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The transfer and stable homotopy theory

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Introduction. The purpose of this paper is to give a proof of the following splitting theorem in stable homotopy theory. We assume all spaces are localized at a fixed prime p. Let $\mathscr{S}_k$ be the symmetric group on $\{1, \ldots, k\}$, $Q(\,.\,) = \lim \Omega^n \Sigma^n(\,.\,)$, and $Q_k S^0$, $k \in \mathbb{Z}$, denote the components of QS^0.

THEOREM. *There exists a map* $t\colon Q_0 S^0 \to QB\mathscr{S}_p$ *such that*

$$Q_0 S^0 \xrightarrow{t} QB\mathscr{S}_p \xrightarrow{\bar{f}} Q_0 S^0$$

is an equivalence at p *for any map* $f\colon B\mathscr{S}_p \to Q_0 S^0$ *with* f_* *non-trivial on* $H_{2p-3}(\,.\,; \mathbb{Z}/p)$. *Here* $\bar{f}$ *denotes the infinite loop map induced by* f. *The result continues to hold if* $B\mathscr{S}_p$ *is replaced by* $B\mathbb{Z}/p$.

The proof contains two main ideas. First the well-known homology equivalence $\phi_\infty\colon B\mathscr{S}_\infty \to Q_0 S^0$ (4, 12) is used to approximate $Q_0 S^0$. Second, the transfer morphism for generalized cohomology theories is used to construct t. We then proceed to show that $\bar{f} \cdot t$ induces an isomorphism in $H_*(\,.\,; \mathbb{Z}/p)$ by applying the results of our study (13) of the transfer in the homology of symmetric groups.

This theorem was announced in (11). Since then other proofs have been given by Segal (21) and the first author (10). The transfer has been studied by Roush (20), Becker and Gottlieb (2), Becker and Schultz (3), Dold (7) and others. It is now well treated in the literature. Adams (1) has established the uniqueness of $\bar{f}$ (up to equivalence). The theorem itself has found several applications (17, 19, 20) and the techniques of the proof have been used by the second author to study G and G/O (18).

In view of all this, we have chosen to limit our exposition to a direct self-contained proof of the theorem relying only on (13). For further applications see ((11), 1·8, 2·4, 3·6–7). In Section 1, we recall the definition of the stable transfer for coverings and show that in cohomology it induces the ordinary transfer homomorphism for the cohomology of coverings. Section 2 is devoted to the proof of the theorem.

1. *Preliminaries on the transfer.* In this section, we use Boardman's 'little cubes' spaces to define the transfer for finite coverings. The resulting definition is more concise and transparent than the one given in (11). We also give a proof of the basic fact that the stable transfer induces the ordinary transfer homomorphism in cohomology.

A 'little n-cube' $c\colon \operatorname{int} I^n \to \operatorname{int} I^n$ is a linear embedding with edges parallel to the coordinate axes (i.e. $c = d_1 \times \ldots \times d_n$, where $d_i\colon (0,1) \to (0,1)$ is a linear map

$$d_i(t) = (1-t)\,a_i + t b_i, \quad 0 \leqslant a_i < b_i \leqslant 1). \qquad ((5); (14), 4\cdot 1)$$

Let $\mathscr{C}_n(k) = \{(c_1, \ldots, c_k) | c_i \text{ a little } n\text{-cube}, \ \operatorname{Im} c_i \cap \operatorname{Im} c_j = \varnothing, \ i \neq j\}$.

$(c_1, \ldots, c_k)$ can be regarded as a continuous function

$$(c_1, \ldots, c_k): \operatorname{int} I^n \amalg \ldots \amalg \operatorname{int} I^n \to \operatorname{int} I^n.$$

Given the compact-open topology $\mathscr{C}_n(k)$ is $(n-2)$ connected and $\mathscr{S}_k$ acts freely on the left by permuting little cubes

$$\tau(c_1, \ldots, c_k) = (c_{\tau^{-1}(1)}, \ldots, c_{\tau^{-1}(k)}).$$

An inclusion $\sigma_n: \mathscr{C}_n(k) \to \mathscr{C}_{n+1}(k)$ is defined by

$$\sigma_n(c_1, \ldots, c_k) = (c_1 \times 1, \ldots, c_k \times 1),$$

where $1 = \mathrm{id}$ on $(0, 1)$. Let $\mathscr{C}_\infty(k) = \bigcup_n \mathscr{C}_n(k)$.

