussion of which is deferred until section 8 and the Appendix.

While L is essential for theoretical purposes, and its precise form is dictated by the uniqueness of adjoints, there is a more homotopical passage from prespectra to spectra that is also very useful. It will be discussed in section 6.

Just as in the nonequivariant case [99,II], the formal relationship between G-spaces and G-spectra is also given by a pair of adjoint functors. We have a zeroth space functor $G\mathcal{P}\mathcal{A} \to G\mathcal{T}$ which assigns $D_0 = D(0)$ to a G-prespectrum D. The restriction of this funtor to $G\mathcal{S}\mathcal{A}$ is denoted

$$\Omega^\infty : G\mathcal{S}\mathcal{A} \to G\mathcal{T}.$$

Spaces and maps in the image of this functor (or G-homotopic to spaces or maps in this image) are called infinite loop G-spaces and G-maps.

For a G-space X, there is a suspension G-prespectrum $\{\Sigma^V X\}$ with structural maps the natural identifications $\Sigma^{W-V}\Sigma^V X \cong \Sigma^W X$ for $V \subset W$ in $\mathcal{A}$. We define the suspension G-spectrum functor

$$\Sigma^\infty : G\mathcal{T} \to G\mathcal{S}\mathcal{A}$$

by $\Sigma^\infty X = L\{\Sigma^V X\}$ and have the following result.

<u>Proposition 2.3</u>. The functor Σ^∞ is left adjoint to Ω^∞. That is,

$$G\mathcal{T}(X, \Omega^\infty E) \cong G\mathcal{S}\mathcal{A}(\Sigma^\infty X, E) \quad \text{for } X \in G\mathcal{T} \text{ and } E \in G\mathcal{S}\mathcal{A}.$$

A similar conclusion holds on the prespectrum level. The composite $\Omega^\infty\Sigma^\infty X$ is also denoted QX; thus

$$QX = \operatorname*{colim}_{V \subset \mathcal{A}} \Omega^V \Sigma^V X.$$

This is the usual nonequivariant space QX with a G action derived from the G-actions on both X and the universe U.

Again, just as in the nonequivariant case [99,II], the category $G\mathcal{S}\mathcal{A}$ is independent, up to equivalence, of the choice of $\mathcal{A}$. Further, $G\mathcal{S}U$ depends only on the isomorphism class of U. This allows us to use whatever choices happen to be convenient for any particular application or construction. The following pair of results make this invariance precise.

<u>Proposition 2.4</u>. Let $\mathcal{A} \subset \mathcal{B}$ be indexing sets in U and let

$$\phi : G\mathcal{S}\mathcal{B} \to G\mathcal{S}\mathcal{A}$$

be the functor obtained by forgetting those indexing spaces in $\mathcal{B}$ but not in $\mathcal{A}$. Then ϕ has a left adjoint

$$\psi : G\mathcal{S}\mathcal{A} \to G\mathcal{S}\mathcal{B}$$

such that the unit $E \to \phi\psi E$ and counit $\psi\phi F \to F$ of the adjunction are natural isomorphisms. In particular, $G\mathcal{S}\mathcal{A}$ is equivalent to $G\mathcal{S}U$ for any indexing set $\mathcal{A}$ in U.

<u>Proof</u>. For $V \in \mathcal{B}$, $(\psi E)(V) = \operatorname{colim} \Omega^{W-V}EW$, where W runs over those indexing spaces in $\mathcal{A}$ which contain V. The structural homeomorphisms $\tilde{\sigma}$ are evident and the remaining verifications are easy.

<u>Proposition 2.5</u>. Let $f : U \to U'$ be a G-linear isometric isomorphism. Then there are functors

$$f^* : G\mathcal{S}U' \to G\mathcal{S}U \quad \text{and} \quad f_* = (f^{-1})^* : G\mathcal{S}U \to G\mathcal{S}U'$$

which are inverse isomorphisms of categories.

<u>Proof</u>. For $V \subset U$, $(f^*E)(V) = E(fV)$. The structural maps are

$$\Sigma^{W-V}E(fV) = E(fV) \wedge S^{W-V} \xrightarrow{1 \wedge S^f} E(fV) \wedge S^{fW-fV} \xrightarrow{\sigma} E(fW)$$

for $V \subset W \subset U$. The rest is trivial.

<u>Remark 2.6</u>. Clearly f^*, but not f_*, is defined when f is only a G-linear isometry, not necessarily an isomorphism. In the next chapter, we shall generalize the previous result by obtaining functors f_* left adjoint to f^* for general G-linear isometries f.

Since equivalences of categories, such as those in the previous propositions, play an important role in our theory, some categorical remarks are in order.

Remarks 2.7. Let $S:\mathcal{A} \to \mathcal{B}$ and $T:\mathcal{B} \to \mathcal{A}$ be functors. We say that S and T are inverse isomorphisms if $ST = 1$ and $TS = 1$. We say that S and T are inverse equivalences if ST and TS are naturally isomorphic to the respective identity functors. We say that S and T are adjoint equivalences if we have an adjunction

$$\mathcal{B}(SA,B) \cong \mathcal{A}(A,TB), \quad A \in \mathcal{A} \text{ and } B \in \mathcal{B},$$

whose unit $\eta:A \to TSA$ and counit $\varepsilon:STB \to B$ are natural isomorphisms. It follows from the uniqueness of inverse morphisms that S is also right adjoint to T, the adjunction

$$\mathcal{A}(TB,A) \cong \mathcal{B}(B,SA)$$

having unit $\varepsilon^{-1}:B \to STB$ and counit $\eta^{-1}:TSA \to A$. By [92,p.91], any equivalence of categories S is part of such an adjoint equivalence (S,T,η,ε).

§3. The functors $E\wedge X$, $F(X,E)$, E/H, and E^H; homotopy theory

We first define the smash product $E\wedge X$ and function G-spectrum $F(X,E)$ of a G-space X and G-spectrum E. This gives the foundational parts of homotopy theory, such as cofibration and fibration sequences, the dual $\lim^1$ exact sequences, and homotopy colimits and limits, by standard arguments. We then define the orbit and fixed point spectra associated to a G-spectrum.

Definition 3.1. Let $D \in G\mathcal{P}\mathcal{A}$ and $X \in G\mathcal{T}$. Define $D\wedge X \in G\mathcal{P}\mathcal{A}$ by letting

$$(D\wedge X)(V) = DV\wedge X \quad \text{for } V \in \mathcal{A}$$

and

$$\sigma = \sigma\wedge 1:\ \Sigma^{W-V}(DV\wedge X) \cong (\Sigma^{W-V}DV)\wedge X \to DW\wedge X \quad \text{for } V \subset W \text{ in } \mathcal{A}.$$

For $E \in G\mathcal{S}\mathcal{A}$, define $E\wedge X \in G\mathcal{S}\mathcal{A}$ by use of L and ℓ:

$$E\wedge X = L(\ell E\wedge X).$$

Define $X\wedge D$ and $X\wedge E$ by symmetry and observe that these are naturally isomorphic to $D\wedge X$ and $E\wedge X$.

In particular, we now have cylinders $E\wedge I^+$, cones $CE = E\wedge I$, and suspensions $\Sigma E = E\wedge S^1$, where G acts trivially on I^+, I, and S^1. We also have the generalized suspension $\Sigma^V E = E\wedge S^V$ determined by a representation V.

Definition 3.2. Let $D \in G\mathcal{P}\mathcal{A}$ and $X \in G\mathcal{T}$. Define $F(X,D) \in G\mathcal{P}\mathcal{A}$ by letting

$$F(X,D)(V) = F(X,DV) \quad \text{for } V \in \mathcal{A}$$

and letting $\tilde{\sigma}:F(X,D)(V) \to \Omega^{W-V}F(X,D)(W)$ be the composite

$$F(X,DV) \xrightarrow{F(1,\tilde{\sigma})} F(X,\Omega^{W-V}DW) \cong \Omega^{W-V}F(X,DW)$$

for $V \subset W$ in $\mathcal{A}$. If D is a spectrum, then so is $F(X,D)$, hence $F(X,?)$ restricts to a functor $G\mathcal{S}\mathcal{A} \to G\mathcal{S}\mathcal{A}$.

In particular, we have the free path spectrum $F(I^+,E)$, the path spectrum $PE = F(I,E)$, and the loop spectrum $\Omega E = F(S^1,E)$. We also have the generalized loop spectrum $\Omega^V E = F(S^V,E)$.

It is useful to topologize the set $\mathcal{P}\mathcal{A}(D,D')$ of (non-equivariant) maps $D \to D'$ as a subspace of the product $\times_{V \in \mathcal{A}} F(DV,D'V)$. Here D and D' are G-prespectra regarded as nonequivariant spectra by neglect of stucture; G acts on $\mathcal{P}\mathcal{A}(D,D')$ by conjugation, and the trivial map gives a G-trivial basepoint. Clearly $G\mathcal{P}\mathcal{A}(D,D')$ may be topologized as the fixed point space $\mathcal{P}\mathcal{A}(D,D')^G$. Of course, these topologies apply equally well to the spectrum level hom sets. We have the following analogs of standard adjunctions on the level of G-spaces.

Proposition 3.3. There are natural homeomorphisms

$$G\mathcal{P}\mathcal{A}(D\wedge X,D') \cong G\mathcal{T}(X,\mathcal{P}\mathcal{A}(D,D')) \cong G\mathcal{P}\mathcal{A}(D,F(X,D'))$$

and

$$G\mathcal{S}\mathcal{A}(E\wedge X,E') \cong G\mathcal{T}(X,\mathcal{S}\mathcal{A}(E,E')) \cong G\mathcal{S}\mathcal{A}(E,F(X,E'))$$

for $X \in G\mathcal{T}$, $D,D' \in G\mathcal{P}\mathcal{A}$, and $E,E' \in G\mathcal{S}\mathcal{A}$.

It follows that the functor $E\wedge X$ preserves colimits in each of its variables and that the functor $F(X,E)$ preserves limits in E and converts colimits in X to limits. We should also record the following isomorphisms.

Proposition 3.4. For $X,Y \in G\mathcal{T}$ and $E \in G\mathcal{S}\mathcal{A}$, there are natural isomorphisms

$$E\wedge S^0 \cong E, \quad F(S^0,E) \cong E$$

$$(E\wedge X)\wedge Y \cong E\wedge(X\wedge Y),\quad F(X\wedge Y,E)\cong F(X,F(Y,E)).$$

These tie in with the commutativity, associativity, and unit isomorphisms of the smash product on $G\mathcal{T}$ to produce a system of coherent natural isomorphisms.

Since $E\wedge I^+$ is now defined, we have a notion of homotopy, namely a map $h\colon E\wedge I^+ \to E'$ or, equivalently on passage to adjoints, $\tilde{h}\colon E\to F(I^+,E')$. Homotopy is an equivalence relation which respects composition, and we have the homotopy category $hG\mathcal{S}\mathcal{A}$ of G-spectra and homotopy classes of maps. Similarly, we have homotopy categories $hG\mathcal{T}$ and $hG\mathcal{P}\mathcal{A}$. We shall write $\pi(X,Y)_G$ for the set of homotopy classes of maps $X\to Y$ in any of these categories, relying on context to determine which is intended.

The basic machinery of homotopy theory, including cofibration and fibration sequences, the homotopy invariance of pushouts of cofibrations and of colimits of sequences of cofibrations (and the dual assertions), the dual $\lim^1$ exact sequences, and the entire theory of homotopy colimits and limits, applies equally well in all of these homotopy categories. The proofs are the same as in the nonequivariant space level context and can in fact be given uniformly in an appropriate general framework of topological categories and continuous functors. The starting point is that all of our categories have underlying hom sets in the category of based spaces and have continuous composition. A functor $F\colon\mathcal{U}\to\mathcal{V}$ between such topological categories is said to be continuous if

$$F\colon\mathcal{U}(X,Y)\to\mathcal{V}(FX,FY)$$

is a continuous based map for all $X,Y\in\mathcal{U}$. Continuous functors are automatically homotopy preserving and so pass to homotopy categories.

All of the functors we have introduced (or will introduce) are continuous and all of our adjunction isomorphisms are homeomorphisms. Therefore all of our functors and adjunctions pass to homotopy categories. Except where the functor $L\colon G\mathcal{P}\mathcal{A}\to G\mathcal{S}\mathcal{A}$ is involved, these assertions are easily verified by direct inspection. The unpleasant point-set topological nature of L makes its direct examination quite difficult, and a little categorical sophistication provides a pleasant way of checking its properties without using or even knowing its precise construction. For example, the continuity of L and the fact that the spectrum level isomorphisms in Proposition 3.3 are homeomorphisms are formal consequences of the continuity of $\ell\colon G\mathcal{S}\mathcal{A}\to G\mathcal{P}\mathcal{A}$ and the obvious equality

(*) $$F(X,\ell E) = \ell F(X,E)\quad\text{for } X\in G\mathcal{T} \text{ and } E\in G\mathcal{S}\mathcal{A}.$$

To see this, note that L is given on hom sets by the composite

$$\mathcal{P}(D,D')\xrightarrow{\mathcal{P}(1,\eta)}\mathcal{P}(D,\ell LD')\cong\mathcal{S}(LD,LD').$$

Since $\mathcal{P}(1,\eta)$ is certainly continuous, L will be continuous if the adjunction bijection is a homeomorphism. Now the adjunction is given by the continuous bijection

$$\mathcal{S}(LD,E) = \mathcal{P}(\ell LD,\ell E)\xrightarrow{\mathcal{P}(\eta,1)}\mathcal{P}(D,\ell E).$$

To see that $\mathcal{P}(\eta,1)^{-1}$ is continuous, consider the diagram

$$\begin{tikzcd}
\mathcal{P}(D,\ell F(X,E)) \arrow[d, equal] & \mathcal{S}(LD,F(X,E)) \arrow[l, "\mathcal{P}(\eta,1)"'] \arrow[r] & \mathcal{T}(X,\mathcal{S}(LD,E)). \arrow[d, "\mathcal{T}(1,\mathcal{P}(\eta,1))"] \\
\mathcal{P}(D,F(X,\ell E)) \arrow[rr] & & \mathcal{T}(X,\mathcal{P}(D,\ell E))
\end{tikzcd}$$

The unlabeled arrows are bijections given by Proposition 3.3 (with topologies ignored). As noted by Kelly [72, p. 173] in a very general categorical context, the diagram commutes. By the diagram, $\mathcal{T}(1,\mathcal{P}(\eta,1))$ is a bijection. With $X = \mathcal{P}(D,\ell E)$, it follows immediately that $\mathcal{P}(\eta,1)^{-1}$ is continuous. It is now easy to check the homeomorphism claim in Proposition 3.3.

Again, the obvious equality (*) displayed above formally implies the very unobvious natural isomorphism

$$L(D\wedge X)\cong(LD)\wedge X\quad\text{for } X\in G\mathcal{T} \text{ and } D\in G\mathcal{P}\mathcal{A}.$$

Indeed, this implication is immediate from the following standard categorical fact about adjoint functors [92, p. 97], which we shall apply over and over again.

<u>Lemma 3.5</u>. Let $S,S'\colon\mathcal{A}\to\mathcal{B}$ be left adjoints of $T,T'\colon\mathcal{B}\to\mathcal{A}$ respectively. Then there is a one-to-one correspondence, called conjugation, between natural transformations $\alpha\colon S\to S'$ and $\beta\colon T'\to T$. Explicitly, α and β are conjugate if and only if the following diagram commutes for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$:

$$\begin{tikzcd}
\mathcal{A}(A,T'B) \arrow[r, phantom, "\cong"] \arrow[d, "\beta_*"'] & \mathcal{B}(S'A,B) \arrow[d, "\alpha^*"] \\
\mathcal{A}(A,TB) \arrow[r, phantom, "\cong"] & \mathcal{B}(SA,B)
\end{tikzcd}$$

Moreover, α is a natural isomorphism if and only if β is a natural isomorphism.

We shall often apply this in the context of diagrams

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "S"] \arrow[d, shift right, "P"'] & \mathcal{B} \arrow[l, shift left, "T"] \arrow[d, shift right, "P'"'] \\ \mathcal{A} \arrow[r, shift left, "S'"] \arrow[u, shift right, "Q"'] & \mathcal{B} \arrow[l, shift left, "T'"] \arrow[u, shift right, "Q'"'] \end{tikzcd}$$

of adjoint functors, using it to deduce that $P'S \cong S'P$ if and only if $QT' \cong TQ'$. In the motivating example preceding the lemma, $(S,T) = (L,\ell) = (S',T')$ while (P,Q) and (P',Q') are $(?) \wedge X$ and $F(X,?)$ on the prespectrum and spectrum levels respectively. As in this example, it is very often the case that the isomorphism is completely obvious for the right adjoints and most useful for the left adjoints. In fact, in some cases the right adjoints will be of little interest except as tools allowing simple formal proofs of information needed about left adjoints. Further examples of conjugate pairs appear in the following result.

Proposition 3.6. For $X,Y \in G\mathcal{J}$ and $E \in G\mathcal{S}\mathcal{A}$, there are natural isomorphisms

$$\Sigma^\infty(X \wedge Y) \cong (\Sigma^\infty X) \wedge Y \quad \text{and} \quad \Omega^\infty F(Y,E) \cong F(Y,\Omega^\infty E).$$

The isomorphisms ϕ and ψ of Proposition 2.4 and f_* and f^* of Proposition 2.5 commute with both functors $(?) \wedge X$ and $F(Y,?)$.

We next define orbit and fixed point spectra. We shall study these and other change of groups functors systematically in the next chapter, hence we content ourselves here with little more than the bare definitions. One's first thought is simply to set $(E^H)(V) = (EV)^H$. However, one quickly sees that there is no way to obtain a homeomorphism between $(EV)^H$ and $\Omega^{W-V}(EW)^H$ unless H acts trivially on $W-V$. Thus one first defines E^H for spectra indexed on an H-trivial universe and then uses a change of universe functor (as in Remark 2.6) to extend the definition to spectra indexed on general universes.

Definition 3.7. (i) Let D be a G-prespectrum indexed on an H-trivial G-universe U. Let $WH = NH/H$, where NH is the normalizer of H in G, and observe that U is a WH-universe. Define the orbit and fixed point WH-spectra D/H and D^H by

$$(D/H)(V) = (DV)/H \quad \text{and} \quad (D^H)(V) = (DV)^H,$$

the structural maps being obtained from those of D by passage to H-orbits or to H-fixed point sets. The action of WH is evident. If D is a G-spectrum, then

D^H is a WH-spectrum. For a G-spectrum E, define $E/H = L(\ell E/H)$.
(ii) For a general G-universe U, let $i\colon U^H \to U$ be the inclusion and observe that U^H is a WH-universe, complete if U is G-complete. For a spectrum $E \in G\mathcal{S}U$, define

$$E^H = (i^*E)^H \in WH\mathcal{S}U^H.$$

It is important to observe that $E^H(V) = (EV)^H$ whenever H acts trivially on V, for example when $V \subset \mathbf{R}^\infty \subset U$. We have not included the obvious symmetric definition, $E/H = (i^*E)/H$, since this appears to us to be useless. Such a construction, involving the composite of a right and left adjoint, tends to result in spectra which cannot be analyzed effectively, and we have no useful definition of orbits of H-spectra indexed on non-trivial H-universes.

Of course, we ignore the WH action in situations where it is irrelevant. We record the spectrum level analogs of the basic adjunctions relating equivariant and nonequivariant maps.

Proposition 3.8. Let U be a G-trivial universe and let $\varepsilon^*\colon \mathcal{S}U \to G\mathcal{S}U$ be the functor which assigns trivial G action to a spectrum. For spectra F and G-spectra E, there are natural isomorphisms

$$\mathcal{S}U(E/G,F) \cong G\mathcal{S}U(E,\varepsilon^*F) \quad \text{and} \quad G\mathcal{S}U(\varepsilon^*F,E) \cong \mathcal{S}U(F,E^G).$$

Moreover, for G-spaces X, the adjoint of the quotient G-map $\Sigma^\infty X \to \varepsilon^*\Sigma^\infty(X/G)$ is a natural isomorphism

$$(\Sigma^\infty X)/G \cong \Sigma^\infty(X/G).$$

Proof. The last statement follows by conjugation from the obvious equality $\varepsilon^*\Omega^\infty F = \Omega^\infty \varepsilon^* F$.

§4. The functors $\Lambda^Z\Sigma^\infty$; sphere spectra and homotopy groups

The functor Σ^∞ is left adjoint to the zero$^{\underline{th}}$ space functor Ω^∞ from spectra to spaces. For any Z in our indexing set $\mathcal{A}$, we also have the $Z^{\underline{th}}$ space functor, which we denote by

$$\Omega^\infty\Lambda_Z\colon G\mathcal{S}\mathcal{A} \to G\mathcal{J}.$$

We next construct its left adjoint, which we think of as a shift desuspension functor and denote by

$$\Lambda^Z\Sigma^\infty\colon G\mathscr{J} \longrightarrow G\mathscr{S}\mathscr{A}.$$

In section 7, we shall construct an adjoint pair of functors Λ_Z and Λ^Z from $G\mathscr{S}\mathscr{A}$ to itself and shall see that the functors $\Omega^\infty\Lambda_Z$ and $\Lambda^Z\Sigma^\infty$ are indeed composites, as indicated by our choice of notation. However, we have immediate need of the composites, whereas the functors Λ_Z and Λ^Z will only play a peripheral role. In fact, they will turn out to be equivalent to Σ^Z and Ω^Z respectively, and their chief role will be to aid in the proof that Σ^Z and Ω^Z become adjoint equivalences on passage to the stable category.

Definition 4.1. For $X \in G\mathscr{J}$ and $Z \in \mathscr{A}$, define $\Lambda^Z\Sigma^\infty X \in G\mathscr{S}\mathscr{A}$ by letting

$$\Lambda^Z\Sigma^\infty X = L\{\Sigma^{V-Z}X\},$$

where $\{\Sigma^{V-Z}X\}$ denotes the prespectrum whose $V^{\underline{th}}$ space is $\Sigma^{V-Z}X$ if $V \supset Z$ and $\{*\}$ otherwise and whose structural maps are the evident identifications

$$\Sigma^{W-V}\Sigma^{V-Z}X \cong \Sigma^{W-Z}X$$

for $Z \subset V \subset W$ and the inclusion of the basepoint otherwise.

Proposition 4.2. For $Z \in \mathscr{A}$, $X \in G\mathscr{J}$, and $E \in G\mathscr{S}\mathscr{A}$, there is a natural isomorphism

$$G\mathscr{J}(X,\Omega^\infty\Lambda_Z E) \cong G\mathscr{S}\mathscr{A}(\Lambda^Z\Sigma^\infty X,E).$$

For $Y \in G\mathscr{J}$, there is a natural isomorphism

$$\Lambda^Z\Sigma^\infty(X\wedge Y) \cong (\Lambda^Z\Omega^\infty X)\wedge Y.$$

For $Z \cong Z'$ in $\mathscr{A}$, there is a natural isomorphism

$$\Lambda^Z\Sigma^\infty X \cong \Lambda^{Z'}\Sigma^\infty X.$$

For $V \subset W$ in $\mathscr{A}$, there is a natural isomorphism

$$\Lambda^V\Sigma^\infty X \cong \Lambda^W\Sigma^\infty\Sigma^{W-V}X.$$

Proof. The adjunction is easily checked on the prespectrum level and follows on the spectrum level. The remaining three isomorphisms can also be checked directly on the prespectrum level, but it is more amusing to obtain them by application of our conjugation trick codified in Lemma 3.5. For the first,

$$F(X,\Omega^\infty\Lambda_Z E) = F(X,EZ) = \Omega^\infty\Lambda_Z F(X,E).$$

For the second, choose $W \in \mathscr{A}$ which contains both Z and Z'. Any isomorphism $W-Z \cong W-Z'$ induces a natural isomorphism

$$EZ \xrightarrow{\tilde\sigma} \Omega^{W-Z}EW \cong \Omega^{W-Z'}EW \xrightarrow{\tilde\sigma^{-1}} EZ'.$$

For the third, the structural homeomorphisms $\tilde\sigma\colon EV \to \Omega^{W-V}EW$ specify a natural isomorphism $\Omega^\infty\Lambda_V \cong \Omega^{W-V}\Omega^\infty\Lambda_W$.

Now recall our standing assumption that any G-universe contains a canonical copy of R^∞ with trivial G action. We assume (or arrange by use of Proposition 2.4) that all indexing sets contain $\{R^n\}$, and we write $\Lambda^n\Sigma^\infty$ for shift desuspension by R^n. We use these functors to define canonical sphere spectra.

Definition 4.3. Define sphere G-spectra $S^n \in G\mathscr{S}\mathscr{A}$ by

$$S^n = \Sigma^\infty S^n = \Sigma^n\Sigma^\infty S^0 \quad\text{and}\quad S^{-n} = \Lambda^n\Sigma^\infty S^0 \quad\text{for } n \geq 0.$$

Abbreviate $S = S^0$. For $H \subset G$, define generalized sphere G-spectra S^n_H by

$$S^n_H = G/H^+ \wedge S^n$$

Observe that Propositions 3.6 and 4.2 imply isomorphisms

$$\Sigma^m S^n_H \cong S^{m+n}_H \quad\text{for } m \geq 0 \text{ and } n \in Z;$$

They also imply isomorphisms

$$S^n_H \cong \Sigma^\infty S^n_H \cong \Sigma^n\Sigma^\infty S^0_H \quad\text{and}\quad S^{-n}_H \cong \Lambda^n\Sigma^\infty S^0_H \quad\text{for } n \geq 0.$$

Definition 4.4. (i) For $H \subset G$, $n \in Z$, and $E \in G\mathscr{S}\mathscr{A}$, define

$$\pi^H_n E = \pi(S^n_H,E)_G.$$

Observe that these are Abelian groups since S^n_H is a double suspension.

(ii) A map $f\colon E \to E'$ of G-spectra is said to be a weak equivalence if $f_*\colon \pi^H_n E \to \pi^H_n E'$ is an isomorphism for all $n \in Z$ and $H \subset G$.

(iii) A G-spectrum E is said to be n-connected if $\pi^H_q E = 0$ for $q \leq n$ and all $H \subset G$; E is said to be connective if it is (-1)-connected and to be bounded below if it is n-connected for some integer n.

Proposition 4.5. Let E be a G-spectrum and let $K \subset H \subset G$.

(i) For $n \geq 0$, $\pi_n^H E \cong \pi_n^H E_0$ and $\pi_{-n}^H E \cong \pi_0^H E R^n$.

(ii) $\pi_n^K E$ is the same for E regarded as an H-spectrum as for E regarded as a G-spectrum.

(iii) $\pi_n^H E$ is naturally isomorphic to $\pi_n(E^H)$.

Proof. Part (i) follows by adjunction from the alternative description of S_H^n in Definition 4.3. Parts (ii) and (iii) are immediate consequences of part (i).

The relations between spectrum level and space level homotopy groups are much less complete equivariantly than non-equivariantly. In particular, it is not clear that the $\pi_n^H E$ determine the $\pi_n^H EV$ for non-trivial representations V. We shall return to this point in section 7, where we shall prove the following fundamental result.

Theorem 4.6. A map $f: E \to E'$ of G-spectra is a weak equivalence if and only if it is a spacewise weak equivalence. If E and E' are connective, then these conditions hold if and only if $\Omega^\infty f: \Omega^\infty E \to \Omega^\infty E'$ is a weak equivalence.

By (i) of the proposition, the rest will follow once we show that the V^{th} space functors $\Omega^\infty \Lambda_V$ preserve weak equivalences.

Returning to the adjoint shift desuspension functors, we show next that the $\Lambda^V \Sigma^\infty X$ may be viewed as building blocks out of which arbitrary spectra can be constructed.

Proposition 4.7. For $E \in G\mathcal{S}\mathcal{A}$, there is a natural isomorphism

$$E \cong \operatorname*{colim}_{V \subset \mathcal{A}} \Lambda^V \Sigma^\infty EV.$$

More generally, for $D \in G\mathcal{P}\mathcal{A}$, there is a natural isomorphism

$$LD \cong \operatorname*{colim}_{V \subset \mathcal{A}} \Lambda^V \Sigma^\infty DV,$$

where the (spectrum level) colimit is taken over the maps

$$\Lambda^W \Sigma^\infty \sigma : \Lambda^V \Sigma^\infty DV \cong \Lambda^W \Sigma^\infty \Sigma^{W-V} DV \longrightarrow \Lambda^W \Sigma^\infty DW.$$

If $D \in G\mathcal{I}\mathcal{A}$, then this colimit can be computed on the prespectrum level, without application of the functor L.

Proof. For any G-prespectrum D, we have natural isomorphisms

$$\begin{aligned} G\mathcal{S}\mathcal{A}(\operatorname{colim} \Lambda^V \Sigma^\infty D, E) &\cong \lim G\mathcal{S}\mathcal{A}(\Lambda^V \Sigma^\infty DV, E) \\ &\cong \lim G\mathcal{T}(DV, EV) \\ &\cong G\mathcal{P}\mathcal{A}(D, \ell E) \cong G\mathcal{S}\mathcal{A}(LD, E). \end{aligned}$$

The isomorphism $LD \cong \operatorname{colim} \Lambda^V \Sigma^\infty DV$ follows by the Yoneda lemma. By the use of L in the definition of the spectra $\Lambda^V \Sigma^\infty DV$, the Z^{th} space of the prespectrum level colimit is

$$(\operatorname*{colim}_V \Lambda^V \Sigma^\infty DV)(Z) = \operatorname*{colim}_V \operatorname*{colim}_{W \supset V, Z} \Omega^{W-Z} \Sigma^{W-V} DV.$$

Due to the maps $\sigma: \Sigma^{W-V} DV \to DW$ appearing in the colimit system, the terms with $W = V$ are cofinal in the double colimit. When D is an inclusion G-prespectrum, the right side is thus already

$$(LD)(Z) = \operatorname*{colim}_{W \supset Z} \Omega^{W-Z} DW,$$

before application of L. This proves the last statement.

This result is most useful for inclusion prespectra, since the last statement then gives a concrete description of the colimit. We apply the result in conjunction with the following observations about maps into colimits.

Lemma 4.8. For a compact G-space K and inclusion G-prespectrum D,

$$G\mathcal{S}\mathcal{A}(\Lambda^V \Sigma^\infty K, LD) \cong \operatorname*{colim}_{W \supset V} G\mathcal{T}(K, \Omega^{W-V} DW).$$

For a directed system $\{E_i\}$ of inclusions of G-spectra which contains a cofinal sequence,

$$G\mathcal{S}\mathcal{A}(\Lambda^V \Sigma^\infty K, \operatorname*{colim}_i E_i) \cong \operatorname*{colim}_i G\mathcal{S}\mathcal{A}(\Lambda^V \Sigma^\infty K, E_i).$$

For any directed system $\{E_i\}$ of G-spectra which contains a cofinal sequence,

$$hG\mathcal{S}(\Lambda^V \Sigma^\infty K, \operatorname*{tel}_i E_i) \cong \operatorname*{colim}_i hG\mathcal{S}\mathcal{A}(\Lambda^V \Sigma^\infty K, E_i).$$

Proof. Since $(LD)(V) = \operatorname{colim} \Omega^{W-V} DW$, the first part is obvious. For the second part, we observe that the prespectrum level colimit of the E_i is already a spectrum and thus

$$G\mathscr{S}\mathscr{A}(\Lambda^V\Sigma^\infty K, \operatorname{colim} E_i) \cong G\mathscr{T}(K, \operatorname{colim} E_iV) \cong \operatorname{colim} G\mathscr{T}(K, E_iV).$$

The second isomorphism requires the specified restriction on the limit system since a map from a compact space into an arbitrary colimit need not factor through one of the terms. For the last part, we observe that $\mathrm{tel}\, E_i$ is the colimit of its system of inclusions of partial telescopes and apply the second part on the homotopy level.

The previous two results have the following consequence.

Corollary 4.9. For an inclusion G-prespectrum D,

$$\pi_n^H(LD) \cong \operatorname*{colim}_{V \subset \mathscr{A}} \pi_n^H(\Lambda^V\Sigma^\infty DV)$$

and the natural map

$$\mathrm{tel}\, \Lambda^V\Sigma^\infty DV \to \operatorname{colim} \Lambda^V\Sigma^\infty DV \cong LD$$

is a weak equivalence.

The corollary will lead to descriptions of the homology and cohomology groups of LD in terms of those of the spaces DV. No such description need hold for general prespectra, and for this reason only inclusion prespectra are of calculational as opposed to theoretical interest.

By a "compact" G-spectrum, we understand one of the form $\Lambda^V\Sigma^\infty K$ for a compact G-space K. By Lemma 4.8, maps in $G\mathscr{S}\mathscr{A}$ with compact domain are colimits of space level maps. When the codomain is of the form $\Lambda^W\Sigma^\infty X$, we can express such maps as shift desuspensions of space level maps.

Lemma 4.10. Let $f:\Lambda^V\Sigma^\infty K \to \Lambda^W\Sigma^\infty X$ be a map of G-spectra, where K is a compact G-space and X is any G-space. Then there exists $Z \in \mathscr{A}$ and a G-map $g:\Sigma^{Z-V}K \to \Sigma^{Z-W}X$ such that the following diagram commutes.

$$\begin{tikzcd}
\Lambda^V\Sigma^\infty K \arrow[r, "f"] \arrow[d, phantom, "\cong" rotate=-90] & \Lambda^W\Sigma^\infty X \arrow[d, phantom, "\cong" rotate=-90] \\
\Lambda^Z\Sigma^\infty\Sigma^{Z-V}K \arrow[r, "\Lambda^Z\Sigma^\infty g"] & \Lambda^Z\Sigma^\infty\Sigma^{Z-W}X
\end{tikzcd}$$

Proof. The isomorphisms come from Proposition 4.2. By Lemma 4.8 and the definition of $\Lambda^W\Sigma^\infty X$,

$$G\mathscr{S}\mathscr{A}(\Lambda^V\Sigma^\infty K, \Lambda^W\Sigma^\infty X) \cong \operatorname*{colim}_{Z \supset V,W} G\mathscr{T}(K, \Omega^{Z-V}\Sigma^{Z-W}X).$$

Here f is realized by the adjoint of some g, and the result follows upon unraveling the definitions.

The homotopy category of those G-spectra of the form $\Lambda^V\Sigma^\infty K$, where K is a finite G-CW complex, is the appropriate equivariant analog of the Spanier-Whitehead S-category. It should be viewed as a halfway house between the world of spaces and the world of spectra.

§5. G-CW spectra and the stable category

We here give the theory of G-CW spectra. Modulo the use of two filtrations, to allow induction in the presence of spheres of negative dimension, the theory is essentially the same as on the space level. In particular, the cellular approximation theorem and Whitehead's theorem are proven exactly as on the space level [115,140] (or as in the nonequivariant case [107]).

Given Whitehead's theorem for G-CW spectra and the fact that every G-spectrum is weakly equivalent to a G-CW spectrum, we can construct the equivariant stable category from the homotopy category of G-spectra by formally inverting its weak equivalences. We include a general categorical discussion of this procedure.

We continue to work in $G\mathscr{S}\mathscr{A}$ for a fixed indexing set $\mathscr{A}$ in a G-universe U. We write * for the trivial G-spectrum (each EV a point) and write Cf for the cofibre $E \cup_f CD$ of a map $f:D \to E$ of G-spectra.

Definitions 5.1. A G-cell spectrum is a spectrum $E \in G\mathscr{S}\mathscr{A}$ together with a sequence of subspectra E_n and maps $j_n:J_n \to E_n$ such that J_n is a wedge of sphere spectra S_H^q, $E_0 = *$, $E_{n+1} = Cj_n$ for $n \geq 0$, and E is the union of the E_n. The map from the cone on a wedge summand of J_n into E is called a cell. The restriction of j_n to a wedge summand is called an attaching map. The sequence $\{E_n\}$ is called the sequential filtration of E. E is said to be finite if it contains only finitely many cells and to be finite dimensional if it contains cells in only finitely many dimensions. A map $f:E \to F$ of G-cell spectra is said to be sequentially cellular if $f(E_n) \subset F_n$ for all $n \geq 0$. A subspectrum A of a G-cell spectrum E is said to be a cell subspectrum if A is a G-cell spectrum such that $A_n \subset E_n$ and the composite of each cell $CS_H^q \to A_n \subset A$ and the inclusion of A in E is a cell of E with image in E_n; thus A is just the union of some of the cells of E.

Note that the cells of G-cell spectra need not be attached only to cells of lower dimension.

Definitions 5.2. A G-CW spectrum is a G-cell spectrum such that each attaching map $S_H^q \to E_n$ factors through a cell subspectrum containing only cells of dimension $\leq q$. The n-skeleton E^n is then defined to be the union of the cells of E of dimension $\leq n$, and E is the union of its skeleta. A map $f:E \to F$ of G-CW spectra is said to be cellular if it preserves the skeletal filtration $\{E^n\}$ and to be bicellular if it also preserves the sequential filtration $\{E_n\}$. A cell subspectrum A of a G-CW spectrum E is necessarily a G-CW spectrum such that the inclusion $A \to E$ is bicellular; in particular, this applies to $A = E^n$ for any integer n and also to $A = E_n$ for $n \geq 0$.

As said before, the use of two filtrations is essential for inductive arguments. However, the sequential filtration, which describes the order in which cells are attached, can be chosen in many different ways. In fact, for any map $f: E \to F$ between G-cell spectra, there is a sequential filtration of E with respect to which f is sequentially cellular (and for this reason the distinction between cellular and bicellular maps of G-CW spectra is not very important). The following lemma, which will be proven in the Appendix, makes it easy to verify the last assertion. Recall that a compact G-spectrum is one of the form $\Lambda^V\Sigma^\infty K$ for some compact G-space K and some $V \in \mathcal{A}$.

Lemma 5.3. If C is a compact G-spectrum and E is a G-cell spectrum, then any map $f: C \to E$ factors through a finite cell subspectrum of E. Any G-cell spectrum is the union of its finite cell subspectra.

Recall the discussion of G-CW complexes from section 1. We have the following relationship between G-CW complexes and G-CW spectra.

Lemma 5.4. The functors $\Lambda^n\Sigma^\infty$ carry based G-CW complexes and cellular maps to G-CW spectra and cellular maps. They carry based G-spaces of the homotopy type of ordinary G-CW complexes to G-spectra of the homotopy type of G-CW spectra.

Proof. Since $\Lambda^n\Sigma^\infty$ preserves G-spheres, cofibres, and unions, the first statement is immediate. For X as in the second statement, ΣX is equivalent to a based G-CW complex, and $\Sigma^\infty X \cong \Lambda\Sigma^\infty\Sigma X$ by Proposition 4.2.

We shall shortly generalize the second assertion to $\Lambda^V\Sigma^\infty$ for general V; the generalization is not obvious since the behavior of these functors on G-spheres is not obvious. We have the following consequence of the previous result.

Lemma 5.5. If E is a G-CW spectrum and X is a G-CW complex, then $E \wedge X$ has the homotopy type of a G-CW spectrum.

Proof. For $q \geq 0$, $S^q_H \wedge X \cong \Sigma^\infty(S^q_H \wedge X)$ and $S^{-q}_H \wedge X \cong \Lambda^q\Sigma^\infty(S^0_H \wedge X)$, hence the functor $(?) \wedge X$ carries spheres S^q_H to G-CW homotopy types. Since cofibres and unions over sequences of cofibrations preserve G-CW homotopy types (by standard arguments with homotopy pushouts and telescopes), the conclusion follows.

We have an analogous result for passage to orbits.

Lemma 5.6. (i) If E is a G-CW spectrum indexed on a G-trivial universe, then E/G is naturally a CW spectrum with one cell for each cell of E.
(ii) If E is a G-CW spectrum indexed on an H-trivial universe, then E/H has the homotopy type of a WH-CW spectrum.

Proof. Passage to orbits commutes with the functors $\Lambda^n\Sigma^\infty$ and with cofibres and colimits. For (i), it suffices to observe that (G/K)/G is a point. For (ii), it suffices to observe that (G/K)/H is the double coset space $H\backslash G/K$, which is triangulable as a WH-CW complex.

Returning to the general theory, we record the following analogs of standard space level facts. All parts of the following lemma are also true for G-cell spectra.

Lemma 5.7. (i) A wedge of G-CW spectra is a G-CW spectrum.
(ii) If A is a cell subspectrum of a G-CW spectrum D, E is a G-CW spectrum, and $f: A \to E$ is a cellular map, then the pushout $E \cup_f D$ is a G-CW spectrum which contains E as a cell subspectrum.
(iii) If E is a G-CW spectrum, then so are $E \wedge I^+$, CE, and ΣE; $E \wedge (\partial I)^+$ is a cell subspectrum of $E \wedge I^+$ and E is a cell subspectrum of CE.

Theorem 5.8 (Cellular approximation). Let A be a cell subspectrum of a G-CW spectrum D, let E be a G-CW spectrum, and let $f:D \to E$ be a map which is cellular when restricted to A. Then f is homotopic rel A to a cellular map. In particular, any map $D \to E$ is homotopic to a cellular map and any two homotopic cellular maps are cellularly homotopic.

For the proof, we may assume that $f|A$ is bicellular and proceed by induction over the sequential filtration. The result quickly reduces to the case of a single cell of D not in A, and there the result can be reduced to the space level by use of Propositions 4.2 and 4.5. A similar induction gives the following homotopy extension and lifting property.

Theorem 5.9 (HELP). Let A be a cellular subspectrum of a G-cell spectrum D and let $e:E \to F$ be a weak equivalence of G-spectra. Suppose that $hi_1 = eg$ and $hi_0 = f$ in the following diagram:

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \wedge I^+ \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& F & & E \arrow[ll, "e"'] & \\
D \arrow[rr, "i_0"'] \arrow[ur, "f"] & & D \wedge I^+ \arrow[ul, dashed, "\tilde{h}"'] & & D \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

Then there exist $\tilde{g}$ and $\tilde{h}$ such that the diagram commutes. In particular, the inclusion of A in D is a cofibration.

Whitehead's theorem is an immediate consequence.

Theorem 5.10 (Whitehead). If $e:E \to F$ is a weak equivalence of G-spectra, then $e_*: \pi(D,E)_G \to \pi(D,F)_G$ is a bijection for every G-cell spectrum D. If E and F are themselves G-cell spectra, then e is an equivalence.

Let $G\mathscr{C}\mathscr{A}$ denote the category of G-CW spectra and cellular maps and let $hG\mathscr{C}\mathscr{A}$ denote its homotopy category. Lemma 5.7 implies that $hG\mathscr{C}\mathscr{A}$ has arbitrary homotopy colimits, and we have enough information to be able to quote Brown's representability theorem [19,107].

Theorem 5.11 (Brown). A contravariant set-valued functor T on $hG\mathscr{C}\mathscr{A}$ is representable as $TE = \pi(E,F)_G$ for some G-CW spectrum F if and only if T takes wedges to products and takes homotopy pushouts to weak pullbacks.

Here homotopy pushouts are double mapping cylinders and weak pullbacks satisfy the existence part but not the uniqueness part of the universal property of pullbacks. Either by quotation of Brown's theorem or by direct construction, we obtain the following result. It asserts that G-spectra can be replaced functorially by weakly equivalent G-CW spectra.

Theorem 5.12. There is a functor $\Gamma: hG\mathscr{S}\mathscr{A} \to hG\mathscr{C}\mathscr{A}$ and a natural weak equivalence $\gamma: \Gamma E \to E$ for $E \in G\mathscr{S}\mathscr{A}$.

We can now define the equivariant stable category $\overline{hG\mathscr{S}\mathscr{A}}$ by formally inverting the weak equivalences in $hG\mathscr{S}\mathscr{A}$, but it may be worthwhile to first give a little general categorical discussion of this inversion procedure.

Let $\mathscr{H}$ be a category with a collection $\mathscr{E}$ of morphisms, called the weak equivalences. Assume that all isomorphisms are in $\mathscr{E}$ and that $\mathscr{E}$ is closed under composition. A localization of $\mathscr{H}$ at $\mathscr{E}$ is a category $\mathscr{E}^{-1}\mathscr{H}$ with the same objects as $\mathscr{H}$ together with a functor $L: \mathscr{H} \to \mathscr{E}^{-1}\mathscr{H}$ such that L is the identity function on

objects, L takes weak equivalences to isomorphisms, and L is universal with respect to the latter property.

Say that an object $X \in \mathscr{H}$ is cocomplete if $e_*: \mathscr{H}(X,Y) \to \mathscr{H}(X,Z)$ is a bijection for every weak equivalence $e:Y \to Z$. Thus Whitehead's theorem asserts that CW objects are cocomplete in homotopy categories. A cocompletion of an object $Y \in \mathscr{H}$ is a weak equivalence $\gamma: \Gamma Y \to Y$, where ΓY is cocomplete. If every Y admits a cocompletion, then a formal argument shows that any collection of choices of ΓY for $Y \in \mathscr{H}$ yields a functor $\Gamma: \mathscr{H} \to \mathscr{H}$ such that γ is natural. When this holds, we can define $\mathscr{E}^{-1}\mathscr{H}$ by

$$\mathscr{E}^{-1}\mathscr{H}(Y,Y') = \mathscr{H}(\Gamma Y, \Gamma Y'),$$

with composition in $\mathscr{E}^{-1}\mathscr{H}$ inherited from $\mathscr{H}$ and with $L = \Gamma$ on morphisms. In fact, this is how localizations are usually shown to exist, since set theoretical difficulties arise in any attempt to give a more direct construction.

If every $Y \in \mathscr{H}$ admits a cocompletion $\Gamma Y \in \mathscr{C}$ for some subcategory $\mathscr{C}$ of $\mathscr{H}$, then each cocomplete object X is isomorphic to the object $\Gamma X \in \mathscr{C}$. In our homotopy categories, this says that the cocomplete objects are precisely those of the homotopy type of CW objects.

For categories $\mathscr{H}$ and $\mathscr{G}$ with weak equivalences and cocompletions as above, a functor $S: \mathscr{H} \to \mathscr{G}$ which takes weak equivalences to weak equivalences induces a functor $S: \mathscr{E}^{-1}\mathscr{H} \to \mathscr{E}^{-1}\mathscr{G}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathscr{H} \arrow[r, "S"] \arrow[d, "L"] & \mathscr{G} \arrow[d, "L"] \\
\mathscr{E}^{-1}\mathscr{H} \arrow[r, "S"] & \mathscr{E}^{-1}\mathscr{G}
\end{tikzcd}$$

If S fails to preserve weak equivalences, then we agree once and for all that $S: \mathscr{E}^{-1}\mathscr{H} \to \mathscr{E}^{-1}\mathscr{G}$ shall denote the functor so obtained from the composite $S\Gamma: \mathscr{H} \to \mathscr{G}$, which obviously does preserve weak equivalences.

In view of the ubiquitous role played by adjoint functors in our theory, we shall make much use of the following observations.

Lemma 5.13. Let $\mathscr{H}$ and $\mathscr{G}$ be categories with weak equivalences and cocompletions of all objects. Let $S: \mathscr{H} \to \mathscr{G}$ be left adjoint to $T: \mathscr{G} \to \mathscr{H}$. Then S takes cocomplete objects to cocomplete objects if and only if T takes weak equivalences to weak equivalences. When these conditions hold, the induced functors $S: \mathscr{E}^{-1}\mathscr{G} \to \mathscr{E}^{-1}\mathscr{H}$ and $T: \mathscr{E}^{-1}\mathscr{H} \to \mathscr{E}^{-1}\mathscr{G}$ are again left and right adjoints. If S and T are adjoint equivalences between $\mathscr{H}$ and $\mathscr{G}$ both of which preserve cocomplete objects (and thus weak equivalences), then the induced functors are adjoint equivalences between $\mathcal{E}^{-1}\mathcal{H}$ and $\mathcal{E}^{-1}\mathcal{J}$.

Our applications of the first statement will go in both directions. A key example is the adjoint pair consisting of $\Lambda^V\Sigma^\infty$ and the V^{th} space functor. By Theorem 4.6, the latter functor preserves weak equivalences. We are entitled to the following consequence.

Corollary 5.14. The functors $\Lambda^V\Sigma^\infty : G\mathcal{J} \to G\mathcal{S}\mathcal{A}$ take G-spaces of the homotopy type of G-CW complexes to G-spectra of the homotopy type of G-CW spectra.

For applications in the other direction, the left adjoint S will always preserve cofibres and unions and will therefore preserve CW homotopy types (= cocomplete objects) if and only if it takes spheres to CW homotopy types. Of course, by "spheres" we understand those objects allowable as domains of attaching maps of CW objects, these being the same objects in terms of which weak equivalences are to be defined. Possible alternative collections of equivariant spheres will be discussed in section 8.

§6. The stable category, cohomology, and the cylinder construction

We have constructed the equivariant stable category $\bar{h}G\mathcal{S}\mathcal{A}$ by formally inverting the weak equivalences of $hG\mathcal{S}\mathcal{A}$. The functor $\Gamma : \bar{h}G\mathcal{S}\mathcal{A} \to hG\mathcal{C}\mathcal{A}$ is an equivalence, and this allows us the freedom both to use natural constructions on the spectrum level (such as products and function spectra) which fail to preserve CW homotopy types and to provide spectra with CW structures whenever desirable. Of course, Brown's representability theorem is inherited from $hG\mathcal{C}\mathcal{A}$. The category $\bar{h}G\mathcal{S}\mathcal{A}$ has arbitrary homotopy colimits and homotopy limits with all the standard properties familiar from spaces. In particular, it has cofibration and fibration sequences and dual Milnor $\lim^1$ exact sequences. It will follow from the desuspension theorem that cofibration sequences give rise to long exact homotopy sequences (for all $H \subset G$) and are equivalent (up to sign) to fibration sequences. Details of the last will be given in III§2, where duality in $\bar{h}G\mathcal{S}\mathcal{A}$ will be studied.

We shall prove in the next chapter that $\bar{h}G\mathcal{S}\mathcal{A}$ is a closed symmetric monoidal category. This means that it has a smash product functor, $D \wedge E$, which is associative, commutative, and unital with unit S (up to coherent natural isomorphisms) and that it has a function spectrum functor, $F(D,E)$, adjoint to the smash product.

We write $[D,E]_G$ for the set of morphisms in $\bar{h}G\mathcal{S}\mathcal{A}$ from D to E. If D is a G-CW spectrum, this is naturally isomorphic to $\pi(D,E)_G$. In particular, $\pi_n^H E = [S_H^n, E]_G$. The adjunction referred to above gives

$$[C \wedge D, E]_G \cong [C, F(D,E)]_G.$$

Recall that $\bar{h}G\mathcal{J}$ was obtained from $hG\mathcal{J}$ by inverting the based maps which are weak equivalences. Since Σ^∞ preserves G-CW homotopy types by Lemma 5.4, we have adjoint functors Σ^∞ and Ω^∞ relating $\bar{h}G\mathcal{J}$ to $\bar{h}G\mathcal{S}\mathcal{A}$. More generally, Corollary 5.14 gives that the functors $\Lambda^V\Sigma^\infty$ preserve G-CW homotopy types, and we thus have

$$[\Lambda^V\Sigma^\infty X, E]_G \cong [X, EV]_G.$$

Theorem 1.2 and Lemma 5.5 imply that the functors Σ^V preserve G-CW homotopy types. Thus Σ^V and Ω^V induce adjoint endofunctors of $\bar{h}G\mathcal{J}$ and $\bar{h}G\mathcal{S}\mathcal{A}$. We have referred to $\bar{h}G\mathcal{S}\mathcal{A}$ as an equivariant stable category, and the word "stable" is justified by the following fundamental result.

Theorem 6.1 (desuspension). For all $V \in \mathcal{A}$, the natural adjunction maps

$$\eta : E \to \Omega^V\Sigma^V E \quad \text{and} \quad \varepsilon : \Sigma^V\Omega^V E \to E$$

in $\bar{h}G\mathcal{S}\mathcal{A}$ are isomorphisms. Therefore Σ^V and Ω^V are inverse self-equivalences of the category $\bar{h}G\mathcal{S}\mathcal{A}$.

Thus, in $\bar{h}G\mathcal{S}\mathcal{A}$, we can desuspend by arbitrary representations $V \in \mathcal{A}$, and we adopt the alternative notation Σ^{-V} for Ω^V. Of course, using $V = R^2$, it follows that $\bar{h}G\mathcal{S}\mathcal{A}$ is an additive category.

A proof of the theorem could be based on the isomorphisms

$$E \cong E \wedge S \cong E \wedge (\Sigma^V\Lambda^V\Sigma^\infty S^0) \cong \Sigma^V E \wedge \Lambda^V\Sigma^\infty S^0$$

implied by Proposition 4.2 and basic properties of the smash product. (Technically, this argument, like any other, depends on Corollary 5.14.) This would give that Σ^V is an equivalence with inverse obtained by smashing with $\Lambda^V\Sigma^\infty S^0$, and categorical nonsense could then be used to derive the asserted adjoint equivalence. We shall give a different proof in the next section. Namely, we shall define endofunctors Λ^V and Λ_V of $\bar{h}G\mathcal{S}\mathcal{A}$ which are inverse adjoint equivalences and we shall prove that Λ^V is naturally equivalent to Ω^V. By the uniqueness of adjoints, it will follow that Λ_V is equivalent to Σ^V and thus that Ω^V and Σ^V are inverse adjoint equivalences.

Just as in the nonequivariant case, the stable category $\bar{h}G\mathcal{S}U$ is equivalent to the category of cohomology theories on G-spectra. Here the latter should be interpreted as $RO(G;U)$-graded, where $RO(G;U)$ is the free Abelian group generated by those specific irreducible representations which generate the universe U. For $a = V - W \in RO(G;U)$, where V and W are sums of distinct irreducibles in our basis for RO(G;U), we set $S^a = \Sigma^{-W}S^V$ and define $\Sigma^a E = E \wedge S^a$. For G-spectra Y, we then define

$$E_a Y = [S^a, Y \wedge E]_G \quad \text{and} \quad E^a Y = [Y, \Sigma^a E]_G.$$

We shall not give a formal axiomatic definition of what we mean by a cohomology theory on G-spectra. Suffice it to say that the zeroth term necessarily satisfies the hypotheses of Brown's representability theorem and the entire theory then takes the form just given. (We shall study such theories in detail in [90], some of the results of which have been announced in [88].)

For a based G-space X and a G-spectrum E, we define

$$\tilde{E}_* X = E_*(\Sigma^\infty X) \quad \text{and} \quad \tilde{E}^* X = E^*(\Sigma^\infty X).$$

If D is an inclusion G-prespectrum, the equivalence $LD \simeq \mathrm{tel}\, \Lambda^V \Sigma^\infty DV$ of Corollary 4.9 gives rise to a $\lim^1$ exact sequence

$$0 \to \lim{}^1 [\Sigma\Lambda^V\Sigma^\infty DV, \Sigma^a E]_G \to [LD, \Sigma^a E]_G \to \lim [\Lambda^V \Sigma^\infty DV, \Sigma^a E]_G \to 0.$$

This may be rewritten in the form

$$0 \to \lim{}^1 \tilde{E}^{a+V-1}(DV) \to E^a(LD) \to \lim \tilde{E}^{a+V}(DV) \to 0.$$

It expresses the cohomology of the G-spectrum LD in terms of the cohomology of the G-spaces DV.

In the rest of this section, we shall be concerned with the represented equivalent of the category of RO(G;U)-graded cohomology theories on G-spaces. We specify that such a theory should consist (at least!) of representable set-valued functors $\tilde{E}^V$ on $\overline{h}G\mathcal{J}$ for indexing spaces V in U together with natural isomorphisms

$$\tilde{E}^V X \cong \tilde{E}^W(\Sigma^{W-V} X) \quad \text{for} \quad V \subset W.$$

For $V \subset W \subset Z$, the evident composite isomorphism should agree with the given isomorphism for $\Sigma^{Z-V}X$. Clearly the entire theory is determined by its values on the indexing spaces A_i of an indexing sequence A in U; given V, we choose the minimal i such that $V \subset A_i$ and have $\tilde{E}^V X \cong \tilde{E}^{a_i}(\Sigma^{a_i - V} X)$. It is convenient to restrict attention to such a sequence so as to avoid consideration of diagrams like those of Definition 2.1(ii) and of non-sequentially indexed colimits and telescopes in the discussion to follow.

Now let $\tilde{E}^{a_i}$ be represented by EA_i. The suspension isomorphisms give rise to isomorphisms $EA_i \to \Omega^{b_i} EA_{i+1}$ in $\overline{h}G\mathcal{J}$, where $B_i = A_{i+1} - A_i$. Either taking each EA_i to be a G-CW complex or allowing structural maps $\tilde{\sigma}$ in $\overline{h}G\mathcal{J}$, we conclude that $\{EA_i\}$ is a G-prespectrum indexed on A such that each $\tilde{\sigma}$ is a weak equivalence. We call such prespectra ΩG-prespectra. Consideration of maps of cohomology theories on G-spaces leads us to introduce the classical notion of maps of prespectra.

Definition 6.2. A w-map $f: D \to D'$ of G-prespectra indexed on $A = \{A_i\}$ consists of G-maps $f_i: DA_i \to D'A_i$ such that the following diagrams are G-homotopy commutative:

$$\begin{tikzcd} \Sigma^{b_i} DA_i \arrow[r, "\Sigma^{b_i} f_i"] \arrow[d, "\sigma"] & \Sigma^{b_i} D'A_i \arrow[d, "\sigma'"] \\ DA_{i+1} \arrow[r, "f_{i+1}"] & D'A_{i+1} \end{tikzcd}$$

Two w-maps f and f' are spacewise homotopic if $f_i \simeq f_i'$ for all i (with no compatibility requirement on the homotopies). Let $wG\mathcal{P}A$ denote the category of G-prespectra and spacewise homotopy classes of w-maps. Here w stands for weak (or Whitehead [142]).

Henceforward in this section we write D_i and σ_i for the i^{th} spaces and structural maps of G-prespectra indexed on A.

To obtain the precise form of the represented equivalent of the category of cohomology theories on G-spaces, and for several other purposes, we need the following elementary construction.

Construction 6.3. Construct a CW-approximation functor $\Gamma: wG\mathcal{P}A \to wG\mathcal{P}A$ and a natural weak equivalence $\gamma: \Gamma \to 1$ by applying any given CW-approximation functor $\Gamma: hG\mathcal{J} \to hG\mathcal{J}$ spacewise. Thus, for $D \in G\mathcal{P}A$, $\Gamma_i D = \Gamma(D_i)$ and $\gamma_i: \Gamma_i D \to D_i$ is the given weak equivalence. The structural map $\sigma_i: \Sigma^{b_i}\Gamma_i D \to \Gamma_{i+1} D$ is that G-map, unique up to homotopy, such that $\gamma_{i+1} \circ \sigma_i \simeq \sigma_i \circ \Sigma^{b_i}\gamma_i$. For a w-map $f: D \to D'$, $\Gamma_i f$ is characterized up to homotopy by $\gamma_i \circ \Gamma_i f \simeq f_i \circ \gamma_i$ and satisfies $\Gamma_{i+1} f \circ \sigma_i \simeq \sigma_i' \circ \Sigma^{b_i}\Gamma_i f$ because the composites of these maps with γ_{i+1} are homotopic.

Remark 6.4. By Theorem 1.1, $\Omega^V X$ has the homotopy type of a G-CW complex if X does. Given this, it is clear that the functor Γ takes ΩG-prespectra to

ΩG-prespectra for which the maps $\tilde{\sigma}_i$ are actual equivalences.

As explained in section 5, use of Γ allows us to construct a category $\overline{w}G\mathcal{P}A$ by formally inverting the spacewise weak equivalences of $wG\mathcal{P}A$. It follows easily from the discussion above that the resulting full subcategory $\overline{w}\Omega G\mathcal{P}A$ of ΩG-prespectra is the represented equivalent of the category of cohomology theories on G-spaces. Observe that if D' (but not necessarily D) is an ΩG-prespectrum, then

$$\overline{w}G\mathcal{P}A(D,D') = \lim\, [D_i, D'_i]_G\ ,$$

where the limit is taken with respect to the composites

$$[D_{i+1}, D'_{i+1}]_G \xrightarrow{\Omega^{b_i}} [\Omega^{b_i} D_{i+1}, \Omega^{b_i} D'_{i+1}]_G \xrightarrow{[\tilde{\sigma}_i, (\tilde{\sigma}'_i)^{-1}]} [D_i, D'_i]_G\ .$$

Let $\overline{w}G\mathcal{S}A$ denote the full subcategory of $\overline{w}\Omega G\mathcal{P}A$ the objects of which are G-spectra.

The following result implies that the evident forgetful transformation from cohomology theories on G-spectra to cohomology theories on G-spaces has as its represented equivalent the forgetful transformation $\overline{h}G\mathcal{S}A \to \overline{w}G\mathcal{S}A$ specified on morphisms by restriction of maps to component spaces:

$$[E,E']_G \to \lim [E_i, E'_i]_G.$$

Note that this is well-defined by Theorem 4.6, which ensures that the A_ith space functor from G-spectra to G-spaces preserves weak equivalences.

<u>Proposition 6.5</u>. There are adjoint equivalences

$$Z: \overline{w}\Omega G\mathcal{P}A \to \overline{w}G\mathcal{S}A \quad\text{and}\quad z: \overline{w}G\mathcal{S}A \to \overline{w}\Omega G\mathcal{P}A$$

Here z is the evident forgetful functor and factors through the full subcategory $\overline{w}\Omega G\mathcal{Q}A$ of inclusion ΩG-prespectra. Just as in section 2, Z is constructed in two steps. Recall Theorem 2.2.

<u>Lemma 6.6</u>. The pair (L,ℓ) induces adjoint equivalences

$$L: \overline{w}\Omega G\mathcal{Q}A \to \overline{w}G\mathcal{S}A \quad\text{and}\quad \ell: \overline{w}G\mathcal{S}A \to \overline{w}\Omega G\mathcal{Q}A.$$

<u>Proof</u>. For an inclusion ΩG-prespectrum D, each of the natural maps

$$\eta_i: D_i \to L_i D = \operatorname{colim} \Omega^{a_j - a_i} D_j$$

is clearly a weak equivalence. Thus $\eta: D \to \ell L D$ is an isomorphism in $\overline{w}\Omega G\mathcal{Q}A$. Since $\varepsilon: L\ell E \to E$ is an isomorphism for a G-spectrum E, this implies the conclusion.

Note that L fails to define a functor $\overline{w}G\mathcal{Q}A \to \overline{w}G\mathcal{S}A$. However, restriction to ΩG-prespectra is not needed for the second step.

<u>Lemma 6.7</u>. There are adjoint equivalences

$$K: \overline{w}G\mathcal{P}A \to \overline{w}G\mathcal{Q}A \quad\text{and}\quad k: \overline{w}G\mathcal{Q}A \to \overline{w}G\mathcal{P}A$$

which restrict to adjoint equivalences on the respective full subcategories of ΩG-prespectra.

Here k is the forgetful functor, hence $z = k\ell$. With $Z = LK$, the proposition will be an immediate consequence. The following cylinder construction, which is a variant of that given in [95], gives K and the proof of Lemma 6.7. The construction has many other uses, and we give more precise information than is required for the cited proof.

<u>Construction 6.8</u>. (i) Let D be a G-prespectrum indexed on A. We construct the cylinder G-prespectrum KD, a w-map $\iota: D \to KD$, and a map $\pi: KD \to D$ such that $\pi \circ \iota = 1$ and $\iota \circ \pi \simeq 1$ spacewise. Thus, let $K_j D$ be the (partial) telescope of the sequence

$$\Sigma^{a_j} D_0 \xrightarrow{\Sigma^{a_j - a_1}\sigma_0} \Sigma^{a_j - a_1} D_1 \xrightarrow{\Sigma^{a_j - a_2}\sigma_1} \Sigma^{a_j - a_2} D_2 \longrightarrow \cdots \longrightarrow D_j\ .$$

Equivalently, $K_0 D = D_0$ and $K_{j+1}D$ is the double mapping cylinder

$$\Sigma^{b_j} K_j D \cup_{\Sigma^{b_j}\iota_j} (\Sigma^{b_j} D_j \wedge I^+) \cup_{\sigma_j} D_{j+1}\ .$$

There are evident cofibrations $\sigma_j: \Sigma^{b_j} K_j D \to K_{j+1} D$, inclusions $\iota_j: D_j \to K_j D$, and quotient maps $\pi_j: K_j D \to D_j$ such that

$$\pi_j \circ \iota_j = 1\ ,\ \iota_j \circ \pi_j \simeq 1 \text{ via a canonical homotopy,}$$

$$\pi_{j+1} \circ \sigma_j = \sigma_j \circ \Sigma^{b_j}\pi_j\ ,\ \text{and}\ \iota_{j+1} \circ \sigma_j \simeq \sigma_j \circ \Sigma^{b_j}\iota_j\ ;$$

here the last homotopy is also canonical, being given by

$$\iota_{j+1} \circ \sigma_j = \iota_{j+1} \circ \sigma_j \circ \Sigma^{b_j}\pi_j \circ \Sigma^{b_j}\iota_j = \iota_{j+1} \circ \pi_{j+1} \circ \sigma_j \circ \Sigma^{b_j}\iota_j \simeq \sigma_j \circ \Sigma^{b_j}\iota_j.$$

(ii) Let $f:D \to D'$ be a w-map together with specified homotopies $h_j:\sigma_j' \circ \Sigma^{b_j} f_j \simeq f_{j+1} \circ \sigma_j$. We construct an associated map $Kf:KD \to KD'$ such that $\iota \circ f = Kf \circ \iota$ and $\pi \circ Kf \simeq f \circ \pi$ spacewise. Proceeding inductively, let $K_0 f = f_0$ and assume given $K_i f$ for $i \le j$. The equations

$$K_{j+1}f \circ \sigma_j = \sigma_j' \circ \Sigma^{b_j} K_j f \quad \text{and} \quad K_{j+1}f \circ \iota_{j+1} = \iota_{j+1} \circ f_{j+1}$$

specify $K_{j+1}f$ on the ends of the double mapping cylinder $K_{j+1}D$ and we set

$$(K_{j+1}f)[x,t] = \begin{cases} [(\Sigma^{b_j} f_j)(x), 2t] & \text{if } 0 \le t \le 1/2 \\ h_j(x, 2t-1) & \text{if } 1/2 \le t \le 1 \end{cases}$$

for $x \in \Sigma^{b_j} D_j$ and $t \in I$. By (i), we have a canonical homotopy

$$\pi_{j+1} \circ K_{j+1}f \simeq \pi_{j+1} \circ K_{j+1}f \circ \iota_{j+1} \circ \pi_{j+1} = \pi_{j+1} \circ \iota_{j+1} \circ f_{j+1} \circ \pi_{j+1} = f_{j+1} \circ \pi_{j+1}.$$

If f is a map, we take h to be constant or redefine

$$(K_{j+1}f)[x,t] = [(\Sigma^{b_j} f_j)(x), t]$$

and find that $\pi \circ Kf = f \circ \pi$ by trivial direct calculation; with the redefinition, K becomes a functor $G\mathcal{P}A \to G\mathcal{Q}A$.

(iii) The map $K_j f$ of (ii) is characterized up to homotopy by the condition $K_j f \circ \iota_j \simeq \iota_j \circ f_j$. Indeed, if $g \circ \iota_j \simeq \iota_j \circ f_j$, then

$$g \simeq g \circ \iota_j \circ \pi_j \simeq \iota_j \circ f_j \circ \pi_j = K_j f \circ \iota_j \circ \pi_j \simeq K_j f.$$

In particular, the homotopy class of $K_j f$ is independent of the choice of h_j, K preserves spacewise homotopies, and $K\pi \simeq \pi:KKD \to KD$ spacewise, the last since

$$K_j \pi \circ \iota_j \simeq \iota_j \circ \pi_j \simeq 1 = \pi_j \circ \iota_j \quad \text{on } K_j D.$$

Of course, $ZD = LKD$ is a well-defined G-spectrum for any G-prespectrum D. Moreover, $\kappa = \eta\iota: D \to ZD$ is always a well-defined w-map, although it is not a spacewise weak equivalence unless D is an ΩG-prespectrum. We also have a natural map $L\pi:ZD \to LD$, and $L\pi$ is a weak equivalence if D is an inclusion G-prespectrum. In particular, for G-spectra E, we have a natural weak equivalence $\varepsilon \circ L\pi:ZE \to E$ of G-spectra. In the nonequivariant case, the homomorphism

$$[ZD,E] \longrightarrow \lim\,[Z_i D, E_i] \xrightarrow{\kappa^*} \lim\,[D_i, E_i]$$

induced by κ is the starting point of McClure's analysis in $[H_\infty$, VII] of the calculational relationship between spectrum level and space level maps. Much of his work goes over to the equivariant case with only notational changes.

The construction Z admits the following spectrum level reinterpretation, which should be compared with Proposition 4.7 and Corollary 4.9.

Proposition 6.9. For $D \in G\mathcal{P}A$, there is a canonical weak equivalence

$$\theta:\mathrm{tel}\ \Lambda^{a_i}\Sigma^\infty D_i \to ZD,$$

where the telescope is constructed with respect to the maps

$$\Lambda^{a_{i+1}}\Sigma^\infty\sigma:\Lambda^{a_i}\Sigma^\infty D_i \cong \Lambda^{a_{i+1}}\Sigma^\infty\Sigma^{b_i} D_i \to \Lambda^{a_{i+1}}\Sigma^\infty D_{i+1},$$

and the restriction of θ to $\Lambda^{a_i}\Sigma^\infty D_i$ is adjoint to the i^{th} component of the w-map $\kappa:D \to ZD$. Moreover, the following diagram commutes:

$$\begin{tikzcd} \mathrm{tel}\ \Lambda^{a_i}\Sigma^\infty D_i \arrow[r, "\theta"] \arrow[d] & ZD \arrow[d, "L\pi"] \\ \mathrm{colim}\ \Lambda^{a_i}\Sigma^\infty D_i \arrow[r, phantom, "\cong"] & LD \end{tikzcd}$$

Proof The canonical homotopies specifying κ as a w-map give canonical homotopies for the construction of θ from the specified restrictions, and Lemma 4.8 and Corollary 4.9 imply that θ is a weak equivalence. The last statement follows easily from $\pi \circ \iota = 1$ and thus $L\pi \circ \kappa = \eta$.

§7. Shift desuspension and weak equivalences

This section is devoted to various related pieces of unfinished business. In section 4, we promised to decompose $\Lambda^Z\Sigma^\infty$ as a composite of Λ^Z and Σ^∞ and to prove that the Z^{th} space functor $\Omega^\infty\Lambda_Z$ preserves weak equivalences. In section 6, we promised a proof of the desuspension theorem. We shall also obtain a result relating the connectivity of G-spectra to the connectivity of their component G-spaces.

We begin with the definition of Λ^Z for an indexing space Z in a G-universe U. We may assume that $U = U' + Z^\infty$, where Z^∞ denotes the sum of countably many copies of Z, and we may take

$$\mathcal{A} = \{V + Z^n \mid V \in \mathcal{A}' \text{ and } n \geq 0\}$$

as our indexing set in U, where $\mathcal{A}'$ is any indexing set in U'.

Definition 7.1. Define functors $\Lambda^z: G\mathcal{P}\mathcal{A} \to G\mathcal{P}\mathcal{A}$ and $\Lambda_z: G\mathcal{P}\mathcal{A} \to G\mathcal{P}\mathcal{A}$ as follows. For $D \in G\mathcal{P}\mathcal{A}$, define

$$\tilde{\sigma}_+ : D(V + Z^{n-1}) \to \Omega^z D(V + Z^n)$$

to be the composite of the given $\tilde{\sigma}$ and the reinterpretation homeomorphism which regards the loop coordinate as the $(n+1)^{st}$ copy rather than the n^{th} copy of Z (that is, as the complement of $V + Z^n$ in $V + Z^{n+1}$ rather than the complement of $V + Z^{n-1}$ in $V + Z^n$). Similarly, define

$$\tilde{\sigma}_- : D(V + Z^n) \to \Omega^z D(V + Z^{n+1})$$

to be the composite of the given $\tilde{\sigma}$ and the reinterpretation homeomorphism which regards the loop coordinate as the n^{th} copy rather than the $(n+1)^{st}$ copy of Z. Define

$$(\Lambda^z D)(V + Z^n) = \begin{cases} \Omega^z DV & \text{if } n = 0, \\ D(V + Z^{n-1}) & \text{if } n \geq 1 \end{cases}$$

where the loop coordinate is understood as the first copy of Z in Z^∞. By the diagram of Definition 2.1(ii), the only structural maps we need to define are

$$\tilde{\sigma}: (\Lambda^z D)(V + Z^n) \to \Omega^{W-V}(\Lambda^z D)(W + Z^n)$$

for $V \subset W$ in $\mathcal{A}'$ and

$$\tilde{\sigma}: (\Lambda^z D)(V + Z^n) \to \Omega^z(\Lambda^z D)(V + Z^{n+1}).$$

In terms of the given $\tilde{\sigma}$ for D, the former is to be $\Omega^z\tilde{\sigma}$ if $n = 0$ and $\tilde{\sigma}$ if $n \geq 1$; the latter is to be the identity if $n = 0$ and $\tilde{\sigma}_+$ if $n \geq 1$. Define

$$(\Lambda_z D)(V + Z^n) = D(V + Z^{n+1})$$

with structural maps the given $\tilde{\sigma}$ for $V + Z^n \subset W + Z^n$ and $\tilde{\sigma}_-$ for $V + Z^n \subset V + Z^{n+1}$. These functors preserve G-spectra and so restrict to endofunctors of $G\mathcal{S}\mathcal{A}$.

Lemma 7.2. The functors Λ_z and Λ^z are inverse adjoint equivalences of the category $G\mathcal{S}\mathcal{A}$.

Proof. $\Lambda_z\Lambda^z$ is the identity functor on both $G\mathcal{P}\mathcal{A}$ and $G\mathcal{S}\mathcal{A}$. For $D \in G\mathcal{P}\mathcal{A}$, the maps $\tilde{\sigma}: DV \to \Omega^z D(V + Z)$ and the identity maps of $D(V + Z^n)$ for $n \geq 1$ specify a natural map $D \to \Lambda^z\Lambda_z D$ which is an isomorphism when D is a spectrum. The adjointness is trivial.

Lemma 7.3. The functor $\Lambda^z\Sigma^\infty: G\mathcal{T} \to G\mathcal{S}\mathcal{A}$ is isomorphic to the composite of Λ^z and Σ^∞.

Proof. Since Λ^z and Σ^∞ are left adjoint to Λ_z and Ω^∞, the composite $\Lambda^z \circ \Sigma^\infty$ is left adjoint to the composite $\Omega^\infty \circ \Lambda_z$. The latter composite is obviously the Z^{th} space functor $\Omega^\infty\Lambda_z$, and the conclusion follows by the uniqueness of adjoints.

If $Y \perp Z$ in U, we can assume that U has the form $U'' + Y^\infty + Z^\infty$ and we can take

$$\mathcal{A} = \{V + Y^m + Z^n \mid V \in \mathcal{A}'', \; m \geq 0, \; n \geq 0\}$$

as our indexing set, where $\mathcal{A}''$ is an indexing set in U''. The functors Λ_y, Λ_z, Λ^y, and Λ^z are then all defined. Upon further restriction to indexing spaces with $m = n$, Λ_{y+z} and Λ^{y+z} are also defined. Trivial verifications give the following result.

Lemma 7.4. The following relations hold for $Y \perp Z$:

$$\Lambda^y\Lambda_z = \Lambda_z\Lambda^y \quad\text{and}\quad \Lambda^z\Lambda_y = \Lambda_y\Lambda^z;$$

$$\Lambda_y\Lambda_z = \Lambda_{y+z} = \Lambda_z\Lambda_y \quad\text{and}\quad \Lambda^y\Lambda^z = \Lambda^{y+z} = \Lambda^z\Lambda^y.$$

To prove the desuspension theorem, we want to relate the functors Λ^z and Ω^z. It is convenient to restrict to an indexing sequence $A = \{V_n + Z^n\}$, where $\{V_n\}$ is an indexing sequence in U'. Write $A_n = V_n + Z^n$ and let $B_n = C_n + Z$ denote the complement of A_n in A_{n+1}, where C_n is the complement of V_n in V_{n+1}. For $E \in G\mathcal{S}A$, we have

$$(\Omega^z E)_i = \Omega^z(E_i)$$

with structural homeomorphisms the composites

$$\Omega^z E_i \xrightarrow{\Omega^z\tilde{\sigma}} \Omega^z\Omega^{b_i}E_{i+1} \xrightarrow{\tau} \Omega^{b_i}\Omega^z E_{i+1},$$

where τ reverses the order of the loop coordinates. Using $\tilde{\sigma}$ to identify $E(V_n + Z^{n-1})$ with $\Omega^z E(V_n + Z^n)$, we find that $\Lambda^z E$ can be written in the form

$$(\Lambda^Z E)_i = \Omega^Z(E_i)$$

with structural homeomorphisms

$$\Omega^Z E_i \xrightarrow{\Omega^Z\tilde{\sigma}} \Omega^Z\Omega^{b_i}E_{i+1} \cong \Omega^{b_i}\Omega^Z E_{i+1},$$

the last isomorphism being given by twisting the first loop coordinate Z past the loop coordinate C_i and then identifying. Clearly these differ by the twisting of Z past Z that occurs for Ω^Z but not for Λ^Z.

Define a natural w-map $d:\Omega^Z E \to \Lambda^Z E$ by letting $d_{2i} = 1:\Omega^Z E_{2i} \to \Omega^Z E_{2i}$ and letting d_{2i-1} be the composite

$$\Omega^Z E_{2i-1} \cong \Omega^Z\Omega^Z\Omega^{c_{2i}}E_{2i} \xrightarrow{\tau} \Omega^Z\Omega^Z\Omega^{c_{2i}}E_{2i} \cong \Omega^Z E_{2i-1}.$$

Here τ twists the two loop coordinates Z and the isomorphisms are given by $\Omega^Z\tilde{\sigma}$ together with the identification $\Omega^{b_{2i}} \cong \Omega^Z\Omega^{c_{2i}}$. Not only is d a natural w-map, but the requisite homotopies are also natural. Indeed, one finds easily that they are consequences of a homotopy from the composite

$$S^Z\wedge S^Z\wedge S^Z \xrightarrow{\tau\wedge 1} S^Z\wedge S^Z\wedge S^Z \xrightarrow{1\wedge\tau} S^Z\wedge S^Z\wedge S^Z$$

to the identity map. The composite of $1\wedge\tau$ and the homotopy

$$h_t(z,z',z'') = (1-t)(z',z,z'') + t(z,z'',z')$$

from $\tau\wedge 1$ to $1\wedge\tau$ serves the purpose.

The kind of situation just displayed occurs often enough to deserve a name. We follow May and Thomason [110].

Definition 7.5. A preternatural transformation $d:D \to D'$ between functors from any category $\mathcal{C}$ to the category $G\mathcal{P}A$ consists of a natural transformation d in the category of G-prespectra and w-maps together with a natural choice of homotopies. Thus, for $C \in \mathcal{C}$ and $i \geq 0$, there is given a map $d_i:D_iC \to D_i'C$ and a homotopy $h_i:\sigma_i'\Sigma^{b_i}d_i \simeq d_{i+1}\sigma_i$, both d_i and h_i being natural in C. A preternatural homotopy $j:d \simeq d'$ is a preternatural transformation $j:D\wedge I^+ \to D'$ which restricts to d and d' at the ends of the cylinder, where the functor $D\wedge I^+$ is specified by $(D\wedge I^+)(C) = DC\wedge I^+$.

Preternaturality gives precise sense to the word "canonical" used in various places in Construction 6.8. Thus $\iota:1 \to K$ is preternatural and $\iota\circ\pi$ is

preternaturally homotopic to the identity transformation $K \to K$. The following result is immediate from part (ii) of the cited construction.

Proposition 7.6. Let $d:D \to D'$ be a preternatural transformation. Then there is a natural transformation $Kd:KD \to KD'$ of functors $\mathcal{C} \to G\mathcal{P}A$ such that the following diagram of G-prespectra is preternaturally homotopy commutative:

$$\begin{tikzcd}
KDC \arrow[r, "Kd"] \arrow[d, "\pi"] & KD'C \arrow[d, "\pi"] \\
DC \arrow[r, "d"] & D'C
\end{tikzcd}$$

If d is a spacewise (weak) equivalence, then so is Kd.

The following observation gives force to the notion of preternatural homotopy.

Lemma 7.7. For $D \in G\mathcal{P}A$ and $X \in G\mathcal{J}$, $K(D\wedge X)$ is naturally isomorphic to $(KD)\wedge X$. If $j:d \simeq d'$ is a preternatural homotopy, then $Kj:Kd \simeq Kd'$ is a natural homotopy.

By application of the functor L, we obtain the following spectrum level consequence of the proposition.

Corollary 7.8. Let $d:D \to D'$ be a preternatural transformation of functors $\mathcal{C} \to G\mathcal{P}A$. Then there are natural transformations

$$LDC \xleftarrow{L\pi} ZDC \xrightarrow{Zd} ZD'C \xrightarrow{L\pi} LD'C.$$

If D and D' take values in $G\mathcal{QA}$, the maps $L\pi$ are weak equivalences.

Of course, Zd is a weak equivalence if d is a spacewise weak equivalence. In particular, this applies to $d:\Omega^Z \to \Lambda^Z$, where Ω^Z and Λ^Z are regarded as functors $G\mathcal{SA} \to G\mathcal{SA}$. We can now deduce the desuspension theorem by passage to $\bar{h}G\mathcal{S}A$. By Theorem 1.2 and Lemma 5.5, the functor Σ^Z preserves G-CW homotopy types. Therefore Ω^Z preserves weak equivalences by Lemma 5.13. By the corollary, it follows that Λ^Z preserves weak equivalences and thus Λ_Z preserves G-CW homotopy types. However, for the adjoint equivalence of Lemma 7.2 to imply the corresponding adjoint equivalence after passage to $\bar{h}G\mathcal{S}A$, we also need the opposite assertion, that Λ_Z preserves weak equivalences and thus Λ^Z preserves G-CW homotopy types. This is an immediate consequence of Theorem 4.6, and the proof of the desuspension theorem is now complete modulo the proof of that result, to which we shall turn shortly. We first display the resulting equivalence between the adjoint pairs (Σ^Z,Ω^Z) and (Λ_Z,Λ^Z) more explicitly.

Theorem 7.9. There is a natural equivalence $\delta\colon \Omega^z \to \Lambda^z$ of functors $\overline{h}G\mathcal{S}\mathcal{Q} \to \overline{h}G\mathcal{S}\mathcal{Q}$ with conjugate natural equivalence $\zeta\colon \Lambda_z \to \Sigma^z$. The following diagrams commute, and all maps η and ε in them are also natural equivalences.

$$\begin{tikzcd} E \arrow[r, "\eta"] \arrow[d, "\eta"'] & \Omega^z\Sigma^z E \arrow[d, "\delta"] \\ \Lambda^z\Lambda_z E \arrow[r, "\Lambda^z\zeta"] & \Lambda^z\Sigma^z E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E & \Sigma^z\Omega^z E \arrow[l, "\varepsilon"'] \\ \Lambda_z\Lambda^z E \arrow[u, "\varepsilon"] & \Lambda_z\Omega^z E \arrow[l, "\Lambda_z\delta"'] \arrow[u, "\zeta"'] \end{tikzcd}$$

Moreover, the following diagrams commute for $Y \perp Z$.

$$\begin{tikzcd} \Omega^y\Omega^z E \arrow[r, "\delta"] \arrow[dd, equal] & \Lambda^y\Omega^z E \arrow[d, "\Lambda^y\delta"] \\ & \Lambda^y\Lambda^z E \arrow[d, equal] \\ \Omega^{y+z}E \arrow[r, "\delta"] & \Lambda^{y+z}E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Lambda_y\Lambda_z E \arrow[r, "\Lambda_y\zeta"] \arrow[dd, equal] & \Lambda_y\Sigma^z E \arrow[d, "\zeta"] \\ & \Sigma^y\Sigma^z E \arrow[d, equal] \\ \Lambda_{y+z}E \arrow[r, "\zeta"] & \Sigma^{y+z}E \end{tikzcd}$$

Proof. Of course, $\delta = (L\pi)(Zd)(L\pi)^{-1}$ as in Corollary 7.8, and ζ is characterized by commutativity of either of the first two diagrams. The transitivity of δ follows from Lemma 7.7 and the transitivity up to preternatural homotopy of d. The point is that $d\colon \Omega^{y+z} \to \Lambda^{y+z}$ involves transposition of the $Y + Z$ loop coordinates whereas $(\Lambda^y d)d$ involves transposition of the two Y coordinates and the two Z coordinates. These transpositions are canonically homotopic.

This implies information relating the $Z^{\underline{th}}$ space and geometric suspension functors on G-spectra. This information makes no reference to the shift functors. Nonequivariantly, it was at the starting point of McClure's work in [H_∞,VII§1].

Corollary 7.10. For $Z \in \mathcal{Q}$, there is a natural equivalence $\zeta_0\colon EZ \to (\Sigma^z E)_0$. For $Y \perp Z$, the following (adjoint) diagrams commute.

$$\begin{tikzcd} EZ \arrow[r, "\zeta_0"] \arrow[dd, "\tilde{\sigma}"'] & (\Sigma^z E)_0 \arrow[d, "\eta_0"] \\ & (\Omega^y\Sigma^y\Sigma^z E)_0 \arrow[d, equal] \\ \Omega^y E(Y+Z) \arrow[r, "\Omega^y\zeta_0"] & \Omega^y(\Sigma^{y+z}E)_0 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \Sigma^y EZ \arrow[r, "\Sigma^y\zeta_0"] \arrow[dd, "\sigma"'] & \Sigma^y(\Sigma^z E)_0 \arrow[d, "\mathrm{adj}(\eta_0)"] \\ & (\Sigma^y\Sigma^z E)_0 \arrow[d, equal] \\ E(Y+Z) \arrow[r, "\zeta_0"] & (\Sigma^{y+z}E)_0 \end{tikzcd}$$

Proof. An easy chase from the diagrams of the theorem gives the commutative diagram

$$\begin{tikzcd} \Lambda_z E \arrow[rr, "\zeta"] \arrow[d, "\eta"'] & & \Sigma^z E \arrow[d, "\eta"] \\ \Lambda^y\Lambda_y\Lambda_z E \arrow[d, equal] & & \Omega^y\Sigma^y\Sigma^z E \arrow[d, equal] \\ \Lambda^y\Lambda_{y+z}E \arrow[r, "\Lambda^y\zeta"] & \Lambda^y\Sigma^{y+z}E & \Omega^y\Sigma^{y+z}E. \arrow[l, "\delta"'] \end{tikzcd}$$

Since $\eta_0 = \tilde{\sigma}$ on the left and $d_0 = 1$, this gives the first of the desired adjoint pair of diagrams on passage to zero$^{\underline{th}}$ spaces.

Turning to the proof of Theorem 4.6, we insert the following lemma to avoid circularity.

Lemma 7.11. If G acts trivially on Z, then the functor Λ_z preserves weak equivalences of G-spectra.

Proof. For any G-trivial $W \subset U$ of dimension n, consideration of $E(W + R^n)$, where the sum need not be direct, shows that EW is naturally G-homeomorphic to ER^n. It follows easily that a weak equivalence $E \to E'$ induces a weak equivalence $(\Lambda_z E)(R^q) \to (\Lambda_z E')(R^q)$ for all q. The conclusion follows immediately from Proposition 4.5.

In this proof, and below, we implicitly use Proposition 2.4 to define $\Lambda_z E$ on general indexing spaces in U.

We can now prove Theorem 4.6, which we first rephrase. The idea of the following argument is due to Henning Hauschild.

Theorem 7.12. Let $f\colon E \to E'$ be a map of G-spectra such that $f_*\colon \pi_n^H E \to \pi_n^H E'$ is an isomorphism for all integers n and all closed subgroups H of G. Then $(fV)_*\colon \pi_n^H EV \to \pi_n^H E'V$ is an isomorphism for all $n \geq 0$, all H, and all indexing spaces V.

Proof. We work in $G\mathcal{S}U$ for definiteness. Being compact Lie, G contains no infinite descending chain of closed subgroups, and a standard argument shows that it suffices to prove the conclusion for H under the inductive assumption that the conclusion holds for all proper closed subgroups of H. Write $V = Z + W$, where W is the complement of the fixed point set $Z = V^H$. By Proposition 4.5, we may as well regard E as an H-spectrum rather than a G-spectrum. By Proposition 2.4, we may then expand H-indexing sets to include Z and W. We then have

$$EV = (\Lambda_z E)(W).$$

Since Λ_z preserves weak equivalences by the lemma, we may now replace E and E' by $\Lambda_z E$ and $\Lambda_z E'$ and consider only those H-representations W with $W^H = \{0\}$. If

$H = e$, then $W = \{0\}$ and the conclusion is immediate. Thus assume $H \neq e$. By our inductive hypothesis, $fW: EW \to E'W$ is a weak K-equivalence for all proper closed subgroups K of H. Thus we need only consider π^H_*. Let SW and DW denote the unit sphere and unit disc in W and consider the fibre sequence arising from the identification $S^W \cong DW^+/SW^+$:

$$\begin{array}{ccccc} E_0 \cong F(S^W,EW) & \longrightarrow & F(DW^+,EW) & \longrightarrow & F(SW^+,EW) \\ {\scriptstyle f_0 \,\cong\, (fW)_*}\downarrow & & \downarrow{\scriptstyle (fW)_*} & & \downarrow{\scriptstyle (fW)_*} \\ E_0' \cong F(S^W,E'W) & \longrightarrow & F(DW^+,E'W) & \longrightarrow & F(SW^+,E'W). \end{array}$$

By Proposition 4.5, f_0 is a weak equivalence since f is. Since DW is H-contractible, $F(DW^+,EW) \simeq EW$. By the five lemma, $(fW)_*: \pi^H_n EW \to \pi^H_n E'W$ will be an isomorphism for all $n \geq 1$ provided that

$$(fW)_*: F(SW^+,EW) \to F(SW^+,E'W)$$

induces an isomorphism on π^H_n for $n \geq 0$. Since

$$\pi^H_0 EW = \pi^H_0 F(DW^+,EW) \to \pi^H_0 F(SW^+,EW)$$

need not be surjective, this five lemma argument won't handle $\pi^H_0 EW \to \pi^H_0 E'W$, but we need only replace E and E' by $\Lambda_1 E$ and $\Lambda_1 E'$ and use the natural isomorphism

$$\pi^H_0 EW = \pi^H_0 \Omega E(W + R) = \pi^H_1 E(W + R) = \pi^H_1(\Lambda_1 E)(W)$$

to deduce the required isomorphism on π^H_0. Of course,

$$\pi^H_n F(SW^+,EW) = [S^n, F(SW^+,EW)^H] = [\Sigma^n(SW^+),EW]_H ,$$

hence what must be shown is that

$$(fW)_*: [\Sigma^n(SW^+),EW]_H \to [\Sigma^n(SW^+),E'W]_H$$

is an isomorphism for $n \geq 0$. Since SW has no H-fixed points, we may replace it by an equivalent H-CW complex all of whose cells have domains of the form $H/K \times e^m$, where K is a proper subgroup. Then $\Sigma^n(SW^+)$ inherits a structure of H-CW complex with a single H-trivial vertex and one $(m+n)$-cell of type K for each m-cell of SW of type K. The successive quotients of the skeletal filtration of $\Sigma^n(SW^+)$ are all wedges of spheres $H/K^+ \wedge S^q$. By induction over skeleta, the desired isomorphism follows from the inductively known isomorphisms

$$(fW)_*: [H/K^+ \wedge S^q, EW]_H \cong \pi^K_q EW \to \pi^K_q E'W \cong [H/K^+ \wedge S^q, E'W]_H.$$

We have the following useful characterization of connective G-spectra.

Proposition 7.13. The following conditions on a G-spectrum E are equivalent.

(i) E is connective.

(ii) ER^n is G-connected for $n \geq 1$.

(iii) ER^n is G-$(n-1)$-connected for $n \geq 1$.

(iv) $\pi^H_q EV = 0$ for all V and all $q < \dim V^H$.

Proof. Trivially (iv) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii), and (ii) $\Longleftrightarrow$ (i) by Proposition 4.5. We must prove that (i) implies (iv). In view of the equivalence of Λ_V and Σ^V, we have the following isomorphisms for $q \geq 0$:

$$\pi^H_q EV = [S^q_H, \Omega^\infty \Lambda_V E]_G \cong [\Sigma^\infty S^q_H, \Sigma^V E]_G = \pi^H_q \Sigma^V E .$$

Write $V = Z + W$, where W is the complement of $Z = V^H$, and let $m = \dim Z$. Regarding E as an H-spectrum and expanding H-indexing sets, we find

$$\pi^H_q(\Sigma^V E) = \pi^H_q(\Sigma^Z \Sigma^W E) \cong \pi^H_{q-m}(\Sigma^W E) .$$

Since E is connective, so is $\Sigma^W E$ because the proposition below implies that E and therefore also $\Sigma^W E$ is weakly equivalent to an H-CW spectrum with no cells of negative dimension. This proves the result.

Of course, there is an analogous implied conclusion for n-connected G-spectra for any n. We quoted the following useful result in the previous proof.

Proposition 7.14. For a G-CW spectrum E, E/E^n is n-connected. For any G-spectrum E, there is a G-CW spectrum $\Gamma_n E$ with no q-cells for $q \leq n$ and a map $\gamma_n: \Gamma_n E \to E$ such that

$$\gamma_{n*}: [D, \Gamma_n E]_G \to [D,E]_G$$

is an isomorphism for all G-CW spectra D with no q-cells for $q \leq n$.

Proof. Since each S^{m+1}_H is m-connected, the first statement results by induction and passage to colimits from the cofibration sequences

$$E^m/E^n \to E^{m+1}/E^n \to E^{m+1}/E^m \text{ for } m \geq n.$$

The second statement results from application of Brown's representability theorem to the functor $[?,E]_G$ defined on the homotopy category of G-CW spectra with no q-cells for $q \leq n$.

§8. Special kinds of G-prespectra and G-spectra

We here collect miscellaneous results of a technical or philosophical nature. The only common denominator is that they all deal with special kinds of prespectra or spectra, beyond the inclusion prespectra and CW-spectra already introduced.

The functor L of Theorem 2.2 was essential to the transport of colimits and various functors - all left adjoints - from the prespectrum to the spectrum level. In all of these applications, L has the unfortunate effect of converting relatively well-understood prespectra into spectra about which we know nothing beyond the desired formal properties. Our first objective here is to ameliorate this situation by describing certain cases in which the effects of L are reasonably well understood. We consider two questions. First, when do our prespectrum level left adjoints produce inclusion prespectra? Of course, when they do, L is given by a simple and easily understood passage to colimits. Second, under what conditions does a functor constructed by use of L preserve subspectra? Specifically, when does it preserve spacewise injections, spacewise closed inclusions, and intersections of closed subspectra? In this connection, we shall prove the following result in the Appendix.

Lemma 8.1. A cofibration of spectra is a spacewise closed inclusion.

This applies in particular to the inclusion of a subcomplex in a CW-spectrum.

To answer our questions, we need the following kinds of prespectra.

Definitions 8.2. (i) A prespectrum D is a Σ-inclusion prespectrum if each $\sigma:\Sigma^{W-V}DV \to DW$ is a closed inclusion. Note that Σ-inclusion prespectra are inclusion prespectra, but not conversely as the example of actual spectra makes clear.
(ii) A prespectrum D is an injection prespectrum if each $\tilde{\sigma}:DV \to \Omega^{W-V}DW$ is an injection. Of course, inclusion prespectra are injection prespectra, but not conversely.

As we shall make explicit below, most of our prespectrum-level left adjoints fail to preserve either spectra or inclusion prespectra but do preserve both injection and Σ-inclusion prespectra. Thus, when restricted to Σ-inclusion prespectra, these functors yield inclusion prespectra and therefore have good calculational behavior.

The point of injection prespectra is that L does not mangle these quite as badly as it does general prespectra. Again, the following result will be proven in the Appendix.

Proposition 8.3. (i) The functor L preserves finite products.
(ii) When restricted to injection prespectra, L preserves all finite limits.
(iii) The map $\eta:D \to \ell LD$ is an injection (resp., inclusion) if D is an injection (resp., inclusion) prespectrum.

Properties (i) and (ii) are unusual for a functor which is a left adjoint. Property (ii) is the key to answering our questions about subspectra since these can all be rephrased as questions about finite limits. In any reasonable category, such as any of our categories of spaces, prespectra, or spectra, a map $f: D \to D'$ is an injection if and only if the diagram

$$\begin{tikzcd} D \arrow[r, "1"] \arrow[d, "1"'] & D \arrow[d, "f"] \\ D \arrow[r, "f"] & D' \end{tikzcd}$$

is a pullback. The intersection of subobjects D' and D'' of D may be described as the pullback

$$\begin{tikzcd} D' \cap D'' \arrow[r] \arrow[d] & D'' \arrow[d] \\ D' \arrow[r] & D \end{tikzcd}$$

A map $f:D \to D'$ is a closed inclusion if and only if there is a pair of maps $D' \rightrightarrows D''$ such that the diagram

$$D \xrightarrow{f} D' \rightrightarrows D''$$

is an equalizer; more precisely, this holds if and only if the canonical diagram

$$D \xrightarrow{f} D' \rightrightarrows D' \cup_D D'$$

is an equalizer. We would like to use this fact in conjunction with Proposition 8.3 to conclude that Lf is a closed inclusion when $f: D \to D'$ is a closed inclusion between injection prespectra. However, this conclusion is actually false, the problem being that $D'\cup_D D'$ need not be an injection prespectrum. Let us say that f is a good closed inclusion if $D'\cup_D D'$ is an injection prespectrum. The following observation is easily checked by inspection and shows that goodness is automatic when D is a spectrum.

Lemma 8.4. Let D be a spectrum (regarded as a prespectrum).
(i) If D' and D'' are injection prespectra, $f: D \to D'$ is a closed inclusion, and $g: D \to D''$ is an injection, then $D' \cup_D D''$ is an injection prespectrum.
(ii) If D' and D'' are spectra (regarded as prespectra) and $f: D \to D'$ and $g: D \to D''$ are closed inclusions, then $D \cup_D D''$ is an inclusion prespectrum.

As an aside, D'/D is hardly ever an injection prespectrum here. Note that $D = D' \cap D''$ in $D' \cup_D D''$; it now follows that $LD = LD' \cap LD''$ in $LD' \cup_{LD} LD''$.

The discussion above has the following consequences.

Proposition 8.5. (i) When restricted to injection prespectra, the functor L preserves injections, good closed inclusions, and intersections of closed subobjects.
(ii) Let F be a prespectrum level functor which preserves pushouts and injection prespectra. If F preserves injections, closed inclusions, and intersections of closed subprespectra, then the spectrum level functor $LF\ell$ preserves injections, closed inclusions, and intersections of closed subspectra.

It remains to survey the behavior of particular functors of interest. Intersections refer to closed subobjects in the following examples.

Examples 8.6 (i) The functors $\Lambda^z\Sigma^\infty$ from spaces to spectra are obtained by application of L to Σ-inclusion prespectra and certainly preserve injections, closed inclusions, and intersections.
(ii) The prespectrum level functor $D \wedge X$ preserves Σ-inclusion and injection prespectra but fails to preserve inclusion prespectra or spectra. It preserves injections, closed inclusions, and intersections in either variable, hence so does the spectrum level functor $E \wedge X$.
(iii) The orbit prespectrum functor D/G (where D is indexed on a G-trivial universe) preserves Σ-inclusion prespectra, injections, closed inclusions, and pullback diagrams one leg of which is a closed inclusion (such as intersections). However, it fails to preserve injection prespectra, and D/G is almost never an injection prespectrum even when D is a spectrum. For this reason, passage to orbits on the spectrum level is quite badly behaved in general. The situation is saved by Lemma 5.6, which shows that the orbit spectrum of a G-CW spectrum has a natural induced structure as a CW spectrum.

We next consider possible cofibration or CW-complex restrictions in the definition of G-prespectra. To many experts, such conditions will have been conspicuous by their absence. We work with an indexing sequence $\{A_i\}$, with $B_i = A_{i+1} - A_i$, and adopt the notations of section 6.

Definition 8.7. A G-prespectrum D is Σ-cofibrant if each $\sigma_i: \Sigma^{b_i} D_i \to D_{i+1}$ is a cofibration; D is cofibrant if each adjoint $\tilde{\sigma}_i$ is a cofibration.

A trivial induction gives the following result.

Lemma 8.8. If D is Σ-cofibrant, then any w-map $D \to D'$ is spacewise homotopic to an actual map.

Results of Lewis [84] imply the following result.

Lemma 8.9. If D is Σ-cofibrant and the diagonal maps $D_i \to D_i \times D_i$ are cofibrations, then D is cofibrant.

The diagonal condition holds if the D_i are G-CW complexes.

Lemma 8.10. If D is cofibrant and each $\tilde{\sigma}_i$ is an equivalence, then $\eta: D \to LD$ is a spacewise equivalence.

Proof. Each $\tilde{\sigma}_i$ is an inclusion of a strong deformation retract by the standard argument, and the rest of the proof is exactly the same as in May [95,p.469].

Definition 8.11. A Σ-cofibrant G-prespectrum D is a G-CW prespectrum if each D_i has cofibred diagonal and the homotopy type of a G-CW complex.

We could only ask for actual G-CW complexes and cellular structure maps by artificially choosing G-CW structures on the G-spheres S^{b_i}; compare Remarks 1.3.

Lemma 8.12. If D is a G-CW prespectrum, then LD has the homotopy type of a G-CW spectrum.

Proof. Since the functors $\Lambda^{a_i}\Sigma^\infty$ preserve G-CW homotopy types, this is immediate from the isomorphism $LD \cong \operatorname{colim} \Lambda^{a_i}\Sigma^\infty D_i$.

Constructions 6.3 and 6.8 show how to replace general G-prespectra by G-CW prespectra.

Proposition 8.13. For any G-prespectrum D, KD is Σ-cofibrant and $K\Gamma D$ is a G-CW prespectrum. If D is an ΩG-prespectrum, then $\eta: K\Gamma D \to LK\Gamma D = Z\Gamma D$ is a spacewise equivalence.

Here the last statement follows from Lemma 8.10 and Remarks 6.4 and leads to the following reassuring result.

Proposition 8.14. If $E \in G\mathscr{S}U$ is a G-CW spectrum, then E is equivalent to $Z\Gamma E$ and each component space EV of E has the homotopy type of a G-CW complex.

Proof. We have the w-map $\gamma:\Gamma E \to E$ of Construction 6.3 and thus an actual map

$$K\Gamma E \xrightarrow{K\gamma} KE \xrightarrow{\pi} E$$

by Construction 6.8. By application of L, there results a weak equivalence $Z\Gamma E \to E$. Since $Z\Gamma E$ has the homotopy type of a G-CW spectrum, this map and thus all of its component maps are equivalences by the Whitehead theorem for G-CW spectra. The component spaces of $Z\Gamma E$ have the homotopy types of G-CW complexes by the previous result.

Thus our G-CW spectra can be replaced by equivalent G-spectra of the form LD for an ΩG-CW prespectrum D. The latter gadgets may be viewed as arising by elementary constructions from the sort of G-prespectra that occur "in nature", for example as representing objects for cohomology theories on G-spaces.

As explained in the preamble, this discussion gives the beginnings of a comparison between Adams' approach to the (nonequivariant) stable category and ours. We end with the beginnings of a comparison between Boardman's approach and ours. Boardman's starting point is that CW-spectra should be the colimits of their finite subcomplexes, as ours are, and that finite CW-spectra should be shift desuspensions of finite CW-complexes. We shall prove the appropriate equivariant version of this characterization of finite CW-spectra.

One's first guess is that a finite G-CW spectrum should have the form $\Lambda^Z\Sigma^\infty X$ for a finite G-CW complex X. However, the usual notion of a G-CW complex is not appropriate: we need something more general.

It has occurred to several people that the proper collection of spheres for the definition of G-CW complexes might be the collection of spaces $(G/H)^+ \wedge S^V$, where V runs through the representations of G. In our context, V would run through a given indexing set $\mathcal{A}$. (Actually, there are good grounds for using the more general collection of spaces $G^+ \wedge_H S^V$, where V runs through the representations of H, but the former collection is appropriate for the discussion to follow.) The corresponding spectrum level collection of spheres would consist of all $\Lambda^W\Sigma^\infty((G/H)^+\wedge S^V)$ with V and W in $\mathcal{A}$.

The general approach to cellular theory presented in [107] makes clear that most standard results go over without change to the generalized G-CW complexes and G-CW spectra obtained by allowing these generalized spheres as the domains of attaching maps. (On the space level, we insist on based attaching maps here.) The one major exception is the cellular approximation theorem, which already fails for generalized spheres. The Whitehead theorem goes through in both contexts but is no

stronger than the form already obtained since Theorem 1.2 and Corollary 5.14 imply that generalized G-CW complexes or spectra have the homotopy types of ordinary G-CW complexes or spectra.

The functors $\Lambda^Z\Sigma^\infty$ clearly take generalized G-CW complexes to generalized G-CW spectra since they preserve generalized spheres, cofibres, and unions.

Proposition 8.15. Any finite generalized G-CW spectrum E is isomorphic to $\Lambda^Z\Sigma^\infty Y$ for some finite generalized G-CW complex Y and some $Z \in \mathcal{A}$.

Proof. Proceeding by induction on the number of cells of E, we may assume that E is the cofibre of a map $f:\Lambda^V\Sigma^\infty K \to D$, where K is a generalized sphere and D is isomorphic to $\Lambda^W\Sigma^\infty X$ for some finite generalized G-CW complex X and some $W \in \mathcal{A}$. By Lemma 4.10, we may write f in the form

$$\Lambda^Z\Sigma^\infty g:\Lambda^Z\Sigma^\infty\Sigma^{Z-V}K \to \Lambda^Z\Sigma^\infty\Sigma^{Z-W}X$$

for some Z containing V and W and some space level map g. If $K = (G/H)^+\wedge S^t$, we may take T to be orthogonal to Z - V and identify the domain of g with $(G/H)^+\wedge S^{Z-V+t}$. Let Y be the cofibre of g. Then E is isomorphic to $\Lambda^Z\Sigma^\infty Y$. Since $\Sigma^{Z-W}X$ is evidently a generalized G-CW complex with one cell for each cell of X, Y is a generalized G-CW complex.

The essential point is that we must use generalized G-CW complexes even if we start with ordinary G-CW spectra. However, using Theorem 1.2 to triangulate the domains of attaching maps and then approximating them by cellular maps, we find by an easy induction that any generalized finite G-CW complex has the homotopy type of an ordinary finite G-CW complex.

Corollary 8.16. Any finite G-CW spectrum E is G-homotopy equivalent to $\Lambda^Z\Sigma^\infty X$ for some finite G-CW complex X and some $Z \in \mathcal{A}$.

II. Change of universe, smash products, and change of groups

by L. G. Lewis, Jr. and J. P. May

We continue our study of the equivariant stable category with a number of deeper or more specifically equivariant constructions and theorems.

In fact, we have different stable categories of G-spectra indexed on different G-universes, and much of our work will concern change of universe. We have already observed that a G-linear isometry $f: U \to U'$ gives rise to a forgetful functor $f^*: G\mathcal{S}U' \to G\mathcal{S}U$. In section 1, we construct a left adjoint $f_*: G\mathcal{S}U \to G\mathcal{S}U'$ and show that the functors f_* (or f^*) given by different G-linear isometries become canonically equivalent upon passage to stable categories.

We use these change of universe functors to construct smash products and function spectra in section 3. It is a very easy matter to write down explicit "external" adjoint smash product and function spectra functors

$$\wedge : G\mathcal{S}U \times G\mathcal{S}U' \longrightarrow G\mathcal{S}(U \oplus U') \quad \text{and} \quad F: (G\mathcal{S}U')^{op} \times G\mathcal{S}(U \oplus U') \longrightarrow G\mathcal{S}U$$

for any pair of G-universes U and U'. Taking $U = U'$, we obtain "internal" adjoint smash product and function spectra functors by composing with

$$f_*: G\mathcal{S}(U \oplus U) \longrightarrow G\mathcal{S}U \quad \text{and} \quad 1 \times f^*: (G\mathcal{S}U)^{op} \times G\mathcal{S}U \longrightarrow G\mathcal{S}U \times G\mathcal{S}(U \oplus U)$$

for any chosen G-linear isometry $f: U \oplus U \to U$. After passage to the stable category, the functors f_* (and f^*) for varying f become canonically equivalent, and this freedom to use varying G-linear isometries makes it easy to prove that the smash product on the stable category is unital, commutative, and associative up to coherent natural isomorphism.

In this application of change of universe, the relevant universes are G-isomorphic. In section 2, we consider an inclusion $i: U' \to U$ of non-isomorphic G-universes and prove that, for suitably restricted spectra $E' \in G\mathcal{S}U'$ and arbitrary spectra $F' \in G\mathcal{S}U'$,

$$i_*: [E',F']_G \longrightarrow [i_*E',i_*F']_G$$

is an isomorphism. The main case of interest is the inclusion $i: U^N \to U$, where N is a normal subgroup of G. Here i_* is an isomorphism whenever E' is an N-free G-CW spectrum. Since $i_*\Sigma^\infty = \Sigma^\infty: G\mathcal{T} \to G\mathcal{S}U$, we find in particular that if X is an N-free G-CW complex and Y is any G-space, then $[\Sigma^\infty X,\Sigma^\infty Y]_G$ is the same when computed in U^N as when computed in U. For finite X and Y, this

result is due to Adams [3]. Its virtue is that it allows us to work in U to construct stable maps $X \to Y$, pull them back along i_* to maps between suspension spectra indexed on U^N, and then pass to orbits over N to obtain stable maps $X/N \to Y/N$. This is precisely how we shall construct stable transfer maps in chapter IV. Along the same lines, we show that any N-free G-CW spectrum $E \in G\mathcal{S}U$ is equivalent to i_*D for an N-free G-CW spectrum $D \in G\mathcal{S}U^N$. Up to equivalence, D is uniquely determined by E. An explicit model for D is $E\mathcal{F}(N)^+ \wedge i^*E$, where $\mathcal{F}(N)$ is the family of subgroups of G which intersect N trivially and $E\mathcal{F}$ denotes the universal $\mathcal{F}$-space associated to a family $\mathcal{F}$. A much more useful explicit model is the twisted half smash product $E\mathcal{F}(N) \ltimes E$ to be constructed in chapter VI.

In section 4, we consider various change of group functors associated to a homomorphism $\alpha: H \to G$ of compact Lie groups. For a G-universe U regarded by pullback as an H-universe, we have a forgetful functor

$$\alpha^*: G\mathcal{S}U \longrightarrow H\mathcal{S}U.$$

We construct left and right adjoints

$$G \ltimes_\alpha (?): H\mathcal{S}U \longrightarrow G\ U \quad \text{and} \quad F_\alpha[G,?): H\mathcal{S}U \longrightarrow G\mathcal{S}U$$

to α^* and study their propeties. When α is an inclusion, we use the notations $G \ltimes_H D$ and $F_H[G,D)$; these are the free and cofree G-spectra generated by an H-spectrum D. When α is the trivial homomorphism $H \to e$, $e \ltimes_\alpha D = D/H$ and $F_\alpha[e,D) = D^H$. In general, if $N = \mathrm{Ker}(\alpha)$ and $J = H/N \subset G$, then

$$G \ltimes_\alpha D = G \ltimes_J (D/N) \quad \text{and} \quad F_\alpha[G,D) = F_J[G,D^N).$$

Section 5 contains some elementary space level geometry needed in the proofs of our main change of groups isomorphisms in sections 6 and 7.

For $H \subset G$, the adjunction

$$[G \ltimes_H D,E]_G \cong [D,E]_H,$$

$D \in H\mathcal{S}U$ and $E \in G\mathcal{S}U$, implies an isomorphism

$$E_G^*(G \ltimes_H D) \cong E_H^*(D)$$

on the level of represented cohomology theories. In section 6, we prove that the free G-spectrum $G \ltimes_H D$ is naturally equivalent to the cofree G-spectrum $F_H[G,\Sigma^L D)$, where L is the tangent H-representation at the identity coset of

G/H (and thus $L = 0$ if G is finite). This equivalence amounts to a complementary adjunction

$$[E, G \ltimes_H D]_G \cong [E, \Sigma^L D]_H$$

and implies an isomorphism

$$E_*^G(G \ltimes_H D) \cong E_*^H(\Sigma^L D)$$

on the level of represented homology theories. When $D = \Sigma^\infty Y$ for an H-space Y, such an isomorphism was first obtained by Wirthmüller [144].

For a normal subgroup N of G with quotient homomorphism $\varepsilon: G \to J$, where $J = G/N$, we have the adjunction

$$[D/N, E]_J \cong [D, \varepsilon^* E]_G;$$

here $D \in G\mathcal{S}U^N$ and $E \in J\mathcal{S}U^N$ for a G-universe U. Assuming that D is N-free, we can combine this with the change of universe isomorphism i_* associated to the inclusion $i: U^N \to U$ to obtain

$$[D/N, E]_J \cong [i_* D, \varepsilon^\# E]_G,$$

where $\varepsilon^\# E$ is defined to be $i_* \varepsilon^* E \in G\mathcal{S}U$. Thus $\varepsilon^\# E$ is the G-spectrum obtained by regarding the J-spectrum E as a G-spectrum by pullback along ε^* and then building in the representations of U not in U^N by means of i_*. Remember here that any N-free G-spectrum indexed on U is equivalent to one of the form $i_* D$ for a uniquely determined N-free G-spectrum D indexed on U^N. In section 7, we prove that the orbit J-spectrum D/N is equivalent to the fixed point J-spectrum $(\Sigma^{-A} i_* D)^N$, where A is the adjoint representation of G on the tangent space at the identity element of N (and thus $A = 0$ if G is finite). The N-fixed point functor $G\mathcal{S}U \to J\mathcal{S}U^N$ is the composite of i^* and the N-fixed point functor $G\mathcal{S}U^N \to J\mathcal{S}U^N$ and is thus the right adjoint of $\varepsilon^\#: J\mathcal{S}U^N \to G\mathcal{S}U$. Our equivalence therefore amounts to a complementary isomorphism

$$[E, D/N]_J \cong [\varepsilon^\# E, \Sigma^{-A} i_* D]_G.$$

When $D = \Sigma^\infty X$ for an N-free G-CW complex X and $E = \Sigma^\infty Y$ for a J-CW complex Y, the last two isomorphisms specialize to give

$$[\Sigma^\infty X/N, \Sigma^\infty Y]_J \cong [\Sigma^\infty X, \Sigma^\infty \varepsilon^* Y]_G$$

and

$$[\Sigma^\infty Y, \Sigma^\infty X/N]_J \cong [\Sigma^\infty \varepsilon^* Y, \Sigma^{-A}\Sigma^\infty X]_G,$$

where the left sides are computed in the J-universe U^N and the right sides are computed in the G-universe U. When G is a finite group and X and Y are finite complexes, these isomorphisms are due to Adams [3] (who in turn credits us for the special case $N = G$).

The isomorphisms of the previous paragraph cry out for interpretations in terms of cohomology and homology analogous to those of the paragraph before. One problem is that, while it is obvious that a spectrum $E_G \in G\mathcal{S}U$ may be viewed as a spectrum $E_H \in H\mathcal{S}U$ for $H \subset G$, it is less obvious how to construct from E_G a spectrum $E_J \in J\mathcal{S}U^N$ for $J = G/N$. In cases like K-theory, cobordism, and cohomotopy, however, we have cohomology theories for all G. We discuss such families in section 8, describing when the isomorphisms of the previous paragraph lead to isomorphisms of the form

$$E_J^*(D/N) \cong E_G^*(i_* D)$$

and

$$E_*^J(D/N) \cong E_*^G(\Sigma^{-A} i_* D)$$

for N-free G-spectra $D \in G\mathcal{S}U^N$.

We give a different perspective on the relationship between J-spectra and G-spectra in section 9. There is a naive construction of J-prespectra from G-prespectra obtained simply by passing to N-fixed points spacewise. This was exploited by Caruso and May [24,103] in their study of the analog of the Segal conjecture for general equivariant cohomology theories and by Araki [4] in his study of localizations of equivariant cohomology theories (compare V§6 below). Following ideas of Costenoble, we give a reinterpretation of this construction in terms of actual fixed point spectra and describe algebraically the resulting passage from G-cohomology theories to J-cohomology theories. However, our main concern will be to demonstrate that the stable homotopy category $\bar{h}J\mathcal{S}U^N$ of J-spectra is equivalent to the full subcategory of the stable homotopy category $\bar{h}G\mathcal{S}U$ of G-spectra whose objects are those G-spectra D such that $\pi_*^H(D) = 0$ unless H contains N.

§1. Change of universe functors

Let U and U' be G-universes and let $f: U \to U'$ be a G-linear isometry. We have observed that there is a change of universe functor

$$f^*\colon G\mathscr{S}U' \longrightarrow G\mathscr{S}U$$

specified by letting $(f^*E')(V) = E'(fV)$, with structural maps

$$\Sigma^{W-V}E'(fV) = E'(fV)\wedge S^{W-V} \xrightarrow{1\wedge f} E'(fV)\wedge S^{fW-fV} \xrightarrow{\sigma} E'(fW).$$

It is vital to our work that f^* has a left adjoint

$$f_*\colon G\mathscr{S}U \longrightarrow G\mathscr{S}U'$$

even when f fails to be an isomorphism.

Definition 1.1. For $D \in G\mathscr{P}U$, define $f_*D \in G\mathscr{P}U'$ as follows. For an indexing space $V' \subset U'$, let $V = f^{-1}(V') \subset U$, so that f maps V onto $V' \cap f(U) \subset V'$. Define

$$(f_*D)(V') = DV\wedge S^{V'-fV}$$

and define the structural map associated to $V' \subset W'$ to be the following composite, where $W = f^{-1}W'$:

$$DV\wedge S^{V'-fV}\wedge S^{W'-V'} \cong DV\wedge S^{fW-fV}\wedge S^{W'-fW}$$
$$\xrightarrow{1\wedge f^{-1}\wedge 1} DV\wedge S^{W-V}\wedge S^{W'-fW} \xrightarrow{\sigma\wedge 1} DW\wedge S^{W'-fW}.$$

For $E \in G\mathscr{S}U$, define $f_*E = Lf_*(\ell E) \in G\mathscr{S}U'$.

Proposition 1.2. For a G-linear isometry $f\colon U \to U'$ and for $E \in G\mathscr{S}U$ and $E' \in G\mathscr{S}U'$, there is a natural isomorphism

$$G\mathscr{S}U(E,f^*E') \cong G\mathscr{S}U'(f_*E,E').$$

Moreover, for $f'\colon U' \to U''$, $(f'f)^* \cong f^*f'^*$ and $(f'f)_* \cong f'_*f_*$.

Proof. For $D \in G\mathscr{P}U$ and $D' \in G\mathscr{P}U'$, define

$$\omega\colon G\mathscr{P}U(D,f^*D') \longrightarrow G\mathscr{P}U'(f_*D,D')$$

by letting $\omega(k)$, $k\colon D \to f^*D'$, have V'^{th} component map the composite

$$DV\wedge S^{V'-fV} \xrightarrow{kV\wedge 1} D'(fV)\wedge S^{V'-fV} \xrightarrow{\sigma} D'V'.$$

Then ω is an isomorphism; for $k'\colon f_*D \to D'$, $\omega^{-1}(k')$ has V^{th} component map

$$k'(fV)\colon DV = (f_*D)(fV) \longrightarrow D'(fV).$$

This gives the adjunction, and the last statement is clear.

We record the basic point-set level and formal properties of the functors f_*. For the first, compare I.8.5.

Lemma 1.3. The prespectrum level functor f_* preserves Σ-inclusion and injection prespectra. Both it and the spectrum level functor f_* preserve injections, closed inclusions, and intersections of closed subobjects.

Proposition 1.4. For $X \in G\mathscr{T}$ and $E \in G\mathscr{S}U$, there is a natural isomorphism

$$f_*(E\wedge X) \cong (f_*E)\wedge X.$$

For isomorphic indexing spaces $V \subset U$ and $V' \subset U'$, there are natural isomorphisms

$$f_*(\Lambda^V\Sigma^\infty X) \cong \Lambda^{fV}\Sigma^\infty X \cong \Lambda^{V'}\Sigma^\infty X.$$

Therefore f_* carries G-CW spectra to G-CW spectra.

Proof. The first and second isomorphisms follow by conjugation from the evident equalities

$$f^*F(X,E') = F(X,f^*E') \quad\text{and}\quad (f^*E')(V) = E'(fV),$$

$E' \in G\mathscr{S}U'$; the last is given by I.4.2. These isomorphisms and the fact that f_* is a left adjoint imply that f_* preserves spheres and commutes with wedges, cofibres, and colimits.

What really matters about the functors f_* is that, up to equivalence, they are independent of the choice of f. The proof of this fact depends on the theory of twisted half smash products to be presented in chapter VI, but we shall explain the basic idea here.

Recall that U is topologized as the colimit of its indexing spaces. Let $\mathscr{I}(U,U')$ denote the function G-space of linear isometries $U \to U'$, with G acting by conjugation. Thus a G-linear isometry is a G-fixed point of $\mathscr{I}(U,U')$. We shall exploit the following result, in which U' must be a G-universe but U could be any real G-inner product space.

Lemma 1.5. If there is at least one G-linear isometry $f\colon U \to U'$, then $\mathscr{I}(U,U')$ is G-contractible.

Proof. Write U' as a sum over various representations V of sequences V_i, $i \geq 1$, with each V_i a copy of V. Let $\alpha\colon U' \to U'$ map V_i identically onto V_{2i} and let $\beta\colon U' \to U' \oplus U'$ map V_{2i} and V_{2i-1} identically onto i_1V_i and i_2V_i respectively, where i_1 and i_2 are the canonical injections $U' \to U' \oplus U'$. Thus β is an isomorphism and $\beta\alpha = i_1$. Define paths $H_1\colon I \to \mathscr{I}(U',U')$ from the identity to α and $H_2\colon I \to \mathscr{I}(U,U \oplus U)$ from i_1 to i_2 by normalizing the obvious linear paths and define $H\colon I \times \mathscr{I}(U,U') \to \mathscr{I}(U,U')$ by

$$H(t,g) = \begin{cases} H_1(2t) \circ g & \text{if} \quad 0 \leq t \leq 1/2 \\ \beta^{-1} \circ (g + f) \circ H_2(2t-1) & \text{if} \quad 1/2 \leq t \leq 1. \end{cases}$$

Then H is a homotopy from the identity to the constant map at $\beta^{-1} \circ i_2 \circ f$, and all maps in sight are G-maps.

Now suppose given a G-map $\chi\colon X \to \mathscr{I}(U,U')$, where X is any (unbased) G-space. In Chapter VI, we shall construct a twisted half smash product functor

$$\chi \ltimes (?)\colon G\mathscr{S}U \longrightarrow G\mathscr{S}U'.$$

The construction will also be functorial in χ, viewed as a space over $\mathscr{I}(U,U')$. The functor $\chi \ltimes (?)$ will come with a right adjoint twisted function spectrum functor

$$F[\chi,?)\colon G\mathscr{S}U' \longrightarrow G\mathscr{S}U,$$

and this will be contravariantly functorial in χ. Appropriate analogs of Lemma 1.3 and Proposition 1.4 will hold.

When X is compact, the definitions are quite straightforward, and the reader can get a quick idea by reading the first few pages of VI§2 (through 2.7). When X is a single point with image f, the definitions specialize to give

$$\chi \ltimes E = f_*E \quad \text{and} \quad F[\chi,E) = f^*E,$$

hence we agree to write

$$\chi \ltimes E = \chi_*E \quad \text{and} \quad F[\chi,E) = \chi^*E$$

in general in what follows. The only property of twisted half smash products

relevant to the present discussion is the following one, whose easy proof is given in VI.2.16.

Lemma 1.6. Let X be a subcomplex and G-deformation retract of a finite G-CW complex Y. Let $\psi\colon Y \to \mathscr{I}(U,U')$ be a G-map with restriction χ to X. For G-CW spectra $E \in G\mathscr{S}U$, the inclusion $i\colon X \to Y$ induces a natural G-homotopy equivalence

$$i_*\colon \chi_*E \longrightarrow \psi_*E.$$

By conjugation, for $E' \in G\mathscr{S}U'$, $i^*\colon \psi^*E' \to \chi^*E'$ is an isomorphism in the stable category $\bar{h}G\mathscr{S}U$.

The previous two lemmas allow us to draw the following conclusions about our change of universe functors.

Theorem 1.7. The functors $f_*\colon G\mathscr{S}U \to G\mathscr{S}U'$ induced by varying G-linear isometries $f\colon U \to U'$ become canonically and coherently naturally equivalent on passage to the stable categories $\bar{h}G\mathscr{S}U$ and $\bar{h}G\mathscr{S}U'$. The same conclusion holds for the functors $f^*\colon G\mathscr{S}U' \to G\mathscr{S}U$.

Proof. Given G-linear isometries $f,g\colon U \to U'$, Lemma 1.5 implies that there is a G-path $h\colon I \to \mathscr{I}(U,U')$ connecting them. For G-CW spectra E, Lemma 1.6 then gives natural G-homotopy equivalences

$$f_*E \xrightarrow{i_{0*}} h_*E \xleftarrow{i_{1*}} g_*E.$$

If $j\colon I \to \mathscr{I}(U,U')$ is another G-path from f to g, Lemma 1.5 implies that there is a G-homotopy $k\colon I \times I \to \mathscr{I}(U,U')$ from h to j through G-paths from f to g. By Lemma 1.6 again, inclusions of vertices and faces of I^2 give a commutative diagram of natural G-homotopy equivalences

$$\begin{tikzcd}
f_*E \arrow[r, "i_{0*}"] \arrow[d, "i_{0*}"'] & (f\pi)_*(E) \arrow[d] & f_*E. \arrow[l, "i_{1*}"'] \arrow[d, "i_{0*}"] \\
h_*E \arrow[r] & k_*E & j_*E \arrow[l] \\
g_*E \arrow[u, "i_{1*}"] \arrow[r, "i_{0*}"] & (g\pi)_*(E) \arrow[u] & g_*E \arrow[l, "i_{1*}"'] \arrow[u, "i_{1*}"']
\end{tikzcd}$$

Here π is the projection $I \to \{*\}$. By a trivial inspection of definitions (see VI.1.7), $(f\pi)_*(E) = I^+ \wedge f_*E$ and the top and bottom horizontal composites $i_{1*}^{-1}i_{0*}$ are identity maps. Thus the equivalence $f_*E \simeq g_*E$ is independent of the choice of h. Clearly any desired coherence relations as f varies can be proven by the same method. These conclusions for the functors f^* follow by conjugation.

We shall need no further information about change of universe for the study of smash products, but we shall need the following formal complement to the preceding proof in the next section.

Corollary 1.8. Let h be a G-path connecting G-linear isometries $f,g\colon U \to U'$. Write η and ε for the units and counits of the adjunctions determined by f, g, and h. Then the following diagrams commute, and all maps in them other than the η and ε are equivalences.

$$\begin{tikzcd}
f^*f_*E \arrow[r, "f^*i_{0*}"] & f^*h_*E \\
E \arrow[u, "\eta"] \arrow[r, "\eta"] \arrow[d, "\eta"'] & h^*h_*E \arrow[u, "i_0^*"'] \arrow[d, "i_1^*"] \\
g^*g_*E \arrow[r, "g^*i_{1*}"] & g^*h_*E
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
f_*f^*E \arrow[d, "\varepsilon"'] & f_*h^*E \arrow[l, "f_*i_0^*"'] \arrow[d, "i_{0*}"] \\
E & h_*h^*E \arrow[l, "\varepsilon"'] \\
g_*g^*E \arrow[u, "\varepsilon"] & g_*h^*E \arrow[l, "g_*i_1^*"] \arrow[u, "i_{1*}"']
\end{tikzcd}$$

If f is an isomorphism, then η and ε for f are isomorphisms, hence η and ε for g and h are (compatible) equivalences.

The last sentence will have particularly useful consequences.

§2. Families and change of universe isomorphisms

Let $i\colon U' \to U$ be an inclusion of non-isomorphic G-universes. Thus more representations occur in U than in U'. We seek conditions which guarantee that

$$i_*\colon [E,F]_G \longrightarrow [i_*E, i_*F]_G$$

is nevertheless an isomorphism. The results are best expressed in terms of families.

Definitions 2.1. (i) A family $\mathcal{F}$ in G is a set of subgroups which is closed under conjugation and passage to subgroups.

(ii) An unbased G-space is an $\mathcal{F}$-space if the isotropy group of each of its points is in $\mathcal{F}$; a based G-space is an $\mathcal{F}$-space if the isotropy group of each of its points other than the basepoint is in $\mathcal{F}$.

(iii) A G-CW spectrum is an $\mathcal{F}$-CW spectrum if the domains of its attaching maps are all of the form S^n_H with $H \in \mathcal{F}$.

(iv) A map of based or unbased G-spaces or of G-spectra is said to be a (weak) $\mathcal{F}$-equivalence if it is a (weak) H-equivalence for all $H \in \mathcal{F}$.

We say that $\mathcal{F}$-objects are "$\mathcal{F}$-isotropic". (The term "$\mathcal{F}$-free" occurs in the literature.) Of course, if X is an $\mathcal{F}$-space and a G-CW complex, then it is an $\mathcal{F}$-CW complex; that is, the domains of its attaching maps are of the form $G/H \times S^n$ with $H \in \mathcal{F}$. Unless the indexing universe is G-trivial, the component spaces of $\mathcal{F}$-CW spectra need not be $\mathcal{F}$-spaces, and we have no useful definition of $\mathcal{F}$-spectra in the absence of a given cell structure.

There is an $\mathcal{F}$-Whitehead theorem, its proof being exactly the same induction starting from cells as the proof of the usual Whitehead theorem (see I.5.10). It reads as follows on the spectrum level.

Theorem 2.2. If $e\colon E \to F$ is a weak $\mathcal{F}$-equivalence of G-spectra, then $e_*\colon [D,E]_G \to [D,F]_G$ is an isomorphism for every $\mathcal{F}$-CW spectrum D. If E and F are themselves $\mathcal{F}$-CW spectra, then e is a G-homotopy equivalence.

Examples of families are legion. We introduce notations for those of interest to us here.

Definitions 2.3. (i) For G-universes U and U', let

$$\mathcal{E}(U,U') \subset \mathcal{F}(U,U')$$

denote the families consisting respectively of those H such that U is H-isomorphic to U' and of those H such that there exists an H-linear isometry $U \to U'$.

(ii) For a normal subgroup N of G, let $\mathcal{F}(N)$ denote the family of subgroups H of G such that $H \cap N = e$ and let $\mathcal{F}[N]$ denote the family of subgroups H which do not contain N; $\mathcal{F}(N) \subset \mathcal{F}[N]$ unless $N = e$.

There are several equalities relating these families.

Lemma 2.4. (i) If U' is a sub G-universe of U, then

$$\mathcal{E}(U,U') = \mathcal{F}(U,U').$$

(ii) If U is a complete G-universe and N is a normal subgroup of G, then

$$\mathcal{E}(U,U^N) = \mathcal{F}(U,U^N) = \mathcal{F}(N).$$

Proof. (i) If there is an H-linear isometry $U \to U'$, then U' contains copies of all the H-irreducible representations appearing in U. Since the converse clearly holds, U and U' are H-isomorphic.

(ii) If $f: U \to U^N$ is an H-linear isometry and $g \in H \cap N$, then $f(gv) = f(v)$ for all $v \in U$. Since G acts effectively on U and f is an injection, $g = e$ and thus $H \in \mathscr{F}(N)$. Conversely, if $H \in \mathscr{F}(N)$, then H maps isomorphically onto a subgroup of G/N. Since U is G-complete, U^N is G/N-complete and thus both U and U^N are H-complete and therefore H-isomorphic.

The interest in $\mathscr{F}(N)$ is that an $\mathscr{F}(N)$-space is precisely the same thing as an N-free G-space. By analogy, we say that an $\mathscr{F}(N)$-CW spectrum is an N-free G-CW spectrum. We need one more definition before we can state our basic change of universe theorem.

Definition 2.5. Let $i: U' \to U$ be an inclusion of G-universes. A U'-representation of a spectrum $E \in G\mathscr{S}U$ is a spectrum $E' \in G\mathscr{S}U'$ together with an equivalence $i_*E' \to E$. For example, by Proposition 1.4, suspension spectra in $G\mathscr{S}U$ are represented by the corresponding suspension spectra in $G\mathscr{S}U'$, and similarly for shift desuspensions by representations in U'.

Theorem 2.6. Let $i: U' \to U$ be an inclusion of G-universes and let $\mathscr{F} = \mathscr{F}(U,U')$.

(i) If $E' \in G\mathscr{S}U'$ is an $\mathscr{F}$-CW spectrum and $F' \in G\mathscr{S}U'$ is any G-spectrum, then

$$i_*: [E',F']_G \longrightarrow [i_*E',i_*F']_G$$

is an isomorphism.

(ii) If $E \in G\mathscr{S}U$ is an $\mathscr{F}$-CW spectrum, then E admits a U'-representation by an $\mathscr{F}$-CW spectrum $E' \in G\mathscr{S}U'$. Moreover, E' is unique up to equivalence and can be chosen to have cells in canonical bijective correspondence with the cells of E.

Proof. (i) The transformation i_* is the composite

$$[E',F']_G \xrightarrow{\eta_*} [E',i^*i_*F']_G \cong [i_*E',i_*F']_G.$$

By Corollary 1.8 (and (i) of Lemma 2.4), $\eta: F' \to i^*i_*F'$ is a weak $\mathscr{F}$-equivalence, hence η_* is an isomorphism by the $\mathscr{F}$-Whitehead theorem.

(ii) We construct E' and an equivalence $i_*E' \to E$ by induction up a sequential filtration of E, starting with $E_0' = *$. Assume that E_n' and an equivalence $i_*E_n' \simeq E_n$ have been constructed. Let E_{n+1} be the cofibre of $k: K \to E_n$, where K is a wedge of spheres $S_H^q \in G\mathscr{S}U$ with $H \in \mathscr{F}$. Let J be the corresponding wedge of spheres $S_H^q \in G\mathscr{S}U'$ and note that $i_*J \cong K$ by Proposition 1.4. Let $j: J \to E_n'$ be the map, unique up to homotopy, such that $i_*(j)$ corresponds to k and let E_{n+1}' be its cofibre. There results an equivalence $i_*E_{n+1}' \simeq E_{n+1}$, and we obtain E' by passage to colimits. The uniqueness of E' is implied by (i).

Restricted to spaces, part (i) can be interpreted as follows.

Corollary 2.7. If X is an $\mathscr{F}$-CW complex and Y is any G-space, then $[\Sigma^\infty X,\Sigma^\infty Y]_G$ is the same when computed in the universe U' as when computed in the universe U.

Our main interest is in the following specialization.

Theorem 2.8. Let N be a normal subgroup of G, let U be a complete G-universe, and let $i: U^N \to U$ be the inclusion.

(i) If $E' \in G\mathscr{S}U'$ is an N-free G-CW spectrum and $F' \in G\mathscr{S}U^N$ is any G-spectrum, then

$$i_*: [E',F']_G \longrightarrow [i_*E',i_*F']_G$$

is an isomorphism.

(ii) If $E \in G\mathscr{S}U$ is an N-free G-CW spectrum, then E admits a U^N-representation by an N-free G-CW spectrum E'. Moreover, E' is unique up to equivalence and can be chosen to have cells in canonical bijective correspondence with the cells of E.

(iii) If $E \in G\mathscr{S}U$ is a finite N-free G-CW spectrum, then E is equivalent to $\Lambda^Z\Sigma^\infty X$ (and thus to $\Sigma^{-Z}\Sigma^\infty X$) for some finite N-free G-CW complex X and some G/N-representation Z.

Proof. In view of Lemma 2.4(ii), parts (i) and (ii) are immediate. For (iii), we may apply (ii) to represent E as i_*E' for a finite N-free G-CW spectrum $E' \in G\mathscr{S}U^N$. By I.8.16 and its proof, E' has the specified form, and the conclusion for E follows from Proposition 1.4.

Corollary 2.9. If X is an N-free G-CW complex and Y is any G-space, then $[\Sigma^\infty X,\Sigma^\infty Y]_G$ is the same when computed in the universe U^N as when computed in the universe U.

When G is a finite group and X and Y are finite complexes, the corollary is due to Adams [3,5.5].

In sum, these results assert that N-free G-CW spectra live in the N-trivial G-universe U^N. We now have all the information about change of universe needed for our study of change of groups. However, for our study of twisted half smash products and to place our results in proper perspective, we go on to relate the observations above to the universal $\mathscr{F}$-spaces introduced by Palais [118]. (Later authors call these classifying spaces; we prefer to follow Palais in reserving the term classifying space for the resulting orbit spaces.)

Definition 2.10. An $\mathcal{F}$-space E is said to be universal if, for any (unbased) $\mathcal{F}$-CW complex X, there is a unique homotopy class of G-maps $X \to E\mathcal{F}$. We require $E\mathcal{F}$ to have the homotopy type of a G-CW complex, and this ensures that $E\mathcal{F}$ is unique up to equivalence. An alternative characterization is that $(E\mathcal{F})^H$ be empty if $H \notin \mathcal{F}$ and nonempty and contractible if $H \in \mathcal{F}$.

For example, EG is universal for the trivial family $\mathcal{F} = \{e\}$. As explained by tom Dieck [37], iterated joins of orbits can be used to construct $E\mathcal{F}$ for general families $\mathcal{F}$, and an attractive conceptual construction has been given by Elmendorf [52]. For the families of interest here, we already have universal $\mathcal{F}$-spaces on hand.

Lemma 2.11. $\mathcal{J}(U,U')$ is universal for the family $\mathcal{F}(U,U')$.

Proof. By Lemma 1.5, $\mathcal{J}(U,U')^H$ is contractible if it is nonempty, and $\mathcal{J}(U,U')$ has the homotopy type of a G-CW complex by Waner [140,4.9] and VI.2.19 below (Z_n there being G-paracompact, completely regular, and GELC).

This fact will imply that the twisted half smash product functors $\chi \ltimes (?)\colon G\mathcal{S}U \to G\mathcal{S}U'$ determined by varying G-maps $\chi\colon X \to \mathcal{J}(U,U')$ become canonically equivalent upon passage to stable categories, where they will be denoted $X \ltimes (?)$. Of course, this is a generalization of Theorem 1.7, which deals with the case $X = \{*\}$.

Smash products with spaces $E\mathcal{F}^+$ give functorial models for the U'-representations of Theorem 2.6, as we proceed to explain. Clearly the Cartesian product of an unbased $\mathcal{F}$-space and an arbitrary G-space is an unbased $\mathcal{F}$-space and the smash product of a based $\mathcal{F}$-space and an arbitrary based G-space is a based $\mathcal{F}$-space. Using I.5.5, we see that the smash product of an $\mathcal{F}$-CW complex and a G-CW spectrum or of a G-CW complex and an $\mathcal{F}$-CW spectrum has the homotopy type of an $\mathcal{F}$-CW spectrum. More generally, using Theorem 3.8 below, the smash product of an $\mathcal{F}$-CW spectrum and a G-CW spectrum has the homotopy type of an $\mathcal{F}$-CW spectrum.

Lemma 2.12. Let $\pi\colon E\mathcal{F}^+ \to S^0$ be the natural projection. For any G-CW spectrum F, $E\mathcal{F}^+\wedge F$ has the homotopy type of an $\mathcal{F}$-CW spectrum and

$$\pi\wedge 1\colon E\mathcal{F}^+\wedge F \longrightarrow S^0\wedge F = F$$

is a weak $\mathcal{F}$-equivalence. If F is an $\mathcal{F}$-CW spectrum, then $\pi\wedge 1$ is a G-homotopy equivalence.

Proof. The first statement holds since π is an H-homotopy equivalence for $H \in \mathcal{F}$ and the second statement follows by the $\mathcal{F}$-Whitehead theorem.

Of course, the evident space level analog also holds.

Now recall the discussion of localizations of categories from I§5. The functor $E\mathcal{F}^+\wedge(?)$ can be viewed as the localization at the collection $\mathcal{E}\mathcal{F}$ of weak $\mathcal{F}$-equivalences (of based G-spaces or of G-spectra). To make this precise, let $\mathcal{F}\mathcal{J}$ and $\mathcal{F}\mathcal{S}U$ denote the categories of G-spaces of the weak homotopy type of $\mathcal{F}$-CW complexes (with basepoint) and of G-spectra of the weak homotopy type of $\mathcal{F}$-CW spectra. These have homotopy categories (prefix h) and localizations at their weak equivalences (prefix $\bar{h}$); here weak equivalences coincide with weak $\mathcal{F}$-equivalences.

Proposition 2.13. The functors $E\mathcal{F}^+\wedge(?)$ induce equivalences

$$(\mathcal{E}\mathcal{F})^{-1}(\bar{h}G\mathcal{J}) \longrightarrow \bar{h}\mathcal{F}\mathcal{J} \quad\text{and}\quad (\mathcal{E}\mathcal{F})^{-1}(\bar{h}G\mathcal{S}U) \longrightarrow \bar{h}\mathcal{F}\mathcal{S}U.$$

Proof. Via CW-approximation, this is an immediate consequence of Lemma 2.12 and the $\mathcal{F}$-Whitehead theorem.

These observations shed further light on Theorem 2.6.

Proposition 2.14. Let $i\colon U' \to U$ be an inclusion of G-universes and let $\mathcal{F} = \mathcal{F}(U,U')$. Then i_* induces an equivalence

$$(\mathcal{E}\mathcal{F})^{-1}(\bar{h}G\mathcal{S}U') \longrightarrow (\mathcal{E}\mathcal{F})^{-1}(\bar{h}G\mathcal{S}U).$$

If $F \in G\mathcal{S}U$ is an $\mathcal{F}$-CW spectrum, then $E\mathcal{F}^+\wedge i^*F$ is a U'-representation of F.

Proof. The first statement follows immediately from Theorem 2.6. For the second statement (in which CW-approximation of i^*F is of course understood), Corollary 1.8 implies that $\varepsilon\colon i_*i^*F \to F$ is an $\mathcal{F}$-equivalence and the conclusion follows from Lemma 2.12 and the commutative diagram

$$\begin{array}{ccccc}
i_*(E\mathcal{F}^+\wedge i^*F) & \xrightarrow{i_*(\pi\wedge 1)} & i_*i^*F & \xrightarrow{\varepsilon} & F \\
\wr\| & & & & \| \\
E\mathcal{F}^+\wedge i_*i^*F & \xrightarrow{1\wedge\varepsilon} & E\mathcal{F}^+\wedge F & \xrightarrow{\pi\wedge 1} & F\,.
\end{array}$$

The problem with this last result is that, as a composite of a left and a right adjoint, the functor $E\mathcal{F}^+\wedge i^*F$ of F is impossible to analyze calculationally. In a very real sense, the essential point of the theory of twisted half smash products in Chapter VI is the construction of a functor $X \ltimes F$ which is an analyzable left adjoint and yet is equivalent to the composite functor $X^+ \wedge i^* F$, where X is any $\mathcal{J}$-CW complex.

§3. Smash products and function spectra

Throughout this section, let $\mathcal{A}$ and $\mathcal{A}'$ be indexing sets in G-universes U and U' and let

$$\mathcal{A} \oplus \mathcal{A}' = \{V \oplus V' \mid V \in \mathcal{A} \text{ and } V' \in \mathcal{A}\}$$

be the resulting indexing set in the G-universe $U \oplus U'$. We shall first define and discuss "external" smash products and function spectra and then use changes of universe to obtain "internal" functors when $U = U'$ and $\mathcal{A} = \mathcal{A}'$. The external functors have good properties before passage to the stable category and are central to the theory of extended powers. The internal functors on the stable category are central to all of stable homotopy and cohomology theory.

Definition 3.1. For $D \in G\mathcal{P}\mathcal{A}$ and $D' \in G\mathcal{P}\mathcal{A}'$, define

$$D \wedge D' \in G\mathcal{P}(\mathcal{A} \oplus \mathcal{A}')$$

by $(D \wedge D')(V \oplus V') = DV \wedge D'V'$ with structural maps

$$\Sigma^{(W-V)\oplus(W'-V')} DV \wedge D'V' \cong \Sigma^{W-V} DV \wedge \Sigma^{W'-V'} D'V' \xrightarrow{\sigma \wedge \sigma'} DW \wedge D'W'.$$

For $E \in G\mathcal{S}\mathcal{A}$ and $E' \in G\mathcal{S}\mathcal{A}'$, define

$$E \wedge E' = L(\ell E \wedge \ell E') \in G\mathcal{S}(\mathcal{A} \oplus \mathcal{A}').$$

Inspection and use of I.8.5 give the following result.

Proposition 3.2. The functor $D \wedge D'$ preserves injection and Σ-inclusion prespectra (but not inclusion prespectra or spectra). Both the prespectrum and spectrum level functors preserve injections, closed inclusions, and intersections of closed subobjects. If E and E' are closed subspectra of F and F', then

$$E \wedge E' = (E \wedge F') \cap (F \wedge E').$$

Recall the discussion of G-spaces of nonequivariant morphisms of G-prespectra given above I.3.3.

Definitions 3.3. For $D'' \in G\mathcal{P}(\mathcal{A} \oplus \mathcal{A}')$ and $V \in \mathcal{A}$, define

$$D''[V] \in G\mathcal{P}\mathcal{A}'$$

by $D''[V](V') = D''(V \oplus V')$ with structural maps induced by those of D''. Other structural maps of D'' give a map of prespectra

$$D''[V] \longrightarrow \Omega^{W-V} D''[W]$$

for $V \subset W$, and this map is an isomorphism if D'' is a spectrum. For $D' \in G\mathcal{P}\mathcal{A}'$, define

$$F(D', D'') \in G\mathcal{P}\mathcal{A}$$

by $F(D', D'')(V) = \mathcal{P}\mathcal{A}'(D', D''[V])$ with structural maps

$$\mathcal{P}\mathcal{A}'(D', D''[V]) \longrightarrow \mathcal{P}\mathcal{A}'(D', \Omega^{W-V} D''[W]) \cong \Omega^{W-V} \mathcal{P}\mathcal{A}'(D', D''[W]),$$

where the $(V')^{\underline{th}}$ component of the last homeomorphism is induced by the double adjunction homeomorphism

$$F(D'V', \Omega^{W-V} D''(W \oplus V')) \cong \Omega^{W-V} F(D'V', D''(W \oplus V')).$$

Clearly $F(D', D'')$ is a G-spectrum if D'' is a G-spectrum.

We have the following analogs of the adjunctions in I.3.3.

Proposition 3.4. There are natural homeomorphisms

$$G\mathcal{P}(\mathcal{A} \oplus \mathcal{A}')(D \wedge D', D'') \cong G\mathcal{P}\mathcal{A}(D, F(D', D''))$$

for $D \in G\mathcal{P}\mathcal{A}$, $D' \in G\mathcal{P}\mathcal{A}'$, and $D'' \in G\mathcal{P}(\mathcal{A} \oplus \mathcal{A}')$ and

$$G\mathcal{S}(\mathcal{A} \oplus \mathcal{A}')(E \wedge E', E'') \cong G\mathcal{S}\mathcal{A}(E, F(E', E''))$$

for $E \in G\mathcal{S}\mathcal{A}$, $E' \in G\mathcal{S}\mathcal{A}'$, and $E'' \in G\mathcal{S}(\mathcal{A} \oplus \mathcal{A}')$.

Of course, there is a symmetric definition of $F(E, E'') \in G\mathcal{S}\mathcal{A}'$ for $E \in G\mathcal{S}\mathcal{A}$ and $E'' \in G\mathcal{S}(\mathcal{A} \oplus \mathcal{A}')$ and a corresponding adjunction. The functor $E \wedge E'$ preserves colimits in both variables while $F(E', E'')$ preserves limits in E'' and converts colimits in E' to limits. The following results are obvious by inspection of the prespectrum level definitions.

Proposition 3.5. The functors $E\wedge E'$ and $F(E',E'')$ are independent of the choices of indexing sets and G-universes in the sense that they commute up to coherent natural isomorphism with the isomorphisms of categories of G-spectra of I.2.4 and I.2.5.

Of course, use of ϕ and ψ also transports our constructions to indexing sets not of the form $\mathscr{A}\oplus\mathscr{A}'$ in $U\oplus U'$.

Proposition 3.6. Let $V\in\mathscr{A}$ and $V'\in\mathscr{A}'$. For based G-spaces Y and Y', there is a natural isomorphism

$$\Lambda^V\Sigma^\infty Y\wedge\Lambda^{V'}\Sigma^\infty Y' \cong \Lambda^{V\oplus V'}\Sigma^\infty(Y\wedge Y').$$

For $E\in G\mathscr{S}\mathscr{A}$ and $E'\in G\mathscr{S}\mathscr{A}'$, there are natural isomorphisms

$$(E\wedge Y)\wedge E' \cong (E\wedge E')\wedge Y \cong E\wedge(E'\wedge Y).$$

This implies that smash products of equivariant cells behave appropriately.

Lemma 3.7. For a p-cell $e^p_H\in G\mathscr{S}\mathscr{A}$ and q-cell $e^q_J\in G\mathscr{S}\mathscr{A}'$, there is a canonical isomorphism of pairs

$$(e^p_H\wedge e^q_J, e^p_H\wedge S^{q-1}_J \cup S^{p-1}_H\wedge e^q_J) \cong (L^+\wedge e^{p+q}, L^+\wedge S^{p+q-1})$$

in $G\mathscr{S}(\mathscr{A}\oplus\mathscr{A}')$, where $L = G/H\times G/J$.

If $H = G$ or $J = G$, the right side is an equivariant cell and its boundary. If G is finite, then $G/H\times G/J$ is a disjoint union of orbits and the right side is a wedge of equivariant cells and their boundaries, although this description depends on double coset choices and is thus not canonical. For general compact Lie groups G, the right side is at least equivalent to a G-CW spectrum and a subcomplex (by I.1.2 and I.5.5). These observations and the commutation of smash products with pushouts and sequential colimits lead to the following result. Say that a G-CW spectrum is G-trivial if no spheres S^n_H with $H\neq G$ appear as domains of its attaching maps.

Theorem 3.8. Let $E\in G\mathscr{S}\mathscr{A}$ and $E'\in G\mathscr{S}\mathscr{A}'$ be G-CW spectra. Then $E\wedge E'$ may be given the sequential filtration

$$(E\wedge E')_n = \bigcup_{p+q=n} E_p\wedge E'_q,\ n\geq 0,$$

and the skeletal filtration

$$(E\wedge E')^n = \bigcup_{p+q=n} E^p\wedge(E')^q,\ n\in Z,$$

both being functorial with respect to bicellular maps of their variables. If E or E' is G-trivial or if G is finite, then $E\wedge E'$ is a G-CW spectrum with respect to these filtrations. In general, $E\wedge E'$ has the homotopy type of a G-CW spectrum.

Clearly $E\wedge E'$ preserves homotopies in both variables and therefore preserves G-CW homotopy types. By I.5.13, this implies the following result.

Corollary 3.9. Let E' have the homotopy type of a G-CW spectrum. Then the functor $F(E',E'')$ preserves weak equivalences in the variable E'', and $E\wedge E'$ and $F(E',E'')$ induce an adjoint pair of functors relating the stable categories $\bar{h}G\mathscr{S}\mathscr{A}$ and $\bar{h}G\mathscr{S}(\mathscr{A}\oplus\mathscr{A}')$.

Here we must apply CW-approximation to E and E' to transport $E\wedge E'$ and $F(E',E'')$ to functors on stable categories.

Remarks 3.10. The restriction to a single group G in sight is the simplest way to set up notations, but there is an alternative viewpoint that is sometimes useful. We might begin with groups G and G' acting on U and U' and thus $G\times G'$ acting on $U\oplus U'$. In this setting, the result on CW structures becomes more precise because the obvious identifications

$$G/H\times(G'/H') = (G\times G')/(H\times H')$$

imply that smash products preserve spheres and cells. Actually, this setting can be included in our original one by noting that the G and G' universes U and U' are both $G\times G'$-universes via the projections. Conversely, with $G = G'$, use of $\Delta\colon G\to G\times G$ shows that our original setting can be obtained by change of groups from the setting with two groups in sight.

Returning to our initial context, we observe next that the smash product is associative and commutative in the appropriate sense. The associativity is immediately obvious on the prespectrum level and passes to the spectrum level via L. The commutativity must take account of the transposition isomorphism

$$t\colon U\oplus U'\longrightarrow U'\oplus U$$

and asserts that $t_*(E\wedge E')$ is naturally isomorphic to $E'\wedge E$ in $G\mathcal{S}(\mathcal{A}'\oplus\mathcal{A})$. The verification is again immediate.

Henceforward, we restrict attention to a single G-universe U and the indexing sets $\mathcal{A}^n = \mathcal{A}\oplus\cdots\oplus\mathcal{A}$ in U^n. It is useful to consider the graded category whose n^{th} term is $G\mathcal{S}\mathcal{A}^n$, with $G\mathcal{S}\mathcal{A}^0 = G\mathcal{J}$. The external smash product and function spectrum functors

$$G\mathcal{S}\mathcal{A}^m \times G\mathcal{S}\mathcal{A}^n \longrightarrow G\mathcal{S}\mathcal{A}^{m+n} \quad\text{and}\quad (G\mathcal{S}\mathcal{A}^m)^{op}\times G\mathcal{S}\mathcal{A}^{m+n} \longrightarrow G\mathcal{S}\mathcal{A}^n$$

give this category a closed symmetric monoidal structure. Further discussion of this point of view will be given in [107], and we turn here to the internalization process. Observe that, by virtue of our conventions on G-universes, there is at least one G-linear isometry $U^n \to U$ for each n and thus each $\mathcal{J}(U^n,U)$ is G-contractible by Lemma 1.5.

Definition 3.11. Choose a G-linear isometry $f\in\mathcal{J}(U\oplus U,U)$. Define internal smash product and function spectrum functors

$$G\mathcal{S}\mathcal{A}\times G\mathcal{S}\mathcal{A}\longrightarrow G\mathcal{S}\mathcal{A} \quad\text{and}\quad (G\mathcal{S}\mathcal{A})^{op}\times G\mathcal{S}\mathcal{A}\longrightarrow G\mathcal{S}\mathcal{A}$$

by $E\wedge E' = f_*(E\wedge E')$ and $F(E,E') = F(E,f^*E')$ for $E,E'\in G\mathcal{S}\mathcal{A}$.

Composition of the (f_*,f^*) adjunction of Proposition 1.2 with the adjunction of Proposition 3.4 yields an adjunction

$$G\mathcal{S}\mathcal{A}(E\wedge E',E'') \cong G\mathcal{S}\mathcal{A}(E,F(E',E'')).$$

The functors $E\wedge E'$ and $F(E',E'')$ behave properly with respect to colimits and limits. The functor $E\wedge E'$ commutes properly with smash products with spaces and with the functors $\Lambda^V\Sigma^\infty$ and preserves G-CW homotopy types. After CW-approximation of the variables E and E', $E\wedge E'$ and $F(E',E'')$ pass to an adjoint pair of endofunctors on the stable category $\bar{h}G\mathcal{S}\mathcal{A}$.

Before passage to the stable category, the functors $E\wedge E'$ and $F(E',E'')$ depend non-trivially on the choice of f, and $E\wedge E'$ fails to be unital, commutative, and associative. Thus these internal functors only become interesting after passage to $\bar{h}G\mathcal{S}\mathcal{A}$. Here they are independent of f by Theorem 1.7, and the following definition gives unity, commutativity, and associativity isomorphisms. Recall that $(f'f)_* \cong f'_*f_*$ for composable G-linear isometries f and f'.

Definitions 3.12. Let $E,E',E''\in G\mathcal{S}\mathcal{A}$ and $Y\in G\mathcal{J}$.

(i) Define a natural isomorphism

$$E\wedge Y \simeq E\wedge\Sigma^\infty Y$$

by passage to $\bar{h}G\mathcal{S}\mathcal{A}$ from the composite

$$E\wedge Y \simeq (fi)_*(E\wedge Y) \cong f_*i_*(E\wedge Y) \cong f_*(E\wedge\Sigma^\infty Y).$$

Here $i\colon U\to U\oplus U$ includes U onto the first summand. The first equivalence is given by Theorem 1.7, and $i_*(E\wedge Y)\cong E\wedge\Sigma^\infty Y$ by an easy direct inspection of definitions on the prespectrum level. When $Y = S^0$, there results a unit isomorphism $E \simeq E\wedge S$.

(ii) Define a natural commutativity isomorphism

$$\gamma\colon E\wedge E' \simeq E'\wedge E$$

by passage to $\bar{h}G\mathcal{S}\mathcal{A}$ from the composite

$$f_*(E\wedge E') \simeq (ft)_*(E\wedge E') \cong f_*t_*(E\wedge E') \cong f_*(E'\wedge E).$$

Here t is the transposition on $U\oplus U$, and the first equivalence is given by Theorem 1.7.

(iii) Define a natural associativity isomorphism

$$(E\wedge E')\wedge E'' \simeq E\wedge(E'\wedge E'')$$

by passage to $\bar{h}G\mathcal{S}\mathcal{A}$ from the composite

$$\begin{aligned} f_*(f_*(E\wedge E')\wedge E') &\cong f_*(f\oplus 1)_*(E\wedge E'\wedge E'')\\ &\cong (f(f\oplus 1))_*(E\wedge E'\wedge E'')\\ &\simeq (f(1\oplus f))_*(E\wedge E'\wedge E'')\\ &\cong f_*(1\oplus f)_*(E\wedge E'\wedge E'') \cong f_*(E\wedge f_*(E'\wedge E'')).\end{aligned}$$

The first and last isomorphisms result from a simple comparision of definitions, and the middle equivalence is given by Theorem 1.7.

These isomorphisms are independent of the choices of paths and commute properly with the natural isomorphism relating f_* to g_* for any other G-linear isometry $g\colon U^2\to U$. Using the contractibility of the $\mathcal{J}(U^n,U)$, one can prove the commutativity of the various coherence diagrams relating these isomorphisms and so obtain the following result. Details may be found in [107], but the ideas are amply illustrated in the proof of Theorem 1.7.

Theorem 3.13. The stable category $\bar{h}G\mathscr{S}\mathscr{A}$ is a closed symmetric monoidal category with respect to the internal smash product and function spectrum functors.

This result is exploited systematically for the study of cohomology theories in [107] and [90] and is the starting point for our discussion of Spanier-Whitehead duality in the next chapter.

We end with a few remarks about the relationship between smash product and fixed point functors. The obvious fact that $(X\wedge Y)^G = X^G\wedge Y^G$ for based G-spaces X and Y is a calculational commutation relation between left and right adjoints. There is no reason to expect this relationship to extend to spectra, and in fact it doesn't.

Remarks 3.14. Let $i\colon U^G \to U$ be the inclusion. By definition, for $E \in G\mathscr{S}U$, $E^G = (i^*E)^G \in \mathscr{S}U^G$. This functor has $\varepsilon^\# = i_*\varepsilon^*\colon \mathscr{S}U^G \to G\mathscr{S}U$ as left adjoint, where ε^* assigns trivial G action to a spectrum (or space; we are thinking of $\varepsilon\colon G \to e$). The counit of the adjunction is a natural G-map $\varepsilon^\# E^G \to E$. This is the spectrum level analog of the obvious inclusion $\varepsilon^* Y^G \to Y$ for a G-space Y. We have the following commutation relations.

(i) It is obvious that $\Omega^\infty(E^G) = (\Omega^\infty E)^G$ and it follows from I.3.5 that $\varepsilon^*\Sigma^\infty X \cong \Sigma^\infty\varepsilon^* X$ for a space X. Applying this to $X = Y^G$ and passing to adjoints, we obtain a natural map

$$\zeta\colon \Sigma^\infty(Y^G) \longrightarrow (\Sigma^\infty\varepsilon^* Y^G)^G \longrightarrow (\Sigma^\infty Y)^G.$$

(Here $\Sigma^\infty\colon \mathscr{T} \to \mathscr{S}U^G$ on the left and $\Sigma^\infty\colon G\mathscr{T} \to G\mathscr{S}U$ in the middle and on the right.)

(ii) Proposition 1.4 implies an isomorphism

$$\varepsilon^\#(D\wedge X) \cong (\varepsilon^\# D)\wedge(\varepsilon^* X)$$

for $D \in \mathscr{S}U^G$. Applying this to $D = E^G$ and $X = Y^G$ and passing to adjoints, we obtain a natural map

$$\nu\colon E^G\wedge Y^G \longrightarrow ((\varepsilon^\# E^G)\wedge(\varepsilon^* Y^G))^G \longrightarrow (E\wedge Y)^G.$$

In contrast to the space level, ν is generally not an isomorphism.

(iii) By an easy inspection (see (VI.3.1(ii)), the functor $\varepsilon^\#$ commutes with external smash products. That is,

$$(i \oplus i)_*\varepsilon^*(D\wedge D') \cong i_*\varepsilon^* D\wedge i_*\varepsilon^* D'$$

for $D,D' \in \mathscr{S}U^G$. For a G-linear isometry $f\colon U \oplus U \to U$, we have $f\circ(i \oplus i) = i\circ f^G$,

and we deduce that $\varepsilon^\#$ commutes with internal smash products. That is

$$\varepsilon^\# f_*^G(D\wedge D') \cong f_*(\varepsilon^\# D\wedge\varepsilon^\# D').$$

Applying this to $D = E^G$ and $D' = E'^G$ for $E,E' \in G\mathscr{S}U$ and passing to adjoints, we obtain a natural map

$$\omega\colon f_*^G(E^G\wedge E'^G) \longrightarrow [f_*(\varepsilon^\# E^G\wedge\varepsilon^\# E'^G)]^G \longrightarrow [f_*(E\wedge E')]^G.$$

Again, ω is generally not an isomorphism. (In practice, of course, we omit the isometries from the notation in this comparison of internal smash products.)

(iv) The maps ζ, ν, and ω are related by the following commutative diagram, where the unlabeled isomorphisms are given by Definition 3.12(i).

$$\begin{tikzcd}
E^G\wedge Y^G \arrow[rr, "\nu"] & & (E\wedge Y)^G \\
f_*^G(E^G\wedge\Sigma^\infty(Y^G)) \arrow[u, "\cong"] \arrow[r, "f_*^G(1\wedge\zeta)"] & f_*^G(E^G\wedge(\Sigma^\infty Y)^G) \arrow[r, "\omega"] & {[f_*(E\wedge\Sigma^\infty Y)]^G} \arrow[u, "\cong"']
\end{tikzcd}$$

§4. Change of groups functors and isomorphisms

Let $\alpha\colon H \to G$ be a homomorphism of compact Lie groups. An indexing set $\mathscr{A}$ in a G-universe U may also be regarded as an indexing set in U regarded as an H-universe via α, and there results a forgetful functor

$$\alpha^*\colon G\mathscr{S}\mathscr{A} \longrightarrow H\mathscr{S}\mathscr{A}.$$

Of course, we have analogous forgetful functors $\alpha^*\colon G\mathscr{T} \to H\mathscr{T}$ and $\alpha^*\colon G\mathscr{P}\mathscr{A} \to H\mathscr{P}\mathscr{A}$. We shall usually omit α^* from the notation, regarding G-objects as H-objects by neglect of structure.

We shall construct left and right adjoints to α^*. On the space level, such functors are given by

$$G^+\wedge_\alpha Y \quad\text{and}\quad F_\alpha(G^+,Y)$$

for $Y \in H\mathscr{T}$. Here $G^+\wedge_\alpha Y$ is the quotient space $G^+\wedge Y/(\sim)$, where

$$(g\alpha(h),y) \sim (g,hy) \quad\text{for } g \in G,\ h \in H, \text{ and } y \in Y;$$

G acts via its left action on itself. The space $F_\alpha(G^+,Y)$ is the function space of left H-maps $G \to Y$ with left G action induced by the right action of G on itself. When α is an inclusion, we use the notations

$$G^+\wedge_H Y \quad \text{and} \quad F_H(G^+,Y)$$

for these constructions. When α is a projection onto a quotient group with kernel N, we have

$$G^+\wedge_\alpha Y \cong Y/N \quad \text{and} \quad F_\alpha(G^+,Y) \cong Y^N.$$

In the general case, if α has kernel N and $J = H/N$, we have

$$G^+\wedge_\alpha Y \cong G\wedge_J(Y/N) \quad \text{and} \quad F_\alpha(G^+,Y) \cong F_J(G^+,Y^N).$$

Thus the definition for general α is logically redundant; however, it will be quite convenient to have the general notation on hand.

We have already defined orbit and fixed point spectra in I.3.7, and, for simplicity, we shall define the general spectrum level change of group functors as the appropriate composites.

We need some standard space level G-homeomorphisms. For $\alpha\colon H \to G$ and for a G-space X and H-space Y, define a G-homeomorphism

$$\zeta\colon G^+\wedge_\alpha (Y\wedge X) \longrightarrow (G^+\wedge_\alpha Y)\wedge X$$

by

$$\zeta(g,y\wedge x) = (g,y)\wedge gx \quad \text{and} \quad \zeta^{-1}((g,y)\wedge x) = (g,y\wedge g^{-1}x)$$

for $g \in G$, $y \in Y$, and $x \in X$. We shall continue to write ζ for cognate G-homeomorphisms differing by transpositions, such as

$$G^+\wedge_\alpha (X\wedge Y) \cong X\wedge(G^+\wedge_\alpha Y).$$

There is an analogous G-homeomorphism

$$\phi\colon F(X,F_\alpha(G^+,Y)) \longrightarrow F_\alpha(G^+,F(X,Y));$$

it is specified on $f\colon X \to F_\alpha(G^+,Y)$ and $f'\colon G^+ \to F(X,Y)$ by

$$\phi(f)(g)(x) = f(g^{-1}x)(g) \quad \text{and} \quad \phi^{-1}(f')(x)(g) = f'(g)(gx).$$

Definitions 4.1. Let H be a subgroup of G and let $D \in H\mathcal{P}\mathcal{A}$, where $\mathcal{A}$ is an indexing set in a G-universe.

(i) Define $G\ltimes_H D \in G\mathcal{P}\mathcal{A}$ by letting

$$(G\ltimes_H D)(V) = G^+\wedge_H DV$$

and letting the structural map associated to $V \subset W$ be the composite

$$(G^+\wedge_H DV)\wedge S^{W-V} \xrightarrow{\zeta^{-1}} G^+\wedge_H(DV\wedge S^{W-V}) \xrightarrow{1\wedge\sigma} G^+\wedge_H DW.$$

For $E \in H\mathcal{S}\mathcal{A}$, define $G\ltimes_H E = L(G\ltimes_H \ell E) \in G\mathcal{S}\mathcal{A}$.

(ii) Define $F_H[G,D) \in G\mathcal{P}\mathcal{A}$ by letting

$$F_H[G,D)(V) = F_H(G^+,DV)$$

and letting the adjoint structural map associated to $V \subset W$ be the composite

$$F_H(G^+,DV) \xrightarrow{F(1,\tilde{\sigma})} F_H(G^+,\Omega^{W-V}DW) \xrightarrow{\phi^{-1}} \Omega^{W-V}F_H(G^+,DW).$$

If D is an H-spectrum, then $F_H[G,D)$ is a G-spectrum.

These functors may be viewed as particularly simple examples of twisted half smash products and twisted function spectra, the maps ζ^{-1} and ϕ^{-1} encoding twisting by the action map $\gamma\colon G \to \mathcal{I}(U,U)$, $\gamma(g)(v) = gv$. They should not be confused with the untwisted smash product and function prespectra $G^+\wedge D$ and $F(G^+,D)$ of I§3, which admit no G action. Our choice of notations is intended to accentuate this distinction.

By inspection on the prespectrum level and use of I.8.5, we have the following observations, which should be contrasted with those for orbit spectra in I.8.6(iii).

Lemma 4.2. The prespectrum level functor $G\ltimes_H D$ preserves Σ-inclusion prespectra and injection prespectra. Both it and the spectrum level functor $G\ltimes_H E$ preserve injections, closed inclusions, and pullback diagrams one leg of which is a closed inclusion (such as intersections of closed subobjects).

We make no further explicit references to prespectra below, and we generally use the letter D for H-spectra and the letter E for G-spectra. We think of $G\ltimes_H D$ as the G-spectrum freely generated by D; dually, $F_H[G,D)$ is "cofree". The expected adjunctions make this precise.

Proposition 4.3. For $H \subset G$, there are natural isomorphisms

$$G\mathcal{S}\mathcal{A}(G \ltimes_H D, E) \cong H\mathcal{S}\mathcal{A}(D,E)$$

and

$$H\mathcal{S}\mathcal{A}(E,D) \cong G\mathcal{S}\mathcal{A}(E, F_H[G,D)),$$

where $E \in G\mathcal{S}\mathcal{A}$ and $D \in H\mathcal{S}\mathcal{A}$, $\mathcal{A}$ being an indexing set in a G-universe.

We use this in conjunction with the following more precise form of the adjunctions for orbit and fixed point spectra given in I.3.8.

<u>Proposition 4.4</u>. For a normal subgroup N of G with quotient map $\varepsilon: G \to J$, $J = G/N$, there are natural isomorphisms

$$J\mathcal{S}\mathcal{A}(D/N,E) \cong G\mathcal{S}\mathcal{A}(D,\varepsilon^*E)$$

and

$$G\mathcal{S}\mathcal{A}(\varepsilon^*E,D) \cong J\mathcal{S}\mathcal{A}(E,D^N),$$

where $E \in J\mathcal{S}\mathcal{A}$ and $D \in G\mathcal{S}\mathcal{A}$, $\mathcal{A}$ being an indexing set in a J-universe. If M is a second normal subgroup of G and $\delta: G/M \to G/MN$ is the quotient map, then

$$(\varepsilon^*E)/M \cong \delta^*(E/(MN/N)).$$

The last statement is easily checked by conjugation and is recorded for use in section 7.

As usual, these adjunctions pass to stable categories. (See Lemmas 4.12 and 4.13 below.) There the orbit adjunction combines with the change of universe results of Theorem 2.6 and Corollary 2.7 to give the following conclusions.

<u>Theorem 4.5</u>. Let $J = G/N$ with quotient map $\varepsilon: G \to J$. Let U be a complete G-universe with inclusion $i: U^N \to U$. If $D \in G\mathcal{S}U^N$ is an N-free G-spectrum and $E \in J\mathcal{S}U^N$, then

$$[D/N,E]_J \cong [i_*D, \varepsilon^\# E]_G,$$

where $\varepsilon^\# E = i_*\varepsilon^*E$. In particular, if X is an N-free G-CW complex and Y is any J-CW complex, then

$$[\Sigma^\infty X/N, \Sigma^\infty Y]_J \cong [\Sigma^\infty X, \Sigma^\infty \varepsilon^* Y]_G,$$

where the left side is computed in the universe U^N and the right side is computed in the universe U.

When G is a finite group and X and Y are finite complexes, the last statement is due to Adams [3,5.3].

Returning to our main theme, we combine definitions to obtain the desired general change of groups functors.

<u>Definition 4.6</u>. Let $\alpha: H \to G$ be a homomorphism of compact Lie groups and set $N = \mathrm{Ker}(\alpha)$ and $J = H/N \subset G$. Let $D \in H\mathcal{S}\mathcal{A}$, where $\mathcal{A}$ is an indexing set in a G-universe. Define

$$G \ltimes_\alpha D = G \ltimes_J (D/N) \quad \text{and} \quad F_\alpha[G,D) = F_J[G,D^N).$$

We have the desired composite adjunctions, and our functors are suitably compatible with their space level analogs.

