Topology and its Applications 157 (2010) 902–915

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# A freeness theorem for $RO(\mathbb{Z}/2)$-graded cohomology

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ARTICLE INFO

*Article history:*
Received 25 August 2009
Received in revised form 7 December 2009
Accepted 7 December 2009

*MSC:*
55N91
55R40

*Keywords:*
Algebraic topology
Equivariant topology
Equivariant homology and cohomology
Homology of classifying spaces

ABSTRACT

In this paper it is shown that the $RO(\mathbb{Z}/2)$-graded cohomology of a certain class of $\text{Rep}(\mathbb{Z}/2)$-complexes, which includes projective spaces and Grassmann manifolds, is always free as a module over the cohomology of a point when the coefficient Mackey functor is $\underline{\mathbb{Z}/2}$.



## 1. Introduction

In non-equivariant topology, it is a triviality that spaces built of only even dimensional cells will have free cohomology, regardless of the chosen coefficient ring. It is just as easy to see that every space has free cohomology when the coefficient ring is taken to be $\mathbb{Z}/2$. Analogous results are not so clear in the equivariant setting.

In [4], it is shown that the ordinary $RO(\mathbb{Z}/p)$-graded homology of a $\mathbb{Z}/p$-space built of only even dimensional cells is free as a module over the homology of a point, regardless of which Mackey functor is chosen for coefficients. The goal of this paper is to establish a similar result for the ordinary $RO(\mathbb{Z}/2)$-graded cohomology of $\mathbb{Z}/2$-spaces without the restriction to cells of even degrees, but with the assumption of using constant $\underline{\mathbb{Z}/2}$ Mackey functor coefficients. Here is the main result:

**Theorem.** *If $X$ is a connected, locally finite, finite dimensional* $\text{Rep}(\mathbb{Z}/2)$*-complex, then $H^{*,*}(X;\underline{\mathbb{Z}/2})$ is free as an $H^{*,*}(pt;\underline{\mathbb{Z}/2})$-module.*

(The bigrading will be explained in Section 2.)

The projective spaces and Grassmann manifolds associated to representations of $\mathbb{Z}/2$ are examples of such $\text{Rep}(\mathbb{Z}/2)$-complexes. In these particular cases, the free generators of the cohomology modules are in bijective correspondence with the Schubert cells. The precise degrees of the cohomology generators is typically unknown, much like in [4].

Section 2 provides some of the background and notation required for the rest of the paper. Most of this information can be found in [6] and [4] but is reproduced here for convenience. Section 3 holds the main freeness theorem. As applications of the freeness theorem, Section 4 exhibits some techniques for calculating the ordinary cohomology of $\text{Rep}(G)$-complexes. The importance of such calculations lies in their potential applications toward understanding $RO(G)$-graded equivariant characteristic classes.

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doi:10.1016/j.topol.2009.12.006

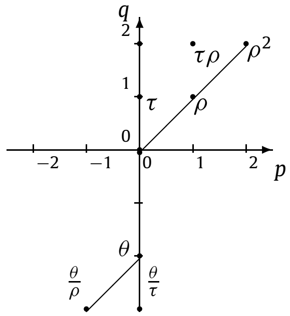


**Fig. 2.1.** $H^{*,*}(pt;\underline{\mathbb{Z}/2})$.

## 2. Preliminaries

This section contains some of the basic machinery and notations that will be used throughout the paper. In this section, $G$ can be any finite group unless otherwise specified.

Given a $G$-representation $V$, let $D(V)$ and $S(V)$ denote the unit disk and unit sphere, respectively, in $V$ with action induced by that on $V$. A Rep$(G)$**-complex** is a $G$-space $X$ with a filtration $X^{(n)}$ where $X^{(0)}$ is a disjoint union of $G$-orbits and $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching cells of the form $D(V_\alpha)$ along maps $f_\alpha : S(V_\alpha) \to X^{(n-1)}$ where $V_\alpha$ is an $n$-dimensional real representation of $G$. The space $X^{(n)}$ is referred to as the $n$**-skeleton** of $X$, and the filtration is referred to as a **cell structure**. If the filtration is finite, then $X$ is a **finite dimensional** Rep$(G)$-complex. If there are finitely many cells of each dimension, then $X$ is **locally finite**.

For the precise definition of a Mackey functor when $G = \mathbb{Z}/2$, the reader is referred to [5] or [3]. A summary of the important aspects of a Mackey functor is given here. The data of a Mackey functor are encoded in a diagram like the one below

$$\overset{t^*}{\circlearrowright} M(\mathbb{Z}/2) \underset{i^*}{\overset{i_*}{\rightleftarrows}} M(e).$$

The maps must satisfy the following four conditions.

1. $(t^*)^2 = id$.
2. $t^* i^* = i^*$.
3. $i_*(t^*)^{-1} = i_*$.
4. $i^* i_* = id + t^*$.

According to [6], each Mackey functor $M$ uniquely determines an ordinary $RO(G)$-graded cohomology theory characterized by:

1. $$H^n(G/H;M) = \begin{cases} M(G/H) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$
2. The map $H^0(G/K;M) \to H^0(G/H;M)$ induced by $i : G/H \to G/K$ is the transfer map $i^*$ in the Mackey functor.

A $p$-dimensional real $\mathbb{Z}/2$-representation $V$ decomposes as $V = (\mathbb{R}^{1,0})^{p-q} \oplus (\mathbb{R}^{1,1})^q = \mathbb{R}^{p,q}$ where $\mathbb{R}^{1,0}$ is the trivial 1-dimensional real representation of $\mathbb{Z}/2$ and $\mathbb{R}^{1,1}$ is the nontrivial 1-dimensional real representation of $\mathbb{Z}/2$. Thus the $RO(\mathbb{Z}/2)$-graded theory is a bigraded theory, one grading measuring dimension and the other measuring the number of "twists". In this case, we write $H^V(X;M) = H^{p,q}(X;M)$ for the $V$th graded component of the $RO(\mathbb{Z}/2)$-graded equivariant cohomology of $X$ with coefficients in a Mackey functor $M$.

In this paper, $G$ will typically be $\mathbb{Z}/2$ and the Mackey functor will almost always be constant $M = \underline{\mathbb{Z}/2}$ which has the following diagram

$$\overset{id}{\circlearrowright} \mathbb{Z}/2 \underset{id}{\overset{0}{\rightleftarrows}} \mathbb{Z}/2.$$

With these constant coefficients, the ordinary $RO(\mathbb{Z}/2)$-graded cohomology of a point is given by the picture in Fig. 2.1.

Every lattice point in the picture that is inside the indicated cones represents a copy of the group $\mathbb{Z}/2$. The **top cone** is a polynomial algebra on the nonzero elements $\rho \in H^{1,1}(pt;\underline{\mathbb{Z}/2})$ and $\tau \in H^{0,1}(pt;\underline{\mathbb{Z}/2})$. The nonzero element $\theta \in H^{0,-2}(pt;\underline{\mathbb{Z}/2})$ in the **bottom cone** is infinitely divisible by both $\rho$ and $\tau$. The cohomology of $\mathbb{Z}/2$ is easier to describe: $H^{*,*}(\mathbb{Z}/2;\underline{\mathbb{Z}/2}) = \mathbb{Z}/2[t,t^{-1}]$ where $t \in H^{0,1}(\mathbb{Z}/2;\underline{\mathbb{Z}/2})$. Details can be found in [3] and [2].

Throughout this paper, $H^{*,*}(-)$ denotes the ordinary $RO(\mathbb{Z}/2)$-graded cohomology of a $\mathbb{Z}/2$-space and $H^*(-)$ denotes non-equivariant cohomology. The reader should be aware that it is common to denote $RO(G)$-graded cohomology by $H^*_G(-)$ for an arbitrary group $G$. In this paper, we always have $G = \mathbb{Z}/2$ in mind and so suppress the $G$ in the notation and replace the single $RO(G)$-grading super script with an integral bidegree.

In addition, coefficients will always be taken to be the Mackey functor $\underline{\mathbb{Z}/2}$ in the equivariant theory and $\mathbb{Z}/2$ in the non-equivariant theory, and these coefficients will be suppressed as well.

A useful tool is the following exact sequence of [1] relating the ordinary $RO(\mathbb{Z}/2)$-graded cohomology of a $\mathbb{Z}/2$-space $X$ to the non-equivariant cohomology of $X$.