Let $\Pi: E \to B$ be an N-fold covering with E connected and B a CW complex. Let $H = \pi_1(E)$, $G = \pi_1(B)$. Then $\Pi: E \to B$ is isomorphic to the covering $p: X/H \to X/G$, where X is a universal covering of B and p is the projection. X can be given the structure of a CW complex so that G (and hence H) act freely and cellularly on X from the right (22). Let $G/H = \{\tau_1 H, \ldots, \tau_N H\}$ be the left cosets of H in G. Then G acts on the left of G/H as a group of permutations and thus defines a representation $\rho: G \to \mathscr{S}_N$ by

$$g\tau_i = \tau_{\rho(g)(i)} h_i \quad \text{for some } h_i \in H. \tag{1·1}$$

Since G acts freely and cellularly on X and since $\mathscr{C}_\infty(N)$ is a contractible $\mathscr{S}_N$-space, there is a ρ-equivariant map $\rho_*: X \to \mathscr{C}_\infty(N)$ which is unique up to a ρ-equivariant homotopy (6). (Here, we view G acting on the left of X by $gx = xg^{-1}$.) Define

$$\Phi: X/G \to \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (X/H)^N$$

by $\Phi(\tilde{x}) = (\rho_*(x); \overline{x\tau_1}, \overline{x\tau_2}, \ldots, \overline{x\tau_N})$, where $\tilde{x}$ is the class of x in X/G and $\overline{x\tau_i}$ is the class of $x\tau_i$ in X/H. Here $\mathscr{S}_N$ acts on the left of $(X/H)^N$ by permuting coordinates:

$$\sigma(x_1, \ldots, x_N) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(N)}).$$

Thus, $\zeta^{-1}(w; x_1, \ldots, x_N) = (\zeta^{-1}w; x_{\zeta(1)}, \ldots, x_{\zeta(N)})$ describes the action of $\mathscr{S}_N$ on $\mathscr{C}_\infty(N) \times (X/H)^N$.

LEMMA 1·2. *Φ is well defined and a change of coset ordering does not change the homotopy class of Φ.*

Proof. Since H acts on X from the right, $\overline{x\tau_i h} = \overline{x\tau_i}$ and Φ is independent of the choice of coset representative. To see that Φ is well defined, we note that

$$\begin{aligned}\Phi(\widetilde{xg}) &= (\rho_*(xg); \overline{xg\tau_1}, \overline{xg\tau_2}, \ldots, \overline{xg\tau_N}) \\ &= (\rho(g^{-1}) . \rho_*(x); \ \overline{x\tau}_{\rho(g)(1)}, \ldots, \overline{x\tau}_{\rho(g)(N)}) = (\rho_*(x); \overline{x\tau_1}, \ldots, \overline{x\tau_N}) = \Phi(\tilde{x}).\end{aligned}$$

Before examining the effect of a change in the order of the cosets, we note that Φ may be defined by the use of *any* ρ-equivariant map $\rho_*: X \to \mathscr{C}_\infty(N)$ without changing the homotopy class of Φ.

Now let $\tau_i' H = \tau_{\epsilon(i)} H$, $\epsilon \in \mathscr{S}_N$, be a reordering of the cosets. Then $\rho' = \epsilon^{-1}\rho\epsilon$ and the map $\rho'_* = \epsilon^{-1}.\rho_*: X \to \mathscr{C}_\infty(N)$ is ρ'-equivariant. With this choice of ρ'_*, $\Phi' = \Phi$, for

$$\Phi'(x) = (\epsilon^{-1}\rho_*(x); \overline{x\tau_1'}, \ldots, \overline{x\tau_N'})$$
$$= (\epsilon^{-1}\rho_*(x); \overline{x\tau_{\epsilon(1)}}, \ldots, \overline{x\tau_{\epsilon(N)}}) = (\rho_*(x); \overline{x\tau_1}, \ldots, \overline{x\tau_N}) = \Phi(x).$$