<u>Theorem 4.7</u>. For $\alpha: H \to G$, there are natural isomorphisms

$$G\mathcal{S}\mathcal{A}(G \ltimes_\alpha D, E) \cong H\mathcal{S}\mathcal{A}(D, \alpha^*E)$$

and

$$H\mathcal{S}\mathcal{A}(\alpha^*E, D) \cong G\mathcal{S}\mathcal{A}(E, F_\alpha[G,D)),$$

where $E \in G\mathcal{S}\mathcal{A}$ and $D \in H\mathcal{S}\mathcal{A}$, $\mathcal{A}$ being an indexing set in a G-universe. Moreover, for $X \in G\mathcal{J}$, $Y \in H\mathcal{J}$, and $Z \in \mathcal{A}$, there are natural isomorphisms

$$G \ltimes_\alpha \Lambda^Z \Sigma^\infty Y \cong \Lambda^Z \Sigma^\infty (G^+ \wedge_\alpha Y) \quad \text{and} \quad \alpha^* \Lambda^Z \Sigma^\infty X \cong \Lambda^Z \Sigma^\infty \alpha^* X.$$

<u>Proof</u>. Composites of adjoint pairs are adjoint pairs. The last statement follows by conjugation from the evident equalities

$$F_\alpha[G,DZ) = (F_\alpha[G,D))(Z) \quad \text{and} \quad \alpha^*(EZ) = (\alpha^*E)(Z).$$

Stated another way, the adjunctions mean that there are natural H-maps

$$\eta: D \longrightarrow G \ltimes_\alpha D \quad \text{and} \quad \varepsilon: F_\alpha[G,D) \longrightarrow D$$

such that H-maps $f\colon D \to E$ and $g\colon E \to D$ uniquely determine G-maps $\tilde{f}$ and $\tilde{g}$ which make the following diagrams of H-spectra commute.

$$\begin{tikzcd} D \arrow[r, "\eta"] \arrow[dr, "f"'] & G \ltimes_\alpha D \arrow[d, "\tilde{f}"] \\ & E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_\alpha[G,D] \arrow[r, "\varepsilon"] & D \\ E \arrow[u, "\tilde{g}"] \arrow[ur, "g"'] & \end{tikzcd}$$

When $D = E$ and f and g are the identity map, we write

$$\xi\colon G \ltimes_\alpha E \longrightarrow E \quad \text{and} \quad \nu\colon E \longrightarrow F_\alpha[G,E]$$

for the resulting "action" and "coaction" G-maps. As on the space level, these maps admit simple reinterpretations.

Lemma 4.8. For a G-spectrum E, the G-maps ζ and ϕ characterized by the commutative diagrams of H-spectra

$$\begin{tikzcd} E \arrow[r, equal] \arrow[d, "\eta"'] & S^0 E \arrow[d, "\eta\wedge 1"] \\ G \ltimes_\alpha E \arrow[r, "\zeta"] & (G/\alpha H)^+ \wedge E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(S^0,E) \arrow[r, equal] & E \\ F((G/\alpha H)^+,E) \arrow[u, "F(\eta,1)"] \arrow[r, "\phi"] & F_\alpha[G,E] \arrow[u, "\varepsilon"'] \end{tikzcd}$$

are isomorphisms, where $\eta\colon S^0 \to (G/\alpha H)^+$ is the obvious H-map. Moreover, the following diagrams of G-spectra commute, where $\xi\colon (G/\alpha H)^+ \to S^0$ is the obvious G-map.

$$\begin{tikzcd} G \ltimes_\alpha E \arrow[r, "\zeta"] \arrow[d, "\xi"'] & (G/\alpha H)^+ \wedge E \arrow[d, "\xi\wedge 1"] \\ E \arrow[r, equal] & S^0 \wedge E \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F((G/\alpha H)^+,E) \arrow[r, "\phi"] & F_\alpha[G,E] \\ F(S^0,E) \arrow[u, "F(\xi,1)"] \arrow[r, equal] & E \arrow[u, "\nu"'] \end{tikzcd}$$

Here one thinks of G as acting from the left on $G \ltimes_\alpha E$ and diagonally on $(G/\alpha H)^+ \wedge E$. Again, just as on the space level, the isomorphisms ζ and ϕ generalize.

Lemma 4.9. For a G-spectrum E and H-spectrum D, the G-maps ζ, ϕ, and κ characterized by the commutative diagrams of H-spectra

$$\begin{tikzcd} & D\wedge E \arrow[dl, "\eta"'] \arrow[dr, "\eta\wedge 1"] & \\ G \ltimes_\alpha (D\wedge E) \arrow[rr, "\zeta"] & & (G \ltimes_\alpha D)\wedge E \end{tikzcd} \quad , \quad \begin{tikzcd} & F(E,D) & \\ F(E,F_\alpha[G,D]) \arrow[ur, "F(1,\varepsilon)"] \arrow[rr, "\phi"] & & F_\alpha[G,F(E,D)] \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

and

$$\begin{tikzcd} & F(D,E) & \\ F(G \ltimes_\alpha D,E) \arrow[ur, "F(\eta,1)"] \arrow[rr, "\kappa"] & & F_\alpha[G,F(D,E)] \arrow[ul, "\varepsilon"'] \end{tikzcd}$$

are all isomorphisms.

In fact, these isomorphisms are all purely formal consequences of the adjunctions, via application of the Yoneda lemma. They can also be checked directly, by passage from spaces to prespectra to spectra. Similarly, we have transitivity isomorphisms.

Lemma 4.10. For $\alpha\colon H \to G$ and $\beta\colon K \to H$ and for a K-spectrum C, the dotted arrow G-maps specified by K-commutativity of the diagrams

$$\begin{tikzcd} C \arrow[r, "\eta"] \arrow[d, "\eta"'] & H \ltimes_\beta C \arrow[d, "\eta"] \\ G \ltimes_{\alpha\beta} C \arrow[r, dashed] & G \ltimes_\alpha H \ltimes_\beta C \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F_\beta[H,C] \arrow[r, "\varepsilon"] & C \\ F_\alpha[G,F_\beta[H,C]] \arrow[u, "\varepsilon"] \arrow[r, dashed] & F_{\alpha\beta}[G,C] \arrow[u, "\varepsilon"'] \end{tikzcd}$$

are isomorphisms. For a G-spectrum E, $\beta^*\alpha^*E = (\alpha\beta)^*(E)$.

These results imply relations between G-sphere and H-sphere spectra.

Lemma 4.11. For $\alpha\colon H \to G$ and $\beta\colon K \to H$ and for $n \in Z$,

$$(G/\alpha\beta K)^+ \wedge S^n \cong G \ltimes_{\alpha\beta} S^n \cong G \ltimes_\alpha H \ltimes_\beta S^n \cong G \ltimes_\alpha ((H/\beta K)^+ \wedge S^n),$$

where we have implicitly used the relations $\alpha^* S^n = S^n$ and $\beta^* S^n = S^n$.

Since the functor $G \ltimes_\alpha (?)$ commutes with wedges, cofibres, and colimits, this has the following immediate consequence.

Lemma 4.12. If D is an H-CW spectrum, then $G \ltimes_\alpha D$ inherits a canonical structure as a G-CW spectrum.

The analog for the functor α^* is weaker.

Lemma 4.13. If E is a G-CW spectrum, then $\alpha^* E$ has the homotopy type of an

H-CW spectrum. If G is finite, then α^*E is itself an H-CW spectrum.

Proof. Again, we need only see what α^* does to sphere spectra. Since α^* commutes with $\Lambda^n\Sigma^\infty$, it suffices to consider the behavior of α^* on space level spheres. For $K \subset G$, $\alpha^* S^q_K$ has the homotopy type of an H-CW complex by I.1.1. If G is finite, $\alpha^*(G/K)$ is a disjoint union of H-orbits and $\alpha^* S^q_K$ is a wedge of H-spheres S^q_L for various L.

In the discussion above, we consistently worked in a fixed G-universe regarded as an H-universe by pullback along $\alpha\colon H \to G$. Inspection on the level of right adjoints and use of conjugation gives the expected behavior with respect to change of universe.

Lemma 4.14. Let $f\colon U \to U'$ be a G-linear isometry between G-universes. Then there are natural isomorphisms

$$f_*\alpha^*E \cong \alpha^* f_*E \qquad f^*\alpha^*E' \cong \alpha^* f^* E'$$

$$f_*(G \ltimes_\alpha D) \cong G \ltimes_\alpha f_*D \qquad f^*F_\alpha[G,D') \cong F_\alpha[G,f^*D')$$

for $E \in G\mathscr{S}U$, $E' \in G\mathscr{S}U'$, $D \in H\mathscr{S}U$, and $D' \in H\mathscr{S}U'$.

We conclude this section with a discussion of the free and cofree G-spectra

$$G \ltimes E = G \ltimes_e E \quad \text{and} \quad F[G,E) = F_e[G,E)$$

generated by a nonequivariant spectrum $E \in \mathscr{S}\mathscr{A}$, where $\mathscr{A}$ is an indexing set in a G-universe U. We are specializing the theory above to the trivial inclusion $e \to G$ and we have the adjunctions

$$G\mathscr{S}\mathscr{A}(G \ltimes E,F) \cong \mathscr{S}\mathscr{A}(E,F) \quad \text{and} \quad \mathscr{S}\mathscr{A}(F,E) \cong G\mathscr{S}\mathscr{A}(F,F[G,E))$$

for $E \in \mathscr{S}\mathscr{A}$ and $F \in G\mathscr{S}\mathscr{A}$. Just as G-spaces can be defined in terms of maps $G^+\wedge X \to X$, so G-spectra can be defined in terms of maps $G \ltimes E \to E$. We summarize the properties of these constructions in the following remarks, all of which can easily be verified by passage from spaces to prespectra to spectra.

Remarks 4.15 (i) The unit and multiplication of G induce natural maps $\eta\colon E \to G \ltimes E$ and $\mu\colon G \ltimes G \ltimes E \to G \ltimes E$ such that $(G \ltimes (?),\mu,\eta)$ is a monad in the category $\mathscr{S}\mathscr{A}$. (See e.g. [92,97] for the relevant categorical definitions.) Moreover, a G-spectrum $E \in G\mathscr{S}\mathscr{A}$ is precisely the same thing as an algebra over this monad. That is, a G-structure on $E \in \mathscr{S}\mathscr{A}$ determines and is determined by a map $\xi\colon G \ltimes E \to E$ such that the evident unit and associativity diagrams commute. Similarly, a map of G-spectra is the same thing as a map of $G \ltimes (?)$-algebras.

(ii) The dual assertions also hold. There are natural maps $\varepsilon\colon F[G,E) \to E$ and $\nu\colon F[G,E) \to F[G,F[G,E))$ such that $(F[G,?),\nu,\varepsilon)$ is a comonad in $\mathscr{S}\mathscr{A}$ and a G-spectrum is the same thing as an $F[G,(?))$-coalgebra. Here (η,ε) and (μ,ν) are conjugate pairs in the sense of I.3.5.

(iii) If G is finite, then $G \ltimes (?)$ and its relation to G actions can be visualized as follows. If $g \in G$ is regarded as a linear isomorphism $U \to U$, then $g_*\colon \mathscr{S}\mathscr{A} \to \mathscr{S}\mathscr{A}$ is defined as in I.2.5 and $g_*h_* = (gh)_*$. Here $g^{-1}V = V$ for an indexing space V in U, and a trivial comparison of definitions shows that

$$G \ltimes E = \bigvee_{g\in G} g_*E.$$

An action of G on E is specified by maps $\xi_g\colon g_*E \to E$ such that $\xi_e = 1$ and $\xi_g \circ g_*\xi_h = \xi_{gh}$. The action of g on E comes via maps $g_*E \to E$ (as compared to the space level maps $X \to X$) because of the role played by the action of G on the universe U. Dually

$$F[G,E) = \times_{g\in G}\, g^*E$$

and the coaction map $E \to F[G,E)$ of a G-spectrum admits a similar description in terms of maps $E \to g^*E$.

(iv) If N is a normal subgroup of G with quotient group J and $E \in \mathscr{S}\mathscr{A}$, where $\mathscr{A}$ is a J-indexing set regarded as a G-indexing set, then

$$J \ltimes E = (G \ltimes E)/N \in J\mathscr{S}\mathscr{A}$$

by Lemma 4.10 applied to the trivial composite $e \to G \to J$. In particular, with $N = G$ and $J = e$,

$$E = (G \ltimes E)/G \in \mathscr{S}\mathscr{A}$$

when $\mathscr{A}$ is an indexing set in a G-trivial universe.

(v) Let $K = G \times H$ and let $\mathscr{A}$ be an indexing set in a K-universe U. Then $G \ltimes (?)\colon \mathscr{S}\mathscr{A} \to G\mathscr{S}\mathscr{A}$ restricts to a functor $H\mathscr{S}\mathscr{A} \to K\mathscr{S}\mathscr{A}$, and similarly for H. Moreover, for $E \in \mathscr{S}\mathscr{A}$, there are natural isomorphisms of K-spectra

$$H \ltimes G \ltimes E \cong K \ltimes E \cong G \ltimes H \ltimes E.$$

For $F \in K\mathscr{S}\mathscr{A}$, the G action $G \ltimes F \to F$ is a map of H-spectra and the H action $H \ltimes F \to F$ is a map of G-spectra. The interpretation is that K-spectra are precisely the same as G-spectra in the category of H-spectra or H-spectra in the category of G-spectra.

In chapter VI, we shall use the last observations to study Σ_j-spectra in the category of G-spectra. In particular, we shall construct extended powers of G-spectra, the full generality presenting no greater difficulty than the case of nonequivariant spectra.

§5. Space level constructions

For simplicity, we separate out the elementary space level constructions and lemmas required in the proofs of our main change of groups theorems. Let H be a (closed) subgroup of G and let L be the tangent H-space at $eH \in G/H$. We write $L = L(H)$ or $L = L(H,G)$ when necessary for clarity. The following map t will be the geometric heart of the equivalence

$$\omega\colon F_H[G,\Sigma^L) \longrightarrow G \ltimes_H D$$

constructed in the next section.

Construction 5.1. Let $j\colon G/H \to V$ be an embedding of G/H in a G-representation V. The inclusion of the tangent space at eH embeds L as a sub H-representation of V. Let W be the orthogonal complement of L, so that $V = L \oplus W$ as an H-space. The normal bundle of the embedding j is $G \times_H W \to G/H$, hence we may extend j to an embedding $\tilde{j}\colon G \times_H W \to V$ of a normal tube. Collapsing the complement of its image to the basepoint, we obtain a G-map

$$t\colon S^V \longrightarrow G \ltimes_H S^W.$$

We need unit and transitivity properties of t. The inclusion $\{0\} \to L$ induces $e\colon S^0 \to S^L$, and we also write e for the induced H-map $S^W \to S^V = S^L \wedge S^W$. Note that e is null H-homotopic if L contains a trivial representation, and this is so if and only if $WH = NH/H$ is infinite. For an H-space Y, write $\eta\colon Y \to G \ltimes_H Y$ for the natural inclusion. The restriction of $\tilde{j}$ to W gives rise to an H-map $S^W \to S^V$ which is H-homotopic to e.

Lemma 5.2. The following diagram is H-homotopy commutative.

$$\begin{tikzcd} & S^W \arrow[dl, "e"'] \arrow[dr, "\eta"] & \\ S^V \arrow[rr, "t"] & & G \ltimes_H S^W \end{tikzcd}$$

For the transitivity relation, let $K \subset H \subset G$ and fix an identification of K-spaces

$$L(K) = L(K,H) \oplus L(H),$$

where $L(K)$ and $L(H)$ are taken with ambient group G.

Construction 5.3. Let $i\colon H/K \to V'$ be an H-embedding of H/K in a G-representation V' and write $V' = L(K,H) \oplus W'$ as a K-space. Let $\tilde{i}\colon H \times_K W' \to V'$ be an embedding of a normal H-tube. Let j and $\tilde{j}$ be as in Construction 5.1. Define a G-embedding

$$k\colon G/K = G \times_H (H/K) \longrightarrow V' \oplus V$$

by

$$k(g,hK) = (gi(hK), j(gH)).$$

Then define an embedding of a normal G-tube

$$\tilde{k}\colon G \times_K (W' \oplus W) = G \times_H (H \times_K (W' \oplus W)) \longrightarrow V' \oplus V$$

by

$$\tilde{k}(g,h,(w',w)) = (g\tilde{i}(h,w'), \tilde{j}(g,hw)).$$

With these notations, Construction 5.1 gives three maps t related by a transitivity diagram.

Lemma 5.4. The following diagram commutes.

$$\begin{tikzcd} S^{V'} \wedge S^V = S^{V' \oplus V} \arrow[rr, "t"] \arrow[d, "1 \wedge t"] & & G \ltimes_K S^{W'+W} = G \ltimes_H [H \ltimes_K (S^{W'} \wedge S^W)] \\ S^{V'} \wedge (G \ltimes_H S^W) \arrow[r, "\zeta^{-1}"] & G \ltimes_H (S^{V'} \wedge S^W) \arrow[r, "1 \ltimes (t \wedge 1)"] & G \ltimes_H [(H \ltimes_K S^{W'}) \wedge S^W] \arrow[u, "1 \ltimes \zeta^{-1}"'] \end{tikzcd}$$

The following map u will be the geometric heart of the equivalence

$$\psi\colon G \ltimes_H D \longrightarrow F_H[G, \Sigma^L D)$$

inverse to ω constructed in the next section.

Construction 5.5. Let $H \times H$ act on G by $(h_1,h_2)g = h_1 g h_2^{-1}$ and act on $L \times H$ by $(h_1,h_2)(\lambda,h) = (h_1\lambda, h_1 h h_2^{-1})$. We think of the first and second factors H as acting from the left and right, respectively. Choose an embedding $f\colon L \to G$ of L as a slice at e such that

$$f(h\lambda) = hf(\lambda)h^{-1} \quad \text{and} \quad f(-\lambda) = f(\lambda)^{-1}$$

(say by use of the exponential map). Via $\tilde{f}(\lambda,h) = f(\lambda)h$, f extends to an $(H\times H)$-embedding $\tilde{f}\colon L \times H \to G$ with image an open neighborhood of e. Collapsing its complement to the basepoint, we obtain an $(H\times H)$-map $u\colon G^+ \to S^L \wedge H^+$. If H has finite index in G, $u\colon G^+ \to H^+$ maps all points of $G - H$ to the disjoint basepoint. For any H-space Y, we obtain an induced left H-map

$$u\colon G \ltimes_H Y \longrightarrow (S^L \wedge H^+) \wedge_H Y = S^L \wedge Y.$$

We are interested particularly in the case $Y = S^W$, where

$$u\colon G \ltimes_H S^W \longrightarrow S^L \wedge S^W = S^V.$$

Again, we need unit and transitivity properties of u.

Lemma 5.6. The following diagram commutes for an H-space Y.

$$\begin{tikzcd} & Y \arrow[dl, "\eta"'] \arrow[dr, "e\wedge 1"] & \\ G \ltimes_H Y \arrow[rr, "u"] & & S^L \wedge Y \end{tikzcd}$$

Lemma 5.7. The following diagram is K-homotopy commutative for a K-space Z.

$$\begin{tikzcd} G \ltimes_H (H \ltimes_K Z) \arrow[r, "u"] \arrow[d, phantom, "\cong"] & S^{L(H)} \wedge (H \ltimes_K Z) \arrow[d, "1\wedge u"] \\ G \ltimes_K Z \arrow[r, "u"] & S^{L(K)} \wedge Z \cong S^{L(H)} \wedge S^{L(K,H)} \wedge Z \end{tikzcd}$$

Proof. Given a K-slice $f'\colon L(K,H) \to H$ and an H-slice $f\colon L(H) \to G$, let f_1 be the K-slice specified by the composite

$$L(K) \cong L(H) \times L(K,H) \xrightarrow{f \times f'} G \times H \xrightarrow{m} G,$$

where m is the product on G. (The fact that $f_1(-\lambda) \neq f_1(\lambda)^{-1}$ is immaterial.) Then f_1 is K-isotopic rel $\{0\}$ to the slice used to construct the bottom map u and, if we use f_1 instead, the diagram commutes by a trivial verification.

The following three lemmas will be used in the next section to construct the spectrum level map ψ from the space level map u and to prove that the composites $\psi\omega$ and $\omega\psi$ are homotopic to the respective identity maps.

Lemma 5.8. For a G-space X and H-space Y, the following diagram is commutative if H has finite index in G and is H-homotopy commutative in general.

$$\begin{tikzcd} G \ltimes_H (X \wedge Y) \arrow[r, "\zeta"] \arrow[d, "u"] & X \wedge (G \ltimes_H Y) \arrow[d, "1\wedge u"] \\ S^L \wedge X \wedge Y \arrow[r, phantom, "\cong"] & X \wedge S^L \wedge Y \end{tikzcd}$$

Proof. Both u and $(1\wedge u)\zeta$ map all points with G coordinate not in the neighborhood $\tilde{f}(L \times H)$ to the basepoint. For $\lambda \in L$, we have

$$(1\wedge u)\zeta(\tilde{f}(\lambda,h), x\wedge y) = f(\lambda)hx \wedge \lambda \wedge hy$$

and

$$u(\tilde{f}(\lambda,h), x\wedge y) = \lambda \wedge hx \wedge hy.$$

The H-contractibility of L implies that $f\colon L \to G$ is H-homotopic to the constant map at e, and the conclusion follows.

Lemma 5.9. The following composite is H-homotopic to the identity.

$$S^V \xrightarrow{t} G \ltimes_H S^W \xrightarrow{u} S^V.$$

Proof. The H-embedding $\tilde{f}\colon L \times H \to G$ of Construction 5.5 and the G-embedding $\tilde{j}\colon G \times_H W \to V$ of Construction 5.1 induce an embedding

$$k\colon V = L \times W = (L \times H) \times_H W \longrightarrow G \ltimes_H W \longrightarrow V.$$

The composite ut is k^{-1} on $k(V)$ and collapses the complement of $k(V)$ to the basepoint. Clearly k is H-isotopic to the identity, and we apply the Pontryagin-Thom construction to an isotopy to obtain the desired homotopy.

Lemma 5.10. For an H-space Y, the following diagram is H-homotopy commutative.

$$\begin{tikzcd}
\Sigma^V Y \arrow[r, "\Sigma^V\eta"] \arrow[d, "\Sigma^V\eta"'] & \Sigma^V(G \ltimes_H Y) = (G \ltimes_H Y)\wedge S^V \arrow[r, "1\wedge t"] & (G \ltimes_H Y)\wedge(G \ltimes_H S^W) \arrow[r, "\zeta^{-1}"] & G \ltimes_H \Sigma^W(G \ltimes_H Y) \arrow[d, "1 \ltimes \Sigma^W u"] \\
\Sigma^V(G \ltimes_H Y) & G \ltimes_H(Y\wedge S^V) \arrow[l, "\zeta"'] & G \ltimes_H(Y\wedge S^V) \arrow[l, "1 \ltimes (1\wedge\sigma\wedge 1)"'] \arrow[r, phantom, "\cong"] & G \ltimes_H \Sigma^W(S^L\wedge Y)
\end{tikzcd}$$

At the bottom, $S^V = S^L\wedge S^W$ and $\sigma: S^L \to S^L$ maps λ to $-\lambda$.

Proof. The composite around the right maps all points with V coordinate not in $\tilde{j}(f(L) \times W)$ to the basepoint. It maps the point $y\wedge\tilde{j}(f(\lambda)\wedge w)$ to $(f(\lambda),y)\wedge f(\lambda)(\lambda,w)$. Again, the H-contractibility of L implies that an H-homotopic map is obtained if $f(\lambda)$ is replaced by the identity element of G. Thus the composite around the right is H-homotopic to $\eta\wedge ut$ and the conclusion follows from the previous lemma.

The following observation reinterprets the "sign" σ appearing in the previous result.

Lemma 5.11. The map $\sigma\wedge 1: S^L\wedge S^L \to S^L\wedge S^L$ is H-homotopic to the transposition map.

Proof. Multiplication by the block matrices $\begin{pmatrix} -t & 1-t \\ 1-t & t \end{pmatrix}$ for $t \in I$ gives the homotopy.

§6. A generalization of Wirthmüller's isomorphism

Fix a G-universe U into which G/H (and any other orbits used) embeds. All spectra are to be indexed on U. Recall the notations of Construction 5.1 and take $V \subset U$ there. Note that our specification $S^V = S^L\wedge S^W$ forces $S^{-V} = S^{-W}\wedge S^{-L}$. Transposition of S^W and S^{-L} gives a natural H-isomorphism

$$\Sigma^W\Sigma^{-V}D = D\wedge S^{-V}\wedge S^W = D\wedge S^{-W}\wedge S^{-L}\wedge S^W \cong D\wedge S^{-L} = \Sigma^{-L}D.$$

Definition 6.1. For a G-spectrum E, define $\bar{t}$ to be the composite G-map

$$E \cong \Sigma^{-V}E\wedge S^V \xrightarrow{1\wedge t} \Sigma^{-V}E\wedge(G \ltimes_H S^W) \xrightarrow{\zeta^{-1}} G \ltimes_H \Sigma^W\Sigma^{-V}E \cong G \ltimes_H \Sigma^{-L}E.$$

For an H-spectrum D, define ω to be the composite G-map

$$F_H[G,\Sigma^L D) \xrightarrow{\bar{t}} G \ltimes_H \Sigma^{-L}F_H[G,\Sigma^L D) \xrightarrow{1 \ltimes \Sigma^{-L}\varepsilon} G \ltimes_H D.$$

The purpose of this section is to prove the following result.

Theorem 6.2. For H-spectra D, the map $\omega: F_H[G,\Sigma^L D) \to G \ltimes_H D$ is a natural equivalence of G-spectra.

Corollary 6.3. There are G-equivalences $F(G/H^+,S) \simeq G \ltimes_H S^{-L}$ and

$$\Sigma^\infty G/H^+ \simeq F_H[G,S^L) \simeq F(G \ltimes_H S^{-L},S).$$

Proof. Take $D = S^{-L}$ and use the isomorphism ϕ of lemma 4.8 for the first equivalence. Take $D = S^0$ for the second. The last is the case $D = S^{-L}$ and $E = S$ of the isomorphism κ of Lemma 4.9, with $F(S^{-L},S) \simeq S^L$.

Remark 6.4. As will be discussed in V§9, a Mackey functor is an additive contravariant functor from $\mathcal{O}G$ to the category of Abelian groups, where $\mathcal{O}G$ is the full subcategory of orbit spectra in $\bar{h}G\mathcal{S}$. This notion is the starting point for the construction of ordinary RO(G)-graded cohomology theories; see [88,90]. When G is finite, $\mathcal{O}G$ is self dual. In the case of general compact Lie groups G, the corollary suggests that one should study not $\mathcal{O}G$ but the self dual full subcategory of $\bar{h}G\mathcal{S}$ containing both the orbit spectra and their duals.

The theorem is most useful in its represented form.

Corollary 6.5. For G-spectra X and H-spectra D,

$$\omega_*: [X,\Sigma^L D]_H \cong [X,F_H[G,\Sigma^L D)]_G \longrightarrow [X,G \ltimes_H D]_G$$

is a natural isomorphism.

Replacing D by $Y\wedge E$ for a G-spectrum E and H-spectrum Y, using the isomorphism ζ to replace $G \ltimes_H(Y\wedge E)$ by $(G \ltimes_H Y)\wedge E$, and letting X run through the G-sphere spectra, we obtain the following homological consequence.

Corollary 6.6. For G-spectra E and H-spectra Y,

$$E^H_*(\Sigma^L Y) \cong E^G_*(G \ltimes_H Y).$$

Here E^H_* denotes the RO(G;U)-graded homology theory on H-spectra obtained by regarding E as an H-spectrum. Wirthmüller's original isomorphism is obtained by specializing to the case of suspension H-spectra Y.

We need some observations about the cylinder construction in order to define the inverse to ω. Consider the categorical functor L of I.2.2 and the cylinder functor $Z = LK$ of I.6.8 from H-prespectra to H-spectra. For H-prespectra D, I.6.8 gives a natural map $\pi: KD \to D$. When D is an inclusion H-prespectrum, for example an H-spectrum, $\tilde{\pi} = L\pi$ is a weak equivalence. When D is an H-CW spectrum, ZD has the homotopy type of an H-CW spectrum by I.8.12 and I.8.14 and $\tilde{\pi}$ is an H-homotopy equivalence. By inspection of I.6.8, we find easily that there are natural isomorphisms which make the following diagrams commute, where Y is an H-space.

$$\begin{tikzcd} K(G \ltimes_H D) \arrow[d, "\cong"'] \arrow[dr, "\pi"] & \\ G \ltimes_H KD \arrow[r, "1\times\pi"'] & G \ltimes_H D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} K(D\wedge Y) \arrow[d, "\cong"'] \arrow[dr, "\pi"] & \\ (KD)\wedge Y \arrow[r, "\pi\wedge 1"'] & D\wedge Y \end{tikzcd}$$

When D is an H-spectrum, $L(G \ltimes_H \ell D)$ and $L(\ell D\wedge Y)$ give the spectrum level functors $G \ltimes_H D$ and $D\wedge Y$, where ℓ is the forgetful functor from spectra to prespectra, and the diagrams imply the following conclusions.

Lemma 6.7. Let D be an H-CW spectrum and Y be an H-CW complex.
(i) $\tilde{\pi}: Z(G \ltimes_H D) \to G \ltimes_H D$ is an equivalence of G-spectra.
(ii) $\tilde{\pi}: Z(D\wedge Y) \to D\wedge Y$ is an equivalence of H-spectra.

Definition 6.8. Let D be an H-prespectrum indexed on U. For an indexing G-space W, Construction 5.5 gives an H-map

$$u: (G \ltimes_H D)(W) = G \ltimes_H DW \longrightarrow S^L\, DW \cong \Sigma^L DW.$$

If H has finite index in G, then Lemma 5.8 implies that the maps u specify a map of H-prespectra $G \ltimes_H D \to \Sigma^L D$ (and the details to follow are unnecessary). In general, Lemma 5.8 implies that the maps u specify a w-map, in the sense of I.6.2. Moreover, the homotopies are natural, so that $u: G \ltimes_H D \to \Sigma^L D$ specifies a preternatural transformation of functors on H-prespectra. Now let D be an H-spectrum. By I.7.8, we obtain a diagram of actual natural transformations of H-spectra

$$G \ltimes_H D \xleftarrow{\ \tilde{\pi}\ } Z(G \ltimes_H \ell D) \xrightarrow{\ Zu\ } Z(\Sigma^L \ell D) \xrightarrow{\ \tilde{\pi}\ } \Sigma^L D.$$

By Lemma 6.7, this diagram specifies a well-defined natural map $\mu: G \ltimes_H D \to \Sigma^L D$ in the stable category of H-spectra. Let

$$\psi: G \ltimes_H D \longrightarrow F_H[G, \Sigma^L D)$$

be the G-map such that $\varepsilon\psi = \mu$ as an H-map.

In the two most important special cases, μ can be written directly in terms of the space level maps u.

Lemma 6.9. For H-spaces Y, the following diagram of H-spectra commutes.

$$\begin{tikzcd} \Sigma^\infty(G \ltimes_H Y) \arrow[r, "\cong"] \arrow[d, "\Sigma^\infty u"'] & G \ltimes_H \Sigma^\infty Y \arrow[d, "\mu"] \\ \Sigma^\infty(S^L \wedge Y) \arrow[r, "\cong"] & \Sigma^L \Sigma^\infty Y \end{tikzcd}$$

Note that the bottom isomorphism involves transposition of S^L and Y.

Lemma 6.10. For G-spectra E, the following diagram of H-spectra commutes.

$$\begin{tikzcd} G/H^+ \wedge E \arrow[r, "\cong"] \arrow[d, "u\wedge 1"'] & G \ltimes_H E \arrow[d, "\mu"] \\ S^L \wedge E \arrow[r, "\cong"] & \Sigma^L E \end{tikzcd}$$

These are both easy verifications from the definitions and naturality diagrams. Similarly, we have the spectrum level analog of Lemma 5.8.

Lemma 6.11. For H-spectra D and G-spectra E, the following diagram of H-spectra commutes.

$$\begin{tikzcd} G \ltimes_H (E\wedge D) \arrow[r, "\zeta"] \arrow[d, "\mu"'] & E\wedge(G \ltimes_H D) \arrow[d, "1\wedge\mu"] \\ \Sigma^L(E\wedge D) \arrow[r, equal] & E\wedge\Sigma^L D \end{tikzcd}$$

We insert the unit and transitivity properties of ω and μ implied by Lemmas 5.2, 5.4, 5.6, 5.7 and diagram chasing. Note that $e: S^0 \to S^L$ induces a map $e: S^{-L} \to S^{-L}\wedge S^L = S$.

Lemma 6.12. The following diagrams of H-spectra commute.

$$\begin{tikzcd} \Sigma^{-L}F_H[G,\Sigma^L D) \arrow[r, "\Sigma^{-L}\varepsilon"] \arrow[d, "1\wedge e"'] & \Sigma^{-L}\Sigma^L D = D \arrow[d, "\eta"] \\ F_H[G,\Sigma^L D) \arrow[r, "\omega"] & G \ltimes_H D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & D \arrow[dl, "\eta"'] \arrow[dr, "1\wedge e"] & \\ G \ltimes_H D \arrow[rr, "\mu"] & & \Sigma^L D \end{tikzcd}$$

Lemma 6.13. Let $K \subset H \subset G$ and let C be a K-spectrum.

(a) The following diagram of G-spectra commutes.

$$\begin{tikzcd} F_H[G,\Sigma^{L(H)}F_K[H,\Sigma^{L(K,H)}C]] \arrow[r, phantom, "\cong"] \arrow[d, "{F[1,\Sigma^{L(H)}\omega]}"'] & F_K[G,\Sigma^{L(K)}C] \arrow[d, "\omega"] \\ F_H[G,\Sigma^{L(H)}(H \ltimes_K C)] \arrow[r, "\omega"] & G \ltimes_H H \ltimes_K C \cong G \ltimes_K C \end{tikzcd}$$

(b) The following diagram of K-spectra commutes.

$$\begin{tikzcd} G \ltimes_K C \arrow[r, phantom, "\cong"] \arrow[d, "\mu"'] & G \ltimes_H H \ltimes_K C \arrow[d, "\mu"] \\ \Sigma^{L(K)}C & \Sigma^{L(H)}H \ltimes_K C \arrow[l, "\Sigma^{L(H)}\mu"'] \end{tikzcd}$$

We begin the proof of Theorem 6.2 with the following reduction.

Lemma 6.14. If $\omega: F_K[H,S^{L(K,H)}) \to H \ltimes_K S$ is an H-equivalence for all $K \subset H \subset G$, then $\omega: F_H[G,\Sigma^{L(H)}D) \to G \ltimes_H D$ is a G-equivalence for all H-CW spectra D.

Proof. With $C = S$, the vertical arrows of the diagram of Lemma 6.13 (a) are assumed to be equivalences. We conclude from the diagram that ω is a G-equivalence when D is $H \ltimes_K S$ or any of its suspensions by G-representations (since the suspension coordinate shuffles through all of the constructions). Now consider the homomorphism

$$\omega_*: [X,\Sigma^L D]_H \cong [X,F_H[G,\Sigma^L D)]_G \longrightarrow [X,G \ltimes_H D]_G$$

for a G-spectrum X. By induction over cells from the case $\Sigma^n\Sigma^\infty H/K^+$, this is an isomorphism when D is finite. By passage to colimits, it is thus an isomorphism for any D when X is a finite G-CW spectrum. Letting X run through the G-sphere spectra, we conclude that ω is a (weak) G-equivalence for any D.

Proof of Theorem 6.2. We shall prove more precisely that ω and ψ are inverse G-equivalences between $G \ltimes_H D$ and $F_H[G,\Sigma^L D)$. We first show that $\psi\omega$ is G-homotopic to the identity map of $F_H[G,\Sigma^L D)$, and it suffices to show that $\varepsilon\psi\omega = \mu\omega$ is H-homotopic to ε. Now $\omega = (1 \ltimes \Sigma^{-L}\varepsilon)\bar{t}$ and an easy chase of naturality diagrams shows that

$$\mu(1 \ltimes \Sigma^{-L}\varepsilon) = \varepsilon\mu: G \ltimes_H \Sigma^{-L}F_H[G,\Sigma^L D) \longrightarrow \Sigma^L D.$$

Thus $\mu\omega = \varepsilon\mu\bar{t}$. A diagram chase from Lemmas 5.9, 6.9, and 6.11 shows that the composite

$$E \xrightarrow{\bar{t}} G \ltimes_H \Sigma^{-L}E \xrightarrow{\mu} E$$

is H-homotopic to the identity for any G-spectrum E. The crucial point is that $ut \simeq 1$, but sign watchers will see that the transposition of S^W and S^{-L} in the definition of $\bar{t}$ is neutralized by the transposition of S^L and S^W coming from our application of Lemma 6.9 to $Y = S^W$.

To show that $\omega\psi$ is G-homotopic to the identity map of $G \ltimes_H D$, it suffices to show that $\omega\psi\eta$ is H-homotopic to η. Note first that

$$\omega\psi = (1 \ltimes \Sigma^{-L}\varepsilon)\bar{t}\psi = (1 \ltimes \Sigma^{-L}\varepsilon)(1 \ltimes \Sigma^{-L}\psi)\bar{t} = (1 \ltimes \Sigma^{-L}\mu)\bar{t}$$

by the naturality of $\bar{t}: E \to G \ltimes_H \Sigma^{-L}E$ and the relation $\mu = \varepsilon\psi$. Note next that the following commutative diagram re-expresses the V^{th} suspension of this composite; we abbreviate $E = G \ltimes_H D$ for legibility.

$$\begin{tikzcd} \Sigma^V E \arrow[r, "\Sigma^V\bar{t}"] \arrow[d, equal] & \Sigma^V(G \ltimes_H \Sigma^{-L}E) \arrow[r, "\Sigma^V(1\ltimes\Sigma^{-L}\mu)"] & \Sigma^V(G \ltimes_H D) \\ E \wedge S^V \arrow[d, "1\wedge t"'] & G \ltimes_H \Sigma^V\Sigma^{-L}E \arrow[u, "\zeta"] \arrow[r, "1\ltimes\Sigma^V\Sigma^{-L}\mu"] & G \ltimes_H \Sigma^V D \arrow[u, "\zeta"] \\ E \wedge (G \ltimes_H S^W) \arrow[r, "\zeta^{-1}"] & G \ltimes_H \Sigma^W E \arrow[u, "1\ltimes\sigma'"] \arrow[r, "1\ltimes\Sigma^W\mu"] & G \ltimes_H \Sigma^V D \arrow[u, "1\ltimes(1\wedge\sigma\wedge 1)"'] \end{tikzcd}$$

Here $\sigma': E \wedge S^W = E \wedge S^L \wedge S^{-L} \wedge S^W \to E \wedge S^{-L} \wedge S^L \wedge S^W = E \wedge S^{-L} \wedge S^V$ is the transposition and $1 \wedge \sigma \wedge 1$ on $D \wedge S^V = D \wedge S^L \wedge S^W$ is given by the sign map $\sigma(\lambda) = -\lambda$. The bottom right square commutes by Lemma 5.11 and the top right square commutes by naturality. The left part of the diagram is seen to commute by writing out $\bar{t}$, using naturality diagrams, and checking carefully which suspension coordinates get permuted. By Lemmas 5.10 and 6.9 and the diagram, we conclude that $\omega\psi\eta$ is H-homotopic to η when $D = \Sigma^\infty Y$. Thus, for such D, ω and ψ are inverse G-equivalences. By Lemma 6.14 and the case $Y = S^0$, we conclude that ω is a G-equivalence for any D. Since $\omega\psi \simeq 1$ for any D, it follows that ψ and ω are inverse G-equivalences for any D.

We conclude this section with a few technical naturality properties of ω. These played a role in our work with McClure [89] comparing various forms of the Segal conjecture and will be needed in Chapter V. We need a definition, which is in fact a first instance of basic notions to be studied in detail later.

Definition 6.15. The transfer associated to the projection $G/H \to pt$ is the composite

$$\tau\colon S = \Sigma^{-V}S^V \xrightarrow{\Sigma^{-V}t} \Sigma^{-V}(G \ltimes_H S^W) \xrightarrow{\Sigma^{-V}(1\ltimes e)} \Sigma^{-V}(G \ltimes_H S^V) \cong \Sigma^\infty G/H^+.$$

The equivariant Euler characteristic $\chi(G/H)$ is the composite $\xi\tau\colon S \to S$, $\xi\colon \Sigma^\infty G/H^+ \to S$; it is to be regarded as an element of $\pi_0^G(S)$.

The name Euler characteristic is justified since a theorem of Hopf (to be given an equivariant generalization in III§7) implies that the composite

$$S^V \xrightarrow{t} G \ltimes_H S^W \xrightarrow{1\ltimes e} G \ltimes_H S^V \xrightarrow{\xi} S^V$$

has nonequivariant degree the classical Euler characteristic $\chi(G/H)$; compare Becker and Gottlieb [10,2.4].

The next result shows how ω relates to the G action ξ and G coaction ν of a G-spectrum E.

Lemma 6.16. The following diagram of G-spectra commutes.

$$\begin{tikzcd}
F_H[G,E] \arrow[rr, "{F[1,e]}"] & & F_H[G,\Sigma^L E] \arrow[r, "\omega"] & G \ltimes_H E \arrow[dl, "\zeta"'] \arrow[dd, "\xi"] \\
 & S \wedge E \arrow[r, "\tau\wedge 1"] & G/H^+ \wedge E \arrow[dr, "\xi"] & \\
E \arrow[uu, "\nu"] \arrow[ur, equal] \arrow[rrr, "\chi(G/H)"] & & & E
\end{tikzcd}$$

Proof. The bottom part commutes by the definition of $\chi(G/H)$ and the triangle commutes by Lemma 4.8. Using $\varepsilon\nu = 1$ and naturality diagrams, we find easily that the commutativity of the top part reduces to the commutativity of the following diagram, which can be checked by a careful inspection of definitions.

$$\begin{tikzcd}
E \arrow[rr, "\bar{t}"] \arrow[d, equal] & & G \ltimes_H \Sigma^{-L}E \arrow[d, "1\ltimes \Sigma^{-L}e"] \\
S \wedge E \arrow[r, "\tau\wedge 1"] & G/H^+ \wedge E & G \ltimes_H E \arrow[l, "\zeta"']
\end{tikzcd}$$

On the level of represented functors,

$$\nu_*\colon [X,E]_G \longrightarrow [X,F_H[G,E)]_G \cong [X,E]_H$$

is the evident forgetful homomorphism, and we may reformulate the previous result as follows.

Corollary 6.17. The following diagram commutes for G-spectra X and E.

$$\begin{tikzcd}
{[X,E]_H} \arrow[rr, "e_*"] & & {[X,\Sigma^L E]_H} \arrow[r, "\omega_*"] & {[X,G \ltimes_H E]_G} \arrow[dl, "\zeta"'] \arrow[dd, "\xi_*"] \\
 & {[X,S\wedge E]_G} \arrow[r, "(\tau\wedge 1)_*"] & {[X,G/H^+\wedge E]_G} \arrow[dr, "\xi_*"] & \\
{[X,E]_G} \arrow[uu, "\nu_*"] \arrow[ur, equal] \arrow[rrr, "\chi(G/H)_*"] & & & {[X,E]_G}
\end{tikzcd}$$

Similarly, it is worth recording the represented form of the transitivity diagram of Lemma 6.13 (a).

Corollary 6.18. The following diagram commutes for G-spectra X and K-spectra C.

$$\begin{tikzcd}
{[X,\Sigma^{L(K)}C]_K} \arrow[r, "\omega_*"] \arrow[d, "(\Sigma^{L(H)}\omega)_*"'] & {[X,G \ltimes_K C]_G} \arrow[d, "\cong" sloped, phantom] \\
{[X,\Sigma^{L(H)}H \ltimes_K C]_H} \arrow[r, "\omega_*"] & {[X,G \ltimes_H H \ltimes_K C]_G}
\end{tikzcd}$$

Remark 6.19. For H-spectra D, the composite

$$G \ltimes_K D \cong G \ltimes_H H \ltimes_K D \xrightarrow{1\ltimes\xi} G \ltimes_H D$$

is the natural projection of G-spectra. The naturality diagram relating ω and $1 \ltimes \xi$ can be used in conjunction with Corollaries 6.17 and 6.18 to obtain the following commutative diagram. (Here e refers to the K-map $S^0 \to S^{L(K,H)}$.)

$$\begin{tikzcd}
{[X,G \ltimes_K D]_G} \arrow[rr, equal] & & {[X,G \ltimes_H H \ltimes_K D]_G} \arrow[r, "(1\ltimes\xi)_*"] & {[X,G \ltimes_H D]_G} \\
{[X,\Sigma^{L(K)}D]_K} \arrow[u, "\omega_*"] & {[X,\Sigma^{L(H)}D]_K} \arrow[l, "(\Sigma^{L(H)}e)_*"'] & {[X,\Sigma^{L(H)}D]_H} \arrow[l, "\nu_*"'] \arrow[r, "\chi(H/K)_*"] & {[X,\Sigma^{L(H)}D]_H} \arrow[u, "\omega_*"']
\end{tikzcd}$$

Remark 6.20. For K-spectra C, the composite

$$H \ltimes_K C \xrightarrow{\eta} G \ltimes_H H \ltimes_K C \cong G \ltimes_K C$$

is the natural inclusion of H-spectra, and the unit diagram for $H \ltimes_K C$ can be used in conjunction with the transitivity diagram of Lemma 6.13 (b) and a naturality diagram to obtain the following commutative diagram of K-spectra. (Here e refers to the H-map $S^0 \to S^{L(H)}$.)

$$\begin{tikzcd} H \ltimes_K C \arrow[r, "\eta"] \arrow[d, "\mu"] & G \ltimes_H H \ltimes_K C \cong G \ltimes_K C \arrow[d, "\mu"] \\ \Sigma^{L(K,H)} C \arrow[r, "e"] & \Sigma^{L(K)} C \end{tikzcd}$$

§7. A generalization of Adams' isomorphism

In parallel with the description just given of the relation between the functors $G \ltimes_H (?)$ and $F_H[G,?)$ on H-spectra for $H \subset G$, we here discuss the relation between the functors $(?)/N$ and $(?)^N$ on G-spectra, where N is a normal subgroup of G with quotient group $J = G/N$. This connection is less general in that it applies only to N-free G-spectra and is more complicated in that it involves changes of indexing universe. We fix a complete G-universe U and let $i: U^N \to U$ be the inclusion of its N-fixed point subuniverse. We may regard U^N as a J-universe and, as such, it is clearly complete.

The group G acts on N by conjugation, fixing $e \in N$. Thus the tangent space of N at e is a G-representation. We denote it A, or $A(N)$ or $A(N,G)$ when necessary for clarity, and call it the adjoint representation of G derived from N.

For a J-spectrum E indexed on U^N, let

$$\varepsilon^{\#}E = i_*\varepsilon^*E, \quad \varepsilon: G \longrightarrow J.$$

The functor $\varepsilon^{\#}: J\mathscr{S}U^N \to G\mathscr{S}U$ is left adjoint to the N-fixed point functor $G\mathscr{S}U \to J\mathscr{S}U^N$ (by Propositions 1.2 and 4.4; compare Remarks 3.14). For an N-free G-spectrum D indexed on U^N, we shall construct a map

$$\tau: \varepsilon^{\#}(D/N) \longrightarrow \Sigma^{-A} i_* D$$

of G-spectra indexed on U, and the purpose of this section is to prove the following result.

Theorem 7.1. For N-free G-spectra D, the adjoint

$$\tilde{\tau}: D/N \longrightarrow (\Sigma^{-A} i_* D)^N$$

of τ is a natural equivalence of J-spectra indexed on U^N.

Corollary 7.2. If $D \in G\mathscr{S}U^N$ is N-free and $E \in J\mathscr{S}U^N$, then

$$\tilde{\tau}_*: [E, D/N]_J \longrightarrow [E, (\Sigma^{-A} i_* D)^N]_J \cong [\varepsilon^{\#}E, \Sigma^{-A} i_* D]_G$$

is a natural isomorphism. In particular, if X is an N-free G-CW complex and Y is any J-CW complex, then

$$[\Sigma^\infty Y, \Sigma^\infty X/N]_J \cong [\Sigma^\infty Y, \Sigma^{-A}\Sigma^\infty X]_G,$$

where the left side is computed in the universe U^N and the right side is computed in the universe U.

When G is a finite group and X and Y are finite complexes, the last statement is due to Adams [3,§5], and his work motivated our work in this section.

Remarks 7.3 (i) If N is not Abelian, its conjugation action on itself is non-trivial. If N is also not finite, then N acts non-trivially on A. In this case, the desuspension functor Σ^{-A} is only defined on $\bar{h}G\mathscr{S}U$, not on $\bar{h}G\mathscr{S}U^N$.
(ii) When $A = 0$, one might naively hope to replace $(i_*D)^N$ by D^N in Theorem 7.1, but this fails hopelessly. For example, if $G = N$ is finite and $D = \Sigma^\infty G^+ \in G\mathscr{S}U^G$, then $D^G = *$, whereas the theorem gives that $(i_*D)^G = S$. This fact helps explain our care in defining N-free G-spectra indexed on non-trivial universes. They are formed from cells involving only N-free orbits G/H and, as we have just observed, it does not follow that their N-fixed point spectra are trivial.

The remainder of this section is devoted to the construction of τ and the proof of Theorem 7.1. In fact, τ is an example of the transfers to be defined in chapter IV. We think of the projection $i_*D \to i_*(D/N)$ as a kind of stable bundle. When D is $\Sigma^\infty Y^+$ for an unbased N-free G-space Y, this map is the stabilization of the "equivariant bundle" $Y \to Y/N$, and we begin by explaining the appropriate way to think about group actions in this space level situation.

In our approach to the usual stable transfer map, it is essential that a bundle $Y \to B$ with fibre F and structural group Π has an associated principal Π-bundle X. We construct the transfer by first constructing a stable pretransfer Π-map $S \to \Sigma^\infty F^+$ in a complete Π-universe, next smashing with X^+, then using the change of universe isomorphism of Theorem 2.8 to pull back to a Π-trivial universe, and finally passing to orbits over Π. The use of change of universe is vital since we cannot pass to orbits when working in a non-trivial universe. The essential observations are just that X is Π-free and that $Y = X \times_\Pi F$ and $B = X \times_\Pi *$.

For our N-free G-space Y, it would seem that the fibre and structure group are both N and the bundle is principal. However, there is no obvious way to give $Y \times_N N$ a G-action making it G-homeomorphic to Y. This illustrates the problem, to be discussed in detail in IV§1, of deciding just what one should mean by an "equivariant bundle". In the present situation, we can resolve the difficulty by working not with G but with the semidirect product $\Gamma = G \times_c N$, where $c: G \to \mathrm{Aut}(N)$ is the conjugation action of G on N. Thus, in Γ,

$$(h,m)(g,n) = (hg, g^{-1}mgn) \quad \text{for } h,g \in G \text{ and } m,n \in N.$$

We agree to write Π for N regarded as the normal subgroup $e \times_c N$ of Γ, so that $G = \Gamma/\Pi$. We have both the obvious quotient homomorphism $\varepsilon: \Gamma \to G$ specified by $\varepsilon(g,n) = g$ and the twisted quotient homomorphism $\theta: \Gamma \to G$ specified by $\theta(g,n) = gn$. The latter restricts on Π to the identity homomorphism of N. Let θ^*Y denote Y regarded as a Γ-space via θ and note that Y is Π-free. We embed G as the subgroup $G \times_c e$ of Γ and observe that both ε and θ restrict to the identity map $G \to G$. Let Γ act on N by

$$(g,n)m = gnmg^{-1} \quad \text{for } g \in G \text{ and } m,n \in N.$$

Then N is Γ-homeomorphic to the orbit Γ/G via the map sending n to the coset of (e,n). Observe that we have the composite G-homeomorphism

$$Y \cong (\Gamma \times_G Y)/\Pi \cong (\theta^*Y \times \Gamma/G)/\Pi \cong (\theta^*Y \times N)/\Pi.$$

(This would also hold for ε^*Y, but we want the free Π-action.) Thus, when thinking of $Y \to Y/N$ as an equivariant bundle, we regard the Γ-space θ^*Y as the total space of its associated principal bundle, the Γ-space $N \cong \Gamma/G$ as its fibre, and the subgroup Π of Γ as its structural group.

We must generalize this description of Y as $(\theta^*Y \times N)/\Pi$ to the spectrum level. For $D \in G\mathcal{S}U^N$, we have $i_*\theta^*D \in \Gamma\mathcal{S}U$, where U is regarded as a Γ-universe via $\varepsilon: \Gamma \to G$ (and not θ); since $N \subset G$ acts trivially on U^N, the Γ-actions on U^N via ε and θ agree. It is easy to see that $i_*\theta^*D$ is Π-free when D is N-free.

<u>Lemma 7.4</u>. Let D be an N-free G-spectrum indexed on U^N. Then there are natural isomorphisms

$$(i_*\theta^*D \wedge N^+)/\Pi \cong i_*D \quad \text{and} \quad (i_*\theta^*D)/\Pi \cong i_*(D/N)$$

of G-spectra indexed on U.

<u>Proof</u>. Since $N \cong \Gamma/G$, Lemma 4.8 gives that $E \wedge N^+$ is Γ-isomorphic to $\Gamma \ltimes_G E$ for $E \in \Gamma\mathcal{S}U$. Since the composite $G \subset \Gamma \xrightarrow{\varepsilon} G$ is the identity, Lemma 4.10 gives that $(\Gamma \ltimes_G E)/\Pi \cong E$ for $E \in G\mathcal{S}U$. Since $i_*\theta^*D = i_*D$ when regarded as a G-spectrum, because the composite $G \subset \Gamma \xrightarrow{\theta} G$ is the identity, this proves the first isomorphism. For the second, observe that $(i_*\theta^*D)/\Pi$ is isomorphic to $i_*((\theta^*D)/\Pi)$ and that $(\theta^*D)/\Pi$ is isomorphic to D/N (by Lemma 4.14 and the last statement of Proposition 4.4).

We now understand how to think of $i_*D \to i_*(D/N)$ as a stable bundle. We next need a stable Γ-map $t: S \to \Sigma^{-A}\Sigma^\infty N^+$. For this, we must work in a complete Γ-universe U'. The geometric source of the "dimension-shifting pretransfer" t is Construction 5.1 applied to the orbit $\Gamma/G \cong N$. The tangent bundle there is just $N \times A$ in this case. To see this, let Γ act on A via $\varepsilon: \Gamma \to G$. We obtain a Γ-trivialization of the tangent bundle of Γ/G by sending an element (n,a) of $N \times A$ to $d(n)(a)$, where $d(n)$ is the differential at e of left translation by n. Embedding Γ/G in a sufficiently large Γ-representation V and taking $W = V - A$, so that W too is a Γ-representation, we see that Construction 5.1 gives a Γ-map

$$t: S^V \longrightarrow \Gamma/G^+ \wedge S^{V-A}.$$

Applying Σ^∞ and desuspending by V, we obtain the desired map

$$t: S \longrightarrow \Sigma^{-A}\Sigma^\infty N^+$$

of Γ-spectra indexed on U'.

We must still mix our stable bundle with our stable pretransfer. For this purpose, we observe that we may take our complete G-universe U to be $(U')^\Pi$. Let $j: U \to U'$ be the resulting inclusion of Γ-universes.

<u>Construction 7.5</u>. Let D be an N-free G-spectrum indexed on U^N and observe that $j_*i_*\theta^*D$ is a Π-free Γ-spectrum indexed on U'. By Theorem 2.8, the map

$$1 \wedge t: j_*i_*\theta^*D \cong j_*i_*\theta^*D \wedge S \longrightarrow j_*i_*\theta^*D \wedge \Sigma^{-A}\Sigma^\infty N^+ \cong j_*\Sigma^{-A}(i_*\theta^*D \wedge N^+)$$

of Π-free Γ-spectra indexed on U' is represented by a map

$$\hat{\tau}: i_*\theta^*D \longrightarrow \Sigma^{-A}(i_*\theta^*D \wedge N^+)$$

of Π-free Γ-spectra indexed on $U = (U')^{\Pi}$. On passage to orbits over Π and use of the identifications of Lemma 7.4, there results a "dimension-shifting transfer" G-map

$$\tau\colon i_*(D/N) \longrightarrow \Sigma^{-A} i_* D.$$

Here we have used that Σ^{-A} commutes with both j_* and passage to orbits over Π since Π acts trivially on A.

<u>Proof of Theorem 7.1</u>. Since $\tilde{\tau}\colon D/N \to (\Sigma^{-A} i_* D)^N$ is natural in D and since it suffices to prove that $\tilde{\tau}$ induces isomorphisms on homotopy groups, it is clear (from I.5.3) that D may be assumed to be finite. Since all functors in sight either preserve cofibre sequences or convert them to fibre sequences, it suffices inductively to assume that $D = G/H^+ \wedge S^n$, where $n \in Z$ and $H \cap N = e$. Since suspensions by trivial representations commute with all functors in sight, it suffices to take $n = 0$.

To handle the case $D = \Sigma^\infty G/H^+$, we require a more explicit description of the transfer G-map

$$\tau\colon (\Sigma^\infty G/H^+)/N \longrightarrow \Sigma^{-A}\Sigma^\infty G/H^+.$$

Since N is normal in G, $HN = \{hn \mid h \in H \text{ and } n \in N\}$ is a subgroup of G, and it is clearly a closed subgroup. Since $H \cap N = e$, each element of HN has a unique expression as a product hn, and of course $(G/H)/N \cong G/HN$.

<u>Lemma 7.6</u>. The map $\tau\colon \Sigma^\infty G/HN^+ \to \Sigma^{-A}\Sigma^\infty G/H^+$ in $\bar{h}G\mathscr{S}U$ can be identified with the map

$$1 \ltimes t\colon G \ltimes_{HN} S \longrightarrow G \ltimes_{HN} (HN \ltimes_H S^{-A}),$$

where $t\colon S \to HN \ltimes_H S^{-A}$ is obtained by applying Construction 5.1 to HN/H.

<u>Proof</u>. The tangent H-space of HN/H at eH is just A regarded as a representation of H, so the statement makes sense. Define a monomorphism $\alpha\colon HN \to \Gamma$ by $\alpha(hn) = (hn, n^{-1})$. Clearly $\theta\colon \Gamma \to G$ induces a HN-isomorphism $HN/H \to \alpha^*\Gamma/G$. By Lemma 4.8, we have isomorphisms

$$\Sigma^\infty\theta^* G/H^+ \cong \Gamma/\alpha(HN)^+ \wedge S \cong \Gamma \ltimes_\alpha S$$

and, since $N \cong \Gamma/G$ and $\Sigma^{-A}\Sigma^\infty HN/H^+ \cong HN \ltimes_H S^{-A}$,

$$\Sigma^\infty\theta^* G/H^+ \wedge \Sigma^{-A}\Sigma^\infty N^+ \cong \Gamma/\alpha(HN)^+ \wedge \Sigma^{-A}\Sigma^\infty\Gamma/G^+ \cong \Gamma \ltimes_\alpha (HN \ltimes_H S^{-A}).$$

Under these identifications, the map

$$1 \wedge t\colon \Sigma^\infty\theta^* G/H^+ \wedge S \longrightarrow \Sigma^\infty\theta^* G/H^+ \wedge \Sigma^{-A}\Sigma^\infty N^+$$

in $\bar{h}\Gamma\mathscr{S}U'$ can be identified with the map

$$1 \ltimes t\colon \Gamma \ltimes_\alpha S \longrightarrow \Gamma \ltimes_\alpha (HN \ltimes_H S^{-A}),$$

where $t\colon S \to HN \ltimes_H S^{-A}$ is the map in $\bar{h}HN\mathscr{S}U'$ obtained by applying Construction 5.1 to the group HN and orbit HN/H (rather than to the group Γ and orbit Γ/G). Since $\alpha(HN) \cap \Pi = e$, U and $\alpha^* U'$ are equivalent HN-universes. It follows that $j_* t = t$, where the left map t is that of the statement of the lemma. By Lemma 4.14, application of j_* to the map

$$\hat{\tau} = 1 \ltimes t\colon \Gamma \ltimes_\alpha S \longrightarrow \Gamma \ltimes_\alpha (HN \ltimes_H S^{-A})$$

in $\bar{h}\Gamma\mathscr{S}U$ gives the corresponding map $1 \ltimes t$ in $\bar{h}\Gamma\mathscr{S}U'$. According to Construction 7.5, τ is obtained from $\hat{\tau}$ by passage to orbits over Π. Since the composite $HN \xrightarrow{\alpha} \Gamma \xrightarrow{\varepsilon} G$ is just the inclusion of HN in G, Lemma 4.10 gives that $(\Gamma \ltimes_\alpha E)/\Pi \cong G \ltimes_{HN} E$ for $E \in HN\mathscr{S}U$. The conclusion follows.

If we view τ as a transfer in the sense of chapter IV, then the identification of the lemma becomes an application of Axiom 6 of IV§4 to the homomorphism of pairs $\alpha\colon (HN, e) \to (\Gamma, \Pi)$.

It is now an easy matter to prove that $\tilde{\tau}$ is an equivalence when $D = \Sigma^\infty G/H^+$, $H \cap N = e$, by using the Wirthmüller isomorphism of Theorem 6.2. We may view A as the tangent H-representation of HN/H at the coset eH, and Definition 6.8 gives an HN-equivalence

$$\psi\colon HN \ltimes_H S^{-A} \longrightarrow F_H[HN, S).$$

If $\varepsilon\colon F_H[HN,S) \to S$ is the evaluation H-map, then $\varepsilon\psi = \mu\colon HN \ltimes_H S^{-A} \to S$ is the stable H-map derived from the map u of Construction 5.5. Lemma 5.9 implies that μt is the identity H-map $S \to S$ (where t is as in Lemma 7.6). Thus $\varepsilon\psi t = 1$ and ψt is the coaction HN-map $\nu\colon S \to F_H[HN,S)$. Clearly $\tilde{\tau}$ will be a J-equivalence if the adjoint of the composite

$$G \ltimes_{HN} S \xrightarrow{1 \ltimes t} G \ltimes_{HN} (HN \ltimes_H S^{-A}) \xrightarrow{1 \ltimes \psi} G \ltimes_{HN} F_H[HN, S)$$

is a J-equivalence, and this composite is just $1 \ltimes \nu$.

Let L be the tangent HN-space of G/HN at the coset eHN and let

$K = HN/N \subset J$. Then K is a copy of H and, since $G/HN \cong J/K$, L is the pullback of the tangent K-space of J/K at eK along the quotient homomorphism $HN \to K$. Definition 6.8 gives the G-equivalences ψ in the following commutative diagram.

$$
\begin{tikzcd}
G \ltimes_{HN} S \arrow[r, "1 \ltimes \nu"] \arrow[d, "\psi"] & G \ltimes_{HN} F_H[HN,S] \arrow[d, "\psi"] \\
F_{HN}[G,S^L] \arrow[r, "{F[1,\Sigma^L \nu]}"] \arrow[dr, "\nu"'] & F_{HN}[G,\Sigma^L F_H[HN,S]] \arrow[d, "\cong"] \\
 & F_H[G,S^L]
\end{tikzcd}
$$

The unlabeled isomorphism is given by Lemma 4.10, and the bottom map ν is the G-map characterized by the requirement that $\varepsilon\nu = \varepsilon\colon F_{HN}[G,S^L) \to S^L$ as an H-map.

It suffices to show that the adjoint J-map of $\nu\psi$ is an equivalence. The domain of this adjoint is $\Sigma^\infty J/K^+$ since

$$G \ltimes_{HN} S \cong \Sigma^\infty G/HN^+ \cong i_*\Sigma^\infty G/HN^+ \cong \varepsilon^\# \Sigma^\infty J/K^+.$$

Noting that $i_* S^L = S^L$ in $H\mathcal{S}U$, we see that its target is

$$F_H[G,S^L)^N \cong (i^* F_H[G,i_*S^L))^N \cong F_H[G,i^*i_*S^L)^N \cong F_K[J,i^*i_*S^L).$$

Here Lemma 4.14 gives the middle isomorphism, and Lemma 4.10 gives the last isomorphism since the composite $H \subset G \to J$ can be identified with the inclusion $K \subset J$. With these identifications, we find easily that the adjoint of $\nu\psi$ is the composite

$$\Sigma^\infty J/K^+ \xrightarrow{\ \psi\ } F_K[J,S^L) \xrightarrow{F[1,\eta]} F_K[J,i^*i_*S^L),$$

where ψ is the J-equivalence of Definition 6.8 and $\eta\colon S^L \to i^*i_*S^L$ is the unit of the (i_*,i^*)-adjunction. Since $H \cap N = e$, U^N and U are isomorphic as H (or K) universes, hence η is a K-equivalence by Corollary 1.8. This completes the proof of Theorem 7.1.

§8. Coherent families of equivariant spectra

To begin with, adopt the notations of the previous section. Thus N is a normal subgroup of G, $\varepsilon\colon G \to J = G/N$ is the quotient homomorphism, A is the adjoint representation of G derived from N, U is a complete G-universe, and $i\colon U^N \to U$ is the inclusion. The functor

$$\varepsilon^\# = i_*\varepsilon^*\colon J\mathcal{S}U^N \longrightarrow G\mathcal{S}U$$

is left adjoint to the N-fixed point functor. Recall that $\mathcal{F}(N)$ is the family of subgroups H of G such that $H \cap N = e$ and that the N-free G-spectra are those G-spectra (weakly) equivalent to $\mathcal{F}(N)$-CW spectra. Up to equivalence, any N-free G-spectrum in $G\mathcal{S}U$ is of the form i_*D for some N-free G-spectrum $D \in G\mathcal{S}U^N$, by Theorem 2.8.

<u>Theorem 8.1</u>. Let $E_J \in J\mathcal{S}U^N$ and $E_G \in G\mathcal{S}U$ and assume given an $\mathcal{F}(N)$-equivalence $\xi\colon \varepsilon^\# E_J \to E_G$. Then, for any N-free G-spectrum $D \in G\mathcal{S}U^N$,

$$E_J^*(D/N) \cong E_G^*(i_*D) \quad\text{and}\quad E^J_*(D/N) \cong E^G_*(\Sigma^{-A} i_*D).$$

In particular, for any N-free G-space X,

$$\tilde{E}_J^*(X/N) \cong \tilde{E}_G^*(X) \quad\text{and}\quad \tilde{E}^J_*(X/N) \cong \tilde{E}^G_*(\Sigma^{-A}X),$$

where the left sides are computed in the J-universe U^N and the right sides are computed in the G-universe U.

<u>Proof</u>. The cohomology statement is immediate from Theorems 2.2 and 4.5. For the homology statement, Theorem 2.2 implies that

$$1 \wedge \xi\colon \Sigma^{-A} i_*(D \wedge \varepsilon^* E_J) \cong (\Sigma^{-A} i_* D) \wedge \varepsilon^\# E_J \longrightarrow (\Sigma^{-A} i_* D) \wedge E_G$$

is a G-equivalence, and the conclusion follows from Corollary 7.2 (since the functor $\varepsilon^\#$ carries J-spheres to G-spheres by Proposition 1.4 and Lemma 4.14).

Of course, $\Sigma^{-A}X$ is abusive notation for $\Sigma^{-A}\Sigma^\infty X$. It is interesting that a spectrum level context is needed even to make sense out of the second isomorphism when A is non-zero.

<u>Example 8.2</u>. Using subscripts to denote the relevant group, we see from Proposition 1.4 and Theorem 4.7 that there is a natural isomorphism $\varepsilon^\# \Sigma_J^\infty X \cong \Sigma_G^\infty X$ of G-spectra for J-spaces X. In particular, with $X = S^0$, $\varepsilon^\# S_J \cong S_G$. Thus the theorem relates J-homotopy and cohomotopy to G-homotopy and cohomotopy for any quotient group J of G.

This example is misleading in that the full strength of the theorem is usually required: one usually has only an $\mathcal{F}(N)$-equivalence and not a G-equivalence between $\varepsilon^\# E_J$ and E_G. There is an illuminating alternative description of what it means to have such an equivalence.

<u>Lemma 8.3</u>. A G-map $\xi\colon \varepsilon^\# E_J \to E_G$ is an $\mathcal{F}(N)$-equivalence if and only if the

composite G-map

$$\varepsilon^* E_J \xrightarrow{\varepsilon^*\zeta} \varepsilon^*[(i^*E_G)^N] \xrightarrow{\iota} i^*E_G$$

is an $\mathcal{F}(N)$-equivalence, where

$$\zeta\colon E_J \longrightarrow (E_G)^N \equiv (i^*E_G)^N$$

is the J-map adjoint to ξ and ι is the evident inclusion.

Proof. For $H \in \mathcal{F}(N)$, $\varepsilon\colon G \to J$ restricts to an isomorphism $H \to \varepsilon(H)$, and U^N and U are isomorphic as H-universes. Unraveling the composite adjunction $\varepsilon^\#$, we see that ξ is the composite

$$\varepsilon^\# E_J = i_*\varepsilon^* E_J \xrightarrow{i_*\varepsilon^*\zeta} i_*\varepsilon^*[(i^*E_G)^N] \xrightarrow{i_*\iota} i_*i^*E_G \longrightarrow E_G,$$

where the last map is the counit of the (i_*,i^*)-adjunction and is an $\mathcal{F}(N)$-equivalence by Corollary 1.8. Since Theorem 2.8 (i) implies that a map f in $G\mathscr{S}U^N$ is an $\mathcal{F}(N)$-equivalence and only if the map i_*f in $G\mathscr{S}U$ is an $\mathcal{F}(N)$-equivalence, the conclusion follows.

Example 8.4. For $E_G \in G\mathscr{S}U$, let $E \in \mathscr{S}U^G$ denote i^*E_G with G-action ignored. We think of E as the underlying nonequivariant spectrum of E_G and have the inclusion $\iota\colon (E_G)^G \equiv (i^*E_G)^G \to E$. The G-spectrum E_G is said to be split if there exists a map $\zeta\colon E \to (E_G)^G$ such that $\iota\zeta \simeq 1\colon E \to E$. (If $\iota\zeta$ were an equivalence, not necessarily the identity, we could precompose ζ with $(\iota\zeta)^{-1}$ and so obtain a new ζ for which $\iota\zeta \simeq 1$.) By the case $N = G$ of the lemma and theorem (with $\mathcal{F}(G) = \{e\}$), we then have

$$E^*(D/G) \cong E^*_G(i_*D) \quad \text{and} \quad E_*(D/G) \cong E^G_*(\Sigma^{-A}i_*D)$$

for a free G-spectrum $D \in G\mathscr{S}U^G$, where A is the adjoint representation of G. The first of these isomorphisms generalizes a result of Kosniowski [75]. Both generalize results in May and McClure [108, Lemmas 12 and 16].

What is so special about the case $N = G$ is that e is canonically both a subgroup and a quotient group of G. In general in Theorem 8.1, we need quite different spectra E_G and E_J. As we shall explain in the next section, there is a general procedure for constructing a J-spectrum E_J from a given G-spectrum E_G, but E_J so constructed will usually not be related to E_G in the manner prescribed in Theorem 8.1. In fact, this fails for such familiar examples as K-theory and cobordism. The failure is important since the constructed quotient group theories

play a major role in the study of the "completion conjecture" in equivariant cohomology theory (see [24,103,109]). However, in K-theory and cobordism, we have preassigned G-spectra for all G, and these are so related as to guarantee the applicability of Theorem 8.1 to any G and J. In the rest of this section, we develop a setting in which such coherent families of equivariant spectra can be studied.

Choose a complete G-universe U_G for each compact Lie group G. Choose an H-linear isometry $j_\alpha\colon \alpha^*U_G \to U_H$ for each homomorphism $\alpha\colon H \to G$. We have adjunctions

$$[j_{\alpha*}D',D]_H \cong [D',j^*_\alpha D]_H \quad \text{and} \quad [\alpha^*E,D']_H \cong [E,F_\alpha[G,D')]_G,$$

where $D' \in H\mathscr{S}\alpha^*U_G$, $D \in H\mathscr{S}U_H$, and $E \in G\mathscr{S}U_G$. We define

$$\alpha^\# E = j_{\alpha*}\alpha^*E \in H\mathscr{S}U_H \quad \text{and} \quad \alpha_\# D = F_\alpha[G,j^*_\alpha D) \in G\mathscr{S}U_G$$

and have the composite adjunction

$$[\alpha^\# E,D]_H \cong [E,\alpha_\# D]_G.$$

If $\beta\colon K \to H$ is another homomorphism, then $(\alpha\beta)^* = \beta^*\alpha^*$ trivially, $j_{\beta*}j_{\alpha*} \cong j_{\alpha\beta*}$ since $j_\beta j_\alpha \simeq j_{\alpha\beta}$, and $j_{\alpha*}\beta^* \cong \beta^*j_{\alpha*}$ by inspection on the adjoint level. We therefore have natural isomorphisms of functors

$$\beta^\#\alpha^\# \cong (\alpha\beta)^\# \quad \text{and} \quad \alpha_\#\beta_\# \cong (\alpha\beta)_\#.$$

Clearly $1^\#$ and $1_\#$ are identity functors, $1\colon G \to G$. If $\alpha\colon H \to G$ is an inclusion, we may take j_α to be an isomorphism, so that $\alpha^\# E$ is essentially just α^*E and $\alpha_\# D$ is essentially just $F_H[G,D)$. If $\varepsilon\colon G \to J = G/N$ is a quotient homomorphism, we may take j_ε to be the composite of an isomorphism $\varepsilon^*U_J \to (U_G)^N$ and the inclusion $(U_G)^N \to U_G$. Thus $\varepsilon^\#$ is essentially the same as the functor entering into Theorem 8.1 and $\varepsilon_\#$ is essentially just the N-fixed point spectrum functor.

Definition 8.5. Let $\mathcal{H}$ be any subcategory of the category of compact Lie groups and their homomorphisms. A $\mathcal{H}$-spectrum consists of a G-spectrum $E_G \in G\mathscr{S}U_G$ for each $G \in \mathcal{H}$ and an H-map $\xi_\alpha\colon \alpha^\# E_G \to E_H$ for each homomorphism $\alpha\colon H \to G$ in $\mathcal{H}$ such that the following conditions hold.

(a) $\xi_1\colon E_G = 1^\# E_G \to E_G$ is the identity map for each G.

(b) The following diagram commutes for $\alpha\colon H \to G$ and $\beta\colon K \to H$.

$$\begin{tikzcd} \beta^{\#}\alpha^{\#}E_G \arrow[r, "\beta^{\#}\xi_\alpha"] \arrow[d, phantom, "\cong" rotate=90] & \beta^{\#}E_H \arrow[d, "\xi_\beta"] \\ (\alpha\beta)^{\#}E_G \arrow[r, "\xi_{\alpha\beta}"] & E_K \end{tikzcd}$$

(c) If $\alpha\colon H \to G$ is an inclusion, then ξ_α is an H-equivalence.
Let $\zeta_\alpha\colon E_G \to \alpha_{\#}E_H$ be the adjoint of ξ_α. Then ζ_1 is the identity map and the following diagram commutes.

$$\begin{tikzcd} E_G \arrow[r, "\zeta_{\alpha\beta}"] \arrow[d, "\zeta_\alpha"'] & (\alpha\beta)_{\#}E_K \arrow[d, phantom, "\cong" rotate=90] \\ \alpha_{\#}E_H \arrow[r, "\alpha_{\#}\zeta_\beta"] & \alpha_{\#}\beta_{\#}E_K \end{tikzcd}$$

The definition requires only appropriate functoriality and behavior on inclusions. Nothing is assumed about quotient homomorphisms, but we have the following implication.

Proposition 8.6. Let $\varepsilon\colon G \to J = G/N$ be a quotient homomorphism. For $H \in \mathcal{F}(N)$, consider the following commutative diagram.

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[d, "\delta"'] & G \arrow[d, "\varepsilon"] \\ K \arrow[r, "\beta"] & J \end{tikzcd}$$

Here α is the inclusion, $K = \varepsilon(H)$, δ is the isomorphism obtained by restriction of ε, and β is the inclusion induced by α. If $\{E_G\}$ is a $\mathcal{H}$-spectrum, where the displayed diagram and δ^{-1} are in $\mathcal{H}$ for each $H \in \mathcal{F}(N)$, then $\xi_\varepsilon\colon \varepsilon^{\#}E_J \to E_G$ is an $\mathcal{F}(N)$-equivalence.

Proof. We have the following commutative diagram for $H \in \mathcal{F}(N)$.

$$\begin{tikzcd} \delta^{\#}\beta^{\#}E_J \arrow[r, phantom, "\cong"] \arrow[d, "\delta^{\#}\xi_\beta"'] & (\beta\delta)^{\#}E_J \arrow[r, equal] \arrow[d, "\xi_{\beta\delta}"] & (\varepsilon\alpha)^{\#}E_J \arrow[r, phantom, "\cong"] \arrow[d, "\xi_{\varepsilon\alpha}"] & \alpha^{\#}\varepsilon^{\#}E_J \arrow[d, "\alpha^{\#}\xi_\varepsilon"] \\ \delta^{\#}E_K \arrow[r, "\xi_\delta"] & E_H \arrow[r, equal] & E_H & \alpha^{\#}E_G \arrow[l, "\xi_\alpha"'] \end{tikzcd}$$

Since α and β are inclusions, ξ_α and $\delta^{\#}\xi_\beta$ are H-equivalences. Since δ is an isomorphism in $\mathcal{H}$, ξ_δ is an H-equivalence with inverse

$$\delta^{\#}\xi_{\delta^{-1}}\colon E_H = \delta^{\#}(\delta^{-1})^{\#}E_H \longrightarrow \delta^{\#}E_K.$$

Therefore $\alpha^{\#}\xi_\varepsilon$ is an H-equivalence.

The hypothesis $\delta^{-1} \in \mathcal{H}$ is essential. We have examples where everthing else holds and the conclusion fails because ξ_δ fails to be an H-equivalence.

Examples 8.7. (i) If $\mathcal{H}$ is the category whose objects are the two groups e and G and whose non-identity morphisms are just $e \to G$, $G \to e$, and $G \to e \to G$, then a $\mathcal{H}$-spectrum is exactly the same thing as a split G-spectrum.
(ii) $\{S_G\}$ is a $\mathcal{H}$-spectrum, where $\mathcal{H}$ is the category of all compact Lie groups and homomorphisms. The $\xi_\alpha\colon \alpha^{\#}S_H \to S_G$ are the canonical isomorphisms given by Proposition 1.4 and Theorem 4.7; compare Lemma 4.11.

Clearly suitably coherent families of RO(G)-graded cohomology theories on G-spaces give rise to coherent families of equivariant spectra (modulo the usual problems with $\lim^1$ terms as we pass from space level to spectrum level data). In particular, the representing spectra for equivariant K-theory give a $\mathcal{H}$-spectrum. Again suitably defined equivariant Thom spectra (as in chapter X) give $\mathcal{H}$-spectra. It would take us too far afield to go into detail here.

Warning 8.8. There are important examples of families of spectra $\{E_G\}$ which depend canonically but not functorially on G. In particular, there are several such families of equivariant Eilenberg-MacLane spectra [88,90].

§9. The construction of (G/N)-spectra from G-spectra

Let N be a normal subgroup of G with quotient group J, $\varepsilon\colon G \to J$, and let U be a G-universe with N-fixed J-universe U^N, $i\colon U^N \to U$; U need not be complete in this section. We shall calculate $\bar{h}J\mathcal{S}U^N$ in terms of $\bar{h}G\mathcal{S}U$.

To establish an appropriate conceptual context, it is useful to begin more generally with complements to some of the ideas in section 2. Let $\mathcal{F}$ be a family of subgroups of G and let $\mathcal{F}' = \{H \mid H \notin \mathcal{F}\}$ be its complementary "cofamily". Define $\tilde{E}\mathcal{F}$ to be the cofibre of the canonical projection $\pi\colon E\mathcal{F}^+ \to S^0$ and let $\iota\colon S^0 \to \tilde{E}\mathcal{F}$ be the inclusion. If $H \in \mathcal{F}$, then $(E\mathcal{F}^+)^H \simeq S^0$ and $(\tilde{E}\mathcal{F})^H \simeq *$. If $H \in \mathcal{F}'$, then $(E\mathcal{F}^+)^H = *$ and $(\tilde{E}\mathcal{F})^H = S^0$.

Say that a map $f\colon X \to Y$ of G-CW complexes or G-CW spectra is an $\mathcal{F}'$-equivalence if

$$1\wedge f\colon \tilde{E}\mathcal{F}\wedge X \longrightarrow \tilde{E}\mathcal{F}\wedge Y$$

is a G-equivalence. On the space level, but not on the spectrum level, this just means that f^H is an equivalence for all $H \in \mathcal{F}'$. Say that a map $f\colon X \to Y$

between general G-spaces or G-spectra is a weak $\mathcal{F}'$-equivalence if $1\wedge\Gamma f$ is a G-equivalence, where $\Gamma f\colon \Gamma X \to \Gamma Y$ is a G-CW approximation of f. On the space level, but not on the spectrum level, this just means that $1\wedge f$ is a weak G-equivalence (because here $\tilde{E}\mathcal{F}\wedge\Gamma X$ is weakly G-equivalent to $\tilde{E}\mathcal{F}\wedge X$). The point is that, on the stable category level, the functor $\tilde{E}\mathcal{F}\wedge X$ must be interpreted as $\tilde{E}\mathcal{F}\wedge\Gamma X$. The careful reader may want to insert CW approximations explicitly in some of the arguments below, as dictated by the discussion in and above I.5.13. It would not do simply to assume that all given spectra are CW homotopy types since some of the functors we will use, such as passage to N-fixed point spectra, need not preserve CW homotopy types.

Let $\mathcal{E}\mathcal{F}'$ denote the collection of weak $\mathcal{F}'$-equivalences in $\bar{h}G\mathscr{T}$ or $\bar{h}G\mathscr{S}U$. The content of Proposition 2.13 is that the localization of $\bar{h}G\mathscr{S}U$ obtained by adjoining formal inverses to its weak $\mathcal{F}$-equivalences can be calculated as

$$(\mathcal{E}\mathcal{F})^{-1}\bar{h}G\mathscr{S}U(X,Y) = [E\mathcal{F}^{+}\wedge X, E\mathcal{F}^{+}\wedge Y]_G.$$

The cofamily analog reads as follows.

Proposition 9.1. The localization of $\bar{h}G\mathscr{S}U$ at its weak $\mathcal{F}'$-equivalences can be calculated as

$$(\mathcal{E}\mathcal{F}')^{-1}\bar{h}G\mathscr{S}U(X,Y) = [\tilde{E}\mathcal{F}\wedge X, \tilde{E}\mathcal{F}\wedge Y]_G.$$

The construction of localizations of categories in terms of "cocompletions" of objects following I.5.12 can be dualized to a construction of localizations in terms of "completions" of objects. In that language, the following result asserts that, with respect to $\mathcal{E}\mathcal{F}'$, the objects $\tilde{E}\mathcal{F}\wedge X$ are complete and the maps

$$\iota\wedge 1\colon X = S^0\wedge X \longrightarrow \tilde{E}\mathcal{F}\wedge X$$

are completions. These assertions imply the previous result.

Proposition 9.2. Let X be a G-CW complex or a G-CW spectrum. Then the map $\iota\wedge 1\colon X \to \tilde{E}\mathcal{F}\wedge X$ is an $\mathcal{F}'$-equivalence and the following three statements are equivalent.

(i) $\iota\wedge 1$ is a G-equivalence.

(ii) $f^{*}\colon [Z,X]_G \to [Y,X]_G$ is an isomorphism for every weak $\mathcal{F}'$-equivalence $f\colon Y \to Z$.

(iii) $\pi_*^H(X) = 0$ for all $H \in \mathcal{F}$.

Proof. The first statement holds since $1\wedge\iota\colon \tilde{E}\mathcal{F} \to \tilde{E}\mathcal{F}\wedge\tilde{E}\mathcal{F}$ is a G-equivalence by a

check on fixed point sets. Observe that $[E\mathcal{F}^{+}\wedge X, \tilde{E}\mathcal{F}\wedge X']_G = 0$ for any X and X' since $E\mathcal{F}$ only has cells of orbit type G/H with $H \in \mathcal{F}$ and $\tilde{E}\mathcal{F}$ is H-contractible for such H. In view of the cofibre sequence

$$E\mathcal{F}^{+}\wedge Y \longrightarrow Y \xrightarrow{\iota\wedge 1} \tilde{E}\mathcal{F}\wedge Y \longrightarrow \Sigma(E\mathcal{F}^{+}\wedge Y),$$

it follows that

$$(\iota\wedge 1)^{*}\colon [\tilde{E}\mathcal{F}\wedge Y, \tilde{E}\mathcal{F}\wedge X]_G \longrightarrow [Y, \tilde{E}\mathcal{F}\wedge X]_G$$

is an isomorphism for any Y. It is clear from this that (i) implies (ii). If $H \in \mathcal{F}$, then $G/H^{+}\wedge\tilde{E}\mathcal{F}$ is G-contractible, hence $G/H^{+}\wedge Y \to *$ is an $\mathcal{F}'$-equivalence for any Y. This shows that (ii) implies (iii). Assume (iii). When X is a G-CW complex, $\iota\wedge 1\colon X \to \tilde{E}\mathcal{F}\wedge X$ is a G-equivalence by another check on fixed point sets. When X is a G-CW spectrum, we claim that $\pi_*^H(E\mathcal{F}^{+}\wedge X) = 0$ for all $H \subset G$, so that $\iota\wedge 1$ is a G-equivalence. If $H \in \mathcal{F}$, then $E\mathcal{F}^{+}$ is H-equivalent to S^0 and our claim holds by hypothesis. If $H \in \mathcal{F}'$, let $\mathcal{F}|H$ be the family of subgroups of H in $\mathcal{F}$ and observe that $E\mathcal{F}$ regarded as an H-space is a model for $E(\mathcal{F}|H)$. It is thus H-equivalent to an $(\mathcal{F}|H)$-CW complex. Since $\pi_*^K(X) = 0$ for $K \in \mathcal{F}|H$, $\pi_*^H(W\wedge X) = \tilde{X}_*^H(W) = 0$ for any $(\mathcal{F}|H)$-CW complex W.

We shall need a space level observation that has no direct spectrum level analog. For a G-space X, let $X_{\mathcal{F}}$ denote the G-subspace $\{x \mid G_x \in \mathcal{F}'\}$. For example, if $\mathcal{F} = \{e\}$, then $X_{\mathcal{F}}$ is the singular set of X. If X is a G-CW complex, then $X_{\mathcal{F}}$ is a subcomplex.

Proposition 9.3. For G-CW complexes X and Y, the inclusions $X_{\mathcal{F}} \to X$ and $S^0 \to \tilde{E}\mathcal{F}$ induce bijections

$$[X, \tilde{E}\mathcal{F}\wedge Y]_G \longrightarrow [X_{\mathcal{F}}, \tilde{E}\mathcal{F}\wedge Y]_G \longleftarrow [X_{\mathcal{F}}, Y]_G.$$

Proof. The first bijection follows easily from the facts that the cells of X not in $X_{\mathcal{F}}$ are of orbit type G/H with $H \in \mathcal{F}$ and that $\tilde{E}\mathcal{F}^H$ is H-contractible for such H. The second bijection is obvious from the fact that $\tilde{E}\mathcal{F}^H = S^0$ for $H \in \mathcal{F}'$.

With this discussion as preamble, we return to our normal subgroup N with quotient group J and consider the family $\mathcal{F}[N]$ of subgroups of G which do not contain N. Here, for a G-space X, $X_{\mathcal{F}[N]} = X^N$. By Proposition 9.2, for a G-space or G-spectrum X, $\iota\wedge 1\colon \Gamma X \to \tilde{E}\mathcal{F}[N]\wedge\Gamma X$ is a G-equivalence (which means that $\iota\wedge 1\colon X \to \tilde{E}\mathcal{F}[N]\wedge X$ is an isomorphism in $\bar{h}G\mathscr{S}U$) if and only if $\pi_*^H(X) = 0$ unless H contains N. Let us say that X is concentrated over N when this holds. We shall show that, from the point of view of equivariant homotopy theory, G-spaces or G-spectra concentrated over N are completely equivalent to J-spaces or J-spectra. We begin on the space level, where the result is trivial.

Proposition 9.4. For J-spaces W and G-spaces X concentrated over N, there is a natural isomorphism

$$[W,X^N]_J \cong [\tilde{E}\mathcal{F}[N]\wedge W,X]_G.$$

The unit of this adjunction is the identity $W = (\tilde{E}\mathcal{F}[N]\wedge W)^N$ and the counit is the natural weak G-equivalence

$$\tilde{E}\mathcal{F}[N]\wedge X^N \longrightarrow \tilde{E}\mathcal{F}[N]\wedge X \simeq X$$

induced by the inclusion $X^N \to X$. Therefore, for J-spaces W and W',

$$[W,W']_J \cong [\tilde{E}\mathcal{F}[N]\wedge W,\tilde{E}\mathcal{F}[N]\wedge W']_G.$$

We have implicitly regarded J-spaces as G-spaces via ε here, and we must use the change of group and universe functor $\varepsilon^\# = i_*\varepsilon^*\colon J\mathcal{S}U^N \to G\mathcal{S}U$ left adjoint to the N-fixed point functor to express the spectrum level analog.

Theorem 9.5. For J-spectra $E \in J\mathcal{S}U^N$ and G-spectra $D \in G\mathcal{S}U$ concentrated over N, there is a natural isomorphism

$$[E,D^N]_J \cong [\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E,D]_G.$$

The unit $\eta\colon E \to (\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E)^N$ of this adjunction is an isomorphism in $\bar{h}J\mathcal{S}U^N$ and the counit $\varepsilon\colon \tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N) \to D$ is an isomorphism in $\bar{h}G\mathcal{S}U$. Therefore, for J-spectra E and E',

$$[E,E']_J \cong [\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E,\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E']_G.$$

Proof. Proposition 9.2 gives the required isomorphism as

$$(\iota\wedge 1)^*\colon [\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E,D]_G \longrightarrow [\varepsilon^\# E,D]_G \cong [E,D^N]_J.$$

The unit and counit of this adjunction are determined by the unit $E \to (\varepsilon^\# E)^N$ and counit $\varepsilon^\#(D^N) \to D$ of the $(\varepsilon^\#,(?)^N)$-adjunction as the composites

$$E \longrightarrow (\varepsilon^\# E)^N \xrightarrow{(\iota\wedge 1)^N} (\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E)^N$$

and

$$\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N) \longrightarrow \tilde{E}\mathcal{F}[N]\wedge D \xrightarrow{(\iota\wedge 1)^{-1}} D.$$

Of course, by our adjunction, the composite

$$\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N) \xrightarrow{1\wedge\varepsilon^\#\eta} \tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N))^N \xrightarrow{\ \varepsilon\ } \tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N)$$

is the identity map for each D. We will prove in Proposition 9.10 below that η is an isomorphism in $\bar{h}J\mathcal{S}U^N$ for each E. We will construct an isomorphism (different from ε) between D and $\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(D^N)$ in Proposition 9.11. By the naturality of ε and the relation $\varepsilon(1\wedge\varepsilon^\#\eta) = 1$, it will follow that ε is an isomorphism in $\bar{h}G\mathcal{S}U$ for each D.

Combining with Proposition 9.2, we can restate the last assertion as a sharpened analog of the case $\mathcal{F} = \mathcal{F}[N]$ of Proposition 2.13.

Corollary 9.6. The functor $\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\#(?)$ induces an equivalence

$$\bar{h}J\mathcal{S}U^N \longrightarrow (\mathcal{E}\mathcal{F}[N]')^{-1}\bar{h}G\mathcal{S}U,$$

the target being equivalent to the full subcategory of $\bar{h}G\mathcal{S}U$ whose objects are the G-spectra concentrated over N. Under this identification of the target, the inverse functor is $(?)^N$.

The rest of the proof of Theorem 9.5 depends on the "spacewise N-fixed point functor" from G-prespectra to J-prespectra.

Definition 9.7. Let $\{A_n \mid n \geq 0\}$ be an indexing sequence in the G-universe U and let $B_n = A_{n+1} - A_n$. Note that $\{A_n^N\}$ is an indexing sequence in U^N. Write $D_n = DA_n$ for $D \in G\mathcal{P}A$ and $E_n = E(A_n^N)$ for $E \in J\mathcal{P}A^N$. Define a functor $\Phi^N\colon G\mathcal{P}A \to J\mathcal{P}A^N$ by letting $(\Phi^N D)_n = (D_n)^N$, with structural maps

$$(D_n)^N\wedge S^{B_n^N} = (D_n\wedge S^{B_n})^N \xrightarrow{\sigma_n^N} (D_{n+1})^N.$$

Define $\Phi^N\colon G\mathcal{S}A \to J\mathcal{S}A^N$ by letting $\Phi^N D = L\Phi^N K\Gamma D$, where $K\Gamma D$ is the G-CW prespectrum canonically weakly equivalent to D (see I.6.3, I.6.8, and I.8.12).

We have used indexing sequences here to avoid ambiguities resulting from the fact that different indexing spaces in U can have the same N-fixed point indexing space in U^N. We have usually passed from a functor on prespectra to a functor on spectra by use of the adjunction (L,ℓ). In the case of Φ^N, this would be inappropriate since it is not the formal properties but the homotopical properties of Φ^N that we wish to retain on the spectrum level. Observe that Φ^N carries G-CW prespectra to J-CW prespectra, on which the functor $L: J\mathcal{P}A^N \to J\mathcal{S}A^N$ is given simply by passage to colimits, as above I.2.2. (See I.8.7-I.8.14.)

The functor Φ^N must not be confused with the actual fixed point functor $G\mathcal{S}U \to J\mathcal{S}U^N$. After change of indexing sets, the latter is specified as a functor $G\mathcal{S}A \to J\mathcal{S}A^N$ by letting $(D^N)_n = (\Omega^{A_n - A_n^N} D_n)^N$. The direct prespectrum level definition of Φ^N is uninteresting when applied to spectra. (Compare Adams [3,§7].) However, there is a simple relationship between Φ^N and $(?)^N$, and we shall use this relationship to complete the proof of Theorem 9.5. We index G-prespectra and spectra on A and J-prespectra and spectra on A^N in the rest of this section. The following result is the spectrum level version of a cohomological observation of Costenoble.

Theorem 9.8. (i) For G-CW prespectra (or, more generally, Σ-inclusion G-prespectra) D, there is a natural weak equivalence of J-spectra

$$\xi: (\tilde{E}\mathcal{F}[N]\wedge LD)^N \longrightarrow L\Phi^N D.$$

(ii) For G-CW spectra D, there is a natural weak equivalence of J-spectra

$$\xi: (\tilde{E}\mathcal{F}[N]\wedge D)^N \longrightarrow \Phi^N D.$$

Proof. (i) As observed above I.3.5, we have an isomorphism

$$\tilde{E}\mathcal{F}[N]\wedge LD \equiv L(\tilde{E}\mathcal{F}[N]\wedge LD) \cong L(\tilde{E}\mathcal{F}[N]\wedge D).$$

Since D is a Σ-inclusion G-prespectrum, so is $\tilde{E}\mathcal{F}[N]\wedge D$. Thus L on the right is given by passage to colimits (as above I.2.2) and the n^{th} space of $(\tilde{E}\mathcal{F}[N]\wedge LD)^N$ is G-homeomorphic to

$$(\operatorname*{colim}_q \Omega^{A_q - A_n^N}(\tilde{E}\mathcal{F}[N]\wedge D_q))^N \cong \operatorname*{colim}_q(\Omega^{A_q - A_n^N}(\tilde{E}\mathcal{F}[N]\wedge D_q))^N.$$

On the other hand, the n^{th} space of $L\Phi^N D$ is

$$\operatorname*{colim}_q \Omega^{A_q^N - A_n^N} D_q^N = \operatorname*{colim}_q \Omega^{A_q^N - A_n^N}(\tilde{E}\mathcal{F}[N]\wedge D_q)^N.$$

The inclusions $S^{A_q^N - A_n^N} \to S^{A_q - A_n^N}$ induce a map from the first colimit to the second, and this map is a weak J-equivalence by application of adjunctions and Proposition

9.3 to the calculation of homotopy groups. As n varies, these maps specify the desired weak J-equivalence

$$\xi: (\tilde{E}\mathcal{F}[N]\wedge LD)^N \longrightarrow L\Phi^N D.$$

(ii) We deduce (ii) by applying (i) to $K\Gamma D$, obtaining

$$(\tilde{E}\mathcal{F}[N]\wedge D)^N \simeq (\tilde{E}\mathcal{F}[N]\wedge LK\Gamma D)^N \xrightarrow{\ \xi\ } L\Phi^N K\Gamma D \equiv \Phi^N D.$$

We can use this result to relate fixed points of spaces to fixed points of spectra.

Corollary 9.9. Let X be a G-CW complex. Then there are natural weak equivalences of J-spectra

$$\Sigma^\infty(X^N) \simeq (\tilde{E}\mathcal{F}[N]\wedge\Sigma^\infty X)^N \simeq \Phi^N\Sigma^\infty X$$

(where Σ^∞ refers to J on the left and to G in the middle and on the right).

Proof. Let $\{\Sigma^{A_n}X\}$ denote the suspension G-CW prespectrum of X, so that $\Sigma^\infty X = L\{\Sigma^{A_n}X\}$. Obviously $\Phi^N\{\Sigma^{A_n}X\}$ is the suspension J-CW prespectrum $\{\Sigma^{A_n^N}X^N\}$. The equivalences result by application of part (i) of the theorem to $\{\Sigma^{A_n}X\}$ and of part (ii) to $\Sigma^\infty X$.

Of course, we can apply the results above to the study of H-fixed point W_GH-spaces and spectra associated to G-spaces and spectra, where $W_GH = N_GH/H$.

Proposition 9.10. (i) If E is a J-prespectrum, then $E = \Phi^N i_*\varepsilon^*E$, where $i_*: G\mathcal{P}A^N \to G\mathcal{P}A$ is the prespectrum level change of universe functor.
(ii) If E is a J-CW spectrum, then

$$\eta: E \longrightarrow (\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E)^N$$

is a weak J-equivalence.

Proof. (i) $i_*\varepsilon^*E$ has n^{th} space $E_n\wedge S^{A_n - A_n^N}$, hence its N-fixed point set is E_n back again.
(ii) We may as well assume that $E = LE'$, where E' is a J-CW prespectrum. By part (i) and the discussion above I.3.5, Theorem 9.8(i) gives a weak J-equivalence

$$\xi: (\tilde{E}\mathcal{F}[N]\wedge\varepsilon^\# E)^N \cong (\tilde{E}\mathcal{F}[N]\wedge Li_*\varepsilon^*E')^N \longrightarrow L\Phi^N i_*\varepsilon^*E' = E.$$

The map η is given on n^{th} spaces by the composite

$$\begin{array}{l}
\operatorname{colim} \Omega^{A_q^N - A_n^N} E'_q \\
\quad \Big\downarrow \, (\operatorname{colim} \Omega^{A_q^N - A_n^N} \eta_n)^N \\
(\operatorname{colim} \Omega^{A_q^N - A_n^N}(\Omega^{A_q - A_q^N} \Sigma^{A_q - A_q^N} E'_q))^N \\
\quad \Big\downarrow \, (\operatorname{colim} \Omega^{A_q - A_n^N}(\iota \wedge 1))^N \\
(\operatorname{colim} \Omega^{A_q - A_n^N}(\tilde{E}\mathcal{F}[N] \wedge E'_q \wedge S^{A_q - A_q^N}))^N
\end{array}$$

and it follows by inspection that $\xi\eta$ is the identity map of E.

Proposition 9.11. (i) If D is a G-CW prespectrum, then there is a spacewise equivalence of G-prespectra

$$\tilde{E}\mathcal{F}[N] \wedge D \longrightarrow \tilde{E}\mathcal{F}[N] \wedge i_* \varepsilon^* \Phi^N D.$$

(ii) If D is a G-spectrum concentrated over N, then D is isomorphic to $\tilde{E}\mathcal{F}[N] \wedge \varepsilon^{\#}(D^N)$ in $\bar{h}G\mathcal{S}A$.

Proof. (i) For each n, the inclusion $\tilde{E}\mathcal{F}[N] \wedge D_n^N \to \tilde{E}\mathcal{F}[N] \wedge D_n$ is a G-equivalence with homotopy inverse r_n, say. Let s'_n be the composite G-equivalence

$$\tilde{E}\mathcal{F}[N] \wedge D_n \xrightarrow{r_n} \tilde{E}\mathcal{F}[N] \wedge D_n^N \xrightarrow{1 \wedge e} \tilde{E}\mathcal{F}[N] \wedge D_n^N \wedge S^{A_n - A_n^N},$$

where $e\colon S^0 \to S^{A_n - A_n^N}$ is the inclusion. It is easily checked that the s'_n specify a w-map (in the sense of I.6.2)

$$s'\colon \tilde{E}\mathcal{F}[N] \wedge D \longrightarrow \tilde{E}\mathcal{F}[N] \wedge i_* \varepsilon^* \Phi^N D.$$

By I.8.8, s' is spacewise homotopic to an actual map s.

(ii) By hypothesis, $D \simeq \tilde{E}\mathcal{F}[N] \wedge D$ in $\bar{h}G\mathcal{S}A$. Thus we have the chain of isomorphisms in $\bar{h}G\mathcal{S}A$

$$D \simeq LK\Gamma D \simeq \tilde{E}\mathcal{F}[N] \wedge LK\Gamma D = L(\tilde{E}\mathcal{F}[N] \wedge K\Gamma D)$$

$$\xrightarrow{Ls} L(\tilde{E}\mathcal{F}[N] \wedge i_* \varepsilon^* \Phi^N K\Gamma D) = \tilde{E}\mathcal{F}[N] \wedge \varepsilon^{\#} \Phi^N D$$

$$\xleftarrow{1 \wedge \varepsilon^{\#} \xi} \tilde{E}\mathcal{F}[N] \wedge \varepsilon^{\#}(\tilde{E}\mathcal{F}[N] \wedge D)^N \simeq \tilde{E}\mathcal{F}[N] \wedge \varepsilon^{\#}(D^N).$$

We have now proven Theorem 9.5. Clearly a large part of the point is the failure of smash products to commute with passage to fixed points on the spectrum level. However, as the following results show, we do have such commutation relations for J-spectra concentrated over N.

Proposition 9.12. (i) For G-CW spectra D and G-CW complexes X, there is a natural weak J-equivalence

$$(\tilde{E}\mathcal{F}[N] \wedge D \wedge X)^N \simeq (\tilde{E}\mathcal{F}[N] \wedge D)^N \wedge X^N.$$

(ii) For G-CW spectra D and D', there is a natural weak J-equivalence

$$(\tilde{E}\mathcal{F}[N] \wedge D \wedge D')^N \simeq (\tilde{E}\mathcal{F}[N] \wedge D)^N \wedge (\tilde{E}\mathcal{F}[N] \wedge D')^N.$$

Proof. (i) For G-prespectra D and G-spaces X, we clearly have

$$\Phi^N(D \wedge X) = (\Phi^N D) \wedge X^N.$$

Application of this with D replaced by $K\Gamma D$ and use of Theorem 9.8 (ii) gives part (i).

(ii) Let $A \oplus A$ be the indexing sequence $\{A_n \oplus A_n\}$ in $U \oplus U$. Let $f\colon U \oplus U \to U$ be a G-linear isometric isomorphism and let A' be the indexing sequence $\{f(A_n \oplus A_n)\}$ in U. We have the external smash product and change of universe functors

$$\wedge\colon G\mathcal{P}A \times G\mathcal{P}A \longrightarrow G\mathcal{P}(A \oplus A) \quad \text{and} \quad f_*\colon G\mathcal{P}(A \oplus A) \longrightarrow G\mathcal{P}A'.$$

For G-prespectra $D, D' \in G\mathcal{P}A$, we find easily that

$$\Phi^N f_*(D \wedge D') = f_*^N(\Phi^N D \wedge \Phi^N D')$$

in $J\mathcal{P}(A')^N$. Applying this with D and D' replaced by $K\Gamma D$ and $K\Gamma D'$, noting that $f_*(K\Gamma D \wedge K\Gamma D')$ is equivalent to $K\Gamma f_*(D \wedge D')$, and using Theorem 9.8 (ii), we obtain (ii) from the definition of internal smash products in section 3.

The discussion above makes clear that, with a change of notation, a compulsively reasonable way to construct a J-spectrum $E_J \in J\mathcal{S}U^N$ from a G-spectrum $E_G \in G\mathcal{S}U$ is to set $E_J = (\tilde{E}\mathcal{F}[N] \wedge E_G)^N$. This is equivalent to setting $E_J = \Phi^N E_G$, and the construction commutes with smash products and preserves ring spectra. For a J-spectrum X, we have

$$E_J^*(X) = (\tilde{E}\mathcal{J}[N]\wedge E_G)^*(\varepsilon^\# X).$$

For J-spaces X, we recall that $\varepsilon^\#\Sigma^\infty X \cong \Sigma^\infty\varepsilon^* X$. We can work with RO(J)-graded cohomology here, using the homomorphism $\varepsilon^*: RO(J) \to RO(G)$ to interpret the grading on the right. The following calculation of $E_J^*(X)$ is due to Costenoble, who proved it by direct inspection of colimits from the prespectrum level definition of Φ^N.

Proposition 9.13. Let $E_G \in G\mathcal{S}U$ be a ring spectrum and set $E_J = (\tilde{E}\mathcal{J}[N]\wedge E_G)^N \in J\mathcal{S}U^N$. For a finite J-CW spectra X, $E_J^*(X)$ is the localization of $E_G^*(\varepsilon^\# X)$ obtained by inverting the Euler classes $\chi_V \in E_G^V(S^0)$ of those representations V of G such that $V^N = 0$.

Proof. Here χ_V is the image of the unit $1 \in E_G^V(S^V)$ under e^*, $e: S^0 \to S^V$. The statement means that, for $\alpha \in RO(J)$,

$$E_J^\alpha(X) = \operatorname{colim} E_G^{\varepsilon^*\alpha+V}(\varepsilon^\# X),$$

where the colimit runs over the indexing G-spaces $V \subset U - U^N$; for $V \subset W$, the map

$$E_G^{\varepsilon^*\alpha+V}(\varepsilon^\# X) \longrightarrow E_G^{\varepsilon^*\alpha+W}(\varepsilon^\# X)$$

of the colimit system is multiplication by χ_{W-V}. For the proof, observe that the colimit of the spheres S^V is a model for $\tilde{E}\mathcal{J}[N]$, as we see by passage to colimits from the cofibration sequences $S(V)^+ \to D(V)^+ \to S^V$, where S(V) and D(V) are the unit sphere and disk in V. Therefore $E_J^\alpha(X)$ is the colimit over V of the groups

$$(S^V\wedge E_G)^{\varepsilon^*\alpha}(\varepsilon^\# X) \cong E_G^{\varepsilon^*\alpha+V}(\varepsilon^\# X).$$

III. Equivariant duality theory

by L. G. Lewis, Jr. and J. P. May

We here give a thorough treatment of duality in the equivariant stable category. For a G-spectrum E, define the dual of E to be the function G-spectrum

$$D(E) = F(E,S).$$

For a based G-space X, define $D(X) = D(\Sigma^\infty X)$. We are concerned with the calculational relationship between E and DE and with the concrete identification of DX.

Nevertheless, our starting point will be purely categorical. In courses since 1970, the second author has emphasized the analogy between the stable category and the category of modules over a commutative ring R. In this comparison, $X\wedge Y$ corresponds to $M\otimes N$, S corresponds to R, F(X,Y) corresponds to Hom(M,N), and finite CW spectra correspond to finitely generated projective R-modules. The analogy is illuminating since many of the central facts about duality theory read the same way in the stable category as in the module category, where they are transparently obvious. In a very pretty paper [47], Dold and Puppe carried this analogy much futher. By discussing duality in the appropriate categorical framework, they showed that many of these facts admit purely formal common derivations. We present our version of their categorical discussion in section 1.

We return to the equivariant stable category in section 2. Because our category of G-spectra has canonical fibration sequences as well as canonical cofibration sequences and has canonical function spectra, the behavior of duality with respect to cofibration sequences is immediately apparent. The central, obvious, fact is that, for a map $f: X \to Y$ and G-spectrum Z, the function G-spectrum F(Cf,Z) is isomorphic to the fibre of $f^*: F(Y,Z) \to F(X,Z)$. In earlier treatments of the stable category, point set level fibration sequences and function spectra did not exist, hence the comparison between cofibre sequences and duality was intrinsically less precise, involving use of maps not uniquely determined up to homotopy. Our treatment uses only canonical natural maps. Given these observations, the basic results about duality directly generalize from orbit spectra $\Sigma^\infty G/H^+$ to arbitrary finite G-CW spectra X, the duals of orbit spectra having been computed in II§6. In particular, we obtain a natural equivalence

$$F(X,E) \simeq DX\wedge E$$

and thus an isomorphism $E^*(X) \cong E_*(DX)$. This section also includes an analysis of the behavior of duality with respect to change of groups.

In section 3, we give a quick derivation of the standard products in homology and cohomology theory and explain the interpretation of duality in terms of slant products. Turning from the spectrum to the space level, we describe V-dualities (for representations V) and give wholly space level interpretations of the basic notions in duality theory. In particular, we show that a G-map $\varepsilon: Y \wedge X \to S^V$ is a V-duality if and only if each of its fixed point maps $\varepsilon^H: Y^H \wedge X^H \to (S^V)^H$ is an n_H-duality, where $n_H = \dim V^H$.

In section 4, we prove an equivariant version of the original Spanier-Whitehead duality theorem [130,132] for G-spaces and G-pairs nicely embedded in G-representations. Our treatment is particularly clean, elementary, and precise. It makes no use of simplicial decompositions or of ordinary homology and gives very simple concrete descriptions of the relevant duality maps. Even nonequivariantly, we find this direct homotopical treatment far more efficient and aesthetically satisfactory then the classical one. Equivariantly, it gives much greater generality than could be hoped for from any argument dependent on triangulations. We make absolutely no claim to originality. Our treatment is essentially that of Dold and Puppe [47] and their students Henn and Hommel [63,64,65] whose nonequivariant work generalizes effortlessly to the equivariant context (as Dold and Puppe realized [47,§7]).

In section 5, we specialize to obtain equivariant generalizations of the classical results of Milnor and Spanier [113] and Atiyah [6] on the duality between smooth manifolds and the Thom complexes of their normal bundles. We combine these results with the Thom isomorphism to obtain equivariant Poincaré duality in section 6.

In sections 7 and 8, we lay the foundations for our study of the transfer in chapter IV and of the Burnside ring in chapter V. The essential starting points for these studies are the pretransfer $\tau(X) \in \pi_0^G(X)$ and the Euler characteristic $\chi(X) \in \pi_0^G(pt)$ associated to a compact G-ENR X. These are special cases of the trace $\tau(f) \in \pi_0^G(X)$ and Lefschetz constant $\chi(f) \in \pi_0^G(pt)$ associated to a G-map $f: X \to X$. We begin our study of these notions in the general categorical context of section 1. We then specialize to the context of finite G-spectra and prove a basic additivity formula for the behavior of traces on cofibre sequences. In section 8, we use this result to compute $\tau(f)$ and $\chi(f)$ in terms of the nonequivariant traces of the fixed point maps f^H and the equivariant transfers $\tau(G/H) \in \pi_0^G(G/H)$ and Euler characteristics $\chi(G/H) \in \pi_0^G(pt)$.

We have gone to considerable trouble in writing down explicit duality maps in this chapter. It is our feeling that the literature on duality, both equivariant and nonequivariant, leaves a great deal to be desired in terms of precision. The resulting pedantry may somewhat obscure the exposition, but its absence would surely obscure the mathematics.

§1. Categorical duality theory

Category theorists have long recognized the convenient unifying role played by the concept of a "closed category". Such a category $\mathcal{C}$ comes equipped with a unit object S, a product $\wedge: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, and an internal hom functor $F: \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. It is required that $\wedge$ be unital (with unit S), associative, and commutative up to coherent natural isomorphism and that there be a natural adjunction

$$\mathcal{C}(X \wedge Y, Z) \cong \mathcal{C}(X, F(Y,Z)).$$

There are coherence theorems to the effect that all diagrams relating these data that reasonably can be expected to commute do in fact commute [73,74]. As is customary, we do not introduce notation for the unit and associativity isomorphisms. We write $\gamma: X \wedge Y \to Y \wedge X$ for the commutativity isomorphism and we write

$$\eta: X \longrightarrow F(Y, X \wedge Y) \quad \text{and} \quad \varepsilon: F(X,Y) \wedge X \longrightarrow Y$$

for the unit and counit of the adjunction. Of course, ε is to be viewed as an evaluation map. We define the dual DX of an object X to be $F(X,S)$.

As indicated by our biased choice of notations, we are thinking of the stable category of G-spectra. However, it is useful to think in terms of such more familiar and elementary examples as the category of modules over a commutative ring R. The essential point is that there is enough information in the purely categorical setup to make it well worthwhile to treat duality first there before turning to details special to the stable category.

Various useful natural transformations are implicit in the structure of a closed category $\mathcal{C}$. First, we have the pairing

$$\wedge: F(X,Y) \wedge F(X',Y') \longrightarrow F(X \wedge X', Y \wedge Y')$$

whose adjoint is the composite

$$F(X,Y) \wedge F(X',Y') \wedge X \wedge X' \xrightarrow{1 \wedge \gamma \wedge 1} F(X,Y) \wedge X \wedge F(X',Y') \wedge X' \xrightarrow{\varepsilon \wedge \varepsilon} Y \wedge Y'.$$

In particular, with $Y = Y' = S$, $\wedge$ specializes to a pairing

$$\wedge: DX \wedge DX' \longrightarrow D(X \wedge X').$$

The map $\eta: Z \to F(S, Z \wedge S) \cong F(S,Z)$ is always an isomorphism (its inverse being $\varepsilon: F(S,Z) \cong F(S,Z) \wedge S \to Z$), and $\wedge$ also specializes to give a natural map

$$\nu\colon F(X,Y)\wedge Z \xrightarrow{1\wedge\eta} F(X,Y)\wedge F(S,Z) \xrightarrow{\wedge} F(X,Y\wedge Z).$$

In the stable category, this general nonsense map lies at the heart of Spanier-Whitehead duality relating homology and cohomology. We obtain a natural map

$$\rho\colon X \longrightarrow DDX$$

by taking the adjoint of the composite

$$X\wedge DX \xrightarrow{\gamma} DX\wedge X \xrightarrow{\varepsilon} S.$$

We obtain a natural map

$$\mu\colon F(X\wedge Y,Z) \longrightarrow F(X,F(Y,Z))$$

by applying adjunction twice to the evaluation map

$$\varepsilon\colon F(X\wedge Y),Z)\wedge X\wedge Y \longrightarrow Z.$$

It is simple to check from the adjunction that μ is an isomorphism for arbitrary objects X, Y, and Z. Of course $\wedge$, ν, and ρ need not be isomorphisms in general, and a central theme of this section is the discussion of conditions on objects which ensure that they are isomorphisms.

Definition 1.1. An object X of $\mathscr{C}$ is said to be finite if there exists a "coevaluation map" $\eta\colon S\to X\wedge DX$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & X\wedge DX \arrow[d, "\gamma"] \\ F(X,X) & DX\wedge X \arrow[l, "\nu"'] \end{tikzcd}$$

commutes. We shall see that this implies that the map ν in the diagram is an isomorphism, so that the coevaluation map is characterized as the composite $\gamma\nu^{-1}\eta$.

Dold and Puppe [47] call finite objects "strongly dualizable" (and give a more complicated but equivalent definition). We prefer the term finite since in practice there is always something finite about them. Observe that any retract of a finite object is finite.

To see the intuition, consider the category of modules over a commutative ring R. If X is a free R-module on the finite basis $\{e_1,\cdots,e_n\}$ with dual basis $\{f_1,\cdots,f_n\}$, then we obtain a coevaluation map $\eta\colon R\to X\otimes \mathrm{Hom}(X,R)$ by

setting $\eta(1)=\sum_i e_i\otimes f_i$. Here the diagram of the definition asserts that $x=\sum_i f_i(x)e_i$ for all $x\in X$. It is an observation called the "dual basis theorem" that an R-module is finite if and only if it is finitely generated and projective.

We shall see in the next section that finite G-CW spectra, and thus also their wedge summands, are finite objects in the stable category of G-spectra. In marked contrast to the nonequivariant case, a wedge summand of a finite G-CW spectrum need not have the homotopy type of a finite G-CW spectrum. We conjecture that, up to equivalence, the finite G-spectra are precisely the wedge summands of finite G-CW spectra. It is illuminating to think of finite G-CW spectra and their wedge summands as analogous to finitely generated free modules and finitely generated projective modules.

For a finite object X, the functor $(?)\wedge DX$ is right adjoint to the functor $(?)\wedge X$ and is therefore isomorphic to the functor $F(X,?)$. The following two results explain these conclusions and give some consequences.

Proposition 1.2. Let X be a finite object of $\mathscr{C}$. Then DX is a finite object of $\mathscr{C}$ and the composites

$$X\cong S\wedge X \xrightarrow{\eta\wedge 1} X\wedge DX\wedge X \xrightarrow{1\wedge\varepsilon} X\wedge S\cong X$$

and

$$DX\cong DX\wedge S \xrightarrow{1\wedge\eta} DX\wedge X\wedge DX \xrightarrow{\varepsilon\wedge 1} S\wedge DX\cong DX$$

are identity maps. Therefore, for any objects W and Z of $\mathscr{C}$,

$$\varepsilon_{\#}\colon \mathscr{C}(W,Z\wedge DX) \longrightarrow \mathscr{C}(W\wedge X,Z)$$

is an isomorphism with inverse $\eta_{\#}$, where, for $f\colon W\to Z\wedge DX$ and $g\colon W\wedge X\to Z$, $\varepsilon_{\#}(f)$ and $\eta_{\#}(g)$ are the composites

$$W\wedge X \xrightarrow{f\wedge 1} Z\wedge DX\wedge X \xrightarrow{1\wedge\varepsilon} Z\wedge S\cong Z$$

and

$$W\cong W\wedge S \xrightarrow{1\wedge\eta} W\wedge X\wedge DX \xrightarrow{g\wedge 1} Z\wedge DX.$$

Proof. By an easy diagram chase, the composite

$$S \xrightarrow{\eta} X \wedge DX \xrightarrow{\gamma} DX \wedge X \xrightarrow{1 \wedge \rho} DX \wedge DDX$$

is a coevaluation map for DX. The essential point is that the diagram

$$\begin{tikzcd}
S \arrow[r, "\eta"] \arrow[d, "\eta"'] & F(X,X) \arrow[d, "F(1,\rho)"] \\
F(DX,DX) \arrow[r, phantom, "\cong"] & F(X,DDX)
\end{tikzcd}$$

commutes, where the isomorphism is the composite

$$F(DX,DX) \xrightarrow{\mu^{-1}} F(DX \wedge X, S) \xrightarrow{F(\gamma,1)} F(X \wedge DX, S) \xrightarrow{\mu} F(X,DDX).$$

The composite $(1 \wedge \varepsilon)(\eta \wedge 1)$ is the identity map of X since the diagram

$$\begin{tikzcd}
X \cong S \wedge X \arrow[r, "\eta \wedge 1"] \arrow[d, "\eta \wedge 1"'] & X \wedge DX \wedge X \arrow[r, "1 \wedge \varepsilon"] \arrow[d, "\gamma \wedge 1"] & X \wedge S \cong X \\
F(X,X) \wedge X \arrow[urr, bend right, "\varepsilon"'] & DX \wedge X \wedge X \arrow[l, "\nu \wedge 1"'] &
\end{tikzcd}$$

commutes and its bottom composite $\varepsilon(\eta \wedge 1)$ is the identity map. Modulo transpositions, the proof that $(\varepsilon \wedge 1)(1 \wedge \eta)$ is the identity map of DX is identical.

Proposition 1.3. (i) If X is finite, then $\rho: X \to DDX$ is an isomorphism.
(ii) If either X or Z is finite, then

$$\nu: F(X,Y) \wedge Z \longrightarrow F(X, Y \wedge Z)$$

is an isomorphism.
(iii) If both X and X' are finite or if X is finite and $Y = S$ (or X' is finite and $Y' = S$), then

$$\wedge : F(X,Y) \wedge F(X',Y') \longrightarrow F(X \wedge X', Y \wedge Y')$$

is an isomorphism.

Proof. If X is finite, then easy diagram chases show that ρ^{-1} is given by the composite

$$DDX \cong S \wedge DDX \xrightarrow{\eta \wedge 1} X \wedge DX \wedge DDX \xrightarrow{1 \wedge \gamma} X \wedge DDX \wedge DX \xrightarrow{1 \wedge \varepsilon} X \wedge S \cong X$$

and ν^{-1} is given by the composite

$$F(X, Y \wedge Z) \cong F(X, Y \wedge Z) \wedge S \xrightarrow{1 \wedge \eta} F(X, Y \wedge Z) \wedge X \wedge DX$$

$$\xrightarrow{\varepsilon \wedge 1} Y \wedge Z \wedge DX \xrightarrow{\gamma} DX \wedge Y \wedge Z \xrightarrow{\nu \wedge 1} F(X,Y) \wedge Z.$$

This case of (ii) (and symmetry) implies (iii) by virtue of the commutative diagram

$$\begin{tikzcd}
F(X,Y) \wedge F(X',Y') \arrow[r, "\wedge"] \arrow[d, "\nu"] & F(X \wedge X', Y \wedge Y') \arrow[d, "\mu"] \\
F(X, Y \wedge F(X',Y')) \arrow[r, "F(1, F(1,\gamma)\nu\gamma)"] & F(X, F(X', Y \wedge Y')).
\end{tikzcd}$$

In turn, this implies that ν is an isomorphism when Z is finite by virtue of the commutative diagram

$$\begin{tikzcd}
F(X,Y) \wedge Z \arrow[r, "1 \wedge \rho"] \arrow[d, "\nu"] & F(X,Y) \wedge DDZ \arrow[r, "\wedge"] \arrow[d, "\nu"] & F(X \wedge DZ, Y) \arrow[d, "\mu"] \\
F(X, Y \wedge Z) \arrow[r, "F(1, 1 \wedge \rho)"] & F(X, Y \wedge DDZ) \arrow[r, "F(1, \nu\gamma)"] & F(X, F(DZ, Y)).
\end{tikzcd}$$

The next two results give useful information about the duals of maps between finite objects. Their proofs are easy diagram chases from the results above.

Proposition 1.4. Let X be a finite object of $\mathcal{C}$. Then the natural composite

$$\delta : DY \wedge X \xrightarrow{\gamma} X \wedge DY \xrightarrow{\rho \wedge 1} DDX \wedge DY \xrightarrow{\wedge} D(DX \wedge Y)$$

is an isomorphism for any object Y. When $Y = X$, δ is a canonical self-duality isomorphism for $DX \wedge X$ and the following diagram commutes.

$$\begin{tikzcd}
S \arrow[r, "\eta"] \arrow[d, "\eta"', "\cong"] & X \wedge DX \arrow[r, "\gamma"] & DX \wedge X \arrow[rr, "\varepsilon"] \arrow[d, "\delta"] & & S \arrow[d, "\cong"', "\eta"] \\
DS \arrow[rr, "D\varepsilon"] & & D(DX \wedge X) \arrow[r, "D\gamma"] & D(X \wedge DX) \arrow[r, "D\eta"] & DS
\end{tikzcd}$$

That is, if we regard δ as an identification, then we may regard $\gamma\eta$ as the dual of ε and ε as the dual of $\gamma\eta$.

Proposition 1.5 The dual $Df: DY \to DX$ of a map $f: X \to Y$ is uniquely characterized by commutativity of the diagram

$$\begin{tikzcd} DY \wedge X \arrow[r, "1 \wedge f"] \arrow[d, "Df \wedge 1"'] & DY \wedge Y \arrow[d, "\varepsilon"] \\ DX \wedge X \arrow[r, "\varepsilon"] & S. \end{tikzcd}$$

If X is finite, then Df coincides with the composite

$$DY \cong DY \wedge S \xrightarrow{1 \wedge \eta} DY \wedge X \wedge DX \xrightarrow{1 \wedge f \wedge 1} DY \wedge Y \wedge DX \xrightarrow{\varepsilon \wedge 1} S \wedge DX \cong DX.$$

If X and Y are both finite, then the following diagram commutes, and its commutativity also uniquely characterizes Df.

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & Y \wedge DY \arrow[d, "1 \wedge Df"] \\ X \wedge DX \arrow[r, "f \wedge 1"] & Y \wedge DX \end{tikzcd}$$

We need usable criteria for recognizing when a given object Y is isomorphic to DX for a given object X. In general, we have the obvious criterion that the functors $\mathscr{C}(?,Y)$ and $\mathscr{C}(? \wedge X,S)$ must be isomorphic. For finite X, there are more illuminating criteria, and these criteria also yield illuminating alternative characterizations of finiteness.

Theorem 1.6. The following data relating objects X and Y of $\mathscr{C}$ determine one another.

(i) Maps $\varepsilon: Y \wedge X \to S$ and $\eta: S \to X \wedge Y$ such that the composites

$$X \cong S \wedge X \xrightarrow{\eta \wedge 1} X \wedge Y \wedge X \xrightarrow{1 \wedge \varepsilon} X \wedge S \cong X$$

and

$$Y \cong Y \wedge S \xrightarrow{1 \wedge \eta} Y \wedge X \wedge Y \xrightarrow{\varepsilon \wedge 1} S \wedge Y \cong Y$$

are identity maps.

(ii) A map $\varepsilon: Y \wedge X \to S$ such that the function

$$\varepsilon_{\#}: \mathscr{C}(W, Z \wedge Y) \longrightarrow \mathscr{C}(W \wedge X, Z)$$

is a bijection for all W and Z, where $\varepsilon_{\#}(f)$ is the composite

$$W \wedge X \xrightarrow{f \wedge 1} Z \wedge Y \wedge X \xrightarrow{1 \wedge \varepsilon} Z \wedge S \cong Z.$$

(iii) A map $\eta: S \to X \wedge Y$ such that the function

$$\eta_{\#}: \mathscr{C}(W \wedge X, Z) \longrightarrow \mathscr{C}(W, Z \wedge Y)$$

is a bijection for all W and Z, where $\eta_{\#}(g)$ is the composite

$$W \cong W \wedge S \xrightarrow{1 \wedge \eta} W \wedge X \wedge Y \xrightarrow{g \wedge 1} Z \wedge Y.$$

Further, maps η and ε satisfy these properties for the pair (X,Y) if and only if the maps $\gamma\eta$ and $\varepsilon\gamma$ satisfy these properties for the pair (Y,X). If X is finite and $Y = DX$, then the canonical maps η and ε satisfy these properties. Conversely, suppose given such maps for the pair (X,Y). Then the adjoint $\tilde{\varepsilon}: Y \to DX$ is an isomorphism with inverse the composite

$$DX \cong DX \wedge S \xrightarrow{1 \wedge \eta} DX \wedge X \wedge Y \xrightarrow{\varepsilon \wedge 1} S \wedge Y \cong Y$$

and X is finite with coevaluation map the composite

$$S \xrightarrow{\eta} X \wedge Y \xrightarrow{1 \wedge \tilde{\varepsilon}} X \wedge DX.$$

Proof. Given (i), $\varepsilon_{\#}$ and $\eta_{\#}$ of (ii) and (iii) are inverse bijections. Given (ii), set $W = S$ and $Z = X$ and specify η by requiring $\varepsilon_{\#}(\eta)$ to be the identity map of X. Then (i) holds. Similarly (iii) determines (i). The symmetry statement is easily checked and we have already seen that the canonical duality maps for finite X satisfy the stated properties. The isomorphism claim of the last statement is another easy diagram chase, and the defining diagram

$$\begin{tikzcd} S \arrow[r, "\eta"] \arrow[d, "\eta"'] & X \wedge Y \arrow[r, "1 \wedge \tilde{\varepsilon}"] & X \wedge DX \arrow[d, "\gamma"] \\ F(X,X) \arrow[rr, "\nu"] & & DX \wedge X \end{tikzcd}$$

for a coevaluation is seen to commute by passing to adjoints and using that $(\eta \wedge 1)(1 \wedge \varepsilon) = 1$.

Warning 1.7. Given a pair X and Y of dual objects, there are in general many choices of pairs of maps η and ε displaying the duality. In particular, given any map ε such that $\varepsilon_{\#}$ in (ii) is a bijection, we obtain another such map by composing ε with any automorphism of S.

Remark 1.8. In the context of the theorem, there is an alternative version of Proposition 1.4 that is sometimes useful. If (η,ε) and (η',ε') represent (X,Y) and (X',Y') as dual finite pairs, then

$$(1\wedge\gamma\wedge 1)(\eta\wedge\eta')\colon S = S\wedge S \longrightarrow X\wedge X'\wedge Y\wedge Y'$$

and

$$(\varepsilon\wedge\varepsilon')(1\wedge\gamma\wedge 1)\colon Y\wedge Y'\wedge X\wedge X' \longrightarrow S\wedge S = S$$

represent $X\wedge X'$ and $Y\wedge Y'$ as a dual finite pair. In particular, this applies to (η,ε) and $(\gamma\eta,\varepsilon\gamma)$, giving a derived duality between $X\wedge Y$ and $Y\wedge X$. The diagram

$$\begin{tikzcd}
Y\wedge X\wedge S \arrow[r, "\varepsilon\wedge 1"] \arrow[d, "1\wedge 1\wedge\eta"'] & S\wedge S \arrow[r, "(1\wedge\gamma\wedge 1)(\eta\wedge\gamma\eta)"] \arrow[d, equal] & X\wedge Y\wedge Y\wedge X \arrow[d, "1\wedge 1\wedge\varepsilon"] \\
Y\wedge X\wedge X\wedge Y \arrow[r, "(\varepsilon\wedge\varepsilon\gamma)(1\wedge\gamma\wedge 1)"'] & S\wedge S \arrow[r, "\eta\wedge 1"'] & X\wedge Y\wedge S
\end{tikzcd}$$

commutes, and in this sense η and ε are characterized as duals to one another.

Now suppose given two closed categories $\mathcal{C}$ and $\mathcal{D}$ with unit objects S and T. A (lax) monoidal functor $\Phi\colon\mathcal{C}\to\mathcal{D}$ is a functor Φ together with a map $\lambda\colon T\to\Phi S$ and a natural transformation

$$\phi\colon \Phi X\wedge\Phi Y \longrightarrow \Phi(X\wedge Y)$$

such that the following coherence diagrams commute.

$$\begin{tikzcd}
\Phi X \arrow[r, "\cong"] \arrow[d, "\cong"'] & \Phi(S\wedge X) \\
T\wedge\Phi X \arrow[r, "\lambda\wedge 1"'] & \Phi S\wedge\Phi X \arrow[u, "\phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi X\wedge\Phi Y \arrow[r, "\gamma"] \arrow[d, "\phi"'] & \Phi Y\wedge\Phi X \arrow[d, "\phi"] \\
\Phi(X\wedge Y) \arrow[r, "\Phi\gamma"'] & \Phi(Y\wedge X)
\end{tikzcd}
\qquad
\begin{tikzcd}
\Phi X\wedge\Phi Y\wedge\Phi Z \arrow[r, "1\wedge\phi"] \arrow[d, "\phi\wedge 1"'] & \Phi X\wedge\Phi(Y\wedge Z) \arrow[d, "\phi"] \\
\Phi(X\wedge Y)\wedge\Phi Z \arrow[r, "\phi"'] & \Phi(X\wedge Y\wedge Z)
\end{tikzcd}$$

(See Lewis [73] for a categorical discussion.) We say that Φ is a strict monoidal functor if λ and ϕ are isomorphisms. In practice, λ is usually an isomorphism even if ϕ is not.

Proposition 1.9. Let $\Phi\colon\mathcal{C}\to\mathcal{D}$ be a monoidal functor such that $\lambda\colon T\to\Phi S$ is an

isomorphism. Let X be a finite object of $\mathcal{C}$ such that

$$\phi\colon \Phi X\wedge\Phi DX \longrightarrow \Phi(X\wedge DX)$$

is an isomorphism. Then ΦX is a finite object of $\mathcal{D}$, the natural map $\Phi DX\to D\Phi X$ is an isomorphism, and

$$\phi\colon \Phi X\wedge\Phi Y \longrightarrow \Phi(X\wedge Y)$$

is an isomorphism for every object Y of $\mathcal{C}$.

Proof. For any object X of $\mathcal{C}$, we have the composite

$$\varepsilon\colon \Phi DX\wedge\Phi X \xrightarrow{\phi} \Phi(DX\wedge X) \xrightarrow{\Phi\varepsilon} \Phi S \xrightarrow{\lambda^{-1}} T.$$

Its adjoint $\tilde{\varepsilon}\colon \Phi DX\to D\Phi X$ is the natural map referred to in the statement. Under our hypotheses on X, we also have the composite

$$\eta\colon T \xrightarrow{\lambda} \Phi S \xrightarrow{\Phi\eta} \Phi(X\wedge DX) \xrightarrow{\phi^{-1}} \Phi X\wedge\Phi DX,$$

and it is easily checked that η and ε satisfy the conditions of Theorem 1.6 (i). For the last part, the dotted arrow composite in the following diagram is easily checked to be inverse to ϕ.

$$\begin{tikzcd}
\Phi(X\wedge Y) \cong T\wedge\Phi(X\wedge Y) \arrow[r, "\eta\wedge 1"] \arrow[d, dashed, "\phi^{-1}"'] & \Phi X\wedge\Phi DX\wedge\Phi(X\wedge Y) \arrow[d, "1\wedge\phi"] \\
\Phi X\wedge\Phi Y \cong \Phi X\wedge\Phi(S\wedge Y) & \Phi X\wedge\Phi(DX\wedge X\wedge Y) \arrow[l, "1\wedge\Phi(\varepsilon\wedge 1)"']
\end{tikzcd}$$

Dold and Puppe say that a finite object X is "Φ-flat" if the map $\phi\colon \Phi X\wedge\Phi Y\to\Phi(X\wedge Y)$ is an isomorphism for all Y or, equivalently, for $Y = DX$.

Remarks 1.10. For any closed category $\mathcal{C}$, $\mathcal{C}\times\mathcal{C}$ and $\mathcal{C}^{op}$ are closed categories in an evident way. The functor $\wedge\colon\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is strict monoidal with respect to the evident isomorphisms

$$S\cong S\wedge S \quad\text{and}\quad 1\wedge\gamma\wedge 1\colon X\wedge X'\wedge Y\wedge Y' \longrightarrow X\wedge Y\wedge X'\wedge Y'.$$

The functor $D\colon\mathcal{C}^{op}\to\mathcal{C}$ is lax monoidal with respect to

$$\eta\colon S \xrightarrow{\cong} DS \quad\text{and}\quad \wedge\colon DX\wedge DY \longrightarrow D(X\wedge Y);$$

D is strict monoidal when restricted to the full subcategory of finite objects.

§2. Duality for G-spectra

We first discuss cofibration and fibration sequences in the G-stable category $\overline{h}G\mathscr{S}U$ and show that finite G-CW spectra are finite objects in the categorical sense of the previous section. We then relate duality to the change of groups isomorphisms proven in the previous chapter.

The definition and basic properties of cofibration and fibration sequences are the same for G-prespectra and G-spectra as they are for G-spaces. We define the cone $CX = X \wedge I$ with I being given the basepoint 1. We define the path object $PX = F(I,X)$ with I being given the basepoint 0. For a map $f\colon X \to Y$, we define the homotopical cofibre and fibre of f by

$$Cf = Y \cup_f CX \quad \text{and} \quad Ff = X \times_f PY.$$

We form the sequence

$$\text{(C)} \qquad X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \longrightarrow \cdots,$$

i and π being the canonical inclusion and quotient map. Each successive pair of maps in (C) is equivalent to a map followed by the inclusion of its target in its cofibre, the inclusion being a cofibration. There results a long exact sequence upon application of the functor $[?,Z]_G$ for any Z. Similarly, we form the sequence

$$\text{(F)} \qquad \cdots \longrightarrow \Omega^2 X \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\iota} \Omega Ff \xrightarrow{-\Omega p} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\iota} Ff \xrightarrow{p} X \xrightarrow{f} Y,$$

ι and p being the canonical inclusion and projection. Each successive pair of maps in (F) is equivalent to a map preceded by the projection of its fibre to its source, the projection being a fibration. There results a long exact sequence upon application of the functor $[Z,?]_G$ for any Z. In particular, fibration sequences give rise to long exact sequences of homotopy groups. The essential new fact for G-spectra, as opposed to G-spaces, is that cofibration sequences also give rise to long exact sequences upon application of $[Z,?]_G$.

<u>Lemma 2.1</u>. Let $f\colon X \to Y$ be a map of G-spectra. Then the following sequence is exact for any G-spectrum Z.

$$[Z,X]_G \xrightarrow{f_*} [Z,Y]_G \xrightarrow{i_*} [Z,Cf]_G.$$

Therefore the following sequence is also exact, where ∂ is the composite $\pi_n^H Cf \xrightarrow{\pi_*} \pi_n^H \Sigma X \xrightarrow{\Sigma_*^{-1}} \pi_{n-1}^H X$.

$$\cdots \longrightarrow \pi_n^H X \xrightarrow{f_*} \pi_n^H Y \xrightarrow{i_*} \pi_n^H Cf \xrightarrow{\partial} \pi_{n-1}^H X \longrightarrow \cdots$$

<u>Proof</u>. Let $i_*(\alpha) = 0$, $\alpha\colon Z \to Y$. We can construct β and γ which make the following diagram homotopy commutative.

$$\begin{tikzcd}
Z \arrow[r, equal] & Z \arrow[r, "i"] \arrow[d, "\alpha"] & CZ \arrow[r, "\pi"] \arrow[d, dashed, "\beta"] & \Sigma Z \arrow[r, equal] \arrow[d, dashed, "\gamma"] & \Sigma Z \arrow[d, "\Sigma\alpha"] \\
X \arrow[r, "f"] & Y \arrow[r, "i"] & Cf \arrow[r, "\pi"] & \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y
\end{tikzcd}$$

If $\gamma = \Sigma\gamma'$, then $f_*(\gamma') = \alpha$.

We next describe how cofibration and fibration sequences hook up.

<u>Definition 2.2</u>. For a map $f\colon X \to Y$ of G-spaces, define

$$\eta\colon Ff \longrightarrow \Omega Cf \quad \text{and} \quad \varepsilon\colon \Sigma Ff \longrightarrow Cf$$

by

$$\eta(x,\omega)(t) = \varepsilon(x,\omega,t) = \begin{cases} \omega(2t) & \text{if } 0 \leq t \leq 1/2 \\ (x,2t-1) & \text{if } 1/2 \leq t \leq 1, \end{cases}$$

where $x \in X$, $\omega \in PY$, and $t \in I$, with $\omega(1) = f(x)$. Define analogous η and ε spacewise for a map $f\colon X \to Y$ of G-prespectra. For a map $f\colon X \to Y$ of G-spectra, note that

$$\Omega\ell X = \ell\Omega X,\ F(\ell f) = \ell F(f),\ \Sigma X = L\Sigma\ell X, \text{ and } Cf = LC(\ell f),$$

and define

$$\varepsilon = L\varepsilon\colon \Sigma Ff = L\Sigma F(\ell f) \longrightarrow LC(\ell f) = Cf;$$

then define $\eta\colon Ff \to \Omega Cf$ to be the (spectrum level) adjoint of ε.

<u>Lemma 2.3</u>. For a map $f\colon X \to Y$ (of G-spaces, G-prespectra, or G-spectra), the

following diagram is homotopy commutative.

$$\begin{tikzcd}
 & & \Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\varepsilon"] & \Sigma\Omega Y \arrow[r, "\Sigma\iota"] \arrow[d, "\varepsilon"] & \Sigma Ff \arrow[r, "\Sigma p"] \arrow[d, "\varepsilon"] & \Sigma X \arrow[d, equal] \\
\Omega Y \arrow[r, "\iota"] \arrow[d, equal] & Ff \arrow[r, "p"] \arrow[d, "\eta"] & X \arrow[r, "f"] \arrow[d, "\eta"] & Y \arrow[r, "i"] \arrow[d, "\eta"] & Cf \arrow[r, "\pi"] & \Sigma X \\
\Omega Y \arrow[r, "\Omega i"] & \Omega Cf \arrow[r, "\Omega\pi"] & \Omega\Sigma X \arrow[r, "\Omega\Sigma f"] & \Omega\Sigma Y & &
\end{tikzcd}$$

The proof is elementary. Up to one change of sign, the top row is obtained by applying Σ to part of (F) and the bottom row is obtained by applying Ω to part of (C). By the desuspension theorem, the five lemma, and the Whitehead theorem, the previous lemmas have the following immediate consequence.

Theorem 2.4. For a map $f\colon X \to Y$ of G-spectra,

$$\eta\colon Ff \longrightarrow \Omega Cf \quad \text{and} \quad \varepsilon\colon \Sigma Ff \longrightarrow Cf$$

are isomorphisms in the G-stable category.

Of course, passage to the stable category entails use of CW-approximation, but there is no loss of homotopical information. In the language of triangulated categories [121,107], the conclusion is that cofiberings and fiberings give two distinct triangulations of the stable category, the negative of a cofibration triangle being a fibration triangle and conversely.

Passage to function objects converts cofiberings to fiberings.

Lemma 2.5. For a map $f\colon X \to Y$ (of G-spaces, G-prespectra, or G-spectra) and an object Z, the sequence

$$F(\Sigma X,Z) \xrightarrow{\pi^*} F(Cf,Z) \xrightarrow{i^*} F(Y,Z) \xrightarrow{f^*} F(X,Z)$$

is isomorphic to the sequence

$$\Omega F(X,Z) \xrightarrow{\iota} F(f^*) \xrightarrow{p} F(Y,Z) \xrightarrow{f^*} F(X,Z).$$

Proof. Modulo reversal of the I coordinate dictated by our choices of basepoints,

we may identify $PF(X,Z)$ with $F(CX,Z)$. Since the functor $F(?,Z)$ converts pushouts to pullbacks, there results an identification of $F(Cf,Z)$ with $F(f^*)$. This identification converts i^* to p and restricts via π^* and ι to the negative of the standard identification of $F(\Sigma X,Z)$ and $\Omega F(X,Z)$.

In particular, the dual of a cofibering is a fibering, and its negative is thus another cofibering. An easy induction on the number of cells, together with our calculation of $D(G/H^+)$ in II.6.3, gives the following consequence.

Corollary 2.6. In the G-stable category, the dual of a finite G-CW spectrum is isomorphic to a finite G-CW spectrum.

We say that a G-spectrum is finite if it is a finite object of $\bar{h}G\mathscr{S}U$ in the sense of Definition 1.1. Recall the criteria for recognizing such objects from Theorem 1.6.

Theorem 2.7. Any finite G-CW spectrum is a finite G-spectrum, hence so is any wedge summand of a finite G-CW spectrum.

Proof. We first show that orbits are finite G-spectra. Define

$$\varepsilon\colon (G \ltimes_H S^{-L}) \wedge \Sigma^\infty G/H^+ \longrightarrow S$$

to be the adjoint of the equivalence

$$\psi\colon G \ltimes_H S^{-L} \longrightarrow F_H[G,S) \cong F(G/H^+,S) = D(G/H^+).$$

The map $\varepsilon_\#$ of Theorem 1.6 (ii) may be described as the dotted arrow composite in the diagram

$$\begin{tikzcd}
{[W, Z \wedge (G \ltimes_H S^{-L})]_G} \arrow[r, "(1\wedge\psi)_*"] \arrow[d, dashed, "\varepsilon_\#"'] & {[W, Z \wedge D(G/H^+)]_G} \arrow[r, "\gamma_*"] & {[W, D(G/H^+) \wedge Z]_G} \arrow[d, "\nu_*"] \\
{[W \wedge \Sigma^\infty G/H^+, Z]_G} \arrow[rr, equal] & & {[W, F(G/H^+, Z)]_G.}
\end{tikzcd}$$

Using the definition of ω in II.6.1 and naturality diagrams, we find easily that the following diagram commutes.

$$\begin{tikzcd}
F_H[G,S)\wedge Z \cong D(G/H^+)\wedge Z \arrow[r, "\nu"] \arrow[d, "\omega\wedge 1"'] & F(G/H^+,Z) \cong F_H[G,Z) \arrow[d, "\omega"] \\
(G\ltimes_H S^{-L})\wedge Z \arrow[r, "\zeta^{-1}"] & G\ltimes_H \Sigma^{-L}Z
\end{tikzcd}$$

Since both maps ω are equivalences, ν is an equivalence. Thus ν_* and $\varepsilon_\#$ are bijections and $\Sigma^\infty G/H^+$ is finite by Theorem 1.6. It follows that all suspensions of $\Sigma^\infty G/H^+$ are finite. Now consider the canonical evaluation map $\varepsilon\colon DX\wedge X\to S$. Clearly

$$\varepsilon_\#\colon [W,Z\wedge DX]_G \longrightarrow [W\wedge X,Z]_G$$

is natural in X. By the five lemma and the results above, the cofibre of a map between finite objects is itself finite. The conclusion follows by induction.

As said before, we conjecture that, up to equivalence, all finite G-spectra are wedge summands of finite G-CW spectra. Proposition 1.3 specializes to give the following conclusions.

Proposition 2.8. (i) If X is a finite G-spectrum, then $\rho\colon X\to DDX$ is an equivalence.

(ii) If either X or Z is a finite G-spectrum, then

$$\nu\colon F(X,Y)\wedge Z \longrightarrow F(X,Y\wedge Z)$$

is an equivalence.

(iii) If both X and X' are finite G-spectra or if X is a finite G-spectrum and $Y = S$ (or X' is finite and $Y' = S$), then

$$\wedge\colon F(X,Y)\wedge F(X',Y') \longrightarrow F(X\wedge X',Y\wedge Y')$$

is an equivalence.

We single out the case $Y = S$ of (ii) since it is the basic spectrum level duality theorem.

Corollary 2.9. if X is a finite G-spectrum, then

$$\nu\colon DX\wedge E \longrightarrow F(X,E)$$

is an equivalence for any G-spectrum E. Therefore

$$\nu_*\colon E_*(DX) = \pi_*^G(DX\wedge E) \longrightarrow \pi_*^G F(X,E) = E^*(X)$$

is an isomorphism.

Here and later, homology and cohomology may be interpreted in either the Z-graded or RO(G;U)-graded sense.

Remark 2.10. We have been working implicitly in a fixed G-universe U. Duality for the orbit G/H requires that G/H embed in U. Provided that we restrict attention to G-CW spectra built up out of orbits which embed in U, we need not insist that U be complete.

We shall return to homological interpretations of duality in the next section, but we first record the behavior of duality with respect to change of groups.

Proposition 2.11. Let H be a subgroup of G and let L be the tangent representation of H at the identity coset of G/H. Let X and Y be dual finite H-spectra with duality maps $\varepsilon\colon Y\wedge X\to S$ and $\eta\colon S\to X\wedge Y$ and let ε and η correspond to the G-maps

$$\tilde\varepsilon\colon G\ltimes_H(Y\wedge X)\longrightarrow S \quad\text{and}\quad \tilde\eta\colon S\longrightarrow G\ltimes_H(X\wedge\Sigma^{-L}Y)$$

under the isomorphisms of II.4.3 and II.6.5. Then $G\ltimes_H X$ and $G\ltimes_H\Sigma^{-L}Y$ are dual finite G-spectra with duality maps

$$(G\ltimes_H\Sigma^{-L}Y)\wedge(G\ltimes_H X) \cong G\ltimes_H((G\ltimes_H\Sigma^{-L}Y)\wedge X) \xrightarrow{1\ltimes(\mu\wedge 1)} G\ltimes_H(Y\wedge X) \xrightarrow{\tilde\varepsilon} S$$

and

$$S \xrightarrow{\tilde\eta} G\ltimes_H(X\wedge\Sigma^{-L}Y) \xrightarrow{1\ltimes(\eta\wedge 1)} G\ltimes_H((G\ltimes_H X)\wedge\Sigma^{-L}Y) \cong (G\ltimes_H X)\wedge(G\ltimes_H\Sigma^{-L}Y),$$

where $\mu\colon G\ltimes_H\Sigma^{-L}Y\to Y$ is the H-map given by II.6.8 and $\eta\colon X\to G\ltimes_H X$ is the natural inclusion of H-spectra.

Proof. For G-spectra E and E', II.4.3, II.4.9, II.6.5, and Theorem 1.6 give the chain of isomorphisms

$$[E\wedge(G\ltimes_H X),E']_G \cong [G\ltimes_H(E\wedge X),E']_G$$

$$\cong [E\wedge X,E']_H \cong [E,E'\wedge Y]_H$$

$$\cong [E,G\ltimes_H(E'\wedge\Sigma^{-L}Y)]_G \cong [E,E'\wedge(G\ltimes_H\Sigma^{-L}Y)]_G.$$

Thus $G\ltimes_H X$ and $G\ltimes_H\Sigma^{-L}Y$ are G-dual. We obtain the specified descriptions of the duality maps by taking $E = G\ltimes_H\Sigma^{-L}Y$ and $E' = S$ and tracing the identity map of $G\ltimes_H\Sigma^{-L}Y$ back through the chain and by taking $E = S$ and $E' = G\ltimes_H X$ and tracing the identity map of $G\ltimes_H X$ through the chain.

Proposition 2.12. Let N be a normal subgroup of G with quotient homomorphism $\varepsilon\colon G \to J = G/N$ and let A be the adjoint representation of G derived from N. Let U be a complete G-universe and let $i\colon U^N \to U$ be the inclusion. Let $X \in G\mathscr{S}U^N$ be an N-free G-spectrum. If i_*X is a finite G-spectrum, then its dual is N-free and therefore has the form i_*Y for an N-free G-spectrum $Y \in G\mathscr{S}U^N$. Let $\varepsilon\colon i_*Y\wedge i_*X \to S$ and $\eta\colon S \to i_*X\wedge i_*Y$ be duality maps and let ε and η correspond to the J-maps

$$\tilde{\varepsilon}\colon (Y\wedge X)/N \longrightarrow S \quad\text{and}\quad \tilde{\eta}\colon S \longrightarrow (X\wedge\Sigma^A Y)/N$$

under the isomorphisms of II.4.5 and II.7.2. Then X/N and $(\Sigma^A Y)/N$ are dual finite J-spectra with duality maps

$$(\Sigma^A Y)/N\wedge(X/N) \cong ((\Sigma^A Y)/N\wedge X)/N \xrightarrow{\tilde{\tau}/N} (Y\wedge X)/N \xrightarrow{\tilde{\varepsilon}} S$$

and

$$S \xrightarrow{\tilde{\eta}} (X\wedge\Sigma^A Y)/N \xrightarrow{(\xi\wedge 1)/N} ((X/N)\wedge\Sigma^A Y)/N \cong (X/N)\wedge(\Sigma^A Y)/N.$$

Here $\tilde{\tau}$ is characterized (via II.2.8(i) and the freeness of X) by

$$i_*(\tilde{\tau}) = \tau\wedge 1\colon i_*(\Sigma^A Y)/N\wedge i_*X \longrightarrow i_*Y\wedge i_*X,$$

where τ is the dimension-shifting transfer of II.7.5, and $\xi\colon X \to X/N$ is the projection. Moreover, $i_*\xi$ and τ are dual G-maps.

Proof. The following diagram commutes, where $\pi\colon E\mathscr{F}^+ \to S^0$ is the projection.

$$\begin{tikzcd}
E\mathscr{F}(N)^+\wedge D(i_*X) \arrow[rr, "\nu"] \arrow[dr, "\pi\wedge 1"'] & & F(i_*X,E\mathscr{F}(N)^+) \arrow[dl, "F(1,\pi)"] \\
& D(i_*X) &
\end{tikzcd}$$

Since i_*X is finite, ν is an equivalence. Since i_*X is N-free, II.2.2 gives that

$$(\Sigma^\infty\pi)_*\colon [\Sigma^n i_*X,\Sigma^\infty E\mathscr{F}(N)^+]_G \longrightarrow [\Sigma^n i_*X,S]_G$$

is an isomorphism for any n. This implies that $F(1,\pi)$ is an equivalence. Therefore $\pi\wedge 1$ is an equivalence and $D(i_*X)$ is N-free (where of course we mean that a G-CW approximation of $D(i_*X)$ has the homotopy type of an N-free G-CW spectrum). By II.2.8, $D(i_*X)$ is equivalent to i_*Y for an N-free $Y \in G\mathscr{S}U^N$. Now let $\varepsilon^\# = i_*\varepsilon^*\colon J\mathscr{S}U^N \to G\mathscr{S}U$. For $E,E' \in J\mathscr{S}U^N$, II.4.5, Theorem 1.6, and II.7.2 give the following three isomorphisms, where G-maps are computed in U, J-maps are computed in U^N, and we implicitly use the commutation of i_* with smash products.

$$[E\wedge X/N,E']_J \cong [\varepsilon^\# E\wedge i_*X,\varepsilon^\# E']_G$$

$$\cong [\varepsilon^\# E,\varepsilon^\# E'\wedge i_*Y]_G \cong [E,E'\wedge(\Sigma^A Y)/N]_J.$$

Thus X/N and $(\Sigma^A Y)/N$ are dual. We obtain the specified descriptions of the duality maps by taking $E = (\Sigma^A Y)/N$ and $E' = S$ and tracing back the identity map of $(\Sigma^A Y)/N$ and by taking $E = S$ and $E' = X/N$ and tracing through the identity map of X/N. Proposition 1.9 implies that $\varepsilon^\#(X/N)$ and $\varepsilon^\#((\Sigma^A Y)/N)$ are dual in $\bar{h}G\mathscr{S}U$, and the last statement follows by the characterization of dual maps in Proposition 1.5.

Proposition 2.13. With N, J, and A as in the previous proposition, let X be a finite N-free G-CW complex. Then the G-dual of X is equivalent to $\Sigma^{-Z}\Sigma^\infty Y$ for some finite N-free G-CW complex Y and some representation Z of J. Moreover, the J-dual of X/N is equivalent to $\Sigma^{-Z}\Sigma^\infty(\Sigma^A Y)/N$.

Proof. The G-dual of X is a finite N-free G-CW spectrum by the proof of Corollary 2.6, and the first statement follows from II.2.8(iii). The previous proposition implies the second statement.

When G is finite, A disappears and this result is due to Adams [3,8.5] (who ascribes the case $N = G$ to us).

§3. Slant products and V-duality of G-spaces

We wish to interpret duality in terms of slant products, and we digress to define the basic products in homology and cohomology theory. Consider variable spectra X and X' and coefficient spectra E and E'. We have the evident pairings

(1) $$1\wedge\gamma\wedge 1\colon X\wedge E\wedge X'\wedge E'\longrightarrow X\wedge X'\wedge E\wedge E'$$

and

(2) $$\wedge\colon F(X,E)\wedge F(X',E')\longrightarrow F(X\wedge X',E\wedge E').$$

On passage to homotopy groups, these give rise to the external products

(1') $$E_*(X)\otimes E_*(X')\longrightarrow (E\wedge E')_*(X\wedge X')$$

and

(2') $$E^*(X)\otimes E'^*(X')\longrightarrow (E\wedge E')^*(X\wedge X').$$

We also have the slant products defined by commutativity of the diagrams

(3) $$\begin{tikzcd} F(X\wedge X',E)\wedge X\wedge E' \arrow[rr, "\mu\wedge 1\wedge 1"] \arrow[d, "/"'] & & F(X,F(X',E))\wedge X\wedge E' \arrow[d, "\varepsilon\wedge 1"] \\ F(X',E\wedge E') & & F(X',E)\wedge E' \arrow[ll, "\nu"'] \end{tikzcd}$$

and

(4) $$\begin{tikzcd} X\wedge X'\wedge E\wedge F(X,E') \arrow[rr, "\gamma"] \arrow[d, "\backslash"'] & & X'\wedge E\wedge F(X,E')\wedge X. \arrow[dll, "1\wedge 1\wedge\varepsilon"] \\ X'\wedge E\wedge E' & & \end{tikzcd}$$

On passage to homotopy groups, these give rise to the homological slant products

(3') $$/\colon E^*(X\wedge X')\otimes E'_*(X)\longrightarrow (E\wedge E')^*(X')$$

and

(4') $$\backslash\colon E_*(X\wedge X')\otimes E'^*(X)\longrightarrow (E\wedge E')_*(X').$$

In view of the artificial and hard to remember appearance of (3') and (4'), it is not surprising that no two authors seem to have chosen precisely the same definition of the slant products. It is convenient and sensible to rewrite these products in their adjoint forms

(3") $$/\colon E^*(X\wedge X')\longrightarrow \mathrm{Hom}(E'_*(X),(E\wedge E')^*(X'))$$

and

(4") $$\backslash\colon E_*(X\wedge X')\longrightarrow \mathrm{Hom}(E'^*(X),(E\wedge E')_*(X')).$$

In these forms, the variables are clearly seen to be written in their most natural order.

By coherence, any well formulated diagram involving the transformations (1) through (4) will automatically commute. The translation of such diagrams to formulas relating the products (1') through (4') is immediate. In practice, of course, $E = E'$ is a ring spectrum and we obtain internal products by composition with the multiplication $E\wedge E\to E$. Similarly, $\wedge$ and $\backslash$ are most often applied with $X = X'$ a suspension spectrum and are then composed with Δ^* and Δ_* to obtain the cup and cap products

$$\cup\colon E^*(X)\otimes E^*(X)\longrightarrow E^*(X) \quad\text{and}\quad \cap\colon E_*(X)\otimes E^*(X)\longrightarrow E_*(X).$$

Now consider a map $\varepsilon\colon Y\wedge X\to S$ with adjoint $\tilde{\varepsilon}\colon Y\to DX$. We may view ε as an element of the G-cohomotopy group $\pi^0_G(Y\wedge X)$. Similarly, we view a map $\eta\colon S\to X\wedge Y$ as an element of the G-homotopy group $\pi_0^G(X\wedge Y)$.

Proposition 3.1. Let $\varepsilon\colon Y\wedge X\to S$ be a map of G-spectra and let $\varepsilon' = \varepsilon\gamma\colon X\wedge Y\to S$. Let E be a ring G-spectrum and, for $H\subset G$, regard G-spectra as H-spectra by neglect of structure. The slant products

$$\varepsilon/(?)\colon E^H_*(Y)\longrightarrow E^*_H(X) \quad\text{and}\quad \varepsilon'/(?)\colon E^H_*(X)\longrightarrow E^*_H(Y)$$

are induced by the respective composite H-maps

$$Y\wedge E\xrightarrow{\tilde{\varepsilon}\wedge 1} DX\wedge E\xrightarrow{\nu} F(X,E) \quad\text{and}\quad X\wedge E\xrightarrow{\tilde{\varepsilon}'\wedge 1} DY\wedge E\xrightarrow{\nu} F(Y,E).$$

If X and Y are finite G-spectra, then $\varepsilon/(?)$ and $\varepsilon'/(?)$ are both isomorphisms for all E, all H, and all gradings in $RO(H;U)$ if and only if either is an isomorphism for $E = S$, all H, and all integer gradings.

Proof. Note that $\tilde{\varepsilon}'$ coincides with the composite

$$X\xrightarrow{\rho} DDX\xrightarrow{D\tilde{\varepsilon}} DY.$$

The first statement is proven by easy diagram chases. If $\varepsilon/(?)\colon \pi_n^H Y\to\pi_n^H DX$ is an isomorphism for all H and n, then $\tilde{\varepsilon}\colon Y\to DX$ is an equivalence by Whitehead's theorem, and similarly for $\varepsilon'/(?)$. The rest follows from Theorem 1.6.

Scholium 3.2. In an exercise [131, p. 462], Spanier defined duality nonequivariantly by requiring both $\pi_*Y \cong \pi^*X$ and $\pi_*X \cong \pi^*Y$. The redundancy has been copied by several later authors.

The previous result admits a dual.

Proposition 3.3. Let $\eta: S \to X \wedge Y$ be a map of finite G-spectra and let $\eta' = \gamma\eta: S \to Y \wedge X$. The slant products

$$\eta'\backslash(?): E^*_H(Y) \longrightarrow E^H_*(X) \quad \text{and} \quad \eta\backslash(?): E^*_H(X) \longrightarrow E^H_*(Y)$$

are induced by the respective composite H-maps

$$F(Y,E) \xrightarrow{\nu^{-1}} DY \wedge E \xrightarrow{\tilde{\eta}\wedge 1} X \wedge E \quad \text{and} \quad F(X,E) \xrightarrow{\nu^{-1}} DX \wedge E \xrightarrow{\tilde{\eta}'\wedge 1} Y \wedge E,$$

where the adjoints are taken with respect to the canonical isomorphisms

$$\varepsilon_\#: [S, X \wedge Y]_H \longrightarrow [DY, X]_H \quad \text{and} \quad \varepsilon_\#: [S, Y \wedge X]_H \longrightarrow [DX, Y]_H.$$

The homomorphisms $\eta'\backslash(?)$ and $\eta\backslash(?)$ are both isomorphisms for all E, all H, and all gradings in $RO(H;U)$ if and only if either is an isomorphism for $E = S$, all H, and all integer gradings.

We now shift our focus from G-spectra to G-spaces. By I.8.16, finite G-CW spectra are homotopy equivalent to desuspensions of finite G-CW complexes. In particular, if X is a finite G-CW complex, then $D(X)$ is equivalent to $\Sigma^{-V}\Sigma^\infty Y$ for a finite G-CW complex Y and a G-representation V. This is what brings "n-duality" into the classical nonequivariant theory and "V-duality" into the equivariant theory. Note that the suspension spectrum of a finitely dominated G-CW complex is a wedge summand of a finite G-CW spectrum and is therefore finite.

Definition 3.4. Let X and Y be based G-spaces whose suspension spectra are finite and let V be a representation of G. A G-map $\varepsilon: Y \wedge X \to S^V$ is said to be a V-duality if

$$\varepsilon/(?): \pi^H_n(\Sigma^\infty Y) \longrightarrow \pi^{-n}_H(\Sigma^{-V}\Sigma^\infty X)$$

is an isomorphism for all $H \subset G$ and all $n \in Z$ or, equivalently, if

$$(\varepsilon\gamma)/(?): \pi^H_n(\Sigma^\infty X) \longrightarrow \pi^{-n}_H(\Sigma^{-V}\Sigma^\infty Y)$$

is an isomorphism for all $H \subset G$ and all $n \in Z$.

The first slant product can be defined directly or by applying the definitions above to

$$\bar{\varepsilon} = \Sigma^{-V}\Sigma^\infty\varepsilon: (\Sigma^{-V}\Sigma^\infty Y) \wedge \Sigma^\infty X \simeq \Sigma^{-V}\Sigma^\infty(Y \wedge X) \longrightarrow \Sigma^{-V}\Sigma^\infty S^V \simeq S,$$

and similarly for the second. The equivalence of the two conditions is part of Proposition 3.1, which also shows that these slant product isomorphisms imply many others. There is a dual definition based on Proposition 3.3. Of course, if ε is a V-duality, then the adjoint of $\bar{\varepsilon}$ is an equivalence $\Sigma^{-V}\Sigma^\infty Y \to DX$ and $\Sigma^\infty X$ and $\Sigma^{-V}\Sigma^\infty Y$ are dual finite G-spectra.

We shall need a purely space level criterion for determining when a given map ε is a V-duality. There is a dual criterion for maps η.

Lemma 3.5. Let $\eta: S^V \to X \wedge Y$ and $\varepsilon: Y \wedge X \to S^V$ be G-maps such that the following diagrams become commutative upon passage to suspension spectra and the stable category.

$$\begin{tikzcd} S^V \wedge X \arrow[r, "\eta\wedge 1"] \arrow[dr, "\gamma"'] & X \wedge Y \wedge X \arrow[d, "1\wedge\varepsilon"] \\ & X \wedge S^V \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} Y \wedge S^V \arrow[r, "1\wedge\eta"] \arrow[d, "\gamma"'] & Y \wedge X \wedge Y \arrow[d, "\varepsilon\wedge 1"] \\ S^V \wedge Y \arrow[r, "\sigma\wedge 1"] & S^V \wedge Y \end{tikzcd}$$

Here σ is the sign map, $\sigma(v) = -v$, and the γ are transpositions. Then the suspension spectra of X and Y are finite and ε is a V-duality. Conversely, if ε is a V-duality, then there is a stable G-map η such that the specified diagrams commute stably, and η is uniquely characterized by the commutativity of either of the diagrams.

Proof. Define $\bar{\varepsilon}$ as above and let

$$\bar{\eta} = \Sigma^{-V}\Sigma^\infty\eta: S \simeq \Sigma^{-V}\Sigma^\infty S^V \longrightarrow \Sigma^{-V}\Sigma^\infty(X \wedge Y) \simeq \Sigma^\infty X \wedge \Sigma^{-V}\Sigma^\infty Y.$$

Easy diagram chases show that stable commutativity of the given diagrams is equivalent to commutativity of the diagrams

$$\begin{tikzcd} \Sigma^\infty X \arrow[r, "\bar{\eta}\wedge 1"] \arrow[dr, equal] & \Sigma^\infty X\wedge\Sigma^{-V}\Sigma^\infty Y\wedge\Sigma^\infty X \arrow[d, "1\wedge\bar{\varepsilon}"] \\ & \Sigma^\infty X \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \Sigma^{-V}\Sigma^\infty Y \arrow[r, "1\wedge\bar{\eta}"] \arrow[dr, equal] & \Sigma^{-V}\Sigma^\infty Y\wedge\Sigma^\infty X\wedge\Sigma^{-V}\Sigma^\infty Y \arrow[d, "\bar{\varepsilon}\wedge 1"] \\ & \Sigma^{-V}\Sigma^\infty Y \end{tikzcd}$$

In the second chase, we use the fact, implied by II.5.11, that the composite

$$S \simeq S^{-V}\wedge S^V \xrightarrow{\gamma} S^V\wedge S^{-V} \xrightarrow{\sigma\wedge 1} S^V\wedge S^{-V} \simeq S$$

is the identity (where the equivalences are given by duality maps, S^{-V} being canonically equivalent to DS^V). The conclusion follows from the equivalence of (i) and (ii) of Theorem 1.6.

The reader may prefer the following nonequivariant criterion. This result has long been folklore when G is finite; we first learned it from Frank Connolly.

Theorem 3.6. Let X be a G-space such that $\Sigma^\infty X$ is a finite G-spectrum with dual of the form $\Sigma^{-V'}\Sigma^\infty Y'$ for some representation V' and G-space Y'. Let Y be a G-space dominated by a finite G-CW complex. Then a G-map $\varepsilon\colon Y\wedge X\to S^V$ is a V-duality if and only if its fixed point map $\varepsilon^H\colon Y^H\wedge X^H\to (S^V)^H$ is an n_H-duality for each $H\subset G$, where $n_H = \dim(V^H)$.

Proof. Clearly ε is a V-duality if and only if $\Sigma^{V'}\varepsilon\colon \Sigma^{V'}Y\wedge X\to S^{V+V'}$ is a $(V+V')$-duality. Since $\Sigma^{V'}Y$ is finitely dominated and $\Sigma^{-(V+V')}\Sigma^\infty(\Sigma^V Y')$ is dual to $\Sigma^\infty X$, we may assume without loss of generality that $V = V'$. Let $\zeta\colon \Sigma^\infty Y\to\Sigma^\infty Y'$ be the adjoint of $\Sigma^\infty\varepsilon$ with respect to the adjunction given by the duality of $\Sigma^\infty X$ and $\Sigma^{-V}\Sigma^\infty Y'$. Explicitly, ζ is the composite

$$\Sigma^\infty Y \xrightarrow{1\wedge\bar{\eta}} \Sigma^\infty Y\wedge\Sigma^\infty X\wedge\Sigma^{-V}\Sigma^\infty Y' \xrightarrow{\Sigma^\infty\varepsilon\wedge 1} \Sigma^\infty S^V\wedge\Sigma^{-V}\Sigma^\infty Y' \simeq \Sigma^\infty Y'.$$

(Compare Theorem 1.6(iii).) Clearly ε is a V-duality if and only if ζ is a G-equivalence. After suitable suspension, say by W, $\Sigma^W\zeta$ is represented by $\Sigma^\infty\lambda$ for a space level G-map $\lambda\colon \Sigma^W Y\to\Sigma^W Y$ (by I.4.10). Thus ε is a V-duality if and only if λ is a stable G-equivalence. Applying exactly the same argument on fixed point spaces and using II.3.14 to check compatability of the equivariant and nonequivariant situations, we find that ε^H is an n_H-duality if and only if λ^H is a stable equivalence. Since Y is finitely dominated, it follows from the space level G-suspension and G-Whitehead theorems (e.g. Hauschild [59] or Namboodiri [115,§2]) that λ is a stable G-equivalence if and only if each λ^H is a stable equivalence.

The hypothesis on X is certainly satisfied if X is a finite G-CW complex,

and we shall see in the next section that it is satisfied more generally if X is a compact G-ENR.

We should also record a purely space level description of dual maps. On the spectrum level, if (η,ε) and (η',ε') display finite pairs (E,F) and (E',F'), as duals and if $f\colon E\to E'$ is a map, then Df is the composite

$$F'\simeq F'\wedge S \xrightarrow{1\wedge\eta} F'\wedge E\wedge F \xrightarrow{1\wedge f\wedge 1} F'\wedge E'\wedge F \xrightarrow{\varepsilon'\wedge 1} S\wedge F \simeq F.$$

Consistency with the spectrum level duality maps $(\bar{\eta},\bar{\varepsilon})$ associated to space level duality maps (η,ε) forces the following specification.