**Lemma 2.1** *(Forgetful long exact sequence). Let $X$ be a based $\mathbb{Z}/2$-space. Then for every $q$ there is a long exact sequence*

$$\cdots \longrightarrow H^{p,q}(X) \xrightarrow{\cdot\rho} H^{p+1,q+1}(X) \xrightarrow{\psi} H^{p+1}(X) \xrightarrow{\delta} H^{p+1,q}(X) \longrightarrow \cdots.$$

The map $\cdot\rho$ is multiplication by $\rho \in H^{1,1}(pt; \underline{\mathbb{Z}/2})$ and $\psi$ is the forgetful map to non-equivariant cohomology with $\mathbb{Z}/2$ coefficients.

## 3. The freeness theorem

Computing the $RO(G)$-graded cohomology of a $G$-space $X$ is typically quite a difficult task. However, if $X$ has a filtration $X^{(0)} \subseteq X^{(1)} \subseteq \cdots$, then we can take advantage of the long exact sequences in cohomology arising from the cofiber sequences $X^{(n)} \subseteq X^{(n+1)} \to X^{(n+1)}/X^{(n)}$. These long exact sequences paste together as an exact couple in the usual way, giving rise to a spectral sequence associated to the filtration.

If $X$ is a Rep$(G)$-complex, then $X$ has a natural filtration coming from the cell structure. If in addition $X$ is connected, the quotient spaces $X^{(n+1)}/X^{(n)}$ are wedges of $(n+1)$-spheres with action determined by the type of cells that were attached. Examples of this sort appear throughout the paper.

For the remainder of the paper, we will only be interested in the case $G = \mathbb{Z}/2$ and always take coefficients to be $\underline{\mathbb{Z}/2}$. These choices will be implicit in our notation.

Given a filtered $\mathbb{Z}/2$-space $X$, for each fixed $q$ there is a long exact sequence

$$\cdots H^{*,q}\big(X^{(n+1)}/X^{(n)}\big) \longrightarrow H^{*,q}\big(X^{(n+1)}\big) \longrightarrow H^{*,q}\big(X^{(n)}\big) \longrightarrow H^{*+1,q}\big(X^{(n+1)}/X^{(n)}\big) \cdots$$

and so there is one spectral sequence for each integer $q$. The specifics are given in the following proposition.

**Proposition 3.1.** *Let $X$ be a filtered $\mathbb{Z}/2$-space. Then for each $q \in \mathbb{Z}$ there is a spectral sequence with*

$$E_1^{p,n} = H^{p,q}\big(X^{(n+1)}, X^{(n)}\big)$$

*converging to $H^{p,q}(X)$.*

The construction of the spectral sequence is completely standard. See, for example, Proposition 5.3 of [7].

It is convenient to plot the $RO(\mathbb{Z}/2)$-graded cohomology in the plane with $p$ along the horizontal axis and $q$ along the vertical axis, and this turns out to be a nice way to view the cellular spectral sequences as well. When doing so, the differentials on each page of the spectral sequence have bidegree $(1,0)$ in the plane, but reach farther up the filtration on each page. It is important to keep track of at what stage of the filtration each group arises. In practice, this can be done by using different colors for group that arise at different stages of the filtration.

It is often quite difficult to determine the effect of all of the attaching maps in the cell attaching long exact sequences. If $X$ is locally finite, then the cells can be attached one-at-a-time, in order of dimension. This simplicity will make it easier to analyze the differentials in the spectral sequence of the 'one-at-a-time' cellular filtration, even when the precise impact of the attaching maps are not a priori known.

**Lemma 3.1.** *Let $B$ be a* Rep$(\mathbb{Z}/2)$*-complex with free cohomology that is built only of cells of dimension strictly less than $p$. Suppose $X$ is obtained from $B$ by attaching a single $(p,q)$-cell and let $\nu$ denote the generator for the cohomology of $X/B \cong S^{p,q}$. Then after an appropriate change of basis either*

1. *all attaching maps to the top cone of $\nu$ are zero (that is, $d(a) = 0$ for all $a$ with $a \in H^{*,q_a}(B)$ with $q_a \geqslant q-1$),*
2. *the cell attaching 'kills' $\nu$ and a free generator in dimension $(p-1,q)$, or*
3. *all nonzero differentials hit the bottom cone of $\nu$.*

**Proof.** Consider the cellular spectral sequence associated to attaching a single $(p,q)$-cell to $B$. The effects of attaching such a cell can cause the lower dimensional generators to hit either the 'top cone' or the 'bottom cone' of the newly attached free generator $\nu$ of degree $(p,q)$.

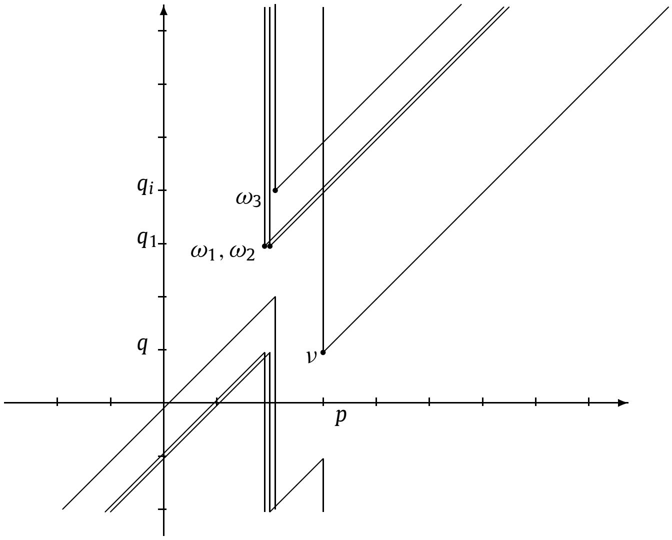


**Fig. 3.1.** The $E_1$ page of the cellular spectral sequence attaching a single $(p,q)$-cell to $B$.

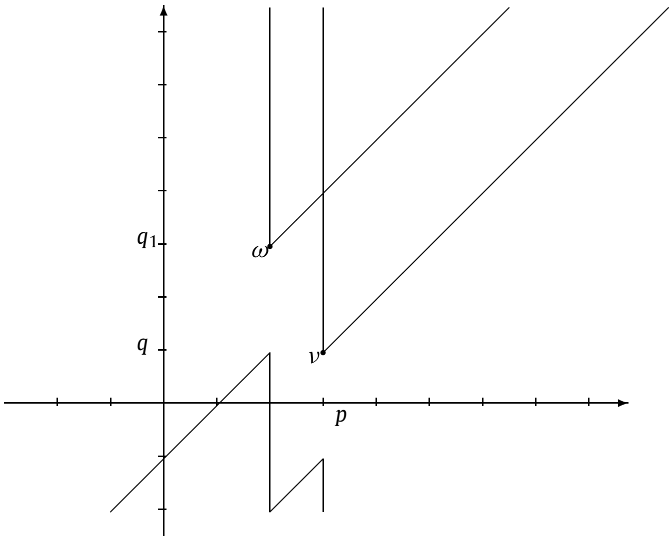


**Fig. 3.2.** The nonzero portion of the same spectral sequence, after a change of basis.

Suppose first that all nonzero differentials hit the top cone. Then any free generator $\omega_i$ having a nonzero differential in the spectral sequence must have degree $(p_i, q_i)$ where $p_i = p-1$ and $q_i \geqslant q$. For illustrative purposes, the $E_1$ page of the cellular spectral sequence of an example of this type is pictured in Fig. 3.1. In this example, there are two generators $\omega_1$ and $\omega_2$ with bidegree $(p-1, q_1)$ and one generator $\omega_3$ with bidegree $(p-1, q_i)$.

Here, only the generator associated to the $(p,q)$-cell and the generators with nonzero differentials are shown. Each of the $\omega_i$ satisfies $d(\omega_i) = \tau^{n_i}\nu$ for integers $n_i$. Relabeling if necessary, we can arrange so that the $\omega_i$ satisfy $n_1 \leqslant n_2 \leqslant \cdots$.

Let $A = \langle \omega_i \rangle$, the $H^{*,*}(pt)$-span of the $\omega_i$'s. A change of basis can be performed on $A$, after which we may assume $d(\omega_1) = \tau^{n_1}\nu$ and $d(\omega_i) = 0$ for $i > 1$. Indeed, $\{\tau^{n_i - n_1}\omega_1 + \omega_i\}$ is a basis for $A$ and $d(\tau^{n_i - n_1}\omega_1 + \omega_i) = \tau^{n_1}\nu$ if $i = 1$ and is zero otherwise. (In effect, the attaching map can 'slide' off of all the $\omega_i$ except for the one for which $q_i$ is minimal.)

If $\omega_1$ happens to be in dimension $(p-1, q)$, then the newly attached cell 'kills' $\omega_1$ and $\nu$. Otherwise the nonzero portion of the spectral sequence is illustrated in Fig. 3.2.

After taking cohomology, the spectral sequence collapses, as in Fig. 3.3.