Remark. The earliest precursor of the map Φ seems to be the homomorphism $\Psi: G \to \mathscr{S}_N \int H$ defined by Evens (9) via the formula $\Psi(g) = (\rho(g); h_1, \ldots, h_n)$ with ρ, h_i as in (1·1); ψ is well-defined up to conjugation. Letting EG be a universal cover of BG, we construct Φ for the cover $BH \simeq EG/H \to EG/G = BG$ and show that Φ may be taken for $B\Psi$. First note that

$$B(\mathscr{S}_N \textstyle\int H) \simeq \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (EG/H)^N = [\mathscr{C}_\infty(N) \times (EG)^N]/(\mathscr{S}_N \textstyle\int H).$$

It is easily checked that the map $\Phi': EG \to \mathscr{C}_\infty(N) \times (EG)^N$ given by

$$\Phi'(x) = (\rho_*(x): x\tau_1, \ldots, x\tau_N)$$

is Ψ-equivariant. Upon dividing by G and $\mathscr{S}_N \int H$ respectively, we obtain the pretransfer $\Phi: BG \to \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (EG/H)^N$. This is all that is necessary to identify the homotopy class of Φ with that of $B\Psi$.

Let $(\,.\,)^+$ denote the addition of a disjoint basepoint. The *pretransfer*

$$T: (X/G)^+ \to \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} ((X/H)^+)^N$$

is the unique pointed map extending Φ.

We turn now to defining the transfer for a generalized cohomology theory. If Y is a pointed space $\Omega^n Y = (Y, *)^{(I^n, \dot{I}^n)}$. The Dyer–Lashof map for $\Omega^n Y$

$$\theta_n^k: \mathscr{C}_n(k) \times_{\mathscr{S}_k} (\Omega^n Y)^k \to \Omega^n Y$$

is defined as follows: if $(c_1, \ldots, c_k) \in \mathscr{C}_n(k)$, $(f_1, \ldots, f_k) \in (\Omega^n Y)^k$ and $u \in I^n$, then

$$\theta_n^k((c_1, \ldots, c_k), (f_1, \ldots, f_k)) = \begin{cases} f_i c_i^{-1}(u) & \text{if} \quad u \in \operatorname{Im} c_i \\ * & \text{if} \quad u \in I^n - \bigcup_j \operatorname{Im} c_j. \end{cases}$$

If $\mathbf{Y} = \{Y_q\}$ is an Ω-spectrum (all Ω-spectra are assumed strict, i.e. $Y_i = \Omega Y_{i+1}$; May (15) shows these spectra are adequate), then the maps θ_n^k fit together (via σ_n) to give a map

$$\theta_\infty^k: \mathscr{C}_\infty(k) \times_{\mathscr{S}_k} (Y_q)^k \to Y_q.$$

The ($\mathbf{Y}$-cohomology) *transfer* morphism

$$p^!: H^q(X/H; \mathbf{Y}) \to H^q(X/G: \mathbf{Y})$$

is defined by

$$p^!(\alpha) = [\theta_\infty^N \cdot (1 \times_{\mathscr{S}_N} a^N) \cdot T]:$$

$$(X/G)^+ \xrightarrow{T} \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} ((X/H^+)^N \xrightarrow{1 \times \mathscr{S}_N a^N} \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (Y_q)^N \xrightarrow{\theta_\infty^N} Y_q,$$

where $a: (X/H)^+ \to Y_q$ represents α. The following is now clear.