Lemma 3.7. Let (η,ε) and (η',ε') be a V-duality and a V'-duality displaying stably finite pairs (X,Y) and (X',Y') as duals and let $f\colon X\to X'$ be a map. Define Df to be the composite

$$Y'\wedge S^V \xrightarrow{1\wedge\eta} Y'\wedge X\wedge Y \xrightarrow{1\wedge f\wedge 1} Y'\wedge X'\wedge Y \xrightarrow{\varepsilon'\wedge 1} S^{V'}\wedge Y.$$

Then the following diagram of spectra commutes.

$$\begin{tikzcd} \Sigma^{-V'}\Sigma^\infty Y' \simeq \Sigma^{-V'}\Sigma^\infty Y'\wedge\Sigma^{-V}\Sigma^\infty S^V \arrow[d, "D\Sigma^\infty f"'] \arrow[r, phantom, "\simeq"] & \Sigma^{-(V\oplus V')}\Sigma^\infty(Y'\wedge S^V) \arrow[d, "\Sigma^{-(V\oplus V')}\Sigma^\infty Df"] \\ \Sigma^{-V}\Sigma^\infty Y \simeq \Sigma^{-V'}\Sigma^\infty S^{V'}\wedge\Sigma^{-V}\Sigma^\infty Y \arrow[r, phantom, "\simeq"] & \Sigma^{-(V\oplus V')}\Sigma^\infty(S^{V'}\wedge Y) \end{tikzcd}$$

Further, the following diagrams of spaces commute stably, and the commutativity of either diagram characterizes Df.

$$\begin{tikzcd} Y'\wedge S^V\wedge X \arrow[r, "1\wedge 1\wedge f"] \arrow[dd, "Df\wedge 1"'] & Y'\wedge S^V\wedge X' \arrow[d, "1\wedge\gamma"] \\ & Y'\wedge X'\wedge S^V \arrow[d, "\varepsilon'\wedge 1"] \\ S^{V'}\wedge Y\wedge X \arrow[r, "1\wedge\varepsilon"] & S^{V'}\wedge S^V \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} S^{V'}\wedge S^V \arrow[r, "\eta'\wedge 1"] \arrow[d, "1\wedge\eta"'] & X'\wedge Y'\wedge S^V \arrow[dd, "1\wedge Df"] \\ S^{V'}\wedge X\wedge Y \arrow[d, "\gamma\wedge 1"'] & \\ X\wedge S^{V'}\wedge Y \arrow[r, "f\wedge 1\wedge 1"] & X'\wedge S^{V'}\wedge Y \end{tikzcd}$$

Also, the following composites specify a $(V+V')$-duality relating the pairs $X\wedge X'$ and $Y\wedge Y'$.

$$S^{V\oplus V'} \cong S^V\wedge S^{V'} \xrightarrow{\eta\wedge\eta'} X\wedge Y\wedge X'\wedge Y' \xrightarrow{1\wedge\gamma\wedge 1} X\wedge X'\wedge Y\wedge Y'$$

and

$$Y\wedge Y'\wedge X\wedge X' \xrightarrow{1\wedge\gamma\wedge 1} Y\wedge X\wedge Y'\wedge X' \xrightarrow{\varepsilon\wedge\varepsilon'} S^V\wedge S^{V'} \cong S^{V\oplus V'}.$$

Proof. Strictly speaking, $V'\oplus V$ in the first diagram should be interpreted as $k(V'\oplus V)$, where k is the linear isometry used to internalize smash products. The equivalences are given by I.7.9, II.1.4, II.3.6, and II.3.12(i), and the commutativity of the diagram is just a naturality statement. Both space level diagrams commute by virtue of the first diagram of Lemma 3.5, and the rest is clear.

§4. Duality for compact G-ENR's

A G-space X is a G-ENR (Euclidean neighborhood retract) if it can be embedded as a retract of an open subset of some G-representation V. It is a standard fact that a separable metric G-space is a G-ENR if and only if it is a finite dimensional G-ANR, and it was observed by Kwasik [79] that any separable metric G-ANR has the homotopy type of a G-CW complex. Jaworowski [68,2.1] gave a convenient characterization of G-ENR's. In the compact case, his criterion is simply that each fixed point subspace must be a nonequivariant ENR. In particular, any finite G-CW complex is a compact G-ENR. While compact G-ENR's have the homotopy types of G-CW complexes, they need not have the homotopy types of finite G-CW complexes, even stably. For example, locally linear compact topological G-manifolds are compact G-ENR's, but several authors have produced examples of such manifolds no suspension of which has the homotopy type of a finite G-CW complex. However, compact G-ENR's are clearly finitely dominated and are therefore wedge summands of finite G-CW complexes stably. Their suspension G-spectra are thus finite (in the categorical sense studied in the previous sections). A pair (X,A) will be called a G-ENR pair if X and A are both G-ENR's; this is true if and only if X is a G-ENR and the inclusion $A\subset X$ is a G-cofibration. We shall construct explicit space level V-dualities for compact G-ENR's X and compact G-ENR pairs (X,A).

For a pair of unbased spaces (X,A), defined $C(X,A)$ to be the unreduced mapping cone $X\cup CA$ with cone point $1\in I$ as basepoint. By definition or convention, $C(X,\emptyset)=X^+$. If X has a nondegenerate basepoint $*$, then $C(X,*)$, which is just X with a whisker grown from its basepoint, is based G-homotopy equivalent to X. We shall prove the following version of the Spanier-Whitehead duality theorem.

Theorem 4.1. Let (X,A) be a G-ENR pair with X embedded as a neighborhood retract in a representation V. Then $C(X,A)$ and $C(V-A,V-X)$ are V-dual. In particular, X^+ and $C(V,V-X)$ are V-dual and, if X has basepoint $*$, X and $C(V-*,V-X)$ are V-dual.

The classical Spanier-Whitehead theorem asserts that if a based finite polyhedron is simplicially embedded in S^{n+1} (with respect to some triangulation), then X and $S^{n+1}-X$ are n-dual. We sketch how to think of theorem 4.1 from the point of view of complementary embeddings in spheres.

Remark 4.2. Think of S^{V+1} as the unreduced suspension of S^V. Embed $C(X,A)$ in S^{V+1} by using the given embedding of X at level $1/3$ and letting the cone coordinate run from $1/3$ to 1. Clearly $C(V-A,V-X)$ is equivalent to $C(S^V-A,S^V-X)$. Embed the latter in S^{V+1} by embedding $V-A$ at level $2/3$ and letting the cone coordinate run from $2/3$ to 0. (Draw a picture!) With these embeddings, it is intuitively clear that $C(S^V-A,S^V-X)$ and $C(X,A)$ are G-deformation retracts of each other's complements.

We need two preliminary lemmas about mapping cones to specify the duality maps to be used in the proof of Theorem 4.1. For a G-map $f\colon (X,A)\to(Y,B)$, let $C(f)$ denote the induced map of mapping cones. Clearly C is a homotopy-preserving functor from pairs of G-spaces to based G-spaces. Say that a G-space X is normal if any two disjoint closed G-subspaces A and B of X have disjoint open G-neighborhoods. The proof of Urysohn's lemma goes through equivariantly to show that there is then a G-map $X\to I$ which takes the value 0 on A and 1 on B. Just as nonequivariantly, if the topology on X is given by a G-invariant metric, then X is normal.

Lemma 4.3. If $U\subset A\subset X$, then the excision

$$(X-U,A-U)\longrightarrow(X,A)$$

induces a based G-homotopy equivalence on passage to mapping cones under either of the following two hypotheses.

(i) $\bar U\subset \mathring{A}$ and X is normal.

(ii) The inclusions $A-U\to X-U$ and $A\to X$ are G-cofibrations.

Proof. (i) Let $u\colon X\to I$ satisfy $u(x)=0$ if $x\notin\mathring{A}$ and $u(x)=1$ if $x\in\bar U$. It is easy to check that the space

$$(X-A)\cup\{(a,s)\mid a\in A-U \text{ and } s\geq u(x)\}$$

embeds as a G-deformation retract in both $C(X-U,A-U)$ and $C(X,A)$.

(ii) Collapsing the cones $C(A-U)$ and CA to a point gives G-equivalences from $C(X-U,A-U)$ and $C(X,A)$ to

$$(X-U)/(A-U) = X/A.$$

Observe that $C(X,A)\wedge Y^+$ is naturally G-homeomorphic to $C(X\times Y, A\times Y)$. We shall need a generalization of this fact.

Lemma 4.4. For pairs of G-spaces (X,A) and (Y,B), define

$$\alpha\colon C(X,A)\wedge C(Y,B) \longrightarrow C(X\times Y,\ X\times B\cup A\times Y)$$

by the following formulas, where $x\in X$, $y\in Y$, $(a,s)\in CA$ and $(b,t)\in CB$.

$$\alpha(x\wedge y) = (x,y)$$

$$\alpha((a,s)\wedge y) = ((a,y),s)$$

$$\alpha(x\wedge(b,t)) = ((x,b),t)$$

$$\alpha((a,s)\wedge(b,t)) = ((a,b),\ \max(s,t)).$$

If A or B is empty, then α is a G-homeomorphism. In general, α is a G-homotopy equivalence under either of the following two hypotheses.
(i) A is open in X, B is open in Y, and $(X\times B)\cup(A\times Y)$ is normal.
(ii) One of the inclusions $A\to X$ and $B\to Y$ is a G-cofibration.

Proof. The homeomorphism part is clear so assume that A and B are nonempty. Let Z denote the double mapping cylinder

$$(X\times B\times\{0\})\cup(A\times B\times I)\cup(A\times Y\times\{1\}).$$

Let $\pi\colon Z\to(X\times B)\cup(A\times Y)$ be the obvious quotient map and let ρ be the composite of π and the inclusion of $(X\times B)\cup(A\times Y)$ in $X\times Y$. Then π induces a map

$$C(\pi)\colon C(X\times Y,Z)\longrightarrow C(X\times Y, X\times B\cup A\times Y),$$

where $C(X\times Y,Z)$ denotes the unreduced mapping cone of ρ. Clearly $C(\pi)$ is a G-equivalence whenever π is a G-equivalence over $X\times Y$. The map α is the composite of $C(\pi)$ and the G-homeomorphism

$$\beta\colon C(X,A)\wedge C(Y,B)\longrightarrow C(X\times Y,Z)$$

specified by

$$\beta(x\wedge y) = (x,y)$$

$$\beta((a,s)\wedge y) = ((a,y,1),s)$$

$$\beta(x\wedge(b,t)) = ((x,b,0),t)$$

$$\beta((a,s)\wedge(b,t)) = \begin{cases} ((a,b,1-t/2s),s) & \text{if } s\geq t \\ ((a,b,s/2t),t) & \text{if } s\leq t. \end{cases}$$

Thus it remains to check that π is a G-equivalence over $X\times Y$ under either of our hypotheses (i) and (ii).
(i) Let $u\colon(X\times B)\cup(A\times Y)\to I$ satisfy $u(x,y)=0$ if $x\notin A$ and $u(x,y)=1$ if $y\notin B$. Define $\psi\colon X\times B\cup A\times Y\to Z$ by

$$\psi(x,y) = (x,y,u(x,y)).$$

Then $\pi\psi = 1$ and $\psi\pi\simeq 1$ via an evident homotopy.
(ii) Here we apply the standard fact that the pushout functor preserves G-equivalences when applied to diagrams one leg of which is a G-cofibration.

Henceforward, we shall often use the (categorically incorrect) notation

$$(X,A)\times(Y,B) = (X\times Y, X\times B\cup A\times Y).$$

Observe that α is associative and commutative in the obvious sense.

Construction 4.5. (i) Choose a G-map

$$\varepsilon\colon C(V-A,V-X)\wedge C(X,A)\longrightarrow S^V$$

which makes the following diagram commute up to G-homotopy.

$$\begin{tikzcd} C(V-A,V-X)\wedge C(X,A) \arrow[r, "\alpha"] \arrow[d, "\varepsilon"'] & C[(V-A,V-X)\times(X,A)] \arrow[d, "C(d)"] \\ S^V & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & C(V,V-\{0\}) \end{tikzcd}$$

Here d is the difference map, $d(v,x)=v-x$; B is any closed disc about the origin in V and the unlabeled arrows are the obvious equivalences.
(ii) Let $r\colon N\to X$ be a G-retraction of an open neighborhood N of X in V.

Since $A \subset X$ is a G-cofibration, (X,A) is a G-NDR pair and there is a G-homotopy $h\colon X \times I \to X$ such that $h(x,0) = x$, $h(a,t) = a$ for $a \in A$, and $s = h_1$ restricts to a G-retraction $U \to A$ for some open neighborhood U of A in X. The composite $sr\colon r^{-1}U \to A$ is a G-retraction of an open neighborhood of A in V. Since X is compact, we may arrange that N is contained in a closed disc B about the origin in V. Choose a G-map

$$\eta\colon S^V \longrightarrow C(X,A) \wedge C(V-A,V-X)$$

which makes the following diagram commute up to G-homotopy.

$$\begin{tikzcd}
S^V \arrow[dd, "\eta"'] & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r] & C(V,(V-X) \cup U) & C(N-A,(N-X) \cup (U-A)) \arrow[l, "C(i)"'] \arrow[d, "C(\Delta)"] \\
 & & & C[(N,r^{-1}U) \times (V-A,V-X)] \arrow[d, "C(sr\times 1)"] \\
C(X,A) \wedge C(V-A,V-X) \arrow[rrr, "\alpha"] & & & C[(X,A) \times (V-A,V-X)]
\end{tikzcd}$$

Here $C(i)$ is an equivalence since Lemma 4.3(i) applies to the excision i. Of course, when A is empty, U is also empty and the diagram simplifies accordingly.

We shall use the criterion of Lemma 3.5 to show that the pair (ε,η) gives a V-duality between $C(X,A)$ and $C(V-A,V-X)$.

<u>Lemma 4.6</u>. The following diagram is G-homotopy commutative.

$$\begin{tikzcd}
S^V \wedge C(X,A) \arrow[r, "\eta\wedge 1"] \arrow[dr, "\gamma"'] & C(X,A) \wedge C(V-A,V-X) \wedge C(X,A) \arrow[d, "1\wedge\varepsilon"] \\
 & C(X,A) \wedge S^V
\end{tikzcd}$$

<u>Proof</u>. By an easy inspection, it suffices to show that the following diagram of pairs becomes G-homotopy commutative upon passage to mapping cones.

$$\begin{tikzcd}
(V,V-B) \times (X,A) \arrow[r] \arrow[dd, "\gamma"'] & (V,(V-X) \cup U) \times (X,A) \arrow[d, "e"] & (N-A,(N-X) \cup (U-A)) \times (X,A) \arrow[l, "i\times 1"'] \arrow[d, "\Delta\times 1"] \\
 & (N,r^{-1}U) \times (V,V-\{0\}) \arrow[d, "sr\times 1"] & (N,r^{-1}U) \times (V-A,V-X) \times (X,A) \arrow[l, "1\times d"'] \arrow[d, "sr\times 1\times 1"] \\
(X,A) \times (V,V-B) \arrow[r] & (X,A) \times (V,V-\{0\}) & (X,A) \times (V-A,V-X) \times (X,A) \arrow[l, "1\times d"']
\end{tikzcd}$$

The bottom right square commutes trivially. The map e is specified by $e(v,x) = (x,v-x)$. The homotopy k of maps of pairs specified by

$$k(v,x,t) = (x,v-tx)$$

and the facts that $r = 1$ on X and $s \simeq 1$ rel A show that the left part of the diagram is G-homotopy commutative. For the top right square, define

$$M = \{(n,x) \mid tx + (1-t)n \in N \text{ for } 0 \le t \le 1\} \subset (N-A) \times X$$

and consider the following schematic diagram (in which Δ_Y denotes the diagonal subspace of $Y \times Y$).

$$\begin{tikzcd}
(M,M-\Delta_{X-U}) \subset \arrow[dr, shift left] \arrow[dr, shift right] & ((N-A) \times X,(N-A) \times X - \Delta_{X-U}) \arrow[d, shift left] \arrow[d, shift right] & \supset (N-A,(N-X) \cup (U-A)) \times (X,A) \arrow[dl, shift left] \arrow[dl, shift right] \\
 & (N,r^{-1}U) \times (V,V-\{0\}) &
\end{tikzcd}$$

In each pair of arrows, the first is given by $(n,x) \to (n,n-x)$ and the second by $(n,x) \to (x,n-x)$. The two left arrows are in the same homotopy class by the definition of M. Since the left inclusion is an excision, Lemma 4.3 implies tha the two right arrows induce homotopic maps of mapping cones.

When A is empty, the following lemma completes the proof of Theorem 4.1.

<u>Lemma 4.7</u>. The following diagram is G-homotopy commutative.

$$\begin{tikzcd}
C(V,V-X) \wedge S^V \arrow[r, "1\wedge\eta"] \arrow[d, "\gamma"'] & C(V,V-X) \wedge X^+ \wedge C(V,V-X) \arrow[d, "\varepsilon\wedge 1"] \\
S^V \wedge C(V,V-X) \arrow[r, "\sigma\wedge 1"] & S^V \wedge C(V,V-X)
\end{tikzcd}$$

Proof. By an easy inspection, it suffices to show that the following diagram of pairs is G-homotopy commutative.

$$\begin{tikzcd}
(V,V-X)\times(V,V-B) \arrow[r] \arrow[dd, "(\sigma\times 1)\gamma"'] & (V,V-X)\times(V,V-X) \arrow[r, "1\times i"] \arrow[dd, "g"] & (V,V-X)\times(N,N-X) \arrow[d, "1\times\Delta"] \arrow[ddl, "f"'] \\
 & & (V,V-X)\times N\times(V,V-X) \arrow[d, "1\times r\times 1"] \\
(V,V-B)\times(V,V-X) \arrow[r] & (V,V-\{0\})\times(V,V-X) & (V,V-X)\times X\times(V,V-X) \arrow[l, "d\times 1"']
\end{tikzcd}$$

Here $f(v,n) = (v-n,n)$ and $g(v,v') = (v-v',v)$. On the right,

$$(d\times 1)(1\times r\times 1)(1\times\Delta)(v,n) = (v-r(n),n),$$

and $(v,n,t) \to (v-[(1-t)n + tr(n)],n)$ gives a homotopy from f to this composite. Similarly, we have linear homotopies

$$(v,n,t) \longrightarrow (v-n,(1-t)v + tn) \quad \text{and} \quad (v,v',t) \longrightarrow (tv-v',v)$$

from $g(1\times i)$ to f and from $(\sigma\times 1)\gamma$ to g.

When A is nonempty, points in the diagonal subspace Δ_U obstruct the definition of maps f and g as above. We complete the proof of Theorem 4.1 by deducing the relative case from the absolute case.

Lemma 4.8. The following diagram is at least stably G-homotopy commutative.

$$\begin{tikzcd}
C(V-A,V-X)\wedge S^V \arrow[r, "1\wedge\eta"] \arrow[d, "\gamma"] & C(V-A,V-X)\wedge C(X,A)\wedge C(V-A,V-X) \arrow[d, "\varepsilon\wedge 1"] \\
S^V\wedge C(V-A,V-X) \arrow[r, "\sigma\wedge 1"] & S^V\wedge C(V-A,V-X)
\end{tikzcd}$$

Proof. By Lemmas 3.5 and 4.6, we need only show that the map ε in the diagram is a V-duality. Observe first that there is a natural G-homotopy equivalence

$$\zeta\colon C[C(V,V-A),C(V,V-X)] \longrightarrow \Sigma C(V-A,V-X),$$

where the outer cone in the domain is reduced rather than unreduced. In fact, ζ is just the quotient map obtained by collapsing the cone CV to the basepoint and transposing the two cone coordinates of $CC(V-X)$. We can use the reduced cone on the left since ζ collapses the line through the inner cone point to the

basepoint. Thus we have a based cofibration sequence

$$C(V,V-X) \longrightarrow C(V,V-A) \longrightarrow \Sigma C(V-A,V-X)$$

in which the second map is obtained by collapsing out V and including. Of course, we also have the based cofibration sequence

$$A^+ \longrightarrow X^+ \longrightarrow C(X,A).$$

The following two diagrams are clearly G-homotopy commutative.

$$\begin{tikzcd}
C(V,V-X)\wedge A^+ \arrow[r] \arrow[d] & C(V,V-X)\wedge X^+ \arrow[d, "\varepsilon"] \\
C(V,V-A)\wedge A^+ \arrow[r, "\varepsilon"] & S^V
\end{tikzcd}$$

and

$$\begin{tikzcd}
C(V-A,V-X)\wedge X^+ \arrow[r] \arrow[d] & C(V-A,V-X)\wedge C(X,A) \arrow[d, "\varepsilon"] \\
C(V,V-X)\wedge X^+ \arrow[r, "\varepsilon"] & S^V
\end{tikzcd}$$

A little less obviously, the following diagram is stably G-homotopy commutative up to the sign -1.

$$\begin{tikzcd}
C(V,V-A)\wedge C(X,A) \arrow[r] \arrow[d] & C(V,V-A)\wedge\Sigma A^+ = \Sigma[C(V,V-A)\wedge A^+] \arrow[d, "\Sigma\varepsilon"] \\
\Sigma C(V-A,V-X)\wedge C(X,A) \cong \Sigma[C(V-A,V-X)\wedge C(X,A)] \arrow[r, "\Sigma\varepsilon"] & \Sigma S^V
\end{tikzcd}$$

To see this, replace S^V by $C(V,V-\{0\})$ and collapse out the contractible G-space V. The target ΣS^V is then replaced by $\Sigma\Sigma(V-\{0\})$, where the inside suspension is unreduced. The two resulting maps into the new target are easily checked to differ only by a transposition of suspension coordinates. Since we are working stably (compare Lemma 4.9 below), we may as well assume that V contains a copy of the trivial representation. This allows us to choose a G-fixed basepoint in $V-\{0\}$ and pass to the reduced suspension, whereupon the assertion becomes clear. By the three diagrams above and an easy comparison of long exact sequences, ε is a V-duality in the relative case because it is a V-duality in the absolute case.

The previous proof used instances of two useful general naturality properties of ε.

Lemma 4.9. If $(X,A) \subset (X',A')$ is an inclusion of compact G-ENR pairs in V, then the following diagram commutes.

$$\begin{tikzcd} C(V-A',V-X') \wedge C(X,A) \arrow[r] \arrow[d] & C(V-A',V-X') \wedge C(X',A') \arrow[d, "\varepsilon"] \\ C(V-A,V-X) \wedge C(X,A) \arrow[r, "\varepsilon"] & S^V \end{tikzcd}$$

Lemma 4.10. Let (X,A) be a compact G-ENR pair in V and let $V \subset V'$ with orthogonal complement W. Then the diagram

$$\begin{tikzcd} \Sigma^W C(V-A,V-X) \arrow[r, phantom, "\simeq"] \arrow[d, "\zeta"] & C(V-A,V-X) \wedge C(W,W-\{0\}) \arrow[d, "\alpha"] \\ C(V'-A,V'-X) & C((V-A,V-X) \times (W,W-\{0\})) \arrow[l, "\text{excision}"'] \end{tikzcd}$$

specifies a G-homotopy equivalence ζ such that the following diagram is G-homotopy commutative.

$$\begin{tikzcd} \Sigma^W C(V-A,V-X) \wedge C(X,A) \arrow[r, phantom, "\cong"] \arrow[d, "\zeta \wedge 1"] & \Sigma^W [C(V-A,V-X) \wedge C(X,A)] \arrow[d, "\Sigma^W \varepsilon"] \\ C(V'-A,V'-X) \wedge C(X,A) \arrow[r, "\varepsilon"] & S^{V'} \cong \Sigma^W S^V \end{tikzcd}$$

The explicit geometric nature of the maps ε and η often allows explicit evaluation of dual maps. We give two examples.

Lemma 4.11. The dual of the projection $\xi: X^+ \to S^0$ is the "Pontryagin-Thom" map

$$t: S^V \xleftarrow{\simeq} C(V,V-B) \longrightarrow C(V,V-X).$$

Proof. Of course, S^0 is 0-self dual via $S^0 = S^0 \wedge S^0$. According to Lemma 3.7, the dual of ξ is the composite

$$S^V \xrightarrow{\eta} X^+ \wedge C(V,V-X) \xrightarrow{\xi \wedge 1} C(V,V-X).$$

A simple inspection shows that this agrees with t.

Lemma 4.12. Define the "Thom diagonal" $\Delta: C(V,V-X) \to X^+ \wedge C(V,V-X)$ to be the composite

$$C(V,V-X) \xleftarrow[\simeq]{C(i)} C(N,N-X) \xrightarrow{C(\Delta)} C(N \times (V,V-X))$$
$$\xrightarrow{C(r \times 1)} C(X \times (V,V-X)) \xrightarrow{\alpha^{-1}} X^+ \wedge C(V,V-X).$$

Then the following diagram is G-homotopy commutative.

$$\begin{tikzcd} S^V \wedge X^+ \arrow[r, "\gamma"] \arrow[d, "\eta \wedge 1"] & X^+ \wedge S^V \arrow[r, "\Delta \wedge 1"] & X^+ \wedge X^+ \wedge S^V \\ X^+ \wedge C(V,V-X) \wedge X^+ \arrow[rr, "1 \wedge \Delta \wedge 1"] & & X^+ \wedge X^+ \wedge C(V,V-X) \wedge X^+ \arrow[u, "1 \wedge 1 \wedge \varepsilon"'] \end{tikzcd}$$

Proof. This follows from the definitions of η and ε, the relation $(1 \wedge \varepsilon)(\eta \wedge 1) \simeq \gamma$, and the transitivity diagram

$$\begin{tikzcd} C(V,V-X) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X^+ \wedge C(V,V-X) \arrow[d, "1 \wedge \Delta"] \\ X^+ \wedge C(V,V-X) \arrow[r, "\Delta \wedge 1"] & X^+ \wedge X^+ \wedge C(V,V-X) \end{tikzcd}$$

We must justify the terminology used in the lemmas.

Remark 4.13. It is often the case that $V-N$ is a G-deformation retract of $V-X$, $V-U$ is a G-deformation retract of $V-A$, and the inclusion of $V-N$ in $V-U$ is a G-cofibration. Under these circumstances, we have natural G-equivalences

$$C(V-A,V-X) \longleftarrow C(V-U,V-N) \longrightarrow (V-U)/(V-N).$$

When A is empty, $V/(V-N)$ is G-homeomorphic to the 1-point compactification N^c of N. The "Pontryagin-Thom" map of Lemma 4.11 corresponds under these equivalences to the Pontryagin-Thom map $t: S^V \to N^c$ obtained by collapsing the complement of

N to the basepoint. The "Thom diagonal" of Lemma 4.12 corresponds to the Thom diagonal $\Delta\colon N^c \to X^+ \wedge N^c$ obtained from the composite

$$N^+ \xrightarrow{\Delta} N^+ \wedge N^+ \xrightarrow{r \wedge 1} X^+ \wedge N^+$$

by use of the natural based map $N^+ \to N^c$. In the manifold context of the next section, N will be a tubular neighborhood of X and r will be the bundle projection.

§5. Duality for smooth G-manifolds

It is now a very easy matter to compute the duals of smooth G-manifolds and to be rather more precise about the duality maps than is customary. We shall need the precision in our discussion of Poincaré duality. We begin with the closed case. Let $T\eta$ denote the Thom complex of a G-vector bundle η.

Theorem 5.1. Let M be a smooth compact G-manifold without boundary smoothly embedded in a G-representation V. Let ν be the normal bundle and let N be a tubular neighborhood of M in V. Let

$$\eta\colon S^V \xrightarrow{t} N^c \cong T\nu \xrightarrow{\Delta} M^+ \wedge T\nu$$

be the composite of the Pontryagin-Thom map and the Thom diagonal. Let $s\colon M \to \nu$ be the zero section, observe that the normal bundle of the composite

$$M \xrightarrow{\Delta} M \times M \xrightarrow{s \times 1} \nu \times M$$

is

$$\Delta^*\nu(s \otimes 1) \oplus \nu(\Delta) \cong \nu \oplus \tau \cong M \times V,$$

and let

$$\varepsilon\colon T\nu \wedge M^+ \xrightarrow{t} M^+ \wedge S^V \xrightarrow{\xi \wedge 1} S^V$$

be the composite of the Pontryagin-Thom map associated to a tubular neighborhood of $(s \times 1)\,\Delta$ and the canonical projection. Then η and ε display M^+ and $T\nu$ as V-duals.

Proof. As observed in Remark 4.13, $C(V,V-M)$ is canonically G-equivalent to $T\nu$. Write η' and ε' for the duality maps of Construction 4.5. It is easy to check that η' corresponds to η under the equivalence. Rather than try to obtain the

analogous (and much less obvious) comparison of ε' and ε directly, we note that, by the dual version of Lemma 3.5, it suffices to prove that the following diagram is G-homotopy commutative.

$$\begin{tikzcd} S^V \wedge M^+ \arrow[r, "\eta \wedge 1"] \arrow[dr, "\gamma"'] & M^+ \wedge T\nu \wedge M^+ \arrow[d, "1 \wedge \varepsilon"] \\ & M^+ \wedge S^V \end{tikzcd}$$

The composite $(1 \wedge \varepsilon)(\eta \wedge 1)$ is easily seen to be the Pontryagin-Thom map associated to a tubular neighborhood of the embedding

$$M \xrightarrow{\Delta} M \times M \xrightarrow{i \times 1} V \times M,$$

where i is our given embedding of M in V. For a radial embedding $r\colon V \to V$ of V as a sufficiently small open ball around the origin, a second, isotopic, tubular neighborhood $M \times V \to V \times M$ is specified by sending (m,v) to $(r(v) + i(m),m)$. Clearly the Pontryagin-Thom map for this second tubular neighborhood is homotopic to the switch map.

The last few results of the previous section imply the following addenda.

Corollary 5.2. The dual of the projection $\xi\colon M^+ \to S^0$ is the Pontryagin-Thom map $t\colon S^V \to T\nu$.

Corollary 5.3. The Thom diagonal fits into the stably G-homotopy commutative diagram

$$\begin{tikzcd} S^V \wedge M^+ \arrow[r, "\gamma"] \arrow[d, "\eta \wedge 1"'] & M^+ \wedge S^V \arrow[r, "\Delta \wedge 1"] & M^+ \wedge M^+ \wedge S^V \arrow[d, "1 \wedge 1 \wedge \varepsilon"] \\ M^+ \wedge T\nu \wedge M^+ \arrow[rr, "1 \wedge \Delta \wedge 1"] & & M^+ \wedge M^+ \wedge T\nu \wedge M^+. \end{tikzcd}$$

Similarly, we have the Atiyah duality theorem for smooth G-manifolds with boundary.

Theorem 5.4. Let M be a smooth compact G-manifold with boundary. Let $V = V' \times \mathbb{R}$, where $(M,\partial M)$ is properly embedded in $(V' \times [0,\infty), V' \times \{0\})$, and let ν' and ν be the normal bundles of ∂M in V' and of M in V. Then the cofibration sequence

$$T\nu' \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'$$

is V-dual to the cofibration sequence

$$\Sigma(\partial M^+) \longleftarrow M/\partial M \longleftarrow M^+ \longleftarrow \partial M^+.$$

Proof. Since our embedding is proper, we may assume that it restricts to the obvious embedding $\partial M \times [0,4] \to V' \times [0,4]$ on a boundary collar $\partial M \times [0,4]$ inside M. (Use of $[0,4]$ rather than $[0,1]$ will aid in the check that all three maps in the two sequences are dual to one another.) Glue another boundary collar $\partial M \times [-1,0]$ on the outside of M to obtain a manifold $\hat{M}$ embedded in $V' \times [-1,\infty)$, and let $\hat{N}$ be a tubular neighborhood of $\hat{M}$ in $V' \times [-1,\infty)$. If

$$N = \hat{N} \cap (V' \times (-1,\infty)) \quad \text{and} \quad N' = \hat{N} \cap (V' \times (-1,1)),$$

then N is the normal bundle of $\hat{M} - \partial\hat{M}$ and N' is the normal bundle of $\partial M \times (-1,1)$. Let

$$M_1 = M - (\partial M \times [0,3)) \quad \text{and} \quad N_1 = \hat{N} \cap (V' \times (2,\infty)).$$

Then N_1 is the normal bundle of $M - (\partial M \times [0,2])$. Via a direction reversing homeomorphism from the interval $[0,1]$ to the interval $[0,3]$, we obtain an identification of $\Sigma(\partial M^+)$ with $M/(\partial M \cup M_1)$ under which the boundary map

$$M/\partial M \simeq M^+ \cup C(\partial M^+) \longrightarrow \Sigma(\partial M^+)$$

becomes homotopic to the quotient map

$$M/\partial M \longrightarrow M/(\partial M \cup M_1).$$

The horizontal arrows are equivalences in the commutative diagram

$$\begin{tikzcd}
C(V-(\partial M \cup M_1), V-M) \arrow[d] & C(V-(N' \cup N_1), V-N) \arrow[l] \arrow[r] \arrow[d] & (V-(N' \cup N_1))/(V-N) \arrow[d] \\
C(V-\partial M, V-M) \arrow[d] & C(V-N', V-N) \arrow[l] \arrow[r] \arrow[d] & (V-N')/(V-N) \arrow[d] \\
C(V, V-M) \arrow[d] & C(V, V-N) \arrow[l] \arrow[r] \arrow[d] & V/(V-N) \arrow[d] \\
C(V, V-\partial M) & C(V, V-N') \arrow[l] \arrow[r] & V/(V-N')
\end{tikzcd}$$

By Theorem 4.1 and our comparison of $\Sigma(\partial M^+)$ to $M/(\partial M \cup M_1)$, the left column is V-dual to the sequence

$$\Sigma(\partial M^+) \longleftarrow M/\partial M \longleftarrow M^+ \longleftarrow \partial M^+.$$

By inspection, the right column is equivalent to the sequence

$$T\bar{\nu} \longrightarrow T\nu \longrightarrow T\nu/T\nu' \longrightarrow \Sigma T\nu'.$$

where $\bar{\nu}$ is the pullback of ν' along the projection $\partial M \times [1,2] \to \partial M$. Since there is an evident equivalence $T\bar{\nu} \simeq T\nu$ compatible with the inclusions $T\bar{\nu} \to T\nu$ and $T\nu' \to T\nu$, this proves the result.

As in Theorem 5.1, it is easy to write down explicit descriptions of the relevant duality maps η and ε. For example, $\eta\colon S^V \to M/\partial M \wedge T\nu$ can again be described as the composite of the Pontryagin-Thom map $t\colon S^V \to T\nu/T\nu'$ and the Thom diagonal $\Delta\colon T\nu/T\nu' \to M/\partial M \wedge T\nu$. We leave the remaining cases to the reader.

In our treatment of Poincaré duality, we shall need a technical result on the relationship between local and global duality on manifolds.

Proposition 5.5. Let M be a smooth compact G-manifold smoothly (and properly) embedded in a G-representation V with normal bundle ν. Let $x \in M - \partial M$ have isotropy group H, let Z be the fibre at x of the normal bundle of $Gx \subset M$ and let W be the fibre at x of ν. Extension of $G/H \cong Gx \subset M$ to a slice $G \times_H Z \to M$ gives rise to a "local Thom map" $t_x\colon M/\partial M \to G^+ \wedge_H S^Z$ (where $M/\partial M$ is interpreted as M^+ if ∂M is empty). The bundle inclusion $G \times_H W \to \nu$ gives rise to an induced map $j_x\colon G^+ \wedge_H S^W \to T\nu$. The maps t_x and j_x are dual to one another.

Proof. We shall prove that the following diagram commutes.

$$\begin{tikzcd}
(G^+ \wedge_H S^W) \wedge M/\partial M \wedge T\nu \arrow[r, "1 \wedge t_x \wedge 1"] & (G^+ \wedge_H S^W) \wedge (G^+ \wedge_H S^Z) \wedge T\nu \arrow[d, "\varepsilon\gamma \wedge 1"] \\
(G^+ \wedge_H S^W) \wedge S^V \arrow[u, "1 \wedge \eta"] \arrow[r, "j_x \wedge 1"] & T\nu \wedge S^V & S^V \wedge T\nu \arrow[l, "\gamma"']
\end{tikzcd}$$

(Compare Lemma 3.7.) Here η is the composite Δt, as above, and ε is the explicit duality map of Proposition 2.11. This makes sense since if L is the tangent H-representation of G/H at the identity coset, then $L \oplus Z \oplus W$ is H-isomorphic to V, so that $\Sigma^L S^Z$ and S^W are V-dual H-spectra. Note that ε depends on a choice of H-equivalence $\Sigma^L S^Z \wedge S^W \to S^V$, and we are free to insert a sign if we choose (see Warning 1.7); $\varepsilon\gamma$ is again a duality map by Theorem 1.6. A little diagram chase shows that the composite around the top can be written in the form

$$(G^+ \wedge_H S^W) \wedge S^V \xrightarrow{\zeta^{-1}} G^+ \wedge_H (S^W \wedge S^V) \xrightarrow{1 \wedge_H \alpha} G^+ \wedge_H (T\nu \wedge S^V) \xrightarrow{\xi} T\nu \wedge S^V,$$

where ζ is the G-homeomorphismn specified above II.4.1 and ξ is the G-action map. The H-map α is specified by commutativity of the diagram

$$\begin{tikzcd}
S^W \wedge S^V \arrow[r, "1\wedge t"] \arrow[d, "\alpha"'] & S^W \wedge T\nu/T\nu' \arrow[r, "1\wedge\Delta"] & S^W \wedge M/\partial M \wedge T\nu \arrow[r, "1\wedge t_x \wedge 1"] & S^W \wedge (G^+ \wedge_H S^Z) \wedge T\nu \arrow[d, "\gamma\wedge 1"] \\
T\nu \wedge S^V & S^V \wedge T\nu \arrow[l, "\gamma"'] \arrow[r, equal] & S^L \wedge S^Z \wedge S^W \wedge T\nu & (G^+ \wedge_H S^Z) \wedge S^W \wedge T\nu, \arrow[l, "u\wedge 1\wedge 1"']
\end{tikzcd}$$

where u is as defined in II.5.5. Let $\pi\colon \nu \to M$ be the bundle projection. Clearly $\alpha(w \wedge v) = *$ unless v is in the copy of ν in the chosen tubular neighborhood of M, $\pi(v)$ is in the slice $G \times_H Z$, and, in there, is in the slice $L \times Z$ used in II.5.5. Therefore α factors through $1\wedge t'$, where $t'\colon S^V \to S^L \wedge S^Z \wedge S^W = S^V$ is a Pontryagin-Thom map H-homotopic to the identity determined by the various embeddings in sight. Regard t' as an identification. Then, for $w, w' \in W$, $\lambda \in L$, and $z \in Z$,

$$\alpha(w \wedge \lambda \wedge z \wedge w') = (\lambda, z, w') \wedge (\lambda \wedge z \wedge w),$$

where $(\lambda, z, w') \in T\nu$ is to be interpreted as the point w' in the fibre of ν at the point $(\lambda, z) \in L \times Z \subset G \times_H Z \subset M$. On the other hand, $j_x \wedge 1$ can be written as the composite

$$(G^+ \wedge_H S^W) \wedge S^V \xrightarrow{\zeta^{-1}} G^+ \wedge_H (S^W \wedge S^V) \xrightarrow{1\wedge_H \beta} G^+ \wedge_H (T\nu \wedge S^V) \xrightarrow{\xi} T\nu \wedge S^V,$$

where

$$\beta(w \wedge \lambda \wedge z \wedge w') = (x, w) \wedge (\lambda \wedge z \wedge w').$$

Since $L \times Z$ is H-contractible, this neighborhood in M contracts to x. Thus, up to the sign introduced by interchange of w and w' (see II.5.11), α and β are H-homotopic. We may alter ε (and thus α) by the same sign and so obtain the result.

Remark 5.6. We have restricted the material above to smooth G-manifolds for simplicity and our own security. However, using the foundational material developed by Lashof and Rothenberg [82] and its generalization from finite groups to compact Lie groups, it seems that one can carry out everything above and in the next section in the more general context of locally linear compact topological G-manifolds. The essential point is that such manifolds are compact G-ENR's and have slices, normal bundles, tubular neighborhoods, and boundary collars with the usual properties.

As noted by Atiyah [6], Theorem 5.4 implies a duality theorem for the Thom complexes of vector bundles over closed manifolds.

Theorem 5.7. Let M be a smooth compact G-manifold without boundary smoothly embedded with normal bundle ν in a G-representation V. Let ξ be a real G-vector bundle over M and let η be a complementary bundle with respect to a G-representation W, $\xi \oplus \eta \approx M \times W$. Then $T\xi$ is $(V \oplus W)$-dual to $T(\nu \oplus \eta)$.

Proof. We may give ξ a smooth structure. If $D\xi$ and $S\xi$ denote the unit disc and unit sphere bundles of ξ then $D\xi$ is a smooth G-manifold with boundary $S\xi$ and $T\xi = D\xi/S\xi$. Thus $T\xi$ is Z-dual to $T\omega$, where ω is the normal bundle of a suitably nice embedding of $D\xi$ in a G-representation Z. Let $\pi\colon D\xi \to M$ be the projection. Clearly the tangent bundle of $D\xi$ is $\pi^*(\tau \oplus \xi)$, where τ is the tangent bundle of M, and of course $\pi^*(\tau \oplus \xi) \oplus \pi^*(\nu \oplus \eta)$ is the trivial bundle $\underline{V \oplus W}$ over $D\xi$. This implies that $\omega + \underline{V \oplus W}$ is stably equivalent to $\pi^*(\nu \oplus \eta) \oplus \underline{Z}$. Interpreting duality maps stably, $T\xi$ is $(V \oplus W \oplus Z)$-dual to $\Sigma^{W\oplus V} T\omega$ and thus to $\Sigma^Z T\pi^*(\nu \oplus \eta)$. Therefore $T\xi$ is $(V \oplus W)$-dual to $T\pi^*(\nu \oplus \eta)$. Since π is a G-homotopy equivalence, $T\pi^*(\nu \oplus \eta)$ is G-homotopy equivalent to $T(\nu \oplus \eta)$.

§6. The equivariant Poincaré duality theorem

We begin by describing the Thom isomorphism of an oriented G-vector bundle ξ over a G-space Y. If $i\colon G/H \to Y$ is the inclusion of an orbit, then $i^*\xi$ is of the form $G \times_H W \to G/H$, where W is the fibre H-representation at $i(eH)$. Thus $T(i^*\xi) = G^+ \wedge_H S^W$. Let E be a commutative ring G-spectrum. (We index G-spectra on a fixed complete G-universe.) Let $\varepsilon\colon RO(G) \to Z$ be the augmentation.

Definition 6.1. An E-orientation of ξ is an element α of $RO(G)$ such that $\varepsilon(\alpha)$ is the fibre dimension of ξ together with a class $\mu \in \tilde{E}^{\alpha}_G(T\xi)$ such that the restriction of μ to $\tilde{E}^*_G(T(i^*\xi)) \cong \tilde{E}^*_H(S^W)$ is a $\pi^H_*(E)$ generator for each orbit inclusion $i\colon G/H \to Y$ with fibre representation W.

The definition makes sense since $\tilde{E}^*_H(S^W)$, regarded as graded over $RO(H)$, is a free $\pi^H_*(E)$-module on one generator, where $\pi^H_*(E)$ is also understood in the $RO(H)$-graded sense.

If Y is G-connected, there is an obvious preferred choice for α, namely the fibre G-representation V at any fixed point of Y. Here any fibre H-representation W as above is isomorphic to the restriction of V to H. For general Y and ξ, there is no obvious preferred choice for α, and the existence of an orientation implies restrictions on the coefficients $\pi^H_*(E)$. If $\mu \in E^{\alpha}_G(T\xi)$ is an orientation of ξ and W is the fibre H-representation at $i(eH)$ for an orbit inclusion $i: G/H \to Y$, then $i^*(\mu) \in \tilde{E}^{\alpha}_{H}(S^W) = \pi^H_{W-\alpha}(E)$ must be a unit. If $\alpha \neq w$, the existence of such a unit forces a certain amount of periodicity in the theory.

If $A \subset Y$ is a G-cofibration, the relative Thom diagonal

$$\Delta: T\xi/T(\xi|A) \longrightarrow Y/A \wedge T\xi$$

gives rise to a cup product

$$E^*(Y,A) \otimes \tilde{E}^*(T\xi) \longrightarrow E^*(T\xi, T(\xi|A))$$

(where $T(\xi|A)$ is a point if A is empty). An easy homotopical proof of the following equivariant Thom isomorphism theorem is given in X§5.

Theorem 6.2. Let $\mu \in \tilde{E}^{\alpha}_G(T\xi)$ be an orientation of the G-vector bundle ξ over Y. Then, for a G-cofibration $A \subset Y$,

$$\cup \mu: E^{\beta}(Y,A) \longrightarrow E^{\alpha+\beta}(T\xi, T(\xi|A))$$

is an isomorphism for all $\beta \in RO(G)$.

Now specialize to the context of smooth G-manifolds.

Definition 6.3. A smooth G-manifold M is said to be E-orientable if its tangent bundle τ is E-orientable, and an orientation μ of τ is also called an orientation of M. If M has boundary, then the smooth boundary collar theorem shows that the normal bundle of the embedding $\partial M \to M$ is trivial. Giving it the canonical orientation $\iota \in \tilde{E}^1_G(\Sigma(\partial M^+))$, we obtain a unique orientation $\partial\mu$ of ∂M such that $\partial\mu \oplus \iota$ is the restriction of μ to ∂M.

Let $\mu \in \tilde{E}^{\alpha}_G T(\tau)$ be an orientation of M. Embed M in a G-representation V and let $\tau^{\perp}$ denote the normal bundle of the embedding. Let $\nu \in \tilde{E}^{V-\alpha}_G(T(\tau^{\perp}))$ be the unique orientation such that $\nu \oplus \mu$ is the canonical orientation in $E^V(\Sigma^V M^+)$. We obtain the Poincaré duality isomorphism by combining Atiyah duality, Spanier-Whitehead duality, and the Thom isomorphism.

Definition 6.4. If M is a closed E-oriented smooth G-manifold, then the composite

$$D: E^{\beta}_G(M) \longrightarrow \tilde{E}^{V-\alpha+\beta}_G(T(\tau^{\perp})) \longrightarrow E^G_{\alpha-\beta}(M)$$

of the Thom and Spanier-Whitehead duality isomorphisms is the Poincaré duality isomorphism. The element $[M] = D(1)$ in $E_{\alpha}(M)$ is called the fundamental class associated to the orientation. If M is a compact E-oriented smooth G-manifold with boundary, then the analogous composites

$$D: E^{\beta}_G(M) \longrightarrow \tilde{E}^{V-\alpha+\beta}_G(T(\tau^{\perp})) \longrightarrow E^G_{\alpha-\beta}(M, \partial M)$$

and

$$D: E^{\beta}_G(M, \partial M) \longrightarrow E^{V-\alpha+\beta}_G(T(\tau^{\perp})), T(\tau^{\perp}|\partial M)) \longrightarrow E^G_{\alpha-\beta}(M)$$

are the relative Poincaré duality isomorphisms. The element $[M] = D(1)$ in $E_{\alpha}(M, \partial M)$ is called the fundamental class associated to the orientation.

Theorem 5.4 and a direct comparison of definitions give the following result.

Proposition 6.5. The Poincaré duality isomorphisms are all given by the cap product with the fundamental class. In the case of manifolds with boundary, the following diagram commutes.

$$\begin{tikzcd}
\cdots \arrow[r] & E^{\beta-1}_G(\partial M) \arrow[r, "\delta"] \arrow[d, "D"] & E^{\beta}_G(M,\partial M) \arrow[r] \arrow[d, "D"] & E^{\beta}_G(M) \arrow[r] \arrow[d, "D"] & E^{\beta}_G(\partial M) \arrow[r] \arrow[d, "D"] & \cdots \\
\cdots \arrow[r] & E^G_{\alpha-\beta}(\partial M) \arrow[r] & E^G_{\alpha-\beta}(M) \arrow[r] & E^G_{\alpha-\beta}(M,\partial M) \arrow[r] & E^G_{\alpha-1-\beta}(\partial M) \arrow[r] & \cdots
\end{tikzcd}$$

The fundamental class of an E-oriented G-manifold admits a local description, as in the nonequivariant case.

Definition 6.6. Let M be a smooth compact G-manifold. For x M-∂M, let $t_x: M/\partial M \to G^+ \wedge_H S^Z$ be a local Thom map at x (as in Proposition 5.5), where H is the isotropy group of x and Z is the fibre at x of the normal bundle of $Gx \subset M$. An E-fundamental class of M is an element α of RO(G) such that $\varepsilon(\alpha) = \dim M$ and an element $[M] \in E^G_{\alpha}(M, \partial M)$ such that the image of $[M]$ under the composite

$$E^G_{\alpha}(M, \partial M) \xrightarrow{t_{x*}} \tilde{E}^G_{\alpha}(G^+ \wedge_H S^Z) \cong \tilde{E}^H_{\alpha}(S^{L+Z})$$

is a $\pi^H_*(E)$-generator of $\tilde{E}^H_*(S^{L+Z})$ for each $x \in M-\partial M$. If $\partial M = \phi$, the same definition applies with $M/\partial M$ interpreted as M^+.

Proposition 6.7. Let M be a smooth compact G-manifold smoothly (and properly) embedded in a G-representation V. Then the Spanier-Whitehead-Atiyah duality isomorphism

$$\tilde{E}_G^{V-\alpha}(T(\tau^{\perp})) \cong E_\alpha^G(M/\partial M)$$

restricts to a bijective correspondence between E-orientations of $\tau^{\perp}$ (and thus of τ and M) and E-fundamental classes of M.

Proof. Let $t_x\colon M/\partial M \to G^+\wedge_H S^Z$ be as in Definition 6.6, let W be the fibre of $\tau^{\perp}$ at x, and let L be the tangent space of G/H at eH. Thus V is isomorphic as an H-representation to $L + Z + W$. If $j_x\colon G^+\wedge_H S^W \to T(\tau^{\perp})$ is induced by the inclusion of $G\times_H W$ in $\tau^{\perp}$, then the top square of the following diagram commutes by Proposition 5.5 and the bottom square commutes by Proposition 2.11.

$$\begin{tikzcd}
\tilde{E}_G^{V-\alpha}(T(\tau^{\perp})) \arrow[r, "\cong"] \arrow[d, "j_x^*"] & \tilde{E}^G(M/\partial M) \arrow[d, "t_{x*}"] \\
\tilde{E}_G^{V-\alpha}(G^+\wedge_H S^W) \arrow[r, "\cong"] \arrow[d, phantom, "\wr\wr"] & \tilde{E}_\alpha^G(G^+\wedge_H S^Z) \arrow[d, phantom, "\wr\wr"] \\
\tilde{E}_H^{V-\alpha}(S^W) \arrow[r, phantom, "\cong"] & \tilde{E}_\alpha^H(S^{L+Z})
\end{tikzcd}$$

The result follows from the diagram and the definitions.

Scholium 6.8. In the nonequivariant case, the previous result is given in Switzer [137,14.18]. As Stong observed, Switzer's proof fails because it relies on [137,14.9], which is false. Our proof escapes the difficulty in [137] by connecting global orientations and fundamental classes directly rather than via local orientations and fundamental classes.

§7. Trace maps and their additivity on cofibre sequences

To begin with, we return to the categorical context of section 1 and assume given a closed symmetric monoidal category $\mathscr{C}$ with unit S, product $\wedge$, and internal hom functor F. We introduce a general categorical notion of a trace map.

Definition 7.1. Let X and C be objects of $\mathscr{C}$, with X finite, and let $f\colon X \to X$ and $\Delta\colon X \to X\wedge C$ be morphisms of $\mathscr{C}$. Define the trace of f with respect to Δ, denoted $\tau(f)$, to be the composite

$$S \xrightarrow{\eta} X\wedge DX \xrightarrow{\gamma} DX\wedge X \xrightarrow{1\wedge f} DX\wedge X \xrightarrow{1\wedge\Delta} DX\wedge X\wedge C \xrightarrow{\varepsilon\wedge 1} S\wedge C \cong C.$$

If $C = S$ and Δ is the unit isomorphism $X \cong X\wedge S$, then $\tau(f)$ is denoted $\chi(f)$ and called the trace (or Lefschetz constant) of f. If f is the identity map, $\chi(f)$ is denoted $\chi(X)$ and called the Euler characteristic of X. If $C = X$, we call Δ a diagonal map; here, if f is the identity map, $\tau(f)$ is denoted $\tau(X)$ and called the transfer (or pretransfer) of X.

To see the intuition, the reader should check for himself that the "trace" $\chi(f)$ of a linear transformation $f\colon X \to X$ on a finite dimensional vector space X really is the usual trace and that, if X is graded, the "Euler characteristic" $\chi(X)$ really is the usual Euler characteristic. The verification will show the essential role played by the transposition γ, with its associated sign.

In the context of G-spectra, the transfer and Euler characteristics will be central to the work of the next two chapters. There only the identity map f will be used, but we shall work with general maps in this chapter with a view towards applications in equivariant fixed point theory.

In practice, C is usually a "coalgebra" and X is a "right C-comodule". This means that C has a coassociative coproduct $\Delta\colon C \to C\wedge C$ with a two-sided counit (or augmentation) $\xi\colon C \to S$ and that the following diagrams commute.

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X\wedge C \arrow[d, "1\wedge\Delta"] \\
X\wedge C \arrow[r, "\Delta\wedge 1"] & X\wedge C\wedge C
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
X \arrow[d, "\Delta"'] \arrow[dr, equal] & \\
X\wedge C \arrow[r, "1\wedge\xi"] & X\wedge S.
\end{tikzcd}$$

The second diagram clearly implies that $\xi\circ\tau(f) = \chi(f)$ for any map $f\colon X \to X$. We shall refer to our given map $\Delta\colon X \to X\wedge C$ as a coaction of C on X even when we don't assume the extra data just specified.

Since $(f\wedge 1)\eta = (1\wedge Df)\eta$ and $\varepsilon(1\wedge f) = \varepsilon(Df\wedge 1)$, easy diagram chases show that the same map $\tau(f)$ is obtained by inserting any one of the four composites

$$DX\wedge X \underset{Df\wedge 1}{\overset{1\wedge f}{\rightrightarrows}} DX\wedge X \xrightarrow{1\wedge\Delta} DX\wedge X\wedge C \quad\text{and}\quad DX\wedge X \xrightarrow{1\wedge\Delta} DX\wedge X\wedge C \underset{Df\wedge 1\wedge 1}{\overset{1\wedge f\wedge 1}{\rightrightarrows}} DX\wedge X\wedge C$$

between $\gamma\eta$ and ε. This fact aids in the verification of the formal properties of the trace, which we catalog in the following series of lemmas.

Lemma 7.2 (Unit property). $\chi(f) = f$ for any map $f\colon S \to S$.

Lemma 7.3 (Fixed point property). Let C coact on X and let $f\colon X \to X$ and $h\colon C \to C$ be such that $(f\wedge h)\Delta = \Delta f$. Then $h\tau(f) = \tau(f)$.

Lemma 7.4 (Invariance under retraction). Let C coact on X and D coact on Y and let $k\colon X \to Y$, $k'\colon Y \to X$, and $h\colon C \to D$ be such that $k'k = 1$ and $(k\wedge h)\Delta = \Delta k$. Then $h\tau(f) = \tau(kfk')$ for any map $f\colon X \to X$.

In particular, we can take $C = D$ and give Y the coaction $(k\wedge 1)\Delta k'$ induced by the coaction on X. When k is an equivalence, we view this as a homotopy invariance property.

Lemma 7.5 (Commutation with products). Let C coact on X and D coact on Y and give $X\wedge Y$ the coaction

$$X\wedge Y \xrightarrow{\Delta\wedge\Delta} X\wedge C\wedge Y\wedge D \xrightarrow{1\wedge\gamma\wedge 1} X\wedge Y\wedge C\wedge D$$

by $C\wedge D$. Then $\tau(f\wedge g) = \tau(f)\wedge\tau(g)\colon S \cong S\wedge S \to C\wedge D$ for any maps $f\colon X \to X$ and $g\colon Y \to Y$.

In practice, and in the next lemma, $\mathscr{C}$ is an additive category and the functor $\wedge$ is bilinear. We then write $\vee$ for the biproduct (which is the wedge sum in our categories of spectra and the direct sum in the usual categories of modules). Here $D(X\vee Y) \cong DX\vee DY$ and $X\vee Y$ is finite if X and Y are finite.

Lemma 7.6 (Commutation with sums). Let C coact on X and Y and give $X\vee Y$ the coaction

$$X\vee Y \xrightarrow{\Delta\vee\Delta} (X\wedge C)\vee(Y\wedge C) \cong (X\vee Y)\wedge C.$$

Then, for any map $h\colon X\vee Y \to X\vee Y$, $\tau(h) = \tau(f) + \tau(g)$, where f and g are the restrictions $X \to X$ and $Y \to Y$ of h.

The force of this result is that the cross terms $X \to Y$ and $Y \to X$ of h make no contribution to $\tau(h)$, as one would expect of a trace function.

Proposition 1.9 has the following immediate consequence.

Proposition 7.7. Let $\phi\colon \mathscr{C} \to \mathscr{D}$ be a monoidal functor whose unit map $\lambda\colon T \to \phi S$ is an isomorphism. Let X be a finite object of $\mathscr{C}$ such that $\phi\colon \phi X\wedge\phi DX \to \phi(X\wedge DX)$ is an isomorphism. Let C coact on X and give ϕX the coaction

$$\phi X \xrightarrow{\phi\Delta} \phi(X\wedge C) \xrightarrow{\phi^{-1}} \phi X\wedge\phi C$$

by ϕC. Then the following diagrams commute for any map $f\colon X \to X$.

$$\begin{tikzcd} T \arrow[r, "\chi(\phi f)"] \arrow[d, "\lambda"'] & T \arrow[d, "\lambda"] \\ \phi S \arrow[r, "\phi\chi(f)"'] & \phi S \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T \arrow[r, "\tau(\phi f)"] \arrow[d, "\lambda"'] & \phi C \arrow[d, equal] \\ \phi S \arrow[r, "\phi\tau(f)"'] & \phi C \end{tikzcd}$$

We now focus attention on the category $\bar{h}G\mathscr{S}U$ of G-spectra, where we either assume that U is a complete G-universe or restrict attention to G-CW spectra built up out of cells $G/H^+\wedge e^n$ such that G/H embeds in U.

Lemma 7.8. For any integer n, $\chi(S^n) = (-1)^n$. If a G-spectrum C coacts on a finite G-spectrum X and $\Sigma^n X$ is given the coaction

$$\Sigma^n X \xrightarrow{\Sigma^n\Delta} \Sigma^n(X\wedge C) \simeq (\Sigma^n X)\wedge C,$$

Then $\tau(\Sigma^n f) = (-1)^n\tau(f)$ for any map $f\colon X \to X$.

Proof. Under the equivalence $S^{-n} \simeq DS^n$ adjoint to the standard equivalence $S^{-n}\wedge S^n \simeq S^0$, $\chi(S^n)$ corresponds to the transposition

$$S \simeq S^n\wedge S^{-n} \xrightarrow{\gamma} S^{-n}\wedge S^n \simeq S,$$

which has degree $(-1)^n$. By Lemma 7.5, $\tau(\Sigma^n f) = \chi(S^n)\tau(f)$.

Proposition 7.9 (Invariance under change of groups). Let $H \subset G$ and let X be a finite H-spectrum with a coaction by an H-spectrum C. Give $G\ltimes_H X$ the coaction

$$G\ltimes_H X \xrightarrow{1\ltimes\Delta} G\ltimes_H(X\wedge C) \xrightarrow{1\ltimes(1\wedge\eta)} G\ltimes_H(X\wedge(G\ltimes_H C)) \cong (G\ltimes_H X)\wedge(G\ltimes_H C)$$

by $G\ltimes_H C$. Then the following diagram of G-spectra commutes for any H-map $f\colon X \to X$, where $\tau(G\ltimes_H S)$ is defined with respect to the diagonal coaction of $G\ltimes_H S \cong \Sigma^\infty G/H^+$ on itself.

$$\begin{tikzcd} & S \arrow[dl, "\tau(G\ltimes_H S)"'] \arrow[dr, "\tau(1\ltimes f)"] & \\ G\ltimes_H S \arrow[rr, "1\ltimes\tau(f)"] & & G\ltimes_H C \end{tikzcd}$$

Proof. Let $Y = DX$. Proposition 2.11 gives explicit maps displaying $G\ltimes_H X$ and $G\ltimes_H\Sigma^{-L}Y$ as dual finite G-spectra. We note that the composite

$$G\ltimes_H(X\wedge\Sigma^{-L}Y) \xrightarrow{1\ltimes(\eta\wedge 1)} G\ltimes_H((G\ltimes_H X)\wedge\Sigma^{-L}Y) \cong (G\ltimes_H X)\wedge(G\ltimes_H\Sigma^{-L}Y)$$

appearing there coincides with the composite

$$G\ltimes_H(X\wedge\Sigma^{-L}Y) \xrightarrow{1\ltimes(1\wedge\eta)} G\ltimes_H(X\wedge(G\ltimes_H\Sigma^{-L}Y)) \cong (G\ltimes_H X)\wedge(G\ltimes_H\Sigma^{-L}Y)$$

since these two G-maps give the same H-map when precomposed with $\eta\colon X\wedge\Sigma^{-L}Y \to G\ltimes_H(X\wedge\Sigma^{-L}Y)$. Now direct inspection of definitions and easy diagram chases show that, with the notation of Proposition 2.11, $\tau(1\ltimes f)$ is the composite G-map

$$S \xrightarrow{\tilde{\eta}} G \ltimes_H (X \wedge \Sigma^{-L}Y) \xrightarrow{1 \ltimes \psi} G \ltimes_H (Y \wedge X \wedge (G \ltimes_H C)) \cong G \ltimes_H (Y \wedge X) \wedge (G \ltimes_H C) \xrightarrow{\tilde{\varepsilon} \wedge 1} G \ltimes_H C,$$

where ψ is the composite H-map

$$X \wedge \Sigma^{-L}Y \xrightarrow{1 \wedge \eta} X \wedge (G \ltimes_H \Sigma^{-L}Y) \xrightarrow{\gamma} (G \ltimes_H \Sigma^{-L}Y) \wedge X \xrightarrow{1 \wedge f} (G \ltimes_H \Sigma^{-L}Y) \wedge X$$

$$\xrightarrow{1 \wedge \Delta} (G \ltimes_H \Sigma^{-L}Y) \wedge X \wedge C \xrightarrow{1 \wedge 1 \wedge \eta} (G \ltimes_H \Sigma^{-L}Y) \wedge X \wedge (G \ltimes_H C) \xrightarrow{\mu \wedge 1 \wedge 1} Y \wedge X \wedge (G \ltimes_H C).$$

Visibly, this last composite coincides with

$$X \wedge \Sigma^{-L}Y \xrightarrow{1 \wedge \mu\eta} X \wedge Y \xrightarrow{(1 \wedge \Delta)(1 \wedge f)\gamma} Y \wedge X \wedge C \xrightarrow{1 \wedge 1 \wedge \eta} Y \wedge X \wedge (G \ltimes_H C).$$

Here the composite H-map

$$\Sigma^{-L}Y \xrightarrow{\eta} G \ltimes_H \Sigma^{-L}Y \xrightarrow{\mu} Y$$

agrees with the composite

$$Y \wedge S^{-L} \xrightarrow{1 \wedge \eta} Y \wedge (G \ltimes_H S^{-L}) \xrightarrow{1 \wedge \mu} Y \wedge S^0.$$

(Actually, $\mu\eta: S^{-L} \to S^0$ is the dual of $\Sigma^{-L}e$, $e: S^0 \to S^L$; compare II.6.12.) We may express $\tau(G \ltimes_H S)$ as the composite

$$S \xrightarrow{\tilde{\eta}} G \ltimes_H S^{-L} \xrightarrow{1 \ltimes \mu\eta} G \ltimes_H S.$$

Now a further easy diagram chase, which uses that the composite

$$G \ltimes_H C \xrightarrow{1 \ltimes \eta} G \ltimes_H (G \ltimes_H C) \cong G/H^+ \wedge (G \ltimes_H C) \xrightarrow{\xi \wedge 1} G \ltimes_H C$$

is the identity, gives the conclusion.

The rest of this section will be devoted to the proof of the following result, whose space level implications will be discussed in the next section.

<u>Theorem 7.10</u> (Additivity on cofibre sequences). Let X and Y be finite G-spectra coacted on by a G-spectrum C and let $f: X \to X$, $g: Y \to Y$, and $k: X \to Y$ be G-maps such that the diagrams

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ X \arrow[r, "k"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\Delta"'] & Y \arrow[d, "\Delta"] \\ X \wedge C \arrow[r, "k \wedge 1"] & Y \wedge C \end{tikzcd}$$

commute up to G-homotopy. Let Z denote the cofibre $Y \cup_k CX$ and construct G-maps $h: Z \to Z$ and $\Delta: Z \to Z \wedge C$ such that the middle squares commute strictly and the right squares commute up to G-homotopy in the diagrams

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "f"'] & Y \arrow[r, "i"] \arrow[d, "g"] & Z \arrow[r, "\pi"] \arrow[d, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X \arrow[r, "k"] & Y \arrow[r, "i"] & Z \arrow[r, "\pi"] & \Sigma X \end{tikzcd}$$

and

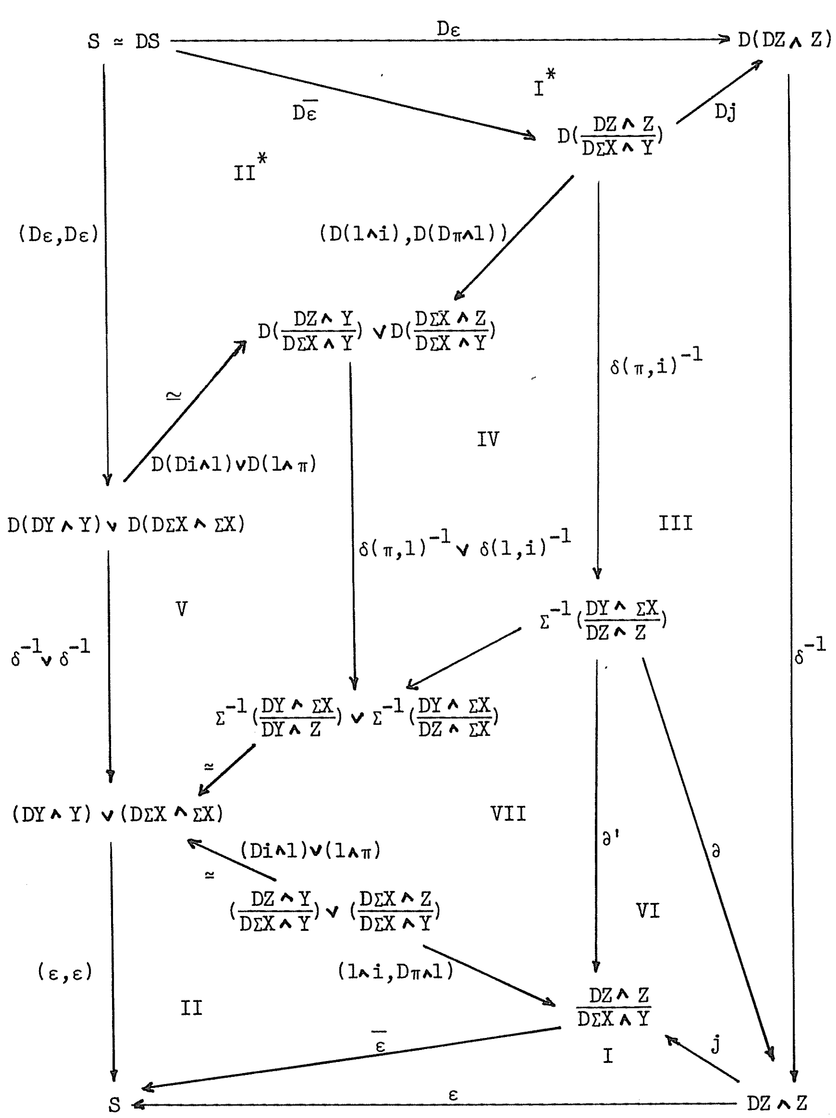
$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "\Delta"'] & Y \arrow[r, "i"] \arrow[d, "\Delta"] & Z \arrow[r, "\pi"] \arrow[d, "\Delta"] & \Sigma X \arrow[d, "\Sigma\Delta"] \\ X \wedge C \arrow[r, "k \wedge 1"] & Y \wedge C \arrow[r, "i \wedge 1"] & Z \wedge C \arrow[r, "\pi \wedge 1"] & \Sigma X \wedge C, \end{tikzcd}$$

where i and π are the natural inclusion and quotient maps. Then $\tau(g) = \tau(f) + \tau(h)$. In particular, with $C = S$, $\chi(g) = \chi(f) + \chi(h)$.

<u>Proof</u>. It is important to note that no compatibility between f and g and the coactions is required. By abuse of notation, we agree to write B/A for the cofibre of a map $A \to B$, not necessarily a cofibration, throughout this proof (except in the case of Z). There will always be a canonical relevant map $A \to B$ in sight. We write $i: B \to B/A$ and $\pi: B/A \to \Sigma A$ generically for the resulting canonical inclusion and quotient map. The construction of h and Δ is a standard use of homotopies, exactly as on the space level. Since $\tau(\Sigma f) = -\tau(f)$, it suffices to prove that $\tau(h) = \tau(g) + \tau(\Sigma f)$. By Proposition 1.4, $\tau(h)$ coincides with the composite

$$S \simeq DS \xrightarrow{D\varepsilon} D(DZ \wedge Z) \xrightarrow{\delta^{-1}} DZ \wedge Z \xrightarrow{1 \wedge h} DZ \wedge Z \xrightarrow{1 \wedge \Delta} DZ \wedge Z \wedge C \xrightarrow{\varepsilon \wedge 1} S \wedge C \simeq C,$$

and similarly for $\tau(g)$ and $\tau(\Sigma f)$. In the special case of Euler characteristics, the last few maps can be replaced by $\varepsilon: DZ \wedge Z \to S$. We claim that all parts of the diagram on the following page commute in the G-stable category. This will prove the result for Euler characteristics. To handle the general case, we need only expand the bottom left corner of the diagram by replacing $\varepsilon: DZ \wedge Z \to S$ by $(\varepsilon \wedge 1)(1 \wedge \Delta)(1 \wedge h): DZ \wedge Z \to C$, and similarly for Y and ΣX. The maps $1 \wedge g$ and $1 \wedge h$ induce self-maps of the three cofibres displayed (in quotient notation) in diagrams I and II, and the maps $1 \wedge \Delta$ induce coactions by C on these cofibres. It is easy to verify that these induced maps fit into naturality diagrams with respect to the maps $Di \wedge 1$, $1 \wedge \pi$, $1 \wedge i$, $D\pi \wedge 1$, and j appearing in the cited diagrams. These naturality diagrams and the diagrams obtained by smashing I and II with C make clear that the general case will follow from the special case.



To prove that our diagram commutes, we require precise control on maps between fibres (or, equivalently by the discussion at the start of section 2, between cofibres), and we work with our explicit dual function spectra. The following diagrams commute on the level of spectra since we have a spectrum level adjunction.

$$
\begin{array}{ccc}
DY\wedge X \xrightarrow{1\wedge k} DY\wedge Y & DZ\wedge Y \xrightarrow{1\wedge i} DZ\wedge Z & D\Sigma X\wedge Z \xrightarrow{1\wedge \pi} D\Sigma X\wedge \Sigma X \\
Dk\wedge 1 \downarrow \qquad\quad \downarrow \varepsilon & Di\wedge 1 \downarrow \qquad\quad \downarrow \varepsilon & D\pi\wedge 1 \downarrow \qquad\quad \downarrow \varepsilon \\
DX\wedge X \xrightarrow{\ \varepsilon\ } S & DY\wedge Y \xrightarrow{\ \varepsilon\ } S & DZ\wedge Z \xrightarrow{\ \varepsilon\ } S
\end{array}
$$

The composite $\varepsilon(D\pi\wedge i)\colon D\Sigma X\wedge Y \to S$ is equal to $\varepsilon(1\wedge\pi)(1\wedge i)$ and is thus the trivial map of spectra. If j denotes the inclusion of $DZ\wedge Z$ in the cofibre of $D\pi\wedge i$, there results a canonical extension $\bar{\varepsilon}$ of ε over j. This gives I in our main diagram, and II commutes by the second and third diagrams above. Diagrams I^* and II^* are the duals of diagrams I and II.

Diagram III is an identification up to equivalence of the map Dj as the homotopy fibre of $Di\wedge\pi\colon DZ\wedge Z \to DY\wedge\Sigma X$. To obtain this diagram, recall the general maps $\delta\colon DB\wedge A \to D(DA\wedge B)$ of Proposition 1.4. These maps are only natural on the stable category level, because the commutativity isomorphism for the smash product and the remaining maps specifying δ involve inverting weak equivalences. However, on the spectrum level, δ is given by a chain of natural maps going in the right direction and natural weak equivalences going in the wrong direction (the latter coming from II.3.12 and the proof of II.1.7). Given spectrum level maps $\alpha\colon A \to A'$ and $\beta\colon B' \to B$, we can work our way stepwise along the chain to obtain a canonical chain of weak equivalences representing $\delta(\alpha,\beta)$ in the comparison of fibration sequences

$$
\begin{array}{ccc}
\Sigma^{-1}\left(\dfrac{DB'\wedge A'}{DB\wedge A}\right) & \xrightarrow{\delta(\alpha,\beta)} & D\left(\dfrac{DA\wedge B}{DA'\wedge B'}\right) \\
\partial\downarrow & & \downarrow Dk \\
DB\wedge A & \xrightarrow{\ \delta\ } & D(DA\wedge B) \\
D\beta\wedge\alpha\downarrow & & \downarrow D(D\alpha\wedge\beta) \\
DB'\wedge A' & \xrightarrow{\ \delta\ } & D(DA'\wedge B')
\end{array}
$$

On the left, ∂ should be interpreted as the projection of the homotopy fibre of $D\beta\wedge\alpha$ onto $DB\wedge A$, but we can use section 2 to replace it by a map in a cofibre sequence, as indicated by the notation. This diagram specializes to give III. Our precise construction of $\delta(\alpha,\beta)$ shows that it is natural in α and β, and IV is just a naturality diagram. Its unlabeled bottom arrow is given by the two canonical comparisons of homotopy fibres, or of desuspensions of cofibres, given by the commutative diagram

$$\begin{tikzcd}
DY\wedge Y \arrow[r, phantom, "\simeq"] \arrow[dr, "1\wedge i"'] & \Sigma^{-1}C(1\wedge\pi) \arrow[d] & \Sigma^{-1}C(Di\wedge\pi) \arrow[l] \arrow[r] \arrow[d] & \Sigma^{-1}C(Di\wedge 1) \arrow[d] \arrow[r, phantom, "\simeq"] & D\Sigma X\wedge\Sigma X \arrow[dl, "D\pi\wedge 1"] \\
& DY\wedge Z \arrow[d, "1\wedge\pi"'] & DZ\wedge Z \arrow[l, "Di\wedge 1"'] \arrow[r, "1\wedge\pi"] \arrow[d, "Di\wedge\pi"] & DZ\wedge\Sigma X \arrow[d, "Di\wedge 1"] & \\
& DY\wedge\Sigma X \arrow[r, equal] & DY\wedge\Sigma X \arrow[r, equal] & DY\wedge\Sigma X &
\end{tikzcd}$$

Of course, the desuspended cofibre on the left is just $DY\wedge Y$ and that on the right is canonically equivalent to $D\Sigma X\wedge\Sigma X$. This fact gives the unlabeled equivalence at the bottom of V, and that diagram can then be checked to be an identification of canonical maps δ.

We define $\partial' = j\partial$ to obtain diagram VI. Then ∂' is the boundary map of the natural cofibre sequence

$$\frac{DZ\wedge Z}{D\Sigma X\wedge Y} \longrightarrow \frac{DY\wedge\Sigma X}{D\Sigma X\wedge Y} \longrightarrow \frac{DY\wedge\Sigma X}{DZ\wedge Z}\ .$$

Here ΣX and DY are equivalent to the cofibres of $i\colon Y\to Z$ and $D\pi\colon D\Sigma X\to DZ$, and the commutativity of VII is a special case of the following lemma.

Lemma 7.11. Let $f\colon A\to X$ and $g\colon B\to Y$ be maps of G-spectra and let $i\colon X\to Cf$ and $j\colon Y\to Cg$ be the inclusions into their cofibres. Then the boundary map

$$\partial\colon \Sigma^{-1}C(i\wedge j)\longrightarrow C(f\wedge g)$$

in the natural cofibre sequence

$$C(f\wedge g)\longrightarrow C(if\wedge jg)\longrightarrow C(i\wedge j)$$

is the sum of the two natural composites

$$\Sigma^{-1}C(i\wedge j)\xrightarrow{\Sigma^{-1}C(i\wedge 1)}\Sigma^{-1}C(1_{Cf}\wedge j)\simeq Cf\wedge B\cong C(f\wedge 1_B)\xrightarrow{C(1\wedge g)}C(f\wedge g)$$

and

$$\Sigma^{-1}C(i\wedge j)\xrightarrow{\Sigma^{-1}C(1\wedge j)}\Sigma^{-1}C(i\wedge 1_{Cg})\simeq A\wedge Cg\cong C(1_A\wedge g)\xrightarrow{C(f\wedge 1)}C(f\wedge g).$$

Proof. We may assume that f and g are inclusions of subcomplexes in G-CW spectra and replace cofibres by actual quotients. The given cofibre sequence then becomes

$$\frac{X\wedge Y}{A\wedge B}\longrightarrow\frac{Cf\wedge Cg}{A\wedge B}\longrightarrow\frac{Cf\wedge Cg}{X\wedge Y}\ .$$

Here $Cf\wedge Cg$ is equivalent to $(X\wedge Y)/(X\wedge B\cup A\wedge Y)$, hence the last quotient is equivalent to $\Sigma(X\wedge B\cup A\wedge Y)$. The boundary map in question agrees under the equivalence with the composite

$$X\wedge B\cup A\wedge Y\longrightarrow X\wedge Y\longrightarrow X\wedge Y/A\wedge B.$$

This factors through

$$\frac{X\wedge B\cup A\wedge Y}{A\wedge B} = \left(\frac{X\wedge B}{A\wedge B}\right)\vee\left(\frac{A\wedge Y}{A\wedge B}\right)$$

in the obvious fashion, and translation back to cofibres gives the conclusion.

§8. Space level analysis of trace maps

We here use the additivity of the trace on cofibre sequences to analyze its behavior on suspension spectra, and we agree to use the same letter for a map of spaces and its induced map of suspension spectra. The essential point is to determine equivariant traces in terms of non-equivariant traces and equivariant Euler characteristics of orbit spaces. As we shall see in chapter V, the latter are typical basis elements of the Burnside ring $A(G)\cong\pi_0^G(pt)$, where π_0^G denotes unreduced stable G-homotopy.

Suppose given a compact G-ENR pair (X,A) and a map of pairs $f\colon(X,A)\to(X,A)$ with induced map $C(f)\colon C(X,A)\to C(X,A)$ of unreduced mapping cones. Let $i\colon A\to X$ be the inclusion. The coactions $\Delta\colon X^+\to X^+\wedge X^+$ and $(1\wedge i^+)\Delta\colon A^+\to A^+\wedge X^+$ are compatible and induce the coaction

$$C(\Delta)\colon C(X,A)\longrightarrow C(X\times X,\ A\times X)\cong C(X,A)\wedge X^+$$

of X^+ on $C(X,A)$. We can use Construction 4.5 to give an explicit space level map which represents the stable map $\tau(C(f))$. We assume the notations of the cited construction, so that $r\colon N\to X$ is a retraction of a neighborhood N of X in some representation V and $s\colon X\to X$ restricts to a retraction $U\to A$ of a neighborhood U of A in X.

Lemma 8.1. The following diagram is G-homotopy commutative.

$$\begin{tikzcd}
S^V \arrow[d, "\tau(C(f))"'] & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r] & C(V,(V-X)\cup U) & C(N-A,(N-X)\cup(U-A)) \arrow[l, "\simeq"'] \arrow[d, "C(\phi)"] \\
S^V\wedge X^+ & C(V\times X,(V-B)\times X) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & C(V\times X,(V-\{0\})\times X) & C[(V-A,V-X)\times(X,A)] \arrow[l, "C(\psi)"']
\end{tikzcd}$$

Here $\phi(n) = (n, sr(n))$ for $n \in N-A$ and $\psi(v,x) = (v-f(x), f(x))$ for $(v,x) \in (V-A) \times X$.

From this description, it is easy to derive a criterion for the vanishing of $\tau(C(f))$ in terms of the fixed points of f.

<u>Proposition 8.2</u>. If f has no fixed points in the complement of A, then $\tau(C(f)) = 0$ and therefore $\chi(C(f)) = 0$. If, further, $f \simeq 1$ as a map of pairs, then $\tau(g) = 0$ and therefore $\chi(g) = 0$ for every stable G-map $g\colon \Sigma^\infty C(X,A) \to \Sigma^\infty C(X,A)$.

<u>Proof</u>. If f has no fixed points in $X-A$, then

$$\psi\colon (V-A, V-X) \times (X,A) \longrightarrow (V \times X, (V - \{0\}) \times X)$$

factors through $((V - \{0\}) \times X, (V - \{0\}) \times X)$. If $f \simeq 1$, then $C(\psi) \simeq C(\omega)$, where $\omega(v,x) = (v-x, x)$. For any map g, $\tau(g)$ factors through the map obtained by applying $\Sigma^{-V}\Sigma^\infty$ to the bottom row of the diagram in Lemma 8.1 with $C(\psi)$ replaced by $C(\omega)$.

We shall calculate $\tau(C(f))$ in general, and we begin by calculating $\tau(f) \in \pi_0^G(G/H)$ for a self-map f of an orbit space $G/H^+ \wedge S^n$, where G/H^+ coacts on $G/H^+ \wedge S^n$ via the $n^{\underline{th}}$ suspension of the diagonal on G/H^+. By homotopy invariance (Lemma 7.4), we could just as well view $G/H^+ \wedge S^n$ as $C(G/H \times e^n, G/H \times S^{n-1})$ coacted on by $(G/H \times e^n)^+$. As usual, let $WH = NH/H$, where NH is the normalizer of H in G, and observe that $WH = (G/H)^H$. Write $\tau(X) = \tau(\Sigma^\infty X^+)$ for an unbased G-space X.

<u>Lemma 8.3</u>. Let $f\colon G/H^+ \wedge S^n \to G/H^+ \wedge S^n$ be any G-map. Let $e(f^H) = 0$ if WH is infinite and $e(f) = \chi(f^H)/|WH|$ if WH is finite, where $\chi(f^H)$ is the nonequivariant trace of the H-fixed point WH-map $f^H\colon WH^+ \wedge S^n \to WH^+ \wedge S^n$. Then $\tau(f) = (-1)^n e(f)\tau(G/H)$.

<u>Proof</u>. If WH contains a circle group, then G/H has a fixed point free self G-map homotopic to the identity and $\tau(G/H)$, $\tau(f)$, and $\chi(f^H)$ are all zero. Thus assume that WH is finite. Passage to H-fixed points gives a bijection

$$[G/H^+ \wedge S^n, G/H^+ \wedge S^n]_G \longrightarrow [WH^+ \wedge S^n, WH^+ \wedge S^n]_{WH} \cong [S^n, WH^+ \wedge S^n].$$

If $n > 0$, this is the integral group ring $Z[WH]$; if $n = 0$, it is WH and may be viewed as a subset of $Z[WH]$. We may write $f = \sum_{w \in WH} n_w w$, where $w \in WH$ is regarded as a self G-map of $G/H^+ \wedge S^n$. By Lemma 7.6, $\tau(f) = \Sigma\, n_w \tau(w)$. If

$w \neq e$, the map w is fixed-point free and $\tau(w) = 0$, and of course $\tau(e) = (-1)^n \tau(G/H)$ by Lemma 7.8. Clearly the nonequivariant trace $\chi(f^H)$ is equal to $n_e|WH|$, and this implies the result.

We wish to compute $\tau(C(f))$ for a map of pairs $f\colon (X,A) \to (X,A)$, but it is useful to proceed in greater generality and compute $\tau(f)$ for all maps $f\colon X/A \to X/A$ and not just those induced by maps of pairs. We let X^+ coact on X/A via the canonical equivalence $C(X,A) \simeq X/A$.

<u>Theorem 8.4</u>. Let (X,A) be a pair of G-CW complexes such that X/A is finite and let $f\colon X/A \to X/A$ be a cellular G-map. Let

$$j_i\colon G/H_i \subset G/H_i \times e^{n_i} \longrightarrow X$$

be the composite of the inclusion of an orbit and the $i^{\underline{th}}$ characteristic map for some enumeration of the cells of $X-A$. Let $f_i\colon G/H_i^+ \wedge S^{n_i} \to G/H_i \wedge S^{n_i}$ be the map induced by f on the $i^{\underline{th}}$ wedge summand of the $n_i^{\underline{th}}$ skeletal subquotient of X/A. Then

$$\tau(f) = \sum_i (-1)^{n_i} e(f_i)(j_i)_* \tau(G/H).$$

<u>Proof</u>. The conclusion follows inductively by use of homotopy invariance (Lemma 7.4), additivity on wedges (Lemma 7.6), the previous lemma, and Theorem 7.10 applied to the inclusions of skeleta $(X/A)^{n-1} \to (X/A)^n$, the restrictions of f to skeleta, and the coactions $(X/A)^n \to (X/A)^n \wedge X^+$ obtained by restriction from any cellular approximation of the diagonal coaction $X/A \to (X/A) \wedge X^+$.

<u>Remark 8.5</u>. When A is empty, the theorem applies to compute $\tau(f^+)$ for a map $f\colon X \to X$. When X is based and A is the base vertex, it applies to compute $\tau(f)$ for a based map $f\colon X \to X$. These two elements of $\pi_0^G(X)$ differ by the summand of $\tau(f^+)$ coming from the base vertex of X. We have a corresponding distinction between the traces $\chi(f^+)$ and $\chi(f)$ in $\pi_0^G(pt)$.

<u>Remark 8.6</u>. If X is a finite G-CW spectrum and $f\colon X \to X$ is a cellular G-map, then the same formal argument applies to give the formula $\chi(f) = \sum_i \chi(f_i)$ in $\pi_0^G(S)$, where $f_i\colon G/H_i^+ \wedge S^{n_i} \to G/H_i^+ \wedge S^{n_i}$ is the map induced by f on the $i^{\underline{th}}$ wedge summand of the $n_i^{\underline{th}}$ skeletal subquotient of X. Since the f_i here are spectrum level maps, Lemma 8.3 is not sufficient to compute the $\chi(f_i)$. It suffices by Lemma 7.8 to consider the case $n_i = 0$, and we shall determine all stable maps $G/H^+ \to G/H^+$ in V§9; at least when G is finite, we shall also determine the traces of all such maps; see V.9.8.

Theorem 8.4 does not apply to general compact G-ENR's since these need not have the homotopy types of finite G-CW complexes. It also suffers from an evident lack of invariance. We want a calculation of $\tau(f)$ intrinsic to the structure of X as a G-space, irrespective of possible cellular decompositions. For simplicity, we restrict attention to the absolute case.

Let X be a compact G-ENR and consider its G-subspaces

$$X_{(H)} = \{x \mid G_x \text{ is conjugate to } H\}.$$

The orbit space $X_{(H)}/G$ breaks up as a disjoint union of path connected subspaces M, called the orbit type components of X/G. If X is a smooth G-manifold, then each M is a smooth manifold. Let $\pi: X \to X/G$ be the quotient map. Given an orbit type component M, let L be the closure of $\pi^{-1}(M)$ in X, let $\partial L = L - \pi^{-1}(M)$, and let $K = L/\partial L$. These subquotients K of X play a role analogous to that of the subquotient spheres $G/H^+\wedge S^n$ of a finite G-CW complex. To obtain an analog of the skeletal filtration, observe that X has only a finite number of orbit types and these are partially ordered by inclusion. We may enumerate them as $G/H_1,\cdots,G/H_n$, with $(H_i) < (H_j)$ only if $i > j$. The closed subspaces

$$X_j = \{x \mid x \text{ has orbit type } G/H_i \text{ with } i \leq j\}$$

give an increasing filtration of X. Any G-map $f: X \to X$ preserves this filtration and so induces quotient maps $X_j/X_{j-1} \to X_j/X_{j-1}$. Visibly, X_j/X_{j-1} is the wedge of the G-spaces K determined as above by the orbit type components M of type G/H_j. Thus f induces a based map $f_K: K \to K$ for each such K. Moreover, the coaction of X_j^+ on $C(X_j,X_{j-1})$ induces a coaction of X^+ on each K via the equivalence $C(X_j,X_{j-1}) \simeq X_j/X_{j-1}$ and the inclusion $X_j \to X$. We therefore have a trace $\tau(f_K) \in \pi_0^G(X)$.

<u>Theorem 8.7</u>. Let X be a compact G-ENR and let $f: X \to X$ be any G-map. Let $\pi: X \to X/G$ be the quotient map and let

$$j_m: G/H \subset \pi^{-1}(M) \subset X$$

be the inclusion of an orbit in the orbit type component M. Let $f_K: K \to K$ be the G-map induced by f on the quotient $K = L/\partial L$, where L is the closure of $\pi^{-1}(M)$ in X and $\partial L = L - \pi^{-1}(M)$. If WH is infinite, let $e(f_K) = 0$. If WH is finite, let $e(f_K) = \chi(f_K^H)/|WH|$, where $\chi(f_K^H)$ is the nonequivariant trace of the H-fixed point WH-map $f_K^H: K^H \to K^H$. Then

$$\tau(f) = \sum_M e(f_K)(j_M)_*\tau(G/H).$$

<u>Proof</u>. By induction up the orbit types, Lemma 7.4, Lemma 7.6, and Theorem 7.10 give that $\tau(f) = \Sigma\tau(f_K)$. Here $\tau(f_K)$ is computed with respect to the coaction above of X^+ on K. Since $K = L/\partial L$, we also have a coaction by L^+, and it is clear that $\tau(f_K)$ in $\pi_0^G(X)$ is obtained by application of i_*^+ to $\tau(f_K)$ in $\pi_0^G(L)$, where $i: L \to X$ is the inclusion. Applied to the pair $(L,\partial L)$, the following result computes the latter trace and so completes the proof. It also gives further information about $e(f_K)$.

<u>Lemma 8.8</u>. Let (X,A) be a compact G-ENR pair such that $X-A$ has constant orbit type G/H, A has no orbits of type G/J with $(J) \leq (H)$, and $(X-A)/G$ is connected. Let $K = X/A$, give K the evident coaction by X^+, and let $f: K \to K$ be any based G-map. If WH is infinite, let $e(f) = 0$. If WH is finite, let $e(f) = \chi(f^H)/|WH|$. Then $\tau(f) = e(f)j_*\tau(G/H)$, where $j: G/H \to X$ is the inclusion of any orbit in $X-A$. Moreover, if the nonequivariant Euler characteristic $\chi(G/H)$ is non-zero, then $e(f) = \chi(f)/\chi(G/H)$, where $\chi(f)$ is the nonequivariant trace of f. If f is the identity map, then $e(f)$ coincides with the reduced Euler characteristic of the orbit space K/G.

<u>Proof</u>. The pair (X,A) is a retract of some finite G-CW pair (Y,B), and it is clear that this remains true if we excise from Y all cells of orbit type G/J with $(J) < (H)$. Since any orbit of $Y-B$ not of type G/H must map to A under the retraction, we may as well assume that all cells of such orbit types are in B. We also may as well delete from Y any cells of $Y-B$ whose images in $(Y-B)/G$ are not in the path component containing the image of $(X-A)/G$. In this way we arrive at maps of pairs

$$i: (X,A) \longrightarrow (Y,B) \quad \text{and} \quad r: (Y,B) \longrightarrow (X,A)$$

such that $ri = 1$, (Y,B) is a finite G-CW pair, $Y-B$ has constant orbit type G/H, B has no orbits of type G/J with $(J) \leq (H)$, and $(Y-B)/G$ is connected. Of course, i and r display K as a retract of Y/B. By Lemma 7.4, $i_*\tau(f) = \tau(ifr)$ in $\pi_0^G(Y)$ and thus $\tau(f) = r_*\tau(ifr)$ in $\pi_0^G(X)$. By Theorem 8.4,

$$\tau(ifr) = \Sigma(-1)^{n_i}e(g_i)(j_i)_*\tau(G/H),$$

where the sum runs over the cells $G/H \times e^{n_i}$ of $Y-B$, j_i is the inclusion of an orbit of $G/H \times e^{n_i}$ in Y, and g_i is the self map of $G/H^+\wedge S^{n_i}$ induced by ifr. Since $(Y-B)/G$ is connected, the inclusions j_i are all homotopic, at least if each is adjusted by an appropriate isomorphism of G/H, and the attaching maps of the cells may be adjusted so as to eliminate these isomorphisms. Thus we may replace all j_i by the composite ij. Since $ri = 1$, we conclude that

$$\tau(f) = nj_*\tau(G/H),$$

where $n = \sum_i (-1)^{n_i} e(g_i)$. If WH is infinite, then $\tau(G/H) = 0$ and we may as well set $n = 0$. Thus assume that WH is finite. We must show that $n = e(f)$. If we compose $\tau(f)$ with ξ_*^+, where $\xi\colon X \to pt$ is the trivial map, we find from the previous equation that

$$\chi(f) = n\chi(G/H).$$

This is true equivariantly, but all of our constructions remain unchanged if we forget the G actions, so it is also true nonequivariantly. If $\chi(G/H) \neq 0$, this gives $n = \chi(f)/\chi(G/H)$. We use change of groups to calculate n in general. The inclusion of (X^H, A^H) in (X,A) is an NH-map and induces a G-map

$$k\colon (G \times_{NH} X^H, G \times_{NH} A^H) \longrightarrow (X,A)$$

whose quotient map is a G-homeomorphism $G^+ \wedge_{NH} K^H \to K$. Thus $\tau(f) = \tau(1 \wedge_{NH} f^H)$, where the coaction of X^+ on $G^+ \wedge_{NH} K^H$ is the composite of $1 \wedge k$ and the natural coaction of $(G \times_{NH} X^H)^+$, as in Proposition 7.9. By that result, $\tau(f)$ is the composite

$$S \xrightarrow{\tau(G/NH)} G \ltimes_{NH} S \xrightarrow{1 \ltimes_{NH} \tau(f^H)} G \ltimes_{NH} \Sigma^\infty (X^H)^+ \cong \Sigma^\infty (G \times_{NH} X^H)^+ \xrightarrow{k^+} \Sigma^\infty X^+.$$

Now (X^H, A^H) is a compact WH-ENR pair, $X^H - A^H$ is WH-free, and $(X^H - A^H)/WH$ is homeomorphic to $(X-A)/G$ and is therefore connected. Exactly as in the first few steps, we conclude that

$$\tau(f^H) = mj_*^H \tau(WH)$$

for some integer m ($j^H\colon WH \to X^H - A^H \subset X^H$ being the inclusion of an orbit). Composing with ξ_*^H and noting that the nonequivariant Euler characteristic $\chi(WH)$ is $|WH|$ and thus non-zero, we see that $m = \chi(f^H)/|WH|$. Applying Proposition 7.9 again, we find that $\tau(G/H)$ is the composite

$$S \xrightarrow{\tau(G/NH)} G \ltimes_{NH} S \xrightarrow{1 \ltimes_{NH} \tau(WH)} G \ltimes_{NH} \Sigma^\infty WH^+ \cong \Sigma^\infty (G \times_{NH} WH)^+ \xrightarrow{(k')^+} \Sigma^\infty G/H^+.$$

Here k' is the canonical G-homeomorphism $G \times_{NH} WH \to G/H$. Clearly $jk' = k(1 \times_{NH} j^H)$, and we see that $m = n$ by comparing our descriptions of $\tau(f)$. Finally, suppose that f is the identity map. We must show that $e(f) = \tilde{\chi}(K/G)$, and we know that $e(f) = \tilde{\chi}(K^H)/|WH|$. Since WH acts freely on $K^H - *$ and K^H/WH is homeomorphic to K/G, $|WH|\tilde{\chi}(K/G) = \tilde{\chi}(K^H)$ by a standard homological argument; see e.g. tom Dieck [44,5.2.10].

IV. Equivariant Transfer

by L. G. Lewis, Jr. and J. P. May

We here study the transfer associated to equivariant bundles. The first problem, to be discussed in section 1, is to decide exactly what we should mean by a "G-bundle". There are at least three reasonable notions, of which the most restrictive has so far been much the most important and the least restrictive is the one that yields the most notationally simple and conceptually clear treatment of the transfer. We begin by specifying the latter notion and its cited specialization.

Throughout this chapter, we shall assume given an extension of compact Lie groups

$$1 \longrightarrow \pi \longrightarrow \Gamma \longrightarrow G \longrightarrow 1.$$

When we use the letter Γ, we often think of it as shorthand notation for the entire extension. All (unbased) Γ-spaces are to have the homotopy types of Γ-CW complexes. By a "G-bundle with total group Γ, structural group π, and fibre F", we understand a G-map

$$\xi\colon X \times_\pi F \longrightarrow X/\pi$$

induced by the projection $F \to *$, where X is a π-free Γ-space and F is any Γ-space. All group actions are to be left actions, and $X \times_\pi F$ denotes the orbit G-space $(X \times F)/\pi$. We think of X as the associated principal bundle of ξ (as will be discussed in section 1). The most important examples are the (G,π)-bundles, for which $\Gamma = G \times \pi$ and F is a π-space regarded as a Γ-space by pullback along the projection $\Gamma \to \pi$. Various examples of (G,π)-bundles and of G-bundles which are not (G,π)-bundles are given in section 1. We occasionally refer to the (G,π)-bundle case as the classical case.

We shall construct and analyze the transfer G-map

$$\tau(\xi)\colon \Sigma^\infty (X/\pi)^+ \longrightarrow \Sigma^\infty (X \times_\pi F)^+$$

associated to a G-bundle ξ. To do so, we require that F have the homotopy type of a compact Γ-ENR, and we then say that F is a "finite Γ-space". The main examples are the finite Γ-CW complexes, but the extra generality causes no difficulty and has applications (as explained in III§4). Nonequivariantly, one usually studies the transfer in terms of its induced homomorphism in cohomology for some given theory, but the utility and power of regarding it as a stable map are by now well understood. For example, this viewpoint makes commutation with cohomology operations obvious. Things work the same way equivariantly.

The first step in the definition of $\tau(\xi)$ is to construct a Γ-map $\tau(F)\colon S \to \Sigma^\infty F^+$. This may be thought of as the transfer for a bundle over a point, but it is a Γ-map whereas the actual transfer is a G-map. We call $\tau(F)$ the pretransfer to emphasize the distinction. In fact, $\tau(F)$ arises directly from Spanier-Whitehead duality and has already been studied in III§§7-8. We review its properties in section 2.

The trick to the construction of the transfer is to insert the Γ-map $\tau(F)$ fibrewise into ξ so as to obtain the desired G-map $\tau(\xi)$. We accomplish this sleight of hand by a change of universe. We assume given a complete Γ-universe U, fixed throughout the chapter. We regard U^π as both a complete G-universe and a π-trivial Γ-universe, and we let $i\colon U^\pi \to U$ be the inclusion. The Γ-map $\tau(F)$ is computed in U. The functor $\Sigma^\infty\colon \Gamma\mathcal{J} \to \Gamma\mathcal{S}U$ is the composite of $i_*\colon \Gamma\mathcal{S}U^\pi \to \Gamma\mathcal{S}U$ and $\Sigma^\infty\colon \Gamma\mathcal{J} \to \Gamma\mathcal{S}U^\pi$, so we have a map of Γ-spectra indexed on U

$$i_*\Sigma^\infty X^+ \simeq X^+\wedge S \xrightarrow{1\wedge\tau(F)} X^+\wedge\Sigma^\infty F^+ \simeq i_*\Sigma^\infty(X\times F)^+.$$

Since X is π-free, II.2.8 ensures that this map can be represented in the form $i_*\tilde{\tau}$ for a uniquely determined map $\tilde{\tau}\colon \Sigma^\infty X^+ \to \Sigma^\infty(X\times F)^+$ of Γ-spectra indexed on U^π. When working in a π-trivial universe, it is legitimate to pass to orbits over π, and $\tau(\xi)$ is just $\tilde{\tau}/\pi$.

We generalize this definition a bit in section 3. The quoted change of universe theorem applies not only to $\Sigma^\infty X^+$ but to any π-free Γ-spectrum D indexed on U^π. Associated to D we have the "stable G-bundle"

$$\xi\colon D\wedge_\pi F^+ \longrightarrow D/\pi$$

and an associated transfer G-map

$$\tau(\xi)\colon D/\pi \longrightarrow D\wedge_\pi F^+.$$

Since it simplifies notations, costs no extra effort, and has useful applications, we shall work in the context of these stable G-bundles throughout the chapter. If he chooses, the reader can disregard the generality by viewing D as simply a shorthand notation for $\Sigma^\infty X^+$. An example of the utility of the more general context is that it obviates the need for any special consideration of relative bundles. If A is a Γ-subspace of X, then the cofibre of $\Sigma^\infty A^+ \to \Sigma^\infty X^+$ is $\Sigma^\infty C(X,A)$, where $C(X,A)$ is the unreduced mapping cone. Since the cofibre of any map of π-free Γ-spectra is a π-free Γ-spectrum, we immediately obtain the transfer for the bundle pair

$$(X\times_\pi F, A\times_\pi F) \longrightarrow (X,A).$$

The naturality of the transfer with respect to the boundary homomorphism of the pair is merely an instance of its naturality with respect to maps of π-free Γ-spectra.

Section 3 also gives some of the basic properties of the transfer, namely those taken as axioms by the first author in [85], points out several easy generalizations of the definition, and compares it to others in the literature. A major advantage of our definition is its convenience for the proofs of calculationally useful properties. Since the two essential components, Spanier-Whitehead duality on the fibre and the structure of the associated principal bundle, are neatly separated, the properties of the transfer all follow directly from the properties of the pretransfer and of the change of universe and passage to orbits functors. Thus all of the real work has already been carried out in the previous chapters. The major disadvantage of our definition is that it is intrinsically restricted to G-bundles rather than G-fibrations.

We prove the basic calculational properties of the transfer in sections 4-7. In section 4, we give a deferred axiom on change of groups and a number of related results. In particular, we describe the relationship between the transfer and our generalized Wirthmüller and Adams isomorphisms. In section 5, we show that the transfer of a product of two bundles is the smash product of their transfers and give an Euler characteristic formula for the composite $\xi\tau$. In section 6, we give sum decomposition theorems and the double coset formula for the computation of the composite $\tau\xi$. In section 7, we prove transitivity relations. While all of these results are proven as statements about commutative diagrams in the stable category of G-spectra, we shall include discussions of their consequences in equivariant cohomology. Of course, there are analogous consequences in equivariant homology.

Nonequivariantly, these results sharpen theorems due to Becker and Gottlieb [10,11], Dold [46], and Feshbach [53,54] by eliminating finiteness conditions on the base space. (However, we shall see in Warning 6.11 that the finiteness hypothesis necessarily reappears in some key applications. We are much indebted to Feshbach for elucidation of this point.) Equivariantly, the Euler characteristic formula similarly sharpens results of Nishida [117] and Waner [141], but all of the rest is new.

The last two sections have a different flavor. In them, we rework and simplify the first author's paper [85] on the axiomatization of the transfer in light of our present much better understanding of stable bundles. Generalizing a bit further from section 3, we start with a general finite Γ-spectrum E rather than one of the form $\Sigma^\infty F^+$. We also start with a module G-spectrum j_G over a ring G-spectrum k_G and assume that our π-free Γ-spectrum D comes with a "coaction" $D \to X^+\wedge D$ by a π-free Γ-space X. With these data, we show in section 8 how to construct a transfer-like homomorphism

$$\tau\colon j_G^*(D \wedge_\pi E) \longrightarrow j_G^*(D/\pi)$$

from any cohomology class $\theta \in k_G^0(X \ltimes_\pi DE)$ and show that these "cohomological transports" enjoy many transfer-like properties.

There is a universal choice of X which coacts on every D, namely the universal π-free Γ-space $E\mathcal{F}(\pi)$ of II.2.10. In section 9, we show that any family of homomorphisms τ, as above, which is defined, natural, and stable for all D arises from a uniquely determined class $\theta \in k_G^0(E\mathcal{F}(\pi) \ltimes_\pi DE)$, where k_G is the function ring spectrum $F(j_G, j_G)$. We deduce from this classification theorem for "j_G^*-transforms" that the standard transfer in j_G^*-cohomology is uniquely characterized by the cohomological versions of the axioms of sections 3 and 4. By constructing transfers for stable bundles from transfers for ordinary bundles, we obtain an analogous axiomatization of the transfer for bundles over finite dimensional base spaces (or, modulo $\lim^1$ terms, over general base spaces).

Throughout, cohomology is to be understood in the $RO(G)$-graded sense; on spaces, j_G^* means unreduced cohomology.

§1. Types of equivariant bundles

A principal (G,π)-bundle $X \to X/\pi$ is a principal π-bundle and a G-map such that G acts on X through π-bundle maps. It is usual to let G act from the right and π from the left, but with both actions on the left it is clear that X is just a π-free $(G \times \pi)$-space. For a π-space F, $\xi\colon X \times_\pi F \to X/\pi$ is the associated (G,π)-bundle with fibre F. For our purposes, this is the appropriate definition of a (G,π)-bundle. We display X and F explicitly in the total space because F and X play clearly separated roles in our treatment of the transfer and because we find it quite convenient not to insist that π act effectively on F.

However, when π does act effectively on F, there is a more intrinsic description of (G,π)-bundles. Consider a bundle $\xi\colon Y \to B$ with fibre F and structural group π such that ξ is also a G-map between G-spaces. With G ignored, the associated principal π-bundle of ξ is the subspace X of the function space Y^F consisting of the admissible homeomorphisms $\psi\colon F \to \xi^{-1}(b)$ for $b \in B$. Admissibility means that the composite of ψ and the homeomorphism $\phi\colon \xi^{-1}(b) \to F$ obtained by use of any coordinate chart coincides with action by an element of π. The π action on F and the G action on Y induce a $(G \times \pi)$-action on X if and only if the composite $g\psi\colon F \to \xi^{-1}(b) \to \xi^{-1}(gb)$ is admissible whenever ψ is admissible, and this is what it means for ξ to be a (G,π)-bundle. (Of course, π must act effectively on F if it is to act freely on X.)

For example, define a finite G-cover $\xi\colon Y \to B$ to be a G-map which is also a finite cover. Such a ξ is necessarily a (G,Σ_n)-bundle, where n is the cardinality of the fibre F. Taking $F = \{1,2,\cdots,n\}$ and identifying Y^F with Y^n, we see in this case that X is the space of those n-tuples of points of Y which together comprise a fibre $\xi^{-1}(b)$, with G acting diagonally and Σ_n acting by permutations.

Again, the G-vector bundles that arise in equivariant differential topology are $(G,U(n))$-bundles or $(G,O(n))$-bundles, and it is this sort of bundle which is used to define equivariant K-theory.

Thus (G,π)-bundles are central to equivariant topology, and we refer the reader to [81, 82, and 125] for further discussion. However, this is really a quite restrictive notion, and there are numerous examples of maps which clearly ought to be G-bundles of some sort but clearly fail to be (G,π)-bundles for any π. Perhaps the most naive example is the projection $\xi\colon X \times F \to X$, where X and F are arbitrary G-spaces and G acts diagonally on their product. The only reasonable candidate for the structural group of ξ is the trivial group, but ξ fails to be a (G,e)-bundle unless G acts trivially on F. Of course, such projections are precisely the G-bundles with total group G and structural group e, as defined in the introduction.

Returning to the context of a G-map $\xi\colon Y \to B$ which is a bundle with fibre F and structural group π, where π acts effectively on F, it is natural to require the composites

$$F \xrightarrow{\psi} \xi^{-1}(b) \xrightarrow{g} \xi^{-1}(gb) \xrightarrow{\phi} F$$

which entered into our description of (G,π)-bundles to be elements of some fixed group Γ of homeomorphisms of F. Clearly Γ must contain π, and it imposes the minimal sensible rigidity on the situation to require π to be normal in Γ with quotient group G. We claim that ξ is a G-bundle with total group Γ, structural group π, and fibre F if and only if, for each $y \in \Gamma$ with image $g \in G$ and each admissible homeomorphism $\psi\colon F \to \xi^{-1}(b)$, the composite

$$F \xrightarrow{y^{-1}} F \xrightarrow{\psi} \xi^{-1}(b) \xrightarrow{g} \xi^{-1}(gb)$$

is admissible. Indeed, if this condition holds, then we can specify an action of Γ on X by $y\psi = g\,\psi\,y^{-1}$. The usual evaluation homeomorphism $X \times_\pi F \to Y$ over $X/\pi \cong B$ is then clearly a G-map. Conversely, if $\xi\colon X \times_\pi F \to X/\pi$ is a G-bundle with total group Γ, structural group π, and fibre F, then the admissible homeomorphisms ψ are of the form $\psi(f) = (x,f)$ for some fixed $x \in X$, and the composite above is the admissible homeomorphism determined by yx.

Tom Dieck [36] and Nishida [117] have studied a type of bundle intermediate between (G,π)-bundles and our G-bundles with structural group π. They assume that the given extension is split, so that $\Gamma = G \times_{\gamma}\pi$ for some homomorphism $\gamma\colon G \to \mathrm{Aut}(\pi)$ with smooth adjoint $G \times \pi \to \pi$. Explicitly, the product on Γ is given by

$$(h,m)(g,n) = (hg, \gamma(g^{-1})(m)n)$$

for $h,g \in G$ and $m,n \in \pi$. Modulo adjustment for our consistent use of left group actions, they define a principal (G,γ,π)-bundle to be the orbit projection $X \to X/\pi$ of a π-free Γ-space, and they define the associated (G,γ,π)-bundle with fibre a Γ-space F exactly as we did in the introduction. When γ is trivial and $\Gamma = G \times \pi$ acts on F through its projection to π, this notion specializes to that of a (G,π)-bundle. Of course, our definition of a G-bundle is obtained from this one by dropping the requirement that the extension be split. Our motivation is primarily simplicity rather than generality. The splitting, if present, would be completely irrelevant to most of our work.

Motivation for the intermediate notion is given in II§7. For an N-free G-space X, where N is a normal subgroup of G, the projection $X \to X/N$ is a principal N-bundle and a G-map but not a (G,N)-bundle since the actions of G and N fail to commute. The failure is measured by the conjugation action c of G on N and, with π a copy of N, $X \to X/N$ is a principal (G,c,π)-bundle.

We have just defined "principal (G,γ,π)-bundles", and we show next how to interpret them as associated (G,γ,π)-bundles with fibre π. Without the splitting, a π-free Γ-space need have no G-action and so can't be the total space of an associated G-bundle. However, it is the total space of an associated (Γ,c,π)-bundle, and this is a useful point of view even in the split case since it keeps track of the full Γ-action.

<u>Remarks 1.1</u> (i) Let $\Gamma = G \times_{\gamma}\pi$ and let Γ act on π via

$$(g,n)m = \gamma(g)(nm)$$

for $g \in G$ and $m,n \in \pi$. For a principal (G,γ,π)-bundle X, the orbit projection $X \to X/\pi$ can be interpreted as the associated (G,γ,π)-bundle $X \times_{\pi}\pi \to X/\pi$. To see this, define a map $\psi\colon X \times \pi \to X$ by the formula

$$\psi(x,n) = (e,n^{-1})x.$$

With the usual diagonal action of Γ on $X \times \pi$ and the action of Γ through the projection $\Gamma \to G \subset \Gamma$ on the target copy of X, ψ is a Γ-map by an easy

verification. Since π acts trivially on the target, ψ factors through a G-homeomorphism $X \times_{\pi}\pi \to X$ over X/π.

(ii) For a general extension $G = \Gamma/\pi$, G is not a subgroup of Γ and we define $\Omega = \Gamma \times_c \pi$, where c is the conjugation action of Γ on its normal subgroup π. We also define $\theta\colon \Omega \to \Gamma$ by $\theta(y,n) = yn$ for $y \in \Gamma$ and $n \in \pi$. Clearly θ restricts on π to its inclusion in Γ. Thus, if X is a π-free Γ-space, then $\theta^{*}X$ is a π-free Ω-space. We conclude from (i) that the orbit projection $X \to X/\pi$ can be interpreted as the associated (Γ,c,π)-bundle $(\theta^{*}X) \times_{\pi}\pi \to X/\pi$ (as was proven directly in II§7.)

Observe that a given G-map $\xi\colon Y \to B$ which is a bundle with fibre F and compact Lie structural group may well admit various descriptions as a G-bundle with total group Γ and structural group π. Our results in section 4 will have the effect of ensuring that all such descriptions lead to the same transfer G-map.

§2. <u>The pretransfer</u>

The bundle context is irrelevant and Γ can be any compact Lie group in this section. As in the introduction, we consider a finite Γ-space F. We index Γ-spectra on any Γ-universe into which F embeds. This ensures that $\Sigma^{\infty}F^{+}$ is a finite Γ-spectrum and so allows us to apply duality theory. Specialization of III.7.1 gives our definition of the pretransfer $\tau(F)\colon S \to \Sigma^{\infty}F^{+}$.

<u>Definition 2.1.</u> Define the pretransfer $\tau(F)$ to be the composite

$$S \xrightarrow{\gamma\eta} DF^{+}\wedge F^{+} \xrightarrow{1\wedge\Delta} DF^{+}\wedge F^{+}\wedge F^{+} \xrightarrow{\varepsilon\wedge 1} S\wedge F^{+} \simeq \Sigma^{\infty}F^{+}.$$

Specialization of III.8.1 gives an explicit space level description of $\tau(F)$.

<u>Lemma 2.2.</u> Let $r\colon N \to F$ be a retraction of an open neighborhood of F in some representation V and let B be a disc in V which contains N. Then the following diagram is Γ-homotopy commutative.

$$\begin{tikzcd}
S^{V} \arrow[d, "\tau(F)"'] & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r] & C(V,V-F) \\
 & & C(N,N-F) \arrow[u, "\simeq"'] \arrow[d, "C(\psi)"] \\
S^{V}\wedge F^{+} & C(V\times F,(V-B)\times F) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & C(V\times F,(V-\{0\})\times F)
\end{tikzcd}$$

Here the unlabeled arrows are inclusions or projections and ψ is specified by

$\psi(n) = (n-r(n),r(n))$ for $n \in N$.

As observed in III.8.2, this description implies that $\tau(F) = 0$ if the identity map of F is Γ-homotopic to a fixed point free map.

When F is a smooth Γ-manifold we can take N to be a tubular neighborhood of F and r to agree with the projection of the normal bundle. The description of $\tau(F)$ then simplifies.

Lemma 2.3. If F is a smooth closed Γ-manifold smoothly embedded in V with normal bundle ν, then $\tau(F)$ is the composite Γ-map

$$S^V \xrightarrow{t} T\nu \xrightarrow{e} T(\nu \oplus \tau) \cong S^V \wedge F^+,$$

where t is the Pontryagin-Thom map and e is induced by the inclusion of ν in $\nu \oplus \tau$.

This shows that our definition of the pretransfer agrees with that of Becker and Gottlieb [10]. A similar comparison of definitions goes through for manifolds with boundary. It is a standard and useful observation that e could just as well be replaced by the map induced by any vector field on F, not necessarily the zero field. Again, this implies that $\tau(F) = 0$ if F admits a nowhere zero Γ-invariant vector field.

Remarks 2.4. (i) For a smooth closed Γ-manifold F, the dual of $\tau(F)$ can be described as the composite

$$T\nu \xrightarrow{\Delta} F^+ \wedge T\nu \xrightarrow{\gamma} T\nu \wedge F^+ \xrightarrow{\varepsilon} S^V,$$

where Δ is the Thom diagonal. One way to show this is to use the description of ε in III.5.1 to verify that the composite

$$T\nu \wedge S^V \xrightarrow{1\wedge\tau(F)} T\nu \wedge S^V \wedge F^+ \xrightarrow{1\wedge\gamma} T\nu \wedge F^+ \wedge S^V \xrightarrow{\varepsilon\wedge 1} S^V \wedge S^V$$

is Γ-homotopic to $(\varepsilon\gamma\Delta)\wedge 1$.

(ii) Let Γ/Λ be an orbit embedded in V and let L and W be the tangent representation of Λ at the identity coset of Γ/Λ and the orthogonal complement of the image of L in V. Then $\tau(\Gamma/\Lambda)$ is the Γ-map

$$S^V \xrightarrow{t} \Gamma^+ \wedge_\Lambda S^W \xrightarrow{1\wedge e} \Gamma^+ \wedge_\Lambda S^V \cong S^V \wedge (\Gamma/\Lambda)^+$$

(as in II.5.1 and II.6.15) and its dual is the Γ-map

$$\Gamma^+ \wedge_\Lambda S^W \xrightarrow{1\wedge e} \Gamma^+ \wedge_\Lambda S^V \xrightarrow{\xi} S^V,$$

where ξ is the Γ-action map. The verification can be made either by use of III.2.11 and 3.7 or by use of part (i).

Returning to the context of general finite Γ-spaces, we catalog some elementary facts about the pretransfer.

Lemma 2.5. Let E and F be finite Γ-spaces.

(i) If F is a point, then $\tau(F) = 1: S \to \Sigma^\infty F^+ = S$.

(ii) If $k: E \to F$ is a Γ-homotopy equivalence, then $k\tau(E) = \tau(F)$.

(iii) $\tau(E \times F) = \tau(E)\wedge\tau(F): S \simeq S\wedge S \to \Sigma^\infty(E \times F)^+ \simeq \Sigma^\infty E^+ \wedge \Sigma^\infty F^+$.

(iv) $\tau(E \amalg F) = \tau(E) + \tau(F): S \to \Sigma^\infty(E \amalg F)^+ \simeq \Sigma^\infty E^+ \vee \Sigma^\infty F^+$.

Proof. These hold by III.7.2, 7.4, 7.5, and 7.6.

In (iii), the smash product is internal. We also need the external analog. No new proof is required. Both statements are formal consequences of the fact that the relevant smash product is a (strict) monoidal functor; compare III.1.9, 1.10, and 7.7.

Lemma 2.6. Let F_i be a finite Γ_i-space, $i = 1$ or 2. Then

$$\tau(F_1 \times F_2) = \tau(F_1)\wedge\tau(F_2): S \simeq S\wedge S \to \Sigma^\infty(F_1 \times F_2)^+ \simeq \Sigma^\infty F_1^+ \wedge \Sigma^\infty F_2^+,$$

where the external smash product $\bar{h}\Gamma_1\mathscr{S}U_1 \times \bar{h}\Gamma_2\mathscr{S}U_2 \to \bar{h}(\Gamma_1 \times \Gamma_2)\mathscr{S}(U_1 \oplus U_2)$ is understood, U_i being a Γ_i-universe into which F_i embeds.

We also need a consistency statement for the behavior of the pretransfer with respect to change of groups and change of universe. This again is a formal consequence of the cited categorical observations.

Lemma 2.7. Let $\alpha: \Lambda \to \Gamma$ be a homomorphism of compact Lie groups. Let F be a finite Γ-space and let α^*F denote F regarded as a finite Λ-space by pullback. Let V and U be Λ and Γ universes into which α^*F and F embed and let $j: \alpha^*U \to V$ be a Λ-linear isometry. The functors $\Sigma^\infty\alpha^*$ and $j_*\alpha^*\Sigma^\infty$ from $\bar{h}\Gamma\mathscr{J}$ to $\bar{h}\Lambda\mathscr{S}V$ are naturally equivalent, and $\tau(\alpha^*F): S \to \Sigma^\infty(\alpha^*F)^+$ agrees under the equivalence with $j_*\alpha^*\tau(F): j_*\alpha^*S \to j_*\alpha^*\Sigma^\infty F^+$.

In particular, with $\Lambda = \Gamma$, the pretransfer is preserved under change of universe. Again, with $\Lambda \subset \Gamma$ and $V = U$, the pretransfer for F regarded as a Γ-space is also the pretransfer for F regarded as a Λ-space. In our study of transitivity, we shall need another invariance property with respect to change of groups. It is a special case of III.7.9.

<u>Lemma 2.8</u>. Let J be a finite Λ-space, where $\Lambda \subset \Gamma$. The following diagram of Γ-spectra commutes.

$$\begin{tikzcd}
\Sigma^\infty(\Gamma/\Lambda)^+ \arrow[d, "\cong"] & S \arrow[l, "\tau(\Gamma/\Lambda)"'] \arrow[r, "\tau(\Gamma\times_\Lambda J)"] & \Sigma^\infty(\Gamma\times_\Lambda J)^+ \arrow[d, "\cong"] \\
\Gamma\ltimes_\Lambda S \arrow[rr, "1\ltimes\tau(J)"] & & \Gamma\ltimes_\Lambda\Sigma^\infty J^+
\end{tikzcd}$$

The most substantial of our axioms for the transfer will be a direct consequence of the following additivity theorem.

<u>Theorem 2.9</u>. If F is the pushout of a Γ-cofibration $f_1\colon F_3 \to F_1$ and a Γ-map $f_2\colon F_3 \to F_2$, where the F_k are finite Γ-spaces, then

$$\tau(F) = j_1\tau(F_1) + j_2\tau(F_2) - j_3\tau(F_3),$$

where $j_k\colon \Sigma^\infty F_k^+ \to \Sigma^\infty F^+$ is induced by the natural map $F_k \to F$.

<u>Proof</u>. We may replace F by the double mapping cone of f_1 and f_2 and so embed $F_1 \amalg F_2$ in F with quotient ΣF_3^+. The natural equivalence $C(F, F_1\amalg F_2) \to \Sigma F_3^+$ then fits into the commutative diagram

$$\begin{tikzcd}
C(F,F_1\amalg F_2) \arrow[r, "\Delta"] \arrow[d] & C(F,F_1\amalg F_2)\wedge F^+ \arrow[dr] & \\
\Sigma F_3^+ \arrow[r, "\Sigma\Delta"] & \Sigma F_3^+\wedge F_3^+ \arrow[r, "1\wedge j_3"] & \Sigma F_3^+\wedge F^+ .
\end{tikzcd}$$

The conclusion follows from III.7.10 applied to the cofibre sequence

$$\Sigma^\infty F_1^+ \vee \Sigma^\infty F_2^+ \to \Sigma^\infty F^+ \to \Sigma^\infty\Sigma F_3^+ \to \Sigma^\infty\Sigma F_1^+ \vee \Sigma^\infty\Sigma F_2^+.$$

As pointed out to us by Albrecht Dold, a simple direct proof of this result is possible based on the concrete space level description of the pretransfer in Lemma 2.2 and the fact that the trace of a map depends only on its fixed point set (compare III.8.1 and 8.2).

We need two consequences of Theorem 2.9, the first of which is a special case of III.8.4. Recall that $\chi(F) = \xi\tau(F)$, where $\xi\colon \Sigma^\infty F^+ \to S$ is induced by the projection $F \to pt$.

<u>Theorem 2.10</u>. Let F be a finite Γ-CW complex and let

$$j_i\colon \Gamma/\Lambda_i \subset \Gamma/\Lambda_i \times e^{n_i} \longrightarrow F$$

be the composite of the inclusion of an orbit and the i^{th} characteristic map for some enumeration of the cells of F. Then $\tau(F) = \sum_i (-1)^{n_i} j_i\tau(\Gamma/\Lambda_i)$, hence $\chi(F) = \sum_i (-1)^{n_i}\chi(\Gamma/\Lambda_i)$.

There is a more invariant analog applicable to a general compact Γ-ENR F. Recall that the path components M of the various orbit spaces $F_{(\Lambda)}/\Gamma$ are called the orbit type components of F/Γ, where

$$F_{(\Lambda)} = \{x \mid \Gamma_x \text{ is conjugate to } \Lambda\}.$$

Let $\bar{M}$ be the closure of M in F/Γ and let $\partial M = \bar{M}-M$. Define the nonequivariant internal Euler characteristic $\chi(M)$ of M in F/Γ to be the reduced Euler characteristic of $\bar{M}/\partial M$; equivalently (by the nonequivariant case of III.7.10), $\chi(M) = \chi(\bar{M}) - \chi(\partial M)$. If F happens to be a finite Γ-CW complex, then $\chi(M)$ is the signed sum of the number of cells with interior contained in the inverse image of M in F. In this case, the previous result directly implies the following one, as we see by grouping together equal summands obtained by including different orbits in the same orbit type component. The general case holds by III.8.7 and the last sentence of III.8.8.

<u>Theorem 2.11</u>. Let F be a compact Γ-ENR and let $j_m\colon \Gamma/\Lambda \to F$ be the inclusion of an orbit in the orbit type component M. Then $\tau(F) = \Sigma\chi(M)j_m\tau(\Gamma/\Lambda)$, hence $\chi(F) = \Sigma\chi(M)\chi(\Gamma/\Lambda)$.

Extraneous terms in the previous two theorems are eliminated by the following consequence of III.8.2 (or of Remark 2.4 (ii)).

<u>Lemma 2.12</u>. If $W\Lambda = N_\Gamma\Lambda/\Lambda$ is infinite, then $\tau(\Gamma/\Lambda) = 0$.

<u>Remark 2.13</u>. Our observations about the vanishing of the pretransfer sometimes imply that the transfer is zero in a situation in which one would like to use it to prove an isomorphism. In many cases, $\tau(F)$ vanishes because some group containing a circle acts freely on F. In Lemma 2.12, for example, $\tau(\Gamma/\Lambda)$ factors as the composite

$$S \xrightarrow{\tau(\Gamma/N)} \Sigma^\infty\Gamma/N^+ \cong \Gamma\ltimes_N S \xrightarrow{1\ltimes_N\tau(W\Lambda)} \Gamma\ltimes_N\Sigma^\infty W\Lambda^+ \cong \Sigma^\infty\Gamma/\Lambda^+,$$

where $N = N_\Gamma\Lambda$, and $\tau(W\Lambda) = 0$ because $W\Lambda$ acts freely on itself. In these situations, II.7.5 offers a substitute for the transfer. Assume that Λ is normal in Γ and that Λ acts freely on F. Let A be the adjoint representation of Γ derived from Λ and let $\Omega = \Gamma/\Lambda$. By II.7.2, there is an isomorphism

$$[S,\Sigma^\infty F/\Lambda^+]_\Omega \cong [S,\Sigma^{-A}\Sigma^\infty F^+]_\Gamma,$$

where the right side is computed in a complete Γ-universe and the left side is computed in its Λ-fixed point Ω-universe. Let

$$\tau(F,\Lambda)\colon S \longrightarrow \Sigma^{-A}\Sigma^\infty F^+$$

be the image of $\tau(F/\Lambda)$ under this isomorphism. (If $A = \{0\}$, this is just $\tau(F)$.) These "dimension-shifting pretransfers" enjoy many of the properties of the standard pretransfer, such as the evident analogs of the properties in 2.5(ii), 2.5(iv), 2.6, 2.9, 2.10 and 2.11 above. The procedure of the next section for obtaining the transfer from the pretransfer applies equally well in the present context, and we shall have occasion to use the resulting dimension-shifting transfers in the next chapter.

§3. The definition and axiomatic properties of the transfer

As in the introduction, consider an extension $G = \Gamma/\pi$, a π-free Γ-spectrum D indexed on U^π, where U is a complete Γ-universe, and a finite Γ-space F. We let $D\wedge_\pi F^+$ denote $(D\wedge F^+)/\pi$ and think of the orbit map

$$\xi\colon D\wedge_\pi F^+ \longrightarrow D/\pi$$

as a stable G-bundle. When $D = \Sigma^\infty X^+$ for a π-free Γ-space X, ξ is obtained from the G-bundle $X\times_\pi F \to X/\pi$ by adjoining disjoint basepoints and applying Σ^∞. By II.2.8 (together with II.1.4 and II.3.12 (i)), the inclusion $i\colon U^\pi \to U$ induces an isomorphism

$$i_*\colon [D,D\wedge F^+]_\Gamma \longrightarrow [i_*D,i_*(D\wedge F^+)]_\Gamma \cong [i_*D\wedge S,i_*D\wedge\Sigma^\infty F^+]_\Gamma.$$

Definition 3.1. Let $\tilde{\tau}\colon D \to D\wedge F^+$ be the Γ-map such that $i_*(\tilde{\tau})$ is $1\wedge\tau(F)$, where $\tau(F)\colon S \to \Sigma^\infty F^+$ is the pretransfer Γ-map. Define the transfer

$$\tau(\xi)\colon D/\pi \longrightarrow D\wedge_\pi F^+$$

to be the G-map obtained from $\tilde{\tau}$ by passage to orbits over π.

We generally abbreviate $\tau(\xi)$ to τ; it is a map of G-spectra indexed on the complete G-universe U^π. When $G = e$, $\tau(\xi)$ is the nonequivariant transfer associated to the nonequivariant stable bundle ξ. The definition specializes to give the transfer G-map associated to a (G,π)-bundle with fibre F; here the π-space F is regarded as a $(G\times\pi)$-space by pullback along the projection $G\times\pi\to\pi$.

Remarks 3.2. (i) Note that we require no finiteness condition on the base spectrum. Adams [2, p. 207-208] has advertised the desirability of having transfer maps for bundles over infinite dimensional CW-complexes. When $D = \Sigma^\infty X^+$ for a finite CW-complex X and F is a compact smooth π-manifold, our definition of transfer coincides with that of Becker and Gottlieb [10] and its equivariant version due to Nishida [117] (as we shall explain shortly).

(ii) Becker and Gottlieb [11] used suspension ex-spectra and fibrewise duality to generalize their earlier transfer from bundles to fibrations with finite fibres and finite dimensional base spaces; see also Dold [46]. By introducing a stable category of ex-spectra, Clapp [26] generalized their construction so as to allow infinite dimensional base spaces. On the other hand, by using suspension ex-G-spectra and equivariant fibrewise duality, Waner [141] generalized their construction to G-fibrations with finite fibres and finite dimensional base spaces. Generalizations of our equivariant stable categories to ex-G-spectra is perfectly feasible and allows the evident simultaneous generalization of the work of Clapp and Waner. When restricted to bundles, these definitions of the transfer also agree with that given here.

(iii) The definition admits numerous variants and generalizations. We can replace $\tau(F)$ by the trace, or twisted pretransfer, $\tau(\phi)$ of III.7.1 for any self-map ϕ of F. More generally, we can replace $\tau(F)$ by any stable map $S\to\Sigma^\infty F^+$. If Λ is a second normal subgroup of Γ and F is Λ-free, we can replace $\tau(F)$ by the dimension-shifting pretransfer $\tau(F,\Lambda)\colon S\to\Sigma^{-A}\Sigma^\infty F^+$ of Remark 2.13 (which is what we did in II§7 and will do again in V§11). In fact, with a little care in defining $D\wedge_\pi E$, we can even replace $\tau(F)$ by an arbitrary map $E\to E'$ of Γ-spectra indexed on U. This may seem altogether silly, but we shall actually get some mileage out of such generality in sections 8 and 9. Again, we shall indicate how to twist the transfer by any self-map of the total space over the base space in Example 8.3(vi) (as comes most naturally out of the ex-spectrum approach). More importantly, such generality will play an essential role in our axiomatization of the transfer. We defer all further discussion of generalizations to section 8.

As the following remarks explain, we earlier found a less elementary, but more explicit, construction of the transfer. While it bears a closer resemblance to other definitions in the literature, we shall make no use of it in this chapter since the new definition generally allows simpler proofs.

Remarks 3.3. (i) In chapter VI, we shall construct the twisted half smash product functor $X \ltimes_\pi (?)\colon \bar{h}\Gamma\mathscr{S}U \to \bar{h}G\mathscr{S}U^\pi$, where X is a π-free Γ-space. For Γ-spaces Y, there is a natural isomorphism

$$\Sigma^\infty(X \times_\pi Y)^+ \cong X \ltimes_\pi \Sigma^\infty Y^+.$$

Before we knew the results of II§2, we took

$$1 \ltimes_\pi \tau(F)\colon \Sigma^\infty(X/\pi)^+ \cong X \ltimes_\pi S \to X \ltimes_\pi \Sigma^\infty F^+ \cong \Sigma^\infty(X \times_\pi F)^+$$

as our definition of the transfer associated to $X \times_\pi F \to X/\pi$. This approach is reflected in Lewis [85] and still has its advantages. For example, it lends itself more readily to precise calculational analysis as in chapter VIII.

(ii) Generalizing (i), let E be any Γ-CW spectrum indexed on U. By VI.1.11, $X \ltimes E$ is then a π-free Γ-spectrum indexed on U^π, and by VI.1.5 we have a natural identification of G-spectra

$$(X \ltimes E) \wedge_\pi Y^+ \cong X \ltimes_\pi (E \wedge Y^+)$$

for Γ-spaces Y. The collapse map $F \to *$ induces a stable G-bundle

$$\xi\colon X \ltimes_\pi (E \wedge F^+) \longrightarrow X \ltimes_\pi E,$$

and the transfer $\tau(\xi)$ is just the obvious map

$$1 \ltimes_\pi (1 \wedge \tau(F))\colon X \ltimes_\pi E \longrightarrow X \ltimes_\pi (E \wedge F^+)$$

induced by $\tau(F)$. In fact, II.2.8 and VI.1.17 imply a natural equivalence of Γ-spectra

$$i_*(X \ltimes E) \simeq X^+ \wedge E$$

under which i_* carries $1 \ltimes (1 \wedge \tau(F))$ to $1 \wedge 1 \wedge \tau(F)\colon X^+ \wedge E \to X^+ \wedge E \wedge F^+$, so that $1 \ltimes (1 \wedge \tau(F))$ gives an explicit description of the map $\tilde{\tau}$ of Definition 3.1 in the present situation.

(iii) We can recover the full generality of Definition 3.1 by the method of (ii). Recall from II.2.10 that there is a universal π-free Γ-space $E\mathscr{F}(\pi)$. For any π-free Γ-spectrum D indexed on U^π, II.2.8 and VI.1.17 imply a Γ-equivalence

$$E\mathscr{F}(\pi) \ltimes i_*D \simeq E\mathscr{F}(\pi)^+ \wedge D,$$

and the projection $E\mathscr{F}(\pi)^+ \wedge D \to D$ is a Γ-equivalence by II.2.12. By (ii) applied with $E = i_*D$ and by the naturality of the transfer (Axiom 1 below), $\tau\colon D/\pi \to D \wedge_\pi F^+$ agrees under the resulting composite equivalence with

$$1 \ltimes_\pi (1 \wedge \tau(F))\colon E\mathscr{F}(\pi) \ltimes_\pi i_*D \longrightarrow E\mathscr{F}(\pi) \ltimes_\pi (i_*D \wedge F^+).$$

(iv) Returning to the context of (i), suppose that X is compact. Then the simple prespectrum level description of the functor $X \ltimes (?)$ of VI.2.5 applies. Using this and the description of the pretransfer for smooth manifolds in Lemma 2.3, it is easy to check the agreement of definitions claimed in Remark 3.2 (i). The point is that the complementary sphere bundles used in VI.2.5 are exactly the same as those used by Becker and Gottlieb (and Nishida).

We single out certain of the most basic properties of the transfer in the following list of axioms.

Axiom 1. Naturality. The following diagram commutes for a map $f\colon D \to D'$ of π-free Γ-spectra.

$$\begin{tikzcd} D/\pi \arrow[r, "f/\pi"] \arrow[d, "\tau"'] & D'/\pi \arrow[d, "\tau"] \\ D \wedge_\pi F^+ \arrow[r, "f \wedge_\pi 1"] & D' \wedge_\pi F^+ \end{tikzcd}$$

Axiom 2. Stability. The following diagram commutes for a representation V of G regarded by pullback as a representation of Γ.

$$\begin{tikzcd} \Sigma^V(D/\pi) \arrow[r, phantom, "\cong"] \arrow[d, "\Sigma^V\tau"'] & (\Sigma^V D)/\pi \arrow[d, "\tau"] \\ \Sigma^V(D \wedge_\pi F^+) \arrow[r, phantom, "\cong"] & (\Sigma^V D) \wedge_\pi F^+ \end{tikzcd}$$

Axiom 3. Normalization. With $F = *$, the transfer associated to the identity bundle $D/\pi \to D/\pi$ is the identity map.

Axiom 4. Fibre invariance. The following diagram commutes for an equivalence $k\colon F \to F'$ of finite Γ-spaces.

$$\begin{tikzcd} & D/\pi \arrow[dl, "\tau"'] \arrow[dr, "\tau"] & \\ D\wedge_\pi F^+ \arrow[rr, "1\wedge_\pi k"] & & D\wedge_\pi (F')^+ \end{tikzcd}$$

Axiom 5. Additivity on fibres. If F is the pushout of a Γ-cofibration $F_3 \to F_1$ and a Γ-map $F_3 \to F_2$, where the F_k are finite Γ-spaces, and if τ_k is the transfer associated to $D\wedge_\pi F_k^+ \to D/\pi$ and $j_k\colon D\wedge_\pi F_k^+ \to D\wedge_\pi F^+$ is induced by the canonical map $F_K \to F$, then

$$\tau = j_1\tau_1 + j_2\tau_2 - j_3\tau_3.$$

There is a sixth axiom, on change of groups. It is more technical to state and will be explained in the next section. The axioms above are immediate consequences of Definition 3.1 together with Lemma 2.5 and Theorem 2.9.

Remarks 3.4. (i) In Axiom 1, the cofibre sequence

$$D \longrightarrow D' \longrightarrow Cf \longrightarrow \Sigma D$$

is a sequence of Γ-maps between π-free Γ-spectra, hence the transfer maps the cofibre sequence of $f/\pi\colon D/\pi \to D'/\pi$ to the cofibre sequence of $f\wedge_\pi 1\colon D\wedge_\pi F^+ \to D'\wedge_\pi F^+$. When f comes from an inclusion $A \to X$ of π-free Γ-spaces, this immediately gives a relative transfer compatible with cofibre sequences in the bundle context (when combined with the case $V = R$ of Axiom 2).

(ii) Given a G-spectrum k and a filtration of D, we obtain spectral sequences for the computation of $k^*(D/\pi)$ and of $k^*(D\wedge_\pi F^+)$. The transfer induces a map of the relevant exact couples and thus a map of spectral sequences. If D is a Γ-CW spectrum with its skeletal filtration, the conclusion is that τ^* maps what should be thought of as the Serre spectral sequence for $k^*(D\wedge_\pi F^+)$ to the Atiyah-Hirzebruch spectral sequence for $k^*(D/\pi)$. In the situation of nonequivariant bundles, $G = e$ and $D = \Sigma^\infty X^+$, Prieto [119] has identified τ^* on the E_2 level.

(iii) In Axiom 2, we are using that the smash product of a π-free Γ-spectrum and a Γ-space (or Γ-spectrum) is a π-free Γ-spectrum. We shall see in Corollary 5.3 that this axiom is actually a special case of the behavior of τ on products. When $D = \Sigma^\infty X^+$, it may be viewed as relating the transfer associated to X to the transfer associated to the bundle pair $(X \times BV, X \times SV)$, where BV and SV are the unit disc and unit sphere in V.

(iv) Axioms 4 and 5 are deceptively simple looking. As we shall explain in section 6, they directly imply equivariant generalizations of Feshbach's

decomposition theorem [53, V.14] and double coset formula [53, II.11].

§4. The behavior of the transfer with respect to change of groups

Before we can study the behavior of transfer with respect to change of groups and smash products, we must slightly generalize its definition and consider its behavior with respect to change of universe. Thus consider a Γ-universe U', not necessarily complete, and a π-trivial Γ-universe U'' contained in $(U')^\pi$. Let $i\colon U'' \to U'$ be the inclusion. Recall from II.2.3 that $\mathscr{E}(U',U'')$ denotes the family of subgroups Λ of Γ such that U' and U'' are Λ-isomorphic. For an $\mathscr{E}(U',U'')$-spectrum D indexed on U'' and a finite Γ-space F, II.2.6 gives an isomorphism

$$i_*\colon [D,D\wedge F^+]_\Gamma \longrightarrow [i_*D, i_*(D\wedge F^+)]_\Gamma \cong [i_*D\wedge S, i_*D\wedge\Sigma^\infty F^+]_\Gamma.$$

Exactly as in Definition 3.1 we define the transfer $\tau\colon D/\pi \to D\wedge_\pi F^+$ associated to the stable G-bundle $D\wedge_\pi F^+ \to D/\pi$ to be the G-map $\tilde{\tau}/\pi$, where $i_*(\tilde{\tau}) = 1\wedge\tau(F)$. Here we must assume that F embeds in U', so that $\tau(F)$ is defined (see Lemma 2.2).

Several of our constructions will force consideration of this more general form of the transfer, and in such cases we can make use of the following comparison to the original form. Observe that there necessarily exists a Γ-linear isometry $j\colon U' \to U$, where U is our given complete Γ-universe, and that j restricts to a Γ-linear isometry $U'' \to U^\pi$.

Lemma 4.1. The following diagram commutes in $\bar{h}G\mathscr{S}U^\pi$ for a π-free $\mathscr{E}(U',U'')$-spectrum D in $\Gamma\mathscr{S}U''$.

$$\begin{tikzcd} j_*(D/\pi) \arrow[r, phantom, "\cong"] \arrow[d, "j_*\tau"'] & (j_*D)/\pi \arrow[d, "\tau"] \\ j_*(D\wedge_\pi F^+) \arrow[r, phantom, "\cong"] & (j_*D)\wedge_\pi F^+ \end{tikzcd}$$

The proof is immediate from Lemma 2.7 and inspection of definitions. Actually, just as long as j exists and both transfers are defined, we need not assume that U is complete.

Turning to change of groups, we assume given a map of extensions of compact Lie groups

$$\begin{tikzcd} 1 \arrow[r] & \rho \arrow[r] \arrow[d, "\alpha"] & \Lambda \arrow[r] \arrow[d, "\alpha"] & H \arrow[r] \arrow[d, "\alpha"] & 1 \\ 1 \arrow[r] & \pi \arrow[r] & \Gamma \arrow[r] & G \arrow[r] & 1 \end{tikzcd}$$

such that $\alpha: H \to G$ is an inclusion or, equivalently, $\alpha^{-1}(\pi) \subset \rho$ in Λ. We say that $\alpha: (\Lambda,\rho) \to (\Gamma,\pi)$ is a homomorphism of pairs with quotient inclusion $H \subset G$. If α^*U denotes our complete Γ-universe regarded via α as a Λ-universe, then α^*U^π is ρ-trivial and, for any $\Psi \subset \Lambda$ with $\Psi \cap \rho = e$, α^*U^π and α^*U are isomorphic as Ψ-universes since $\alpha(\Psi) \cap \pi = e$. Therefore, if D is a ρ-free Λ-spectrum indexed on α^*U^π, then D is an $\mathcal{E}(\alpha^*U,\alpha^*U^\pi)$-spectrum. Of course, α^*U need not be Λ-complete, but we nevertheless have a transfer

$$\tau: D/\rho \longrightarrow D\wedge_\rho \alpha^*F^+$$

associated to $D\wedge_\rho\alpha^*F^+ \to D/\rho$, where α^*F denotes a finite Γ-space F regarded as a Λ-space (since α^*F certainly embeds in α^*U). The condition $\alpha^{-1}(\pi) \subset \rho$ also implies that $\Gamma \ltimes_\alpha D$ is a π-free Γ-spectrum (since its cells are of type $\Gamma/\alpha(\Psi)$ where $\Psi \cap \rho = e$), hence we also have a transfer

$$\tau: (\Gamma \ltimes_\alpha D)/\pi \longrightarrow (\Gamma \ltimes_\alpha D)\wedge_\pi F^+.$$

By II.4.10, we have a natural isomorphism

$$G \ltimes_H (E/\rho) \xrightarrow{\cong} (\Gamma \ltimes_\alpha E)/\pi$$

in $G\mathcal{S}U^\pi$ for E in $\Lambda\mathcal{S}\alpha^*U^\pi$, and this permits a comparison of the above two transfers.

Axiom 6. The following diagram commutes in $\bar{h}G\mathcal{S}U^\pi$ for a homomorphism of pairs $\alpha: (\Lambda,\rho) \to (\Gamma,\pi)$ with quotient inclusion $H \subset G$ and a ρ-free Λ-spectrum D indexed on α^*U^π.

$$\begin{tikzcd}
G \ltimes_H (D/\rho) \arrow[rr, "\cong"] \arrow[d, "1\ltimes\tau"'] & & (\Gamma \ltimes_\alpha D)/\pi \arrow[d, "\tau"] \\
G \ltimes_H (D \wedge_\rho \alpha^*F^+) \arrow[r, "\cong"] & (\Gamma \ltimes_\alpha (D\wedge\alpha^*F^+))/\pi \arrow[r, phantom, "\cong"] & (\Gamma \ltimes_\alpha D)\wedge_\pi F^+
\end{tikzcd}$$

The proof is immediate from Lemma 2.7 and the commutation of i_* with the relevant change of group functors given by II.4.14. We have the following homological interpretation (in which we retain the hypotheses of the axiom).

Proposition 4.2. Let E be a G-spectrum indexed on U^π. Then the following diagrams are commutative, where L is the tangent H-representation at the identity coset of G/H.

$$\begin{tikzcd}
E_H^*(D\wedge_\rho \alpha^*F^+) \arrow[r, phantom, "\cong"] \arrow[d, "\tau^*"'] & E_G^*((\Gamma \ltimes_\alpha D)\wedge_\pi F^+) \arrow[d, "\tau^*"] \\
E_H^*(D/\rho) \arrow[r, phantom, "\cong"] & E_G^*((\Gamma \ltimes_\alpha D)/\pi)
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_*^H(\Sigma^L D/\rho) \arrow[r, phantom, "\cong"] \arrow[d, "\tau_*"'] & E_*^G((\Gamma \ltimes_\alpha D)/\pi) \arrow[d, "\tau_*"] \\
E_*^H(\Sigma^L D\wedge_\rho \alpha^*F^+) \arrow[r, phantom, "\cong"] & E_*^G((\Gamma \ltimes_\alpha D)\wedge_\pi F^+)
\end{tikzcd}$$

Proof. These are immediate consequences of the natural isomorphisms of II.4.3 and II.6.5 together with Axiom 6 (and Axiom 2, which allows us to regard $\Sigma^L\tau$ on the bottom left as a transfer).

We single out some special cases in the context of (G,π)-bundles.

Examples 4.3. (i) Suppose that ρ is a subgroup of π, $\Gamma = G \times \pi$, $\Lambda = G \times \rho$, and $\alpha: \Lambda \to \Gamma$ is the obvious inclusion. For $E \in \Lambda\mathcal{S}U^\pi$,

$$(\Gamma \ltimes_\alpha E)/\pi \cong (\pi \ltimes_\rho E)/\pi \cong E/\rho$$

by II.4.10. Let F be a finite π-space. Axiom 6 asserts that, for a ρ-free $(G\times\rho)$-spectrum D, the transfer associated to $D\wedge_\rho F^+ \to D/\rho$ agrees with the transfer associated to $(\pi \ltimes_\rho D)\wedge_\pi F^+ \to D/\rho$. In particular, if $\xi: X \times_\pi F \to X/\pi$ is a (G,π)-bundle whose structural group reduces to ρ, so that $X \cong \pi \times_\rho Y$ for a principal (G,ρ)-bundle Y, then the transfer for ξ regarded as a (G,π)-bundle agrees with the transfer for ξ regarded as a (G,ρ)-bundle.

(ii) Suppose that $\pi = \rho/\sigma$ for a normal subgroup σ of ρ, $\Gamma = G \times \pi$, $\Lambda = G \times \rho$, and $\alpha: \Lambda \to \Gamma$ is the obvious quotient homomorphism. For $E \in \Lambda\mathcal{S}U^\pi$,

$$(\Gamma \ltimes_\alpha E)/\pi \cong (E/\sigma)/\pi \cong E/\rho$$

by II.4.10. Let F be a finite π-space. Axiom 6 asserts that, for a ρ-free $(G\times\rho)$-spectrum D, the transfer associated to $D\wedge_\rho F^+ \to D/\rho$ agrees with the transfer associated to $(D/\sigma)\wedge_\pi F^+ \to D/\rho$. In particular, this says that, when ρ acts ineffectively on F with kernel σ, the transfer for a (G,ρ)-bundle with fibre F agrees with the transfer for the same bundle regarded as a $(G,\rho/\sigma)$-bundle.

(iii) Suppose that $\rho = \pi$, $\Gamma = G \times \pi$, $\Lambda = H \times \pi$, and $\alpha\colon \Lambda \to \Gamma$ is the inclusion determined by an inclusion $H \subset G$. For $E \in \Lambda\mathcal{S}U^{\pi}$,

$$(\Gamma \ltimes_{\alpha} E)/\pi \cong (G \ltimes_H E)/\pi.$$

Axiom 6 asserts that, for a π-free $(H \times \pi)$-spectrum D, the G-map $G \ltimes_H \tau$ extending the transfer H-map associated to the stable (H,π)-bundle $D \wedge_{\pi} F^{+} \to D/\pi$ agrees with the transfer G-map associated to the extended (G,π)-bundle $(G \ltimes_H D) \wedge_{\pi} F^{+} \to (G \ltimes_H D)/\pi$. The homological interpretation is of particular interest in this case.

There is another, simpler, compatibility property for $H \subset G$. The transfer G-map associated to a (G,π)-bundle may be viewed as an H-map and, as such, it may be identified with the transfer H-map associated to ξ regarded as an (H,π)-bundle. This property generalizes as follows to arbitrary homomorphisms $\alpha\colon H \to G$. Here $\alpha^{*}U^{\pi}$ need not be H-complete (in contrast to the situation in Axiom 6), hence we must bring in a change of universe as in Lemma 4.1.

<u>Proposition 4.4</u>. Assume given a map of extensions

$$\begin{tikzcd}
1 \arrow[r] & \pi \arrow[r] \arrow[d, equal] & \Lambda \arrow[r] \arrow[d, "\alpha"] & H \arrow[r] \arrow[d, "\alpha"] & 1 \\
1 \arrow[r] & \pi \arrow[r] & \Gamma \arrow[r] & G \arrow[r] & 1.
\end{tikzcd}$$

Let U' be a (complete) Λ-universe and let $j\colon \alpha^{*}U \to U'$ be a Λ-linear isometry. Let D be a π-free Γ-spectrum indexed on U^{π}. Then $j_{*}\alpha^{*}D$ is a π-free Λ-spectrum indexed on $(U')^{\pi}$ and the following diagram commutes.

$$\begin{tikzcd}
j_{*}\alpha^{*}(D/\pi) \arrow[r, phantom, "\cong"] \arrow[d, "j_{*}\alpha^{*}\tau"'] & (j_{*}\alpha^{*}D)/\pi \arrow[d, "\tau"] \\
j_{*}\alpha^{*}(D \wedge_{\pi} F^{+}) \arrow[r, phantom, "\cong"] & (j_{*}\alpha^{*}D) \wedge_{\pi} F^{+}.
\end{tikzcd}$$

<u>Proof</u>. If $\Psi \subset \Gamma$ and $\Psi \cap \pi = e$, then Γ/Ψ is triangulable as a finite π-free Λ-CW complex. It follows as in II.4.13 that $\alpha^{*}D$, and thus $j_{*}\alpha^{*}D$, is a π-free Λ-spectrum. The rest is clear from Lemmas 2.7 and 4.1 by inspection of definitions.

As in Lemma 4.1, as long as we make hypotheses which ensure that the transfers are defined, we need not assume that U and U' are complete here.

We again single out some special cases in the context of (G,π)-bundles.

<u>Examples 4.5</u>. (i) Let F be a finite π-space. Taking $\alpha\colon \Lambda \to \Gamma$ to be

$\alpha \times 1\colon H \times \pi \to G \times \pi$ for some $\alpha\colon H \to G$, letting $D = \Sigma^{\infty}X^{+}$ for a principal (G,π)-bundle X, and using that the functors $j_{*}\alpha^{*}\Sigma^{\infty}$ and $\Sigma^{\infty}\alpha^{*}$ are isomorphic, we see that application of $j_{*}\alpha^{*}$ to the transfer G-map associated to the (G,π)-bundle $\xi\colon X \times_{\pi} F \to X/\pi$ gives the transfer H-map associated to ξ regarded as an (H,π)-bundle via α. When $\alpha\colon H \to G$ is an inclusion, we may set $U' = U$ and take j to be the identity; this gives the case mentioned before the proposition.

(ii) For a different special case of (i), let $\varepsilon\colon G \to J$ be a quotient homomorphism with kernel N and take $\alpha\colon \Lambda \to \Gamma$ to be $\varepsilon \times 1\colon G \times \pi \to J \times \pi$. Here our original U is a complete $(G \times \pi)$-universe and we may take U^{N} as our complete $(J \times \pi)$-universe and let $j\colon U^{N} \to U$ be the resulting inclusion of $(G \times \pi)$-universes. If E is a π-free $(J \times \pi)$-spectrum indexed on $U^{N \times \pi}$, then $j_{*}\alpha^{*}E$ is an N-trivial π-free $(G \times \pi)$-spectrum indexed on U^{π} and the two resulting transfers are related by $j_{*}\alpha^{*}\tau = \tau$. In particular, with $N = G$, this relates the nonequivariant transfer of $\xi\colon E \wedge_{\pi} F^{+} \to E/\pi$ to the equivariant transfer of ξ regarded as a G-trivial stable (G,π)-bundle.

The previous example is of particular interest when E is obtained by passage to orbits over N from an $(N \times \pi)$-free $(G \times \pi)$-spectrum. In this situation, but reverting more generally to our original extension $G = \Gamma/\pi$, we have the following analog of Proposition 4.2. As usual, F is a finite Γ-space and U is our complete Γ-universe.

<u>Proposition 4.6</u>. Let $G = \Gamma/\pi$ and $J = G/N$. Let $\Lambda \supset \pi$ be the inverse image of N in Γ, so that $J = \Gamma/\Lambda$. Let $E_J \in J\mathcal{S}U^{\Lambda}$ and $E_G \in G\mathcal{S}U^{\pi}$ and assume given an $\mathcal{F}(N)$-equivalence $\varepsilon^{\#}E_J \to E_G$ (where $\varepsilon^{\#} = j_{*}\varepsilon^{*}$, $\varepsilon\colon G \to J$ and $j\colon U^{\Lambda} = (U^{\pi})^{N} \subset U^{\pi}$). Let D be a Λ-free Γ-spectrum indexed on U^{Λ}. Then the following diagrams are commutative, where A is the adjoint G-representation derived from N.

$$\begin{tikzcd}
E_J^{*}(D \wedge_{\Lambda} F^{+}) \arrow[r, phantom, "\cong"] \arrow[d, "\tau^{*}"'] & E_G^{*}((j_{*}D) \wedge_{\pi} F^{+}) \arrow[d, "\tau^{*}"] \\
E_J^{*}(D/\Lambda) \arrow[r, phantom, "\cong"] & E_G^{*}((j_{*}D)/\pi)
\end{tikzcd}$$

and

$$\begin{tikzcd}
E_{*}^{J}(D/\Lambda) \arrow[r, phantom, "\cong"] \arrow[d, "\tau_{*}"'] & E_{*}^{G}((\Sigma^{-A}j_{*}D)/\pi) \arrow[d, "\tau_{*}"] \\
E_{*}^{J}(D \wedge_{\Lambda} F^{+}) \arrow[r, phantom, "\cong"] & E_{*}^{G}((\Sigma^{-A}j_{*}D) \wedge_{\pi} F^{+})
\end{tikzcd}$$

<u>Proof</u>. The stable J-bundle $D \wedge_{\Lambda} F^{+} \to D/\Lambda$ is obtained by passage to orbits over N from the stable G-bundle $D \wedge_{\pi} F^{+} \to D/\pi$. While U^{Λ} need not be G-complete, D is

an $\mathcal{E}(U,U^{\Lambda})$-spectrum since it is Λ-free, and there is thus a transfer G-map $\tau\colon D/\pi \to D\wedge_{\pi}F^{+}$. It is immediate from the definitions that τ/N is the transfer J-map of $D\wedge_{\Lambda}F^{+} \to D/\Lambda$. Lemma 4.1 identifies $j_{*}\tau$ as a transfer. The diagrams are thus direct consequences of the naturality of the isomorphisms of II.8.1 (and Axiom 2, which allows us to regard $\Sigma^{-A}\tau$ as a transfer).

Example 4.7. When $\Gamma = G \times \pi$ in the proposition, $\Lambda = N \times \pi$. Here F is a finite π-space and D is an $(N \times \pi)$-free $(G \times \pi)$-spectrum, we can identify $D\wedge_{\Lambda}F^{+} \to D/\Lambda$ with $(D/N)\wedge_{\pi}F^{+} \to D/(N \times \pi)$, and we have a map of stable (G,π)-bundles $(j_{*}D)\wedge_{\pi}F^{+} \to j_{*}(D/N)\wedge_{\pi}F^{+}$. Comparing with Examples 4.5 (ii), we find that the identifications of transfers of the proposition factor through the resulting naturality diagrams.

The basic source of suitably related pairs of spectra E_J and E_G is explained in II.8.6. In particular, when $N = G$, Proposition 4.6 relates the transfer in E_G^{*} and E_{*}^{G} to the transfer in E^{*} and E_{*}, where E_G is a split G-spectrum with associated nonequivariant spectrum E as in II.8.4.

To give a concrete illustration, let $H \subset G$ and consider the projection $\xi\colon EG \times G/H \to EG$. The orbit bundle ξ/G is the natural bundle $BH \to BG$. Writing EH for EG regarded as an H-space and using the evident bundle map from ξ to $G/H \to *$, we obtain the following special case.

Corollary 4.8. If k_G is a split G-spectrum with associated nonequivariant spectrum k, then the following diagram commutes.

$$\begin{array}{ccccc}
k_H^{*}(*) & \xrightarrow{\varepsilon^{*}} & k_H^{*}(EH) & \cong & k^{*}(BH) \\
\downarrow{\tau^{*}} & & \downarrow{\tau^{*}} & & \downarrow{\tau^{*}} \\
k_G^{*}(*) & \xrightarrow{\varepsilon^{*}} & k_G^{*}(EG) & \cong & k^{*}(BG).
\end{array}$$

In the case of complex K-theory, this diagram (on skeleta) is due to Nishida [117], who also verified that the left most transfer may then be identified with the standard induction homomorphism $R(H) \to R(G)$.

§5. Product and Euler characteristic formulas

The basic topological fact here is the commutation of transfer with products. A special case will imply an Euler characteristic formula for the evaluation of the composite $\xi\cdot\tau$ for a (stable) G-bundle ξ. The relevant Euler characteristic depends on ξ and not just its fibres, but we show that the determination of when this composite induces an isomorphism on cohomology does reduce to questions about Euler characteristics of fibres.

We begin with a result on external products.

Theorem 5.1. For $i = 1$ and $i = 2$, assume given an extension $G_i = \Gamma_i/\pi_i$, a complete Γ_i-universe U_i, a π_i-free Γ_i-spectrum D_i indexed on $U_i^{\pi_i}$, and a finite Γ_i-space F_i. Then the following diagram of $(G_1 \times G_2)$-spectra indexed on $U^{\pi_1} \oplus U^{\pi_2}$ commutes, where all of the smash products are external.

$$\begin{array}{ccc}
D_1/\pi_1 \wedge D_2/\pi_2 & \cong & (D_1\wedge D_2)/(\pi_1 \times \pi_2) \\
\downarrow{\tau\wedge\tau} & & \downarrow{\tau} \\
(D_1 \wedge_{\pi_1} F_1^{+})\wedge(D_2 \wedge_{\pi_2} F_2^{+}) & \cong & (D_1\wedge D_2)\wedge_{\pi_1\times\pi_2}(F_1 \times F_2)^{+}
\end{array}$$

Proof. While $U_1 \oplus U_2$ need not be a complete $(\Gamma_1 \times \Gamma_2)$-universe, $D_1\wedge D_2$ is an $\mathcal{E}(U_1 \oplus U_2, U_1^{\pi_1} \oplus U_2^{\pi_2})$-spectrum since its cells are given by orbits $(\Gamma_1 \times \Gamma_2)/(\Lambda_1 \times \Lambda_2)$ with $\Lambda_i \cap \pi_i = e$. Thus the right hand transfer is defined as in the previous section. The conclusion follows from Lemma 2.6 and the commutation of external smash products with the change of universe functors used to define the three transfers in sight.

We use change of groups to internalize this result.

Theorem 5.2. With the hypotheses of Theorem 5.1, assume further that $G_1 = G_2 = G$ and define Γ to be the equalizer of the projections $\Gamma_i \to G$, so that the following is a map of extensions.

$$\begin{array}{ccccccccc}
1 & \to & \pi_1 \times \pi_2 & \to & \Gamma & \to & G & \to & 1 \\
 & & \| & & \downarrow{\Delta} & & \downarrow{\Delta} & & \\
1 & \to & \pi_1 \times \pi_2 & \to & \Gamma_1 \times \Gamma_2 & \to & G \times G & \to & 1
\end{array}$$

Choose the Γ_i-universes U_i so that $U_1^{\pi_1} = U_2^{\pi_2} = U^{\pi_1\times\pi_2}$ for a complete Γ-universe U. Then the internal smash product $D_1\wedge D_2$ is a $(\pi_1 \times \pi_2)$-free Γ-spectrum indexed on $U^{\pi_1\times\pi_2}$ and the following diagram of G-spectra indexed on $U^{\pi_1\times\pi_2}$ commutes, where all of the smash products are internal.

$$\begin{array}{ccc}
D_1/\pi_1 \wedge D_2/\pi_2 & \cong & (D_1\wedge D_2)/\pi_1 \times \pi_2 \\
\downarrow{\tau\wedge\tau} & & \downarrow{\tau} \\
(D_1 \wedge_{\pi_1} F_1^{+})\wedge(D_2 \wedge_{\pi_2} F_2^{+}) & \cong & (D_1\wedge D_2)\wedge_{\pi_1\times\pi_2}(F_1 \times F_2)^{+}
\end{array}$$

Proof. The G-universe $U^{\pi_1\times\pi_2}$ has complementary Γ_1, Γ_2, and Γ summands in U_1, U_2, and U, hence we can extend the G-linear isometry $U^{\pi_1\times\pi_2} \oplus U^{\pi_1\times\pi_2} \to U^{\pi_1\times\pi_2}$

used to internalize the smash product to a Γ-linear isometry $j\colon \Delta^*(U_1 \oplus U_2) \to U$. Then the internal smash products here are obtained from the external smash products of the theorem by applying the functor $j_*\Delta^*$. We obtain the conclusion by applying $j_*\Delta^*$ to the diagram of the theorem and using Proposition 4.4 to identify the resulting arrow $j_*\Delta^*\tau$ as a transfer.

Of course, we can always construct the required universes U_i by adding appropriate summands to the complete G-universe $U^{\pi_1 \times \pi_2}$. In the context of (G,π)-bundles, we start with stable (G,π_i)-bundles D_i and construct the stable $(G,\pi_1 \times \pi_2)$-bundle $D_1 \wedge D_2$. If $D_i = \Sigma^\infty X_i^+$, then $D_1 \wedge D_2 \simeq \Sigma^\infty (X_1 \times X_2)^+$. Here $\Gamma_i = G \times \pi_i$, $\Gamma = G \times \pi_1 \times \pi_2$ and $\Delta\colon \Gamma \to \Gamma_1 \times \Gamma_2$ is induced by the diagonal map of G.

Either specializing to the case where one of the stable G-bundles is an identity map and quoting Axiom 3 or arguing by direct inspection of definitions, we obtain the following generalization of Axiom 2. We revert to our usual context of π-free Γ-spectra D indexed on U^π and finite Γ-spaces F, where U is a complete Γ-universe.

<u>Corollary 5.3</u>. Let E be any G-spectrum indexed on U^π. Then $D \wedge E$ is a π-free Γ-spectrum indexed on U^π and the following diagram of G-spectra commutes, where all smash products are internal.

$$\begin{tikzcd}
(D\wedge E)/\pi \arrow[r, phantom, "\cong"] \arrow[d, "\tau"'] & (D/\pi)\wedge E \arrow[d, "\tau\wedge 1"] \\
(D\wedge E)\wedge_\pi F^+ \arrow[r, phantom, "\cong"] & (D\wedge_\pi F^+)\wedge E
\end{tikzcd}$$

We want to use this result to obtain homological formulas involving cup and cap products. Here we need appropriate diagonal maps. While these are obviously present when $D = \Sigma^\infty X^+$, it is again simple and useful to proceed in greater generality. The following definition makes sense for any compact Lie group Γ.

<u>Definition 5.4</u>. A coaction of a Γ-space X on a Γ-spectrum D is a map $\Delta\colon D \to X^+ \wedge D$ of Γ-spectra such that the following counit and coassociativity diagrams commute.

$$\begin{tikzcd}
D \arrow[r, "\Delta"] \arrow[dr, equal] & X^+\wedge D \arrow[d, "\varepsilon\wedge 1"] \\
& S^0 \wedge D
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
D \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X^+\wedge D \arrow[d, "\Delta\wedge 1"] \\
X^+\wedge D \arrow[r, "1\wedge\Delta"] & X^+\wedge X^+\wedge D
\end{tikzcd}$$

For a (commutative) ring Γ-spectrum k_Γ and a k_Γ-module Γ-spectrum j_Γ, there

results a cup product

$$\cup\colon k_\Gamma^*(X) \otimes j_\Gamma^*(D) \longrightarrow j_\Gamma^*(D)$$

and a cap product

$$\cap\colon j_*^\Gamma(D) \otimes k_\Gamma^*(X) \longrightarrow j_*^\Gamma(D)$$

arising from the external products (2') and (4') of III§3, and these endow $j_\Gamma^*(D)$ and $j_*^\Gamma(D)$ with $k_\Gamma^*(X)$-module structures.

Of course, we are free to replace X^+ by $\Sigma^\infty X^+$ (or to view X^+ as shorthand notation for $\Sigma^\infty X^+$). We are interested in coactions by π-free Γ-spaces on π-free Γ-spectra, and in this case we continue to write Δ for the induced composites

$$D \xrightarrow{\Delta} X^+\wedge D \xrightarrow{q\wedge 1} (X/\pi)^+\wedge D \quad \text{and} \quad D \xrightarrow{\Delta} X^+\wedge D \xrightarrow{1\wedge q} X^+\wedge D/\pi,$$

where the q's are the quotient maps, and for the induced maps

$$D/\pi \longrightarrow (X/\pi)^+\wedge D/\pi \quad \text{and} \quad D\wedge_\pi F^+ \longrightarrow (X\times_\pi F)^+\wedge(D\wedge_\pi F^+).$$

Observe that the first two of these are maps of π-free Γ-spectra which both induce the third map on passage to orbits over π. The fourth map arises in the evident way by use of the diagonal on F. The naturality of the transfer, the previous corollary, and elementary chases give diagrams relating Δ, τ, and ξ.

<u>Corollary 5.5</u>. Let $\Delta\colon D \to X^+\wedge D$ be a coaction of a π-free Γ-space X on a π-free Γ-spectrum D. Then the following diagram of G-spectra commutes.

$$\begin{tikzcd}
(X/\pi)^+\wedge D/\pi \arrow[d, "\tau\wedge 1"'] & D/\pi \arrow[l, "\Delta"'] \arrow[r, "\tau"] & D\wedge_\pi F^+ \arrow[d, "\Delta"] & D/\pi \arrow[l, "\tau"'] \arrow[r, "\Delta"] & (X/\pi)^+\wedge D/\pi \arrow[d, "1\wedge\tau"] \\
(X\times_\pi F)^+\wedge D/\pi & & (X\times_\pi F)^+\wedge(D\wedge_\pi F^+) \arrow[ll, "1\wedge\xi"'] \arrow[rr, "\xi\wedge 1"] & & (X/\pi)^+\wedge(D\wedge_\pi F^+)
\end{tikzcd}$$

There result formulas relating the transfer to cup products and cap products.

<u>Corollary 5.6</u>. Let k_G be a (commutative) ring G-spectrum and let j_G be a k_G-module G-spectrum. Then

$$\tau^*(w)\cup y = \tau^*(w\cup\xi^*(y)) \quad \text{for } w\in k_G^*(X\times_\pi F) \text{ and } y\in j_G^*(D/\pi)$$

$$x\cup\tau^*(z) = \tau^*(\xi^*(x)\cup z) \quad \text{for } x\in k_G^*(X/\pi) \text{ and } z\in j_G^*(D\wedge_\pi F^+)$$

$$u\cap\tau^*(w) = \xi_*(\tau_*(u)\cap w) \quad \text{for } u\in j_*^G(D/\pi) \text{ and } w\in k_G^*(X\times_\pi F)$$

$$\tau_*(u)\cap\xi^*(x) = \tau_*(u\cap x) \quad \text{for } u\in j_*^G(D/\pi) \text{ and } x\in k_G^*(X/\pi)$$

With w taken to be the identity element, the first formula implies an Euler characteristic formula for the computation of $\xi^*\tau^*$.

Definition 5.7. Define the Euler characteristic $\chi(\xi)$ associated to the G-bundle $\xi\colon X\times_\pi F\to X/\pi$ to be $\tau^*(1)\in k_G^0(X/\pi)$, where k_G is any given ring spectrum. In represented form, $\chi(\xi)$ is the composite

$$\Sigma^\infty(X/\pi)^+ \xrightarrow{\tau} \Sigma^\infty(X\times_\pi F)^+ \xrightarrow{\rho} S \xrightarrow{e} k_G,$$

where ρ is the collapse map and e is the unit of k_G.

Theorem 5.8. Let the π-free Γ-space X coact on the π-free Γ-spectrum D and let j_G be a module G-spectrum over a ring G-spectrum k_G. Then the composite

$$j_G^*(D/\pi) \xrightarrow{\xi^*} j_G^*(D\wedge_\pi F^+) \xrightarrow{\tau^*} j_G^*(D/\pi)$$

is multiplication by the Euler characteristic $\chi(\xi)\in k_G^0(X/\pi)$.

We shall shortly restrict to the space level and discuss conditions which ensure that $\chi(\xi)$ is a unit, but we should first display some examples of coactions on stable G-bundles to which the general considerations above apply.

Examples 5.9. (i) Of course, a Γ-space X coacts on $\Sigma^\infty X^+$. If A is a Γ-subspace of X and $C(X,A)$ is the unreduced mapping cone, then X coacts on $\Sigma^\infty C(X,A)$ via

$$\Sigma^\infty C(X,A) \xrightarrow{\Sigma^\infty C(\Delta)} \Sigma^\infty C(X\times X, X\times A) \cong X^+\wedge\Sigma^\infty C(X,A).$$

More generally, if A_1 and A_2 are Γ-subspaces of X with union A, then the diagonal of X and the Γ-equivalence of III.4.4 give a Γ-map

$$C(X,A) \xrightarrow{C(\Delta)} C(X\times X, X\times A_2\cup A_1\times X) \simeq C(X,A_1)\wedge C(X,A_2).$$

Via diagrams like those of Corollary 5.5, we can prove the evident relative versions

of the formulas of Corollary 5.6.

(ii) If Δ is a coaction of X on D and E is any Γ-spectrum, then $\Delta\wedge 1$ is a coaction of X on $D\wedge E$. The case of interest is when X is a π-free Γ-space, D is a π-free Γ-spectrum, and E is a π-trivial Γ-spectrum, as in Corollary 5.3.

(iii) As pointed out in Remark 3.3 (iii), the projection $E\mathcal{F}(\pi)^+\wedge D\to D$ is a Γ-equivalence for any π-free Γ-spectrum D, by II.2.12. The inverse of this equivalence specifies a coaction ψ of $E\mathcal{F}(\pi)$ on D. This example plays a universal role. For any π-free Γ-space X, there is a unique Γ-map $\lambda\colon X\to E\mathcal{F}(\pi)$, and $(\lambda\wedge 1)\Delta=\psi$ whenever X coacts on D via Δ (as we see by composing with the projection).

Since $\chi(\xi)$ is only defined for space level G-bundles $\xi\colon X\times_\pi F\to X/\pi$, we ignore the more general context henceforward in this section. We want to determine when $\chi(\xi)\in k_G^0(X/\pi)$ is a unit. Observe that, by the previous example, this holds for all X (and a given fixed F and k_G) if and only if it holds for $X=E\mathcal{F}(\pi)$, in which case $\tau^*\xi^*$ is an isomorphism for every stable G-bundle $\xi\colon D\wedge_\pi F^+\to D/\pi$.

