LECTURES ON TOPOLOGICAL HOCHSCHILD HOMOLOGY AND CYCLOTOMIC SPECTRA

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Abstract. These are lecture notes for series of talks given by the second author in Copenhagen and Bloomington. Most of the results are based on the paper [NS17]. We also discuss the construction of non-commutative Witt vectors and how to realize Eilenberg MacLane spectra as Thom spectra.

Contents

1. Introduction

1.1. **Acknowledgements.** We would like to thank Andrew Blumberg for a helpful discussion of how coherent the equivalence for THH of Thom spectra (Proposition 4.7) is and in general for many helpful discussions and explanations. We also thank Nitu Kitchloo for informing us about the possibility of realizing $H\mathbb{Z}/p^n$ as a Thom spectrum (Theorem A.3). These notes profited a lot from many fruitful discussions with Benjamin Antieau, David Gepner, Peter Scholze and Akhil Mathew. We also thank Lars Hesselholt for many comments in the talks and about a first draft and in general for his interest and encouragement.

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2. Hochschild and cyclic homology

Let R be an associative and unital ring. The following classical definition was first given by Hochschild around 1945.

Definition 2.1. *The Hochschild homology groups* $\mathrm{HH}_*(R)$ *of* R *are defined as the homology groups of the chain complex*

$$\mathrm{HH}(R) = \qquad \big(\ldots \to R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} R \to R \otimes_{\mathbb{Z}} R \to R\big)$$

whose differential is given by

$$d(x_0 \otimes ... \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i ... \otimes x_i x_{i+1} \otimes ... + (-1)^n x_n x_0 \otimes ... \otimes x_{n-1} \ .$$

Example 2.2. For $R = \mathbb{Q}$ we get that $R^{\otimes n} = \mathbb{Q}$ and the differentials alternate between 0 and id. Thus,

$$\mathrm{HH}_*(\mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } * = 0 \\ 0 & \text{else} \end{cases}$$

For R a commutative ring we find that

$$\begin{aligned} \mathrm{HH}_0(R) &= R \\ \mathrm{HH}_1(R) &= \frac{\ker(R^{\otimes 2} \to R)}{\mathrm{im}(R^{\otimes 3} \to R^{\otimes 2})} \cong \Omega^1_{R/\mathbb{Z}} \end{aligned}$$

where $\Omega^1_{R/\mathbb{Z}}$ are the Kähler differentials of R relative to $\mathbb{Z}$. The first equality is clear. For the second part note that $\ker(R^{\otimes 2} \to R)$ is given by $R^{\otimes 2}$ and an explicit isomorphism is given by

$$[x \otimes y] \in \frac{R^{\otimes 2}}{R^{\otimes 3}} \quad \mapsto \quad x \cdot dy \in \Omega^1_{R/\mathbb{Z}}$$

with inverse $x \cdot dy \mapsto [x \otimes y]$. We leave it to the reader to check that these maps are well-defined.

Lemma 2.3. *If* R *is a commutative ring, then* $\mathrm{HH}_*(R)$ *admits the structure of a graded commutative ring in which all odd elements square to zero.* [1]

Proof. The chain complex is associated to the simplicial commutative ring

$$\cdots \;\substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow}\; R \otimes_{\mathbb{Z}} R \otimes_{\mathbb{Z}} R \;\substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow}\; R \otimes_{\mathbb{Z}} R \;\substack{\textstyle\rightarrow\\[-1ex]\textstyle\rightarrow}\; R \ .$$

where the face maps are given by the summands in the definition of the differential (and its clear that all are ring maps). But in such a situation the Eilenberg-Zilber map equips the chain complex $\mathrm{HH}(R)$ with the structure of a CDGA and thus the homology groups are graded commutative. Concretely we have

$$(r_0 \otimes ... \otimes r_m) \cdot (r'_0 \otimes ... \otimes r'_n) = \sum_{(\mu,\nu)} \mathrm{sign}(\mu,\nu) s_\mu(r_i) s_\nu(r'_i)$$

where the sum runs over all shuffles of and s_μ and s_ν are the simplicial degeneracy maps associated to the respective shuffles. Explicitly, an (m,n)-shuffle $\sigma = (\mu,\nu)$

[1] We thank Lars Hesselholt for suggesting this statement.

of the set $\{1, \ldots, m+n\}$ is a permutation such that σ is increasing on each of the subsets $\{1, \ldots, m\}$ and $\{m+1, \ldots, n+m\}$. The summand associated to σ equals

$$\mathrm{sign}(\sigma)(r_0 r_0') \otimes (r_1 \otimes \ldots \otimes r_m \otimes r_1' \otimes \ldots \otimes r_n')_{\sigma^{-1}}$$

where $(a_1 \otimes \ldots \otimes a_{m+n})_\rho = (a_{\rho(1)} \otimes \ldots \otimes a_{\rho(m+n)})$.

Finally it is generally true for CDGA's induced from simplicial commutative rings that the squares of odd elements are zero. This can be seen by reduction to the universal case: the free simplicial commutative ring on a generator in degree n gives rise to the graded commutative ring $H_*(K(\mathbb{Z}, n), \mathbb{Z})$ with the Pontryagin ring structure by the Dold-Thom theorem. The claim is that in this ring the degree n generator squares to zero for odd n. This can be checked with $\mathbb{F}_2$-coefficients, in which case it follows by dualizing from the fact that the cohomology $H^*(K(\mathbb{Z}, n), \mathbb{F}_2)$ is a polynomial algebra on primitive generators. Thus the homology is a divided power algebra. □

Theorem 2.4 (Hochschild-Kostant-Rosenberg)**.** *For R a commutative and smooth algebra over $\mathbb{Z}$ (or more generally a localization of $\mathbb{Z}$) the map*

$$\Omega^*_{R/\mathbb{Z}} \to \mathrm{HH}_*(R),$$

induced from $\Omega^1_{R/\mathbb{Z}} \xrightarrow{\simeq} \mathrm{HH}_1(R)$ and the fact that $\Omega^*_{R/\mathbb{Z}}$ is free and $\mathrm{HH}_*(R)$ is an R-algebra, is an isomorphism. □

Example 2.5.

$$\mathrm{HH}_*(\mathbb{Q}[x_1, ..., x_n]) = \mathbb{Q}[x_1, ..., x_n] \otimes \Lambda(dx_1, ..., dx_n)$$

Note that at this point this is an isomorphism of graded algebras, there is no differential on Hochschild homology that corresponds to the de Rham differential. This will be corrected soon and lead to negative cyclic homology.

Note also that in the rational case there is even a chain level statement, namely that the chain complex is formal, so that the Hochschild chain complex is naturally quasi-isomorphic to the de Rham complex with zero differential.

For $R = \mathbb{F}_p$ we find that

$$\mathrm{HH}_*(\mathbb{F}_p) = \mathbb{F}_p$$

as in the case of $\mathbb{Q}$. This is not the 'correct' result though, since $\mathbb{F}_p$ is not flat over $\mathbb{Z}$ and thus the tensor product does not agree with the derived one. We want to work as derived as possible and therefore modify our definition of $\mathrm{HH}(R)$ to account for this. To do this we note that if $R_\bullet$ is a DGA (as opposed to an ordinary ring)[2] then Definition 2.1 still makes sense if we interpret this chain complex as a double complex and take its total complex (using direct sums). This way we have defined $\mathrm{HH}_*(R_\bullet)$ and $\mathrm{HH}_*(R_\bullet)$ for $R_\bullet$ a DGA over $\mathbb{Z}$. For a non-flat ring R we replace R by a flat resolution $R_\bullet$ which is a DGA and define $\mathrm{HH}(R)$ as $\mathrm{HH}(R_\bullet)$. It is straightforward to see that this does, up to quasi-isomorphism, not depend on the choice of $R_\bullet$. This derived variant of Hochschild Homology is sometimes called *Shukla homology.* But we will suppress the distinction and only work with the derived variant from now on.

[2]Note that a simplicial ring gives rise to a DGA, thus we can think of simplicial rings as well.

Proposition 2.6. *We get that* $\mathrm{HH}_*(\mathbb{F}_p) = \mathbb{F}_p[x, x^2/2!, x^3/3!, ..] = \mathbb{F}_p\langle x\rangle$ *is the free divided power algebra on a single generator* x *in degree 2.*[3]

Proof. This can be easily seen using that the cotangent complex of $\mathbb{F}_p$ relative to $\mathbb{Z}$ is in degree 1 and the symmetric powers are then given by divided powers, but let us give an explicit argument.

We first claim that by definition $\mathrm{HH}(R)$ is always equivalent to the chain complex $R \otimes^{\mathrm{L}}_{R\otimes^{\mathrm{L}} R^{\mathrm{op}}} R$. This follows easily from using the standard resolution of R as a R-R bimodule to compute this derived tensor product and is left as an exercise. Now in our concrete case we first compute $\mathbb{F}_p \otimes^{\mathrm{L}} \mathbb{F}_p$ by using the flat resolution of $\mathbb{F}_p$ by the CDGA

$$\mathbb{Z}[\varepsilon]/\varepsilon^2 \qquad d\varepsilon = p \qquad |\varepsilon| = 1$$

and obtain

$$\mathbb{F}_p \otimes^{\mathrm{L}} \mathbb{F}_p \simeq \mathbb{F}_p[\varepsilon]/\varepsilon^2 =: A\ .$$

Thus we have to compute the derived tensor product $\mathbb{F}_p \otimes^{\mathrm{L}}_A \mathbb{F}_p$. To this end we resolve $\mathbb{F}_p$ as an A-algebra by

$$A\langle x\rangle = \frac{A[x_1, x_2, ..]}{x_i x_j = \binom{i+j}{i} x_{i+j}} \qquad dx_i = \varepsilon x_{i-1} \qquad |x_i| = 2i.$$

It is an easy exercise to check that this is a well defined CDGA and a resolution of $\mathbb{F}_p$. Tensoring this over A with $\mathbb{F}_p$ we get the desired result. Note that we have even shown that $\mathrm{HH}(\mathbb{F}_p)$ is as a CDGA quasi isomorphic to its homology. $\square$

We now come to the structure on the Hochschild complex which resembles the de Rham differential. This has been introduced by Connes 1982. For R an associative ring (or a DGA) there is extra structure on $\mathrm{HH}(R)$, namely for every n a morphism

$$B : \mathrm{HH}(R)_n \to \mathrm{HH}(R)_{n+1}$$

with $dB + Bd = 0$ and $B^2 = 0$ defined as

$$B(r_0 \otimes \cdots \otimes r_n) = \sum_{\sigma \in C_{n+1}} (-1)^{n\sigma(0)} \big(1 \otimes r_\sigma - (-1)^n r_\sigma \otimes 1\big)$$

where C_{n+1} is the group of cyclic permutations of $\{0, .., n\}$ and $r_\sigma = r_{\sigma(0)} \otimes ... \otimes r_{\sigma(n)}$. If R is commutative then B is a derivation on $\mathrm{HH}_*(R)$.

Exercise 2.7. *Check that* B *has the desired properties.*

This operator B equips the complex $\mathrm{HH}(R)$ with the structure of a DG module over the DG algebra

$$A = \mathbb{Z}[b]/b^2 = H_*(\mathbb{T}, \mathbb{Z}) \qquad |b| = 1,$$

where $\mathbb{T} = U(1)$ is the circle group. In particular the Hochschild homology groups inherit a degree-increasing differential (which we will also denote by B). For R a commutative ring this operator $B : \mathrm{HH}_*(R) \to \mathrm{HH}_{*+1}(R)$ turns out to be a derivation.

[3] Writing $\mathbb{F}_p[x, x^2/2!, x^3/3!, ..]$ is very sloppy notation. Really the free divided power algebra $\mathbb{F}_p\langle x\rangle$ has generators $x_1, x_2, x_3, ..$ (with $x_1 = x$) and the relations that $x_i x_j = \binom{i+j}{i} x_{i+j}$. We denote the generators by $x^i/i!$ to indicate this.

Remark 2.8. The operator B is not a derivation on the chain complex $\mathrm{HH}(R)$ on the nose, but only up to coherent chain homotopies. This is a bit tricky to verify and most easily done by comparison with the topological setting. We will assume this here so that we really get ring structures on the groups $\mathrm{HC}^-_*(R)$ and $\mathrm{HP}_*(R)$ that will be defined below.

In the example of a smooth algebra R the operator on $\mathrm{HH}_*(R)$ induced from B corresponds under the HKR-isomorphism to the de Rham differential.

Example 2.9. For $\mathrm{HH}(\mathbb{F}_p)$ the Connes operator acts trivially on Hochschild homology for degree reasons. But we can take the mod p-reduction, i.e. $\mathrm{HH}(\mathbb{F}_p)/\!\!/p = \mathrm{HH}(\mathbb{F}_p) \otimes^L_{\mathbb{Z}} \mathbb{F}_p$.[4] As an algebra we get that the homology groups are given by

$$\mathbb{F}_p[\varepsilon]/\varepsilon^2 \otimes \mathbb{F}_p\langle x\rangle \qquad |\varepsilon| = 1, |x| = 2.$$

We claim that the B-operator takes x_i to zero and εx_i to $(i+1)ux_{i+1}$ for some unit u (independent of i). In particular, by rescaling $x'_{i+1} = u^{i+1}x_{i+1}$, we obtain a description $\mathrm{HH}_*(\mathbb{F}_p)/\!\!/p = \mathbb{F}_p[\varepsilon]/\varepsilon^2 \otimes \mathbb{F}_p\langle x'\rangle$ with $B(\varepsilon x'_i) = (i+1)x'_{i+1}$.

Proof. The fact that $Bx_i = 0$ follows since these generators lift to the unreduced $\mathrm{HH}_*(\mathbb{F}_p)$, where B acts as 0 for degree reasons. To see the other part, we have to approach $\mathrm{HH}_*(\mathbb{F}_p)$ from a perspective compatible with the cyclic structure. We again replace $\mathbb{F}_p$ by the graded commutative differential $\mathbb{Z}$-algebra $\mathbb{Z}[\varepsilon]/\varepsilon^2$. We obtain a double complex computing $\mathrm{HH}_*(\mathbb{F}_p)$ from the cyclic bar complex for $\mathbb{Z}[\varepsilon]/\varepsilon^2$, the second differential comes from $d\varepsilon = p$.

The reduced version of the complex, i.e. modulo degenerate elements of the form $r_0 \otimes \ldots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \ldots \otimes r_n$ for $0 \le i \le n-1$, computes the same homology. This has one generator in each total degree, of the form $1, \varepsilon, 1\otimes\varepsilon, \varepsilon\otimes\varepsilon, \ldots$. Either from comparison with the above description or from explicit computation, we see that mod p, all of these are cycles in the total complex. The fact that ε is in odd degree gives slightly different signs than stated above in some of the formulas, by the Koszul sign rule.

However, by definition $B\varepsilon = 1\otimes\varepsilon + \varepsilon\otimes 1$, which is $1\otimes\varepsilon$ modulo degenerates. This translates to $B\varepsilon = ux_1$ in our earlier description for some unit u. Since B is a derivation, this immediately shows $B(\varepsilon x_i) = ux_1x_i = (i+1)ux_i$. ☐

For the next definition recall that we have defined the algebra $A = \mathbb{Z}[b]/b^2$ which we consider as a DGA with b in degree 1 and zero differential.

Definition 2.10. *The cyclic homology of R is defined as*

$$\mathrm{HC}_*(R) := \mathrm{Tor}^A_*(\mathbb{Z}, \mathrm{HH}(R)) \ ,$$

the negative cyclic homology as[5]

$$\mathrm{HC}^-_*(R) := \mathrm{Ext}^*_A(\mathbb{Z}, \mathrm{HH}(R))$$

and the periodic homology as the localization

$$\mathrm{HP}_*(R) := \mathrm{HC}^-_*(R)[t^{-1}]$$

where $\mathrm{HC}^-_*(R)$ is considered as a module over

$$\mathrm{Ext}^*_A(\mathbb{Z},\mathbb{Z}) = \mathbb{Z}[t] \qquad |t| = -2 \ .$$

[4]Note that this is quasi-isomorphic to $\mathrm{HH}(\mathbb{F}_p[\varepsilon]/\varepsilon^2\,/\mathbb{F}_p)$.

[5]Here Ext means the mapping space in the derived category of DG-modules over A. Thus these Ext-groups are just singly graded and not bigraded. The same applies for the Tor-groups.

Note that A is canonically a Hopf-algebra with counit $\mathbb{Z}$ and $\Delta(b) = b \otimes 1 + 1 \otimes b$ which leads to ring structures on $\mathrm{HC}^-_*(R)$ and $\mathrm{HP}_*(R)$ for commutative rings.

Proposition 2.11. *For R a smooth algebra over $\mathbb{Q}$ we find that*

$$\mathrm{HC}^-_n(R) \cong Z^n(\mathrm{Spec}\, R) \oplus \prod_{i \geq 1} H^{n+2i}_{dR}(\mathrm{Spec}\, R)$$
$$\mathrm{HP}_*(R) = H^*_{\mathrm{dR}}(\mathrm{Spec}\, R)(\!(t)\!) \qquad |t| = -2$$

Proof. To see this we want to invoke the strong form of the HKR isomorphism which holds over $\mathbb{Q}$, namely that the Hochschild chain complex $\mathrm{HH}(R)$ is as a chain complex with the Hochschild differential d as differential quasi-isomorphic to $\Omega^*(R)$ with zero differential. This is a formality result. Moreover this quasi-isomorphism sends the B-operator to the de Rham differential.