There is a class $\omega_1 \frac{\theta}{\tau^m}$ that, potentially, could satisfy $\rho \cdot \omega_1 \frac{\theta}{\tau^m} = \nu$. However, for degree reasons, $\rho \cdot \omega_1 \frac{\theta}{\tau^{m+1}} = 0$ and since $\rho$ and $\tau$ commute, $\rho \cdot \omega_1 \frac{\theta}{\tau^m} = 0$. This means $\nu$ determines a nonzero class in $H^{*,*}(X)$ that is not in the image of $\cdot\rho$. If $B$ is based, then $X$ is based, and so, by the forgetful long exact sequence, $\nu$ determines a nonzero class in non-equivariant cohomology. Then since $\tau$ maps to 1 in non-equivariant cohomology, $\tau^n\nu$ is nonzero for all $n$. But, as the picture indicates,

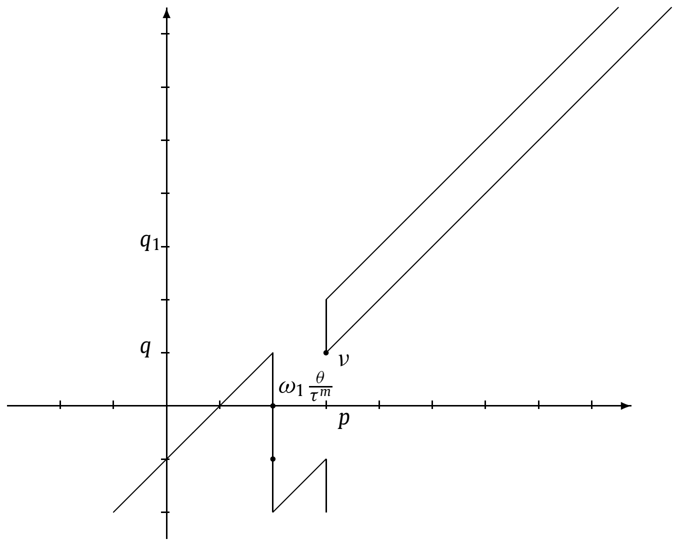


**Fig. 3.3.** The $E_2 = E_\infty$ page of the above spectral sequence.

$\tau^n \nu$ is zero for large enough $n$. This contradiction implies that there could not have been any nonzero differentials hitting the top cone of $\nu$.

This argument is independent of whether there are any differentials hitting the bottom cone, and so there simply cannot be any nonzero differentials on the top cone. □

Differentials hitting the bottom cone can cause a shifting in degree of the cohomology generators.

**Theorem 3.1.** *Suppose $X$ is a* Rep($\mathbb{Z}/2$)*-complex formed by attaching a single $(p, q)$-cell to a space $B$. Suppose also that $\tilde{H}^{*,*}(B)$ is a free $H^{*,*}(pt)$-module with a single generator $\omega$ of dimension strictly less than $p$. Then $H^{*,*}(X)$ is a free $H^{*,*}(pt)$-module. In particular, one of the following must hold*:

1. $H^{*,*}(X) \cong H^{*,*}(pt)$.
2. $H^{*,*}(X) \cong H^{*,*}(B) \oplus \Sigma^\nu H^{*,*}(pt)$, *where the degree of $\nu$ is $(p, q)$.*
3. $H^{*,*}(X)$ *is free with two generators $a$ and $b$.*

*In* 3 *above, the degrees of the generators $a$ and $b$ are $(p - n - 1, q - n - 1)$ and $(p, q - m - 1)$ where $d(\omega) = \frac{\theta}{\rho^n \tau^m} \nu$.*

**Proof.** Under these hypotheses, there is a cofiber sequence of the form $B \stackrel{i}{\hookrightarrow} X \stackrel{j}{\twoheadrightarrow} S^{p,q}$. Denote by $\nu$ the generator of $H^{*,*}(S^{p,q})$.

If $d(\omega) = \nu$ then 1 holds and $H^{*,*}(X)$ is free. If $d(\omega) = 0$, then 2 holds and again $H^{*,*}(X)$ is free. The remaining case is $d(\omega) \neq 0$. By the previous lemma, $d(\omega)$ is in the bottom cone of $\nu$. That is $d(\omega) = \frac{\theta}{\rho^n \tau^m} \nu$ for some $n$ and $m$. Recall that $\nu$ has dimension $(p, q)$ and so $\omega$ has dimension $(p - n - 1, q - n - m - 2)$. The $E_1$ page of the cellular spectral sequence is given in Fig. 3.4.

After taking cohomology, the spectral sequence collapses, and what remains is pictured in Fig. 3.5.

Let $a$ be the generator in degree $(p - n - 1, q - n - 1)$ and $b$ the generator in dimension $(p, q - m - 1)$. For degree reasons, $b$ is not in the image of $\cdot\rho$ and so determines a nonzero class in non-equivariant cohomology. Thus, $\tau^i b$ is nonzero for all $i$, and so we have that $b_i = \tau^i b$. In particular, $\rho^{n+1} a$ and $\tau^m b$ generate $H^{p,q}(X)$. Consider the portion of the long exact sequence associated to the cofiber sequence $B \stackrel{i}{\hookrightarrow} X \stackrel{j}{\twoheadrightarrow} S^{p,q}$ given below:

$$\cdots \longrightarrow H^{p,q}\big(S^{p,q}\big) \xrightarrow{j^*} H^{p,q}(X) \xrightarrow{i^*} H^{p,q}(B) \longrightarrow 0.$$

Since $i^*(\rho^{n+1} a) = i^*(\tau^m b) = \rho^{n+1} \tau^m \omega$, exactness implies that $j^*(\nu) = \rho^{n+1} a + \tau^m b$. Also $j^*$ is an $H^{*,*}(pt)$-module homomorphism, and so $j^*(\frac{\theta}{\rho^{n+1}} \nu) = \theta a$ and $j^*(\frac{\theta}{\tau^m} \nu) = \theta b$. In particular, we can create a map $f$ from a free module with generators $\alpha$ and $\beta$ in degrees $(p - n - 1, q - n - 1)$ and $(p, q - m - 1)$ respectively to $\tilde{H}^{p,q}(X)$ with $f(\alpha) = a$ and $f(\beta) = b$. This $f$ is an isomorphism. □

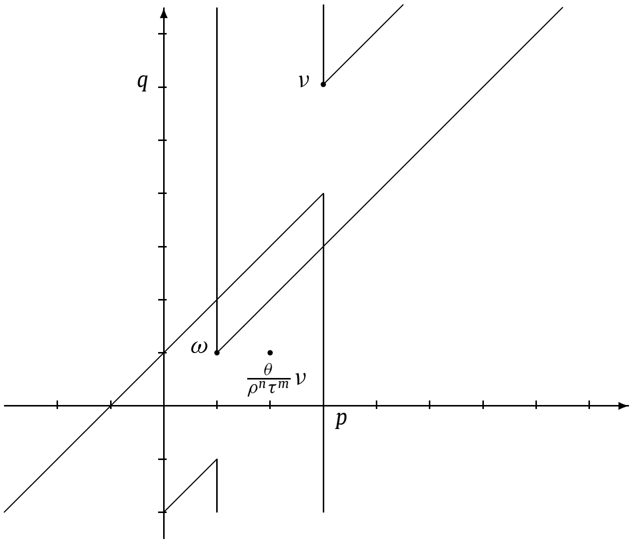


**Fig. 3.4.** The $E_1$ page of the cellular spectral sequence with a single nonzero differential hitting the bottom cone of an attached $(p,q)$-cell.

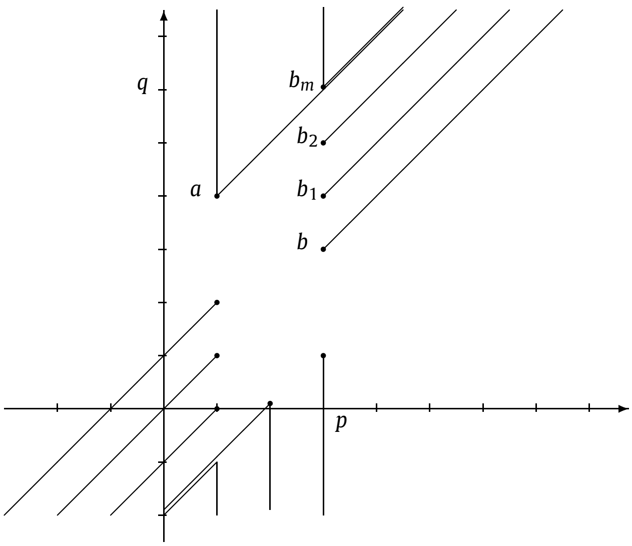


**Fig. 3.5.** The $E_2 = E_\infty$ page of the cellular spectral sequence with a single nonzero differential hitting the bottom cone of an attached $(p,q)$-cell.