We need to relate $\chi(\xi)$ to $\chi(F)$, and for this purpose we must first study the case when the base G-space X/π is an orbit G/H. Here X must be an orbit Γ/Λ, but there are in general many possible choices of Λ for a given choice of H. Since X is π-free, $\Lambda\in\mathcal{F}(\pi)$; that is, $\Lambda\cap\pi=e$. The composite $\Lambda\subset\Gamma\to G$ maps Λ isomorphically onto H. Let $\alpha\colon H\to\Gamma$ denote the composite of the inverse of this isomorphism and the inclusion of Λ in Γ. We call α the fibre representation of the orbit Γ/Λ (and think of it as defined up to conjugation in Γ). For a general π-free Γ-space X and base orbit $G/H\subset X/\pi$, we say that α is the fibre representation of X at G/H if the pullback of X over G/H is Γ-homeomorphic to Γ/Λ.

Our finite Γ-space F has the Euler characteristic $\chi(F)=\xi^*\tau(F)\in\pi_\Gamma^0(S)$, $\xi\colon F\to\mathrm{pt}$. If $\alpha\colon H\to\Gamma$ is a fibre representation, then, regarding F as a finite H-space via α, we obtain an Euler characteristic $\chi(\alpha^*F)\in\pi_H^0(S)$. Equivalently, $\chi(\alpha^*F)$ is the image of $\chi(F)$ under $\alpha^*\colon\pi_\Gamma^0(S)\to\pi_H^0(S)$. We continue to write $\chi(\alpha^*F)$ for its image in $k_H^0(S)\cong k_G^0(G/H)$ when k_G is a ring G-spectrum.

Lemma 5.10. Let $\Lambda\in\mathcal{F}(\pi)$ determine the fibre representation $\alpha\colon H\to\Gamma$. Then the G-bundle $\xi\colon(\Gamma/\Lambda)\times_\pi F\to\Gamma/\Lambda\pi$ may be identified with the extended G-map $1\times_H\varepsilon\colon G\times_H\alpha^*F\to G/H$ and the following diagram of G-spectra commutes.

$$\begin{tikzcd}
G\ltimes_H S \arrow[r, phantom, "\cong"] \arrow[d, "1\times\tau(\alpha^*F)"'] & \Sigma^\infty(\Gamma/\Lambda\pi)^+ \arrow[d, "\tau(\xi)"] \\
G\ltimes_H\Sigma^\infty\alpha^*F^+ \arrow[r, phantom, "\cong"] & \Sigma^\infty((\Gamma/\Lambda)\times_\pi F)^+
\end{tikzcd}$$

Therefore $\chi(\xi)=\chi(\alpha^*F)$ in the ring $\pi_G^0(G/H)\cong\pi_H^0(S)$.

Proof. Since $(\Gamma/\Lambda) \times F$ is Γ-homeomorphic to $\Gamma \times_\Lambda F$ and $(\Gamma \times_\Lambda F)/\pi$ is G-homeomorphic to $G \times_H \alpha^* F$, the identification is clear. The diagram results by application of Axiom 6 to the inclusion of pairs $(\Lambda, e) \to (\Gamma, \pi)$ with quotient inclusion $H \subset G$, with D in Axiom 6 taken to be the sphere Λ-spectrum (which of course is e-free).

Example 5.11. Let $\Gamma = G \times \pi$ and let F be a π-space. Here any $\Lambda \in \mathcal{F}(\pi)$ has the form $\Lambda = \{(h, \rho(h)) | h \in H\}$ for some subgroup H of G and homomorphism $\rho: H \to \pi$. The corresponding fibre representation $\alpha: H \to \Gamma$ is given by $\alpha(h) = (h, \rho(h))$, and of course $\alpha^* F = \rho^* F$. This recovers the usual description of (G,π)-bundles over orbits as extended bundles $G \times_H \rho^* F$.

We insert two criteria for an element of $k_G^0(B)$ to be a unit.

Lemma 5.12. Let k_G be a ring G-spectrum and let $u \in k_G^0(B)$, where B is a G-CW complex. Then u is a unit if either of the following conditions hold.

(i) The image of u in $k_G^0(G/H)$ is a unit for each orbit $G/H \subset B$.

(ii) B is G-connected with basepoint $*$ and the image of u in $k_G^0(*)$ is a unit.

Proof. First note that condition (ii) implies condition (i) since, if B is G-connected with basepoint $*$, then any inclusion $G/H \to B$ is homotopic to the projection $G/H \to *$ (because there is an H-path connecting $*$ to the image of the orbit eH). Thus assume (i). We show that u acts isomorphically on $k_G^*(B)$. Since u acts on the Milnor $\lim^1$ exact sequence for $k_G^*(B)$, we may as well assume that B has finite dimension n. The result certainly holds for $n = 0$, and an easy Mayer-Vietoris sequence argument from the pushout diagram describing B in terms of its n-cells and (n-1)-skeleton gives the conclusion by induction on n.

We obtain the following result by combining Lemmas 5.10 and 5.12.

Theorem 5.13. Let $B = X/\pi$ for a G-bundle $\xi: X \times_\pi F \to X/\pi$ and let k_G be a ring G-spectrum. Then $\chi(\xi) \in k_G^0(B)$ is a unit if any of the following conditions hold.

(i) $\chi(\alpha^* F) \in k_H^0(S)$ is a unit for each $\alpha: H \to \Gamma$ such that G/H embeds in B with fibre representation α.

(ii) B is G-connected with basepoint $*$ and $\chi(\alpha^* F) \in k_G^0(S)$ is a unit, where $\alpha: G \to \Gamma$ is the fibre representation at $*$.

(iii) B is G-free and the classical nonequivariant Euler characteristic $\chi(F) \in k_e^0(S)$ is a unit.

Part (iii) is the special case of (i) in which only the trivial isotropy group and thus only the classical Euler characteristic appears. When B is finite and

G-trivial, (ii) is due to Nishida [117, 4.7]. The most important case of (ii) occurs when G acts trivially on F, so that $\chi(\alpha^* F)$ is the image of the classical Euler characteristic of F under the unit $Z \to k_G^0(S)$.

Of course, in practice one usually localizes appropriately to obtain the unit conditions of the theorem. Nonequivariantly, when $G = e$, connectivity is a negligible hypothesis and (i) and (ii) have the same content. Equivariantly, G-connectivity is an annoyingly strong hypothesis. One is stuck with (i) and should localize so as to invert all $\chi(\alpha^* F)$. We shall not pursue the relevant techniques here. Application of (i) in ordinary RO(G)-graded cohomology gives the following conclusion; see [89, Thm B].

Theorem 5.14. If G is a finite p-group and $\xi: Y \to B$ is a finite G-cover whose fibre F has cardinality prime to p, then the composite G-map

$$\Sigma^\infty B^+ \xrightarrow{\tau} \Sigma^\infty Y^+ \xrightarrow{\xi} \Sigma^\infty B^+$$

becomes an equivalence upon localization at p. In fact, the conclusion applies to any stable G-bundle of the form $\xi: D \wedge_\pi F^+ \to D/\pi$.

§6. The sum decomposition and double coset formulas

We here exploit the homotopy invariance and additivity on fibres axioms to prove Feshbach's sum decomposition and double coset theorems. We also discuss the equivariant analogs of Feshbach's applications of the double coset formula, this being an area where much further work remains to be done.

We shall use the same letter for a map of spaces and for its induced map of suspension spectra. We shall also use the same letter for a map of finite π-spaces and for its induced map of stable bundles with these spaces as fibres. For a π-free Γ-spectrum D and a subgroup Λ of Γ, we write $\xi(\Lambda,\Gamma)$ for the stable bundle $D \wedge_\pi (\Gamma/\Lambda)^+ \to D/\pi$ and $\tau(\Lambda,\Gamma)$ for its associated transfer map.

With these notations, the following two decomposition theorems are immediate consequences of the pretransfer level assertions of Theorems 2.10 and 2.11.

Theorem 6.1. Let F be a finite Γ-CW complex and let

$$j_i: \Gamma/\Lambda_i \subset \Gamma/\Lambda_i \times e^{n_i} \longrightarrow F$$

be the composite of the inclusion of an orbit and the $i^{\underline{th}}$ characteristic map for some enumeration of the cells of F. Then, for any π-free Γ-spectrum D,

$$\tau = \sum_i (-1)^{n_i} j_i \tau(\Lambda_i,\Gamma)\colon D/\pi \longrightarrow D\wedge_\pi F^+.$$

As above Theorem 2.11, write F/Γ as a disjoint union of orbit type path components M and let $\chi(M)$ be the internal Euler characteristic of M in F/Γ.

Theorem 6.2. Let F be a compact Γ-ENR and let $j_m\colon \Gamma/\Lambda \to F$ be the inclusion of an orbit in the orbit type component M. Then, for any π-free Γ-spectrum D,

$$\tau = \sum_M \chi(M) j_m \tau(\Lambda,\Gamma)\colon D/\pi \longrightarrow D\wedge_\pi F^+.$$

Let σ and ρ be subgroups of π and let $\xi(\sigma,\pi)$ be the evident quotient map $D/\sigma \to D/\pi$. We would like to obtain a double coset formula for the computation of $\tau(\rho,\pi)\xi(\sigma,\pi)$, where $\tau(\rho,\pi)$ is the transfer associated to $\xi(\rho,\pi)$. Here for the first time our general context causes real difficulty. It is not clear what equivariance $\xi(\sigma,\pi)$ has or that it is the sort of stable bundle for which our methods provide a transfer. We shall return to the general context at the end of the section, but until then we shall take $\Gamma = G\times\pi$ and work in the context of stable (G,π)-bundles. Remember that a principal (G,π)-bundle is the same thing as a π-free $(G\times\pi)$-space.

Thus consider a π-free $(G\times\pi)$-spectrum D indexed on U^π, where U is a complete $(G\times\pi)$-universe. As in II.4.8 and II.4.15, the π-action $\phi\colon \pi\ltimes D \to D$ is a $(G\times\pi)$-map and gives rise to the following commutative diagram of π-free $(G\times\pi)$-spectra.

$$\begin{tikzcd} \pi\ltimes_\sigma D \arrow[d, "\phi"'] \arrow[r, phantom, "\cong"] & D\wedge(\pi/\sigma)^+ \arrow[d, "1\wedge\varepsilon"] \\ D \arrow[r, phantom, "\cong"] & D\wedge S^0 \end{tikzcd}$$

By II.4.10, we have an isomorphism of G-spectra

$$D/\sigma \cong (\pi\ltimes_\sigma D)/\pi,$$

and $\xi(\sigma,\pi)$ is just the composite G-map

$$D/\sigma \cong (\pi\ltimes_\sigma D)/\pi \xrightarrow{\phi/\pi} D/\pi$$

or, equivalently,

$$D/\sigma \cong D\wedge_\pi(\pi/\sigma)^+ \xrightarrow{1\wedge_\pi\varepsilon} D/\pi.$$

Thus $\xi(\sigma,\pi)$ is a stable (G,π)-bundle and has a transfer $\tau(\sigma,\pi)$.

Theorem 6.3. Let ρ and σ be subgroups of π and let $\sigma\backslash\pi/\rho$ be the double coset space regarded as the space of orbits under σ of the homogeneous space π/ρ. Let $\{m\}$ be a set of representatives in π for the orbit type component manifolds M of $\sigma\backslash\pi/\rho$ and let $\chi(M)$ be the internal Euler characteristic of M in $\sigma\backslash\pi/\rho$. Then, for any π-free $(G\times\pi)$-spectrum D, the composite

$$D/\sigma \xrightarrow{\xi} D/\pi \xrightarrow{\tau} D/\rho$$

is the sum over M of $\chi(M)$ times the composite

$$D/\sigma \xrightarrow{\tau} D/\rho^m\cap\sigma \xrightarrow{\xi} D/\rho^m \xrightarrow{c_m} D/\rho.$$

Here $\rho^m = m\rho m^{-1}$ and c_m is induced by the left π-map $\pi/\rho^m \to \pi/\rho$ given by right multiplication by m. In symbols,

$$\tau(\rho,\pi)\circ\xi(\sigma,\pi) = \sum_M \chi(M) c_m\circ\xi(\rho^m\cap\sigma,\rho^m)\circ\tau(\rho^m\cap\sigma,\sigma).$$

Proof. The isotropy group under σ of the point $m\rho \in \pi/\rho$ is $\rho^m\cap\sigma$, and we let

$$j_m\colon \sigma/\rho^m\cap\sigma \xrightarrow{\cong} \sigma m\rho \subset \pi/\rho$$

be the inclusion, $s(\rho^m\cap\sigma) \to sm\rho$ for $s\in\sigma$. Observe that j_m coincides with the composite

$$\sigma/\rho^m\cap\sigma \subset \pi/\rho^m \xrightarrow{c_m} \pi/\rho.$$

With the isomorphism again coming from II.4.10, we see from our description of $\xi(\sigma,\pi)$ that we have a commutative diagram

$$\begin{tikzcd} D\wedge_\sigma(\pi/\rho)^+ \arrow[d, "\xi"'] \arrow[r, phantom, "\cong"] & (\pi\ltimes_\sigma D)\wedge_\pi(\pi/\rho)^+ \arrow[r, "\phi\wedge_\pi 1"] & D/\rho \arrow[d, "\xi(\rho,\pi)"] \\ D/\sigma \arrow[rr, "\xi(\sigma,\pi)"] & & D/\pi \end{tikzcd}$$

By Axiom 6 (in the context of Example 4.3 (i)) and Axiom 1,

$$\tau(\rho,\pi)\xi(\sigma,\pi) = (\phi\wedge_\pi 1)\tau(\xi).$$

By application of the previous decomposition theorem to ξ,

$$\tau(\xi) = \sum_M \chi(M) j_m \tau(\rho^m \cap \sigma, \sigma).$$

By a simple diagram chase from our initial observations,

$$(\phi \wedge_\pi 1) j_m = c_m \xi(\rho^m \cap \sigma, \rho^m).$$

We obtain the conclusion by combining these relations.

Of course, if ρ has finite index in π, then M is the point $\sigma m \rho$ and $\chi(M) = 1$. Here the result has the same form as the classical double coset formula in the cohomology of finite groups.

The most important case occurs when $D = \Sigma^\infty E(G,\pi)^+$, where $E(G,\pi)$ is the universal principal (G,π)-bundle. Here $E(G,\pi)/\rho$ is a classifying G-space $B(G,\rho)$ for any $\rho \subset \pi$ and the result takes the following form.

Corollary 6.4. The composite

$$\Sigma^\infty B(G,\sigma)^+ \xrightarrow{\xi} \Sigma^\infty B(G,\pi)^+ \xrightarrow{\tau} \Sigma^\infty B(G,\rho)^+$$

is the sum over M of $\chi(M)$ times the composite

$$\Sigma^\infty B(G,\sigma)^+ \xrightarrow{\tau} \Sigma^\infty B(G,\rho^m \cap \sigma)^+ \xrightarrow{\xi} \Sigma^\infty B(G,\rho^m)^+ \xrightarrow{c_m} \Sigma^\infty B(G,\rho)^+.$$

Remarks 6.5. Taking $G = e$ (and restricting to skeleta), we obtain the main theorem of Feshbach's paper [53]. Feshbach later gave a separate argument for the generalization of the double coset formula to Borel cohomology, namely $k^*_\pi(Y) = k^*(E\pi \times_\pi Y)$ for a π-space Y and nonequivariant theory k [54,II.2]. In fact, one need only apply Theorem 6.3 to $\Sigma^\infty(E\pi \times Y)^+$ to read off this generalization.

It is to be emphasized that the double coset formula depends only on the structure of fibres and thus has the same form and can in principal be exploited in the same way equivariantly as nonequivariantly. In particular, the following consequence of Lemma 2.12 serves to eliminate terms.

Proposition 6.6. If $W\rho = N_\pi\rho/\rho$ is not finite, then

$$\tau(\rho,\pi): D/\pi \longrightarrow D/\rho$$

is zero for every π-free $(G \times \pi)$-spectrum D.

For example, if σ is a torus in the double coset theorem, then all terms vanish except those indexed on fixed point orbit type components M, that is, those with $\rho^m \cap \sigma = \sigma$; of course, $\tau(\sigma,\sigma) = 1$, and all terms vanish if there is no m such that $\sigma \subset \rho^m$. If $\sigma \subset \rho \subset \pi$ and σ is a maximal torus in π, then the fixed point subspace $\{m\rho \mid \sigma \subset \rho^m\}$ of $\sigma\backslash\pi/\rho$ is the finite set $W_\pi\sigma/W_\rho\sigma$ and each $\chi(M) = 1$. Indeed, if $\sigma \subset \rho^m$, then $m^{-1}\sigma m$ is a maximal torus in ρ and is thus equal to σ^n for some $n \in \rho$, so that $mn \in N_\pi(\sigma)$ and $m \in N_\pi(\sigma)\rho$. Therefore, as proven by Feshbach [53,54] when $G = e$, the following result is an immediate specialization of the double coset theorem.

Corollary 6.7. Let D be any π-free $(G \times \pi)$-spectrum.

(i) If ρ is the normalizer of a maximal torus σ in π, then

$$\tau(\rho,\pi)\xi(\sigma,\pi) = \xi(\sigma,\rho): D/\sigma \longrightarrow D/\rho,$$

hence $\mathrm{Im}\,\xi(\sigma,\rho)^* = \mathrm{Im}\,\xi(\sigma,\pi)^*$ in $j^*(D/\sigma)$ for any G-spectrum j.

(ii) If ρ is a maximal torus in π, then

$$\tau(\rho,\pi)\xi(\rho,\pi) = \Sigma c_m: D/\rho \longrightarrow D/\rho,$$

where the sum ranges over a set $\{m\}$ of coset representatives for $W_\pi\rho$.

(iii) If ρ is normal and of finite index in π, then

$$\tau(\rho,\pi)\xi(\rho,\pi) = \Sigma c_m: D/\rho \longrightarrow D/\rho,$$

where the sum ranges over a set $\{m\}$ of coset representatives for π/ρ. (For example, ρ might be the identity component of π.)

Still following Feshbach, we insert a definition which will allow us to state a best possible reduction theorem for the computation of $j^*(D/\pi)$ in terms of $j^*(D/\rho)$ for a subgroup ρ of π.

Definition 6.8. An element $y \in j^*(D/\rho)$ is said to be stable (or π-stable) if

$$\xi(\rho \cap \rho^m, \rho)^*(y) = \xi(\rho \cap \rho^m, \rho^m)^* c_m^*(y)$$

for all $m \in \pi$; $j^*(D/\rho)^S$ denotes the set of stable elements. Obviously $\xi(\rho,\pi)c_m = \xi(\rho^m,\pi)$, hence

$$\operatorname{Im}\xi(\rho,\pi)^* \subset j^*(D/\rho)^S.$$

Theorem 6.9. Let the π-free $(G \times \pi)$-space X coact on the π-free $(G \times \pi)$-spectrum D and let j_G be a module G-spectrum over a ring G-spectrum k_G. Let $\rho \subset \pi$ and consider $\xi = \xi(\rho,\pi)$. Suppose that $\chi(\xi) \in k_G^0(X/\pi)$ is a unit. Then

$$\xi^*: j^*(D/\pi) \longrightarrow j^*(D/\rho)^S$$

is an isomorphism.

Proof. For $y \in j^*(D/\rho)^S$, an immediate calculation from the double coset and Euler characteristic formulas shows that $\xi^*\tau^*(y) = uy$, where

$$u = \Sigma\chi(M)\chi(\xi(\rho^m \cap \rho,\rho)) \in k_G^0(X/\rho).$$

Applying this relation to $1 = \xi^*(1) \in k_G^0(X/\rho)^S$, we find that $\xi^*\tau^*(1) = u$, so that u is a unit. Since $\tau^*\xi^*(z) = \chi(\xi)z$ for $z \in j_G^*(D/\pi)$, the conclusion follows.

With $B = X/\pi$ and $F = \pi/\rho$, Theorem 5.13 gives three criteria for the requisite unit condition. While the first criterion is unfortunately the one relevant to classifying spaces $B(G,\pi)$ and thus to the theory of equivariant characteristic classes, the last two lead to the following omnibus theorem. When $G = e$, it contains Feshbach's results in [53] and their improvements along the lines of [54,II.2].

Theorem 6.10. Let j_G be a G-spectrum. Let X be a principal (G,π)-bundle, let $\rho \subset \pi$, and consider $\xi = \xi(\rho,\pi)\colon X/\rho \to X/\pi$. Assume that X/π is either G-connected with trivial fibre representation $G \to \pi$ at the basepoint or G-free.

(i) If ρ is the normalizer of a maximal torus in π, then

$$\xi^*: j_G^*(X/\pi) \longrightarrow j_G^*(X/\rho)^S$$

is an isomorphism.

(ii) If ρ is the inverse image in the normalizer of a maximal torus of a p-Sylow subgroup of the quotient Weyl group, then

$$\xi^*: j_G^*(X/\pi) \longrightarrow j_G^*(X/\rho)^S$$

is an isomorphism if j_G is p-local.

(iii) If ρ is a maximal torus in π with Weyl group W, then

$$\xi^*: j_G^*(X/\pi) \longrightarrow j_G^*(X/\rho)^W$$

is an isomorphism if j_G is localized away from $|W|$.

(iv) If ρ is normal and of finite index in π, then

$$\xi^*: j_G^*(X/\pi) \longrightarrow j_G^*(X/\rho)^{\pi/\rho}$$

is an isomorphism if j_G is localized away from $|\pi/\rho|$.

Moreover, with the assumption on the base space X/π dropped, ξ^* in (iii) and (iv) is still an epimorphism. The same conclusions apply with X replaced by any π-free $(G \times \pi)$-spectrum on which X coacts.

Proof. In (i), $\chi(\pi/\rho) = 1$. In (ii), $\chi(\pi/\rho)$ is prime to p. The last statement holds since Corollary 6.7 implies that $\xi^*\tau^*(y) = wy$ for stable elements y, where $w = |W|$ or $w = |\pi/\rho|$. For (iii), note that this formula already holds for W-fixed elements and thus implies that all such elements are stable. The rest is clear from Theorem 6.9.

Warning 6.11. When we say that a nonequivariant spectrum j is T-local for a set of primes T, we mean that its homotopy groups are all T-local or, equivalently, that the natural map $j \to j_T$ is an equivalence, where $j_T = j \wedge MZ_T$. Here MZ_T is the Moore spectrum $\Sigma^{-1}\Sigma^\infty S_T^1$, S_T^1 being the localization of S^1 at T. If we start with a general spectrum j and form its localization j_T, then $j_T^*(B) = j^*(B) \otimes Z_T$ for finite CW complexes B but not for general CW complexes, because localization at T fails to commute with infinite products and inverse limits. Our unit criteria of Lemma 5.12 depended heavily on the wedge axiom and so apply only to theories represented by local spectra, not to localized theories $j^*(?) \otimes Z_T$. Thus parts (ii) through (iv) of Theorem 6.10 do not apply to $j^*(?) \otimes Z_T$ without a finiteness condition on the base spaces. In fact, the (nonequivariant) K-theory of classifying spaces of finite groups provides obvious counterexamples. Note, however, that the surjectivity in parts (iii) and (iv) clearly does apply to $j^*(?) \otimes Z_T$ in full generality, by the proofs given. The moral seems to be that one should learn how to exploit part (i) rather than the more familiar but less general part (iii). This warning applies verbatim equivariantly.

Returning to our original context, we ask how generally there is a double coset formula for non-trivial extensions Γ. There is a reasonably satisfactory answer when Γ is a split extension $G \times_\gamma \pi$. Here we restrict attention to subgroups σ and ρ of π invariant under the action of G given by γ. We define $\Gamma_\sigma = G \times_\gamma \sigma$ and identify Γ_σ/σ with G. Here again II.4.8 and II.4.10 give isomorphisms of G-spectra over D/π

$$D/\sigma \cong (\Gamma \ltimes_{\Gamma_\sigma} D)/\pi \cong D\wedge_\pi (\Gamma/\Gamma_\sigma)^+$$

for a π-free Γ-spectrum D. Thus $\xi(\sigma,\pi)\colon D/\sigma \to D/\pi$ is a stable G-bundle and has a transfer $\tau(\sigma,\pi)$. We need some observations and notations to state the double coset formula in this context (and we shall leave verifications to the reader).

Define an action of G on $\sigma\backslash\pi/\rho$ by the formula

$$g(\sigma m\rho) = \sigma\gamma(g)(m)\rho$$

for $g \in G$ and $m \in \pi$. Then the orbit map $\sigma\backslash\pi/\rho \to (\sigma\backslash\pi/\rho)/G$ can be identified with the map $\sigma\backslash\pi/\rho \to \Gamma_\sigma\backslash\Gamma/\Gamma_\rho$ which sends $\sigma m\rho$ to $\Gamma_\sigma m\Gamma_\rho$. When γ is trivial, G acts trivially and this map is an identification.

For $m \in \pi$, define $\Gamma_m = \Gamma_\sigma \cap (\Gamma_\rho)^m$; explicitly,

$$\Gamma_m = \{(g,\gamma(g^{-1})(m)rm^{-1}) \mid r \in \rho \text{ and } \gamma(g^{-1})(m)rm^{-1} \in \sigma\}.$$

Define $H_m = \Gamma_m/\sigma\cap\rho^m$ and note that H_m embeds in $\Gamma_\sigma/\sigma = G$; when γ is trivial, this embedding is an isomorphism.

For a π-free Γ-spectrum D, $D/\sigma\cap\rho^m$ is an H_m-spectrum but not a G-spectrum. By II.4.8 and II.4.10, we have a stable G-bundle

$$G \ltimes_{H_m} (D/\sigma\cap\rho^m) \cong D\wedge_\sigma(\Gamma_\sigma/\Gamma_m)^+ \longrightarrow D/\sigma,$$

and we let

$$\tau'(\sigma\cap\rho^m,\sigma)\colon D/\sigma \longrightarrow G \ltimes_{H_m} (D/\sigma\cap\rho^m)$$

be its transfer. Since $(\Gamma_\rho)^m/\rho^m \cong G$, D/ρ^m is a G-spectrum and the H_m-map $\xi(\sigma\cap\rho^m,\rho^m)\colon D/\sigma\cap\rho^m \to D/\rho^m$ extends to a G-map

$$\xi'(\sigma\cap\rho^m,\rho^m)\colon G \ltimes_{H_m} (D/\sigma\cap\rho^m) \longrightarrow D/\rho^m.$$

With these notations, we have the following generalization of Theorem 6.3.

<u>Theorem 6.12</u>. Let $\pi = G \times_\gamma \pi$ and let σ and ρ be γ-invariant subgroups of π. Let $\{m\}$ be a set of representatives in π for the orbit type component manifolds M of the orbit space $(\sigma\backslash\pi/\rho)/G$ and let $\chi(M)$ be the internal Euler characteristic of M in $(\sigma\backslash\pi/\rho)/G$. Then, for any π-free Γ-spectrum D, the composite

$$D/\sigma \xrightarrow{\xi(\sigma,\pi)} D/\pi \xrightarrow{\tau(\rho,\pi)} D/\rho$$

is the sum over M of $\chi(M)$ times the composite

$$D/\sigma \xrightarrow{\tau'(\sigma\cap\rho^m,\sigma)} G \ltimes_{H_m} (D/\sigma\cap\rho^m) \xrightarrow{\xi'(\sigma\cap\rho^m,\rho^m)} D/\rho^m \xrightarrow{c_m} D/\rho.$$

<u>Proof</u>. In outline, the argument is the same as the proof of Theorem 6.3. One has a commutative diagram

$$\begin{tikzcd}
D\wedge_\sigma(\Gamma/\Gamma_\rho)^+ \cong (\Gamma \ltimes_{\Gamma_\sigma} D)\wedge_\pi(\Gamma/\Gamma_\rho)^+ \arrow[r, "\phi\wedge_\pi 1"] \arrow[d, "\xi"'] & D\wedge_\pi(\Gamma/\Gamma_\rho)^+ \cong D/\rho \arrow[d, "\xi(\rho,\pi)"] \\
D/\sigma \arrow[r, "\xi(\sigma,\pi)"] & D/\pi
\end{tikzcd}$$

and one evaluates $\tau(\xi)$ by use of Theorem 6.2. Since the fibre of ξ is Γ/Γ_ρ regarded as a Γ_σ-space, it is the double coset space $\Gamma_\sigma\backslash\Gamma/\Gamma_\rho$ rather than $\sigma\backslash\pi/\rho$ which enters at this point. The remaining details are a bit tedious but straightforward and are left to the reader.

There is an analog of Corollary 6.4 obtained by taking D to be $\Sigma^\infty E(G,\gamma,\pi)^+$, where $E(G,\gamma,\pi)$ is the universal principal (G,γ,π)-bundle. Here $E(G,\gamma,\pi)/\rho$ is a classifying G-space $B(G,\gamma,\rho)$ for any γ-invariant $\rho \subset \pi$ (but of course ρ^m and $\sigma\cap\rho^m$ are generally not γ-invariant).

<u>Remark 6.13</u>. A check of details shows that the only properties of $\sigma \subset \pi$ and $\Gamma_\sigma \subset \Gamma$ used in proving Theorem 6.12 are that σ is normal in Γ_σ, $\Gamma_\sigma \cap \pi = \sigma$, and $(\Gamma_\sigma)\pi = \Gamma$. We say that such a group Γ_σ is a π-complement of σ in Γ; it is not uniquely determined. Provided only that both σ and ρ have π-complements, the statement and proof of the theorem remain valid without the requirement that Γ be a split extension. The only subtle point here is the specification of the action of G on $\sigma\backslash\pi/\rho$. One identifies G with Γ_σ/σ, uses $\Gamma = (\Gamma_\rho)\pi$ to write an element of G in the form $yn\sigma$ for $y \in \Gamma_\rho$ and $n \in \pi$, and sets $(yn\sigma)(\sigma m\rho) = \sigma(ynmy^{-1})\rho$.

The following remarks apply in the generality of the previous one.

<u>Remarks 6.14</u>. (i) By the transitivity theorem of the next section, the transfer $\tau'(\sigma\cap\rho^m,\sigma)$ factors as the composite

$$D/\sigma \xrightarrow{\tau} G \ltimes_{H_m} (D/\sigma) \xrightarrow{1\ltimes\tau(\sigma\cap\rho^m,\sigma)} G \ltimes_{H_m} (D/\sigma\cap\rho^m).$$

Here τ is specified in terms of the pretransfer G-map $\tau(G/H_m)$ as

$$D/\sigma \simeq D/\sigma\wedge S \xrightarrow{1\wedge\tau(G/H_m)} (D/\sigma)\wedge\Sigma^\infty(G/H_m)^+ \simeq G\ltimes_{H_m}(D/\sigma).$$

(ii) If we forget about G in Theorem 6.3, then it and Theorem 6.12 give two different calculations of $\tau(\rho,\pi)\xi(\sigma,\pi)$ as a nonequivariant map. The first uses $\sigma\backslash\pi/p$ directly while the second uses $(\sigma\backslash\pi/\rho)/G$ and compensates by use of the change of groups $G\ltimes_{H_m}(?)$. When G/H_m is finite, $G\ltimes_{H_m}E$ is nonequivariantly equivalent to $|G/H_m|$ copies of E and (i) shows that $\tau'(\sigma\cap\rho^m,\sigma)$ is $|G/H_m|$ copies of $\tau(\sigma\cap\rho^m,\sigma)$. This is compensated for by the difference in the Euler characteristics of the two formulas. When G/H_m is not finite, one can apply Theorem 2.10 nonequivariantly to τ in (i) to obtain the first formula from the second. In both cases, the idea is that the second decomposition results from the first by assembling nonequivariant transfers into equivariant maps.
(iii) In (i), $\tau'(\sigma\cap\rho^m,\sigma)$ vanishes if $W_{\Gamma}\Gamma_m$ is not finite and, perhaps more usefully, $\tau(\sigma\cap\rho^m,\sigma)$ vanishes if $W_{\Gamma_m\sigma}\Gamma_m$ is not finite. The latter group can be identified with the subgroup

$$\{s(\sigma\cap\rho^m)\,|\,s\in\sigma \text{ and } ysy^{-1}s^{-1}\in\sigma\cap\rho^m \text{ for all } y\in\Gamma_m\}$$

of $W_\sigma(\sigma\cap\rho^m)$.

§7. Transitivity relations

Dold studied the transitivity of his fixed point transfer in [46,§7]. Consideration of transitivity in terms of the Becker-Gottlieb definition seems not to appear in the literature. In our context, it seems most natural to consider transitivity for stable bundles built up from bundles of fibres.

To handle equivariance, we assume given a commutative diagram

$$\begin{tikzcd}
\sigma \arrow[r, equal] \arrow[d] & \sigma \arrow[d] & \\
\rho \arrow[r] \arrow[d, "\varepsilon"] & \Omega \arrow[r] \arrow[d, "\varepsilon"] & G \arrow[d, equal] \\
\pi \arrow[r] & \Gamma \arrow[r] & G
\end{tikzcd}$$

in which both the rows and the columns are extensions of compact Lie groups. In the classical case, these extensions are all to be trivial; that is, $\Gamma = G\times\pi$, $\rho = \pi\times\sigma$, and $\Omega = G\times\pi\times\sigma$. To fix universes, let U' be a complete Ω-universe and let $U = (U')^\sigma$. Then U is a complete Γ-universe and $U^\pi = (U')^\rho$ is a complete G-universe.

Let P be a finite σ-free Ω-CW complex with orbit space $K = P/\sigma$ and let

J be any finite Ω-CW complex. (In this section, we use cell structures so we must at least restrict to finite Ω-CW homotopy types.) Define $F = P\times_\sigma J$. The resulting Γ-bundle $\zeta\colon F\to K$ with total group Ω, structural group σ, and fibre J is to be our bundle of fibres. In the classical case, P is to be a π-free $(\pi\times\sigma)$-CW complex and J a σ-CW complex, so that F and K are finite π-complexes and ζ is a (π,σ)-bundle with fibre J regarded by pullback as a $(G\times\pi,\sigma)$-bundle. (In practice, π does not act effectively on K, but this causes no problems by Example 4.3 (ii).)

As usual, we assume given a π-free Γ-spectrum D indexed on U^π. Since F and K are finite Γ-spaces, our Γ-bundle ζ induces a map

$$\begin{tikzcd}
D\wedge_\pi F^+ \arrow[rr, "1\wedge_\pi\zeta"] \arrow[dr, "\xi"'] & & D\wedge_\pi K^+ \arrow[dl, "\xi'"] \\
& D/\pi &
\end{tikzcd}$$

of stable G-bundles, and of course ξ and ξ' have transfer G-maps $\tau(\xi)$ and $\tau(\xi')$. With these notations and hypotheses, our transitivity theorem reads as follows.

Theorem 7.1. (i) The following diagram commutes in $\bar{h}\Gamma\mathscr{S}U$.

$$\begin{tikzcd}
& S \arrow[dl, "\tau(K)"'] \arrow[dr, "\tau(F)"] & \\
\Sigma^\infty K^+ \arrow[rr, "\tau(\zeta)"] & & \Sigma^\infty F^+
\end{tikzcd}$$

(ii) The G-map $1\wedge_\pi\zeta\colon D\wedge_\pi F^+\to D\wedge_\pi K^+$ is a stable G-bundle with total group Ω, structural group ρ, and fibre J and the following diagram commutes in $\bar{h}G\mathscr{S}U^\pi$.

$$\begin{tikzcd}
& D/\pi \arrow[dl, "\tau(\xi')"'] \arrow[dr, "\tau(\xi)"] & \\
D\wedge_\pi K^+ \arrow[rr, "\tau(1\wedge_\pi\zeta)"] & & D\wedge_\pi F^+
\end{tikzcd}$$

Proof. (i) If P is given as a pushout of a cellular inclusion $P_3\to P_1$ and a cellular map $P_3\to P_2$, where the P_i are all σ-free Ω-CW complexes, then F and K inherit pushout structures and the map $J\to *$ induces a map from the pushout diagram for F to the pushout diagram for K. By additivity and naturality, the result for ζ will follow from the result for its pullbacks over the P_i/σ. By induction on the number of cells of P and by further use of additivity and homotopy invariance to handle spheres and cells, we see that the result will hold for all P if it holds for $P = \Omega/\Psi$, where $\Psi\cap\sigma = e$. Let Ω/Ψ have fibre representation $\alpha\colon\Lambda\cong\Psi\subset\Omega$, where $\Lambda\subset\Gamma$. Then, by Lemma 5.10,

$$\tau(\zeta)\colon \Sigma^\infty(\Omega/\Psi\sigma)^+ \longrightarrow \Sigma^\infty((\Omega/\Psi)\times_\sigma J)^+$$

can be identified with

$$1 \ltimes_\Lambda \tau(\alpha^* J)\colon \Gamma \ltimes_\Lambda S \longrightarrow \Gamma \ltimes_\Lambda \Sigma^\infty \alpha^* J^+.$$

Therefore (i) holds in this case by Lemma 2.8.

(ii) Let ε^*D denote D regarded as an Ω-spectrum. We observe first that $\varepsilon^*D\wedge P^+$ is a ρ-free Ω-spectrum. In fact, by consideration of product cells, the verification quickly reduces to the space level observation that $\varepsilon^*(\Gamma/\Lambda)\times\Omega/\Psi$ is ρ-free if $\Lambda\cap\pi = e$ and $\Psi\cap\sigma = e$ (by a simple check from our initial diagram of extensions). Since the quotient homomorphism $\Omega\to G$ factors through Γ, we find by use of II.4.10 that the G-map

$$1\wedge_\pi\zeta\colon D\wedge_\pi F^+ \longrightarrow D\wedge_\pi K^+$$

can be identified with the evident G-bundle

$$(\varepsilon^*D\wedge P^+)\wedge_\rho J^+ \longrightarrow (\varepsilon^*D\wedge P^+)/\rho.$$

Let $i\colon U^\pi\to U$, $i'\colon U^\pi = (U')^\rho \to U'$, and $j\colon U = (U')^\sigma\to U'$ be the inclusions, so that $i' = ji$. If we smash the diagram of part (i) with i_*D, we obtain a commutative diagram

$$\begin{tikzcd} & i_*D\wedge S \arrow[dl, "1\wedge\tau(K)"'] \arrow[dr, "1\wedge\tau(F)"] & \\ i_*D\wedge\Sigma^\infty K^+ \arrow[rr, "1\wedge\tau(\zeta)"] & & i_*D\wedge\Sigma^\infty F^+ \end{tikzcd}$$

in $\bar{h}\Gamma\mathscr{S}U$. Referring back to Definition 3.1, we see that this diagram arises by application of i_* to a diagram of the form

$$\begin{tikzcd} & D \arrow[dl, "\tilde{\tau}'"'] \arrow[dr, "\tilde{\tau}"] & \\ D\wedge K^+ \arrow[rr, "\mu"] & & D\wedge F^+ \end{tikzcd}$$

in $\bar{h}\Gamma\mathscr{S}U^\pi$ and that $\tilde{\tau}'/\pi = \tau(\xi')$ and $\tilde{\tau}/\pi = \tau(\xi)$. Similarly, $\tau(\zeta) = \tilde{\tau}(\zeta)/\sigma$ for a map $\tilde{\tau}(\zeta)$ in $\bar{h}\Omega\mathscr{S}U$ such that

$$j_*\tilde{\tau}(\zeta) = 1\wedge\tau(J)\colon \Sigma^\infty P^+\wedge S \longrightarrow \Sigma^\infty P^+\wedge\Sigma^\infty J^+$$

in $\bar{h}\Omega\mathscr{S}U'$. Therefore $\mu = \nu/\sigma$ for a map ν such that

$$i'_*\nu = 1\wedge 1\wedge\tau(J)\colon i'_*\varepsilon^*D\wedge P^+\wedge S \longrightarrow i'_*\varepsilon^*D\wedge P^+\wedge\Sigma^\infty J^+.$$

By Definition 3.1 again, $\mu/\pi = \nu/\rho$ is the transfer $\tau(1\wedge_\pi\zeta)$, and the desired diagram is obtained from the previous one by passage to orbits over π.

In view of Theorems 6.1 and 6.2, the following multiplicativity formulas for the Euler classes of bundles are easy consequences of part (i) of the theorem and the naturality of the transfer on pullbacks. Nonequivariantly, results like this are usually proven using the Serre spectral sequence.

Corollary 7.2. Let $j_i\colon \Gamma/\Lambda_i \subset \Gamma/\Lambda_i\times e^{n_i}\to K$ be the inclusion of an orbit in the i^{th} cell for some enumeration of the cells of K and let $\zeta_i\colon F_i\to\Gamma/\Lambda_i$ be the pullback of $\zeta\colon F\to K$ along j_i. Then $\chi(F) = \sum_i(-1)^{n_i}\chi(\zeta_i)\tau(\Gamma/\Lambda_i)$.

Corollary 7.3. Let $j_m\colon \Gamma/\Lambda\to K$ be the inclusion of an orbit in the orbit type component M and let $\zeta_m\colon F_m\to\Gamma/\Lambda$ be the pullback of $\zeta\colon F\to K$ along j_m. Then $\chi(F) = \sum_M \chi(M)\chi(\zeta_m)\tau(\Gamma/\Lambda)$.

Here Lemma 5.10 explains how to compute the relevant Euler characteristics of bundles over orbits.

We give some examples of situations in which part (ii) of the theorem applies.

Examples 7.4. (i) If we prove Corollary 5.3 directly, then it and transitivity imply Theorem 5.2. Here the bundle of fibres is just the projection $F_1\times F_2\to F_2$, its structural group σ being trivial, and we start with the extension $\pi_1\times\pi_2\to\Gamma\to G$ used in Theorem 5.2. In fact, for any $(\pi_1\times\pi_2)$-free Γ-spectrum D, not necessarily of the form $D_1\wedge D_2$, we obtain transitivity for the transfers associated to the diagram

$$\begin{tikzcd} D\wedge_{\pi_1\times\pi_2}(F_1\times F_2)^+ \arrow[r] \arrow[d] & D\wedge_{\pi_1\times\pi_2}F_2^+ \arrow[d, phantom, "\cong" rotate=90] \\ D/\pi_1\times\pi_2 & D/\pi_1\wedge_{\pi_2}F_2^+ \arrow[l] \end{tikzcd}$$

There is also a version of this example for the external smash products, relating the external version of Corollary 5.3 and transitivity to Theorem 5.1.

(ii) In the classical case, with $P = \pi$ and J a σ-CW complex, we obtain transitivity for the diagram of transfers associated to the diagram

$$\begin{tikzcd} D\wedge_\pi(\pi\times_\sigma J)^+ \arrow[r, phantom, "\cong"] \arrow[d] & D\wedge_\sigma J^+ \arrow[d] \\ D/\pi & D/\sigma. \arrow[l, "\xi(\sigma,\pi)"'] \end{tikzcd}$$

(iii) For a general Γ, suppose that $\sigma \subset \pi$ has a π-complement $\Gamma_\sigma \subset \Gamma$, as in Remark 6.13. Replacing $\sigma \subset \rho \subset \Omega$ in our original diagram of extensions with $\Gamma_\sigma \subset \pi \times \Gamma_\sigma \subset \Gamma \times \Gamma_\sigma$, taking $P = \Gamma$ with $(\Gamma \times \Gamma_\sigma)$-action given by the left action of Γ and right action of Γ_σ, and letting J be a finite Γ_σ-CW complex, we obtain transitivity for the diagram of transfers associated to the diagram

$$\begin{tikzcd} D\wedge_\pi(\Gamma \times_{\Gamma_\sigma} J)^+ \arrow[d] \arrow[r, phantom, "\cong"] & D\wedge_\sigma J^+ \arrow[d] \\ D/\pi & D/\sigma. \arrow[l, "\xi(\sigma,\pi)"'] \end{tikzcd}$$

(iv) The theorem implies the transitivity of the transfers associated to any composite of finite G-covers. To see this, suppose given finite G-covers

$$(*) \qquad Y \xrightarrow{\lambda} Z \xrightarrow{\psi} B, \qquad \xi = \psi\lambda.$$

Let ψ have fibre $K = \{1,\cdots,n\}$ and λ have fibre $J = \{1,\cdots,m\}$ and let $F = K \times J$. The projection $\zeta\colon F \to K$ may be viewed as a $(\Sigma_n\int\Sigma_m, \Sigma_m)$-bundle, where the wreath product $\Sigma_n\int\Sigma_m$ acts on F via

$$(\sigma,\tau_1,\cdots,\tau_n)(k,j) = (\sigma(k),\tau_k(j))$$

and acts on K through the projection $\Sigma_n\int\Sigma_n \to \Sigma_n$. While ξ is of course a (G,Σ_{mn})-bundle, the factorization $\xi = \psi\lambda$ implies a reduction of its structural group to $\Sigma_n\int\Sigma_m$. In fact, the associated principal $(G,\Sigma_n\int\Sigma_m)$-bundle X of ξ is the space of mn-tuples $(y_{k,j}) \in Y^F$ such that the $y_{k,j}$ together comprise a fibre $\xi^{-1}(b)$ for some $b \in B$ and the $y_{k,j}$ for fixed k together comprise a fibre $\lambda^{-1}(z_k)$ for some $z_k \in \psi^{-1}(b)$. It is easy to check that $(*)$ may be identified with the composite

$$X \times_{\Sigma_n\int\Sigma_m} F \longrightarrow X \times_{\Sigma_n\int\Sigma_m} K \longrightarrow X/\Sigma_n\int\Sigma_m.$$

(v) Thinking nonequivariantly, with $G = e$, suppose given a σ-bundle $\lambda\colon Y \to Z$ and a π-bundle $\psi\colon Z \to B$ (where of course the indicated groups are the structure groups). Let $\zeta\colon F \to K$ be the σ-bundle obtained by restricting λ to a fibre K of ψ. The theorem requires the π action on K to lift appropriately to F. Without some such restriction, we would have no control over the structure group of the composite "bundle" $\psi\lambda$, which might not even be a compact Lie group. (It would require some work even to determine exactly when $\psi\lambda$ is necessarily a bundle.)

§8. Cohomological transports

As usual, we assume given $G = \Gamma/\pi$ and a complete Γ-universe U. We shall construct and study certain transfer-like homomorphisms

$$j_G^*(D\wedge_\pi E) \longrightarrow j_G^*(D/\pi),$$

where D is a π-free Γ-spectrum indexed on U^π and E is a finite Γ-spectrum indexed on U. With $E = \Sigma^\infty F^+$ for a finite Γ-space F, the transfer will be a special case.

We must first make sense of the relevant smash products. Let $i\colon U^\pi \to U$ be the inclusion. Since D is π-free, $i_*D\wedge E$ is π-free. By II.1.8, the natural map $i_*i^*E \to E$ is an $\mathscr{F}(\pi)$-equivalence. By II.2.2, it thus induces a Γ-equivalence

$$i_*(D\wedge i^*E) \simeq i_*D\wedge i_*i^*E \longrightarrow i_*D\wedge E.$$

Therefore $D\wedge i^*E$ is an explicit model for the π-free Γ-spectrum which represents $i_*D\wedge E$ in the universe U^π. The existence and uniqueness (up to equivalence) of such a spectrum was proven in II.2.8 (ii). By abuse of notation, we agree to write $D\wedge E$ for $D\wedge i^*E$ throughout this section and the next. We shall usually be dealing with the G-spectrum $D\wedge_\pi E$ obtained by passage to orbits over π. Since passage to orbits presupposes use of U^π, it should be easy to remember that $D\wedge E$ is indexed on U^π and not on U. It follows directly from II.2.8 and the study of smash products in II§3 that $D\wedge E$ inherits good formal properties from $i_*D\wedge E$. We catalog what we need. (For this, E need not be finite.)

Lemma 8.1. (i) $D\wedge E$ is functorial in D with respect to maps in $\bar{h}\Gamma\mathscr{S}U^\pi$ and functorial in E with respect to maps in $\bar{h}\Gamma\mathscr{S}U$.

(ii) For π-free $D_i \in \Gamma\mathscr{S}U^\pi$ and for $E_i \in \Gamma\mathscr{S}U$, there is a natural equivalence

$$(D_1\wedge D_2)\wedge(E_1\wedge E_2) \simeq (D_1\wedge E_1)\wedge(D_2\wedge E_2).$$

(iii) For $C \in \Gamma\mathscr{S}U^\pi$, there is a natural equivalence

$$(D\wedge E)\wedge C \simeq (D\wedge C)\wedge E;$$

if $C \in G\mathscr{S}U^\pi$, then this equivalence passes to orbits to give

$$(D\wedge_\pi E)\wedge C \simeq (D\wedge C)\wedge_\pi E.$$

(iv) For Γ-spaces F, there is a natural equivalence

$$D\wedge\Sigma^\infty F^+ \simeq D\wedge F^+.$$

(v) For π-free Γ-spaces X, there is a natural equivalence

$$\Sigma^\infty X^+\wedge E \simeq X \ltimes E.$$

(vi) A coaction $\Delta: D \to X^+\wedge D$ induces a coaction, also denoted Δ,

$$(D\wedge E)\wedge C \xrightarrow{(\Delta\wedge 1)\wedge 1} ((X^+\wedge D)\wedge E)\wedge C \simeq X^+\wedge((D\wedge E)\wedge C),$$

where $C \in \Gamma\mathscr{S}U^\pi$ and the equivalence comes from two applications of (iii).

Proof. Only (iv) and (v) require comment. For (iv), simply note that

$$i_*(D\wedge F^+) \cong (i_*D)\wedge F^+ \simeq i_*D\wedge\Sigma^\infty F^+$$

by II.1.4 and II.3.12. In (v), $X \ltimes E \in \Gamma\mathscr{S}U^\pi$ denotes the twisted half smash product to be constructed in chapter VI, and it is essential not to confuse this with the ordinary half smash product $X^+\wedge E \in \Gamma\mathscr{S}U$. By VI.1.17, II.1.4, and II.3.12,

$$i_*(X \ltimes E) \simeq X^+\wedge E \simeq i_*\Sigma^\infty X^+\wedge E,$$

and it follows that

$$X \ltimes E \simeq \Sigma^\infty X^+\wedge i^*E \simeq X^+\wedge i^*E.$$

We shall make considerable use of the spectra $X \ltimes E$, but we only need the formal properties following from the equivalences just displayed, not the explicit construction.

We can now define the homomorphisms we wish to study.

Definition 8.2. Let $j_G \in G\mathscr{S}U^\pi$ be a (left) module spectrum over a (not necessarily commutative) ring spectrum $k_G \in G\mathscr{S}U^\pi$, let $\Delta: D \to X^+\wedge D$ be a coaction of a π-free Γ-space X on a π-free Γ-spectrum $D \in \Gamma\mathscr{S}U^\pi$, and let $E \in \Gamma\mathscr{S}U$ be a finite Γ-spectrum with canonical duality map $\eta: S \to E\wedge DE$. Define the intertwining δ of Δ and η to be the composite G-map

$$\delta: D/\pi \cong D\wedge_\pi S \xrightarrow{\Delta\wedge_\pi \gamma\eta} (X^+\wedge D)\wedge_\pi(DE\wedge E) \longrightarrow (X \ltimes_\pi DE)\wedge(D\wedge_\pi E),$$

where the second map is induced by the equivalence of Lemma 8.1 (ii). For $\theta \in k_G^0(X \ltimes_\pi DE)$, define the "cohomological transport of θ"

$$\tau(\theta): j_G^*(D\wedge_\pi E) \longrightarrow j_G^*(D/\pi)$$

by sending a class $f: D\wedge_\pi E \to \Sigma^\alpha j_G$ of degree $\alpha \in RO(G)$ to the class

$$D/\pi \xrightarrow{\delta} (X \ltimes_\pi DE)\wedge(D\wedge_\pi E) \xrightarrow{\theta\wedge f} k_G\wedge\Sigma^\alpha j_G \longrightarrow \Sigma^\alpha j_G,$$

where the last map is given by the action of k_G on j_G. Similarly, define the "homological transport of θ"

$$\tau_*(\theta): j_*^G(D/\pi) \longrightarrow j_*^G(D\wedge_\pi E)$$

by sending a class $f: S^\alpha \to D/\pi\wedge j_G$ to the class

$$S^\alpha \xrightarrow{f} D/\pi\wedge j_G \xrightarrow{(\theta\wedge 1)\delta\wedge 1} k_G\wedge(D\wedge_\pi E)\wedge j_G \xrightarrow{\gamma\wedge 1} (D\wedge_\pi E)\wedge k_G\wedge j_G \longrightarrow (D\wedge_\pi E)\wedge j_G.$$

We shall often apply $\tau(\theta)$ with D replaced by $D\wedge C$ for some $C \in G\mathscr{S}U^\pi$, and we then use Lemma 8.1 (iii) to rewrite it in the form

$$\tau(\theta): j_G^*((D\wedge_\pi E)\wedge C) \longrightarrow j_G^*(D/\pi\wedge C).$$

We shall not discuss homological transports in detail since their behavior is parallel to that of cohomological transports; we delete the adjective henceforward.

In our earlier work, we always restricted attention to commutative ring spectra. We allow non-commutativity here since we shall exploit the function ring spectrum $k_G = F(j_G,j_G)$ in the next section. For calculational, as opposed to theoretical, applications, commutativity is essential, and some of our results below will require it. Again, the most important case calculationally is $D = \Sigma^\infty X^+$, but the case of interest for the theory of the next section is $X = E\mathscr{F}(\pi)$ with its natural coaction on arbitrary D given in Example 5.9 (iii).

Transports include all of the generalizations of the transfer mentioned in Remarks 3.2 (iii).

Examples 8.3 (i) We may take $k_G = S$ for any j_G, using the unit equivalence $S\wedge j_G \simeq j_G$ as the module action. Here $\tau(\theta)$ is just the induced homomorphism $[(\theta\wedge 1)\delta]^*$.

(ii) Let $f: S \to E$ be any map in $\bar{h}\Gamma\mathscr{S}U$ (such as a pretransfer $\tau: S \to \Sigma^\infty F^+$) and define $\theta(f): X \ltimes_\pi DE \to S$ to be the composite

$$X \ltimes_\pi DE \xrightarrow{1\ltimes_\pi Df} X \ltimes_\pi S \simeq \Sigma^\infty(X/\pi)^+ \xrightarrow{\rho} S,$$

where ρ is the collapse map. A little diagram chase shows that

$$(\theta(f)\wedge 1)\delta = 1\wedge_\pi f\colon D/\pi = D\wedge_\pi S \longrightarrow D\wedge_\pi E.$$

(iii) With $E = \Sigma^\infty F^+$, we obtain a transport

$$\tau(\theta)\colon j_G^*(D\wedge_\pi F^+) \longrightarrow j_G^*(D/\pi)$$

for each class $\theta \in k_G^0(X \ltimes_\pi DF^+)$. We sometimes refer to $\tau(\theta)$ as the generalized transfer determined by θ. By (i) and (ii), the standard transfer of Definition 3.1 and the twisted transfer of Remarks 3.2 (iii) are special cases.

(iv) For $\alpha \in RO(\Gamma)$, we can replace E by $\Sigma^\alpha E$ and so obtain transports

$$\tau(\theta)\colon j_G^*(D\wedge_\pi \Sigma^\alpha E) \longrightarrow j_G^*(D/\pi)$$

from classes $\theta \in k_G^0(X \ltimes_\pi \Sigma^\alpha E)$. Applying this and (ii) to the dimension shifting pretransfers $S \to \Sigma^{-A}\Sigma^\infty F^+$ of Remark 2.13, we obtain the dimension shifting transfers of Remark 3.2 (iii). If $\alpha \in RO(G)$, then Σ^α permutes through our constructions and we obtain

$$\tau(\theta)\colon j_G^{*-\alpha}(D\wedge_\pi E) \longrightarrow j_G^*(D/\pi)$$

from $\theta \in k_G^\alpha(X \ltimes_\pi DE)$. All of our results below apply to these dimension shifting transports since they are obtained by specialization of our original framework. However, signs due to transpositions of suspension coordinates may be needed in some of the resulting cohomological formulas.

(v) With $E = S$, hence $DE = S$, we obtain a transport

$$\tau(\theta)\colon j_G^*(D/\pi) \longrightarrow j_G^*(D/\pi)$$

from each $\theta \in k_G^0(X/\pi)$. Clearly $\tau(\theta)$ is just left multiplication by θ with respect to the product induced by $\Delta\colon D/\pi \to X/\pi^+ \wedge D/\pi$. We shall shortly use this example to obtain a general Euler characteristic formula.

(vi) This section may be viewed as a disguised form of manipulations of Spanier-Whitehead duality for ex-spectra. The spectrum $X \ltimes_\pi DF^+$ is the ex-spectrum over X/π Spanier-Whitehead dual to the suspension ex-spectrum associated to the bundle $X \times_\pi F \to X/\pi$. Upon making this rigorous (which we shall not do here), we see that any map $X \times_\pi F \to X \times_\pi F$ over X/π induces a self map of $X \ltimes_\pi DF^+$. Composing with $\theta(\tau)\colon X \ltimes_\pi DF^+ \to S$, where τ is the pretransfer and $\theta(\tau)$ is as in (ii), we see that we can twist the standard transfer by any self-map of the bundle.

In the rest of this section, we run through some generalizations and analogs of the properties of the transfer developed in the previous sections. Since the proofs are easy and are much the same as those already given, they will be sketched or omitted. We shall not spell out hypotheses unduly; we simply assume whatever data are needed to make sense of the statements. We begin with naturality properties.

<u>Proposition 8.4</u>. Let $\theta \in k_G^0(X \ltimes_\pi DE)$, where X coacts on D.

(i) If X' coacts on D' and if $f\colon D' \to D$ and $g\colon X' \to X$ are Γ-maps such that $(g\wedge f)\Delta' \simeq \Delta f$, then the following diagram commutes, where $\theta' = (g \ltimes_\pi 1)^*(\theta)$.

$$\begin{tikzcd}
j_G^*(D\wedge_\pi E) \arrow[r, "(f\wedge_\pi 1)^*"] \arrow[d, "\tau(\theta)"'] & j_G^*(D'\wedge_\pi E) \arrow[d, "\tau(\theta')"] \\
j_G^*(D/\pi) \arrow[r, "(f/\pi)^*"] & j_G^*(D'/\pi)
\end{tikzcd}$$

(ii) If $\phi\colon E \to E'$ is a Γ-map, then the following diagram commutes, where $\theta' = (1 \ltimes_\pi D\phi)^*(\theta)$.

$$\begin{tikzcd}
j_G^*(D\wedge_\pi E') \arrow[rr, "(1\wedge_\pi \phi)^*"] \arrow[dr, "\tau(\theta')"'] & & j_G^*(D\wedge_\pi E) \arrow[dl, "\tau(\theta)"] \\
& j_G^*(D/\pi) &
\end{tikzcd}$$

(iii) If k_G' acts on j_G' and if $\mu\colon k_G \to k_G'$ is a ring map and $\nu\colon j_G \to j_G'$ is μ-equivariant, then the following diagram commutes.

$$\begin{tikzcd}
j_G^*(D\wedge_\pi E) \arrow[r, "\nu_*"] \arrow[d, "\tau(\theta)"'] & (j_G')^*(D\wedge_\pi E) \arrow[d, "\tau(\mu_*\theta)"] \\
j_G^*(D/\pi) \arrow[r, "\nu_*"] & (j_G')^*(D/\pi)
\end{tikzcd}$$

We next record analogs of the main change of groups results in section 4. Recall the discussion above Axiom 6.

<u>Proposition 8.5</u>. Let $\alpha\colon (\Lambda,\rho) \to (\Gamma,\pi)$ be a homomorphism of pairs with quotient inclusion $H \subset G$. Let D be a ρ-free Λ-spectrum indexed on $\alpha^* U^\pi$ and coacted on by a ρ-free Λ-space X. If

$$\theta \in k_H^0(X \ltimes_\rho D(\alpha^* E)) \cong k_G^0((\Gamma \times_\alpha X) \ltimes_\pi DE),$$

then the following diagram commutes.

$$\begin{tikzcd} j_H^*(D\wedge_\rho \alpha^*E) \arrow[r, phantom, "\cong"] \arrow[d, "\tau(\theta)"'] & j_G^*((\Gamma \ltimes_\alpha D)\wedge_\pi E) \arrow[d, "\tau(\theta)"] \\ j_H^*(D/\rho) \arrow[r, phantom, "\cong"] & j_G^*((\Gamma \ltimes_\alpha D)/\pi) \end{tikzcd}$$

Proof. For $E \in \Gamma\mathscr{S}U$, II.4.9 and II.4.10 imply a natural isomorphism

$$G \ltimes_H (D\wedge_\rho \alpha^*E) \cong (\Gamma \ltimes_\alpha D)\wedge_\pi E$$

in $G\mathscr{S}U^\pi$. This gives the isomorphisms of the statement. The coaction of X on D induces a coaction of $\Gamma \times_\alpha X$ on $\Gamma \ltimes_\alpha D$, namely the composite

$$\Gamma \ltimes_\alpha D \xrightarrow{1\ltimes_\alpha \Delta} \Gamma \ltimes_\alpha (X^+\wedge D) \xrightarrow{1\ltimes_\alpha(\eta\wedge 1)} \Gamma \ltimes_\alpha ((\Gamma \times_\alpha X)^+\wedge D) \cong (\Gamma\times_\alpha X)^+\wedge(\Gamma \ltimes_\alpha D),$$

where the isomorphism is given by II.4.9. Since the Λ-universe α^*U need not be complete, we must extend Definition 8.2 just as at the start of section 4 in order to make sense of $\tau(\theta)$ on the left. The conclusion follows by inspection of definitions.

There are also analogs of Lemma 4.1 and Proposition 4.4, which we leave to the reader. The analog of Proposition 4.6 is perhaps more interesting and reads as follows.

Proposition 8.6. Let $G = \Gamma/\pi$ and $J = G/N$. Let $\Lambda \supset \pi$ be the inverse image of N in Γ, so that $J = \Gamma/\Lambda$. Let $k_J \in J\mathscr{S}U^\Lambda$ be a ring spectrum and $j_J \in J\mathscr{S}U^\Lambda$ be a k_J-module spectrum. Assume given $\mathscr{F}(N)$-equivalences of G-spectra $\mu\colon \varepsilon^\# k_J \to k_G$ and $\nu\colon \varepsilon^\# j_J \to j_G$ such that μ is a ring map and ν is μ-equivariant. Let D be a Λ-free Γ-spectrum indexed on U^Λ and coacted on by a Λ-free Γ-space X. If

$$\theta \in k_J^0(X \ltimes_\Lambda DE) \cong k_G^0(X\ltimes_\pi DE),$$

Then the following diagram commutes (where $i\colon U^\Lambda \subset U^\pi$).

$$\begin{tikzcd} j_J^*(D\wedge_\Lambda E) \arrow[r, phantom, "\cong"] \arrow[d, "\tau(\theta)"'] & j_G^*((i_*D)\wedge_\pi E) \arrow[d, "\tau(\theta)"] \\ j_J^*(D/\Lambda) \arrow[r, phantom, "\cong"] & j_G^*((i_*D)/\pi) \end{tikzcd}$$

Proof. The isomorphisms come from II.8.1. Application of i_* to the coaction of X on D gives a coaction of X on i_*D. The conclusion follows by inspection of definitions.

Turning to products, we easily see that Theorem 5.1 directly generalizes to transforms by use of the obvious external pairings. We record the analog of the more useful internal product formula of Theorem 5.2.

Theorem 8.7. Assume given extensions $G = \Gamma_i/\pi_i$, $i = 1$ and $i = 2$, and let Γ be the equalizer of the projections $\Gamma_i \to G$. Let U_i be a complete Γ_i-universe so chosen that $U_1^{\pi_1} = U_2^{\pi_2} = U^{\pi_1\times\pi_2}$ for a complete Γ-universe U. Let D_i be a π_i-free Γ_i-spectrum indexed on $U_i^{\pi_i}$ and coacted on by a π_i-free Γ_i-space X_i, let E_i be a finite Γ_i-spectrum indexed on U_i, and let h_G and j_G be module spectra over a commutative ring spectrum k_G indexed on $U^{\pi_1\times\pi_2}$. If $\theta_i \in k_G^0(X_i \ltimes_{\pi_i} DE_i)$ and if

$$\theta_1\wedge\theta_2 \in k_G^0((X_1\times X_2)\ltimes_{\pi_1\times\pi_2} D(E_1\wedge E_2))$$

is their external product, then the following diagram commutes

$$\begin{tikzcd} h_G^*(D_1\wedge_{\pi_1}E_1)\otimes j_G^*(D_2\wedge_{\pi_2}E_2) \arrow[r, "\wedge"] \arrow[d, "\tau(\theta_1)\otimes\tau(\theta_2)"'] & (h_G\wedge j_G)^*((D_1\wedge D_2)\wedge_{\pi_1\times\pi_2}(E_1\wedge E_2)) \arrow[d, "\tau(\theta_1\wedge\theta_2)"] \\ h_G^*(D_1/\pi_1)\otimes j_G^*(D_2/\pi_2) \arrow[r, "\wedge"] & (h_G\wedge j_G)^*((D_1\wedge D_2)/(\pi_1\times\pi_2)) \end{tikzcd}$$

When $h_G = j_G = k_G$, the product on k_G can be used to replace $h_G\wedge j_G$ by k_G on the right.

Proof. Let $\Delta\colon \Gamma \to \Gamma_1\times\Gamma_2$ be the natural homomorphism and let $j\colon \Delta^*(U_1\oplus U_2)\to U$ be a Γ-linear isometry as in the proof of Theorem 5.2. The smash product $E_1\wedge E_2$ must be understood as obtained by application of the functor $j_*\Delta^*\colon (\Gamma_1\times\Gamma_2)\mathscr{S}(U_1\oplus U_2)\to \Gamma\mathscr{S}U$ to the obvious external smash product. With this interpretation, we have a natural equivalence of G-spectra

$$(D_1\wedge_{\pi_1}E_1)\wedge(D_2\wedge_{\pi_2}E_2) \simeq (D_1\wedge D_2)\wedge_{\pi_1\times\pi_2}(E_1\wedge E_2)$$

and, with E_1 and E_2 finite, a natural equivalence of Γ-spectra

$$D(E_1\wedge E_2)\simeq DE_1\wedge DE_2.$$

This makes sense of all the external products used. The coactions of the X_i on the D_i induce a coaction of $X_1 \times X_2$ on $D_1 \wedge D_2$ in the evident way, and the rest of the argument is like the proof of Theorem 5.2.

If k_G were not commutative, we would have to regard $\theta_1 \wedge \theta_2$ as a class in $(k_G \wedge k_G)$-cohomology. We would still obtain the diagram, but it would not be possible to use the product on k_G to replace $k_G \wedge k_G$ by k_G when $h_G = j_G = k_G$.

The first part of the following result is a direct generalization of Corollary 5.3, but the remaining parts are more useful.

Proposition 8.8. Let $\theta \in k_G^0(X \ltimes_\pi DE)$ and let $C \in G\mathscr{S}U^\pi$.

(i) Regard $F(C, j_G)$ as a k_G-module spectrum via the natural map $k_G \wedge F(C, j_G) \to F(C, k_G \wedge j_G)$ and the action of k_G on j_G. Then the following diagram commutes, where the isomorphisms are given by the smash product and function spectrum adjunction.

$$\begin{array}{ccc} j_G^*((D\wedge_\pi E)\wedge C) & \cong & F(C, j_G)^*(D\wedge_\pi E) \\ \downarrow{\scriptstyle \tau(\theta)} & & \downarrow{\scriptstyle \tau(\theta)} \\ j_G^*(D/\pi \wedge C) & \cong & F(C, j_G)^*(D/\pi) \end{array}$$

(ii) The following diagram commutes, where k_G is given its natural left action on itself in defining $\tau(\theta)$ on the left.

$$\begin{array}{ccc} k_G^*(D\wedge_\pi E) \otimes j_G^*(C) & \xrightarrow{\wedge} & j_G^*((D\wedge_\pi E)\wedge C) \\ \downarrow{\scriptstyle \tau(\theta)\otimes 1} & & \downarrow{\scriptstyle \tau(\theta)} \\ k_G^*(D/\pi) \otimes j_G^*(C) & \xrightarrow{\wedge} & j_G^*(D/\pi \wedge C) \end{array}$$

(iii) If k_G is commutative, the following diagram also commutes.

$$\begin{array}{ccc} k_G^*(C) \otimes j_G^*(D\wedge_\pi E) & \xrightarrow{\wedge} & j_G^*(C\wedge(D\wedge_\pi E)) \\ \downarrow{\scriptstyle 1\otimes\tau(\theta)} & & \downarrow{\scriptstyle \tau(\theta)} \\ k_G^*(C) \otimes j_G^*(D/\pi) & \longrightarrow & j_G^*(C\wedge D/\pi) \end{array}$$

When k_G is commutative, these diagrams lead to analogs of all of the formulas of Corollary 5.6. In the absence of commutativity, only some of the formulas work.

We need appropriate Euler characteristics to obtain an Euler characteristic formula. Here we restrict attention to generalized transfers, taking $E = \Sigma^\infty F^+$.

Definition 8.9. For $\theta \in k_G^0(X \ltimes_\pi DF^+)$, define

$$\chi(\theta) = (1 \ltimes_\pi D\rho)^*(\theta) \in k_G^0(X/\pi),$$

where $\rho: F^+ \to S$ is the collapse map. Here we use the identification $X \ltimes_\pi S \cong \Sigma^\infty(X/\pi)^+$ and recall that, on spaces, k_G^* is understood to mean unreduced cohomology.

Proposition 8.10. Let $\theta \in k_G^0(X \ltimes_\pi D(F^+))$. Then the composite

$$j_G^*(D/\pi) \xrightarrow{\xi^*} j_G^*(D\wedge_\pi F^+) \xrightarrow{\tau(\theta)} j_G^*(D/\pi)$$

is multiplication by $\chi(\theta) \in k_G^0(X/\pi)$. Moreover, if $\chi(\theta)$ is a unit for some class θ, then the diagram

$$j_G^*(D/\pi) \xrightarrow{\xi^*} j_G^*(D\wedge_\pi F^+) \underset{\pi_2^*}{\overset{\pi_1^*}{\rightrightarrows}} j_G^*(D\wedge_\pi (F\times F)^+)$$

is an equalizer, where the π_i are induced by the projections $F \times F \to F$.

Proof. By naturality, $\tau(\theta)\xi^*$ is the transport $\tau(\chi(\theta))$ associated to the map $\chi(\theta)$, and $\tau(\chi(\theta))$ is multiplication by $\chi(\theta)$ by Example 8.3 (v). For the second statement, suppose that $\chi(\theta)$ is a unit and define

$$f = \chi(\theta)^{-1}\tau(\theta)\colon j_G^*(D\wedge_\pi F^+) \longrightarrow j_G^*(D/\pi)$$

and

$$g = \chi(\theta)^{-1}\tau(\theta)\colon j_G^*(D\wedge_\pi (F\times F)^+) \longrightarrow j_G^*(D\wedge_\pi F^+).$$

Here the latter $\tau(\theta)$ is defined since X coacts on $D\wedge F^+$ (compare Lemma 8.1). Then $f\xi^* = 1$, $g\pi_1^* = 1$, and $g\pi_2^* = \xi^* f$, the last by naturality. This proves that the diagram is in fact a split equalizer in the sense of [92, p. 145].

In view of the criteria for $\chi(\theta)$ to be a unit given in Lemma 5.12, it is important to understand the behavior of generalized transfers for bundles over orbits. With the preliminaries in and above Lemma 5.10, Proposition 8.5 implies the following generalization of that result.

Lemma 8.11. Let $\Lambda \in \mathfrak{J}(\pi)$ determine the fibre representation $\alpha: H \to \Gamma$. If

$$\theta \in k_H^0(D(\alpha^* F^+)) \cong k_G^0(\Gamma/\Lambda \ltimes_\pi D(F^+)),$$

then the following diagram commutes.

$$\begin{array}{ccc} j_H^*(\alpha^* F) & \cong & j_G^*(\Gamma/\Lambda \times_\pi F) \\ \Big\downarrow{\scriptstyle\tau(\theta)} & & \Big\downarrow{\scriptstyle\tau(\theta)} \\ j_H^*(*) & \cong & j_G^*(\Gamma/\Lambda\pi) \end{array}$$

This can be used in conjunction with the evident generalization of Theorem 5.13.

Theorem 8.12. Let $\theta \in k_G^0(X \ltimes_\pi DF^+)$. With the Euler characteristic of $\xi: X \times_\pi F \to X/\pi$ and of its fibres replaced by the Euler characteristic $\chi(\theta) \in k_G^0(X/\pi)$ and its pullbacks along inclusions of orbits, the criteria for $\chi(\xi)$ to be a unit in Theorem 5.13 apply equally well to $\chi(\theta)$.

Additivity on fibres is a distinguishing property of the standard transfer and its dimension-shifting analogs, hence we cannot expect versions of the results of section 6 to hold for generalized transfers. (However, the second part of Proposition 8.10 is closely related to such results as Theorem 6.9.)

The main input of the transitivity theorem of section 7 was the transitivity of pretransfers given by Lemma 2.8. That input was processed by means of facts which can be generalized to the context of transports, provided that we first generalize the latter notion.

Definition 8.13. Let K and E be finite Γ-spectra and observe that there is a duality map

$$\eta': K \simeq S \wedge K \xrightarrow{\eta\wedge 1} E \wedge DE \wedge K \xrightarrow{\gamma} DE \wedge K \wedge E.$$

Exactly as in Definition 8.2, if X coacts on D there results an intertwining

$$\delta: D \wedge_\pi K \longrightarrow (X \ltimes_\pi (DE \wedge K)) \wedge (D \wedge_\pi E)$$

of $\Delta: D \to X^+ \wedge D$ and η', and any class $\theta \in k_G^0(X \ltimes_\pi (DE \wedge K))$ induces a cohomological transport

$$\tau(\theta): j_G^*(D \wedge_\pi E) \longrightarrow j_G^*(D \wedge_\pi K).$$

Everything done so far in this section can be generalized to this context by simply inserting K wherever appropriate. To study transitivity, observe that there is a canonical composite

$$\omega: X \ltimes_\pi DE \xrightarrow{1\ltimes(1\wedge\eta)} X \ltimes_\pi (DE \wedge K \wedge DK) \longrightarrow (X \ltimes_\pi (DE \wedge K)) \wedge (X \ltimes_\pi DK),$$

The second map being induced by the diagonal of X and a transposition. Assume given

$$\theta \in k_G^0(X \ltimes_\pi (DE \wedge K)) \quad \text{and} \quad \psi \in k_G^0(X \ltimes_\pi DK)$$

and define $\psi \circ \theta = \omega^*(\theta \wedge \psi) \in k_G^0(X \ltimes_\pi DE)$. Note that k_G need not be assumed to be commutative here. With these notations and hypotheses, we have a transitivity statement.

Proposition 8.14. The following diagram commutes.

$$\begin{array}{ccc} j_G^*(D \wedge_\pi E) & \xrightarrow{\tau(\theta)} & j_G^*(D \wedge_\pi K) \\ & {\scriptstyle\tau(\psi\circ\theta)}\searrow \quad \swarrow{\scriptstyle\tau(\psi)} & \\ & j_G^*(D/\pi) & \end{array}$$

A large part of the proof of Theorem 7.1 consists of the verification that, in the special case there, the relevant generalized transport $\tau(\theta)$ can be reevaluated as an ordinary transport (as in Definition 8.2). This part of the argument can be generalized, but we desist.

§9. Classification of transforms and uniqueness of transfers

We first show that all families of homomorphisms

$$(*) \qquad \tau: j_G^*(D \wedge_\pi E) \longrightarrow j_G^*(D/\pi)$$

which are suitably natural and stable in D arise from cohomological transports. We then use this result to give an axiomatic characterization of the transfer in j_G^*-cohomology.

Definition 9.1. Let $j_G \in G\mathscr{S}U^\pi$ and let $E \in \Gamma\mathscr{S}U$ be finite. A j_G^*-transform for

E is a family of homomorphisms (*), one for each π-free $D \in \Gamma\mathscr{S}U^{\pi}$, which satisfy the following cohomological versions of Axioms 1 and 2.

Axiom C.1. Naturality. The following diagram commutes for any map $f\colon D \to D'$ of π-free Γ-spectra.

$$\begin{tikzcd} j_G^*(D'\wedge_\pi E) \arrow[r, "(f\wedge_\pi 1)^*"] \arrow[d, "\tau"'] & j_G^*(D\wedge_\pi E) \arrow[d, "\tau"] \\ j_G^*(D'/\pi) \arrow[r, "(f/\pi)^*"] & j_G^*(D/\pi) . \end{tikzcd}$$

Axiom C.2. Stability. The following diagram commutes for any representation V of G and any π-free Γ-spectrum D.

$$\begin{tikzcd} j_G^{*+V}((\Sigma^V D)\wedge_\pi E) \arrow[r, phantom, "\cong"] \arrow[d, "\tau"'] & j_G^*(D\wedge_\pi E) \arrow[d, "\tau"] \\ j_G^{*+V}((\Sigma^V D)/\pi) \arrow[r, phantom, "\cong"] & j_G^*(D/\pi) . \end{tikzcd}$$

Here the unlabeled isomorphisms are given by suspension and application of the natural equivalence

$$(D\wedge C)\wedge_\pi E \simeq (D\wedge_\pi E)\wedge C$$

for $C \in G\mathscr{S}U^{\pi}$ (of Lemma 8.1) to $C = S^V$. More generally, we can use this equivalence to extend a j_G^*-transform τ to a family of homomorphisms

$$\tau\colon\ j_G^*((D\wedge_\pi E)\wedge C) \longrightarrow j_G^*(D/\pi\wedge C)$$

natural in both C and D. We use this extension to relate transforms under a product axiom. Let j_G be a k_G-module spectrum and suppose given j_G^* and k_G^*-transforms for E, both denoted τ.

Axiom C.2'. Commutation with products. The following diagram commutes for any π-free $D \in \Gamma\mathscr{S}U^{\pi}$ and any $C \in G\mathscr{S}U^{\pi}$.

$$\begin{tikzcd} k_G^*(D\wedge_\pi E) \otimes j_G^*(C) \arrow[r, "\wedge"] \arrow[d, "\tau\otimes 1"'] & j_G^*((D\wedge_\pi E)\wedge C) \arrow[d, "\tau"] \\ k_G^*(D/\pi) \otimes j_G^*(C) \arrow[r, "\wedge"] & j_G^*(D/\pi\wedge C) \end{tikzcd}$$

We say that a k_G^*-transform for E is multiplicative if Axiom C.2' holds for the product action of k_G on itself; here Axioms C.1 and C.2' imply Axiom C.2.

Proposition 9.2. Let τ be a j_G^*-transform for E. If k_G is the function ring spectrum $F(j_G, j_G)$, then the homomorphisms

$$\tau\colon\ k_G^*(D\wedge_\pi E) \cong j_G^*((D\wedge_\pi E)\wedge j_G) \longrightarrow j_G^*(D/\pi\wedge j_G) \cong k_G^*(D/\pi)$$

specify a multiplicative k_G^*-transform such that Axiom C.2' holds for j_G considered as a k_G-module under evaluation.

Proof. It is clear that τ for k_G^* satisfies Axioms C.1 and C.2 since τ for j_G^* does so. Observe that Axiom C.2 implies the corresponding axiom with V replaced by any $\beta \in RO(G)$. For the action of k_G on j_G, the diagram of Axiom C.2' can be rewritten in the following form, where $\alpha,\beta \in RO(G)$.

$$\begin{tikzcd} j_G^{\alpha}((D\wedge_\pi E)\wedge j_G) \otimes j_G^{\beta}(C) \arrow[r, "\wedge"] \arrow[d, "\tau\otimes 1"'] & j_G^{\alpha+\beta}((D\wedge_\pi E)\wedge C) \arrow[d, "\tau"] \\ j^{\alpha}(D/\pi\wedge j_G) \otimes j_G^{\beta}(C) \arrow[r, "\wedge"] & j^{\alpha+\beta}(D/\pi\wedge C) \end{tikzcd}$$

For $f\colon (D\wedge_\pi E)\wedge j_G \to \Sigma^{\alpha} j_G$ and $g\colon C \to \Sigma^{\beta} j_G$, $f\wedge g$ is the composite

$$(D\wedge_\pi E)\wedge C \xrightarrow{1\wedge g} (D\wedge_\pi E)\wedge \Sigma^{\beta} j_G \cong \Sigma^{\beta}((D\wedge_\pi E)\wedge j_G) \xrightarrow{\Sigma^{\beta} f} \Sigma^{\alpha+\beta} j_G,$$

and the axioms imply that $\tau(f\wedge g) = \tau(f)\wedge g$. If we replace C by $C\wedge j_G$ here, we obtain the diagram of Axiom C.2' for the action of k_G on itself.

Recall from Example 5.9 (iii) that $E\mathscr{J}(\pi)$ coacts naturally on π-free Γ-spectra D. Propositions 8.4 and 8.8 imply that we have the following examples of transforms.

Proposition 9.3. Let k_G be a ring spectrum and let $\theta \in k_G^0(E\mathscr{J}(\pi) \ltimes_\pi DE)$. For any k_G-module spectrum j_G, the cohomological transports

$$\tau(\theta)\colon j_G^*(D\wedge_\pi E) \longrightarrow j_G^*(D/\pi)$$

specify a j_G^*-transform for E such that Axiom C.2' holds.

We shall prove that all examples are of this form. Observe that $E\mathcal{J}(\pi) \ltimes DE$ is a π-free Γ-spectrum with π-orbit G-spectrum $E\mathcal{J}(\pi) \ltimes_\pi DE$.

<u>Definition 9.4</u>. Let $k_G \in G\mathscr{S}U^\pi$ be a ring spectrum and let $E \in \Gamma\mathscr{S}U$ be finite. Define the test class

$$\phi \in k_G^0((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E)$$

to be the composite of the unit $e\colon S \to k_G$ and the cohomotopy test class

$$(E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E \simeq E\mathcal{J}(\pi) \ltimes_\pi (DE\wedge E) \xrightarrow{1\ltimes_\pi \varepsilon} E\mathcal{J}(\pi) \ltimes_\pi S \simeq \Sigma^\infty B\mathcal{J}(\pi)^+ \xrightarrow{\rho} S,$$

where the equivalences are given by Lemma 8.1, ε is the canonical duality map, and ρ is the collapse map. Define the characteristic class

$$\theta(\tau) \in k_G^0(E\mathcal{J}(\pi) \ltimes_\pi DE)$$

of a k_G^*-transform τ for E to be the image of ϕ under

$$\tau\colon k_G^0((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E) \longrightarrow k_G^0(E\mathcal{J}(\pi) \ltimes_\pi DE).$$

<u>Theorem 9.5</u>. Let j_G be a module spectrum over a ring spectrum $k_G \in G\mathscr{S}U^\pi$ and let $E \in \Gamma\mathscr{S}U$ be finite. If τ is the k_G^*-transform for E given by the transports of a class $\theta \in k_G^0(E\mathcal{J}(\pi) \ltimes_\pi DE)$, then $\theta = \theta(\tau)$. Conversely, if θ is the characteristic class of a given k_G^*-transform for E and if Axiom C.2' holds for a j_G^*-transform τ for E, then $\tau = \tau(\theta)$. Therefore multiplicative k_G^*-transforms for E are in canonical bijective correspondence with elements of $k_G^0(E\mathcal{J}(\pi) \ltimes_\pi DE)$ and also, if $k_G = F(j_G, j_G)$, with j_G^*-transforms for E.

<u>Proof</u>. The last clause follows from Proposition 9.2. With $D = E\mathcal{J}(\pi) \ltimes DE$, Definition 8.2 gives an intertwining map

$$\delta\colon E\mathcal{J}(\pi) \ltimes_\pi DE \longrightarrow (E\mathcal{J}(\pi) \ltimes_\pi DE)\wedge((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E),$$

and an easy chase shows that $(1\wedge\phi)\delta \simeq 1$, where ϕ is the cohomotopy test class. The first statement follows by a comparison of definitions. For the converse, Definition 8.13 (with $K = E$) and Lemma 8.1 give an intertwining map

$$\delta\colon D\wedge_\pi E \longrightarrow (E\mathcal{J}(\pi) \ltimes_\pi (DE\wedge E))\wedge(D\wedge_\pi E) \simeq ((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E)\wedge(D\wedge_\pi E),$$

and another easy chase shows that $(\phi\wedge 1)\delta \simeq 1$ here. The axioms imply the commutativity of the following diagram.

$$\begin{tikzcd}
k_G^0((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E) \otimes j_G^*(D\wedge_\pi E) \arrow[r, "\tau\otimes 1"] \arrow[d] & k_G^0(E\mathcal{J}(\pi) \ltimes_\pi DE) \otimes j_G^*(D\wedge_\pi E) \arrow[d] \\
j_G^*(((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E) \wedge (D\wedge_\pi E)) \arrow[r, "\tau"] \arrow[d, "\delta^*"] & j_G^*((E\mathcal{J}(\pi) \ltimes_\pi DE) \wedge (D\wedge_\pi E)) \arrow[d, "\delta^*"] \\
j_G^*(D\wedge_\pi E) \arrow[r, "\tau"] & j_G^*(D/\pi)
\end{tikzcd}$$

If $\phi \in k_G^0((E\mathcal{J}(\pi) \ltimes DE)\wedge_\pi E)$ is the test class and $f \in j_G^*(D\wedge_\pi E)$, then $\delta^*(\phi\wedge f) = f$ by inspection and $\delta^*(\tau(\phi)\wedge f) = \tau(\theta(\tau))(f)$ by Definitions 8.2 and 9.4.

There is a dimension-shifting analog of j_G^*-transforms and an analogous classification of them in terms of dimension-shifting transports.

Taking $E = \Sigma^\infty F^+$, we may view the theorem as giving a classification of generalized transfers for stable bundles with fibre F. We shall use this result to prove that the standard transfer in j_G^*-cohomology is characterized by the cohomological versions of Axioms 1 through 6. To make sense of Axioms 3 through 5, we assume given a j_G^*-transform τ for $\Sigma^\infty F^+$ for each finite Γ-space F; these axioms then read as follows. We require finite Γ-spaces to be equivalent to finite Γ-CW complexes throughout the rest of this section.

<u>Axiom C.3. Normalization</u>. For any π-free Γ-spectrum D, $\tau\colon j_G^*(D/\pi) \to j_G^*(D/\pi)$ is the identity homomorphism.

<u>Axiom C.4. Fibre invariance</u>. The following diagram commutes for any π-free Γ-spectrum D and any equivalence $k\colon F \to F'$ of finite Γ-spaces.

$$\begin{tikzcd}
j_G^*(D\wedge_\pi (F')^+) \arrow[rr, "(1\wedge_\pi k)^*"] \arrow[dr, "\tau"'] & & j_G^*(D\wedge_\pi F^+) \arrow[dl, "\tau"] \\
& j_G^*(D/\pi) &
\end{tikzcd}$$

<u>Axiom C.5. Additivity on fibres</u>. If F is the pushout of a Γ-cofibration $F_3 \to F_1$ and a Γ-map $F_3 \to F_2$, where the F_i are finite Γ-spaces and if $i_k\colon D\wedge_\pi F_k^+ \to D\wedge_\pi F^+$ is induced by the canonical map $F_k \to F$, then

$$\tau = \tau i_1^* + \tau i_2^* - \tau i_3^*\colon j_G^*(D\wedge_\pi F^+) \longrightarrow j_G^*(D/\pi).$$

To make sense of Axiom 6, we assume given a j_H^*-transform τ for each finite Λ-space F whenever (Λ,ρ) is a subpair of (Γ,π) with quotient inclusion $H \subset G$. Since we assume that Λ is contained in Γ, our complete Γ-universe U is also complete as a Λ-universe, in contrast to the more general situation studied in section 4. In the context of (G,π)-bundles, where $\Gamma = G \times \pi$, it is sensible and sufficient to restrict attention to the case $H = G$ and $\Lambda = G \times \rho$.

Axiom C.6. Change of group invariance. If (Λ,ρ) is a subpair of (Γ,π) with quotient inclusion $H \subset G$, then the following diagram commutes for any ρ-free Λ-spectrum D and any finite Γ-space F.

$$\begin{array}{ccc} j_H^*(D\wedge_\rho F^+) & \cong & j_G^*((\Gamma \ltimes_\Lambda D)\wedge_\pi F^+) \\ \downarrow \tau & & \downarrow \tau \\ j_H^*(D/\rho) & \cong & j_G^*((\Gamma \ltimes_\Lambda D)/\pi) \end{array}$$

Definition 9.6. Let j_G be a G-spectrum. A j_G^*-transfer is a collection of homomorphisms

$$\tau\colon j_H^*(D\wedge_\rho F^+) \longrightarrow j_H^*(D/\rho),$$

one for each subpair (Λ,ρ) of (Γ,Γ) with quotient inclusion $H \subset G$, each ρ-free Λ-spectrum D, and each finite Λ-space F, such that Axioms C.1 through C.6 hold for each theory j_H^* (as ambient theory). If $j_G = k_G$ is a ring spectrum and Axiom C.2' also holds for each k_H^* (for the action of k_H on itself), we say that the transfer is multiplicative.

Here we let j_H denote j_G regarded as an H-spectrum and we observe that $F(j_G,j_G)$ regarded as an H-spectrum is $F(j_H,j_H)$. Proposition 9.2 extends immediately to the context of transfers.

Proposition 9.7. Let τ be a j_G^*-transfer and let $k_G = F(j_G,j_G)$. Then the homomorphisms

$$\tau\colon k_H^*(D\wedge_\rho F^+) \cong j_H^*((D\wedge_\rho F^+)\wedge j_H) \longrightarrow j_H^*(D/\rho\wedge j_H) \cong k_H^*(D/\rho)$$

specify a multiplicative k_G^*-transfer.

We shall prove that there is only one j_G^*-transfer.

Theorem 9.8. The only j_G^*-transfer is the standard one.

This is a consequence of an apparently weaker result.

Theorem 9.9. If k_G is a ring spectrum, then the only multiplicative k_G^*-transfer is the standard one.

To deduce Theorem 9.8, we simply note that Theorems 9.5 and 9.9 imply that any given j_G^*-transfer and the standard j_G^*-transfer are determined by the same characteristic classes in $k_H^0(E\mathscr{F}(\rho) \ltimes_\rho DF^+)$, $k_H = F(j_H,j_H)$, and are therefore equal.

Proof of Theorem 9.9. Assume given a multiplicative k_G^*-transfer τ. Proceeding inductively, using the descending chain condition on closed subgroups of Γ, it suffices to prove that the given transfer

$$\tau\colon k_G^*(D\wedge_\pi F^+) \longrightarrow k_G^*(D/\pi)$$

is the standard transfer for all π-free Γ-spectra D and all finite Γ-spaces F under the inductive hypothesis that

$$\tau\colon k_H^*(D\wedge_\rho F^+) \longrightarrow k_H^*(D/\rho)$$

is the standard transfer for all ρ-free Λ-spectra D and all finite Λ-spaces F when (Λ,ρ) is a proper subpair of (Γ,π) with quotient inclusion $H \subset G$. The cohomological version of Theorem 6.1 is an immediate consequence of our cohomological axioms, hence it suffices to prove that τ agrees with the standard transfer when F is an orbit Γ/Λ. (It is for this step that we require F to be equivalent to a finite Γ-CW complex.) If $\Lambda = \Gamma$, the conclusion is immediate from Axiom C.3. Thus assume that $\Lambda \neq \Gamma$, let $\rho = \Lambda \cap \pi$, and let $H = \Lambda/\rho$. By Theorem 9.5,

$$\tau = \tau(\theta)\colon k_G^*(D\wedge_\pi \Gamma/\Lambda^+) \longrightarrow k_G^*(D/\pi),$$

where θ is the characteristic class of τ, namely the image of the test class $\phi \in k_G^0((E\mathscr{F}(\pi) \ltimes D\Gamma/\Lambda^+)\wedge_\pi \Gamma/\Lambda^+)$ under

$$\tau\colon k_G^0((E\mathscr{F}(\pi) \ltimes D\Gamma/\Lambda^+)\wedge_\pi \Gamma/\Lambda^+) \longrightarrow k_G^0(E\mathscr{F}(\pi) \ltimes_\pi D\Gamma/\Lambda^+).$$

The standard transfer admits the same description. Embed Γ/Λ in a Γ-representation V and let L be the tangent representation of Λ at the identity coset of Γ/Λ. Then the dual of Γ/Λ^+ is $\Gamma \ltimes_\Lambda S^{-L}$. By II.4.9 (and Lemma 8.1), we have a canonical equivalence

$$E\mathcal{F}(\pi) \ltimes (\Gamma \ltimes_\Lambda S^{-L}) \simeq \Gamma \ltimes_\Lambda (E\mathcal{F}(\pi) \ltimes S^{-L}),$$

and $E\mathcal{F}(\pi) \ltimes S^{-L}$ is a ρ-free Λ-spectrum. By Axiom C.6, the last transfer can be identified with

$$\tau\colon k_H^0((E\mathcal{F}(\pi) \ltimes S^{-L}) \wedge_\rho \Gamma/\Lambda^+) \longrightarrow k_H^0(E\mathcal{F}(\pi) \ltimes_\rho S^{-L}),$$

and similarly for the standard transfer. The desired equality of transfers is now immediate from the induction hypothesis.

In principle, the proof gives an inductive construction of the standard transfer that makes no use of the pretransfer.

In practice, one generally starts with a j_G^*-transfer defined only when $D = \Sigma^\infty X^+$ for a π-free Γ-space X, perhaps with X restricted to be finite or finite dimensional. Here one must work with pairs (X,A) or with nondegenerately based π-free Γ-spaces Y. If τ is defined in the based context, we define τ on unbased pairs (X,A) by considering the unreduced cone $C(X,A)$. If τ is defined in the unbased context, we define τ for based spaces Y by exploiting the canonical equivalence

$$C(E\mathcal{F}(\pi) \times Y,\ E\mathcal{F}(\pi) \times \{*\}) \longrightarrow C(Y,\{*\}) \simeq Y.$$

(The infinite dimensionality of $E\mathcal{F}(\pi)$ is of no concern since we shall be working modulo $\lim^1$ terms.) Thus the based and unbased contexts are essentially equivalent. We shall discuss the based context for technical convenience, although the motivating applications deal with unbased pairs.

We claim that, modulo $\lim^1$ terms, a j_G^*-transform τ for E defined on based π-free (finite dimensional) Γ-complexes Y, that is, on $D = \Sigma^\infty Y$, extends uniquely to a j_G^*-transform defined on all D. To see this, let $\{W_n\}$ be an indexing sequence in the Γ-universe U (as in I§2). If $V_n = W_n^\pi$, then $\{V_n\}$ is an indexing sequence in U^π. For $D \in \Gamma\mathcal{S}U^\pi$, let D_n denote the component space $D(V_n)$. Since π acts trivially on U^π, the D_n are π-free if D is π-free. By I.4.9, we have a canonical equivalence

$$D \simeq \mathrm{tel}\ \Sigma^{-V_n}\Sigma^\infty D_n$$

and thus, if D is π-free, a canonical equivalence

$$D \wedge_\pi E \simeq \mathrm{tel}\ \Sigma^{-V_n}\Sigma^\infty D_n \wedge_\pi E$$

for $E \in \Gamma\mathcal{S}U$. We define

$$\hat{j}_G^*(D \wedge_\pi E) = \lim j_G^*(\Sigma^{-V_n}\Sigma^\infty D_n \wedge_\pi E)$$

and have a $\lim^1$ exact sequence

$$0 \longrightarrow \lim^1 j_G^{\alpha+V_n-1}(\Sigma^\infty D_n \wedge_\pi E) \longrightarrow j_G^\alpha(D \wedge_\pi E) \longrightarrow \hat{j}_G^\alpha(D \wedge_\pi E) \longrightarrow 0$$

When restricting attention to finite dimensional Y, we may replace the D_n by suitably large skeleta of the component spaces of a G-CW Ω-prespectrum approximating D (as in I§8) without altering any of the groups in sight. We define $\hat{j}_G$-transforms and $\hat{j}_G^*$-transfers exactly as in Definitions 9.1 and 9.6, replacing all represented cohomology groups by inverse limit cohomology groups. (Of course, using the natural map $j_G^* \to \hat{j}_G^*$, we could replace $\hat{j}_G^*$ by j_G^* in the domains of all homomorphisms.)

<u>Proposition 9.10</u>. A j_G^*-transform for E defined on $D = \Sigma^\infty Y$ for all based π-free (finite-dimensional) Γ-spaces Y induces a unique $\hat{j}_G^*$-transform for E defined on all π-free Γ-spectra D. The analogous statement for j_G^*-transfers also holds.

The proof is immediate by passage to limits. Our arguments above go through unchanged to give inverse limit versions of the classification of transforms and the axiomatization of the transfer. Using the proposition and restricting back to spaces, we obtain an axiomatization of the transfer for based bundles $Y \wedge_\pi F^+ \to Y/\pi$ or for unbased bundle pairs $(X \times_\pi F,\ A \times_\pi F) \to (X/\pi,\ A/\pi)$.

<u>Theorem 9.11</u>. The only j_G^*-transfer defined for based bundles with finite dimensional base spaces (or for unbased bundle pairs with finite dimensional base spaces) is the standard one.

The reader is referred to [85] for a version of the axiomatization of the transfer which applies when given a fixed rather than a variable fibre. Here Axioms C.3 - C.5 no longer make sense and a more sophisticated normalization axiom is used in their place.

V. The Burnside ring and splittings in equivariant homology theory

by L. G. Lewis, Jr., J. P. May, and J. E. McClure

When G is a finite group, the Burnside ring $A(G)$ is the Grothendieck ring obtained from the semi-ring of isomorphism classes of finite G-sets; addition and multiplication are given by disjoint union and Cartesian product. While $A(G)$ is a classical object, it is a fundamental insight of Segal [126] that $A(G)$ is isomorphic to the zero$\underline{\text{th}}$ stable homotopy group $\pi_0^G(S)$. In a series of papers [40-44], tom Dieck defined and studied the Burnside ring of a compact Lie group and generalized the isomorphism $A(G) \cong \pi_0^G(S)$ to that context.

Tom Dieck defined $A(G)$ in terms of the nonequivariant Euler characteristics of the fixed point spaces of compact G-ENR's (our finite G-spaces). We find it more illuminating to work with the equivariant Euler characteristics of finite G-spaces. The point is that these are defined as elements of $\pi_0^G(S)$, and their use clarifies the isomorphism $A(G) \cong \pi_0^G(S)$. (Tom Dieck defined equivariant Euler classes in $R(G) \cong K_0^G(S)$, but these invariants depend only on $\pi_0(G)$ and are too weak to serve as a basis for the construction of $A(G)$.)

We study equivariant Euler characteristics in section 1. Additivity on cofibre sequences reduces their calculation to the case of orbit spaces G/H, where a connection with degrees of maps between spheres is easily established. This allows the analysis of equivariant Euler characteristics in terms of nonequivariant Euler characteristics of fixed point spaces.

We define the Burnside ring $A(G)$, prove that it is isomorphic to $\pi_0^G(S)$, and observe that the isomorphism commutes with various natural homomorphisms in section 2. We analyze the prime ideals of $A(G)$ in section 3. We relate its idempotent elements to the perfect subgroups of G in section 4. Most of the results in these sections are due to Dress [48] when G is finite and to tom Dieck [40-44] in general, but we include several useful addenda. Section 4 ends with the generalization to compact Lie groups of a result of Araki [4] for finite groups which states that $e_L^G A(G)$ is isomorphic to $e_1^{WL}A(WL)$, where e_L^G is the idempotent determined by a perfect subgroup L (and 1 denotes the trivial perfect subgroup of $WL = NL/L$).

We study localizations of $A(G)$ in section 5, following Dress [48] and others [44,4,56,147] when G is finite and tom Dieck [40,41,44] in general. Here the two cases differ sharply. Say that a finite group is p-perfect if it admits no normal subgroup of index p. We allow the case $p = 0$, agreeing that any finite group is 0-perfect. Each p-perfect subgroup L of a finite group G determines a primitive idempotent $e_L^G \in A(G)_{(p)}$, and $A(G)_{(p)}$ is the product over conjugacy classes (L) of its subrings $e_L^G A(G)_{(p)}$, these being the localizations of $A(G)_{(p)}$ at its maximal ideals. Moreover, as pointed out by Araki [4], $e_L^G A(G)_{(p)}$ is isomorphic to $e_1^{WL}A(WL)_{(p)}$. (Of course, localization at (0) is rationalization.)

The situation for general compact Lie groups is much less satisfactory. Here $A(G)_{(p)}$ is not the product of its localizations at its maximal ideals and these localizations are not determined by idempotents. Nevertheless, we show that the rings $A(G)_{(p)}$, and also all modules over these rings, are determined sheaf theoretically by their localizations at maximal ideals and that these localizations can be computed in a reasonable algebraic way. We are grateful to Spencer Bloch for tutorials on the relevant sheaf theory. The ring $A(G)_{(0)}$ is absolutely flat (equivalently, von Neumann regular), and we include some general algebraic information to give a feel for such rings. We are grateful to Irving Kaplansky for tutorials on this material.

From our point of view, the force of this algebraic analysis is its implications in homology theory. We explain in section 6 how techniques of induction and reduction to subquotients simplify the calculation of the pieces into which homology and cohomology groups split along the splittings of $A(G)$ and its localizations.

By the work in chapter II, we can associate a J-spectrum E_J to a G-spectrum E_G for any subquotient group J of G. Generalizing a result of Araki [4] from finite groups to compact Lie groups, we prove that if L is a perfect subgroup of G, then

$$e_L^G E_G^*(X) \cong e_1^{WL} E_{WL}^*(X^L)$$

for any finite G-CW complex X. When G is finite, we reprove and concatenate results of Araki [4] and tom Dieck [44] to show that if L is a p-perfect subgroup of G and if VL is a p-Sylow subgroup of WL (with VL the trivial group in the case $p = 0$), then

$$e_L^G E_G^*(X)_{(p)} \cong E_{VL}^*(X^L)_{(p)}^{\mathrm{inv}},$$

where $E_{VL}^*(X^L)_{(p)}^{\mathrm{inv}}$ is the kernel of the difference of projections homomorphism

$$\pi_1^* - \pi_2^* : E_{WL}^*(WL/VL^+ \wedge X^L)_{(p)} \longrightarrow E_{WL}^*(WL/VL^+ \wedge WL/VL^+ \wedge X^L)_{(p)}.$$

Since $E_G^*(X)_{(p)}$ is the product over (L) of these groups, this result gives a complete determination of $E_G^*(X)_{(p)}$ in terms of groups $E_J^*(X^L)_{(p)}^{\mathrm{inv}}$ for appropriate subquotient p-groups J of G. Observe that all reference to $A(G)_{(p)}$ and its idempotents has disappeared from this final calculational conclusion. The analog in homology is also valid, and here X need not be finite. It should be remembered that E_J depends on the representation of J as a subquotient of G and not just on J as an abstract group. In particular, with $p = 0$, we obtain a complete determination of the rationalization of E^*_G and E^G_* in terms of various nonequivariant subquotient cohomology and homology theories. (Actually, it suffices for this to localize away from $|G|$ rather than to rationalize.)

The results described so far give little information about localizations of homology and cohomology theories at prime ideals of $A(G)$ when G is a general compact Lie group. After giving some preliminaries concerning universal $(\mathcal{F}',\mathcal{F})$-spaces for pairs $\mathcal{F}\subset\mathcal{F}'$ of families of subgroups of G in section 7, we explain in section 8 how one might construct such localizations topologically by concentrating theories between pairs of families. Following tom Dieck [42,44] in homology, we also give analogous constructions of the parts of homology and cohomology theories determined by idempotents e^G_L in $A(G)$ and of the localizations of homology and cohomology theories at multiplicative sets of Euler classes of representations of G.

The last three sections concentrate on equivariant stable homotopy theory. Except for reliance on section 7, which gives key lemmas on adjacent pairs of families, they are largely independent of the rest of the chapter. We begin section 9 by explaining an alternative approach to the isomorphism $A(G)\cong\pi^0_G(S)$. This is based on the case $Y = S^0$ of the splitting

$$\tilde{\pi}^G_*(Y)\cong\sum_{(H)}\tilde{\pi}_*(EWH^+\wedge_{WH}\Sigma^{Ad(WH)}Y^H),$$

which holds for arbitrary based G-spaces Y. Here $Ad(G)$ denotes the adjoint representation of G and the sum runs over the conjugacy classes of subgroups of G. The splitting results by combining a theorem of tom Dieck [38] with our generalized Adams isomorphism of II.7.2. We then use this splitting to analyze the full subcategory $\mathcal{O}G$ of the stable category whose objects are the orbit spectra $\Sigma^\infty G/H^+$. Together with earlier work in this book, this analysis completes the technical preliminaries needed to fill in all details of our announcement [88]. There we explained how to use the equivariant stable category to construct ordinary RO(G)-graded cohomology theories with coefficients in Mackey functors, where a Mackey functor is a contravariant additive functor $\mathcal{O}G\to\mathcal{A}b$. As explained in [88], the particular Mackey functor displayed in Proposition 9.10 leads to a transfer homomorphism

$$\tau\colon H^*(X/G;Z)\longrightarrow H^*(X/H;Z)$$

in ordinary cohomology for any G-CW homotopy type X and any $H\subset G$. We shall return to the study of these theories in [90].

We prove a generalization of the splitting of $\tilde{\pi}^G_*(Y)$ cited above in section 10. We assume given a normal subgroup Π of a compact Lie group Γ with quotient group G and give a splitting theorem for the computation of $[\Sigma^n\Sigma^\infty X,\Sigma^\infty Y]_\Gamma$ for based G-spaces X and Γ-spaces Y. There is one summand for each Γ-conjugacy class of subgroups Λ of Π, and that summand is

$$[\Sigma^n\Sigma^\infty X,\ \Sigma^\infty E^+\wedge_{W'}\Sigma^A Y^\Lambda]_{W/W'},$$

where $W = W_\Gamma\Lambda(= N_\Gamma\Lambda/\Lambda)$, $W' = W_\Pi\Lambda$, E is the universal $W_\Pi\Lambda$-free $W_\Gamma\Lambda$-space, and A is the adjoint representation of $W_\Gamma\Lambda$ derived from $W_\Pi\Lambda$. As we explain in section 11, this splitting implies a decomposition of the Π-fixed point G-spectrum $(\Sigma^\infty Y)^\Pi$ as the wedge over (Λ) of the suspension G-spectra of the G-spaces

$$G^+\wedge_{W/W'}(E^+\wedge_{W'}\Sigma^A Y^\Lambda).$$

This analysis generalizes our comparison between the equivariant and nonequivariant forms of the Segal conjecture in [89]. It will be used to obtain an interesting generalization of the Segal conjecture, concerning universal Π-free Γ-spaces, in [106].

§1. Equivariant Euler characteristics

Recall from III.1.1 that a "finite G-spectrum" is one for which duality works. The finite G-CW spectra and their wedge summands are the main (and, we conjecture, the only) examples. Recall too that a "finite G-space" is one of the homotopy type of a compact G-ENR; its suspension G-spectrum is finite. Specialization of III.7.1 gives our definition of Euler characterisitics.

Definition 1.1. Define the Euler characteristic $\chi(X)\in\pi^G_0(S)$ of a finite G-spectrum X to be the composite

$$S\xrightarrow{\gamma\eta}DX\wedge X\xrightarrow{\varepsilon}S.$$

For an unbased finite G-space X, define $\chi(X)=\chi(\Sigma^\infty X^+)$. For a based finite G-space X, define $\tilde{\chi}(X)=\chi(\Sigma^\infty X)$.

It will follow from our results below that $\chi(X)$ is the classical Euler characteristic of X when $G = e$. By III.8.1, we have an explicit topological description of $\chi(X)$ when X is a compact G-ENR.

Lemma 1.2. Let $r\colon N \to X$ be a retraction of an open neighborhood of X in some representation V and let B be a disc in V which contains N. Then the following diagram is G-homotopy commutative.

$$\begin{tikzcd}
S^V & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r] & C(V,V-X) \\
 & & C(N,N-X) \arrow[u, "\simeq"'] \arrow[d, "C(\omega)"] \\
S^V \arrow[from=uu, "\chi(X)"'] & C(V,V-B) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & C(V,V-\{0\})
\end{tikzcd}$$

Here the unlabeled arrows are inclusions or projections and ω is specified by $\omega(n) = n-r(n)$ for $n \in N$.

As usual, it follows that $\chi(X) = 0$ if the identity map of X is G-homotopic to a fixed point free map.

We catalog some elementary facts about Euler characteristics.

Lemma 1.3. Let X and Y be finite G-spectra.

(i) $\chi(S)$ is the identity element of $\pi_0^G(S)$.

(ii) $\chi(*) = 0$, where $*$ is the trivial G-spectrum.

(iii) $\chi(X) = \chi(Y)$ if X is G-equivalent to Y.

(iv) $\chi(X \wedge Y) = \chi(X)\chi(Y)$.

(v) $\chi(X \vee Y) = \chi(X) + \chi(Y)$.

(vi) $\chi(\Sigma^n X) = (-1)^n \chi(X)$.

Proof. These hold by III.7.2, 7.4, 7.5, 7.6, and 7.8.

The key property of Euler characteristics is their additivity on cofibre sequences, which follows from III.7.10.

Theorem 1.4. If Z is the cofibre of a map $X \to Y$ of finite G-spectra, then $\chi(Z) = \chi(Y) - \chi(X)$.

By induction on the number of cells, this has the following consequence.

Theorem 1.5. If X is a finite G-CW spectrum and $\nu(H,n)$ is the number of n-cells of orbit type G/H in X, then

$$\chi(X) = \sum_n \sum_{(H)} (-1)^n \nu(H,n)\chi(G/H).$$

For a G-space X, $X_{(H)} = \{x \mid (G_x) = (H)\}$. Define the nonequivariant internal Euler characteristic $\chi(X_{(H)}/G)$ to be the sum of the internal Euler

characteristics $\chi(M)$, where M runs over the path components of $X_{(H)}/G$. The $\chi(M)$ were specified above IV.2.11, and that result implies the following one.

Theorem 1.6. If X is a compact G-ENR, then

$$\chi(X) = \sum_{(H)} \chi(X_{(H)}/G)\chi(G/H).$$

These results focus attention on $\chi(G/H)$. Recall that $WH = NH/H$, where NH is the normalizer of H in G.

Lemma 1.7. If WH is infinite, then $\chi(G/H) = 0$ and $\chi((G/H)^K) = 0$ for all K. If WH is finite and G/H embeds in a representation V, then $\chi(G/H)$ is represented by a G-map $f\colon S^V \to S^V$ such that $\deg(f^K) = |(G/H)^K|$ for each K such that WK is finite.

Proof. Of course, $(G/H)^K$ is nonempty if and only if $(K) \leq (H)$. Since the tangent space of $WH = (G/H)^H$ at e is the H-fixed point space of the tangent space $L(H)$ of G/H at eH, WH is finite if and only if $L(H)$ contains no positive dimensional trivial summand. If $(K) \leq (H)$ and WK is finite, then WH is finite since $L(H)$ is a summand of $L(K)$. Since WH acts freely on $(G/H)^K$, the first statement is now clear (compare IV.2.12). Thus assume that WH and WK are finite. As in IV.2.4(ii) and II.6.15, $\chi(G/H)$ is represented by the composite

$$f\colon S^V \xrightarrow{t} G \ltimes_H S^W \xrightarrow{1 \ltimes e} G \ltimes_H S^V \cong (G/H)^+ \wedge S^V \xrightarrow{\xi \wedge 1} S^V,$$

where $V = L(H) \oplus W$. If K is not subconjugate to H, then f^K is clearly trivial. Thus assume that $(K) \leq (H)$. Conjugating if necessary, we may assume that $K \subset H$. By Bredon [18,II.5.7], $(G/H)^K$ has finitely many WK orbits and is thus a finite set. Its tangent space $L(H)^K$ at eH is therefore zero and $V^K = W^K$. Thus f^K is the composite

$$S^{V^K} \xrightarrow{t} [(G/H)^K]^+ \wedge S^{V^K} \xrightarrow{\xi \wedge 1} S^{V^K},$$

where ξ collapses $(G/H)^K$ to a point and t^K is obtained by embedding $(G/H)^K$ in V^K, extending to an embedding of small copies of V^K around the points of $(G/H)^K$, and collapsing out the complement. It is obvious from this description that the degree of f^K is the cardinality of $(G/H)^K$.

Define a homomorphism of rings $d_K\colon \pi_0^G(S) \to Z$ by representing an element of $\pi_0^G(S)$ by a G-map $f\colon S^V \to S^V$ and taking the degree of $f^K\colon S^{V^K} \to S^{V^K}$. The results above have the following consequence relating equivariant and nonequivariant Euler characteristics.

Corollary 1.8. Let X be a finite G-space. Then

$$d_K\chi(X) = \chi(X^K).$$

Proof. The previous results and standard arguments with nonequivariant Euler characteristics give

$$d_K\chi(X) = \sum_{(K)\leq(H)} \chi(X_{(H)}/G)\chi(G/H^K) = \sum_{(K)\leq(H)} \chi(X^K_{(H)}) = \chi(X^K).$$

This leads to a criterion for the equality of the equivariant Euler characteristics of two finite G-spaces. Since the following result is a special case of more general ones [44,45,60], we shall content ourselves with a sketch of its elementary obstruction theoretic proof.

Proposition 1.9. Let V be a complex representation of G. Two G-maps $f,g\colon S^V \to S^V$ are G-homotopic if and only if $\deg(f^H) = \deg(g^H)$ for all H such that WH is finite.

Proof. Necessity is obvious. Assume the equality of degrees. By induction up the orbit types of V, it suffices to show that, for each $H \subset G$, $f \simeq g\colon S^{V^H} \to S^{V^H}$ as WH-maps under the inductive hypothesis that $f \simeq g\colon T^{V^H} \to T^{V^H}$ as WH-maps, where T^{V^H} denotes the union over isotropy groups $J \supset H$, $J \neq H$, of the spaces S^{V^J}. Since the inclusion $T^{V^H} \to S^{V^H}$ is a WH-cofibration and WH acts freely on the complement, the obstructions lie in the groups

$$H^i(S^{V^H}/WH, T^{V^H}/WH;\ \pi_i(S^{V^H})).$$

These groups are zero unless $i = \dim V^H$ and WH is finite, in which case the group is Z and maps isomorphically under projection on orbits to

$$H^i(S^{V^H}, T^{V^H}; Z).$$

(The verification of these claims is easy since our assumption that V is complex ensures that $\dim V^J \leq \dim V^H - 2$ when J is an isotropy group of V which properly contains H.) The obstruction to a WH-homotopy maps under the projection on orbits to the obstruction to a nonequivariant homotopy, and the latter is of course $\deg f^H - \deg g^H$.

Corollary 1.10. For finite G-spaces X and Y, $\chi(X) = \chi(Y)$ if and only if $\chi(X^H) = \chi(Y^H)$ for all H such that WH is finite.

Now recall the classical congruences for nonequivariant Euler characteristics.

Lemma 1.11. Let X be a finite G-space.
(i) If G is a torus, then $\chi(X^G) = \chi(X)$.
(ii) If G is a finite p-group, then $\chi(X^G) \equiv \chi(X) \bmod p$.

Proof. Theorem 1.6 applies as it stands to the computation of the nonequivariant Euler characteristic of a finite G-space X, and of course $X_{(G)}/G = X^G$. In (i), $\chi(G/H) = 0$ for all proper subgroups H. In (ii), $\chi(G/H) \equiv 0 \bmod p$ for all proper subgroups H.

Following tom Dieck and Petrie [45], we use Corollary 1.8 to obtain a general set of congruences relating the Euler characteristics $\chi(X^K)$. Let $\mu(C)$ denote the number of generators of a finite cyclic group C.

Proposition 1.12. Let V be a complex representation of a finite group G and let $f\colon S^V \to S^V$ by any G-map. Then

$$\sum_{(C)} [G:NC]\mu(C)\deg(f^C) \equiv 0 \bmod |G|,$$

where the sum ranges over the conjugacy classes of cyclic subgroups of G.

Corollary 1.13. Let V be a complex representation of a compact Lie group G and let $f\colon S^V \to S^V$ be a G-map. Let H be a subgroup of G such that WH is finite. Then

$$\sum_{(K)} [NH:NH \cap NK]\mu(K/H)\deg(f^K) \equiv 0 \bmod |WH|,$$

where the sum ranges over the NH-conjugacy classes of groups K such that $H \subset K \subset NH$ and K/H is cyclic.

Proof. Apply the proposition to the WH-map f^H.

Corollary 1.14. Let X be a finite G-space. Let H be a subgroup of G such that WH is finite. Then

$$\sum_{(K)} [NH:NH \cap NK]\mu(K/H)\chi(X^K) \equiv 0 \bmod |WH|,$$

where the index of summation is as in the previous corollary.

Proof. Apply the previous corollary to a G-map f which represents $\chi(X)$ and use Corollary 1.8.

For completeness, we recall the proof of Proposition 1.12 from [43]. Let $b(G,V) \in \tilde{K}_G(S^V)$ be the Bott class [8,125]. Since $\tilde{K}_G(S^V)$ is a free $R(G)$-module on the generator $b(G,V)$, there exists $\alpha \in R(G)$ such that

$$f^*b(G,V) = \alpha b(G,V), \quad \text{where} \quad f^*: \tilde{K}_G(S^V) \longrightarrow \tilde{K}_G(S^V).$$

Let $\alpha(C) \in R(C)$ be the restriction of α. Since $b(G,V)$ restricts to $b(C,V) \in \tilde{K}_C(S^V)$, we also have

$$f^*b(C,V) = \alpha(C)b(C,V), \quad \text{where} \quad f^*: \tilde{K}_C(S^V) \longrightarrow \tilde{K}_C(S^V).$$

Write $V = V^C \oplus V_C$ as a C-space and let $e: S^{V^C} \to S^V$ be the inclusion. Then

$$e^*f^* = (f^C)^*e^*, \quad \text{where} \quad e^*: \tilde{K}_C(S^V) \longrightarrow \tilde{K}_C(S^{V^C}).$$

If $\lambda_{-1}(W)$ denotes the alternating sum $\Sigma(-1)^k\lambda^k(W)$, then

$$e^*b(C,V) = \lambda_{-1}(V_C)b(C,V^C),$$

and of course $(f^C)^*$ is multiplication by $\deg(f^C)$. Therefore

$$\alpha(C)\lambda_{-1}(V_C) = \deg(f^C)\lambda_{-1}(V_C)$$

in $R(C)$. Evaluating characters on a generator $x \in C$ and using that $\lambda_{-1}(V_C)(x) \neq 0$, we see that

$$\alpha(x) = \alpha(C)(x) = \deg(f^C).$$

At this point, we use our hypothesis that G is finite. Standard representation theory gives the relation

$$\sum_{x \in G} \alpha(x) \equiv 0 \bmod |G|.$$

Grouping elements of G as generators of cyclic groups and conjugates thereof, we find immediately that

$$\sum_{x \in G} \alpha(x) = \sum_{(C)} [G:NC]\mu(C)\deg(f^C).$$

<u>Remark 1.15</u>. Tom Dieck [40,44] defined the Euler characteristic of a compact G-ENR X to be the element

$$\chi(X) = \Sigma(-1)^i H^i(X;C) \in R(G),$$

where $H^i(X;C)$ has its induced action by G (which depends only on the finite group G/G_0). Our definition is related to his by $e_*\chi(X) = \chi(X)$, where $e_*: \pi_G^0(S) \to K_G^0(S) = R(G)$ is the unit homomorphism for equivariant K-theory. A character-theoretic proof based on [40, Prop. 22] should be possible. Alternatively, since both Euler characteristics are appropriately additive, it suffices to check the formula when X is an orbit or, more generally, a smooth closed G-manifold. Now $e_*\chi(X) \in K_G^0(S)$ is the product of $1 \in K_G^0(S)$ and $\chi(X) = \tau^*\xi^*(1) \in \pi_G^0(S)$ and is thus the transfer of $1 \in K_G^0(X)$. As shown by Nishida [117, 5.1 and 5.3], the transfer $\tau: K_G(X) \to K_G(S)$ is computable in terms of the topological index in such a way that the relation $\tau(1) = \chi(X)$ is easily verified.

§2. <u>The Burnside ring and</u> $\pi_0^G(S)$

We exploit our study of Euler characteristics to define $A(G)$ and prove that it is isomorphic to $\pi_0^G(S)$. While our initial definition is a bit different from his, we emphasize that most of the results in this section and the next two are due to tom Dieck [40-44].

<u>Definition 2.1</u>. Define the Burnside ring $A(G)$ to be the set of equivalence classes of finite G-spaces under the equivalence relation specified by $X \sim Y$ if $\chi(X) = \chi(Y)$ in $\pi_0^G(S)$. The addition and multiplication on $A(G)$ are induced by disjoint union and Cartesian product. We write $[X]$ for the equivalence class of a finite G-space X; $-[X] = [K \times X]$, where K is any compact ENR with trivial G-action such that $\chi(K) = -1$ nonequivariantly.

By Lemma 1.3 and Corollary 1.10, $A(G)$ is a well-defined commutative ring and χ specifies a well-defined ring monomorphism $A(G) \to \pi_0^G(S)$. We shall see shortly that χ is an isomorphism.

<u>Remark 2.2</u>. We are working implicitly in a G-universe U which contains all irreducible representations of G. We define $A(G;U)$ similarly for a general universe U, but restricting to compact G-ENR's which embed in U. Everything in this section applies equally well in the more general context.

We need notations for certain sets of subgroups of G.

<u>Notations 2.3</u>. Let $\mathcal{C}G$ denote the set of closed subgroups of G and $\mathcal{F}G$ denote the set of those closed subgroups H such that WH is finite. Let ΓG and ΦG denote the sets of conjugacy classes of subgroups in $\mathcal{C}G$ and $\mathcal{F}G$, respectively.

We record some finiteness results concerning these sets. Part (i) was pointed out by Palais [118, 1.7.27] and parts (ii) and (iii) are due to tom Dieck [41; 44, 5.10.8 and 5.9.9].

Theorem 2.4. (i) ΓG is a countable set.

(ii) ΦG is a finite set if and only if the Weyl group WT acts trivially on the maximal torus T.

(iii) There exists an integer b such that $|WH/W_0H| < b$ for every $H \in \mathcal{C}G$, where W_0H denotes the component of the identity in WH; therefore the set of orders $|WH|$ for $(H) \in \Phi G$ has a least common multiple $n(G)$.

We begin our study of $A(G)$ by recording some direct consequences of the results of the previous section. Recall that the degrees of maps of fixed point sets give ring homomorphisms $d_H: \pi_0^G(S) \to Z$.

Proposition 2.5. For $H \subset G$, define $\phi_H = d_H \circ \chi : A(G) \to Z$. Then $\phi_H[X] = \chi(X^H)$ for a finite G-space X.

Proposition 2.6. Additively, $A(G)$ is the free Abelian group on the basis $\{[G/K] \mid (K) \in \Phi G\}$. For a compact G-ENR X,

$$[X] = \sum_{(K)\in\Phi G} \chi(X_{(K)}/G)[G/K].$$

Proof. By Theorem 1.6, we need only check that the $[G/K]$ are linearly independent. If $\Sigma n_K[G/K] = 0$ and some $n_K \neq 0$, we may choose an (H) which is maximal among those (K) such that $n_K \neq 0$. This leads to the contradiction

$$0 = \phi_H(\Sigma n_K[G/K]) = n_H|WH| \neq 0.$$

It follows, of course, that $A(G)$ is the Grothendieck ring of finite G-sets when G is finite. It also follows that $A(G)$ could just as well have been defined in terms of finite G-CW complexes rather than compact G-ENR's.

Proposition 2.7. For any $x \in A(G)$ and any $(K) \in \Phi(G)$,

$$x \cdot [G/K] = \phi_K(x)[G/K] + \sum_{(H)<(K)} n_H[G/H].$$

If $x = [G/J]$, then the integers n_H are all non-negative. If T is a maximal torus in G, then $[G/T]^2 = |WT|[G/T]$.

Proof. It suffices to consider $x = [X]$. Here the product is $[X \times G/K]$, and it is clear that only orbits type G/H with $(H) \leq (K)$ appear in $X \times G/K$. The

coefficient of $[G/K]$ is computed by applying ϕ_K. If $X = G/J$, then

$$n_H = \chi((G/J \times G/K)_{(H)}/G).$$

Here $(G/J \times G/K)_{(H)}/G \cong (G/J \times G/K)_H/NH \subset (G/J \times G/K)^H/NH$ (where $X_{(H)} = \{x \mid (G_x) = (H)\}$ and $X_H = \{x \mid G_x = H\}$). Since $(G/J)^H$ and $(G/K)^H$ are finite sets, n_H is just the cardinality of a finite set. For the last statement, $(H) \leq (T)$ and WH finite imply $(H) = (T)$.

We require a little topological algebra to proceed further. For a metric space X with bounded metric, we give the set $\mathcal{S}X$ of closed subsets of X the Hausdorff metric

$$d(A,B) = \max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(A,b)\}.$$

The space $\mathcal{S}X$ is complete or compact if X is so. In particular, starting with a bi-invariant metric on G, we topologize $\mathcal{C}G$ as a subspace of $\mathcal{S}G$ and then topologize $\mathcal{F}G$ as a subspace of $\mathcal{C}G$.

Lemma 2.8. (i) $\mathcal{C}G$ is a closed subspace of $\mathcal{S}G$.

(ii) $\mathcal{F}G$ is a closed subspace of $\mathcal{C}G$.

(iii) The action of G on $\mathcal{C}G$ by conjugation is continuous.

(iv) With the orbit space topology, ΓG and ΦG are totally disconnected compact metric spaces, and ΦG is a closed subspace of ΓG.

Proof. We give a sketch, referring the reader to tom Dieck [44,5.6.1] for details. Part (i) is a direct verification from the definitions. Part (ii) follows from the facts that subgroups in small neighborhoods of a subgroup H are subconjugate to H [18,II.5.6] and that if $H \supset K$ with $K \in \mathcal{F}G$ then $H \in \mathcal{F}G$. Part (iii) is clear and part (iv) follows by countability (Theorem 2.4(i)) and part (ii).

If G is finite, $A(G)$ is effectively studied by using the ring homomorphisms ϕ_K to embed it in a product of copies of Z. For general compact Lie groups G, such a product is too big to be of much use and we use the topology on ΦG to obtain the appropriate substitute.

Definition 2.9. Give Z the discrete topology and define $C(G)$ to be the ring of continuous (= locally constant) functions $\Phi G \to Z$. Observe that, by the compactness of ΦG, such a function takes only finitely many values. For a G-map $f: S^V \to S^V$, the degree function, $d(f)(K) = \deg f^K$, is clearly continuous. Let $d: \pi_0^G(S) \to C(G)$ be the resulting ring homomorphism and define

$$\phi = d\chi\colon A(G) \longrightarrow C(G).$$

Since χ is a monomorphism by definition and d is a monomorphism by Proposition 1.9, ϕ is a monomorphism.

We shall prove the following result in section 5.

Lemma 2.10. Upon tensoring with Q, ϕ becomes an isomorphism of rings.

Granting this, we obtain the following conclusions.

Theorem 2.11. Consider the commutative diagram

$$\begin{tikzcd} A(G) \arrow[rr, "\chi"] \arrow[dr, "\phi"'] & & \pi_0^G(S). \arrow[dl, "d"] \\ & C(G) & \end{tikzcd}$$

Additively, $C(G)$ is the free Abelian group generated by $\{\alpha_K \mid (K) \in \Phi G\}$, where α_K is the unique element such that $|WK|\alpha_K = \phi[G/K]$. A continuous function $\alpha\colon \Phi G \to Z$ is in the image of ϕ if and only if, for each $(H) \in \Phi(G)$,

$$\sum_{(K)} [NH\colon NH \cap NK]\mu(K/H)\alpha(K) \equiv 0 \bmod |WH|,$$

where the sum runs over the NH-conjugacy classes of groups K such that $H \subset K \subset NH$ and K/H is cyclic and where $\mu(K/H)$ denotes the number of generators of K/H. Moreover, χ is an isomorphism of rings.

Proof. By Corollary 1.13, any $\alpha \in \operatorname{Im} d$, hence any $\alpha \in \operatorname{Im}\phi$, satisfies the specified congruences. Since χ and d are monomorphisms, the last statement will therefore follow from the second. Since WK acts freely on the finite set $(G/K)^H$ when $(H) \leq (K)$ and WH is finite, $|WK|$ divides $\phi[G/K]$ and α_K is present in $C(G)$. We must show that any $\alpha \in C(G)$ is an integral linear combination of the α_K. By the promised Lemma 2.10, after tensoring with Q we can write $\alpha = \Sigma q_K\alpha_K$, $q_K \in Q$. Let (H) be maximal such that $q_H \neq 0$. Then $\alpha_K(H) = 0$ if $(K) \neq (H)$ and $q_K \neq 0$, while $\alpha_H(H) = 1$. Thus $\alpha(H) = q_H$ and q_H is an integer. Repeating the argument on $\alpha - q_H\alpha_H$ and iterating, we conclude that all $q_K \in Z$. Finally, suppose given α satisfying the specified congruences. We may write $\alpha = \Sigma n_K\alpha_K$, $n_K \in Z$. To prove that $\alpha \in \operatorname{Im}\phi$, it suffices to prove that $|WK|$ divides n_K for each (K). Choose (H) maximal with $n_H \neq 0$. The coefficient of $\alpha(H)$ in the $H^{\underline{th}}$ congruence is one and the remaining $\alpha(K)$ are zero by the maximality of (H). Therefore $\alpha(H) = n_H$ is divisible by $|WH|$. Repeating the argument on $\alpha - n_H\alpha_H$ and iterating, we conclude that n_K is divisible by $|WK|$ for each (K).

We conclude from Theorem 2.4 (iii) that $n(G)$ annihilates $C(G)/A(G)$. When G is finite, this fact, and thus Lemma 2.10 and the theorem, are easily proven directly from the congruences. We shall give a different proof that χ is an isomorphism in section 9.

Remark 2.12. We defined $A(G)$ in terms of finite G-spaces and observed that we could instead have restricted to finite G-CW complexes. It is obvious that we could just as well have worked with based finite G-spaces or based finite G-CW complexes and reduced Euler characteristics; in this context, additive inverses are given by suspensions. It is also clear that we could have started with finite G-CW spectra and their Euler characteristics. Finally, after the fact, we see that $A(G)$ could have been defined in terms of finite G-spectra X and their Euler characteristics. Although we have no a priori formula for the calculation of $\chi(X)$ as a linear combination of the $\chi(G/H)$ in this generality, there necessarily is such a formula for each X since $\chi(X)$ is an element of $\pi_0^G(S)$.

There are various maps relating Burnside rings which correspond appropriately to maps between stable homotopy groups (or cohomotopy groups).

Definition 2.13. Let $\alpha\colon H \to G$ be a homomorphism of compact Lie groups. Define a ring homomorphism $\alpha^*\colon A(G) \to A(H)$ by sending a finite G-space X to X regarded as a finite H-space via α. Since $X^K = X^{\alpha(K)}$ for $K \subset H$, $\phi_K\alpha^* = \phi_{\alpha(K)}$. When α is an inclusion, we write $\alpha^* = r_H^G$ (or r for short) and call it the restriction. When $H \subset G$ and $g \in G$, we write $c_g\colon A(H) \to A(gHg^{-1})$ for the isomorphism induced by the conjugation isomorphism $gHg^{-1} \to H$.

Lemma 2.14. The following diagram commutes for $\alpha\colon H \to G$.

$$\begin{tikzcd} A(G) \arrow[r, "\chi"] \arrow[d, "\alpha^*"'] & \pi_G^0(S) \arrow[d, "\alpha^*"] \\ A(H) \arrow[r, "\chi"] & \pi_H^0(S). \end{tikzcd}$$

Definition 2.15. For $H \subset G$, define induction $\tau_H^G\colon A(H) \to A(G)$ (or τ for short) by sending a finite H-space Y to $G \times_H Y$. Clearly $(G \times_H Y)^K$ is empty and thus $\phi_K\tau_H^G = 0$ if $K \subset G$ is not subconjugate to H. By inspection, if $K \subset H$ and W_GK is finite (so that $(G/H)^K$ is a finite set), then $(G \times_H Y)^K$ is homeomorphic to the disjoint union over a set $\{g\}$ of coset representatives of $(G/H)^K$ of the spaces $Y^{g^{-1}Kg}$, hence $\phi_K\tau_H^G = \sum_g \phi_{g^{-1}Kg}$.

We record the basic algebraic properties of τ_H^G and then its relationship to cohomotopy. The former can be derived by direct inspection and use of the homomorphisms ϕ_K or by quotation of results about the transfer in cohomotopy.

Lemma 2.16. (i) $\tau_H^G\colon A(H) \to A(G)$ is a morphism of $A(G)$-modules, where $A(G)$ acts on $A(H)$ through r_H^G (Frobenius reciprocity).

(ii) The composite $\tau_H^G r_H^G\colon A(G) \to A(G)$ is multiplication by $[G/H]$.

(iii) For subgroups H and K of G, the composite $r_K^G \tau_H^G\colon A(H) \to A(K)$ is the sum, over a set $\{g\}$ of representatives in G for the orbit type component manifolds M of $K\backslash G/H$, of $\chi(M)$ times the composite

$$A(H) \xrightarrow{c_g} A(H^g) \xrightarrow{r} A(H^g \;\; K) \xrightarrow{\tau} A(K),$$

where $H^g = gHg^{-1}$ (double coset formula).

Lemma 2.17. Let $K \subset H \subset G$ and $g \in G$ and let $\pi\colon G/K \to G/H$ and $c_g\colon G/H^g \to G/H$ be the canonical G-maps. Then the following diagrams commute.

$$\begin{tikzcd} A(H) \arrow[r, "\chi"] \arrow[d, "r"'] & \pi_H^0(S) \cong \pi_G^0(G/H) \arrow[d, "\pi^*"] \\ A(K) \arrow[r, "\chi"] & \pi_K^0(S) \cong \pi_G^0(G/K) \end{tikzcd} \qquad \begin{tikzcd} A(K) \arrow[r, "\chi"] \arrow[d, "\tau"'] & \pi_K^0(S) \cong \pi_G^0(G/K) \arrow[d, "\tau"] \\ A(H) \arrow[r, "\chi"] & \pi_H^0(S) \cong \pi_G^0(G/H) \end{tikzcd}$$

$$\begin{tikzcd} A(H) \arrow[r, "\chi"] \arrow[d, "c_g"'] & \pi_H^0(S) \cong \pi_G^0(G/H) \arrow[d, "c_g^*"] \\ A(H^g) \arrow[r, "\chi"] & \pi_{H^g}^0(S) \cong \pi_G^0(G/H^g) \end{tikzcd}$$

Definition 2.18. Let N be a normal subgroup of G with quotient group J. Define a ring homomorphism $\psi\colon A(G) \to A(J)$ by sending a finite G-space X to the finite J-space X^N. Since $\psi\varepsilon^* = 1$, $\varepsilon\colon G \to J$, ψ is a split epimorphism. If $N \subset H \subset G$ and $K = H/N \subset J$, then $(X^N)^K = X^H$ and thus $\phi_K\psi = \phi_H$. The kernel of ψ is spanned by those $[G/L]$, $(L) \in \Phi G$, such that L does not contain N; equivalently (by an argument like the proof of Proposition 2.6), $\alpha \in \operatorname{Ker}\psi$ if and only if $\phi_H(\alpha) = 0$ for all $(H) \in \Phi G$ such that H contains N.

Lemma 2.19. Let N be a normal subgroup of G with quotient group J and define $\psi\colon \pi_G^0(S^0) \to \pi_J^0(S^0)$ by passage to N-fixed points on representative maps of G-spheres. Then the following diagram commutes.

$$\begin{tikzcd} A(G) \arrow[r, "\chi"] \arrow[d, "\psi"'] & \pi_G^0(S) \arrow[d, "\psi"] \\ A(J) \arrow[r, "\chi"] & \pi_J^0(S) \end{tikzcd}$$

§3. Prime ideals in $A(G)$

For $H \subset G$, let $q(H,p)$ denote the prime ideal $\phi_H^{-1}(p)$ of $A(G)$, where $(p) \subset Z$; here p is zero or a prime number. When necessary for clarity, we write $q(H,p;G)$ instead of $q(H,p)$ to indicate the ambient group.

Proposition 3.1. Let q be a prime ideal of $A(G)$ and let $p = \operatorname{char} A(G)/q$. Then there exists a unique $(K) \in \Phi G$ such that $q = q(K,p)$ and $|WK| \not\equiv 0 \bmod p$.

Proof. Of course, the second condition is vacuous if $p = 0$. Consider

$$\{(H) \mid [G/H] \notin q\} \subset \Phi G.$$

This set is non-empty since it contains (G), hence it contains a minimal element (K). By Proposition 2.7 and minimality,

$$x \cdot [G/K] \equiv \phi_K(x)[G/K] \bmod q$$

for any $x \in A(G)$. Since $[G/K] \notin q$, $x \equiv \phi_K(x) \bmod q$ and thus $q = q(K,p)$. Of course, $\phi_K[G/K] = |WK| \not\equiv 0 \bmod p$. If (J) also satisfies $q = q(J,p)$ and $|WJ| \not\equiv 0 \bmod p$, then

$$\phi_K[G/J] \equiv \phi_J[G/J] \not\equiv 0 \bmod p$$

and $(G/J)^K$ is non-empty. Similarly $(G/K)^J$ is non-empty, and this implies $(J) = (K)$.

Corollary 3.2. (i) Every proper containment of prime ideals of $A(G)$ is of the form $q(K,0) \subset q(K,p)$ for some $(K) \in \Phi G$ and $p > 0$.

(ii) $A(G)$ is Noetherian if and only if ΦG is finite.

(iii) Any ring homomorphism from $A(G)$ to an integral domain factors through ϕ_K for some $(K) \in \Phi G$.

(iv) If $(J) \notin \Phi G$, then $\phi_J = \phi_K$ for some $(K) \in \Phi G$.

Proof. Part (i) is clear and implies (ii) since a Noetherian ring can have only finitely many minimal prime ideals [16,II§4.3 Cor 3 to Prop 14]. Part (iii) holds since the kernel of such a ring homomorphism is a prime ideal; (iv) is a special case of (iii).

Theorem 2.4 (ii) shows that $A(G)$ generally fails to be Noetherian when G is not finite.

We next study when $q(H,p) = q(J,p)$ for $(H) \neq (J)$. For $p = 0$, the answer is entirely satisfactory.

Proposition 3.3. Let H and J be closed subgroups of G.

(i) If $H \lhd J$ and J/H is a torus, then $q(H,0) = q(J,0)$.

(ii) If (K) is the unique element of ΦG such that $q(H,0) = q(K,0)$, then, up to conjugation, $H \lhd K$ and K/H is a torus.

(iii) If $(H) \in \Phi G$ and $(J) \in \Phi G$, then $q(H,0) = q(J,0)$ if and only if $(H) = (J)$.

Proof. Part (i) follows from Lemma 1.11 and Proposition 2.5 and part (iii) is immediate from Proposition 3.1. In view of (i) and the uniqueness of (K), to prove (ii) it suffices to construct K such that $H \lhd K$, K/H is a torus, and WK is finite. We claim that the inverse image in NH of a maximal torus T in WH is a group K as desired, and we need only check that WK is finite. Let S be a maximal torus of WK and let L be the inverse image of S in NK. Since $L/K = S$, conjugation of K by an element of L is homotopic to an inner automorphism and therefore, by [35, 38.1], equal to an inner automorphism. Since H is normal in K, it is thus also normal in L. The extension $1 \to T \to L/H \to S \to 1$ shows that L/H is a torus. By the maximality of T, $S = e$ and WK is finite.

The analog for $p > 0$ is less satisfactory. Before stating it, we record an obvious consequence of Proposition 3.1.

Corollary 3.4. Fix $p > 0$ and let $\Phi(G;p)$ be the subset of ΦG consisting of those (K) such that $|WK| \not\equiv 0 \bmod p$. Then every prime ideal of $A(G)$ of residual characteristic p has the form $q(K,p)$ for precisely one $(K) \in \Phi(G;p)$.

Remark 3.5. We may topologize $\Phi(G;p)$ as a subspace of ΦG, but it need not be a closed subspace. To see this, consider the sequence of subgroups $K_n = D_n \times D_{2n} \times D_{3n}$ of the subgroup $K = O(2) \times O(2) \times O(2)$ of $O(6)$, where D_n denotes the dihedral group of order $2n$. The normalizer of D_n in $O(2)$ is D_{2n}. In $O(6)$, $NK_n = K_{2n}$ and $|WK_n| \not\equiv 0 \bmod 3$. However, K is the closure of the union of the K_n, and $|WK| \equiv 0 \bmod 3$. (We are indebted to Dale Peterson and David Vogan for pointing out this example.)

Proposition 3.6. Let H and J be closed subgroups of G and fix $p > 0$.

(i) If $H \lhd J$ and J/H is an extension of a torus by a finite p-group, then $q(H,p) = q(J,p)$.

(ii) If $(H) \in \Phi(G)$ and $|WH| \equiv 0 \bmod p$, then there exists K such that $(K) \in \Phi G$, $H \lhd K$, and K/H is a finite p-group.

(iii) If $(H) \in \Phi G$ and $|H/H_0| \not\equiv 0 \bmod p$, where H_0 is the component of the identity in H, and if (K) is the unique element of $\Phi(G;p)$ such that $q(H,p) = q(K,p)$, then, up to conjugation, $H \lhd K$ and K/H is a finite p-group.

Proof. Part (i) follows from Lemma 1.11 and Proposition 2.5. For (ii), the inverse image in NH of a p-Sylow subgroup of WH is a group K as required. For (iii), it suffices by uniqueness to show that this group K satisfies $|WK| \not\equiv 0 \bmod p$ when $|H/H_0| \not\equiv 0 \bmod p$. Since K/H is a p-Sylow subgroup of WH, it suffices to show that $NK \subset NH$. Thus let $g \in NK$ and consider $H^g = gHg^{-1}$. The groups H, H^g, and K all have the same identity component, hence $H/(H \cap H^g)$ is a quotient of H/H_0 and thus has order prime to p. However, $H/(H \cap H^g)$ is also isomorphic to a subgroup of the p-group K/H^g. Therefore $H = H^g$.

The following result has no counterpart for general compact Lie groups and plays a major role in many applications.

Proposition 3.7. Let G be finite and, for $H \subset G$, let H_p be the (unique) smallest normal subgroup of H such that H/H_p is a p-group. For $(K) \in \Phi(G,p)$, $q(H,p) = q(K,p)$ if and only if, up to conjugacy, $K_p \subset H \subset K$. Therefore $q(H,p) = q(J,p)$ if and only if $(H_p) = (J_p)$.

Proof. Given H, let K be the inverse image in NH_p of a p-Sylow subgroup of WH_p. Then $H_p \lhd K$ and K/H_p is a p-group, hence $H_p = K_p$. It suffices to show that $|WK| \not\equiv 0 \bmod p$, and this will hold if $NK \subset NH_p$. If $g \in NK$, then $H_p^g \lhd K$ and K/H_p^g is a p-group. By minimality, $H_p^g = H_p$.

For a general compact Lie group G and subgroup H, Propositions 3.3(ii) and 3.6(i) show that, if (K) is the unique element of ΦG such that $q(H,0) = q(K,0)$, then $q(H,p) = q(K,p)$ for every prime p. Given $(H) \in \Phi G$ and a fixed prime p, one can reach the unique element $(K) \in \Phi(G;p)$ such that $q(H,p) = q(K,p)$ by transfinite iteration of Proposition 3.6(ii). That is, one can start with $H_1 = H$ and construct an expanding sequence $\{H_i\}$ in ΦG such that $H_i \lhd H_{i+1}$ and H_{i+1}/H_i is a p-group. One may not reach an H_n such that $|WH_n| \not\equiv 0 \bmod p$ after finitely many steps. For example, the dihedral groups D_{2^i} in $O(2)$ all satisfy $|WD_{2^i}| \equiv 0 \bmod 2$. In that case, one can pass to the closure J of the union of the H_i. Certainly $q(J,p) = q(H_i,p)$ for all i since $X^{H_i} = X^J$ for any given finite G-CW complex X and all sufficiently large i. One can then again apply Proposition 3.6(ii) iteratively, starting with J, and so on. Eventually one must reach K such that $|WK| \not\equiv 0 \bmod p$. This discussion raises an open problem.

Question 3.8. If $H \subset J \subset K$ and $q(H,p) = q(K,p)$, is $q(J,p) = q(H,p)$?

We thought the answer was yes until the final proofreading of this book, when we noticed a gap in our proof. The answer is clearly yes if G is finite, by Proposition 3.7. In general, we can enlarge K as above to arrange that $(K) \in \Phi(G,p)$. We can also apply Proposition 3.3(ii) to obtain $H \lhd H'$ such that H'/H is a torus.

(If we enlarge J to J' this way, we need not have $H' \subset J'$.) Then $H' \cap J \lhd H'$, $H'/H' \cap J$ is a torus, and thus

$$q(H,p) = q(H',p) = q(H' \cap J,p).$$

As above, we can find a possibly transfinite ascending chain connecting H' to K. The intersection of this chain with J connects $H' \cap J$ to $K \cap J = J$. We would like to say that all groups L in the new chain define the same prime ideal $q(L,p)$, but we do not see a way around the fact that the closure of a union $\bigcup(H_i \cap J)$ may be quite different from the intersection of J with the closure of the union of the H_i.

Warning 3.9. For $H \subset G$, ΦH and ΦG need have little to do with one another. For example, there may be no $(K) \in \Phi G$ such that $(K) \leq (H)$, as the inclusion of a (finite) subgroup in a circle makes clear. Thus one must consider general conjugacy classes when comparing prime ideals in $A(G)$ and in $A(H)$.

Remark 3.10. Proposition 3.1, Corollary 3.2, and part (i) of both Propositions 3.3 and 3.6 remain valid for the Burnside ring $A(G;U)$ of a general G-universe U. Here the $[G/H]$ such that G/H embeds in U form an additive basis. None of the rest of the results above generalize to this context since they all start with a given subgroup H and proceed by constructing another subgroup K, and G/K need not embed in U when G/H does. In particular, when G is finite, G/H_p need not embed in U when G/H does; see Namboodiri [115,§8] for an example and a context in which this failure matters.

§4. Idempotent elements in $A(G)$

We begin by recalling the relationship between idempotents and prime ideals in a general commutative ring A. Of course, for an idempotent e, eA is the localization of A obtained by inverting e. Recall that the prime ideal spectrum Spec A is defined to be the set of prime ideals of A topologized by letting the closed subsets be those of the form

$$V(I) = \{P \mid P \supset I\}$$

for an ideal I. The open sets $D(a) = \{P \mid a \notin P\}$ for $a \in A$ form a basis for this topology. Later we shall also make use of the subspace Max A of maximal ideals.

As explained, for example, in Bourbaki [16, II.4.3 Prop. 15], the following three sets are in bijective correspondence for any given finite indexing set J.

(1) The set of families $\{e_j \mid j \in J\}$ of non-zero orthogonal idempotents with sum 1.

(2) The set of partitions $\{U_j \mid j \in J\}$ of Spec A as a disjoint union of nonempty open subsets.

(3) The set of families $\{I_j \mid j \in J\}$ of non-zero ideals such that A is the direct sum of the I_j.

Given $\{e_j\}$, the corresponding $\{U_j\}$ and $\{I_j\}$ are specified by

$$U_j = V((1-e_j)) \quad \text{and} \quad I_j = (e_j).$$

In particular, Spec A is connected if and only if 0 and 1 are the only idempotent elements of A.

This motivates us to compute the set of components of Spec $A(G)$. We denote this set πSpec $A(G)$ and topologize it as a quotient space of Spec $A(G)$. We need some notations and recollections.

Notations 4.1. Recall that a compact Lie group is said to be perfect if it is equal to the closure of its commutator subgroup. Let $\mathcal{P}G$ denote the set of perfect closed subgroups of G and topologize $\mathcal{P}G$ as a subspace of $\mathcal{C}G$. Let ΠG denote the set of conjugacy classes of groups in $\mathcal{P}G$ and give ΠG its topology as an orbit space of $\mathcal{P}G$. Let $G^{(1)}$ denote the closure of the commutator subgroup of G and, inductively, let $G^{(k+1)} = (G^{(k)})^{(1)}$. Recall that G is said to be solvable if it is an extension of a torus by a finite solvable group and that this holds if and only if there exists k such that $G^{(k)} = e$.

Tom Dieck proved the following facts about these notions [42; 44§§5.9 and 5.11].

Theorem 4.2. (i) G has a unique minimal normal subgroup G_a such that G/G_a is solvable, and G_a is perfect.

(ii) There exists an integer $k = k(G)$ such that $H^{(k)} = H_a$ for every closed subgroup H of G.

(iii) The functions $H \to H^{(1)}$ and $H \to H_a$ from $\mathcal{C}G$ to itself are continuous, and $\mathcal{P}G$ is a closed subspace of $\mathcal{C}G$.

(iv) ΠG is a closed subspace of ΓG and is thus a totally disconnected compact metric space.

(v) If G is a finite extension of a torus, $1 \to T \to G \to F \to 1$, and E is the kernel of the conjugation homomorphism $F \to \mathrm{Aut}\, T$, then ΠG is finite if and only if F/E is solvable, in which case $\mathcal{P}G$ is already finite.

The last part implies that ΠG is infinite for most classical Lie groups G. The next three results are also due to tom Dieck [42; 44, §5.11]; we include slight variants of his proofs.

Lemma 4.3. For a given $H \subset G$, all prime ideals $q(H,p)$ are in the same component of $\mathrm{Spec}\, A(G)$ as $q(H_a,0)$.

Proof. There is a finite normal sequence

$$H_a = H_k \lhd H_{k-1} \lhd \cdots \lhd H_1 = H$$

such that each H_j/H_{j+1} is either a torus or a cyclic group of prime order. By Propositions 3.3 and 3.6, $q(H_j,p_j) = q(H_{j+1},p_j)$ for some p_j (possibly 0). Since the closure of $q(H,0)$ in $\mathrm{Spec}\, A(G)$ clearly contains all $q(H,p)$, and similarly for the other H_j, the conclusion follows.

Proposition 4.4. Define $\beta\colon \Pi G \to \pi\, \mathrm{Spec}\, A(G)$ by sending a conjugacy class (L) to the component containing $q(L,0)$. Then β is a homeomorphism. Therefore G is solvable if and only if $A(G)$ contains no non-trivial idempotent elements.

Proof. Define $\alpha\colon \Phi G \to \Pi G$ by $\alpha(H) = (H_a)$. Then the solid arrow part of the following diagram commutes by the lemma, where q sends $((H),(p))$ to $q(H,p)$.

$$\begin{tikzcd}
\Phi G \times \mathrm{Spec}\, Z \arrow[rr, "q"] \arrow[d, "\pi_1"'] & & \mathrm{Spec}\, A(G) \arrow[d] \arrow[dl, dashed, "\gamma"'] \\
\Phi G \arrow[r, "\alpha"] & \Pi G \arrow[r, "\beta"] & \pi\mathrm{Spec}\, A(G)
\end{tikzcd}$$

Define $\gamma q(H,p) = (H_a)$. To see that γ is a well-defined function, it suffices to check that $(H_a) = (K_a)$, where (K) is the unique element of ΦG such that $q(H,p) = q(K,p)$ and $|WK| \not\equiv 0 \bmod p$. Since $J_a = J'_a$ if $J \lhd J'$ and J'/J is solvable and since the function sending H to H_a is continuous, this follows from the procedure for reaching (K) from (H) described above Question 3.8. Clearly γ makes the diagram commute. We shall see in Proposition 5.7 below that the surjection q is a continuous closed map. Since $\alpha\pi_1$ is continuous by Theorem 4.2, it follows that γ is continuous. Since ΠG is totally disconnected, it follows further that γ factors through a continuous map $\delta\colon \pi\mathrm{Spec}\, A(G) \to \Pi G$. Obviously $\beta\delta$ and $\delta\beta$ are the respective identity functions. Since ΠG is Hausdorff and $\pi\mathrm{Spec}\, A(G)$ is compact, δ is a homeomorphism with inverse β.

With Lemma 4.3, this implies an algebraic description of the support of an idempotent in terms of perfect subgroups.

Corollary 4.5. For an idempotent $e \in A(G)$ and a subgroup H of G, $\phi_H(e) = 1$ if and only if $\phi_{H_a}(e) = 1$.

Let e_L, or e_L^G, denote the idempotent corresponding to a component $\beta(L)$, where L is a perfect subgroup of G. Thus

$$\phi_H(e_L) = \begin{cases} 1 & \text{if } (H_a) = (L) \\ 0 & \text{if } (H_a) \neq (L). \end{cases}$$

Of course, if L is normal in G, then $(H_a) = (L)$ if and only if $H_a = L$. In particular, writing 1 for the trivial perfect subgroup,

$$\phi_H(e_1) = \begin{cases} 1 & \text{if } H \text{ is solvable} \\ 0 & \text{otherwise} \end{cases}$$

It is in general quite difficult to express e_L in terms of the standard basis $\{[G/K] \mid (K) \in \Phi G\}$. When G is finite, Yoshida [147,3.1] has given a precise formula. Araki [4, Cor B] used Yoshida's formula to prove that $e_L^G A(G)$ is isomorphic to $e_1^{WL} A(WL)$. We give a variant of Araki's argument which avoids use of Yoshida's formula and so works to prove this isomorphism for general compact Lie groups.

Theorem 4.6. Let L be a perfect subgroup of G and set $N = NL$ and $W = WL$. Then restriction from G to N and passage to L-fixed points induce isomorphisms of rings

$$e_L^G A(G) \xrightarrow{\rho} e_L^N A(N) \xrightarrow{\psi} e_1^W A(W).$$

Proof. Observe first that if $(K) \in \Phi G$ is the conjugacy class such that $q(L,0) = q(K,0)$, then, up to conjugacy, L is a normal subgroup of K and thus K is a subgroup of N. Therefore $W_G N$ is finite (even though $W_G L$ need not be).

Since $\phi_H r_N^G = \phi_H$ for $H \subset N$, it is clear that

$$r_N^G(e_L^G) = e_L^N + f_{L,G}^N,$$

where $f_{L,G}^N$ is an idempotent orthogonal to e_L^N. We define

$$\rho(e_L^G \alpha) = e_L^N r_N^G(\alpha)$$

for $\alpha \in A(G)$. By Definition 2.15, $\phi_K \tau_N^G = 0$ if K is not subconjugate to N and

$$\phi_K \tau_N^G = \sum_g \phi_{g^{-1}Kg}$$

if $K \subset N$ and $W_G K$ is finite, where g runs through a set of coset representatives for $(G/N)^K$. It follows easily from the defining formulas for the idempotents and consideration of G-conjugacy versus N-conjugacy that

$$\tau_N^G(e_L^N) = e_L^G.$$

Define an additive homomorphism $\tau\colon e_L^N A(N) \to e_L^G A(G)$ by letting

$$\tau(e_L^N\beta) = e_L^G \tau_N^G(e_L^N\beta)$$

for $\beta \in A(N)$. We claim that $\tau = \rho^{-1}$. By Frobenius reciprocity,

$$(\tau\rho)(e_L^G\alpha) = e_L^G\tau_N^G(e_L^N r_N^G(\alpha)) = e_L^G e_L^G\alpha = e_L^G\alpha.$$

By the double coset formula,

$$\begin{aligned}(\rho\tau)(e_L^N\beta) &= e_L^N r_N^G(e_L^G\tau_N^G(e_L^N\beta)) = e_L^N r_N^G \tau_N^G(e_L^N\beta)\\ &= \sum_g \chi(M) e_L^N \tau^N_{N^g\cap N}\, r^{N^g}_{N^g\cap N}\, c_g(e_L^N\beta)\end{aligned}$$

where g runs through a set of representatives in G for the orbit type component manifolds M of $N\backslash G/N$. We may take one g to be the identity element; the corresponding M is the identity component of the finite group $W_G N$, and it follows that the corresponding summand is $e_L^N\beta$. We claim that the remaining summands are zero. Clearly

$$c_g(e_L^N) = e_{L^g}^{N^g}.$$

It is easily checked by calculating ϕ_H's that

$$r^{N^g}_{N^g\cap N}(e^{N^g}_{L^g}) = \begin{cases} e^{N^g\cap N}_{L^g} & \text{if } L^g \subset N \\ 0 & \text{if } L^g \not\subset N. \end{cases}$$

In the case $L^g \subset N$, $N^g \cap N$ is the normalizer of L^g in N and we know that

$$\tau^N_{N^g\cap N}(e^{N^g\cap N}_{L^g}) = e^N_{L^g}$$

and

$$r^N_{N^g\cap N}(e^N_{L^g}) = e^{N^g\cap N}_{L^g} + f^{N^g\cap N}_{L^g,N}.$$

The last formula implies that

$$e^{N^g\cap N}_{L^g}\, r^N_{N^g\cap N}(e^N_{L^g}) = e^{N^g\cap N}_{L^g}.$$

Let $x = r^{N^g}_{N^g\cap N} c_g(\beta)$. By the formulas above and Frobenius reciprocity,

$$\begin{aligned} e_L^N \tau^N_{N^g\cap N}\, r^{N^g}_{N^g\cap N}\, c_g(e_L^N\beta) &= e_L^N \tau^N_{N^g\cap N}(e^{N^g\cap N}_{L^g} x)\\ &= e_L^N e^N_{L^g} \tau^N_{N^g\cap N}(e^{N^g\cap N}_{L^g} x) = 0\end{aligned}$$

for those g such that $L^g \subset N$ and $L^g \neq L$, that is, $g \neq e$.

For the second isomorphism of the theorem, observe that the restriction of $\psi\colon A(N) \to A(W)$ to $e_L^N A(N)$ is a monomorphism since $\operatorname{Ker}\psi$ consists of those β such that $\phi_H(\beta) = 0$ for all $(H) \in \Phi N$ with $L \subset H$ and e_L^N annihilates such elements. Since $\phi_K\psi = \phi_H$ for $K = H/L \subset W$, $\psi(e_L^N) = e_1^W$ and ψ sends $e_L^N A(N)$ to $e_1^W A(W)$. Let $\varepsilon\colon N \to W$ be the quotient homomorphism. Since $\phi_H\varepsilon^* = \phi_{\varepsilon(H)}$ for $H \subset N$, $\varepsilon^*(e_1^W) = e_L^N + f_L^N$, where f_L^N is an idempotent orthogonal to e_L^N. Define $\zeta\colon e_1^W A(W) \to e_L^N A(N)$ by letting $\zeta(e_1^W\gamma) = e_L^N\varepsilon^*(\gamma)$ for $\gamma \in A(W)$. Clearly $\psi\zeta$ is the identity, hence ψ is an isomorphism with inverse ζ.

§5. Localizations of $A(G)$ and of $A(G)$-modules

We begin by calculating the localizations of $A(G)$ at its prime ideals.

Proposition 5.1. (i) For $(K) \in \Phi G$, the localization of $A(G)$ at $q(K,0)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/q(K,0))_{(0)} = Q.$$

(ii) For $(K) \in \Phi(G;p)$, let $\Phi(G;K,p)$ be the subspace of ΦG consisting of those (J) such that $q(J,p) = q(K,p)$ and let

$$I(K,p) = \bigcap_{(J)\in\Phi(G;K,p)} q(J,0).$$

Then the localization of $A(G)$ at $q(K,p)$ is the canonical homomorphism

$$A(G) \longrightarrow (A(G)/I(K,p))_{(p)}.$$

Proof. With an evident interpretation of notations, we may view (i) as the special case $p = 0$ of (ii). It will follow from Proposition 5.7 below that $\Phi(G;K,p)$ is a closed subspace of ΦG. We first show that $I(K,p)$ coincides with the kernel

$$I' = \{\alpha \mid \text{there exists } \beta \notin q(K,p) \text{ such that } \alpha\beta = 0\}$$

of the localization of $A(G)$ at $q(K,p)$. Clearly $I' \subset I(K,p)$ since $\beta \notin q(K,p)$

implies $\beta \notin q(J,0)$ for $(J) \in \Phi(G;K,p)$. To show that $I(K,p) \subset I'$, we let $\alpha \in I(K,p)$ and construct $\beta \notin q(K,p)$ such that $\alpha\beta = 0$. For each $(J) \in \Phi(G;K,p)$, $\phi_J(\alpha) = 0$ and we can choose an open neighborhood U_J of (J) in ΦG such that $\phi_H(\alpha) = 0$ for all $(H) \in U_J$. The complement C of the union of the U_J is a closed subspace of the complement of $\Phi(G;K,p)$. For $(L) \in C$, there is an element β_L of $q(L,0)$ which is not in $q(K,p)$, by Corollary 3.2(i). Then $\phi_M(\beta_L) = 0$ for all (M) in some open neighborhood of (L) in C. Since C is compact, some finite product of the β_L will be an element $\beta \notin q(K;p)$ such that $\phi_L(\beta) = 0$ for all $(L) \in C$. Clearly $\alpha\beta = 0$.

We show next that $q(K,p)$ is the only prime ideal of residual characteristic p which contains $I(K,p)$. Indeed, suppose that $q(H,p) \neq q(K,p)$. For each $(J) \in \Phi(G;K,p)$, there exists an element γ_J in $q(J,0)$ which is not in $q(H,p)$. Since $\Phi(G;K,p)$ is compact, some finite product of the γ_J will be an element γ in $I(K,p)$ but not in $q(H,p)$. Thus, if $x \notin q(K,p)$, then the ideal generated by the image of x in $(A(G)/I(K,p))_{(p)}$ cannot be contained in any proper prime ideal.

For $(H) \in \Phi G$, let Z_H denote Z regarded as an $A(G)$-module via ϕ_H. The proposition and its proof have the following immediate consequences.

Corollary 5.2. (i) $\phi_K\colon A(G) \to Z_K$ induces an isomorphism upon localization at $q(K,0)$, and $(A(G)_{q(K,0)})_{q(H,0)} = 0$ if $(H) \neq (K)$.

(ii) $\underset{(J)}{\times} \phi_J\colon A(G) \to \underset{(J)\in\Phi(G;K,p)}{\times} Z_J$ induces a monomorphism upon localization at $q(K,p)$, and $(A(G)_{q(K,p)})_{q(H,p)} = 0$ if $(H) \notin \Phi(G;K,p)$.

We must remember here that infinite products need not commute with localization. We record another useful vanishing result along these lines.

Lemma 5.3. Let H and K be subgroups of G with $(K) \in \Phi G$ and regard $A(H)$ as a $A(G)$-module via r^G_H.

(i) $A(H)_{q(K,0)} \neq 0$ if and only if $(K) \leq (H)$.

(ii) If $p > 0$ and G is finite, $A(H)_{q(K,p)} \neq 0$ if and only if $(K_p) \leq (H)$.

(iii) In general, if $p > 0$, $A(H)_{q(K,p)} \neq 0$ if and only if there is a subgroup L of H such that $q(L,p) = q(K,p)$.

Proof. It is easily checked by use of Propositions 3.3 and 3.7 that (i) and (ii) are implied by (iii). Clearly $A(H)_{q(K,p)} \neq 0$ if and only if

$$(A(G) - q(K,p)) \cap \operatorname{Ker} r^G_H = \phi.$$

Now $\alpha \in \operatorname{Ker} r^G_H$ if and only if $\phi_L(\alpha) = 0$ for all $L \subset H$ while $\alpha \in A(G) - q(K,p)$ if and only if $\phi_K(\alpha) \not\equiv 0 \bmod p$. Obviously these conditions are contradictory if $q(L,p) = q(K,p)$ for some $L \subset H$. Conversely, assume that $q(L,p) \neq q(K,p)$ for all $L \subset H$. For each $(L) \in \Gamma G$ such that $(L) \leq (H)$, choose $\beta_L \in q(K,p)$ such that $\phi_L(\beta_L) = 0$. Then $\phi_M(\beta_L) = 0$ for all (M) in some neighborhood of (L). Since the set of $(L) \in \Gamma G$ with $(L) \leq (H)$ is easily seen to be a closed and hence compact subset of ΓG, some finite product of the β_L is an element of $(A(G) - q(K,p)) \cap \operatorname{Ker} r^G_H$. (Question 3.8 is relevant here.)

In the rest of this section, we shall be concerned with localizations of $A(G)$ at sets of integer primes. For a subring R of the rational numbers, we adopt the notations

$$A(G;R) = A(G) \otimes_Z R \quad\text{and}\quad C(G;R) = C(G) \otimes_Z R.$$

The prime ideals of $A(G;Q) = A(G)_{(0)}$ and of $A(G;Z_{(p)}) = A(G)_{(p)}$ will be denoted by the same names $q(K,0)$ and $q(K,p)$ as in $A(G)$. Clearly

$$A(G)_{q(K,0)} \cong A(G;Q)_{q(K,0)} \quad\text{and}\quad A(G)_{q(K,p)} \cong A(G;Z_{(p)})_{q(K,p)}.$$

When G is finite, the previous results admit convenient reinterpretations in terms of idempotents in $A(G)_{(p)}$. Recall from Proposition 3.7 that H_p denotes the smallest normal subgroup of H such that H/H_p is a p-group. We allow $p = 0$, in which case we set $H_0 = H$. Gluck [56] and Yoshida [147] gave explicit formulas in terms of the standard basis $\{[G/H]\}$ for the idempotents appearing in the following analog of Theorem 4.6; the isomorphisms ρ and ψ are due to Araki [4] (whose proof relied on Yoshida's formulas).

Theorem 5.4. Let G be finite. Let L be a p-perfect subgroup of G (or any subgroup if $p = 0$) and set $N = NL$ and $W = WL$. Then the idempotent $e^G_L \in C(G)_{(p)}$ such that

$$\phi_H(e^G_L) = \begin{cases} 1 & \text{if } (H_p) = (L) \\ 0 & \text{if } (H_p) \neq (L) \end{cases}$$

lies in $A(G)_{(p)}$. Restriction from G to N and passage to L-fixed points induce isomorphisms of rings

$$e^G_L A(G)_{(p)} \xrightarrow{\rho} e^N_L A(N)_{(p)} \xrightarrow{\psi} e^W_1 A(W)_{(p)},$$

and $e_L^G A(G)_{(p)}$ is also isomorphic to the localization of $A(G)$ at $q(L,p)$. Moreover,

$$A(G)_{(p)} = \underset{(L)}{\times} e_L^G A(G)_{(p)}.$$

Proof. When $p = 0$, the result is an easy consequence of Lemma 2.10. It is clear from section 3 that Spec $A(G)_{(p)}$ is the disjoint union of open and closed subsets each containing exactly one of the prime ideals $q(L,p)$ and all of the prime ideals $q(H,0)$ such that $q(H,p) = q(L,p)$. For any idempotent $e \in A(G)_{(p)}$, $\phi_H(e)$ is obviously the same (either 0 or 1) for each H defining a given $q(L,p)$. The presence in $A(G)_{(p)}$ of the cited idempotents of $C(G)_{(p)}$ now follows from the standard commutative algebra reviewed at the start of section 4. The homomorphisms ρ and ψ are defined and shown to be isomorphisms precisely as in the proof of Theorem 4.6. The evident homomorphism $e_L^G A(G)_{(p)} \to A(G)_{q(L,p)}$ is an isomorphism by Proposition 5.1. The last statement holds since the identity element of $A(G)_{(p)}$ is the sum over (L) of the orthogonal idempotents e_L^G.

When W is a p-group, for example when $L = G_p$, e_1^W is the identity element.

As observed by Araki [4], there is an analog of this result for $A(G;R)$ for any $R \subset Q$. For example, the statements about rationalization remain valid if we only invert all primes dividing the order of G. Rather than pursue this, we return to general compact Lie groups G. Here we cannot expect localizations at maximal ideals to be determined by idempotents. We shall see shortly that Spec $A(G;Q)$ is homeomorphic to ΦG. If we had idempotents $e_L^G \in A(G;Q)$ as in the finite case, then points of Spec $A(G;Q)$ would be open and closed subsets and ΦG would be discrete. Although $A(G)_{(p)}$ is not the product of its localizations at maximal ideals, it can nevertheless be computed in terms of these localizations. To show this, we must recall some elementary sheaf theoretical facts.

Let A be a commutative ring. Then A is isomorphic to the ring $\Gamma(\text{Spec } A,\tilde{A})$ of global sections of the structural sheaf $\tilde{A}$ over Spec A; see e.g. Hartshorne [58, II.2.2]. More generally, an A-module M is isomorphic to $\Gamma(\text{Spec } A,\tilde{M})$ [58, II.5.1]. Here $\tilde{M}$ is the sheaf over Spec A associated to M. Explicitly, for an open subset U of Spec A, $\tilde{M}(U)$ is the set of functions $s\colon U \to \coprod_{P \in U} M_P$ such that $s(P) \in M_P$ and s is locally constant in the sense that, for each $P \in U$, there is an open neighborhood V of P in U, an element $m \in M$, and an element $a \in A$ such that, for each $Q \in V$, $a \notin Q$ and $s(Q) = m/a$ in M_Q. If $\lambda_P\colon M \to M_P$ denotes the localization of M at P, then the formula $\psi(m)(P) = \lambda_P(m)$ for $m \in M$ specifies a natural isomorphism $\psi\colon M \to \Gamma(\text{Spec } A,\tilde{M})$. The intuition is that to know a point of M it suffices to know its localizations at all prime ideals.

Now let $i\colon \text{Max } A \to \text{Spec } A$ be the inclusion of the subspace of maximal ideals. Using open sets of Max A, we can specify a sheaf $\tilde{M}$ over Max A exactly as above. By the same proof as for Spec A [58, II.5.1], the stalk of $\tilde{M}$ at P is M_P. We have the direct image sheaf $i_*\tilde{M}$ over Spec A [58, p.65]. Using distinguishing subscripts Max and Spec, we see that restriction of local sections

$$\tilde{M}_{\text{Spec}}(U) \longrightarrow \tilde{M}_{\text{Max}}(U \cap \text{Max } A) = (i_*\tilde{M}_{\text{Max}})(U)$$

specifies a map of sheaves $\tilde{M}_{\text{Spec}} \to i_*\tilde{M}_{\text{Max}}$. Thus we have a map of ringed spaces [58, p.72]

$$(\text{Max } A,\tilde{A}) \longrightarrow (\text{Spec } A,\tilde{A})$$

and an associated ring homomorphism

$$i^*\colon \Gamma(\text{Spec } A,\tilde{A}) \longrightarrow \Gamma(\text{Max } A,\tilde{A}),$$

and similarly in the module context. For any A-module M, we have a homomorphism $\psi\colon M \to \Gamma(\text{Max } A,\tilde{M})$ such that the diagram

$$\begin{tikzcd} & M \arrow[dl, "\psi"'] \arrow[dr, "\psi"] & \\ \Gamma(\text{Spec } A,\tilde{M}) \arrow[rr, "i^*"] & & \Gamma(\text{Max } A,\tilde{M}) \end{tikzcd}$$

commutes. Since any open subset of Spec A which contains a maximal ideal P also contains all prime ideals $Q \subset P$ (this being obvious for the basic open subsets $D(a)$), the local constancy condition makes clear that two global sections on Spec A coincide if they coincide on Max A. Thus i^* is always a monomorphism. It is usually not an isomorphism. However, it is an isomorphism when A has the property that if a prime ideal Q is contained in two ideals P and P', then either $P \subset P'$ or $P' \subset P$. In general, the only possible way to extend a section $s \in \Gamma(\text{Max } A,\tilde{M})$ to a section $t \in \Gamma(\text{Spec } A,\tilde{M})$ is to let $t(Q)$ be the image of $s(P)$ in $M_Q = (M_P)_Q$ whenever Q is a prime ideal contained in a maximal ideal P. The specified condition on prime ideals ensures that t is well-defined and locally constant.