Now in order to calculate

$$\mathrm{HC}^-_*(R) = \mathrm{Ext}^*_A(\mathbb{Z}, \mathrm{HH}(R))$$

we resolve $\mathbb{Z}$ as a coalgebra over A by the co-DG-algebra

$$C := A\{x_0, x_1, x_2, ...\}$$

with A-basis x_k for x_k in degree $2k$, comultiplication

$$\Delta(x_k) = \sum_{i+j=k} x_i \otimes x_j$$

and differential $dx_k = bx_{k-1}$. Mapping out of that we immediately get the CDGA

$$\underline{\mathrm{Hom}}_A(C, \Omega^*(R)) = \Omega^*(R)[[t]] \qquad |t| = -2,$$

with t^i dual to x_i, and differential given by $d(xt^i) = d_{dR}(x)t^{i+1}$. Taking homology we get the first formula which immediately implies the second. □

Proposition 2.12. *For the ring $R = \mathbb{F}_p$ we find that*

$$\mathrm{HC}^-_*(\mathbb{F}_p) = \mathbb{Z}_p[t]\langle\!\langle x \rangle\!\rangle/(xt - p) \qquad |x| = 2, |t| = -2$$
$$\mathrm{HP}_*(\mathbb{F}_p) = \mathbb{Z}_p[t^{\pm}]\langle\!\langle x \rangle\!\rangle/(xt - p) = \big(\mathbb{Z}_p\langle\!\langle y \rangle\!\rangle/(y - p)\big)[t^{\pm}] \qquad |y| = 0$$

where $R\langle\!\langle x \rangle\!\rangle$ for a ring R means the divided power series algebra*, i.e. the completion of $R\langle x \rangle$ at the multiplicative filtration defined by the divided powers of x, and in the graded case we take, as usual, homogenous elements, i.e. perform the completion degreewise.*

Proof. For every ring R there is a spectral sequence induced from the canonical filtration (aka Postnikov) filtration on $\mathrm{HH}(R)$. It is multiplicative and takes the form

$$E_2 = \mathbb{Z}[t] \otimes \mathrm{HH}_*(R) \Rightarrow \mathrm{HC}^-_*(R)$$

where t is in degree -2. Moreover it converges conditionally since the Postnikov limit commutes with the derived mapping space giving the Ext-groups. [6]

[6] In the language of $\mathbb{T}$-actions that we will use later this is just the homotopy fixed point spectral sequence.

$$
\begin{array}{ccccccc|c}
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 4 \\
 & & & & & & & 3 \\
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 2 \\
 & & & & & & & 1 \\
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 0 \\
\hline
-6 & -5 & -4 & -3 & -2 & -1 & 0 &
\end{array}
$$

For the case $R = \mathbb{F}_p$ it degenerates at the E_2-page, i.e. there can not be any further differentials, since everything is even. As a result it strongly converges and we see that $\mathrm{HC}^-_*(\mathbb{F}_p)$ has a descending, complete filtration with associated graded $\mathbb{F}_p\langle x\rangle[t]$. The connective cover of $\mathrm{HC}^-(\mathbb{F}_p)$ can be represented by a simplicial commutative ring (as $\mathrm{HH}(R)$ is a simplicial commutative ring and the B-operator is compatible), thus $\mathrm{HC}^-_*(\mathbb{F}_p)$ has a divided power structure on positive degree homotopy groups (see e.g. [Ric09, Definition 4.1]). Thus for every pair of lifts of x and t we get a map

$$\mathbb{Z}\langle x\rangle[t] \to \mathrm{HC}^-_*(\mathbb{F}_p).$$

The result follows if we can show the following two things: first that there are lifts of x and t to $\mathrm{HC}^-_*(\mathbb{F}_p)$ (abusively also denoted by x and t) such that $xt = p$ and secondly that the resulting map

$$\mathbb{Z}\langle x\rangle[t]/(xt-p) \to \mathrm{HC}^-_*(\mathbb{F}_p)$$

induces an isomorphism on the associated graded, where we filter the left hand side by the divided powers of x. If we assume the first assertion, then the second is straightforward using the relation

$$px^{(n)} = txx^{(n)} = (n+1)tx^{(n+1)} \ .$$

and the fact that the edge homomorphism $\mathrm{HC}^-_*(\mathbb{F}_p) \to \mathrm{HH}_*(\mathbb{F}_p)$ preserves divided power structures since it arises from a map of simplicial commutative rings.

Now we come to the extension problem. First of all, it suffices to check that that there are lifts of x and t such that $xt = p$ modulo higher filtration. This is true since then we write

$$xt = p + \sum_{k\geq 2} a_k t^k x^{(k)}$$

and then we replace x by $x' = x - \sum_{k\geq 2} a_k t^{k-1}x^{(k)}$ to find $x't = p$. Clearly we can replace $\mathrm{HH}(\mathbb{F}_p)$ by its 2-truncation to check this. i.e. it suffices to show that in $\mathrm{Ext}_A(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p))$ the relation $xt = p$ holds, where the elements x, t are the images under the canonical map $\mathrm{Ext}_A(\mathbb{Z}, \mathrm{HH}(\mathbb{F}_p)) \to \mathrm{Ext}_A(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p))$. We have that $\tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(x)$. By the same spectral sequence as above the homotopy groups of $\mathrm{Ext}_A(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p))$ are in degree 0 either given by $\mathbb{F}_p \oplus \mathbb{F}_p$ or by $\mathbb{Z}/p^2$ depending on the extension behavior. The idea is to consider the mod p reduction to distinguish the two.

First we form the mod p reduction of $\tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)$ to obtain a chain complex with homology groups

$$\Lambda(\varepsilon, x) \qquad |\varepsilon| = 1, |x| = 2.$$

By Example 2.9 and the fact that the map $\mathrm{HH}(\mathbb{F}_p) \to \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)$ preserves the B-operator, we get that the B-operator is determined by $B\varepsilon = x$. As before we see that

$$\mathrm{Ext}_A\big(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)\big)/\!\!/p = \mathrm{Ext}_A(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)/\!\!/p)$$

can be computed by a multiplicative spectral sequence with E_2-page

$$\mathbb{F}_p[t] \otimes \Lambda(\varepsilon, x)\ .$$

$$\begin{array}{ccccccc|c}
 & & & & & & & 4 \\
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 3 \\
\mathbb{F}_p & \sim & \mathbb{F}_p & \sim & \mathbb{F}_p & \sim & \mathbb{F}_p & 2 \\
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 1 \\
\mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & & \mathbb{F}_p & 0 \\
\hline
-6 & -5 & -4 & -3 & -2 & -1 & 0 &
\end{array}$$

The first differential is described by the B-operator. Thus we find that $d_2(\varepsilon) = xt$ and d_2 is zero on the other generators. Thus the E_3-page is given by

$$\frac{\mathbb{F}_p[t] \otimes \Lambda(x, y)}{(xt, xy)}.$$

with y in degree $(0, 3)$ represented by εx. In particular in degree 0 we get

$$\mathrm{Ext}^0_A(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)/\!\!/p) = \mathbb{F}_p$$

which forces $\mathrm{Ext}^0_A\big(\mathbb{Z}, \tau_{\leq 2}\mathrm{HH}(\mathbb{F}_p)\big) = \mathbb{Z}/p^2$ and finishes the proof. $\square$

Note that

$$\mathrm{HC}^-_0(\mathbb{F}_p) = \mathrm{HP}_0(\mathbb{F}_p) = \mathbb{Z}_p\langle\!\langle y \rangle\!\rangle/(y - p)$$

is the object where we freely adjoin divided powers of p to $\mathbb{Z}_p$. Despite the fact that $\mathbb{Z}_p$ already has divided powers of p this is not isomorphic to $\mathbb{Z}_p$ since for example terms like $y^{(p)}$ are newly added. This ring has a lot of torsion, for example, the element $y^{(p)} - p^p/p!$ is p-torsion. In fact one can show that one has $\mathbb{Z}_p\langle y \rangle/(y - p) = \mathbb{Z}_p\langle z \rangle/z$ where $z = y - p$. Also note that the cyclic homology is the periodic version of the derived de Rham cohomology of $\mathbb{F}_p$ relative $\mathbb{Z}$. In some sense the above computation should be considered as a warning that Hochschild homology of $\mathbb{F}_p$-algebras relative to $\mathbb{Z}$ can be a very pathological object.

We end by noting that from the concrete description of the above definitions one can construct a long exact sequence

$$\to \mathrm{HC}_{*-1}(R) \to \mathrm{HC}^-_*(R) \to \mathrm{HP}_*(R) \to \ldots$$

which is very useful in practice. For example it can be used to work out the cyclic homologies in the given examples.

3. Topological Hochschild homology

We now want to phrase the results of the last section in the language of ∞-categories and thereby generalize them to homotopy theory. This will eventually lead to the definition of topological cyclic homology. We assume that the reader is familiar with the basic language of ∞-categories as developed in [Lur09]. We will also use the theory of symmetric monoidal ∞-categories as in [Lur16]. The upshot of these books is that there is a good theory of ∞-categories and it generalizes ordinary category theory. The reader can try to use this slogan as a rough black box.

The most important examples of symmetric monoidal ∞-categories are the symmetric monoidal ∞-category Sp of spectra and the ∞-categorical refinement $\mathcal{D}\mathbb{Z}$ of the derived category of $\mathbb{Z}$. In both cases we write the tensor product as $\otimes$ (and not as a smash product as is often done in Sp) or maybe as $\otimes_{\mathbb{Z}}$ and $\otimes_{\mathbb{S}}$ if we want to be more careful about the basis.

Definition 3.1. *We define an (ordinary) category* $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$*. The objects are finite sets. A morphism from* S *to* T *is given by a map* $S \to T$ *together with a linear ordering on the preimages for each* $t \in T$*. Composition is defined by composition of maps with lexicographic ordering on preimages. The category* $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ *becomes symmetric monoidal by disjoint union.*

Note that in $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ the singleton $[1]$ naturally becomes an associative algebra.

Exercise 3.2. *Check that the category of symmetric monoidal functors* $\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$ *to an (ordinary) symmetric monoidal category* $\mathcal{C}$ *is equivalent to the category of associative algebra objects in* $\mathcal{C}$*. The equivalence is induced by evaluation on* $[1] \in \mathrm{Ass}^{\otimes}_{\mathrm{act}}$*.*

The nerve $N\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ is a symmetric monoidal ∞-category. The following is our definition of an algebra object in a symmetric monoidal ∞-category.

Definition/Proposition 3.3. *An algebra object in a symmetric monoidal* ∞*-category* $\mathcal{C}$ *is a symmetric monoidal functor*

$$N\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$$

The ∞*-category* $\mathrm{Alg}(\mathcal{C})$ *is the* ∞*-category of symmetric monoidal functors.*

Using the definition of algebras as in [Lur16] the equivalence to our Definition/Proposition 3.3 is [Lur16, Proposition 2.2.4.9].

Example 3.4. A (flat) DGA R gives rise to a functor $N\mathrm{Ass}^{\otimes}_{\mathrm{act}} \to \mathcal{D}\mathbb{Z}$ which sends S to $R^{\otimes S}$ considered as an object of $\mathcal{D}(\mathbb{Z})$. For a non-flat ring or DGA we choose a flat resolution to get the corresponding functor.

There is a functor $\Delta^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ which sends a finite ordered set S to the set of cuts

$$\mathrm{Cut}(S) = \{S = S_0 \sqcup S_1 \mid S_0 < S_1\}/\sim \ .$$

where the cuts $\emptyset \sqcup S$ and $S \sqcup \emptyset$ are identified. This functor sends $\langle n \rangle \in \Delta$ to a set isomorphic to $\langle n \rangle$. For a map $f : S \to T$ of finite ordered sets the induced map $\mathrm{Cut}(T) \to \mathrm{Cut}(S)$ is given by taking preimages of cuts. For a given cut $S_0 < S_1$ of S the preimage is given by the set of all cuts $T_0 < T_1$ such that $f(S_0) \subseteq T_0$ and $f(S_1) \subseteq T_1$. If S is a non-trivial cut the set of all such T has an ordering by the maximal element in T_0. The preimage of the trivial cut consists of all $T_0 < T_1$ such that $\max(T_0)$ is either $\geq f(S)$ or $< f(S)$ which is ordered such that $\max(f(S))$ is the minimal element. This ordering is really more natural in the language of cyclic sets that will be relevant soon.

In fact this functor describes the cyclic bar construction

$$\cdots \Rrightarrow R \otimes R \otimes R \Rrightarrow R \otimes R \rightrightarrows R\ .$$

where $\otimes$ denotes the tensor product in $\mathrm{Ass}^{\otimes}_{\mathrm{act}}$ (disjoint union) and R is the associative algebra $\langle 1 \rangle$.

[TODO: If $[0] \to S$ is any map in Δ, the induced $\mathrm{Cut}(S) \to \mathrm{Cut}([0])$ has $\mathrm{Cut}(S)$ as preimage (since $\mathrm{Cut}([0])$ has one point). Something seems wrong.]

Definition 3.5. *Let $R \in \mathrm{Alg}(\mathcal{C})$ be an algebra in a symmetric monoidal ∞-category $\mathcal{C}$. Then we have the cyclic bar construction*

$$\mathrm{HH}(R/\mathcal{C})_\bullet : \quad N\Delta^{\mathrm{op}} \to N\mathrm{Ass}^{\otimes}_{\mathrm{act}} \xrightarrow{R} \mathcal{C}$$

an we can take its realization

$$\mathrm{HH}(R/\mathcal{C}) := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} \mathrm{HH}(R_{/\mathcal{C}})_\bullet\ .$$

if this colimit exists in $\mathcal{C}$.

Proposition 3.6. *Let R be a DGA considered as an algebra in $\mathcal{D}(\mathbb{Z})$. Then the Hochschild complex* $\mathrm{HH}(R)$ *(derived if necessary) is equivalent to the Hochschild complex* $\mathrm{HH}(R/\mathcal{C})$ *relative to* $\mathcal{C} = \mathcal{D}(\mathbb{Z})$ *.*

Proof. The object $\mathrm{HH}(R)_\bullet$ is equivalent to the simplicial diagram which lead to the Hochschild complex. Thus the claim follows from the fact that taking the total complex of the associated double complex of a simplicial chain complex is a model for the ∞-categorical colimit over Δ^{op}. □

Definition 3.7. *Let $R \in \mathrm{Alg}(\mathrm{Sp})$ be an associative ring spectrum. Then* $\mathrm{THH}(R)$ *is defined as* $\mathrm{HH}(R/\mathrm{Sp})$.

Note that ordinary rings R can be considered as associative ring spectra by considering the associated Eilenberg Mac-Lane spectra HR. Thus we in particular have a definition of $\mathrm{THH}(R)$ and $\mathrm{THH}_*(R)$ for an ordinary ring R. More informally we can write this definition as

$$\mathrm{THH}(R) = \left| \cdots \Rrightarrow HR \otimes_{\mathbb{S}} HR \otimes_{\mathbb{S}} HR \Rrightarrow HR \otimes_{\mathbb{S}} HR \rightrightarrows HR \right|\ .$$

There is a generalization of the fact that for a commutative ring $\mathrm{HH}_*(R)$ is commutative as well. This generalization is even better since it gives extra structure on the chain complex $\mathrm{HH}(R)$:

Proposition 3.8. *If R is a commutative algebra in a symmetric monoidal ∞-category $\mathcal{C}$ then* $\mathrm{HH}(R/\mathcal{C})$ *is a commutative algebra in $\mathcal{C}$ as well.*

Proof. We will give a semi-formal argument: the Hochschild complex is levelwise a commutative algebra object and all maps preserve this commutative algebra structure. The geometric realization in the category of commutative algebras is formed in the underlying category, so that the underlying geometric realization inherits a commutative algebra structure. □

Example 3.9. For $R = \mathbb{S}$ we get that $\mathrm{THH}(\mathbb{S}) = \mathbb{S}$.

There is a functor $H : \mathcal{D}\mathbb{Z} \to \mathrm{Sp}$ which generalizes taking the Eilenberg MacLane spectrum of an abelian group. In fact $\mathcal{D}\mathbb{Z}$ is as an ∞-category equivalent to the ∞-category of $H\mathbb{Z}$-module spectra and then this functor is just forgetting the $H\mathbb{Z}$ action. We have that $\pi_*(HC_\bullet) = H_*(C_\bullet)$ for every $C_\bullet \in \mathcal{D}\mathbb{Z}$. This functor preserves all colimits and is lax symmetric monoidal, i.e. there are coherent maps $H(C_\bullet) \otimes_{\mathbb{S}} H(D_\bullet) \to H(C_\bullet \otimes_{\mathbb{Z}} D_\bullet)$. As a formal consequence it sends algebras to algebras and we get for every $R_\bullet \in \mathrm{Alg}(\mathcal{D}\mathbb{Z})$ a canonical map

$$\mathrm{THH}(HR_\bullet) \to H(\mathrm{HH}(R_\bullet))$$

and therefore an induced map $\mathrm{THH}_*(R) \to \mathrm{HH}_*(R)$ for every ordinary ring which is a map of graded commutative rings if R is commutative.

Example 3.10. For R a $\mathbb{Q}$-algebra the map $\mathrm{THH}_*(R) \to \mathrm{HH}_*(R)$ is an isomorphism. This follows since $H\mathbb{Q} \otimes_{\mathbb{S}} H\mathbb{Q} \simeq H\mathbb{Q} = H\mathbb{Q} \otimes_{\mathbb{Z}} H\mathbb{Q}$

Proposition 3.11. *For a classical ring R the map*

$$\mathrm{THH}_i(R) \to \mathrm{HH}_i(R)$$

is 3-connected, i.e. an isomorphism in degrees $i \leq 2$ and surjective in degree 3.