**Theorem 3.2** *(Freeness theorem). If X is a connected, locally finite, finite dimensional* $\mathrm{Rep}(\mathbb{Z}/2)$*-complex, then* $H^{*,*}(X;\underline{\mathbb{Z}/2})$ *is free as an* $H^{*,*}(pt;\underline{\mathbb{Z}/2})$*-module.*

**Proof.** Since $X$ is locally finite, the cells can be attached one-at-a-time. Order the cells $\alpha_1, \alpha_2, \ldots$ so that their degrees satisfy $p_i \leqslant p_j$ if $i \leqslant j$ and $q_i \leqslant q_j$ if $p_i = p_j$ and $i \leqslant j$. We can proceed by induction over the spaces in the 'one-at-a-time' cell filtration $X^{(0)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq X$, with the base case obvious since $X$ is connected.

First, suppose that $H^{*,*}(X^{(n)})$ is a free $H^{*,*}(pt)$-module and that $X^{(n+1)}$ is obtained by attaching a single $(p,q)$-cell and that $X^{(n)}$ has no $p$-cells. Denote by $\nu$ the free generator of $H^{*,*}(X^{(n+1)}/X^{(n)}) \cong H^{*,*}(S^{p,q})$. Consider the spectral sequence of the filtration $X^{(n)} \subseteq X^{(n+1)}$. An example is pictured in Fig. 3.6 to aid in the discussion.

As before, a change of basis allows us to focus on a subset $\omega_1, \ldots, \omega_n$ of the free generators of $H^{*,*}(X^{(n)})$ whose differentials hit the bottom cone of $\nu$ and that satisfy

1. $d(\omega_i) \neq 0$ for all $i$,
2. $|\omega_i| > |\omega_j|$ when $i > j$,
3. $|\omega_i^G| > |\omega_j^G|$ when $i > j$,

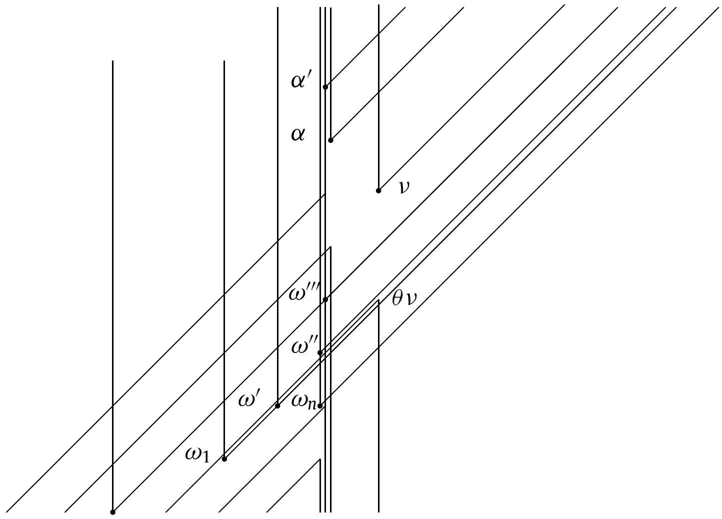


**Fig. 3.6.** The spectral sequence of a filtration for attaching a single $(p,q)$-cell to a space with free cohomology.

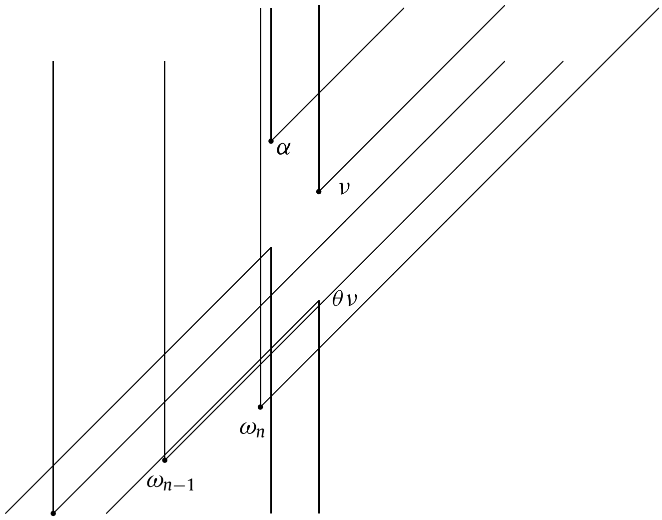


**Fig. 3.7.** The nonzero portion of the above spectral sequence, after a change of basis.

and all other basis elements have zero differentials to the bottom cone of $\nu$. This is similar to what is referred to in [4] as a ramp of length $n$. Also, we can change the basis again so that there is only one free generator, $\alpha$, of $H^{*,*}(X^{(n)})$ with a nonzero differential to the top cone of $\nu$. Then, after this change of basis, the nonzero portion of the spectral sequence of the filtration looks like the one in Fig. 3.7

As above, $\alpha$ cannot support a nonzero differential, and we can see that each of the $\omega_i$'s will shift up in $q$-degree and $\nu$ will shift down in $q$-degree. That is, the $\omega_i$'s and $\nu$ each give rise to free generators in the cohomology of $H^{*,*}(X^{(n+1)})$, but in different bidegree than their predecessors. Thus, $H^{*,*}(X^{(n+1)})$ is again free.

Now suppose that $X^{(n+1)}$ is obtained by attaching a $(p,q)$-cell $\nu'$ and that $X^{(n)}$ has a single $p$-cell $\nu$ already. Then by the previous case, the generator for $\nu$ was either shifted down, killed off, or was left alone at the previous stage. In any case, because of our choice of ordering of the cells, the generator for $\nu$ cannot support a differential to the generator for $\nu'$. Thus, the only nonzero differentials to $\nu'$ are from strictly lower dimensional cells. Thus, we are reduced again to the previous case and $H^{*,*}(X^{(n+1)})$ is free. By induction, $H^{*,*}(X)$ is free. □

## 4. Real projective spaces and Grassmann manifolds

In this section, $G = \mathbb{Z}/2$ exclusively, and the coefficient Mackey will always be $M = \underline{\mathbb{Z}/2}$ and will be suppressed from the notation.

Since each representation $\mathbb{R}^{p,q}$ has a linear $\mathbb{Z}/2$-action, there is an induced action of $\mathbb{Z}/2$ on $G_n(\mathbb{R}^{p,q})$, the **Grassmann manifold** of $n$-dimensional linear subspaces of $\mathbb{R}^{p,q}$. These Grassmann manifolds play a central role in the classification of equivariant vector bundles, and so it is important to understand their cohomology. As a special case we have the real projective spaces $\mathbb{P}(\mathbb{R}^{p,q}) = G_1(\mathbb{R}^{p,q})$.

The usual Schubert cell decomposition endows the Grassmann manifolds with a Rep($\mathbb{Z}/2$)-cell structure. However, the number of twists in each cell is dependent upon the flag of subrepresentations of $\mathbb{R}^{p,q}$ that is chosen. A **flag symbol** $\varphi$ is a sequence of integers $\varphi = (\varphi_1, \ldots, \varphi_q)$ satisfying $1 \leqslant \varphi_1 < \cdots < \varphi_q \leqslant q$. A flag symbol $\varphi$ determines a flag of subrepresentations $V_0 = 0 \subset V_1 \subset \cdots \subset V_p = \mathbb{R}^{p,q}$ satisfying $V_{\varphi_i}/V_{\varphi_i - 1} = \mathbb{R}^{1,1}$ for all $i = 1, \ldots, q$, and all other quotients of consecutive terms are $\mathbb{R}^{1,0}$. For concreteness, we also require that $V_i$ is obtained from $V_{i-1}$ by adjoining a coordinate basis vector. For example, there is a flag in $\mathbb{R}^{5,3}$ determined by the flag symbol $\varphi = (1, 3, 4)$ of the form $\mathbb{R}^{0,0} \subset \mathbb{R}^{1,1} \subset \mathbb{R}^{2,1} \subset \mathbb{R}^{3,2} \subset \mathbb{R}^{4,3} \subset \mathbb{R}^{5,3}$.

A **Schubert symbol** $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a sequence of integers such that $1 \leqslant \sigma_1 < \sigma_2 < \cdots < \sigma_n \leqslant p$. Given a Schubert symbol $\sigma$ and a flag symbol $\varphi$, let $e(\sigma, \varphi)$ be the set of planes $\ell \in G_n(\mathbb{R}^{p,q})$ for which $\dim(\ell \cap V_{\sigma_i}) = 1 + \dim(\ell \cap V_{\sigma_i - 1})$, where $V_0 \subset \cdots \subset V_n$ is the flag determined by $\varphi$. Then $e(\sigma, \varphi)$ is the interior of a cell $D(W)$ for some representation $W$. The dimension of the cell is determined by the Schubert symbol $\sigma$ just as in non-equivariant topology, but the number of twists depends on both $\sigma$ and the flag symbol $\varphi$.