Returning to $A(G)$, observe that every prime ideal of $A(G)_{(0)}$ is maximal and that a prime ideal of $A(G)_{(p)}$ is contained in at most one other prime ideal. Combining the discussion above with Proposition 5.1, we arrive at the following procedure for computing modules over these rings in terms of their localizations at maximal ideals and for computing these localizations.

Theorem 5.5. Let $A = A(G)_{(p)}$, where p is zero or a prime, and let M be an A-module. Then the canonical homomorphism $\psi: A \to \Gamma(\mathrm{Max}\, A, \tilde{A})$ is an isomorphism of rings and the canonical homomorphism $\psi: M \to \Gamma(\mathrm{Max}\, A, \tilde{M})$ is an isomorphism of A-modules. Moreover

(i) If $A = A(G)_{(0)}$ and $(K) \in \Phi G$, then $M_{q(K,0)} = M/q(K,0)M$.

(ii) If $A = A(G)_{(p)}$ and $(K) \in \Phi(G;p)$, then $M_{q(K,p)} = M/I(K,p)M$.

Of course, for a particular M, it may happen that $M_{q(k,0)}$ or $M_{q(K,p)}$ is zero for all but finitely many (K). When this holds, $\Gamma(\mathrm{Max}\, A, \tilde{M})$ is just the product of the localizations of M at the $q(K,0)$ or $q(K,p)$.

We next determine the spaces Spec $A(G;R)$ and Max $A(G;R)$ in terms of the space ΦG for $R \subset Q$. We use the following general algebraic result, much of which is in Bourbaki [16, II§4 Exer 17] or tom Dieck [44, 5.7.10].

Proposition 5.6. Let $C(X,R)$ be the ring of continuous functions from a totally disconnected compact Hausdorff space X to a discrete commutative ring R.

(i) The function $q: X \times \mathrm{Spec}\, R \to \mathrm{Spec}\, C(X,R)$ specified by

$$q(x,P) = \{\alpha \mid \alpha(x) \in P\}$$

is a homeomorphism.

(ii) If R is an integral domain, then evaluation at x induces an isomorphism from the localization of $C(X,R)$ at $q(x,P)$ to the localization of R at P.

Proof. (i) Let $e: R \to C(X,R)$ embed R as the subring of constant functions. Then $e^*: \mathrm{Spec}\, C(X,R) \to \mathrm{Spec}\, R$ is continuous and $e^* q(x,P) = P$. Let $Q \in \mathrm{Spec}\, C(X,R)$ and let $Y = \bigcap_{\alpha \in Q} \alpha^{-1}(P)$, where $P = e^* Q$. We claim that Y consists of a single point x and that $Q = q(x,P)$. Given the claim, the rest of the proof requires only an elementary check that q and its inverse are continuous. We first show that Y is non-empty. The topology on X has a basis of open and closed subsets U, and such U have characteristic functions $\kappa(U) \in C(X,R)$ which take the value 1 on U and 0 off U. Suppose that Y is empty and choose $\beta_x \in Q$ such that $\beta_x(x) \notin P$ for each $x \in X$. Then choose an open and closed neighborhood U_x of x such that $\beta_x(y) = \beta_x(x)$ for $y \in U_x$. Choose a finite subcover $\{U_i \mid 1 \leq i \leq k\}$ of the cover $\{U_x\}$ and let β_i be the i^{th} β_x. Set $s_i = \beta_i(U_i)$, $t_i = \prod_{j \neq i} s_j$, and $t = s_i t_i = \prod_j s_j$ and define $\alpha_i = t_i \beta_i \kappa(U_i)$. Then $\alpha_i(y) = t \notin P$ for $y \in U_i$ and $\alpha_i(y) = 0$ for $y \notin U_i$. Since

$$\kappa(U \cup V) = \kappa(U) + \kappa(V) - \kappa(U \cap V),$$

we find inductively that $t\kappa(U_1 \cup \cdots \cup U_j) \in Q$ for $1 \leq j \leq k$. Thus $e(t) \in Q$ and $t \in P$. This contradiction establishes that Y contains some element x. Clearly $e(P) \subset Q \subset q(x,P)$. If $\alpha \in q(x,P)$, we can choose an open and closed neighborhood U of x such that $\alpha(y) = \alpha(x) \in P$ for $y \in U$. Then

$$\kappa(U) \cdot \alpha = \kappa(U) \cdot e\alpha(x) \in Q \quad \text{and} \quad \kappa(U) \notin q(x,P),$$

hence $\alpha \in Q$. Since $q(x,P) \neq q(y,P)$ if $x \neq y$ (as we see by considering characteristic functions of disjoint open and closed neighborhoods of x and y), $Y = \{x\}$.

(ii) the kernel of the localization $C(X,R) \to C(X,R)_{q(x,P)}$ is

$$I(x,P) = \{\alpha \mid \exists\, \beta \notin q(x,P) \ni \alpha\beta = 0\} \subset q(x,0).$$

In fact, equality holds here. If $\alpha \in q(x,0)$, then $\alpha(y) = 0$ for all y in some open and closed neighborhood U of x and thus $\alpha\kappa(U) = 0$ and $\alpha \in I(x,P)$. For any $\beta \in C(X,R)$, $\beta \equiv e\beta(x) \bmod q(x,0)$, hence e induces an isomorphism $R \to C(X,R)/q(x,0)$. The conclusion follows.

Observe that $C(G,R) = C(\Phi G,R)$ for $R \subset Q$.

Proposition 5.7. For $R \subset Q$, the function

$$q: \Phi G \times \mathrm{Spec}\, R \longrightarrow \mathrm{Spec}\, A(G;R)$$

is a continuous closed surjection. For $R = Q$, it is a homeomorphism. For $R = Z_{(p)}$, q induces a homeomorphism $\Psi(G;p) \to \mathrm{Max}\, A(G;Z_{(p)})$, where $\Psi(G;p)$ is the quotient space of ΦG obtained by identifying each closed subspace $\Phi(G;K,p)$, $(K) \in \Phi(G;p)$, to a single point.

Proof. The map $\phi^*: \mathrm{Spec}\, C(G) \to \mathrm{Spec}\, A(G)$ is a surjection. It is closed by inspection or by the observation that $C(G)$ is integral over any subring because any $\alpha \in C(G)$ is a linear combination of idempotent functions. (See [9, Exer 1 (p. 67)] for the fact that ι^* is closed when $\iota: A \to A'$ is an inclusion such that A' is integral over A.) The same holds for any $R \subset Q$, and the rest is clear.

Remark 5.8. The composite $\Phi(G;p) \subset \Phi G \to \Psi(G;p)$ is evidently a continuous bijection. However, since $\Phi(G;p)$ need not be a closed subspace of ΦG (by Remark 3.5), this composite need not be a homeomorphism.

Proposition 5.7 leads to the promised proof of Lemma 2.10, which asserts that

$$\phi \otimes 1: A(G;Q) \longrightarrow C(G;Q)$$

is an isomorphism. It is now clear that $\phi \otimes 1$ induces a homeomorphism on passage to Spec. By Propositions 5.1(i) and 5.6(ii), $A(G;Q)$ and $C(G;Q)$ localize to Q at their respective prime ideals $q(K,0)$, hence $\phi \otimes 1$ localizes to an isomorphism. In view of the natural isomorphism $A \cong \Gamma(\mathrm{Spec}\ A, \tilde{A})$, a ring homomorphism $A \to A'$ is an isomorphism if and only if it induces a homeomorphism on passage to Spec and an isomorphism upon localization at corresponding prime ideals.

As observed by tom Dieck [41], this has the following consequence.

Corollary 5.9. The natural map $A(G) \to C(G;Q)$ is the inclusion of $A(G)$ in its total quotient ring and $\phi: A(G) \to C(G)$ is the inclusion of $A(G)$ in its integral closure in $C(G;Q)$.

Proof. Every non zero divisor of $C(G;Q)$ is a unit, hence $C(G;Q)$ is its own total quotient ring. Any $\alpha \in C(G;Q)$ which is integral over the image of ϕ takes values in Z.

$A(G;Q)$ is an example of an absolutely flat (or von Neumann regular) commutative ring. To give a feel for the nature of such rings, we record without proof the results of some exercises in Bourbaki [16; I§2 Exer 16-18, II§3 Exer 9, II§4 Exer 16-17] and Kaplansky [71; Exer 22 (p. 64)].

Proposition 5.10. Let A be a commutative ring.

(i) The following are equivalent for an element $\alpha \in A$.
 (a) $\alpha \in (\alpha^2)$.
 (b) (α) is a direct summand of the A-module A.
 (c) $A/(\alpha)$ is a flat A-module.
 (d) $(\alpha) \cap I = \alpha I$ for all ideals I.

(ii) The following conditions on A are equivalent.
 (a) Every element $\alpha \in A$ satisfies the conditions in (i).
 (b) Every finitely generated ideal of A is a direct summand.
 (c) Every A-module is flat.
 (d) A has no non-zero nilpotent elements and every prime ideal of A is maximal.
 (e) The localization of A at each of its maximal ideals P is the field A/P.

A ring satisfying these conditions is said to be absolutely flat.

(iii) If A is absolutely flat, then every finitely generated submodule of a projective A-module is a direct summand and every projective A-module is a direct sum of cyclic submodules.

(iv) If A is absolutely flat, then Spec A is a totally disconnected compact Hausdorff space, two ideals I and J are equal if and only if $V(I) = V(J)$, and the following conditions on an ideal I are equivalent.
 (a) The closed set $V(I)$ is open.
 (b) I is finitely generated.
 (c) I is generated by an idempotent element.
 (d) A/I is a projective A-module.

By Proposition 5.6 and criterion (d) or (e) of (ii), $C(X,R)$ is absolutely flat for any totally disconnected compact Hausdorff space X and field R.

§6. Localizations of equivariant homology and cohomology theories

In section 4, we showed how to split $A(G)$ and thus every $A(G)$-module in terms of idempotents associated to perfect subgroups of G. In section 5, we showed how to calculate $A(G)_{(p)}$ and any module over $A(G)_{(p)}$ in terms of localizations at the ideals $q(K,p)$ and gave algebraic descriptions of these localizations. We here specialize to homology and cohomology groups and use the topology to obtain further information. We work in the stable category $\bar{h}G\mathscr{S}U$, where U is a complete G-universe.

An idempotent $e \in A(G)$ may be viewed as an idempotent map $e: S \to S$. Via $Y \simeq S \wedge Y$, e induces an idempotent map $e: Y \to Y$ for each G-spectrum Y. Of course, idempotent maps induce splittings just as they do in the nonequivariant stable category. We let eY be the telescope of the countable iterate of e and find that Y is equivalent to $eY \vee (1-e)Y$. Since telescopes commute with smash products, eY is equivalent to $eS \wedge Y$ and $e(Y \wedge Z)$ is equivalent to $Y \wedge eZ$. For any G-spectrum X, $[X,eY]_G$ is just the $A(G)$-submodule $e[X,Y]_G$ of $[X,Y]_G$. If $K \lhd H \subset G$ with $J = H/K$, we let Y_J be the J-spectrum $(\tilde{E}\mathscr{F}[K] \wedge Y)^K$, as explained and justified in II§9. With these notations, we have a stable category level analog of Theorem 4.6; when G is finite, it is implicit in Araki [4].

Theorem 6.1. Let L be a perfect subgroup of G with associated idempotent $e_L^G \in A(G)$ and set $N = NL$ and $W = WL$. For G-spectra X and Y, there are natural isomorphisms

$$[X,e_L^G Y]_G \xrightarrow{\rho} [X,e_L^N Y]_N \xrightarrow{\psi} [X_W,e_1^W Y_W]_W.$$

Proof. Let $\pi: G/N^+\wedge X \to X$ be the projection and $\tau: X \to G/N^+\wedge X$ be the associated transfer (defined in IV§3). With $A(G)$ and $A(N)$ replaced by $[X,Y]_G$ and $[G/N^+\wedge X,Y]_G \cong [X,Y]_N$ and with r_N^G and τ_N^G replaced by π^* and τ^*, the isomorphism ρ and its inverse τ are constructed and shown to be inverses by verbatim repetition of the proof of Theorem 4.6. The relevant formulas, Frobenius reciprocity and the double coset formula, are still available by IV§§5,6.

If $\mathcal{F}[L]$ is the family of subgroups of N which do not contain L, then $\pi_*^K(e_L^N Y) = 0$ for $K \in \mathcal{F}[L]$ since $r_K^N(e_L^N) = 0$. In the language of II§9, this says that $e_L^N Y$ is concentrated over L. By II.9.2 and II.9.5, the functor $(?)_W: \bar{h}N\mathcal{J}U \to \bar{h}W\mathcal{J}U^N$ induces an isomorphism

$$\psi: [X,e_L^NY]_N \longrightarrow [X_W,(e_L^NY)_W]_W.$$

We claim that $(e_L^NY)_W$ can be identified with $e_1^W(Y_W)$. By II.9.12, the functor $(?)_W$ commutes with smash products, so it suffices to check our claim for $Y = S$. By II.9.5, the functor $(?)_W$ is an equivalence from a full subcategory of $\bar{h}N\mathcal{J}U$ to $\bar{h}W\mathcal{J}U^N$ and therefore preserves splittings induced by idempotent maps. Thus $(e_L^NS)_W \simeq (e_L^N)_W S_W$, $(e_L^N)_W: S_W \to S_W$. By II.9.9, S_W can be identified with the sphere W-spectrum and, by Lemma 2.19, the map $\psi: A(G) \to A(W)$ agrees under this identification with $(?)_W: \pi_0^N(S) \to \pi_0^W(S_W)$. Therefore $(e_L^N)_W = e_1^W$ by the proof of Theorem 4.6.

Remark 6.2. Under the change of groups isomorphism

$$[X,e_L^NY]_N \cong [X,F_N[G,e_L^NY)]_G,$$

the isomorphism ρ coincides with $(\hat{\rho})_*$, where

$$\hat{\rho} = \rho(1): e_L^GY \longrightarrow F_N[G,e_L^NY).$$

The formula $r_N^G(e_L^G) = e_L^N + f_{L,G}^N$ in the proof of Theorem 4.6 implies that e_L^NY is a wedge summand of e_L^GY regarded as an N-spectrum, and $\hat{\rho}$ is characterized as the G-map whose composite with the evaluation N-map $F_N[G,e_L^NY) \to e_L^NY$ is the resulting projection. Clearly $\hat{\rho}$ is an equivalence. Its inverse is the composite

$$F_N[G,e_L^NY) \xrightarrow{i_*} F_N[G,e_L^GY) \cong F(G/N^+,e_L^GY) \xrightarrow{\tau^*} F(S^0,e_L^GY) \cong e_L^GY,$$

where $i: e_L^NY \to e_L^GY$ is the inclusion, $\tau: S^0 \to G/N^+$ is the pretransfer of II.6.15, and the middle isomorphism is given by II.4.9.

Remembering that $e_L^G(Z\wedge Y) \simeq Z\wedge e_L^GY$ and $(Z\wedge Y)_W \simeq Z_W\wedge Y_W$ for G-spectra Y and Z and that $(\Sigma^\infty X)_W \simeq \Sigma^\infty(X^L)$ for G-spaces X (by II.9.9), we obtain the following homological consequence.

Corollary 6.3. Let E_G be a G-spectrum, write E_N for E_G regarded as an N-spectrum, and let E_W be the associated W-spectrum. Let X be a G-space. For $\alpha \in RO(G)$, let $\beta = r_N^G(\alpha) \in RO(N)$ and $\gamma = \beta^L \in RO(W)$. Then there are natural isomorphisms

$$e_L^GE_\alpha^G(X) \xrightarrow{\rho} e_L^NE_\beta^N(X) \xrightarrow{\psi} e_1^WE_\gamma^W(X^L)$$

and

$$e_L^GE_G^\alpha(X) \xrightarrow{\rho} e_L^NE_N^\beta(X) \xrightarrow{\psi} e_1^WE_W^\gamma(X^L).$$

With X^L replaced by X_W, these assertions generalize to G-spectra X.

Just as Theorem 5.4 on p-perfect subgroups of a finite group G parallels Theorem 4.6, so we have a stable category level analog of Theorem 5.4 which parallels Theorem 6.1. The proof is exactly the same except that we replace the arbitrary G-spectrum Y in the discussion above by its localization $Y_{(p)} = Y\wedge MZ_{(p)}$, where $MZ_{(p)}$ is the Moore G-spectrum (obtained by applying i_*, $i: U^G \to U$, to the nonequivariant Moore spectrum with trivial G-action).

Theorem 6.4. Let G be finite. Let L be a p-perfect subgroup of G (or any subgroup if $p = 0$) with associated idempotent $e_L^G \in A(G)_{(p)}$ and set $N = NL$ and $W = WL$. For G-spectra X and Y, there are natural isomorphisms

$$[X,e_L^GY_{(p)}]_G \xrightarrow{\rho} [X,e_L^NY_{(p)}]_N \xrightarrow{\psi} [X_W,e_1^W(Y_{(p)})_W]_W.$$

Here $(Y_{(p)})_W \simeq (Y_W)_{(p)}$, as follows from either II.9.9 and II.9.12 or the standard algebraic characterization of localization. There is an evident analog of Remark 6.2. Before stating the homological consequence analogous to Corollary 6.3, we interpolate an easy induction theorem. It is due to Kosniowski [76] when G is finite and to tom Dieck [44, 7.5.3] in general. We agree to write $[X,Y]^G = [X,Y]_G$ when convenient.

Theorem 6.5. Let p be zero or a prime and let $(K) \in \Phi G$ if $p = 0$ and $(K) \in \Phi(G;p)$ if $p > 0$. For G-spectra X and Y, define $([X,Y]^K_{q(K,p)})^{inv}$ to be the kernel of the difference of projections homomorphism

$$[G/K^+\wedge X,Y]^G_{q(K,p)} \longrightarrow [G/K^+\wedge G/K^+\wedge X,Y]^G_{q(K,p)}.$$

Then the projection $G/K^+\wedge X \to X$ induces an isomorphism

$$[X,Y]^G_{q(K,p)} \longrightarrow ([X,Y]^K_{q(K,p)})^{inv}.$$

In particular, for a G-spectrum E_G,

$$E^G_*(X)_{q(K,p)} \cong E^K_*(X)^{inv}_{q(K,p)} \quad \text{and} \quad E^*_G(X)_{q(K,p)} \cong E^*_K(X)^{inv}_{q(K,p)}.$$

Proof. The assumption on (K) ensures that $|WK|$ is prime to p and thus that $[G/K]$ is a unit in $A(G)_{q(K,p)}$. When G is finite, the conclusion follows by Dress' induction theory [49, 50] from the observation that the induction homomorphism

$$\tau^G_K\colon A(K)_{q(K,p)} \longrightarrow A(G)_{q(K,p)}$$

is onto since $\tau^G_K r^G_K$ is multiplication by $[G/K]$. The proof in general is an application of the extension of Dress' theory to the compact Lie case that is implicit in section 9 below. We have a complex

$$0 \longrightarrow [X,Y]^G \xrightarrow{d_0} [G/K^+\wedge X,Y]^G \xrightarrow{d_1} [G/K^+\wedge G/K^+\wedge X,Y]^G \xrightarrow{d_2} \cdots .$$

Here $d^n = \sum_{i=0}^{n}(-1)^i \pi_i^*$ where, for $0 \leq i \leq n$,

$$\pi_i\colon (\bigwedge_{j=0}^{n} G/K^+)\wedge X \longrightarrow (\bigwedge_{j=0}^{n-1} G/K^+)\wedge X$$

is the projection which collapses the $i^{\underline{th}}$ copy of G/K to a point. When localized at $q(K,p)$, this complex acquires the contracting homotopy specified by $s^n = [G/H]^{-1}\tau_0^*$. We replace Y by $\Sigma^\alpha E_G$ to obtain the result about $E^*_G(X)$. We replace X by S^0 and Y by $X\wedge\Sigma^\alpha E_G$ to obtain the result about $E^G_*(X)$.

Remark 6.6. One could use Wirthmüller's isomorphism to identify $E^K_*(X)$ with $E^G_*(G/K^+\wedge X)$ (with a dimension shift) and define "$E^K_*(X)^{inv}_{q(K,p)}$" to be the kernel of the difference of transfers associated to the projections $G/K^+\wedge G/K^+\wedge X$. Using a complex with differentials given by transfers, one could prove that $E^G_*(X)_{q(K,p)}$ is also isomorphic to "$E^K_*(X)^{inv}_{q(K,p)}$". This approach is taken in [44, 7.5.3], where $E^K_*(X)$ is actually defined to be $E^G_*(G/K^+\wedge X)$; in general, this definition is incompatible with our preferred definition in terms of representing spectra.

We shall use the following special case in conjunction with Theorem 6.4.

Example 6.7. Let K be a p-Sylow subgroup of a finite group G. Certainly $|WK|$ is prime to p, so the induction theorem applies. Since K is a p-group, $K_p = 1$ (the trivial subgroup) and $q(K,p) = q(1,p)$. Thus localization at $q(K,p)$ is the same as localization at p followed by multiplication by e^G_1. Since $H_p = 1$ if and only if H is a p-group, we see immediately that $r^G_K(e^G_1)$ is the identity element of $A(K)_{(p)}$ and $r^G_K(e^G_L) = 0$ for any non-trivial p-perfect subgroup L of G. Therefore localization of E^*_K and E^K_* at $q(K,p)$ is the same as localization at p, and similarly for the complexes used in the proof of the induction theorem. We write $E^*_K(X)^{inv}_{(p)}$ and $E^K_*(X)^{inv}_{(p)}$ for the resulting invariants.

We can now put things together to prove the reduction to subquotient p-groups promised in the introduction.

Theorem 6.8. Let G be finite. Let L be a p-perfect subgroup of G (or any subgroup if $p = 0$), set $N = NL$ and $W = WL$, and let $V = VL$ be a p-Sylow subgroup of W (or the trivial subgroup if $p = 0$). Let E_G be a G-spectrum, write E_N for E_G regarded as an N-spectrum, let E_W be the associated W-spectrum, and write E_V for E_W regarded as a V-spectrum. Let X be a G-space. For $\alpha \in RO(G)$, let $\beta = r^G_N(\alpha) \in RO(N)$, $\gamma = \beta^L \in RO(W)$, and $\delta = r^W_V(\gamma) \in RO(V)$. Then there are natural isomorphisms

$$e^G_L E^G_\alpha(X)_{(p)} \xrightarrow{\rho} e^N_L E^N_\beta(X)_{(p)} \xrightarrow{\psi} e^W_1 E^W_\gamma(X^L)_{(p)} \cong E^V_\delta(X^L)^{inv}_{(p)}$$

and, if X is finite,

$$e^G_L E^\alpha_G(X)_{(p)} \xrightarrow{\rho} e^N_L E^\beta_N(X)_{(p)} \xrightarrow{\psi} e^W_1 E^\gamma_W(X^L)_{(p)} \cong E^\delta_V(X^L)^{inv}_{(p)}.$$

Therefore, with X finite in the case of cohomology,

$$E^G_*(X)_{(p)} \cong \times_{(L)} E^{VL}_*(X^L)^{inv}_{(p)} \quad \text{and} \quad E^*_G(X)_{(p)} \cong \times_{(L)} E^*_{VL}(X^L)^{inv}_{(p)}.$$

With X^L replaced by X_W, these assertions generalize to G-spectra X, and the assertions about rationalization only require localization away from $|G|$.

Proof. The first statement makes clear how to interpret the grading in the statement about products. We must restrict to finite X in cohomology since localized spectra do not represent algebraic localizations of cohomology groups in general (since localization fails to commute with infinite products and thus with lim and $\lim^1$). The conclusions follow from Theorem 6.4, Theorem 6.5, Example 6.7, and the splitting of $A(G)_{(p)}$ in Theorem 5.4.

Remarks 6.9. (i) If $p = 0$ or if $p > 0$ and p is prime to $|G|$, then the VL are trivial groups and the δ are integers. Thus, in these cases, the theorem reduces calculations to the study of Z-graded nonequivariant homology and cohomology theories which, by II.9.13, are related to the originally given theory in terms of localizations obtained by inverting Euler classes.

(ii) Here is an example to show the necessity of the finiteness assumption on X in the case of cohomology. Let $p = 0$ and $G = L = N$ and consider $X = EG$. Since $(EG)^G$ is empty, $E_1^*((EG)^G)_{(0)} = 0$ no matter what E_G^* is. However, $e_G^G E_G^*(EG)_{(0)}$ need not be zero. For an explicit example, consider complex K-theory; $K_G^0(EG)$ is the completion of $R(G)$ at its augmentation ideal. If G is cyclic of order 2, $\hat{R}(G) = Z \oplus \hat{Z}_2$ additively and $[G]$ acts by multiplication by 2 on Z and as zero on $\hat{Z}_2$. Since $e_G^G = 1-1/2[G]$, $e_G^G K_G^0(EG)_{(0)} \cong \hat{Z}_2 \otimes Q$.

§7. Preliminaries on universal $(\mathcal{F}',\mathcal{F})$-spaces and adjacent pairs

Recall the language of families and universal $\mathcal{F}$-spaces from II.2.1 and II.2.10. For an inclusion $\mathcal{F} \subset \mathcal{F}'$ of families in G, there is a G-map $E\mathcal{F} \to E\mathcal{F}'$, unique up to G-homotopy, and we define $E(\mathcal{F}',\mathcal{F})$ to be its (unreduced) mapping cone with cone point as basepoint. If $H \notin \mathcal{F}'$, then $E(\mathcal{F}',\mathcal{F})^H$ is the basepoint, so that $E(\mathcal{F}',\mathcal{F})$ is a based $\mathcal{F}'$-space. If $H \in \mathcal{F}'-\mathcal{F}$, then $E(\mathcal{F}',\mathcal{F})^H$ is equivalent to S^0. If $H \in \mathcal{F}$, then $E(\mathcal{F}',\mathcal{F})^H$ is contractible. When $\mathcal{F}$ is empty, $E\mathcal{F}$ is empty; we interpret $E(\mathcal{F},\phi)$ as $E\mathcal{F}^+$. When $\mathcal{F}'$ is the family of all subgroups of G, denoted $\mathcal{F}_\infty$, we may take $E\mathcal{F}'$ to be a point; we write $E(\mathcal{F}_\infty,\mathcal{F}) = \tilde{E}\mathcal{F}$ as in II§9. We say that the pair $(\mathcal{F}',\mathcal{F})$ is adjacent if $\mathcal{F}'-\mathcal{F}$ consists of a single conjugacy class. Easy cofibre sequence arguments yield the following observations. Note that an inclusion of pairs $(\mathcal{E}',\mathcal{E}) \to (\mathcal{F}',\mathcal{F})$ induces a G-map $E(\mathcal{E}',\mathcal{E}) \to E(\mathcal{F}',\mathcal{F})$.

Lemma 7.1. For $\mathcal{F} \subset \mathcal{F}'$, $E(\mathcal{F}',\mathcal{F})$ is G-equivalent to $(E\mathcal{F}')^+\wedge\tilde{E}\mathcal{F}$.

Lemma 7.2. For $\mathcal{F} \subset \mathcal{F}' \subset \mathcal{F}''$, the cofibre of the induced map $E(\mathcal{F}',\mathcal{F}) \to E(\mathcal{F}'',\mathcal{F})$ is equivalent to $E(\mathcal{F}'',\mathcal{F}')$.

We concentrate based G-spaces between $\mathcal{F}$ and $\mathcal{F}'$ by smashing with $E(\mathcal{F}',\mathcal{F})$, and similarly for G-spectra.

Lemma 7.3. If X is an $\mathcal{F}$-CW spectrum, then $E(\mathcal{F}',\mathcal{F})\wedge X$ is G-contractible.

Proof. The projections $E\mathcal{F}^+\wedge X \to X$ and $(E\mathcal{F}')^+\wedge X \to X$ are both G-equivalences by II.2.12, and the conclusion follows.

For G-spaces X, the commutation of smash products with passage to fixed points allows sharper statements. Recall that $X_{\mathcal{F}}$ denotes $\{x \mid G_x \notin \mathcal{F}\}$ and that, if X is a G-CW complex, then $X_{\mathcal{F}}$ is a subcomplex. The G-Whitehead theorem implies the following results.

Lemma 7.4. If $f\colon X \to Y$ is a map of G-CW complexes such that $f^H\colon X^H \to Y^H$ is an equivalence for $H \in \mathcal{F}'-\mathcal{F}$, then $1\wedge f\colon E(\mathcal{F}',\mathcal{F})\wedge X \to E(\mathcal{F}',\mathcal{F})\wedge Y$ is a G-equivalence. In particular, the inclusion $X_{\mathcal{F}} \to X$ induces a G-equivalence $E(\mathcal{F}',\mathcal{F})\wedge X_{\mathcal{F}} \to E(\mathcal{F}',\mathcal{F})\wedge X$.

The following formalization of induction up orbit types provides us with an operationally analogous result for maps of G-spectra. It could be stated in terms of weak equivalences, but the represented form is more convenient for reference in the applications. Let $\mathcal{F}|H$ denote the family of subgroups of H in $\mathcal{F}$ and observe that $E\mathcal{F}$ regarded as an H-space is $E(\mathcal{F}|H)$.

Proposition 7.5. Let $\mathcal{F} \subset \mathcal{F}'$ and let $f\colon Y \to Z$ be a map of G-spectra.

(i) If $f_*\colon [X,Y]_H \to [X,Z]_H$ is an isomorphism for all $H \in \mathcal{F}$ and all H-spectra X, then

$$(1\wedge f)_*\colon [X,W\wedge Y]_G \longrightarrow [X,W\wedge Z]_G$$

is an isomorphism for all G-spectra X and all $\mathcal{F}$-spectra W, such as $W = \Sigma^\infty E\mathcal{F}^+$.

(ii) If $(1\wedge f)_*\colon [X,\tilde{E}(\mathcal{F}|H)\wedge Y]_H \to [X,\tilde{E}(\mathcal{F}|H)\wedge Z]_H$ is an isomorphism for all $H \in \mathcal{F}'-\mathcal{F}$ and all H-spectra X, then

$$(1\wedge f)_*\colon [X,E(\mathcal{F}',\mathcal{F})\wedge Y]_G \longrightarrow [X,E(\mathcal{F}',\mathcal{F})\wedge Z]_G$$

is an isomorphism for all G-spectra X.

(iii) If $(1\wedge f)_*\colon [X,E(\mathcal{F}_2,\mathcal{F}_1)\wedge Y]_G \to [X,E(\mathcal{F}_2,\mathcal{F}_1)\wedge Z]_G$ is an isomorphism for all adjacent families $(\mathcal{F}_2,\mathcal{F}_1)$ with $\mathcal{F} \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}'$ and all G-spectra X, then

$$(1\wedge f)_*\colon [X,E(\mathcal{F}',\mathcal{F})\wedge Y]_G \longrightarrow [X,E(\mathcal{F}',\mathcal{F})\wedge Z]_G$$

is is isomorphism for all G-spectra X.

Provided X is finite, (i)-(iii) also hold for localized hom sets. Moreover, (i)-(iii), but not their localized versions, remain valid if the spectra or spaces smashed with Y and Z are instead smashed with X.

Proof. By the $\lim^1$ exact sequence, we may assume that X is finite throughout. By Wirthmüller's isomorphism and the standard change of groups isomorphism, the hypothesis of (i) implies that

$$(1\wedge f)_*\colon [X, G/H^+\wedge Y]_G \longrightarrow [X, G/H^+\wedge Z]_G$$

and

$$f_*\colon [G/H^+\wedge X, Y]_G \longrightarrow [G/H^+\wedge X, Z]_G$$

are isomorphisms for all $H\in\mathcal{F}$ and all X. By induction on the number of cells for finite W and passage to colimits in general, we still have such isomorphisms when G/H^+ is replaced by an $\mathcal{F}$-CW spectrum W. This proves both versions of (i), the alternate version also being a special case of the $\mathcal{F}$-Whitehead theorem, II.2.2. Using that $E(\mathcal{F}',\mathcal{F}) \simeq (E\mathcal{F}')^+\wedge\tilde{E}\mathcal{F}$ and that, as an H-space, $\tilde{E}\mathcal{F} = \tilde{E}(\mathcal{F}|H)$ is H-contractible if $H\in\mathcal{F}$, we see that (ii) and its alternate version follow by applying (i) and its alternate version with $\mathcal{F}$ replaced by $\mathcal{F}'$, W taken to be $(E\mathcal{F}')^+$, and f replaced by $1\wedge f\colon \tilde{E}\mathcal{F}\wedge Y \to \tilde{E}\mathcal{F}\wedge Z$ or by $F(1,f)\colon F(\tilde{E}\mathcal{F},Y)\to F(\tilde{E}\mathcal{F},Z)$.

We prove (iii) by applying transfinite induction to the collection $\mathcal{C}$ of families $\mathcal{E}$, ordered by inclusion, such that $\mathcal{F}\subset\mathcal{E}\subset\mathcal{F}'$ and

$$(1\wedge f)_*\colon [X, E(\mathcal{E},\mathcal{F})\wedge Y]_G \longrightarrow [X, E(\mathcal{E},\mathcal{F})\wedge Z]_G$$

is an isomorphism for all X. Since G has only countably many conjugacy classes of subgroups, any totally ordered subset of $\mathcal{C}$ has a cofinal sequence $\{\mathcal{E}_i\}$. If $\mathcal{E}=\cup\,\mathcal{E}_i$, then tel $E\mathcal{E}_i$ is a universal $\mathcal{E}$-space and tel $E(\mathcal{E}_i,\mathcal{F})$ is G-equivalent to $E(\mathcal{E},\mathcal{F})$. Thus $\mathcal{E}\in\mathcal{C}$ by an easy colimit argument. Therefore $\mathcal{C}$ has a maximal element $\mathcal{E}$. If $\mathcal{E}$ were not equal to $\mathcal{F}'$ and H was a subgroup of G minimal among those in $\mathcal{F}'-\mathcal{E}$, then the long exact sequence associated to $\mathcal{F}\subset\mathcal{E}\subset\mathcal{E}\cup(H)$ by virtue of Lemma 7.2 and the hypothesis on adjacent families would imply that $\mathcal{E}\cup(H)$ was in $\mathcal{C}$, contradicting the maximality of $\mathcal{E}$. The alternate version of (iii) is similar, using the $\lim^1$ exact sequence to handle the telescope. As usual, for the localized versions, one must remember that the $\lim^1$ exact sequence is no longer available.

Observe that $\mathcal{F}|H$ is the family $\mathcal{P}=\mathcal{P}(H)$ of all proper subgroups of H when $(\mathcal{F}',\mathcal{F})$ is an adjacent pair with $\mathcal{F}'-\mathcal{F}=(H)$. In practice, we apply (ii) to adjacent pairs to verify the hypothesis of (iii).

We now have all of the preliminaries needed for the next section, but we shall need further information for the proof of the splitting theorem in section 10. We concentrate on space level hom sets in the rest of this section.

Lemma 7.6. For G-CW complexes X and Y, the projection $q\colon X\to X/X_{\mathcal{F}'}$ induces a bijection

$$q^*\colon [X/X_{\mathcal{F}'}, E(\mathcal{F}',\mathcal{F})\wedge Y]_G \longrightarrow [X, E(\mathcal{F}',\mathcal{F})\wedge Y]_G$$

and the inclusion $j\colon X_{\mathcal{F}}\to X$ induces a bijection

$$j^*\colon [X, E(\mathcal{F}',\mathcal{F})\wedge Y]_G \longrightarrow [X_{\mathcal{F}}, E(\mathcal{F}',\mathcal{F})\wedge Y]_G.$$

Proof. Since $E(\mathcal{F}',\mathcal{F})$ is an $\mathcal{F}'$-space, any G-map $f\colon X\to E(\mathcal{F}',\mathcal{F})\wedge Y$ must send $X_{\mathcal{F}'}$ to the basepoint, and similarly for homotopies. The cells of X not in $X_{\mathcal{F}}$ are of orbit type G/H with $H\in\mathcal{F}$; since $E(\mathcal{F}',\mathcal{F})^H$ is contractible for such H, the statement about j^* holds by induction up the relative skeleta of $(X, X_{\mathcal{F}})$.

Note that, apart from the base vertex, all cells of $(X_{\mathcal{F}}/X_{\mathcal{F}'})$ are of orbit type G/H with $H\in\mathcal{F}'-\mathcal{F}$. We apply the lemma to obtain information about passage to H-fixed points when $(\mathcal{F}',\mathcal{F})$ is an adjacent pair with $\mathcal{F}'-\mathcal{F}=(H)$, and we let $N = NH$ and $W = WH$.

Lemma 7.7. If $\mathcal{F}'-\mathcal{F}=(H)$, then $E(\mathcal{F}',\mathcal{F})^H$ is W-equivalent to EW^+.

Proof. Since $E\mathcal{F}^H$ is empty, $E(\mathcal{F}',\mathcal{F})^H = [(E\mathcal{F}')^H]^+$. Certainly $(E\mathcal{F}')^H$ is contractible, and it is W-free since $(E\mathcal{F}')^M$ is empty if H is a proper subgroup of $M\subset N$.

Lemma 7.8. If $\mathcal{F}'-\mathcal{F}=(H)$ and X is a G-CW complex such that every isotropy group G_x which properly contains H has at least one element in $N-H$, then, for any Y, passage to H-fixed points specifies a bijection

$$\phi_H\colon [X, E(\mathcal{F}',\mathcal{F})\wedge Y]_G \longrightarrow [X^H, EW^+\wedge Y^H]_W.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccc}
[X/X_{\mathcal{F}'}, E(\mathcal{F}',\mathcal{F})\wedge Y]_G & \xrightarrow{q^*} & [X, E(\mathcal{F}',\mathcal{F})\wedge Y]_G \\
\downarrow{\scriptstyle \phi_H} & & \downarrow{\scriptstyle \phi_H} \\
[X^H/X^H_{\mathcal{F}'}, EW^+\wedge Y^H]_W & \xrightarrow{(q^H)^*} & [X^H, EW^+\wedge Y^H]_W,
\end{array}$$

in which we identify $(X/X_{\mathscr{F}'})^H$ with its homeomorph $X^H/X^H_{\mathscr{F}'}$. Our hypothesis on X ensures that every element of $X^H_{\mathscr{F}'}$ is fixed by some nonidentity element of W. Therefore $(q^H)^*$ is a bijection (by Lemma 7.6 applied to $E(\{1\},\phi) = EW^+$). Since q^* is also a bijection, it suffices to prove that ϕ_H on the left is a bijection. Thus assume that $X_{\mathscr{F}'} = \{*\}$. Then $X^H = X^H_{\mathscr{F}}$ and ϕ_H factors as the composite

$$[X,E(\mathscr{F}',\mathscr{F})\wedge Y]_G \xrightarrow{j^*} [X_{\mathscr{F}},E(\mathscr{F}',\mathscr{F})\wedge Y]_G \xrightarrow{\phi_H} [X^H_{\mathscr{F}},EW^+\wedge Y^H]_W.$$

Since j^* is a bijection by Lemma 7.6, we may assume that $X = X_{\mathscr{F}}$. Then all cells of X, apart from the base vertex, are of orbit type G/H and the action map $G^+\wedge_N X^H \to X$ is therefore a G-homeomorphism. This clearly implies that ϕ_H is a bijection (compare Bredon [18, II.5.12]).

Scholium 7.9. The hypothesis on X is necessary. For example, if H is a proper subgroup of G such that $H = N$ and if $X = Y = S^0$, then ϕ_H has domain the 1-point set $\pi_0 E(\mathscr{F}',\mathscr{F})^G$ and range the 2-point set $\pi_0 S^0$. The lemma was stated without such an hypothesis in [38, Satz 1].

We shall also need an elaboration of the lemma.

Lemma 7.10. Let Π be a normal subgroup of a compact Lie group Γ with quotient group G. Let $(\mathscr{F}',\mathscr{F})$ be an adjacent pair in Γ such that $\mathscr{F}'-\mathscr{F} = (\Psi)$, where $\Pi \subset \Psi \subset \Gamma$. Then

$$\phi_\Pi\colon [X,E(\mathscr{F}',\mathscr{F})\wedge Y]_\Gamma \longrightarrow [X^\Pi,E(\mathscr{F}',\mathscr{F})^\Pi\wedge Y^\Pi]_G$$

is a bijection for any Γ-CW complexes X and Y.

Proof. Here $E(\mathscr{F}',\mathscr{F})^\Pi = E(\mathscr{F}'/\Pi,\mathscr{F}/\Pi)$, where $\mathscr{F}/\Pi$ is the family

$$\{\Omega/\Pi \mid \Pi \subset \Omega \subset \Gamma \text{ and } \Omega \in \mathscr{F}\}$$

in G, and similarly for $\mathscr{F}'$; clearly $(\mathscr{F}'/\Pi,\mathscr{F}/\Pi)$ is an adjacent pair with $\mathscr{F}'/\Pi - \mathscr{F}/\Pi = (\Psi/\Pi)$. Observe that $(X_{\mathscr{F}'})^\Pi = (X^\Pi)_{\mathscr{F}'/\Pi}$ as a G-subcomplex of X^Π. Let

$$W = N_\Gamma(\Psi)/\Psi \cong N_G(\Psi/\Pi)/(\Psi/\Pi)$$

and consider the following commutative diagram.

$$\begin{tikzcd}
{[X,E(\mathscr{F}',\mathscr{F})\wedge Y]_\Gamma} \arrow[rr, "\phi_\Pi"] & & {[X^\Pi,E(\mathscr{F}',\mathscr{F})^\Pi\wedge Y^\Pi]_G} \\
{[X/X_{\mathscr{F}'},E(\mathscr{F}',\mathscr{F})\wedge Y]_\Gamma} \arrow[u, "q^*"] \arrow[rr, "\phi_\Pi"] \arrow[dr, "\phi_\Psi"'] & & {[X^\Pi/X^\Pi_{\mathscr{F}'},E(\mathscr{F}',\mathscr{F})^\Pi\wedge Y^\Pi]_G} \arrow[u, "(q^\Pi)^*"'] \arrow[dl, "\phi_{\Psi/\Pi}"] \\
& {[X^\Psi/X^\Psi_{\mathscr{F}'},E(\mathscr{F}',\mathscr{F})^\Psi\wedge Y^\Psi]_W} &
\end{tikzcd}$$

The maps q^* and $(q^\Pi)^*$ are bijections by Lemma 7.6 and the maps ϕ_Ψ and $\phi_{\Psi/\Pi}$ are bijections by Lemma 7.8.

§8. Concentration of homology and cohomology theories between families

The results of section 6 give a quite satisfactory picture of localizations when G is finite. The lack of a unique smallest subgroup defining a given prime ideal $q(K,p)$ prevents us from obtaining such a nice picture when G is a general compact Lie group. However, by focusing on the largest subgroup defining $q(K,p)$, we can obtain an interesting topological interpretation of localization at $q(K,p)$ that may work in general. We can also obtain an analogous interpretation of multiplication by an idempotent of $A(G)$ determined by a perfect subgroup of G. These results generalize the isomorphisms ψ of Theorems 6.1 and 6.8, which deal with idempotents of $A(G)$ or, if G is finite, $A(G)_{(p)}$ determined by normal perfect or p-perfect subgroups. As an aside, we give an analogous generalization of II.9.13, concerning localizations at Euler classes.

These results all deal with concentrations of homology and cohomology theories between families. There are two natural ways to concentrate a cohomology theory between $\mathscr{F}$ and $\mathscr{F}'$, but there is only one natural way to so concentrate a homology theory.

Definitions 8.1. Let $\mathscr{F} \subset \mathscr{F}'$ and let E_G and X be G-spectra.

(i) Define $E[\mathscr{F}',\mathscr{F}]^G$ and $E[\mathscr{F}',\mathscr{F}]^*_G$ to be the homology and cohomology theories represented by $E(\mathscr{F}',\mathscr{F})\wedge E_G$.

(ii) Define

$$E^G_*[\mathscr{F}',\mathscr{F}](X) = E^G_*(E(\mathscr{F}',\mathscr{F})\ X) \quad\text{and}\quad E^*_G[\mathscr{F}',\mathscr{F}](X) = E^*_G(E(\mathscr{F}',\mathscr{F})\ X).$$

In homology, these notions clearly coincide,

$$E[\mathscr{F}',\mathscr{F}]^G_*(X) = E^G_*[\mathscr{F}',\mathscr{F}](X).$$

In cohomology they are quite different, and $E^*_G[\mathcal{F}',\mathcal{F}]$ is represented by the function G-spectrum $F(E(\mathcal{F}',\mathcal{F}),E_G)$.

The second definition is implicit in Conner and Floyd [35] and explicit in tom Dieck [44]. The first definition is new and is forced by our work here and in II§9. Lemma 7.2 implies the existence of long exact sequences connecting these theories, and Lemmas 7.3 and 7.4 imply vanishing and excision properties of the theories given in (ii). Proposition 7.5 implies invariance properties that are the crux of our work in this section.

In the next few results, which are essentially due to tom Dieck [42] (but see Scholium 8.5), e is a fixed idempotent element of $A(G)$. We state results in stable category terms, but the interest is in their homological and cohomological interpretations.

Lemma 8.2. Let $\mathcal{F}\subset\mathcal{F}'$ be families such that $\phi_H(e) = 1$ for $H\in\mathcal{F}'-\mathcal{F}$. For all G-spectra X and Y, multiplication by e is an isomorphism on the groups $[X,E(\mathcal{F}',\mathcal{F})\wedge Y]_G$ and $[E(\mathcal{F}',\mathcal{F})\wedge X,Y]_G$.

Proof. We may assume that $\mathcal{F}$ and $\mathcal{F}'$ are adjacent with $\mathcal{F}'-\mathcal{F}=(H)$. By Proposition 7.5(ii), it suffices to show that multiplication by e, that is, by $r^G_H(e)$, is an isomorphism on $[X,\tilde{E}\mathcal{P}\wedge Y]_H$ and $[\tilde{E}\mathcal{P}\wedge X,Y]_H$, where $\mathcal{P}$ is the family of proper subgroups of H. Since $\phi_H(e) = 1$, $r^G_H(e) = [H/H] + \sum_{(J)<(H)} a_J[H/J]$ for some integers a_J. Since multiplication by $[H/J]$ is obtained by smashing maps with the composite

$$S \xrightarrow{\tau} \Sigma^\infty H/J^+ \xrightarrow{\pi} S$$

and since $\tilde{E}\mathcal{P}$ is J-contractible if $J\neq H$, multiplication by $[H/J]$ is zero and multiplication by $r^G_H(e)$ is the identity.

Lemma 8.3. Let $\mathcal{E} = \{H \mid \phi_J(e) = 0 \text{ for all } J\subset H\} = \{H\mid r^G_H(e) = 0\}$. For G-spectra X and Y and a G-CW complex W, the inclusion $W_{\mathcal{E}}\to W$ induces isomorphisms

$$e[X,W_{\mathcal{E}}\wedge Y]_G \longrightarrow e[X,W\wedge Y]_G \quad\text{and}\quad e[W\wedge X,Y]_G \longrightarrow e[W_{\mathcal{E}}\wedge X,Y]_G.$$

Proof. It suffices to show that $e[X,Z\wedge Y]_G = 0$ and $e[Z\wedge X,Y]_G = 0$ for all $\mathcal{E}$-CW complexes Z. If $Z = \Sigma^n G/H^+$, this holds because $r^G_H(e) = 0$. The conclusions follow for finite Z by induction and for general Z by colimit and $\lim^1$ exact sequence arguments.

Proposition 8.4. Let $\mathcal{E} = \{H\mid r^G_H(e) = 0\}$, $\mathcal{F}' = \{H\mid \phi_K(e) = 1 \text{ for some } K\supset H\}$, and $\mathcal{F} = \mathcal{F}'\cap\mathcal{E}$. For G-spectra X and Y, there are natural isomorphisms

$$e[E(\mathcal{F}',\mathcal{F})\wedge X,Y]_G \cong e[X,Y]_G \cong e[X,E(\mathcal{F}',\mathcal{F})\wedge Y]_G.$$

Proof. Since $\phi_K(1-e) = 1$ if $K\notin\mathcal{F}'$, Lemma 8.2 gives that multiplication by $1-e$ is an isomorphism on the groups $[\tilde{E}\mathcal{F}'\wedge X,Y]_G$ and $[X,\tilde{E}\mathcal{F}'\wedge Y]_G$. Therefore multiplication by e is zero on these groups. Since $\mathcal{F}\subset\mathcal{E}$, $(E\mathcal{F})_{\mathcal{E}}$ is empty and Lemma 8.3 gives that $e[E\mathcal{F}^+\wedge X,Y]_G = 0$ and $e[X,E\mathcal{F}^+\wedge Y]_G = 0$. The conclusion follows from the long exact sequences obtained from the cofibre sequences

$$(E\mathcal{F}')^+ \longrightarrow S^0 \longrightarrow \tilde{E}\mathcal{F}' \quad\text{and}\quad E\mathcal{F}^+ \longrightarrow (E\mathcal{F}')^+ \longrightarrow E(\mathcal{F}',\mathcal{F}).$$

Scholium 8.5. We would like to use Lemma 8.2 to conclude that multiplication by e is an isomorphism on the groups involving $E(\mathcal{F}',\mathcal{F})$. We have

$$\mathcal{F}'-\mathcal{F} = \{H \mid J\subset H\subset K \text{ for some } J \text{ and } K \text{ such that } \phi_J(e) = \phi_K(e) = 1\}.$$

Clearly $\mathcal{F}'-\mathcal{F}$ contains $\{H\mid \phi_H(e) = 1\}$, but the inclusion can be proper. For example, if $e = 1-e_L$ where L is a perfect subgroup of G other than 1 or G_a, then $\phi_1(e) = \phi_G(e) = 1$ but $\phi_L(e) = 0$. This point is missed in [42, §5], where it is assumed that $\mathcal{F}'-\{H\mid\phi_H(e) = 1\}$ is a family. Fortunately, this assumption is warranted when $e = e_L$ for a perfect subgroup L of G. In this case

$$\mathcal{F}'-\mathcal{F} = \{H\mid \phi_H(e_L) = 1\}$$

because $J\subset H\subset K$ and $(J_a) = (L) = (K_a)$ imply that $(H_a) = (L)$.

Using the scholium and Lemmas 8.2 and 8.3, we obtain the following homological implications of Proposition 8.4.

Corollary 8.6. If X is a G-CW complex and E_G is a G-spectrum, then, with $\mathcal{E} = \{H\mid r^G_H(e_L) = 0\}$, $\mathcal{F}' = \{H\mid\phi_K(e_L) = 1 \text{ for some } K\supset H\}$, and $\mathcal{F} = \mathcal{F}'\cap\mathcal{E}$,

$$e_L E^G_*(X) \cong E[\mathcal{F}',\mathcal{F}]^G_*(X_{\mathcal{E}})$$

and

$$E[\mathcal{F}',\mathcal{F}]^*_G(X_{\mathcal{E}}) \cong e_L E^*_G(X) \cong E^*_G[\mathcal{F}',\mathcal{F}](X_{\mathcal{E}}).$$

There is an analogous sequence of results for localizations, but here, because of the limitations of the localized version of Proposition 7.5, we are forced to use only our first definition of concentrated theories. In the next few statements, p is zero or a prime, K is a subgroup of G with $|WK|$ finite and prime to p, and q is the prime ideal $q(K,p)$ of $A(G)$. We agree to write $[X,Y]^G = [X,Y]_G$ for notational convenience.

Lemma 8.7. Let $\mathscr{F} \subset \mathscr{F}'$ be families such that $q(H,p) = q$ for $H \in \mathscr{F}' - \mathscr{F}$. For finite G-CW spectra X and general G-spectra Y, the natural map

$$[X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_{(p)} \longrightarrow [X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_q$$

is an isomorphism.

Proof. Let $f \in A(G) - q$. It suffices to show that multiplication by f is an isomorphism on $[X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_{(p)}$. The argument is precisely the same as the proof of Lemma 8.2; the essential point is that if $q(H,p) = q$, then $r^G_H(f) = \phi_H(f)[H/H] + \sum_{(J)<(H)} a_J[H/J]$ with $\phi_H(f)$ prime to p.

Lemma 8.8. Let $\mathscr{E} = \{H \mid q(J,p) \neq q \text{ for all } J \subset H\} = \{H \mid A(H)_q = 0\}$. Let X and Y be G-CW spectra and W be a G-CW complex. If X or W is finite, the inclusion $W_{\mathscr{E}} \to W$ induces an isomorphism

$$[X, W_{\mathscr{E}}\wedge Y]^G_q \longrightarrow [X, W\wedge Y]^G_q;$$

if W is finite, the inclusion $W_{\mathscr{E}} \to W$ induces an isomorphism

$$[W\wedge X, Y]^G_q \longrightarrow [W_{\mathscr{E}}\wedge X, Y]^G_q.$$

Proof. Lemma 5.3 gives the agreement of the two specifications of $\mathscr{E}$. It suffices to show that, for an $\mathscr{E}$-CW complex Z, $[X, Z\wedge Y]^G_q = 0$ if X or Z is finite and $[Z\wedge X, Y]^G_q = 0$ if Z is finite. This is clear if $Z = \Sigma^n G/H^+$ and follows by induction if Z is finite. If X is finite, we can pass to colimits to conclude that $[X, Z\wedge Y]^G_q = 0$.

Proposition 8.9. Let $q = q(K,p)$ with $|WK|$ finite and prime to p and let $\mathscr{E} = \{H \mid q(J,p) \neq q \text{ for all } J \subset H\}$, $\mathscr{F}' = \{H \mid (H) \leq (K)\}$, and $\mathscr{F} = \mathscr{E} \cap \mathscr{F}'$. Assume that G is finite or that the answer to Question 3.8 is yes. Then for finite G-CW spectra X and general G-spectra Y, there are natural isomorphisms

$$[X,Y]^G_q \cong [X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_{(p)} \cong ([X, \tilde{E}(\mathscr{F}|K)\wedge Y]^K_{(p)})^{\mathrm{inv}}.$$

Proof. The second isomorphism is immediate from Theorem 6.5 and the first. By our definitions,

$$\mathscr{F}' - \mathscr{F} = \{H \mid (H) \leq (K) \text{ and } q(J,p) = q \text{ for some } J \subset H\}.$$

In view of the maximality of (K) among conjugacy classes (H) such that $q(H,p) = q$, a yes answer to Question 3.8 implies that

$$\mathscr{F}' - \mathscr{F} = \{H \mid q(H,p) = q\}.$$

Thus $[X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_{(p)} \cong [X, E(\mathscr{F}',\mathscr{F})\wedge Y]^G_q$ by Lemma 8.7. Since $(E\mathscr{F})_{\mathscr{E}}$ is empty, Lemma 8.8 implies that $[X, (E\mathscr{F})^+\wedge Y]^G_q = 0$. We claim that $[X, \tilde{E}\mathscr{F}'\wedge Y]^G_q = 0$. The first isomorphism will follow from the claim and the evident long exact sequences. To prove the claim, it suffices to show that $[X, E(\mathscr{F}_2,\mathscr{F}_1)\wedge Y]^G_q = 0$ if $\mathscr{F}' \subset \mathscr{F}_1 \subset \mathscr{F}_2$ and $(\mathscr{F}_2,\mathscr{F}_1)$ is an adjacent pair, say $\mathscr{F}_2 - \mathscr{F}_1 = (J)$. Since $(J) \not\leq (K)$, $q(J,p) \neq q$. By Lemma 8.7 again,

$$[X, E(\mathscr{F}_2,\mathscr{F}_1)\wedge Y]^G_{(p)} \cong [X, E(\mathscr{F}_2,\mathscr{F}_1)\wedge Y]^G_{q(J,p)}$$

hence

$$[X, E(\mathscr{F}_2,\mathscr{F}_1)\wedge Y]^G_q \cong ([X, E(\mathscr{F}_2,\mathscr{F}_1)\wedge Y]^G_{q(J,p)})_q.$$

This is zero since $(A(G)_{q(J,p)})_q = 0$ by Corollary 5.2.

Remembering that $\tilde{E}\mathscr{F} = E(\mathscr{F}_\infty,\mathscr{F})$, we obtain the following homological consequence.

Corollary 8.10. Let X be a G-CW complex and E_G be a G-spectrum. Then

$$E^G_*(X)_q \cong E[\mathscr{F}',\mathscr{F}]^G_*(X_{\mathscr{E}})_{(p)} \cong (E[\mathscr{F}_\infty,\mathscr{F}|K]^K_*(X_{\mathscr{E}})_{(p)})^{\mathrm{inv}}$$

and, if X is finite,

$$E_G^*(X)_q \cong E[\mathscr{F}',\mathscr{F}]_G^*(X_{\mathscr{E}})_{(p)} \cong (E[\mathscr{F}_\infty,\mathscr{F}|K]_K^*(X_{\mathscr{E}})_{(p)})^{\mathrm{inv}}.$$

Remarks 8.11. (i) By Lemma 5.3, we see that $X_{\mathscr{E}} = G\cdot X^K$ if $p = 0$ and $X_{\mathscr{E}} = G\cdot X^{K_p}$ if G is finite and $p > 0$.
(ii) Even for finite X, the corollary fails if we try to use the usual version, $E_G^*[\mathscr{F}',\mathscr{F}](X)$, of cohomology concentrated between $\mathscr{F}$ and $\mathscr{F}'$. For an explicit

counterexample, let $p = 0$, let $G = K$ be cyclic of order 2, and let X be a point. Here $\mathcal{F}' = \mathcal{F}_\infty$ and $\mathcal{E} = \mathcal{F} = \{1\}$; $E\mathcal{F} = EG$ and we write $\tilde{E}\mathcal{F} = \tilde{E}G$. We have an exact sequence

$$E_G^{-1}(EG)_{(0)} \longrightarrow \tilde{E}_G^0(\tilde{E}G)_{(0)} \longrightarrow E_G^0(pt)_{(0)} \longrightarrow E_G^0(EG)_{(0)} \longrightarrow \tilde{E}_G^1(\tilde{E}G)_{(0)}.$$

Take E_G^* to be complex K-theory K_G^*. We have $K_G^{-1}(EG) = 0$, $K_G^0(pt) = R(G)$, and $K_G^0(EG) = \hat{R}(G) = Z + \hat{Z}_2$. If we had $\tilde{K}_G^*(\tilde{E}G)_{(0)} \cong K_G^*(pt)_q$, then, since $R(G)_q = Q$, we would have the impossible exact sequence

$$0 \longrightarrow Q \longrightarrow Q \oplus Q \longrightarrow Q \oplus (\hat{Z}_2 \otimes Q) \longrightarrow 0.$$

We complete this section with an analogous result, due in homology to tom Dieck [44], which relates localizations defined by Euler classes to concentrations between families. A special case was given in II.9.13.

Let $F = F_G$ be a module spectrum over a ring spectrum $E = E_G$. We have Euler classes $\chi_V = e^*(1) \in E_G^V(pt)$, where $e\colon S^0 \to S^V$ is the inclusion and $1 \in \tilde{E}_G^V(S^V) = E_G^0(pt)$ is the identity element. Clearly $\chi_{V+W} = \chi_V\chi_W$ and $\chi_V = 0$ if V contains a trivial summand. Let $U' \subset U$ be the sum of countably many copies of each of a chosen set of representations of G; U' need not be a subuniverse since it need not contain a trivial representation. Let $S = \{\chi_V \mid V \subset U'\}$. Then S is a multiplicative subset of the $RO(G)$-graded ring $E_G^*(pt)$. Let $\mathcal{E}$ be the family of subgroups H of G such that U' contains an H-trivial representation. With these notations and hypotheses, we have the following results.

<u>Lemma 8.12</u>. Let X and Y be G-CW spectra and W be a G-CW complex. If X or W is finite, the inclusion $W_{\mathcal{E}} \to W$ induces an isomorphism

$$S^{-1}(W_{\mathcal{E}} \wedge Y \wedge F)_G^*(X) \longrightarrow S^{-1}(W \wedge Y \wedge F)_G^*(X);$$

if W is finite, the inclusion $W_{\mathcal{E}} \to W$ induces an isomorphism

$$S^{-1}(Y \wedge F)_G^*(W \wedge X) \longrightarrow S^{-1}(Y \wedge F)_G^*(W_{\mathcal{E}} \wedge X).$$

<u>Proof</u>. The image of χ_V in $E_G^*(G/H) = E_H^*(pt)$ is zero if $V^H \neq 0$, so $S^{-1}E_H^*(pt) = 0$ if $H \in \mathcal{E}$. The rest is as in Lemma 8.8.

<u>Theorem 8.13</u>. For finite G-CW spectra X and general G-spectra Y, there is a natural isomorphism

$$S^{-1}(Y \wedge F)_G^*(X) \cong (\tilde{E}\mathcal{E} \wedge Y \wedge F)_G^*(X)$$

<u>Proof</u>. As in the proof of II.9.13, the colimit over $V \subset U'$ of the spheres S^V is a model for $\tilde{E}\mathcal{E}$ and the maps $S^V \to S^W$ of the colimit system are induced by inclusions $e\colon S^0 \to S^{W-V}$ and so induce multiplication by χ_{W-V}

$$(Y \wedge F)_G^V(X) \cong [X, S^V \wedge Y \wedge F]_G \longrightarrow [X, S^W \wedge Y \wedge F]_G \cong (Y \wedge F)_G^W(X).$$

The conclusion follows (in all gradings by the generality of X and Y).

<u>Corollary 8.14</u>. For G-CW complexes, there are natural isomorphisms

$$S^{-1}F_*^G(X) \cong F[\mathcal{F}_\infty, \mathcal{F}]_*^G(X_{\mathcal{E}})$$

and, if X is finite,

$$S^{-1}F_G^*(X) \cong F[\mathcal{F}_\infty, \mathcal{F}]_G^*(X_{\mathcal{E}}).$$

<u>Remark 8.15</u>. Again, the analog for $F_G^*[\mathcal{F}_\infty, \mathcal{F}]$ fails. With X a point and $U' = U - U^G$, K_G^* already provides a counterexample when G is cyclic of order 2.

§9. <u>Equivariant stable homotopy groups and Mackey functors</u>

One can reprove the isomorphism $A(G) \cong \pi_0^G(S)$ by specializing the following general splitting theorem for equivariant stable homotopy groups. Let $Ad(G)$ denote the adjoint representation of G.

<u>Theorem 9.1</u>. For based G-spaces Y, there is a natural isomorphism

$$\tilde{\pi}_*^G(Y) \cong \sum_{(H)\in \Gamma G} \tilde{\pi}_*(EWH^+ \wedge_{WH} \Sigma^{Ad(WH)} Y^H).$$

Note that the sum here is over ΓG, not ΦG. However, WH is finite if and only if $Ad(WH) = 0$, and the space $EWH^+ \wedge_{WH} \Sigma^{Ad(WH)} Y^+$ is clearly connected if $Ad(WH) \neq 0$.

<u>Corollary 9.2</u>. For based G-spaces Y, there is a natural isomorphism

$$\tilde{\pi}_0^G(Y) \cong \sum_{(H)\in \Phi G} H_0(WH; \tilde{\pi}_0(Y^H)).$$

<u>Proof</u>. For a finite group W and based W-space X, the standard filtration of EW leads to a spectral sequence converging from $H_*(W; \tilde{\pi}_*(X))$ to $\tilde{\pi}_*(EW^+ \wedge_W X)$. In total

degree 0, the spectral sequence collapses to an isomorphism $H_0(W;\tilde{\pi}_0X) \cong \tilde{\pi}_0(EW^+ \wedge_W X)$.

Of course, for any W-module M, $H_0(W;M) \cong M/(IW)M$, where IW is the augmentation ideal of the group ring $Z[W]$.

We shall prove a generalization of Theorem 9.1 in the next section and we shall there write down an explicit homomorphism θ which gives the isomorphism. A diagram chase from the definition of θ leads to explicit generators for the group $\tilde{\pi}_0^G(Y)$. The relevant chase is a specialization of that at the end of section 11 and will therefore be omitted.

Corollary 9.3. $\tilde{\pi}_0^G(Y)$ is the free Abelian group generated by the composites

$$S \xrightarrow{\tau} \Sigma^\infty G/H^+ \xrightarrow{\Sigma^\infty \tilde{y}} \Sigma^\infty Y,$$

where τ is the transfer, $\tilde{y}$: $G/H^+ \to Y$ is the based G-map such that $\tilde{y}(eH) = y$, H runs over a set of representatives for the conjugacy classes in ΦG, and y runs over a set of representatives in Y^H for the non-basepoint components of Y^H/WH.

With $Y = S^0$, this says that $\pi_0^G(S)$ is the free Abelian group generated by the Euler characteristics $\chi(G/H)$ for $(H) \in \Phi G$. This shows directly that χ: $A(G) \to \pi_0^G(S)$ is an epimorphism, which was the substantive step in the proof that χ is an isomorphism.

There is a useful conceptual variant of the previous corollary.

Corollary 9.4. Let X be an unbased G-space. For $H \subset G$, the group $[\Sigma^\infty G/H^+, \Sigma^\infty X^+]_G$ is isomorphic to the free Abelian group generated by the equivalence classes $[\phi,\chi]$ of diagrams

$$G/H \xleftarrow{\phi} G/K \xrightarrow{\chi} X$$

of space-level G-maps, where $K \subset H$ and W_HK is finite. Here (ϕ,χ) is equivalent to (ϕ',χ') if there is a G-homeomorphism ξ: $G/K \to G/K'$ such that $\phi'\xi = \phi$ and $\chi'\xi \simeq \chi$ in the diagram

$$\begin{tikzcd} & G/K \arrow[dl, "\phi"'] \arrow[dr, "\chi"] \arrow[dd, "\xi"] & \\ G/H & & X. \\ & G/K' \arrow[ul, "\phi'"] \arrow[ur, "\chi'"'] & \end{tikzcd}$$

Proof. We would obtain the same equivalence relation if we allowed $\phi'\xi \simeq \phi$. Map the specified group to $[\Sigma^\infty G/H^+, \Sigma^\infty X^+]$ by sending $[\phi,\chi]$ to the composite $(\Sigma^\infty\chi^+)\tau(\phi)$. Clearly this gives a well-defined homomorphism. Since any G-map

ϕ: $G/K \to G/H$ is a composite of a conjugation c_g: $G/K \to G/g^{-1}Kg$ and a canonical projection π: $G/g^{-1}Kg \to G/H$, where $g^{-1}Kg \subset H$, we see that each equivalence class contains a representative of the form

$$G/H \xleftarrow{\pi} G/K \xrightarrow{\chi} X.$$

Since $\pi c_g = \pi$ if $g \in H$, we see that all equivalence classes are still represented if we restrict attention to one group K in each conjugacy class $(K) \in \Phi H$. Of course, the G-homotopy class of χ is determined by a component of X^K. If χ and χ' correspond to components with the same image in $\pi_0(X^K/W_HK)$, then $\chi' \simeq \chi c_g$ for some $g \in N_HK$. Thus we obtain a complete set of representatives if we restrict attention to one χ corresponding to each component of X^K/W_HK. Under the isomorphism

$$[\Sigma^\infty G/H^+, \Sigma^\infty X^+]_G \cong [S, \Sigma^\infty X^+]_H,$$

$(\Sigma^\infty\chi^+)\tau(\pi)$ corresponds to the composite

$$S \xrightarrow{\tau} \Sigma^\infty H/K^+ \xrightarrow{\Sigma(\tilde{x}^+)} \Sigma^\infty X^+,$$

where $x = \chi(eK)$ (as we see by II.6.15 or IV.5.10), hence our homomorphism is an isomorphism by Corollary 9.3.

In the rest of this section, we use this corollary to study the stable orbit category $\mathcal{O}G$, namely the full subcategory of $\bar{h}G\mathscr{S}U$ whose objects are the suspension spectra $\Sigma^\infty G/H^+$. We would like to express composition in $\mathcal{O}G$ in concrete space level terms. In principle, this could be done by use of the double coset formula, but the details are unmanageable in the compact Lie case and unilluminating in the finite case. In the latter case, $\mathcal{O}G$ embeds as a full subcategory of a category with a purely algebraic definition and an obvious composition law. To explain this, we need a categorical construction. Recall that a category $\mathcal{C}$ is preadditive if its hom sets are Abelian groups and composition is bilinear; $\mathcal{C}$ is additive if, in addition, it has finite coproducts (which are then also finite products; see [92, VIII§2]).

Construction 9.5. Let $\mathcal{C}$ be a category which satisfies Axioms (M1) and (M2) of Dress [49, p. 201]:

(M1) $\mathcal{C}$ is small, in the sense that it has a set of isomorphism classes of objects, and has finite limits and coproducts; in particular, it has pullbacks, a terminal object, and an initial object.

(M2) In a commutative diagram

$$\begin{tikzcd} A' \arrow[r, "i'"] \arrow[d] & C' \arrow[d] & B' \arrow[l, "j'"'] \arrow[d, dash] \\ A \arrow[r, "i"] & A \amalg B & B \arrow[l, "j"'] \end{tikzcd}$$

in which i and j are the canonical maps, the squares are pullbacks if and only if i' and j' represent C' as the coproduct of A' and B'.

We construct an additive category $S\mathcal{C}$, which we think of as a kind of stabilization of $\mathcal{C}$, in two steps. First, define a category $S^+\mathcal{C}$ (sometimes called the category of spans of $\mathcal{C}$) by letting $S^+\mathcal{C}$ have the same objects as $\mathcal{C}$ and letting its morphisms from C to D be the equivalence classes $[\phi,\chi]$ of diagrams $C \xleftarrow{\phi} P \xrightarrow{\chi} D$ in $\mathcal{C}$, where $(\phi,\chi) \sim (\phi',\chi')$ if there is an isomorphism $\xi\colon P \to P'$ in $\mathcal{C}$ such that $\phi'\xi = \phi$ and $\chi'\xi = \chi$. The composite of $[\phi,\chi]$ and $[\psi,\omega]\colon D \to E$ is displayed as $[\phi\tilde{\psi},\omega\tilde{\chi}]$ in the diagram

$$\begin{tikzcd} & & R \arrow[dl, "\tilde{\psi}"'] \arrow[dr, "\tilde{\chi}"] & & \\ & P \arrow[dl, "\phi"'] \arrow[dr, "\chi"] & & Q \arrow[dl, "\psi"'] \arrow[dr, "\omega"] & \\ C & & D & & E, \end{tikzcd}$$

in which the square is a pullback. Define a sum on this hom set by letting $[\phi,\chi] + [\phi',\chi']$ be $[(\phi,\phi'),(\chi,\chi')]$, where

$$(\phi,\phi')\colon P \amalg P' \longrightarrow C \quad \text{and} \quad (\chi,\chi')\colon P \amalg P' \longrightarrow D$$

are the maps in $\mathcal{C}$ canonically induced by the given pairs. Clearly this sum is commutative and associative. Its zero element is the unique diagram $C \leftarrow P \rightarrow D$ in which P is the initial object of $\mathcal{C}$. Composition in $S^+\mathcal{C}$ is bilinear since pullbacks in $\mathcal{C}$ preserve coproducts (this being part of Axiom (M2)). The desired category $S\mathcal{C}$ is obtained from $S^+\mathcal{C}$ by the Grothendieck construction. That is, the objects of $S\mathcal{C}$ are the objects of $S^+\mathcal{C}$ (and thus of $\mathcal{C}$), and its morphisms are formal differences of morphisms of $S^+\mathcal{C}$. By construction, $S\mathcal{C}$ is a preadditive category, and Axiom (M2) implies that coproducts in $\mathcal{C}$ become coproducts in $S\mathcal{C}$, so that $S\mathcal{C}$ is additive.

Proposition 9.6. Let G be a finite group and let $\mathcal{F}G$ be the category of finite G-sets and G-maps. Then $S\mathcal{F}G$ is isomorphic to the full subcategory of the stable category whose objects are the suspension G-spectra $\Sigma^\infty F^+$ for $F \in \mathcal{F}G$.

Proof. Clearly $\mathcal{F}G$ satisfies Axioms (M1) and (M2). Define a functor $S^+\mathcal{F}G \to \bar{h}G\mathcal{S}U$ by sending F to $\Sigma^\infty F^+$ and sending a morphism $[\phi,\chi]$ to $(\Sigma^\infty\chi^+)\circ\tau(\phi)$. More explicitly, given the diagram $E \xleftarrow{\phi} P \xrightarrow{\chi} F$ of finite G-sets, we may view ϕ as a disjoint union of finite G-covers (provided we allow the empty cover, whose transfer is necessarily zero), and ϕ has a well-defined transfer $\tau(\phi)\colon \Sigma^\infty E^+ \to \Sigma^\infty P^+$. Our functor is well-defined by the naturality and transitivity of the transfer, and it extends uniquely to an additive functor $S\mathcal{F}G \to \bar{h}G\mathcal{S}U$. The extended functor is an isomorphism on hom sets by Corollary 9.4.

Remarks 9.7. The inclusion $S\mathcal{F}G \to \bar{h}G\mathcal{S}U$ sends disjoint unions to wedges and Cartesian products to smash products. It also preserves duality. To see this, observe that $S\mathcal{F}G$ is self-dual. Indeed, the contravariant functor $D\colon S\mathcal{F}G \to S\mathcal{F}G$ which is the identity on objects and sends a map $[\phi,\chi]\colon E \to F$ to the map $[\chi,\phi]\colon F \to E$ provides an explicit isomorphism between $S\mathcal{F}G$ and its opposite category. (In fact, this works for any category $\mathcal{C}$ as in Construction 9.5.) To see that D corresponds to Spanier-Whitehead duality in $\bar{h}G\mathcal{S}U$, let

$$\eta = [\xi,\Delta]\colon * \longrightarrow F \times F \quad \text{and} \quad \varepsilon = [\Delta,\xi]\colon F \times F \longrightarrow *,$$

where $\xi\colon F \to *$ is the projection to a point. The diagram

$$\begin{tikzcd} & & & F \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & & & \\ & & F \times F \arrow[dl, "\xi\times 1"'] \arrow[dr, "\Delta\times 1"] & & F \times F \arrow[dl, "1\times\Delta"'] \arrow[dr, "1\times\xi"] & & \\ & F & & F \times F \times F & & F & \end{tikzcd}$$

and its symmetric analog with $\Delta \times 1$ and $1 \times \Delta$ and $\xi \times 1$ and $1 \times \xi$ interchanged demonstrate that η and ε display $\Sigma^\infty F^+$ as a self-dual finite G-spectrum. An equally trivial diagram chase shows that $[\phi,\chi]$ and $[\chi,\phi]$ are dual maps.

Remarks 9.8. For a map $f\colon F \to F$ in $S\mathcal{J}G$, III.7.1 specifies a trace $\tau(f)\colon * \to F$ with respect to Δ and a Lefschetz constant $\chi(f)\colon * \to *$. To compute these maps, it suffices by additivity to consider the case when F is an orbit G/H and f is given by a diagram $G/H \xleftarrow{\psi} G/K \xrightarrow{\omega} G/H$. Here, by use of the duality maps η and ε and inspection of the diagram

$$\begin{tikzcd}
 & & & ? \arrow[dl] \arrow[dr] & & & \\
 & & G/K \arrow[dl, "\psi"'] \arrow[dr, "{(\psi,1)}"] & \text{pullback} & G/K \arrow[dl, "{(\omega,1)}"'] \arrow[dr, "\omega"] & & \\
 & G/H \arrow[dl, "\xi"'] \arrow[dr, "\Delta"] & & G/H \times G/H \arrow[dl, "1\times\psi"'] \arrow[dr, "1\times\omega"] & & G/H \arrow[dl, "\Delta"'] \arrow[dr, "\xi"] & \\
* & & G/H \times G/H & & G/H \times G/H & & *,
\end{tikzcd}$$

we find easily that $\chi(f) = 0$ if $\psi \neq \omega$ and $\chi(f) = [\xi\psi, \xi\psi]$ if $\psi = \omega$. Similarly, $\tau(f) = 0$ if $\psi \neq \omega$ and $\tau(f) = [\xi\psi, \psi]$ if $\psi = \omega$.

The original motivation for the introduction of the categories $S^+\mathcal{J}G$ and $S\mathcal{J}G$ was not this connection with the stable category but rather the connection with Mackey functors. Lindner [91] defined $S^+\mathcal{J}G$, observed that it has finite products, and proved that a product-preserving contravariant functor $S^+\mathcal{J}G \to \mathscr{A}b$ is precisely the same thing as a Mackey functor in the sense of Dress [49,50]. Since preserving products is equivalent to preserving addition on hom sets, such a functor extends uniquely to an additive contravariant functor $S\mathcal{J}G \to \mathscr{A}b$. This description of Mackey functors is the starting point of the first author's systematic algebraic study of Mackey functors in [87]. Together with Proposition 9.6, it implies the following claim from [88]. A direct computational proof based on double coset decompositions is also possible.

Proposition 9.9. For a finite group G, a Mackey functor determines and is determined by an additive contravariant functor $\mathscr{O}G \to \mathscr{A}b$.

As in [88], for a compact Lie group G we define a Mackey functor to be an additive contravariant functor $\mathscr{O}G \to \mathscr{A}b$. In this generality, our lack of control over composition in $\mathscr{O}G$ makes it difficult to construct Mackey functors algebraically. Together with formal properties of the stable category already developed, the following example completes the construction of the orbit transfer

$$\tau\colon H^*(X/H;Z) \longrightarrow H^*(X/G;Z)$$

in ordinary cohomology that we outlined in [88].

Proposition 9.10. There is a unique Mackey functor $\underline{Z}\colon \mathscr{O}G \to \mathscr{A}b$ with the following properties.

(i) On objects, $\underline{Z}(\Sigma^\infty G/H^+) = Z$ for all $H \subset G$.

(ii) On maps $\Sigma^\infty G/H^+ \to \Sigma^\infty (G/H')^+$ induced by G-maps $f\colon G/H \to G/H'$, $\underline{Z}(\Sigma^\infty f^+)$ is the identity homomorphism.

(iii) On transfer maps $\tau\colon \Sigma^\infty(G/H')^+ \to \Sigma^\infty G/H^+$ induced by G-maps $G/H \to G/H'$ given by inclusions $H \subset H'$, $\underline{Z}(\tau)$ is multiplication by the Euler characteristic $\chi(H'/H)$.

Proof. Actually, by application of the double coset formula to the pullback diagram

$$\begin{tikzcd}
G \times_H (H'/H) \arrow[r] \arrow[d] & G/H, \arrow[d] \\
G/H \arrow[r] & G/H'
\end{tikzcd}$$

one can show that (iii) follows from (i) and (ii), but we shall give a different argument. The uniqueness of $\underline{Z}$ is immediate from Corollary 9.4. The problem is to show that the specifications (i)-(iii) are compatible with composition in $\mathscr{O}G$. We show this indirectly by displaying $\underline{Z}$ as a quotient Mackey functor of the Burnside ring Mackey functor $\underline{A}\colon \mathscr{O}G \to \mathscr{A}b$ specified by

$$\underline{A}(\Sigma^\infty G/H^+) = A(H) \cong \pi_H^0(S) \cong \pi_G^0(\Sigma^\infty G/H^+).$$

Here the last form of the definition displays the functoriality of $\underline{A}$. Define $\underline{I}(\Sigma^\infty G/H^+)$ to be the kernel $I(H)$ of the augmentation $\varepsilon\colon A(H) \to Z$. Here, for a finite H-space Y, $\varepsilon[Y]$ is the nonequivariant Euler characteristic of Y. We claim that $\underline{I}$ is a subfunctor of $\underline{A}$. To see this, it suffices to check that if $\alpha\colon \Sigma^\infty G/K^+ \to \Sigma^\infty G/H^+$ is a G-map, then $\alpha^*(I(H)) \subset I(K)$. By Corollary 9.4, it suffices to check this when α is induced by a map π or c_g as in Lemma 2.17 and when α is a transfer. In the first case, this is immediate from the cited lemma. In the case of a transfer, this follows from the lemma and the formula

$$\chi(Y)\chi(H/K) = \chi(H \times_K Y)$$

for $K \subset H$ and a finite K-space Y; the formula holds by the multiplicativity on bundles of nonequivariant Euler characteristics. Now define $\underline{Z} = \underline{A}/\underline{I}$. Clearly $\underline{Z}$ satisfies (i) and (ii), and (iii) holds by the formula just cited.

§10. Normal subgroups in equivariant stable homotopy theory

We here formulate and prove our generalization of Theorem 9.1. Fix a normal subgroup Π of a compact Lie group Γ and let $E\mathscr{F}(\Pi;\Gamma)$ denote the universal Π-free Γ-space (denoted $E\mathscr{F}(\Pi)$ in the previous chapter, but we want to indicate the total group here). Let $Ad(\Pi;\Gamma)$ denote the adjoint representation of Γ derived from Π, that is, the tangent space of Π at e with Γ action induced by the conjugation action of Γ on Π. For based Γ-spaces X and Y and integers n, we agree to write

$$\{X,Y\}_n^\Gamma = [\Sigma^n\Sigma^\infty X, \Sigma^\infty Y]_\Gamma,$$

where the right side is computed in a complete Γ-universe U. Let $G = \Gamma/\Pi$ and regard G-spaces as Γ-spaces by pullback. With these notations, we have the following splitting theorem.

Theorem 10.1. Let X be a based G-space and Y be a based Γ-space. Assume either that X is a finite G-CW complex or that Π is finite. Then there is a natural isomorphism

$$\{X,Y\}_*^\Gamma \cong \sum_{(\Lambda)} \{X, E\mathscr{F}(W_\Pi\Lambda;W_\Gamma\Lambda)^+ \wedge_{W_\Pi\Lambda} \Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)} Y^\Lambda\}_*^{W_\Gamma\Lambda/W_\Pi\Lambda},$$

where the sum runs over the Γ-conjugacy classes of subgroups Λ of Π.

We shall shortly display $W_\Gamma\Lambda/W_\Pi\Lambda$ as a subgroup of G and so fix an action of this group on X; the remaining group actions are the evident ones. If we set $\Pi = \Gamma$ and rename it G, then Theorem 10.1 reduces to a slight generalization of Theorem 9.1. Of course, the adjoint representations here are all zero when G is finite. As will be explained elsewhere [106], Theorem 10.1 then implies an interesting generalization of the Segal conjecture.

We shall write down an explicit natural homomorphism θ from the sum displayed in Theorem 10.1 to $\{X,Y\}_*^\Gamma$; θ is defined for any X, and we shall prove it to be an isomorphism when X is finite. When Π is finite, θ specifies a morphism of cohomology theories for each fixed Y, and the $\lim^1$ exact sequence will imply that θ is an isomorphism for all X. In general, the sum is infinite, hence the wedge axiom fails, and we cannot expect θ to be an isomorphism for all X.

Virtually all of our work proceeds one Λ at a time, and, in an attempt to avoid unreadable clusters of symbols, we fix abbreviated notations to be adhered to throughout this section and the next.

Notations 10.2. Fix $\Lambda \subset \Pi$, where Π is a normal subgroup of Γ.

(i) Let N denote the normalizer of Λ in Γ and W denote N/Λ.

(ii) Let $N' = N \cap \Pi$, so that N' is the normalizer of Λ in Π, and let $W' = N'/\Lambda$.

(iii) Let L denote the tangent N-representation at the identity coset of Γ/N.

(iv) Let A denote the adjoint representation of W derived from W'.

(v) Let E denote the universal W'-free W-space.

Of course, N' is a normal subgroup of N, W' is a normal subgroup of W, and, since Π is normal in Γ, the product $N\Pi$ is a subgroup of Γ.

Lemma 10.3. (i) The identifications

$$W/W' \cong N/N' = N/N \cap \Pi \cong N\Pi/\Pi$$

display W/W' as a subgroup of $G = \Gamma/\Pi$.

(ii) L contains no Λ-trivial summands; that is, $L^\Lambda = 0$.

(iii) $N\Pi$ has finite index in Γ, hence W/W' has finite index in G.

Proof. (i) is clear. For (ii), it suffices to show that the path component of eN in $(\Gamma/N)^\Lambda$ consists of the single point eN. Let $\alpha\colon I \to (\Gamma/N)^\Lambda$ be a path starting at eN. We may lift α to a path $\tilde\alpha\colon I \to \Gamma$ starting at e. For $t \in I$, $\alpha_t = \tilde\alpha_t N$ is Λ-fixed, hence $\lambda\tilde\alpha_t \in \tilde\alpha_t N$ and thus $\tilde\alpha_t^{-1}\lambda\tilde\alpha_t \in N$ for all $\lambda \in \Lambda$. Define $\bar\alpha\colon \Lambda \times I \to N$ by $\bar\alpha_t(\lambda) = \tilde\alpha_t^{-1}\lambda\tilde\alpha_t$. Since $\bar\alpha$ is a homotopy through homomorphisms, [35, 38.1] implies that $\bar\alpha_1(\lambda) = n^{-1}\bar\alpha_0(\lambda)n$ for some $n \in N$ and all $\lambda \in \Lambda$. Since $\bar\alpha_0(\lambda) = \lambda$ and $n \in N$, $\bar\alpha_1(\lambda) \in \Lambda$ and thus $\tilde\alpha_1 \in N$ and $\alpha_1 = eN$. For (iii), observe that $L = L' + L''$ as an N-representation, where L' is the tangent $N\Pi$-representation at $eN\Pi \in \Gamma/N\Pi$ and L'' is the tangent N-representation at $eN \in N\Pi/N$. Since $\Gamma/N\Pi \cong G/(N\Pi/\Pi)$, Π and thus also Λ acts trivially on L'. By (ii), $L^\Lambda = 0$. Therefore $L' = (L')^\Lambda = 0$ and $N\Pi$ has finite index in Γ.

The homomorphism θ is the sum over Γ-conjugacy classes (Λ) of homomorphisms

$$\theta_\Lambda\colon \{X, E^+ \wedge_{W'} \Sigma^A Y^\Lambda\}_*^{W/W'} \longrightarrow \{X,Y\}_*^\Gamma.$$

The definition of θ_Λ will use the change of group and universe functor

$$\varepsilon^\# = j_*\varepsilon^*\colon \bar{h}W\mathscr{S}U^\Lambda \longrightarrow \bar{h}N\mathscr{S}U,$$

where $\varepsilon\colon N \to W$ is the quotient homomorphism and $j\colon U^\Lambda \to U$ is the inclusion of the Λ-fixed point N-universe in our complete Γ-universe U. Recall that $\varepsilon^\#$ preserves suspension spectra in the sense that $\varepsilon^\#\Sigma^\infty \cong \Sigma^\infty\varepsilon^*$ on $\bar{h}W\mathscr{T}$.

Definition 10.4. Define θ_Λ by commutativity of the diagram

$$\begin{tikzcd}
\{X, E^+\wedge_{W'}\Sigma^A Y^\Lambda\}^{W/W'}_* \arrow[r, "\tilde{\tau}_*"] \arrow[d, "\theta_\Lambda"'] & \{X, E^+\wedge Y^\Lambda\}^W_* \arrow[r, "\lambda"] & \{X, \Sigma^L(E^+\wedge Y)\}^N_* \arrow[d, "\omega"] \\
\{X,Y\}^\Gamma_* & \{X, (\Gamma\times_N E)^+\wedge Y\}^\Gamma_* \arrow[l, "(\rho^+\wedge 1)_*"'] & \{X, \Gamma^+\wedge_N(E^+\wedge Y)\}^\Gamma_*. \arrow[l, "\zeta_*"']
\end{tikzcd}$$

Here $\tilde{\tau}_*$ is a generalized Adams isomorphism given by II.7.2, ω is a Wirthmuller isomorphism given by II.6.5, ζ is the standard Γ-homeomorphism

$$\Gamma^+\wedge_N(E^+\wedge Y) \longrightarrow (\Gamma\times_N E)^+\wedge Y,$$

ρ is the collapse map $\Gamma\times_N E\to\{*\}$, and λ is the composite

$$\{X, E^+\wedge Y^\Lambda\}^W_* \xrightarrow{\varepsilon^\#} \{X, E^+\wedge Y^\Lambda\}^N_* \xrightarrow{i_*} \{X, \Sigma^L(E^+\wedge Y)\}^N_*,$$

where $i\colon E^+\wedge Y^\Lambda = (\Sigma^L(E^+\wedge Y))^\Lambda \to \Sigma^L(E^+\wedge Y)$ is the inclusion of the Λ-fixed point set.

We wish to prove that $\theta = \underset{(\Lambda)}{\Sigma}\theta_\Lambda$ is an isomorphism when X is finite. For each fixed X, θ is a morphism of homology theories in Y. As in Proposition 7.5, it suffices to prove that θ is an isomorphism when Y is replaced by $E(\mathcal{F}',\mathcal{F})\wedge Y$ for any adjacent pair of families $(\mathcal{F}',\mathcal{F})$ in Γ. The first part of the following lemma shows that, in this situation, the domain of θ_Λ is zero for all but one of our summands (Λ).

Lemma 10.5. Let $(\mathcal{F}',\mathcal{F})$ be an adjacent pair of families in Γ with $\mathcal{F}'-\mathcal{F}=(\Omega)$.
(i) If $(\Omega\cap\Pi)\neq(\Lambda)$, then $E^+\wedge E(\mathcal{F}',\mathcal{F})^\Lambda$ is W-contractible.
(ii) If $(\Omega\cap\Pi)=(\Lambda)$, then $(\Gamma\times_N E)^\Omega$ is nonequivariantly contractible.

Proof. (i). Consider $E^+\wedge E(\mathcal{F}',\mathcal{F})^\Lambda$ as an N-space. For $K\subset N$, E^K is empty unless $K\cap\Pi\subset\Lambda$ and $E(\mathcal{F}',\mathcal{F})^{\Lambda K}$ is contractible unless $(\Lambda K)=(\Omega)$. Thus $E^+\wedge E(\mathcal{F}',\mathcal{F})^\Lambda$ is N-contractible unless there exists $K\subset N$ such that both $K\cap\Pi\subset\Lambda$ and $(\Lambda K)=(\Omega)$. Replacing Ω by a conjugate if necessary, we may assume that $\Omega=\Lambda K$. Then $\Lambda\subset\Omega$ and thus $\Lambda\subset\Omega\cap\Pi$. Conversely,

$$\Omega\cap\Pi = \Lambda K\cap\Pi = \Lambda(K\cap\Pi)\subset\Lambda\Lambda=\Lambda.$$

and thus $\Lambda=\Omega\cap\Pi$.
(ii) We may assume that $\Lambda=\Omega\cap\Pi$. Since Π is normal in Γ, this implies that Ω is contained in N. Let $(\gamma,x)\in(\Gamma\times_N E)^\Omega$, $\gamma\in\Gamma$ and $x\in E$. Then $\gamma^{-1}\Omega\gamma$ is contained in the isotropy group N_x of x and

$$\gamma^{-1}\Lambda\gamma = (\gamma^{-1}\Omega\gamma)\cap\Pi\subset N_x\cap\Pi\subset\Lambda.$$

Thus $\gamma\in N$ and $(\gamma,x)=(e,\gamma x)$ with $\gamma x\in E^\Omega$. It follows that $(\Gamma\times_N E)^\Omega$ is homeomorphic to the contractible space E^Ω.

When X is a finite G-CW complex, we have

$$\{X,Y\}^\Gamma_n = \begin{cases} \operatorname*{colim}_{V\subset U}[\Sigma^V\Sigma^n X, \Sigma^V Y]_\Gamma & \text{if } n\geq 0 \\ \operatorname*{colim}_{V\subset U}[\Sigma^V X, \Sigma^V\Sigma^{-n}Y]_\Gamma & \text{if } n<0, \end{cases}$$

and similarly for the other groups appearing in Definition 10.4. In view of that definition and the previous lemma, we need only check that the component maps λ and $(\rho^+\wedge 1)_*$ of θ_Λ are isomorphisms when Y is replaced by $E(\mathcal{F}',\mathcal{F})\wedge Y$, where $\mathcal{F}'-\mathcal{F}=(\Omega)$ with $(\Omega\cap\Pi)=(\Lambda)$. It is an easy matter to interpret these maps in terms of space level colimits, and we claim that they are both colimits of bijections. For $(\rho^+\wedge 1)_*$, this is immediate from Lemma 7.4 and the contractibility of $(\Gamma\times_N E)^\Omega$.

Thus it remains to consider λ. Passage to Λ-fixed points on representative maps gives a homomorphism

$$\phi\colon \{X, \Sigma^L(E^+\wedge E(\mathcal{F}',\mathcal{F})\ Y)\}^N_* \longrightarrow \{X, E^+\wedge E(\mathcal{F}',\mathcal{F})^\Lambda\ Y^\Lambda\}^W_*.$$

An immediate inspection shows that $\phi\lambda$ is the identity, hence it suffices to show that ϕ is an isomorphism. As an N-space, $E(\mathcal{F}',\mathcal{F})$ is $E(\mathcal{F}'|N,\mathcal{F}|N)$. Of course, the pair $(\mathcal{F}'|N,\mathcal{F}|N)$ need not be adjacent; $(\Omega)|N$ is a disjoint union of N-conjugacy classes (Ψ), and it is clear by inspection of fixed points that $E(\mathcal{F}'|N,\mathcal{F}|N)$ is equivalent to the wedge over (Ψ) of the spaces $E(\mathcal{E}',\mathcal{E})$, where $(\mathcal{E}',\mathcal{E})$ is an adjacent pair in N with $\mathcal{E}'-\mathcal{E}=(\Psi)$. The map ϕ breaks up as a corresponding sum. Since Ψ is Γ-conjugate to Ω and $\Omega\cap\Pi$ is Γ-conjugate to Λ, $\Psi\cap\Pi$ is Γ-conjugate to Λ, say $\Psi\cap\Pi=\gamma\Lambda\gamma^{-1}$. Clearly $E^+\wedge E(\mathcal{E}',\mathcal{E})$ is N-contractible unless $\Psi\cap\Pi\subset\Lambda$, in which case $\gamma\in N$ and $\Psi\cap\Pi=\Lambda$. Thus, with $E(\mathcal{F}',\mathcal{F})$ replaced by $E(\mathcal{E}',\mathcal{E})$, the source and target of ϕ are zero unless $\Psi\cap\Pi=\Lambda$, in which case $\Lambda\subset\Psi\subset N$ and ϕ is a colimit of bijections by Lemma 7.10.

§11. Fixed point spectra of suspension spectra

We here reinterpret Theorem 10.1 as a calculation of fixed point spectra. Precisely, with the notations above that theorem, we have the following spectrum level splitting theorem.

<u>Theorem 11.1</u>. Let $G = \Gamma/\Pi$. For based Γ-spaces Y, there is a natural equivalence of G-spectra

$$(\Sigma^\infty Y)^\Pi \simeq \bigvee_{(\Lambda)} \Sigma^\infty(G^+\wedge_{W_\Gamma\Lambda/W_\Pi\Lambda}[E\mathscr{F}(W_\Pi\Lambda;W_\Gamma\Lambda)^+\wedge_{W_\Pi\Lambda}\Sigma^{Ad(W_\Pi\Lambda;W_\Gamma\Lambda)}Y^\Lambda]),$$

where the wedge runs over the Γ-conjugacy classes of subgroups Λ of Π.

Here Σ^∞ on the left refers to Γ and Σ^∞ on the right refers to G. If we set $\Pi = \Gamma$ and rename it G, then Theorem 11.1 is a reinterpretation of Theorem 9.1. For finite G, we proved this reinterpretation in [89], where we used it to prove the equivalence of the equivariant and nonequivariant forms of the Segal conjecture and to derive a generalization of the Segal conjecture in the context of classifying G-spaces $B(G,\Pi)$ for principal (G,Π)-bundles. In this connection, we conjectured the following result [89, Remark 11].

<u>Corollary 11.2</u>. Let G and Π be finite groups. Then there is an equivalence of G-spectra

$$S^\Pi \simeq \bigvee_{(\Lambda)} \Sigma^\infty B(G,W_\Pi\Lambda)^+,$$

where S is the sphere $(G\times\Pi)$-spectrum and the wedge runs over the conjugacy classes of subgroups Λ of Π.

<u>Proof</u>. Set $\Gamma = G\times\Pi$ and $Y = S^0$ in Theorem 9.1 and note that $W_\Gamma\Lambda = G\times W_\Pi\Lambda$.

We shall write down an explicit map ξ from the wedge sum in Theorem 11.1 to $(\Sigma^\infty Y)^\Pi$, and we shall deduce that ξ is an equivalence from a diagram chase relating ξ to θ. Again, we work one Λ at a time, and we adopt Notations 10.2. A little care is needed in fixing universes in which to work. Let U be a complete Γ-universe. We index Γ-spectra on U, G-spectra on U^Π, and W-spectra on U^Λ. We shall be using a transfer associated to W'-free W-spectra, and such spectra would normally be indexed on the complete W/W'-universe $U^{N'} = (U^\Lambda)^{W'}$. However, in view of Lemma 10.3(i), the universe U^Π, being $N\Pi/\Pi$-complete, must also be W/W'-complete. Thus the inclusion $U^\Pi \subset U^{N'}$ is a W-equivalence and we may index our W'-free W-spectra on U^Π. We have a triangle of inclusions

$$\begin{tikzcd} & U^\Pi \arrow[dl, "i"'] \arrow[dr, "k"] & \\ U^\Lambda \arrow[rr, "j"] & & U \end{tikzcd}$$

We shall write ε generically for quotient homomorphisms and $\varepsilon^\#$ for corresponding change of group and universe functors on stable categories. Thus we have

$$\varepsilon^\# = i_*\varepsilon^*: \bar{h}(W/W')\mathscr{S}U^\Pi \longrightarrow \bar{h}W\mathscr{S}U^\Lambda$$

$$\varepsilon^\# = j_*\varepsilon^*: \bar{h}W\mathscr{S}U^\Lambda \longrightarrow \bar{h}N\mathscr{S}U$$

$$\varepsilon^\# = k_*\varepsilon^*: \bar{h}G\mathscr{S}U^\Pi \longrightarrow \bar{h}\Gamma\mathscr{S}U$$

Upon restriction to W/W'-spectra, the last functor agrees with the composite of the first two. In each case, $\varepsilon^\#$ is left adjoint to the appropriate fixed point functor going the other way. Moreover, each composite $\varepsilon^\#\Sigma^\infty$ is isomorphic to $\Sigma^\infty\varepsilon^*$; thus, when applied to suspension spectra, the functors $\varepsilon^\#$ will be omitted from the notations.

The map ξ is defined to be the wedge sum of maps

$$\xi_\Lambda: \Sigma^\infty(G^+\wedge_{W/W'}(E^+\wedge_{W'}\Sigma^AY^\Lambda)) \longrightarrow (\Sigma^\infty Y)^\Pi$$

of G-spectra indexed on U^Π. To define ξ_Λ, it suffices to specify its adjoint map

$$\tilde{\xi}_\Lambda: \Sigma^\infty(G^+\wedge_{W/W'}(E^+\wedge_{W'}\Sigma^AY^\Lambda)) \longrightarrow \Sigma^\infty Y$$

of Γ-spectra indexed on U. To simplify notation, consider a general W/W'-space Z; we are thinking of $Z = E^+\wedge_{W'}\Sigma^AY^\Lambda$. In view of Lemma 10.3(i), we have a Γ-homeomorphism

$$G^+\wedge_{W/W'}Z \cong \Gamma^+\wedge_{N\Pi}Z.$$

The projection $\Gamma^+\wedge_N Z \to \Gamma^+\wedge_{N\Pi}Z$ may be viewed as a based $(\Gamma,N\Pi)$-bundle with fibre $N\Pi/N$ since it can be written in the form

$$(\Gamma^+\wedge Z)\wedge_{N\Pi}(N\Pi/N)^+ \longrightarrow (\Gamma^+\wedge Z)/N\Pi,$$

where $\Gamma\times N\Pi$ acts on $\Gamma^+\wedge Z$ via the left Γ action on Γ, the right $N\Pi$ action on Γ, and the $N\Pi$ action on Z derived from Lemma 10.3(i). By IV.3.1, we have a transfer map of Γ-spectra indexed on U

$$\tau: \Sigma^\infty(\Gamma^+\wedge_{N\Pi}Z) \longrightarrow \Sigma^\infty(\Gamma^+\wedge_N Z).$$

Here Σ^∞ commutes with change of groups; that is,

$$\Sigma^\infty(\Gamma^+\wedge_N Z) \cong \Gamma\times_N\Sigma^\infty Z,$$

and similarly for the domain of τ. Now suppose that $Z = (\Sigma^AF)/W'$, where F is a based W'-free W-space. By II.7.5 (and the relation $i_*\Sigma^\infty \cong \Sigma^\infty$), we have a dimension-shifting transfer map

$$\tau\colon \Sigma^\infty Z \longrightarrow \Sigma^\infty F$$

of W-spectra indexed on U^Λ. Applying the change of group and universe functor associated to $\varepsilon\colon N \to W$, we may view τ as a map of N-spectra indexed on U. Taking $F = E^+\wedge Y^\Lambda$, we define $\tilde{\xi}_\Lambda$ by commutativity of the following diagram of stable Γ-maps. By an abuse justified by the commutativity relations relating Σ^∞ to all functors in sight, we suppress Σ^∞ from the notation.

$$\begin{tikzcd}
G\wedge_{W/W'}(E'\wedge_{W'}\Sigma^A Y^\Lambda) \cong \Gamma^+\wedge_{N\Pi}(E^+\wedge_{W'}\Sigma^A Y^\Lambda) \arrow[r, "\tau"] \arrow[d, "\tilde{\xi}_\Lambda"'] & \Gamma^+\wedge_N(E^+\wedge_{W'}\Sigma^A Y^\Lambda) \arrow[d, "1\wedge\tau"] \\
Y & (\Gamma\times_N E)^+\wedge Y \cong \Gamma^+\wedge_N(E^+\wedge Y) \arrow[l, "\rho\wedge 1"'] & \Gamma^+\wedge_N(E^+\wedge Y^\Lambda) \arrow[l, "1\wedge 1\wedge i"']
\end{tikzcd}$$

To prove Theorem 11.1, it suffices to check that the G-map ξ induces an iosmorphism on $\pi_*^{K/\Pi}$ for all subgroups K of Γ which contain Π. By the Π-fixed point adjunction,

$$\pi_n^{K/\Pi}(D^\Pi) = [\Sigma^n\Sigma^\infty\Gamma/K^+, D^\Pi]_G \cong [\Sigma^n\Sigma^\infty\Gamma/K^+, D]_\Gamma = \pi_n^K(D)$$

for any $n \in Z$ and $D \in \Gamma\mathscr{A}U$. For a G-space T and G-map $\zeta\colon \Sigma^\infty T \to D^\Pi$ with adjoint Γ-map $\tilde{\zeta}\colon \Sigma^\infty T \to D$, the following diagram commutes, where X is Γ/K^+ or any other G-space.

$$\begin{tikzcd}
{[\Sigma^n\Sigma^\infty X, \Sigma^\infty T]_G} \arrow[r, "\zeta_*"] \arrow[d, "\varepsilon^\#"'] & {[\Sigma^n\Sigma^\infty X, D^\Pi]_G} \arrow[d, phantom, "\cong"] \\
{[\Sigma^n\Sigma^\infty X, \Sigma^\infty T]_\Gamma} \arrow[r, "\tilde{\zeta}_*"] & {[\Sigma^n\Sigma^\infty X, D]_\Gamma}.
\end{tikzcd}$$

In particular, $(\xi_\Lambda)_*$ agrees up to isomorphism with $(\tilde{\xi}_\Lambda)_*\varepsilon^\#$. We shall prove that the following diagram commutes.

$$\begin{tikzcd}
\{X, E^+\wedge_{W'}\Sigma^A Y^\Lambda\}_*^{W/W'} \arrow[r, "\omega"] \arrow[d, "\theta_\Lambda"'] & \{X, G^+\wedge_{W/W'}(E^+\wedge_{W'}\Sigma^A Y^\Lambda)\}_*^G \arrow[d, "\varepsilon^\#"] \\
\{X, Y\}_*^\Gamma & \{X, G^+\wedge_{W/W'}(E^+\wedge_{W'}\Sigma^A Y^\Lambda)\}_*^\Gamma \arrow[l, "(\tilde{\xi}_\Lambda)_*"']
\end{tikzcd}$$

Here ω is a Wirthmüller isomorphism given by II.6.5; no representation occurs in this instance of ω by Lemma 10.3(iii). Granting the diagram, it is clear that Theorem 10.1 implies Theorem 11.1.

To chase our diagram, abbreviate $Z = E^+\wedge_{W'}\Sigma^A Y^\Lambda$ and $F = E^+\wedge Y^\Lambda$. We may expand the diagram as follows.

$$\begin{tikzcd}
\{X,Z\}_*^{W/W'} \arrow[rr, "\omega"] \arrow[dd, "\varepsilon^\#"'] \arrow[ddr, "\varepsilon^\#"] & & \{X, G^+\wedge_{W/W'}Z\}_*^G \arrow[r, "\varepsilon^\#"] & \{X, G^+\wedge_{W/W'}Z\}_*^\Gamma \arrow[d, phantom, "\cong"] \\
 & & & \{X, \Gamma^+\wedge_{N\Pi}Z\}_*^\Gamma \arrow[d, "\tau_*"] \\
\{X,Z\}_*^W \arrow[r, "\varepsilon^\#"] \arrow[d, "\tau_*"'] & \{X,Z\}_*^N \arrow[r, "(1\wedge e)_*"] \arrow[d, "\tau_*"] & \{X, \Sigma^L Z\}_*^N \arrow[r, "\omega"] \arrow[d, "(\Sigma^L\tau)_*"] & \{X, \Gamma^+\wedge_N Z\}_*^\Gamma \arrow[d, "(1\wedge\tau)_*"] \\
\{X,F\}_*^W \arrow[r, "\varepsilon^\#"] \arrow[dd] & \{X,F\}_*^N \arrow[r, "(1\wedge e)_*"] \arrow[dr, "(1\wedge i\wedge e)_*"'] & \{X, \Sigma^L F\}_*^N \arrow[r, "\omega"] \arrow[d, "\Sigma^L(1\wedge i)_*"] & \{X, \Gamma^+\wedge_N F\}_*^\Gamma \arrow[d, "(1\wedge 1\wedge i)_*"] \\
 & & \{X, \Sigma^L(E^+\wedge Y)\}_*^N \arrow[r, "\omega"] & \{X, \Gamma^+\wedge_N(E^+\wedge Y)\}_*^\Gamma \arrow[d, phantom, "\cong"] \\
\{X,Y\}_*^\Gamma & & & \{X, (\Gamma\times_N E)^+\wedge Y\}_*^\Gamma \arrow[lll, "(\rho\wedge 1)_*"']
\end{tikzcd}$$

The four little squares are just naturality diagrams, the two little triangles clearly commute, and the unlabeled arrow is defined by commutativity of the bottom subdiagram. Since $\tau_*\varepsilon^\# = \tilde{\tau}_*$, the left vertical composite is θ_Λ. The composite from the top right to the bottom left is $(\tilde{\xi}_\Lambda)_*$. Thus it only remains to check that the top right subdiagram commutes, and for this Z can be any W/W'-space. Recalling that ω is induced by a map of the same name (as in II.6.2) and that $\varepsilon^\# = k_*\varepsilon^*$ in this subdiagram, we see that it suffices to prove that the following diagram of G-spectra commutes

$$\begin{tikzcd}
k_*F_{W/W'}[G, \Sigma^\infty Z) \arrow[r, "k_*\omega"] \arrow[d, "\nu"] & k_*(G \ltimes_{W/W'}\Sigma^\infty Z) \arrow[d, phantom, "\cong"] \\
F_N[\Gamma, \Sigma^\infty Z) \arrow[d, "{F[1,1\wedge e)}"'] & \Gamma \ltimes_{N\Pi}\Sigma^\infty Z \arrow[d, "\tau"] \\
F_N[\Gamma, \Sigma^L\Sigma^\infty Z) \arrow[r, "\omega"] & \Gamma \ltimes_N \Sigma^\infty Z
\end{tikzcd}$$

Here ν is the Γ-map which coextends the N-map

$$k_*F_{W/W'}[G, \Sigma^\infty Z) \longrightarrow k_*\Sigma^\infty Z \cong \Sigma^\infty Z$$

obtained by applying k_* to the evaluation W/W'-map regarded as an N-map. Since the maps ω here are equivalences, this diagram will commute if it commutes with the maps ω replaced by their inverses ψ. By II.6.8, the maps ψ are coextensions of maps μ, so that the last diagram will commute in $\bar{h}\Gamma\mathscr{S}U$ if the following diagram commutes in $\bar{h}N\mathscr{S}U$.

$$\begin{tikzcd}
k_*\Sigma^\infty Z & & & k_*(G \ltimes_{W/W'} \Sigma^\infty Z) \arrow[lll, "k_*\mu"'] \arrow[d, "\cong"] \\
\Sigma^\infty Z \arrow[u, "\cong"'] \arrow[dr, "\tau"] \arrow[dd, "1\wedge e"'] & & & \Gamma \ltimes_{N\Pi} \Sigma^\infty Z \arrow[lll, "\mu"'] \arrow[dl, "1\ltimes\tau"'] \arrow[dd, "\tau"] \\
& N\Pi \ltimes_N \Sigma^\infty Z \arrow[dl, "\mu"] & \Gamma \ltimes_{N\Pi}(N\Pi \ltimes_N \Sigma^\infty Z) \arrow[l, "\mu"'] \arrow[dr, phantom, "\cong"] & \\
\Sigma^L\Sigma^\infty Z & & & \Gamma \ltimes_N \Sigma^\infty Z \arrow[lll, "\mu"']
\end{tikzcd}$$

Here the top rectangle commutes in view of the space level identification of the maps μ given in II.6.9. The trapezoid in the middle commutes by the naturality of μ. The trapezoid at the bottom commutes by the transitivity of μ given in II.6.13(b). The right triangle commutes by IV.4.3(iii). The left triangle commutes by the space level identifications of μ and τ (see II.6.9 and II.5.5 for μ and IV.3.1 and II.6.15 for τ) and the space level homotopy of II.5.9; alternatively, this triangle is implied by II.6.16.

VI. Twisted half smash products and extended powers

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The extended powers of spectra used in $[H_\infty]$ are special cases of a general three stage construction in equivariant stable homotopy theory. One starts with a functor from some category to the category of G-spectra, one forms a twisted half smash product defined on G-spectra, and one then passes to orbits with respect to G or to some subgroup H of G. In particular, with $H = G = \Sigma_j$, $D_j\colon \mathscr{S}R^\infty \to \mathscr{S}R^\infty$ is the composite of the three functors

$E \to E^{(j)}$	from	$\mathscr{S}R^\infty$ to $\Sigma_j\mathscr{S}(R^\infty)^j$
$E \to E\Sigma_j \ltimes E$	from	$\Sigma_j\mathscr{S}(R^\infty)^j$ to $\Sigma_j\mathscr{S}R^\infty$
$E \to E/\Sigma_j$	from	$\Sigma_j\mathscr{S}R^\infty$ to $\mathscr{S}R^\infty$.

The first and third functors have already been defined, and this chapter is devoted to the construction and analysis of twisted half-smash products of G-spaces and G-spectra. As usual, G is to be a compact Lie group throughout. When we turn to extended powers, this generality will make it a simple matter to replace G by $G \times \Sigma_j$ and so obtain the extended powers D_jE of a G-spectrum E via composite functors

$$G\mathscr{S}U \xrightarrow{(?)^{(j)}} (G \times \Sigma_j)\mathscr{S}U^j \xrightarrow{E\Sigma_j\ltimes(?)} (G \times \Sigma_j)\mathscr{S}U \xrightarrow{(?)/\Sigma_j} G\mathscr{S}U.$$

On the level of spaces, we have the definition

$$X \ltimes Y = X^+ \wedge Y.$$

If G acts on both spaces, it acts diagonally on $X \ltimes Y$. Now replace Y by a G-spectrum E. Then $X^+ \wedge E$ is a perfectly good G-spectrum. We insist that this is not what ought to be meant by a twisted half smash product, and we agree to reserve the notation $X \ltimes E$ for genuine twisted half smash products. These will be given by a very different construction. Nevertheless, our insistence on the distinction notwithstanding, it will turn out that twisted half-smash products are in fact equivalent to composites of change of universe functors and untwisted half-smash products $X^+ \wedge (?)$. This fact in no way diminishes the importance and utility of the new construction: the relevant change of universe functors are simple to define but impossible to analyze effectively.

Recall from II§1 that, for G-universes U and U', $\mathscr{I}(U,U')$ denotes the function G-space of linear isometries $U \to U'$. As pointed out in II.2.11,

$\mathscr{I}(U,U')$ is a universal $\mathscr{F}$-space, where $\mathscr{F} = \mathscr{F}(U,U')$ is the family of subgroups H of G for which there exists an H-linear isometry $U \to U'$. The "twisting" in our construction is encoded by a G-map $\chi: X \to \mathscr{I}(U,U')$. For there to be such a map, X must be an $\mathscr{F}$-space, and we agree once and for all that X is always required to have the homotopy type of a G-CW complex and thus of an $\mathscr{F}$-CW complex. There is then a unique homotopy class of G-maps χ. For $E \in G\mathscr{S}U$, we shall construct $\chi \ltimes E \in G\mathscr{S}U'$. Homotopic G-maps will lead to canonically equivalent spectrum level functors, and we shall use the notation $X \ltimes E$ for the resulting functor on the level of stable categories. This dichotomy between spectrum level and stable category level functors and maps must be kept in mind throughout the theory to follow; in particular, we use the notation $\cong$ for isomorphisms of spectra and the notation $\simeq$ for isomorphisms in the relevant stable category.

For clarity, we catalog the properties of our functors in section 1, deferring the detailed constructions and proofs until section 2. We study coherence diagrams relating smash products and twisted half smash products in section 3. Throughout the first three sections, we work with G-spectra for our fixed ambient group G.

The homotopical behavior of our functors $\chi \ltimes E$ is governed by the nature of the family $\mathscr{F}(U,U')$. For example, if there is a G-linear isometry $f: U \to U'$, then $\mathscr{I}(U,U')$ is G-contractible and there is a natural equivalence

$$(*) \qquad \chi \ltimes E \simeq X^{+} \wedge f_{*}E.$$

This degeneracy will play a crucial role in our theory, for instance in our study of cellular chains in Chapter VIII. The point is that $\chi \ltimes E$ is given by a fairly complicated construction, whereas $X^{+} \wedge f_{*}E$ is quite simple to analyze. In the application to cellular chains, G will be the trivial group, another group π will act on everything in sight, and the previous result will compute the underlying nonequivariant spectrum of the π-spectrum $\chi \ltimes E$. More generally, we can apply the result to compute the underlying G-spectrum of the $(G \times \pi)$-spectrum $\chi \ltimes E$ when $G \times \pi$ acts on everything in sight and there is a G-linear isometry $U \to U'$. We study this situation in detail in section 4, the essential point being an analysis of the behavior of the action of π with respect to the G-equivalence (*).

Finally, in section 5, permutation groups make their first explicit appearance in our development of equivariant stable homotopy theory. Here we discuss the extended powers $\chi \ltimes_{\pi} E^{(j)}$ for a G-spectrum E and subgroup π of Σ_j.

The reader is urged to skip sections 3 and 4 on a first reading. These sections are full of technical coherence diagrams. While essential to the theory, it cannot be pretended that this material makes interesting reading.

§1. Statements of results about $\chi \ltimes E$

We begin with the basic formal properties of $\chi \ltimes E$. The reader is assumed to be familiar with the definitions of I§2.

Theorem 1.1. Let $\mathcal{A}$ and $\mathcal{A}'$ be indexing sets in G-universes U and U', let X be an (unbased) G-space, and let $\chi: X \to \mathscr{I}(U,U')$ be any G-map. Let $E \in G\mathscr{S}\mathcal{A}$ and $E' \in G\mathscr{S}\mathcal{A}'$. Then there are functorial constructions of G-spectra

$$\chi \ltimes E \in G\mathscr{S}\mathcal{A}' \qquad \text{and} \qquad F[\chi,E') \in G\mathscr{S}\mathcal{A}.$$

For fixed χ, these functors are left and right adjoint:

$$G\mathscr{S}\mathcal{A}'(\chi \ltimes E,E') \cong G\mathscr{S}\mathcal{A}(E,F[\chi,E')).$$

The functor $\chi \ltimes E$ preserves colimits in both variables. The functor $F[\chi,E')$ preserves limits in E' and converts colimits in χ to limits.

Functoriality in χ refers to the category of G-spaces over $\mathscr{I}(U,U')$, and colimits in χ are to be taken in this sense. We have the following behavior with respect to subobjects; compare I§8.

Proposition 1.2. The functor $\chi \ltimes E$ preserves injections, closed inclusions, and intersections of closed subobjects in both variables. If A is a closed G-subspace of X, $\alpha = \chi|A$, and D is a closed G-subspectrum of E, then

$$\alpha \ltimes D = (\chi \ltimes D) \cap (\alpha \ltimes E) \subset \chi \ltimes E.$$

The constructions enjoy the following basic invariance properties.

Proposition 1.3. The functors $\chi \ltimes E$ and $F[\chi,E')$ are independent of the choices of indexing sets and G-universes in the sense that they commute up to coherent natural isomorphism with the various isomorphisms of categories of G-spectra specified in I.2.4 and I.2.5.

The most elementary examples are those with X a point.

Examples 1.4. A G-map $\{*\} \to \mathscr{I}(U,U')$ is clearly just a choice of a G-linear isometry $f: U \to U'$, and we shall see that $f \ltimes E = f_{*}E \in G\mathscr{S}U'$ for $E \in G\mathscr{S}U$ and $F[f,E') = f^{*}E' \in G\mathscr{S}U$ for $E' \in G\mathscr{S}U'$.

Perhaps the main interest in the right adjoints $F[\chi,E']$ is that their existence and properties yield simple proofs of the results we want about $\chi \ltimes E$. In particular, the commutation isomorphisms involving $F[\chi,E']$ in the following result are easily checked, and the isomorphisms involving $\chi \ltimes E$ follow formally, by I.3.5.

Proposition 1.5. For a G-map $\chi: X \to \mathcal{J}(U,U')$ and for $Y \in G\mathcal{J}$, $E \in G\mathscr{S}\mathcal{U}$, and $E' \in G\mathscr{S}\mathcal{U}'$, there are natural isomorphisms

$$\chi \ltimes (E \wedge Y) \cong (\chi \ltimes E) \wedge Y \quad\text{and}\quad F(Y,F[\chi,E']) \cong F[\chi,F(Y,E')];$$

$$\chi \ltimes \Sigma^\infty Y \cong \Sigma^\infty(X^+ \wedge Y) \quad\text{and}\quad F(X^+,\Omega^\infty E') \cong \Omega^\infty F[\chi,E'].$$

The following commutation relation is a bit more delicate.

Proposition 1.6. For a G-map $\chi: X \to \mathcal{J}(U,U')$ with X compact, for $V \in \mathcal{U}$ and $V' \in \mathcal{U}'$ with $V \cong V'$ as G-modules, and for $Y \in G\mathcal{J}$, there is a natural isomorphism of G-spectra

$$\chi \ltimes \Lambda^V\Sigma^\infty Y \cong \Lambda^{V'}\Sigma^\infty(X^+ \wedge Y).$$

For noncompact X, the conclusion holds after passage to the stable category.

Of course, there may be no $V' \in \mathcal{U}'$ with $V \cong V'$. A more general result applies to compute $\chi \ltimes \Lambda^V\Sigma^\infty Y$ for such V, but its statement requires more notation; see Proposition 2.15.

The first isomorphisms of Proposition 1.5 have the following analogs for the variable χ, the right adjoint isomorphism again being easy and implying the other.

Proposition 1.7. Let $\chi: X \to \mathcal{J}(U,U')$ be a G-map, W be a G-space, and $\pi: X \times W \to X$ be the projection. For $E \in G\mathscr{S}\mathcal{U}$ and $E' \in G\mathscr{S}\mathcal{U}'$, there are natural isomorphisms

$$(\chi\pi) \ltimes E \cong (\chi \ltimes E) \wedge W^+ \quad\text{and}\quad F[\chi\pi,E'] \cong F(W^+,F[\chi,E']).$$

We have one last set of commutation relations. The first of each of the following pairs will be immediate from the constructions and will imply the second by I.3.5.

Proposition 1.8. Let $\chi: X \to \mathcal{J}(U,U')$ be a G-map and $\alpha: H \to G$ be a map of Lie groups. For $D \in G\mathscr{S}\mathcal{U}$, $D' \in G\mathscr{S}\mathcal{U}'$, $E \in H\mathscr{S}\mathcal{U}$, and $E' \in H\mathscr{S}\mathcal{U}'$, there are natural

isomorphisms

$$\chi \ltimes \alpha^*D \cong \alpha^*(\chi \ltimes D) \quad\text{and}\quad F[\chi,F_\alpha[G,E']) \cong F_\alpha[G,F[\chi,E']);$$

$$F[\chi,\alpha^*D') \cong \alpha^*F[\chi,D') \quad\text{and}\quad \chi \ltimes (G \ltimes_\alpha E) \cong G \ltimes_\alpha (\chi \ltimes E);$$

If $X = G \times_\alpha Y$ and χ is the G-map extending an H-map $\psi: Y \to \mathcal{J}(U,U')$, there are natural isomorphisms

$$\chi \ltimes D \cong G \ltimes_\alpha (\psi \ltimes \alpha^*D) \quad\text{and}\quad F[\chi,F_\alpha[G,E')) \cong F_\alpha[G,F[\psi,E')).$$

See II§4 for the relevant definitions. The following remarks are based on II.4.15 and will play an important role in the technical work of section 4.

Remarks 1.9. The special cases with $H = e$ give isomorphisms

$$\chi \ltimes (G \ltimes E) \cong G \ltimes (\chi \ltimes E) \quad\text{and}\quad F[\chi,F[G,E')) \cong F[G,F[\chi,E'))$$

for $E \in \mathscr{S}\mathcal{U}$ and $E' \in \mathscr{S}\mathcal{U}'$. On the right sides, χ is viewed simply as a map, with G actions ignored. We have the following compatibility diagrams for the monad structure maps:

$$\begin{tikzcd} & \chi \ltimes G \ltimes E \arrow[d, phantom, "\cong" rotate=90] \\ \chi \ltimes E \arrow[ur, "1 \ltimes \eta"] \arrow[r, "\eta"'] & G \ltimes \chi \ltimes E \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \chi \ltimes G \ltimes G \ltimes E \arrow[r, "1 \ltimes \mu"] \arrow[d, phantom, "\cong" rotate=90] & \chi \ltimes G \ltimes E \arrow[d, phantom, "\cong" rotate=90] \\ G \ltimes G \ltimes \chi \ltimes E \arrow[r, "\mu"] & G \ltimes \chi \ltimes E. \end{tikzcd}$$

Moreover, for a G-spectrum E, the G action on $\chi \ltimes E$ is given by

$$1 \ltimes \xi: G \ltimes \chi \ltimes E \cong \chi \ltimes G \ltimes E \longrightarrow \chi \ltimes E,$$

where ξ gives the G action on E. The dual assertions are valid for the comonad $F[G,?)$. As suggested by our notations, a comparison of definitions shows that if $\gamma: G \to \mathcal{J}(U,U)$ is specified by $\gamma(g)(u) = gu$, then

$$\gamma \ltimes E = G \ltimes E \quad\text{and}\quad F[\gamma,E) = F[G,E).$$

The results on hand suffice to analyze the behavior of $\chi \ltimes E$ with respect to cellular structures. The essential point is the following lemma on equivariant cells.

<u>Lemma 1.10</u>. Let $\varepsilon\colon G/H \times e^p \to \mathscr{I}(U,U')$ be a G-map with restriction σ to $G/H \times S^{p-1}$, $p \geq 1$, and let $e_J^q \in G\mathscr{S}\mathscr{A}$ be the q-cell $(G/J)^+ \wedge CS^q$, $q \in Z$. Let $L = G/H \times G/J$. Then there is a canonical isomorphism of pairs

$$(\varepsilon \ltimes e_J^q,\ \varepsilon \ltimes S_J^{q-1} \cup \sigma \ltimes e_J^q) \cong (L^+ \wedge CS^{p+q-1}, L^+ \wedge S^{p+q-1}).$$

<u>Proof</u>. This is immediate from Propositions 1.2, 1.5, and 1.6 by use of any fixed chosen space level homeomorphisms

$$((e^p)^+ \wedge CS^{q-1},\ (e^p)^+ \wedge S^{q-1} \cup (S^{p-1})^+ \wedge CS^{q-1}) = (CS^{p+q-1}, S^{p+q-1})$$

for $p \geq 1$ and $q \geq 1$; the case $q = 1$ accounts for all non-positive dimensional cells by virtue of Proposition 1.6 applied to the respective canonical shift desuspension functors $\Lambda^r\Sigma^\infty$.

The discussion above II.3.8 applies verbatim here, and we obtain the following result.

<u>Theorem 1.11</u>. Let $\chi\colon X \to \mathscr{I}(U,U')$ be a G-map, where X is a G-CW complex, and let E be a G-CW spectrum. Then $\chi \ltimes E$ may be given the sequential filtration

$$(\chi \ltimes E)_n = \bigcup_{p+q=n} \chi_p \ltimes E_q,\ n \geq 0,$$

and the skeletal filtration

$$(\chi \ltimes E)^n = \bigcup_{p+q=n} \chi_p \ltimes E^q,\ n \in Z,$$

where χ_p is the restriction of χ to the p-skeleton of X and $\{E_q | q \geq 0\}$ and $\{E^q | q \in Z\}$ are the sequential and skeletal filtrations of E. These filtrations are functorial with respect to cellular maps of X over $\mathscr{I}(U,U')$ and bicellular maps of E. If X or E is G-trivial or if G is finite, then $\chi \ltimes E$ is a G-CW spectrum with respect to these filtrations. In general, $\chi \ltimes E$ has the homotopy type of a G-CW spectrum.

<u>Proof</u>. Since E_0 is trivial, so is $(\chi \ltimes E)_0 = \chi_0 \ltimes E_0$. For $n \geq 1$, $(\chi \ltimes E)_n$ is given by the pushout diagram

$$\begin{array}{ccc} \vee(\varepsilon_H^p \ltimes S_J^{r-1} \cup \sigma_H^{p-1} \ltimes e_J^r) & \subset & \vee(\varepsilon_H^p \ltimes e_J^r), \\ \Big\downarrow{\scriptstyle f_{n-1}} & & \Big\downarrow \\ (\chi \ltimes E)_{n-1} & \longrightarrow & (\chi \ltimes E)_n \end{array}$$

where the wedge runs over the p-cells of X paired with the sequential (n-p)-cells of E (of any dimensions r); ε_H^p and σ_H^{p-1} are restrictions of χ to cells and their boundaries, and the map f_{n-1} is obtained by functoriality from the cells of X and E. The conclusions follow.

It is clear from the case $Y = I^+$ of Proposition 1.5 that $\chi \ltimes E$ preserves homotopies in the variable E. We conclude that it preserves G-CW homotopy types. By I.5.13, the following result is an immediate consequence.

<u>Corollary 1.12</u>. If $\chi\colon X \to \mathscr{I}(U,U')$ is a G-map, where X is a G-CW complex, then the functor $F[\chi,E')$ preserves weak equivalences in E' and the pair $\chi \ltimes E$ and $F[\chi,E')$ induces an adjoint pair of functors relating the stable categories $\overline{h}G\mathscr{S}\mathscr{A}$ and $\overline{h}G\mathscr{S}\mathscr{A}'$.

We remind the reader that functors such as $\chi \ltimes E$ which need not preserve weak equivalences pass to the stable category by composition of the given functor with CW-approximation.

It is easy to check that $\chi \ltimes E$ preserves homotopies over $\mathscr{I}(U,U')$ in its space variable. However, it is vital to our theory that we can deal with more general homotopies, in particular those arising from homotopical properties of $\mathscr{I}(U,U')$ itself. The following is the last of our list of results to be proven in the next section. Although not difficult, it is the technical heart of our entire theory.

<u>Theorem 1.13</u>. Let X be a G-CW complex and let $i_t\colon X \to X \times I$, $t = 0$ and $t = 1$, be the standard inclusions. Let $\psi\colon X \times I \to \mathscr{I}(U,U')$ be a G-map and let E be a G-CW spectrum. Then

$$i_t \ltimes 1\colon (\psi \cdot i_t) \ltimes E \longrightarrow \psi \ltimes E$$

is an equivalence.

Thus homotopic G-maps $X \to \mathscr{I}(U,U')$ induce equivalent functors. Since any two such G-maps are in fact homotopic, by the $\mathscr{I}$-universality of $\mathscr{I}(U,U')$, we obtain the following generalization of II.1.7 (which is the case $X = \{*\}$).

<u>Theorem 1.14</u>. The functors $\chi \ltimes (?)\colon G\mathscr{S}U \to G\mathscr{S}U'$ induced by varying G-maps $\chi\colon X \to \mathscr{I}(U,U')$ become canonically and coherently naturally equivalent on passage to the stable categories $\overline{h}G\mathscr{S}U$ and $\overline{h}G\mathscr{S}U'$. The same conclusion holds for the functors $F[\chi,?)\colon G\mathscr{S}U' \to G\mathscr{S}U$.

As usual, the second statement follows from the first by conjugation. Henceforward, we denote the stable category level functors by

$$X \ltimes E \quad \text{and} \quad F[X,E').$$

We show next that these functors depend only on the G-homotopy type of X.

Theorem 1.15. Let $f\colon X' \longrightarrow X$ be an equivalence of G-CW complexes and let $\chi\colon X \to (U,U')$ be a G-map. For a G-CW spectrum E,

$$f \ltimes 1\colon (\chi \circ f) \ltimes E \longrightarrow \chi \ltimes E$$

is an equivalence. Therefore $f \ltimes 1$ and $F[f,1)$ induce natural isomorphisms between functors on stable categories.

Proof. Let $g\colon X \longrightarrow X'$ be an inverse equivalence with homotopies $h\colon 1 \simeq fg$ and $h'\colon 1 \simeq gf$. We have the following commutative diagrams:

$$\begin{tikzcd}
(\chi fg) \ltimes E \arrow[r, "i_1 \ltimes 1"] \arrow[d, "g\ltimes 1"'] & (\chi h) \ltimes E \arrow[ddl, "h\ltimes 1"'] \\
(\chi f) \ltimes E \arrow[d, "f\ltimes 1"'] & \\
\chi \ltimes E \arrow[r, equal] & \chi \ltimes E \arrow[uu, "i_0 \ltimes 1"']
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
(\chi fgf) \ltimes E \arrow[r, "i_1 \ltimes 1"] \arrow[d, "f\ltimes 1"'] & (\chi fh') \ltimes E \arrow[ddl, "h'\ltimes 1"'] \\
(\chi fg) \ltimes E \arrow[d, "g\ltimes 1"'] & \\
(\chi f) \ltimes E \arrow[r, equal] & (\chi f) \ltimes E \arrow[uu, "i_0 \ltimes 1"']
\end{tikzcd}$$

The maps $i_0 \ltimes 1$ and $i_1 \ltimes 1$ are equivalences, hence so are $h \ltimes 1$ and $h' \ltimes 1$. Thus $g \ltimes 1$ is injective and surjective on homotopy groups by the first and second diagrams respectively. By the first diagram, $f \ltimes 1$ is also a weak equivalence and thus an equivalence.

In the next three results, we calculate the twisted functors $X \ltimes E$ and $F[X,E)$ as composites of change of universe functors and untwisted functors $X^+ \wedge (?)$ and $F(X^+,?)$.

Theorem 1.16. Let $f\colon U \to U'$ be a G-linear isometry. Then there are natural isomorphisms

$$X \ltimes E \simeq X^+ \wedge f_*E \quad \text{and} \quad F[X,E') \simeq F(X^+, f^*E')$$

of functors on stable categories.

Proof. We may calculate $X \ltimes E$ by use of the G-map $\omega(f)$ which sends all of X to the point f. By Proposition 1.7,

$$\omega(f) \ltimes E \cong X^+ \wedge f_*E \quad \text{and} \quad F[\omega(f),E') \cong F(X^+, f^*E').$$

That is, twisted half smash products with trivial twisting maps are ordinary half smash products.

Of course, there may be no G-linear isometry $f\colon U \to U'$. In practice, it is very often the case that there is a G-linear isometry in the other direction, and this returns us to the context studied in II§2.

Theorem 1.17. Let $i\colon U' \to U$ be an inclusion of G-universes. Then there are natural isomorphisms

$$X \ltimes E \simeq X^+ \wedge i^*E \quad \text{and} \quad F[X,E') \simeq F(X^+, i_*E')$$

of functors on stable categories.

Proof. Consider $i_*\colon \mathcal{I}(U,U') \to \mathcal{I}(U,U)$. For $\chi\colon X \to \mathcal{I}(U,U')$, an easy inspection of definitions (compare Proposition 3.1 below) gives

$$i_*(\chi \ltimes E) \cong (i_*\chi) \ltimes E.$$

Applying Theorem 1.16 to the identity map $U \to U$, we conclude that

$$i_*(X \ltimes E) \simeq X^+ \wedge E.$$

Thus $X \ltimes E \in G\mathcal{S}U'$ is a U'-representation of $X^+ \wedge E \in G\mathcal{S}U$. By uniqueness (see II.2.6), it suffices to check that $X^+ \wedge i^*E$ is also a U'-representation of $X^+ \wedge E$. To see this, consider the map

$$i_*(X^+ \wedge i^*E) \cong X^+ \wedge i_*i^*E \xrightarrow{1 \wedge \varepsilon} X^+ \wedge E.$$

By II.1.8 and II.2.4, ε is an $\mathcal{F}$-equivalence, where $\mathcal{F} = \mathcal{F}(U,U')$. Since X is an $\mathcal{F}$-CW homotopy type, $1 \wedge \varepsilon$ is a G-equivalence. (We are implicitly using CW approximation and the $\mathcal{F}$-Whitehead theorem; compare II.2.2 and II.2.13.) The second isomorphism follows from the first, the equivalence $1 \wedge \varepsilon$, and the Yoneda lemma.

If there are no G-linear isometries connecting U to U', we use a little trick to concoct some.

Theorem 1.18. Let $i\colon U' \to U \oplus U'$ and $f\colon U \to U \oplus U'$ be the evident inclusions. Then there are natural isomorphisms

$$X \ltimes E \simeq X^+ \wedge i^* f_* E \quad \text{and} \quad F[X,E') \simeq F(X^+, f^* i_* E')$$

of functors on stable categories.

Proof. Observe that the families $\mathcal{J}(U,U')$ and $\mathcal{J}(U \oplus U', U')$ are equal and consider $f^*: \mathcal{J}(U \oplus U', U') \to \mathcal{J}(U,U')$. For $\psi: X \to \mathcal{J}(U \oplus U', U')$, another easy inspection (compare Proposition 3.1 again) gives

$$(f^*\psi) \ltimes E \cong \psi \ltimes f_* E.$$

We can use $f^*\psi$ to compute $X \ltimes E$, and application of the previous theorem to i gives the conclusion.

We have another curious description of $X \ltimes E$.

Theorem 1.19. Let $\mathcal{J} = \mathcal{J}(U,U')$ and let $E\mathcal{J}$ be a universal $\mathcal{J}$-space (such as $\mathcal{J}(U,U')$). Then there are natural isomorphisms

$$X \ltimes E \simeq X^+ \wedge (E\mathcal{J} \ltimes E) \quad \text{and} \quad F[X,E') \simeq F(X^+, F[E\mathcal{J}, E'))$$

of functors on stable categories.

Proof. Since X has the homotopy type of an $\mathcal{J}$-CW complex, the projection $\pi_1: X \times E\mathcal{J} \to X$ is a G-homotopy equivalence (compare II.2.12). Let $\chi: X \to \mathcal{J}(U,U')$ and $\omega: E\mathcal{J} \to \mathcal{J}(U,U')$ be G-maps. We have equivalences

$$\chi \ltimes E \xleftarrow{\pi_1 \ltimes 1} (\chi\pi_1) \ltimes E \simeq (\omega\pi_2) \ltimes E \cong X^+ \wedge (\omega \ltimes E)$$

given by Theorem 1.15, Theorem 1.13, and Proposition 1.7, respectively.

We conclude by pointing out how various claims in $[H_\infty$, I§1] follow from the results above.

Remarks 1.20. When G acts trivially on U', we define $X \ltimes_G E = (X \ltimes E)/G$. With $X \ltimes_G E$ here corresponding to $W \ltimes_\pi E$ there, $[H_\infty$, I.1.1] follows from Proposition 1.5 and I.3.8 (which shows that passage to orbits commutes with Σ^∞). Part (i) of $[H_\infty$, I.1.2] holds since $W \ltimes_\pi (?)$ is a left adjoint, part (ii) results from Proposition 1.5, and part (iii) follows from the retract of mapping cylinders characterization of cofibrations and parts (i) and (ii). Proposition 1.8 and II.4.10 together imply $[H_\infty$, I.1.4]. The maps ι, α, β and δ promised in $[H_\infty$, I§1] will be constructed in section 3 below.

§2. Constructions of $\chi \ltimes E$; proofs

We shall need several variant constructions of $\chi \ltimes E$, with different emphases and virtues. In all of its guises, the core of the construction is the use of Thom complexes to codify the changes of indexing spaces dictated by χ. We codify this core in the following definition and lemmas.

Definition 2.1. Let $\chi: X \to \mathcal{J}(U,U')$ be a G-map and suppose given indexing spaces V in U and V' in U' such that $\chi(X)(V) \subset V'$. Consider the map of G-bundles over X

$$\begin{tikzcd} X \times V \arrow[rr] \arrow[dr] & & X \times V' \arrow[dl] \\ & X & \end{tikzcd}$$

specified by $(x,v) \to (x, \chi(x)(v))$. Since the $\chi(x)$ are G-linear isometries, the image is a G-subbundle and has an orthogonal complement $\zeta(V,V')$. Let $\eta(V,V')$ be the sphere G-bundle obtained from $\zeta(V,V')$ by one-point compactification of fibres and let $T(V,V')$ be the Thom complex obtained from $\eta(V,V')$ by identifying all of the points at infinity.

We write $T(X;V,V')$ instead of $T(V,V')$ when the base space requires explicit identification. Note that

$$T(X;V,V') \cong G^+ \wedge_H T(Y;V,V')$$

when $X = G \times_H Y$ and χ is the G-map extending an H-map $\psi: Y \to \mathcal{J}(U,U')$.

Lemma 2.2. If $V \subset W$ and $V' \subset W'$ are indexing spaces in U and U' and $\chi: X \to \mathcal{J}(U,U')$ is a G-map such that $\chi(X)(V) \subset V'$ and $\chi(X)(W) \subset W'$, then there are canonical homeomorphisms

$$T(V,V') \wedge S^V \cong X^+ \wedge S^{V'}$$

and

$$T(V,V') \wedge S^{W'-V'} \cong T(W,W') \wedge S^{W-V}.$$

Proof. The first homeomorphism is obvious from the definition. For the second, there is an evident isomorphism

$$\zeta(V,V') \oplus \varepsilon' \cong \zeta(W,W') \oplus \varepsilon,$$

where $\varepsilon: X \times (W-V) \to X$ and $\varepsilon': X \times (W'-V') \to X$ are the projections. Explicitly, the fibres over x of these Whitney sums are

$$(V'-\chi(x)(V)) \oplus (W'-V') \quad \text{and} \quad (W'-\chi(x)(W)) \oplus (W-V).$$

Upon applying $\chi(x)$ to $W-V$ in the second sum, we obtain an isomorphism with $W'-\chi(x)(V)$, which is exactly the first sum. The required homeomorphism follows on passage to Thom spaces.

We shall also make use of a relative addendum.

Lemma 2.3. If $V \subset W$ and $V' \subset W'$ are indexing spaces in U and U', X is a closed G-subspace of Y, and $\chi\colon Y \to \mathcal{J}(U,U')$ is a G-map such that $\chi(X)(V) \subset V'$ and $\chi(Y)(W) \subset W'$, then there is a canonical closed inclusion

$$T(X;V,V') \wedge S^{W'-V'} \subset T(Y;W,W') \wedge S^{W-V}.$$

Proof. The left side is homeomorphic to the subspace $T(X;W,W') \wedge S^{W-V}$ of the right side.

These observations suggest the following definitions.

Definitions 2.4. A G-map $\chi\colon X \to \mathcal{J}(U,U')$ is said to be compact if for each indexing space V in U there is an indexing space V' in U' such that $\chi(X)(V) \subset V'$. The G-map χ is said to be filtered if X is the union of an expanding sequence of closed G-subspaces X_i such that the restriction of χ to each X_i is compact. If X itself is compact, then any χ is compact. If X is the union of an expanding sequence of compact G-subspaces, then any χ is filtered.

For filtered χ, we have the following simple definition.

Definition 2.5. Let $\chi\colon X \to \mathcal{J}(U,U')$ be a filtered G-map. Choose indexing sequences $A = \{A_i\}$ in U and $A' = \{A'_i\}$ in U' such that $\chi(X_i)(A_i) \subset A'_i$ for each i and let $B_i = A_{i+1}-A_i$ and $B'_i = A'_{i+1}-A'_i$. For $D \in G\mathcal{P}A$, define $\chi \ltimes D \in G\mathcal{P}A'$ by

$$(\chi \ltimes D)(A'_i) = T(X_i;A_i,A'_i) \wedge DA_i$$

with structural maps

$$\begin{array}{c} T(X_i;A_i,A'_i) \wedge DA_i \wedge S^{b'_i} \\ \cap \\ T(X_{i+1};A_{i+1},A'_{i+1}) \wedge DA_i \wedge S^{b_i} \\ \big\downarrow {\scriptstyle 1\wedge\sigma} \\ T(X_{i+1};A_{i+1},A'_{i+1}) \wedge DA_{i+1}, \end{array}$$

the inclusion being given by Lemma 2.3 (and transpositions).

Of course, for $E \in G\mathcal{S}A$, we could then define

$$\chi \ltimes E = L(\chi \ltimes \ell E) \in G\mathcal{S}A'.$$

This definition, which was May's original one, has the merits of brevity and concreteness, and many of the claims of the previous section could be verified directly from it. Further, as we shall show at the end of the section, the restriction to filtered G-maps results in no real loss of generality. On the prespectrum level, this is the definition of choice, and it is the one used, for example, in McClure's work in $[H_\infty$, VII].

Nevertheless, this is not the best possible definition. For one thing, the tying of indexing sequences to χ in this noninvariant way makes functoriality in χ awkward to handle. For another, there seems to be no way to define an adjoint $F[\chi,D') \in G\mathcal{P}A$ for $D' \in G\mathcal{P}A'$. One attempts to construct such prespectra by setting

$$F[\chi,D')(A_i) = F(T(X_i;A_i,A'_i),D'A'_i)$$

and finds that the maps of Lemma 2.3 go the wrong way to allow the specification of structural maps.

The definitive definition of $\chi \ltimes E$ is as a spectrum level colimit. When χ is compact and is filtered by $X_i = X$ for all i, we need only use Lemma 2.2 in Definition 2.5, and the specification above works perfectly well to define an adjoint $F[\chi,D')$. We shall give a more elaborate coordinate free reformulation of our functors $\chi \ltimes D$ and $F[\chi,D')$ for compact χ. This will allow us to define spectrum level functors for general χ by passage to colimits and limits over compact restrictions. It would be awkward and unprofitable to give a prespectrum level version of the new definition, and we shall make no attempt to do so. When χ is filtered, the new definition of $\chi \ltimes E$ will be equivalent to the old one.

For our coordinate free definition, we require a formalism in terms of which we can discuss pairings of spaces V in U with spaces V' in U' such that $\chi(X)(V) \subset V'$ for a given compact χ. The following notion is analogous to that of a Galois connection.

Definitions 2.6. Let $\mathcal{A}$ and $\mathcal{A}'$ be indexing sets in U and U' and regard $\mathcal{A}$ and $\mathcal{A}'$ as partially ordered by inclusion. A connection $(\mu,\nu)\colon \mathcal{A} \to \mathcal{A}'$ is a pair of order preserving functions $\mu\colon \mathcal{A} \to \mathcal{A}'$ and $\nu\colon \mathcal{A}' \to \mathcal{A}$ such that $\mu V \subset V'$ if and only if $V \subset \nu V'$, where $V \in \mathcal{A}$ and $V' \in \mathcal{A}'$. It follows that $\mu(0) = 0$ and the following conditions hold.

(i) $V \subset \nu\mu V$ and $\mu V = \mu\nu\mu V$ for $V \in \mathcal{A}$.
(ii) $\mu\nu V' \subset V'$ and $\nu\mu\nu V' = \nu V'$ for $V' \in \mathcal{A}'$.
A connection $(\mu,\nu):\mathcal{A} \to \mathcal{A}'$ is a subconnection of a connection $(\sigma,\tau):\mathcal{B} \to \mathcal{B}'$ if $\mathcal{A} \subset \mathcal{B}$, $\mathcal{A}' \subset \mathcal{B}'$, and both $\mu V \subset \sigma V$ for all $V \in \mathcal{A}$ and $\tau V' \subset \nu V'$ for all $V' \in \mathcal{A}'$.

This notion allows the following basic definition.

Definitions 2.7. Let $\chi: X \to \mathcal{J}(U,U')$ be a compact G-map. Say that a connection $(\mu,\nu):\mathcal{A} \to \mathcal{A}'$ is a χ-connection if

$$\chi(X)(V) \subset \mu V \quad \text{for all} \quad V \in \mathcal{A}$$

or, equivalently, if

$$\chi(X)(\nu V') \subset V' \quad \text{for all} \quad V' \in \mathcal{A}'.$$

Fix a χ-connection (μ,ν). For $D \in G\mathcal{P}\mathcal{A}$, define $\chi \ltimes D \in G\mathcal{P}\mathcal{A}'$ by

$$(\chi \ltimes D)(V') = T(\nu V', V') \wedge D(\nu V')$$

with structural maps

$$\begin{array}{c} T(\nu V',V') \wedge D(\nu V') \wedge S^{W'-V'} \\ \cong \\ T(\nu W',W') \wedge D(\nu V') \wedge S^{\nu W'-\nu V'} \\ \downarrow 1\wedge\sigma \\ T(\nu W',W') \wedge D(\nu W'). \end{array}$$

Similarly, for $D' \in G\mathcal{P}\mathcal{A}'$, define $F[\chi,D') \in G\mathcal{P}\mathcal{A}$ by

$$F[\chi,D')(V) = F(T(V,\mu V),D'(\mu V))$$

with structural maps

$$\begin{array}{c} F(T(V,\mu V),D'(\mu V)) \\ \downarrow F(1,\tilde{\sigma}) \\ F(T(V,\mu V),\Omega^{\mu W-\mu V}D'(\mu W)) \\ \cong \\ \Omega^{W-V}F(T(W,\mu W),D'(\mu W)). \end{array}$$

The homeomorphisms are given by Lemma 2.2 and adjunction, and the transitivity diagram of I.2.1 is easily checked. The functor $F[\chi,?)$ preserves G-spectra. For $E \in G\mathcal{S}\mathcal{A}$, define

$$\chi \ltimes E = L(\chi \ltimes \ell E) \in G\mathcal{S}\mathcal{A}'.$$

We retain the notations of the definition in the following three lemmas. These give the desired adjunctions and the invariance properties necessary for our later passage to colimits and limits.

Lemma 2.8. The functor $\chi \ltimes (?)$ is left adjoint to the functor $F[\chi,?)$ on the prespectrum level and thus also on the spectrum level.

Proof. Let $D \in G\mathcal{P}\mathcal{A}$ and $D' \in G\mathcal{P}\mathcal{A}'$. Given a map $f: \chi \ltimes D \to D'$, define a map $g: D \to F[\chi,D')$ by letting its $V^{\underline{th}}$ component be the dotted arrow composite in the diagram

$$\begin{array}{ccc} \Omega^{\nu\mu V-V}D(\nu\mu V) & \xrightarrow{\Omega^{\nu\mu V-V}\tilde{f}} & \Omega^{\nu\mu V-V}F(T(\nu\mu V,\mu V),D'(\mu V)), \\ \tilde{\sigma}\uparrow & & \cong \\ DV & \dashrightarrow & F(T(V,\mu V),D'(\mu V)) \end{array}$$

where $\tilde{f}$ is the adjoint of the $(\mu V)^{\underline{th}}$ component of f and the homeomorphism is given by adjunction and Lemma 2.2. Conversely, given $g: D \to F[\chi,D')$, define $f: \chi \ltimes D \to D'$ by letting its $(V')^{\underline{th}}$ component be the dotted arrow composite in the diagram

$$\begin{array}{ccc} T(\nu V',V') \wedge D(\nu V') & \dashrightarrow & D'V' \\ \cong & & \uparrow\sigma \\ T(\nu V',\mu\nu V') \wedge D(\nu V') \wedge S^{V'-\mu\nu V'} & \xrightarrow{\tilde{g}\wedge 1} & D'(\mu\nu V') \wedge S^{V'-\mu\nu V'} \end{array},$$

where $\tilde{g}$ is the adjoint of the $(\nu V')^{\underline{th}}$ component of g and the homeomorphism is again given by Lemma 2.2. A pleasant verification, to which (i) and (ii) in Definitions 2.6 provide the key, shows that these are well-defined inverse bijections of hom sets.

Lemma 2.9. The functor $\chi \ltimes D$ preserves injection and Σ-inclusion prespectra (but not inclusion prespectra or spectra), and the conclusions of Proposition 1.2 hold on both the prespectrum and spectrum level. In particular, for a closed G-subspace A of X and $\alpha = \chi|A$ and for $E \in G\mathcal{S}\mathcal{A}$ and $E' \in G\mathcal{S}\mathcal{A}'$, there are natural closed inclusions and projections

$$\alpha \ltimes E \to \chi \ltimes E \quad \text{and} \quad F[\chi,E') \to F[\alpha,E').$$

Proof. The χ-connection (μ,ν) is also an α-connection. The prespectrum level assertions hold by inspection and imply the spectrum level assertions by I.8.3 and I.8.5.

Recall the change of indexing set equivalences ϕ and ψ from I.2.4.

Lemma 2.10. Let $(\mu,\nu):\mathcal{A}\to\mathcal{A}'$ be a χ-subconnection of a χ-connection $(\sigma,\tau):\mathcal{B}\to\mathcal{B}'$. For $E\in G\mathcal{S}\mathcal{A}$, $E'\in G\mathcal{S}\mathcal{A}'$, $F\in G\mathcal{S}\mathcal{B}$, and $F'\in G\mathcal{S}\mathcal{B}'$, there are natural isomorphisms of G-spectra

$$\chi \ltimes_\tau \psi E \cong \psi'(\chi \ltimes_\nu E) \quad\text{and}\quad F_\sigma[\chi,\psi' E'] \cong \psi F_\mu[\chi,E'];$$

$$\chi \ltimes_\nu \phi F \cong \phi'(\chi \ltimes_\tau F) \quad\text{and}\quad F_\mu[\chi,\phi' F'] \cong \phi F_\sigma[\chi,F'].$$

Here the subscripts indicate the relevant connections.

Proof. Since ϕ and ψ are inverse adjoint equivalences, I.3.5 shows that any one of these isomorphisms implies all the others. The following is the $V^{\underline{th}}$ component, $V\in\mathcal{A}$, of a natural isomorphism $F_\mu[\chi,\phi'F'] \to \phi F_\sigma[\chi,F']$:

$$\begin{array}{c} F(T(V,\mu V),F'(\mu V)) \\ \downarrow F(1,\tilde{\sigma}) \\ F(T(V,\mu V),\Omega^{\sigma V-\mu V}F'(\sigma V)) \\ \cong \\ F(T(V,\sigma V),F'(\sigma V)). \end{array}$$

As usual, the homeomorphism is given by Lemma 2.2 and adjunction.

The following examples give content to our formalism.

Examples 2.11. Let $\chi: X \to \mathcal{J}(U,U')$ be a compact G-map.

(i) Let $\mathcal{A}$ and $\mathcal{A}'$ be the standard indexing sets in U and U'. Define

$$\mu V = \sum_{x\in X} \chi(x)(V) \quad \text{for} \quad V \in \mathcal{A}$$

and

$$\nu V' = \bigcap_{x\in X} \chi(x)^{-1}(V') \quad \text{for} \quad V'\in\mathcal{A}'.$$

Then (μ,ν) is a χ-connection, called the canonical χ-connection. It is a subconnection of any other χ-connection $\mathcal{A}\to\mathcal{A}'$.

(ii) Let (μ,ν) and (μ',ν') be χ-connections $\mathcal{A}\to\mathcal{A}'$, where $\mathcal{A}$ is closed

under intersections and $\mathcal{A}'$ is closed under sums. Then (μ,ν) and (μ',ν') are both subconnections of the χ-connection $(\mu+\mu',\nu\cap\nu')$ specified by

$$(\mu+\mu')(V) = \mu V + \mu' V \quad\text{and}\quad (\nu\cap\nu')(V') = \nu V' \cap \nu' V'.$$

(Contrary to our usual conventions, sums need not be orthogonal here.)

(iii) If A and A' are indexing sequences in U and U' such that $\chi(X)(A_i) \subset A'_i$ for all i, then $A_i \leftrightarrow A'_i$ specifies a χ-connection $A \to A'$. The resulting functor $\chi \ltimes (?): G\mathcal{P}A \to G\mathcal{P}A'$ coincides with that specified in Definition 2.5 (with $X_i = X$ for all i).

(iv) For arbitrary indexing sets $\mathcal{A}$ in U and $\mathcal{A}'$ in U', choose indexing sequences $A\subset\mathcal{A}$ and $A'\subset\mathcal{A}'$ such that $\chi(X)(A_i)\subset A'_i$ for all i and define

$$\mu V = A'_i \quad\text{for the smallest } i \text{ such that } V\subset A_i$$

and

$$\nu V' = A_i \quad\text{for the largest } i \text{ such that } A'_i \subset V'.$$

Then (μ,ν) is a χ-connection $\mathcal{A}\to\mathcal{A}'$ which extends the χ-connection $A \to A'$ of (iii).

(v) If $X = \{*\}$ and $\chi = f: U \to U'$, the canonical χ-connection reduces to

$$\mu V = f(V) \quad\text{and}\quad \nu V' = f^{-1}(V').$$

The identifications of Examples 1.4 follow by comparison with II.1.1.

(vi) For any indexing set $\mathcal{A}$ in U, $(1,1):\mathcal{A}\to\mathcal{A}$ is a γ-connection, where $\gamma: G \to \mathcal{J}(U,U)$ is the action map. Direct inspection gives the last assertion of Remarks 1.9.

We put things together to prove the following invariance result.

Proposition 2.12. Let $\chi: X \to \mathcal{J}(U,U')$ be a compact G-map. Then, up to canonical natural isomorphism, the functors $\chi \ltimes E$ and $F[\chi,E']$ between $G\mathcal{S}\mathcal{A}$ and $G\mathcal{S}\mathcal{A}'$ are independent of the choice of connection used to define them (and of the choice of $\mathcal{A}$ and $\mathcal{A}'$).

Proof. We use an argument supplied by Elmendorf. Let $(\mu,\nu):\mathcal{A}\to\mathcal{A}'$ be any χ-connection. Choose an indexing sequence A' in $\mathcal{A}'$ and let $A_i = \nu A'_i$. By discarding some of the A'_i, we can arrange that $A_i \neq A_{i+1}$ for all i. Since A_0 need not be 0, $A = \{A_i\}$ need not be an indexing sequence. Nevertheless, we can apply Example 2.11 (iv) to obtain a χ-connection $(\sigma,\tau):\mathcal{B}\to\mathcal{B}'$ from A and

A', where $\mathscr{B}$ and $\mathscr{B}'$ are the standard indexing sets in U and U'. It is easily checked that (μ,ν) is a subconnection of (σ,τ), and of course so is the canonical χ-connection $\mathscr{B} \to \mathscr{B}'$. Let $\phi\colon G\mathscr{S}\mathscr{B} \to G\mathscr{S}\mathscr{A}$ and $\psi\colon G\mathscr{S}\mathscr{A} \to G\mathscr{S}\mathscr{B}$ be the change of indexing set equivalences. Two applications of Lemma 2.10 give that, for $E \in G\mathscr{S}\mathscr{A}$, $\chi \ltimes E$ constructed by use of the given connection is naturally isomorphic to $\phi(\chi \ltimes \psi E)$, where $\chi \ltimes \psi E$ is constructed by use of the canonical connection. Using subsequences of unions of sequences, it is not hard to check that this isomorphism is independent of the choice of A'.

Turning to the study of general maps χ, we can now make the following final definition.

Definition 2.13. For a G-map $\chi\colon X \to \mathscr{J}(U,U')$ and for $E \in G\mathscr{S}\mathscr{A}$ and $E' \in G\mathscr{S}\mathscr{A}'$, define

$$\chi \ltimes E = \operatorname*{colim}_{\alpha}\, \alpha \ltimes E \in G\mathscr{S}\mathscr{A}' \quad\text{and}\quad F[\chi,E') = \lim_{\alpha} F[\alpha,E') \in G\mathscr{S}\mathscr{A},$$

where the colimit and limit are taken over all compact restrictions $\alpha\colon A \subset X \to \mathscr{J}(U,U')$ of χ with A closed in X.

In more detail, choose an α-connection (μ_α,ν_α) for each α and use these connections to define the functors $\alpha \ltimes E$. If $A \subset B \subset X$ and $\beta = \chi|B$ is compact, then (μ_α,ν_α) and (μ_β,ν_β) are both α-connections. The previous proposition yields an isomorphism from $\alpha \ltimes E$ to $\alpha \ltimes E$ defined with respect to (μ_β,ν_β), and Lemma 2.9 yields an inclusion of the latter in $\beta \ltimes E$. The colimit is taken over the resulting composites $\alpha \ltimes E \to \beta \ltimes E$. If $B \subset C \subset X$ and $\gamma = \chi|C$ is also compact, the resulting composite $\alpha \ltimes E \to \beta \ltimes E \to \gamma \ltimes E$ agrees with the correspondingly described map $\alpha \ltimes E \to \gamma \ltimes E$ by the coherence of the isomorphisms and inclusions involved and a naturality diagram for their commutation. Thus the colimit is well-defined, and a similar check shows that, up to canonical natural isomorphism, it is independent of the initial choices of connections.

When $\mathscr{A}$ and $\mathscr{A}'$ are the standard indexing sets, the construction becomes completely canonical. We take (μ_α,ν_α) to be the canonical α-connection and see that (μ_α,ν_α) is then a subconnection of (μ_β,ν_β). This allows direct use of Lemmas 2.9 and 2.10 to obtain the maps of the colimit system. One can reobtain the construction for general $\mathscr{A}$ and $\mathscr{A}'$ by transporting this canonical construction along the equivalences ϕ and ψ.

Writing the domain of each compact α as the colimit of its compact subspaces and reducing the resulting double colimits to a single colimit, we find that we could just as well take colimits over compact subspaces A rather than over the larger set of compact restrictions. When X is a G-CW complex, we could and

should take colimits over the finite subcomplexes of X.

The derivation of Theorem 1.1 from the compact case consists of elementary categorical nonsense. Its adjunction results from the calculation

$$\begin{aligned} G\mathscr{S}\mathscr{A}'(\chi \ltimes E,E') &\cong \lim G\mathscr{S}\mathscr{A}'(\alpha \ltimes E,E') \\ &\cong \lim G\mathscr{S}\mathscr{A}(E,F[\alpha,E')) \\ &\cong G\mathscr{S}\mathscr{A}(E,F[\chi,E')), \end{aligned}$$

where the first and last isomorphisms are given by the definition of colimits and limits. The compatibility of the adjunctions of Lemma 2.8 with the maps of the limit systems that is required to deduce the middle isomorphism results from the fact that the maps

$$\alpha \ltimes E \to \beta \ltimes E \quad\text{and}\quad F[\beta,E') \to F[\alpha,E')$$

give conjugate natural transformations in the sense of I.3.5. The statements about colimits in χ in Theorem 1.1 are standard exercises in colimits commuting with colimits and limits commuting with limits.

The derivation of Proposition 1.2 requires some point-set topology. By the definitions of colimits of spectra and prespectra,

$$\chi \ltimes E = L \operatorname{colim} \ell(\alpha \ltimes E)$$

and

$$(\operatorname{colim} \ell(\alpha \ltimes E))(V') = \operatorname{colim}((\alpha \ltimes E)(V')).$$

Each $\alpha \ltimes E \to \beta \ltimes E$ is a closed inclusion, space level colimits of directed systems of inclusions preserve finite limits, and the forgetful functor ℓ preserves limits. Therefore the compact case of Proposition 1.2 implies its conclusions on the level of colimits of prespectra. Since these colimits are easily checked to be injection prespectra and L preserves finite limits on such prespectra, the desired spectrum level conclusions follow.

Most of Proposition 1.3 has already been checked and the rest is trivial. The commutation relations of Propositions 1.5, 1.7, and 1.8 are easy. For example, the crucial isomorphism

$$\chi \ltimes \Sigma^\infty Y \cong \Sigma^\infty(X^+ \wedge Y)$$

follows either from the first homeomorphism of Lemma 2.2 and inspection of definitions or from I.3.5 and the observation that $F[\chi,E')$ has zero$\frac{\text{th}}{}$ space $F(X^+,E_0')$.

By I.3.5 again, the following lemma implies the first assertion of Proposition 1.6 (which only needs χ to be compact).

Lemma 2.14. Let $\chi\colon X \to \mathscr{I}(U,U')$ be a compact G-map. For $V \in \mathscr{A}$ and $V' \in \mathscr{A}'$ with $V \cong V'$ and for $E' \in G\mathscr{S}\mathscr{A}'$, $F[\chi,E')(V)$ is naturally homeomorphic to $F(X^+,EV')$.

Proof. Fix a χ-connection (μ,ν). By I.4.2, we may as well asume $V' \perp \mu V$. Using structural homeomorphisms, adjunctions, $S^V \cong S^{V'}$, and Lemma 2.2, we obtain the chain of natural homeomorphisms

$$\begin{aligned}
F[\chi,E')(V) &= F(T(V,\mu V),E'(\mu V))\\
&\cong F(T(V,\mu V),\Omega^{V'}E'(\mu V + V'))\\
&\cong F(T(V,\mu V) \wedge S^V,E'(\mu V + V'))\\
&\cong F(X^+ \wedge S^{\mu V},E'(\mu V + V'))\\
&\cong F(X^+,\Omega^{\mu V}E'(\mu V + V')) \cong F(X^+,E'V').
\end{aligned}$$

When χ is not compact, we may not be able to choose our connections (μ_α,ν_α) so that $V' \perp \mu_\alpha V$ for all α. To get around this, choose a linear isometry $f\colon U' \rightarrow U'$ with image orthogonal to V'. We can choose connections $(\mu_\alpha',\nu_\alpha')$ for $f\circ\chi$ with $V' \perp \mu_\alpha' V$ for all α. The homeomorphisms just proven then pass to limits to yield

$$F[f\circ\chi,E')(V) \cong F(X^+,E'V').$$

By I.3.5, it follows that $f\circ\chi \ltimes \Lambda^V\Sigma^\infty Y \cong \Lambda^{V'}\Sigma^\infty(\chi \ltimes Y)$. By use of a path connecting f to the identity in $\mathscr{I}(U',U')$ and quotation of Theorem 1.13 (to be proven shortly), the left side becomes isomorphic to $\chi \ltimes \Lambda^V\Sigma^\infty Y$ on passage to the stable category.

The following analog of Proposition 1.6 computes $\chi \ltimes \Lambda^V\Sigma^\infty Y$ for general indexing spaces V as a colimit of appropriate shift desuspensions.

Proposition 2.15. Let $\chi\colon X \to \mathscr{I}(U,U')$ be a G-map and let $V \in \mathscr{A}$ and $Y \in G\mathscr{T}$. If χ is compact and $\chi(X)(V) \subset V'$, there is a natural isomorphism of G-spectra

$$\chi \ltimes \Lambda^V\Sigma^\infty Y \cong \Lambda^{V'}\Sigma^\infty(T(V,V') \wedge Y).$$

For general χ, there is a natural isomorphism of G-spectra

$$\chi \ltimes \Lambda^V\Sigma^\infty Y \cong \operatorname{colim} \Lambda^{V_\alpha'}\Sigma^\infty(T(A;V,V_\alpha') \wedge Y),$$

where the colimit is taken over the compact restrictions $\alpha\colon A \to \mathscr{I}(U,U')$ of χ and the V_α' are so chosen that $\alpha(A)(V) \subset V_\alpha'$.

Proof. In the compact case, we may choose a χ-connection (μ,ν) with $\mu V = V'$; in general, we may choose α-connections (μ_α,ν_α) with $\mu_\alpha V' = V_\alpha'$. For $E' \in G\mathscr{S}\mathscr{A}'$, various adjunctions and definitions give the following chain of natural isomorphisms of hom sets, and the conclusion follows by the Yoneda lemma:

$$\begin{aligned}
G\mathscr{S}\mathscr{A}'(\chi \ltimes \Lambda^V\Sigma^\infty Y,E') &\cong G\mathscr{S}\mathscr{A}(\Lambda^V\Sigma^\infty Y,F[\chi,E'))\\
&\cong G\mathscr{T}(Y,F[\chi,E')(V))\\
&= G\mathscr{T}(Y,\lim F(T(A;V,\mu_\alpha V),E'(\mu_\alpha V))\\
&\cong \lim G\mathscr{T}(T(A;V,\mu_\alpha V) \wedge Y,E'(\mu_\alpha V))\\
&\cong \lim G\mathscr{S}\mathscr{A}'(\Lambda^{\mu_\alpha V}\Sigma^\infty(T(A;V,\mu_\alpha V) \wedge Y),E')\\
&\cong G\mathscr{S}\mathscr{A}'(\operatorname{colim} \Lambda^{\mu_\alpha V}\Sigma^\infty T(A;V,\mu_\alpha V) \wedge Y,E').
\end{aligned}$$

The only result of section 1 yet to be proven is Theorem 1.13. We shall deduce it from the following special case of Theorem 1.15.

Lemma 2.16. Let X be a subcomplex of a G-CW complex Y such that the inclusion $i\colon X \rightarrow Y$ is an equivalence. Let $\psi\colon Y \to \mathscr{I}(U,U')$ be a compact G-map and let $\chi = \psi\circ i$. Let E be a G-CW spectrum. Then

$$i \ltimes 1\colon \chi \ltimes E \to \psi \ltimes E$$

is an equivalence.

Proof. Let $(\mu,\nu)\colon \mathscr{A} \to \mathscr{A}'$ be a ψ-connection. By standard homotopical properties of G-bundles, the maps

$$T(X;V,\mu V) \to T(Y;V,\mu V)$$

induced by i are equivalences. Thus, for $D \in G\mathcal{P}\mathcal{A}$ and $D' \in G\mathcal{P}\mathcal{A}'$,

$$i \ltimes 1\colon \chi \ltimes D \to \psi \ltimes D \quad\text{and}\quad F[1,i)\colon F[\psi,D') \to F[\chi,D')$$

are spacewise equivalences. In particular, for $E' \in G\mathcal{S}\mathcal{A}'$,

$$F[1,i)\colon F[\psi,E') \to F[\chi,E')$$

is a weak equivalence. We cannot pass so readily to the spectrum level on the twisted half smash product side. For $E \in G\mathcal{S}\mathcal{A}$, $\chi \ltimes \ell E$ and $\psi \ltimes \ell E$ need not be inclusion prespectra, and L is only known to preserve spacewise (weak) equivalences when restricted to inclusion prespectra. However, by Corollary 1.12, our G-CW hypotheses allow us to to pass to the stable category. Here $i \ltimes 1\colon \chi \ltimes E \to \psi \ltimes E$ is conjugate to the natural isomorphism $F[i,1)$ and is thus a natural isomorphism by I.3.5.

When $\psi\colon X \times I \to \mathcal{I}(U,U')$ is compact, Theorem 1.13 follows immediately from the lemma. For the general case, we have the following commutative diagram for any compact G-spectrum K, where β runs through the retrictions of ψ to $A \times I$ for finite subcomplexes A of X:

$$\begin{array}{ccc}
\operatorname{colim} G\mathcal{S}\mathcal{A}'(K,\beta i_t \ltimes E) & \longrightarrow & G\mathcal{S}\mathcal{A}'(K,\psi i_t \ltimes E) \\
\downarrow{\scriptstyle \operatorname{colim}(i_t \ltimes 1)_*} & & \downarrow{\scriptstyle (i_t \ltimes 1)_*} \\
\operatorname{colim} G\mathcal{S}\mathcal{A}'(K,\beta \ltimes E) & \longrightarrow & G\mathcal{S}\mathcal{A}'(K,\psi \ltimes E).
\end{array}$$

The horizontal arrows are isomorphisms by I.5.3. Letting K run through the sphere G-spectra and passing to homotopy classes, we see that $i_t \ltimes 1$ on the right is a weak equivalence and therefore an equivalence.

We shall need one more result. While somewhat technical in nature, it sheds considerable light on the inner workings of our definition of $\chi \ltimes E$. The crux of our applications of twisted half smash products is the construction of G-maps $\chi \ltimes E \to E'$. In view of our lack of a convenient coordinate-free prespectrum level description of $\chi \ltimes E$, it is not obvious how to recognize such maps in elementary space level terms. The following result gives a quite simple and concrete solution to this problem.

Proposition 2.17. Let $\chi\colon X \to \mathcal{I}(U,U')$ be a G-map. For $V \in \mathcal{A}$ and $V' \in \mathcal{A}'$, let $A(V,V') \subset X$ denote the G-subspace consisting of points x such that $\chi(x)(V) \subset V'$. Then, for $E \in G\mathcal{S}\mathcal{A}$ and $E' \in G\mathcal{S}\mathcal{A}'$, a map $\xi\colon \chi \ltimes E \to E'$ in $G\mathcal{S}\mathcal{A}'$ determines and is determined by maps $\xi(x)\colon E \to \chi(x)^*(E')$ in $G\mathcal{S}\mathcal{A}$ for points

$x \in X$ such that the functions

$$\zeta\colon T(A(V,V');V,V') \wedge EV \to E'V'$$

specified by $\zeta([x,v'] \wedge y) = \sigma(\xi(x)(y) \wedge v')$ for $x \in A(V,V')$, $v' \in V' - \chi(x)(V)$, and $y \in EV$ are continuous.

Proof. Given ξ, define $\xi(x)$ to be the adjoint of the composite

$$\chi(x)_*(E) \to \chi \ltimes E \xrightarrow{\xi} E'.$$

Note that $\xi(x)$ has V^{th} map $EV \to E'(\chi(x)(V))$, so the definition of the functions ζ makes sense. To check continuity, let α be the restriction of χ to any compact subspace of $A(V,V')$ for some fixed V and V'. By (iii) and (iv) of Examples 2.11, we may choose an α-connection (μ_α,ν_α) such that $\nu_\alpha V' = V$. With this choice, ζ is precisely the V^{th} map of the map of prespectra

$$\alpha \ltimes \ell E \xrightarrow{\eta} \ell L(\alpha \ltimes \ell E) = \ell(\alpha \ltimes E) \xrightarrow{\ell\xi} \ell E',$$

as we see by restricting α to points. Conversely, assume given maps $\xi(x)$ with the prescribed continuity property. Let $\alpha = \chi|A$ be any compact restriction of χ and let (μ_α,ν_α) be an α-connection. For $V' \in \mathcal{A}'$, $\alpha(A)(\nu V') \subset V'$, hence $A \subset A(\nu V',V')$. There results a map

$$T(A;\nu V',V') \wedge EV \subset T(A(\nu V',V');\nu V',V') \wedge EV \xrightarrow{\zeta} E'V'.$$

Since the definition of ζ gives complete control of these maps on points, it is easy to check that they specify a map of prespectra $\alpha \ltimes \ell E \to \ell E'$. By application of L, there results a map of spectra $\alpha \ltimes E \to E'$. These maps are compatible as α varies because of the evident compatibility on restriction to points of X. They thus pass to colimits to yield $\chi \ltimes E \to E'$. It is clear that these two constructions are inverse to one another.