PROPOSITION 1·3. *The transfer $p^!$ is well defined and natural with respect to morphisms of strict Ω-spectra.*

Next we consider a stable version of the transfer. Let $f\colon (X/G)^+ \to Q((X/H)^+)$ be the composite

$$(X/G)^+ \xrightarrow{T} \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} ((X/H)^+)^N \xrightarrow{1 \times_{\mathscr{S}_N} i} \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (Q((X/H)^+))^N \xrightarrow{\theta_\infty^N} Q((X/H)^+),$$

where $i\colon (X/H)^+ \to Q((X/H)^+)$ is the inclusion, i.e.

$$[f] = p^![i] \in H^0(X/H;\, \mathbf{Y})$$

for the Ω-spectrum $\mathbf{Y}$ with $Y_k = Q(\Sigma^k((X/H)^+))$. If $\mathbf{\Sigma}^\infty X$ denotes the suspension spectrum of X, then the stable adjoint of f

$$\operatorname{adj} f\colon \mathbf{\Sigma}^\infty(X/G)^+ \to \mathbf{\Sigma}^\infty(X/H)^+$$

is a map of suspension spectra.

Proposition 1·4. *If* $\mathbf{E}$ *is an* Ω*-spectrum, then*

$$\begin{tikzcd} H^*(X/H;\, \mathbf{E}) \arrow[r, "p^!"] \arrow[d, "\approx"', "\sigma"] & H^*(X/G\colon \mathbf{E}) \arrow[d, "\approx"', "\sigma"] \\ H^*(\mathbf{\Sigma}^\infty(X/H)^+;\, \mathbf{E}) \arrow[r, "(\operatorname{adj} f)^*"] & H^*(\mathbf{\Sigma}^\infty(X/G)^+;\, \mathbf{E}) \end{tikzcd}$$

commutes, where σ *denotes the suspension isomorphism.*

Proof. Let $[a] \in H^q(X/H;\, \mathbf{E})$ and consider the homotopy commutative diagram

$$\begin{tikzcd} (X/G)^+ \arrow[r, "T"] \arrow[ddr, "f"'] & \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (Q((X/H)^+))^N \arrow[r, "1 \times_{\mathscr{S}_N} a^N"] \arrow[d, "1 \times_{\mathscr{S}_N} i^N"] & \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (E_q)^N \arrow[r, "\theta_\infty^N"] \arrow[d, "1 \times_{\mathscr{S}_N} i^N"] & E_q \\ & \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (Q((X/H)^+))^N \arrow[r, "1 \times_{\mathscr{S}_N} Q(a)^N"] \arrow[d, "\theta_\infty^N"] & \mathscr{C}_\infty(N) \times_{\mathscr{S}_N} (Q(E_q))^N \arrow[r, "\theta_\infty^N"] & Q(E_q) \arrow[u, "r"'] \\ & Q((X/H)^+) \arrow[urr, "Q(a)"'] & & \end{tikzcd}$$

where r is the retraction map. The top row represents $p^!([a])$ and so its stable adjoint $\mathbf{\Sigma}^\infty((X/G)^+) \to \mathbf{E}$ is a map of degree q which represents $p^!([a])$. The stable adjoint of the bottom route is a map

$$\mathbf{\Sigma}^\infty(X/G)^+ \xrightarrow{\operatorname{adj} f} \mathbf{\Sigma}^\infty(X/H)^+ \xrightarrow{\mathbf{\Sigma}^\infty a} \mathbf{E}$$

of degree q which represents $(\operatorname{adj} f)^*\,(\sigma a)$. This completes the proof.

We now investigate $\operatorname{adj} f$ in case $\mathbf{E}$ is an Eilenberg–MacLane spectrum $\mathbf{K}(A)$, where A is an abelian group. The nth space of the strict Ω-spectrum $\mathbf{K}(A)$ is denoted by $K(A,n)$. We can find a homotopy equivalence $l\colon K(A,n) \to K'$, where K' is an abelian topological monoid. We may also assume that l carries the constant loop to the unit of K'. The homotopy equivalence l converts the Dyer–Lashof maps for $K(A,n)$ into a particularly simple form. Let $SP^k(X) = X^k/\mathscr{S}_k$ be the symmetric product.