For example, consider $G_2(\mathbb{R}^{5,3})$, $\sigma = (3, 5)$, and $\varphi = (1, 3, 4)$. Then $e(\sigma, \varphi)$ consists of planes $\ell$ which have a basis with echelon form given by the matrix below.

$$\begin{matrix} - & + & - & - & + \end{matrix}$$
$$\begin{pmatrix} * & * & 1 & 0 & 0 \\ * & * & 0 & * & 1 \end{pmatrix}.$$

Here, the action of $\mathbb{Z}/2$ on the columns, as determined by $\varphi$, has been indicated by inserting the appropriate signs above the matrix. After acting, this becomes the following.

$$\begin{matrix} - & + & - & - & + \end{matrix}$$
$$\begin{pmatrix} -* & * & -1 & 0 & 0 \\ -* & * & 0 & -* & 1 \end{pmatrix}.$$

We require the last nonzero entry of each row to be 1, and so we scale the first row by $-1$.

$$\begin{matrix} - & + & - & - & + \end{matrix}$$
$$\begin{pmatrix} * & -* & 1 & 0 & 0 \\ -* & * & 0 & -* & 1 \end{pmatrix}.$$

There are five coordinates which can be any real numbers, three of which the $\mathbb{Z}/2$-action of multiplication by $-1$, so this is a $(5, 3)$-cell. Through a similar process, we can obtain a cell structure for $G_n(\mathbb{R}^{p,q})$ given any flag $\varphi$. The type of cell determined by the Schubert symbol $\sigma$ and the flag $\varphi$ is given by the following proposition. Here, $\underline{\sigma_i} = \{1, \ldots, \sigma_i\}$ and $\sigma(i) = \{\sigma_1, \ldots, \sigma_i\}$.

**Proposition 4.1.** *Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a Schubert symbol and $\varphi = (\varphi_1, \ldots, \varphi_q)$ be a flag symbol for $\mathbb{R}^{p,q}$. The cell $e(\sigma, \varphi)$ of $G_n(\mathbb{R}^{p,q})$ is of dimension $(a, b)$ where $a = \sum_{i=1}^{n} (\sigma_i - i)$ and $b = \sum_{\sigma_i \in \varphi} |\underline{\sigma_i} \setminus (\varphi \cup \sigma(i))| + \sum_{\sigma_i \notin \varphi} |(\underline{\sigma_i} \cap \varphi) \setminus \sigma(i)|$.*

**Proof.** The formula for $a$ is exactly the same as in the non-equivariant case. The one for $b$ follows since the number of twisted coordinates in each row is exactly the number of $*$ coordinates for which the action is opposite to that on the coordinate containing the 1 in that echelon row. □

**Corollary 4.1.** *Real and complex projective spaces and Grassmann manifolds have free $RO(\mathbb{Z}/2)$-graded cohomology with $\underline{\mathbb{Z}/2}$ coefficients.*

**Proposition 4.2.** *If $V \subseteq V'$ is an inclusion of representations and $\varphi \subseteq \varphi'$ is an extension of flag symbols for $V$ and $V'$, then there is a cellular inclusion $G_n(V) \hookrightarrow G_n(V')$.*

The following theorem guarantees that the cohomology of Grassmann manifolds have cohomology generators in bijective correspondence with the Schubert cells.

**Theorem 4.1.** *$H^{*,*}(G_n(\mathbb{R}^{u,v}))$ is a free $H^{*,*}(pt)$-module with generators in bijective correspondence with the Schubert cells.*

**Proof.** Since $G_n(\mathbb{R}^{u,v})$ has a Rep($\mathbb{Z}/2$)-complex structure, we know $H^{*,*}(G_n(\mathbb{R}^{u,v}))$ is free by the freeness theorem, Theorem 3.2. Let $\{\omega_1, \ldots, \omega_k\}$ be a set of free generators. Then $k \leqslant m$ where $m$ is the number of Schubert cells.

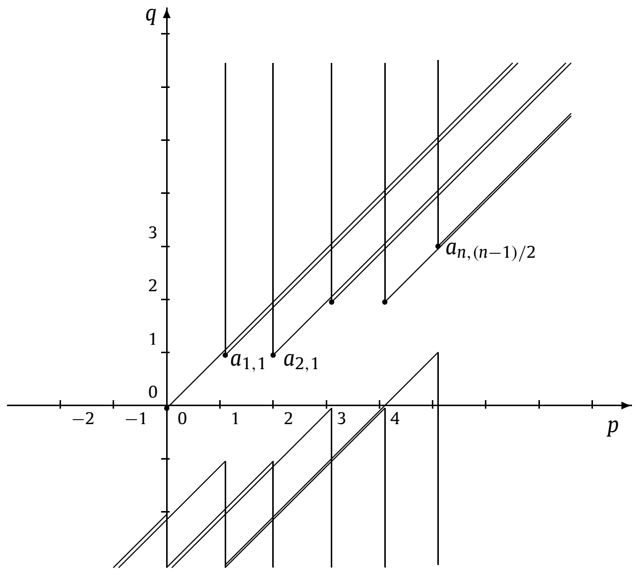


**Fig. 4.1.** The $E_1$ page of the cellular spectral sequence for $\mathbb{RP}^n_{tw}$ for $n$ odd.

These spaces are based, so we can appeal to the forgetful long exact sequence Lemma 2.1. By freeness and finite dimensionality, the multiplication by $\rho$ map is an injection for large enough $q$. Thus the forgetful map to non-equivariant cohomology is surjective. Since $H^*(G_n(\mathbb{R}^{u,v}))$ is free with generators $a_1, \dots, a_m$ in bijective correspondence with the Schubert cells, $H^{*,*}(G_n(\mathbb{R}^{u,v}))$ has a set of elements, $\{\alpha_1, \dots, \alpha_m\}$, with $\psi(\alpha_i) = a_i$. We can uniquely express each $\alpha_i$ as $\alpha_i = \sum_{j=1}^k \rho^{e_{ij}} \tau^{f_{ij}} \omega_j$. We can ignore any terms that have $\rho$ in them since $\psi(\rho) = 0$. This gives a new set of elements, $\bar{\alpha}_i = \sum_{j=1}^k \epsilon_{ij} \tau^{f_{ij}} \omega_j$, where $\epsilon_{ij} = 0$ or 1 and $\psi(\bar{\alpha}_i) = a_i$. Since $\psi(\tau) = 1$, we have that $\sum_{j=1}^k \epsilon_{ij} \psi(\omega_j) = a_i$. Since linear combinations of the linearly independent $\omega_j$'s map to the linearly independent $a_i$'s, there are at least as many $\omega_j$'s as there are $a_i$'s. That is, $k \geqslant m$. □

The above theorem is enough to determine the additive structure of the $RO(\mathbb{Z}/2)$-graded cohomology of the real projective spaces.

Recall that $\mathcal{U} = (\mathbb{R}^{2,1})^\infty$ is a complete universe in the sense of [6]. Denote by $\mathbb{RP}^\infty_{tw} = \mathbb{P}(\mathcal{U})$, the space of lines in the complete universe $\mathcal{U}$.

Denote by $\mathbb{RP}^n_{tw} = \mathbb{P}(\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor})$, the equivariant space of lines in $\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor}$. For example, $\mathbb{RP}^3_{tw} = \mathbb{P}(\mathbb{R}^{4,2})$, $\mathbb{RP}^4_{tw} = \mathbb{P}(\mathbb{R}^{5,2})$, and $\mathbb{RP}^1_{tw} = S^{1,1}$. There are natural cellular inclusions $\mathbb{RP}^n_{tw} \hookrightarrow \mathbb{RP}^{n+1}_{tw}$, the colimit of which is $\mathbb{RP}^\infty_{tw}$.

**Lemma 4.1.** *$\mathbb{RP}^n_{tw}$ has a* Rep$(\mathbb{Z}/2)$*-structure with cells in dimension* $(0,0), (1,1), (2,1), (3,2), (4,2), \dots, (n, \lceil \frac{n}{2} \rceil)$.