The only remaining piece of unfinished business is to connect up our two definitions of $\chi \ltimes E$ and to discuss the generality of filtered G-maps. For the moment, write $\chi \ltimes_f D$ for the functor specified in Definition 2.5.

Proposition 2.18. Let $\chi\colon X \to \mathcal{I}(U,U')$ be a filtered G-map and let A and A' be indexing sequences in U and U' such that $\chi(X_i)(A_i) \subset A'_i$ for each i. For $D \in G\mathcal{P}A$, $L(\chi \ltimes_f D)$ is naturally isomorphic to $\chi \ltimes LD$.

Proof. Inspection of Definition 2.5 shows that

$$\chi \ltimes_f D = \operatorname{colim}_n \chi_n \ltimes_f D,$$

where $\chi_n: X_n \to \mathcal{J}(U,U')$ is the sub filtered G-map of χ obtained by filtering X_n by the X_i for $i < n$ with higher filtrations constant at X_n. On the other hand, $\chi \ltimes LD$ as defined in Definition 2.13 is also a colimit, and the compact maps χ_n are cofinal in the limit system used there. A χ_n-connection $A \to A'$ is specified by $A_0 \leftrightarrow A_0'$, $A_i \to A_n'$ and $A_i' \to A_0$ for $i \leq i < n$, and $A_i \leftrightarrow A_i'$ for $i \geq n$. With this connection, we obviously have

$$(\chi_n \ltimes_f D)(A_i) = (\chi_n \ltimes D)(A_i) \quad \text{for } i \geq n.$$

Since they differ only in their initial terms, $\chi_n \ltimes_f D$ and $\chi_n \ltimes D$ yield precisely the same spectrum on application of L. This gives a compatible family of identifications

$$L(\chi_n \ltimes_f D) \cong L(\chi_n \ltimes D) \cong \chi_n \ltimes LD,$$

where the second isomorphism follows by I.3.5 from the obvious equality

$$F[\chi_n, \ell E') = \ell F[\chi_n, E') \quad \text{for } E' \in G\mathcal{S}\mathcal{A}.$$

Since L commutes with colimits, the conclusion follows.

Remarks 2.19. Write U and U' as infinite sums $\sum_{k \geq 1} (V_k)^\infty$ and $\sum_{k \geq 1} (V_k')^\infty$, where the V_k and V_k' are irreducible representations. Then define subspaces Z_n of $\mathcal{J}(U,U')$ for $n \geq 1$ by

$$Z_n = \{f \mid f(V_1^k + \cdots + V_k^k) \subset (V_1')^{nk} + \cdots + (V_k')^{nk} \quad \text{for all } k \geq 1\}$$

and let Z be the union of the Z_n. We refer to Z as the space of bounded linear isometries $U \to U'$. The inclusions $\zeta_n: Z_n \to \mathcal{J}(U,U')$ are obviously compact, hence the inclusion $\zeta: Z \to \mathcal{J}(U,U')$ is filtered. By inspection, if $f \in Z$, then Z is closed under the homotopies used to prove II.1.5, and it is easy to see that if there exists an H-linear isometry $U \to U'$, then there exists one in Z. Therefore II.2.11 applies to show that the inclusion of Z in $\mathcal{J}(U,U')$ is a G-homotopy equivalence. Since any composite $X \to Z \to \mathcal{J}(U,U')$ may be filtered by the inverse images of the Z_n, restriction to filtered maps would result in no real loss of generality. Indeed, there would be no substantive changes in the theory as a whole if we were to replace $\mathcal{J}(U,U')$ by Z throughout.

§3. Relations between smash products and twisted half smash products

In this rather technical (but elementary) section, we consider various natural maps relating smash products to twisted half smash products and produce precursors of the coherence diagrams for extended powers displayed in [H_∞, I §2]. Two kinds of proof are used. The simpler pattern just follows the standard general prescription of first checking things on the prespectrum level for compact G-maps, then passing to the spectrum level by use of L, and finally passing to colimits over compact restrictions. The results requiring only such simple arguments are presented first. The later constructions and diagrams depend on application of Theorem 1.13 to homotopies produced by the homotopical properties of $\mathcal{J}(U,U')$. We shall indicate explicitly all places where Theorem 1.13 is used, and we tacitly assume its G-CW hypotheses at those places.

The following result introduces three of the four maps we wish to study.

Proposition 3.1. (i) Let $\chi: X \to \mathcal{J}(U,U')$ be a G-map and let $x \in X$ be a G-fixed point, so that $e = \chi(x)$ is a G-linear isometry. For $E \in G\mathcal{S}\mathcal{A}$, the inclusion $i: \{x\} \to X$ induces a natural map

$$\iota = i \ltimes 1: e_* E = \chi(x) \ltimes E \to \chi \ltimes E$$

in $G\mathcal{S}\mathcal{A}'$, where $\mathcal{A}$ and $\mathcal{A}'$ are indexing sets in U and U'.

(ii) Let $\chi_1: X_1 \to \mathcal{J}(U_1,U_1')$ and $\chi_2: X_2 \to \mathcal{J}(U_2,U_2')$ be G-maps and define $\chi_1 \oplus \chi_2$ to be the composite

$$X_1 \times X_2 \xrightarrow{\chi_1 \times \chi_2} \mathcal{J}(U_1,U_1') \times \mathcal{J}(U_2,U_2') \xrightarrow{s} \mathcal{J}(U_1 \oplus U_2, U_1' \oplus U_2'),$$

where s is the direct sum map. For $E_i \in G\mathcal{S}\mathcal{A}_i$, there is a natural isomorphism

$$\alpha: (\chi_1 \ltimes E_1) \wedge (\chi_2 \ltimes E_2) \to (\chi_1 \oplus \chi_2) \ltimes (E_1 \wedge E_2)$$

in $G\mathcal{S}(\mathcal{A}_1' \oplus \mathcal{A}_2')$, where $\mathcal{A}_i$ and $\mathcal{A}_i'$ are indexing sets in U_i and U_i'.

(iii) Let $\chi: X \to \mathcal{J}(U,U')$ and $\chi': X' \to \mathcal{J}(U',U'')$ be G-maps and define $\chi' \circ \chi$ to be the composite

$$X' \times X \xrightarrow{\chi' \times \chi} \mathcal{J}(U',U'') \times \mathcal{J}(U,U') \xrightarrow{c} \mathcal{J}(U,U''),$$

where c denotes composition. For $E \in G\mathcal{S}\mathcal{A}$, there is a natural isomorphism

$$\beta\colon \chi' \ltimes (\chi \ltimes E) \to (\chi' \odot \chi) \ltimes E$$

in $G\mathscr{S}\mathcal{A}''$, where $\mathcal{A}$ and $\mathcal{A}''$ are indexing sets in U and U''.

Proof. Part (i) is trivial. For (ii), assume first that χ_1 and χ_2 are compact. If $(\mu_i,\nu_i)\colon \mathcal{A}_i \to \mathcal{A}_i'$ are χ_i-connections, then their sum gives a $(\chi_1 \oplus \chi_2)$-connection $\mathcal{A}_1 \oplus \mathcal{A}_2 \to \mathcal{A}_1' \oplus \mathcal{A}_2'$ for which the prespectrum level isomorphism α is immediate by inspection of the relevant Thom complexes. Passage to spectra is immediate and passage to non-compact maps χ_i follows by the cofinality of sums of compact G-maps among all compact restrictions of $\chi_1 \oplus \chi_2$. Part (iii) is similar, using composites of connections.

Observe that the operation $\odot$ is given by compositions

$$(f' \odot \chi)(x) = f' \circ \chi(x) \quad \text{or} \quad (\chi' \odot f)(x') = \chi'(x') \circ f$$

when X' or X is a single point with image f' or f. Such special cases will occur ubiquitously in our work.

The maps just defined are compatible with their evident space level counterparts, an assertion made precise in the following result. It should be kept in mind that the same notation Σ^∞ is used for the functor from G-spaces to G-spectra for all choices of indexing sets and G-universes.

Proposition 3.2. The following diagrams of spectra commute

(i)
$$\begin{tikzcd}
\Sigma^\infty Y = \Sigma^\infty(\{x\}^+ \wedge Y) \arrow[r, "\Sigma^\infty(i^+ \wedge 1)"] \arrow[d, "\cong"] & \Sigma^\infty(X^+ \wedge Y) \arrow[d, "\cong"] \\
e_*\Sigma^\infty Y = e \ltimes \Sigma^\infty Y \arrow[r, "\iota"] & \chi \ltimes \Sigma^\infty Y
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
\Sigma^\infty(X_1^+ \wedge Y_1 \wedge X_2^+ \wedge Y_2) \arrow[r, "\Sigma^\infty(1 \wedge t \wedge 1)"] \arrow[d, "\cong"] & \Sigma^\infty(X_1^+ \wedge X_2^+ \wedge Y_1 \wedge Y_2) \arrow[d, "\cong"] \\
(\chi_1 \ltimes \Sigma^\infty Y_1) \wedge (\chi_2 \ltimes \Sigma^\infty Y_2) \arrow[r, "\alpha"] & (\chi_1 \oplus \chi_2) \ltimes (\Sigma^\infty Y_1 \wedge \Sigma^\infty Y_2)
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
\Sigma^\infty(X'^+ \wedge X^+ \wedge Y) \arrow[r, equal] \arrow[d, "\cong"] & \Sigma^\infty((X' \times X)^+ \wedge Y) \arrow[d, "\cong"] \\
\chi' \ltimes \chi \ltimes \Sigma^\infty Y \arrow[r, "\beta"] & (\chi' \odot \chi) \ltimes \Sigma^\infty Y
\end{tikzcd}$$

Analogous assertions hold with all functors Σ^∞ replaced by suitable functors $\Lambda^V\Sigma^\infty$.

Proof. The isomorphisms are given by Proposition 1.5, (i) is just a naturality diagram, and (ii) and (iii) are easily checked by our general prescription. The last assertion is based on Propositions 1.6 and 2.15 and is not much more difficult.

Our main goal in this section is to give precursors and analogs of most of the coherence diagrams of [H_∞, I.2.8-I.2.15]. Upon specialization to extended powers and use of operads, these will lead to proofs of the cited diagrams in VII§1. (The precursor of [H_∞, I.2.11] is just an obvious pair of naturality diagrams, and [H_∞, I.2.15] admits no non-tautologous precursor on the level at which we are presently working.) In most of these results, we shall not state precise hypotheses. We merely require the specified diagrams to be defined, the notations being taken from Proposition 3.1 (or Definition 3.4 below) or being cognate in ways which should be obvious from context. The following result gives precursors of [H_∞, I.2.8, I.2.9, and I.2.13].

Lemma 3.3. The following commutativity and associativity diagrams relating α and β commute on the level of spectra.

(i)
$$\begin{tikzcd}
(\chi_1 \ltimes E_1) \wedge (\chi_2 \ltimes E_2) \arrow[rr, phantom, "\cong"] \arrow[d, "\alpha"] & & t_*[(\chi_2 \ltimes E_2) \wedge (\chi_1 \ltimes E_1)] \arrow[d, "t_*\alpha"] \\
(\chi_1 \oplus \chi_2) \ltimes (E_1 \wedge E_2) \arrow[d, "\cong"] & & t_*[(\chi_2 \oplus \chi_1) \ltimes (E_2 \wedge E_1)] \arrow[d, "\beta"] \\
(\chi_1 \oplus \chi_2) \ltimes t_*(E_2 \wedge E_1) \arrow[dr, "\beta"] & & [t \odot (\chi_2 \oplus \chi_1)] \ltimes (E_2 \wedge E_1) \\
& [(\chi_1 \oplus \chi_2) \odot t] \ltimes (E_2 \wedge E_1) \arrow[ur, "t \ltimes 1"] &
\end{tikzcd}$$

Here, in $t \ltimes 1$, t is the transposition map $X_1 \times X_2 \to X_2 \times X_1$.

(ii)
$$\begin{tikzcd}
(\chi_1 \ltimes E_1) \wedge (\chi_2 \ltimes E_2) \wedge (\chi_3 \ltimes E_3) \arrow[r, "\alpha \wedge 1"] \arrow[d, "1 \wedge \alpha"] & {[(\chi_1 \oplus \chi_2) \ltimes (E_1 \wedge E_2)] \wedge (\chi_3 \ltimes E_3)} \arrow[d, "\alpha"] \\
(\chi_1 \ltimes E_1) \wedge [(\chi_2 \oplus \chi_3) \ltimes (E_2 \wedge E_3)] \arrow[r, "\alpha"] & (\chi_1 \oplus \chi_2 \oplus \chi_3) \ltimes (E_1 \wedge E_2 \wedge E_3)
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
\chi'' \ltimes \chi' \ltimes \chi \ltimes E \arrow[r, "\beta"] \arrow[d, "1 \ltimes \beta"] & (\chi'' \odot \chi') \ltimes \chi \ltimes E \arrow[d, "\beta"] \\
\chi'' \ltimes (\chi' \odot \chi) \ltimes E \arrow[r, "\beta"] & (\chi'' \odot \chi' \odot \chi) \ltimes E
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
(\chi_1' \ltimes \chi_1 \ltimes E_1) \wedge (\chi_2' \ltimes \chi_2 \ltimes E_2) \arrow[r, "\beta \wedge \beta"] \arrow[d, "\alpha"] & (\chi_1' \odot \chi_1 \ltimes E_1) \wedge (\chi_2' \odot \chi_2 \ltimes E_2) \arrow[d, "\alpha"] \\
(\chi_1' \oplus \chi_2') \ltimes [(\chi_1 \ltimes E_1) \wedge (\chi_2 \ltimes E_2)] \arrow[d, "1 \ltimes \alpha"] & {[(\chi_1' \odot \chi_1) \oplus (\chi_2' \odot \chi_2)] \ltimes (E_1 \wedge E_2)} \arrow[d, "(1 \times t \times 1) \ltimes 1"] \\
(\chi_1' \oplus \chi_2') \ltimes [(\chi_1 \oplus \chi_2) \ltimes (E_1 \wedge E_2)] \arrow[r, "\beta"] & {[(\chi_1' \oplus \chi_2') \odot (\chi_1 \oplus \chi_2)] \ltimes (E_1 \wedge E_2)}
\end{tikzcd}$$

Proof. In (i), $t \ltimes 1$ makes sense since a trivial diagram chase gives that $(\chi_1 \oplus \chi_2) \circ t = [t \circ (\chi_2 \oplus \chi_1)]t$. A similar verification is needed for (iv). The proofs are all easy and proceed via our general prescription.

We next construct the fourth map we wish to study. We take $G = H \times \pi$ here in order to facilitate comparison with the maps already given. In practice, we replace G by $G \times \pi$. The case relevant to the applications of [H_∞] is $H = e$.

Definition 3.4. Let $G = H \times \pi$. Let $\chi\colon X \to \mathcal{J}(U,U')$ be a G-map, where X is a G-CW complex, and let $f\colon U \oplus U \to U$ and $f'\colon U' \oplus U' \to U'$ be G-linear isometries. For G-CW spectra $E, F \in G\mathcal{S}\mathcal{A}$, we shall construct a natural map

$$\delta\colon \chi \ltimes f_*(E \wedge F) \to f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)]$$

in $\overline{h}G\mathcal{S}\mathcal{A}'$, where $\mathcal{A}$ and $\mathcal{A}'$ are indexing sets in U and U'. Consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "\chi"] \arrow[d, "\Delta"'] & \mathcal{J}(U,U') \arrow[r, "f^*"] \arrow[d, "\Delta"] & \mathcal{J}(U \oplus U, U') \\
X \times X \arrow[r, "\chi \times \chi"] & \mathcal{J}(U,U') \times \mathcal{J}(U,U') \arrow[r, "s"] & \mathcal{J}(U \oplus U, U' \oplus U') \arrow[u, "f^!_*"']
\end{tikzcd}$$

Since f^* and $f^!_* s\Delta$ are both G-maps, there is a G-homotopy $h\colon f^* \simeq f^!_* s\Delta$. Composing with χ and applying Theorem 1.13, we obtain an isomorphism

$$(\chi \circ f) \ltimes (E \wedge F) \simeq [f' \circ (\chi \oplus \chi)\Delta] \ltimes (E \wedge F)$$

in $\overline{h}G\mathcal{S}\mathcal{A}'$. The required map δ is defined by commutativity of the diagram

$$\begin{tikzcd}
\chi \ltimes f_*(E \wedge F) \arrow[r, "\delta"] \arrow[d, "\beta"'] & f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)]. \arrow[d, "f^!_*\alpha"] \\
(\chi \circ f) \ltimes (E \wedge F) \arrow[d, "\cong"'] & f^!_*[(\chi \oplus \chi) \ltimes (E \wedge F)] \arrow[d, "\beta"] \\
{[f' \circ (\chi \oplus \chi)\Delta] \ltimes (E \wedge F)} \arrow[r, "\Delta \ltimes 1"] & {[f' \circ (\chi \oplus \chi)] \ltimes (E \wedge F)}
\end{tikzcd}$$

The following mixed space and spectrum level analog of δ played an important role in McClure's work in [H_∞].

Definition 3.5. Let $\pi\colon X \times X \to X$ be the projection on the first coordinate and observe that $\pi\Delta = 1$, hence $\chi\pi\Delta = \chi$. For $E \in G\mathcal{S}\mathcal{A}$ and $Y \in G\mathcal{J}$, define

$$\delta = \Delta \ltimes 1\colon \chi \ltimes (E \wedge Y) \to (\chi\pi) \ltimes (E \wedge Y) \cong (\chi \ltimes E) \wedge (X^+ \wedge Y),$$

where the isomorphism is given by Propositions 1.5 and 1.7 (and transposition). Explicitly, the prespectrum level version of this map is induced from the standard reduced diagonal maps

$$T(X;V,V') \longrightarrow T(X;V,V') \wedge X^+$$

obtained by use of the bundle projections from the diagonal maps of the total spaces of the bundles $\eta(V,V')$ of Definition 2.1.

We have the following compatibility assertions.

Proposition 3.6. The following diagrams commute in the stable category.

(i)

$$\begin{tikzcd}
\Sigma^\infty(X^+ \wedge Y \wedge Z) \arrow[r, "\Sigma^\infty\delta"] \arrow[d, "\cong"'] & \Sigma^\infty(X^+ \wedge Y \wedge X^+ \wedge Z) \arrow[d, "\cong"] \\
\chi \ltimes f_*(\Sigma^\infty Y \wedge \Sigma^\infty Z) \arrow[r, "\delta"] & f^!_*[(\chi \ltimes \Sigma^\infty Y) \wedge (\chi \ltimes \Sigma^\infty Z)],
\end{tikzcd}$$

where the top map δ is $(1 \wedge t \wedge 1) \circ (\Delta \wedge 1 \wedge 1)$.

(ii)

$$\begin{tikzcd}
\chi \ltimes (E \wedge Y) \arrow[r, "\delta"] \arrow[dd, "\cong"'] & (\chi \ltimes E) \wedge (X^+ \wedge Y) \arrow[d, "\cong"] \\
 & f^!_*[(\chi \ltimes E) \wedge \Sigma^\infty(X^+ \wedge Y)] \arrow[d, "\cong"] \\
\chi \ltimes f_*(E \wedge \Sigma^\infty Y) \arrow[r, "\delta"] & f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes \Sigma^\infty Y)],
\end{tikzcd}$$

where the unit isomorphism of II.3.12(i) is used twice.

Proof. With $E = \Sigma^\infty Y$ and $F = \Sigma^\infty Z$, the equivalence appearing in Definition 3.4 reduces under the isomorphisms of Proposition 1.5 to the trivial equivalence induced by the standard inclusions $X \to X \times I \leftarrow X$, independent of the choice of h. Here the remaining components of δ are evaluated by Proposition 3.2, and part (i) follows. Part (ii) requires more work, and the argument provides a good illustration of how to exploit the full generality of Theorem 1.13 and Corollary 1.14. Let i and i' include U and U' as the first summands of $U \oplus U$ and $U' \oplus U'$. By II.3.12(i) and various uses of isomorphisms β, one finds that diagram (ii) reduces to the following one:

$$\begin{tikzcd}
\chi \ltimes (E \wedge Y) \arrow[rr, "\Delta \ltimes 1"] \arrow[d, "\cong"] & & (\chi\pi) \ltimes (E \wedge Y) \arrow[d, "\cong"] \\
(\chi \odot f \odot i) \ltimes (E \wedge Y) \arrow[d, "\cong"] & & (f' \odot i' \odot \chi\pi) \ltimes (E \wedge Y) \arrow[d, equal] \\
{[f' \odot (\chi \oplus \chi)\Delta \odot i] \ltimes (E \wedge Y)} \arrow[rr, "\Delta \ltimes 1"] & & {[f' \odot (\chi + \chi) \odot i] \ltimes (E \wedge Y)}
\end{tikzcd}$$

The bottom right equality holds since $i' \odot \chi\pi = (\chi \oplus \chi) \odot i$. The equivalences all come from application of Theorem 1.13 to homotopies derived from properties of $\mathcal{I}(U,U')$, and the commutativity requires use of Theorem 1.15 to process a similarly derived homotopy between homotopies. In detail, let $h\colon f^* \simeq f^!_* s\Delta$ be used in the construction of δ and let $j\colon 1 \simeq (fi)^*$ and $k\colon 1 \simeq (f'i')_*$ be homotopies derived from paths in $\mathcal{I}(U,U)$ and $\mathcal{I}(U',U')$ connecting 1 to fi and 1 to $f'i'$. Let

$$\tilde{h} = i^*h\colon (fi)^* = i^*f^* \simeq i^*f^!_*s\Delta = f^!_*i^*s\Delta = f^!_*i^!_* = (f'i')_*.$$

Then $\tilde{h}$, j, and k are homotopies between maps from $\mathcal{I}(U,U')$ to itself such that $j_1 = \tilde{h}_0$, $\tilde{h}_1 = k_1$, and $j_0 = k_0$. Thus these homotopies together specify a G-map

$$m\colon \mathcal{I}(U,U') \times \partial\Delta_2 \to \mathcal{I}(U,U'),$$

where Δ_2 is a 2-simplex. This is an (H,π)-bundle map, and so is the projection onto the first coordinate. Regarding Δ_2 as the cone on $\partial\Delta_2$ we see that any homotopy between these G-maps provided by the universality of $\mathcal{I}(U,U')$ yields an extension n of m over Δ_2. With $\mathcal{I}(U,U')$ abbreviated to W, n maps the following commutative diagram into W, the arrows being given by the faces of I and Δ_2:

$$\begin{tikzcd}
& & W \arrow[dl] \arrow[dr] & & \\
& W \times I \arrow[r] & W \times \Delta_2 & W \times I \arrow[l] & \\
W \arrow[ur] \arrow[rr] & & W \times I \arrow[u] & & W. \arrow[ul] \arrow[ll]
\end{tikzcd}$$

Using χ to map in the corresponding diagram with W replaced by X and quoting Theorem 1.15, we obtain a commutative diagram of twisted half smash products and equivalences between them for any variable spectrum. Applying this to $E \wedge Y$ and applying naturality to the right leg of the triangle, we obtain the desired commutative diagram in the form

$$\begin{tikzcd}
& \chi \ltimes (E \wedge Y) \arrow[rr, "\Delta \ltimes 1"] \arrow[dl, "\simeq"'] \arrow[d, "\simeq"] & & (\chi\pi) \ltimes (E \wedge Y) \arrow[d, "\simeq"] \\
(\chi \odot f \odot i) \ltimes (E \wedge Y) \arrow[r, "\simeq"] & {[f' \odot (\chi \oplus \chi)\Delta \odot i] \ltimes (E \wedge Y)} \arrow[rr, "\Delta \ltimes 1"] & & (f' \odot i' \odot \chi\pi) \ltimes (E \wedge Y).
\end{tikzcd}$$

The following two results will yield $[H_\infty$, I.2.10 and I.2.12$]$ by specialization.

Lemma 3.7. The following commutativity and associativity diagrams commute in the stable category.

(i)
$$\begin{tikzcd}
\chi \ltimes f_*(E \wedge F) \arrow[r, "\delta"] \arrow[d, "1 \wedge \gamma"'] & f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)] \arrow[d, "\gamma"] \\
\chi \ltimes f_*(F \wedge E) \arrow[r, "\delta"] & f^!_*[(\chi \ltimes F) \wedge (\chi \ltimes E)]
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
\chi \ltimes f_*[f_*(D \wedge E) \wedge F] \arrow[r, "\delta"] \arrow[d, "\cong"] & f^!_*[[\chi \ltimes f_*(D \wedge E)] \wedge (\chi \ltimes F)] \arrow[d, "f^!_*{[\delta \wedge 1]}"] \\
\chi \ltimes f_*[D \wedge f_*(E \wedge F)] \arrow[d, "\delta"'] & f^!_*[f^!_*[(\chi \ltimes D) \wedge (\chi \ltimes E)] \wedge (\chi \ltimes F)] \arrow[d, "\cong"] \\
f^!_*[(\chi \ltimes D) \wedge [\chi \ltimes f_*(E \wedge F)]] \arrow[r, "f^!_*{[1 \wedge \delta]}"] & f^!_*[(\chi \ltimes D) \wedge f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)]]
\end{tikzcd}$$

Proof. The commutativity isomorphisms γ and the unlabeled associativity isomorphisms are as specified in II.3.12. Identifying the four corners of the first diagram as single twisted half smash products by use of isomorphisms α and β, one finds that it reduces to

$$\begin{tikzcd}
(\chi \odot f) \ltimes (E \wedge F) \arrow[r, phantom, "\simeq"] \arrow[d, "\cong"] & {[f' \odot (\chi \oplus \chi)\Delta] \ltimes (E \wedge F)} \arrow[r, "\Delta \ltimes 1"] \arrow[d, "\cong"] & {[f' \odot (\chi \oplus \chi)] \ltimes (E \wedge F)} \arrow[d, "\cong"] \\
(\chi \odot ft) \ltimes (E \wedge F) \arrow[r, phantom, "\simeq"] & {[f't' \odot (\chi \oplus \chi))\Delta] \ltimes (E \wedge F)} \arrow[r, "\Delta \ltimes 1"] & {[f't' \odot (\chi \oplus \chi)] \ltimes (E \wedge F).}
\end{tikzcd}$$

Here $t' \odot (\chi \oplus \chi) = (\chi \oplus \chi) \odot t$, t and t' being the transpositions on $U \oplus U$ and $U' \oplus U'$. The right square is a naturality diagram. The top and bottom left equivalences are induced by homotopies

$$h\colon f^* \simeq f^!_*s\Delta \quad \text{and} \quad t^*h\colon (ft)^* \simeq t^*f^!_*s\Delta = (f't')_*s\Delta.$$

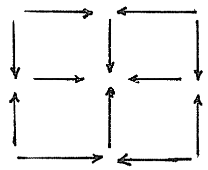
The left and middle vertical equivalences are induced by homotopies

$$f^* \simeq (ft)^* \quad \text{and} \quad f^!_*s\Delta \simeq (f't')_*s\Delta$$

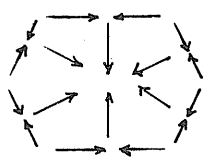
given by paths $f \to ft$ in $\mathcal{I}(U \oplus U,U)$ and $f' \to f't'$ in $\mathcal{I}(U' \oplus U',U')$. Clearly these four homotopies together specify a G-map

$$\mathcal{I}(U,U') \times \partial I^2 \to \mathcal{I}(U \oplus U,U').$$

By the universality of $\mathcal{I}(U \oplus U,U')$, this map extends over the square. Application of Theorem 1.15 yields a commutative diagram of equivalences of the shape



which gives the commutativity of the left square on passage to the stable category. The argument for the associativity diagram is precisely similar except that one has six homotopies to contend with, quotes the universality of $\mathcal{J}(U^3,U')$, and applies Theorem 1.15 to a diagram of the shape



<u>Lemma 3.8</u>. Let $\chi(x) = e$ for a G-fixed point $x \in X$. Then the following diagram commutes in the stable category:

$$\begin{tikzcd}
(ef)_*(E\wedge F) \arrow[r, phantom, "\simeq"] & {[f'(e \oplus e)]_*(E \wedge F)} \\
e_*f_*(E \wedge F) \arrow[u, "\beta"] \arrow[r, "\delta"] \arrow[d, "\iota"'] & f^!_*(e_*E \wedge e_*F) \arrow[u, "\beta o f^!_* \alpha"'] \arrow[d, "f^!_*(\iota \wedge \iota)"] \\
\chi \ltimes f_*(E \wedge F) \arrow[r, "\delta"] & f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)].
\end{tikzcd}$$

<u>Proof</u>. The top equivalence is derived from any path in $\mathcal{J}(U \oplus U,U')$ connecting ef to $f'(e \oplus e)$, and the commutativity of the top square is immediate from the definition of δ. The bottom square is a naturality diagram.

The following result is a precursor of $[H_\infty$, I.2.14].

<u>Lemma 3.9</u>. The following diagrams commute in the stable category.

(i)
$$\begin{tikzcd}
{[\chi_1 \ltimes f_{1*}(E_1\wedge F_1)]\wedge[\chi_2 \ltimes f_{2*}(E_2\wedge F_2)]} \arrow[r, "\alpha"] \arrow[d, "\delta\wedge\delta"'] & (\chi_1 \oplus \chi_2) \ltimes [f_{1*}(E_1\wedge F_1)\wedge f_{2*}(E_2\wedge F_2)] \arrow[d, phantom, "\cong"] \\
f^!_{1*}(\tilde{E}_1 \wedge \tilde{F}_1)\wedge f^!_{2*}(\tilde{E}_2 \wedge \tilde{F}_2) \arrow[d, phantom, "\cong"] & (\chi_1 \oplus \chi_2) \ltimes g_*(E_1 \wedge E_2 \wedge F_1 \wedge F_2) \arrow[d, "\delta"] \\
g^!_*(\tilde{E}_1 \wedge \tilde{E}_2 \wedge \tilde{F}_1 \wedge \tilde{F}_2) \arrow[r, "g^!_*(\alpha\wedge\alpha)"] & g^!_*[[(\chi_1 \oplus \chi_2) \ltimes (E_1 \wedge E_2)]\wedge[(\chi_1 \oplus \chi_2) \ltimes (F_1 \wedge F_2)]]
\end{tikzcd}$$

Here $\chi_i\colon X_i \to \mathcal{J}(U_i,U_i^!)$, $f_i\colon U_i \oplus U_i \to U_i$, and $f_i^!\colon U_i^! \oplus U_i^! \to U_i^!$ are given and we

have abbreviated notations by setting

$$\tilde{E}_1 = \chi_1 \ltimes E_1,\ \tilde{F}_1 = \chi_1 \ltimes F_1,\ \tilde{E}_2 = \chi_2 \ltimes E_2,\ \tilde{F}_2 = \chi_2 \ltimes F_2,$$

$$g = (f_1 \oplus f_2)(1 \oplus t \oplus 1)\colon U_1 \oplus U_2 \oplus U_1 \oplus U_2 \to U_1 \oplus U_2,$$

and $g' = (f_1^! + f_2^!)(1 \oplus t \oplus 1)$. The unlabeled isomorphisms combine the commutativity of the external smash product with usages of α and β.

(ii)
$$\begin{tikzcd}
\chi' \ltimes \chi \ltimes f_*(E \wedge F) \arrow[rr, "\beta"] \arrow[d, "1\ltimes\delta"'] & & (\chi' \odot \chi) \ltimes f_*(E \wedge F) \arrow[d, "\delta"] \\
\chi' \ltimes f^!_*[(\chi \ltimes E) \wedge (\chi \ltimes F)] \arrow[dr, "\delta"'] & & f^{!!}_*[[(\chi' \odot \chi) \ltimes E]\wedge[(\chi' \odot \chi) \ltimes F] \\
& f^{!!}_*[(\chi' \ltimes \chi \ltimes E) \wedge (\chi' \ltimes \chi \ltimes F)] \arrow[ur, "f''_*(\beta\wedge\beta)"'] &
\end{tikzcd}$$

Here $\chi\colon X \to \mathcal{J}(U,U')$, $\chi'\colon X \to \mathcal{J}(U',U'')$, and concomitant isometries f, f', and f'' are given.

<u>Proof</u>. Identifying vertices as single twisted half smash products and using naturality diagrams and the diagram of Lemma 3.3, one finds that part (i) reduces to the comparison of two equivalences

$$[(\chi_1 \odot f_1) \oplus (\chi_2 \odot f_2)] \ltimes (E_1\wedge F_1\wedge E_2\wedge F_2) \simeq (f_1^! s\Delta\chi_1 \oplus f_2^! s\Delta\chi_2) \ltimes (E_1\wedge F_1\wedge E_2\wedge F_2).$$

Here the evident sum of the homotopies $f_i^* \simeq f^!_{i*}s\Delta$ used to obtain the first equivalence determines (via transposition) an appropriate homotopy for obtaining the second. With this choice, the two equivalences are exactly the same. Similarly, identifying vertices of the second diagram as single twisted half smash products and using naturality diagrams based on the definition of δ together with the diagrams of Lemma 3.3, one finds that part (ii) reduces to proving the commutativity of a diagram of equivalences

$$\begin{tikzcd}
& (\chi' \odot \chi \odot f) \ltimes (E \wedge F) \arrow[dl, phantom, "\simeq"] \arrow[dr, phantom, "\cong"] & \\
(\chi' \odot f' s\Delta\chi) \ltimes (E \wedge F) \arrow[rr, phantom, "\simeq"] & & (f'' s\Delta'')(\chi' \odot \chi) \ltimes (E \wedge F).
\end{tikzcd}$$

Here $c(s\Delta' \times s\Delta) = s\Delta'' c\colon \mathcal{J}(U',U'') \times \mathcal{J}(U,U') \to \mathcal{J}(U + U,U'' + U'')$, where Δ, Δ', and Δ'' are the diagonals of $\mathcal{J}(U,U')$, $\mathcal{J}(U',U'')$, and $\mathcal{J}(U,U'')$. In terms of given homotopies

$$h\colon f^* \simeq f_*^! s\Delta, \quad h\colon \mathscr{I}(U,U') \times I \to \mathscr{I}(U \oplus U, U')$$

$$h'\colon (f')^* \simeq f_*'' s\Delta', \quad h'\colon \mathscr{I}(U',U'') \times I \to \mathscr{I}(U' \oplus U', U'')$$

$$h''\colon f^* \simeq f_*'' s\Delta'', \quad h''\colon \mathscr{I}(U,U'') \times I \to \mathscr{I}(U \oplus U, U''),$$

The three homotopies $\mathscr{I}(U',U'') \times \mathscr{I}(U,U') \times I \to \mathscr{I}(U \oplus U, U'')$ inducing the equivalences above are $c(1 \times h)$, $c(h' \times s\Delta)$, and $h''(c \times 1)$. These give a triangle of homotopies and, as usual, one can fill the triangle by universality and deduce the conclusion by Theorem 1.15.

Finally, the following observation relating δ to the transfer was used by McClure in [H_∞, IX§7]. Here it is convenient to pass to orbits over π.

<u>Lemma 3.10</u>. Assume that U is π-complete and that $U' = U^\pi$. Then the following diagram commutes for any finite π-space Y.

$$\begin{tikzcd} \chi \ltimes_\pi f_*(E \wedge F) \arrow[r, "\delta"] \arrow[d, "\tau"'] & f_*^![(\chi \ltimes_\pi E) \wedge (\chi \ltimes_\pi F)] \arrow[d, "f_*^![1 \wedge \tau]"] \\ \chi \ltimes_\pi (f_*(E \wedge F) \wedge Y^+) \arrow[r, "\delta"] & f_*^![(\chi \ltimes_\pi E) \wedge (\chi \ltimes_\pi (F \wedge Y^+))] \end{tikzcd}$$

<u>Proof</u>. Our assumptions on U and U' ensure that $\mathscr{I}(U,U')$ and X in Definition 3.4 are π-free, so that δ there is a map of π-free G-spectra, $G = H \times \pi$, and they also allow the indicated passages to orbits over π. On the bottom left, note that $f_*(?) \wedge Y^+ = f_*(? \wedge Y^+)$. The transfers are associated to "stable bundles" obtained by collapsing Y to the non-basepoint of S^0, as in IV.3.3(ii). (Here we use Proposition 1.5 to identify $\chi \ltimes_\pi (F \wedge Y^+)$ with $[(\chi \ltimes F) \wedge Y^+]/\pi)$. The map $f_*^![1 \wedge \tau]$ on the right is itself a transfer, by IV.5.3, the map δ on the bottom is a map of the relevant stable bundles, and the diagram commutes by the naturality of the transfer.

§4. <u>Untwisting G-homotopies and π-actions</u>

We now change our point of view. So far, G has been a perfectly arbitrary compact Lie group. We retain that convention, but consider G-spectra as the underlying ground objects of $(G \times \pi)$-spectra. While π will later be restricted to be a permutation group, it can be any compact Lie group in this section. See II.4.15(v) for generalities about $(G \times \pi)$-spectra, which we think of as π-spectra in the category of G-spectra.

We assume that U is a $(G \times \pi)$-universe on which π acts effectively and that U' is a $(G \times \pi)$-universe on which π acts trivially. We assume also that there exists a G-linear isometry $f\colon U \to U'$. These assumptions ensure that $\mathscr{I}(U,U')$ is π-free and G-contractible. By II.2.11, $\mathscr{I}(U,U')$ is a universal $\mathscr{F}(U,U')$-space. In practice, U' is G-complete, and a slight modification of the proof of II.2.4(ii) shows that $\mathscr{F}(U,U')$ then coincides with the family $\mathscr{F}(\pi)$ of subgroups of $G \times \pi$ which intersect π trivially. As explained in IV§1, this means that $\mathscr{I}(U,U')$ is the total space of a universal principal (G,π)-bundle.

Let X be an $\mathscr{F}(U,U')$-CW complex. If U' is G-complete, this just means that X is a π-free $(G \times \pi)$-CW complex and thus the total space of a principal $(G \times \pi)$-bundle. In general, there may be further restrictions on the isotropy groups of X. In any case, there is a unique homotopy class of $(G \times \pi)$-maps $\chi\colon X \to \mathscr{I}(U,U')$. We are interested in the calculational analysis of the twisted half smash products $\chi \ltimes E \in G\mathscr{S}\mathscr{A}'$ for $E \in G\mathscr{S}\mathscr{A}$, where $\mathscr{A}$ and $\mathscr{A}'$ are fixed chosen indexing sets in U and U'.

Our basic tool is the natural pair of equivalences of G-spectra

$$\text{(A)} \qquad \chi \ltimes E \xrightarrow{i_0} h \ltimes E \xleftarrow{i_1} \omega(f) \ltimes E = X^+ \wedge f_* E$$

provided by Theorem 1.16, where $f\colon U \to U'$ is any chosen G-linear isometry, $\omega(f)$ maps all of X to the point f, and $h\colon \chi \simeq \omega(f)$ is a G-homotopy provided by the G-contractibility of $\mathscr{I}(U,U')$. As in this diagram, we shall generally abbreviate maps of the form $i \ltimes 1$ to i throughout this section. We think of h as an "untwisting G-homotopy" and (A) as an untwisting G-equivalence between the twisted half smash product $\chi \ltimes E$ and the untwisted half smash product $X^+ \wedge f_* E$.

Of course, f and h are not π-maps. For this reason, $f_* E$ and $h \ltimes E$ are not π-spectra. The calculational utility of (A) depends crucially on the analysis of its behavior with respect to the action of π. It is this analysis and the analysis of the behavior of (A) with respect to the transformations of the previous section that are our subjects here. When we come to studying cellular chains in chapter VIII, we shall find it an easy matter to compute $C_*(X^+ \wedge f_* E)$, and the present analysis will allow us to deduce the structure of $C_*(\chi \ltimes E)$ as a π-complex. In this application we shall take $G = e$ and π finite, but the relevant geometry all works in full generality.

We emphasize before we begin that it is essential for the precise calculational control we want that the diagrams in all of the following lemmas commute on the level of spectra, before passage to the stable category. Lettered diagrams will generally be used to express the stable category level implications of these technical diagrams.

We first note that, up to a suitable diagram of equivalences, the equivalence (A) is independent of the various choices made. Thus let $\chi': X \to \mathcal{J}(U,U')$ be another $(G \times \pi)$-map and let $h': \chi' \simeq \omega(f')$ be another untwisting G-homotopy. Let $j: \chi \simeq \chi'$ be a $(G \times \pi)$-homotopy and let $k: \omega(f) \simeq \omega(f')$ be a G-homotopy provided by a G-path in $\mathcal{J}(U,U')$ connecting f to f'. Then there is a G-map

$$s: X \times I^2 \to \mathcal{J}(U,U')$$

which restricts to h, k, h', and j on the four sides of the square. There results a natural diagram of G-equivalences

(B)
$$\begin{tikzcd}
\chi \ltimes E \arrow[r, "i_0"] \arrow[d, "i_0"'] & j \ltimes E \arrow[d] & \chi' \ltimes E \arrow[l, "i_1"'] \arrow[d, "i_0"] \\
h \ltimes E \arrow[r] & s \ltimes E & h' \ltimes E \arrow[l] \\
X^+ \wedge f_* E \arrow[r, "i_0"] \arrow[u, "i_1"] & k \ltimes E \arrow[u] & X^+ \wedge f_*^! E \arrow[l, "i_1"'] \arrow[u, "i_1"']
\end{tikzcd}$$

in which the maps of the top row are $(G \times \pi)$-equivalences. It is in this sense that the equivalence (A) is independent of the choices of χ, f, and h.

The essential idea behind our analysis of π actions is to take seriously the discussion in II.4.15 and Remarks 1.9. This shows that π actions on G-spectra can be entirely understood in terms of morphisms of G-spectra. That is, we concentrate on the π-free $(G \times \pi)$-spectra $\pi \ltimes E$ generated by G-spectra E. To allow quotation of Theorem 1.13, we always assume that E has the homotopy type of a G-CW spectrum. By abuse, we agree to let π denote both our given group and the G-map $\pi \to \mathcal{J}(U,U)$ specified by the action of π on U. The abuse is justified by the last sentence of Remarks 1.9, which shows that the two possible interpretations of $\pi \ltimes E$ yield one and the same spectrum. We specialize (A) to a calculation of the underlying G-spectrum of $\pi \ltimes E$ by choosing an untwisting G-homotopy $d: \pi \simeq \omega(1)$, where $1: U \to U$ is the identity. This gives natural equivalences of G-spectra

(C) $$\pi \ltimes E \xrightarrow{i_0} d \ltimes E \xleftarrow{i_1} \omega(1) \ltimes E = \pi^+ \wedge E.$$

The following result computes the monad structure G-maps $\eta: E \to \pi \ltimes E$ and $\mu: \pi \ltimes \pi \ltimes E \to \pi \ltimes E$ in terms of the monad structure G-maps

$$\eta: E = \{1\}^+ \wedge E \to \pi^+ \wedge E \quad \text{and} \quad \mu: (\pi \times \pi)^+ \wedge E \to \pi^+ \wedge E$$

induced by the inclusion $\eta: \{1\} \to \pi$ and product $\mu: \pi \times \pi \to \pi$. Since η is a cofibration, we may assume that $d(1,t) = 1$ for all t.

Lemma 4.1. For a G-spectrum E, the following diagrams commute:

$$\begin{tikzcd}
E \arrow[r, "i_0"] \arrow[d, "\eta"'] & I^+ \wedge E \arrow[d, "\eta\ltimes 1"] & E \arrow[l, "i_1"'] \arrow[d, "\eta"] \\
\pi \ltimes E \arrow[r, "i_0"] & d \ltimes E & \pi^+ \wedge E \arrow[l, "i_1"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & \pi^+ \wedge \pi \ltimes E \arrow[ddll, "i_1 \times 1"'] \arrow[d, "\beta"] \arrow[ddrr, "1 \times i_0"] & & \\
 & & (\omega(1) \Phi \pi) \ltimes E \arrow[dl, "i_1 \times 1"'] \arrow[dr, "1 \times i_0"] & & \\
d \ltimes \pi \ltimes E \arrow[r, "\beta"'] & (d \Phi \pi) \ltimes E \arrow[dr] & & (\omega(1) \Phi d) \ltimes E \arrow[dl] & \pi^+ \wedge d \ltimes E \arrow[l, "\beta"] \\
 & & T \ltimes E & & \\
\pi \ltimes \pi \ltimes E \arrow[uu, "i_0 \times 1"] \arrow[r, "\beta"] \arrow[d, "\mu"'] & (\pi \Phi \pi) \ltimes E \arrow[r, "i_0"] \arrow[uu, phantom, "i_0 \times 1"] & d(\mu \times 1) \ltimes E \arrow[u] \arrow[d, "\mu \times 1"] & & (\pi \times \pi)^+ \wedge E \arrow[ll, "i_1"'] \arrow[uu, "1 \wedge i_1"'] \arrow[d, "\mu \wedge 1"] \\
\pi \ltimes E \arrow[rr, "i_0"] & & d \ltimes E & & \pi^+ \wedge E \arrow[ll, "i_1"']
\end{tikzcd}$$

Here $T: \pi \times \pi \times \Delta_2 \to \mathcal{J}(U,U)$ is a G-map which restricts to $d \Phi \pi$, $\omega(1) \Phi d$, and $d(\mu \times 1)$ on the faces of Δ_2. All maps not labeled η or μ are G-equivalences.

Proof. Since $\pi \Phi \mu = \pi \Phi \pi$ and $\omega(1) \Phi \mu = \omega(1) \Phi \omega(1)$, $d(\mu \times 1)$ is a homotopy between $\pi \Phi \pi$ and $\omega(1) \Phi \omega(1)$, and of course the notation $d \Phi \pi$ suppresses a transposition. The rest is immediate by naturality and Theorem 1.15.

The roof of the preceding diagram is just the natural equivalence $\pi \ltimes \pi \ltimes E \simeq \pi^+ \wedge \pi^+ \wedge E$ obtained by composing two equivalences (C). When E is a $(G \times \pi)$-spectrum, we relate ξ to $\pi^+ \wedge E$ by attaching the following monad action and naturality diagrams to the left sides of the previous diagrams.

Lemma 4.2. For a $(G \times \pi)$-spectrum E, the following diagrams commute:

$$\begin{tikzcd}
 & E \arrow[dl, equal] \arrow[d, "\eta"] \\
E & \pi \ltimes E \arrow[l, "\xi"']
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
\pi^+ \wedge E \arrow[d, "i_1"'] & \pi^+ \wedge \pi \ltimes E \arrow[l, "1 \wedge \xi"'] \arrow[d, "i_1 \times 1"] \\
d \ltimes E & d \ltimes \pi \ltimes E \arrow[l, "1 \ltimes \xi"'] \\
\pi \ltimes E \arrow[u, "i_0"] \arrow[d, "\xi"'] & \pi \ltimes \pi \ltimes E \arrow[l, "1 \ltimes \xi"'] \arrow[u, "i_0 \times 1"'] \arrow[d, "\mu"] \\
E & \pi \ltimes E \arrow[l, "\xi"']
\end{tikzcd}$$

Of course, these observations apply equally well with U replaced by U', and we let $d'\colon \pi \simeq \omega(1)$ be an untwisting G-homotopy $\pi \times I \to \mathcal{J}(U',U')$.

Remark 4.3. Since π acts trivially on U', $\pi = \omega(1)$ and we may take d' to be the constant homotopy. Then (C) trivializes and may as well be replaced by the natural identification

$$\pi \ltimes E' = \pi^+ \wedge E' \quad \text{for } E' \in G\mathcal{S}\mathcal{Q}'.$$

To exploit free π-spectra for the study of $\chi \ltimes E$, we need the following explicit description of the isomorphism

$$\text{(D)} \qquad \pi \ltimes \chi \ltimes E \xrightarrow{\cong} \chi \ltimes \pi \ltimes E$$

discussed in Remarks 1.9.

Lemma 4.4. Define a G-homeomorphism $\phi\colon \pi \times X \to X \times \pi$ by the formula $\phi(\sigma,x) = (\sigma x,\sigma)$ for $\sigma \in \pi$ and $x \in X$. Then the isomorphism (D) is the composite

$$\pi \ltimes \chi \ltimes E \xrightarrow{\beta} (\pi \Phi \chi) \ltimes E \xrightarrow{\phi} (\chi \Phi \pi) \ltimes E \xrightarrow{\beta^{-1}} \chi \ltimes \pi \ltimes E.$$

Proof. Since χ is a π-map, we have for $u \in U$ that

$$(\chi \Phi \pi)\phi(\sigma,x)(u) = (\chi \Phi \pi)(\sigma x,\sigma)(u) = \chi(\sigma x)(\sigma u)$$

$$= [\sigma\chi(x)](\sigma u) = \sigma[\chi(x)(u)] = (\pi \Phi \chi)(\sigma,x)(u).$$

Thus ϕ (i.e., $\phi \ltimes 1$) makes sense. By passage to adjoints, it is easy to see that this isomorphism is the one to which the recommended proof of Proposition 1.8 leads.

The following result relates (A), (C), and (D) via a diagram of the form

$$\text{(E)} \qquad \begin{array}{ccccc} \pi \ltimes \chi \ltimes E & \overset{(C)}{\simeq} \pi^+ \wedge \chi \ltimes E & \overset{(A)}{\simeq} & \pi^+ \wedge X^+ \wedge f_*E & = (\pi \times X)^+ \wedge f_*E \\ {\scriptstyle (D)}\Big\| \wr & & & & \Big\downarrow {\scriptstyle \phi \wedge 1} \\ \chi \ltimes \pi \ltimes E & \overset{(A)}{\simeq} X^+ \wedge f_*(\pi \ltimes E) & \overset{(C)}{\simeq} & X^+ \wedge f_*(\pi^+ \wedge E) & = (X \times \pi)^+ \wedge f_*E \end{array}$$

Here the bottom right identification comes from Proposition 1.5 and is obtained by smashing X^+ with the natural isomorphism

$$f_*(\pi^+ \wedge E) \xrightarrow{\beta} [f \Phi \omega(1)] \ltimes E = [\omega(1) \Phi f] \ltimes E \xrightarrow{\beta^{-1}} \pi^+ \wedge f_*E.$$

The top equivalence (C) trivializes by Remark 4.3.

Lemma 4.5. For a G-spectrum E, the following diagram commutes:

$$\begin{tikzcd}
\pi \ltimes \chi \ltimes E \arrow[r, "i_0 \times 1"] \arrow[d, "\beta"] & d' \ltimes \chi \ltimes E \arrow[d, "\beta"] & \pi^+ \wedge \chi \ltimes E \arrow[l, "i_1 \times 1"'] \arrow[r, "1 \wedge i_0"] \arrow[d, "\beta"] & \pi^+ \wedge h \ltimes E \arrow[d, "\beta"] & \pi^+ \wedge X^+ \wedge f_*E \arrow[l, "1 \wedge i_1"'] \arrow[d, "\beta"] \\
(\pi \Phi \chi) \ltimes E \arrow[r, "i_0 \times 1"] \arrow[ddrr, "i_0"] \arrow[ddddd, "\phi"] & (d' \Phi \chi) \ltimes E \arrow[dr] & (\omega(1) \Phi \chi) \ltimes E \arrow[l, "i_1 \times 1"'] \arrow[r, "1 \times i_0"] & (\omega(1) \Phi h) \ltimes E \arrow[dl] & (\omega(1) \Phi \omega(f)) \ltimes E \arrow[l, "1 \times i_1"'] \arrow[ddll, "i_1"'] \arrow[ddddd, "\phi"] \\
 & & M \ltimes E & & \\
 & & j \ltimes E \arrow[u] \arrow[d, "\phi \times 1"] & & \\
 & & k \ltimes E \arrow[d] & & \\
 & & N \ltimes E & & \\
(\chi \Phi \pi) \ltimes E \arrow[r, "i_0 \times 1"] \arrow[uurr, "i_0"] \arrow[d, "\beta^{-1}"] & (h \Phi \pi) \ltimes E \arrow[ur] \arrow[d, "\beta^{-1}"] & (\omega(f) \Phi \pi) \ltimes E \arrow[l, "i_1 \times 1"'] \arrow[r, "1 \times i_0"] \arrow[d, "\beta^{-1}"] & (\omega(f) \Phi d) \ltimes E \arrow[ul] \arrow[d, "\beta^{-1}"] & (\omega(f) \Phi \omega(1)) \ltimes E \arrow[l, "1 \times i_1"'] \arrow[uull, "i_1"'] \arrow[d, "\beta^{-1}"] \\
\chi \ltimes \pi \ltimes E \arrow[r, "i_0 \times 1"] & h \ltimes \pi \ltimes E & X^+ \wedge f_*(\pi \ltimes E) \arrow[l, "i_1 \times 1"'] \arrow[r, "1 \wedge f_* i_0"] & X^+ \wedge f_*(d \ltimes E) & X^+ \wedge f_*(\pi^+ \wedge E) \arrow[l, "1 \wedge f_* i_1"']
\end{tikzcd}$$

All maps in the diagram are isomorphisms or G-equivalences and, modulo the identification $f_*(\pi^+ \wedge E) = \pi^+ \wedge f_*E$, the right vertical composite is

$$\phi \wedge 1\colon (\pi \times X)^+ \wedge f_*E \to (X \times \pi)^+ \wedge f_*E.$$

Proof. The map $k\colon X \times \pi \times I \to \mathcal{J}(U,U')$ is an untwisting G-homotopy $\chi \Phi \pi \simeq \omega(f) \Phi \omega(1)$, hence $j = k(\phi \times 1)$ is an untwisting G-homotopy $\pi \Phi \chi \simeq \omega(1) \Phi \omega(f)$; here both right side composites are constant at f. The G-maps

$$M\colon \pi \times X \times \Delta_2 \to \mathcal{J}(U,U') \quad \text{and} \quad N\colon X \times \pi \times \Delta_2 \to \mathcal{J}(U,U')$$

restrict on the boundary of Δ_2 to the three homotopies dictated by the sources of the unlabeled arrows with targets $M \times E$ and $N \times E$. As usual, these maps are obtained by universality, and the rest follows by naturality and Theorem 1.15.

For a $(G \times \pi)$-spectrum E, the diagram

$$\text{(F)} \qquad f_*E \xleftarrow{f_*\xi} f_*(\pi \ltimes E) \xrightarrow{f_* i_0} f_*(d \times E) \xleftarrow{f_* i_1} f_*(\pi^+ \wedge E) = \pi^+ \wedge f_*E$$

should be viewed as providing a calculational substitute for an action of π on

f_*E. Here we attach the diagram of the following lemma to the bottom left of that of the preceding lemma to obtain a diagram of the form

$$\text{(G)}\qquad \begin{tikzcd} \pi \ltimes \chi \ltimes E \arrow[r, "\overset{(C)}{\cong}", phantom] \arrow[d, "\xi"'] & \pi^+ \wedge \chi \ltimes E \arrow[r, "\overset{(A)}{\cong}", phantom] & \pi^+ \wedge X^+ \wedge f_*E = (\pi \times X)^+ \wedge f_*E \arrow[d, "\phi"] \\ \chi \ltimes E \arrow[r, "\overset{(A)}{\cong}", phantom] & X^+ \wedge f_*E & X^+ \wedge \pi^+ \wedge f_*E = (X \times \pi)^+ \wedge f_*E \arrow[l, "1\wedge(F)"'] \end{tikzcd}$$

Lemma 4.6. For a $(G \times \pi)$-spectrum E, the following diagram commutes:

$$\begin{tikzcd} \chi \ltimes \pi \ltimes E \arrow[r, "i_0 \times 1"] \arrow[d, "1\ltimes\xi"'] & h \ltimes \pi \ltimes E \arrow[d, "1\ltimes\xi"] & X^+ \wedge f_*(\pi \ltimes E) \arrow[l, "i_1 \times 1"'] \arrow[d, "1\wedge f_*\xi"] \\ \chi \ltimes E \arrow[r, "i_0"] & h \ltimes E & X^+ \wedge f_*E \arrow[l, "i_1"'] \end{tikzcd}$$

For (G), recall that the π action map ξ of $\chi \ltimes E$ is the composite of $1 \ltimes \xi$ and the isomorphism (D).

Unfortunately, we are not quite finished with equivariance diagrams. While the previous results suffice to compute the cellular chains $C_*(X \ltimes_\pi E)$ when $G = e$, Bruner's work in $[H_\infty]$ required a precise homotopical analysis of the successive quotients $(X^n \ltimes_\pi E)/(X^{n-1} \ltimes_\pi E)$, where X^n is the n-skeleton of X. This analysis will be based on the following result, which is just an explicit description of a special case of the last isomorphism of Proposition 1.8.

Lemma 4.7. Let K be a G-space and let $\kappa\colon K \to \mathcal{J}(U,U')$ be a G-map. Let $\bar{\kappa}\colon \pi \times K \to \mathcal{J}(U,U')$ be the associated $(G \times \pi)$-map. Define $\theta\colon \pi \times K \to \pi \times K \times \pi$ by $\theta(\sigma,k) = (\sigma,k,\sigma^{-1})$. Then, for a $(G \times \pi)$-spectrum E, the composite

$$\bar{\kappa} \ltimes E \xrightarrow{\theta} (\pi \Phi \kappa \Phi \pi) \ltimes E \xrightarrow{\beta^{-1}} \pi \ltimes \kappa \ltimes \pi \ltimes E \xrightarrow{1 \ltimes 1 \ltimes \xi} \pi \ltimes \kappa \ltimes E$$

is an isomorphism of $(G \times \pi)$-spectra; its inverse is the $(G \times \pi)$-map

$$\bar{\eta}\colon \pi \ltimes \kappa \ltimes E \to \bar{\kappa} \ltimes E$$

obtained by freeness from the G-map $\eta\colon \kappa \ltimes E \to \bar{\kappa} \ltimes E$.

Proof. The idea here is that $\bar{\kappa} \ltimes E$ and $\pi \ltimes \kappa \ltimes E$ may be viewed as analogs of $(\pi \times K)^+ \wedge Y$ with two different π actions, namely the diagonal action and the left action on π, where Y is a π-space. Our maps are analogs of the standard inverse π-isomorphisms present in that situation. Of course, η is induced from the inclusion $K = \{1\} \times K \subset \pi \times K$. The map θ (i.e., $\theta \ltimes 1$) is well-defined since, for $\sigma \in \pi$, $k \in K$, and $u \in U$,

$$(\pi \Phi \kappa \Phi \pi)\ \theta(\sigma,k)(u) = (\pi \Phi \kappa \Phi \pi)(\sigma,k,\sigma^{-1})(u)$$
$$= \sigma[\kappa(k)(\sigma^{-1}u)] = [\sigma\kappa(k)](u) = \bar{\kappa}(\sigma,k)(u).$$

Thus the composite $(1 \ltimes 1 \ltimes \xi)\beta^{-1}\theta$ is certainly a well-defined G-map. That it is also a π-map is not obvious but will drop out from the fact that it is the inverse isomorphism to the $(G \times \pi)$-map $\bar{\eta}$. In terms of the isomorphism (D), $\bar{\eta}$ is the composite

$$\pi \ltimes \kappa \ltimes E \xrightarrow{1 \ltimes \eta} \pi \ltimes \bar{\kappa} \ltimes E \xrightarrow{(D)} \bar{\kappa} \ltimes \pi \ltimes E \xrightarrow{1 \times \xi} \bar{\kappa} \ltimes E.$$

Let $\psi = \phi(1 \times \eta)\colon \pi \times K \to (\pi \times K) \times \pi$, so that $\psi(\sigma,k) = (\sigma,k,\sigma)$. The evaluation of (D) in Lemma 4.4 implies that $\bar{\eta}$ is the composite

$$\pi \ltimes \kappa \ltimes E \xrightarrow{\beta} (\pi \Phi \kappa) \ltimes E \xrightarrow{\psi} (\bar{\kappa} \Phi \pi) \ltimes E \xrightarrow{\beta^{-1}} \bar{\kappa} \ltimes \pi \ltimes E \xrightarrow{1 \ltimes \xi} \bar{\kappa} \ltimes E.$$

By trivial computations, the composite

$$\pi \times K \xrightarrow{\psi} \pi \times K \times \pi \xrightarrow{\theta \times 1} \pi \times K \times \pi \times \pi \xrightarrow{1 \times 1 \times \mu} \pi \times K \times \pi$$

and the analogous composite with the roles of θ and ψ reversed both reduce to $1 \times 1 \times \eta$, $\eta\colon \{1\} \to \pi$. Using this observation, the monad identities $\xi\eta = 1$ and $\xi(1 \ltimes \xi) = \xi\mu$, the associativity of β, and the naturality of β, θ, and ψ, we can now check that our two maps are inverse isomorphisms by easy diagram chases.

We also need a diagram of the form

$$\text{(H)}\qquad \begin{tikzcd} \pi \ltimes \kappa \ltimes E \arrow[r, "\overset{(C)}{\cong}", phantom] \arrow[d, "\bar{\eta}"'] & \pi^+ \wedge \kappa \ltimes E \arrow[r, "\overset{(A)}{\cong}", phantom] & \pi^+ \wedge K^+ \wedge f_*E = (\pi \times K)^+ \wedge f_*E \arrow[d, "\psi \wedge 1"] \\ \bar{\kappa} \ltimes E \arrow[r, "\overset{(A)}{\cong}", phantom] & (\pi \times K)^+ \wedge f_*E & (\pi \times K)^+ \wedge (\pi^+ \wedge f_*E) = (\pi \times K \times \pi)^+ \wedge f_*E. \arrow[l, "1\wedge(F)"'] \end{tikzcd}$$

By the description of $\bar{\eta}$ in terms of the π action $\xi = (1 \ltimes \xi) \circ (D)$ on $\bar{\kappa} \ltimes E$ and the definition $\psi = \phi(1 \times \eta)$ in the previous proof, we see that such a diagram is obtained by attaching the following diagram to the top row of diagram (G) of Lemmas 4.5 and 4.6, with χ replaced by $\bar{\kappa}$.

<u>Lemma 4.8</u>. For a G-spectrum E, the following diagram commutes:

$$
\begin{tikzcd}
\pi \ltimes \kappa \ltimes E \arrow[r, "i_0\times 1"] \arrow[d, "1\ltimes\eta"'] & d' \ltimes \kappa \ltimes E \arrow[d, "1\ltimes\eta"] & \pi^+ \wedge \kappa \ltimes E \arrow[l, "i_1\times 1"'] \arrow[r, "1\wedge i_0"] \arrow[d, "1\wedge\eta"] & \pi^+ \wedge j \ltimes E \arrow[d, "1\wedge(\eta\times 1)"] & \pi^+ \wedge K^+ \wedge f_*E \arrow[l, "1\wedge i_1"'] \arrow[d, "1\wedge\eta\wedge 1"] \\
\pi \ltimes \bar{\kappa} \ltimes E \arrow[r, "i_0\times 1"] & d' \ltimes \bar{\kappa} \ltimes E & \pi^+ \wedge \bar{\kappa} \ltimes E \arrow[l, "i_1\times 1"'] \arrow[r, "1\wedge i_0"] & \pi^+ \wedge h \ltimes E & \pi^+ \wedge (\pi \times K)^+ \wedge f_*E. \arrow[l, "1\wedge i_1"']
\end{tikzcd}
$$

Here h is any untwisting G-homotopy for $\bar{\kappa}$, hence $j = h(\eta \times 1)$ is an untwisting G-homotopy for κ.

Returning to a $(G \times \pi)$-spectrum E, we combine Lemma 4.1, diagrams (G) and (H), and the fact that $\bar{\eta}$ is a π-map to obtain the following schematic diagram. It calculates the maps of the inner square in terms of simple space level maps and our substitute (F) for a π action on f_*E.

$$
\begin{tikzcd}[column sep=small]
 & (\pi\times\pi\times K)^+\wedge f_*E \arrow[rrr, "(\mu\times 1)\wedge 1"] \arrow[dr, phantom, "\cong"] \arrow[dd, "(1\times\psi)\wedge 1"'] & & & (\pi\times K)^+\wedge f_*E \arrow[dl, phantom, "\cong"] \arrow[dd, "\psi\wedge 1"] \\
 & & \pi\ltimes\pi\ltimes\kappa\ltimes E \arrow[r, "\mu"] \arrow[ddd, "1\times\bar{\eta}"] & \pi\ltimes\kappa\ltimes E \arrow[ddd, "\bar{\eta}"] & \\
 & (\pi\times\pi\times K\times\pi)^+\wedge f_*E \arrow[d, equal] & & & (\pi\times K\times\pi)^+\wedge f_*E \arrow[d, equal] \\
\text{(I)} & (\pi\times\pi\times K)^+\wedge\pi^+\wedge f_*E \arrow[dd, "1\wedge(F)"'] & & & (\pi\times K)^+\wedge\pi^+\wedge f_*E \arrow[dd, "1\wedge(F)"] \\
 & & \pi\ltimes\bar{\kappa}\ltimes E \arrow[r, "\xi"] & \bar{\kappa}\ltimes E & \\
 & (\pi\times\pi\times K)^+\wedge f_*E \arrow[ur, phantom, "\cong"] \arrow[dr, "\omega\wedge 1"'] & & & (\pi\times K)^+\wedge f_*E \arrow[ul, phantom, "\cong"] \\
 & & (\pi\times K\times\pi)^+\wedge f_*E = (\pi\times K)^+\wedge\pi^+\wedge f_*E \arrow[urr, "1\wedge(F)"'] & &
\end{tikzcd}
$$

Here $\psi(\sigma,k) = (\sigma,k,\sigma)$ at the top and $\omega(\sigma,\tau,k) = (\sigma\tau,k,\sigma)$ at the bottom left. The reader may find it instructive to replace f_*E by a π-space Y and chase the outer diagram on elements. He will find that K is a dummy variable and this is just the standard conversion of π actions on $(\pi \times K)^+\wedge Y$ mentioned at the start of the proof of Lemma 4.7.

The application to $(X^n \ltimes_\pi E)/(X^{n-1} \ltimes_\pi E)$ goes as follows.

<u>Proposition 4.9</u>. Let the n-skeleton X^n of the π-free $(G \times \pi)$-CW complex X be given by the pushout diagram

$$
\begin{tikzcd}
\pi \times K_{n-1} \arrow[r] \arrow[d] & X^{n-1}, \arrow[d] \\
\pi \times L_n \arrow[r] & X^n
\end{tikzcd}
$$

where K_{n-1} is a disjoint union of G-spheres S^{n-1}_H and L_n is the disjoint union of the corresponding G-cells. Observe that L_n/K_{n-1} is G-homeomorphic to $(X^n/\pi)/(X^{n-1}/\pi)$. Let χ_n denote the restriction to X^n of the $(G \times \pi)$-map $\chi\colon X \to \mathcal{J}(U,U')$ and let λ_n and κ_{n-1} be the G-maps obtained by restriction of χ_n to L_n and K_{n-1}, so that $\bar{\lambda}_n$ and $\bar{\kappa}_{n-1}$ are the restrictions of χ_n to $\pi \times L_n$ and $\pi \times K_{n-1}$. For $(G \times \pi)$-spectra E, there are natural isomorphisms of $(G \times \pi)$-spectra

$$(\chi_n \ltimes E)/(\chi_{n-1} \ltimes E) \cong (\bar{\lambda}_n \ltimes E)/(\bar{\kappa}_{n-1} \ltimes E) \cong (\pi \ltimes \lambda_n \ltimes E)/(\pi \ltimes \kappa_{n-1} \ltimes E).$$

When π acts trivially on U', there result natural isomorphisms

$$(\chi_n \ltimes_\pi E)/(\chi_{n-1} \ltimes_\pi E) \cong (\bar{\lambda}_n \ltimes_\pi E)/(\bar{\kappa}_{n-1} \ltimes_\pi E) \cong (\lambda_n \ltimes E)/(\kappa_{n-1} \ltimes E).$$

By restriction of χ, h and $\omega(f)$ and passage to quotients, diagram (A) induces an equivalence of G-spectra

$$(\lambda_n \ltimes E)/(\kappa_{n-1} \ltimes E) \simeq (L_n^+ \wedge f_*E)/(K_{n-1}^+ \wedge f_*E) \cong (L_n/K_{n-1}) \wedge f_*E.$$

<u>Proof</u>. The functor $\chi \ltimes E$ preserves pushouts over $\mathcal{J}(U,U')$ in χ, hence the first pair of isomorphisms is immediate from Lemma 4.7. Since passage to orbits over π is a left adjoint, it preserves pushouts and quotients and, by II.4.15 (iv), $\pi \times E/\pi = E$ for a G-spectrum E. The second pair of isomorphisms follows. The final G-equivalence is obvious and can be viewed as obtained by passage to orbits over π from an analogous $(G \times \pi)$-equivalence

$$(\pi \ltimes \lambda_n \ltimes E)/(\pi \ltimes \kappa_{n-1} \ltimes E) \simeq (\pi \ltimes L_n^+ \wedge f_*E)/(\pi \ltimes K_{n-1}^+ \wedge f_*E).$$

Calculationally, the left π actions on these $(G \times \pi)$-spectra correspond to the natural diagonal π action on $(\chi_n \ltimes E)/(\chi_{n-1} \ltimes E)$, an assertion made precise by diagram (I).

Finally, we turn to the relationship between the maps ι, α, β, and δ and untwisting G-homotopies. Of course, ι is only a G-map since $f = \chi(x)\colon U \to U'$ is only a G-linear isometry. (We use the letter f for consistency with diagram (A).) On the other hand, α and β are obviously $(G \times \pi)$-isomorphisms, a fact we have already exploited in the case of β, and Definition 3.4 constructs

δ as a $(G \times \pi)$-map provided that we start with $(G \times \pi)$-linear isometries

$$f_2\colon U \oplus U \to U \quad \text{and} \quad f_2'\colon U' \oplus U' \to U'.$$

(We add subscripts here to avoid confusion with our fixed f.) Observe that the discussion of smash products in II§3 applies here with G replaced by $G \times \pi$ since $\mathcal{J}(U,U)$ and $\mathcal{J}(U',U')$ are $(G \times \pi)$-contractible.

In the following result, we use the explicit spectrum level definition of δ instead of passing to the stable category. Otherwise our notations are slight variants of those of Proposition 3.1 and Definition 3.4, with G replaced by $G \times \pi$.

Proposition 4.10. For a G-spectrum E, the following diagrams commute, where all homotopies are untwisting G-homotopies.

(i)

$$\begin{tikzcd}
f_*E \arrow[r, "i_0"] \arrow[d, "\iota"] & I^+ \wedge f_*E \arrow[d, "i \times 1"] & f_*E \arrow[l, "i_1"'] \arrow[d, "i \wedge 1"] \\
\chi \ltimes E \arrow[r, "i_0"] & h \ltimes E & X^+ \wedge f_*E \arrow[l, "i_1"']
\end{tikzcd}$$

Here $i\colon \{x\} \subset X$ and $h\colon \chi \simeq \omega(f)$ is assumed to satisfy $h(x,t) = f$ for all t.

(ii)

$$\begin{tikzcd}
(\chi \ltimes E) \wedge (\chi' \ltimes E') \arrow[r, "i_0 \wedge i_0"] \arrow[dd, "\alpha"'] & (h \ltimes E) \wedge (h' \ltimes E') \arrow[d, "\alpha"] & X^+ \wedge f_*E \wedge (X')^+ \wedge f'_*E' \arrow[l, "i_1 \wedge i_1"'] \arrow[d, "1 \wedge t \wedge 1"] \\
 & (h \oplus h') \ltimes (E \wedge E') & (X \times X')^+ \wedge f_*E \wedge f'_*E' \arrow[d, "1 \wedge \alpha"] \\
(\chi \oplus \chi') \ltimes (E \wedge E') \arrow[r, "i_0"] \arrow[ur, "i_0 \times i_0"] & j \ltimes (E \wedge E') \arrow[u, "w"] & (X \times X')^+ \wedge (f \oplus f')_*(E \wedge E') \arrow[l, "i_1"'] \arrow[ul, "i_1 \times i_1"']
\end{tikzcd}$$

Here $w = (1 \times t \times 1)(1 \times 1 \times \Delta)\colon X \times X' \times I \to X \times I \times X' \times I$, h and h' are given untwisting G-homotopies $\chi \simeq \omega(f)$ and $\chi' \simeq \omega(f')$ and $j = (h \oplus h')w$ is the resulting untwisting G-homotopy $\chi \oplus \chi' \simeq \omega(f \oplus f')$.

(iii)

$$\begin{tikzcd}
\chi' \ltimes \chi \ltimes E \arrow[ddd, "\beta"'] & h' \ltimes \chi \ltimes E \arrow[l, "i_0"'] \arrow[d, "\beta"] & X'^+ \wedge f'_*(\chi \ltimes E) \arrow[l, "i_1"'] \arrow[r, "1 \wedge f'_* i_0"] \arrow[d, "1\ \beta"] & X'^+ \wedge f'_*(h \ltimes E) \arrow[d, "1\ \beta"] & X'^+ \wedge f'_*(X^+ \wedge f_*E) \arrow[l, "1 \wedge f'_* i_1"'] \arrow[dd, equal] \\
 & (h' \Phi \chi) \ltimes E \arrow[dr, "1 \times i_0"] & X'^+ \wedge (f' \Phi \chi) \ltimes E \arrow[l, "i_1 \times 1"'] \arrow[r, "1 \wedge (1 \times i_0)"] & X'^+ \wedge (f' \Phi h) \ltimes E \arrow[dl, "i_1 \times 1"'] & \\
 & & (h' \Phi h) \ltimes E & & (X' \times X)^+ \wedge f'_* f_*E \arrow[d, "1 \wedge \beta"] \\
(\chi' \Phi \chi) \ltimes E \arrow[rr, "i_0"] \arrow[uur, "i_0 \times 1"] & & h'' \ltimes E \arrow[u, "w"] & & (X' \times X)^+ \wedge (f'f)_*E \arrow[ll, "i_1"'] \arrow[uul, "1 \times i_1"']
\end{tikzcd}$$

Here $w = (1 \times t \times 1)(1 \times 1 \times \Delta)\colon X' \times X \times I \to X' \times I \times X \times I$, h and h' are given untwisting G-homotopies $\chi \simeq \omega(f)$ and $\chi' \simeq \omega(f')$, and $h'' = (h' \Phi h)w$ is the resulting untwisting G-homotopy $\chi' \Phi \chi \simeq \omega(f'f)$.

(iv)

$$\begin{tikzcd}
\chi \ltimes f_{2*}(E \wedge F) \arrow[r, "i_0"] \arrow[d, "\delta"] & h \ltimes f_{2*}(E \wedge F) \arrow[d, "\delta"] & X^+ \wedge f_*f_{2*}(E \wedge F) \arrow[l, "i_1"'] \arrow[d, "\delta"] \\
f'_{2*}[(\chi \ltimes E) \wedge (\chi \ltimes F)] \arrow[r, "f'_{2*}(i_0 \wedge i_0)"] & f'_{2*}[(h \ltimes E) \wedge (h \ltimes F)] & f'_{2*}[X^+ \wedge f_*E \wedge X^+ \wedge f_*E] \arrow[l, "f'_{2*}(i_1 \wedge i_1)"']
\end{tikzcd}$$

Here the right hand map δ is evaluated as the formal composite

$$\begin{tikzcd}
X^+ \wedge f_*f_{2*}(E \wedge F) \arrow[r, "1 \wedge \beta"] \arrow[d, "\delta"] & X^+ \wedge (ff_2)_*(E \wedge F) \arrow[r, "1 \wedge i_0"] & X^+ \wedge j \ltimes (E \wedge F) & X^+ \wedge g_*(E \wedge F) \arrow[l, "1 \wedge i_1"'] \arrow[d, "\Delta \wedge 1"] \\
f'_{2*}(X^+ \wedge f_*E \wedge X^+ \wedge f_*F) \arrow[rr, "f'_{2*}(1 \wedge t \wedge 1)"] & & (X \times X)^+ \wedge f'_{2*}(f_*E \wedge f_*F) \arrow[r, "1 \wedge \beta f'_{2*} \alpha"] & (X \times X)^+ \wedge g_*(E \wedge F),
\end{tikzcd}$$

where $g = f_2'(f \oplus f)$ and j is a path in $\mathcal{J}(U \oplus U, U')$ from ff_2 to g.

Proof. The assumption in (i) ensures that $h(i \times 1) \ltimes E = I^+ \wedge f_*E$ and is justified since i is a cofibration. The top right identification in (iii) is given by Proposition 1.5 and can be expressed in terms of isomorphisms β. Now parts (i)-(iii) are immediate. In part (iv), the maps δ are to be interpreted as formal composites, as in the second diagram, and the first diagram is to be interpreted as commutative on the spectrum level when suitably expanded and filled in. Since we have already given a plethora of such diagrams, we leave the details as an exercise for the interested reader.

Remark 4.11. If we redefine δ by replacing $\Delta\colon X \to X \times X$ by a $(G \times \pi)$-cellular approximation Δ', then, in the stable category, the new map is equal to the old map in view of the spectrum level commutative diagram

$$\begin{tikzcd}
{[f_2' \oplus (\chi \oplus \chi)\Delta] \ltimes (E \wedge F)} \arrow[r, "i_0"] \arrow[d, "\Delta"] & {[f_2' \oplus (\chi \oplus \chi)k] \ltimes (E \wedge F),} \arrow[dl, "k"'] \\
{[f_2' \oplus (\chi \oplus \chi)] \ltimes (E \wedge F)} & {[f_2' \oplus (\chi \oplus \chi)\Delta'] \ltimes (E \wedge F)} \arrow[l, "\Delta'"'] \arrow[u, "i_1"']
\end{tikzcd}$$

where $k\colon \Delta \simeq \Delta'$. That is, we must actually replace Δ by $\Delta' i_1^{-1} i_0$. Of course, all claims about diagrams commuting in the stable category remain valid for the new δ. In the diagram of (iv) above, Δ is now replaced by Δ', and this will allow calculation of δ on cellular chains when $G = e$.

§5. Extended powers of G-spectra

While the full generality of the previous section has its uses, the application to extended powers is based on restriction to the case when π is taken to be a subgroup of a symmetric group Σ_j. We fix a complete G-universe U and regard it as a Σ_j-trivial $(G \times \Sigma_j)$-universe. With left action by permutations, U^j is a $(G \times \Sigma_j)$-universe with effective Σ_j action. We regard $\mathcal{I}(U^j,U)$ as a left G and right Σ_j-space, with $(f\sigma)(u) = f(\sigma u)$. It is a universal principal (G,Σ_j)-bundle.

We fix an indexing set $\mathcal{A}$ in the G-universe U and we let $\mathcal{A}^j = \{V^j \mid V \in \mathcal{A}\}$ be the resulting "diagonal" indexing set in the $(G \times \Sigma_j)$-universe U^j.

We also fix a subgroup π of Σ_j and a π-free $(G \times \pi)$-CW complex X. In line with our conventions on $\mathcal{I}(U^j,U)$, we regard X as a left G and right π-space. Let $\chi\colon X \to \mathcal{I}(U^j,U)$ be any chosen $(G \times \pi)$-map.

For $E \in G\mathcal{S}\mathcal{A}$, we define $\chi \ltimes_\pi E^{(j)}$, the χ^{th} extended power of E, to be the orbit spectrum $(\chi \ltimes E^{(j)})/\pi$. We write $X \ltimes_\pi E^{(j)}$ for $\chi \ltimes_\pi E^{(j)}$ regarded as a spectrum in the stable category $\bar{h}G\mathcal{S}\mathcal{A}$. It is independent of χ by Theorem 1.14.

The essential, obvious, point here is that the j-fold external smash power $E^{(j)} \in G\mathcal{S}\mathcal{A}^j$ is a $(G \times \pi)$-spectrum under permutations. It is worthwhile to be precise about this. For $\sigma \in \pi$ and G-prespectra D_r, $1 \leq r \leq j$, permutation of smash products (from the left)

$$\sigma\colon D_1V \wedge \cdots \wedge D_jV \to D_{\sigma^{-1}(1)}V \wedge \cdots \wedge D_{\sigma^{-1}(j)}V$$

specifies the external commutativity isomorphism

$$\sigma\colon \sigma_*(D_1 \wedge \cdots \wedge D_j) \to D_{\sigma^{-1}(1)} \wedge \cdots \wedge D_{\sigma^{-1}(j)}.$$

Here σ_* enters because the permutation above implicitly also permuted indexing spaces. This passes to spectra via L. The π action

$$\pi \ltimes E^{(j)} = \bigvee_{\sigma \in \pi} \sigma_* E^{(j)} \to E^{(j)}$$

has σ^{th} component the isomorphism σ just specified.

As in the previous sections, extended powers are best studied in appropriate equivariant categories. That is, we keep track of equivariance and only pass to orbits at the last possible moment. This results in greater precision and generality and simplifies various proofs. It is also essential to the key definitions of the next chapter. In line with this, it is sensible to work more generally with

$$\chi \ltimes (E_1 \wedge \cdots \wedge E_j) \in G\mathcal{S}\mathcal{A}^j$$

for $E_r \in G\mathcal{S}\mathcal{A}$ rather than just with $\chi \ltimes E^{(j)}$. For $\sigma \in \pi$, right multiplication $\sigma\colon X \to X$ leads to the dotted arrow composite

$$\begin{tikzcd}
\chi \ltimes (E_1 \wedge \cdots \wedge E_j) \arrow[r, "1 \ltimes \sigma^{-1}"] \arrow[d, dotted, "\sigma^{-1}"'] & \chi \ltimes \sigma_* (E_{\sigma(1)} \wedge \cdots \wedge E_{\sigma(j)}). \arrow[d, "\beta"] \\
\chi \ltimes (E_{\sigma(1)} \wedge \cdots \wedge E_{\sigma(j)}) & (\chi \oplus \sigma) \ltimes (E_{\sigma(1)} \wedge \cdots \wedge E_{\sigma(j)}) \arrow[l, "\sigma \ltimes 1"']
\end{tikzcd}$$

When all $E_r = E$, this is the left action by σ^{-1} on $\chi \ltimes E^{(j)}$. Clearly it might happen that some but not all of the E_r are equal and that π permutes blocks of equal variables among themselves. We can still pass to orbits and so obtain more general variants of extended powers. Since these variants play a role in our definition of operad ring spectra, we give a notation which allows us to be precise.

Notations 5.1. Let p be a partition of the set $\underline{j} = \{1,\cdots,j\}$ into k π-invariant subsets. We may think of p as a function $\underline{j} \to \underline{k}$, $p^{-1}(s)$ being the s^{th} subset. In the most interesting case, π is the group of all permutations which fix each set $p^{-1}(s)$ and is denoted $\pi[p]$. Suppose given spectra F_s for $1 \leq s \leq k$. If $E_r = F_{p(r)}$ for $1 \leq r \leq j$, then

$$\chi \ltimes_\pi (\bigwedge_{r=1}^{j} E_r) = \chi \ltimes_\pi (\bigwedge_{r=1}^{j} F_{p(r)})$$

is defined. Of course, the uniqueness assertion of Theorem 1.14 still applies, and there are evident analogs of Propositions 5.2 and 5.3 below. We shall say that a map is defined or a diagram commutes "with all possible equivariance" if whenever variables coalesce with respect to a partition p, the map is a $\pi[p]$-map or the diagram commutes in the stable category of $(G \times \pi[p])$-spectra and so passes via orbits to a commutative diagram in $\bar{h}G\mathcal{S}\mathcal{A}$.

Formally, the stable category level functor $X \ltimes_\pi E^{(j)}$ is the composite

$$\bar{h}G\mathcal{SA} \xrightarrow{(?)^{(j)}} \bar{h}(G \times \pi)\mathcal{SA}^j \xrightarrow{X \ltimes (?)} \bar{h}(G \times \pi)\mathcal{SA} \xrightarrow{(?)/\pi} \bar{h}G\mathcal{SA}.$$

This formulation raises a technical question. As usual, we must approximate E by a G-CW spectrum before passage to the stable category. However, it is important to know that we don't have to reapproximate $E^{(j)}$ by a $(G \times \pi)$-CW spectrum. We shall prove the following result at the end of the section.

Proposition 5.2. If E is a G-CW spectrum, then $E^{(j)}$ has the $(G \times \pi)$-homotopy type of a $(G \times \pi)$-CW spectrum.

By Theorem 1.11 and I.5.6, we already know that the functor $\chi \ltimes (?)$ preserves $(G \times \pi)$-CW homotopy types and the functor $(?)/\pi$ carries $(G \times \pi)$-CW spectra to G-CW spectra.

In many respects, the depth of the construction $\chi \ltimes_\pi E^{(j)}$ lies in the initial j^{th} power functor. This fails to commute with wedges, pushouts, smash products with spaces, and most other constructions of interest, the deviations leading to much of the calculational power of the construction. The commutation relations that do hold are crucial.

Proposition 5.3. For a based G-space Y and for $V \in \mathcal{A}$, there is a natural isomorphism

$$(\Lambda^V \Sigma^\infty Y)^{(j)} \cong \Lambda^{V^j} \Sigma^\infty (Y^{(j)})$$

in $(G \times \pi)\mathcal{SA}^j$ and there are natural isomorphisms

$$\chi \ltimes_\pi (\Sigma^\infty Y)^{(j)} \cong \Sigma^\infty (X^+ \wedge_\pi Y^{(j)})$$

and

$$\chi \ltimes_\pi (\Lambda^V \Sigma^\infty Y)^{(j)} \cong \operatorname{colim} \Lambda^{\mu_\alpha V^j} \Sigma^\infty (T(A;V^j,\mu_\alpha V^j) \wedge_\pi Y^{(j)})$$

in $G\mathcal{SA}$, where the colimit is taken over the compact restrictions $\alpha\colon A \subset X \to \mathcal{I}(U^j,U)$ of χ and $\alpha(A)(V^j) \subset \mu_\alpha V^j$.

Proof. The first isomorphism is an immediate verification from II.3.6. The others hold before passage to orbits by Propositions 1.5 and 2.15 and follow as written since passage to orbits commutes with colimits and with Σ^∞, by I.3.8.

Of course, on passage to stable categories, the shift desuspensions Λ^V here may as well be replaced by the geometric desuspensions Ω^V.

Remarks 5.4. (i) If the bundle $\zeta(V^j,\mu_\alpha V^j)$ over A of Definition 2.1 has total space Z, then the space

$$T(A;V^j,\mu_\alpha V^j) \wedge_\pi Y^{(j)}$$

is the quotient of the Thom complex of $Z \times_\pi Y^{(j)} \to A \times_\pi Y^{(j)}$ by the Thom complex of $Z \times_\pi \{*\} \to A \times_\pi \{*\}$. Of course, when $G = e$, this relative Thom complex has a Thom isomorphism whenever the larger bundle named is orientable (see e.g. IX§5 below).

(ii) Since X is a π-free $(G \times \pi)$-CW complex, the colimit may be taken over its finite subcomplexes. In practice, the skeleta of X are finite and the colimit may be taken over them or over any conveniently chosen cofinal subsequence of skeleta.

(iii) When $G = e$, $U = R^\infty$, and $V = R^n$, $\mu_\alpha V^j$ is simply some $R^{q(\alpha)}$ with $q(\alpha)$ sufficiently large that $\alpha(A)(R^n) \subset R^{q(\alpha)}$. Here $T(A;R^{nj},R^{q(\alpha)})$ is the Thom complex of the π-bundle $\varepsilon_{q(\alpha)} - n\zeta_j$ over A, where ε_q is the q-dimensional trivial π-bundle $A \times R^q$ and ζ_j is the π-bundle $A \times (R^1)^j$ with diagonal π action on the total space. Explicit computation is quite practicable; see Bruner [H_∞, V§2] for examples.

The following result allows us to recognize maps of spectra defined on extended powers in terms of concrete space level data; it is related to McClure's work in chapter [H_∞,VII].

Proposition 5.5. Let $E, E' \in G\mathcal{SA}$. A map $\xi\colon \chi \ltimes_\pi E^{(j)} \to E'$ in $G\mathcal{SA}$ determines and is determined by maps $\xi(x)\colon (\ell E)^{(j)^\pi} \to \chi(x)^*(\ell E')$ in $G\mathcal{PA}$ for points $x \in X$ such that $\xi(x) \circ \tau = \xi(x\tau)$ for $\tau \in \Sigma_j$ and the functions

$$\zeta\colon T(A(V,W);\, V,W) \wedge EV_1 \wedge \cdots \wedge EV_j \to E'W,\ V = V_1 \oplus \cdots \oplus V_j,$$

specified by $\zeta([x,w] \wedge y_1 \wedge \cdots \wedge y_j) = \sigma(\xi(x)(y_1 \wedge \cdots \wedge y_j \wedge w)$ for $x \in X$ such that $\chi(x)(V) \subset W$, $w \in W - \chi(x)(V)$, and $y_r \in EV_r$ are continuous.

Proof. By adjunction, the prespectrum level maps $\xi(x)$ determine and are determined by spectrum level maps $L\xi(x)\colon E^{(j)} \to \chi(x)^*(E')$, and the continuity of L (discussed above I.3.5) implies that the given continuity statement is equivalent to the continuity statement of Proposition 2.17. Thus that result implies that a G-map $\xi\colon \chi \ltimes E^{(j)} \to E'$ determines and is determined by G-maps $\xi(x)$ making the functions ζ continuous. The formula $\xi(x) \circ \tau = \xi(x\tau)$ makes sense since

$$\chi(x\tau)(V_1 \oplus \cdots \oplus V_j) = \chi(x)(V_{\tau^{-1}(1)} \oplus \cdots \oplus V_{\tau^{-1}(j)})$$

and is equivalent to the π-equivariance of the corresponding map ξ.

We have left one unfinished piece of business.

Proof of Proposition 5.2. $E^{(j)}$ has the sequential and skeletal filtrations obtained iteratively from II.3.8. Inductively, it suffices to prove that, for $n \geq 0$, the domain of the attaching map for the construction of $(E^{(j)})_{n+1}$ from $(E^{(j)})_n$ has the $(G \times \pi)$-homotopy type of a $(G \times \pi)$-CW spectrum. This domain may be written as the wedge of boundaries of distinct j-fold smash products of cells of E, and the wedge summands may be grouped into orbits under the action of π. It suffices to consider a single orbit, and we choose an orbit representative

$$F = \partial(e_{H_1}^{q_1} \wedge \cdots \wedge e_{H_j}^{q_j}).$$

Let $\rho \subset \pi$ be the subgroup of elements of π which fix F. Then it is easy to see that the given orbit is isomorphic as a $(G \times \pi)$-spectrum to

$$(G \times \pi) \ltimes_{G \times \rho} F.$$

By II.4.12, the extension of group actions functors preserve CW homotopy types, hence it suffices to consider the $(G \times \rho)$-spectrum F. Since cells are smash products with orbits, we may write

$$F = [(G/H_1) \times \cdots \times (G/H_j)]^+ \wedge \partial(e^{q_1} \wedge \cdots \wedge e^{q_j}).$$

Let $\{K_1, \cdots, K_i\}$ be a subsequence of $\{H_1, \cdots, H_j\}$ consisting of one group for each distinct cell in the sequence $\{e_{H_1}^{q_1}, \cdots, e_{H_j}^{q_j}\}$ and let the cell corresponding to K_s appear a_s times in this sequence. Of course, the K_s need not be distinct since the same group may index more than one cell (a fact obscured by our notations, which ignore the source of our cells in the structure of E). Let $K = K_1^{a_1} \times \cdots \times K_i^{a_i}$ and embed K in G^j according to the positions in which corresponding cells appear. The elements (σ, k) for $\sigma \in \rho$ and $k \in K$ specify a subgroup L of the wreath product $\rho \int G$, and we have an identification of $(G \times \rho)$-spaces

$$G/H_1 \times \cdots \times G/H_q = (\rho \int G)/L.$$

Here the right side is a $(G \times \rho)$-space by pullback along the inclusion $i\colon G \times \rho \to \rho \int G$ given by transposition and the diagonal on G. Now II.4.8 implies an isomorphism of $(G \times \rho)$-spectra

$$F \cong i^*[(\rho \int G) \ltimes_L \partial(e^{q_1} \wedge \cdots \wedge e^{q_j})],$$

where e^{q_r} is viewed as an H_r-spectrum. Again by II.4.12, it suffices to consider the L-spectrum $F' = \partial(e^{q_1} \wedge \cdots \wedge e^{q_j})$. We agree to write $e^{q_r} = \Lambda^{m_r}\Sigma^\infty e^{n_r}$, where m_r and n_r are specfied by

$$e^{q_r} = \Lambda^0\Sigma^\infty e^{q_r} \text{ if } q_r \geq 1 \text{ and } e^{q_r} = \Lambda^{-1-q_r}\Sigma^\infty e_1 \text{ if } q_r \leq 0.$$

Let $V = R^{m_1} \oplus \cdots \oplus R^{m_j} \subset (R^\infty)^j$. The definition of ρ implies that V is a ρ-space and thus a K-trivial L-space, and II.3.6 implies an isomorphism of L-spectra

$$F' \cong \Lambda^V\Sigma^\infty\partial(e^{n_1} \wedge \cdots \wedge e^{n_j}).$$

By I.5.14, the functor $\Lambda^V\Sigma^\infty$ preserves L-CW homotopy types. The space level boundary here can easily be subdivided to a ρ-CW complex and thus a K-trivial L-CW complex and the conclusion follows.

In the case $G = e$, we shall say more about the cell structure on $E^{(j)}$ and on $\chi \ltimes_\pi E^{(j)}$ in VIII§2.

VII. Operad ring spectra

by L. G. Lewis Jr., J. P. May, and M. Steinberger

The notions of structured ring spectra with which $[H_\infty]$ was primarily concerned presupposed sequences of extended j-fold powers with suitable interrelationships as j varies. The appropriate framework for defining such sequences is given by the notion of an operad originally introduced for the study of iterated loop spaces. We recall the definition in section 1 and use it to complete the derivation of the coherence diagrams promised in $[H_\infty$, I§2]. In particular, we shall see that all of the diagrams occurring in the definition of an operad lift to natural commutative diagrams relating the associated extended powers of spectra.

This last fact leads in section 2 to the definition of an action of an operad $\mathcal{C}$ on a spectrum E in terms of maps $\xi_j\colon \mathcal{C}_j \ltimes E^{(j)} \to E$. Actually, there are two variants of the definition. For $\mathcal{C}$-spectra, the diagrams relating the maps ξ_j are required to commute on the level of spectra. For $\overline{h}\mathcal{C}$-spectra, or "$\mathcal{C}$-spectra up to homotopy", these diagrams are only required to commute in the stable category. Of course, $\mathcal{C}$-spectra are $\overline{h}\mathcal{C}$-spectra by neglect of structure. When $\mathcal{C}$ is an E_∞ operad, we call $\mathcal{C}$-spectra E_∞ ring spectra and $\overline{h}\mathcal{C}$-spectra H_∞ ring spectra. This notion of an E_∞ ring spectrum is equivalent to that originally specified by May, Quinn, and Ray [99], and this notion of an H_∞ ring spectrum is equivalent to that specified in $[H_\infty$, I§3]. The former notion is appropriate to infinite loop space theory and, in particular, to the study of oriented bundle and fibration theories. Such applications were the theme of [99]. The latter notion is appropriate to applications in stable homotopy theory, as was the theme in $[H_\infty]$. From the stable homotopy point of view, the main interest in E_∞ ring spectra is that, as discussed in $[H_\infty$, I§4], the machinery of multiplicative infinite loop space theory constructs a plethora of interesting examples of E_∞ ring spectra and thus of H_∞ ring spectra. We conclude section two with a discussion of A_∞ ring spectra; Robinson's paper [124] on the Kunneth and universal coefficients theorems in generalized cohomology theories is based on the use of such spectra.

The rest of this chapter is concerned with free operad ring spectra. The free $\mathcal{C}$-space CX generated by a based space X was constructed in [97,§2], and this construction has played a central role in iterated loop space theory and its applications. In section 3, we construct a spectrum CE from a unital spectrum E and a suitable "coefficient system" $\mathcal{C}$. In section 4, we show that CE is the free $\mathcal{C}$-spectrum generated by E when $\mathcal{C}$ is an operad.

Ralph Cohen [34] observed that the formal properties of our construction CE lead to an extremely simple conceptual proof of the stable splittings

$$\Sigma^\infty CX \simeq \bigvee_{r\geq 1} \Sigma^\infty(\mathcal{C}_r^+ \wedge_{\Sigma_r} X^{(r)})$$

of Snaith [129] and F. Cohen, May, and Taylor [30]. Going further, and following up later conversations with R. Cohen, we show in section 5 that his proof can also be applied to obtain the multiplicative properties of the generalized James maps which give the splitting. These properties were first proven by Caruso, F. Cohen, May, and Taylor [23,32] using space level combinatorial constructions of the James maps. As explained in [32], these properties yield an algorithm for the calculation of the homological behavior of the James maps. This algorithm was effectively exploited by Kuhn [77] and was the main technical tool in his later proof of the Whitehead conjecture [78].

In view of the work in the previous chapters, we find it simple and natural to continue working in the equivariant context throughout this chapter. Thus G is again to be an arbitrary compact Lie group, and all spaces and spectra are to be G-spaces and G-spectra, even when we neglect to mention this assumption. In particular, this means that we are generalizing the cited splitting theorems and analysis of James maps to the equivariant setting, as was first done by Caruso and May working on the space level. As we shall point out at the end of section 5, this implies a generalization of Snaith's stable splittings of spaces $\Omega^n\Sigma^n X$ to stable splittings of equivariant loop spaces $\Omega^V\Sigma^V X$ when G is finite.

In this chapter, we use τ rather than γ for the commutativity isomorphism $E\wedge F \simeq F\wedge E$.

§1. Operads and extended powers

Definition 1.1. A G-operad $\mathcal{C}$ is a sequence of $G\times\Sigma_j$-spaces $\mathcal{C}_j$ for $j \geq 0$, with G acting on the left and Σ_j acting on the right and with $\mathcal{C}_0$ a single point $*$, together with G-maps

$$\gamma\colon \mathcal{C}_k \times \mathcal{C}_{j_1} \times\cdots\times \mathcal{C}_{j_k} \to \mathcal{C}_j,\quad j = j_1+\cdots+j_k,$$

for all $k \geq 0$ and $j_s \geq 0$ and a G-fixed unit element $1\in \mathcal{C}_1$. These data are subject to the following axioms.

(i) The following associativity diagram commutes, where $j = j_1+\cdots+j_k$, $i = i_1+\cdots+i_j$, $g_s = j_1+\cdots+j_s$, and $h_s = i_{g_{s-1}+1}+\cdots+i_{g_s}$:

$$\begin{tikzcd}
\mathcal{C}_k\times\mathcal{C}_{j_1}\times\cdots\times\mathcal{C}_{j_k}\times\mathcal{C}_{i_1}\times\cdots\times\mathcal{C}_{i_j} \arrow[r, "\gamma\times 1"] \arrow[d, "\text{shuffle}"] & \mathcal{C}_j\times\mathcal{C}_{i_1}\times\cdots\times\mathcal{C}_{i_j} \arrow[dr, "\gamma"] & \\
 & & \mathcal{C}_i \\
\mathcal{C}_k\times\left(\prod_{s=1}^{k}\mathcal{C}_{j_s}\times\mathcal{C}_{i_{g_{s-1}+1}}\times\cdots\times\mathcal{C}_{i_{g_s}}\right) \arrow[r, "1\times(\prod_{s=1}^{k}\gamma)"] & \mathcal{C}_k\times\mathcal{C}_{h_1}\times\cdots\times\mathcal{C}_{h_k} \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

(ii) The following unit diagrams commute:

$$\begin{tikzcd} \mathcal{C}_k \times \{1\}^k \arrow[d, "\cap"'] \arrow[r, phantom, "="] & \mathcal{C}_k \\ \mathcal{C}_k \times \mathcal{C}_1^k \arrow[ur, "\gamma"'] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \{1\} \times \mathcal{C}_j \arrow[d, "\cap"'] \arrow[r, phantom, "="] & \mathcal{C}_j \\ \mathcal{C}_1 \times \mathcal{C}_j \arrow[ur, "\gamma"'] & \end{tikzcd}$$

(iii) The following equivariance diagrams commute, where $\sigma \in \Sigma_k$, $\tau_s \in \Sigma_{j_s}$, $\sigma(j_1,\cdots,j_k) \in \Sigma_j$ permutes k blocks of letters as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ is the standard block sum permutation:

$$\begin{tikzcd}[column sep=large] \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \arrow[r, "\sigma \times \sigma^{-1}"] \arrow[d, "\gamma"] & \mathcal{C}_k \times \mathcal{C}_{j_{\sigma(1)}} \times \cdots \times \mathcal{C}_{j_{\sigma(k)}} \arrow[d, "\gamma"] \\ \mathcal{C}_j \arrow[r, "\sigma(j_{\sigma(1)},\cdots,j_{\sigma(k)})"] & \mathcal{C}_j \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=large] \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \arrow[r, "1 \times \tau_1 \times \cdots \times \tau_k"] \arrow[d, "\gamma"] & \mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k} \arrow[d, "\gamma"] \\ \mathcal{C}_j \arrow[r, "\tau_1 \oplus \cdots \oplus \tau_k"] & \mathcal{C}_j \end{tikzcd}$$

We have gone to the trouble to write out diagrams for clarity in the generalization to diagrams of extended powers to be given shortly. Henceforward in this chapter, all operads are understood to be G-operads.

Definition 1.2. An operad $\mathcal{C}$ is said to be Σ-free if the $\mathcal{C}_j$ are the total spaces of principal (G,Σ_j)-bundles. It is said to be an E_∞ operad if, in addition, these bundles are universal. $\mathcal{C}$ is said to be cellular if the $\mathcal{C}_j$ are $(G \times \Sigma_j)$-CW complexes.

Now let U be a G-universe (as specified in I§2).

Remark 1.3. There is a more general notion of an $E_\infty U$ operad, for which $\mathcal{C}_j$ is required to be a universal $\mathcal{F}(U^j,U)$-space and thus to be $(G \times \pi)$-equivalent to $\mathcal{I}(U^j,U)$; compare II.2.11.

Example 1.4. Let $\mathcal{L}_j = \mathcal{I}(U^j,U)$ with its usual $G \times \Sigma_j$ action. Let $1 \in \mathcal{L}_1$ be

the identity map and define structure maps γ as composites

$$\mathcal{L}_k \times \mathcal{L}_{j_1} \times \cdots \times \mathcal{L}_{j_k} \xrightarrow{1 \times (\mathrm{sum})} \mathcal{L}_k \times \mathcal{I}(U^j,U^k) \xrightarrow{\text{composition}} \mathcal{L}_j.$$

Then $\mathcal{L}$ is an $E_\infty U$ operad and is an E_∞ operad if U is complete. $\mathcal{L}$ is called the linear isometries operad of U.

More examples of operads are given in section 5 and in [62], where equivariant infinite loop space theory is developed. When $G = e$, more examples appear in [97,99, and 134]. In particular, there is a cellular operad $\mathcal{B}$ such that $\mathcal{B}_j$ is precisely the canonical universal Σ_j-bundle $E\Sigma_j$; see [97,p. 161].

We use operads to define canonical extended powers. As usual, fix an indexing set $\mathcal{A}$ in U.

Definition 1.5. Let $\mathcal{C}$ be a Σ-free operad and assume given a morphism of operads $\chi\colon \mathcal{C} \to \mathcal{L}$ with j^{th} component χ_j. For $E \in G\mathcal{S}\mathcal{A}$ and $\pi \subset \Sigma_j$, define

$$D_\pi(\mathcal{C},E) = \chi_j \ltimes_\pi E^{(j)} \quad \text{and} \quad D_j(\mathcal{C},E) = \chi_j \ltimes_{\Sigma_j} E^{(j)}.$$

In particular, with the convention $E^{(0)} = S$, $D_0(\mathcal{C},E) = S$ for all $\mathcal{C}$ and E. As they stand, these are well-defined spectra in $G\mathcal{S}\mathcal{A}$. When working in the stable category $\overline{h}G\mathcal{S}\mathcal{A}$, the same symbols will denote the corresponding constructions obtained from the composite of χ_j and any chosen Σ_j-free $(G \times \Sigma_j)$-CW approximation $\Gamma\mathcal{C}_j \to \mathcal{C}_j$. These constructions for different E_∞ operads $\mathcal{C}$ are canonically equivalent, hence in this case we often abbreviate notation to

$$D_\pi E = D_\pi(\mathcal{C},E) \quad \text{and} \quad D_j E = D_j(\mathcal{C},E).$$

The notation agrees with the space level notation in Cohen, May, and Taylor [30]. As there, the definition is also useful when $\mathcal{C}$ is only a coefficient system; this notion will be recalled and the extra generality exploited in sections 3-5. By VI.5.3, we have the fundamental relation

$$\Sigma^\infty D_\pi(\mathcal{C},Y) \cong D_\pi(\mathcal{C},\Sigma^\infty Y).$$

In practice, $\mathcal{C}$ is a product operad $\mathcal{C}' \times \mathcal{L}$ and the map $\chi\colon \mathcal{C} \to \mathcal{L}$ is just the projection. This use of products results in no real loss of generality since the other projection $\mathcal{C} \to \mathcal{C}'$ has j^{th} map a weak $(G \times \Sigma_j)$-equivalence if $\mathcal{C}'$ is Σ-free.

When G is finite, we can take $\Gamma\mathcal{C}_j$ to be the realization of the total singular complex of $\mathcal{C}_j$. We see that this construction takes G-spaces to G-CW complexes by looking at orbits of simplices in the total singular complex. By functoriality and preservation of products, the $\Gamma\mathcal{C}_j$ are the spaces of a cellular operad $\Gamma\mathcal{C}$, and the natural weak equivalences $\Gamma\mathcal{C}_j \to \mathcal{C}_j$ specify a morphism of operads. Thus restriction to cellular operads results in no real loss of generality. This argument fails for general compact Lie groups because passage through the simplicial category results in loss of continuity and so fails to carry G-spaces to G-spaces. We believe that a more sophisticated construction would obviate the difficulty. In any case, CW approximation is only used on passage to the stable category, where any resulting homotopies cause no difficulty by virtue of VI.1.13 and 1.14. In particular, they cause no difficulty in transporting the diagrams to follow to the stable category. When $G = e$, the E_∞ operad $\mathcal{D}$ may be taken as a CW-approximation of $\mathcal{L}$ (the structural maps only agreeing up to appropriately equivariant homotopy, of course).

We turn to the coherence data promised in [H_∞, I§2] and needed to make sense of the definitions in the next section. In fact, three of the four families of maps ι_j, $\alpha_{j,k}$, $\beta_{j,k}$, and δ_j discussed in [H_∞, I§2] are special cases of a single general family derived from the maps α and β of VI§3 and the structural maps γ of operads. We assume given a Σ-free operad $\mathcal{C}$ and a morphism of operads $\chi\colon \mathcal{C} \to \mathcal{L}$ throughout the rest of this section. We adopt the notation

$$\mathcal{C}_j \ltimes E_1 \wedge \cdots \wedge E_j = \chi_j \ltimes E_1 \wedge \cdots \wedge E_j$$

and similarly for sums and composites of the χ_j; here the notation is ambiguous, because products of the $\mathcal{C}_j$'s can be domains of both sums and composites, but which operation is intended should be clear from context (compare VI.3.1).

<u>Definition 1.6</u>. (i) For $k > 0$, $j_s > 0$, and $j = j_1 + \cdots + j_k$ and for G-spectra $E_{s,r} \in G\mathcal{S}\mathcal{A}$, $1 \le s \le k$ and $1 \le r \le j_s$, define a map ζ as the dotted arrow composite in the diagram

$$\begin{tikzcd}
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \bigwedge_{r=1}^{j_s} E_{s,r}\right) \arrow[r, "1 \ltimes \alpha"] \arrow[d, dotted, "\zeta"] & \mathcal{C}_k \ltimes (\mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k}) \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,r} \arrow[d, "\beta"] \\
\mathcal{C}_j \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,r} & (\mathcal{C}_k \times \mathcal{C}_{j_1} \times \cdots \times \mathcal{C}_{j_k}) \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,r}. \arrow[l, "\gamma \ltimes 1"']
\end{tikzcd}$$

(Here empty smash products are to be interpreted as S.)

(ii) Assume given a G-fixed point $c_j \in \mathcal{C}_j$, with $c_1 = 1$, and let $f_j = \chi_j(c_j)\colon U^j \to U$. For a G-spectrum $E \in G\mathcal{S}\mathcal{A}$, define

$$\iota_j\colon f_{j*}E^{(j)} \to D_j(\mathcal{C},E)$$

$$\alpha_{j,k}\colon f_{2*}[D_j(\mathcal{C},E) \wedge D_k(\mathcal{C},E)] \to D_{j+k}(\mathcal{C},E)$$

and

$$\beta_{j,k}\colon D_j(\mathcal{C},D_k(\mathcal{C},E)) \to D_{jk}(\mathcal{C},E)$$

by passage to orbits from the following instances of restrictions of maps ζ:

$$\{1\} \ltimes \{c_j\} \ltimes E^{(j)} \to \mathcal{C}_1 \ltimes \mathcal{C}_j \ltimes E^{(j)} \xrightarrow{\zeta} \mathcal{C}_j \ltimes E^{(j)}$$

$$\{c_2\} \ltimes (\mathcal{C}_j \ltimes E^{(j)} \wedge \mathcal{C}_k \ltimes E^{(k)}) \to \mathcal{C}_2 \ltimes (\mathcal{C}_j \ltimes E^{(j)} \wedge \mathcal{C}_k \ltimes E^{(k)}) \xrightarrow{\zeta} \mathcal{C}_{j+k} \ltimes E^{(j+k)}$$

$$\mathcal{C}_j \ltimes (\mathcal{C}_k \ltimes E^{(k)})^{(j)} \xrightarrow{\zeta} \mathcal{C}_{jk} \ltimes E^{jk}$$

Of course, ι_j is just $\{c_j\} \ltimes E^{(j)} \to \mathcal{C}_j \ltimes E^{(j)}$, but the more complicated description makes diagrams involving it special cases of general diagrams involving ζ. The equivariance needed to make sense of $\alpha_{j,k}$ and $\beta_{j,k}$ is contained in the following result, which shows that all diagrams in the definition of an operad pass to coherence diagrams relating extended powers.

<u>Proposition 1.7</u>. The following diagrams of G-spectra commute, their variables being arbitrary G-spectra; unspecified notations are taken from the definition of an operad.

(i)
$$\begin{tikzcd}
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \left(\bigwedge_{r=g_{s-1}+1}^{g_s} \mathcal{C}_{i_r} \ltimes \bigwedge_{q=1}^{i_r} E_{r,q}\right)\right) \arrow[r, "\zeta"] \arrow[d, "1 \ltimes (\bigwedge_{s=1}^{k} \zeta)"'] & \mathcal{C}_j \ltimes \left(\bigwedge_{r=1}^{j} \mathcal{C}_{i_r} \ltimes \bigwedge_{q=1}^{i_r} E_{r,q}\right) \arrow[d, "\zeta"] \\
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{h_s} \ltimes \bigwedge_{r=g_{s-1}+1}^{g_s} \bigwedge_{q=1}^{i_r} E_{r,q}\right) \arrow[r, "\zeta"] & \mathcal{C}_i \ltimes \bigwedge_{r=1}^{j} \bigwedge_{q=1}^{i_r} E_{r,q}
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \{1\} \ltimes E_s\right) = \mathcal{C}_k \ltimes \bigwedge_{s=1}^{k} E_s \arrow[d] & \\
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_1 \ltimes E_s\right) \arrow[ur, "\zeta"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\{1\} \ltimes \mathcal{C}_j \ltimes \bigwedge_{r=1}^{j} E_r = \mathcal{C}_j \ltimes \bigwedge_{r=1}^{j} E_r \arrow[d] & \\
\mathcal{C}_1 \ltimes \mathcal{C}_j \ltimes \bigwedge_{r=1}^{j} E_r \arrow[ur, "\zeta"'] &
\end{tikzcd}$$

(iii)
$$\begin{tikzcd}
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \bigwedge_{r=1}^{j_s} E_{s,r}\right) \arrow[r, "\sigma^{-1}"] \arrow[d, "\zeta"] & \mathcal{C}_k \ltimes \bigwedge_{s=1}^{k} \mathcal{C}_{j_{\sigma(s)}} \ltimes \bigwedge_{r=1}^{j_{\sigma(s)}} E_{\sigma(s),r} \arrow[d, "\zeta"] \\
\mathcal{C}_j \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,r} \arrow[r, "\sigma(j_1,\cdots,j_k)^{-1}"] & \mathcal{C}_j \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_{\sigma(s)}} E_{\sigma(s),r}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \bigwedge_{r=1}^{j_s} E_{s,r}\right) \arrow[r, "1\times(\tau_1^{-1}\wedge\cdots\wedge\tau_k^{-1})"] \arrow[d, "\zeta"'] & \mathcal{C}_k \ltimes \left(\bigwedge_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \bigwedge_{r=1}^{j_s} E_{s,\tau_s(r)}\right) \arrow[d, "\zeta"] \\
\mathcal{C}_j \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,r} \arrow[r, "\tau_1^{-1}\oplus\cdots\oplus\tau_k^{-1}"] & \mathcal{C}_j \ltimes \bigwedge_{s=1}^{k} \bigwedge_{r=1}^{j_s} E_{s,\tau_s(r)}
\end{tikzcd}$$

Proof. In the last diagrams, the left actions by permutations are as specified above VI.5.1. Inverses appear naturally because of our convention of using right actions by permutation groups on the component spaces of operads. The verifications require only the diagrams defining operads, the diagrams of VI.3.3 and the diagrams in many variables they imply, and naturality diagrams.

On passage to the stable category, these spectrum level diagrams lead to some of the diagrams displayed in $[H_\infty, \text{I§2}]$ and to many others when some but not all of the variable spectra are equal. Before discussing this, we introduce the maps δ_j. We shall be working in the stable category $\overline{h}G\mathcal{S}\mathcal{A}$ in the rest of this section. We use the notation $\wedge$ for both internal and external smash products, relying on context to determine the intended interpretation (compare II.3.11).

Definition 1.8. For G-spectra E and F, define

$$\delta_j \colon D_j(\mathcal{C}, E\wedge F) \longrightarrow D_j(\mathcal{C},E)\wedge D_j(\mathcal{C},F)$$

by passage to orbits from the composite

$$\mathcal{C}_j \ltimes f_{2*}(E\wedge F)^{(j)} \cong \mathcal{C}_j \ltimes g_{j,2*}(E^{(j)}\wedge F^{(j)}) \xrightarrow{\delta} f_{2*}(\mathcal{C}_j \ltimes E^{(j)} \wedge \mathcal{C}_j \ltimes F^{(j)}),$$

where $g_{j,2}$ is the composite of f_2^j and the evident shuffle $\nu_j \colon U^j \oplus U^j \cong (U\oplus U)^j$ and the isomorphism is obtained by use of α, β, and the commutativity of the external smash product. The map δ is constructed in VI.3.4 by use of the pullback along χ_j of a $(G\times\Sigma_j)$-homotopy $h\colon \mathcal{L}_j \times I \to \mathcal{L}_{2j}$ between the (G,Σ_j)-bundle maps $e,f\colon \mathcal{L}_j \to \mathcal{L}_{2j}$ specified by

$$e(g) = \gamma(g;f_2^j)\nu_j \quad \text{and} \quad f(g) = \gamma(f_2;g,g).$$

If $\mathcal{C}$ is an E_∞ operad, but not in general otherwise, the requisite homotopy can be obtained in $\mathcal{C}$ rather than in $\mathcal{L}$.

The rest of this section is devoted to the proofs of the following omnibus coherence assertion and of a result about transfer promised in $[H_\infty]$. The reader is invited to skip to the next section.

Recall that a G-space X is said to be G-connected if all of its fixed point sets X^H are connected.

Theorem 1.9. The transformations ι_j, $\alpha_{j,k}$, $\beta_{j,k}$, and δ_j are natural with respect to morphisms of operads over $\mathcal{L}$ and are compatible under Σ^∞ with their space level analogs. With D_jE replaced by $D_j(\mathcal{C},E)$, all diagrams displayed in Lemmas I.2.8 through I.2.14 of $[H_\infty]$ commute in the stable category $\overline{h}G\mathcal{S}\mathcal{A}$ provided that, for I.2.8, I.2.11, and I.2.14, each $\mathcal{C}_j$ is G-connected; the diagrams displayed in I.2.15 commute in $\overline{h}G\mathcal{S}\mathcal{A}$ if $\mathcal{C}$ is an E_∞ operad.

Proof. The naturality in $\mathcal{C}$ is evident. The compatibility with Σ^∞ is implied by VI.3.2 and 3.6 and is expressed by the diagrams displayed above $[H_\infty, \text{I.2.6}]$ and the following diagram, whose commutativity is implied by VI.3.6 (ii):

$$\begin{tikzcd}
D_j(\mathcal{C}, E\wedge Y) \arrow[r, "\delta_j"] \arrow[d, phantom, "\cong" rotate=90] & D_j(\mathcal{C},E)\wedge D_j(\mathcal{C},Y) \arrow[d, phantom, "\cong" rotate=90] \\
D_j(\mathcal{C}, E\wedge\Sigma^\infty Y) \arrow[r, "\delta_j"] & D_j(\mathcal{C},E)\wedge D_j(\mathcal{C},\Sigma^\infty Y).
\end{tikzcd}$$

Here the top map δ_j is obtained by passage to orbits from

$$\mathcal{C}_j \ltimes (E\wedge Y)^{(j)} \cong \mathcal{C}_j \ltimes (E^{(j)}\wedge Y^{(j)}) \xrightarrow{\delta} (\mathcal{C}_j \ltimes E^{(j)})\wedge(\mathcal{C}_j \ltimes Y^{(j)}),$$

where δ is as specified in VI.3.5.

Proof of $[H_\infty, \text{I.2.8}]$. The commutativity relation $\alpha_{k,j}\circ\tau \simeq \alpha_{j,k}$ is deduced from the diagram

$$\begin{tikzcd}
\mathcal{C}_2 \ltimes (\mathcal{C}_j \ltimes E^{(j)})\wedge(\mathcal{C}_k \ltimes E^{(k)}) \arrow[r, "\sigma"] \arrow[d, "\zeta"'] & \mathcal{C}_2 \ltimes (\mathcal{C}_k \ltimes E^{(k)})\wedge(\mathcal{C}_j \ltimes E^{(j)}) \arrow[d, "\zeta"] \\
\mathcal{C}_{j+k} \ltimes E^{(j+k)} \arrow[r, "\sigma(k{,}j)"] & \mathcal{C}_{j+k} \ltimes E^{(j+k)}
\end{tikzcd}$$

of Proposition 1.7 (iii), where $\sigma\in\Sigma_2$ is the transposition. Here $\sigma(k,j)$ disappears on passage to orbits. When restricted to $\{c_2\sigma\}$, the top map σ is the trivial part of the commutativity isomorphism τ specified in II.3.12 (ii), and we have $\alpha_{k,j}\circ\sigma = \alpha^!_{j,k}$, where $\alpha^!_{j,k}$ is defined with respect to the basepoint $c_2\sigma$. A G-path from c_2 to $c_2\sigma$ induces an equivalence τ' from the domain of $\alpha_{j,k}$ to the domain of $\alpha^!_{j,k}$ which satisfies $\alpha^!_{j,k}\circ\tau' \simeq \alpha_{j,k}$ and agrees with the nontrivial part of the commutativity isomorphism. Since $\tau = \sigma\tau'$, $\alpha_{k,j}\circ\tau \simeq \alpha_{j,k}$ follows. The associativity relation

$$\alpha_{i+j,k} \circ (\alpha_{i,j} \wedge 1) = \alpha_{i,j+k} \circ (1 \wedge \alpha_{j,k})$$

is similar. Both composites here are restrictions of special cases of traversal of the diagram of Proposition 1.7 (i) in the counterclockwise direction. Traversing these diagrams clockwise, we see that these composites are given by three variable maps $\alpha_{i,j,k}$ and $\alpha'_{i,j,k}$ defined with respect to the basepoints $\gamma(c_2; c_2, 1)$ and $\gamma(c_2; 1, c_2)$. A G-path in $\mathcal{C}_3$ connecting these points gives an equivalence α of domains which satisfies $\alpha'_{i,j,k} \simeq \alpha_{i,j,k} \circ \alpha$ and agrees with the associativity isomorphism of II.3.12 (iii) (use of which is implicit in the asserted "equality").

Proofs of $[H_\infty$, I.2.9 and I.2.13]. These are both special cases of Proposition 1.7 (i) and thus already commute on the level of spectra.

Proofs of $[H_\infty$, I.2.10 and I.2.12]. These are easily deduced from VI.3.7 and VI.3.8. The operad structure is irrelevant here, and these lemmas apply to all j-fold extended powers $X \ltimes_{\Sigma_j} E^{(j)}$.

Proof of $[H_\infty$, I.2.11]. The relation $\alpha_{j,k} \circ (\iota_j \wedge \iota_j) = \iota_{j+k}$ would be a special case of Proposition 1.7 (i) if c_{j+k} were $\gamma(c_2; c_j, c_k)$. A G-path in $\mathcal{C}_{j+k}$ connecting these points leads to an equivalence which implies the required equality in the stable category. Similarly, the second diagram of $[H_\infty$, I.2.11] would be a special case of Proposition 1.7 (i) if the iterate $\alpha_{k,\dots,k}$ were interpreted as the restriction of

$$\zeta\colon \mathcal{C}_j \ltimes (\mathcal{C}_k \ltimes E^{(k)})^{(j)} \longrightarrow \mathcal{C}_{jk} \ltimes E^{(jk)}$$

to $c_j \in \mathcal{C}_j$. This interpretation is valid in the stable category.

Proof of $[H_\infty$, I.2.14]. The diagrams here result from laborious chases based on the diagrams of VI.3.9, both of its diagrams being needed for each of the diagrams of $[H_\infty$, I.2.14], and various naturality diagrams. The source of the commutativity isomorphism $1 \wedge \tau \wedge 1$ appearing in the first diagram of $[H_\infty$, I.2.14] is the reduction of δ to the equivalence induced by a path between isometries when it is applied to a one point domain G-space.

Proof of $[H_\infty$, I.2.15]. Identifying all vertices of the asserted diagram as extended $(ij + ik)^{\underline{th}}$ powers of E and noting which permutations occur in arranging that the variables E appear in the same order at each vertex, we find after another laborious chase that the diagram in question reduces to traversal of the exterior of the following diagram of domain G-spaces, in which the unlabeled arrows are obtained by restricting to c_2 in all variables $\mathcal{C}_2$:

$$\begin{tikzcd}
\mathcal{C}_i \times (\mathcal{C}_j \times \mathcal{C}_k)^i \arrow[rrr, "\Delta \times \nu_i^{-1}"] \arrow[d] & & & \mathcal{C}_i^2 \times \mathcal{C}_j^i \times \mathcal{C}_k^i \arrow[d] \\
\mathcal{C}_i \times \mathcal{C}_2^i \times (\mathcal{C}_j \times \mathcal{C}_k)^i \arrow[r, "\gamma \times 1"] \arrow[d, phantom, "\cong" rotate=-90] & \mathcal{C}_{2i} \times (\mathcal{C}_j \times \mathcal{C}_k)^i \arrow[r, "\nu_i \times \nu_i^{-1}"] \arrow[dd, "\gamma"'] & \mathcal{C}_{2i} \times \mathcal{C}_j^i \times \mathcal{C}_k^i \arrow[dd, "\gamma"'] & \mathcal{C}_2 \times \mathcal{C}_i^2 \times \mathcal{C}_j^i \times \mathcal{C}_k^i \arrow[l, "\gamma \times 1 \times 1"'] \arrow[d, phantom, "\cong" rotate=-90] \\
\mathcal{C}_i \times (\mathcal{C}_2 \times \mathcal{C}_j \times \mathcal{C}_k)^i \arrow[d, "1 \times \gamma^i"'] & & & \mathcal{C}_2 \times \mathcal{C}_i \times \mathcal{C}_j^i \times \mathcal{C}_i \times \mathcal{C}_k^i \arrow[d, "1 \times \gamma \times \gamma"] \\
\mathcal{C}_i \times \mathcal{C}_{j+k}^i \arrow[r, "\gamma"] & \mathcal{C}_{ij+ik} \arrow[r, "\nu_i(j,k,\dots,j,k)"] & \mathcal{C}_{ij+ik} & \mathcal{C}_2 \times \mathcal{C}_{ij} \times \mathcal{C}_{ik} \arrow[l, "\gamma"']
\end{tikzcd}$$

After passage to extended powers, the bottom part of the diagram commutes by Proposition 1.7, while the homotopy used to transport the upper part of the diagram to a commutative diagram in the stable category is precisely the same as the $\mathcal{C}$ level of the homotopy used to define δ_i. The need for a $\mathcal{C}$ level homotopy forces the restriction to E_∞ operads.

Finally, we prove a generalization of the promised result $[H_\infty$,II.1.5] about transfer. We need some notations. Let $J = (j_1, \dots, j_k)$ be a partition of j, so that $j = j_1 + \dots + j_k$, and let $\Sigma_J = \Sigma_{j_1} \times \dots \times \Sigma_{j_k} \subset \Sigma_j$. Let $\mathcal{C}$ be an E_∞ operad and define

$$D_J(\mathcal{C}, E) = \chi_j \ltimes_{\Sigma_J} E^{(j)}.$$

Choose a G-fixed point $c_k \in \mathcal{C}_k$. In Definition 1.6, restrict to the point c_k and set all $E_{s,r} = E$. The resulting specialization of ζ is a $(G \times \Sigma_j)$-equivalence. On passage to orbits over Σ_J, it gives a G-equivalence

$$\zeta_J\colon D_{j_1}(\mathcal{C}, E) \wedge \dots \wedge D_{j_k}(\mathcal{C}, E) \longrightarrow D_J(\mathcal{C}, E).$$

Of course, if we compose ζ_J with the evident projection

$$\xi\colon D_J(\mathcal{C}, E) \longrightarrow D_j(\mathcal{C}, E),$$

we obtain a map α_J as in Definition 1.6(ii). Now ξ is a "stable bundle" and so has the transfer τ given by IV.3.3. We specified a seemingly different map τ_J in $[H_\infty$,II.1.4].

Theorem 1.10. The following diagram commutes for any E_∞ operad $\mathcal{C}$ and G-spectrum E.

$$\begin{tikzcd} & D_{j_1}(\mathcal{C},E)\wedge\cdots\wedge D_{j_k}(\mathcal{C},E) \arrow[d, "\zeta_J"] \\ D_j(\mathcal{C},E) \arrow[ur, "\tau_J"] \arrow[r, "\tau"'] & D_J(\mathcal{C},E) \end{tikzcd}$$

Proof. Let ${}^{(k)}E$ denote the wedge of k copies of E. The j-fold external smash product $({}^{(k)}E)^{(j)}$ decomposes $(G\times\Sigma_j)$-equivariantly as a wedge of terms, one of which may be identified with

$$\Sigma_j \ltimes_{\Sigma_J} E^{(j)} \cong E^{(j)}\wedge(\Sigma_j/\Sigma_J)^+.$$

Let $\Delta\colon E\to{}^{(k)}E$ be the diagonal and let $\pi_J\colon ({}^{(k)}E)^{(j)}\to\Sigma_j\ltimes_{\Sigma_J}E^{(j)}$ be the projection. By $[H_\infty, II.1.4]$, $\zeta_J\tau_J$ is obtained by applying the functor $\mathcal{C}_j\ltimes_{\Sigma_J}(?)$ to the composite

$$\tau_J^!\colon E^{(j)} \xrightarrow{\Delta^{(j)}} ({}^{(k)}E)^{(j)} \xrightarrow{\pi_J} \Sigma_j\ltimes_{\Sigma_J}E^{(j)}$$

and using the identification $\mathcal{C}_j\ltimes_{\Sigma_j}(\Sigma_j\ltimes_{\Sigma_J}E^{(j)})\cong\mathcal{C}_j\ltimes_{\Sigma_J}E^{(j)}$. In view of IV.3.3, it suffices to identify $\tau_J^!$ with the map

$$1\wedge\tau\colon E^{(j)}\simeq E^{(j)}\wedge S\to E^{(j)}\wedge\Sigma^\infty(\Sigma_j/\Sigma_J)^+\simeq\Sigma_j\ltimes_{\Sigma_J}E^{(j)}.$$

The diagonal Δ is homotopic to

$$1\wedge\rho\colon E\simeq\Sigma^{-1}E\wedge S^1\longrightarrow\Sigma^{-1}E\wedge{}^{(k)}S^1\simeq{}^{(k)}E,$$

where $\rho\colon S^1\to{}^{(k)}S^1$ is the k-fold pinch map of the circle. It is now easily seen that $\tau_J^!$ for a general G-spectrum E is obtained from $\tau_J^!$ for $E = S$ by smashing with $E^{(j)}$. The map $\tau_J^!$ for $E = S$ is easily identified with the pretransfer $\tau\colon S\to\Sigma^\infty(\Sigma_j/\Sigma_J)^+$ as described in II.6.15. In fact, if we take the Σ_j-representation V there to be $\mathbb{R}^j$ with the permutation action and embed Σ_j/Σ_J as the orbit of a point with isotropy group Σ_J, then the Pontryagin-Thom map $S^V\to(\Sigma_j/\Sigma_J)^+\wedge S^V$ is Σ_j-homotopic to the composite

$$(S^1)^{(j)} \xrightarrow{\rho^{(j)}} ({}^{(k)}S^1)^{(j)} \xrightarrow{\pi_J} (\Sigma_j/\Sigma_J)^+\wedge(S^1)^{(j)}.$$

§2. Actions of operads on spectra

Again, let $\mathcal{A}$ be an indexing set in a G-universe U regarded as a Σ_j-trivial $(G\times\Sigma_j)$-universe for all $j\geq 0$. Let $\mathcal{C}$ be a Σ-free operad together with a morphism of operads $\chi\colon\mathcal{C}\to\mathcal{L}$. When working in the stable category, we tacitly assume that $\mathcal{C}$ is cellular (or agree to apply cellular approximation without change of notation). Recall the many variable generalization of extended powers in VI.5.1.

Definitions 2.1. Let $E\in G\mathcal{S}\mathcal{A}$ and regard E as a Σ_j-trivial $(G\times\Sigma_j)$-spectrum for all $j\geq 0$. A structure of $\mathcal{C}$-spectrum on E consists of maps of $(G\times\Sigma_j)$-spectra

$$\xi_j\colon\mathcal{C}_j\ltimes E^{(j)}\longrightarrow E,\quad j\geq 0,$$

such that the composite $\xi_1\iota_1\colon E\to\mathcal{C}_1\ltimes E\to E$ is the identity map and the following diagram of spectra commutes, where $k\geq 0$, $j_s\geq 0$, and $j = j_1+\cdots+j_k$:

$$\begin{tikzcd} \mathcal{C}_k\ltimes(\bigwedge_{s=1}^{k}\mathcal{C}_{j_s}\ltimes E^{(j_s)}) \arrow[r, "\zeta"] \arrow[d, "1\ltimes(\bigwedge_{s=1}^{k}\xi_{j_s})"'] & \mathcal{C}_j\ltimes E^{(j)} \arrow[d, "\xi_j"] \\ \mathcal{C}_k\ltimes E^{(k)} \arrow[r, "\xi_k"] & E \end{tikzcd}$$

A structure of $\bar{h}\mathcal{C}$-spectrum on E consists of maps ξ_j in the stable category $\bar{h}(G\times\Sigma_j)\mathcal{S}\mathcal{A}$ such that $\xi_1\iota_1\simeq 1$ and the above diagram commutes with all possible equivariance; that is, it commutes in $\bar{h}(G\times\pi)\mathcal{S}\mathcal{A}$ for the largest subgroup π of Σ_j for which all maps in the diagram are defined in $\bar{h}(G\times\pi)\mathcal{S}\mathcal{A}$. Here we embed Σ_k and $\Sigma_{j_1}\times\cdots\times\Sigma_{j_k}$ in Σ_j via

$$\sigma\longrightarrow\sigma(j_1,\ldots,j_k)\quad\text{and}\quad(\tau_1,\ldots,\tau_k)\longrightarrow\tau_1\oplus\cdots\oplus\tau_k.$$

For example, if $j_s = i$ for all s, then $\pi = \Sigma_k\int\Sigma_i$.

We have evident notions of morphisms of $\mathcal{C}$-spectra and of $\bar{h}\mathcal{C}$-spectra, and passage to stable categories evidently induces a functor from the category of $\mathcal{C}$-spectra to the category of $\bar{h}\mathcal{C}$-spectra.

If $\mathcal{C}_1$ is G-contractible, then $\iota_1\colon E\to\mathcal{C}_1\ltimes E$ is an equivalence by VI.1.15 and, modulo this equivalence, ξ_1 is required to be the identity. On passage to orbits, two special cases of our diagram become

$$\begin{tikzcd}
D_j(\mathcal{C},E)\wedge D_k(\mathcal{C},E) \arrow[rr, "\alpha_{j,k}"] \arrow[d, "\xi_j\wedge\xi_k"'] & & D_{j+k}(\mathcal{C},E) \arrow[d, "\xi_{j+k}"] \\
E\wedge E \arrow[r, "\iota_2"] & D_2(\mathcal{C},E) \arrow[r, "\xi_2"] & E
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
D_j(\mathcal{C},D_k(\mathcal{C},E)) \arrow[r, "\beta_{jk}"] \arrow[d, "D_j(\mathcal{C},\xi_k)"'] & D_{jk}(\mathcal{C},E) \arrow[d, "\xi_{jk}"] \\
D_jE \arrow[r, "\xi_j"] & E.
\end{tikzcd}$$

Comparing with the definition, $[H_\infty$, I.3.1] read G-equivariantly, of an H_∞ ring spectrum, we see immediately that if $\mathcal{C}$ is an E_∞ operad, then an $\bar{h}\mathcal{C}$ spectrum is an H_∞ ring spectrum by neglect of structure. In fact, there is no loss of information.

Proposition 2.2. An H_∞ ring spectrum is an $\bar{h}\mathcal{C}$-spectrum for any E_∞ operad $\mathcal{C}$.

Proof. We must show that the particular diagrams just displayed imply the general diagram of Definition 2.1. Let $\{i_1,\cdots,i_m\}$ be the set of distinct j_s, let $p\colon \underline{k}\to\underline{m}$ be the partition $p(s)=q$ if $j_s=i_q$, and let $k_q=|p^{-1}(q)|$. Then

$$k=k_1+\cdots+k_m \quad\text{and}\quad j=k_1i_1+\cdots+k_mi_m.$$

Let $F_q=\mathcal{C}_{i_q}\ltimes E^{(i_q)}$. As in VI.5.1, we have

$$\mathcal{C}_k\ltimes\Big(\bigwedge_{s=1}^{k}\mathcal{C}_{j_s}\ltimes E^{(j_s)}\Big)=\mathcal{C}_k\ltimes\Big(\bigwedge_{s=1}^{k}F_{p(s)}\Big).$$

Let $\psi\in\Sigma_k$ be the permutation needed to rearrange $\bigwedge_{s=1}^{k}F_{p(s)}$ as $\bigwedge_{q=1}^{m}F_q^{(k_q)}$. Then Proposition 1.7 and diagrams implied by the definition of an H_∞ ring spectrum yield the commutativity of each interior cell of the following diagram. In fact, this much requires only that the $\mathcal{C}_j$ be G-connected. However, when $\mathcal{C}$ is an E_∞ operad, $\alpha_{k_1,\cdots,k_m}$ is an equivalence by VI.1.15 and the commutativity of the interior cells implies the commutativity with all possible equivariance of the outer square.

$$\begin{tikzcd}
\mathcal{C}_k\ltimes\big(\bigwedge_{s=1}^{k}F_{p(s)}\big) \arrow[rrrrr, "\zeta"] \arrow[dr, "\psi"] \arrow[ddddd, "1\ltimes(\bigwedge_{s=1}^{k}\xi_{j_s})"'] & & & & & \mathcal{C}_j\ltimes E^{(j)} \arrow[dl, "\psi(j_1,\cdots,j_k)"'] \arrow[ddddd, "\xi_j"] \\
& \mathcal{C}_k\ltimes\big(\bigwedge_{q=1}^{m}F_q^{(k_q)}\big) \arrow[rrr, "\zeta"] \arrow[ddd, "1\ltimes(\bigwedge_{q=1}^{m}\xi_{i_q}^{(k_q)})"'] & & & \mathcal{C}_j\ltimes E^{(j)} \arrow[ddddr, "\xi_j"] & \\
& & \bigwedge_{q=1}^{m}\mathcal{C}_{k_q}\ltimes F_q^{(k_q)} \arrow[ul, "\alpha_{k_1,\cdots,k_m}"'] \arrow[r, "\bigwedge_{q=1}^{m}\beta_{k_qi_q}"] \arrow[d, "\bigwedge_{q=1}^{m}1\ltimes\xi_{i_q}^{(k_q)}"'] & \bigwedge_{q=1}^{m}\mathcal{C}_{k_qi_q}\ltimes E^{(k_qi_q)} \arrow[ur, "\alpha_{k_1i_1,\cdots,k_qi_q}"] \arrow[d, "\bigwedge_{q=1}^{m}\xi_{k_qi_q}"] & & \\
& & \bigwedge_{q=1}^{m}\mathcal{C}_{k_q}\ltimes E^{(k_q)} \arrow[r, "\bigwedge_{q=1}^{m}\xi_{k_q}"] \arrow[dl, "\alpha_{k_1,\cdots,k_m}"] & \bigwedge_{q=1}^{m}E \arrow[ddrr, "\phi"] & & \\
& \mathcal{C}_k\ltimes E^{(k)} \arrow[drrrr, "\xi_k"] & & & & \\
\mathcal{C}_k\ltimes E^{(k)} \arrow[ur, "\psi"] \arrow[rrrrr, "\xi_k"] & & & & & E
\end{tikzcd}$$

The basic formal properties of H_∞ ring spectra, as developed in $[H_\infty$, I.3.2-3.4], directly generalize to $\bar{h}\mathcal{C}$-spectra provided only that each $\mathcal{C}_j$ is G-connected. In particular, the following result holds.

Proposition 2.3. Assume that each $\mathcal{C}_j$ is G-connected. The sphere spectrum is an $\bar{h}\mathcal{C}$-spectrum and the smash product of $\bar{h}\mathcal{C}$-spectra is an $\bar{h}\mathcal{C}$-spectrum. An $\bar{h}\mathcal{C}$-spectrum is a ring spectrum whose unit and product are maps of $\bar{h}\mathcal{C}$-spectra.

The following result compares the present definition of a $\mathcal{C}$-spectrum to that originally given by May, Quinn, and Ray [99, IV.1.1]. We regard elements of $\mathcal{C}_j$ as linear isometries $U^j\to U$ via χ_j.

Proposition 2.4. Let $E\in G\mathcal{S}\mathcal{A}$. A structure of $\mathcal{C}$-spectrum on E determines and is determined by maps of G-spaces

$$\xi_j(c)\colon EV_1\wedge\cdots\wedge EV_j \longrightarrow Ec(V_1\oplus\cdots\oplus V_j)$$

for $j\geq 0$, $c\in\mathcal{C}_j$, and $V_r\in\mathcal{A}$ which satisfy the following properties; here, when $j=0$, $\xi_0(*)$ is interpreted as a map $S^0\to E_0$.

(i) The following diagram is commutative, where $c\in\mathcal{C}_k$, $d_s\in\mathcal{C}_{j_s}$, $j=j_1+\cdots+j_k$, $b=\gamma(c;d_1,\ldots,d_j)\in\mathcal{C}_j$, and $W_s=d_s(V_{j_1+\cdots+j_{s-1}+1}\oplus\cdots\oplus V_{j_1+\cdots+j_s})$, with $W_s=\{0\}$ if $j_s=0$:

$$\begin{tikzcd}
EV_1\wedge\cdots\wedge EV_j \arrow[r, "\xi_j(b)"] \arrow[d, "\xi_{j_1}(d_1)\wedge\cdots\wedge\xi_{j_k}(d_k)"'] & Eb(V_1\oplus\cdots\oplus V_j) \arrow[d, equal] \\
EW_1\wedge\cdots\wedge EW_k \arrow[r, "\xi_k(c)"] & Ec(W_1\oplus\cdots\oplus W_k)
\end{tikzcd}$$

(ii) $\xi_1(1)\colon EV\to EV$ is the identity map, where $1\in\mathcal{C}_1$ is the unit.

(iii) The following diagram is commutative, where $c\in\mathcal{C}_j$ and $\tau\in\Sigma_j$:

$$\begin{tikzcd}
EV_1\wedge\cdots\wedge EV_j \arrow[r, "\xi_j(c\tau)"] \arrow[d, "\tau"'] & Ec\tau(V_1\oplus\cdots\oplus V_j) \arrow[d, equal] \\
EV_{\tau^{-1}(1)}\wedge\cdots\wedge EV_{\tau^{-1}(j)} \arrow[r, "\xi_j(c)"] & Ec(V_{\tau^{-1}(1)}\oplus\cdots\oplus V_{\tau^{-1}(j)})
\end{tikzcd}$$

(iv) For $V_r\in\mathcal{A}$ and $W\in\mathcal{A}$, let $V=V_1\oplus\cdots\oplus V_j$ and let $A(V,W)\subset\mathcal{C}_j$ be the subspace of those c such that $c(V)\subset W$; then the function

$$\zeta\colon T(A(V,W);V,W)\wedge EV_1\wedge\cdots\wedge EV_j\longrightarrow EW$$

specified by $\zeta([c,w]\wedge y_1\wedge\cdots\wedge y_j)=\sigma(\xi_j(c)(y_1\wedge\cdots\wedge y_j)\wedge w)$ for $c\in A(V,W)$, $w\in W-c(V)$, and $y_r\in EV_r$ is continuous.

(v) For $V_r\subset W_r$ and $c\in\mathcal{C}_j$, the following diagram is commutative, where $V=V_1\oplus\cdots\oplus V_j$ and $W=W_1\oplus\cdots\oplus W_j$:

$$\begin{tikzcd}
EV_1\wedge S^{W_1-V_1}\wedge\cdots\wedge EV_j\wedge S^{W_j-V_j} \arrow[r, phantom, "\cong"] \arrow[d, "\sigma\wedge\cdots\wedge\sigma"'] & EV_1\wedge\cdots\wedge EV_j\wedge S^{(W_1-V_1)\oplus\cdots\oplus(W_j-V_j)} \arrow[d, "\xi_j(c)\wedge S^c"] \\
EW_1\wedge\cdots\wedge EW_j \arrow[r, "\xi_j(c)"] & E(cW) \quad E(cV)\wedge S^{cW-cV} \arrow[l, "\sigma"']
\end{tikzcd}$$

Proof. Parts (iii) and (v) are simply the space level transcriptions of the assertion that the $\xi_j(c)$ are maps of prespectra $(\ell E)^{(j)}\to c^*(\ell E)$ such that $\xi_j(c)\circ\tau=\xi_j(c\tau)$, and (iv) is the continuity condition of VI.5.5. Thus that result and a comparison of (i) to the diagram of Definition 2.1 imply the conclusion.

The continuity condition (iv) is stronger than that prescribed in [99], but the change has no effect on the results there.

Remarks 2.5. For a G-operad $\mathcal{C}$, the fixed point spaces $\mathcal{C}_j^G$ comprise a nonequivariant operad under the fixed point maps γ^G and induced Σ_j actions. Moreover, if $\mathcal{C}$ is an E_∞ G-operad, then $\mathcal{C}^G$ is an E_∞ operad, in the sense that each $\mathcal{C}_j^G$ is Σ_j-free and contractible. Recall that, for a G-spectrum E, E^G is obtained by first restricting to indexing spaces in the fixed point universe U^G and then passing to fixed points spacewise. It is clear from the proposition that E^G is a $\mathcal{C}^G$-spectrum if E is a $\mathcal{C}$-spectrum. Thus the fixed point spectra of E_∞ ring G-spectra are E_∞ ring spectra. An elaboration of this observation shows that E^G is an $\bar{h}\mathcal{C}^G$-spectrum if E is an $\bar{h}\mathcal{C}$-spectrum.

Remark 2.6. Due to the intricacies in our construction of extended powers, our first definition of $\mathcal{C}$-spectra is hard to interpret on the prespectrum level. In constrast, the space level maps $\xi_j(c)$ and conditions (i)-(iv) make perfect sense with E replaced by an arbitrary prespectrum $D\in G\mathcal{P}\mathcal{A}$. Moreover, with the resulting notion of a $\mathcal{C}$-prespectrum, it is clear that the functor L carries $\mathcal{C}$-prespectra to $\mathcal{C}$-spectra.

As in [H_∞, I.3.7], we also have space level analogs of the notions of $\mathcal{C}$-spectra and $\bar{h}\mathcal{C}$-spectra.

Definition 2.7. A $\mathcal{C}_0$-space is a G-space X with basepoint 0 together with based $(G\times\Sigma_j)$-maps $\xi_j\colon\mathcal{C}_j\ltimes X^{(j)}\to X$ such that $\xi_1\iota_1=1$ and the diagram of Definition 2.1 commutes with E replaced by X. (The map $\xi_0\colon S^0\to X$ then gives X a second basepoint 1.) An $\bar{h}\mathcal{C}_0$-space is a space X with maps ξ_j in $\bar{h}(G\times\Sigma_j)\mathcal{J}$ such that $\xi_1\iota_1=1$ in $\bar{h}G\mathcal{J}$ and the same diagram commutes in $\bar{h}(G\times\pi)\mathcal{J}$ for the largest possible $\pi\subset\Sigma_j$. A $\mathcal{C}$-space or $\bar{h}\mathcal{C}$-space Y is defined similarly, but without 0 and with the ξ_j having domains $\mathcal{C}_j\times Y^j$; $Y^+=Y\amalg\{0\}$ is then a $\mathcal{C}_0$-space or $\bar{h}\mathcal{C}_0$-space.

The comparison between space level and spectrum level definitions proceeds exactly as in [H_∞, I.3.8-3.10], hence we omit the details. As there, we obtain natural maps $\mathcal{C}_j \ltimes (E_0)^{(j)} \to (\mathcal{C}_j \ltimes E^{(j)})_0$ by applying $\mathcal{C}_j \ltimes (?)$ to the evaluation map $\varepsilon: \Sigma^\infty\Omega^\infty E \to E$ and passing to adjoints.

Proposition 2.8. Let X be a $\mathcal{C}_0$-space. Then $\Sigma^\infty X$ is a $\mathcal{C}$-spectrum with structural maps

$$\mathcal{C}_j \ltimes (\Sigma^\infty X)^{(j)} \cong \Sigma^\infty(\mathcal{C}_j \ltimes X^{(j)}) \xrightarrow{\Sigma^\infty \xi_j} \Sigma^\infty X.$$

Let E be a $\mathcal{C}$-spectrum. Then $E_0 = \Omega^\infty E$ is a $\mathcal{C}_0$-space with structural maps

$$\mathcal{C}_j \ltimes (E_0)^{(j)} \longrightarrow (\mathcal{C}_j \ltimes E^{(j)})_0 \xrightarrow{(\xi_j)_0} E_0.$$

Moreover, $\eta: X \to QX = \Omega^\infty\Sigma^\infty X$ is a map of $\mathcal{C}_0$-spaces and $\varepsilon: \Sigma^\infty\Omega^\infty E \to E$ is a map of $\mathcal{C}$-spectra. Therefore Σ^∞ and Ω^∞ restrict to an adjoint pair of functors relating the categories of $\mathcal{C}_0$-spaces and $\mathcal{C}$-spectra. The same conclusions apply to $\overline{h}\mathcal{C}_0$-spaces and $\overline{h}\mathcal{C}$-spectra.

The $\mathcal{C}$ action on $\mathcal{C}_0$-spaces and $\mathcal{C}$-spectra is thought of as multiplicative. On the spectrum level, an additive structure is implicit and gives rise to an additive H-space structure on the zero$\underline{\text{th}}$ space level. To express the full structure present, one uses a second, additive, operad action on $\mathcal{C}_0$-spaces suitably related to the given multiplicative action. We shall not be precise about this here, referring the reader to [99,102,134] for details. The basic ideas run as follows. There is a canonical E_∞ operad $\mathcal{K}$ which acts on infinite loop spaces. There is an associated free $\mathcal{K}$-space functor K, and KX is a $\mathcal{C}_0$-space if X is a $\mathcal{C}_0$-space. A $(\mathcal{K},\mathcal{C})$-space is a $\mathcal{K}$-space and a $\mathcal{C}_0$-space X such that the additive action $KX \to X$ is a map of $\mathcal{C}_0$-spaces. There is also a natural map $KX \to QX$ of $\mathcal{C}_0$-spaces. If E is a $\mathcal{C}$-spectrum, the previous result implies that $\varepsilon_0: QE_0 \to E_0$ is a map of $\mathcal{C}_0$-spaces. Since the additive action on E_0 is the composite $KE_0 \to QE_0 \to E_0$, it follows that E_0 is a $(\mathcal{K},\mathcal{C})$-space. An up to homotopy reworking of this discussion presents no difficulty and leads to the conclusion that the zero$\underline{\text{th}}$ space of an $\overline{h}\mathcal{C}$-spectrum is an $\overline{h}(\mathcal{K},\mathcal{C})$-space. When $\mathcal{C}$ is an E_∞ operad, $(\mathcal{K},\mathcal{C})$-spaces are called E_∞ ring spaces and $\overline{h}(\mathcal{K},\mathcal{C})$-spaces are called H_∞ ring spaces. (Warning: in [99] and [102], $\mathcal{C}$ is used for operads which act additively and $\mathcal{H}$ is used for operads which act multiplicatively.)

Remarks 2.9. There is also a generalization to arbitrary operads $\mathcal{C}$ over $\mathcal{L}$ of the notion of an H_∞^d ring spectrum introduced in $[H_\infty]$ for the study of homology and cohomology operations. An $\overline{h}\mathcal{C}^d$-spectrum is a spectrum E together with maps

$$\xi_j: \mathcal{C}_j \ltimes (\Sigma^{dq}E)^{(j)} \longrightarrow \Sigma^{djq}E$$

for some fixed d and all $j \geq 0$ and $q \in Z$. There is a natural equivalence of $\mathcal{C}_k \ltimes (\bigvee_q \Sigma^{dq}E)^{(k)}$ with a wedge of smash products of spectra $\mathcal{C}_j \ltimes (\Sigma^{dq}E)^{(j)}$, and, by use of the product $\xi_2\iota_2$ on E, the maps ξ_j induce maps

$$\xi_k: \mathcal{C}_k \ltimes (\bigvee_q \Sigma^{dq}E)^{(k)} \longrightarrow \bigvee_q \Sigma^{dq}E.$$

We require $\bigvee_q \Sigma^{dq}E$ to be an $\overline{h}\mathcal{C}$-ring spectrum with respect to these maps; compare $[H_\infty$, I§4 and VII§6]. While this notion is not of much practical value except in the case of E_∞ operads, its use will be convenient in the discussion of the Thom isomorphism in IX§7.

If one is uninterested in commutativity, one can develop an analogous theory by dropping all reference to permutation groups. As in [97,§3], we refer to non-Σ operads when no Σ_j actions are considered. We say that a non-Σ operad $\mathcal{C}$ is an A_∞ operad if each $\mathcal{C}_j$ is G-contractible. By an A_∞ ring spectrum, we understand a $\mathcal{C}$-spectrum for some A_∞ operad $\mathcal{C}$ over $\mathcal{L}$. Of course, an E_∞ ring spectrum is an A_∞ ring spectrum by neglect of structure. Simple chases based on (i) and (ii) of Proposition 1.7 and the obvious relation $f\wedge g = (f\wedge 1)(1\wedge g)$ give the following result.

Lemma 2.10. Let $\mathcal{C}$ be an A_∞ operad over $\mathcal{L}$ and let $E \in G\mathcal{S}\mathcal{A}$. Then G-maps $\xi_j: \mathcal{C}_j \ltimes E^{(j)} \to E$ give E a structure of $\mathcal{C}$-spectrum (or of $\overline{h}\mathcal{C}$-spectrum) if $\xi_1\iota = 1$ and the following diagrams commute (or commute in $\overline{h}G\mathcal{S}\mathcal{A}$), $1 \leq i \leq k$.

$$\begin{tikzcd}
\mathcal{C}_k \ltimes (E^{(i-1)} \wedge (\mathcal{C}_j \ltimes E^{(j)}) \wedge E^{(k-i)}) \arrow[r, "\gamma"] \arrow[d, "1\ltimes(1\wedge\xi_j\wedge 1)"'] & \mathcal{C}_{j+k-1} \ltimes E^{(j+k-1)} \arrow[d, "\xi_{j+k-1}"] \\
\mathcal{C}_k \ltimes E^{(k)} \arrow[r, "\xi_k"] & E
\end{tikzcd}$$

That is, we need consider only those diagrams in Definitions 2.1 with $j_s = 1$ and $\mathcal{C}_1$ replaced by $1 \in \mathcal{C}_1$ for all but one index s. Nonequivariantly, the interest in A_∞ ring spectra arises from the theory of A_∞ ring spaces [100] and from their central role in Robinson's very interesting paper [124]. There $\mathcal{L}_j$ is denoted $\mathrm{LIE}((R^\infty)^j, R^\infty)$ and $X \ltimes (E_1 \wedge \cdots \wedge E_j)$ is denoted $X^+ \wedge E_1 \wedge \cdots \wedge E_j$ (by abuse, of course, since the latter has another quite different meaning).

Scholium 2.11. Robinson says that he is working in Boardman's stable category, but he quotes May [99,100] for the existence of interesting examples of A_∞ ring spectra. In fact, May's different but equivalent version of Boardman's category was originally introduced precisely because of its much greater convenience for the construction of structured ring spectra. Robinson is quite precise as to what facts he needs, and we have given all necessary prerequisites for his work. Noting that his Lemma 1.7 applies with $\mathcal{L}$ replaced by any A_∞ operad $\mathcal{C}$, we see immediately from our Lemma 2.10 that an A_∞ ring spectrum in our sense is a "ring spectrum" in his sense [124,2.1]. With this interpretation, his list of examples [124,14.4] of theories having a Künneth spectral sequence is perfectly correct and can be extended.

§3. The constructions CX and CE

Our main objective here is to construct the free $\mathcal{C}$-ring spectrum CE associated to an operad $\mathcal{C}$ augmented over $\mathcal{L}$ and a spectrum E with unit $e\colon S \to E$. We shall also relate this construction to that of the free $\mathcal{C}$-space CX associated to a based space X. The definitions of CE and CX do not employ the full operad structure of $\mathcal{C}$; they are meaningful and useful when $\mathcal{C}$ is only a coefficient system.

Definitions 3.1. (i) Define Λ to be the category of finite based sets $\underline{r} = \{0,1,\dots,r\}$ with basepoint 0 and injective based functions. Say that an injection ϕ is ordered if $a < b$ implies $\phi(a) < \phi(b)$. Any morphism in Λ is the composite of a permutation and an ordered injection. Any ordered injection is a composite of degeneracy operations $\sigma_q\colon \underline{r} \to \underline{r+1}$, $0 \le q \le r$, where

$$\sigma_q(a) = \begin{cases} a & \text{if } a \le q \\ a+1 & \text{if } a > q \end{cases}$$

(ii) A coefficient system $\mathcal{C}$ is a contravariant functor from Λ to the category $G\mathcal{U}$ of compactly generated unbased G-spaces such that $\mathcal{C}_0$ is a single point $*$. For an injection $\phi\colon \underline{r} \to \underline{s}$, write $\phi\colon \mathcal{C}_s \to \mathcal{C}_r$ on elements by $\phi(c) = c\phi$ for $c \in \mathcal{C}_s$. Thus $\mathcal{C}_s$ is regarded as a right Σ_s-space under the permutations. A map of coefficient systems is a natural transformation of functors $\Lambda \to G\mathcal{U}$.

An operad $\mathcal{C}$ is a coefficient system by neglect of structure. Its degeneracy maps $\sigma_q\colon \mathcal{C}_{r+1} \to \mathcal{C}_r$ are given by $c\sigma_q = \gamma(c;s_q)$ for $c \in \mathcal{C}_{r+1}$, where

$$s_q = 1^q \times * \times 1^{r-q} \in \mathcal{C}_1^q \times \mathcal{C}_0 \times \mathcal{C}_1^{r-q}.$$

Many other examples of coefficient systems appear in [30]. There the space CX associated to a coefficient system $\mathcal{C}$ and based space X was defined to be the coend

$$\textstyle\int^\Lambda \mathcal{C}_r \times X^r = \coprod \mathcal{C}_r \times X^r/(\sim),$$

where $(c\phi,y) \sim (c,\phi y)$ for $c \in \mathcal{C}_s$, $\phi\colon \underline{r} \to \underline{s}$, and $y \in X^r$. Here

where $x'_{\phi(a)} = x_a$ and $x'_b = *$ if $b \notin \mathrm{Im}\phi$. Alternatively, CX can be described by the coequalizer diagram

(1) $$\coprod_{\phi\colon \underline{r}\to\underline{s}} \mathcal{C}_s \times X^r \overset{g}{\underset{h}{\rightrightarrows}} \coprod_{r\ge 0} \mathcal{C}_r \times X^r \longrightarrow CX.$$

Here the maps g and h are defined to be

$$1\times\phi\colon \mathcal{C}_s \times X^r \longrightarrow \mathcal{C}_s \times X^s \quad\text{and}\quad \phi\times 1\colon \mathcal{C}_s \times X^r \longrightarrow \mathcal{C}_r \times X^r$$

on the summand $\mathcal{C}_s \times X^r$ indexed on $\phi\colon \underline{r} \to \underline{s}$. We take diagram (1) as our basic description of CX since it generalizes most readily to a description of CE.

Thus, to define CE, we replace disjoint unions by wedges, Cartesian powers X^r by external smash powers $E^{(r)}$, and products $\mathcal{C}_s \times X^r$ by appropriate twisted half smash products $\mathcal{C}_\phi \ltimes E^{(r)}$. To define the latter, we assume given a morphism of coefficient systems $\chi\colon \mathcal{C} \to \mathcal{L}$, where $\mathcal{L}$ is the linear isometries operad associated to our given universe U. For an injection $\phi\colon \underline{r} \to \underline{s}$, we then define $\chi_\phi\colon \mathcal{C}_s \to \mathcal{L}_r$ to be either composite in the commutative diagram

$$\begin{tikzcd} \mathcal{C}_s \arrow[r, "\phi"] \arrow[d, "\chi_s"'] \arrow[dr, "\chi_\phi"] & \mathcal{C}_r \arrow[d, "\chi_r"] \\ \mathcal{L}_s \arrow[r, "\phi"'] & \mathcal{L}_r \end{tikzcd}.$$

We define $\mathcal{C}_\phi \ltimes E^{(r)}$ to be $\chi_\phi \ltimes E^{(r)}$ and observe that ϕ induces a map $\chi_\phi \to \chi_r$ of spaces over $\mathcal{L}_r$ and thus a map

$$\phi \ltimes 1\colon \mathcal{C}_\phi \ltimes E^{(r)} \longrightarrow \mathcal{C}_r \ltimes E^{(r)}$$

in $G\mathcal{S}\mathcal{A}$ for any spectrum $E \in G\mathcal{S}\mathcal{A}$.

Now suppose given a unit $e\colon S \to E$. Thinking of U as a based space with basepoint 0, we see that an injection $\phi\colon \underline{r} \to \underline{s}$ induces a linear isometry $\phi\colon U^r \to U^s$. The spectrum $\phi_* E^{(r)} \in G\mathcal{S}\mathcal{A}^s$ is easily seen to be isomorphic to the external smash product $E_1 \wedge \dots \wedge E_s$, where $E_{\phi(a)} = E$ and $E_b = S$ if $b \notin \mathrm{Im}\phi$. Define

$$\phi = f_1 \wedge \dots \wedge f_s\colon \phi_* E^{(r)} \cong E_1 \wedge \dots \wedge E_s \longrightarrow E^{(s)},$$

where $f_{\phi(a)}$ is the identity of E and $f_b = e$ if $b \notin \mathrm{Im}\phi$.

Definition 3.2. For a coefficient system $\mathcal{C}$ augmented over $\mathcal{L}$ and a unital spectrum $E \in G\mathcal{S}\mathcal{A}$, define $CE \in G\mathcal{S}\mathcal{A}$ by the coequalizer diagram

$$(2) \qquad \bigvee_{\phi: \underline{r} \to \underline{s}} \mathcal{C}_\phi \ltimes E^{(r)} \underset{h}{\overset{g}{\rightrightarrows}} \bigvee_{r \geq 0} \mathcal{C}_r \ltimes E^{(r)} \longrightarrow CE.$$

On the $\phi^{\underline{th}}$ wedge summand, g and h are defined to be

$$1 \ltimes \phi: \mathcal{C}_\phi \ltimes E^{(r)} \cong \mathcal{C}_s \ltimes \phi_* E^{(r)} \longrightarrow \mathcal{C}_s \ltimes E^{(s)},$$

the isomorphism being given by VI.3.1 (iii), and

$$\phi \ltimes 1: \mathcal{C}_\phi \ltimes E^{(r)} \longrightarrow \mathcal{C}_r \ltimes E^{(r)}.$$

Since $S = \mathcal{C}_0 \ltimes E^{(0)}$, CE has an evident unit.

It is easily checked that CX and CE are continuous functors of X, E, and $\mathcal{C}$ (where maps of $\mathcal{C}$ are maps over $\mathcal{L}$ in the case of CE). Compare the remarks following I.3.4. This has the following consequence, which could also be verified by direct inspection of the constructions.

Proposition 3.3. CX and CE are homotopy preserving functors of coefficient systems $\mathcal{C}$ over $\mathcal{L}$, based spaces X, and unital spectra E.

We insert some general observations about maps between coequalizers.

Remark 3.4. If the two rows in the diagram

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[d, "\gamma"] \\ A' \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] & B' \arrow[r] & C' \end{tikzcd}$$

are coequalizer diagrams, then a pair of maps (α, β) such that $g'\alpha = \beta g$ and $h'\alpha = \beta h$ is called a map of coequalizer diagrams. Such a pair determines a map γ such that the right square commutes, and γ is an isomorphism if α and β are isomorphisms. For a given β, there is at most one γ making the right square commute.

If X is a based space, map S^0 to X^+ by sending the non-basepoint of S^0 to the basepoint of X. There results a unit map $e: S = \Sigma^\infty(S^0) \to \Sigma^\infty(X^+)$. We have the following basic consistency result.

Proposition 3.5. There is a natural isomorphism

$$C\Sigma^\infty(X^+) \cong \Sigma^\infty(CX)^+$$

for based spaces X.

Proof. After adjoining disjoint basepoints, the coequalizer diagram (1) can be rewritten in the form

$$(1') \qquad \bigvee_{\phi: \underline{r} \to \underline{s}} \mathcal{C}_s^+ \wedge (X^+)^{(r)} \rightrightarrows \bigvee_{r \geq 0} \mathcal{C}_r^+ \wedge (X^+)^{(r)} \longrightarrow (CX)^+.$$

Since Σ^∞ commutes with coequalizers, wedges, smash powers, and twisted half smash products (see II.3.6 and VI.1.5), application of Σ^∞ converts this to the coequalizer diagram (2) for the unital spectrum $\Sigma^\infty(X^+)$.

While diagrams (1) and (2) provide the most conceptual descriptions of CX and CE, they involve an inconvenient redundancy of domain summands. The only essential summands are those indexed on degeneracy maps and permutations, and the effect of the latter is to collapse out the diagonal actions of Σ_r on $\mathcal{C}_r \times X^r$ and $\mathcal{C}_r \ltimes E^{(r)}$. Thus write $\chi_{r,q}$ for the composite

$$\mathcal{C}_{r+1} \xrightarrow{\sigma_q} \mathcal{C}_r \xrightarrow{\chi_r} \mathcal{C}_r, \quad 0 \leq q \leq r$$

(that is, $\chi_{r,q} = \chi_{\sigma_q}$) and write $\mathcal{C}_{r,q} \ltimes E^{(r)}$ for $\chi_{r,q} \ltimes E^{(r)}$.

Proposition 3.6. The following diagrams are coequalizers:

$$(3) \qquad \coprod_{r \geq 0} \coprod_{0 \leq q \leq r} \mathcal{C}_{r+1} \times X^r \underset{h}{\overset{g}{\rightrightarrows}} \coprod_{r \geq 0} \mathcal{C}_r \times_{\Sigma_r} X^r \longrightarrow CX$$

$$(4) \qquad \bigvee_{r \geq 0} \bigvee_{0 \leq q \leq r} \mathcal{C}_{r,q} \ltimes E^{(r)} \underset{h}{\overset{g}{\rightrightarrows}} \bigvee_{r \geq 0} \mathcal{C}_r \ltimes_{\Sigma_r} E^{(r)} \longrightarrow CE$$

Here the $(r,q)^{\underline{th}}$ domain summand is indexed on $\sigma_q: \underline{r} \to \underline{r+1}$, and g and h are restrictions of the maps g and h of diagrams (1) and (2).

It is generally more useful to describe CX and CE inductively in terms of their natural filtrations. Thus define F_sCX and F_sCE, $s \geq 0$, to be the coequalizers in the following diagrams:

$$\coprod_{0 \leq r \leq s-1} \coprod_{0 \leq q \leq r} \mathcal{C}_{r+1} \times X^r \underset{h}{\overset{g}{\rightrightarrows}} \coprod_{0 \leq r \leq s} \mathcal{C}_r \times_{\Sigma_r} X^r \longrightarrow F_sCX$$

$$\bigvee_{0 \leq r \leq s-1} \bigvee_{0 \leq q \leq r} \mathcal{C}_{r,q} \ltimes E^{(r)} \underset{h}{\overset{g}{\rightrightarrows}} \bigvee_{0 \leq r \leq s} \mathcal{C}_r \ltimes_{\Sigma_r} E^{(r)} \longrightarrow F_sCE$$

Note that $F_0CX = \{*\}$ and $F_0CE = S$. A comparison of the defining coequalizer diagrams yields natural maps

$$\iota_s\colon F_sCX \longrightarrow F_{s+1}CX \quad\text{and}\quad \iota_s\colon F_sCE \longrightarrow F_{s+1}CE,$$

and there result natural isomorphisms

$$CX \cong \operatorname{colim} F_sCX \quad\text{and}\quad CE \cong \operatorname{colim} F_sCE.$$

The space level maps ι_s are always closed inclusions and careful inspection of the prespectrum level constructions shows that the spectrum level maps ι_s are spacewise closed inclusions if $e\colon S \to E$ is a spacewise closed inclusion.

As discussed in [30,§2], the F_sCX can be described inductively by means of pushout diagrams

$$\begin{tikzcd} \mathcal{C}_{s+1} \times_{\Sigma_{s+1}} \sigma(X^s) \arrow[r, phantom, "\subset"] \arrow[d, "f"'] & \mathcal{C}_{s+1} \times_{\Sigma_{s+1}} X^{s+1}. \arrow[d] \\ F_sCX \arrow[r, "\iota_s"] & F_{s+1}CX \end{tikzcd}$$

Here $\sigma(X^s) = \bigcup_{0\le q\le s} \sigma_q(X^s)$ is the subspace of points of X^{s+1} at least one of whose coordinates is the basepoint and $f(c,\sigma_q y) = [c\sigma_q, y]$. Assuming that the basepoint of X is nondegenerate (that is, that $\{*\} \to X$ is a G-cofibration), we conclude that the maps ι_s and the natural maps $F_sCX \to CX$ are G-cofibrations, and this implies that the natural map $\operatorname{tel} F_sCX \to CX$ is a G-homotopy equivalence. We also conclude inductively that if X has the homotopy type of a G-CW complex and the $\mathcal{C}_s$ have the homotopy types of $(G \times \Sigma_s)$-CW complexes, then CX has the homotopy type of a G-CW complex.

We wish to prove analogs of these statements for CE. Since the unit $e\colon S \to E$ cannot be thought of in quite the same way as the inclusion $\{*\} \to X$ and since we have no spectrum level analog of the notion of an NDR-pair, we shall have to proceed more carefully and impose more stringent hypotheses.

With the obvious actions of Σ_s, the map

$$1\wedge e\colon E^{(s)} \wedge S \longrightarrow E^{(s+1)}$$

is Σ_s-equivariant and so induces a Σ_{s+1}-map

$$j\colon \Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S) \longrightarrow E^{(s+1)}.$$

Of course, there is a natural isomorphism

$$\Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S) \cong \bigvee_{0\le q\le s} E^{(q)} \wedge S \wedge E^{(s-q)}.$$

With the obvious actions of Σ_{s-1}, the two maps

$$1\wedge e\wedge 1\colon E^{(s-1)} \wedge S \wedge S \longrightarrow E^{(s)} \wedge S \subset \Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S)$$

and

$$1\wedge 1\wedge e\colon E^{(s-1)} \wedge S \wedge S \longrightarrow E^{(s-1)} \wedge S \wedge E \subset \Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S)$$

are Σ_{s-1}-equivariant and so induce Σ_{s+1}-maps

$$k_1, k_2\colon \Sigma_{s+1} \ltimes_{\Sigma_{s-1}} (E^{(s-1)} \wedge S \wedge S) \longrightarrow \Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S).$$

We define $\sigma(E^{(s)})$ to be the coequalizer in the diagram

$$\Sigma_{s+1} \ltimes_{\Sigma_{s-1}} (E^{(s-1)} \wedge S \wedge S) \overset{k_1}{\underset{k_2}{\rightrightarrows}} \Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)} \wedge S) \longrightarrow \sigma(E^{(s)}).$$

Clearly j coequalizes k_1 and k_2, hence we have an induced Σ_{s+1}-map

$$i_s\colon \sigma(E^{(s)}) \longrightarrow E^{(s+1)}.$$

By inspection on the prespectrum level, i_s is a spacewise closed inclusion if $e\colon S \to E$ is so. The composites

$$\mathcal{C}_{s+1} \ltimes (E^{(q)} \wedge S \wedge E^{(s-q)}) \cong \mathcal{C}_{s,q} \ltimes E^{(s)} \xrightarrow{\sigma_q \ltimes 1} \mathcal{C}_s \ltimes E^{(s)} \longrightarrow F_sCE$$

induce a map

$$f\colon \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} \sigma(E^{(s)}) \longrightarrow F_sCE.$$

<u>Proposition 3.7</u>. For $s \ge 0$, the following diagram is a pushout:

$$\begin{tikzcd} \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} \sigma(E^{(s)}) \arrow[r, "1\ltimes i_s"] \arrow[d, "f"'] & \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} E^{(s+1)} \arrow[d] \\ F_sCE \arrow[r, "\iota_s"] & F_{s+1}CE \end{tikzcd}$$

<u>Proof</u>. The definition of the F_sCE as coequalizers implies that the outer rectangle of the following diagram is a pushout, and the diagram commutes by our definitions of i_s and f:

$$\begin{tikzcd}
\bigvee_{0\leqslant q\leqslant s} \mathcal{C}_{s,q} \ltimes E^{(s)} \arrow[r, phantom, "\cong"] \arrow[d, "\gamma(\sigma_q\times 1)"'] & \bigvee_{0\leqslant q\leqslant s} \mathcal{C}_{s+1} \ltimes \sigma_{q*}E^{(s)} \arrow[r, "(1\ltimes\sigma_q)"] \arrow[d, phantom, "\wr\|"] & \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} E^{(s+1)} \arrow[ddd, "1\ltimes i_s"'] \\
\mathcal{C}_s \ltimes E^{(s)} \arrow[dd] & \mathcal{C}_{s+1} \ltimes (\Sigma_{s+1} \ltimes_{\Sigma_s} (E^{(s)}\ S)) \arrow[d] & \\
 & \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} \sigma(E^{(s)}) \arrow[dl, "f"'] & \\
F_sCE \arrow[rr, "\iota_s"'] & & F_{s+1}CE
\end{tikzcd}$$

The conclusion follows by an elementary diagram chase.