Proof. A map of spectra is 3-connected iff the fibre is 2-connected. Thus it suffices to show that the fibre $F \to \mathrm{THH}(R) \to \mathrm{HH}(R)$ is 2-connected. We can compute this fibre as the geometric realization of the respective fibres on the cyclic Bar complexes. Since all terms are connective we see that $\tau_{\leq 2}F$ only depends on

$$\begin{aligned}
&\tau_{\leq -1}\mathrm{fib}\big(HR \otimes_{\mathbb{S}} HR \otimes_{\mathbb{S}} HR \otimes_{\mathbb{S}} HR \to H(R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R)\big) \ , \\
&\tau_{\leq 0}\mathrm{fib}\big(HR \otimes_{\mathbb{S}} HR \otimes_{\mathbb{S}} HR \to H(R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R)\big) \ , \\
&\tau_{\leq 1}\mathrm{fib}\big(HR \otimes_{\mathbb{S}} HR \to H(R \overset{\mathbb{L}}{\otimes}_{\mathbb{Z}} R)\big) \ , \\
&\tau_{\leq 2}\mathrm{fib}\big(HR \to HR\big) \ .
\end{aligned}$$

We claim that these fibres are all contractible. For the first and last this is clear. For the ones in the middle we can resolve R by a 2-step resolution with free abelian groups since taking cofibres can only increase the connectivity. Then the claim boils down to a computation of the first nontrivial homotopy element in $H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$ and $H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z} \otimes_{\mathbb{S}} H\mathbb{Z}$. These lie in degrees 2 as is well-known. □

Remark 3.12. One can also obtain Proposition 3.11 using the equivalence $\mathrm{HH}(R) \simeq \mathrm{THH}(R) \otimes_{\mathrm{THH}(\mathbb{Z})} H\mathbb{Z}$, which can be shown directly from the cyclic bar construction, together with the computation of $\mathrm{THH}(\mathbb{Z})$ that will be done in Corollary 4.2.

As a result we get a canonical element $x \in \mathrm{THH}_2(\mathbb{F}_p) = \mathrm{HH}_2(\mathbb{F}_p)$ from Proposition 2.12. The next result is the foundational result in the area.

Theorem 3.13 (Bökstedt). *We have that* $\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x]$.

We will give a proof of Bökstedt's theorem in the next section using a theorem of Hopkins-Mahowald about Thom spectra. Note that the canonical map

$$\mathrm{THH}_*(\mathbb{F}_p) = \mathbb{F}_p[x] \to \mathbb{F}_p\langle x\rangle = \mathrm{HH}_*(\mathbb{F}_p)$$

sends x to x. It is thus zero in degrees $\geq 2p$. This is part of why THH is a 'better' invariant in characteristic p than HH relative to $\mathbb{Z}$.

Remark 3.14. In fact Bökstedts theorem is more generally true: for every perfect ring k of characteristic p one has $\mathrm{THH}_*(k) = k[x]$.

We close be a remark that will be relevant for Lemma 4.6 in the next section but can otherwise be safely ignored.

Remark 3.15. So far we have assumed that the ambient ∞-category $\mathcal{C}$ is symmetric monoidal to define Hochschild homology of an algebra object $A \in \mathcal{C}$. But it turns out that the definition extends to the case where $\mathcal{C}$ is only $\mathbb{E}_2$-monoidal (aka braided monoidal). This can be seen as follows: for a braided monoidal 1-category it is a straightforward exercise to write down the cyclic bar complex of an algebra object and left to the reader (see e.g. [Bae94] for a reference). Note also that it does not work in the case of a plain monoidal category since the cyclic bar complex involves the opposite of the multiplication of an algebra object.

Now we want to proceed as in the case of symmetric monoidal ∞-categories and reduce everything to the universal case. Therefore we have to understand the initial $\mathbb{E}_2$-monoidal ∞-category with an associative algebra object. This is given by the $\mathbb{E}_2$-monoidal envelope of the ∞-operad $\mathrm{Ass}^\otimes$ with the canonical map $\mathrm{Ass}^\otimes \simeq \mathbb{E}_1^\otimes \to \mathbb{E}_2^\otimes$ (using the language of [Lur16, Section 2.2.4]). The underlying ∞-category of this envelope is given by the ∞-category U obtained as the pullback

$$\begin{tikzcd} \mathcal{U} \arrow[r] \arrow[d] & \big((\mathbb{E}_2^\otimes)_{\mathrm{act}}\big)_{/1} \arrow[d] \\ N\mathrm{Ass}^\otimes_{\mathrm{act}} \arrow[r] & (\mathbb{E}_2^\otimes)_{\mathrm{act}} \end{tikzcd} .$$

By [Lur16, Proposition 2.2.4.4] this ∞-category has a canonical $\mathbb{E}_2$-monoidal structure and is by [Lur16, Proposition 2.2.4.9] initial among such $\mathbb{E}_2$-monoidal ∞-categories with an algebra object. In other words: the ∞-category of $\mathbb{E}_2$-monoidal functors from $\mathcal{U}$ to any $\mathbb{E}_2$-monoidal ∞-category $\mathcal{C}$ is equivalent to the ∞-category of associative (i.e. $\mathbb{E}_1$) algebra objects in $\mathcal{C}$.

Thus to construct the cyclic bar construction for every algebra in a braided monoidal category it is enough to do it in $\mathcal{U}$. But we claim that $\mathcal{U}$ is actually (equivalent to) the nerve of a 1-category U. To see this note that $(\mathbb{E}_2^\otimes)_{\mathrm{act}}$ is a $(2,1)$-category since the mapping spaces in the $\mathbb{E}_2$-operad, which are given by configurations of points in $\mathbb{R}^2$, are homotopy 1-types (in fact they are equivalent to the classifying spaces of pure braid groups). But this implies that the slice fibration $\big((\mathbb{E}_2^\otimes)_{\mathrm{act}}\big)_{/1} \to (\mathbb{E}_2^\otimes)_{\mathrm{act}}$ has fibres equivalent to 1-types and therefore so has the pullback $\mathcal{U} \to N\mathrm{Ass}^\otimes_{\mathrm{act}}$. This shows that $\mathcal{U}$ is equivalent to the nerve of a 1-category U, since it is a Cartesian fibration over a 1-category whose fibres are 1-categories. This reduces the general case to the construction in 1-categories discussed at the beginning of this remark. With some more care one can also identity the universal category U and give an explicit combinatorial construction of the map from $\Delta^{\mathrm{op}} \to U$.[7].

Finally let us also note that there is a more direct ad hoc construction of Hochschild homology for an algebra A in a $\mathbb{E}_2$-monoidal ∞-category as follows: one just observes that one can use the braiding to turn a left $A \otimes A^{\mathrm{op}}$-module M into a right

[7]There is no extension to a map $\Lambda^{\mathrm{op}} \to U$ as in the symmetric monoidal case (see Section 5). As a result the Hochschild homology in a $\mathbb{E}_2$-category will not carry a circle action.

$(A \otimes A^{\mathrm{op}})^{\mathrm{op}} = A \otimes A^{\mathrm{op}}$-module. Therefore the term $A \otimes_{A \otimes A^{\mathrm{op}}} A$ makes sense and gives a definition of Hochschild homology. But for the cyclic structure one has to work harder, e.g. as explained above.

4. THH of Thom spectra

In this section we want to give a computation of $\mathrm{THH}(\mathbb{Z}/p^n)$ for all n and $\mathrm{THH}(\mathbb{Z})$. This will in particular prove Bökstedt's theorem. The key idea is to observe that $H\mathbb{Z}$ as well as $H\mathbb{Z}/p^n$ are generalized Thom spectra and give a general formula for THH of Thom spectra. This strategy and the general formula is due to Blumberg [Blu10] and Blumberg, Cohen and Schlichtkrull [BCS10]. Let us first state the final result about THH which is for $n = 1$ and $n = \infty$ due to Blumberg, Cohen and Schlichtkrull (in a slightly weaker form without ring structures) and for other n due to Nitu Kitchloo [Kit18].

Theorem 4.1. *We have the following equivalences as $\mathbb{E}_1$-ring spectra:*

1. $\mathrm{THH}(\mathbb{Z}/p) \simeq H\mathbb{F}_p \otimes \Omega S^3$ *for any p.*
2. $\mathrm{THH}(\mathbb{Z}/p^n) \simeq H\mathbb{Z}/p^n \otimes \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}/p^{n-1}, 2)\big)$ *for odd p.*
3. $\mathrm{THH}(\mathbb{Z}/2^n) \simeq H\mathbb{Z}/p^n \otimes \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}/p^{n-2}, 2)\big)$ *for $p = 2$ and $n \geq 3$.*
4. $\mathrm{THH}(\mathbb{Z}) \simeq H\mathbb{Z} \otimes \tau_{\geq 3}\Omega S^3$.

Here the maps $\Omega S^3 \to K(\mathbb{Z}/p^k, 2)$ are given by the Postnikov section $\Omega S^3 \to K(\mathbb{Z}, 2)$ followed by the projection $K(\mathbb{Z}, 2) \to K(\mathbb{Z}/p^k, 2)$ (it is in particular an $\mathbb{E}_1$-map) and the (homotopy) fibre is considered as an unbased space.

From this result the topological Hochschild homology of $\mathbb{Z}$ and $\mathbb{Z}/p^n$ can be computed by elementary methods. Bökstedt's result (Theorem 3.13)immediately follows since the homology of ΩS^3 is polynomial on a degree 2 generator. We will also demonstrate in the next Corollary that one can now simply use the Serre spectral sequence to understand $\mathrm{THH}(\mathbb{Z}/p^n)$ for higher n. The result is in the case $\mathbb{Z}/p^n$ due to Brun [Bru00] but without the exception of $\mathbb{Z}/4$.

Corollary 4.2. *We get the following homotopy groups for all $\mathbb{Z}/p^n$ except $\mathbb{Z}/4$:*

$$\mathrm{THH}_*(\mathbb{Z}/p^n) = \begin{cases} \bigoplus_{i=0}^k \mathbb{Z}/\gcd(i, p^n) & \textit{for } * = 2k \\ \bigoplus_{i=1}^k \mathbb{Z}/\gcd(i, p^n) & \textit{for } * = 2k-1 \\ 0 & \textit{for } * < 0 \end{cases}$$

$$\mathrm{THH}_*(\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}/k & \textit{for } * = 2k-1 \\ 0 & \textit{else} \end{cases}$$

Proof. We start with the case of the integers. We set

$$X = \tau_{\geq 3}\Omega S^3 = \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}, 2)\big)$$

and study the Serre spectral sequence associated to the fibre sequence

$$K(\mathbb{Z}, 1) \to X \to \Omega S^3$$

the E_2-page is $\mathbb{Z}[x] \otimes \Lambda[\varepsilon]$ and the d_2-differential is multiplicatively determined by $d_2(x) = \varepsilon$. This leads to the result.

$$
\begin{tikzcd}
1 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
0 & \mathbb{Z} & & \mathbb{Z} \arrow[ull, "\cdot 1"'] & & \mathbb{Z} \arrow[ull, "\cdot 2"'] & & \mathbb{Z} \arrow[ull, "\cdot 3"'] \\
& 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{tikzcd}
$$

For the case $\mathbb{Z}/p^n$ we set $d = 0$ if $n = 1$, and otherwise $d = n-1$ if p odd, $d = n-2$ if $p = 2$ (recall that we assume $n \geq 3$ if $p = 2$).

We define $X := \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}/p^d, 2)\big)$ and have a fibre sequence

$$K(\mathbb{Z}/p^d, 1) \to X \to \Omega S^3.$$

We first compute the cohomology of the space X with $\mathbb{Z}$ coefficients (recall that we want to compute the homology with $\mathbb{Z}/p^n$-coefficients). The Serre spectral sequence has E_2-page $\mathbb{Z}\langle y\rangle \otimes \mathbb{Z}[t]/p^d t$ with y in bidegree $(2,0)$ and t in bidegree $(0,2)$. For degree reasons there is no differential. But since $\pi_2(X) = p^d\mathbb{Z} \subseteq \pi_2(\Omega S^3)$ we get a non-trivial extension in total degree 2, i.e. that y, t can be represented by elements in the cohomology with $p^d t = y$. The multiplicative structure then resolves all the extension problems and we get

$$H^*(X, \mathbb{Z}) = \mathbb{Z}\langle y\rangle[t]/(y - p^d t).$$

$$
\begin{array}{c|ccccccc}
4 & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} \\
3 & & & & & & & \\
2 & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} & & \mathbb{Z}/p^{n-1} \\
1 & & & & & & & \\
0 & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z} \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

Note that for $d = 0$, i.e. $n = 1$, we can directly see from the spectral sequence that the cohomology is $\mathbb{Z}\langle y\rangle$. So we will assume $d > 0$. Observe also directly from the spectral sequence that the only possible torsion is p-torsion, so we can localize at p.

In $\mathbb{Z}_{(p)}\langle y\rangle[t]$, we can define an element

$$z_k = \sum_i (-1)^i \frac{p^{id} t^i}{i!} y^{(k-i)}$$

where the coefficients lie in $\mathbb{Z}_{(p)}$ since

$$v_p(i!) = \sum_{l \geq 1} \left\lfloor \frac{i}{p^l} \right\rfloor < \frac{i}{p-1} \leq id\ .$$

These z_k form a $\mathbb{Z}_{(p)}[t]$-basis for $\mathbb{Z}_{(p)}\langle y\rangle[t]$, as they are of the form $y^{(k)}$ modulo t. We have $z_1 = y - p^d t$, and

$$z_k z_l = \binom{k+l}{k} z_{k+l}$$

so they behave like divided powers on $y - p^d t$, and we will denote them by $z^{(k)}$. This allows us to identify $\mathbb{Z}_{(p)}\langle y\rangle[t]/(y-p^d t)$ with $\mathbb{Z}_{(p)}\langle z\rangle[t]/z$, which is the $\mathbb{Z}_{(p)}[t]$-module generated by 1 and the $p^{v_p(i)}$-torsion element $z^{(i)}$ in degree $2i$ for each $i \geq 1$.

Together with the fact that the homology groups have no torsion prime to p, we obtain:

$$H^*(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \bigoplus_{i=1}^k \mathbb{Z}/p^{v_p(i)} & \text{for } * = 2k \\ 0 & \text{else} \end{cases}$$

Finally the space X is of finite type, therefore we have a universal coefficient theorem which gives us that

$$H_*(X, \mathbb{Z}/p^n) = \mathrm{Hom}(H^*(X, \mathbb{Z}), \mathbb{Z}/p^n) \oplus \mathrm{Ext}(H^{*+1}(X, \mathbb{Z}), \mathbb{Z}/p^n)$$

which leads to the desired result. □

Remark 4.3. For $X = \mathrm{fib}(\Omega S^3 \to K(\mathbb{Z}/p^d, 2)$, we can see $H^*(X, \mathbb{Z}) = \mathbb{Z}\langle y\rangle[t]/(y - p^d t)$ more naturally from a $\mathbb{Z}$-coefficient Eilenberg-Moore spectral sequence.

We can describe X by the following homotopy pullback diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & K(\mathbb{Z}, 2) \arrow[d, "p^d"] \\ \Omega S^3 \arrow[r] & K(\mathbb{Z}, 2) \end{tikzcd}$$

So X can be written as totalization of the cosimplicial object $\Omega S^3 \times K(\mathbb{Z}, 2)^{\times \bullet} \times K(\mathbb{Z}, 2)$. Since $K(\mathbb{Z}, 2)$ is simply-connected, the usual convergence statement for the Eilenberg-Moore spectral sequence shows that this cosimplicial object gives rise to a spectral sequence converging to $H^*(X, \mathbb{Z})$, with E_1 page given by the chain complex with terms

$$H^*(\Omega S^3 \times K(\mathbb{Z}, 2)^{\times \bullet} \times K(\mathbb{Z}, 2), \mathbb{Z}) \ .$$

Since all the involved spaces have $\mathbb{Z}$-free cohomology of finite type, there is a $\mathbb{Z}$-coefficient Künneth theorem. We can thus identify the E_1-page with the Bar complex with terms

$$H^*(\Omega S^3, \mathbb{Z}) \otimes H^*(K(\mathbb{Z}, 2), \mathbb{Z})^{\otimes \bullet} \otimes H^*(K(\mathbb{Z}, 2), \mathbb{Z}),$$

such that the E_2 page coincides with

$$\mathrm{Tor}_*^{H^*(K(\mathbb{Z},2),\mathbb{Z})}(H^*(\Omega S^3, \mathbb{Z}), H^*(K(\mathbb{Z}, 2), \mathbb{Z})) = \mathrm{Tor}_*^{\mathbb{Z}[x]}(\mathbb{Z}\langle y\rangle, \mathbb{Z}[t]),$$

with x acting on $\mathbb{Z}\langle y\rangle$ by y, and on $\mathbb{Z}[t]$ by $p^d t$.