**Proof.** This follows from Proposition 4.1 using the flag symbol $\varphi = (2, 4, 6, \dots)$. □

**Lemma 4.2.** *$\mathbb{RP}^\infty_{tw}$ has a cell structure with a single cell in dimension $(n, \lceil \frac{n}{2} \rceil)$, for all $n \in \mathbb{N}$.*

**Proof.** The inclusions $\mathbb{RP}^1_{tw} \hookrightarrow \mathbb{RP}^2_{tw} \hookrightarrow \cdots$ are cellular and their colimit is $\mathbb{RP}^\infty_{tw}$. □

**Proposition 4.3.** *As an $H^{*,*}(pt)$-module, $H^{*,*}(\mathbb{RP}^n_{tw})$ is free with a single generator in each degree $(k, \lceil \frac{k}{2} \rceil)$ for $k = 0, 1, \dots, n$.*

**Proof.** Any nonzero differentials in the cellular spectral sequence associated to the cell structure using the flag symbol $\varphi = (2, 4, 6, \dots)$ would decrease the number of cohomology generators below the number of cells. (See Figs. 4.1 and 4.2.) By Theorem 4.1 this is not the case, and so the cohomology generators have degrees matching the dimensions of the cells. □

**Proposition 4.4.** *As an $H^{*,*}(pt)$-module, $H^{*,*}(\mathbb{RP}^\infty_{tw})$ is free with a single generator in each degree $(n, \lceil \frac{n}{2} \rceil)$, for all $n \in \mathbb{N}$.*

**Proof.** $\mathbb{RP}^\infty_{tw}$ is the colimit of the above projective spaces. Thus, any nonzero differential for $\mathbb{RP}^\infty_{tw}$ would induce a nonzero differential at some finite stage. By the above proposition, this is not the case. □

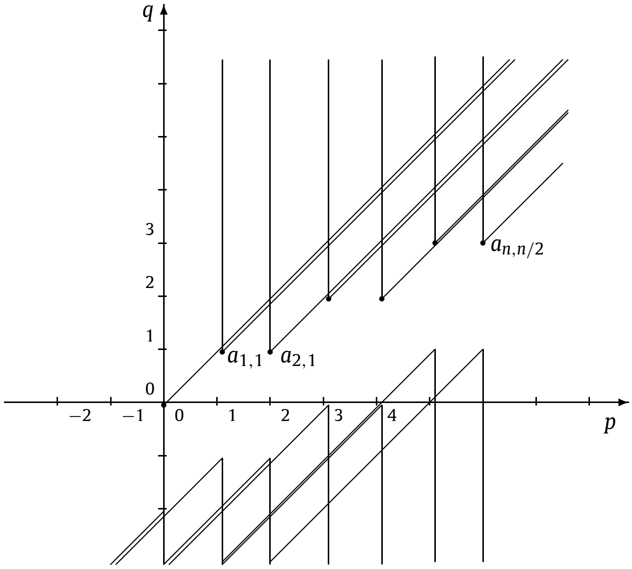


**Fig. 4.2.** The $E_1$ page of the cellular spectral sequence for $\mathbb{RP}^n_{tw}$ for $n$ even.

**Lemma 4.3.** *As an $H^{*,*}(pt)$-module, $H^{*,*}(S^{1,1})$ is free with a single generator $a$ in degree $(1,1)$. As a ring, $H^{*,*}(S^{1,1}) \cong H^{*,*}(pt)[a]/(a^2 = \rho a)$.*

**Proof.** The statement about the module structure is immediate since $S^{1,1} \cong \mathbb{RP}^1_{tw}$.

Since $S^{1,1}$ is a $K(\mathbb{Z}(1),1)$, we can consider $a \in [S^{1,1}, S^{1,1}]$ as the class of the identity and $\rho \in [pt, S^{1,1}]$ as the inclusion. Then $a^2$ is the composite

$$a^2 : S^{1,1} \xrightarrow{\Delta} S^{1,1} \wedge S^{1,1} \xrightarrow{a \wedge a} S^{2,2} \longrightarrow K(\mathbb{Z}/2(2), 2).$$

Similarly, $\rho a$ is the composite

$$\rho a : S^{1,1} \longrightarrow S^{0,0} \wedge S^{1,1} \xrightarrow{\rho \wedge a} S^{2,2} \longrightarrow K(\mathbb{Z}/2(2), 2).$$

The claim is that these two maps are homotopic. Considering the spheres involved as one point compactifications of the corresponding representations, the map $a^2$ is inclusion of $(\mathbb{R}^{1,1})^+$ as the diagonal in $(\mathbb{R}^{2,2})^+$ and $\rho a$ is inclusion of $(\mathbb{R}^{1,1})^+$ as the vertical axis. There is then an equivariant homotopy $H : (\mathbb{R}^{1,1})^+ \times I \to (\mathbb{R}^{2,2})^+$ between these two maps given by $H(x,t) = (tx, x)$. □

From here, we are poised to compute the ring structure of the $RO(\mathbb{Z}/2)$-graded cohomology of each real projective space.

**Theorem 4.2.** *$H^{*,*}(\mathbb{RP}^\infty_{tw}) = H^{*,*}(pt)[a,b]/(a^2 = \rho a + \tau b)$, where $\deg(a) = (1,1)$ and $\deg(b) = (2,1)$.*

**Proof.** It remains to compute the multiplicative structure of the cohomology ring. Denote by $a = a_{(1,1)}$, and $b = a_{(2,1)}$. By Lemma 2.1, the forgetful map $\psi : H^{*,*}(\mathbb{RP}^\infty_{tw}) \to H^*(\mathbb{RP}^\infty)$ maps $\psi(a) = z$ and $\psi(b) = z^2$ where $z \in H^1(\mathbb{RP}^\infty)$ is the ring generator for non-equivariant cohomology. Since $\psi$ is a homomorphism of rings, $\psi(ab) = z^3 \neq 0$, and so the product $ab$ is nonzero in $H^{*,*}(\mathbb{RP}^\infty_{tw})$. Observe that $\rho b$ is also in degree $(3,2)$ in $H^{*,*}(\mathbb{RP}^\infty_{tw})$, but $\psi(\rho b) = 0$ since $\psi(\rho) = 0$. Thus $ab$ and $\rho b$ generate $H^{*,*}(\mathbb{RP}^\infty_{tw})$ in degree $(3,2)$. Also, $\psi(b^2) = z^4$, and so $b^2$ in nonzero in $H^{*,*}(\mathbb{RP}^\infty_{tw})$. This means that $b^2$ is the unique nonzero element of $H^{*,*}(\mathbb{RP}^\infty_{tw})$ in degree $(4,2)$. Inductively, it can be shown that if $n$ is even the unique nonzero element of $R$ in degree $(n, \frac{n}{2})$ is $b^{n/2}$ and that if $n$ is odd, then $ab^{(n-1)/2}$ is linearly independent from $\rho b^{(n-1)/2}$.

Now, $a^2 \in H^{2,2}(\mathbb{RP}^\infty_{tw})$ and so is a linear combination of $\rho a$ and $\tau b$. Since $\psi(a^2) = z^2$, there must be a $\tau b$ term in the expression for $a^2$. Also, upon restriction to $\mathbb{RP}^1_{tw} = S^{1,1}$, $a^2$ restricts to $a^2 = \rho a$. Thus, $a^2 = \rho a + \tau b \in H^{*,*}(\mathbb{RP}^\infty_{tw})$. □

**Theorem 4.3.** *Let $n > 2$. If $n$ is even, then $H^{*,*}(\mathbb{P}(\mathbb{R}^{n,\frac{n}{2}})) = H^{*,*}(pt)[a_{1,1}, b_{2,1}]/\sim$ where the generating relations are $a^2 = \rho a + \tau b$ and $b^k = 0$ for $k \geqslant \frac{n}{2}$. If $n$ is odd, then $H^{*,*}(\mathbb{P}(\mathbb{R}^{n,\frac{n-1}{2}})) = H^{*,*}(pt)[a_{1,1}, b_{2,1}]/\sim$ where the generating relations are $a^2 = \rho a + \tau b$, $b^k = 0$ for $k \geqslant \frac{n+1}{2}$, and $a \cdot b^{(n-1)/2} = 0$.*

**Proof.** Only the multiplicative structure needs to be checked since the cohomology is free and the generators given above are in the correct degrees. Considering the restriction of the corresponding classes $a$ and $b$ in $H^{*,*}(\mathbb{RP}^{\infty}_{tw})$, the relation $a^2 = \rho a + \tau b$ is immediate. The relations $b^k = 0$ for $k > \frac{n}{2}$ when $n$ is even and $b^k = 0$ for $k \geqslant \frac{n+1}{2}$ when $n$ is odd follow for degree reasons. Also, since the class $ab^{(n-1)/2} \in H^{*,*}(\mathbb{RP}^{\infty}_{tw})$ is a free generator, it restricts to zero in $H^{*,*}(\mathbb{P}(\mathbb{R}^{n,\frac{n-1}{2}}))$. Thus $ab^{(n-1)/2} = 0 \in H^{*,*}(\mathbb{P}(\mathbb{R}^{n,\frac{n-1}{2}}))$. □

We can also compute the cohomology of projective spaces associated to arbitrary representations. The following easy lemma will be useful. In particular, it allows us to only consider the projective spaces associated to representations $V \cong \mathbb{R}^{p,q}$ where $q \leqslant p/2$.