<u>Proposition 3.8</u>. let E be a G-CW spectrum with a unit $e: S \to E$ which is the inclusion of a cellular subspectrum. Assume the $\mathcal{C}_s$ are $(G \times \Sigma_s)$-CW complexes. Then the following conclusions hold.

(i) The maps $\iota_s: F_sCE \to F_{s+1}CE$ are G-cofibrations and

$$F_{s+1}CE/F_sCE \cong D_{s+1}(\mathcal{C},E/S).$$

(ii) The natural map tel $F_sCE \to CE$ is a G-homotopy equivalence and CE has the homotopy type of a G-CW spectrum.

<u>Proof</u>. An argument along exactly the same lines as the proof of VI.5.2 shows that the map $i_s: \sigma(E^{(s)}) \to E^{(s+1)}$ is a $(G \times \Sigma_{s+1})$-cofibration with quotient $(E/S)^{(s+1)}$ and that $(E^{(s+1)}, \sigma(E^{(s)}))$ has the homotopy type of a $(G \times \Sigma_{s+1})$-CW pair. It follows that $1 \ltimes i_s$ is a G-cofibration with quotient $D_{s+1}(\mathcal{C},E/S)$ and that

$$(\mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} E^{(s+1)}, \mathcal{C}_{s+1} \ltimes_{\Sigma_{s+1}} \sigma(E^{(s)}))$$

has the homotopy type of a G-CW pair (see I.5.6 and VI.1.11). Now the conclusions follow from the previous proposition by standard arguments.

§4. <u>Pairings and operad actions on CE</u>

We show here that CE is a $\mathcal{C}$-ring spectrum when $\mathcal{C}$ is an operad over $\mathcal{L}$. More generally, we begin by relating pairings of coefficient systems and actions of operads on coefficient systems to the constructions CX and CE.

<u>Definition 4.1</u>. The wedge sum $\vee : \Lambda \times \Lambda \to \Lambda$ is given on objects by $\underline{p} \vee \underline{q} = \underline{p+q}$ and on injections $\phi: \underline{p} \to \underline{p}'$ and $\psi: \underline{q} \to \underline{q}'$ by

$$(\phi \vee \psi)(i) = \begin{cases} \phi(i) & \text{if } 1 \leqslant i \leqslant p \\ \psi(i-p) + p' & \text{if } p < i \leqslant p+q. \end{cases}$$

A pairing $\oplus: (\mathcal{A},\mathcal{B}) \to \mathcal{C}$ of coefficient systems is a natural transformation $\mathcal{A} \times \mathcal{B} \to \mathcal{C} \circ \vee$ of functors from $\Lambda \times \Lambda$ to G-spaces.

If $\mathcal{C}$ is an operad and $c \in \mathcal{C}_2$, then the maps $\gamma(c): \mathcal{C}_p \times \mathcal{C}_q \to \mathcal{C}_{p+q}$ specified by $\gamma(c)(x,y) = \gamma(c;x,y)$ give a pairing $\oplus: (\mathcal{C},\mathcal{C}) \to \mathcal{C}$. In particular, we obtain such a pairing on $\mathcal{L}$ from any isometry $f \in \mathcal{L}_2$. A pairing of coefficient systems over $\mathcal{L}$ is a pairing $\oplus: (\mathcal{A},\mathcal{B}) \to \mathcal{C}$ such that the diagram

$$\begin{tikzcd}
\mathcal{A} \times \mathcal{B} \arrow[r, "\oplus"] \arrow[d] & \mathcal{C} \circ \vee \arrow[d] \\
\mathcal{L} \times \mathcal{L} \arrow[r, "\oplus"] & \mathcal{L} \circ \vee
\end{tikzcd}$$

commutes. That is, for $p,q \geq 0$, the following diagram commutes, where ψ, ω and χ are the respective maps to $\mathcal{L}$.

$$\begin{tikzcd}
\mathcal{A}_p \times \mathcal{B}_q \arrow[r, "\oplus"] \arrow[d, "\psi\oplus\omega"'] & \mathcal{C}_{p+q} \arrow[d, "\chi"] \\
\mathcal{L}_p \times \mathcal{L}_q \arrow[r, "\gamma(f)"] & \mathcal{L}_{p+q}
\end{tikzcd}$$

Such pairings arise from general pairings $(\mathcal{A}',\mathcal{B}') \to \mathcal{C}'$ by crossing with $\mathcal{L}$ and projecting.

The naturality of twisted half smash products and of the isomorphism of VI.1.5 gives the following result.

<u>Lemma 4.2</u>. Let $\oplus: (\mathcal{A},\mathcal{B}) \to \mathcal{C}$ be a pairing over $\mathcal{L}$. Let $D \in (G \times \Sigma_p)\mathscr{S}U^p$ and $D' \in (G \times \Sigma_q)\mathscr{S}U^q$. Then there is a natural map

$$\oplus: (\mathcal{A}_p \ltimes_{\Sigma_p} D) \wedge (\mathcal{B}_q \ltimes_{\Sigma_q} D') \longrightarrow \mathcal{C}_{p+q} \ltimes_{\Sigma_p\times\Sigma_q} (D \wedge D')$$

in $G\mathscr{S}U$. When $D = E^{(p)}$ and $D' = E^{(q)}$ for $E \in G\mathscr{S}U$, we can pass to orbits over Σ_{p+q} on the right. If $X \in G\mathcal{T}$, the following diagram commutes:

$$\begin{tikzcd}
{[\mathcal{A}_p \ltimes_{\Sigma_p} (\Sigma^\infty X)^{(p)}] \wedge [\mathcal{B}_q \ltimes_{\Sigma_q} (\Sigma^\infty X)^{(q)}]} \arrow[r, "\oplus"] \arrow[d, "\cong"'] & \mathcal{C}_{p+q} \ltimes_{\Sigma_{p+q}} (\Sigma^\infty X)^{(p+q)} \arrow[d, "\cong"] \\
\Sigma^\infty[(\mathcal{A}_p^+ \wedge_{\Sigma_p} X^{(p)}) \wedge (\mathcal{B}_q^+ \wedge_{\Sigma_q} X^{(q)})] \arrow[r, "\Sigma^\infty\oplus"] & \Sigma^\infty(\mathcal{C}_{p+q}^+ \wedge_{\Sigma_{p+q}} X^{(p+q)})
\end{tikzcd}$$

This carries over to the constructions CX and CE.

Proposition 4.3. Let $\oplus\colon (\mathcal{A},\mathcal{B}) \to \mathcal{C}$ be a pairing over $\mathcal{L}$, let $X \in G\mathcal{T}$, and let $E \in G\mathcal{S}U$ be a unital spectrum. Then there are natural maps $\oplus\colon AX \times BX \to CX$ of based spaces and $\oplus\colon AE \wedge BE \to CE$ of unital spectra such that the following diagram commutes:

$$\begin{tikzcd}
A\Sigma^\infty(X^+) \wedge B\Sigma^\infty(X^+) \arrow[r, "\oplus"] \arrow[d, "\cong"] & C\Sigma^\infty(X^+) \arrow[d, "\cong"] \\
\Sigma^\infty(AX \times BX)^+ \arrow[r, "\Sigma^\infty\oplus"] & \Sigma^\infty(CX)^+
\end{tikzcd}$$

Proof. By an elementary formal calculation, $AE \wedge BE$ is the coequalizer in the diagram

$$\begin{tikzcd}
\displaystyle\bigvee_{\phi:\underline{p}\to\underline{p}'} \bigvee_{\psi:\underline{q}\to\underline{q}'} [(\mathcal{A}_\phi \ltimes E^{(p)}) \wedge (\mathcal{B}_q \wedge E^{(q)})] \wedge [(\mathcal{A}_p \ltimes E^{(p)}) \wedge (\mathcal{B}_\psi \ltimes E^{(q)})] \arrow[d, shift right, "(g\wedge 1)\vee(1\wedge g)"'] \arrow[d, shift left, "(h\wedge 1)\vee(1\wedge h)"] \\
\displaystyle\bigvee_{p,q\geq 0} (\mathcal{A}_p \ltimes E^{(p)}) \wedge (\mathcal{B}_q \ltimes E^{(q)}) \arrow[d] \\
AE \wedge BE
\end{tikzcd}$$

obtained by combining the coequalizer diagrams defining AE and BE. Thus, to define $\oplus$, it suffices to define a map from this coequalizer diagram to that defining CE, and maps given by the lemma provide the required restrictions to wedge summands. The unit of $AE \wedge BE$ is $e \wedge e\colon S \cong S \wedge S \to AE \wedge BE$, and it is clear that $\oplus$ is unital. The map $\oplus\colon AX \times BX \to CX$ is defined analogously. The isomorphisms in the diagram are given by Proposition 3.5, and its commutativity follows from the lemma and the fact that Σ^∞ preserves coequalizers.

We next consider iterated pairings parametrized by operads.

Definition 4.4. Let $\mathcal{C}$ be an operad and $\mathcal{B}$ be a coefficient system. An action of $\mathcal{C}$ on $\mathcal{B}$ is a collection of maps

$$\gamma\colon \mathcal{C}_k \times \mathcal{B}_{j_1} \times \cdots \times \mathcal{B}_{j_k} \to \mathcal{B}_{j_1+\cdots+j_k}$$

which satisfy the associativity, left unit, and equivariance properties specified in the definition of an operad (see 1.1) and are such that, for each fixed $c \in \mathcal{C}_k$, the maps $\gamma(c)$ specify a k-fold sum on $\mathcal{B}$; that is, the $\gamma(c)$ define a natural transformation $\mathcal{B}^k \to \mathcal{B} \circ \vee$ of functors $\Lambda^k \to \Lambda$. We then say that $\mathcal{B}$ is a $\mathcal{C}$-module.

Remark 4.5. Let $\mathcal{B}$ be a $\mathcal{C}$-module. By use of the associativity diagram of Definition 1.1 (i) applied to the degeneracy operators of $\mathcal{C}$, we can generalize the first equivariance diagram of Definition 1.1 (iii) to arbitrary injections $\phi\colon \underline{k} \to \underline{m}$. Precisely, for $c \in \mathcal{C}_m$ and $b_r \in \mathcal{B}_{j_r}$, $1 \leq r \leq k$, we have

$$\gamma(c\phi; b_1,\ldots,b_k) = \gamma(c; b'_1,\ldots,b'_m)\, \phi(j_1,\ldots,j_k),$$

where $b'_{\phi(r)} = b_r$ and $b'_s = * \in \mathcal{B}_0$ if $s \notin \operatorname{Im}\phi$ and where

$$\phi(j_1,\ldots,j_k)\colon \underline{j} \cong \underline{j}_1 \vee \cdots \vee \underline{j}_k \to \underline{j}'_1 \vee \cdots \vee \underline{j}'_m \cong \underline{j}$$

maps $\underline{j}_r$ identically to $\underline{j}'_{\phi(r)} = \underline{j}_r$, with $j'_s = 0$ if $s \notin \operatorname{Im}\phi$.

Clearly any operad acts on itself. If $\mathcal{C} \to \mathcal{C}'$ is a map of operads and $\mathcal{C}'$ acts on $\mathcal{B}$, then $\mathcal{C}$ acts on $\mathcal{B}$ by pullback. Other examples appear in [32]. If $\mathcal{C}$ is an operad over $\mathcal{L}$ via $\chi\colon \mathcal{C} \to \mathcal{L}$ and $\mathcal{B}$ is a coefficient system over $\mathcal{L}$ via $\psi\colon \mathcal{B} \to \mathcal{L}$, then $\mathcal{B}$ is said to be a $\mathcal{C}$-module over $\mathcal{L}$ if $\mathcal{C}$ acts on $\mathcal{B}$ in such a way that ψ is a map of $\mathcal{C}$-modules. That is, the following diagram commutes for $k \geq 0$ and $j_r \geq 0$.

$$\begin{tikzcd}
\mathcal{C}_k \times \mathcal{B}_{j_1} \times \cdots \times \mathcal{B}_{j_k} \arrow[r, "\gamma"] \arrow[d, "\chi\times\psi\times\cdots\times\psi"'] & \mathcal{B}_{j_1+\cdots+j_k} \arrow[d, "\psi"] \\
\mathcal{L}_k \times \mathcal{L}_{j_1} \times \cdots \times \mathcal{L}_{j_k} \arrow[r, "\gamma"] & \mathcal{L}_{j_1+\cdots+j_k}
\end{tikzcd}$$

The maps ζ of Definition 1.6 (i) directly generalize to give the following analog of Lemma 4.2.

Lemma 4.6. Let $\mathcal{B}$ be a $\mathcal{C}$-module over $\mathcal{L}$. Let $E_{s,r} \in G\mathcal{S}U$, $1 \leq s \leq k$ and $1 \leq r \leq j_s$. Then there is a natural map

$$\zeta\colon \mathcal{C}_k \ltimes \Big(\bigwedge_{1\leq s\leq k} \mathcal{B}_{j_s} \ltimes \bigwedge_{1\leq r\leq j_s} E_{s,r}\Big) \to \mathcal{B}_{j_1+\cdots+j_k} \ltimes \bigwedge_{1\leq s\leq k}\ \bigwedge_{1\leq r\leq j_s} E_{s,r}.$$

When $E_{s,r} = \Sigma^\infty X_{s,r}$ for $X_{s,r} \in G\mathcal{T}$, these maps are compatible with their evident space level analogs under the commutation isomorphisms relating smash products and twisted half smash products to the functor Σ^∞.

Again, this carries over to the constructions CX and CE.

Proposition 4.7. Let $\mathcal{B}$ be a $\mathcal{C}$-module over $\mathcal{L}$, let $X \in G\mathcal{T}$, and let $E \in G\mathcal{S}U$ be a unital spectrum. Then there are natural maps

$$\eta\colon E \longrightarrow CE, \quad \mu\colon CCE \longrightarrow CE, \quad \text{and} \quad \xi\colon CBE \longrightarrow BE$$

with respect to which C is a monad in the category of unital spectra and BE is a C-algebra. That is, the following unit and associativity diagrams commute:

$$\begin{tikzcd} CE \arrow[r, "C\eta"] \arrow[dr, equal] & CCE \arrow[d, "\mu"] & CE \arrow[l, "\eta"'] \arrow[dl, equal] \\ & CE & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} CBE \arrow[d, "\xi"'] & BE \arrow[l, "\eta"'] \arrow[dl, equal] \\ BE & \end{tikzcd}$$

$$\begin{tikzcd} CCCE \arrow[r, "\mu"] \arrow[d, "C\mu"'] & CCE \arrow[d, "\mu"] \\ CCE \arrow[r, "\mu"] & CE \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} CCBE \arrow[r, "\mu"] \arrow[d, "C\xi"'] & CBE \arrow[d, "\xi"] \\ CBE \arrow[r, "\xi"] & BE \end{tikzcd}$$

The analogous assertions hold for CX and BX, and the following diagrams commute:

$$\begin{tikzcd} & C\Sigma^\infty(X^+) \arrow[d, "\cong"] \\ \Sigma^\infty X^+ \arrow[ur, "\eta"] \arrow[r, "\Sigma^\infty\eta"'] & \Sigma^\infty(CX)^+ \end{tikzcd} \quad \begin{tikzcd} CC\Sigma^\infty(X^+) \arrow[r, "\mu"] \arrow[d, "\cong"] & C\Sigma^\infty(X^+) \arrow[d, "\cong"] \\ \Sigma^\infty(CCX)^+ \arrow[r, "\Sigma^\infty\mu"] & \Sigma^\infty(CX)^+ \end{tikzcd} \quad \begin{tikzcd} CB\Sigma^\infty(X^+) \arrow[r, "\xi"] \arrow[d, "\cong"] & B\Sigma^\infty(X^+) \arrow[d, "\cong"] \\ \Sigma^\infty(CBX)^+ \arrow[r, "\Sigma^\infty\xi"] & \Sigma^\infty(BX)^+ \end{tikzcd}$$

Proof. The spectrum CBE is the coequalizer in the diagram

$$\begin{tikzcd} \left[\bigvee_{r\geq 0} \mathcal{C}_r \times \left(\bigvee_{\psi:\underline{s}\to\underline{s}'} \mathcal{B}_\psi \ltimes E^{(s)}\right)^{(r)}\right] \vee \left[\bigvee_{\phi:\underline{r}\to\underline{r}'} \mathcal{C}_\phi \ltimes \left(\bigvee_{s\geq 0} \mathcal{B}_s \ltimes E^{(s)}\right)^{(r)}\right] \arrow[d, shift right, "{[\bigvee(1\ltimes g)]\vee g}"'] \arrow[d, shift left, "{[\bigvee(1\ltimes h)]\vee h}"] \\ \bigvee_{r\geq 0} \mathcal{C}_r \ltimes \left(\bigvee_{s\geq 0} \mathcal{B}_s \ltimes E^{(s)}\right)^{(r)} \arrow[d] \\ CBE \end{tikzcd}$$

(On a technical note, use of the full coequalizer diagram (3.2), with identity maps indexing some of the domain wedge summands, is essential to this description.) To define ξ, it suffices to define a map from this coequalizer diagram to that defining BE. Modulo use of obvious identifications and maps induced by the injections ϕ, maps ζ given in the lemma provide the required restrictions to wedge summands. Remark 4.5 and the fact that the $\gamma(c)$ are natural transformations provide what is needed to show that these maps define a map of coequalizer diagrams. The map μ is the special case of ξ with $\mathcal{B} = \mathcal{C}$ and the map η is

just the composite

$$E \cong \{1\} \ltimes E \subset \mathcal{C}_1 \times E \subset \bigvee_{r\geq 0} \mathcal{C}_r \ltimes E^{(r)} \longrightarrow CE.$$

The unit and associativity diagrams commute by the unit and associativity conditions in the definitions of operads and their actions on coefficient systems. The last statement is checked by further formal comparisons of coequalizer diagrams.

Specializing to $\mathcal{B} = \mathcal{C}$, we obtain the following description of $\mathcal{C}$-ring spectra. It is analogous to the description of $\mathcal{C}$-spaces given in [97, §2].

Proposition 4.8. A $\mathcal{C}$-ring structure on a unital spectrum E determines and is determined by a structure of C-algebra on E. The functor C from unital spectra to $\mathcal{C}$-ring spectra is left adjoint to the forgetful functor. That is, a unital map $f\colon D \to E$ from D to a $\mathcal{C}$-ring spectrum E gives rise to a unique map $\tilde{f}\colon CD \to E$ of $\mathcal{C}$-ring spectra such that $\tilde{f} \circ \eta = f$.

Proof. If $\xi\colon CE \to E$ gives an action of C on E, then its restrictions $\xi_j\colon \mathcal{C}_j \ltimes E^{(j)} \to E$ give an action of $\mathcal{C}$ on E, and conversely by passage to wedges and coequalizers. The rest follows by a standard argument about monads; $\tilde{f}$ can and must be defined to be the composite $\xi \circ Cf$.

§5. Splitting theorems and James maps

To validate Cohen's proof [34] of the generalization in [30] of Snaith's stable splitting theorem [129], we must analyze the spectrum $C(S \vee E)$, where the unit of $S \vee E$ is the inclusion of the wedge summand S. On the space level, we have

$$C(X^+) = \coprod_{r\geq 0} \mathcal{C}_r \times_{\Sigma_r} X^r$$

since the basepoint identifications serve only to identify points some of whose X^+ coordinates are the disjoint basepoint either to the point of filtration zero or to a point none of whose coordinates is the basepoint. The spectrum level analog reads as follows. We adopt the notation $D_j(\mathcal{C}, E)$ of Definition 1.5.

Lemma 5.1. For coefficient systems $\mathcal{C}$ over $\mathcal{L}$ and spectra E, there is a natural isomorphism

$$C(S \vee E) \cong \bigvee_{r\geq 0} D_r(\mathcal{C}, E).$$

Proof. With the notations of Proposition 3.7, but with E there replaced by $S \vee E$ here, we find by the usual decomposition of $(S \vee E)^{(s+1)}$ as a wedge of smash products $E_1 \wedge \cdots \wedge E_{s+1}$, $E_i = E$ or $E_i = S$, that

$$(S \vee E)^{(s+1)} \cong \sigma((S \vee E)^{(s)}) \vee E^{(s+1)}$$

as a Σ_{s+1}-spectrum, the map i_s being the inclusion of the first wedge summand. By an easy inductive argument based on the universal properties of wedges and pushouts, this implies

$$F_s C(S \vee E) \cong \bigvee_{r=0}^{s} D_r(\mathcal{C}, E)$$

for all $s \geq 0$.

We also need to know that the standard equivalence $\Sigma^\infty(X^+) \simeq S \vee \Sigma^\infty X$ is unital.

Lemma 5.2. Let X be a nondegenerately based G-space. Then there are unital G-maps

$$f: \Sigma^\infty(X^+) \longrightarrow S \vee \Sigma^\infty X \quad \text{and} \quad g: S \vee \Sigma^\infty X \longrightarrow \Sigma^\infty(X^+)$$

and unital G-homotopies $h: fg \simeq 1$ and $k: gf \simeq 1$. That is, f, g, and each h_t and k_t are unital G-maps.

Proof. Since $\Sigma^\infty \cong \Lambda\Sigma^\infty\Sigma$ by I.4.2 and these functors commute with wedges, it suffices to construct such a unital G-homotopy equivalence between $\Sigma(X^+)$ and $S^1 \vee \Sigma X$, where $e: S^1 \to \Sigma(X^+)$ is the suspension of $S^0 \to X^+$ and $e: S^1 \to S^1 \vee \Sigma X$ is the inclusion of the first wedge summand. Since these maps e are G-cofibrations, standard arguments about cofibre homotopy equivalences show that it suffices to construct a G-homotopy equivalence $f: \Sigma(X^+) \to S^1 \vee \Sigma X$ such that $fe \simeq e$. The sum of the suspensions of the evident based G-maps $X^+ \to S^0$ and $X^+ \to X$ has the required properties.

Theorem 5.3. Let $\mathcal{C}$ be a coefficient system over $\mathcal{L}$ and let X be a nondegenerately based G-space. Then there is a natural equivalence of G-spectra

$$\Sigma^\infty CX \simeq \bigvee_{r\geq 1} \Sigma^\infty D_r(\mathcal{C}, X).$$

Proof. Applying Lemma 5.2 both to X and to CX and applying Proposition 3.5, Lemma 5.1, and VI.5.3, we obtain the following chain of equivalences of unital G-spectra.

$$\begin{aligned} S \vee \Sigma^\infty CX &\simeq \Sigma^\infty(CX)^+ \cong C\Sigma^\infty(X^+) \simeq C(S \vee \Sigma^\infty X) \\ &\cong \bigvee_{r\geq 0} D_r(\mathcal{C}, \Sigma^\infty X) \cong S \vee [\bigvee_{r\geq 1} \Sigma^\infty D_r(\mathcal{C}, X)]. \end{aligned}$$

We obtain the desired equivalence upon quotienting out S.

Let $j_r: \Sigma^\infty CX \to \Sigma^\infty D_r(\mathcal{C}, X)$ be the r^{th} component of this equivalence. We shall study the multiplicative properties of these "James maps". The following result was proven in [23, 4.6] for the combinatorial James maps defined in [30].

Theorem 5.4. Let $\oplus: (\mathcal{A}, \mathcal{B}) \to \mathcal{C}$ be a pairing over $\mathcal{L}$. Then the following diagram commutes in the stable category, its horizontal arrows being equivalences.

$$\begin{tikzcd}
\Sigma^\infty(AX \times BX) \arrow[r, "\sum_{r\geq 1}\sum_{p+q=r} j_p \wedge j_q"] \arrow[d, "\Sigma^\infty(\oplus)"'] & \bigvee_{r\geq 1}\bigvee_{p+q=r} \Sigma^\infty(D_p(\mathcal{A}, X) \wedge D_q(\mathcal{B}, X)) \arrow[d, "\bigvee_{r\geq 1}\bigvee_{p+q=r}\Sigma^\infty(\oplus)"] \\
\Sigma^\infty CX \arrow[r, "\sum_{r\geq 1} j_r"] & \bigvee_{r\geq 1} \Sigma^\infty D_r(\mathcal{C}, X)
\end{tikzcd}$$

Proof. The top equivalence implicitly uses the equivalences

$$S \vee \Sigma^\infty(AX \times BX) \simeq \Sigma^\infty(AX \times BX)^+ \simeq \Sigma^\infty(AX)^+ \wedge \Sigma^\infty(BX)^+ \simeq (S \vee \Sigma^\infty AX) \wedge (S \vee \Sigma^\infty BX)$$

on the left and a similar equivalence on the right. The result is a diagram chase from Lemma 4.2, Proposition 4.3, and the following diagram, whose commutativity is an easy consequence of Lemma 5.1 and the distributivity of smash products over wedges.

$$\begin{tikzcd}
A(S \vee E) \wedge B(S \vee E) \arrow[r, "\cong"] \arrow[dd, "\oplus"'] & {[\bigvee_{p\geq 0} D_p(\mathcal{A}, E)] \wedge [\bigvee_{q\geq 0} D_q(\mathcal{B}, E)]} \arrow[d, equal] \\
 & \bigvee_{r\geq 0} [\bigvee_{p+q=r} D_p(\mathcal{A}, E) \wedge D_q(\mathcal{B}, E)] \arrow[d, "\bigvee_{r\geq 0}\bigvee_{p+q=r}\oplus"] \\
C(S \vee E) \arrow[r, "\cong"] & \bigvee_{r\geq 0} D_r(\mathcal{C}, E)
\end{tikzcd}$$

Turning to operads and their actions, we obtain the following parameterized version of the previous result. We define

$$D(\mathcal{B}, X) = \bigvee_{r\geq 0} D_r(\mathcal{B}, X).$$

Theorem 5.5. Let $\mathcal{B}$ be a $\mathcal{C}$-module over $\mathcal{L}$. Then $\Sigma^\infty(BX)^+$ is naturally isomorphic as a $\mathcal{C}$-spectrum to $\Sigma^\infty D(\mathcal{B}, X)$, where the action of $\mathcal{C}$ on the G-space $D(\mathcal{B}, X)$ is obtained by passage to orbits and wedges from the maps

$$\xi\wedge 1\colon \mathcal{C}_k^+ \wedge \mathcal{B}_{j_1}^+ \wedge X^{(j_1)} \wedge \cdots \wedge \mathcal{B}_{j_k}^+ \wedge X^{(j_k)} \cong (\mathcal{C}_k \times \mathcal{B}_{j_1} \times \cdots \times \mathcal{B}_{j_k})^+ \wedge X^{(j)} \longrightarrow \mathcal{B}_j^+ \wedge X^{(j)},$$

where $j = j_1 + \cdots + j_k$.

Proof. By Propositions 4.7 and 4.8, BE is naturally a $\mathcal{C}$-spectrum for unital spectra E and $B\Sigma^\infty(X^+)$ is isomorphic as a $\mathcal{C}$-spectrum to $\Sigma^\infty(BX)^+$. As in the diagram of the previous proof, one checks that, under the isomorphism of Lemma 5.1, the action of $\mathcal{C}$ on $B(S \vee E)$ is induced by the maps

$$\zeta\colon \mathcal{C}_k \ltimes (\mathcal{B}_{j_1} \ltimes E^{(j_1)} \wedge \cdots \wedge \mathcal{B}_{j_k} \ltimes E^{(j_k)}) \longrightarrow \mathcal{B}_j \ltimes E^{(j)}$$

of Lemma 4.6. The conclusion then follows from the compatibility statement of that lemma.

Passing to adjoints from the equivalence of $\mathcal{C}$-spectra $\Sigma^\infty(BX)^+ \to \Sigma^\infty D(\mathcal{B},X)$ and throwing away the disjoint basepoint, we obtain a natural map of $\mathcal{C}$-spaces

$$j\colon BX \longrightarrow QD(\mathcal{B},X),$$

which we think of as a total James map. In practice, $\mathcal{B} = \mathcal{B}' \times \mathcal{L}$ and $\mathcal{C} = \mathcal{C}' \times \mathcal{L}$, where $\mathcal{C}'$ acts on $\mathcal{B}'$. Then j fits into the chain of $\mathcal{C}$-maps

(*) $$B'X \longleftarrow BX \longrightarrow QD(\mathcal{B},X) \longrightarrow QD(\mathcal{B}',X).$$

The first and last arrows are induced by the projection $\mathcal{B} \to \mathcal{B}'$ and are equivalences if $\mathcal{B}'$ is Σ-free (in the sense of Definition 1.2). We think of (*) too as a total James map.

Under appropriate hypotheses on the action of $\mathcal{C}'$ on $\mathcal{B}'$, a wholly different combinatorial construction of a total James map $B'X \to QD(\mathcal{B}',X)$ was obtained in [32]. As in (*), the construction involved the formal inverse of a $\mathcal{C}$-map which is an equivalence.

In the most interesting cases, the multiplicative properties of the two constructions formally imply that the resulting James maps are actually equivalent as $\mathcal{C}$-maps. To see this, assume that $\mathcal{B}' = \mathcal{C}'$ and is Σ-free. Then our James map (*) can be viewed as a pair of $\mathcal{C}$-maps

(**) $$C'X \xleftarrow{\simeq} CX \longrightarrow QD(\mathcal{C}',X),$$

the first map being induced by the projection $\mathcal{C} \to \mathcal{C}'$. As explained in and around diagram B in [32, §3] (in which $\mathcal{C}$ and $\mathcal{A}$ correspond to our $\mathcal{B}$ and $\mathcal{C}$), the James map there is equivalent as a $\mathcal{C}$-map to a diagram of the same form as (**) and with the same first map. Since CX is the free $\mathcal{C}$-space generated by X, to see that the second maps of the two diagrams are homotopic as $\mathcal{C}$-maps, it suffices

to check that their restrictions to X are homotopic as maps. It is immediate from the definitions that both restrictions are homotopic to the obvious inclusion $\iota_1\colon X \to D_1(\mathcal{C},X)$.

We reiterate that everything so far has been done with an ambient compact Lie group G acting on all spaces and spectra. Now let V be a real representation of G. As explained in detail in [62], there is a little discs operad $\mathcal{C}_V$ which acts naturally on V-fold loop G-spaces $\Omega^V X$. The natural map $X \to \Omega^V\Sigma^V X$ and the action map of $\Omega^V\Sigma^V X$ give rise to the composite

$$\alpha_V\colon C_V X \longrightarrow C_V\Omega^V\Sigma^V X \longrightarrow \Omega^V\Sigma^V X,$$

and α_V is a $\mathcal{C}_V$-map. If the universe U is written as $\sum_i V_i$, then the little discs in the V_i give rise via products to a little cylinders operad $\mathcal{C}_\infty$ which acts naturally on infinite loop G-spaces. There results a natural $\mathcal{C}_\infty$-map

$$\alpha_\infty\colon C_\infty X \longrightarrow \Omega^\infty\Sigma^\infty X = QX.$$

Results of Hauschild [61] and Segal [127] have the following implication.

Theorem 5.6. Assume that G is finite and let X be G-connected. If V contains a trivial summand, then α_V is a weak G-equivalence; α_∞ is also a weak G-equivalence.

These assertions are false for general compact Lie groups G; see Caruso and Waner [25] for the present state of the art.

By Theorem 5.3, we have the following immediate consequence, which is an equivariant generalization of Snaith's splitting [129].

Theorem 5.7. Assume that G is finite and let X be G-connected. Then there are natural equivalences of G-spectra

$$\Sigma^\infty\Omega^V\Sigma^V X \simeq \bigvee_{r\geq 1} \Sigma^\infty D_r(\mathcal{C}_V,X),$$

where V is assumed to contain a trivial summand, and

$$\Sigma^\infty QX \simeq \bigvee_{r\geq 1} \Sigma^\infty D_r(\mathcal{C}_\infty,X).$$

Of course, our results on operad actions transpose directly to these splittings.

VIII. The homological analysis of extended powers

by J. P. May and M. Steinberger

Here we return to the classical nonequivariant context of $[H_\infty]$ in order to generalize to spectra the standard homological facts about extended powers of spaces.

We gather together various preliminaries on cellular chains and the spectral sequences associated to filtered and bifiltered spectra in section 1. In particular, we show that cellular theory is invariant with respect to changes of universe and compute the cellular chains of external and internal smash products of CW spectra.

In section 2, we use this material and the constructions of chapter VI to study spectral sequences associated to filtrations of extended powers and, in particular, to compute the cellular chains of extended powers.

This material completes the proofs of all results promised in $[H_\infty$, I and III$]$ except the Nishida relations, and these follow from the computation of the Steenrod operations in the mod p homology of D_pE which we give in section 3.

§1. Cellular chains and filtered spectra

Let $\mathcal{A}$ be an indexing set in a universe U, as in I§2. We assume given a canonical copy of R^∞ in U such that its subspaces R^n are in $\mathcal{A}$. We are thinking of $U = (R^\infty)^j$ for any $j > 0$, with R^∞ being the subspace spanned by the vectors $(e_i,\cdots,e_i)$. Thus R^∞ will be Σ_j-invariant when we consider questions of equivariance in the next section.

We quickly review the definition of cellular homology. Let $E \in \mathcal{S}U$ be a CW spectrum. We define

$$C_nE = \pi_n(E^n/E^{n-1})$$

and define $d\colon C_nE \to C_{n-1}E$ to be

$$\Sigma_*^{-1} \circ \pi_n(\partial)\colon \pi_n(E^n/E^{n-1}) \longrightarrow \pi_n(\Sigma(E^{n-1}/E^{n-2})) \longrightarrow \pi_{n-1}(E^{n-1}/E^{n-2}).$$

Here ∂ is the canonical geometric boundary map

$$E^n/E^{n-1} \simeq C(i_{n-1}) \longrightarrow \Sigma E^{n-1} \longrightarrow \Sigma(E^{n-1}/E^{n-1}),$$

where $i_{n-1}\colon E^{n-1} \to E^n$ is the inclusion and $C(i_{n-1})$ is its cofibre. We then

define $H_nE = H_n(C_*E)$ and note that the cellular approximation theorem gives the requisite invariance. If we happen to know some other definition of homology, then we easily see that this definition agrees with it. Compatibility with the usual space level definition is clear by suspension and the following immediate consequence of I.5.4. Recall the functors $\Lambda^n\Sigma^\infty$ from I.4.1; the Λ^n are the shift desuspensions by the canonical indexing spaces R^n. We also have the reindexing functors Λ^n on chain complexes specified by $(\Lambda^nC)_q = C_{q+n}$.

<u>Proposition 1.1</u>. Let X be a based CW complex (with $X^0 = \{*\}$ and based attaching maps). Then $C_*(\Lambda^n\Sigma^\infty X)$ is canonically naturally isomorphic to $\Lambda^n\tilde{C}_*X$, where $\tilde{C}_*X$ denotes the reduced cellular chains of X.

Returning to CW spectra, we note the obvious fact that the utility of cellular chains comes from the identification of C_nE with the free abelian group generated by the n-cells of E. We shall make considerable use of the observation that not all such chain complexes arise from skeletal filtrations. Given any CW spectrum filtered by subcomplexes E_n, we obtain a chain complex C_*E exactly as above, and the construction is functorial on filtration preserving maps. In particular, if E is a CW spectrum and X is a CW complex, we may filter $X^+\wedge E$ by its subspectra $X^+\wedge E^n$. If X is contractible, then the projection $X^+ \to S^0$ and the inclusion $S^0 \to X^+$ induced by any choice of basepoint in X induce inverse isomorphisms of chain complexes between $C_*(X^+\wedge E)$ and C_*E. As a first example of how we shall exploit this observation, we use it to study the effect of change of universe functors on cellular chains. Let $\mathcal{A}'$ be an indexing set in a second universe U'.

<u>Proposition 1.2</u>. Let $f\colon U \to U'$ be a linear isometry. For a CW spectrum $E \in \mathcal{S}\mathcal{A}$, there is a canonical natural isomorphism $C_*E \cong C_*(f_*E)$, where $f_*\colon \mathcal{S}\mathcal{A} \to \mathcal{S}\mathcal{A}'$ is the change of universe functor induced by f. If $g\colon U \to U'$ is another linear isometry and $h\colon I \to \mathcal{J}(U,U')$ is a path from f to g, then the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
 & C_*E \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\
C_*(f_*E) \arrow[dr, "(i_0\times 1)_*"'] & & C_*(g_*E) \arrow[dl, "(i_1\times 1)_*"] \\
 & C_*(h \ltimes E) &
\end{tikzcd}$$

<u>Proof</u>. The functor f_* preserves CW spectra by II.1.4 and we have

$$C_n(f_*E) = \pi_n(f_*E^n/f_*E^{n-1}) = \pi_n(f_*(E^n/E^{n-1})) = [f^*S^n, E^n/E^{n-1}].$$

Here f^* is left as well as right adjoint to f_* because these are inverse equivalences of stable categories (see I.2.7). Thus, for the first part, it suffices to construct a canonical family of equivalences $\phi_n\colon S^n \to f^*S^n$ in $\bar{h}\mathscr{A}\mathcal{a}$ such that the following diagrams commute:

$$\begin{tikzcd} \Sigma S^n \arrow[r, "\Sigma\phi_n"] \arrow[d, phantom, "\cong" rotate=90] & \Sigma f^*S^n \arrow[d, phantom, "\cong" rotate=90] \\ S^{n+1} \arrow[r, "\phi_{n+1}"] & f^*\Sigma S^n. \end{tikzcd}$$

Here $\Sigma S^n \cong S^{n+1}$ by I.4.3 and $f^*\Sigma \cong \Sigma f^*$ since $f^*\Omega \cong \Omega f^*$ by inspection and since Σ and Ω are inverse equivalences of stable categories (see I.6.1). Since Σ is an equivalence of categories, the entire family is uniquely determined by ϕ_0 and induction up and down. Since $f(0) = 0$, we have $\Omega^\infty f^*S = \Omega^\infty S$ and thus

$$\pi_0(f^*S) = \pi_0(\Omega^\infty f^*S) = \pi_0(\Omega^\infty S) = [S^0, QS^0].$$

The natural inclusion $\eta\colon S^0 \to QS^0$ pulls back along these equalities to the desired canonical equivalence $\phi_0\colon S \to f^*S$. For the second statement, filter $h \ltimes E$ by its subcomplexes $h \ltimes E^n$. Then, by II.1.7,

$$h \ltimes E^n/h \ltimes E^{n-1} \cong h \ltimes (E^n/E^{n-1})$$

is equivalent via $i_0 \ltimes 1$ to $f_*(E^n/E^{n-1})$ and via $i_1 \ltimes 1$ to $g_*(E^n/E^{n-1})$. Thus $(i_0 \ltimes 1)_*$ and $(i_1 \ltimes 1)_*$ are certainly isomorphisms. The commutativity of the diagram is easily checked by directly identifying $C_*(h \ltimes E)$ with C_*E by mimicry of the argument just given, with f^* replaced by the right and left adjoint $F[h,?)$ of $h \ltimes (?)$. Here, by inspection (see VI.1.5),

$$\Omega^\infty F[h,S) = F(I^+, \Omega^\infty S) = F(I^+, QS^0),$$

and we let $\phi_0\colon S \to F[h,S)$ be the pullback of the map $S^0 \longrightarrow F(I^+, QS^0)$ with adjoint the composite of $\eta\colon S^0 \to QS^0$ and the projection $I^+ \longrightarrow S^0$.

We next prove the expected behavior of cellular chains of external smash products.

Proposition 1.3. For CW spectra $E \in \mathscr{A}G\mathcal{a}$ and $F \in \mathscr{A}G\mathcal{a}'$, there is a canonical natural isomorphism

$$\kappa\colon C_*E \otimes C_*F \longrightarrow C_*(E \wedge F).$$

Proof. Write $D = E \wedge F$. Then D is a CW spectrum by II.3.8, and we have

$$D^n/D^{n-1} = \bigvee_{p+q=n} (E^p/E^{p-1}) \wedge (F^q/F^{q-1}).$$

By easy comparisons of cofibre sequences, the $(p,q)^{\text{th}}$ restriction of the boundary map $\partial\colon D^n/D^{n-1} \to \Sigma(D^{n-1}/D^{n-2})$ is the wedge sum of the maps

$$\partial'\colon E^p/E^{p-1} \wedge F^q/F^{q-1} \xrightarrow{\partial\wedge 1} \Sigma(E^{p-1}/E^{p-2}) \wedge F^q/F^{q-1} \xrightarrow{1\wedge t} \Sigma(E^{p-1}/E^{p-2} \wedge F^q/F^{q-1})$$

and

$$\partial''\colon E^p/E^{p-1} \wedge F^q/F^{q-1} \xrightarrow{1\wedge\partial} E^p/E^{p-1} \wedge \Sigma(F^{q-1}/F^{q-2}) = \Sigma(E^p/E^{p-1} \wedge F^{q-1}/F^{q-2}).$$

In ∂', t permutes the suspension coordinate S^1 past F^q/F^{q-1}. The switch map enters here and not in ∂'' since we are writing suspension coordinates on the right. Note next that there are three sets of sphere spectra in sight, in $\mathscr{A}\mathcal{a}$, $\mathscr{A}\mathcal{a}'$, and $\mathscr{A}(\mathcal{a} \oplus \mathcal{a}')$. These are related by a canonical system of equivalences $\psi_{p,q}\colon S^p \wedge S^q \to S^{p+q}$ such that the left square of the following diagram commutes up to the sign $(-1)^q$ and the right square commutes:

$$\begin{tikzcd} \Sigma S^{p-1} \wedge S^q \arrow[r, "1\wedge t"] \arrow[d, phantom, "\cong" rotate=90] & \Sigma(S^{p-1} \wedge S^q) \arrow[r, "\Sigma\psi_{p-1,q}"] & \Sigma S^{p+q-1} \arrow[d, phantom, "\cong" rotate=90] & \Sigma(S^p \wedge S^{q-1}) \arrow[l, "\Sigma\psi_{p,q-1}"'] = S^p \wedge \Sigma S^{q-1} \arrow[d, phantom, "\cong" rotate=90] \\ S^p \wedge S^q \arrow[rr, "\psi_{p,q}"] & & S^{p+q} & S^p \wedge S^q \arrow[l, "\psi_{p,q}"'] \end{tikzcd}$$

Indeed, we specify $\psi_{0,0}\colon S \wedge S \to S$ to be the equivalence obtained by pulling back $\eta\colon S^0 \to QS^0$ along the chain of equalities

$$[S\ \ S, S] = [S, F(S,S)] = [S^0, F(S,S)_0] = [S^0, F(S^0, QS^0)] = [S^0, QS^0],$$

the identification $F(S,S)_0 = F(S^0, QS^0)$ following directly from the definition of function spectra in II.3.3. We then specify the $\psi_{p,0}$ by the left square and induction up and down and finally specify the $\psi_{p,q}$ by the right square and induction up and down. Easy chases show that both diagrams commute and that the same $\psi_{p,q}$ are obtained by first defining the $\psi_{0,q}$ and then the $\psi_{p,q}$. Now define

$$\kappa\colon C_pE \otimes C_qE \longrightarrow C_{p+q}(E \wedge F)$$

to be $(-1)^{pq}$ times the composite

$$[S^p, E^p/E^{p-1}]\otimes[S^q, F^q/F^{q-1}] \xrightarrow{\wedge} [S^p\wedge S^q, E^p/E^{p-1}\wedge F^q/F^{q-1}] \xrightarrow{[\psi_{p,q},\iota]} [S^{p+q}, D^{p+q}/D^{p+q-1}],$$

where ι is the inclusion. As should be but isn't well known, introduction of a sign here is the price one pays for writing suspension coordinates on the right. With this definition, easy chases from the information given imply that κ carries the differential $d\otimes 1 + 1\otimes d$ (with the standard sign convention) to the cellular differential.

We shall need a consistency statement relating the previous two results and the simplest cases of the isomorphisms α and β of VI.3.1.

Proposition 1.4. (i) Let $f_1\colon U_1 \to U_1'$ and $f_2\colon U_2 \to U_2'$ be linear isometries and let E_1 and E_2 be CW spectra in $\mathscr{S}\mathcal{U}_1$ and $\mathscr{S}\mathcal{U}_2$. Then the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
C_*(f_{1*}E_1)\otimes C_*(f_{2*}E_2) \arrow[r, "\kappa"] \arrow[d, phantom, "\cong" rotate=90] & C_*(f_{1*}E_1\wedge f_{2*}E_2) \arrow[r, "\alpha_*"] & C_*((f_1\oplus f_2)_*(E_1\wedge E_2)) \arrow[d, phantom, "\cong" rotate=90] \\
C_*(E_1)\otimes C_*(E_2) \arrow[rr, "\kappa"] & & C_*(E_1\wedge E_2)
\end{tikzcd}$$

(ii) Let $f\colon U \to U'$ and $f'\colon U' \to U''$ be linear isometries and let E be a CW spectrum in $\mathscr{S}\mathcal{U}$. Then the following diagram of isomorphisms commutes:

$$\begin{tikzcd}
C_*(f'_*f_*E) \arrow[r, "\beta"] \arrow[d, phantom, "\cong" rotate=90] & C_*((f'\circ f)_*(E)) \arrow[d, phantom, "\cong" rotate=90] \\
C_*(f_*E) \arrow[r, phantom, "\cong"] & C_*E
\end{tikzcd}$$

Proof. By the previous two proofs, these quickly reduce to simple verifications of the compatibility of certain canonical equivalences of zero sphere spectra.

A proof similar to that of Proposition 1.3 gives the following simpler analog.

Proposition 1.5. For CW spectra E and based CW complexes X, there is a canonical natural isomorphism

$$\kappa\colon C_*E\otimes\tilde{C}_*X \longrightarrow C_*(E\wedge X).$$

We put things together to deduce the behavior of internal smash products with respect to cellular chains. The conclusions are what one would expect.

Theorem 1.6. Consider the internal smash product in $\bar{h}\mathscr{S}\mathcal{U}$. Let D, E, F be CW spectra in $\mathscr{S}\mathcal{U}$ and let X be a based CW complex. Then there is a canonical natural isomorphism

$$\kappa\colon C_*E\otimes C_*F \longrightarrow C_*(E\wedge F).$$

Under this isomorphism the natural commutativity and associativity equivalences

$$\tau\colon E\wedge F \simeq F\wedge E \quad\text{and}\quad D\wedge(E\wedge F)\simeq(D\wedge E)\wedge F$$

induce the natural twist isomorphism $\tau\colon C_*E\otimes C_*F \cong C_*F\otimes C_*E$ (with the standard sign) and the natural associativity isomorphism. The unit equivalence $E\wedge X \simeq E\wedge\Sigma^\infty X$ induces the composite natural isomorphism

$$C_*(E\wedge X) \xrightarrow{\kappa^{-1}} C_*E\otimes\tilde{C}_*X \cong C_*E\otimes C_*\Sigma^\infty X \xrightarrow{\kappa} C_*(E\wedge\Sigma^\infty X).$$

Proof. The internal smash product is the composite of the external smash product and a change of universe functor, and κ here is the composite of isomorphisms given in Propositions 1.2 and 1.3. The equivalences for the rest are defined in II.3.12. They result from paths connecting different linear isometries, isomorphisms β, and explicit spectrum level commutativity, associativity, and unit isomorphisms. In view of Propositions 1.2 and 1.4, we need only inspect these last isomorphisms. The commutativity is given by the transposition isomorphism $t_*(E\wedge F) \cong F\wedge E$. When $E = S^p$ and $F = S^q$ and the domain and range are identified with S^{p+q} as in the proofs of Propositions 1.2 and 1.3, this is seen to be a map of degree $(-1)^{pq}$. The associativity is similar. For the unit, one checks that the isomorphism $i_*(E\wedge X) \cong E\wedge\Sigma^\infty X$, $i\colon U \subset U\oplus U$, becomes the identity map when $E = S^p$ and $X = S^q$ and the domain and range are identified with S^{p+q} as in the cited proofs.

We shall prove analogs of these results for extended powers in the next section. A general discussion of filtered and bifiltered spectra will help clarify the arguments.

A CW spectrum E filtered by subcomplexes E_n gives rise to an exact triangle of spectra via the natural boundary map $\partial\colon E_n/E_{n-1} \to \Sigma(E_{n-1}/E_{n-2})$. Application of a given generalized homology theory k_* yields an exact couple and thus a spectral sequence $\{E^r(E;k)\}$ with initial term

$$E^1_{s,t}(E;k) = k_{s+t}(E_s/E_{s-1})$$

and with d^1 induced by ∂. When E_s is the s-skeleton of E,

$$k_{s+t}(E_s/E_{s-1}) = C_s(E;k_t(S))$$

and this is the classical (Dold-Atiyah-Hirzebruch) spectral sequence. We shall be interested in the filtered CW spectra associated to bifiltered CW spectra.

Definition 1.7. A bifiltered CW spectrum is a CW spectrum E with subcomplexes $E_{p,q}$ for integers p and q such that $E_{p-1,q}$ and $E_{p,q-1}$ are subcomplexes of $E_{p,q}$ with intersection $E_{p-1,q-1}$. The associated filtration on E is specified by

$$E_s = \bigcup_{p+q=s} E_{p,q}.$$

In practice, we usually have $E_{p,q} = *$ for $p < 0$ and all q.

Lemma 1.8. Let E be a bifiltered CW spectrum and define

$$E^0_{p,q} = E_{p,q}/E_{p-1,q} \cup E_{p,q-1}.$$

Then there are natural identifications as follows.

(i) $$E_s/E_{s-1} = \bigvee_{p+q=s} E^0_{p,q}$$

(ii) $$(E_{p-1,q} \cup E_{p,q-1})/(E_{p-2,q} \cup E_{p-1,q-1} \cup E_{p,q-2}) = E^0_{p-1,q} \cup E^0_{p,q-1}.$$

(iii) $$(E_{p,q}/E_{p,q-1})/(E_{p-1,q}/E_{p-1,q-1}) = E^0_{p,q} = (E_{p,q}/E_{p-1,q})/(E_{p,q-1}/E_{p-1,q-1}).$$

Moreover, the boundary map $\partial\colon E_s/E_{s-1} \to \Sigma(E_{s-1}/E_{s-2})$ restricts on the wedge summand $E^0_{p,q}$ to the wedge sum of the boundary maps

$$\partial'\colon E^0_{p,q} \longrightarrow \Sigma E^0_{p-1,q} \quad \text{and} \quad \partial''\colon E^0_{p,q} \longrightarrow \Sigma E^0_{p,q-1}$$

associated to the filtered CW spectra

$$\bigcup_p E_{p,q}/E_{p,q-1} \quad \text{and} \quad \bigcup_q E_{p,q}/E_{p-1,q}.$$

Proof. The definition implies a pushout diagram

$$\begin{array}{ccc} \bigvee_{p+q=s} E_{p-1,q} \cup E_{p,q-1} & \longrightarrow & E_{s-1}, \\ \downarrow & & \downarrow \\ \bigvee_{p+q=s} E_{p,q} & \longrightarrow & E_s \end{array}$$

and this implies (i). For any CW pairs (A,C) and (B,D), we have a pushout diagram

$$\begin{array}{ccc} [C \cup (A \vee B)] \cap [(A \cap B) \cup D)] & \longrightarrow & C \cup (A \cap B) \cup D. \\ \downarrow & & \downarrow \\ A \vee B & \longrightarrow & A \cup B \end{array}$$

This implies (ii), and (iii) is clear. The last statement follows from these identifications by an immediate comparison of cofibre sequences.

Of course, our earlier discussion of external smash products is a special case. We single out the following special case for later reference.

Proposition 1.9. Let W be a CW complex with its skeletal filtration, let F be any filtered CW spectrum, and give $W^+ \wedge F$ the induced bifiltration. Then

$$E^1_{s,t}(W^+ \wedge F;k) = \sum_{p+q=s} C_pW \otimes E^1_{q,t}(F;k)$$

and the differential d^1 on the left agrees under the isomorphism with the differential $d \otimes 1 + 1 \otimes d^1$ on the right.

As in the proof of Proposition 1.3, the relevant Kunneth map is defined with a sign in order to make the signs work out in the comparison of differentials. When F is given its skeletal filtration, the associated filtration of $W^+ \wedge F$ is the skeletal filtration of a CW structure by inspection of smash products of cells (as in the proofs of II.3.7 and 3.8).

§2. Spectral sequences and cellular chains of extended powers

It is convenient to adopt the following abbreviated notations for our basic categories of spectra.

Notations 2.1. Let $\mathcal{S} = \mathcal{S}R^\infty$ and, for $j > 0$, let $\mathcal{S}_j = \mathcal{S}R^{\infty j}$. Here $\mathcal{S}_0 = \mathcal{J}$ by convention. Observe that the graded category $\coprod_{j \geq 0} \mathcal{S}_j$ is the natural domain of definition for external smash products. For $\pi \subset \Sigma_j$, let $\pi\mathcal{S}_j$ denote the category of π-spectra in $\mathcal{S}_j$ and let $\mathcal{L}_j$ denote the right π-space $\mathcal{J}(R^{\infty j}, R^\infty)$. With π acting trivially on R^∞, $\pi\mathcal{S}$ is also defined.

Of course, $\overline{h}\mathcal{S}$ is the canonical stable category in which all of the work of $[H_\infty]$ took place.

We revert to the notation of $[H_\infty$, I§1$]$ and start with a free π-CW complex W and a π-map $\omega\colon W \to \mathcal{L}_j$, where $\pi \subset \Sigma_j$. Let $E \in \mathcal{S}$ be a CW spectrum. We shall analyze the cellular chains of $\omega \ltimes_\pi E^{(j)}$ and the spectral sequences associated to filtrations of $\omega \ltimes_\pi E^{(j)}$ induced from π-cellular filtrations of $E^{(j)}$. For generality and notational simplicity, we work with more general π-spectra $F \in \pi\mathcal{S}_j$. We must first discuss cell structures on such spectra since examples like $E^{(j)}$ are not π-CW spectra. The following definition specifies the structure they do have.

Definition 2.2. Let $F \in \pi\mathcal{S}_j$ be a π-spectrum and a CW spectrum with sequential filtration $\{F_n\}$, where $F_0 = *$ and $F_{n+1} = Cj_n$ for $j_n\colon J_n \to F_n$. Then F is said to be a CW spectrum with cellular π action if the following conditions hold.

(i) Each F_n is a π-subspectrum of F.

(ii) Each J_n is a wedge of π-spectra of the form $\pi \ltimes_\rho S^q$, where S^q is the q-sphere spectrum with some action by $\rho \subset \pi$ with respect to which it has the ρ-homotopy type of a ρ-CW spectrum.

(iii) Each j_n is a wedge of π-maps $\pi \ltimes_\rho S^q \to F_n$.

It follows that the skeletal filtration $\{F^n\}$ is also given by π-subspectra. Explicitly, F^{n+1} is the cofibre of the wedge over all n-sphere orbits in all J_m of the attaching maps $\pi \ltimes_\rho S^n \to F^n$.

If F is a π-CW spectrum, then S^q in (ii) is restricted to be the canonical sphere π-spectrum regarded as a ρ-spectrum, so that, by II.4.8, $\pi \ltimes_\rho S^q \cong (\pi/\rho)^+ \wedge S^q$.

Of course, we have an analogous space level dichotomy between π-CW complexes and CW complexes with cellular π action. Here if ρ acts simplicially or differentiably on S^q, then S^q can be subdivided to a ρ-CW complex and the distinction becomes negligible.

The following observation may help explain the force of the definition. It is utterly trivial on the space level. Recall that $\pi \ltimes J = \bigvee_{\sigma\in\pi} \sigma_* J$ and that π actions are given by maps $\xi\colon \pi \ltimes J \to J$ (see II.4.15).

Lemma 2.3. Suppose given a π-spectrum $J \in \pi\mathcal{S}_j$ which is nonequivariantly isomorphic to a wedge of sphere spectra S^q (without action) and contains no non-trivial π-subspectra. Let $\rho \subset \pi$ be the subgroup consisting of those elements which map one of the wedge summands, S^q_ρ say, to itself. Then the π-map $\pi \ltimes_\rho S^q_\rho \to J$ which restricts to the inclusion on S^q_ρ is an isomorphism.

Proof. The maps $\xi_\sigma\colon \sigma_* J \to J$ satisfy $\xi_e = 1$ and $\xi_\sigma \sigma_*\xi_\tau = \xi_{\sigma\tau}$ (see II.4.15). Taking $\tau = \sigma^{-1}$, we find that the composite

$$J = \sigma_*\sigma_*^{-1} J \xrightarrow{\sigma_*\xi_{\sigma^{-1}}} \sigma_* J \xrightarrow{\xi_\sigma} J$$

is the identity. Now $\sigma_* J$ is a wedge of copies of $\sigma_* S^q$ and $\sigma_* S^q$ is isomorphic to S^q. Unraveling our construction of sphere spectra, wedges and maps (see I§§2,4), we see that a map of spectra $S^q \to S^q$ is represented on the space level by a map $S^r \to \vee S^r$ for some suitably large r. The composites above display each wedge summand of J as a retract of $\sigma_* J$. This carries over to the space level, and it follows easily that each action map $\xi_\sigma\colon \sigma_* J \to J$ is a wedge of isomorphisms. If σ runs through a set of coset representatives for ρ in π, then $\pi \ltimes_\rho S^q_\rho$ is isomorphic to the wedge over σ of the spectra $\sigma_* S^q_\rho$ and J is the wedge over σ of the sphere spectra $\xi_\sigma(\sigma_* S^q_\rho)$.

Note in particular that J is a free π-spectrum $\pi \ltimes S^q$ if no non-identity element of π maps a wedge summand to itself. This fact and our assumption that W is π-free lead to the strong conclusion in part (ii) of the following result.

Proposition 2.4. (i) Let $E \in \mathcal{S}$ be a CW spectrum. Then $E^{(j)} \in \pi\mathcal{S}_j$ is a CW spectrum with cellular π action for $\pi \subset \Sigma_j$.

(ii) Let $F \in \pi\mathcal{S}_j$ be a CW spectrum with cellular π action. Then $\omega \ltimes F \in \pi\mathcal{S}$ is a free π-CW spectrum.

(iii) Let $D \in \pi\mathcal{S}$ be a π-CW spectrum. Then $D/\pi \in \mathcal{S}$ is a CW spectrum.

Proof. Part (i) is immediate from the proof of VI.5.2 and part (ii) follows easily from the proof of VI.1.11 and the observation above. Part (iii) is a special case of I.5.6; note for the free case that $\pi \ltimes E/\pi = E$ by II.4.15 (iv).

Corollary 2.5. Let $\{W_p\}$ and $\{F_q\}$ be filtrations by π-subcomplexes of the free π-CW complex W and the CW spectrum with cellular π action $F \in \pi\mathcal{S}_j$. Then $\omega \ltimes_\pi F$ is a bifiltered CW-spectrum with respect to $\{\omega_p \ltimes_\pi F_q\}$, where ω_p is the restriction of ω to W_p.

Proof. By the naturality of the cell structures in the proposition, we need only observe that the intersection condition in Definition 1.7 holds before passage to orbits by VI.1.2 and still holds after passage to orbits by inspection of the cell structures.

We need the following notations to describe the behavior of the resulting spectral sequences.

Notations 2.6. Let k_* be any homology theory defined on the stable category $\bar{h}\mathcal{S}$. Fix a linear isometry $f_j \in \mathcal{I}_j$ and extend k_* to the stable category $\bar{h}\mathcal{S}_j$ by the definition $k_*F = k_*(f_{j*}F)$ for $F \in \mathcal{S}_j$. By II.1.7, k_*F is independent of the choice of f_j up to canonical natural isomorphism. If F is filtered, let $\{E^r(F,k)\}$ denote the spectral sequence obtained from the induced filtration on $f_{j*}F$.

With these notations, the basic calculational results on twisted half smash products read as follows. Their proofs will include discussions of the relevant algebraic π actions.

Theorem 2.7. Let $\pi \subset \Sigma_j$, let W be a free π-CW complex with its skeletal filtration $\{W^p\}$, let $\omega\colon W \to \mathcal{L}_j$ be a π-map with restriction ω_p to W^p, and let $F \in \pi\mathcal{S}_j$ be a CW spectrum with cellular π action and any filtration $\{F_q\}$ by π-subcomplexes. Then

$$E^1_{s,t}(W \ltimes_\pi F;k) \cong \sum_{p+q=s} C_pW \otimes_\pi E^1_{q,t}(F;k)$$

in the resulting spectral sequence, and d^1 on the left agrees under the isomorphism with $d \otimes 1 + 1 \otimes d^1$ on the right. Moreover, if $F = \Sigma^\infty X$ for a filtered based CW complex X with cellular π action, then the isomorphism of spectra

$$\Sigma^\infty(W \ltimes_\pi X) \cong W \ltimes_\pi \Sigma^\infty X$$

induces a natural isomorphism of spectral sequences which, on the E^1-level, is the natural identification

$$C_*W \otimes_\pi \tilde{E}^1_{**}(X;k) = C_*W \otimes_\pi E^1_{**}(\Sigma^\infty X;k).$$

Retaining the hypotheses of the theorem but deleting k from the notation, we have the following complement concerning the maps ι, α, β, and δ of VI.3.1 and 3.4. We assume given appropriate cellular structures and filtrations on all spectra in sight.

Theorem 2.8. (i) Let $\omega(w) = f_j$ and let $i\colon \{w\} \to W$ be the inclusion. Then the following diagram commutes for $F \in \pi\mathcal{S}_j$:

$$\begin{tikzcd}
E^1_{**}(F) \arrow[r, "\iota"] \arrow[d, equal] & E^1_{**}(\omega \ltimes_\pi F) \arrow[d, "\cong"] \\
C_0\{w\} \otimes E^1_{**}(F) \arrow[r, "i_* \otimes 1"] & C_*W \otimes_\pi E^1_{**}(F)
\end{tikzcd}$$

(ii) Let V be a free ρ-CW complex, $\rho \subset \Sigma_k$, and let $\nu\colon V \to \mathcal{L}_k$ be a ρ-map. Then the following diagram commutes for $F_j \in \pi\mathcal{S}_j$ and $F_k \in \rho\mathcal{S}_k$:

$$\begin{tikzcd}
E^1_{**}(\omega \ltimes_\pi F_j) \otimes E^1_{**}(\nu \ltimes_\rho F_k) \arrow[r, "\alpha_* \circ \kappa"] \arrow[d, "\cong"] & E^1_{**}((\omega \oplus \nu) \ltimes_{\pi\times\rho} (F_j \wedge F_k)) \arrow[d, "\cong"] \\
C_*W \otimes_\pi E^1_{**}F_j \otimes C_*V \otimes_\rho E^1_{**}F_k \arrow[r, "(\kappa\otimes\kappa)(1\otimes\tau\otimes 1)"] & C_*(W \times V) \otimes_{\pi\times\rho} E^1_{**}(F_j \wedge F_k)
\end{tikzcd}$$

(iii) Let $\omega^k\colon W^k \to \mathcal{I}(R^{\infty jk}, R^{\infty k})$ be the k-fold direct sum of ω and let ν be as in (ii). Then the following diagram commutes for $F_{jk} \in (\rho\int\pi)\mathcal{S}_{jk}$:

$$\begin{tikzcd}
E^1_{**}(\nu \ltimes_\rho \omega^k \ltimes_{\pi^k} F_{jk}) \arrow[r, "\beta_*"] \arrow[d, "\cong"] & E^1_{**}((\nu \circ \omega^k) \ltimes_{\rho\int\pi} F_{jk}) \arrow[d, "\cong"] \\
C_*V \otimes_\rho C_*(W^k) \otimes_{\pi^k} E^1_{**}F_{jk} \arrow[r, "\kappa\otimes 1"] & C_*(V \times W^k) \otimes_{\rho\int\pi} E^1_{**}F_{jk}
\end{tikzcd}$$

(iv) Let $g_{j,2} = f^j_2 \circ \nu_j\colon R^{\infty 2j} \to R^{\infty j}$, where $\nu_j\colon R^{\infty j} \oplus R^{\infty j} \to (R^\infty \oplus R^\infty)^j$ is the evident shuffle, and let $\Delta'\colon W \to W \times W$ be any π-cellular approximation to the diagonal map. Then the following diagram commutes for $F, F' \in \pi\mathcal{S}_j$:

$$\begin{tikzcd}
E^1_{**}(\omega \ltimes_\pi g_{j,2*}(F \wedge F')) \arrow[rr, "\delta_*"] \arrow[d, "\cong"] & & E^1_{**}[(\omega \ltimes_\pi F) \wedge (\omega \ltimes_\pi F')] \\
C_*W \otimes_\pi E^1_{**}(F \wedge F') & & E^1_{**}(\omega \ltimes_\pi F) \otimes E^1_{**}(\omega \ltimes_\pi F') \arrow[u, "\kappa"] \\
C_*W \otimes_\pi (E^1_{**}F \otimes E^1_{**}F') \arrow[u, "1\times\kappa"] \arrow[rr, "(1\otimes\tau\otimes 1)(\kappa^{-1}\Delta'_*\otimes 1\otimes 1)"] & & (C_*W \otimes_\pi E^1_{**}F) \otimes (C_*W \otimes_\pi E^1_{**}F') \arrow[u, "\cong"]
\end{tikzcd}$$

As will be discussed in the proofs, these diagrams require a bit of interpretation to account for changes of isometries.

When k_* is ordinary integral homology and F is given its skeletal filtration, we have

$$E^1_{**}(F) = E^1_{*,0}(F) = C_*(f_{j*}F) \cong C_*F.$$

Thus the theorems specialize to give the following result, in which we retain their notations and hypotheses. Actually, Propositions 1.2 and 1.4 are also needed here because of the changes of isometries implicit in the diagrams above.

Corollary 2.9. There is a natural isomorphism

$$C_*(W \ltimes_\pi F) \cong C_*W \otimes_\pi C_*F.$$

When $F = \Sigma^\infty X$, this isomorphism is the evident composite

$$C_*(W \ltimes_\pi F) \cong \tilde{C}_*(W^+ \wedge_\pi X) \cong C_*W \otimes_\pi \tilde{C}_*X \cong C_*W \otimes_\pi C_*F.$$

Under this isomorphism and Kunneth isomorphisms, ι_*, α_*, β_*, and δ_* coincide with the natural homomorphisms

$$i_* \otimes 1\colon C_*F = C_0\{w\} \otimes C_*F \longrightarrow C_*W \otimes_\pi C_*F$$

$$1 \otimes \tau \otimes 1\colon (C_*W \otimes_\pi C_*F_j) \otimes (C_*W \otimes_\rho C_*F_k) \longrightarrow (C_*W \otimes C_*V) \otimes_{\pi\times\rho} (C_*F_j \otimes C_*F_k)$$

$$\kappa \otimes 1\colon C_*V \otimes_\rho C_*(W^k) \otimes_{\pi^k} C_*F_{jk} \longrightarrow C_*(V \otimes W^k) \otimes_{\rho\int\pi} C_*F_{jk}$$

and

$$(1 \otimes \tau \otimes 1)(\kappa^{-1}\Delta' \otimes 1 \otimes 1)\colon C_*W \otimes_\pi (C_*F \otimes C_*F') \longrightarrow (C_*W \otimes_\pi C_*F) \otimes (C_*W \otimes_\pi C_*F')$$

Remarks 2.10. (i) When $F = E^{(j)}$, $f_{j*}F$ is the internal j-fold smash power of E. The alternative notation F_j used in some of our statements is meant to suggest this central case. By Theorem 1.6 and its proof, we may identify $C_*(E^{(j)})$ with the j-fold tensor power $C_*(E)^j$ with its natural π action.
(ii) Together with VI.1.2 and VI.4.9, Corollary 2.9 completes the proof of $[H_\infty$, I.1.3]. It also completes the proof of all results of $[H_\infty$, III] other than the Nishida relations.
(iii) When W is contractible and F is given the trivial filtration, $f_q = *$ for $q < 0$ and $F_q = F$ for $q \geq 0$, we clearly have

$$E^2_{s,t}(W \ltimes_\pi F;k) = H_s(\pi;k_rF).$$

This completes the proof of $[H_\infty$, I.2.4].
(iv) When E is given the filtration by a k_* Adams resolution and $E^{(j)}$ is given the induced filtration, Theorem 2.7 is the essential starting point for Bruner's work on the Adams spectral sequence; see $[H_\infty$, IV§5].
(v) If $\mathcal{C}$ is a cellular operad with cellular structure maps γ and $\chi\colon \mathcal{C} \to \mathcal{X}$ is a morphism of operads, then there is an analog of Theorem 2.8 for the calculation of $E^1_{**}\zeta$ and $C_*\zeta$ before passage to orbits but with all possible equivariance, where

ζ is the composite map displayed in VII.1.6. Both that general analog and parts (i) - (iii) of Theorem 2.8 specialize to calculations of ι_j, $\alpha_{j,k}$, and $\beta_{j,k}$ on E^1_{**} and C_* and part (iv) specializes to the analogous calculations for the maps δ_j of VII.1.8.

The rest of this section is devoted to the proofs of Theorems 2.7 and 2.8. The requisite geometry is given in VI§4. The real problem is to keep track of π actions and, as explained there, the way to do this is to work equivariantly, before passage to orbits, regarding π actions as ordinary maps $\pi \ltimes F \to F$.

Our trick of fattening filtrations by taking products with spaces allows us to exploit fully the diagrams of VI§4. We start with our free π-CW complex W and consider such π-spaces as $\pi \times W$, $W \times C$, $\pi \times W \times C$, etc., where C is a contractible CW complex (such as I, Δ_2, etc.). We filter these spaces by their subcomplexes $\pi \times W^p$, $W^p \times C$, $\pi \times W^p \times C$, etc. We proceed similarly for π-spaces $\pi \times K$, $\pi \times K \times C$, etc., where K is a given CW complex without π action. Then, with X, χ, and E there replaced by W, ω, and F here, every single map displayed in VI§4 is a map of bifiltered CW spectra and so induces a map of spectral sequences. Moreover, all of the equivalences there were induced by maps of the form $1 \times i\colon W \times C \to W \times C'$, where i is the inclusion of a contractible subcomplex in a contractible CW complex. Clearly any such equivalence restricts to an equivalence in each bidegree and therefore induces an isomorphism of spectral sequences.

To begin the proof of Theorem 2.7, abbreviate $f_j = f$ and observe that Lemma 1.8 (iii) and the equivalence $\omega \ltimes F \simeq W^+ \wedge f_*F$ of diagram (A) in VI§4 imply equivalences

$$(1)\qquad \frac{\omega_p \ltimes F_q}{\omega_{p-1} \ltimes F_q \cup \omega_p \ltimes F_{q-1}} \simeq \frac{W^p}{W^{p-1}} \wedge \frac{f_*F_q}{f_*f_{q-1}}$$

These give rise via Proposition 1.9 to an isomorphism

$$(2)\qquad E^1_{**}(\omega \ltimes F) \cong C_*W \otimes E^1_{**}F$$

While the equivalences (1) are not π-equivalences and f_*F is not a π-spectrum, VI.4.9 shows that these equivalences nevertheless "pass to orbits" to yield equivalences

$$(3)\qquad \frac{\omega_p \ltimes_\pi F_q}{\omega_{p-1} \ltimes_\pi F_q \cup \omega_p \ltimes_\pi F_{q-1}} \simeq \frac{W^p/\pi}{W^{p-1}/\pi} \wedge \frac{f_*F_q}{f_*F_{q-1}}$$

Since $(W^p/\pi)/(W^{p-1}/\pi)$ is a wedge of p-spheres, one for each π-basis element of $C_p(W/\pi) \otimes Z[\pi] \cong C_pW$, this implies the additive identification

(4) $$E^1_{**}(W \ltimes_\pi F) \cong C_*(W/\pi) \otimes E^1_{**}F \cong C_*W \otimes_\pi E^1_{**}F.$$

We claim first that the π action induced on $E^1_{**}(\omega \ltimes F)$ from the action of π on $\omega \ltimes F$ agrees under the isomorphism (2) with the diagonal action

$$\sigma \otimes w \otimes x \longrightarrow w\sigma^{-1} \otimes \sigma x$$

(the inverse entering since we are using a right π action on W). We claim second that the homomorphism

$$E^1_{**}(\omega \ltimes F) \longrightarrow E^1_{**}(\omega \ltimes_\pi F)$$

induced by passage to orbits agrees under the isomorphisms of (2) and (4) with the natural algebraic quotient map

$$\varepsilon\colon C_*W \ltimes E^1_{**}F \longrightarrow C_*W \ltimes_\pi E^1_{**}F.$$

These claims and the fact that the isomorphism of (2) respects differentials will imply that the isomorphism (4) respects differentials and thus will complete the proof of Theorem 2.7.

We regard algebraic π actions as homomorphisms $Z[\pi] \otimes (?) \to (?)$ and relate them to the topology as follows. Since k_* carries wedges to sums, we have

$$k_*(\pi^+ \wedge E) = \sum_{\sigma \in \pi} k_*E = Z[\pi] \otimes k_*E$$

for $E \in \mathcal{S}$. By naturality, the monad unit and product (above VI.4.1) induce the standard π-module structure on $Z[\pi] \otimes k_*E$. Since π acts trivially on R^∞, we have $\pi^+ \wedge E = \pi \ltimes E$ here (compare VI.4.3).

For $F \in \mathcal{S}_j$ (not necessarily a π-spectrum for the moment), diagram (C) of VI§4 gives $\pi \ltimes F \simeq \pi^+ \wedge F$. This and the identification $f_*(\pi^+ \wedge F) = \pi^+ \wedge f_*E$ give a natural isomorphism

$$k_*(\pi \ltimes F) \cong k_*(\pi^+ \wedge F) = Z[\pi] \otimes k_*F.$$

Here VI.4.1 tells us that the monad unit and product of $\pi \ltimes F$ induce the standard π-module structure on $Z[\pi] \otimes k_*F$.

For $F \in \pi\mathcal{S}_j$, the substitute, diagram (F) of VI§4, for an action of π on f_*F obtained by applying f_* to the equivalence $\pi \ltimes f \simeq \pi^+ \wedge F$ and action $\xi\colon \pi \ltimes F \to F$ induces the composite

$$\xi_*\colon Z[\pi] \otimes k_*F \cong k_*(\pi \ltimes F) \longrightarrow k_*F.$$

Here VI.4.2 tells us that k_*F is a π-module with ξ_* as action.

Now diagram (G) of VI§4 reads as follows on the E^1-level:

$$\begin{tikzcd}
E^1_{**}(\pi \ltimes \omega \ltimes F) \cong Z[\pi] \otimes E^1_{**}(\omega \ltimes F) \arrow[d, "\xi_*"'] & \cong Z[\pi] \otimes C_*W \otimes E^1_{**}F. \arrow[d, "\phi\otimes 1"] \\
E^1_{**}(\omega \otimes F) \cong C_*W \otimes E^1_{**}F & C_*W \otimes Z[\pi] \otimes E^1_{**}F \arrow[l, "1 \otimes \xi_*"']
\end{tikzcd}$$

Here $\phi(\sigma \otimes w) = (w\sigma^{-1} \otimes \sigma)$, and this proves our first claim.

For our second claim, note the commutative algebraic diagram

$$\begin{tikzcd}
C_*W \otimes E^1_{**}F \arrow[r, "\varepsilon"] & C_*W \otimes_\pi E^1_{**}F, \arrow[d, phantom, "\cong" rotate=90] \\
C_*W \otimes E^1_{**}F \arrow[u, "\zeta"] \arrow[r, "\varepsilon \otimes 1"] & (C_*W/\pi) \otimes E^1_{**}F
\end{tikzcd}$$

where $\zeta(\overline{w} \otimes \sigma \otimes x) = \overline{w} \otimes \sigma^{-1} \otimes \sigma x$ for $\overline{w}$ in our π-basis for C_*W, $\sigma \in \pi$, and $x \in E^1_{**}F$. By inspection of VI.4.9, the orbit map from the left side of (1) to the left side of (3) is evaluated in terms of the right sides by passage to quotients from pairs of diagrams of the general form

$$\begin{tikzcd}
\overline{K} \ltimes F \arrow[d, phantom, "\cong" rotate=90] & \pi \ltimes K \ltimes F \arrow[l, "\overline{\eta}"'] \arrow[r, "\varepsilon"] \arrow[d, phantom, "\cong" rotate=90] & \pi \ltimes K \ltimes F/\pi = K \ltimes F \arrow[d, phantom, "\cong" rotate=90] \\
(\pi \times K)^+ \wedge f_*F & (\pi \times K)^+ \wedge f_*F \arrow[l, "\zeta"'] \arrow[r, "\varepsilon \wedge 1"] & K^+ \wedge f_*F,
\end{tikzcd}$$

where ζ and the orbit maps ε are the evident geometric forerunners of ζ and ε in the algebraic diagram displayed above. Here $\overline{\eta}$ is the isomorphism specified in VI.4.7, the left square is diagram (H) of VI§4, and the right square reflects the fact that π acts on $\pi \ltimes K \ltimes F$ only through its action on π. The spaces K consist of the cells or their boundary spheres used to construct the successive skeleta of W/π, as in the statement of VI.4.9. This proves our second claim.

Observe that diagram (B) of VI§4 shows that, up to canonical and coherent natural isomorphism, our identification of E_1 is independent of all choices in the construction of diagram (A).

The last statement of Theorem 2.7 results by commuting Σ^∞ through all steps of the arguments above and noting that all of the equivalences used trivialize on the space level.

The proof of Theorem 2.8 is based on VI.4.10. Part (i) of Theorem 2.8 is immediate from part (i) of VI.4.10. In part (ii), $E^1_{**}(F_j \wedge F_k)$ implicitly refers to $(f_{j+k})_*(F_j \wedge F_k)$, by Notations 2.6. However, upon traversal of the diagram counterclockwise, we naturally land in E^1_{**} of the spectrum $f_{2*}(f_{j*}F_j \wedge f_{k*}F_k)$. Via α, β, and a path connecting $f_2(f_j \oplus f_k)$ to f_{j+k}, we obtain an equivalence

$$f_{2*}(f_{j*}F_j \wedge f_{k*}F_k) \simeq (f_{j+k})_*(F_j \wedge F_k)$$

and an induced isomorphism of spectral sequences. Use of this isomorphism is necessary to make sense of the diagram and, with this interpretation, its commutativity is easily deduced from VI.4.10 (ii). Part (iii) is similar, the relevant equivalence being

$$f_{k*}(f_j^k)_*(F_{jk}) \simeq (f_{jk})_*(F_{jk}).$$

Again, in part (iv), the left isomorphism involves comparison of $f_{j*}\, g_{j,2*}(F \wedge F')$ to $f_{2*}(f_j \oplus f_j)_*(F \wedge F')$, and precisely such a comparison was involved in the second diagram of VI.4.10 (iv). Use of the diagonal approximation is justified by VI.4.11.

§3. Steenrod operations in $D_\pi E$

We specialize the work of the preceding section to the case of the standard π-free CW-complex $W = S^\infty$, where π is the cyclic group of order p embedded as usual in Σ_p. All chain complexes and homology groups are to have mod p coefficients. The cellular chain group $C_i(W)$ is π-free on one generator e_i. We fix a π-map $\omega\colon W \to \mathcal{L}(p)$ and write $D_\pi E$ for $W \ltimes_\pi E^{(p)}$ and $D_\pi^k E$ for $W^k \ltimes_\pi E^{(p)}$, where W^k is the k-skeleton of W. For a CW-spectrum E, we have

$$C_*(D_\pi^k E) \cong C_*(W^k) \otimes_\pi C_*(E)^p.$$

By an easy algebraic argument [96, 1.3], the mod p homology $H_*(D_\pi^k E)$ has a basis consisting of the union of the three sets

(A) $\{e_i \otimes x^p \mid 0 \leq i \leq k\}$, $\{e_0 \otimes x_1 \otimes \cdots \otimes x_p\}$, $\{f_k \otimes x_1 \otimes \cdots \otimes x_p\}$.

Here f_k runs through a basis for the kernel of $d_k\colon C_k(W) \to C_{k-1}(W)$, x runs through a basis for $H_*(E)$, and $x_1 \otimes \cdots \otimes x_p$ runs through a π-basis for the π-module of elements of $H_*(E)^p$ not fixed by π. The elements $e_0 \otimes x^p$ and the elements of the last two sets span the image of $H_*(W^k \ltimes E^{(p)})$.

For an H_∞ ring spectrum E, homology operations in $H_*(E)$ were defined in terms of a map $D_\pi E \to E$. By naturality, the Nishida relations $[H_\infty$, III.1.1] for their commutation with homology Steenrod operations are an immediate consequence of the following calculation of the Steenrod operations in $H_*(D_\pi E)$. We agree to write $P^s = Sq^s$ if $p = 2$.

Theorem 3.1. Let E be a CW spectrum and let $x \in H_q(E)$.

(i) If $p = 2$ and 2^t is sufficiently large, then

$$P^s_*(e_r \otimes x^2) = \sum_i (s - 2i,\ r + q - 2s + 2i + 2^t)\ e_{r-s+2i} \otimes P^i_*(x)^2.$$

(ii) If $p > 2$ and p^t is sufficiently large, then

$$P^s_*(e_r \otimes x^p) = \sum_i (s - pi,\ [\tfrac{r}{2}] + qm - ps + pi + p^t)\ e_{r+2(pi-s)(p-1)} \otimes P^i_*(x)^p$$
$$+ \delta(r)\alpha(q) \sum_i (s-pi-1,\ [\tfrac{r+1}{2}] + qm - ps + pi + p^t)\ e_{r+p+2(pi-s)(p-1)} \otimes P^i_*\beta(x)^p,$$

where $m = (p-1)/2$, $\alpha(q) = -(-1)^{mq}m!$, and $\delta(2n+\varepsilon) = \varepsilon$, $\varepsilon = 0$ or 1.

(iii) If $p > 2$ then $\beta(e_{2r} \otimes x^p) = e_{2r-1} \otimes x^p$.

Our conventions are that $(a,b) = (a + b)!/a!b!$ if $a \geq 0$ and $b \geq 0$, while $(a,b) = 0$ if $a < 0$ or $b < 0$. Since $D_\pi \Sigma^\infty X \cong \Sigma^\infty D_\pi X$ for a based space X, the result holds by Nishida [116] (modulo correction of signs) or [96, 9.4] when $E = \Sigma^\infty X$. Actually, the cited references prove the result with the powers p^t omitted from the binomial coefficients. The two versions are seen to give the same answer by use of the following two facts.

(a) For spaces, $P^i_*(x) = 0$ if $p = 2$ and $q < 2i$ or if $p > 2$ and $q < 2ip$.

(b) $(b, p^t + a - b) = 0$ and, if $p > 2$, $(b - 1, p^t + a - b) = 0$ if $0 \leq a < b < p^t$.

The formulas become cleaner when written in terms of classical binomial coefficients since

$$(a, p^t + b) = \binom{p^t + a + b}{a} = \binom{a + b}{a} \quad \text{for } a < p^t.$$

We shall prove Theorem 3.1 in cohomology, and we may as well assume that H_*E is of finite type. At least if H_*E is bounded below ($H_q(E) = 0$ for all sufficiently small q), $H^*(D_\pi E)$ can be computed from the cochain complex

$$C^*(W) \otimes_\pi H^*(E)^p.$$

We write w_i for the cochain dual to e_i. Then $H^*(D_\pi^k E)$ has a basis consisting of the union of $\{w_i \otimes y^p \mid 0 \leq i \leq k\}$, where y runs through a basis for $H^*(E)$, and the dual basis to the second and third sets in (A). When $p > 2$, the standard sign conventions give

$$\langle w_i \otimes y^p, e_i \otimes x^p \rangle = (-1)^{(i+m)\deg y} \langle y,x \rangle.$$

The standard sign convention $\delta f = (-1)^{\deg f+1} fd$ used in defining coboundaries in terms of boundaries implies

$$\langle \beta y,x \rangle = (-1)^{\deg x} \langle y,\beta x \rangle.$$

With these conventions, a careful check of signs gives the following dualized form of the theorem.

<u>Theorem 3.2</u>. Assume that H_*E is bounded below and of finite type. Modulo the subspace spanned by the dual basis elements of the last two sets in (A), the following relations hold in $H^*(D_\pi^k E)$. Moreover, the cited subspace is closed under Steenrod operations.

(i) For $p = 2$, $0 \leq j < k$, and $y \in H^q(E)$,

$$P^s(w_j \otimes y^2) = \sum_i (s - 2i, j - s + i + q + 2^t)\, w_{j+s-2i} \otimes (P^i y)^2.$$

(ii) For $p > 2$, $0 \leq j < k$, and $y \in H^q(E)$,

$$P^s(w_j \otimes y^p) = \sum_i (s - pi, [\tfrac{j}{2}] - s + i + qm + p^t)\, w_{j+2(s-pi)(p-1)} \otimes (P^i y)^p$$

$$+ \delta(j-1)\alpha(q) \sum_i (s - pi - 1, [\tfrac{j}{2}] - s + i + qm + p^t)\, w_{j-p+2(s-pi)(p-1)} \otimes (\beta P^i y)^p.$$

(iii) For $p > 2$, $\beta(w_{2j-1} \otimes y^p) = w_{2j} \otimes y^p$.

When $E = \Sigma^\infty X$, Theorems 3.1 and 3.2 are known. To prove them for spectra, we first recall from I.4.7 that we have a natural isomorphism

$$E \cong \operatorname{colim} \Lambda^n \Sigma^\infty E_n.$$

Here Λ^n is the shift desuspension and is equivalent to Σ^{-n} by I§7. We agree to also write Λ^n for the desuspension isomorphism $H_q(E) \to H_{q-n}(\Lambda^n E)$. Since the functor D_π commutes with colimits, we may assume that $E = \Lambda^n \Sigma^\infty X$ for a based space X. By I.4.2, we also have a natural isomorphism $\Lambda^n \Sigma^\infty X \cong \Lambda^{n+1}\Sigma^\infty(\Sigma X)$, hence we may as well assume that n is even (to avoid signs) and that X is connected.

Recalling the Thom complexes of VI.2.1, we set

$$T_k = T(W^k; R^{np}, R^{q(k)}).$$

Here $\{q(k)\}$ is an increasing sequence such that $\omega(W^k)(R^{np}) \subset R^{q(k)}$, and we agree to take the $q(k)$ to be even. Thus T_k is the Thom complex of the π-bundle $\eta_k = \varepsilon_{q(k)} - n\zeta_p^k$ over W^k, where ε_q is the q-dimensional trivial π-bundle and ζ_p^k is the evident π-bundle $W^k \times (R^1)^p \to W^k$. By VI.5.3 and VI.5.4, we have a natural isomorphism

$$\text{(B)} \qquad D_\pi \Lambda^n \Sigma^\infty X \cong \operatorname*{colim}_k \Lambda^{q(k)} \Sigma^\infty (T_k \wedge_\pi X^{(p)})$$

Note that if Y^k is the total space of η_k, then $T_k \wedge_\pi X^{(p)}$ is the quotient of the Thom complex of

$$\eta_k \times_\pi 1\colon Y^k \times_\pi X^{(p)} \longrightarrow W^k \times_\pi X^{(p)}$$

by the Thom complex of $Y^k \times_\pi \{*\} \to W^k \times_\pi \{*\}$. Moreover, $\eta_k \times_\pi 1$ is the pullback of η_k/π along the projection $W^k \times_\pi X^{(p)} \to W^k/\pi$.

The Thom complex of the trivial bundle ε_q/π over W^k/π is $\Sigma^q(W^k/\pi^+)$ and has a canonical orientation ι_q of degree q. The Thom complex of $(n\zeta_p^k)/\pi$ is $D_\pi^k S^n$ and the basis element ν_k dual to $e_0 \otimes i_n^p$ is an orientation, where i_n is the fundamental class of S^n. There results an orientation $\mu_k \in H^{q(k)-pn}(T(\eta_k/\pi))$ such that

$$\nu_k \oplus \mu_k = (T\Delta)^*(\nu_k \wedge \mu_k) = \iota_{q(k)}.$$

We give $\eta_k \times_\pi 1$ the pullback of this orientation, also denoted μ_k, and we write ϕ_* for any of the resulting Thom isomorphisms in homology. In particular, we have the relative Thom isomorphism

$$\phi_*\colon \tilde{H}_*(T_k \wedge_\pi X^{(p)}) \longrightarrow \tilde{H}_*(D_\pi^k X).$$

The colimit system in (B) is given by the maps

$$\begin{array}{c} \Lambda^{q(k)}\Sigma^\infty(T_k \wedge_\pi X^{(p)}) \\ \wr\wr \\ \Lambda^{q(k+1)}\Sigma^\infty\Sigma^{q(k+1)-q(k)}(T_k \wedge_\pi X^{(p)}) \\ \Big\downarrow \Lambda^{q(k+1)}\Sigma^\infty i \\ \Lambda^{q(k+1)}\Sigma^\infty(T_{k+1} \wedge_\pi X^{(p)}), \end{array}$$

where i is induced by the inclusion $T_k \wedge S^{q(k+1)-q(k)} \longrightarrow T_{k+1}$ of VI.2.3. From this description, it is easy to check that our Thom isomorphisms fit into the following commutative diagrams.

$$\begin{tikzcd}
\tilde{H}_q(D^k_\pi X) \arrow[r, "\phi_*^{-1}"] \arrow[d] & \tilde{H}_{q+q(k)-pn}(T_k \wedge_\pi X^{(p)}) \arrow[r, "\Lambda^{q(k)}"] & H_{q-pn}(\Lambda^{q(k)}\Sigma^\infty(T_k \wedge_\pi X^{(p)})) \arrow[d] \\
\tilde{H}_q(D^{k+1}_\pi X) \arrow[r, "\phi_*^{-1}"] & \tilde{H}_{q+q(k+1)-pn}(T_{k+1} \wedge_\pi X^{(p)}) \arrow[r, "\Lambda^{q(k+1)}"] & H_{q-pn}(\Lambda^{q(k+1)}\Sigma^\infty(T_{k+1} \wedge_\pi X^{(p)}))
\end{tikzcd}$$

Thus passage to colimits yields the isomorphism

(C) $$\phi_*^{-1}\colon \tilde{H}_q(D_\pi X) \longrightarrow H_{q-pn}(D_\pi \Lambda^n \Sigma^\infty X)$$

Both sides have bases as in (A), and we have so chosen our orientations as to have the following consistency statement.

<u>Lemma 3.3</u>. $\phi_*^{-1}(e_0 \otimes x_1 \otimes \cdots \otimes x_p) = e_0 \otimes \Lambda^n x_1 \otimes \cdots \otimes \Lambda^n x_p$ and

$$\phi_*^{-1}(e_i \otimes x^p) = e_i \otimes (\Lambda^n x)^p.$$

<u>Proof</u>. The isomorphism (B) is a colimit of isomorphisms

(D) $$D^k_\pi \Lambda^n \Sigma^\infty X \cong \Lambda^{q(k)}\Sigma^\infty(T_k \wedge_\pi X^{(p)}).$$

In fact, inspection of the definitions (I.4.1 and VI.2.5) of the functors $\Lambda^n\Sigma^\infty$ and D^k_π shows that the $q(k)^{\underline{th}}$ space of the prespectrum level constructions here are both precisely $T_k \wedge_\pi X^{(p)}$. The spectrum level identification of homology classes given by the arguments of the previous section desuspends on the prespectrum level to the Thom isomorphism just specified.

Thus to prove Theorem 3.1 we need only compute the Steenrod operations in the relative Thom complexes $T_k \wedge_\pi X^{(p)}$. We must first compute the Wu classes of the η_k. Here the Wu classes of an oriented bundle ξ are defined by $w_i(\xi) = \phi^{-1}P^i\phi(1)$, and we write $w(\xi)$ for the total Wu classes $\sum_{i \geq 0} w_i(\xi)$. (By abuse, we are calling Stiefel-Whitney classes Wu classes when $p = 2$.) Recall that

$$H^*(W^k/\pi) = P\{a\}/(a^{k+1}) \quad \text{if } p = 2,$$

where $a = e_1^*$, and that

$$H^*(W^k/\pi) = E\{a\} \otimes P\{b\}/(b^{j+1}, (1-\varepsilon)ab^j) \quad \text{if } p > 2 \text{ and } k = 2j+\varepsilon,$$

where $a = e_1^*$ and $b = e_2^* = \beta a$.

<u>Lemma 3.4</u>. (i) If $p = 2$, $w(\zeta_2^k/\pi) = 1 + a$ and therefore

$$w(\eta_k/\pi) = (1+a)^{2^t - n}$$

for any t such that $2^t > k$.

(ii) If $p > 2$ and $m = (p-1)/2$, $w(\zeta_p^k/\pi) = (1 + b^{p-1})^m$ and therefore

$$w(\eta_k/\pi) = (1 + b^{p-1})^{p^t - mn}$$

for any t such that $2p^t(p-1) > k$.

<u>Proof</u>. Since $\eta_k/\pi = \varepsilon_{q(k)}/\pi - n\zeta_p^k/\pi$ and $(1+a)^{2^t} = 1$ or $(1+b^{p-1})^{p^t} = 1$ for the cited values of t, the second clauses follow from the first. Clearly ζ_p^k/π is the restriction to W^k/π of the bundle ζ_p/π over W/π induced by the regular representation of π. If $p = 2$, ζ_2/π is the sum of a trivial line bundle and the canonical line bundle, hence $w(\zeta_2/\pi) = 1 + a$. If $p > 2$, ζ_p/π is the sum of a trivial line bundle and the bundles induced by the m different nontrivial representations of π in $SO(2)$. If $\alpha\colon W/\pi \to BSO(2)$ classifies one such and $\chi \in H^2(BSO(2))$ is the canonical generator $(= c_1)$, then $\alpha^*(\chi) = \kappa b$ for some non-zero constant $\kappa \in Z_p$. If γ is the universal bundle over $BSO(2)$, then $w(\gamma) = 1 + \chi^{p-1}$. Thus $w(\alpha) = 1 + b^{p-1}$ by naturality since $\kappa^{p-1} = 1$.

$H^*(W^k/\pi)$ acts in the obvious way on $H^*(W^k \times_\pi X^{(p)})$, and the lemma has the following immediate consequence.

<u>Lemma 3.5</u>. Let $y \in H^*(X)$ and let $0 \leq j \leq k$. Let

$$\Delta\colon T_k \wedge_\pi X^{(p)} \longrightarrow D^k_\pi X \wedge (T_k \wedge_\pi X^{(p)})$$

be the Thom diagonal.

(i) If $p = 2$, $\Delta^*[(w_j \otimes y^2) \otimes P^r\mu_k] = (r, 2^t - n - r)w_{j+r} \otimes y^2$

(ii) If $p > 2$, $\Delta^*[(w_j \otimes y^p) \otimes P^r\mu_k] = (r, p^t - mn - r)w_{j+2r(p-1)} \otimes y^p$.

The following lemma, which is proven in [96, 9.3], will allow us to assemble the information above.

<u>Lemma 3.6</u>. For $a \geq 0$, $b \geq 0$, and $c \geq 0$,

$$\sum_r (r,\ a-r)(c-r,\ r+b-c) = (c,\ a+b-c).$$

We put things together to obtain the following analog of Theorem 3.2.

Proposition 3.7. Modulo the subspace spanned by the images under ϕ of the dual basis elements of the last two sets in (A), the following relations hold in $H^*(T_k \wedge_\pi X^{(p)})$. Moreover, the cited subspace is closed under Steenrod operations.

(i) If $p = 2$, $0 \leq j < k$, and $y \in H^q(x)$,

$$P^s\phi(w_j \otimes y^2) = \sum_i (s-2i,\ j-s+i+q-n+2^t)\phi(w_{j+s-2i} \otimes (P^i y)^2).$$

(ii) If $p > 2$, $0 \leq j < k$, and $y \in H^q(X)$,

$$P^s\phi(w_j \otimes y^p) = \sum_i (s-pi, [\tfrac{j}{2}] - s+i+qm-nm+p^t)\phi(w_{j+2(s-pi)(p-1)} \otimes (P^i y)^p)$$

$$+\ \delta(j-1)\alpha(q) \sum_i (s-pi-1, [\tfrac{j}{2}] - s+i+qm-nm+p^t)\phi(w_{j-p+2(s-pi)(p-1)} \otimes (\beta P^i y)^p).$$

(iii) If $p > 2$ and $0 \leq 2j < k$, $\beta\phi(w_{2j-1} \otimes y^p) = \phi(w_{2j} \otimes y^p)$.

Proof By the Cartan formula, $P^s\phi(w_j \otimes y^p)$ is the image under Δ^* of $\sum_r P^{s-r}(w_j \otimes y^p) \otimes P^r\mu_k$. Upon application of Theorem 3.2 (for spaces) and Lemma 3.5, we obtain expressions of the standard form but with the binomial coefficients replaced by

$$\sum_r (r,\ 2^t-n-r)(s-r-2i,\ j-s+r+i+q)$$

if $p = 2$ and by

$$\sum_r (r,\ p^t-mn-r)(s-r-pi,\ [\tfrac{j}{2}] - s+r+i+qm)$$

and

$$\sum_r (r,\ p^t-mn-r)(s-r-pi-1,\ [\tfrac{j}{2}] - s+r+i+qm)$$

if $p > 2$. Lemma 3.6 gives the conclusions of (i) and (ii), and (iii) is clear since $\beta\mu_k = 0$.

By Lemma 3.3, we may think of $\phi_*^{-1}(e_j \otimes x^p)$ as the element $e_j \otimes (\Lambda^n x)^p$ of $H_*(D_\pi^k \Lambda^n \Sigma^\infty X)$, and of course $\Lambda^n x$ has degree $q-n$, $q = \deg x$. Theorem 3.1 for $E = \Lambda^n \Sigma^\infty X$ and thus for general E follows immediately by dualization.

IX. Thom Spectra

by L. G. Lewis, Jr.

In [12], Boardman constructed the Thom spectrum associated to a map $f: Y \to BO$. Intuitively, it is clear that his construction generalizes to one yielding a Thom spectrum from any map $f: Y \to BF$. Primarily because of the pioneering work of Mahowald, this construction has had numerous applications. See [28, 31, 33, 93, 120], for example.

In this chapter, we analyze this construction in detail, producing a well-behaved Thom spectrum associated to any map $f: Y \to BG$, where G is any one of the standard infinite groups or monoids (e.g. SO, O, SU, U, STOP, TOP, SF or F). Our expansion of Boardman's construction to allow monoids involves only the introduction of a technical trick to handle the difficulties which arise because the geometric bar construction yields only a quasifibration when G is only a monoid.

Our main result is that, for sufficiently nice maps $f: Y \to BG$, the Thom spectra Mf have enriched ring structures. In particular, if f is an n-fold loop map $(1 \leq n \leq \infty)$, then Mf is an E_n-ring spectrum. Both n-fold loop spaces and E_n-ring spectra have Dyer-Lashof operations in their mod p homology and we show that when the Thom isomorphism exists (the usual orientability conditions apply), it carries the Dyer-Lashof operations in the homology of Mf to those in the homology of Y.

Our second contribution is a collection of invariance results about Thom spectra. The first step in any construction of the spectrum Mf associated to $f: Y \to BG$ is the selection of a suitable filtration on Y. Intuitively, the spectrum Mf should be independent of the choice of this filtration. It should also depend only on the homotopy class of f. We make these intuitions rigorous. However, some caution must be exercised in using the homotopy invariance. For example, the Thom spectrum Mf derived from an H-map $f: Y \to BG$ has a multiplication, but this multiplication need not be commutative or associative even if Y is. For this, the map f must be a homotopy commutative or associative H-map in the sense of Zabrodsky [148]. Our detailed treatment of the invariance properties facilitates the detection and resolution of such difficulties.

The basic source of interest in Thom spectra is their connection to cobordism, and it is an easy matter to adapt the standard discussions [80,123,135] to our context.

The necessary technical details on bundles and fibrations all appear in Sections 1 and 2. Section 3 contains the definition of the Thom spectrum of a map and several basic examples - in particular, Mahowald's description of the Thom spectrum of a map from a suspension into BG. Section 4 contains our invariance results and a discussion of the relation between H-maps $f\colon Y \to BG$ and ring structures on Mf. Section 5 contains a discussion of the homology and cohomology Thom isomorphisms (roughly following Boardman). In section 6, we show that the extended powers of the Thom spectrum Mf are Thom spectra derived from maps naturally related to f. This result is applied in section 7 to show that the Thom spectra derived from suitable maps $f\colon Y \to BG$ are operad ring spectra.

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§1. Preliminaries on sphere spaces and spherical fibrations

We here define sphere spaces and their associated Thom spaces, discuss the conversion of sphere spaces into spherical fibrations and recall two key relations between pullbacks and colimits.

We begin with the latter. Let $\mathcal{U}/B$ be the category of (compactly generated, weak Hausdorff) spaces over a given space B. Any map $f\colon Y \to B$ yields a pullback functor $f^*\colon \mathcal{U}/B \to \mathcal{U}/Y$ which assigns to a space $g\colon Z \to B$ over B the pullback $f^*g\colon f^*Z \to Y$ of g along f. The category $\mathcal{U}/B$ has colimits; the colimit of a diagram over B consists of the colimit in $\mathcal{U}$ of the diagram together with the natural map into B. Recall that the colimit of a diagram in $\mathcal{U}$ is the maximal weak Hausdorff quotient of the colimit of that diagram in the category of all topological spaces. Here, we transcribe from [83,86] the basic results on the behavior of colimits under pullback functors.

Proposition 1.1. Let $f\colon Y \to B$ be a map in $\mathcal{U}$. The functor $f^*\colon \mathcal{U}/B \to \mathcal{U}/Y$ preserves all colimits if and only if f is an open map. However, for any f, the functor f^* preserves those colimits which are not proper quotients of the corresponding colimits in the category of all topological spaces. In particular, any f^* preserves colimits of directed systems all of whose maps are injections.

An easy consequence of [15, 6.2] provides one major source of open maps. A space B is said to be locally equiconnected, or LEC, if the diagonal $\Delta\colon B \to B \times B$ is a cofibration (see [51,84]).

Proposition 1.2. If $f\colon E \to B$ is a (Hurewicz) fibration and B is LEC, then f is an open map.

Our basic source for open maps is the realization of certain well-behaved maps of simplicial spaces (see [97] for our choice of notation). Easy point-set topology gives the following.

Proposition 1.3. Let $f_*\colon E_* \to B_*$ be a map of simplicial spaces such that each $f_q\colon E_q \to B_q$ is open and E_q maps onto the pullback in all of the commutative diagrams

$$\begin{tikzcd} E_q \arrow[r, "s_i"] \arrow[d, "f_q"'] & E_{q+1} \arrow[d, "f_{q+1}"] \\ B_q \arrow[r, "s_i"] & B_{q+1} \end{tikzcd} \qquad \begin{tikzcd} E_q \arrow[r, "\partial_i"] \arrow[d, "f_q"'] & E_{q-1} \arrow[d, "f_{q-1}"] \\ B_q \arrow[r, "\partial_i"] & B_{q-1} \end{tikzcd}$$

Then the geometric realization of f_* is an open map.

This result can easily be applied to the two-sided bar construction of [98,§7].

Corollary 1.4. Let G be a topological monoid and let X and Y be a left and a right G-space. Then the natural map

$$\pi\colon B(Y,G,X) \longrightarrow B(Y,B,*)$$

is open provided the action map $g\colon X \to X$ is surjective for each g in G.

In our applications, X is S^n and each g is a homotopy equivalence - hence a surjection.

By an S^n-space, we mean a projection $p\colon D \to A$ together with a section $s\colon A \to D$ such that $(p^{-1}(a), s(a))$ has the based weak homotopy type of $(S^n,*)$ for each a A. We think of s as supplying basepoints for the fibres of p and insist that these be non-degenerate. The non-degeneracy is automatic if the fibres are actual spheres and can usually be arranged using Proposition 1.11 (iii) below. From the fact that $ps = 1$, it follows that p is a quotient map and s is a closed inclusion. A map $(F,f)\colon p \to q$ of S^n-spaces, $p\colon D \to A$ and $q\colon E \to B$, is a pair of maps $F\colon D \to E$ and $f\colon A \to B$ such that the diagrams

$$\begin{array}{ccc} D & \xrightarrow{F} & E \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ A & \xrightarrow{f} & B \end{array} \qquad\qquad \begin{array}{ccc} D & \xrightarrow{F} & E \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s} \\ A & \xrightarrow{f} & B \end{array}$$

commute and $F: (p^{-1}(a), s(a)) \to (q^{-1}(f(a)), sf(a))$ is a weak equivalence for each $a \in A$. The map (F,f) is said to be a pullback if the left square above is a pullback. We refer to S^n-spaces as sphere spaces when it is not necessary to mention the dimension.

For an S^n-space $p: D \to A$, the quotient space D/sA is denoted $T(p)$ and called the Thom space of p. Passage to Thom spaces gives a functor T from the category of S^n-spaces to that of based spaces. Since the quotient map $D \to T(p)$ is closed, we have the following observation.

Lemma 1.5. Let $(F,f): p \to q$ be a map of S^n-spaces. If (F,f) is a pullback and f is a closed inclusion, then F is a closed inclusion. If F is a closed inclusion, then so is $T(F,f): T(p) \to T(q)$.

An S^n-quasifibration is an S^n-space whose projection is a quasifibration. An S^n-fibration is an S^n-space $q: E \to B$ such that q satisfies the homotopy lifting property with respect to maps of S^n-spaces; that is, for every S^n-space $p: D \to A$, map $(F,f): p \to q$ of S^n-spaces, and homotopy $h: A \times I \to B$ with $f = hi_0$, there is a map $(H,h): p \times 1 \to q$ of S^n-spaces with $F = Hi_0$. Here $p \times 1$ is the S^n-space $D \times I \to A \times I$. By [98, p. 12], S^n-fibrations are necessarily (Hurewicz) fibrations.

The long exact homotopy sequence of a quasifibration and the homotopy excision theorem suffice to describe the low dimensional homotopy groups of Thom spaces.

Lemma 1.6. Let $p: D \to A$ be an S^n-quasifibration whose section is a cofibration. Then $T(p)$ is $(n-1)$-connected and, if A is path connected, $\pi_n(T(p))$ is either $\mathbf{Z}$ or $\mathbf{Z}/2$. It is $\mathbf{Z}$ if and only if p is integrally oriented.

Proof. If A is path connected, then $\pi_n(T(p))$ must be a cyclic group generated by the map $S^n = T(p|\{a\}) \to T(p)$ coming from the inclusion of any point a of A into A. If p is integrally oriented, then the classifying map of p lifts to $BSF(n)$ and we obtain a map $T(p) \to MSF(n)$ such that the composite $S^n \to T(p) \to MSF(n)$ is a generator of $\pi_n(MSF(n))$. Thus, $\pi_n T(p)$ must be $\mathbf{Z}$. If p is not integrally oriented, then the first Stiefel-Whitney class $\omega_1(p)$ must be non-zero. Using the universal coefficient theorem and the Hurewicz isomorphism, we obtain a map $f: S^1 \to A$ such that $f^*\omega_1(p) \neq 0$. It follows that the Thom space $T(f^*p)$ of the induced spherical fibration over S^1 is $\Sigma^{n-1}\mathbb{RP}^2$. The generator of $\pi_n T(p)$ is the image of the generator of $\pi_n\Sigma^{n-1}\mathbb{RP}^2$ and so must be of order 2.

If $p: D \to A$ and $q: E \to B$ are an S^m- and an S^n-space respectively, then $p \wedge q: D \wedge E \to A \times B$, the fiberwise smash product of p and q, is the S^{m+n}-space whose fibre over (a,b) in $A \times B$ is $p^{-1}(a) \wedge q^{-1}(b)$ (see [98, p 23]). The section of $p \wedge q$ takes (a,b) to the basepoint $s(a) \wedge s(b)$ of $p^{-1}(a) \wedge q^{-1}(b)$. Note that $D \wedge E$ is an abuse of traditional notation - it does not denote the actual smash product of two based spaces, but rather the fibrewise smash product.

We need an observation about smash products.

Lemma 1.7. For sphere spaces $p_i: D_i \to A_i$ $(1 \leq i \leq n)$, $T(p_1 \wedge p_2 \wedge \cdots \wedge p_n)$ is naturally homeomorphic to $T(p_1) \wedge T(p_2) \wedge \cdots \wedge T(p_n)$. For a set of pullbacks of sphere spaces $\{(F_i,f_i): p_i \to q_i \mid 1 \leqslant i \leqslant n\}$,

$$\bigwedge_{i=1}^{n} (F_i,f_i): \bigwedge_{i=1}^{n} p_i \longrightarrow \bigwedge_{i=1}^{n} q_i$$

is also a pullback.

We also need an observation about S^n-spaces on which a finite group π acts. An S^n-space $p: D \to A$ is called a π-S^n-space if D and A are π-spaces and $p: D \to A$ and $s: A \to D$ are π-maps. Note that a pullback of a π-S^n-space along a π-map is a π-S^n-space.

Lemma 1.8. Let $(F,f): (p: D \to A) \to (q: E \to B)$ be a π-map of π-S^n-spaces such that (F,f) is a pullback and π acts trivially on q. Then the orbit map $p/\pi: D/\pi \to A/\pi$ is an S^n-space and the induced map

$$(F/\pi, f/\pi): p/\pi \longrightarrow q$$

is a map of S^n-spaces and a pullback.

We shall frequently need to convert maps into fibrations. We recall this standard procedure here since we are going to use some of its little-known properties. Our notation is that of [98, §3], but our Proposition 1.11 shows how to avoid the whiskering construction used there.

For any B, define ΠB to be the space

$$\Pi B = \{(\theta,r) \in B^{[0,\infty]} \times [0,\infty) \mid \theta(t) = \theta(r) \text{ for } t > r\}.$$

We abbreviate (θ,r) to θ and define the length $\ell: \Pi B \to [0,\infty)$ by $\ell(\theta) = r$ and the end-point projection $\rho\colon \Pi B \to B$ by $\rho(\theta) = \theta(r)$. For any map $f\colon Y \to B$, define $\Gamma f\colon \Gamma Y \to B$ and $\delta\colon Y \to \Gamma Y$ by

$$\Gamma Y = \{(\theta,y) \mid \theta(0) = f(y)\} \subset \Pi B \times Y,$$

$$\Gamma f(\theta,y) = \rho(\theta) \quad \text{and} \quad \delta(y) = (\langle f(y)\rangle, y),$$

where, for $b \in B$, $\langle b\rangle$ is the path of length zero at b. We write $\Gamma_f Y$ for ΓY when it is necessary to specify f. Note that $f = (\Gamma f)\delta$ and that δ is a homotopy equivalence but not necessarily a fibre homotopy equivalence. The construction Γ clearly induces a functor $\mathcal{U}/B \to \mathcal{U}/B$ such that δ is natural. Since ΓY is just the pullback in the diagram

$$\begin{tikzcd} \Gamma Y \arrow[r] \arrow[d] & \Pi B \arrow[d, "e_0"] \\ Y \arrow[r, "f"] & B \end{tikzcd}$$

where e_0 is evaluation at $0 \in [0,\infty]$, Propositions 1.1 and 1.2 describe the behavior of Γ on colimits.

<u>Lemma 1.9</u>. If B is LEC, then the functor $\Gamma\colon \mathcal{U}/B \to \mathcal{U}/B$ preserves colimits.

If $f\colon Y \to B$ has a section $s\colon B \to Y$, then we define the associated section $s\colon B \to \Gamma Y$ by $s(b) = (\langle b\rangle, s(b))$.

<u>Remark 1.10</u>. Let $q\colon E \to B$ be an S^n-space whose section is a cofibration and let $f\colon Y \to B$ be any map. In the pullback diagrams

$$\begin{tikzcd} f^*\Gamma E \arrow[r] \arrow[d, "f^*\Gamma q"'] & \Gamma E \arrow[d, "\Gamma q"] \\ Y \arrow[r, "f"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} (\Gamma f)^*E \arrow[r] \arrow[d, "(\Gamma f)^*q"'] & E \arrow[d, "q"] \\ \Gamma Y \arrow[r, "\Gamma f"] & B \end{tikzcd}$$

we have

$$f^*\Gamma E = \{(y,\theta,e) \mid f(y) = \rho(\theta) \quad \text{and} \quad q(e) = \theta(0)\}$$

$$(\Gamma f)^*q = \{(\theta,y,e) \mid f(y) = \theta(0) \quad \text{and} \quad q(e) = \rho(\theta)\}$$

Obviously, reversal of path coordinates defines a homeomorphism $\gamma\colon f^*\Gamma E \to (\Gamma f)^*E$ such that the section $s\colon Y \to f^*\Gamma E$ is the composite

$$Y \xrightarrow{\delta} \Gamma Y \xrightarrow{s'} (\Gamma f)^*E \xrightarrow{\gamma^{-1}} f^*\Gamma E$$

where s' is the section of $(\Gamma f)^*q$ induced by that of q. Since δ is a cofibration and a homotopy equivalence and $s'\colon \Gamma Y \to (\Gamma f)^*E$ is a cofibration (by the result below), we see that γ induces a natural homotopy equivalence

$$T(f^*\Gamma q) = f^*\Gamma E/sY \longrightarrow (\Gamma f)^*E/s'(\Gamma Y) = T((\Gamma f)^*q).$$

Thus, from the point of view of Thom spaces and cobordism, it is a matter of complete indifference whether we replace q or f by a fibration.

The proposition below is a compilation of the properties of Γ needed in this chapter. The rest of the section is devoted to its proof.

<u>Proposition 1.11</u>. (i) The functor Γ takes spaces over B into fibrations over B and therefore converts homotopy equivalences into fibre homotopy equivalences.
(ii) The functor Γ converts maps over B which are cofibrations into fibrewise cofibrations.
(iii) If $q\colon E \to B$ has a section $s\colon B \to E$ which is a cofibration, then the associated section $s\colon B \to \Gamma E$ is a fibrewise cofibration and $\Gamma q\colon \Gamma E \to B$ is a based fibration (in the sense of [98,§5]). If, further, q is an S^n-quasi-fibration, then Γq is an S^n-fibration.

To prove the proposition, we need a simple observation about representations of NDR-pairs.

<u>Lemma 1.12</u>. Any NDR-pair (X,A) can be represented by a pair of maps

$$\eta\colon X \longrightarrow I \quad \text{and} \quad h\colon X \longrightarrow \Pi X$$

such that $A = \eta^{-1}(0)$, $\eta = \ell h$, $h(x)(0) = x$ for $x \in X$, and $\rho h(x) \in A$ for $x \in X$ with $\eta(x) < 1$.

<u>Proof</u>. It is well-known (e.g. [136]) that we can represent (X,A) as an NDR-pair by maps $\eta\colon X \to I$ and $k\colon X \times I \to X$ such that $A = \eta^{-1}(0)$, $k(x,0) = x$ for $x \in X$ and $k(x,t) \in A$ if $\eta(x) < 1$ and $t \geq \eta(x)$. Define $\tilde{h}\colon X \times I \to X$ by

$$\tilde{h}(x,t) = \begin{cases} k(x,t) & \text{for } t \leq \eta(x) \\ k(x,\eta(x)) & \text{for } t \geq \eta(x). \end{cases}$$

Clearly, $\tilde{h}$ is continuous and it induces a map $h\colon X \to \Pi X$ such that η and h satisfy the required conditions.

Proof of 1.11. (i) This is standard (e.g. [98,3.4] and [98, p. 11,12]).

(ii) Let $q: E \to B$ be a map and let (E,D) be an NDR-pair. Using the lemma above, select $\eta: E \to I$ and $h: E \to \Pi E$ representing (E,D) as an NDR-pair. Define

$$\eta': \Gamma E \longrightarrow I \quad \text{and} \quad h': (\Gamma E) \times I \longrightarrow \Gamma E$$

by

$$\eta'(\theta,e) = \eta(e) \quad \text{for} \quad (\theta,e) \in \Gamma E$$

$$h'(\theta,e,t) = (\phi(\theta,e,t),\ h(e)(t))$$

for $(\theta,e) \in \Gamma E \subset \Pi B \times E$ where $\phi(\theta,e,t) \in \Pi B$ is given by

$$\ell(\phi(\theta,e,t)) = \min(t,\eta(e)) + \ell(\theta)$$

$$\phi(\theta,e,t)(u) = \begin{cases} q(h(e)(r-u)) & \text{for } 0 \leq u \leq r = \min(t,\eta(e)) \\ \theta(u-r) & \text{for } u \geq r \end{cases}$$

The pair (η',h') represents $(\Gamma E,\Gamma D)$ as a fibrewise NDR-pair.

(iii) The section $s: B \to \Gamma E$ of $\Gamma q: \Gamma E \to B$ is just the composite

$$B \xrightarrow{\ \delta\ } \Gamma B \xrightarrow{\ \Gamma s\ } \Gamma E$$

where Γs is derived from the section $s: B \to E$ and is a fibrewise cofibration by part (ii). The map δ, being the inclusion of a fibrewise deformation retract, is trivially a fibrewise cofibration. Thus, $s: B \to \Gamma E$ is a fibrewise cofibration.

To show that Γq is a based fibration, we define a based lifting function $\xi: \Gamma\Gamma E \to \Gamma E$ for it. Using a fibrewise version of Lemma 1.12, select maps $\nu: \Gamma E \to I$ and $H: \Gamma E \to \Pi(\Gamma E)$ representing $(\Gamma E, sB)$ as a fibrewise NDR-pair such that $s(B) = \nu^{-1}(0)$, $\nu = \ell H$, $\rho(H(\theta,e)) \in s(B)$ if $\nu(\theta,e) < 1$ and $H(\theta,e)(0) = (\theta,e)$. Thinking of $\Pi(\Gamma E)$ as a subspace of $\Pi(\Pi B) \times \Pi E$, we decompose H into a pair of maps

$$h: \Gamma E \longrightarrow \Pi\Pi B \quad \text{and} \quad k: \Gamma E \longrightarrow \Pi E.$$

Then ξ is given by

$$\xi(\beta,(\theta,e)) = \begin{cases} (\lambda(r),k(\theta,e)(r)) & \text{for} \quad \nu(\theta,e) \geq r = \min(\tfrac{1}{2},\ell(\beta)) \\ (\beta_t,\ s(\beta_t(0))) & \text{for} \quad \nu(\theta,e) \leq r \end{cases}$$

where $\beta,\theta \in \Pi B$ and $e \in E$ with $q(e) = \theta(0)$ and $\rho(\theta) = \beta(0)$. Here

$$t = \begin{cases} \nu(\theta,e) & \text{for} \quad \ell(\beta) \leq \frac{1}{2} \\ 2\nu(\theta,e)\ell(\beta) & \text{for} \quad \ell(\beta) \geq \frac{1}{2} \end{cases}$$

and $\lambda: I \to \Pi B$ is the map with

$$\ell(\beta(u)) = \ell(\beta) + \ell(h(\theta,e)(u))$$

$$\lambda(u)(v) = \begin{cases} h(\theta,e)(u)(v) & \text{for} \quad 0 \leq v \leq \ell(h(\theta,e)(u)) \\ \beta(v-\ell(h(\theta,e)(u))) & \text{for} \quad v \geq \ell(h(\theta,e)(u)) \end{cases}$$

Also, β_t is the path in ΠB of length t with

$$\beta_t(v) = \beta(\ell(\beta) - t + v) \quad \text{for} \quad v \in [0,\infty].$$

For $\beta \in \Pi B$, define $\tilde{\beta}: (\Gamma q)^{-1}\beta(0) \to (\Gamma\Gamma q)^{-1}\rho(\beta)$ by $\tilde{\beta}(\theta,e) = (\beta(\theta,e))$. Then the composite $\xi\tilde{\beta}: (\Gamma q)^{-1}(\beta(0)) \to (\Gamma q)^{-1}(\rho(\beta))$ is a based map so Γq is a based fibration.

Note that the composites $\xi\tilde{\beta}$ and $\xi(\widetilde{\beta^{-1}})$ are based homotopy inverses. It is easy to construct unbased homotopies exhibiting them as unbased homotopy inverses and to check that these homotopies induce trivial loops at the basepoint. Since the fibres of Γq are nondegenerately based, the unbased homotopies can be deformed to based homotopies.

If $q: E \to B$ is an S^n-quasifibration, then $\delta: E \to \Gamma E$ is a fibrewise weak equivalence (by the long exact homotopy sequences of q and Γq) and so Γq is an S^n-space. Our observations about $\xi\tilde{\beta}$ imply that ξ is an S^n-lifting function in the sense of [98, p. 11], and Γq is an S^n-fibration by [98,3.4].

§2. Preliminaries on $\mathscr{I}$-spaces

We recall the notions of $\mathscr{I}$-spaces (or $\mathscr{I}$-functors) and $\mathscr{I}$-monoids from [99,I]. These provide a convenient language for talking about the usual families of universal sphere bundles and spherical fibrations.

Let $\mathscr{I}$ denote the category of finite or countably infinite dimensional real inner product spaces and their linear isometries. Give inner product spaces the topology of the union of their finite dimensional subspaces and give the set $\mathscr{I}(V,V')$ of linear isometries from V to V' its compactly generated function space topology. An $\mathscr{I}$-space (X,ω) is a continuous functor from $\mathscr{I}$ to the category of non-degenerately based spaces together with a (coherently) unital, associative, and commutative Whitney sum $\omega: X \times X \to X \circ \oplus$. We require $X(f)$ to be a closed inclusion for any morphism f in $\mathcal{J}$ and $X(V)$ to be the colimit of its subspaces $X(V')$ where V' runs over the finite dimensional subspaces of V.

The category of $\mathcal{J}$-spaces has products. An $\mathcal{J}$-monoid is a monoid-valued $\mathcal{J}$-space G such that the monoid products $GV \times GV \to GV$ specify a morphism of $\mathcal{J}$-spaces. In particular, we have the $\mathcal{J}$-monoid F with FV the monoid of based homotopy equivalences of the one-point compactification S^V of V. We restrict attention to $\mathcal{J}$-monoids G with a given morphism of $\mathcal{J}$-monoids $G \to F$. We insist that G be group-like, that is, that $\pi_0 GV$ be a group for each V. We also assume that the spaces GV are LEC; this follows from the nondegeneracy of the unit e of GV when G is group-valued, and it holds for F and SF by [51, II.4 and II.6].

Associated to G, we have $\mathcal{J}$-spaces BG and EG, where $EGV = B(*, GV, S^V)$, and a map of $\mathcal{J}$-spaces $\pi: EG \to BG$. Each $\pi: EGV \to BGV$ is an open map by Corollary 1.4, and our standing assumptions ensure that each π is a spherical quasifibration whose section s is a cofibration (see [98, p. 34] and [84]). Note that, contrary to standard usage, we are writing EGV for the total space of the universal spherical quasifibration rather than of the associated principal quasifibration. If G is group-valued, $\pi: EGV \to BGV$ is a numerable GV-bundle with fibre S^V (e.g. [98, p. 40]).

We restrict attention to finite dimensional real inner product spaces (V, V_i, W, etc) in the rest of this section. The $\mathcal{J}$-space structure of BG gives Whitney sum maps

$$\omega: \prod_{s=1}^{k} BGV_s \longrightarrow BG\Big(\bigoplus_{s=1}^{k} V_s\Big)$$

and evaluation maps

$$\varepsilon: \mathcal{J}(V,V') \times BGV \longrightarrow BGV'.$$

For our characterization of extended powers of Thom spectra, we need the following description of the sphere spaces classified by these maps.

Proposition 2.1. (i) The Whitney sum maps of BG and EG induce pullback diagrams

$$\begin{tikzcd}
EGV_1 \wedge EGV_2 \wedge \cdots \wedge EGV_k \arrow[r, "\omega"] \arrow[d, "\pi \wedge \pi \wedge \cdots \wedge \pi"] & EG\big(\bigoplus_{s=1}^{k} V_s\big) \arrow[d, "\pi"] \\
BGV_1 \times BGV_2 \times \cdots \times BGV_k \arrow[r, "\omega"] & BG\big(\bigoplus_{s=1}^{k} V_s\big)
\end{tikzcd}$$

(ii) If $V_s = \bigoplus_{r=1}^{j_s} W_{s,r}$, then the diagram

$$\begin{tikzcd}
\bigwedge_{s=1}^{k} \Big(\bigwedge_{r=1}^{j_s} EGW_{s,r}\Big) \arrow[rr, "\bigwedge_{s=1}^{k} \omega"] \arrow[dr, "\omega"'] & & \bigwedge_{s=1}^{k} EGV_s \arrow[dl, "\omega"] \\
& EG\big(\bigoplus_{s=1}^{k} V_s\big) &
\end{tikzcd}$$

commutes and covers the corresponding diagram of classifying spaces.

Corollary 2.2. If $Z = V \oplus W$, then the pullback of $\pi: EGZ \to BGZ$ along the inclusion $BGV \subset BGZ$ is the fibrewise smash product of $\pi: EGV \to BGV$ and the trivial map $S^W \to *$, that is, the fibrewise suspension

$$\Sigma^W \pi: \Sigma^W EGV \longrightarrow BGV.$$

Let $\eta(V,V'): E(V,V') \to \mathcal{J}(V,V')$ denote the orthogonal complement sphere bundle to the bundle map

$$(\pi_1, \varepsilon): \mathcal{J}(V,V') \times V \longrightarrow \mathcal{J}(V,V') \times V'$$

over $\mathcal{J}(V,V')$ and let $T(V,V')$ be the associated Thom space (compare VI.2.1).

Proposition 2.3. There is a map

$$\tilde{\varepsilon}: E(V,V') \wedge EGV \longrightarrow EGV'$$

such that the diagram

$$\begin{tikzcd}
E(V,V') \wedge EGV \arrow[r, "\tilde{\varepsilon}"] \arrow[d, "\eta(V,V') \wedge \pi"'] & EGV' \arrow[d, "\pi"] \\
\mathcal{J}(V,V') \times BGV \arrow[r, "\varepsilon"] & BGV'
\end{tikzcd}$$

is a pullback.

Proof. We define $\tilde{\varepsilon}$ simplicially. Let $E(V,V')$ and $\mathcal{J}(V,V')$ also denote the constant simplicial spaces whose spaces are all $E(V,V')$ or $\mathcal{J}(V,V')$ and whose face and degeneracy maps are all the identity. Let B_*G and E_*G be the simplicial $\mathcal{J}$-spaces from which BG and EG are derived (see [98,§7]). Define the simplicial map

$$\gamma_*: E(V,V') \times E_*GV \longrightarrow E_*GV'$$

by

$$\gamma_k((f,x),[g_1,\cdots,g_k]y) = [\varepsilon(f,g_1),\cdots,\varepsilon(f,g_k)]x \wedge f(y)$$

where $k \geq 0$, $g_i \in GV$, $(f,x) \in E(V,V')$ and $y \in S^V$. The map γ_* factors through $E(V,V') \wedge E_*GV$ to give a simplicial map $\tilde{\varepsilon}_*$ such that the diagram

$$\begin{tikzcd}
E(V,V') \wedge E_*GV \arrow[r, "\tilde{\varepsilon}_*"] \arrow[d, "\eta(V,V') \wedge \pi"'] & E_*GV' \arrow[d, "\pi"] \\
\mathcal{J}(V,V') \times B_*GV \arrow[r, "\varepsilon_*"] & B_*GV'
\end{tikzcd}$$

is a pullback in the simplicial category. The geometric realization of this diagram is the diagram of the proposition and is a pullback since realization preserves pullbacks.

We need two coherence results about the maps $\tilde{\varepsilon}$ just defined. The first concerns iterated use of ε. For any V, V' and V'', the diagram

$$\begin{tikzcd}
\mathcal{J}(V',V'') \times \mathcal{J}(V,V') \times BGV \arrow[r, "1 \times \varepsilon"] \arrow[d, "c \times 1"'] & \mathcal{J}(V',V'') \times BGV' \arrow[d, "\varepsilon"] \\
\mathcal{J}(V,V'') \times BGV \arrow[r, "\varepsilon"] & BGV''
\end{tikzcd}$$

commutes where c is composition. If

$$\tilde{c}: E(V',V'') \wedge E(V,V') \longrightarrow E(V,V'')$$

is defined by

$$\tilde{c}((f,x) \wedge (g,y)) = (fg, x \wedge f(y)),$$

then the map of sphere spaces

$$(\tilde{c},c): \eta(V',V'') \wedge \eta(V,V') \longrightarrow \eta(V,V'')$$

is a pullback and the following holds.

Proposition 2.4. For any V, V' and V'', the diagram

$$\begin{tikzcd}
\eta(V',V'') \wedge \eta(V,V') \wedge \pi_V \arrow[r, "1 \wedge (\tilde{\varepsilon},\varepsilon)"] \arrow[d, "(\tilde{c},c) \wedge 1"'] & \eta(V',V'') \wedge \pi_{V'} \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] \\
\eta(V,V'') \wedge \pi_V \arrow[r, "(\tilde{\varepsilon},\varepsilon)"] & \pi_{V''}
\end{tikzcd}$$

commutes.

Our second result relates $\tilde{\varepsilon}$ to inclusions of subspaces. Let $V \subset W$ and $V' \subset W'$ and let $\mathcal{J}((W,V),(W',V'))$ be the subspace of $\mathcal{J}(W,W')$ consisting of those maps which take V into V'. Since BG is an $\mathcal{J}$-space, the diagram

(1)
$$\begin{tikzcd}
 & \mathcal{J}((W,V),(W',V')) \times BGV \arrow[dl] \arrow[dr] & \\
\mathcal{J}(V,V') \times BGV \arrow[d, "\varepsilon"] & \mathcal{J}(V,W') \times BGV \arrow[d, "\varepsilon"] & \mathcal{J}(W,W') \times BGW \arrow[d, "\varepsilon"] \\
BGV' \arrow[r] & BGW' \arrow[r, equal] & BGW'
\end{tikzcd}$$

commutes. Lemma 1.7 and Proposition 2.3 yield three descriptions of the sphere space over $\mathcal{J}((W,V), (W',V'))$ obtained by pulling back $\pi: EGW' \to BGW'$. Let $\bar{\eta}(W,W')$, $\bar{\eta}(V,W')$ and $\bar{\eta}(V,V')$ denote the pullbacks of $\eta(W,W')$, $\eta(V,W')$ and $\eta(V,V')$ respectively along the evident maps of $\mathcal{J}((W,V), (W',V'))$ into $\mathcal{J}(W,W')$, $\mathcal{J}(V,W')$ and $\mathcal{J}(V,V')$. There are obvious isomorphisms

$$\bar{\eta}(V,V') \wedge S^{W'-V'} \cong \bar{\eta}(V,W') \cong \bar{\eta}(W,W') \wedge S^{W-V}.$$

These two isomorphisms and the obvious shuffle map provide the relation among our three descriptions above.

Proposition 2.5. With the notation above, the following diagram of spaces over diagram (1) commutes.

$$\begin{tikzcd}
\bar{\eta}(V,V')\wedge\pi_V\wedge S^{W'-V'} \arrow[r, phantom, "\cong"] \arrow[d] & \bar{\eta}(V,W')\wedge\pi_V \arrow[r, phantom, "\cong"] \arrow[d] & \bar{\eta}(W,W')\wedge\pi_V\wedge S^{W-V} \arrow[d] \\
\eta(V,V')\wedge\pi_V\wedge S^{W'-V'} \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] & \eta(V,W')\wedge\pi_V \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] & \eta(W,W')\wedge\pi_W \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] \\
\pi_{V'}\wedge S^{W'-V'} \arrow[r] & \pi_{W'} \arrow[r, equal] & \pi_{W'}
\end{tikzcd}$$

In the next section, we construct Thom spectra indexed on any universe U (see I§2) from maps $f: Y \to BGU$. Since we consider only one universe U, we abbreviate BGU to BG.

Remark 2.6. The space BG is a homotopy commutative and associative H-space with multiplication

$$\phi: BG \times BG \xrightarrow{\omega} BG(U \oplus U) \xrightarrow{k} BGU = BG$$

where $k: U \oplus U \to U$ is the linear isometry chosen for internalizing smash products of spectra indexed on U (see II§3). Unit homotopies come from paths in $\mathcal{L}_1$ and commutativity and associativity homotopies can be written in the form

$$c: BG \times BG \times I \xrightarrow{\omega\times 1} BG(U \oplus U) \times I \xrightarrow{\tilde{c}} BG \quad \text{and}$$

$$a: BG \times BG \times BG \times I \xrightarrow{\omega(\omega\times 1)\times 1} BG(U \oplus U \oplus U) \times I \xrightarrow{\tilde{a}} BG,$$

where the maps $\tilde{c}$ and $\tilde{a}$ come from paths in $\mathcal{L}_2$ and $\mathcal{L}_3$; here $\mathcal{L}_j = \mathcal{J}(U^j, U)$, as in Chapter VII. This precise view of the homotopies is essential for our discussion of ring structures on Thom spectra in section 4. Since BG is connected, the standard shearing map argument provides a map $\chi: BG \to BG$ giving inverses up to homotopy.

Using the H-space structure on BG we can define sums and products of maps into BG. For any maps $f: Y \to BG$ and $g: Z \to BG$, we abuse notation by writing $f \times g: Y \times Z \to BG$ for the composite of $f \times g: Y \times Z \to BG \times BG$ and ϕ. If $Y = Z$, we define $f + g: Y \to BG$ to be $(f \times g)\Delta$.

§3. The definition and basic examples of Thom spectra

We here define the Thom spectrum Mf derived from a map $f: Y \to BG$ and describe the Thom spectra associated to several standard maps. Above, and in the remainder of the chapter, BG denotes the space BGU of the $\mathcal{J}$-space BG of the previous section. Our Thom spectrum construction produces a functor from the

category $\mathcal{U}/BG$ of spaces over BG to the category $\mathcal{S}\mathcal{A}$ of spectra indexed on any indexing set $\mathcal{A}$ in the universe U.

The first step in the construction of Mf is the selection of a filtration of f. By a filtration of a space Y, we mean a set of closed subspaces $F_\alpha Y$ of Y indexed on a directed set $\{\alpha\}$ such that Y is the colimit of the $F_\alpha Y$. Note that any indexing set $\mathcal{A}$ is directed (by inclusion) and has a cofinal subsequence. A filtration $\{F_V f: F_V Y \to BGV \mid V \in \mathcal{A}\}$ of f is a filtration of Y by subspaces $F_V Y$ indexed on $\mathcal{A}$ such that $f(F_V Y) \subset BGV$; we write $F_V f$ for $f|F_V Y$ regarded as a map into BGV. Two types of filtrations are especially useful.

Definition 3.1 (i). The canonical filtration $\{f_V: Y_V \to BGV | V \in \mathcal{A}\}$ of $f: Y \to BG$ is defined by $Y_V = f^{-1}(BGV)$. The set $\{Y_V\}$ is a filtration of Y by Proposition 1.1. This filtration is natural: any map of spaces over BG preserves it.

(ii) If $\{F_V f: F_V Y \to BGV\}$ is any filtration of $f: Y \to BG$, then the associated filtration of $\Gamma f: \Gamma Y \to BG$ is $\{\Gamma(F_V f): \Gamma(F_V Y) \to BGV\}$. The space $\Gamma(F_V Y)$ is closed in ΓY since $F_V Y$ and ΠBGV are closed in Y and ΠBG, and $\Gamma Y = \operatorname{colim} \Gamma(F_V Y)$ since any path in ΠBG lies in some ΠBGV. The point of the associated filtration is that the maps $\Gamma(F_V f): \Gamma(F_V Y) \to BGV$ are fibrations. Note that the natural map $\delta: f \to \Gamma f$ is filtration preserving. Further, if $\lambda: f \to g$ is filtration preserving with respect to chosen filtrations of f and g, then $\Gamma\lambda: \Gamma f \to \Gamma g$ is filtration preserving with respect to the associated filtrations of Γf and Γg.

Construction 3.2. Let $\{F_V f: F_V Y \to BGV\}$ and $\{F_V g: F_V Z \to BGV\}$ be filtrations of maps $f: Y \to BG$ and $g: Z \to BG$ and let $\lambda: f \to g$ be a filtration preserving map. We construct a Thom prespectrum $T(f, \{f_V Y\})$ and a map of prespectra

$$T\lambda: T(f, \{F_V Y\}) \longrightarrow T(g, \{F_V Z\}).$$

For $V \in \mathcal{A}$, the space $T(f, \{F_V Y\})(V)$ is the Thom space $T(\tau_V)$ of the sphere space $\tau_V: E_V \to F_V Y$ classified by $F_V f$. The map

$$T\lambda: T(f, \{F_V Y\})(V) \longrightarrow T(g, \{F_V Z\})(V)$$

is obtained by passage to Thom spaces from the map of sphere spaces induced by λ when $\pi: EGV \to BGV$ is pulled back over the diagram

$$\begin{tikzcd}
F_V Y \arrow[rr, "\lambda"] \arrow[dr, "F_V f"'] & & F_V Z \arrow[dl, "F_V g"] \\
& BGV &
\end{tikzcd}$$

If $V \subset W$ in $\mathcal{A}$, then the structure map

$$\sigma\colon \Sigma^{W-V}T(f,\{F_VY\})(V) \longrightarrow T(f,\{F_VY\})(W)$$

is obtained by passage to Thom spaces from the map of sphere spaces induced by the inclusion $\iota\colon F_VY \to F_WY$ when $\pi\colon EGW \to BGW$ is pulled back over the diagram

$$\begin{tikzcd} F_VY \arrow[r, "\iota"] \arrow[d, "F_Vf"'] & F_WY \arrow[d, "F_Wf"] \\ BGV \arrow[r] & BGW. \end{tikzcd}$$

Corollary 2.2 permits the identification of $\Sigma^{W-V}T(f,\{F_VY\})(V)$ with the Thom space of the resulting sphere space over F_VY. The naturality of this identification ensures that the maps $T\lambda$ commute with the maps σ. Proposition 2.1 (ii) gives the coherence condition required of the maps σ (see I.2.1). Since the maps $i\colon F_VY \to F_WY$ are closed inclusions, the maps σ are closed inclusions by Lemma 1.5. Thus, $T(f,\{F_VY\})$ is a Σ-inclusion prespectrum (see I.8.2). Define the Thom spectrum $M(f,\{F_VY\})$ to be $LT(f,\{F_VY\})$ and the map $M\lambda$ to be $LT\lambda$. We write Tf and Mf for the Thom prespectrum and spectrum associated to the canonical filtration of f. Clearly, use of these filtrations gives functors

$$T\colon \mathcal{U}/BG \longrightarrow \mathcal{P}\mathcal{A} \quad \text{and} \quad M\colon \mathcal{U}/BG \longrightarrow \mathcal{S}\mathcal{A}\,.$$

Remark 3.3. In the appropriate special cases, this construction produces the classical Thom spectra. In particular, the spectrum $M1$ associated to the identity map $1\colon BG \to BG$ is just MG (cf [99, p. 75]). Further, if Y is an $\mathcal{J}$-space with a right G-action and $q\colon B(Y,G,*) \to BG$ is the standard map, then Mq is just the Thom spectrum $M(G;Y)$ of [99, p. 75] (as follows immediately from [98, 7.8]). Note that any map $f\colon Y \to BG$ specifies a morphism $f\colon f \to 1$ in $\mathcal{U}/BG$ and therefore a canonical map $Mf \to MG$.

It is easy to see that any trivial map $Y \to BG$ yields $\Sigma^\infty Y^+$ as a Thom spectrum. In particular, the inclusion of any point into BG yields the sphere spectrum S and the trivial map of the empty set into BG yields the point spectrum $*$.

Since pullbacks of quasifibrations need not be quasifibrations, the construction Mf need not be well-behaved homotopically when G is only monoid-valued. Let us say that a filtration $\{F_Vf\colon F_VY \to BGV\}$ is "good" if the

projections $\tau_V\colon E_V \to F_VY$ are all quasifibrations and the sections $F_VY \to E_V$ are all cofibrations. These conditions hold automatically when G is group-valued. We say that f itself is good if it admits a good filtration with respect to some indexing set $\mathcal{A}$; good maps yield homotopically well behaved Thom spectra. The obvious way around this concern about f is to use Γ to replace the universal quasifibrations by universal fibrations; this approach was carried out in my thesis [83]. It turns out to be technically simpler to exploit the following observation and Remark 1.10, which show that we may instead replace f by a fibration.

Lemma 3.4 (i). If $f\colon Y \to BG$ is a fibration, then its canonical filtration is good.
(ii) For any filtration of any map $f\colon Y \to BG$, the associated filtration of $\Gamma f\colon \Gamma Y \to BG$ is good.

Proof. If $f\colon Y \to BG$ is a fibration, then so is $f_V\colon Y_V \to BGV$ since it is a pullback of f. Also, $\Gamma(F_Vf)\colon \Gamma(F_VY) \to BGV$ is a fibration for any f. Pulling back a quasifibration along a fibration (f_V or $\Gamma(F_Vf)$) yields a quasifibration by a easy argument using long exact homotopy sequences. Pulling back an NDR-pair (like $(EGV, sBGV)$) along a fibration (like $E_V \to EGV$) yields an NDR-pair by [136, Theorem 12].

For good maps f, Mf and $M\Gamma f$ are stably equivalent by Proposition 4.9 below. When f is not good, we take $M\Gamma f$ to be the Thom spectrum of f for homotopy-theoretic purposes. Nevertheless, we continue to discuss Mf for arbitrary f because of our applications to extended powers.

Remark 3.5. In our discussion of extended powers of Thom spectra, we will encounter π-spaces Y and π-maps $f\colon Y \to BG$ where the finite group π acts trivially on BG. If $\{F_Vf\colon F_VY \to BGV\}$ is a π-invariant filtration of such a map f, then the Thom spectrum $M(f,\{F_VY\})$ is easily seen to be a π-spectrum indexed on the trivial π-universe U. Clearly the canonical filtration is π-invariant. If $\lambda\colon f \to g$ is a π-map preserving π-invariant filtrations, then $M\lambda\colon M(f,\{F_VY\}) \to M(g,\{F_VZ\})$ is easily seen to be a π-map.

The following description of the Thom spectrum of the product $f \times g\colon Y \times Z \to BG$ of two maps $f\colon Y \to BG$ and $g\colon Z \to BG$ (see Remark 2.6) is an easy consequence of the way linear isometries enter into both the definition of the product $\phi\colon BG \times BG \to BG$ and the definition of the smash product of spectra (II§3). The proof is omitted since a much more general result is proved in section 6.

Proposition 3.6. For any maps $f\colon Y \to BG$ and $g\colon Z \to BG$, there is a natural isomorphism

$$M(f \times g) \cong Mf \wedge Mg.$$

We give two basic examples of Thom spectra.

Proposition 3.7. Let $f\colon Y \to BG$ be a map which factors through BGW for some finite dimensional subspace W of U.

(i) The spectrum Mf is naturally isomorphic to $\Lambda^W \Sigma^\infty T$, where T is the Thom space of the sphere space classified by $f\colon Y \to BGW$ and $\Lambda^W \Sigma^\infty$ is the shift desuspension functor of I.4.1.

(ii) The spectrum $M\Gamma f$ is stably equivalent to $\Lambda^W \Sigma^\infty T'$ where T' is the Thom space of the spherical fibration classified by $f\colon Y \to BGW$.

Proof. (i) For $V \supset W$, $Tf(V) = \Sigma^{V-W} T$. Thus, on a cofinal subset of $\mathcal{A}$, Tf agrees with the prespectrum-level version of $\Lambda^W \Sigma^\infty T$ (see I.4.1).

(ii) For $V \supset W$, write $\Gamma(f,V)\colon \Gamma(Y,V) \to BGV$ for the fibration obtained by applying Γ to $f\colon Y \to BGV$. These maps give a cofinal part of the filtration of $\Gamma f\colon \Gamma Y \to BG$ associated to the canonical filtration of f. Let $g\colon \Gamma(Y,W) \to BG$ be the composite of $\Gamma(f,W)$ and the inclusion of BGW into BG and let $\lambda\colon g \to \Gamma f$ be the evident inclusion. Give g the (good) filtration obtained by pulling back the cited filtration of Γf along λ. For $V \supset W$, this coincides with the canonical filtration and the V^{th} sphere space $\tau_V\colon E_V \to \Gamma(Y,W)$ is the fibrewise suspension of τ_W by S^{V-W}. Thus, $Mg = \Lambda^W \Sigma^\infty (E_W/\Gamma(Y,W))$. Since T' is equivalent to $E_W/\Gamma(Y,W)$ by Remark 1.10, Mg is equivalent to $\Lambda^W \Sigma^\infty T'$. On the other hand, since the inclusion $\Gamma(Y,W) \to \Gamma(Y,V)$ is obviously a homotopy equivalence, the V^{th} map of Thom spaces induced by λ is a weak equivalence for $V \supset W$, by comparisons of quasifibrations and cofibrations, and it follows that $M\lambda\colon Mg \to M\Gamma f$ is a stable equivalence.

Our second example is a stabilization of the classical description of the Thom space of a spherical fibration over a suspension.

Proposition 3.8. Let A be a nondegenerately based space, $\tilde{g}\colon A \to G$ be a based map, and $g\colon \Sigma A \to BG$ be the adjoint of the composite of $\tilde{g}$ and the standard map $G \to \Omega BG$.

(i) The canonical filtration of g is good.

(ii) If $f\colon \Sigma^\infty A^+ \to S$ is the map adjoint to the composite

$$A^+ \xrightarrow{\tilde{g}^+} G^+ \longrightarrow F^+ \subset QS^0,$$

then Mg is naturally isomorphic to the pushout $S \cup_f \Sigma^\infty (CA)^+$ in the diagram

$$\begin{tikzcd} \Sigma^\infty A^+ \arrow[r] \arrow[d, "f"'] & \Sigma^\infty (CA)^+ \arrow[d] \\ S \arrow[r] & S \cup_f \Sigma^\infty (CA)^+ \end{tikzcd}$$

(iii) Let $f_0\colon \Sigma^\infty A \to S$ be the map adjoint to a based map $A \to QS^0$ homotopic to the composite

$$A \xrightarrow{\tilde{g}} G \longrightarrow F = Q_1 S^0 \amalg Q_{-1} S^0 \simeq Q_0 S^0 \amalg Q_{-2} S^0$$

where the last map comes from translating the components of QS^0 by a map of degree -1. Then Mg is homotopy equivalent to the cofibre Cf_0 of f_0.

Proof. (i) For $V \in \mathcal{A}$, let $A_V = \tilde{g}^{-1}(GV)$. Then the canonical filtration of g is $\{g_V\colon \Sigma A_V \to BGV\}$. We can write ΣA_V as the pushout

$$\begin{tikzcd} A_V \arrow[r, "\iota"] \arrow[d] & CA_V \arrow[d, "p"] \\ * \arrow[r] & \Sigma A_V \end{tikzcd}$$

Thus, the map $\tau_V\colon E_V \to \Sigma A_V$ classified by g_V is a quasifibration by Hardie's result in [57] and the section $s\colon \Sigma A_V \to E_V$ is a cofibration by [84, Prop. 2.5]. The assumption that A is nondegenerately based is needed to ensure that (CA, A) is an NDR-pair.

(ii) By applying Proposition 4.3 below to the pushout description of ΣA analogous to the one above for ΣA_V, we obtain that $M(g)$ is the pushout in the diagram

$$\begin{tikzcd} Mgp\iota \arrow[r] \arrow[d] & Mgp \arrow[d] \\ S = M(*) \arrow[r] & Mg. \end{tikzcd}$$

From the description in [29, p. 126] of the sphere space classified by g_V, we obtain obvious isomorphisms

$$\Sigma^\infty A^+ \cong Mgp\iota \quad \text{and} \quad \Sigma^\infty (CA)^+ \cong Mgp$$

under which the diagram above becomes that defining $S \cup_f \Sigma^\infty (CA)^+$.

(iii) The pair $(\Sigma^\infty (CA)^+, \Sigma A^+)$ is homotopy equivalent to the pair $(S \vee C\Sigma^\infty A, S \vee \Sigma^\infty A)$ by VII.5.2. Since the maps $\Sigma^\infty A^+ \to \Sigma^\infty (CA)^+$ and $\Sigma^\infty A \to C\Sigma^\infty A$ are cofibrations, the pushout $S \cup_f \Sigma(CA)^+$ is homotopy equivalent to the pushout P in the diagram

$$\begin{tikzcd} S \vee \Sigma^\infty A \arrow[r, "1 \vee \iota"] \arrow[d, "{(1,f_0)}"'] & S \vee C\Sigma^\infty A \arrow[d] \\ S \arrow[r] & P \end{tikzcd}$$

and this is the same as the pushout

$$\begin{tikzcd} \Sigma^\infty A \arrow[r] \arrow[d, "f_0"'] & C\Sigma^\infty A \arrow[d] \\ S \arrow[r] & Cf_0 . \end{tikzcd}$$

The left vertical maps in the two diagrams above are identified with f_0 by VII.5.2.

§4. Invariance properties of Thom spectra

We prove that the spectrum Mf depends essentially only on the homotopy class of the composite

$$Y \xrightarrow{f} BG \longrightarrow BF.$$

In particular, it is independent of the choice of the filtration of f, the choice of G, and the choice of the indexing set $\mathcal{A}$. We also prove that the Thom spectrum functor M preserves all colimits. At the end of this section, we apply our invariance results to study ring structures on Thom spectra.

Recall the change of indexing set functors ϕ and ψ of I.2.4.

Lemma 4.1. For an inclusion $\mathcal{A} \subset \mathcal{B}$ of indexing sets, the following diagrams commute up to coherent natural isomorphisms.

$$\begin{tikzcd} & \mathcal{U}/BG \arrow[dl, "M"'] \arrow[dr, "M"] & \\ \mathcal{S}\mathcal{B} \arrow[rr, "\phi"] & & \mathcal{S}\mathcal{A} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & \mathcal{U}/BG \arrow[dl, "M"'] \arrow[dr, "M"] & \\ \mathcal{S}\mathcal{A} \arrow[rr, "\psi"] & & \mathcal{S}\mathcal{B} \end{tikzcd}$$

The result for ϕ is easily seen on the prespectrum level and the result for ψ follows by adjunction.

Let G and G' be $\mathcal{J}$-monoids augmented over F and let $j: G \to G'$ be an augmentation preserving morphism of $\mathcal{J}$-monoids. Denote the Thom spectrum functors on $\mathcal{U}/BG$ and $\mathcal{U}/BG'$ by M and M' respectively.

Lemma 4.2. Let $f: Y \to BG$ be any map

(i) There is a natural spectrum-level isomorphism

$$Mf \cong M'(Bj)f$$

(ii) There is a natural stable equivalence

$$M'(Bj)\Gamma f \longrightarrow M'\Gamma'((Bj)f)$$

where Γ' is the functor converting maps into BG' into fibrations.

Proof. Part (i) is an immediate consequence of [98, 7.8]. For (ii), define $\nu: \Gamma Y \to \Gamma' Y$ to be the map which takes (θ, y) in ΓY to $((Bj)_*\theta, y)$. Clearly, ν is a map from $(Bj)\Gamma f$ to $\Gamma'((Bj)f)$. It is a homotopy equivalence since both $\delta: Y \to \Gamma Y$ and $\delta': Y \to \Gamma' Y$ are. Filter $(Bj)\Gamma f$ and $\Gamma'((Bj)f)$ by $\{(Bj)\Gamma(f_V): \Gamma(Y_V) \to BG'V\}$ and $\{\Gamma'((Bj)f_V): \Gamma'(Y_V) \to BG'V\}$, where $\{f_V\}$ is the canonical fibration of f. These filtrations of $(Bj)\Gamma f$ and $\Gamma'((Bj)f)$ are good, so Proposition 4.9 below applies.

As a result of this lemma, we may think of all Thom spectra as coming from maps into BF or, if we wish to consider only those Thom spectra arising from groups, from maps into $BTOP$.

The fact that $M(f)$ is independent of the choice of filtration of f follows from the fact that M preserves colimits.

Proposition 4.3. The functors

$$T: \mathcal{U}/BG \to \mathcal{P}\mathcal{A} \quad \text{and} \quad M: \mathcal{U}/BG \longrightarrow \mathcal{S}\mathcal{A}$$

preserve all colimits.

Proof. Since $M = LT$, the result for M follows from that for T. Clearly, canonical filtrations commute with colimits in $\mathcal{U}/BG$. For each V in $\mathcal{A}$, pulling back the universal sphere space $\pi: EGV \to BGV$ preserves colimits by Proposition 1.1 and Corollary 1.4. Obviously, passage to Thom spaces preserves colimits. Thus,

T preserves colimits since colimits in $\mathcal{P}\mathcal{A}$ are formed spacewise.

Recall that Γ also preserves colimits so $T\Gamma$ and $M\Gamma$ preserve colimits (see 1.9).

Corollary 4.4. If $\{F_V f\colon F_V Y \to BGV\}$ is a filtration of $f\colon Y \to BG$, then the natural map

$$M(f, \{F_V Y\}) \longrightarrow Mf$$

is an isomorphism of spectra.

Proof. There is such a natural map because the identity map $1\colon f \to f$ is filtration preserving when the domain is filtered by $F_V f$ and the range is given the canonical filtration. For any W in $\mathcal{A}$, the given filtration of $f\colon Y \to BG$ pulls back to the filtration $\{F_V F_W f\colon F_V F_W Y \to BGV \mid V \in \mathcal{A}\}$ of $F_W f$ where $F_V F_W Y = F_V Y \cap F_W Y$. For $V \supset W$, this filtration is the canonical one; hence the natural map

$$M(F_W f, \{F_V F_W Y\}) \longrightarrow MF_W f$$

is an isomorphism. Thus, in the diagram

$$\begin{tikzcd} \operatorname{colim}_W M(F_W f, \{F_V F_W Y\}) \arrow[r, "\cong"] \arrow[d, "\cong"] & M(f, \{F_V Y\}) \arrow[d] \\ \operatorname{colim}_W MF_W f \arrow[r, "\cong"] & Mf \end{tikzcd}$$

the left vertical map is an isomorphism. The bottom horizontal arrow is an isomorphism by Proposition 4.3, and the proof of that result restricts to show that the top horizontal arrow is also an isomorphism.

Of course, Mf is itself the colimit of the $MF_V f$ for any filtration $\{F_V f\}$ of f, and any map from a compact spectrum into Mf factors through some $MF_V f$; compare I.4.8. In particular, using Lemma 1.6 and Proposition 3.7, we easily obtain the following conclusion about $\pi_* Mf$.

Proposition 4.5. If $f\colon Y \to BG$ is good, then Mf is (-1)-connected and, if Y is path connected, $\pi_0(Mf)$ is either Z or Z_2; it is Z if and only if the composite of f and $BG \to BF$ lifts to BSF.

The first step in understanding the homotopy invariance of Thom spectra is to realize what does not work: M does not induce a functor into the stable category from the category with objects the homotopy classes $\langle f\rangle\colon Y \to BG$ and morphisms $\langle\lambda\rangle\colon \langle f\rangle \to \langle g\rangle$ the homotopy classes $\langle\lambda\rangle\colon Y \to Z$ such that f is homotopic to $g\lambda$. We shall see that the Thom spectra induced by (good) homotopic maps are stably equivalent, but the equivalence depends non-trivially on the choice of the homotopy. An analogous difficulty arises even on the level of Thom spaces.

The only homotopies actually preserved by M are homotopies in $\mathcal{U}/BG$, that is, fibrewise homotopies. Such a homotopy may be described as a map in $\mathcal{U}/BG$ with domain

$$Y \times I \xrightarrow{\pi_1} Y \xrightarrow{f} BG,$$

where π_1 is projection onto the first factor. On the prespectrum level, it is easy to identify the Thom spectrum of such a composite.

Proposition 4.6. The Thom spectrum $Mf\pi_1$ associated to the composite

$$Y \times X \xrightarrow{\pi_1} Y \xrightarrow{f} BG$$

is naturally isomorphic to $Mf \wedge X^+$.

Taking $X = I$ as above, we obtain our result on homotopies.

Corollary 4.7. The functors T and M take homotopies in $\mathcal{U}/BG$ into homotopies in $\mathcal{P}\mathcal{A}$ or $\mathcal{S}\mathcal{A}$ and therefore convert fibrewise homotopy equivalences into homotopy equivalences.

Any map $\lambda\colon (f\colon Y \to BG) \to (g\colon Z \to BG)$ in $\mathcal{U}/BG$ has a mapping cylinder

$$g \cup_\lambda (f \times 1)\colon Z \cup_\lambda (Y \times I) \longrightarrow BG,$$

and λ is a fibrewise cofibration if $g \cup_\lambda (f \times 1)$ is a retract over BG of the cylinder

$$Z \times I \xrightarrow{\pi_1} Z \xrightarrow{g} BG.$$

Cofibrations in $\mathcal{S}\mathcal{A}$ have a similar characterization in terms of mapping cylinders. By Proposition 4.6, M converts the cylinder $g\pi_1$ in $\mathcal{U}/BG$ to the cylinder $Mg \wedge I^+$ in $\mathcal{S}\mathcal{A}$ and thus, by Proposition 4.3, M converts mapping cylinders in $\mathcal{U}/BG$ into mapping cylinders in $\mathcal{S}\mathcal{A}$. This gives the following result.

Corollary 4.8. The functor M converts fibrewise cofibrations into cofibrations in $\mathcal{SA}$.

Recall that Γ converts homotopies into fibrewise homotopies and cofibrations into fibrewise cofibrations by Proposition 1.11.

Working in the stable category, we can obtain much more general invariance results than Corollary 4.7.

Proposition 4.9. If $\lambda\colon (f\colon Y \to BG) \to (g\colon Z \to BG)$ is a map between good spaces over BG such that $\lambda\colon Y \to Z$ is a weak equivalence, then $M\lambda\colon Mf \to Mg$ is a stable equivalence. In particular, if $f\colon Y \to BG$ is good, then $M\delta\colon Mf \to M\Gamma f$ is a stable equivalence.

Proof. Note first that if there exist good filtrations $\{F_V f\colon F_V Y \to BGV \mid V \in \mathcal{A}\}$ and $\{F_V g\colon F_V Z \to BGV \mid V \in \mathcal{A}\}$ such that $\lambda\colon f \to g$ preserves these filtrations and $\lambda\colon F_V Y \to F_V Z$ is a weak equivalence for some cofinal subset of $\mathcal{A}$, then $M\lambda\colon Mf \to Mg$ is a stable equivalence. This follows from an elementary argument using the long exact homotopy sequences for the quasifibrations classified by $F_V f$ and $F_V g$ and the fact that the sections of these quasifibrations are cofibrations. Taking a good filtration on f and the associated filtration on Γf, we obtain from this that $M\delta\colon Mf \to M\Gamma f$ is a stable equivalence. Since $M\Gamma\lambda \circ M\delta = M\delta \circ M\lambda$, it now suffices to prove that $M\Gamma\lambda$ is a stable equivalence. If Y and Z are CW complexes, then λ is a homotopy equivalence, $\Gamma\lambda$ is a fibre homotopy equivalence and $M\Gamma\lambda$ is a homotopy equivalence by Corollary 4.7. If Y and Z are not CW complexes, then let $\theta\colon Y' \to Y$, $\phi\colon Z' \to Z$ and $\lambda'\colon Y' \to Z'$ be functorial CW approximations so that $\phi\lambda' = \lambda\theta$. The map λ' is a homotopy equivalence and $M\Gamma\lambda'\colon M\Gamma(f\theta) \to M\Gamma(g\phi)$ is a homotopy equivalence by Corollary 4.7 again. Thus, it suffices to prove that $M\Gamma\theta\colon M\Gamma(f\theta) \to M\Gamma f$ and $M\Gamma\phi\colon M\Gamma(g\phi) \to M\Gamma g$ are stable equivalences. By functoriality, $\theta\colon Y'_V \to Y_V$ and $\phi\colon Z'_V \to Z_V$ are CW approximations and our initial argument applies to give that $M\Gamma\theta$ and $M\Gamma\phi$ are stable equivalences.

Proposition 4.10. If $f_0, f_1\colon Y \to BG$ are homotopic and both are good, then Mf_0 and Mf_1 are stably equivalent.

Proof. By Proposition 4.9, it suffices to show that $M\Gamma f_0$ and $M\Gamma f_1$ are stably equivalent. Let $f\colon Y \times I \to BG$ be a homotopy between f_0 and f_1. The inclusions $\iota_0\colon f_0 \to f$ and $\iota_1\colon f_1 \to f$ (from $\iota_0, \iota_1\colon Y \to Y \times I$) are homotopy equivalences so the maps $\Gamma\iota_0\colon \Gamma f_0 \to \Gamma f$ and $\Gamma\iota_1\colon \Gamma f_1 \to \Gamma f$ are fibre homotopy equivalences. Thus, by Corollary 4.7, $M\Gamma\iota_0$ and $M\Gamma\iota_1$ are homotopy equivalences and $M\Gamma f_0$ is homotopy equivalent to $M\Gamma f_1$.

Notice how the equivalence above depends on the choice of the homotopy. One concise way of expressing the homotopy invariance of Thom spectra is to say that, for each space Y, the composite $M\Gamma$ induces a functor from the fundamental groupoid of BG^Y (the space of maps from Y to BG) to the homotopy category $h\mathcal{SA}$ (and thus, to the stable category $\bar{h}\mathcal{SA}$).

Recall from Remark 2.6 that BG is a homotopy commutative and associative H-space. Thus, it is reasonable to inquire about the relation between Thom spectra and H-maps $f\colon Y \to BG$. We assume that Y is a group-like H-space (that is, $\pi_0(Y)$ is a group), with a homotopy unit e and f is a good map which preserves units, strictly commutes with products and takes a specified unit homotopy for Y to the standard unit homotopy for BG. If the unit of Y is a nondegenerate basepoint and $f\colon Y \to BG$ is an H-map in the classical sense, so that $f\phi$ is only homotopic to $\phi(f \times f)$, then we can alter the multiplication on Y to an equivalent one for which e is a strict unit and alter f to a homotopic map f' strictly preserving the unit. It is then easy to define a multiplication on $\Gamma_{f'}Y$ so that $\Gamma f'\colon \Gamma_{f'}Y \to BG$ satisfies our conditions. Since $M\Gamma f'$, $M\Gamma f$ and, if f is good, Mf are all stably equivalent, our hypotheses are not unreasonable.

From the diagram

$$\begin{tikzcd} Y \times \{e\} \arrow[r, "1 \times e"] \arrow[dr, "\simeq"'] & Y \times Y \arrow[r, "f \times f"] \arrow[d, "\phi"] & BG \times BG \arrow[d, "\phi"] \\ & Y \arrow[r, "f"] & BG \end{tikzcd}$$

in which the square commutes and the triangle homotopy commutes, we obtain the diagram

$$\begin{tikzcd} Mf \wedge S \arrow[r, "1 \wedge Me"] \arrow[dr, "\simeq"'] & Mf \wedge Mf \arrow[d, "M\phi"] \\ & Mf \end{tikzcd}$$

in $\bar{h}\mathcal{SA}$ which asserts that Mf has a multiplication $M\phi$ and a unit $Me\colon S \to Mf$. Here, $M\phi(f \times f)$ and $M\phi(f \times fe)$ have been identified with $Mf \wedge Mf$ and $Mf \wedge S$ by Proposition 3.6. Also the equivalence $Mf \wedge S \simeq Mf$ coming from the homotopy commutativity of the triangle by Proposition 4.10 has been identified with the standard equivalence between $Mf \wedge S$ and Mf (see II.3.12) by Proposition 6.1 below, using the fact that the unit homotopy for Y covers that for BG.

We would like to say that if Y is homotopy associative or commutative, then so is Mf. That this need not be the case is a striking example of the failure of M to be a functor on homotopy classes. Recall the associativity and commutativity homotopies a and c defined for BG in Remark 2.6.

Proposition 4.11. Let $f\colon Y \to BG$ be a good strict H-map, as above.

(i) If an associativity homotopy A for Y can be found so that the diagram

$$\begin{tikzcd} Y \times Y \times Y \times I \arrow[r, "A"] \arrow[d, "f\times f\times f\times 1"'] & Y \arrow[d, "f"] \\ BG \times BG \times BG \times I \arrow[r, "a"] & BG \end{tikzcd}$$

commutes, then the multiplication on Mf is associative in $\bar{h}\mathcal{SA}$.

(ii) If a commutativity homotopy C for Y can be found so that the diagram

$$\begin{tikzcd} Y \times Y \times I \arrow[r, "C"] \arrow[d, "f\times f\times 1"'] & Y \arrow[d, "f"] \\ BG \times BG \times I \arrow[r, "c"] & BG \end{tikzcd}$$

commutes, then the multiplication on Mf is commutative in $\bar{h}\mathcal{SA}$.

Proof. The maps a and c are derived from maps $\tilde{a}\colon I \to \mathcal{L}(3)$ and $\tilde{c}\colon I \to \mathcal{L}(2)$ and Proposition 6.1 (i) applies to give that the sequence

$$Mf \wedge Mf \cong M(f\phi) \xrightarrow{\iota_0} M(fC) \xleftarrow{\iota_1} M(f\phi\tau) \cong Mf \wedge Mf$$

(where τ is the twist map), and the corresponding sequence for A, are exactly the stable equivalences expressing the commutativity and associativity of the smash product in the stable category $\bar{h}\mathcal{SA}$ (see II§3). The required results follow directly from this observation.

It should be noted that, in the presence of the Thom isomorphism, the homology algebra of Mf will be commutative and associative when Y is - even though Mf itself may not be.

We give a more sophisticated method for obtaining Thom spectra with ring structures in section 7.

If the H-space Y has a homotopy inverse $r\colon Y \to Y$, then there is a simple

description of the smash product $Mf \wedge Mf$, due to Mahowald [93], which is useful in forming resolutions and in homotopy calculations.

Proposition 4.12. Let Y be an H space with a homotopy inverse $r\colon Y \to Y$ and assume that $f\colon Y \to BG$ is a good, strict H-map. Then there is an equivalence

$$Y^+ \wedge Mf \simeq Mf \wedge Mf.$$

Proof. Define $\lambda\colon Y \times Y \to Y \times Y$ by

$$\lambda(x,y) = (x, \phi(rx,y))$$

for $x,y \in Y$. Since r is a homotopy inverse, λ is an equivalence. It therefore induces an equivalence

$$M(\Gamma f\phi\lambda) \longrightarrow M(\Gamma f\phi).$$

By Proposition 3.6 and 4.9, $M(\Gamma f\phi)$ is equivalent to $Mf \wedge Mf$. The composite $\phi\lambda$ is homotopic to the projection $\pi_2\colon Y \times Y \to Y$ onto the second factor, and so, by Propositions 4.6, 4.9, and 4.10, $M(\Gamma f\phi\lambda)$ is equivalent to $Y^+ \wedge Mf$.

§5. The Thom isomorphism

Throughout this section, E is to be a commutative ring spectrum. We follow May [99,III.1] in defining orientations.

Definition 5.1. A good map $f\colon Y \to BG$ is said to be oriented if there exists a class $\mu \in E^0 Mf$ such that μ restricts to a generator of the free π_*E-module $E^*M(f_y)$ for each $y \in Y$, where $f_y\colon \{y\} \to BG$ is f evaluated at y. A general map $f\colon Y \to BG$ is said to be oriented if Γf is oriented.

Remarks 5.2. (i) Let $g\colon Z \to BG$ factor as $Z \xrightarrow{\lambda} Y \xrightarrow{f} BG$. If f is E-oriented, then so is g. Thus, if f is E-oriented, then so are all the terms in any filtration of f.

(ii) Defining the orientability of f in terms of Γf is clearly the correct approach when G is only a monoid. If f is good (as it always is when G is a group), the two possible definitions agree.

(iii) If E is an E_∞-ring spectrum, then the obstruction theoretic approach to orientability developed in [99,III] applies to our generalized Thom spectra. In particular, f is integrally orientable if and only if the composite $Y \xrightarrow{f} BG \longrightarrow BF$ factors through BSF.

We describe two Thom isomorphisms. The first relates the Thom spectrum Mf associated to $f\colon Y \to BG$ and the space Y. The second relates the Thom spectra $M(g+f)$ and Mg where $g+f$ is the sum of two maps g and f from Y into BG (see Remark 2.6). The first isomorphism is actually the special case of the second in which g is the trivial map.

From the diagrams

$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "f"'] & Y \times Y \arrow[r, "\pi_2"] & Y \arrow[dl, "f"] \\ & BG & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "g+f"'] & Y \times Y \arrow[d, "g\times f"] \\ & BG \end{tikzcd}$$

we derive the reduced diagonal maps

$$\Delta\colon Mf \longrightarrow Y^+ \wedge Mf \quad \text{and} \quad \Delta\colon M(g+f) \longrightarrow Mg \wedge Mf$$

by using Proposition 3.6 and 4.6 to identify $M(g\times f)$ and $M(f\pi_2)$. If A is a closed subspace of Y, then these maps pass to quotients to give commuting diagrams

$$\begin{tikzcd} M(f|A) \arrow[r] \arrow[d, "\Delta"] & Mf \arrow[r] \arrow[d, "\Delta"] & Mf/M(f|A) \arrow[d, "\Delta"] \\ A^+ \wedge M(f|A) \arrow[r] & Y^+ \wedge Mf \arrow[r] & Y/A \wedge Mf \end{tikzcd}$$

and

$$\begin{tikzcd} M((g+f)|A) \arrow[r] \arrow[d, "\Delta"] & M(g+f) \arrow[r] \arrow[d, "\Delta"] & M(g+f)/M((g+f)|A) \arrow[d, "\Delta"] \\ M(g|A) \wedge M(f|A) \arrow[r] & Mg \wedge Mf \arrow[r] & (Mg/M(g|A)) \wedge Mf \end{tikzcd}$$

We obtain our Thom maps from these reduced diagonals.

Definition 5.3. For any $\mu \in E^0Mf$, the associated spectrum-level Thom maps are the composites Φ_μ and Φ^μ in the commutative diagrams

$$\begin{tikzcd} M(g+f)\wedge E \arrow[r, "\Delta\wedge 1"] \arrow[d, "\Phi_\mu"] & Mg \wedge Mf \wedge E \arrow[d, "1\wedge\mu\wedge 1"] \\ Mg \wedge E & Mg \wedge E \wedge E \arrow[l, "1\wedge\phi"'] \end{tikzcd}$$

and

$$\begin{tikzcd} F(Mg,E) \arrow[r, "F(1,\tilde{\phi})"] \arrow[d, "\Phi^\mu"] & F(Mg,F(E,E)) \cong F(Mg\wedge E,E) \arrow[d, "F(1\wedge\mu,1)"] \\ F(M(g+f),E) & F(Mg\wedge Mf,E) \arrow[l, "F(\Delta,1)"'] \end{tikzcd}$$

Here $\phi\colon E\wedge E \to E$ is the product and $\tilde{\phi}\colon E \to F(E,E)$ is its adjoint. For a closed subspace A of Y, the relative diagonal gives rise to analogous Thom maps making the appropriate diagrams commute. Replacing $M(g+f)$ by Mf and Mg by Y^+, we obtain Thom maps

$$\Phi_\mu\colon Mf\wedge E \longrightarrow Y^+\wedge E \quad \text{and} \quad \Phi^\mu\colon F(Y^+,E) \longrightarrow F(Mf,E)$$

and their compatible relative counterparts.

The induced E-homology and cohomology Thom maps are

$$\Phi_\mu\colon E_nM(g+f) \cong [S^n, M(g+f)\wedge E] \xrightarrow{(\Phi_\mu)_*} [S^n, Mg\wedge E] \cong E_nMg$$

and

$$\Phi^\mu\colon E^nMg \cong [S^{-n}, F(Mg,E)] \xrightarrow{(\Phi^\mu)^*} [S^{-n}, F(M(g+f),E)] \cong E^nM(g+f)$$

and similarly for Y^+ and Mf and the relative cases.