We can describe a 2-stage resolution of $Z\langle y\rangle$ by free $Z[x]$-modules as

$$\mathbb{Z}[a] \otimes \mathbb{Z}\langle y\rangle \xrightarrow{a} \mathbb{Z}[a] \otimes \mathbb{Z}\langle y\rangle,$$

where x acts by $a + y$. Tensoring with $\mathbb{Z}[t]$ over $\mathbb{Z}[x]$, we see that Tor_* is computed by the chain complex

$$\mathbb{Z}\langle y\rangle[t] \xrightarrow{p^d t - y} \mathbb{Z}\langle y\rangle[t] \ .$$

Since the differential is injective, $\mathrm{Tor}_0 = \mathbb{Z}\langle y\rangle[t]/(y-p^d t)$ and the higher Tor_i vanish.

Now in order to explain the proof of Theorem 4.1 we start by reviewing generalized Thom spectra. To this end, let R be an $\mathbb{E}_\infty$-ring spectrum and $\mathrm{BGL}_1(R)$ be the classifying spaces for invertible, locally trivial R-modules. This space is itself an $\mathbb{E}_\infty$-space. Now let f be a map $X \to \mathrm{BGL}_1(R)$ for a space X. Then this defines a diagram $X \to \mathrm{BGL}_1(R) \to \mathrm{Sp}$ and the Thom spectrum Mf is the colimit of this diagram. If X is an $\mathbb{E}_n$-space and the map f is $\mathbb{E}_n$, then the Thom spectrum Mf is an $\mathbb{E}_n$ ring spectrum. The most important example for us is the case that $X = BG$ for some $\mathbb{E}_1$-group G. Then a map $BG \to \mathrm{BGL}_1(R)$ in particular defines an action of G on R and the Thom spectrum is the homotopy quotient R_{hG}.

Lemma 4.4. *Let $f : X \to \mathrm{BGL}_1(R)$ be a $\mathbb{E}_2$-map with X connected. Then there is an equivalence of $\mathbb{E}_1$-R-algebras*

$$\mathrm{THH}(Mf/R) \simeq M\left(LBX \xrightarrow{h} \mathrm{BGL}_1(R)\right)$$

where h is some map of $\mathbb{E}_1$-spaces that depends on f.

Proof. The Thom spectrum functor $M : \mathcal{S}_{/\mathrm{BGL}_1(R)} \to \mathrm{Mod}_R$ preserves colimits and is symmetric monoidal. Thus it also preserves Hochschild homology. As a result we get that

$$\mathrm{THH}(Mf/R) \simeq M(\mathrm{HH}(X/\mathcal{S})).$$

In other words, we can take the cyclic bar construction of the $\mathbb{E}_2$-space X in the ∞-category $\mathcal{S}$ of spaces. This is by a theorem of Goodwillie-Jones (see Section 8 for a discussion) equivalent to the free loop space $LBX = \mathrm{Map}(S^1, BX)$, which is now only an $\mathbb{E}_1$-group, and comes with an *augmentation map* $h : LBX \to \mathrm{BGL}_1(R)$ since the whole cyclic bar construction takes place over $\mathrm{BGL}_1(R)$. In fact this is equivalent to the cyclic bar construction in spaces since the forgetful functor preserves colimits and tensor products. This augmentation map can by functoriality also be described as a composition

$$LBX \to LB^2\mathrm{GL}_1(R) \to \mathrm{BGL}_1(R)$$

where the first map is the induced map of free loop spaces (which is $\mathbb{E}_1$) and the second map is the augmentation map for the cyclic bar construction of the identity map $\mathrm{BGL}_1(R) \to \mathrm{BGL}_1(R)$. ☐

For what follows we need a slightly non-standard version of the Thom isomorphism. As before we let R be an $\mathbb{E}_\infty$ ring spectrum.

Lemma 4.5 (Thom isomorphism)**.** *Consider a commutative diagram of spaces*

$$\begin{tikzcd} Y \arrow[d, "g"] \arrow[dr, "fg"] & \\ X \arrow[r, "f"'] & B\mathrm{GL}_1 R \end{tikzcd}$$

such that Y is an $\mathbb{E}_n$-space over $B\mathrm{GL}_1 R$ (i.e. Y as well as the map $Y \xrightarrow{fg} B\mathrm{GL}_1 R$ are equipped with $\mathbb{E}_n$-structures), X is a grouplike $\mathbb{E}_{n+1}$-space over $B\mathrm{GL}_1 R$ and for the map $Y \xrightarrow{g} X$ the associated map of Thom spectra $MY \to MX$ refines to a map of $\mathbb{E}_n$-ring spectra where MX and MY inherit the $\mathbb{E}_n$-structures from the $\mathbb{E}_n$-structures on spaces.[8]

[8]Note that we do not require that the map $Y \to X$ is itself a map of $\mathbb{E}_n$-spaces.

Then there is an equivalence of $\mathbb{E}_n$ *ring spectra*

$$MX \otimes MY \xrightarrow{\simeq} MX \otimes Y$$

Proof. Consider the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\Delta"] \arrow[dr, "fg"'] & Y \times Y \arrow[d, "{(fg,0)}"] \\ & B\mathrm{GL}_1 R \end{tikzcd}.$$

Applying the Thom spectrum functor, we get a map

$$MY \to MY \otimes Y$$

which is $\mathbb{E}_n$, since both fg and 0 are $\mathbb{E}_n$-maps.

We can postcompose with the map $MY \xrightarrow{Mg} MX$, which is $\mathbb{E}_n$ by assumption, to obtain a map $MY \to MX \otimes Y$. Now we induce this to an MX-module map. More precisely, we form the composite

$$MX \otimes MY \to MX \otimes MX \otimes Y \xrightarrow{\mu \otimes \mathrm{id}} MX \otimes Y.$$

This is still an $\mathbb{E}_n$-map, since MX was assumed to be $\mathbb{E}_{n+1}$, such that its multiplication is an $\mathbb{E}_n$-map.

Finally, the map is an equivalence. To see this, it is sufficient to check that the corresponding map on spaces,

$$X \times Y \xrightarrow{\mathrm{id}\times\Delta} X \times Y \times Y \xrightarrow{\mathrm{id}\times g\times\mathrm{id}} X \times X \times Y \xrightarrow{\mu\otimes\mathrm{id}} X \times Y,$$

is an equivalence. An explicit inverse is given by the map

$$X \times Y \xrightarrow{\mathrm{id}\times\Delta} X \times Y \times Y \xrightarrow{\mathrm{id}\times(-g)\times\mathrm{id}} X \times X \times Y \xrightarrow{\mu\otimes\mathrm{id}} X \times Y$$

with g replaced by $(-g)$, which means $g : Y \to X$ postcomposed with the inversion map $X \to X$. □

Lemma 4.6. *Let R be an $\mathbb{E}_{n+2}$ ring spectrum (with $n = \infty$ allowed). Then there is a canonical map* $\mathrm{THH}(R) \to R$ *of* $\mathbb{E}_n$ *ring spectra.*[9]

Proof. We observe that if R is an $\mathbb{E}_2$-ring spectrum then by [Lur16, Theorem 2.2.2.4(3)] the slice ∞-category $\mathrm{Sp}_{/R}$ admits an $\mathbb{E}_2$-monoidal structure such that the functor $\mathrm{Sp}_{/R} \to \mathrm{Sp}$ is $\mathbb{E}_2$-monoidal. Moreover $\mathrm{id} : R \to R$ is an $\mathbb{E}_2$-algebra object in the slice and we can form the Hochschild homology in this ∞-category (using Remark 3.15) and get a spectrum $\mathrm{HH}\big((R \to R)\,/(\mathrm{Sp}_{/R})\big)$ over R. But after forgetting the augmentation this spectrum is equivalent to $\mathrm{HH}(R/\mathrm{Sp}) = \mathrm{THH}(R)$ since the forgetful functor$\mathrm{Sp}_{/R} \to \mathrm{Sp}$ preserves colimits and is $\mathbb{E}_2$-monoidal. This gives us the map in the case $n = 0$. For higher values of n we observe that an $\mathbb{E}_{n+2}$ ring spectrum is the same as an $\mathbb{E}_2$-algebra in the symmetric monoidal category of $\mathbb{E}_n$-ring spectra with 'pointwise' tensor product by Dunn additivity. Then we repeat the argument from above in the category of $\mathbb{E}_n$-ring spectra an get a map $\mathrm{HH}(R\,/\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})) \to R$ of $\mathbb{E}_n$-ring spectra. But we claim that $\mathrm{HH}(R\,/\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp}))$ is equivalent to $\mathrm{THH}(R)$. This follows since the forgetful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Sp})_{/R} \to \mathrm{Sp}$ is $\mathbb{E}_2$-monoidal and preserves sifted colimits. □

[9]Note that in contrast to the situation for $n = \infty$ the map $\mathrm{THH}(R) \to R$ is for $n < \infty$ not compatible (at least not obviously) with the circle action on $\mathrm{THH}(R)$ that will be discussed later and the trivial action on R.

The following result is a version of a result of Blumberg [Blu10, Theorem 7.5], see Remark 4.8 below.

Proposition 4.7. *Let $f : X \to \mathrm{BGL}_1(R)$ be an $\mathbb{E}_2$-map with X connected and assume that the $\mathbb{E}_2$-R-algebra structure on Mf extends to an $\mathbb{E}_3$-R-algebra structure (not necessarily coming from a space level structure on X). Then there is an equivalence of $\mathbb{E}_1$-R-algebras*

$$\mathrm{THH}(Mf_{/R}) \simeq Mf \otimes BX$$

Proof. According to Lemma 4.4 we know that

$$\mathrm{THH}(Mf_{/R}) \simeq M(LBX)$$

for some map $h : LBX \to B\mathrm{GL}_1(R)$. The free loop space receives maps

$$X \simeq \Omega BX \to LBX \tag{1}$$

$$BX \to LBX, \tag{2}$$

defined as the inclusion maps of based loops and constant loops respectively. All these spaces and maps inherit at least an $\mathbb{E}_1$ structure from the $\mathbb{E}_1$-structure on BX.

The composite defined by $X \times BX \to LBX \times LBX \xrightarrow{\mu} LBX$ is an equivalence: On homotopy groups, we have a natural fibre sequence $X \to LBX \to BX$, which is split by the constant loop map $BX \to LBX$. This shows that on homotopy groups, the map $X \times BX \to LBX$ defined above induces the isomorphism $\pi_n(X) \times \pi_n(BX) \to \pi_n(LBX)$ obtained from the split short exact sequence

$$0 \to \pi_n(X) \to \pi_n(LBX) \to \pi_n(BX) \to 0 .$$

Note that the map $X \times BX \to LBX$ defined above is not $\mathbb{E}_1$, since the map μ is not. However, by assumption, after applying the Thom spectrum functor, the induced equivalence

$$Mf \otimes M(BX \to B\mathrm{GL}_1 R) \to \mathrm{THH}(Mf_{/R}) \otimes \mathrm{THH}(Mf_{/R}) \xrightarrow{\mu} \mathrm{THH}(Mf_{/R})$$

is $\mathbb{E}_1$, since by assumption the $\mathbb{E}_2$-structure on Mf extends to an $\mathbb{E}_3$-structure, so the $\mathbb{E}_1$-structure on $\mathrm{THH}(Mf_{/R})$ extends to an $\mathbb{E}_2$-structure, so the multiplication map on $\mathrm{THH}(Mf_{/R})$ is $\mathbb{E}_1$.

Now the map $BX \to B\mathrm{GL}_1 R$ appearing in the above description is determined as the composite in the following commutative diagram:

$$\begin{tikzcd} BX \arrow[r] \arrow[dr, "g"'] & LBX \arrow[r] \arrow[d] & LB^2\mathrm{GL}_1 R \arrow[d] \\ & X \arrow[r, "f"] & B\mathrm{GL}_1 R \end{tikzcd}$$

where the vertical maps are the bottom cell maps guaranteed by Lemma 4.6. (This map $LBX \to X$ does not agree in general with the first coordinate of the equivalence $LBX \simeq X \times BX$ given above!)

Since X is $\mathbb{E}_2$, BX is $\mathbb{E}_1$, and the maps to $B\mathrm{GL}_1 R$ in the diagram above are compatible with that structure (for BX, this is seen from the composite along the top), we only need to check that g becomes $\mathbb{E}_1$ on Thom spectra. We will then be able to apply Lemma 4.5 to see that there is an $\mathbb{E}_1$-equivalence $Mf \otimes BX \to Mf \otimes M(BX \to B\mathrm{GL}_1)$ and finish the proof.

But this is clear: The map g is the composite of an $\mathbb{E}_1$ map and the bottom cell map $LBX \to X$. Since X is only $\mathbb{E}_2$, that map is not $\mathbb{E}_1$, but after applying the Thom spectrum functor, it becomes the bottom cell map $\mathrm{THH}(Mf_{/R}) \to Mf$. Since the $\mathbb{E}_2$-structure of Mf extends to an $\mathbb{E}_3$-structure, that map is $\mathbb{E}_1$. □

Remark 4.8. In [Blu10, Theorem 7.5] a weaker version of the last result is proven, namely that there is only an equivalence $\mathrm{THH}(Mf_{/R}) \simeq Mf \otimes BX$ of spectra and not of ring spectra. But for the computation of $\mathrm{THH}_*(\mathbb{F}_p)$ as a ring it is of course essentially that we have an equivalence of ring spectra.

Proof of Theorem 4.1. The result follows from Lemma 4.7 together with the statement that, for p odd or for $p = 2$ and $n = 0$, $H\mathbb{Z}/p^n$ is a Thom spectrum of an $\mathbb{E}_2$-map

$$\mathrm{fib}\big(\Omega^2 S^3 \to K(\mathbb{Z}/p^{n-1}, 1)\big) \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \ ,$$

while for $p = 2$ and $n \geq 3$, $H\mathbb{Z}/2^n$ is a Thom spectrum of an $\mathbb{E}_2$-map

$$\mathrm{fib}\big(\Omega^2 S^3 \to K(\mathbb{Z}/p^{n-2}, 1)\big) \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \ .$$

This will be proven in Section A. We get that THH of $H\mathbb{Z}/p^n$ relative to the p-complete sphere is given by

$$H\mathbb{Z}/p^n \otimes \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}/p^{n-1}, 2)\big) \ ,$$

or

$$H\mathbb{Z}/2^n \otimes \mathrm{fib}\big(\Omega S^3 \to K(\mathbb{Z}/2^{n-2}, 2)\big)$$

for $p = 2$ and $n \geq 3$. But THH relative to the p-complete sphere has the same p-completion as THH relative to the non-complete sphere as is obvious from the definition (e.g. by reducing mod p). Since $\mathrm{THH}(\mathbb{Z}/p^n)$ is a $H\mathbb{Z}/p^n$-module it is automatically p-complete, which proves the result.

For the case $H\mathbb{Z}$ we note that $H\mathbb{Z}$ is a Thom spectrum of an $\mathbb{E}_2$-map

$$\tau_{\geq 2}\Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S})$$

and thus the result immediately follows from Lemma 4.7. □

Remark 4.9. By using a modified version of the same methods, it is possible to obtain Brun's result for $\mathbb{Z}/4$ as well. The issue in finding a Thom spectrum description of $H\mathbb{Z}/4$ was that for any $(1 + 2u) \in \mathbb{Z}_2^\times$, with u odd, the square $(1 + 2u)^2 = 1 + 4u(u + 1)$ is congruent 1 modulo 8, so the associated Thom spectrum on the double cover $\mathrm{fib}(\Omega^2 S^3) \to K(\mathbb{Z}/2, 1)$ is of the form $H\mathbb{Z}/2^n$ for n at least 3.

To solve this problem, we pass to an etale extension $S_2^\wedge[\zeta_3]$ of the 2-adic sphere, where ζ_3 is a third root of unity. The analogue of Theorem A.3 for an $\mathbb{E}_2$-map $\Omega^2 S^3 \to B\mathrm{GL}_1(S_2^\wedge[\zeta_3])$ still holds. For the map represented by the element $(1 + 2\zeta_3) \in B\mathrm{GL}_1(S_2^\wedge[\zeta_3])$, it shows that the Thom spectrum of $\mathrm{fib}(\Omega^2 S^3 \to K(\mathbb{Z}/2, 1)$ is $H\mathbb{Z}/4[\zeta_3]$, since $(1 + 2\zeta_3)^2 = 1 - 4$.

As in the proof of Theorem 4.1, this implies that

$$\mathrm{THH}(\mathbb{Z}/4[\zeta_3]_{/S_2^\wedge[\zeta_3]}) = H\mathbb{Z}/4[\zeta_3] \otimes \mathrm{fib}(\Omega^2 S^3 \to K(\mathbb{Z}/2, 1)),$$

and since $H\mathbb{Z}/4[\zeta_3]$ is as an $S_2^\wedge[\zeta_3]$-algebra just induced up from the $S_2^\wedge$-algebra $H\mathbb{Z}/4$, the left hand side agrees with

$$S_2^\wedge[\zeta_3] \otimes \mathrm{THH}(\mathbb{Z}/4) \ .$$

As the equivalence obtained from the Thom isomorphism is not Galois invariant, it isn't clear whether this descends to an equivalence between $\mathrm{THH}(H\mathbb{Z}/4)$ and

$H\mathbb{Z}/4 \otimes \mathrm{fib}(\Omega^2 S^3 \to K(\mathbb{Z}/2,1)$. However, since all the homotopy groups are finitely generated abelian groups, we can determine the isomorphism type of $\pi_*\mathrm{THH}(\mathbb{Z}/4)$ from the groups

$$\pi_*(S_2^\wedge[\zeta_3] \otimes \mathrm{THH}(\mathbb{Z}/4)) = \pi_*(\mathrm{THH}(\mathbb{Z}/4)) \oplus \pi_*(\mathrm{THH}(\mathbb{Z}/4))$$

simply by counting terms in a decomposition into cyclic summands.