**Lemma 4.4.** $\mathbb{P}(\mathbb{R}^{p,q}) \cong \mathbb{P}(\mathbb{R}^{p,p-q})$.

**Proof.** Consider a basis of $\mathbb{R}^{p,q}$ in which the first $q$ coordinates have the nontrivial action, and a basis of $\mathbb{R}^{p,p-q}$ in which the first $q$ coordinates are fixed by the action. Then the map $f : \mathbb{P}(\mathbb{R}^{p,q}) \to \mathbb{P}(\mathbb{R}^{p,p-q})$ that sends the span of $(x_1, \ldots, x_p)$ to the span of $(x_1, \ldots, x_p)$ is equivariant. It is clearly a homeomorphism. □

**Lemma 4.5.** *If $q \leqslant p/2$, then $\mathbb{P}(\mathbb{R}^{p,q})$ has a cell structure with a single cell in each dimension* $(0,0), (1,1), (2,1), (3,2), (4,2), \ldots, (2q-1,q), (2q,q), \ldots, (p-1,q)$.

For example, $\mathbb{P}(\mathbb{R}^{4,1})$ has a single cell in each dimension $(0,0)$, $(1,1)$, $(2,1)$, and $(3,1)$.

**Proof.** The result follows by Proposition 4.1 using the flag symbol $\varphi = (2, 4, \ldots, 2q)$. □

**Lemma 4.6.** *As an $H^{*,*}(pt)$-module, $H^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$ is free with a single generator in degrees* $(0,0), (1,1), (2,1), (3,2), (4,2), \ldots, (2q,q), (2q+1,q), \ldots, (p-1,q)$.

**Proof.** Using the cell structure in the previous lemma, Theorem 4.1 implies there can be no nonzero differentials in the cellular spectral sequence. □

The ring structure of the other projective spaces can be computed by considering the restriction of $H^{*,*}(\mathbb{RP}^{\infty}_{tw})$ to $H^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$.

**Proposition 4.5.** *$H^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$ is a truncated polynomial algebra over $H^{*,*}(pt)$ on generators in degrees* $(1,1), (2,1), (2q+1,q), (2q+2,q), \ldots, (p-1,q)$, *subject to the relations determined by the restriction of $H^{*,*}(\mathbb{RP}^{\infty}_{tw})$ to $H^{*,*}(\mathbb{P}(\mathbb{R}^{p,q}))$.*

For example, consider $\mathbb{P}(\mathbb{R}^{4,1})$. By the above proposition, $H^{*,*}(\mathbb{P}(\mathbb{R}^{4,1}))$ is generated by classes $a_{1,1}$, $b_{2,1}$, and $c_{3,1}$. The classes $a$ and $b$ in $H^{*,*}(\mathbb{RP}^{\infty}_{tw})$ restrict to $a$ and $b$ respectively, so $a^2 = \rho a + \tau b$ in $H^{*,*}(\mathbb{P}(\mathbb{R}^{4,1}))$. Now, $ab$ has degree $(3,2)$ and so $ab = ?\rho b + ?\tau c$. However, the product $ab$ in $H^{*,*}(\mathbb{RP}^{\infty}_{tw})$ restricts to the class $\tau c$. Since restriction is a map of rings, $ab = \tau c$ in $H^{*,*}(\mathbb{P}(\mathbb{R}^{4,1}))$. Similar considerations show that $bc = 0$ and $c^2 = 0$. Thus $H^{*,*}(\mathbb{P}(\mathbb{R}^{4,1})) = H^{*,*}(pt)[a_{1,1}, b_{2,1}, c_{3,1}]/\sim$, where the generating relations are $a^2 = \rho a + \tau b$, $ab = \tau c$, $bc = 0$, and $c^2 = 0$.

In some cases, the freeness theorem is enough to determine the additive structure of the $RO(\mathbb{Z}/2)$-graded cohomology of Grassmann manifolds.

**Proposition 4.6.** *$G_2(\mathbb{R}^{p,1})$ has a $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure so that $H^{*,*}(G_n(\mathbb{R}^{p,1}))$ is a free $H^{*,*}(pt)$-module on generators whose degrees match the dimensions of the cells.*

**Proof.** Using the flag symbol $\varphi = (2)$, every cell, except the $(0,0)$-cell, has either one or two twists. The cells are in bidegrees so that there can be no dimension shifting in the cellular spectral sequence. The result now follows by Theorem 4.1. □

For example, $H^{*,*}(G_2(\mathbb{R}^{4,1}); \underline{\mathbb{Z}/2})$ is a free $H^{*,*}(pt; \underline{\mathbb{Z}/2})$-module with generators in degrees $(0,0)$, $(1,1)$, $(2,1)$, $(2,1)$, $(3,1)$, and $(4,2)$ (see Fig. 4.3).

Interestingly, there are situations where there must be nonzero differentials in the cellular spectral sequences.

As another example, consider now $X = G_2(\mathbb{R}^{4,2})$. Consider the three flag symbols $\varphi_1 = (2,3)$, $\varphi_2 = (2,4)$, and $\varphi_3 = (3,4)$. The spectral sequences associated to the cell structures with these flag symbols have $E_1$ term given in Figs. 4.4, 4.5, and 4.6 respectively.

The cohomology of $X$ can be deduced by comparing these three cellular spectral sequences. We can see from the picture for $\varphi_2$ that $H^{1,0}(X) = 0$, and so the differential leaving the $(1,0)$ generator in the $\varphi_1$ spectral sequence is nonzero. Thus,

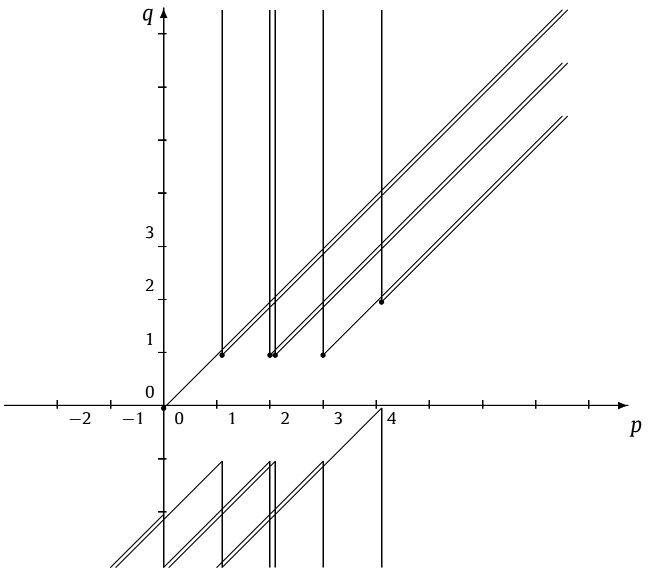


**Fig. 4.3.** $H^{*,*}(G_2(\mathbb{R}^{4,1}))$.

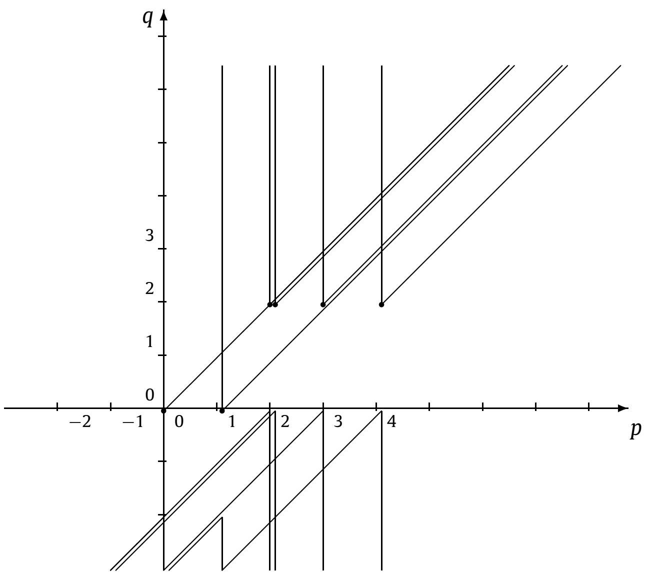


**Fig. 4.4.** The $E_1$ page of the cellular spectral sequence for $G_2(\mathbb{R}^{4,2})$ using $\varphi_1 = (2,3)$.

$H^{1,1}(X) = \mathbb{Z}/2$, $H^{2,1}(X) = \mathbb{Z}/2$ and $H^{2,0}(X) = \mathbb{Z}/2$. In particular, there is a free generator in degree $(1,1)$ and there is a nontrivial differential leaving the $(2,1)$ generators of the spectral sequence for $\varphi_2$. After a change of basis, if necessary, the differential can be adjusted so that it is zero on one of the $(2,1)$ generators and the other generator maps nontrivially. Now from $\varphi_1$ we see that $H^{4,1}(X) = 0$, and so there is a nontrivial differential leaving the $(3,1)$ generator in the $\varphi_3$ spectral sequence. This means that the $(4,2)$ generator in the $\varphi_1$ and $\varphi_2$ spectral sequences must survive. Thus, all differentials in the $\varphi_2$ spectral sequence are known. They are all zero, except for the one leaving the two $(2,1)$ generators, which behaves as described above. That spectral sequence collapses almost immediately to give the cohomology of $G_2(\mathbb{R}^{4,2})$ pictured in Fig. 4.7.