If $A \subset Y$ is a fibrewise cofibration with respect to f, g or $g+f$ as appropriate, then the inclusions $M(f|A) \subset Mf$, $M(g|A) \subset Mg$ and $M((g+f)|A) \subset M(g+f)$ are cofibrations by Corollary 4.8. The corresponding quotient spectra (e.g. $Mf/M(f|A)$) are then equivalent to the cofibres, and the homology and cohomology Thom maps are maps between the long exact sequences of the pairs $(M(g+f), M((g+f)|A))$ and $(Mg, M(g|A))$ or $(Mf, M(f|A))$ and (Y,A).

Remark 5.4. By comparison with the definitions in III§3, we see that Φ_μ and Φ^μ are given by the usual cap and cup product with μ. Explicitly, Φ_μ is the composite

$$E_*M(g+f) \xrightarrow{(\gamma\Delta)_*} E_*(Mf\wedge Mg) \xrightarrow{(?)\backslash\mu} E_*Mg$$

and Φ^μ is the composite

$$E^*Mg \xrightarrow{(?)\wedge\mu} E^*(Mg\wedge Mf) \xrightarrow{\Delta^*} E^*M(g+f).$$

Questions about the behavior of the Thom maps with respect to multiplicative structures quickly reduce to questions about the behavior of μ. For example, if Y is an H-space and f and g are strict H-maps, then $\Phi_\mu\colon M(g+f)\wedge E \to Mg\wedge E$ preserves products if $\mu\colon Mf \to E$ does.

<u>Remark 5.5</u>. In order to ensure the goodness of maps and fibrewise cofibration conditions needed in the relative case, we usually employ the Thom maps Φ_μ and Φ^μ for $\Gamma f\colon \Gamma Y \to BG$ instead of $f\colon Y \to BG$. The maps $\Gamma(g \times f)\colon \Gamma(Y \times Y) \to BG$ and $\Gamma g \times \Gamma f\colon \Gamma_g Y \times \Gamma_f Y \to BG$ do not even have the same domain, so the introduction of Γ produces a minor difficulty in the definition of the reduced diagonal. However, multiplication of paths induces a map $\nu\colon \Gamma_g Y \times \Gamma_f Y \to \Gamma(Y \times Y)$ such that the diagram

$$\begin{tikzcd}
& \Gamma(Y \times Y) \arrow[dr, "\Gamma(g \times f)"] & \\
Y \times Y \arrow[ur, "\delta"] \arrow[dr, "\delta \times \delta"'] & & BG \\
& \Gamma_g Y \times \Gamma_f Y \arrow[uu, "\nu"'] \arrow[ur, "\Gamma g \times \Gamma f"'] &
\end{tikzcd}$$

commutes. The maps δ and $\delta \times \delta$ are both homotopy equivalences, and so ν is an equivalence. Moreover, the maps $\Gamma(g \times f)$ and $\Gamma g \times \Gamma f$ are both good, so we have a stable equivalence

$$M\Gamma(g \times f) \simeq M\Gamma g \times \Gamma f$$

by Proposition 4.9. We use this equivalence to obtain the reduced diagonal

$$\Delta\colon M\Gamma(g+f) \xrightarrow{\Gamma\Delta} M\Gamma(g \times f) \simeq M\Gamma g \wedge M\Gamma f$$

for the Thom maps of $\Gamma(g+f)$.

Of course, our interest in Thom maps comes from the following result.

<u>Theorem 5.6</u>. Let $f\colon Y \to BG$ and $g\colon Y \to BG$ be good maps, A be a closed subspace of Y such that the inclusions of A in Y is a fibrewise cofibration with respect to f, g and $g+f$ and $\mu \in E^0 Mf$ be an orientation for f. Then the spectrum-level Thom maps

$$\Phi_\mu\colon M(g+f)\wedge E \longrightarrow Mg \wedge E$$

$$\Phi_\mu\colon M(g+f)/M((g+f)|A) \wedge E \longrightarrow (Mg/M(g|A)) \wedge E$$

$$\Phi^\mu\colon F(Mg,E) \longrightarrow F(M(g+f),E)$$

and

$$\Phi^\mu\colon F(Mg/M(g|A),E) \longrightarrow F(M(g+f)/M((g+f)|A),E),$$

and the corresponding maps with $g+f$ replaced by f, Mg replaced by Y^+ and $M(g|A)$ replaced by A^+, are stable equivalences. Thus, the corresponding E-homology and cohomology Thom maps are isomorphisms.

<u>Proof</u>. The results for the relative case will follow from the absolute one by 5-lemma arguments. The results for Mf and Y^+ will follow from those for $M(g+f)$ and Mg by taking g to be the trivial map and noting that our two definitions of the Thom map agree up to homotopy under the identification of Mg with $\Sigma^\infty Y^+$ in Remark 3.3. By Proposition 4.9, we can assume that Y is a CW complex and argue inductively over its skeleta Y^n. To arrange that the cofibrations $j_n\colon Y^n \to Y^{n+1}$ are fibrewise cofibrations without cluttering up notations, we rely on the homotopy invariance results of Propositions 4.9 and 4.10 and agree to prefix Γ tacitly to every space and map over BG in the rest of this proof. Thus, $f\colon Y \to BG$ and $f^n\colon Y^n \to BG$ now mean $\Gamma f\colon \Gamma Y \to BG$ and $\Gamma f^n\colon \Gamma Y^n \to BG$.

By I.4.8,

$$\pi_*(Mg \wedge E) = \operatorname{colim} \pi_*(Mg^n \wedge E)$$

and

$$\pi_*(M(g+f)\wedge E) = \operatorname{colim} \pi_*(M(g+f)^n \wedge E).$$

To show that $\Phi_\mu\colon M(g+f)\wedge E \to Mg \wedge E$ is a stable equivalence, it suffices to show that

$$\Phi_{\mu_n}\colon M(g+f)^n \wedge E \longrightarrow Mg^n \wedge E$$

is a stable equivalence for $n \geq 0$. Here $\mu_n \in E^0 Mf^n$ is the orientation of f^n obtained by restriction from μ.

For Φ^μ, note that the maps

$$F(M(g+f)^{n+1},E) \longrightarrow F(M(g+f)^n,E)$$

and

$$F(Mg^{n+1},E) \longrightarrow F(Mg^n,E)$$

are fibrations so, using Milnor's $\lim^1$ sequence for the limit of a sequence of fibrations, we again reduce the problem to proving the equivalence on skeleta.

For $n = 0$, the required equivalences follow directly from the definition of an orientation. For $n > 0$, we have the commuting diagram

$$\begin{tikzcd}
M(g+f)^n \wedge E \arrow[r] \arrow[d, "\Phi_{\mu_n}"] & M(g+f)^{n+1} \wedge E \arrow[r] \arrow[d, "\Phi_{\mu_{n+1}}"] & (M(g+f)^{n+1}/M(g+f)^n \wedge E) \arrow[d, "\Phi_{\mu_{n+1}}"] \\
Mg^n \wedge E \arrow[r] & Mg^{n+1} \wedge E \arrow[r] & (Mg^{n+1}/Mg^n) \wedge E
\end{tikzcd}$$

in which the rows are cofibre sequences and an analogous diagram for Φ^μ in which the rows are fibre sequences. By induction, it suffices to prove that the relative Thom maps $\Phi_{\mu_{n+1}}$ and $\Phi^{\mu^{n+1}}$ of these diagrams are stable equivalences.

Since Y^{n+1} is the cofibre of the attaching map of the $(n+1)$-cells, Mg^{n+1}/Mg^n and $M(g+f)^{n+1}/M(g+f)^n$ have compatible wedge decompositions into pieces each of which comes from an individual cell and we can reduce to the relative case $Y = e^{n+1}$ and $A = S^n \subset e^{n+1}$. Here e^{n+1} is contractible to a point y, and Propositions 4.9 and 4.10 and Remark 3.3 provide stable equivalences

$$Mg^{n+1}/Mg^n \simeq S^{n+1}$$

$$M(g+f)^{n+1}/M(g+f)^n \simeq S^{n+1}$$

and

$$Mf^{n+1} \simeq S.$$

Under these equivalences, the reduced diagonal

$$\Delta\colon M(g+f)^{n+1}/M(g+f)^n \longrightarrow (Mg^{n+1}/Mg^n) \wedge Mf^{n+1}$$

becomes the standard equivalence

$$S^{n+1} \simeq S^{n+1} \wedge S.$$

We conclude that the map $\Phi_{\mu_{n+1}}$ is the composite

$$S^{n+1} \wedge E \simeq S^{n+1} \wedge S \wedge E \xrightarrow{1\wedge\mu_y} S^{n+1} \wedge E \wedge E \xrightarrow{1\wedge\phi} S^{n+1} \wedge E$$

where μ_y is the restriction of μ to the Thom spectrum $M(f|\{y\}) = S$. Since

μ_y is a generator of $\pi_* M(f|\{y\})$, this map is an isomorphism. The proof for $\Phi^{\mu^{n+1}}$ is similar.

Remark 5.7. In X.5.4, inverses of the Thom isomorphisms Φ_μ and Φ^μ are constructed. However, the construction does not provide an alternative proof of the theorem since it uses an orientation on the inverse χf of $f\colon Y \to BG$, which apparently can only be obtained by using that the Thom map is an isomorphism.

§6. Extended powers of Thom spectra

Our basic result here is that the extended powers of the Thom spectrum Mf derived from $f\colon Y \to BG$ are Thom spectra induced by maps constructed naturally from f. The key to the construction of these maps is the action of the linear isometries operad $\mathcal{L}$ on BG. Recall that the operad $\mathcal{L}$ for our universe U has spaces $\mathcal{L}_j = \mathcal{I}(U^j, U)$ and that $\mathcal{L}$ acts on BG via maps $\theta\colon \mathcal{L}_j \times BG^j \to BG$. For a map $\chi\colon X \to \mathcal{L}_j$ and maps $f_r\colon Y_r \to BG$, $1 \leq r \leq j$, we agree to abuse notation by writing $\chi \times \prod_r f_r$ for the composite of θ and this product. The possible equivariance of these maps can be described using the notation of VI.5.1. If π is a subgroup of Σ_j, $p\colon \underline{j} \to \underline{k}$ is a π-equivariant partition of $\underline{j} = \{1,\cdots,j\}$, $\chi\colon X \to \mathcal{L}_j$ is a π-map, and $f_s\colon Y_s \to BG$ are maps, $1 \leq s \leq k$, then $\chi \times \prod_r f_{p(r)}$ is π-equivariant if we let π act trivially on BG. Thus, by Remark 3.5, $M(\chi \times \prod_r Y_{p(r)})$ is a π-spectrum. Also, we have a map

$$\chi \times_\pi \prod_r f_{p(r)}\colon X \times_\pi \prod_{r=1}^{j} Y_{p(r)} \longrightarrow BG$$

obtained by passage to orbits. In particular, for any π-map $\chi\colon X \to \mathcal{L}_j$ and any $f\colon Y \to Bg$, we have the map

$$\chi \times_\pi f^j\colon X \times_\pi Y^j \longrightarrow BG.$$

Clearly these constructions are natural in χ and their variable spaces over BG.

Recall the twisted half smash products of VI§1.

Proposition 6.1. (i) For maps $\chi\colon X \to \mathcal{L}_j$ and $f_r\colon Y_r \to BG$, $1 \leq r \leq i$, there is a spectrum-level isomorphism

$$\chi \ltimes \Big(\bigwedge_{r=1}^{j} Mf_r\Big) \cong M\big(\chi \times \prod_r f_r\big)$$

which is as equivariant as the choices of χ and the f_r allow.

(ii) For $\pi \subset \Sigma_j$, a π-map $\chi\colon X \to \mathcal{L}_j$, a π-equivariant partition $p\colon \underline{j} \to \underline{k}$ and maps $f_s\colon Y_s \to BG$, $1 \le s \le k$, $M(\chi \times_\pi \prod_r f_{p(r)})$ is the orbit spectrum of the π-spectrum $M(\chi \times \prod_r f_{p(r)})$. Thus, the isomorphism of part (i) induces an isomorphism

$$\chi \ltimes_\pi (\bigwedge_{r=1}^{j} Mf_{p(r)}) \cong M(\chi \times_\pi \prod_r f_{p(r)}).$$

In particular, for any $f\colon Y \to BG$,

$$\chi \ltimes_\pi (Mf)^{(j)} \cong M(\chi \times_\pi f^j).$$

Proof. (i) Since M preserves colimits and commutes with the change of indexing set functors ϕ and ψ, we may assume that $\chi\colon X \to \mathcal{L}_j$ is compact and we may select the indexing sets on U^j and U and the connection between them as we please. We may also filter the maps f_r and $\chi \times \prod_r f_r$ as we please. Let $\mathcal{A}$ be the standard indexing set on U. We use the indexing set $\mathcal{A}^j$ on U^j and select any χ-connection $(\mu,\nu)\colon \mathcal{A}^j \to \mathcal{A}$. For $V \in \mathcal{A}$, define $\nu_r V \in \mathcal{A}$ for $1 \le r \le j$ by the equation $\nu V = \nu_1 V \oplus \nu_2 V \oplus \cdots \oplus \nu_j V \in \mathcal{A}^j$. Let $Y' = X \times \prod_r Y_r$ and $f' = \chi \times \prod_r f_r\colon Y' \to BG$. Let $\{(f_r)_V\colon (Y_r)_V \to BGV\}$ be the canonical filtration of f_r for $1 \le r \le j$. Filter f' by $\{F_V f'\colon F_V Y' \to BGV\}$ where $F_V Y' = X \times \prod_{r=1}^{j} (Y_r)_{\nu_r V}$. Since (μ,ν) is a χ-connection, $F_V f'$ does map into BGV as required.

Let R' be the Thom prespectrum $T(f',\{F_V Y'\})$ and let R be the prespectrum-level extended power $\chi \ltimes (\bigwedge_{r=1}^{j} Tf_r)$. For $V \in \mathcal{A}$,

$$RV = T(\nu V,V) \wedge (\bigwedge_{r=1}^{j} (Tf_r)(\nu_r V))$$

where $(Tf_r)(\nu_r V)$ is the Thom space of the sphere space classified by

$$(f_r)_{\nu_r V}\colon (Y_r)_{\nu_r V} \longrightarrow BG(\nu_r V)$$

and $T(\nu V,V)$ is the Thom space defined in VI.2.1. By Lemma 1.7 and Propositions 2.1 and 2.3,

$$RV \cong R'V.$$

Propositions 2.1 (ii) and 2.5 show that the structure maps of R and R' commute with these isomorphisms, so that $R \cong R'$. We obtain our isomorphism of spectra by applying L and quoting Corollary 4.4. The equivariance is obvious on the prespectrum-level.

(ii) That $M(\chi \times_\pi \prod_r f_{p(r)})$ is the orbit spectrum of $M(\chi \times \prod_r f_{p(r)})$ follows

on the prespectrum-level from Lemma 1.8. The rest of (ii) follows from this fact and (i).

To relate our description of the extended powers of a Thom spectrum to the study of operad ring structures on Thom spectra, we must relate the isomorphisms of Proposition 6.1 to the maps α and β of VI.3 and the map ζ of VII.1.6(i). For $k \ge 0$ and $j_s \ge 0$, $1 \le s \le k$, and for maps

$$\chi\colon X \longrightarrow \mathcal{L}_k$$

$$\chi_s\colon X_s \longrightarrow \mathcal{L}_{j_s}$$

$$f_{s,r}\colon Y_{s,r} \longrightarrow BG, \qquad 1 \le s \le k \text{ and } 1 \le r \le j_s,$$

the shuffle map

$$\nu\colon X \times (\prod_{s=1}^{k} X_s \times (\prod_{r=1}^{j_s} Y_{s,r})) \longrightarrow (X \times \prod_{s=1}^{k} X_s) \times \prod_{s=1}^{k}\prod_{r=1}^{j_s} Y_{s,r}$$

induces a map

$$\nu\colon \chi \times \prod_s (\chi_s \times \prod_r f_{s,r}) \longrightarrow \chi \circ (\oplus_s \chi_s) \times (\prod_s\prod_r f_{s,r})$$

in $\mathcal{U}/BG$. Here, as always, we adopt the convention that the empty product is a point and the empty smash product is a zero sphere.

Proposition 6.2. With the notation above, the diagram

$$\begin{array}{ccc}
\chi \ltimes (\bigwedge_{s=1}^{k} \chi_s \ltimes \bigwedge_{r=1}^{j_s} Mf_{s,r}) & \xrightarrow{\cong} & \chi \ltimes (\bigwedge_{s=1}^{k} M(\chi_s \times \prod_r f_{s,r})) \\
\downarrow{\scriptstyle 1\ltimes\alpha} & & \downarrow{\scriptstyle\cong} \\
\chi \ltimes ((\oplus_{s=1}^{k} \chi_s) \ltimes (\bigwedge_{s=1}^{k}\bigwedge_{r=1}^{j_s} Mf_{s,r})) & & M(\chi \times \prod_s(\chi_s \times \prod_r f_{s,r})) \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle M(\nu)} \\
(\chi \circ (\oplus_{s=1}^{k} \chi_s)) \ltimes (\bigwedge_{s=1}^{k}\bigwedge_{r=1}^{j_s} Mf_{s,r}) & \xrightarrow{\cong} & M((\chi \circ (\oplus_{s=1}^{k} \chi_s)) \times (\prod_s\prod_r f_{s,r}))
\end{array}$$

commutes on the spectrum level (and therefore up to all possible equivariance) where the unlabeled isomorphisms are those of Proposition 6.1.

Proof. Again, we may assume that χ and the χ_s are compact and may select whatever indexing sets, connections, and filtrations we like. We make the obvious choices in terms of the given data, trace through the explicit prespectrum-level definitions, and find that both composites in our diagram are given by shuffle maps followed by identifications of smash products of Thom spaces with Thom spaces of fibrewise smash products.

Remark 6.3. Propositions 6.1 and 6.2 are abbreviated versions of general results relating Thom spectra derived from the Whitney sum map $\omega\colon BGV \times BGW \to BG(V+W)$ and the $\mathcal{J}$-space evaluation map $\mathcal{J}(V,W) \times BGV \to BGW$ to external smash products and twisted half-smash products of Thom spectra. These results also relate the maps ι, α, β, and δ of VI§3, when applied to twisted half-smash products of Thom spectra, to maps of Thom spectra analogous to the map $M\nu$ of Proposition 6.2. When G is a group, these more general results are given in X§6. We leave the monoid case to the interested reader since our primary concern is with Thom spectra which are operad ring spectra.

To relate the isomorphisms of Proposition 6.1 to the map ζ of VII.1.6(i), we consider an operad $\mathcal{C}$ argumented over $\mathcal{L}$ by maps $\chi_j\colon \mathcal{C}_j \to \mathcal{L}_j$ for $j \geq 0$. For $k > 0$, $j_s > 0$ $(1 \leq s \leq k)$, and $j = j_1 + j_2 + \cdots + j_k$, and for $f_{s,r}\colon Y_{s,r} \to BG$ $(1 \leq s \leq k,\ 1 \leq r \leq j_s)$, let $\tilde{\zeta}$ be the composite in the commuting diagram

$$\begin{tikzcd}
\mathcal{C}_k \times \prod\limits_{s=1}^{k} \Big(\mathcal{C}_{j_s} \times \prod\limits_{r=1}^{j_s} Y_{s,r}\Big) \arrow[r, "\nu"] \arrow[dr, "\tilde{\zeta}"'] & \Big(\mathcal{C}_k \times \prod\limits_{s=1}^{k} \mathcal{C}_{j_s}\Big) \times \Big(\prod\limits_{s=1}^{k} \prod\limits_{r=1}^{j_s} Y_{s,r}\Big) \arrow[d, "\gamma\times 1"] \\
 & \mathcal{C}_j \times \Big(\prod\limits_{s=1}^{k} \prod\limits_{r=1}^{j_s} Y_{s,r}\Big)
\end{tikzcd}$$

where γ is the structure map of $\mathcal{C}$. Clearly, $\tilde{\zeta}$ provides a map

$$\tilde{\zeta}\colon \chi_k \times \prod_s \Big(\chi_{j_s} \times \prod_r f_{s,r}\Big) \longrightarrow \chi_j \times \Big(\prod_s \prod_r f_{s,r}\Big)$$

in $\mathcal{U}/BG$. The relation between ζ and $M(\tilde{\zeta})$ follows easily from Proposition 6.2 and the naturality of the isomorphisms of Proposition 6.1.

Proposition 6.4. For $k > 0$, $j_s > 0$, and $f_{s,r}\colon Y_{s,r} \to BG$ $(1 \leq s \leq k$ and $1 \leq r \leq j_s)$, the diagram below commutes on the spectrum-level (and so up to all possible equivariance) where the unlabeled isomorphisms are those of Proposition 6.1.

$$\begin{tikzcd}
\mathcal{C}_k \ltimes \Big(\bigwedge\limits_{s=1}^{k} \mathcal{C}_{j_s} \ltimes \Big(\bigwedge\limits_{r=1}^{j_s} Mf_{s,r}\Big)\Big) \arrow[r, "\cong"] \arrow[dd, "\zeta"'] & \mathcal{C}_k \ltimes \Big(\bigwedge\limits_{s=1}^{k} M\Big(\chi_{j_s} \times \prod\limits_r f_{s,r}\Big)\Big) \arrow[d, "\cong"] \\
 & M\Big(\chi_k \times \prod\limits_s \Big(\chi_{j_s} \times \prod\limits_r f_{s,r}\Big)\Big) \arrow[d, "M(\tilde{\zeta})"] \\
\mathcal{C}_j \ltimes \Big(\bigwedge\limits_{s=1}^{k} \bigwedge\limits_{r=1}^{j_s} Mf_{s,r}\Big) \arrow[r, "\cong"] & M\Big(\chi_j \times \Big(\prod\limits_s \prod\limits_r f_{s,r}\Big)\Big)
\end{tikzcd}$$

§7. Thom spectra and operad ring spectra

We now shift our attention from arbitrary maps into BG and their associated Thom spectra to certain special maps into BG which yield Thom spectra with enriched ring structures. Specifically, BG, being an $\mathcal{L}$-space, is a $\mathcal{C}$-space for any operad $\mathcal{C}$ augmented over $\mathcal{L}$, and the maps to be considered are $\mathcal{C}$-maps of $\mathcal{C}$-spaces into BG. The Thom spectra derived from these maps have $\mathcal{C}$ structures. For example, if $\mathcal{C}$ is an E_∞-operad, then the maps being considered are infinite loop maps and the resulting Thom spectra are E_∞-ring spectra. If $\mathcal{C} = \mathcal{C}_n \times \mathcal{L}$, where $\mathcal{C}_n$ is the n^{th} little cubes operad, then the maps are n-fold loop maps and the Thom spectra are E_n-ring spectra.

Operads act on based spaces and the category $\mathcal{J}/BG$ of based spaces over BG replaces $\mathcal{U}/BG$ in this section. All the results on Thom spectra in the previous sections pass over to $\mathcal{J}/BG$ via the forgetful functor $\mathcal{J}/BG \to \mathcal{U}/BG$. In addition, the Thom spectrum Mf derived from a map $f\colon Y \to BG$ in $\mathcal{J}/BG$ has a canonical unit $S \to Mf$ coming from the inclusion of the basepoint $\{*\} \subset Y$.

Let $\mathcal{C}\mathcal{J}/BG$ be the subcategory of $\mathcal{J}/BG$ whose objects are $\mathcal{C}$-maps $f\colon Y \to BG$ and whose morphisms $\lambda\colon (f\colon Y \to BG) \to (g\colon Z \to BG)$ are $\mathcal{C}$-maps $\lambda\colon Y \to Z$ over BG. Related to the operad $\mathcal{C}$ is a monad $C\colon \mathcal{J} \to \mathcal{J}$, which takes a based space Y to the associated free $\mathcal{C}$-space CY. This monad induces a functor $C\colon \mathcal{J}/BG \to \mathcal{C}\mathcal{J}/BG$ which takes $f\colon Y \to BG$ to $Cf\colon CY \to BG$, where Cf is the unique $\mathcal{C}$-map extending f. In fact, $C\colon \mathcal{J}/BG \to \mathcal{C}\mathcal{J}/BG$ is a monad and the category $\mathcal{C}\mathcal{J}/BG$ is just the category of algebras over this monad.

Note that if $f\colon Y \to BG$ is a $\mathcal{C}$-map, then so is $\Gamma f\colon \Gamma Y \to BG$ by Lemma 1.8 of [97, p.7]. Also, if $\lambda\colon f \to g$ is a map in $\mathcal{C}\mathcal{J}/BG$, then so is $\Gamma\lambda\colon \Gamma f \to \Gamma g$. Thus Γ induces a functor from $\mathcal{C}\mathcal{J}/BG$ to $\mathcal{C}\mathcal{J}/BG$ which converts arbitrary $\mathcal{C}$-maps into fibrations which are $\mathcal{C}$-maps.

An operad $\mathcal{C}$ augmented over $\mathcal{L}$ also induces a monad C on the category $\mathcal{S}_e\mathcal{A}$ of unital spectra (see VII§4); the functor C takes a unital spectrum $e: S \to E$ to its associated free $\mathcal{C}$-spectrum CE. If $f: Y \to BG$ is in $\mathcal{J}/BG$, then Mf has a canonical unit and thus an associated free $\mathcal{C}$-spectrum CMf. Mahowald made the key observation that the spectra MCf and CMf should be isomorphic. In fact, more is true; M takes the space-level monad $C: \mathcal{J}/BG \to \mathcal{J}/BG$ to the spectrum level monad $C: \mathcal{S}_e\mathcal{A} \to \mathcal{S}_e\mathcal{A}$.

<u>Theorem 7.1</u>. For any $f: Y \to BG$ in $\mathcal{J}/BG$, there is a natural isomorphism

$$CMf \cong MCf$$

such that the following diagrams commute.

$$\begin{tikzcd} CMf \arrow[r, phantom, "\cong"] & MCf \\ & Mf \arrow[ul, "\eta"] \arrow[u, "M\eta"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} CCMf \arrow[r, "\mu"] \arrow[d, phantom, "\cong" rotate=90] & CMf \arrow[dd, "\cong"] \\ CMCf \arrow[d, phantom, "\cong" rotate=90] & \\ MCCf \arrow[r, "M\mu"] & MCf \end{tikzcd}$$

Here the maps η and μ are the units and multiplications of the two monads. Thus, if f is a map of $\mathcal{C}$-spaces, then Mf is a $\mathcal{C}$-spectrum with action map

$$CMf \cong MCf \xrightarrow{M\xi} Mf$$

where $\xi: CY \to Y$ gives the action of $\mathcal{C}$ on Y. Also, M takes maps of $\mathcal{C}$-spaces over BG into maps of $\mathcal{C}$-spectra.

<u>Proof</u>. Let $\sigma_q: \mathcal{C}_{r+1} \to \mathcal{C}_r$ be the degeneracy map defined in VII§3 for $r \geq 0$ and $0 \leq q \leq r$. If $\chi_{r,q}: \mathcal{C}_{r+1} \to \mathcal{L}_r$ is the composite

$$\chi_{r,q}: \mathcal{C}_{r+1} \xrightarrow{\sigma_q} \mathcal{C}_r \xrightarrow{\chi_r} \mathcal{L}_r$$

and the maps g and h below are defined as in VII.3.6, then the diagram

$$\begin{tikzcd}[column sep=small] \coprod\limits_{r\geq 0} \coprod\limits_{0\leq q\leq r} \mathcal{C}_{r+1} \times Y^r \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] \arrow[dr, "\coprod\limits_{r\geq 0} \coprod\limits_{0\leq q\leq r} \chi_{r,q} \times f^r"'] & \coprod\limits_{r\geq 0} \mathcal{C}_r \times_{\Sigma_r} Y^r \arrow[r] \arrow[d, "\coprod\limits_{r\geq 0} \chi_r \times_{\Sigma_r} f^r"] & CY \arrow[dl, "Cf"] \\ & BG & \end{tikzcd}$$

is a coequalizer diagram over BG by VII.3.6. Thus, the bottom row of the diagram

$$\begin{tikzcd} \bigvee\limits_{r\geq 0} \bigvee\limits_{0\leq q\leq r} \chi_{r,q} \ltimes Mf^{(r)} \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] \arrow[d, "\theta_1", "\cong"'] & \bigvee\limits_{r\geq 0} \mathcal{C}_r \ltimes_{\Sigma_r} Mf^{(r)} \arrow[r] \arrow[d, "\theta_2", "\cong"'] & CMf \arrow[d, "\theta", "\cong"'] \\ \bigvee\limits_{r\geq 0} \bigvee\limits_{0\leq q\leq r} M(\chi_{r,q} \times f^r) \arrow[r, shift left, "Mg"] \arrow[r, shift right, "Mh"'] & \bigvee\limits_{r\geq 0} M(\chi_r \times_{\Sigma_r} f^r) \arrow[r] & MCf \end{tikzcd}$$

is a coequalizer. The top row of the diagram is a coequalizer by VII.3.6 if the maps g' and h' are as indicated there. Proposition 6.1 provides the isomorphisms θ_1 and θ_2 and ensures that

$$\theta_2 h' = (Mh)\theta_1.$$

Proposition 6.2 gives that

$$\theta_2 g' = (Mg)\theta_1.$$

Thus, there is a unique isomorphism

$$\theta: CMf \longrightarrow MCf$$

making the right hand square above commute.

The commutativity of diagrams in the statement of the theorem follows easily from the definitions of the units and multiplications in VII§4 and Propositions 6.1 and 6.4.

If Y is a $\mathcal{C}$-space whose action is given by the map

$$\xi: CY \longrightarrow Y$$

and $f: Y \to BG$ is a $\mathcal{C}$-map, then the diagram

$$\begin{tikzcd} CY \arrow[r, "\xi"] \arrow[dr, "Cf"'] & Y \arrow[d, "f"] \\ & BG \end{tikzcd}$$

commutes and ξ induces a map $M\xi: MCf \to Mf$. Our diagrams for η and μ imply that the map

$$CMf \cong MCf \xrightarrow{M\xi} Mf$$

gives an action of $\mathcal{C}$ on Mf. That M takes $\mathcal{C}$-maps over BG into maps of $\mathcal{C}$-spectra follows from the naturality of the isomorphism $CMf \cong MCf$.

Remark 7.2. (i) If $f: Y \to BG$ is a $\mathcal{C}$-map, then we have already noted that $\Gamma f: \Gamma Y \to BG$ is a $\mathcal{C}$-map. Thus $M\Gamma f$ is a $\mathcal{C}$-spectrum.

(ii) If $f: Y \to BG$ is a $\mathcal{C}$-map, then the action maps

$$\mathcal{C}_j \ltimes_{\Sigma_j} Mf^{(j)} \longrightarrow Mf$$

of the $\mathcal{C}$-spectrum $M(f)$ are just the composites

$$\mathcal{C}_j \ltimes_{\Sigma_j} Mf^{(j)} \cong M(\chi_j \times_{\Sigma_j} f^j) \xrightarrow{M\xi} Mf$$

where $\xi: \mathcal{C}_j \times_{\Sigma_j} Y^j \to Y$ gives the action of $\mathcal{C}$ on Y.

(iii) Recall that the free $\mathcal{C}$-spectrum functor preserves stable equivalences when the units of the spectra in question are cofibrations. If $\{*\} \to Y$ is a fibrewise cofibration over BG and $f: Y \to BG$ is a based map, then the unit of Mf is a cofibration by Corollary 4.8. The standard whiskering construction on Y converts any based map $f: Y \to BG$ into a based map $f': Y' \to BG$ such that $\{*\} \to Y'$ is a fibrewise cofibration over BG, and f' is a $\mathcal{C}$-map if f is by Lemma A.11 of [97, p.170]. It is easy to see that Mf' is just the mapping cylinder of the unit $S \to Mf$, so that we are just converting a map into a cofibration in the usual fashion.

(iv) In order to obtain the isomorphism $CMf \cong MCf$ of Theorem 7.1, we need only assume that $\mathcal{C}$ is a coefficient system augmented over $\mathcal{L}$ (see VII§3).

Remark 7.3. The notion of an $\bar{h}\mathcal{C}$-space Y - that is, a $\mathcal{C}$-space up to homotopy - is defined in VII.2.7, and there is an obvious corresponding notion of an $\bar{h}\mathcal{C}$-map between $\bar{h}\mathcal{C}$-spaces. We might naively hope that an $\bar{h}\mathcal{C}$-map $f: Y \to BG$ from an $\bar{h}\mathcal{C}$-space Y into BG would yield an $\bar{h}\mathcal{C}$-Thom spectrum Mf. However, we encounter the same difficulty here that we encounter with H-spaces and H-maps in Proposition 4.11 - the homotopies must fit properly with the action of the linear isometries operad on BG.

Rather than attempting to describe the general consistency condition needed for $\bar{h}\mathcal{C}$ structures, we discuss here the one case of special interest. Assume that $\mathcal{C}$ is an E_∞-operad, that Y is a $\mathcal{C}$-space, and f and g are $\mathcal{C}$-maps from Y to BG. The sum map $g + f: Y \to BG$ defined in Remark 2.6 is not a $\mathcal{C}$-map, but it is an $\bar{h}\mathcal{C}$-map (see [97, p.8]) and the required homotopies come directly from the universal properties of the spaces $\mathcal{C}_j$. It follows easily that $M(g + f)$ is an $\bar{h}\mathcal{C}$-spectrum and the reduced diagonal map

$$\Delta: M(g + f) \longrightarrow M(g \times f) \cong Mg \wedge Mf$$

of the previous section is a map of $\bar{h}\mathcal{C}$-spectra. Here we use Proposition 3.6 to identify $M(g \times f)$ with $Mg \wedge Mf$.

If Y is a $\mathcal{C}$-space, then so is $Y \times Y$ and the maps

$$\Delta: Y \longrightarrow Y \times Y \quad \text{and} \quad \pi_2: Y \times Y \longrightarrow Y$$

are $\mathcal{C}$-maps. Thus, if $f: Y \to BG$ is a $\mathcal{C}$-map, then the reduced diagonal map

$$\Delta: Mf \longrightarrow Y^+ \wedge Mf$$

is a map of $\mathcal{C}$-spectra, where $\mathcal{C}$ acts diagonally on $Y^+ \wedge Mf$. Assume that E is an $\bar{h}\mathcal{C}^d$-spectrum. Then $Mf \wedge E$ and $Y^+ \wedge E$, or in the setting of Remark 7.3, $M(g + f) \wedge E$ and $Mg \wedge E$, are $\bar{h}\mathcal{C}^d$-spectra with the diagonal $\mathcal{C}$-action. With these observations, it is easy to see that the map μ is the only obstruction to either of the Thom maps Φ_μ of the previous section being maps of $\bar{h}\mathcal{C}^d$-spectra.

Proposition 7.4. Let E be an $\bar{h}\mathcal{C}^d$-spectrum, $f: Y \to BG$ be a $\mathcal{C}$-map and $\mu: Mf \to E$ be a map of $\bar{h}\mathcal{C}$-spectra.

(i) The Thom map

$$\Phi_\mu: Mf \wedge E \longrightarrow Y^+ \wedge E$$

is a map of $\bar{h}\mathcal{C}^d$-spectra and the homology Thom map

$$\Phi_\mu: E_* Mf \longrightarrow E_* Y$$

preserves Dyer-Lashof operations.

(ii) If $\mathcal{C}$ is an E_∞ operad and $g: Y \to BG$ is a $\mathcal{C}$-map, then the Thom map

$$\Phi_\mu: M(g + f) \wedge E \longrightarrow Mg \wedge E$$

is a map of $\bar{h}\mathcal{C}^d$-spectra and the homology Thom map

$$\Phi_\mu: E_* M(g + f) \longrightarrow E_* Mg$$

preserves Dyer-Lashof operations.

Remark 7.5. By $[H_\infty, I.3.6]$, if R is a commutative ring, $f: Y \to BG$ is a good $\mathcal{C}$-map, and $\mu: Mf \to HR$ is a unital map (where HR is the Eilenberg-Mac Lane spectrum for R), then μ is an $\bar{h}\mathcal{C}$-map.

If $f_0\colon Y \to BG$ and $f_1\colon Y \to BG$ are $\mathcal{C}$-maps which are good and are homotopic as $\mathcal{C}$-maps, then it is natural to ask about the relation between the stable equivalence $Mf_0 \simeq Mf_1$ of Proposition 4.10 and the $\mathcal{C}$ structures on Mf_0 and Mf_1. Since the stable equivalence does not come from a map of $\mathcal{C}$-spaces, it is not clear that it respects the $\mathcal{C}$ structures.

Proposition 7.6. If $f_0, f_1\colon Y \to BG$ are $\mathcal{C}$-maps which are good and are homotopic as $\mathcal{C}$-maps, then the stable equivalence

$$Mf_0 \simeq Mf_1$$

of Proposition 4.10 is an equivalence of $\bar{h}\mathcal{C}$-spectra.

Proof. Since $\delta\colon Mf_i \to M\Gamma f_i$ (for $i = 0,1$) is a stable equivalence of $\bar{h}\mathcal{C}$-spectra by Proposition 4.9 and Remark 7.2, it suffices to show that the equivalence between $M\Gamma f_0$ and $M\Gamma f_1$ given in the proof of Proposition 4.10 is a equivalence of $\bar{h}\mathcal{C}$-spectra.

Let $f_t\colon Y \to BG$, $t \in I$, be a homotopy through $\mathcal{C}$ maps from f_0 to f_1. Denote the total space of $\Gamma(f_t)$ by $\Gamma_t Y$. For $r \geq 0$, let

$$D_r = \bigcup_{t\in I} (\Gamma_t Y)^r \subset (\Gamma(Y \times I))^r$$

and let

$$\tilde{\xi}_r\colon \mathcal{C}_r \times_{\Sigma_r} D_r \longrightarrow \Gamma(Y \times I)$$

be the map induced by the action maps $\mathcal{C}_r \times_{\Sigma_r} (\Gamma_t Y)^r \to \Gamma_t Y \subset \Gamma(Y \times I)$. Note that the inclusions $\iota_0\colon (\Gamma_0 Y)^r \to D_r$ and $\iota_1\colon (\Gamma_1 Y)^r \to D_r$ are homotopy equivalences; ι_0 has homotopy inverse $k\colon D_r \to (\Gamma_0 Y)^r$ given by

$$k(\theta_1, y_1, \theta_2, y_2, \cdots, \theta_r, y_r) = (\theta_1', y_1, \theta_2', y_2, \cdots, \theta_r', y_r)$$

where $y_i \in Y$, $\theta_i \in \Pi BG$, and the paths $\theta_i' \in \Pi BG$ have length $1 + \ell(\theta_i)$ and are given by

$$\theta_i'(s) = \begin{cases} f_{st}(y_i) & 0 \leq s \leq 1 \\ \theta_i(s-1) & s \geq 1. \end{cases}$$

Let $Z \subset C\Gamma(Y \times I)$ be the image of the composite

$$\coprod_{r\geq 0} \mathcal{C}_r \times_{\Sigma_r} D_r \subset \coprod_{r\geq 0} \mathcal{C}_r \times_{\Sigma_r} (\Gamma(Y \times I))^r \longrightarrow C\Gamma(Y \times I)$$

and let

$$\tilde{\xi}\colon Z \longrightarrow \Gamma(Y \times I)$$

be the map induced by the maps $\tilde{\xi}_r$. The inclusions $\iota_t\colon (\Gamma_t Y)^r \to D_r$ induce inclusions $\tilde{\iota}_t\colon C\Gamma_t Y \to Z$ such that the diagram

$$\begin{tikzcd}
 & C\Gamma(Y \times I) \arrow[ddd, bend left=90, "C\Gamma f"] & \\
C\Gamma_0 Y \arrow[ur, "C\Gamma\iota_0"] \arrow[r, "\tilde{\iota}_0"'] \arrow[d, "\xi_0"'] & Z \arrow[u] \arrow[d, "\tilde{\xi}"] & C\Gamma_1 Y \arrow[ul, "C\Gamma\iota_1"'] \arrow[l, "\tilde{\iota}_1"] \arrow[d, "\xi_1"] \\
\Gamma_0 Y \arrow[r, "\Gamma\iota_0"] \arrow[dr, "\Gamma f_0"'] & \Gamma(Y \times I) \arrow[d, "\Gamma f"] & \Gamma_1 Y \arrow[l, "\Gamma\iota_1"'] \arrow[dl, "\Gamma f_1"] \\
 & BG &
\end{tikzcd}$$

commutes. The maps $\Gamma\iota_0$, $\Gamma\iota_1$, $C\Gamma\iota_0$, and $C\Gamma\iota_1$ are obviously all fibre homotopy equivalences. Moreover, the maps $\tilde{\iota}_0$ and $\tilde{\iota}_1$ are fibre homotopy equivalences because the homotopy inverses and homotopies for the various $\iota_0\colon (\Gamma_0 Y)^r \to D_r$ fit together to give a homotopy inverse and homotopies for $\tilde{\iota}_0$ and similarly for $\tilde{\iota}_1$. Applying M and the isomorphism of Theorem 7.1, we obtain the diagram

$$\begin{tikzcd}
 & CM\Gamma f & \\
CM\Gamma f_0 \arrow[ur, "CM\Gamma\iota_0"] \arrow[r, "\simeq", "M\tilde{\iota}_0"'] \arrow[d, "\xi_0"'] & M((\Gamma f)\tilde{\xi}) \arrow[u] \arrow[d, "M\tilde{\xi}"] & CM\Gamma f_1 \arrow[ul, "CM\Gamma\iota_1"'] \arrow[l, "\simeq"', "M\tilde{\iota}_1"] \arrow[d, "\xi_1"] \\
M\Gamma f_0 \arrow[r, "\simeq", "M\Gamma\iota_0"'] & M\Gamma f & M\Gamma f_1 \arrow[l, "\simeq"', "M\Gamma\iota_1"]
\end{tikzcd}$$

in which all the maps, except $M(\tilde{\xi})$ and the action maps ξ_0 and ξ_1, are homotopy equivalences. Thus, if $(M\Gamma\iota_1)^{-1}$ is a homotopy inverse for $M\Gamma\iota_1$, then

$$(M\Gamma\iota_1)^{-1} M\Gamma\iota_0\colon M\Gamma f_0 \longrightarrow M\Gamma f_1$$

is a homotopy equivalence and a map of $\bar{h}\mathcal{C}$-spectra.

X. Equivariant Thom spectra

by L. G. Lewis, Jr. and J. P. May

We here present an equivariant analog of the theory of Thom spectra developed in the previous chapter. For technical simplicity, and to avoid unresolved problems with classifying spaces, we shall restrict attention to Thom G-spectra associated to G-vector bundles (as opposed to spherical G-fibrations). On the other hand, while we concentrated on stable fibrations, that is to say on virtual fibrations of virtual dimension zero, in the previous chapter, we shall here study virtual bundles of arbitrary virtual dimension. As a result, the nonequivariant specialization of the present treatment has real advantages over the specialization to vector bundles of the earlier treatment. In fact, we shall introduce a new variant of the usual Grassmannian model for BO that gives a concrete geometric model for $BO \times Z$.

However, we are interested primarily in the equivariant context, and our model for $BO \times Z$ will generalize to give a very simple Grassmannian classifying space representing the functor $KO_G(X)$. While we shall be precise about preliminaries in sections 1 and 2, the reader may wish to consult Segal [125] and Atiyah [7,8] for discussions of equivariant K-theory. We shall concentrate on KO_G for definiteness, but everything we do applies verbatim to KU_G. Perhaps the most interesting new phenomenon we uncover concerns the usual completion map

$$\alpha\colon KO_G(Y) \longrightarrow KO_G(EG \times Y) \cong KO(EG \times_G Y)$$

induced by the projection $EG \times Y \to Y$. We shall prove (in Corollary 6.3) that, for $f \in KO_G(Y)$,

$$EG \ltimes_G Mf \simeq M(\alpha(f)).$$

In particular, with $Y = \{*\}$ and $f = -V \in RO(G) = KO_G(*)$, $Mf = S^{-V}$ and thus

$$EG \ltimes_G S^{-V} \simeq M(\alpha(-V)).$$

Here $\alpha(-V) \in KO(BG)$, and $M(\alpha(-V))$ is often denoted BG^{-V}. Such spectra played an important role in Carlsson's work on the Segal conjecture [22] and in work on its generalizations to theories other than stable cohomotopy; see [24,103,109]. They are singled out for attention in [105].

The organization of this chapter is precisely parallel to that of the previous one. We summarize the relevant information about G-vector bundles in section 1 and about $G\mathcal{J}$-spaces in section 2. We define our Thom G-spectra and give their basic properties in section 3. We study their homotopy invariance properties in section 4. We discuss the Thom isomorphism in section 5. Finally, we relate Thom G-spectra

to twisted half-smash products and to operad actions in section 6. Throughout, G is to be a compact Lie group.

§1. Preliminaries on G-vector bundles

Let $G\mathcal{U}/B$ denote the category of (compactly generated, weak Hausdorff) G-spaces over a given G-space B. Since colimits in $G\mathcal{U}/B$ are colimits in $\mathcal{U}/B$ with the induced G-actions, the conclusions of IX.1.1 on the preservation of colimits apply as stated in the equivariant context.

We could develop equivariant analogs of the results on sphere spaces in IX§1, but we prefer to restrict attention to G-vector bundles, by which we understand $(G,O(n))$-bundles with fibre $\mathbb{R}^n$. See e.g. [81,125] and IV§1 for discussion of such bundles. We may give any G-vector bundle a G-invariant Euclidean metric, and the projection of any such bundle is an open map.

The Thom G-space $T(\xi)$ of a G-vector bundle $\xi\colon E \to B$ is obtained by applying fibrewise one-point compactification to obtain a G-sphere bundle S^ξ and then identifying all the points at infinity. Up to G-homeomorphism, $T(\xi)$ can also be described as the quotient of the unit disc bundle $D(\xi)$ by the unit sphere bundle $S(\xi)$. We think in terms of the diagram

$$\begin{tikzcd}
S(\xi) \arrow[r] \arrow[d, "\xi"'] & D(\xi) \arrow[r] \arrow[d, "\bar{\xi}"] & D(\xi)/S(\xi), \arrow[d, phantom, "\cong" rotate=90] \\
B \arrow[r, "\sigma"] & S^\xi \arrow[r] & T(\xi)
\end{tikzcd}$$

where σ is the section given by the points at ∞ and $\bar{\xi}$ is specified fibrewise by identifying all vectors of length 1 to the point at infinity and expanding the interior of the unit disk radially to fill up $\mathbb{R}^n$.

Maps of G-vector bundles are defined in the evident way, so that a map is given by a pullback diagram. If the map of base spaces of a G-bundle map is a closed inclusion, then so is the map of total spaces and the induced map of Thom spaces.

The product of G-vector bundles is a G-vector bundle, and the product of G-bundle maps is a G-bundle map. We record the behavior of Thom spaces on products.

Lemma 1.1. For G-vector bundles $\xi_i\colon E_i \to B_i$, $i \leq i \leq n$,

$$T(\xi_1 \times \cdots \times \xi_n) \cong T(\xi_1) \wedge \cdots \wedge T(\xi_n).$$

As usual, the Whitney sum $\xi \oplus \psi$ of G-vector bundles ξ and ψ over a given base space is the pullback $\Delta^*(\xi \times \psi)$. There are two standard descriptions of $T(\xi \oplus \psi)$ corresponding to the two descriptions of Thom spaces in the diagram above.

<u>Lemma 1.2</u>. For G-vector bundles ξ and ψ over B,

$$T(S^\xi)^*(\psi)/T(\psi) \cong T(\xi \oplus \psi) \cong T(D\xi)^*(\psi)/T(S\xi)^*(\psi).$$

Here $(S^\xi)^*(\psi)$, $(D\xi)^*(\psi)$, and $(S\xi)^*(\psi)$ denote the G-vector bundles obtained by pulling ψ back along the projections of S^ξ, $D(\xi)$, and $S(\xi)$; $T(\psi)$ is contained in $T(S^\xi)^*(\psi)$ via the G-bundle map induced by pulling $(S^\xi)^*(\psi)$ back to ψ along σ.

Of course, $T(D\xi)^*(\psi)$ is G-homotopy equivalent to $T(\psi)$. When ψ is the trivial bundle $\underline{V}$, that is, the projection $B \times V \to B$, the first identification specializes to

$$T(\xi \oplus \underline{V}) \cong \Sigma^V T(\xi).$$

We also need some observations about the behavior of Thom spaces with respect to change of groups and passage to orbits.

<u>Lemma 1.3</u>. If $\xi: E \to B$ is an H-vector bundle, $H \subset G$, then $G \times_H \xi: G \times_H E \to G \times_H B$ is a G-vector bundle and

$$T(G \times_H \xi) \cong G^+ \wedge_H T(\xi).$$

<u>Lemma 1.4</u>. Let ξ be a $(G \times G')$-vector bundle, let ψ be a G'-trivial $(G \times G')$-vector bundle, and assume given a map $\xi \to \psi$ of $(G \times G')$-vector bundles. Then ξ has an orbit G-vector bundle ξ/G', $\xi \to \psi$ factors through a G-bundle map $\xi/G' \to \psi$, and

$$T(\xi/G') \cong T(\xi)/G'.$$

§2. <u>Preliminaries on $G\mathscr{J}$-spaces</u>

Recall the category $\mathscr{J}$ of inner product spaces and linear isometries from IX§2. Let $G\mathscr{J}$ denote the full subcategory of $\mathscr{J}$ whose objects are G-inner product spaces. For objects V and W of $G\mathscr{J}$, give the space $\mathscr{J}(V,W)$ of linear isometries $V \to W$ the action of G by conjugation.

We define $G\mathscr{J}$-spaces exactly as we defined $\mathscr{J}$-spaces, but with the following equivariance conditions.

(a) Each $X(V)$ is a nondegenerately based G-space.

(b) Each evaluation map $\varepsilon: \mathscr{J}(V,W) \times X(V) \to X(W)$ is a G-map.

(c) Each sum map $\omega: X(V) \times X(W) \to X(V \oplus W)$ is a G-map.

Our treatment of Thom G-spectra is entirely based on an appropriate universal Grassmannian map of $G\mathscr{J}$-spaces $\pi: EO_G \to BO_G$, and a little discussion of nonequivariant Grassmanns will clarify the construction. There is an obvious Grassmann $\mathscr{J}$-space which assigns

$$Gr(V) = O(V \oplus V)/O(V) \times O(V)$$

to a finite dimensional inner product space V (see [99,I§1] for details.) If dim $V = n$, one thinks of $Gr(V)$ as an approximation to $BO(n)$. Consider $V \subset W$, dim $W = n+k$. The embedding $Gr(V) \to Gr(W)$ really involves two distinct kinds of partial stabilization, involving both passage to an approximation to $BO(n+k)$ and an improvement of the degree of approximation. This is better seen in terms of the interpretation of $Gr(V)$ as the space of n-planes in $V \oplus V$. Here we take the first copy of V in $V \oplus V$ as the base plane; application of $f \in O(V \oplus V)$ to V gives the n-plane associated to the coset $f(O(V) \times O(V))$. Now the inclusion of $Gr(V)$ in $Gr(W)$ sends an n-plane A in $V \oplus V$ to the (n+k)-plane $A + (W - V)$ in $W \oplus W$, where $W - V$ sits in the first copy of W. Addition of $W - V$ in the first copy corresponds to stabilization $BO(n) \to BO(n+k)$, while the presence of $W - V$ in the second copy improves the degree of approximation.

It is desirable to have actual classifying spaces on hand at the outset, and this can easily be arranged by replacing the second copy of V by V^∞, the sum of countably many copies of V, in our original description of $Gr(V)$. In terms of planes, we may as well consider n-planes in V^∞, the base n-plane being the first copy of V.

It is also desirable to obtain not BO but $BO \times Z$ as the homotopy type of the "Grassmann" associated to an infinite dimensional inner product space. This can easily be arranged by allowing planes of all dimensions in V^∞ in our $\underline{V^{th}}$ spaces for finite dimensional V. If $BO(q,V)$ denotes the space of q-planes in V^∞ (topologized as $O(V^\infty)/O(A_q) \times O(V^\infty - A_q)$ for any chosen base q-plane A_q in V^∞), then the $\underline{V^{th}}$ space of our classifying $\mathscr{J}$-space is now $\coprod_{q \geq 0} BO(q,V)$ with basepoint $V \in BO(n,V)$, dim $V = n$. In terms of KO-theory, if a map $f: X \to BO(q,V)$ classifies the q-plane bundle ξ, then we think of f as corresponding to $\xi - n$ in $KO(X)$.

We now proceed to the equivariant case. For a finite dimensional G-inner product space V, define $BO_G(q,V)$ to be the G-space of q-planes in V^∞, where G acts by translation of planes. Similarly, define $EO_G(q,V)$ to be the G-space of pairs (A,a), where $A \in BO_G(q,V)$ and $a \in A$. let

$$\pi(q,V)\colon EO_G(q,V) \longrightarrow BO_G(q,V)$$

be the evident projection; it is clearly a G-vector bundle. Define

$$BO_G(V) = \coprod_{q\geq 0} BO_G(q,V) \quad \text{and} \quad EO_G(V) = \coprod_{q\geq 0} EO_G(q,V).$$

Give $BO_G(V)$ and $EO_G(V)$ the basepoints V and $(V,0)$, where the base plane V is the first copy of V in V^∞. For $V \subset W$, embed $EO_G(V)$ in $EO_G(W)$ by sending (A,a) to $(A + (W - V),a)$, where $W - V$ is regarded as a subplane of the base plane W. For an infinite dimensional G-inner product space U, define $EO_G(U)$ to be the colimit of the $EO_G(V)$ for V finite dimensional in U. Note that the unique point of $EO_G(0,\{0\})$ is a canonical representative for the basepoint of $EO_G(U)$. For a linear isometry $k\colon U \to U'$, define $k_*\colon EO_G(U) \to EO_G(U')$ by passage to colimits over $V' \subset U'$ from the restrictions $k_*\colon EO_G(V) \to EO_G(V')$, where $V = k^{-1}(V')$ and k_* sends a point $(A,a) \in EO_G(V)$ to the point $(kA + (V' - kV),ka) \in EO_G(V')$. With the sums ω being given by addition of planes and vectors, EO_G is a $G\mathcal{J}$-space. We specify the G-space $BO_G(U)$ similarly and have the desired universal Grassmannian map $\pi\colon EO_G \to BO_G$ of $G\mathcal{J}$-spaces.

We do not claim that $BO_G(q,V)$ is a classifying G-space for q-plane G-bundles, although this does hold if G is finite and V contains all irreducible representations of G. However, on passage to colimits, we obtain the following basic observation.

Proposition 2.1. Let U be a complete G-universe. Then there is a natural isomorphism

$$\nu\colon [X^+,BO_G(U)]_G \longrightarrow KO_G(X)$$

for finite G-CW complexes X.

Proof. We have added a disjoint basepoint to X since the bracket refers to based G-maps. We may as well assume that X/G is connected. Of course,

$$[X^+,BO_G(U)]_G \cong \operatorname{colim}[X^+,BO_G(V)]_G,$$

where V runs over the indexing G-spaces in U, and a map $X \to BO_G(V)$ lands in some $BO_G(q,V)$. We can describe $KO_G(X)$ as the Abelian group with generators of the form $\xi - \underline{V}$, where ξ is a G-vector bundle over X and $\underline{V}$ is the trivial G-vector bundle $X \times V$, modulo the relations

$$\xi - V = \xi' - V'$$

if and only if there exists Z such that $Z \supset V$, $Z \supset V'$, and

$$(*) \qquad \xi \oplus (\underline{Z - V}) \approx \xi' \oplus (\underline{Z - V'}).$$

For a G-map $f\colon X \to BO_G(q,V)$, let $\xi = f^*\pi(q,V)$ and define $\nu(f) = \xi - V$. If $V \subset W$, $\dim(W - V) = k$, then

$$(i_*f)^*(\pi(q + k,W)) = \xi + (\underline{W - V}),$$

where $i_*\colon BO_G(q,V) \to BO_G(q + k,W)$ is the inclusion, hence $\nu(i_*f) = \nu(f)$. Thus $\nu\colon [X^+,BO_G(U)]_G \to KO_G(X)$ is well defined. If $\nu(f) = \nu(f')$, where $f'\colon X \to BO_G(q',V')$ classifies ξ', and if Z is as in $(*)$, then f and f' become homotopic in $BO_G(r,Z)$, where $r = q + \dim(Z - V) = q' + \dim(Z - V')$. Thus ν is an injection. To see that ν is a surjection, recall from Segal [125, p. 134-135] that any q-plane G-bundle ξ over X embeds as a subbundle of some $\underline{W}$ and is therefore a pullback of the Grassmann G-bundle of vectors in q-planes in W. For any given V, we can expand W to contain V. Thinking of ξ as embedded using the first copy of W in W^∞ and $W - V$ as embedded in the second copy, we see that $\xi \oplus (\underline{W - V})$ is a pullback of $\pi(q + k,W)$, $k = \dim(W - V)$. Thus $\xi - V = \xi \oplus (W - V) - W$ is in the image of ν.

For infinite G-CW complexes X, we define

$$KO_G(X) = [X^+,BO_G(U)]_G.$$

By a standard abuse, we shall sometimes abbreviate $BO_G(U)$ to BO_G.

For any G-universe U, the passage from $G\mathcal{J}$-spaces to $\mathcal{L}(U)$-spaces, where $\mathcal{L}(U)$ is the linear isometries G-operad of U, works exactly as in the nonequivariant case (see [99,I§1]). Thus $BO_G(U)$ is an $\mathcal{L}(U)$-space and in particular a homotopy associative and commutative Hopf G-space: the discussion in IX.2.6 applies verbatim. Since $\pi_0^G(BO_G(U))$ is a group, the standard shearing map argument works equivariantly to show that $BO_G(U)$ admits a G-map $\chi\colon BO_G(U) \to BO_G(U)$ which specifies inverse elements up to G-homotopy, $\phi(1 \times \chi)\Delta \simeq *$.

In the next few results, we restrict attention to finite dimensional G-inner product spaces and say more about the relationship of the map $\pi\colon EO_G \to BO_G$ of $G\mathcal{J}$-spaces to G-vector bundles. We have already used our first observation.

Lemma 2.2. For a G-linear isometry $i\colon V \to W$, $\dim V = n$ and $\dim W = n + k$, the pullback of $\pi(q + k,W)$ along $i_*\colon BO_G(q,V) \to BO_G(q + k,W)$ is $\pi(q,V) \oplus \underline{(W - iV)}$.

Lemma 2.3. The sum maps ω of EO_G and BO_G specify maps of G-vector bundles

$$\pi(q_1,V_1) \times \cdots \times \pi(q_n,V_n) \to \pi(q_1 + \cdots + q_n, V_1 \oplus \cdots \oplus V_n).$$

Of course, the coherence of the ω (as in IX.2.1(ii)) is part of the definition of a $G\mathcal{I}$-space.

As in VI.2.1, let $\zeta(V,V')\colon E(V,V') \to \mathcal{I}(V,V')$ denote the complementary G-vector bundle to the bundle map

$$(\pi_1,\varepsilon)\colon \mathcal{I}(V,V') \times V \to \mathcal{I}(V,V') \times V'$$

over $\mathcal{I}(V,V')$. Let $\zeta(X;V,V')$ denote the pullback of $\zeta(V,V')$ along a given G-map $\chi\colon X \to \mathcal{I}(V,V')$.

Lemma 2.4. The pullback of $\pi(q + k,V')$ along the evaluation map

$$\varepsilon\colon \mathcal{I}(V,V') \times BO_G(q,V) \to BO_G(q + k,V')$$

is $\zeta(V,V') \times \pi(q,V)$, where $k = \dim V' - \dim V$.

Proof. The fibre of $\zeta(V,V')$ over $f\colon V \to V'$ is $V' - fV$ and the fibre of $\pi(q,V)$ over the point $A \in BO_G(q,V)$ is the q-plane A. Define

$$\tilde{\varepsilon}\colon E(V,V') \times EO_G(q,V) \to EO_G(q + k,V')$$

by $\tilde{\varepsilon}(v',a) = f(a) + v'$ for $f\colon V \to V'$, $v' \in V' - fV$, and $a \in A \in BO_G(q,V)$. Then $(\tilde{\varepsilon},\varepsilon)$ is a map of G-vector bundles.

The pullback of $\zeta(V,V'')$ along the composition map

$$c\colon \mathcal{I}(V',V'') \times \mathcal{I}(V,V') \to \mathcal{I}(V,V'')$$

is $\zeta(V',V'') \times \zeta(V,V')$, and c is covered by the G-bundle map

$$\tilde{c}\colon E(V',V'') \times E(V,V') \to E(V,V'')$$

specified by $\tilde{c}(v'',v') = g(v') + v''$ for $f\colon V \to V'$, $g\colon V' \to V''$, $v' \in V' - fV$, and $v'' \in V'' - gV'$.

Lemma 2.5. The following diagram of G-bundle maps commutes, where $k = \dim V' - \dim V$ and $k' = \dim V'' - \dim V'$.

$$\begin{tikzcd}
\zeta(V',V'') \times \zeta(V,V') \times \pi(q,V) \arrow[r, "1 \times (\tilde{\varepsilon},\varepsilon)"] \arrow[d, "(\tilde{c},c) \times 1"'] & \zeta(V',V'') \times \pi(q + k,V') \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] \\
\zeta(V,V'') \times \pi(q,V) \arrow[r, "(\tilde{\varepsilon},\varepsilon)"] & \pi(q + k + k',V'')
\end{tikzcd}$$

As above IX.2.5, let $\mathcal{I}((W,V),(W',V'))$ denote the evident G-space of relative isometries when $V \subset W$ and $V' \subset W'$ and let $\bar{\zeta}(W,W')$, $\bar{\zeta}(V,W')$, and $\bar{\zeta}(V,V')$ denote the pullbacks of $\zeta(W,W')$, $\zeta(V,W')$, and $\zeta(V,V')$ along the obvious maps of $\mathcal{I}((W,V),(W',V'))$ into $\mathcal{I}(W,W')$, $\mathcal{I}(V,W')$, and $\mathcal{I}(V,V')$. We then have isomorphisms

$$\bar{\zeta}(V,V') \oplus \underline{(W' - V')} \cong \bar{\zeta}(V,W') \cong \bar{\zeta}(W,W') \oplus \underline{(W-V)}.$$

Lemma 2.6. The following diagram of G-bundle maps commutes, where $k = \dim V' - \dim V$, $m = \dim W' - \dim W$, and $s = \dim W - \dim V$.

$$\begin{tikzcd}
(\bar{\zeta}(V,V') \times \pi(q,V)) \oplus \underline{(W'-V')} \arrow[r, phantom, "\cong"] \arrow[d] & \bar{\zeta}(V,W') \times \pi(q,V) \arrow[r, phantom, "\cong"] \arrow[d] & \bar{\zeta}(W,W') \times (\pi(q,V) \oplus \underline{(W-V)}) \arrow[d] \\
(\zeta(V,V') \times \pi(q,V)) \oplus \underline{(W'-V')} \arrow[d, "(\tilde{\varepsilon},\varepsilon)\oplus 1"'] & \zeta(V,W') \times \pi(q,V) \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] & \zeta(W,W') \times \pi(q+s,W) \arrow[d, "(\tilde{\varepsilon},\varepsilon)"] \\
\pi(q+k,V') \oplus \underline{(W'-V')} \arrow[r] & \pi(q+m+s,W') \arrow[r, equal] & \pi(q+m+s,W')
\end{tikzcd}$$

The canonical isomorphism $\zeta(V,V') \oplus \underline{V} \cong \underline{V'}$ implies the following parametrized generalization.

Lemma 2.7. Let $\chi\colon X \to \mathcal{I}(V,V')$ and $f\colon Y \to BO_G(q,V)$ be G-maps. Then the following diagram commutes up to canonical G-homotopy, where $\dim V = n$ and $\dim V' = n+k$.

$$\begin{tikzcd}
X \times Y \arrow[r, "\chi \times f"] \arrow[d, "\pi"'] & \mathcal{I}(V,V') \times BO_G(q,V) \arrow[r, "\varepsilon"] & BO_G(q+k,V') \arrow[d] \\
Y \arrow[r, "f"] & BO_G(q,V) \arrow[r] & BO_G(q+n+k,V \oplus V')
\end{tikzcd}$$

Proof. If f classifies ξ, then the composite around the top classifies $(\zeta(X;V,V') \times \xi) \oplus \underline{V}$ and the composite around the bottom classifies $(X \times \xi) \oplus \underline{V}'$. These are canonically equivalent bundles. Composing the equivalence with the evident maps of bundles over the displayed diagram of base spaces, we obtain two classifying diagrams for one of our bundles, say $(X \times \xi) \oplus \underline{V}'$. By an easy equivariant generalization of the classical argument in Milnor and Stasheff [114, p.67-68] (basis vectors there being replaced by irreducible representations here, as in the proof of II.1.5), this information gives rise to an explicit canonical homotopy.

Corollary 2.8. Let U and U' be G-universes such that $U' \oplus U \cong U$. let $\chi\colon X \to \mathcal{J}(U,U')$ and $f\colon Y \to BO_G(U)$ be G-maps. Then the following diagram is G-homotopy commutative, where i and j are the inclusions of U' and U in $U' \oplus U \cong U$.

$$\begin{tikzcd}
X \times Y \arrow[r, "\chi \times f"] \arrow[d, "\pi"'] & \mathcal{J}(U,U') \times BO_G(U) \arrow[r, "\varepsilon"] & BO_G(U') \arrow[d, "i_*"] \\
Y \arrow[r, "f"] & BO_G(U) \arrow[r, "j_*"] & BO_G(U)
\end{tikzcd}$$

Proof. This holds by the previous lemma if X and Y are compact. The general case follows by the canonicity, which gives us compatible homotopies as we run over the compact subspaces of X and Y.

We are thinking of the case $U' = \mathbf{R}^\infty$ with trivial G-action, where U is a complete G-universe. Any isomorphism $U^G \cong \mathbf{R}^\infty \oplus U^G$ induces the required isomorphism $U \cong \mathbf{R}^\infty \oplus U$, and of course the map $j_*\colon BO_G(U) \to BO_G(U)$ is a G-equivalence. The space $BO_G(\mathbf{R}^\infty)$ is just a model for $BO \times Z$ with trivial G action (and coincides with $BO_e(\mathbf{R}^\infty)$). The G-map $i_*\colon BO_G(\mathbf{R}^\infty) \to BO_G(U)$ factors through $BO_G(U)^G$ and is a nonequivariant homotopy equivalence. On the represented level, for a space Y with trivial G-action, the composite

$$KO(Y) \longrightarrow KO_G(Y) \longrightarrow KO(Y)$$

is the identity, where the first map is represented by i_* and the second by the forgetful map. This splitting is compatible with Bott periodicity and shows that the spectrum representing KO_G^* is split in the sense discussed in II§8. As explained there, for a free G-CW complex Y, the composite

$$(*) \qquad KO(Y/G) \xrightarrow{i_*} KO_G(Y/G) \xrightarrow{\pi^*} KO_G(Y), \quad \pi\colon Y \longrightarrow Y/G,$$

is an isomorphism. When Y is compact, this is obvious bundle theoretically; $\pi^* i_*$ sends a stable vector bundle over Y/G to the pullback along $Y \to Y/G$ of the given bundle regarded as a G-trivial G-bundle. Passage from G-bundles over Y to their orbit bundles over Y/G specifies the inverse isomorphism. In the previous corollary, X and therefore $X \times Y$ is necessarily G-free, and $\varepsilon(\chi \times f)$ necessarily factors through $X \times_G Y$ in the present context.

Corollary 2.9. let X and Y be G-CW complexes and let $\chi\colon X \to \mathcal{J}(U,\mathbf{R}^\infty)$ be a G-map, where U contains all irreducible representations of G. Let $\pi\colon X \times Y \to Y$ be the projection. Then, for $f \in KO_G(Y)$, $\pi^*(f)$ agrees under the canonical isomorphism

$$KO_G(X \times Y) \cong KO(X \times_G Y)$$

with the element of $KO(X \times_G Y)$ obtained by passage to orbits from the composite G-map

$$X \times Y \xrightarrow{\chi \times f} \mathcal{J}(U,\mathbf{R}^\infty) \times BO_G(U) \xrightarrow{\varepsilon} BO_G(\mathbf{R}^\infty).$$

When $X = EG$, the corollary describes the standard map

$$\alpha\colon KO_G(Y) \longrightarrow KO(EG \times_G Y)$$

in terms of an evaluation map ε of the $G\mathcal{J}$-space BO_G.

§3. The definition and basic properties of Thom G-spectra

Let U be a G-universe and index G-spectra on some indexing set $\mathcal{A}$ in U (such as the standard one). Our main interest is in the case when U is complete, but we shall also make use of the case $U = \mathbf{R}^\infty$. We shall define the Thom G-spectrum Mf determined by a G-map $f\colon Y \to BO_G(U)$. For convenience, we tacitly assume throughout that Y/G is path connected. Otherwise Y would be the disjoint union of inverse images Y_i of components of Y/G and we would define Mf to be the wedge of the spectra Mf_i obtained from the restrictions $f_i\colon Y_i \to BO_G(U)$.

A filtration of Y is a collection of closed G-subspaces F_jY indexed on a directed set $\{j\}$ such that Y is the colimit of the F_jY. A filtration of f is a filtration of Y indexed on $\mathcal{A}$ (or on any cofinal subset of $\mathcal{A}$) such that f restricts to a map $F_Vf\colon F_VY \to BO_G(V)$ for each V. Since Y/G is connected, each nonempty F_VY maps into a single space $BO_G(q_V,V)$, where $q_V = \dim V + k$ for some fixed integer k depending only on f. (If $\dim V + k < 0$, then F_VY is necessarily empty.) We call k the virtual dimension of f. We define the canonical filtration $\{Y_V\}$ of f by letting $Y_V = f^{-1}BO_G(V)$. Clearly this filtration is natural with respect to maps of G-spaces over $BO_G(U)$.

Constructions 3.1. (i) Let $\{F_V f\colon F_V Y \to BO_G(q_V,V)\}$ be a filtration of the G-map $f\colon Y \to BO_G(U)$. We construct the Thom prespectrum $T(f,\{F_V Y\})$. If $F_V Y$ is empty, let $T(f,\{F_V Y\}) = *$. Otherwise, let $T(f,\{F_V Y\})(V)$ be the Thom space of the q_V-plane bundle classified by $F_V f$. If $V \subset W$, the structure map

$$\sigma\colon \Sigma^{W-V}T(f,\{F_V Y\})(V) \longrightarrow T(f,\{F_V Y\})(W)$$

arises by Lemma 2.2 from the bundle map over the inclusion $F_V Y \to F_W Y$ resulting from the commutative diagram

$$\begin{tikzcd} F_V Y \arrow[r] \arrow[d] & F_W Y \arrow[d] \\ BO_G(q_V,V) \arrow[r] & BO_G(q_W,W), \end{tikzcd}$$

and σ is a closed inclusion. Define the Thom spectrum $M(f,\{F_V Y\})$ to be $LT(f,\{F_V Y\})$. Write Tf and Mf for the Thom prespectrum and spectrum so obtained from the canonical filtration of f.

(ii) Let $\{F_V Y\}$ and $\{F_V Z\}$ be filtrations of G-maps $f\colon Y \to BO_G(U)$ and $g\colon Z \to BO_G(U)$ and let $\lambda\colon Y \to Z$ be a filtration-preserving G-map over $BO_G(U)$. Note that f and g then have the same virtual dimension. Let

$$(T\lambda)(V)\colon T(f,\{F_V Y\})(V) \longrightarrow T(g,\{F_V Z\})(V)$$

be the map of Thom spaces induced by the evident bundle map over $\lambda\colon F_V Y \to F_V Z$. The $(T\lambda)(V)$ specify a map $T\lambda$ of prespectra and give rise to a map $M\lambda = LT\lambda$ of spectra. Restricting attention to the canonical filtrations, we obtain functors

$$T\colon G\mathcal{U}/BO_G(U) \longrightarrow G\mathcal{P}\mathcal{A} \quad\text{and}\quad M\colon G\mathcal{U}/BO_G(U) \longrightarrow G\mathcal{S}\mathcal{A}.$$

For the last statement, we must use wedges to extend our functions to general G-spaces Y, with Y/G not necessarily connected, but we retain our connectivity hypothesis below.

Exactly as in IX.4.3, IX.1.1 and the fact that the projections of vector bundles are open maps imply the following result.

Proposition 3.2. The functors T and M preserve colimits.

In turn, as in IX.4.4, this implies that our Thom spectra are independent of the choice of filtration used to define them.

Corollary 3.3. If $\{F_V Y\}$ is a filtration of $f\colon Y \to BO_G(U)$, then the natural map

$$M(f,\{F_V Y\}) \longrightarrow Mf$$

is an isomorphism of spectra.

As in the nonequivariant case, Mf is thus the colimit of the $MF_V f$ for any filtration $\{F_V Y\}$ of f, and any map from a compact spectrum into Mf factors through some $MF_V f$; compare I.4.8. An easy comparison of definitions gives that the $MF_V f$ are shift desuspensions (as specified in I.4.1) of Thom spaces.

Proposition 3.4. Let $f\colon Y \to BO_G(q,V)$ classify ξ, $\xi = f^*\pi(q,V)$. Then, regarding f as a map to $BO_G(U)$,

$$Mf = \Lambda^V \Sigma^\infty T\xi.$$

In particular, if f is constant at a q-plane A, then

$$Mf = \Lambda^V \Sigma^\infty \Sigma^A Y^+.$$

If $A \subset V$, then $\Lambda^V\Sigma^\infty\Sigma^A Y^+ \cong \Lambda^{V-A}\Sigma^\infty Y^+$ by I.4.2 When $Y = \{*\}$, $\Lambda^V\Sigma^\infty S^A$ should be interpreted as an "(A-V)-sphere spectrum", where $A - V$ is viewed as an element of $RO(G;U)$.

The functor M is invariant under changes of $\mathcal{A}$ (as in I.2.4) and of U (as in I.2.5 and II.1.1).

Lemma 3.5. For indexing sets $\mathcal{A} \subset \mathcal{B}$ in U, the following diagrams of functors commute up to natural isomorphism.

$$\begin{tikzcd}[column sep=small] & G\mathcal{U}/BO_G(U) \arrow[dl, "M"'] \arrow[dr, "M"] & \\ G\mathcal{S}\mathcal{B} \arrow[rr, "\phi"] & & G\mathcal{S}\mathcal{A} \end{tikzcd} \quad\text{and}\quad \begin{tikzcd}[column sep=small] & G\mathcal{U}/BO_G(U) \arrow[dl, "M"'] \arrow[dr, "M"] & \\ G\mathcal{S}\mathcal{A} \arrow[rr, "\psi"] & & G\mathcal{S}\mathcal{B} \end{tikzcd}$$

Lemma 3.6. Let $k\colon U \to U'$ be a G-linear isometry. Then the following diagram commutes up to natural isomorphism.

$$\begin{tikzcd} G\mathcal{U}/BO_G(U) \arrow[r, "k_*"] \arrow[d, "M"] & G\mathcal{U}/BO_G(U') \arrow[d, "M"] \\ G\mathcal{S}U \arrow[r, "k_*"] & G\mathcal{S}U' \end{tikzcd}$$

The proofs are easy inspections of definitions. Another easy inspection, based on Lemma 1.1, shows that the functor M commutes with external products.

Lemma 3.7. Let $\mathcal{A}$ and $\mathcal{A}'$ be indexing sets in U and U'. Then the following diagram commutes up to natural isomorphism.

$$\begin{tikzcd}
G\mathcal{U}/BO_G(U) \times G\mathcal{U}/BO_G(U') \arrow[r, "\omega_*"] \arrow[d, "M\times M"'] & G\mathcal{U}/BO_G(U \oplus U') \arrow[d, "M"] \\
G\mathscr{S}\mathcal{A} \times G\mathscr{S}\mathcal{A}' \arrow[r, "\wedge"] & G\mathscr{S}(\mathcal{A} \oplus \mathcal{A}')
\end{tikzcd}$$

We define both the product on $BO_G(U)$ and the internal smash product on $G\mathscr{S}\mathcal{A}$ in terms of a given G-linear isometry $k\colon U \oplus U \to U$. The previous results then have the following consequence.

Proposition 3.8. For G-maps $f\colon Y \to BO_G(U)$ and $g\colon Z \to BO_G(U)$, there is a natural isomorphism of spectra

$$M(f \times g) \cong Mf \wedge Mg,$$

where the internal smash product in $G\mathscr{S}\mathcal{A}$ is understood.

When Z is a point with image under g the G-plane $A \in BO_G(V)$, $f \to f \times g$ is addition of the element $A - V \in RO(G;U)$. Here we can apply Proposition 3.4.

Corollary 3.9. If $f = f' + (A - V)\colon Y \to BO_G(U)$, then

$$Mf \cong Mf' \wedge \Lambda^V \Sigma^\infty S^A.$$

Starting with any map f' and choosing A and V appropriately, we can arrange that f has virtual dimension zero. Thus all of our Thom spectra are obtained from those associated to maps of virtual dimension zero by smashing with sphere spectra.

When $Y = Z$ in Proposition 3.8, $f + g = (f \times g)\Delta$ and we can use Lemma 1.2 to identify $M(f + g)$. Let us say that a q-plane G-bundle ξ over Y is classifiable if $\xi = f^*\pi(q,V)$ for some indexing G-space V in U and some G-map $f\colon Y \to BO_G(q,V)$. When U is complete and either G is finite or Y is compact, every ξ is classifiable, but in general there need be no V such that $BO_G(q,V)$ is a genuine classifying space. When regarded as a map to $BO_G(U)$, f corresponds to $\xi - V$ in $KO_G(Y)$, hence we agree to let ξ also denote the map $f + V\colon Y \to BO_G(U)$.

Proposition 3.10. Let ξ be a classifiable G-vector bundle over Y and let $g\colon Y \to BO_G(U)$ be a G-map. let $(S^\xi)^*(g)$, $(D\xi)^*(g)$, and $(S\xi)^*(g)$ denote the composite of g and the projection of the based sphere, unit disc, and unit sphere bundles associated to ξ. Then there are equivalences, natural in g,

$$M(S^\xi)^*(g)/M(g) \simeq M(\xi + g) \simeq M(D\xi)^*(g)/M(S\xi)^*(g).$$

Moreover, $M(D\xi)^*(g)$ is equivalent to Mg.

Proof. For a G-space Z, let $p_V\colon Z \to BO_G(0,V)$ be the unique map; it represents $-V \in KO_G(Z)$. For $V \subset W$ and any G-map $h\colon Z \to BO_G(r,W)$, the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[r, "\Delta"] \arrow[d, "h"'] & Z \times Z \arrow[r, "p_V \times h"] & BO_G(0,V) \times BO_G(r,W) \arrow[d, "\omega"] \\
BO_G(r,W) \arrow[rr, "\subset"] & & BO_G(r, V \oplus W)
\end{tikzcd}$$

Let $f\colon Y \to BO_G(q,V)$ classify ξ and let $\{F_W g\colon F_W Y \to BO_G(r_W,W)\}$ be a filtration of g. We restrict attention to W containing V. Let $k\colon U \oplus U \to U$ be the G-linear isometry used to specify both the product on $BO_G(U)$ and the internal smash product functor. Then, with the obvious filtration, the $k(W \oplus W)$th space of $T(f + g)$ is the Thom space of the bundle classified by the dotted arrow composite

$$\begin{tikzcd}
F_W Y \arrow[r, "\Delta"] \arrow[dd, dashed] & F_W Y \times F_W Y \arrow[r, "f \times g"] & BO_G(q,V) \times BO_G(r_W,W) \arrow[d, "\omega"] \\
& & BO_G(q+r_W, V \oplus W) \arrow[d, "(i+1)_*"] \\
BO_G(q+r_W+s_W, k(W \oplus W)) & & BO_G(q+r_W+s_W, W \oplus W), \arrow[ll, "k_*"']
\end{tikzcd}$$

where $i\colon V \subset W$ and $s_W = \dim(W - V)$. Let Z be Y or the total space of one of our sphere or disc bundles over Y and let $h\colon Z \to BO_G(U)$ be one of the obvious filtered G-maps. Then the $k(W \oplus W)$th space of $T(p_V + h)$ is the Thom space of the bundle classified by the dotted arrow composite

$$\begin{tikzcd}
F_W Z \arrow[r, "h"] \arrow[d, dashed] & BO_G(r_W,W) \arrow[r, "\subset"] & BO_G(r_W, V \oplus W) \arrow[d, "(i\oplus 1)_*"] \\
BO_G(r_W+s_W, k(W \oplus W)) & & BO_G(r_W+s_W, W \oplus W). \arrow[ll, "k_*"']
\end{tikzcd}$$

Now Lemma 1.2 implies isomorphisms of prespectra

$$T(f + g) \cong T(p_V + (S^\xi)^*(g))/T(p_V + g)$$

$$\cong T(p_V + (D\xi)^*(g))/T(p_V + (S\xi)^*(g)).$$

Here $p_V + h$ may be thought of as $h - V$ and, up to G-homotopy, f is of the form $p_V + \xi$ when regarded as a map to $BO_G(U)$. The conclusions follow from Corollary 3.9 and Propositions 4.4 and 4.5 below.

By passage to colimits from the proof, we obtain the following more general conclusion.

Proposition 3.11. Let f and g be G-maps $Y \to BO_G(U)$. Let $\{F_V f: F_V Y \to BO_G(q_V, V)\}$ be a filtration of f, let $F_V f$ classify ξ_V, and let g_V be the restriction of g to $F_V Y$. Then there are isomorphisms, natural in f and g,

$$M(f + g) \cong \operatorname*{colim}_V M((S^{\xi_V})^*(g_V) - V)/M(g - V)$$

$$\cong \operatorname*{colim}_V M((D\xi_V)^*(g_V) - V)/M((S\xi_V)^*(g_V) - V)$$

By Corollary 4.3 below, the quotients in the previous two propositions are taken with respect to cofibrations.

Inspection from Lemma 1.3 gives the behavior of Thom spectra with respect to change of groups. Observe that, for $H \subset G$, $BO_G(U)$ regarded as an H-space is exactly $BO_H(U)$.

Proposition 3.12. Let Y be an H-space and let $\tilde{f}: G \times_H Y \to BO_G(U)$ be the G-map induced by an H-map $f: Y \to BO_H(U)$. Then

$$M\tilde{f} \cong G \ltimes_H Mf.$$

Inspection from Lemma 1.4 gives the behavior of Thom spectra with respect to passage to orbits. Observe that if U is a G'-trivial $(G \times G')$-universe, then $BO_{G\times G'}(U)$ is just $BO_G(U)$ endowed with the trivial action by G'.

Proposition 3.13. Let U be a G'-trivial $(G \times G')$-universe, let Y be a $(G \times G')$-space, and let $f: Y \to BO_{G\times G'}(U)$ be a $(G \times G')$-map. There results an orbit G-map $f/G': Y/G' \to BO_G(U)$, and

$$M(f/G') \cong (Mf)/G'.$$

We shall be interested in both the case $G = \{e\}$ (with $U = \mathbf{R}^\infty$) and the case $G' = \pi$, a finite permutation group.

Remark 3.14. In the nonequivariant case, we obtain a second model for the classical Thom spectrum MO by applying the present construction to the inclusion of the zero component in $BO_e(\mathbf{R}^\infty) \simeq BO \times Z$. We obtain a "classical" Thom G-spectrum MO_G similarly, and Mf maps canonically to MO_G for any G-map $f: Y \to BO_G(U)$ of virtual dimension zero.

§4. Homotopy invariance properties of Thom G-spectra

This brief section precisely parallels IX§4 (starting from IX.4.6). As there, we begin with the following observations.

Proposition 4.1. For a G-map $f: Y \to BO_G(U)$ and G-space X, $M(f\pi) \cong Mf \wedge X^+$, where $\pi: Y \times X \to Y$ is the projection.

Corollary 4.2. The functor $M: G\mathcal{U}/BO_G(U) \to G\mathcal{S}\mathcal{A}$ preserves homotopies and therefore carries fibrewise G-homotopy equivalences to G-homotopy equivalences.

By the discussion above IX.4.8, Propositions 3.2 and 4.1 have the following consequence.

Corollary 4.3. The functor M converts fibrewise cofibrations of G-spaces over $BO_G(U)$ into cofibrations of G-spectra.

Even here in our present purely bundle theoretical context, we must replace G-maps $f: Y \to BO_G(U)$ by G-fibrations $\Gamma f: \Gamma Y \to BO_G(U)$ so as to be able to exploit Corollaries 4.2 and 4.3. The point is that if $\lambda: Y \to Z$ is a G-map over $BO_G(U)$ and a G-homotopy equivalence or a G-cofibration, then the induced map $\Gamma\lambda: \Gamma Y \to \Gamma Z$ over $BO_G(U)$ is a fibrewise G-homotopy equivalence or a fibrewise G-cofibration. We can define Γ as in IX§1 or we can use the classical construction based on paths of length one. Either way, we get a functor $\Gamma: G\mathcal{U}/B \to G\mathcal{U}/B$ and a natural transformation $\delta: 1 \to \Gamma$ for any G-space B. For $f: Y \to B$, $\Gamma f: \Gamma Y \to B$ is a G-fibration and $\delta: Y \to \Gamma Y$ is a G-homotopy equivalence. As in the nonequivariant case, a G-homotopy equivalence which is a map of G-fibrations over B is necessarily a fibrewise G-homotopy equivalence. The proof of IX.1.11(ii) applies equivariantly to show that Γ carries G-cofibrations over B to fibrewise G-cofibrations.

As in IX§4, we use Γ as a technical tool in the proof of our basic homotopy invariance results.

Proposition 4.4. Let $f\colon Y \to BO_G(U)$ and $g\colon Z \to BO_G(U)$ be G-maps and let $\lambda\colon Y \to Z$ be a G-map over $BO_G(U)$ which is a weak G-homotopy equivalence. Then $M\lambda\colon Mf \to Mg$ is an isomorphism in the stable category $\bar{h}G\mathscr{S}\mathcal{A}$.

Proof. Observe first that if λ is filtration-preserving with respect to filtrations $\{F_VY\}$ of f and $\{F_VZ\}$ of g such that $\lambda\colon F_VY \to F_VZ$ is a weak equivalence for all V, then I.4.9 and the fact that Mf is independent of the filtration used to define it imply that $M\lambda$ is a weak equivalence. If we give f any filtration and give Γf the associated filtration, then this observation applies to show that $M\delta\colon Mf \to M\Gamma f$ is a weak equivalence, and similarly for g. If Y and Z have the homotopy types of G-CW complexes, then λ is a G-homotopy equivalence and $M\Gamma\lambda$ is a G-homotopy equivalence by Corollary 4.2. Since $(M\delta)(M\lambda) = (M\Gamma\lambda)(M\delta)$ by naturality, $M\lambda$ is thus a weak equivalence in this case. We reduce the general case to this case by use of functorial G-CW approximation (as in IX.4.9). When G is finite, the geometric realization of the total singular complex gives an appropriate approximation functor. For general G, Seymour has constructed an appropriate G-CW approximation functor [128].

Proposition 4.5. If $f_0 \simeq f_1\colon Y \to BO_G(U)$, then Mf_0 is isomorphic to Mf_1 in the stable category $\bar{h}G\mathscr{S}\mathcal{A}$.

Proof. Since the maps $M\delta\colon Mf_i \to M\Gamma f_i$ are weak equivalences and δ is natural, we need only observe that, by Corollary 4.2, the diagram

$$M\Gamma f_0 \xrightarrow{M\Gamma i_0} M\Gamma f \xleftarrow{M\Gamma i_1} M\Gamma f_1$$

displays an equivalence between $M\Gamma f_0$ and $M\Gamma f_1$, where $f\colon Y \times I \to BO_G(U)$ is any G-homotopy from f_0 to f_1.

Recall that $BO_G(U)$ is a homotopy associative and commutative Hopf G-space. Let Y be a Hopf G-space and let $f\colon Y \to BO_G(U)$ be a G-map which preserves units and strictly commutes with products. Then Mf inherits a product $Mf \wedge Mf \to Mf$ with unit $S \to Mf$. Note that f necessarily has virtual dimension zero on the G orbit of the unit component and that the strict commutation with products can always be arranged at the price of replacing f by Γf. If, further, Y is homotopy associative or commutative via a homotopy strictly compatible under f with that for $BO_G(U)$, then the product on Mf is associative or commutative in $\bar{h}G\mathscr{S}\mathcal{A}$. The proofs are the same as those in and above IX.4.11.

§5. The equivariant Thom isomorphism

Let E be a commutative ring G-spectrum and write E_*^H and E_H^* for the associated $RO(H;U)$-graded homology and cohomology theories on H-spectra for

$H \subset G$. (We shall be a little fuzzy about grading here since we intend to be fussy in [90].) Of course, $RO(H;U) = RO(H)$ when U is complete. Modulo the restriction $RO(G;U) \to RO(H;U)$ on gradings,

$$E_G^*(G/H) = E_H^*(*) = \pi_*^H(E).$$

For a G-map $f\colon G/H \to BO_G(U)$ which takes the coset eH to the H-plane $A \in BO_G(q,V)$, we have

$$Mf \cong G \ltimes_H M(f|eH) \cong G \ltimes_H \Lambda^V S^A,$$

by Propositions 3.4 and 3.12. Here $E_H^*(\Lambda^V S^A)$ is a free $\pi_*^H(E)$-module on one generator. Its canonical generator lies in $E_H^{\alpha}(\Lambda^V S^A)$, where $\alpha = A - V$ regarded as an element of $RO(H;U)$. There may also be a generator in $E_H^0(\Lambda^V S^A)$. If so, the generator may be viewed as a unit in $\pi_{-\alpha}^H(E)$ (and thus also in $\pi_{-\alpha}^K(E)$ by restriction for $K \subset H$). This implies that $\pi_0^H(E)$ is isomorphic to $\pi_{-\alpha}^H(E)$, and this implication places a real restriction on the applicability of the natural definition of an orientation in the equivariant context.

Definition 5.1. An E-orientation of a G-map $f\colon Y \to BO_G(U)$ is a class $\mu \in E_G^0 Mf$ such that, for each $y \in Y$, μ restricts to a generator of the free $\pi_*^H(E)$-module $E_H^*(M(f|y))$, where H is the isotropy group of y (so that the orbit Gy is a copy of G/H). For any G-map $\lambda\colon Z \to Y$, μ restricts to an E-orientation of $M(f\lambda)$, hence μ restricts to an E-orientation of each $M(F_Vf)$ for any filtration $\{F_Vf\}$ of f.

It might be objected that requiring orientations to have degree zero is unduly restrictive. However, the restriction is more apparent than real in view of Corollary 3.9 and the fact that we have not restricted ourselves to maps of virtual dimension zero.

Since $f\colon Y \to BO_G(U)$ factors as the composite

$$Y \xrightarrow{\Delta} Y \times Y \xrightarrow{\pi_2} Y \xrightarrow{f} BO_G(U)$$

and since $M(f\pi_2) \cong Y^+ \wedge Mf$ by Proposition 4.1, we have an induced diagonal map $\Delta\colon Mf \to Y^+ \wedge Mf$. More generally, for a second G-map $g\colon Y \to BO_G(U)$, the relation $g + f = (g \times f)\Delta$ implies that we have an induced diagonal map $\Delta\colon M(g+f) \to Mg \wedge Mf$. If A is a closed subspace of Y, then these maps pass to quotients to give relative diagonal maps

$$\Delta\colon Mf/M(f|A) \longrightarrow (Y/A) \wedge Mf$$

and

$$\Delta: M(g+f)/M((g+f)|A) \longrightarrow (Mg/M(g|A)) \wedge Mf.$$

If $A \to Y$ is a fibrewise cofibration with respect to f, g, and $g+f$, then all quotients here are taken with respect to cofibrations, by Corollary 4.3. Modulo the caveat explained in IX.5.5, we can arrange the cited cofibration conditions on $A \to Y$ by use of Γ.

Definition 5.2. Define the Thom maps associated to an element $\mu \in E_G^0 Mf$ to be the composites Φ_μ and Φ^μ in the commutative diagrams

$$\begin{tikzcd}
Mf \wedge E \arrow[r, "\Delta\wedge 1"] \arrow[d, "\Phi_\mu"'] & Y^+ \wedge Mf \wedge E \arrow[d, "1\wedge\mu\wedge 1"] \\
Y^+ \wedge E & Y^+ \wedge E \wedge E \arrow[l, "1\wedge\phi"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
F(Y^+,E) \arrow[r, "F(1,\tilde{\phi})"] \arrow[d, "\Phi^\mu"'] & F(Y^+,F(E,E)) \cong F(Y^+ \wedge E,E) \arrow[d, "F(1\wedge\mu,1)"] \\
F(Mf,E) & F(Y^+ \wedge Mf,E) \arrow[l, "F(\Delta,1)"]
\end{tikzcd}$$

Here $\phi: E \wedge E \to E$ is the product and $\tilde{\phi}: E \to F(E,E)$ is its adjoint. For a closed subspace A of Y, the relative diagonal gives rise to analogous Thom maps which make the appropriate diagrams commute. More generally, there are analogous Thom maps

$$\Phi_\mu: M(g+f) \wedge E \longrightarrow Mg \wedge E \quad \text{and} \quad \Phi^\mu: F(Mg,E) \longrightarrow F(M(g+f),E)$$

and their compatible relative counterparts.

On passage to the functors $[S^\alpha,?]_G$, we obtain homology and cohomology Thom maps

$$\Phi_\mu: E_*^G Mf \longrightarrow E_*^G Y \quad \text{and} \quad \Phi^\mu: E_G^* Y \longrightarrow E_G^* Mf,$$

and similarly in the relative and two map cases. By comparison with the definitions in III§3, we see that Φ_μ and Φ^μ are given by cap and cup product with μ.

Explicitly, Φ_μ is the composite

$$E_*^G Mf \xrightarrow{(\gamma\Delta)_*} E_*^G(Mf \wedge Y^+) \xrightarrow{\backslash\mu} E_*^G Y$$

and Φ^μ is the composite

$$E_G^* Y \xrightarrow{\wedge\mu} E_G^*(Y^+ \wedge Mf) \xrightarrow{\Delta^*} E_*^G Mf.$$

As in the nonequivariant context, questions about the behavior of the Thom maps with respect to multiplicative structures quickly reduce to questions about the behavior of μ. For example, if Y is a Hopf G-space and f and g are strict Hopf G-maps, then $\Phi_\mu: M(g+f) \wedge E \to Mg \wedge E$ preserves products if $\mu: Mf \to E$ preserves products.

Theorem 5.3. Let (Y,A) be a pair of G-CW complexes, let μ be an E-orientation of $f: Y \to BO_G(U)$, and let $A \to Y$ be a fibrewise cofibration with respect to f and to g and $g+f$ for a second G-map $g: Y \to BO_G(U)$. Then the Thom maps

$$\Phi_\mu: M(g+f)/M((g+f)|A) \wedge E \longrightarrow Mg/M(g|A) \wedge E$$

and

$$\Phi^\mu: F(Mg/M(g|A),E) \longrightarrow f(M(g+f)/M((g+f)|A),E)$$

are isomorphisms in the stable category $\bar{h}G\mathscr{S}U$ and therefore induce homology and cohomology Thom isomorphisms.

Proof. By a comparison of cofibration sequences (for Φ_μ) or of fibration sequences (for Φ^μ), the relative case will follow from the absolute case. Thus we may forget about A.

By the first description of $M(g+f)$ in Proposition 3.11 (with the roles of f and g there reversed), the general case will follow from the case when g is not present (that is, the case when g projects Y to the basepoint of $BO_G(U)$). Indeed, if g factors through some $BO_G(q,V)$, this reduction is another comparison of cofibration or fibration sequences. The general reduction for Φ_μ is obvious by passage to colimits; for Φ^μ it follows by passage to limits from a comparison of towers of fibrations by use of the dual $\lim^1$ exact sequence for the computation of π_*^H on such a limit. Thus we may forget about g.

It remains to prove that the Thom maps

$$\Phi_\mu: Mf \wedge E \longrightarrow Y^+ \wedge E \quad \text{and} \quad \Phi^\mu: F(Y^+,E) \longrightarrow F(Mf,E)$$

induce isomorphisms on π_q^H for all $H \subset G$ and all integers q. Let Y^n be the n-skeleton of Y. To arrange that the cofibrations $Y^{n-1} \to Y^n$ are fibrewise cofibrations without cluttering up the notations, we rely on our homotopy invariance results of Propositions 4.4 and 4.5 and agree once and for all to tacitly prefix Γ to every space and map over $BO_G(U)$ in the rest of the proof. Thus Y, Y^n, etc now mean ΓY, ΓY^n, etc. With this convention, another passage to colimits or passage to limits argument shows that it suffices to consider the restrictions $f|Y^n$. By induction and comparisons of cofibration or fibration sequences, it suffices to consider the relative cases obtained by restriction of f to the pairs (Y^n, Y^{n-1}) for $n \geq 0$, where $Y^{-1} = \phi$. By compatible wedge decompositions and pullback along the characteristic maps of cells, it suffices to consider the relative case $(G \times_H e^n, G \times_H S^{n-1})$ for an E-oriented G-map $f\colon G \times_H e^n \to BO_G(U)$, where $e^0 = \{1\}$ and $S^{-1} = \phi$. Here e^n is contractible to a point y and we assume that $f(y) = A \in BO_G(q,V)$. Then Propositions 3.4, 3.12, 4.4 and 4.5 imply stable equivalences

$$Mf/M(f|G \times_H S^{n-1}) \simeq G \ltimes_H (\Lambda^V S^A \wedge S^n) \cong G \ltimes_H (S^n \wedge \Lambda^V S^A)$$

and (using the projection $e^n \to \{y\}$)

$$(G \times_H e^n)/(G \times_H S^{n-1}) \wedge Mf \simeq G \ltimes_H (G/H^+ \wedge S^n \wedge \Lambda^V S^A).$$

Moreover, the relative diagonal map between the left hand G-spectra corresponds under the equivalence to the map induced by the inclusion $S^0 \to G/H^+$ on the right. We conclude that, up to homotopy, the relative Thom G-maps Φ_μ and Φ^μ here are the G-maps obtained by application of the functor $G \ltimes_H S^n \wedge (?)$ to the composite H-map

$$\Lambda^V S^A \wedge E \xrightarrow{\mu \wedge 1} E \wedge E \xrightarrow{\phi} E$$

and by application of the functor $F(G \ltimes_H S^n, ?)$ to the composite H-map

$$E \xrightarrow{\tilde{\phi}} F(E,E) \xrightarrow{F(\mu,1)} F(\Lambda^V S^A, E).$$

On passage to π_q^K for any $K \subset H$, these composites induce multiplication by the unit $\mu \in E_K^0(\Lambda^V S^A) \cong \pi_{V-A}^K(E)$, hence these composites are stable H-equivalences.

Remarks 5.4. (i) If $f\colon Y \to BO_G(U)$ and $g\colon Z \to BO_G(U)$ have E-orientations μ and ν, then the image of $\nu \otimes \mu$ under the external product

$$E_G^* Mg \otimes E_G^* Mf \longrightarrow E_G^*(Mg \wedge Mf) \cong E_G^* M(g \times f)$$

is an orientation $\nu \times \mu$ of $g \times f$. If $Y = Z$, we write $\nu \oplus \mu$ for the induced orientation $\Delta^*(\nu \times \mu)$ of $g + f$.

(ii) If μ is an orientation of $f\colon Y \to BO_G(U)$ and ω is an orientation of $g \oplus f$, then the composite

$$E_G^* Y \xrightarrow{\Phi^\omega} E_G^* M(g+f) \xrightarrow{(\Phi^\mu)^{-1}} E_G^* Mg$$

carries the unit $1 \in E^0 Y$ to the unique orientation ν of g such that $\nu + \mu = \omega$.

(iii) Let $p\colon Y \to BO_G(U)$ be the trivial map at the basepoint. Then $M(p) \cong \Sigma^\infty Y^+$ and we take $1 \in E^0 Y$ as the canonical orientation of p. For any $g\colon Y \to BO_G(U)$, $g + p$ is G-homotopic to g and

$$\Phi_1\colon M(g+p) \wedge E \longrightarrow Mg \wedge E \quad \text{and} \quad \Phi^1\colon F(Mg,E) \longrightarrow F(M(g+p),E)$$

are carried to identity maps under the canonical equivalence of $M(g+p)$ with Mg induced by the right unit homotopy for $BO_G(U)$.

(iv) If μ and ν are orientations of G-maps $f\colon Y \to BO_G(U)$ and $g\colon Y \to BO_G(U)$ and $h\colon Y \to BO_G(U)$ is another G-map, then easy chases show that the following diagrams are commutative.

$$\begin{tikzcd} M(h+g+f) \wedge E \arrow[r, "\Phi_{\nu\oplus\mu}"] \arrow[d, "\Phi_\mu"'] & Mh \wedge E \\ M(h+g) \wedge E \arrow[ur, "\Phi_\nu"'] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} F(Mh,E) \arrow[r, "\Phi^{\nu\oplus\mu}"] \arrow[d, "\Phi^\nu"'] & F(M(h+g+f),E) \\ F(M(h+g),E) \arrow[ur, "\Phi^\mu"'] & \end{tikzcd}$$

(v) The Hopf G-space $BO_G(U)$ has a homotopy inverse G-map $\chi\colon BO_G(U) \to BO_G(U)$, so that $\phi(1 \times \chi)\Delta \simeq p$. For $f\colon Y \to BO_G(U)$, we have $f + \chi f \simeq p$ and thus $M(f + \chi f) \simeq \Sigma^\infty Y^+$. By (ii), for any orientation μ of f, there is a unique orientation $\chi\mu$ of χf such that $\mu \oplus \chi\mu = 1 \in E^0 Y$. By (iii) and (iv), for any $g\colon Y \to BO_G(U)$,

$$\Phi_{\chi\mu}\colon Mg \wedge E \simeq M(g + f + \chi f) \wedge E \longrightarrow M(g+f) \wedge E$$

and

$$\Phi^{\chi\mu}\colon F(M(g+f),E) \longrightarrow F(M(g+f+\chi f),E) \simeq F(Mg,E)$$

are inverse equivalences to Φ_μ and Φ^μ.

If we had an a priori construction of the inverse orientations $\chi\mu$, the previous remarks would supply another proof of the Thom isomorphism.

Remark 5.5. Let ξ be a G-vector bundle over Y and let $\mu \in E_G^{\alpha}(T\xi)$ be an orientation of ξ. Thus, for an orbit inclusion $i: G/H \to Y$ with fibre H-representation W at $i(eH)$, $i^*(\mu)$ is a generator of $\tilde{E}_G^*(T(i^*\xi)) \cong \tilde{E}_H^*(S^W)$. We may think of (ξ,μ) as the element $\xi - \alpha \in KO_G(Y)$ together with the orientation $\mu \in E_G^0(M(\xi - \alpha))$. This makes sense since Proposition 3.4 and Corollary 3.8 imply an isomorphism

$$M(\xi - \alpha) \cong \Sigma^{\infty}T\xi \wedge S^{-\alpha},$$

so that $\tilde{E}_G^{\alpha}(T\xi) = E_G^{\alpha}(\Sigma^{\infty}T\xi) \cong E_G^0(M(\xi - \alpha))$. Theorem 5.3 specializes to show that, for $A \subset Y$,

$$\mu: E_G^{\beta}(Y,A) \longrightarrow E_G^{\alpha+\beta}(T\xi,T(\xi|A))$$

is an isomorphism for all $\beta \in RO(G)$.

§6. Twisted half-smash products and Thom G-spectra

We here prove a general commutation relation between twisted half-smash products and Thom spectra. Since we are now working in a fully equivariant setting, we can give a cleaner treatment than in IX§6; the analogs of the results there will follow by the three step procedure (external smash products, twisted half-smash products, orbit spectra) advertised in the introduction to chapter VI.

Proposition 6.1. Let U and U' be G-universes and let $\chi: X \to \mathcal{J}(U,U')$ and $f: Y \to BO_G(U)$ be G-maps. Then the Thom G-spectrum $M(\varepsilon(\chi \times f))$ associated to the composite

$$X \times Y \xrightarrow{\chi \times f} \mathcal{J}(U,U') \times BO_G(U) \xrightarrow{\varepsilon} BO_G(U')$$

is isomorphic to $\chi \ltimes Mf$.

Proof. Since the functor M commutes with colimits, we may as well assume that χ is compact. Let $\mathcal{A}$ and $\mathcal{A}'$ be our given indexing sets in U and U' and choose a χ-connection $(\mu,\nu):\mathcal{A} \to \mathcal{A}'$ (as in VI.2.7). Let $\{F_Vf: F_VY \to BO_G(q_V,V)\}$ be any filtration of f and observe that the spaces $X \times F_{\nu V'}Y$ then give a filtration of $\varepsilon(\chi \times f)$ since $\chi(X)(\nu V') \subset V'$. On the prespectrum level,

$$(\chi \ltimes Tf)(V') = T(X;V,V') \wedge T(\xi_V), \qquad V = \nu V',$$

where ξ_V is the G-vector bundle classified by F_Vf. Here $T(X;V,V')$ is the Thom space of the G-vector bundle $\zeta(X;V,V')$ of VI.2.1, and lemma 2.4 implies that $\zeta(X;V,V') \times \xi_V$ is canonically isomorphic to the G-vector bundle classified by $\varepsilon(\chi \times f): X \times F_VY \to BO_G(V')$. Thus

$$(\chi \ltimes Tf)(V') \cong T(\varepsilon(\chi \times f))(V'),$$

and it is easy to check that the structural maps σ agree.

Proposition 3.13 gives the following consequence.

Corollary 6.2. When $U' = R^{\infty}$ with trivial G-action,

$$\chi \ltimes_G Mf \cong M(\varepsilon(\chi \times f)/G).$$

Now let U be complete. Then $\mathcal{J}(U,\mathbf{R}^{\infty})$ is a universal principal G-bundle and there is thus a G-map $\chi: EG \to \mathcal{J}(U,\mathbf{R}^{\infty})$, unique up to G-homotopy, and a corresponding twisted half-smash product functor $EG \ltimes (?)$ from $\bar{h}G\mathcal{S}U$ to $\bar{h}\mathcal{S}\mathbf{R}^{\infty}$. By Proposition 2.1 if Y is finite and by definition otherwise,

$$KO_G(Y) = [Y^+,BO_G(U)]_G$$

for G-CW complexes Y. By Proposition 4.5, Mf depends only on the homotopy class of f and thus on f as an element of $KO_G(Y)$. By Corollary 2.9, the transformation

$$\alpha: KO_G(Y) \longrightarrow KO_G(EG \times Y) \cong KO(EG \times_G Y)$$

induced by the projection $EG \times Y \to Y$ is given by $\alpha(f) = \varepsilon(\chi \times f)/G$. Thus Corollary 6.2 specializes to give the following comparison.

Corollary 6.3. For $f \in KO_G(Y)$, $M(\alpha(f)) \simeq EG \ltimes_G Mf$.

We have already observed that the functor M commutes appropriately with external smash products and with passage to orbits (Lemma 3.7 and Proposition 3.13). Starting from a G-universe U regarded as a π-trivial $(G \times \pi)$-universe for the appropriate π, we immediately deduce the equivariant analog of IX.6.1. Using Lemmas 2.5 and 2.6, we also read off the equivariant analogs of IX.6.2 and 6.4; the generalizations alluded to in IX.6.3 would be more natural in the present context.

Remarks 6.4. The relationship between Thom G-spectra and operad ring G-spectra works exactly as explained in IX§7 in the nonequivariant case. One need only replace the classifying space BG used there with the classifying G-space $BO_G(U)$ used here (resolutely ignoring the conflicting uses of the letter G). One must check that the three easy results quoted from [97] remain valid in the equivariant setting, but this presents no difficulty. The essential points are that chapter VII applies as it stands equivariantly and that Proposition 6.1 and its elaborations discussed in the previous paragraph give all requisite properties of Thom G-spectra.

To summarize, if $\mathcal{C}$ is a G-operad over $\mathcal{L}(U)$, Y is a $\mathcal{C}$-space, and $f: Y \to BO_G(U)$ is a $\mathcal{C}$-map, then Mf is a $\mathcal{C}$-spectrum and $\Delta: Mf \to Y^+ \wedge Mf$ is a $\mathcal{C}$-map. If, further, $\mathcal{C}$ is an E_∞ G-operad and $g: Y \to BO_G(U)$ is another $\mathcal{C}$-map, then $M(g + f)$ is an $\bar{h}\mathcal{C}$-spectrum and $\Delta: M(g + f) \to Mg \wedge Mf$ is an $\bar{h}\mathcal{C}$-map. In either case, if E is an $\bar{h}\mathcal{C}^d$-spectrum and if $\mu: Mf \to E$ is an $\bar{h}\mathcal{C}$-map, then the Thom map

$$\Phi_\mu: Mf \wedge E \longrightarrow Y^+ \wedge E \quad \text{or} \quad \Phi_\mu: M(g + f) \wedge E \longrightarrow Mg \wedge E$$

is an $\bar{h}\mathcal{C}^d$-map between $\bar{h}\mathcal{C}^d$-spectra.

Given the present state of our knowledge and calculational expertise, these facts are rather esoteric here in the equivariant context, and it must be kept in mind that the relationship between operad actions and equivariant loop spaces is not clear unless G is finite; compare VII.5.6.

Appendix. Analysis of the passage from prespectra to spectra

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The left adjoint L to the inclusion functor from the category $G\mathcal{S}\mathcal{A}$ of spectra to the category $G\mathcal{P}\mathcal{A}$ of prespectra appears throughout our work. It is required for almost all colimits, for the smash product of a space and a spectrum or of two spectra, for the change of groups functors $G \ltimes_H ?$ and most change of universe functors f_*, and for half smash products. Many of these constructions involve non-inclusion prespectra and so use the more sophisticated form of L. In spite of those frequent appearances, L has not become an old, familiar friend; it remains an unknown and unwelcome intruder into our work. Here, we offer the limited introduction we are able to give to it.

In section I.2, we noted that L is the composite of the left adjoints $L': G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ and $L'': G\mathcal{I}\mathcal{A} \to G\mathcal{S}\mathcal{A}$ to the corresponding inclusion functors. There we described L", the simpler part of L. Here, in section 1, we construct L'. Most of what is known about L' and L, beyond the formal consequences of their being left adjoints, is derived from their behavior on finite limits. Section 2 is devoted to this. In I§8, we characterized the spacewise closed inclusions preserved by L. We return to this in section 3 and also give more information about inclusions, including a proof of I.8.1, which asserts that a cofibration is a closed inclusion. Two bits of unfinished business with cell spectra are disposed of in the final section -- the behavior of maps from compact spectra to cell spectra and the lattice theoretic behavior of cell subspectra.

§1. The construction of the functor L

The functor $L': G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ is essential to all of our work and yet remains quite mysterious. Therefore we give two approaches to it -- an easily followed, formal existence proof giving nothing more than its existence as a left adjoint, and a longer, unattractive construction providing a little more insight into its behavior.

Theorem 1.1. There is a left adjoint $L': G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ to the inclusion functor $\ell': G\mathcal{I}\mathcal{A} \to G\mathcal{P}\mathcal{A}$.

Proof (by Freyd's adjoint functor theorem). As we noted in I§2, the category $G\mathcal{P}\mathcal{A}$ has all (set indexed) limits. It is easy to check that the limit in $G\mathcal{P}\mathcal{A}$ of a diagram in $G\mathcal{I}\mathcal{A}$ is an inclusion prespectrum and so is also the limit in $G\mathcal{I}\mathcal{A}$. Thus, the category $G\mathcal{I}\mathcal{A}$ has all limits and the functor $\ell': G\mathcal{I}\mathcal{A} \to G\mathcal{P}\mathcal{A}$ preserves limits. By Freyd's adjoint functor theorem [92, p. 117], the functor ℓ' must have a left adjoint if it satisfies the following.

<u>Solution Set Condition</u>. For each prespectrum C, there exists a set I and an I-indexed family of maps $f_i: C \to D_i$ into inclusion prespectra D_i such that any map $f: C \to D$ with D in $G\mathcal{I}\mathcal{A}$ factors as $f = gf_i$ for some $i \in I$ and $g: D_i \to D$.

To see that this condition is satisfied, let $f: C \to D$ be a map from a prespectrum C to an inclusion prespectrum D. Define a new prespectrum $\operatorname{im} f$ by

$$(\operatorname{im} f)(V) = \operatorname{image}(f: CV \longrightarrow DV)$$

for each $V \in \mathcal{A}$. The structure maps for D induce structure maps for $\operatorname{im} f$ making it an inclusion prespectrum, and f factors as a composite

$$C \xrightarrow{\tilde{f}} \operatorname{im} f \longrightarrow D.$$

The map $\tilde{f}$ is a spacewise surjection. Thus we may take, as the solution set for C, a family of representatives of the isomorphism classes of spacewise surjective maps from C into inclusion prespectra. This family is a proper set because $\mathcal{A}$ is a set, for each $V \in \mathcal{A}$ there is only a set of isomorphism classes of quotient sets of CV, and for each quotient set there is only a set of possible topologies.

Since the proof just given offers no intuition about L', we turn to a construction of it. Roughly speaking, our method is to define a functor

$$J: G\mathcal{P}\mathcal{A} \longrightarrow G\mathcal{P}\mathcal{A}$$

and a natural transformation

$$\gamma: C \longrightarrow JC \quad \text{for} \quad C \in G\mathcal{P}\mathcal{A},$$

which are first approximations to $L': G\mathcal{P}\mathcal{A} \to G\mathcal{I}\mathcal{A}$ and the unit $\eta': C \to \ell'L'C$ of the (L',ℓ') adjunction. We iterate J, possibly transfinitely, until we obtain L'.

<u>Construction 1.2</u>. Let $A = \{A_n\}$ be an indexing sequence in $\mathcal{A}$. Given $C \in G\mathcal{P}\mathcal{A}$, define JC by

$$JCV = \operatorname{image}(\tilde{\sigma}: CV \longrightarrow \Omega^{A_m-V}CA_m) \quad \text{for} \quad V \in \mathcal{A},$$

where A_m is the smallest indexing space in A properly containing V. Note the obvious surjection $\gamma: CV \to JCV$. If $V \subset W$, then the structure map

$$JCV \longrightarrow \Omega^{W-V}JCV$$

is induced by the composite on the bottom row of the following commuting diagram.

$$\begin{tikzcd}
CV \arrow[rr, "\tilde{\sigma}"] \arrow[d, "\gamma"'] & & \Omega^{W-V}CW \arrow[d, "\Omega^{W-V}\gamma"] \\
JCV \arrow[rr, dashed] \arrow[d, phantom, "\cap"] & & \Omega^{W-V}JCW \arrow[d, phantom, "\cap"] \\
\Omega^{A_m-V}CA_m \arrow[r, "\Omega^{A_m-V}\tilde{\sigma}"] & \Omega^{A_m-V}\Omega^{A_n-A_m}CA_n \arrow[r, phantom, "\cong"] & \Omega^{W-V}\Omega^{A_n-W}CA_n
\end{tikzcd}$$

Here, A_n is the least indexing space in A properly containing W (so that $m \leq n$). Clearly the structure maps for JCV satisfy the coherence condition needed for a prespectrum and the maps γ induce a map $\gamma: C \to JC$. Moreover, J is a functor and γ is a natural transformation.

Recall the notion of an injection prespectrum from I.8.2. The basic properties of J are as follows.

<u>Lemma 1.3</u>. Let $C \in G\mathcal{P}\mathcal{A}$ and $D \in G\mathcal{I}\mathcal{A}$.

(i) $\gamma: C \to JC$ is a spacewise surjection.

(ii) $\gamma: C \to JC$ is a spacewise injection if and only if C is an injection prespectrum, and γ is an isomorphism if and only if C is an inclusion prespectrum.

(iii) Any map $f: C \to D$ factors uniquely through $\gamma: C \to JC$.

(iv) J preserves finite products and, restricted to the full subcategory of injection prespectra, it preserves all finite limits.

<u>Proof</u>. Part (i) is included in the construction. For part (ii), it is obvious that if C is an injection (or inclusion) prespectrum, then the map γ is an injection (or isomorphism). For the converse, assume that γ is an injection. Then for any V and its associated A_m,

$$\tilde{\sigma}: CV \longrightarrow \Omega^{A_m-V}CA_m$$

is an injection. If $V = A_n$, then $m = n+1$ because we require proper containment of V in A_m. Arguing by induction on n, we have that

$$\tilde{\sigma}: CV \longrightarrow \Omega^{A_n-V}CA_n$$

is an injection for any $V \in \mathcal{A}$ and n with $V \subset A_n$. Factoring such a map through

$$\tilde{\sigma}: CV \longrightarrow \Omega^{W-V}CW$$

for any $V \subset W \subset A_n$, we have that the latter map must be an injection and C must be an injection prespectrum. The argument that C is an inclusion prespectrum if γ is an isomorphism is similar.

For (iii), note that $f: C \to D$ must be $\gamma^{-1}Jf\gamma$ by part (ii) and the commutativity of the diagram

$$\begin{tikzcd} C \arrow[r, "\gamma"] \arrow[d, "f"'] & JC \arrow[d, "Jf"] \\ D \arrow[r, "\gamma", "\cong"'] & JD \end{tikzcd} .$$

The first half of part (iv) follows from the structure maps for products of prespectra and the commutativity of products and images. Since finite limits can be formed from finite products and equalizers, for the second half it suffices to prove that J restricted to the category of injection prespectra preserves equalizers. This can be verified by inspection.

Now we iterate J transfinitely.

Definition 1.4. For each ordinal α, define the functor $J_\alpha: G\mathcal{PA} \to G\mathcal{PA}$ and for each pair $\alpha \leq \beta$ of ordinals, define the natural transformation $\gamma^\alpha_\beta: J_\alpha \to J_\beta$ by

(i) J_0 is the identity functor and, for any α, $\gamma^\alpha_\alpha: J_\alpha \to J_\alpha$ is the identity natural transformation.

(ii) For any ordinal α, $J_{\alpha+1}$ is JJ_α and for ordinals $\alpha \leq \beta$, $\gamma^\alpha_{\beta+1} = \gamma\gamma^\alpha_\beta$.

(iii) If β is a limit ordinal and C is any prespectrum, then $J_\beta C = \operatorname{colim}_{\alpha<\beta} J_\alpha C$, where the colimit is taken over the maps $\gamma^{\alpha'}_\alpha$ for $\alpha' \leq \alpha < \beta$. For $\alpha < \beta$, the map $\gamma^\alpha_\beta: J_\alpha C \to J_\beta C$ is the natural map into the colimit. The universality of colimits and the naturality of the maps $\gamma^{\alpha'}_\alpha$ ensure that J_β is a functor and that the maps γ^α_β are natural transformations.

Note that for $\alpha \leq \beta \leq \delta$, $\gamma^\alpha_\delta = \gamma^\beta_\delta\gamma^\alpha_\beta$.

The extension of most parts of Lemma 1.3 to J_α and $\gamma^0_\alpha: C \to J_\alpha C$, for all α, will not be discussed until the next section. However, the following partial extension is needed to complete our second proof of Theorem 1.1.

Lemma 1.5. Let C be in $G\mathcal{PA}$, D be in $G\mathcal{IA}$ and $\alpha \leq \beta \leq \delta$ be ordinals

(i) The map $\gamma^\alpha_\beta: J_\alpha \to J_\beta C$ is a spacewise surjection.

(ii) The map $\gamma^\alpha_{\alpha+1}: J_\alpha C \to J_{\alpha+1}C$ is an isomorphism if and only if $J_\alpha C$ is an inclusion prespectrum. Moreover, if $J_\alpha C$ is an inclusion prespectrum, then $\gamma^\beta_\delta: J_\beta C \to J_\delta C$ is an isomorphism.

(iii) Any map $f: C \to D$ factors uniquely through $\gamma^0_\alpha: C \to J_\alpha C$.

Proof. Part (i) follows from Lemma 1.3(i), the definition of γ^α_β, and an obvious observation about surjections and colimits. The first half of part (ii) follows from Lemma 1.3(ii). For the second half, it suffices to prove by induction on δ that γ^α_δ is an isomorphism if $\delta > \alpha$ and $J_\alpha C$ is an inclusion prespectrum. If δ is a successor ordinal, this follows from Lemma 1.3(ii) and the induction hypothesis. If δ is a limit ordinal, then $J_\delta C$ may be taken to be the colimit of the $J_\beta C$ for $\alpha \leq \beta < \delta$ since this system is cofinal in the defining system. The result follows because this is a colimit over a diagram of isomorphisms. Part (iii) follows from (ii) just as Lemma 1.3(iii) follows from Lemma 1.3(ii).

Proof of Theorem 1.1 (by construction). By [92, p. 81], it suffices to show that for each $C \in G\mathcal{PA}$, there exists an inclusion prespectrum $L'C$ and a map $\eta': C \to L'C$ which is universal with respect to maps from C into inclusion prespectra. The assignment of $L'C$ to C can then be uniquely extended to a functor $L': G\mathcal{PA} \to G\mathcal{IA}$. Moreover, the functor L' must then be left adjoint to ℓ' and η' must be the unit of the adjunction. By Lemma 1.5(iii), for each ordinal α the map $\gamma^0_\alpha: C \to J_\alpha C$ has the proper unique factorization property with respect to maps from C into inclusion prespectra. Thus, it suffices to produce an ordinal α for each C in $G\mathcal{PA}$ such that $J_\alpha C$ is an inclusion prespectra.

Since $\gamma^0_\alpha: C \to J_\alpha C$ is a spacewise surjection, $J_\alpha C$ may be described completely by giving, for each $V \in \mathcal{A}$, the pairs of points in CV to be identified to form $J_\alpha CV$ and the open sets in CV which are the inverse images of open sets in $J_\alpha CV$. If $J_\alpha C$ is not an inclusion prespectrum, then $\gamma^\alpha_{\alpha+1}: J_\alpha C \to J_{\alpha+1}C$ is not an isomorphism so the formation of $J_{\alpha+1}C$ must involve either the identification of more pairs of points in the spaces CV or the elimination of more open sets from the spaces CV. There is only a set of pairs of points that can be identified and a set of open sets that may be omitted. If the cardinality of α is greater than the cardinality of the union of these two sets, then $\gamma^\alpha_{\alpha+1}$ must be an isomorphism. For each C, we select a sufficiently large ordinal α (which may depend on C) and let $L'C$ be $J_\alpha C$ and $\eta': C \to L'C$ be γ^0_α. The second part of Lemma 1.5(ii) ensures that the choice of α, among the sufficiently large ordinals, is immaterial.

Remarks 1.6. This iterative approach to left adjoints has other applications in topology. For example, left adjoints to the inclusions of Hausdorff spaces into all spaces and of compactly generated weak Hausdorff spaces into all compactly generated spaces may be constructed this way. In both cases, the closure of the diagonal in $X \times X$ is used to form an equivalence relation on a space X. The associated quotient space of X is the first approximation to the left adjoint. Also, if the topology on JCV in Construction 1.2 is taken to be the quotient topology from CV instead of the subspace topology from $\Omega^{A_m - V} CA_m$, then J may be iterated to form a left adjoint to the inclusion functor from injection prespectra to prespectra. The functor L' is the composite of this left adjoint and the restriction of L' to the category of injection prespectra.

Examples 1.7. It is natural to inquire about the number of times J must be iterated to obtain L'. The following two examples show that iteration to the first limit ordinal may be required. The first example is quite transparent. The second, more sophisticated example, shows that even restricted to injection prespectra, L' may alter the homotopy groups of the spaces of a prespectrum in quite unpredictable ways. In both examples, we take $G = e$, $U = \mathbb{R}^\infty$ and $\mathcal{A}$ to be the standard indexing sequence $\{\mathbb{R}^n\}_{n \geq 0}$. We abbreviate $C\mathbb{R}^n$ to C_n for any prespectrum C.

(i) Let C be the prespectrum with spaces $C_n = S^n$ and maps $\sigma\colon \Sigma C_n \to C_{n+1}$ being the standard identification $\Sigma S^n \cong S^{n+1}$ for n not a power of 2 and the trivial map for n a power of 2. Clearly $L'C$ must be the point prespectrum, but $J_\alpha C$ is not the point prespectrum for any finite ordinal α.

(ii) We construct an injection prespectrum D such that $L'D$ is the suspension sphere prespectrum $\{S^{n+2}\}_{n \geq 0}$ but the integral homology groups $H_m D_n$ vanish for $m \geq n+2$. This easily implies that the natural map

$$\operatorname{colim} \pi_{n+2}(D_n) \longrightarrow \operatorname{colim} \pi_{n+2}((L'D)_n) \cong \mathbb{Z}$$

is zero instead of an isomorphism as one might hope from the behavior of $L''\colon G\mathcal{Z}\mathcal{A} \to G\mathcal{S}\mathcal{A}$ (see I.4.8 and I.4.9).

Embed S^1 in $\mathbb{R}^3$ as the unit circle in $\mathbb{R}^2 \subset \mathbb{R}^3$. Let $\theta\colon I \to S^1$ be the map

$$\theta(t) = (\cos 2\pi t,\ \sin 2\pi t,\ 0) \quad \text{for} \quad t \in I.$$

Number the rational points of I as $\{r_m\}_{m \geq 1}$. Let $X_0 \subset \mathbb{R}^3$ be the union of $S^1 - \{\theta(r_m)\}_{m \geq 1}$ and the set of points $\{(\cos 2\pi r_m, \sin 2\pi r_m, \frac{1}{m}) \mid m \geq 1\}$; that is, form X_0 from $S^1 \subset \mathbb{R}^3$ by moving the point $\theta(r_m)$ up the distance $\frac{1}{m}$ for each $m \geq 1$. For $n \geq 1$, form the space X_n from X_0 by returning the points associated to the r_m, for $1 \leq m \leq n$, to their proper places in S^1. There are obvious continuous maps $\phi_n\colon X_n \to X_{n+1}$ and $\zeta_n\colon X_n \to S^1$. Moreover, S^1 is the colimit of the X_n and the maps ζ_n are the natural maps into the colimit. We are indebted to Andrew Berner for pointing out the spaces X_n and their colimit. Note

that any pair of distinct points in X_n, $n \geq 0$, can be separated by a pair of disjoint open sets whose union is all of X_n. Thus, any map from a connected space into X_n must be a constant map and $H_m X_n = 0$ for $m \geq 1$.

Let Y_n denote the unreduced suspension of X_n and let $\phi_n\colon Y_n \to Y_{n+1}$ and $\zeta_n\colon Y_n \to S^2$ be the unreduced suspensions of the earlier maps ϕ_n and ζ_n. Our prespectrum D is defined by

$$D_n = \Sigma^n Y_n$$

and

$$\sigma_n\colon \Sigma D_n = \Sigma^{n+1} Y_n \xrightarrow{\Sigma^{n+1}\phi_n} \Sigma^{n+1} Y_{n+1} = D_{n+1}.$$

It is easy to see that $(J_k D)_n = \Sigma^n Y_{n+k}$ if k is an integer and J_k is the k^{th} iterate of J. Thus, if ω is the first limit ordinal, then $J_\omega D$ is the suspension sphere prespectrum $\{S^{n+2}\}_{n \geq 0}$. Since this is an inclusion prespectrum, $L'D = J_\omega D = \{S^{n+2}\}_{n \geq 0}$. Because Y_n is the unreduced suspension of X_n and so has a nondegenerate basepoint, we have suspension isomorphisms

$$H_m(D_n) = H_m(\Sigma^n Y_n) \cong H_{m-n}(Y_n) \cong H_{m-n-1}(X_n) = 0$$

for $m \geq n+2$.

§2. The behavior of L with respect to limits

Almost everything we know about $L\colon G\mathcal{P}\mathcal{A} \to G\mathcal{S}\mathcal{A}$ and $L'\colon G\mathcal{P}\mathcal{A} \to G\mathcal{Z}\mathcal{A}$ that is not a formal consequence of their being left adjoints follows from their behavior on finite limits.

Proposition 2.1. (i) The functors L' and L preserve finite products.

(ii) The functor L'' and the functors L and L' restricted to the full subcategory of injection prespectra preserve all finite limits.

Since $L = L''L'$, it suffices to prove the claims of the proposition for L' and L''. For a given prespectrum (or inclusion prespectrum) C, the inclusion prespectrum $L'C$ (or spectrum $L''C$) is computed spacewise as a colimit of a directed system of spaces. Thus, to prove the proposition, we must investigate the commutativity of finite limits and directed colimits in the category $G\mathcal{U}$ of compactly generated weak Hausdorff G-spaces. We assume familiarity with the basic results on such limits and colimits in the category of sets (see [92, p. 211]).

As a further consequence of our results on finite limits and directed colimits of spaces, we will obtain the nonelementary parts of the following result on the units of our various adjunctions.

Proposition 2.2. (i) The unit $\eta': C \to \ell'L'C$ of the (L', ℓ') adjunction is always a spacewise surjection. It is an injection if and only if C is an injection prespectrum. It is an isomorphism if and only if C is an inclusion prespectrum.

(ii) The unit $\eta'': D \to \ell''L''D$ of the (L'', ℓ'') adjunction is always a spacewise inclusion. It is an isomorphism if and only if D is a spectrum.

(iii) The unit $\eta: C \to \ell LC$ of the (L, ℓ) adjunction is an injection if and only if C is an injection prespectrum and an inclusion if and only if C is an inclusion prespectrum. It is an isomorphism if and only if C is a spectrum.

The surjection assertion in part (i) above follows from the construction of L'. The isomorphism assertions in all three parts are formal consequences of the fact that we are dealing with inclusions of full subcategories [92, p. 88]. The remaining assertions of part (iii) follow from those of parts (i) and (ii) since $\eta: C \to \ell LC$ is the composite

$$C \xrightarrow{\eta'} \ell'L'C \xrightarrow{\ell'\eta''} \ell'\ell''L''L'C = \ell LC.$$

The remaining assertions in parts (i) and (ii) come from the following point set results.

Lemma 2.3. Let $\{X_\alpha, \lambda^\alpha_\beta: X_\alpha \to X_\beta\}$ and $\{Y_\alpha, \mu^\alpha_\beta: Y_\alpha \to Y_\beta\}$ be two directed systems in $G\mathcal{U}$ indexed on the same directed set I.

(i) The natural map

$$\operatorname*{colim}_{\alpha I}(X_\alpha \times Y_\alpha) \longrightarrow (\operatorname*{colim}_{\alpha I} X_\alpha) \times (\operatorname*{colim}_{\beta I} Y_\beta)$$

is an isomorphism.

(ii) If all the maps λ^α_β are injections, then the colimit X of the X_α in $G\mathcal{U}$ is the same as the colimit in the category of all topological spaces (rather than X being a proper quotient of the latter colimit) and the natural maps $X_\alpha \to X$ into the colimit are injections. Moreover, if the maps λ^α_β are all inclusions, then the maps $X_\alpha \to X$ are also inclusions.

Proof. (i) For any space Z in $G\mathcal{U}$, the functors $? \times Z$ and $Z \times ?$ are left adjoints and so preserve colimits. Thus

$$\begin{aligned}(\operatorname*{colim}_\alpha X_\alpha) \times (\operatorname*{colim}_\beta Y_\beta) &\cong \operatorname*{colim}_\alpha(X_\alpha \times (\operatorname*{colim}_\beta Y_\beta))\\ &\cong \operatorname*{colim}_\alpha(\operatorname*{colim}_\beta(X_\alpha \times Y_\beta)\\ &\cong \operatorname*{colim}_{\alpha,\beta}(X_\alpha \times Y_\beta)\end{aligned}$$

where the last colimit is indexed on the directed set $I \times I$ and the last isomorphism is a purely formal result on rearranging iterated colimits [92, p. 227]. Since the diagonal in $I \times I$ is a cofinal subset, the natural map

$$\operatorname*{colim}_\alpha(X_\alpha \times Y_\alpha) \longrightarrow \operatorname*{colim}_{\alpha,\beta}(X_\alpha \times Y_\beta)$$

is an isomorphism. The composite of all these isomorphisms is the isomorphism of part (i).

(ii) Let $G\mathcal{K}$ be the category of compactly generated, not necessarily weak Hausdorff, G-spaces (see [83,146]). Colimits in $G\mathcal{K}$ are the same as the corresponding colimits in the category of all G-spaces; colimits in $G\mathcal{U}$ are formed by taking the largest weak Hausdorff quotient of the corresponding colimit in $G\mathcal{K}$. Part (i) holds, with the same proof, in $G\mathcal{K}$. Let $\{X_\alpha, \lambda^\alpha_\beta: X_\alpha \to X_\beta\}$ be a directed system in $G\mathcal{U}$ with each λ^α_β an injection and let X be the colimit of the X_α in $G\mathcal{K}$. The natural maps $X_\alpha \to X$ are injections by the construction of colimits in $G\mathcal{K}$. Applying part (i) for $G\mathcal{K}$ with $Y_\alpha = X_\alpha$ and $\mu^\alpha_\beta = \lambda^\alpha_\beta$, it is easy to see that the diagonal in $X \times X$ is closed. Thus, X is in $G\mathcal{U}$ and so is also the colimit there. If the maps λ^α_β are, in fact, inclusions, then a typical point-set argument for compactly generated, weak Hausdorff spaces gives that the maps $X_\alpha \to X$ are also inclusions [83, p. 175].

Lemma 2.4. In $G\mathcal{U}$, finite limits commute with colimits of directed systems all of whose maps are injections.

Proof. Arbitrary finite limits can be constructed from finite products and equalizers. Therefore, the previous lemma reduces the proof to showing that equalizers commute with colimits of directed systems all of whose maps are injections. Let $\{X_\alpha, \lambda^\alpha_\beta: X_\alpha \to X_\beta\}$ and $\{Y_\alpha, \mu^\alpha_\beta: Y_\alpha \to Y_\beta\}$ be directed systems in $G\mathcal{U}$ indexed on the same directed set such that the maps λ^α_β and μ^α_β are all injections. Let $\{f_\alpha: X_\alpha \to Y_\alpha\}$ and $\{g_\alpha: X_\alpha \to Y_\alpha\}$ be a pair of maps of directed systems and let $E_\alpha \subset X_\alpha$ be the equalizer of f_α and g_α. The injections λ^α_β induce injections $\nu^\alpha_\beta: E_\alpha \to E_\beta$ making the set $\{E_\alpha, \nu^\alpha_\beta: E_\alpha \to E_\beta\}$ a directed system. Let X and Y be the colimits of the X_α and Y_α respectively, $f,g: X \to Y$ be the maps induced by the f_α and g_α, and $E \subset X$ be the equalizer of f and g. There is a natural map

$$\theta\colon \operatorname*{colim}_{\alpha} E_\alpha \longrightarrow E$$

which we must show is an isomorphism. The point of assuming that the maps λ^α_β and μ^α_β, and thus η^α_β, are injections is that Lemma 2.3(ii) ensures that the colimits of the X_α, Y_α and E_α in $G\mathcal{U}$ are the same as those in $G\mathcal{K}$ (or the category of all G-spaces). Thus, the underlying sets of the colimits are the colimits of the corresponding diagrams in the category of sets. It follows by [92, p. 211] that θ is a bijection. The commuting diagram

$$\begin{tikzcd} \operatorname{colim} E_\alpha \arrow[r, "\theta"] \arrow[d, "\phi"'] & E \arrow[d, phantom, "\cap"] \\ \operatorname{colim} X_\alpha \arrow[r, "="] & X \end{tikzcd},$$

in which ϕ is induced by the inclusions $E_\alpha \subset X_\alpha$, indicates that it suffices to prove that ϕ is an inclusion. For $\alpha \leq \beta$, the diagram

$$\begin{tikzcd} E_\alpha \arrow[r, phantom, "\subset"] \arrow[d, "\nu^\alpha_\beta"'] & X_\alpha \arrow[d, "\lambda^\alpha_\beta"] \\ E_\beta \arrow[r, phantom, "\subset"] & X_\beta \end{tikzcd}$$

is easily seen to be a pullback since μ^α_β is an injection. It follows that, for each α, the diagram

$$\begin{tikzcd} E_\alpha \arrow[r, "\iota_\alpha"] \arrow[d, "\xi_\alpha"'] & X_\alpha \arrow[d, "\zeta_\alpha"] \\ \operatorname{colim} E_\alpha \arrow[r, "\phi"] & \operatorname{colim} X_\alpha = X \end{tikzcd}$$

is a pullback if the vertical arrows are the natural maps into the colimits. Thus, for any closed subset C of $\operatorname{colim} E_\alpha$, $\iota_\alpha(\xi_\alpha^{-1}(C)) = \zeta_\alpha^{-1}(\phi(C))$. That ϕ is a closed inclusion now follows easily from the fact that each ι_α is a closed inclusion.

Remark 2.5. The cautious reader may have noted that we need results on limits and colimits in the category $G\mathcal{J}$ of based spaces for our applications, but that we have worked in the category $G\mathcal{U}$ of unbased spaces to have use of the fact that $Z \times ?$

preserves colimits there. Since limits and directed colimits in $G\mathcal{J}$ are formed by assigning the obvious basepoints to the corresponding constructions in $G\mathcal{U}$, our results apply to $G\mathcal{J}$.

Proof of 2.2. All that remains to be proved is that η'' is a spacewise inclusion and that, for injection prespectra, η' is an injection. That η'' is a spacewise inclusion follows immediately from the last remark in Lemma 2.3(ii). Assume that C is an injection prespectrum. To show that $\eta'\colon C \to \ell' L' C$ is an injection, it suffices to show that $\gamma^0_\alpha\colon C \to J_\alpha C$ is an injection for all α. In fact, we show by transfinite induction on α that $J_\alpha C$ is an injection prespectrum, and for all $\alpha \leq \beta$, $\gamma^\alpha_\beta\colon J_\alpha C \to J_\beta C$ is an injection. It is easy to check that if D is an injection prespectrum, then so is $JD = J_1 D$. Also, by Lemma 1.3(ii), $\gamma = \gamma^0_1\colon D \to JD$ is an injection. Thus, if $J_\beta C$ and $\gamma^\alpha_\beta\colon J_\alpha C \to J_\beta C$ satisfy our conditions, then so do $J_{\beta+1}C = JJ_\beta C$ and $\gamma^\alpha_{\beta+1} = \gamma\gamma^\alpha_\beta$. If δ is a limit ordinal and for all $\alpha \leq \beta < \delta$, $J_\alpha C$ and γ^α_β satisfy our conditions, then the colimit diagrams used to define $J_\delta C$ are directed systems of injections. By Lemma 2.3(ii), the maps $\gamma^\alpha_\beta\colon J_\alpha C \to J_\delta C$ are injections and the colimits used to form $J_\delta C$ do not involve passing to the largest weak Hausdorff quotient. Given this, it is easy to check that $J_\delta C$ is an injection prespectrum.

Proof of 2.1. Part (i) follows immediately from Lemmas 1.3(iv) and 2.3(i), Definition 1.4, and the description of L'' in section I.2. Part (ii) for L'' follows immediately from Lemma 2.4. For L', since only finitely many prespectra are involved in forming a finite limit, we may assume that L' agrees with J_α for some sufficiently large ordinal α. Thus, it suffices to prove, by transfinite induction on α, that J_α restricted to injection prespectra preserves finite limits. Recall from the proof of Proposition 2.2 that J_α preserves injection prespectra so we may assume hereafter that all functors are applied only to injection prespectra. By Lemma 1.3(iv), $J = J_1$ preserves finite limits, and if J_α preserves finite limits, so does $J_{\alpha+1} = JJ_\alpha$. Assume that β is a limit ordinal and J_α preserves finite limits for all $\alpha < \beta$. By the proof of Proposition 2.2, J_β is formed from colimits of directed systems of injections. Therefore, Lemma 2.4 ensures that J_β preserves finite limits.

Remarks 2.6. If C is an injection prespectrum, then, by Proposition 2.2(i), $\eta'\colon C \to \ell' L' C$ is a spacewise continuous bijection. Thus, $L'C$ is formed from C by simply altering the topology of the component spaces CV for $V \in \mathcal{A}$. Proposition 3.11 sheds some light on the nature of this alteration.

§3. Prespectrum and spectrum level closed inclusions

Recall from I§8 the connection between equalizers and (spacewise) closed inclusions, and its use in showing that certain functors preserve closed inclusions. In that discussion, we noted that L need not preserve closed inclusions and introduced a rather ad hoc concept of good closed inclusions to characterize those preserved by L. This failure of L to preserve arbitrary closed inclusions is rooted in the failure of L' to preserve arbitrary equalizers. However, this failure is not merely a misbehavior of L', as opposed to L'', with respect to closed inclusions. There are closed inclusions between Σ-inclusion prespectra which L'' (and so L) does not preserve. Here we give an alternative description of good closed inclusions and tie them to the behavior of pushouts in the category of injection prespectra. Using this, and various lemmas on closed inclusions, we show that a cofibration between spectra must be a closed inclusion.

Definition 3.1. A map $\iota: A \to D$ of injection prespectra is a good closed inclusion if it is a (spacewise) closed inclusion, and if for every $V \subset W$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} AV \arrow[r, "\iota"] \arrow[d, "\tilde{\sigma}_A"'] & DV \arrow[d, "\tilde{\sigma}_D"] \\ \Omega^{W-V}AW \arrow[r, "\Omega^{W-V}\iota"] & \Omega^{W-V}DW \end{tikzcd}$$

is a pullback. Since $\iota: AV \to DV$ is a closed inclusion, it suffices to check that AV maps onto the pullback of $\tilde{\sigma}_D$ and $\Omega^{W-V}\iota$; that is, for each $d \in DV$ and $\omega \in \Omega^{W-V}AW$ with $\tilde{\sigma}_D(d) = (\Omega^{W-V}\iota)(\omega)$, there is an $a \in AV$ with $\iota(a) = d$ and $\tilde{\sigma}_A(a) = \omega$. If A is a spectrum, then any closed inclusion out of A must be good.

The following extension of I.8.4 is our motivation for considering good closed inclusions; its proof is just diagram chasing.

Lemma 3.2. Let A, C, and D be injection prespectra, $\iota: A \to D$ be a good closed inclusion and $f: A \to C$ be any map. Then the pushout $C \cup_f D$ is an injection prespectrum if either of the following conditions holds

i) f is an injection.

ii) For every $V \subset W$ in $\mathcal{A}$ and every pair d, d' in $DV - \iota(AV)$, there is an element t in S^{W-V} such that $\tilde{\sigma}_D(d)(t) \neq \tilde{\sigma}_D(d')(t)$ and at least one of $\tilde{\sigma}_D(d)(t)$ and $\tilde{\sigma}_D(d')(t)$ is in $DW - \iota(AW)$.

Remarks 3.3. (i) Condition (i) of the lemma is a direct extension of I.8.4(i). Using it, one may check that good closed inclusions in the sense of Definition 3.1 are also good in the sense of I§8. In fact, the two notions are equivalent. If $\iota: A \to D$ is a closed inclusion between injection prespectra which is not good, then select $V \subset W$ in $\mathcal{A}$ and $d \in DV$, $\omega \in \Omega^{W-V}AW$ with $\tilde{\sigma}_D(d) = (\Omega^{W-V}\iota)(\omega)$ such that there is no $a \in AV$ with $\iota(a) = d$ and $\tilde{\sigma}_A(a) = \omega$. If $D \cup_A D$ is the prespectrum level pushout, then the map $(D \cup_A D)(V) \to \Omega^{W-V}(D \cup_A D)(W)$ is not an injection because the two canonical images of d in $(D \cup_A D)(V)$ both go to the obvious single image of ω in $\Omega^{W-V}(D \cup_A D)(W)$.

(ii) To see why our pullback condition for good closed inclusions should be tied to the preservation of closed inclusions by L, recall that L is defined in terms of directed colimits and note the use of pullbacks in the proof of Lemma 2.4. In general, if $\{X_\alpha, \lambda^\alpha_\beta: X_\alpha \to X_\beta\}$ and $\{Y_\alpha, \mu^\alpha_\beta: Y_\alpha \to Y_\beta\}$ are directed systems indexed on the same set and $\{\iota_\alpha: X_\alpha \to Y_\alpha\}$ is a collection of closed inclusions giving a map of directed systems, then the induced map $\iota: X \to Y$ between the colimits of the X_α and Y_α need not be a closed inclusion. However, a pullback condition, like that in the proof of Lemma 2.4, relating the ι_α, λ^α_β and μ^α_β ensures that ι is a closed inclusion.

Remarks 3.4. (i) If A and D in Lemma 3.2 are spectra, then condition (ii) of the lemma is almost certainly not satisfied unless ι is the identity map of D or the unique inclusion of the point spectrum into D. On the other hand, if E is an injection prespectrum and $j: X \to Z$ is a space level closed inclusion, then $1 \wedge j: E \wedge X \to E \wedge Z$ is a good closed inclusion satisfying condition (ii). In particular, if E is an injection prespectrum, and CE and $E \wedge I^+$ are the prespectrum level cone and cylinder of E, then the standard maps

$$\iota: E \longrightarrow CE \quad \text{and} \quad \iota_t: E \longrightarrow E \wedge I^+, \quad t \in I$$

are good closed inclusions satisfying condition (ii). Thus, if $f: A \to C$ is a map of injection prespectra, then the prespectrum level mapping cone Cf and mapping cylinder Mf are injection prespectra. A further, definitive, result on the cofibres used to define G-CW spectra is given in Lemma 4.1 below.

(ii) Another way to gain a feel for condition (ii) of the lemma is to consider the special case in which $C = *$ so that the pushout is the quotient prespectrum D/A. The connection between condition (ii) and D/A being an injection prespectrum is quite obvious.

Remarks 3.5. In our proof that a cofibration is a closed inclusion and in our study of CW spectra in the next section, we will want to apply the observation of Remark 3.4(i) on mapping cones and cylinders to a map $f: A \to C$ of spectra. The spectrum level mapping cone Cf is defined to be the spectrum associated by L to the prespectrum level pushout of the maps

$$\ell A \xrightarrow{\ell f} \ell C \quad\text{and}\quad \ell A \xrightarrow{\eta} \ell L \ell A \xrightarrow{\ell L\iota} \ell L((\ell A)\wedge I).$$

Thus, the remark does not apply directly. However, applying L to the prespectrum level pushout of the maps

$$\ell A \xrightarrow{\ell f} \ell C \quad\text{and}\quad \ell A \xrightarrow{\iota} (\ell A)\wedge I$$

produces an isomorphic spectrum; that is, the spectrum level mapping cone or cylinder of $f\colon A \to C$ is isomorphic to the result of applying L to the corresponding prespectrum level construction for $\ell f\colon \ell A \to \ell C$. Remark 3.4(i) applies to the latter construction.

Our proof that a spectrum level cofibration $f\colon X \to Y$ is a closed inclusion turns on the observation that f is a cofibration if and only if the natural map $i\colon Mf \to Y\wedge I^+$ is the inclusion of a retract. To apply this, we must relate f to i and record two formal observations on closed inclusions.

Lemma 3.6. Let $f\colon X \to Y$ be a map in $G\mathcal{J}$, $G\mathcal{PA}$ or $G\mathcal{SA}$ and let Mf and $Y\wedge I^+$ be the mapping cylinder and cylinder in the same category. Then the diagram

$$\begin{tikzcd} X \arrow[r, "\iota_1"] \arrow[d, "f"'] & Mf \arrow[d, "i"] \\ Y \arrow[r, "\iota_1"] & Y\wedge I^+ \end{tikzcd}$$

is a pullback.

This result is easily checked in $G\mathcal{J}$ and follows immediately for $G\mathcal{PA}$. Proposition 2.1(ii), Lemma 3.2, and Remarks 3.5 allow us to derive the spectrum level result from the prespectrum level one.

Lemma 3.7. Let $i\colon Z \to W$ and $r\colon W \to Z$ be maps in $G\mathcal{J}$, $G\mathcal{PA}$ or $G\mathcal{SA}$ with $ri = 1$. Then the diagram

$$\begin{tikzcd} Z \arrow[r, "i"] & W \arrow[r, shift left, "1"] \arrow[r, shift right, "ir"'] & W \end{tikzcd}$$

is an equalizer, so i is a closed inclusion.

Proof. The pair of maps $r\colon W \to Z$ and $1\colon W \to W$ display the diagram above as a split equalizer (see [92, p. 146]).

Lemma 3.8. If the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "i"] \\ Y \arrow[r, "g"] & W \end{tikzcd}$$

is a pullback in $G\mathcal{J}$, $G\mathcal{PA}$ or $G\mathcal{SA}$ and i is a closed inclusion, then so is f.

Proof. If i is a closed inclusion, then it is the equalizer of the obvious pair of maps $\iota_1,\iota_2\colon W \to W\cup_Z W$. If, further, the diagram is a pullback, then it follows formally that f is the equalizer of the pair $\iota_1 g, \iota_2 g\colon Y \to W\cup_Z W$. Thus f is a closed inclusion.

By taking $Z = Mf$ and $W = Y\wedge I^+$ in the last two lemmas, we obtain our promised result on cofibrations.

Proposition 3.9. A cofibration $f\colon X \to Y$ in $G\mathcal{J}$, $G\mathcal{PA}$ or $G\mathcal{SA}$ is a closed inclusion.

Remark 3.10. If Z and W in Lemma 3.7 are injection prespectra, then clearly i is a good closed inclusion. In the context of Lemma 3.8, if Y, Z and W are injection prespectra and $i\colon Z \to W$ is a good closed inclusion, then so is f.

We conclude this section with a result on the behavior of L' with respect to inclusions; it places a strong restriction on the way L' can alter the topology of injection prespectra (see Remark 2.6).

Proposition 3.11. Let A be an inclusion prespectrum, D be an injection prespectrum, and $\iota\colon A \to D$ be a spacewise inclusion. Then the map

$$L'\iota\colon A \cong L'A \longrightarrow L'D$$

is an inclusion.

Proof. It suffices to prove (by transfinite induction on α) that

$$J_\alpha \iota\colon A \cong J_\alpha A \longrightarrow J_\alpha D$$

is a spacewise inclusion for all α. The case $\alpha = 1$ is easy and implies that if the result is true for α, then it is also true for $\alpha + 1$. The result for limit ordinals follows from the following point set lemma.

Lemma 3.12. Let $\{Y_\alpha, \mu_\beta^\alpha\colon Y_\alpha \to Y_\beta\}$ be a directed system of injections in G and let $\{f_\alpha\colon X \to Y_\alpha\}$ be a collection of inclusions with $f_\beta = \mu_\beta^\alpha f_\alpha$ for $\alpha < \beta$. Then the induced map $f\colon X \to Y = \operatorname{colim} Y_\alpha$ is an inclusion.

Proof. Lemma 2.3(ii) ensures that the underlying set of Y is just the union of the underlying sets of the Y_α and that $f\colon X \to Y$ and the natural maps $\zeta_\alpha\colon Y_\alpha \to Y$ into the colimit are injections. Since Y is a compactly generated weak Hausdorff space, it suffices to show that if $K \subset f(X) \subset Y$ is compact Hausdorff, then f induces a homeomorphism between $\tilde{K} = f^{-1}(K)$ and K. Let $g = f|\tilde{K}$ and $g_\alpha = f_\alpha|\tilde{K}$. The maps g_α are closed inclusions since the f_α are inclusions and $g_\alpha(\tilde{K}) = \zeta_\alpha^{-1}(K)$, which is closed. Let C be a closed subset of $\tilde{K}$. Then $\zeta_\alpha^{-1}(g(C)) = g_\alpha(C)$, which is closed for any α, so $g(C)$ is closed. Thus g, being a closed map, is a homeomorphism from $\tilde{K}$ to K.

§4. The point set topology of CW spectra

Recall the definitions of a generalized sphere G-spectrum $S_H^q = G/H^+ \wedge S^q$ from I.4.3 and of G-cell spectra and G-CW spectra from I.5.1 and I.5.2. All of our statements in this section are about G-cell spectra; however, since G-CW spectra are a special type of G-cell spectra, these results are, of course, directly applicable to them. In defining G-cell spectra, we worked purely on the spectrum level, describing the $(n+1)^{st}$ stage E_{n+1} of a sequential filtration $\{E_n\}$ of a G-cell spectrum E as the spectrum level cofibre of an attaching map $j_n\colon J_n \to E_n$ whose domain spectrum J_n is a wedge of sphere G-spectra S_H^q. However, as noted in Remarks 3.5, it is sometimes best to construct a spectrum level cofibre indirectly from the prespectrum level cofibre of an associated map. To do this for G-cell spectra, we recall that the sphere S_H^q is derived from a Σ-inclusion prespectrum denoted $\tilde{S}_H^q$. If $\tilde{J}_n$ is the prespectrum level wedge of the sphere G-prespectra $\tilde{S}_H^q$ associated to the S_H^q in J_n, then the map j_n corresponds to a map $\tilde{j}_n\colon \tilde{J}_n \to E_n$ under the (L,ℓ) adjunction. Let $\tilde{E}_{n+1}$ be the prespectrum level cofibre of $\tilde{j}_n$. As in Remarks 3.5, we have that $E_{n+1} = L\tilde{E}_{n+1}$.

An obvious advantage of describing E_{n+1} in terms of $\tilde{E}_{n+1}$ is that the component spaces $\tilde{E}_{n+1}V$, for $V \in \mathcal{A}$, are relative generalized G-cell complexes made from the spaces E_nV by attaching generalized G-spheres $G/H^+ \wedge S^Z$ (for Z a G-representation) of the sort introduced at the end of I§8. The main results in this section, on the lattice theoretic behavior of cell subspectra and on maps from compact spectra into cell spectra, show that the good behavior of G-cell spectra is

largely due to the good behavior of relative cell complexes. To obtain these results, we must first show that $\tilde{E}_{n+1}$ is an inclusion prespectrum and so is not so badly mangled by L that the structure of its component spaces is lost.

Lemma 4.1. Let $\tilde{j}\colon \tilde{J} \to E$ be a map from a prespectrum level wedge of sphere G-prespectra to a spectrum E and let D be the prespectrum level cofibre of $\tilde{j}$. Then D is an inclusion prespectrum.

Proof. By Remarks 3.4(i), D is an injection prespectrum. Since the sphere G-prespectra $\tilde{S}_H^q$ are Σ-inclusion prespectra, so are $\tilde{J}$ and the prespectrum level cone $C\tilde{J}$. Let $V \subset W$ be in $\mathcal{A}$. In the diagram

$$\begin{tikzcd}
& \tilde{J}V \arrow[rr, "\iota"] \arrow[dl, "\tilde{j}"'] \arrow[dd, "\tilde{\sigma}_{\tilde{J}}"] & & C\tilde{J}V \arrow[dl, "\alpha"'] \arrow[dd, "\tilde{\sigma}_{C\tilde{J}}"] \\
EV \arrow[rr, "\iota'"] \arrow[dd, "\tilde{\sigma}_E"', "\cong"] & & DV \arrow[dd, "\tilde{\sigma}_D"] & \\
& \Omega^{W-V}\tilde{J}W \arrow[rr, "\Omega^{W-V}\iota"] \arrow[dl, "\Omega^{W-V}\tilde{j}"'] & & \Omega^{W-V}C\tilde{J}W \arrow[dl, "\beta"] \\
\Omega^{W-V}EW \arrow[rr, "\Omega^{W-V}\iota'"] & & \Omega^{W-V}DW &
\end{tikzcd}$$

the maps ι, $\Omega^{W-V}\iota$, ι', $\Omega^{W-V}\iota'$, $\tilde{\sigma}_{\tilde{J}}$ and $\tilde{\sigma}_{C\tilde{J}}$ are closed inclusions, the map $\tilde{\sigma}_D$ is an injection and the map $\tilde{\sigma}_E$ is a homeomorphism. To show that $\tilde{\sigma}_D$ is a closed inclusion, it suffices to show that it is a closed map. Let C be a closed subset of DV and $C' = \tilde{\sigma}_D(C)$. It suffices to show that C' is compactly closed. If it is not, then there is a compact Hausdorff subset K of $\Omega^{W-V}DW$ such that $K \cap C'$ is not closed in K. Let $\{y_\alpha\}_{\alpha\in A}$ be a net in $K \cap C'$ converging to a point y_0 in the K-closure of $K \cap C'$ but not in $K \cap C'$. Let $\{x_\alpha\}_{\alpha\in A}$ be the unique net in C with $\tilde{\sigma}_D(x_\alpha) = y_\alpha$ for all α. The space DV is the pushout of $\tilde{j}$ along ι and so is the disjoint union of $\iota'(EV)$ and $\alpha(C\tilde{J}V) - \alpha\iota(\tilde{J}V)$. The net $\{x_\alpha\}_{\alpha\in A}$ must have a subnet, which we denote $\{x_\beta\}_{\beta\in B}$, in either $\iota'(EV)$ or $\alpha(C\tilde{J}V) - \alpha\iota(\tilde{J}V)$. If the subnet lies in $\iota'(EV)$, then the corresponding subnet $\{y_\beta\}_{\beta\in D}$ must lie in the closed subset $\Omega^{W-V}\iota'(\Omega^{W-V}EW)$. Thus, y_0 is also in this set. Using the facts that the maps ι' and $\Omega^{W-V}\iota'$ are closed inclusions, the map $\tilde{\sigma}_E$ is a homeomorphism, and the set C is closed, it is easy to argue that the net $\{x_\beta\}_{\beta\in B}$ must converge to a point $x_0 \in C$ with $y_0 = \tilde{\sigma}_D(x_0)$, contradicting the assumption that y_0 is not in C'. Assume that a subnet $\{x_\beta\}_{\beta\in B}$ lies in $\alpha(C\tilde{J}V) - \alpha\iota(\tilde{J}V)$ and select points $\{z_\beta\}_{\beta\in B}$ with $\alpha(z_\beta) = x_\beta$. The image of the composite

$$K \wedge S^{W-V} \longrightarrow (\Omega^{W-V}DW) \wedge S^{W-V} \xrightarrow{\ \varepsilon\ } DW,$$

where ε is the evaluation map, is compact and can meet only finitely many of the cells added to EW to form DW. Therefore the points $\{z_\beta\}_{\beta\in B}$ must lie in a finite subwedge of the wedge $C\tilde{J}V$ of cones on generalized G-spheres. This subwedge is compact, and the net $\{z_\beta\}_{\beta\in B}$ must have a subnet converging to a point z_0. The corresponding subnets of the nets $\{x_\beta\}_{\beta\in B}$ and $\{y_\beta\}_{\beta\in B}$ must converge to $\alpha(z_0)$ and $\tilde{\sigma}_D\alpha(z_0)$ respectively. But this subnet of $\{y_\beta\}_{\beta\in B}$ also converges to y_0. Since nets have unique limits in the compact Hausdorff space K, $\tilde{\sigma}_D\alpha(z_0) = y_0$. However, $\tilde{\sigma}_D\alpha(z_0)$ must be in C' since $\alpha(z_0)$ must be in the closed set C. This contradicts the assumption that y_0 is not in C'. Thus, C' must be compactly closed and $\tilde{\sigma}_D$ must be a closed inclusion.

Recall that a compact G-spectrum is a spectrum of the form $\Lambda^V\Sigma^\infty K$, where $V \in \mathcal{A}$ and K is a compact Hausdorff G-space. The following result, stated without proof as I.5.3, follows from the lemma and familiar properties of space level cell complexes.

<u>Proposition 4.2</u>. Any map from a compact G-spectrum $\Lambda^V\Sigma^\infty K$ to a G-cell spectrum E factors through a finite cell subspectrum of E. Any G-cell spectrum is the union of its finite cell subspectra.

Before proving this, we derive a corollary ensuring that unions and intersections of cell subspectra are cell subspectra with behavior analogous to that of the subcomplexes of a space level cell complex.

<u>Corollary 4.3</u>. Let E be a G-cell spectrum and $\{E_\alpha\}$ be a collection of cell subspectra. Then $\cap E_\alpha$ and $\cup E_\alpha$ are cell subspectra of E whose cells are, respectively, those of E which appear in all of the E_α and those that appear in at least one of the E_α. Moreover, for any cell subspectrum D of E, the lattice identities

$$D \cap (\cup E_\alpha) = \cup(D \cap E_\alpha) \quad\text{and}\quad D \cup (\cap E_\alpha) = \cap(D \cup E_\alpha)$$

hold.

<u>Proof</u>. The lattice identities follow immediately from the identification of the unions and intersections as cell subspectra containing specified cells. We begin the identification of $\cap E_\alpha$ with that of the special case $E' \cap E''$, where E' and E'' are cell subspectra of E. Let $\{E_n\}$ be a sequential filtration of E. The spectra E' and E'' have sequential filtrations $\{E_n'\}$ and $\{E_n''\}$ such that E_n' (and E_n'') contain exactly those cells of E' (or E'') which are also in E_n.

By Lemma 2.4 and the proof of I.4.8, $E' \cap E''$ is the prespectrum level colimit of the $E_n' \cap E_n''$. Thus, it suffices to prove that $E_n' \cap E_n''$ is a cell subspectrum of E_n with the proper cells. This is trivially true for $n = 0$. Assume the result is true for some $n \geq 0$. To prove the result for $n+1$, we let $\tilde{j}: \tilde{J} \to E_n$, $\tilde{j}': \tilde{J}' \to E_n'$, and $\tilde{j}'': \tilde{J}'' \to E_n''$ be the prespectrum level attaching maps associated to the usual spectrum level attaching maps as in the introduction to this section. Let $\tilde{E}_{n+1}$, $\tilde{E}'_{n+1}$ and $\tilde{E}''_{n+1}$ be the prespectrum level cofibres of the maps $\tilde{j}$, $\tilde{j}'$ and $\tilde{j}''$. Then $E_{n+1} = L\tilde{E}_{n+1}$ and similarly for E'_{n+1} and E''_{n+1}. The prespectra $\tilde{J}'$ and $\tilde{J}''$ are both subwedges of $\tilde{J}$, and $\tilde{J}' \cap \tilde{J}''$ is just the prespectrum level wedge of those spheres $\tilde{S}^q_H$ which appear in both $\tilde{E}'_{n+1}$ and $\tilde{E}''_{n+1}$. Clearly $\tilde{E}'_{n+1} \cap \tilde{E}''_{n+1}$ is the prespectrum level cofibre of $\tilde{J}' \cap \tilde{J}'' \to E_n' \cap E_n''$. By Lemma 3.2, $\tilde{E}_{n+1}$, $\tilde{E}'_{n+1}$ and $\tilde{E}''_{n+1}$ are all inclusion prespectra, so L takes the intersection $\tilde{E}'_{n+1} \cap \tilde{E}''_{n+1}$ to the intersection $E'_{n+1} \cap E''_{n+1}$. Thus, $E'_{n+1} \cap E''_{n+1}$ is the spectrum level cofibre of the map $L(\tilde{J}' \cap \tilde{J}'') \to E_n' \cap E_n''$ and so is a cell subspectrum with the correct cells.

Now consider an arbitrary intersection $\cap E_\alpha$, which may be defined as the limit of all the inclusions $E_\alpha \to E$. Let C be the cell subspectrum of E which is claimed to be isomorphic to $\cap E_\alpha$. The inclusion $\iota: C \to E$ must factor as the composite of a map $f: C \to \cap E_\alpha$ and the natural map $\cap E_\alpha \to E$ because of the universality of $\cap E_\alpha$. Since ι is a closed inclusion, so is f, and to show that f is an isomorphism it suffices to show that it is a spacewise surjection. A point x in $(\cap E_\alpha)(V)$, for $V \in \mathcal{A}$, may be identified with a map $\tilde{x}: \Lambda^V\Sigma^\infty S^0 \to \cap E_\alpha$, and x is in the image of f if and only if $\tilde{x}$ factors through f. Because the spectrum $\Lambda^V\Sigma^\infty S^0$ is compact, the composite of $\tilde{x}$ and the map $\cap E_\alpha \to E$ factors through a finite subspectrum D of E. The spectra $D \cap C$ and $D \cap E_\alpha$ are cell subspectra of D and x comes from a point in $(\bigcap_\alpha(D \cap E_\alpha))(V)$. Thus, it suffices to prove that the corresponding map $D \cap C \to \bigcap_\alpha(D \cap E_\alpha)$ is a spacewise surjection; that is, we may assume that E is finite. In this case, E has only finitely many distinct cell subspectra and $\cap E_\alpha$ is, in essence, a finite intersection. The claimed cell structure for finite intersections follows directly from the initial case of two cell subspectra.

The union $\cup E_\alpha$ is properly defined as the colimit over the diagram of subspectra obtained by considering all possible intersections of the E_α. By the result on intersections, this is a diagram of cell subspectra. Since any cell spectrum is the colimit of its finite cell subspectra, standard results on rearranging iterated colimits [92, p. 227] give that $\cup E_\alpha$ is a cell subspectrum of E with the indicated cells.

Proof of Proposition 4.2. For the first part, let $\{E_n\}$ be a sequential filtration of the G-cell spectrum E. By I.5.9 and Proposition 3.9, the maps $E_n \to E_{n+1}$ and $E_n \to E$ are cofibrations and therefore spacewise closed inclusions. Any map $f\colon \Lambda^V\Sigma^\infty K \to E$ must factor through some E_n by I.4.8, and it suffices to prove that any map $f\colon \Lambda^V\Sigma^\infty K \to E_n$ factors through a finite cell subspectrum of E_n. Our proof essentially parallels the obvious inductive proof of the corresponding result on a map from a compact space into a space level cell complex X. Clearly, our result holds for $E_0 = *$. We assume the result for all integers m, $0 \leq m \leq n$, for some integer n and prove it for $n+1$. Let $j : J \to E_n$ be the attaching map used to form E_{n+1} from E_n and let $g\colon \Lambda^V\Sigma^\infty K \to E_{n+1}/E_n \cong \Sigma J$ be the composite of $f\colon \Lambda^V\Sigma^\infty K \to E_{n+1}$ and the projection $E_{n+1} \to E_{n+1}/E_n$. Under the adjunction of I.4.2, the maps f and g correspond to maps

$$\hat{f}\colon K \longrightarrow E_{n+1}V \quad \text{and} \quad \hat{g}\colon K \longrightarrow (\Sigma J)(V).$$

Let K' be the inverse image under $\hat{f}$ of the image of E_nV in $E_{n+1}V$. The subspace K' is closed and compact, so by the induction hypothesis, the restriction of f to a map $\Lambda^V\Sigma^\infty K' \to E_n$ factors through a finite cell subspectrum C of E_n.

We want to argue that since g is a map from a compact spectrum into a spectrum level wedge ΣJ, it must factor through a finite subwedge. Then we would like to show that f factors through the smallest cell subspectrum of E_{n+1} containing C and the cells associated to this subwedge. To carry out this argument, we let $\tilde{j}\colon \tilde{J} \to E_n$ be the prespectrum level attaching map associated to j and $\tilde{E}_{n+1}$ be the prespectrum level cofibre of $\tilde{j}$ so that $E_{n+1} = L\tilde{E}_{n+1}$ as in the introduction to this section. Since $\tilde{E}_{n+1}$ and $\Sigma\tilde{J}$ are inclusion prespectra yielding E_{n+1} and ΣJ by application of L, there is a $W \in \mathcal{A}$ with $V \subset W$ such that $\hat{f}$ and $\hat{g}$ factor as the composites

$$K \xrightarrow{\hat{f}_1} \Omega^{W-V}\tilde{E}_{n+1}W \longrightarrow \operatorname*{colim}_{V' \supset V} \Omega^{V'-V}\tilde{E}_{n+1}V' = E_{n+1}V$$

and

$$K \xrightarrow{\hat{g}_1} \Omega^{W-V}\Sigma\tilde{J}W \longrightarrow \operatorname*{colim}_{V' \supset V} \Omega^{V'-V}\Sigma\tilde{J}V' = (\Sigma J)(V).$$

Let

$$f_1\colon \Sigma^{W-V}K \longrightarrow \tilde{E}_{n+1}W \quad \text{and} \quad g_1\colon \Sigma^{W-V}K \longrightarrow \Sigma\tilde{J}W$$

be the adjoints of $\hat{f}_1$ and $\hat{g}_1$.

The map g_1 goes from a compact space into a space level wedge and so must factor through a finite subwedge. Let $M \subset J$ and $\tilde{M} \subset \tilde{J}$ be the corresponding subwedges. Tracing back through the adjunctions, we see that g factors through the inclusion $\Sigma M \to \Sigma J$, as we suggested it should. If $n = 0$, this completes the proof.

If $n > 0$, let D_{n+1} be the smallest cell subspectrum of E_{n+1} containing C and the cells associated to M. Clearly D_{n+1} must contain not only the cells from M, but also any cells from E_n hit by the attaching map $M \subset J \to E_n$. To include these cells from E_n, we must also include any cells from E_{n-1} hit by their attaching maps and so forth. Since sphere spectra are compact, the induction hypothesis ensures that only finitely many cells must be added to C to form D_{n+1}. Let $D_n = D_{n+1} \cap E_n$. Since $C \subset D_n$, the map $\Lambda^V\Sigma^\infty K' \to E_n$ factors through D_n. By our definitions of D_{n+1} and D_n, the attaching map $M \subset J \to E_n$ factors through D_n, and D_{n+1} is the cofibre of the resulting map $M \to D_n$. Construct $\tilde{D}_{n+1}$ from D_n and $\tilde{M}$ as a prespectrum level cofibre so that $L\tilde{D}_{n+1} = D_{n+1}$. The restriction of f_1 to $\Sigma^{W-V}K'$ factors through the inclusion $D_nW \subset E_nW \subset \tilde{E}_{n+1}W$ and the map g_1 factors through the inclusion $\Sigma\tilde{M}W \subset \Sigma\tilde{J}W$. Since $\tilde{D}_{n+1}W/D_nW \cong \Sigma\tilde{M}W$, the map f_1 must factor, at least as a set map, through the map $\tilde{D}_{n+1}W \to \tilde{E}_{n+1}W$. This factorization through $\tilde{D}_{n+1}W$ must be continuous because the map $\tilde{D}_{n+1}W \to \tilde{E}_{n+1}W$ is a closed inclusion. Tracing back through the adjunctions, we see that f factors through the finite cell subspectrum D_{n+1}.

For the second part of the proposition, again let $\{E_n\}$ be a sequential filtration of a G-cell spectrum E. Since E is the colimit of the E_n, to show that E is the union, that is colimit, of its finite cell subspectra, it suffices to show that each E_n is the colimit of its finite cell subspectra. For $n = 0$, this is trivial. For $n = 1$, this is the assertion that a wedge is the colimit of its finite subwedges, which is a formal result on rearranging colimits [92, p. 227]. The result for E_{n+1} follows from that for E_n by a similar argument, using cofinality and rearranging colimits.

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Index

Index of Notations - Roman letter

Due to a super abundance of notations beginning with the letter G, such notations are generally arranged under their second character.

References to distinct usages of the same symbol are given on subsequent lines.

Index of Notations - Greek letter

Index of Notations -- nonalphabetic, subscripts and superscripts

Index of Categories

Categories whose names begin in G, h, w, Ω, or some combination thereof, are alphabetized by their remaining letters.

Index of Adjoint Pairs of Functors

The notation

$$(F,G):\mathcal{C} \longrightarrow \mathcal{D}$$

means that $F:\mathcal{C}\to\mathcal{D}$ is left adjoint to $G:\mathcal{D}\to\mathcal{C}$. The pairs are arranged by the names of the left adjoints. Any unnamed functor is an inclusion. Pairs of functors on the space or spectrum level pass to the homotopy categories and then to the categories with weak equivalences inverted.

Index of natural transformations