5. Topological periodic homology

Recall that there was a module structure of $A = \mathbb{Z}[b]/b^2$ on $\mathrm{HH}(R)$ for every ring R. We can consider A as an object in $\mathrm{Alg}(\mathcal{D}\mathbb{Z})$, i.e. an algebraic in the derived ∞-category of $\mathbb{Z}$. Then the A-action on $\mathrm{HH}(R)$ is the same as a $\mathbb{T} := U(1)$-action on $\mathrm{HH}(R)$ in the following sense.

Proposition 5.1. *There is an equivalence of ∞-categories between the ∞-category of modules over A in $\mathcal{D}\mathbb{Z}$ and objects in $\mathcal{D}\mathbb{Z}$ with a $\mathbb{T}$-action, that is the functor ∞-category $\mathrm{Fun}(B\mathbb{T}, \mathcal{D}\mathbb{Z})$.*

Proof. The ∞-category $\mathrm{Fun}(B\mathbb{T}, \mathcal{D}\mathbb{Z})$ is equivalent to the category of modules over the algebra $C_*(\mathbb{T}, \mathbb{Z})$ in $\mathcal{D}(\mathbb{Z})$. Thus the result follows by comparing $C_*(\mathbb{T}, \mathbb{Z})$ with $A = H_*(\mathbb{T}, \mathbb{Z})$. The two are equivalent by formality of the latter over $\mathbb{Z}$ (which is not quite as easy as over $\mathbb{Q}$). □

Warning 5.2. *Both ∞-categories have canonical symmetric monoidal strucutre, namely the tensor product coming from the cocommutative Hopf-algebra structure on A and the pointwise tensor product of chain complexes with an action of $\mathbb{T}$. But the equivalence of ∞-categories is not compatible with symmetric monoidal structures since $C_*(\mathbb{T}, \mathbb{Z})$ is not formal as a co-$\mathbb{E}_\infty$-Hopf algebra. As a result we can not really use the purely algebraic description to see the $\mathbb{E}_\infty$-structure on the chain complexes $\mathrm{HC}^-(R)$ and $\mathrm{HP}(R)$ properly. For the ring structures on the homology groups $\mathrm{HC}^-_*(R)$ and $\mathrm{HP}_*(R)$ it is however good enough.*

It turns out that under this equivalence there is a more general instance of this $\mathbb{T}$-action on the cyclic bar construction.

Proposition 5.3. *For every $R \in \mathrm{Alg}(\mathcal{C})$ the Hochschild homology $\mathrm{HH}(R/\mathcal{C})$ admits a natural $\mathbb{T}$-action which is compatible with the lax symmetric monoidal structure on the functor $\mathrm{HH}(-/\mathcal{C})$.*

Proof. This follows from the following two facts:

- First the diagram $\Delta^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ extends to a cyclic diagram $\Lambda^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ where Λ is Connes' cyclic category.
- Secondly for every such extension of a simplicial object to a cyclic object the geometric realization (aka Δ^{op}-indexed category) of a simplicial objects inherits an $\mathbb{T}$-action.

The first fact is a combinatorial exercise in the definition of Λ and the map $\Delta^{\mathrm{op}} \to \mathrm{Ass}^{\otimes}_{\mathrm{act}}$ and the second follows from the fact that the geometric realization of Λ is $\mathrm{B}\mathbb{T}$ (see [NS17, Appendix T] for details). □

In particular we can apply this to the ∞-category Sp and admit a $\mathbb{T}$-action on $\mathrm{THH}(R)$. This is the generalization of Connes' B-operator to homotopy-land. Now we want to reformulate the definition of HC and HC^- in terms of the $\mathbb{T}$-action. They become

$$\mathrm{HC}^-(R) = \mathrm{HH}(R)^{h\mathbb{T}} \qquad \text{and} \qquad \mathrm{HC}(R) = \mathrm{HH}(R)_{h\mathbb{T}}$$

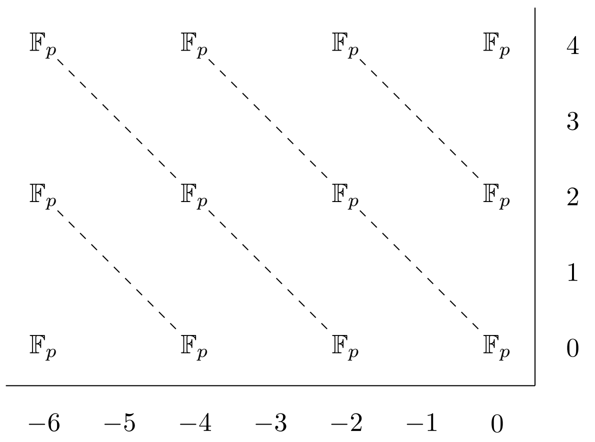
and the equivalence follows from 5.1 once one checks that the $\mathbb{T}$ actions from the cyclic realization corresponds to Connes' operator [Hoy15].

Definition 5.4. *We define for* $R \in \mathrm{Alg}(\mathrm{Sp})$ *the negative cyclic homology as* $\mathrm{TC}^-(R) := \mathrm{THH}(R)^{h\mathbb{T}}$.

We warn the reader that the topological cyclic homology $\mathrm{TC}(R)$ that we will define later is not equivalent to $\mathrm{THH}(R)_{h\mathbb{T}}$ and should also be considered as a variant of $\mathrm{HC}(R)$ but rather as a refinement of $\mathrm{TC}^-(R)$ taking into account Frobenius operators (this will be explained in the next talk). Also the notation $\mathrm{TC}^-(R)$ is not standard.

Proposition 5.5. *The graded ring* $\mathrm{TC}^-_*(\mathbb{F}_p)$ *is isomorphic to* $\mathbb{Z}_p[x,t]/(xt-p)$ *for* $|x| = 2$ *and* $|t| = -2$.

Proof. We consider the homotopy fixed point spectral sequence. Since everything is concentrated in even degrees it degenerates at E_2. The E_∞-term is given by $\mathbb{F}_p[x,t]$, but there are potential extension problems. We proceed as in the proof of Proposition 2.12 (but much easier). In order the get the desired result it suffices to check that $x, t \in E^\infty$ can be lifted to elements $x, t \in \mathrm{TC}^-_*(\mathbb{F}_p)$ such that $xt = p$. To see this we check that the image of $p \in \mathrm{TC}^-_0(\mathbb{F}_p)$ is detected on E_∞ by xt (by possibly changing x or t). By Proposition 3.11 this can be checked for $p \in \mathrm{HC}^-_0(\mathbb{F}_p)$ and is true by the computation of $\mathrm{HC}_0(\mathbb{F}_p)$ as in Proposition 2.12.



Alternatively one can also (as in the proof of Proposition 2.12) consider the mod p-reduction of $\mathrm{THH}(\mathbb{F}_p)/p$ which has an element in degree one whose image under B is the generator x. Chasing through the computation we get that the mod p-reduction of $\mathrm{TC}^-(\mathbb{F}_p)$ has homotopy groups $\mathbb{F}_p[x,t]/xt$. Since these are even there can not be any p-torsion in $\mathrm{TC}^-_*(\mathbb{F}_p)$ which also forces the extension behaviour. $\square$

Remark 5.6. With similar arguments one gets that for a perfect field k of characteristic p that $\mathrm{TC}^-(k) = W(k)[x,t]/(xt-p)$ where $W(k)$ is the ring of p-typicial Witt vectors of k.

Now we want to give an analogue of periodic homology. To this end we have to discuss Tate spectra. Let G be a finite group and X a spectrum with G-action.

Then there is an analogue in stable homotopy theory of the norm map from algebra, namely a map

$$X_{hG} \to X^{hG}$$

whose underlying map $X \to X$ is given by the sum $\sum_{g \in G} \rho_g$ where $\rho_g : X \to X$ is given by the action with g. This in fact works with any stable ∞-category (which has the necessary colimits and limits) instead of the ∞-category of spectra. For example $\mathcal{D}(\mathbb{Z})$ is allowed as well.

The Tate spectrum X^{tG} of G is defined as the cofibre of this map, in particular it comes with a canonical map

$$\mathrm{can} : X^{hG} \to X^{tG}$$

and a nullhomotopy of the composite $X_{hG} \to X^{hG} \to X^{tG}$. In fact there is a similar statement for compact Lie groups, in which case the norm map goes

$$(D_G \otimes X)_{hG} \to X^{hG}$$

where D_G is the dualizing spectrum, concretely $D_G = S^{\mathrm{Ad}_G}$ is the representation sphere of the adjoint representation of G. Abstractly the map $(D_G \otimes X)_{hG} \to X^{hG}$ can be described as the assembly map of the functor

$$-^{hG} : \mathrm{Sp}^{BG} \to \mathrm{Sp}$$

i.e. as the terminal functor $\mathrm{Sp}^G \to \mathrm{Sp}$ which preserves colimits equipped with a transformation to $-^{hG}$. For example in the case of $G = \mathbb{T}$ we find that $D_G = \Sigma\mathbb{S}$ with trivial $\mathbb{T}$-action. In this case we still define the Tate spectrum as the cofibre of the norm map

$$\Sigma X_{h\mathbb{T}} \to X^{h\mathbb{T}} \to X^{t\mathbb{T}}$$

and similarly for arbitrary G. We will need the following statement:

Proposition 5.7 ([NS17, Theorem I.3.1])**.** *The Tate functor* $\mathrm{Sp}^G \to \mathrm{Sp}$ *admits a unique lax symmetric monoidal structure such that the canonical map* $\mathrm{can} : (-)^{hG} \to (-)^{tG}$ *admits the structure of a lax symmetric monoidal transformation.*

Now we can proceed to our higher description of topological periodic homology. The relevant result is the following:

Proposition 5.8. *For a ring R the periodic homology* $\mathrm{HP}_*(R)$ *is isomorphic to the homology of the chain complex* $\mathrm{HH}(R)^{t\mathbb{T}}$.

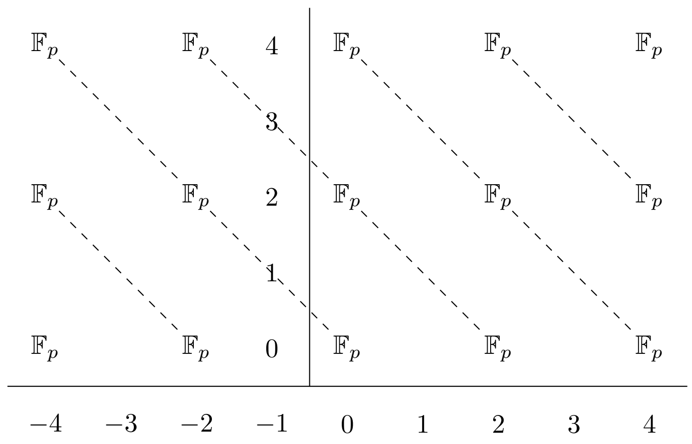
Proof. This follows from the more general fact that Tate spectra of complex oriented ring spectra with an action that preserves the complex orientation can be obtained from homotopy fixed points by inverting the 'Euler class'. The proof of this result uses equivariant homotopy theory and is beyond the scope of this lecture series. □

Definition 5.9. *For R a ring (spectrum) we define topological periodic homology* $\mathrm{TP}_*(R)$ *as homotopy groups of* $\mathrm{THH}(R)^{t\mathbb{T}}$.

Proposition 5.10. *For $R = \mathbb{F}_p$ we have that* $\mathrm{TP}_*(\mathbb{F}_p) \cong \mathbb{Z}_p[t^{\pm}]$ *and the canonical map induces on homotopy groups inversion of t:*

$$\mathrm{TC}^-(\mathbb{F}_p) = \mathbb{Z}_p[x,t]/(xt-p) \to \mathbb{Z}_p[x,t^{\pm}]/(xt-p) = \mathbb{Z}_p[t^{\pm}]$$

Proof. To see this we look at the Tate spectral sequence which is obtained from the Postnikov tower of $\mathrm{THH}(\mathbb{F}_p)$, degenerates at E_2 and has the same extension behavior as the homotopy fixed points spectral sequence since the canonical map $\mathrm{TC}^-(\mathbb{F}_p) \to \mathrm{TP}_*(\mathbb{F}_p)$ induces the obvious inclusion map on spectral sequences.



□

We warn the reader that the name topological periodic homology is a bit unfortunate since $\mathrm{TP}_*(R)$ is not always periodic. For example $\mathrm{TP}_*(\mathbb{Z})$ is not periodic at all. However the name is justified since it is a generalization of periodic homology and in many cases, for example for every ring of characteristic p, it is 2-periodic.

6. The Frobenius

Let us start with some elementary facts from algebra. If A is an abelian group we would like to have a diagonal map i.e. a map

$$A \to (A \otimes ... \otimes A)^{C_p}$$

where p is a prime, similar to the diagonal of sets. We define this map as sending a to $a \otimes ... \otimes a$. Unfortunately this assignment is not additive. But instead we can consider the quotient

$$(A \otimes ... \otimes A)^{C_p}/\text{norms}$$

where norms are elements in the image of the norm map which exists for every C_p-module M and is given by

$$M_{C_p} \to M^{C_p} \qquad m \mapsto \sum_{g \in C_p} gm\ .$$

This is the algebraic version of Tate cohomology, in fact it is isomorphism to the group $\hat{H}^0(C_p, A \otimes ... \otimes A)$.

Exercise 6.1. *Check that the composition* $\Delta_p : A \to (A \otimes ... \otimes A)^{C_p}/\text{norms}$ *given by* $a \mapsto [a \otimes ... \otimes a]$ *is a homomorphism of abelian groups. Moreover show that it exhibits for every abelian group* A *the group* $(A \otimes ... \otimes A)^{C_p}/\text{norms}$ *as the mod* p *reduction of* A.[10]

Now the analogous statements in homotopy theory are as follows:

Theorem 6.2 ([NS17, Theorem III.1.7 and Proposition III.3.1])**.** *For every spectrum* X *there is a map* $\Delta_p : X \to (X \otimes_{\mathbb{S}} ... \otimes_{\mathbb{S}} X)^{tC_p}$ *with the following properties:*

(1) *The map is natural in* X*, that is it extends to a natural transformation between functors* $\mathrm{Sp} \to \mathrm{Sp}$.

[10]By the latter we mean that the target is p-torsion and the induced map $A/p \to (A \otimes ... \otimes A)^{C_p}/\text{norms}$ is an isomorphism.

(2) *The transformation is symmetric monoidal.*

The transformation Δ_p is unique with respect to these properties. Moreover the map Δ_p exhibits for every bounded below spectrum X the spectrum $(X \otimes_{\mathbb{S}} ... \otimes_{\mathbb{S}} X)^{tC_p}$ as the p-completion of X.

We will refer to the map Δ_p as the Tate diagonal. It follows in particular from the fact that the transformation is symmetric monoidal that for an $\mathbb{E}_n$ (with $0 \leq n \leq \infty$) ring spectrum R that the map $R \to (R \otimes ... \otimes R)^{tC_p}$ is a map of $\mathbb{E}_n$-ring spectra.

Example 6.3. For $R = \mathbb{S}$ the map $\mathbb{S} \to (\mathbb{S} \otimes ... \otimes \mathbb{S})^{tC_p} = \mathbb{S}^{tC_p}$ is equivalent to the map obtained as the composition

$$\mathbb{S} \xrightarrow{p^*} \mathrm{map}(BC_p, \mathbb{S}) = \mathbb{S}^{hC_p} \to \mathbb{S}^{tC_p}$$

where $p : BC_p \to \mathrm{pt}$. The statement that this map is a p-completion is equivalent to the Segal conjecture for C_p (Lin's resp. Gunawardena's theorem). In this sense the second part of the last theorem generalizes the Segal conjecture.

Now we want to use these Tate diagonals to construct a certain extra structure on the spectra $\mathrm{THH}(R)$, namely Frobenius operators. We will do this for $\mathbb{E}_\infty$-ring spectra R as input. To this end we recall the following theorem.

Proposition 6.4 (McClure-Schwänzl-Vogt [MSV97])**.** *For an $\mathbb{E}_\infty$-ring spectrum R, the map $R \to \mathrm{THH}(R)$ is initial among all maps from R to an $\mathbb{E}_\infty$-ring spectrum equipped with a $\mathbb{T}$-action (through $\mathbb{E}_\infty$-maps). That is the map $R \to \mathrm{THH}(R)$ induces an equivalence*

$$\mathrm{Map}^{\mathbb{T}}_{\mathbb{E}_\infty}(\mathrm{THH}(R), A) \to \mathrm{Map}_{\mathbb{E}_\infty}(R, A)$$

for each $\mathbb{E}_\infty$-algebra A with a $\mathbb{T}$-action.

Proof. We will use the following two facts about $\mathbb{E}_\infty$-algebras

(1) The coproduct in the category of $\mathbb{E}_\infty$-algebras is the tensor product
(2) Filtered colimits of $\mathbb{E}_\infty$-algebras are formed underlying.