Proposition 1·5. *There is a homotopy commutative diagram*

$$\begin{tikzcd} \mathscr{C}_m(k) \times_{\mathscr{S}_N} (K(A,n))^k \arrow[r, "\theta_m^k"] \arrow[d, "\pi"] & K(A,n) \arrow[r, "l"] & K' \\ SP^k(K(A,n)) \arrow[r, "SP^k l"] & SP^k K' \arrow[ur, "\mu"'] & \end{tikzcd}$$

where π *is projection on the second factor and* μ *is induced by addition.*

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Proof. Let $c_{i,t}$ be a homotopy of the little m-cube c_i to the identity map on $\operatorname{int} I^m$. Clearly we can arrange $c_{i,t}$ to be a homotopy through little m-cubes (e.g. expand c_i linearly to fill $\operatorname{int} I^m$). The desired homotopy between $l\cdot\theta_m^k$ and $\mu\cdot SP^k(l)\cdot\pi$ is given by

$$H((c_1,\ldots,c_k),(f_1,\ldots,f_k),t) = \begin{cases} \Sigma l f_i c_i^{-1}(u) & \text{if } u\in \operatorname{Im} c_{i,t} \text{ for some } i \\ * & \text{if } u\in I^m - \bigcup_j \operatorname{Im}(c_{j,t}). \end{cases}$$

Let us now assume $p\colon X/H\to X/G$ is a covering in the category of CW complexes with X simply connected. The n-cells of X are freely permuted by G acting on the right of X. Let the orbits of the n-cells be indexed by the set J and for each $\alpha\in J$, let $\phi_\alpha\colon(e^n,\dot{e}^n)\to(X^n,X^{n-1})$ be the characteristic map of one of the cells in the orbit α. Then the cells of X have as characteristic maps $\{R_g\cdot\phi_\alpha|\alpha\in J, g\in G\}$, where $R_g(x)=x.g$. The characteristic maps of the n-cells of X/H is the set $\{p_H\cdot R_{\tau_i}\cdot\phi_\alpha\}$, where

$$G=\coprod_{i=1}^N \tau_i H$$

and p_H is the projection $X\to X/H$. Finally the n-cells X/G have as characteristic maps $\{p_G\cdot\phi_\alpha\}$, where p_G is the projection $X\to X/G$.

By abuse of notation, let us denote cells by their characteristic maps. The classical transfer is given by the cochain map

$$\operatorname{tr}^*\colon C^n(X/H;A)\to C^n(X/G;A)$$

defined by
$$(\operatorname{tr}^* u)(p_G\cdot\phi_\alpha)=\sum_{i=1}^N u(p_H\cdot R_{\tau_i}\cdot\phi_\alpha).$$

PROPOSITION 1·6. *Let* $\mathbf{K}(A)$ *denote the Eilenberg–MacLane spectrum. Then*

$$\begin{tikzcd} H^n(X/H;\mathbf{K}(A)) \arrow[r, "p^!"] \arrow[d, "d"] & H^n(X/G;\mathbf{K}(A)) \arrow[r, "l_*"] & {[X/G,K']} \arrow[dll, "d"] \\ H^n(X/H;A) \arrow[r, "\operatorname{tr}^*"] & H^n(X/G;A) & \end{tikzcd}$$

commutes, where d is the natural isomorphism. Thus the transfer $p^!$ agrees with the classical transfer for ordinary cohomology.

Proof. Let $f\colon X/H\to K(A,n)$ represent a cohomology class $[f]\in H^n(X/H;\mathbf{K}(A))$. We may assume that f carries X^{n-1}/H to the base point of $K(A,n)$. Then the natural isomorphism d assigns to $[f]$ the cohomology class of the cocycle u_f whose value on $p_H\cdot R_{\tau_i}\cdot\phi_\alpha$ is the homotopy class $[f\cdot p_H\cdot R_{\tau_i}\cdot\phi_\alpha]\in\pi_n(K(A,n))=A$. By Proposition 1·5, $l*p^![f]$ is represented by the composition

$$X_G\xrightarrow{\Phi}\mathscr{C}_\infty(N)\times_{\mathscr{S}_N}(X/H)^N\xrightarrow{\pi}SP^N(X/H)\xrightarrow{SP^Nf}SP^N(SPK(A,n))$$
$$\xrightarrow{SP^Nl}SP^N(K')\xrightarrow{\mu}K'.$$