By the freeness Theorem 3.2, we know that $H^{*,*}(G_2(\mathbb{R}^{4,2}))$ is free. Counting the $\mathbb{Z}/2$ vector space dimensions in each bidegree reveals that the degrees are the same as those of a free $H^{*,*}(pt)$-module with generators in degrees $(1,1)$, $(2,1)$, $(2,2)$, $(3,2)$, and $(4,2)$. This is the only free $H^{*,*}(pt)$-module with these $\mathbb{Z}/2$ dimensions, and so we have the following computation.

**Proposition 4.7.** *$H^{*,*}(G_2(\mathbb{R}^{4,2}))$ is a free $H^{*,*}(pt)$-module with generators in degrees* $(1,1)$, $(2,1)$, $(2,2)$, $(3,2)$, *and* $(4,2)$.

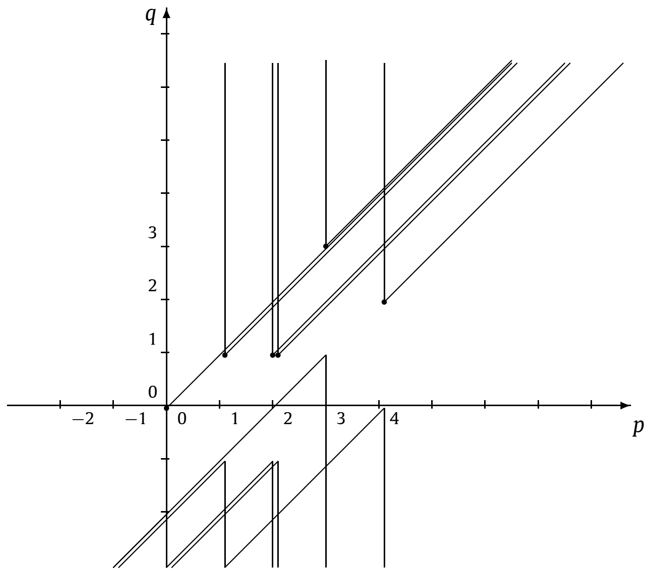


**Fig. 4.5.** The $E_1$ page of the cellular spectral sequence for $G_2(\mathbb{R}^{4,2})$ using $\varphi_2 = (2,4)$.

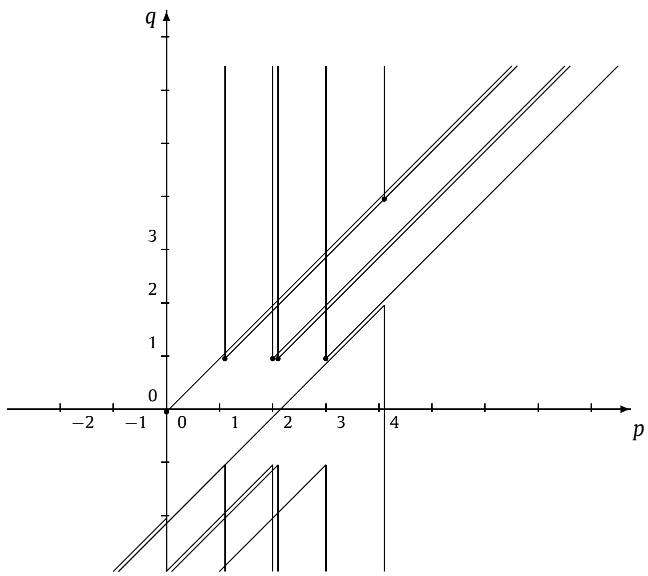


**Fig. 4.6.** The $E_1$ page of the cellular spectral sequence for $G_2(\mathbb{R}^{4,2})$ using $\varphi_3 = (3,4)$.

That is, $H^{*,*}(G_2(\mathbb{R}^{4,2}))$ has free generators as displayed in Fig. 4.8.

It should be noted that in the case of $G_2(\mathbb{R}^{4,1})$, with the proper choice of flag symbols, the cell structure is such that the differentials are all zero, and so the cohomology is free with generators in the same degrees as the dimensions of the cells. This is **not** the case with $G_2(\mathbb{R}^{4,2})$. Regardless of the choice of flag symbol, there are some nonzero differentials which cause some degree shifting of the cohomology generators.

Unfortunately, we cannot play this game indefinitely. For the Grassmann manifolds $G_n(\mathbb{R}^{p,q})$ with $n$ and $q$ small enough, say $n \leqslant 2$ and $q \leqslant 2$, the above techniques can be used to obtain the additive structure of $H^{*,*}(G_n(\mathbb{R}^{p,q}))$. However, there are examples where the precise degrees of the cohomology generators cannot be determined by comparing the cellular spectral sequences for various flag symbols. A serious inquiry into the geometry of the attaching maps in these cell structures may reveal more information.

## Acknowledgements

The work in this paper was originally part of the author's dissertation while at the University of Oregon.

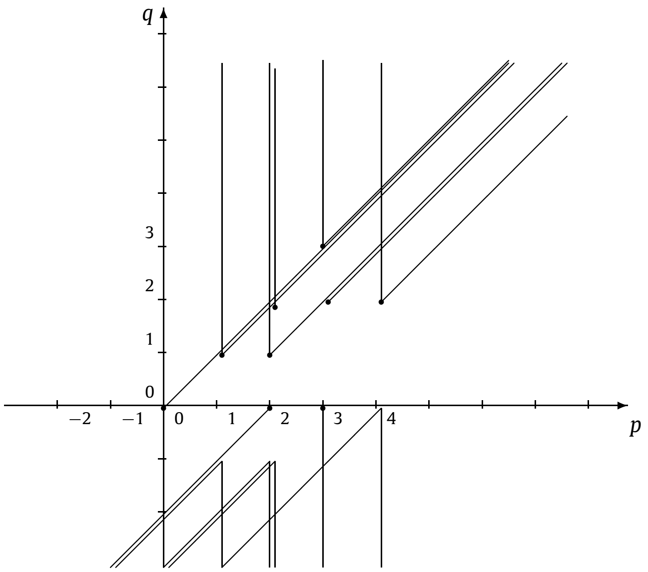


**Fig. 4.7.** $H^{*,*}(G_2(\mathbb{R}^{4,2}))$.

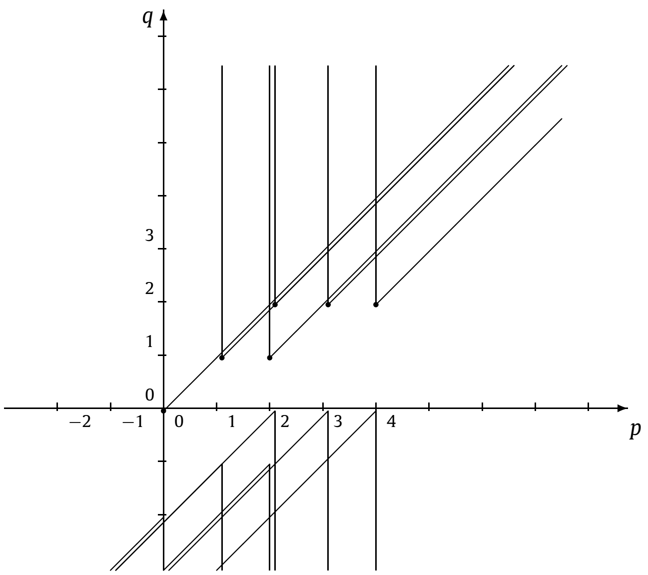


**Fig. 4.8.** $H^{*,*}(G_2(\mathbb{R}^{4,2}))$ with free generators shown.

The author is indebted to Dan Dugger for his guidance and innumerable helpful conversations and also to the referee for suggestions which have greatly improved the flow of the paper.

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