The initial object is by definition the colimit over S^1 of the constant diagram with value R in the category of $\mathbb{E}_\infty$-rings. Using the standard model $\Delta^1/\partial\Delta^1$ of the circle, with $n+1$-simplices in dimension n, we see that this colimit is equivalent to the colimit of the simplicial spectrum given in degree n by the $n+1$-fold coproduct of R with itself, which is $R^{\otimes n+1}$. But this is just the cyclic bar construction. □

Note that there is also an initial $\mathbb{E}_\infty$-ring with a map from R and a C_p-action, namely $R \otimes \ldots \otimes R$. As a result we get from the map $R \to \mathrm{THH}(R)$ a map $R \otimes \ldots \otimes R \to \mathrm{THH}(R)$ which is C_p-equivariant and $\mathbb{E}_\infty$. This map can also be explicitly expressed using a p-fold subdivision. We take the Tate C_p-spectrum on this map and obtain a map $(R \otimes \ldots \otimes R)^{tC_p} \to \mathrm{THH}(R)^{tC_p}$.

Corollary 6.5. *There is a unique $\mathbb{T}$-equivariant $\mathbb{E}_\infty$-map φ_p as in the diagram*

$$\begin{tikzcd} R \arrow[r] \arrow[d, "\Delta_p"] & \mathrm{THH}(R) \arrow[d, dashed, "\varphi_p"] \\ (R \otimes ... \otimes R)^{tC_p} \arrow[r] & \mathrm{THH}(R)^{tC_p} \end{tikzcd}$$

where the $\mathbb{T}$-action on $\mathrm{THH}(R)^{tC_p}$ is given under the isomorphism $\mathbb{T} \cong \mathbb{T}/C_p$ by the residual action.

Proof. This follows from the universal property of $\mathrm{THH}(R)$ (as stated in Proposition 6.4) together with the fact that $\mathrm{THH}(R)^{tC_p}$ admits a $\mathbb{T}/C_p = \mathbb{T}$-action. □

We refer to this map as the Frobenius of $\mathrm{THH}(R)$. In fact this equips $\mathrm{THH}(R)$ with the structure of a cyclotomic spectrum that we will study in the next section. It turns out that there is even a definition of a cyclotomic structure on $\mathrm{THH}(R)$ if R is not $\mathbb{E}_\infty$ but only $\mathbb{E}_1$. We only sketch the construction in the $\mathbb{E}_1$ case, details are given in [NS17, Section III.2]. The idea is to assemble the Tate diagonals into a map of cyclic objects. First of all we note that for every finite subgroup $G \subseteq \mathbb{T}$ we have a map

$$i_G : R^{\otimes G} \to \mathrm{THH}(R)$$

which is given by the inclusion into the colimit defining $\mathrm{THH}(R)$. Moreover this map is G-equivariant for the permutation action on the source and the restricted $\mathbb{T}$-action on the target. We will also need a parametrized version of the Tate diagonal given as follows: for a spectrum X and a group $G \subseteq \mathbb{T}$ we consider the subgroup $\widetilde{G} \subseteq \mathbb{T}$ given as the preimage of G under the p-fold power map $(-)^p : \mathbb{T} \to \mathbb{T}$. In particular we have $C_p \subseteq \widetilde{G}$ and $\widetilde{G}/C_p = G$. Now there is a variant of the Tate diagonal

$$\Delta_p : X^{\otimes G} \to \left(X^{\otimes \widetilde{G}}\right)^{tC_p}$$

which is G-equivariant for the permutation action on the source and the residual action on the target. Then for an $\mathbb{E}_1$-ring spectrum R the map $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ is defined such that for every $G \subseteq \mathbb{T}$ the diagram

$$\begin{tikzcd}
R^{\otimes G} \arrow[r, "i_G"] \arrow[d, "\Delta_p"] & \mathrm{THH}(R) \arrow[d, "\varphi_p"] \\
\left(R^{\otimes \widetilde{G}}\right)^{tC_p} \arrow[r, "i_{\widetilde{G}}^{tC_p}"] & \mathrm{THH}(R)^{tC_p}
\end{tikzcd}$$

commutes G-equivariantly. If we take the coherences for cyclic maps between different G into account properly this in fact leads to a definition of φ_p since $\mathrm{THH}(R)$ is the colimit over $R^{\otimes G}$ for varying G.

Since the map $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ is $\mathbb{T}$-equivariant we can lift it to a map $\mathrm{THH}(R)^{h\mathbb{T}} \to (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$. We want to re-express the target of this map slightly differently as follows. To this end we write the residual action on $\mathrm{THH}(R)^{tC_p}$ as an action of $\mathbb{T}/C_p$ which will later be identified with $\mathbb{T}$ by the p-th power map.

Lemma 6.6. [NS17, Lemma II.4.2] *Let X be a bounded below spectrum with $\mathbb{T}$-action. Then the canonical map*

$$X^{t\mathbb{T}} \to (X^{tC_p})^{h\mathbb{T}/C_p}$$

is a p-completion. If X is moreover p-complete then the map is an equivalence.

Proof. By a connectivity argument one sees that both sides commute with the limit of Postnikov towers. As a result we can assume that X is concentrated in a single degree (using that limits of p-complete objects are p-complete). Then X^{tC_p} is p-torsion and thus $(X^{tC_p})^{h\mathbb{T}}$ is p-complete. Resolving by a 2-term resolution we can assume that X is torsion free, i.e. of the form HM for M a torsion free abelian group (with trivial $\mathbb{T}$-action). Then

$$HM^{t\mathbb{T}} = \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i} HM \qquad \text{and} \qquad HM^{tC_p} = \bigoplus_{i\in\mathbb{Z}} \Sigma^{2i} H(M/p) \; .$$

i.e., $HM^{t\mathbb{T}}$ is just a periodized $HM^{h\mathbb{T}}$, and HM^{tC_p} is a periodized HM^{hC_p}.

We now compare the homotopy fixed point spectral sequence for $(HM^{hC_p})^{h\mathbb{T}}$ with the one for $(HM^{tC_p})^{hC_p}$.

$$\begin{array}{ccccc|c}
 & & & & & 2 \\
 & & & & & 1 \\
A & & A & & A & 0 \\
\hline
-4 & -3 & -2 & -1 & 0 & -1 \\
A/p & & A/p & & A/p & -2 \\
 & & & & & -3 \\
A/p & & A/p & & A/p & -4
\end{array}
\qquad
\begin{array}{ccccc|c}
A/p & & A/p & & A/p & 2 \\
 & & & & & 1 \\
A/p & & A/p & & A/p & 0 \\
\hline
-4 & -3 & -2 & -1 & 0 & -1 \\
A/p & & A/p & & A/p & -2 \\
 & & & & & -3 \\
A/p & & A/p & & A/p & -4
\end{array}$$

The first one has M in degree $(-2k, 0)$ for all $k \geq 0$, and M/p in degree $(-2k, -2l)$ for $k \geq 0$ and $l > 0$. Everything is in even degree, so all differentials are 0. Since we know $(HM^{hC_p})^{h\mathbb{T}} = HM^{h\mathbb{T}}$, we see that $\pi_{2k}((HM^{hC_p})^{h\mathbb{T}})$ is endowed with the filtration

$$p^k M \subseteq p^{k-1} M \subseteq \ldots \subseteq pM \subseteq M$$

in the spectral sequence. In particular, if we truncate the spectral sequence in horizontal degrees $< -2n$, we obtain a spectral sequence computing

$$\pi_i \mathrm{map}^{\mathbb{T}}(S^{2n+1}, HM^{hC_p})$$

to be M in even degrees $-2n \leq i \leq 0$, and M/p^{n+1} in even degrees $< -2n$. (Here we've used an explicit description of $E\mathbb{T}$ as the unit sphere in $\mathbb{C}^\infty$, and of the filtration giving rise to the homotopy fixed point spectral sequence as by unit spheres in $\mathbb{C}^n$)

Now the homotopy fixed point spectral sequence for $(HM^{tC_p})^{h\mathbb{T}}$ has M/p in degrees $(-2k, 2l)$ for $k \geq 0$ and any $l \in \mathbb{Z}$. So again all differentials are 0, and by comparing the truncated version to the truncated version for $(HM^{hC_p})^{h\mathbb{T}}$, we see that $\pi_i \mathrm{map}^{\mathbb{T}}(S^{2n+1}, HM^{tC_p})$ is M/p^{n+1} for every even i.

Since the filtration is complete, this shows that $(HM^{tC_p})^{h\mathbb{T}}$ has homotopy groups $M_p^\wedge$ in each even degree, and that the map

$$HM^{h\mathbb{T}} = (HM^{hC_p})^{h\mathbb{T}} \to (HM^{tC_p})^{h\mathbb{T}}$$

is the p-completion map $M \to M_p^\wedge$ in each negative even degree. The map

$$HM^{t\mathbb{T}} \to (HM^{tC_p})^{h\mathbb{T}}$$

is therefore the p-completion map in each even degree. $\square$

As a result we can consider the map $\mathrm{THH}(R)^{h\mathbb{T}} \to (\mathrm{THH}(R)^{tC_p})^{h\mathbb{T}}$ as a map into the p-completion of $\mathrm{THH}(R)^{t\mathbb{T}}$. If we do this for all primes at once we get a map

$$\varphi : \mathrm{THH}(R)^{h\mathbb{T}} \to \prod_p (\mathrm{THH}(R)^{t\mathbb{T}})^\wedge_p = (\mathrm{THH}(R)^{t\mathbb{T}})^\wedge$$

where the target is the profinite completion. Thus it represents a map $\varphi : \mathrm{TC}^-_*(R) \to \mathrm{TP}_*(R)^\wedge$.

Proposition 6.7. *The map $\varphi : \mathrm{THH}(\mathbb{F}_p)^{h\mathbb{T}} \to (\mathrm{THH}(\mathbb{F}_p)^{t\mathbb{T}})^\wedge$ is given on homotopy groups by the map*

$$\mathbb{Z}_p[t,x]/(tx-p) \to \mathbb{Z}_p[t^\pm]$$

with $x \mapsto \lambda t^{-1}$ and $t \mapsto p\lambda^{-1}t$ where $\lambda \in \mathbb{Z}_p^\times$ is a unit in $\mathbb{Z}_p$. In particular it is an isomorphism in positive degrees and the image in degree $-2n$ is given by $p^n\mathbb{Z}_p$.

Proof. The map has to send x to λt^{-1} and t to μt with $\lambda, \mu \in \mathbb{Z}_p$. The relation gives that $\lambda\mu = p$. Then we look at the commutative diagram

$$\begin{tikzcd}
\pi_{-2}\mathrm{THH}(\mathbb{F}_p)^{h\mathbb{T}} \arrow[r] \arrow[d] & \pi_{-2}\mathrm{THH}(\mathbb{F}_p)^{t\mathbb{T}} \arrow[r] \arrow[d] & \pi_{-2}H\mathbb{F}_p^{t\mathbb{T}} \arrow[d] \\
\pi_{-2}\mathrm{THH}(\mathbb{F}_p) \arrow[r] & \pi_{-2}\mathrm{THH}(\mathbb{F}_p)^{tC_p} \arrow[r] & \pi_{-2}H\mathbb{F}_p^{tC_p}
\end{tikzcd}.$$

Note that the lower left corner is equal to 0, while the clockwise composition

$$\pi_{-2}\mathrm{THH}(H\mathbb{F}_p)^{h\mathbb{T}} = \mathbb{Z}_p \cdot t \to \pi_{-2}H\mathbb{F}_p^{tC_p} \cong \mathbb{F}_p$$

sends t to the residue class $[\mu]$ in $\mathbb{F}_p$. Thus μ is divisible by p and thus $\lambda \cdot \frac{\mu}{p} = 1$ which shows the claim. ☐

We will later see that the unit λ is equal to 1.

Remark 6.8. For k perfect of characteristic p one finds that the map $\varphi : \mathrm{THH}(k)^{h\mathbb{T}} \to (\mathrm{THH}(k)^{t\mathbb{T}})^\wedge$ is given by the map

$$W(k)[x,t]/(xt-p) \to W(k)[t^\pm]$$

with $x \mapsto t^{-1}$ and $t \mapsto pt$ and on π_0 by the Witt vector Frobenius $W(k) \to W(k)$.

7. Topological cyclic homology

Recall that in the last section we have constructed for a ring R maps $\varphi_p : \mathrm{THH}(R) \to \mathrm{THH}(R)^{tC_p}$ which are $\mathbb{T}$-equivariant (and $\mathbb{E}_\infty$ if R was commutative).

Definition 7.1. *A cyclotomic structure on a spectrum X is given by a $\mathbb{T}$-action together with a $\mathbb{T}$-equivariant map $\varphi_p : X \to X^{tC_p}$ for every prime p, called the Frobenius.*

Example 7.2.
(1) For every ring (or ring spectrum) R the spectrum $\mathrm{THH}(R)$ is a cyclotomic spectrum.
(2) Every spectrum X can be considered as a cyclotomic spectrum with the trivial $\mathbb{T}$-action and the map $X \to X^{tC_p}$ for each p given by the composition

$$X \xrightarrow{p^*} \mathrm{map}(BC_p, X) = X^{hC_p} \to X^{tC_p}$$

which can be made $\mathbb{T}$-equivariant by noting that the map $X \xrightarrow{p^*} X^{hC_p}$ admits a canonical $\mathbb{T}$-equivariant refinement given by the factorization through

$(X^{hC_p})^{h\mathbb{T}} = X^{h\mathbb{T}}$ and the fact that $\mathbb{T}$-acts trivially on X. We will write this cyclotomic spectrum as X^{triv}.
(3) The cyclotomic spectra $\mathrm{THH}(\mathbb{S})$ and $\mathbb{S}^{\mathrm{triv}}$ are equivalent.

Recall that for a bounded below cyclotomic spectrum X we can combine all the Frobenii into a map $\varphi : X^{h\mathbb{T}} \to (X^{t\mathbb{T}})^\wedge$. There is of course another map given by the canonical map obtained in the definition of the Tate spectrum composed with the completion $\mathrm{can} : X^{h\mathbb{T}} \to X^{t\mathbb{T}} \to (X^{t\mathbb{T}})^\wedge$ which we will abusively also call the canonical map.

Definition 7.3. *For a bounded below cyclotomic spectrum X we define $\mathrm{TC}(X)$ as the equalizer in the diagram*

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} (X^{t\mathbb{T}})^\wedge \ .$$

For R a connective ring spectrum we will write $\mathrm{TC}(\mathrm{THH}(R))$ as $\mathrm{TC}(R)$ which is the equalizer

$$\mathrm{TC}(R) \longrightarrow \mathrm{TC}^-(R) \underset{\varphi}{\overset{\mathrm{can}}{\rightrightarrows}} \mathrm{TP}(R)^\wedge \ .$$

Note that if R is a commutative ring spectrum then $\mathrm{TC}(R)$ is a commutative ring spectrum as well since both maps can and φ are commutative ring maps.

Example 7.4. Let us compute $\mathrm{TC}_*(\mathbb{F}_p)$. By definition there is a long exact sequence

$$... \to \mathrm{TC}_i(\mathbb{F}_p) \to \mathrm{TC}_i^-(\mathbb{F}_p) \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}_i(\mathbb{F}_p) \to ...$$

We claim that the map $\mathrm{can} - \varphi$ is an isomorphism in all nonzero degrees. This follows from the fact that it is the difference of an isomorphism and a map which is divisible by p between two copies of $\mathbb{Z}_p$ (see Proposition 6.7). In degree 0 it is the zero map. Thus $\mathrm{TC}_*(\mathbb{F}_p)$ is only non-zero in degree 0 and -1 and in both cases given by $\mathbb{Z}_p$. Thus we get that

$$\mathrm{TC}_*(\mathbb{F}_p) = \mathbb{Z}_p[\epsilon]/\epsilon^2 \qquad |\epsilon| = 1 \ .$$

Remark 7.5. In fact using Galois descent one can identify the ring spectrum $\mathrm{TC}(\mathbb{F}_p)$ as an $\mathbb{E}_\infty$-ring with the cochains $C^*(S^1, \mathbb{Z}_p) \simeq \mathrm{map}(\Sigma^\infty_+ S^1, H\mathbb{Z}_p)$.

Remark 7.6. Similarly one finds that for a perfect field k of characteristic p that $\mathrm{TC}_*(k)$ only has two non-trivial homotopy groups: $\mathrm{TC}_0(k) = \mathbb{Z}_p$ and $\mathrm{TC}_{-1}(k)$ is the coequalizer of the maps $\varphi, \mathrm{id} : W(k) \to W(k)$. In particular if k is algebraically closed then $\mathrm{TC}_{-1}(k) = 0$.

Example 7.7. Let us consider the trivial cyclotomic spectrum $H\mathbb{Z}_p^{\mathrm{triv}}$ associated with the Eilenberg-MacLane spectrum $H\mathbb{Z}_p$ (see Example 7.2). Then $\pi_*(H\mathbb{Z}_p^{h\mathbb{T}}) = \mathbb{Z}_p[t]$ with $|t| = -2$ and $\pi_*(H\mathbb{Z}_p^{t\mathbb{T}}) = \mathbb{Z}_p[t^\pm]$. The map can sends t to t and the map φ sends t to pt. This determines both maps since they are multiplicative. We conclude that $\mathrm{can} - \varphi_p$ is an isomorphism in negative degrees and the zero map in degree 0. Thus we find that

$$\mathrm{TC}_*(H\mathbb{Z}_p^{\mathrm{triv}}) = \begin{cases} \mathbb{Z}_p & \text{for } * = -1, 0, 1, 3, 5, 7, \dots \\ 0 & \text{else} \end{cases} \ .$$

Theorem 7.8 ([NS17])**.** *This definition of $\mathrm{TC}(R)$ is for a connective ring spectrum R equivalent to the classical $\mathrm{TC}(R)$. More precisely to Goodwillie's integral TC based on the Bökstedt construction.*

Remark 7.9. For a not necessarily bounded below cyclotomic spectrum we define TC as the equalizer

$$\mathrm{TC}(X) \longrightarrow X^{h\mathbb{T}} \underset{\prod \varphi_p}{\overset{\mathrm{can}}{\rightrightarrows}} \prod (X^{tC_p})^{h\mathbb{T}} \ .$$

By Lemma 6.6 this agrees with Definition 7.3 in the bounded below case. It does however not agree with the classical definition of TC in terms of equivariant spectra. Therefore we will restrict to the connective case here. We are not aware that TC for non-connective ring spectra has any significant applications.