Upon composition with $p_G \cdot \phi_\alpha$, we obtain the composite map

$$(e^n, \dot{e}^n) \xrightarrow{\text{pinch}} (e^n/\dot{e}^n) \xrightarrow{\phi_\alpha} (X^n/X^{n-1}) \xrightarrow{\text{diag}} (X/X^{n-1})^N$$
$$\xrightarrow[R_{\tau_1} \times \ldots \times R_{\tau_N}]{} (X^n/X^{n-1})^N \xrightarrow[(f \cdot p_B)^N]{} (K(A,n))^N \xrightarrow[l^N]{} K'^N \xrightarrow[\mu]{} K'.$$

Since this represents the homotopy class

$$\sum_{i=1}^{N} [l \cdot f \cdot p_H \cdot R_{\tau_i} \cdot \phi_\alpha] = \operatorname{tr}^* (u_f)(p_G \cdot \phi_\alpha),$$

the proof of 1·6 is complete.

Remark 1·7. In the situation of Proposition 1·4, consider the map

$$\operatorname{adj} f\colon \quad \Sigma^\infty((X/G)^+) = \Sigma^\infty(X/G) \vee \Sigma^\infty S^0 \to \Sigma^\infty(X/H) \vee \Sigma^\infty S^0 = \Sigma^\infty((X/H)^+).$$

Since X/H is connected one may pinch to a point $\Sigma^\infty S^0$ in both the source and target and thus obtain a map $\Sigma^\infty(X/G) \to \Sigma^\infty(X/H)$ whose induced cohomology homomorphism is equivalent to a transfer on reduced cohomology groups. We shall call this map the stable reduced transfer and denote it by tr.

Remark 1·8. Consider the situation of Proposition 1·4. It follows from 1·6 and the universal coefficient theorem (if X/G has finite type) that the induced ordinary homology homomorphism $(\operatorname{adj} f)_*$ with $\mathbb{Z}/p$ coefficients (p a prime) is equivalent to the classical ordinary homology transfer. This fact can be proved directly and holds for integer coefficients also.

2. *Proof of the Theorem*. We begin by recalling some preliminaries on the homology of $\mathscr{S}_n$ and $Q_0 S^0$. Throughout this section, all (co)-homology groups are taken with simple coefficients in $\mathbb{Z}/p$.

The homology of $\mathscr{S}_m$ is described in ((13), §1); a $\mathbb{Z}/p$-basis is given by

$$H_* B\mathscr{S}_m = \{e_{I_1} * \ldots * e_{I_s} | \Sigma p^{l(I_i)} \leqslant m, I_i \text{ admissible}\},$$

where $*$ denotes the map in homology induced by the pairing $\mathscr{S}_k \times \mathscr{S}_l \to \mathscr{S}_{k+l}$.

Let $Q_i\colon H_k Q_0 S^0 \to H_{2k+i} Q_0 S^0$ denote the Dyer–Lashof operation derived from the loop product. By (8),

$$H_* Q_0 S^0 = A[Q_I[1] * [-p^{l(I)}] | I \text{ admissible}],$$

where $A[\,.\,]$ denotes the graded free commutative algebra functor and where $*$ denotes the loop product in $Q_0 S^0$ and also the induced product in homology. A *weight grading* can be defined in $H_* Q_0 S^0$ by setting

$$w(Q_I[1]) = p^{l(I)}, \quad w([i]) = 0,$$
$$w(x * y) = w(x) + w(y).$$

Consider the composite

$$\phi_n\colon B\mathscr{S}_n \simeq \mathscr{C}_\infty(n) \times_{\mathscr{S}_n} ([1])^n \to \mathscr{C}_\infty(n) \times_{\mathscr{S}_n} (Q_1 S^0)^n \xrightarrow{\theta^n_\infty} Q_n S^0 \xrightarrow{*[-n]} Q_0 S^0,$$

where θ^n_∞ is the Dyer–Lashof map and $[-n]$ is the basepoint of $Q_{-n} S^0$. In homology ϕ_{n*} is an isomorphism in dimensions $\leqslant \frac{1}{2}n$ (4, 16). Furthermore

$$\phi_m(e_{I_1} * \ldots * e_{I_s}) = Q_{I_1}[1] * \ldots * Q_{I_s} * [-p^{\Sigma l(I_i)}].$$

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LEMMA 2·1. *If $f: B\mathscr{S}_p \to Q_0S^0$ is non-trivial on $H_{2p-3}(.)$ then $f_*(e_{2i(p-1)-\epsilon})$ is a non-zero multiple of $Q_{2i(p-1)-\epsilon}[1]$ modulo terms of weight $> p$ ($\epsilon = 0, 1$).*

Proof. The p-Sylow subgroup $\pi \subset \mathscr{S}_p$ is cyclic of order p; hence $H^*(B\mathscr{S}_p)$ imbeds in $H^*(B\pi) \approx E[x_1] \otimes P[y_2]$ ($P[x_1]$ if $p = 2$). Here $E[\ ]$ denotes the Exterior algebra, $P[\ ]$ denotes the polynomial algebra and the subscript denotes the dimension. The result follows from the known action of the Steenrod operations on $H^*(B\pi)$.

If Y is an infinite loop space and $\alpha: X \to Y$, then there is a natural extension $\bar{\alpha}: Q(X) \to Y$ given by $\bar{\alpha} = r\cdot Q(\alpha)$, where $r: Q(Y) \to Y$ is the retraction (15).

Proof of the Theorem. $\mathscr{S}(p^k, p) = \pi \int \dots \int \pi$ (k factors) is a p-Sylow subgroup of $\mathscr{S}_{p^k}$. Consider the composite

$$\rho_k: \Sigma^\infty B\mathscr{S}_{p^k} \xrightarrow{\mathrm{tr}_k} \Sigma^\infty B\mathscr{S}(p^k, p) \xrightarrow{\Sigma^\infty g_k} \Sigma^\infty QB\mathscr{S}_p \xrightarrow{\Sigma^\infty \bar{f}} \Sigma^\infty Q_0 S^0,$$

where tr_k is the stable reduced transfer (1·7) for the covering $B\mathscr{S}(p^k,p) \to B\mathscr{S}_{p^k}$ and g_k is the composite

$$g_k: B\mathscr{S}(p^k,p) \to B\mathscr{S}_{p^{k-1}} \int \pi \simeq \mathscr{C}_\infty(p^{k-1}) \times_{\mathscr{S}_{p^{k-1}}} (B\pi)^{k-1} \to \mathscr{C}_\infty(p^{k-1}) \times_{\mathscr{S}_{p^{k-1}}} (QB\pi)^{p^{k-1}}$$
$$\xrightarrow{\theta_\infty^{p^{k-1}}} QB\pi \to QB\mathscr{S}_p.$$

Our first task is to show ρ_{k*} is an isomorphism in homology in dimensions $< \frac{1}{2}p^k$. Let $\lambda = [\mathscr{S}(p^k,p): \mathscr{S}_{p^k}]$ and let $i: \mathscr{S}(p^k,p) \to \mathscr{S}_{p^k}$ denote the inclusion. If

$$x = e_{I_1} * \dots * e_{I_s} \in H_* B\mathscr{S}_{p^k},$$

then by Propositions 1·4, 1·6 above and theorem 3·8 (13) we have (in the notation of (13))

$$\mathrm{tr}_{k*}(x) = \hat{e}_{I_1}|\dots|\hat{e}_{I_s} + \Sigma \hat{e}_{I_1'}|\dots|\hat{e}_{I_s'},$$

where $l(I_j') = l(I_j)$ and $i_*(\Sigma\hat{e}_{I_1'}|\dots|\hat{e}_{I_s'}) = (\lambda - 1)x$. Thus

$$g_{k*}\,\mathrm{tr}_{k*}(x) = Q_{J_1}(e_{j_1}) * \dots * Q_{J_s}(e_{j_s}) + \Sigma Q_{J_1'}(e_{j_1'}) * \dots * Q_{J_s'}(e_{j_s'}),$$