Recall that a cyclotomic spectrum is a spectrum X with $\mathbb{T}$-action and a $\mathbb{T}$-equivariant Frobenius for each prime p. There is an obvious ∞-category of cyclotomic spectra CycSp defined as the pullback

$$\begin{tikzcd}
\mathrm{CycSp} \arrow[r] \arrow[d] & \prod_p \mathrm{Fun}(B\mathbb{T},\mathrm{Sp})^{\Delta^1} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\mathrm{Fun}(B\mathbb{T},\mathrm{Sp}) \arrow[r, "\prod(\mathrm{id},(-)^{tC_p})"] & \prod_p \Big(\mathrm{Fun}(B\mathbb{T},\mathrm{Sp}) \times \mathrm{Fun}(B\mathbb{T},\mathrm{Sp})\Big)
\end{tikzcd}$$

It is a stable ∞-category and inherits in a canonical way a symmetric monoidal structure. In particular for every pair of objects (X,Y) in CycSp there is a mapping spectrum $\mathrm{map}(X,Y)$. The idea for the following description of TC is originally due to Kaledin and has first been implemented (at least after p-completion) by Blumberg-Mandell with a totally different definition of cyclotomic spectrum.

Proposition 7.10. *For a cyclotomic spectrum X we have an equivalence* $\mathrm{TC}(X) \simeq \mathrm{map}(\mathbb{S}, X)$*, i.e.* TC *is corepresentable by the sphere. In particular we have an adjunction*

$$(-)^{\mathrm{triv}} \colon \mathrm{Sp} \rightleftarrows \mathrm{CycSp} \colon \mathrm{TC} \ .$$

Proof. In such a pullback of ∞-categories we get that the mapping spectra are pullbacks itself. In other words, for cyclotomic spectra (W,ψ_p) and (X,φ_p) the mapping spectrum in CycSp is given by the pullback

$$\begin{tikzcd}
\mathrm{map}\big((W,\psi_p),(X,\varphi_p)\big) \arrow[r] \arrow[d] & \prod_p \mathrm{map}(\psi_p,\varphi_p) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\
\mathrm{map}(W,X) \arrow[r, "\prod(\mathrm{id},(-)^{tC_p})"] & \prod_p \mathrm{map}(W,X) \times \mathrm{map}(W^{tC_p},X^{tC_p})\ .
\end{tikzcd}$$

The right vertical map in this diagram is the equalizer of the two obvious maps

$$\prod_p \mathrm{map}(W,X) \times \mathrm{map}(W^{tC_p},X^{tC_p}) \underset{(\varphi_p)_*}{\overset{(\psi_p)^*}{\rightrightarrows}} \prod_p \mathrm{map}(W,X^{tC_p}) \ .$$

Thus we get an equalizer diagram

$$\mathrm{map}\big((W,\psi_p),(X,\varphi_p)\big) \longrightarrow \mathrm{map}(W,X) \underset{(\varphi_p)^*}{\overset{(\psi_p)_*}{\rightrightarrows}} \prod_p \mathrm{map}(W,X^{tC_p}).$$

We now specialize this formula to the case $(W,\psi_p) = \mathbb{S}^{\mathrm{triv}}$ and get the result using the equivalence $\mathrm{map}_{\mathrm{Fun}(B\mathbb{T},\mathrm{Sp})}(\mathbb{S},X) = X^{h\mathbb{T}}$. The second statement follows from the first by noting that $X^{\mathrm{triv}} \simeq X \otimes \mathbb{S}^{\mathrm{triv}}$ (where the tensor is the tensor over Sp which every stable ∞-category with all colimits has). $\square$

Now we finally want to finish our computation of $\mathrm{THH}(\mathbb{F}_p)$ with all the relevant structure. Note that the computation of $\mathrm{TC}(\mathbb{F}_p)$ in particular yields that $\mathrm{THH}(\mathbb{F}_p)$ is a $H\mathbb{Z}$ algebra spectrum through the map $H\mathbb{Z} \to H\mathbb{Z}_p = \tau_{\geq 0}\mathrm{TC}(\mathbb{F}_p)$. The $H\mathbb{Z}$-action is compatible with the $\mathbb{T}$-action and the Frobenius $\varphi_p : \mathrm{THH}(\mathbb{F}_p) \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$. More precisely by the adjunction of Proposition 7.10 we get a map

$$H\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$$

of cyclotomic spectra.

Construction 7.11. Consider the full subcategory $\mathrm{CycSp}^{\geq 0} \subseteq \mathrm{CycSp}$ of connective cyclotomic spectra. By 'connective' for a cyclotomic spectrum $(X, (\varphi_p)_{p\in\mathbb{P}})$ we mean that the underlying spectrum X is connective. Then the maps φ_p factor as maps $X \to \tau_{\geq 0}X^{tC_p} \to X^{tC_p}$ and we will abusively also denote the first map by φ_p. Let us denote by F the endofunctor

$$F : \mathrm{Sp}^{\geq 0} \to \mathrm{Sp}^{\geq 0} \qquad X \mapsto \tau_{\geq 0}(X^{tC_p}) \ .$$

For a given connective cyclotomic spectrum (X, φ_p) we define a new cyclotomic spectrum $\mathrm{sh}_p X$ whose underlying object is $F(X)$ and whose p-Frobenius is

$$FX \xrightarrow{F(\varphi_p)} F^2X \ .$$

The other Frobenii are all trivial since the spectrum itself is p-complete. This defines an endofunctor

$$\mathrm{sh}_p : \mathrm{CycSp}^{\geq 0} \to \mathrm{CycSp}^{\geq 0}.$$

The functor sh_p comes with a natural transformation $\mathrm{id} \to \mathrm{sh}_p$, given on underlying objects by the Frobenius $X \xrightarrow{\varphi_p} F(X)$.

Theorem 7.12. *Consider the diagram induced from applying* sh_p *to the map* $H\mathbb{Z}^{\mathrm{triv}} \to \mathrm{THH}(\mathbb{F}_p)$ *of cyclotomic spectra*

$$\begin{tikzcd}
H\mathbb{Z}^{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{THH}(\mathbb{F}_p) \arrow[d, "\simeq"] \\
\mathrm{sh}_p H\mathbb{Z}^{\mathrm{triv}} \arrow[r, "\simeq"] & \mathrm{sh}_p \mathrm{THH}(\mathbb{F}_p) \ .
\end{tikzcd}$$

The right and the lower map are equivalences of $\mathbb{E}_\infty$*-cyclotomic spectra, in particular* $\mathrm{THH}(\mathbb{F}_p) \simeq \mathrm{sh}_p H\mathbb{Z}^{\mathrm{triv}}$.

Proof. Lets compute the homotopy groups of the respective spectra. First for $\mathrm{sh}_p H\mathbb{Z}_p^{\mathrm{triv}}$ the underlying spectrum is by definition given by $\tau_{\geq 0} H\mathbb{Z}_p^{tC_p}$. The spectrum $H\mathbb{Z}_p^{tC_p}$ has homotopy groups $\mathbb{F}_p[t^{\pm}]$ with $|t| = 2$. The spectrum in the lower right is $\tau_{\geq 0}\mathrm{THH}(\mathbb{F}_p)^{tC_p}$. We claim that the homotopy groups of $\mathrm{THH}(\mathbb{F}_p)^{tC_p}$ are given by $\mathbb{F}_p[t^{\pm}]$. Since the lower horizontal map is given by the connected cover of a ring map $H\mathbb{Z}_p^{tC_p} \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ this implies that it has to be an isomorphism on homotopy groups (indeed every graded ring endomorphism of $\mathbb{F}_p[t^{\pm}]$ is an isomorphism).

To analyse the Tate construction $\mathrm{THH}(\mathbb{F}_p)^{tC_p}$ we use the fact that for every $H\mathbb{Z}$-module X with an $\mathbb{T}$-action (i.e. functor $B\mathbb{T} \to \mathrm{Mod}_{H\mathbb{Z}}$) there is an equivalence

$X^{t\mathbb{T}}/p = X^{t\mathbb{T}} \otimes_{H\mathbb{Z}^{tC_p}} H\mathbb{F}_p^{tC_p} \simeq X^{tC_p}$. This follows either by an analysis of Euler classes or by observing that both sides commute with colimits in X and thus reducing to $X = H\mathbb{Z}$ in which case it is obvious (see [NS17, LEM XXX]).

Now we apply this formula to compute $\mathrm{THH}(\mathbb{F}_p)^{tC_p} = \mathrm{THH}(\mathbb{F}_p)^{t\mathbb{T}}/p$ and get

$$\pi_*(\mathrm{THH}(\mathbb{F}_p)^{tC_p}) \simeq \mathrm{TP}_*(\mathbb{F}_p)/p = \mathbb{F}_p[t^{\pm}] \ .$$

Finally we note that the map $\varphi_p : \mathrm{THH}\mathbb{F}_p \to \mathrm{THH}(\mathbb{F}_p)^{tC_p}$ is given on homotopy groups by sending $x \in \pi_2\mathrm{THH}(\mathbb{F}_p)$ to $t \in \pi_2\mathrm{THH}(\mathbb{F}_p)^{tC_p}$ (up to a unit) which follows by comparison with Proposition 6.7. This implies that the right hand map in the diagram of the statement is an equivalence. □

In particular we get that the underlying spectrum $\mathrm{THH}(\mathbb{F}_p)$ is as an $\mathbb{E}_\infty$-spectrum with $\mathbb{T}$-action equivalent to $\tau_{\geq 0}H\mathbb{Z}^{tC_p}$. The cyclotomic structure map is given by

$$\tau_{\geq 0}H\mathbb{Z}^{tC_p} \to (\tau_{\geq 0}H\mathbb{Z}^{tC_p})^{tC_p}$$

and is an equivalence on connective covers (this is not entirely obvious from the above description).

There is a similar result for perfect rings R of characteristic p.[11] To understand this result we have to introduce a bit more notation. Before, we have considered the cyclotomic spectrum HR^{triv} for every ring R (or more generally every spectrum in place of HR). Now if R comes equipped with an endomorphism $f : R \to R$ we can tweak this to obtain a cyclotomic spectrum $(HR, f)^{\mathrm{triv}}$ where the Frobenii are given by $HR \xrightarrow{f} HR \xrightarrow{\mathrm{triv}} HR^{tC_p}$.

Theorem 7.13. *Let R be a commutative and perfect ring of characteristic p. There is a map of cyclotomic spectra $(HW(R), \varphi_p)^{\mathrm{triv}} \to \mathrm{THH}(R)$. In the resulting diagram*

$$\begin{tikzcd}
(HW(R), \varphi_p)^{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{THH}(R) \arrow[d, "\simeq"] \\
\mathrm{sh}_p(HW(R), \varphi_p)^{\mathrm{triv}} \arrow[r, "\simeq"] & \mathrm{sh}_p\mathrm{THH}(R)
\end{tikzcd}$$

the right and the lower map are equivalences of $\mathbb{E}_\infty$-cyclotomic spectra, i.e. $\mathrm{THH}(R) \simeq \mathrm{sh}_p(HW(R), \varphi_p)^{\mathrm{triv}}$.

Proof. Similarly to the case of $\mathbb{F}_p$ one computes that $\tau_{\geq 0}\mathrm{TC}(R) = \mathbb{Z}_p$. This gives a map of $\mathbb{E}_\infty$-rings $H\mathbb{Z}_p \to \mathrm{THH}(R)^{h\mathbb{T}}$. We want to extend this to a map

$$HW(R) \to \mathrm{THH}(R)^{h\mathbb{T}} = \mathrm{TC}^-(R)$$

First one computes that $\mathrm{TC}_0^-(R) = W(R)$ and so the extension exists on the level of π_0. Since $H\mathbb{Z}_p \to HW(R)$ is formally étale all the $\mathbb{E}_\infty$-obstructions vanish and we get a unique extension of $\mathbb{E}_\infty$-rings. Similarly one sees that on π_0 the Frobenius has the desired description and thus concludes from the compatibility with the 'trivial' Frobenius on $H\mathbb{Z}_p$ and the same obstruction theory the fact that there is a map of cyclotomic spectra $(HW(R), \varphi_p)^{\mathrm{triv}} \to \mathrm{THH}(R)$. Now the rest proceeds exactly as in Theorem 7.13. □

[11] We thank Dustin Claussen for pointing this out to us.

8. Loop spaces

In this section we want to explain how to compute TC for spherical group rings $\mathbb{S}[G]$ where G is a group object in the ∞-category of spaces. Every such group object is of the form $G = \Omega Y$ for some pointed, connected space Y. Then K-theory of the ring $\mathbb{S}[G]$ is equivalent to Waldhausen A-theory of the space Y. This is how this computation is relevant in geometric topology. But since we do not talk about K-theory in this note we just concentrate on the TC-computation.

We first recall and reprove the classical and well-known result that we have an equivalence

$$\mathrm{THH}(\mathbb{S}[G]) \simeq \Sigma^\infty_+ LBG$$

for G a group object in the ∞-category of spaces. Under this equivalence the $\mathbb{T}$-action on THH corresponds to the rotation $\mathbb{T}$-action of the circle. The Frobenii of the cyclotomic spectrum $\Sigma^\infty LBG$ correspond to the maps

$$\Sigma^\infty_+ LBG \xrightarrow{\psi_p} \Sigma^\infty_+ LBG^{hC_p} \xrightarrow{\mathrm{can}} \Sigma^\infty_+ LBG^{tC_p} \tag{3}$$

where ψ_p are the maps that are given by precomposing a loop with the p-th power map on S^1. These are $\mathbb{T}$-equivariant. For a direct proof of this fact along the lines of the classical proofs see [NS17, Section IV.3]. Here will prove a slightly more general result about $\mathbb{S}[M]$ for M a monoid object in the ∞-category of spaces. Every such monoid can be delooped to an ∞-category BM with a single object and M as the space of endomorphisms.

For any natural number $n \geq 1$ we consider the category S_n which has n objects $\{0, 1, \ldots, n-1\}$ and morphisms generated by morphisms $i \to i+1$ for every $0 \leq i < n$ where $i+1$ is taken modulo n.

Proposition 8.1. *There is a canonical map from the cyclic Bar construction associated with M in* $\mathrm{Alg}(\mathcal{S})$ *to the (cyclic) object*

$$k \mapsto \mathrm{Fun}(S_k, BM)^\sim$$

which induces an equivalence on geometric realizations.

Proof. The object $\mathrm{Fun}(S_k, BM)^\sim$ is equivalent to the homotopy quotient of M^k by the componentwise action of the grouplike monoid $(M^\times)^k$ of invertible elements in M. In fact the whole diagram $k \mapsto \mathrm{Fun}(S_k, BM)^\sim$ is equivalent to the homotopy quotient of the cyclic Bar construction of M by the monoid

$$\cdots \; M^\times \times M^\times \times M^\times \; M^\times \times M^\times \; M^\times$$

which is codiscrete, i.e. the maps are given by the projections. The point is, that the geometric realization of this simplicial (or cyclic) monoid is trivial. This implies the result since we can first realize and then take the homotopy quotient. □

The statement about the cyclic Bar construction for G now immediately follows since in this case the diagram $k \mapsto \mathrm{Fun}(S_k, BG)^\sim$ is equivalent to the constant diagram

$$k \mapsto \mathrm{Fun}\left(|S_k|, BG\right)^\sim = LBG$$

and the claim follows. To identify the Frobenius an extra argument is needed, see [NS17, Section IV.3]. Also note that the description of Proposition 8.1 immediately generalizes to any ∞-category $\mathcal{C}$ in place of BG and thus gives a definition of Hochschild homology for ∞-categories.

Definition 8.2. *We say that a cyclotomic spectrum $\big(X, (\varphi_p)_{p\in\mathbb{P}}\big)$ admits Frobenius lifts if there are $\mathbb{T}$-equivariant lifts of φ_p to maps*

$$\psi_p : X \to X^{hC_p}$$

which commute for different p, that is there are specified homotopies filling the diagrams

$$\begin{tikzcd}
X \arrow[rr, "\psi_{p_i}"] \arrow[d, "\psi_{p_{i+1}}"'] & & X^{hC_{p_i}} \arrow[d, "(\psi_{p_{i+1}})^{hC_{p_i}}"] \\
X^{hC_{p_{i+1}}} \arrow[r, "(\psi_{p_{i+1}})^{hC_{p_{i+1}}}"'] & \left(X^{hC_{p_i}}\right)^{hC_{p_{i+1}}} \arrow[r, "\simeq"] & \left(X^{hC_{p_{i+1}}}\right)^{hC_{p_i}}
\end{tikzcd}$$

for every $i \geq 1$ where $(p_1, p_2, p_3, \ldots)$ is the list of primes.

Note that we only require the homotopies for adjacent primes to avoid having to give homotopy coherences. But these of course induced homotopies for all pairs of primes by composition. The commutativity of the maps ψ_p implies that for the subgroup $C_n \subseteq \mathbb{T}$ we have a well-defined map

$$\psi_n : X \to X^{hC_n} \tag{4}$$

defined using a decomposition of n into prime factors and the composing the respective ψ_p's.