where $(J_i, j_i) = I_i$, $(J_i', j_i') = I_i'$. By Lemma 2·1, we have

$$\rho_{k*}(x) = \bar{f}_* g_* \mathrm{tr}_*(x) \equiv \lambda\{Q_{J_1}(Q_{j_1}[1] * [-p]) * \dots * Q_{J_s}(Q_{j_s}[1] * [-p])\}$$

modulo terms of higher weight. Now, applying the Cartan formula and observing that $Q_J(.)$ and $*$ do not decrease weight, we have

$$\rho_{k*}(x) \equiv \lambda\{Q_{I_1}[1] * \dots * Q_{I_s}[1] * [-p^{\Sigma l(I_i)}]\}$$

modulo terms of higher weight. Hence ρ_{k*} is an isomorphism in the desired range.

Next we observe that the maps $\psi_k = \Sigma^\infty g_k \cdot \mathrm{tr}_k$ determine a map

$$\psi_\infty: \Sigma^\infty B\mathscr{S}_\infty \to \Sigma^\infty QB\mathscr{S}_p$$

such that $\Sigma^\infty \bar{f}\cdot\psi_\infty$ is a homology equivalence. Let $B\mathscr{S}_n^{(n)}$ denote the n-skeleton (which can be assumed finite) and let $\mathscr{H}_n \subset [\Sigma B^\infty \mathscr{S}_n^{(n)}, \Sigma^\infty QB\mathscr{S}_p]$ be the set of all homotopy classes $[\alpha]$ such that

$$\Sigma^\infty \bar{f}\cdot\alpha: \Sigma^\infty B\mathscr{S}_n^{(n)} \to \Sigma^\infty Q_0 S^0$$

is a homology equivalence up through dimension $\frac{1}{2}n$. We have just shown that $\psi_k \in \mathscr{H}_{p^k}$. Since $[\Sigma^\infty B\mathscr{S}_n^{(n)}, \Sigma^\infty QB\mathscr{S}_p]$ is finite, we see that $\mathscr{H}_n$ is a non-empty finite set. Thus $\varprojlim \mathscr{H}_n \neq \varnothing$. Since $B\mathscr{S}_\infty = \bigcup B\mathscr{S}_n^{(n)}$ the desired ψ_∞ exists.

Applying Ω^∞ and localizing all spaces at p, we have an equivalence

$$Q(B\mathscr{S}_\infty) \xrightarrow{\Omega^\infty\psi_\infty} QQB\mathscr{S}_p \xrightarrow{Q\bar{f}} Q(Q_0S^0).$$

Further, since $\phi_\infty\colon B\mathscr{S}_\infty \to Q_0S^0$ is a homology equivalence, $Q\phi_\infty\colon QB\mathscr{S}_\infty \to Q(Q_0S^0)$ is an equivalence. Thus there exists a map $k\colon Q(QS^0) \to QQB\mathscr{S}_p$ such that $Q\bar{f}\cdot k$ is an equivalence. Now consider the commutative diagram

$$\begin{tikzcd}
Q(Q_0S^0) \arrow[r, "k"] & QQB\mathscr{S}_p \arrow[r, "Q\bar{f}"] \arrow[d, "r"] & Q(Q_0S^0) \arrow[d, "r"] \\
Q_0S^0 \arrow[u, "i"] & QB\mathscr{S}_p \arrow[r, "\bar{f}"] & Q_0S^0,
\end{tikzcd}$$

where $r\cdot i = id$. Let h be a homotopy inverse of $Q\bar{f}\cdot k$ and set $t = r\cdot k\cdot h\cdot i$. Then $\bar{f}\cdot t = \bar{f}\cdot r\cdot k\cdot h\cdot i = r\cdot Q(\bar{f})\cdot k\cdot h\cdot i \simeq r\cdot i = id$ and so $\bar{f}\cdot t$ is an equivalence as desired.

Since g factors through $QB\pi$, the result also holds for $B\pi$. This completes the proof.

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