Example 8.3.
1. The cyclotomic spectrum $\Sigma^\infty_+ LBG$ admits canonical Frobenius lifts by the construction of φ_p in (3). These also evidently commute.
2. More generally for every spherical monoid ring $\mathbb{S}[M]$ associated with an $\mathbb{E}_1$-space M the spectrum $\mathrm{THH}\big(\mathbb{S}[M]\big)$ has canonical Frobenius lifts. The key observation to show this is to observe that in this case the Tate diagonal
$$\Delta_p : \mathbb{S}[M] \to (\mathbb{S}[M] \otimes \ldots \otimes \mathbb{S}[M])^{tC_p}$$
can be lifted through
$$\big(\mathbb{S}[M] \otimes \ldots \otimes \mathbb{S}[M]\big)^{hC_p} = \mathbb{S}[M \times \ldots \times M]^{hC_p}$$
by the space level diagonal $M \to (M \times \ldots \times M)^{hC_p}$. Then one gets the Frobenius lifts ψ_p directly from the construction of φ_p. We refer to [NS17] for details.
3. For every spectrum X the cyclotomic spectrum X^{triv} admits Frobenius lifts, since φ_p was defined as a composite $X \to X^{hC_p} \to X^{tC_p}$ and these maps also clearly commute.

Now let $\big(X, (\varphi_p)_{p\in\mathbb{P}}\big)$ be a cyclotomic spectrum with Frobenius lifts ψ_p. We try to compute $\mathrm{TC}(X)$. We assume that X is bounded below and p-complete to concentrate on a single prime. We get a lift of the map $\varphi : X^{h\mathbb{T}} \to X^{t\mathbb{T}}$ to a map $\psi : X^{h\mathbb{T}} \to X^{h\mathbb{T}}$ by applying $\mathbb{T}$-homotopy fixed points to ψ_p. By definition $\mathrm{TC}(X)$ is given by the outer pullback in the commutative diagram

$$\begin{tikzcd}
\mathrm{TC}(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[r] \arrow[d] & 0 \arrow[d] \\
X^{h\mathbb{T}} \arrow[r, "\psi-\mathrm{id}"] & X^{h\mathbb{T}} \arrow[r, "\mathrm{can}"] & X^{t\mathbb{T}}
\end{tikzcd}$$

since the lower horizontal composition is given by $\varphi - \mathrm{can}$. The right hand square is obtained as a pullback by definition of the Tate construction for the circle. Thus the left diagram is a pullback as well. Now we also have the diagram

$$\begin{tikzcd} X^{h\mathbb{T}} \arrow[r, "\psi-\mathrm{id}"] \arrow[d] & X^{h\mathbb{T}} \arrow[d] \\ X \arrow[r, "\psi_p-\mathrm{id}"] & X \end{tikzcd}$$

where we have abusively denoted the composition $X \xrightarrow{\psi_p} X^{hC_p} \to X$ by ψ_p as well. The key fact is now that this square is also a pullback in the p-complete world, see [NS17, Lemma IV.3.5]. Thus we have proven the following result.

Theorem 8.4. *For a bounded below, p-complete cyclotomic spectrum X with Frobenius lifts we have a pullback square*

$$\begin{tikzcd} \mathrm{TC}(X) \arrow[r] \arrow[d] & \Sigma X_{h\mathbb{T}} \arrow[d, "\mathrm{tr}"] \\ X \arrow[r, "\psi_p-\mathrm{id}"] & X \end{tikzcd} .$$

Corollary 8.5. *For every p-complete spectrum X we find that*

$$\mathrm{TC}_*(X^{\mathrm{triv}}) = X_* \oplus X_{*-1}\left(\mathbb{C}P^\infty_{-1}\right)$$

Proof. The map $\psi_p : X \to X$ is the identity map. Thus the lower horizontal map in the diagram above is given by the zero map and it follows that $\mathrm{TC}(X^{\mathrm{triv}})$ is equivalent to the sum of X and the fibre of the transfer map $\Sigma X_{h\mathbb{T}} \to X$. This fibre is given by the tensor product of X and the fibre of the transfer $\Sigma\mathbb{S}_{h\mathbb{T}} \to \mathbb{S}$ since the transfer and taking fibres both commute with colimits and desuspensions. But this fibre is by Definition $\Sigma\mathbb{C}P^\infty_{-1}$. □

Note that the last corollary in particular applies to $\mathbb{S}_p^{\mathrm{triv}} = \mathrm{THH}(\mathbb{S}_p)$ and thus computes TC of the sphere which is one of the first TC-computations that were done historically, see [BHM93]. Note also that for $H\mathbb{Z}_p^{\mathrm{triv}}$ we get that $(H\mathbb{Z}_p)_*(\mathbb{C}P^\infty_{-1}) = \mathbb{Z}_p$ for $* = -2, 0, 2, 4, \ldots$ and zero else. Thus we easily recover Example 7.7.

The following result has been shown by Bökstedt, Carlsson, Cohen, Goodwillie, Hsiang and Madsen in [BCC+96, Proposition 3.9] for simply connected spaces using an intricate Goodwillie calculus argument. We give an elementary argument for a slightly stronger result.

Proposition 8.6. *For Y a connected space with $\pi_1(Y)$ a p-group, the associated diagram*

$$\begin{tikzcd} \Sigma^\infty_+ LY \arrow[r, "\mathrm{id}-\psi_p"] \arrow[d, "\mathrm{ev}_1"] & \Sigma^\infty_+ LY \arrow[d, "\mathrm{ev}_1"] \\ \Sigma^\infty_+ Y \arrow[r, "0=\mathrm{id}-\mathrm{id}"] & \Sigma^\infty_+ Y \end{tikzcd}$$

is a pullback after p-completion.

Proof. Since all terms are connective, it is enough to check that the induced map on horizontal cofibres is an $\mathbb{F}_p$-homology equivalence. The cofibre of $\mathrm{id} - \psi_p$ on LY coincides with homotopy orbits $(LY)_{h\mathbb{N}}$ under the $\mathbb{N}$-action by ψ_p. This agrees with

$(\psi_p^{-1}LY)_{h\mathbb{Z}}$, i.e. $\mathbb{Z}$-homotopy orbits on the telescope formed from LY by inverting ψ_p. Similarly, the cofibre of the lower row agrees with $Y_{h\mathbb{Z}}$, and since taking homotopy orbits preserves homology equivalences, it is enough to check that the map $\psi_p^{-1}LY \to Y$ is an equivalence on $\mathbb{F}_p$-homology. We check this on the level of spaces. The filtered colimit of fiber sequences

$$\begin{tikzcd}
\Omega Y \arrow[r, "\psi_p"] \arrow[d] & \Omega Y \arrow[r, "\psi_p"] \arrow[d] & \Omega Y \arrow[r] \arrow[d] & \dots \\
LY \arrow[r, "\psi_p"] \arrow[d] & LY \arrow[r, "\psi_p"] \arrow[d] & LY \arrow[r] \arrow[d] & \dots \\
Y \arrow[r, "\mathrm{id}"] & Y \arrow[r, "\mathrm{id}"] & Y \arrow[r] & \dots
\end{tikzcd}$$

yields a fiber sequence

$$\psi_p^{-1}\Omega Y \to \psi_p^{-1}LY \to Y \ .$$

Note that ψ_p acts on $\pi_0\Omega Y = \pi_1(X)$ by the p-th power map. Since $\pi_1(Y)$ is a p-group, i.e. every element is p-torsion, the colimit $\psi_p^{-1}\Omega Y$ is connected, and agrees with $\psi_p^{-1}\Omega\widetilde{Y}$, the colimit on the unit connected component of ΩY.

Loop composition endows $H_*(\Omega\widetilde{Y};\mathbb{F}_p)$ with the structure of a connected Hopf algebra. Since ψ_p can be written as the composite

$$\Omega\widetilde{Y} \xrightarrow{\Delta} (\Omega\widetilde{Y})^{\times p} \xrightarrow{\mu} \Omega\widetilde{Y}$$

with μ the loop composition map, we can write $(\psi_p)_*$ as the p-fold comultiplication map followed by the multiplication map. On every fixed degree, this acts nilpotently (see below). Therefore the $\mathbb{F}_p$-homology of $\psi_p^{-1}\Omega\widetilde{Y}$ vanishes. Now the Serre spectral sequence shows that the map $\psi_p^{-1}LY \to Y$ is an $\mathbb{F}_p$-homology equivalence.

To see that ψ_p acts nilpotently on the homology Hopf algebra $H_*(\Omega\widetilde{Y};\mathbb{F}_p)$, it is easier to argue in the dual Hopf algebra $H^*(\Omega\widetilde{Y};\mathbb{F}_p)$. This is graded-commutative, so the map $(\psi_p)^*$, given by the p-fold comultiplication map followed by the multiplication map, is multiplicative. Since we are in characteristic p, it sends every element to a decomposable element, from which nilpotence follows by induction. $\square$

Note that the condition that $\pi_1(Y)$ is a p-group (in the sense that every element is p-power torsion) is also necessary for the conclusion of Proposition 8.6 to hold. This can be verified by computing π_0 of the horizontal cofibres.

Corollary 8.7. *For Y a connected space with $\pi_1(Y)$ a p-group there is an equivalence*

$$\mathrm{TC}\left(\mathbb{S}[\Omega Y]\right) \simeq \Sigma^\infty_+ Y \oplus \mathrm{fib}\left(\Sigma(\Sigma^\infty_+ LY)_{h\mathbb{T}} \xrightarrow{\mathrm{tr}} \Sigma^\infty_+ LY \xrightarrow{\mathrm{ev}_1} \Sigma^\infty_+ Y\right)$$

Proof. By pasting the pullbacks of Theorem 8.4 and Proposition 8.6 together we obtain a pullback square

$$\begin{tikzcd}
\mathrm{TC}(\mathbb{S}[\Omega Y]) \arrow[r] \arrow[d, "\mathrm{ev}_1"] & \Sigma(\Sigma^\infty_+ LY)_{h\mathbb{T}} \arrow[d, "\mathrm{ev}_1 \circ \mathrm{tr}"] \\
\Sigma^\infty_+ Y \arrow[r, "0"] & \Sigma^\infty_+ Y
\end{tikzcd}$$

which implies the claim. $\square$

9. The relation to Witt vectors

We will now indicate how to compare our notion of cyclotomic spectrum to the classical one. Let us introduce further spectra obtained from the cyclotomic spectrum $\mathrm{THH}(R)$ which were relevant in the equivariant approach. We do this more generally for every p-cyclotomic spectrum X by which we mean a spectrum X with $\mathbb{T}$-action and a $\mathbb{T}$-equivariant 'Frobenius' map $\varphi_p : X \to X^{tC_p}$. [12]

Definition 9.1. *We define spectra $X^{C_{p^n}}$, called the fixed points of X, as the iterated pullback*

$$X^{hC_{p^n}} \times_{\left(X^{tC_p}\right)^{hC_{p^{n-1}}}} X^{hC_{p^{n-1}}} \times \ldots \times_{X^{tC_p}} X \ ,$$

where in each of these pullbacks, the map from the left factor is given by the canonical projection from fixed points to Tate and the map from the right factor is induced by the Frobenius φ_p.

The next proposition explains the relation to the classical equivariant definition of p-cyclotomic spectra and is the only point in this paper where we really need genuine equivariant homotopy theory. We refer the reader to [Sch16] for an introduction. The next statement is not relevant for the rest of the text and can be safely omitted.

Proposition 9.2. *Assume that X is a bounded below p-cyclotomic spectrum. Then the spectra $X^{C_{p^n}}$ are fixed points of a genuine $\mathbb{T}$-spectrum with respect to the family of finite p-subgroups of $\mathbb{T}$.*

Proof. We denote the ∞-category of genuine $\mathbb{T}$-spectra with respect to the family of finite p-subgroups of $\mathbb{T}$ as $\mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p}$. We use the following two adjunctions (with the upper arrows being the left adjoints):

$$\mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p} \underset{B}{\overset{U}{\rightleftarrows}} \mathrm{Sp}^{B\mathbb{T}}$$

$$\mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p} \underset{R}{\overset{\Phi^{C_p}}{\rightleftarrows}} \mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p}$$

where U takes the underlying spectrum of a genuine spectrum, BX is the Borel complete spectrum with $BX^G = BX^{hG}$ and Φ^{C_p} is the geometric fixed points functor. R is the right adjoint to the geometric fixed points which has the effect that

$$(RX)^{C_{p^n}} = X^{C_{p^{n-1}}}.$$

Now the key Lemma that one has to prove is that for a bounded below spectrum with $\mathbb{T}$-action (in the naive sense) the spectrum $(BX)^{\Phi C_p}$ is equivalent to $B(X^{tC_p})$. This is equivalent to the Tate-Orbit Lemma of [NS17, Lemma I.2.1], see also [NS17, Lemma II.6.1] and will not be proven here.

Now for a p-cyclotomic spectrum X (in the sense as defined above) with map $\varphi_p : X \to X^{tC_p}$ we define a genuine $\mathbb{T}$-spectrum $\mathbb{X}$ as the iterated pullback

$$\mathbb{X} := BX \times_{RB(X^{tC_p})} RBX \times_{R^2B(X^{tC_p})} R^2BX \times_{R^3B(X^{tC_p})} \cdots$$

in $\mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p}$ where maps to the right are given by the unit of the adjunction $\Phi^{C_p} \dashv R$ (using that $B(X^{tC_p}) = \Phi^{C_p}BX$) and the maps to left are induced by the Frobenius

[12]Note that in [NS17, Definition II.1.1] a p-cyclotomic spectrum X is a only required to have a C_{p^∞}-action and an C_{p^∞}-equivariant map $X \to X^{tC_p}$. But for reasons that will become clear later we want to work with $\mathbb{T}$-actions here.

φ_p. Then it is straightforward to check that this spectrum has the desired properties using that taking fixed points commutes with all limits in $\mathbb{T}\mathrm{Sp}^{\mathrm{fin}_p}$. $\square$

In fact one can not only recover fixed points $X^{C_{p^n}}$ for p-subgroups of $\mathbb{T}$ but for all finite subgroups. To explain the construction we need the following result.

Lemma 9.3. *For primes $p \neq l$ and X a spectrum with $C_{p^n} \times C_l$-action we have*

$$\left(X^{tC_p}\right)^{tC_l} \simeq \left(X^{tC_l}\right)^{tC_p} \simeq 0$$

and the canonical map

$$\left(X^{hC_{p^n}}\right)^{tC_l} \to \left(X^{tC_l}\right)^{hC_{p^n}}$$

is an equivalence.

Proof. To see that $\left(X^{tC_l}\right)^{tC_p} \simeq 0$, note that l acts invertibly on X^{tC_p}, since it is a module over the p-local $\mathbb{S}^{tC_p}$. In particular, X^{tC_p} is an $\mathbb{S}[1/l]$-module. But $\mathbb{S}[1/l]^{tC_l} = 0$, so

$$(X^{tC_p})^{tC_l} \simeq 0,$$

and analogously $(X^{tC_l})^{tC_p} \simeq 0$.

For the equivalence, note that we can write an arbitrary X as a pullback $X[1/p] \times_{X_{\mathbb{Q}}} X_{(p)}$, so it is sufficient to prove that the canonical map $(X^{hC_{p^k}})^{tC_l} \to (X^{tC_l})^{hC_{p^k}}$ is an equivalence under the assumption that either l or p acts invertibly on X. If l acts invertibly on X, it acts invertibly on $X^{hC_{p^k}}$, so both sides are modules over $S[1/l]^{tC_l} = 0$, thus zero. If p acts invertibly on X, it acts invertibly on X^{tC_l} as well. Now we use that on $S[1/p]$-modules, the functor $(-)^{hC_{p^k}}$ is isomorphic to a finite limit, for example $\lim_{(BC_{p^k})^{2n}}(-)$, with $(BC_{p^k})^{2n}$ the $2n$-skeleton of BC_{p^k} in the standard cell structure. But $(-)^{tC_l}$ commutes with finite limits. $\square$

Construction 9.4. Assume X is a spectrum with $\mathbb{T}$-action and cyclotomic structures for all primes in a set S of primes. Then for each $p \in S$ the spectrum $X^{C_{p^n}}$ is naturally l-cyclotomic for each $l \in S \setminus p$.

Proof. First we note that the spectra $X^{C_{p^n}}$ have a canonical $\mathbb{T}$-action since all the spectra in the defining pullback (see Definition 9.1) have a $\mathbb{T}$-action, where $\mathbb{T}$-acts on $\mathrm{THH}(R)^{hC_{p^n}}$ residually by identifying $\mathbb{T} = \mathbb{T}/C_{p^n}$. Moreover all the maps in the pullback are $\mathbb{T}$-equivariant.

Using that $(X^{C_l})^{tC_p} \simeq 0$ we see that the spectrum X^{tC_l} admits a (trivial) p-cyclotomic structure, such that the map $X \xrightarrow{\varphi_l} X^{tC_l}$ is a map of p-cyclotomic spectra. Using this we define the l-Frobenius on $X^{C_{p^n}}$ as the composite

$$X^{C_{p^n}} \xrightarrow{\varphi_l} (X^{tC_l})^{C_{p^n}} \simeq (X^{C_{p^n}})^{tC_l}.$$

where the latter equivalence follows from the definition of C_{p^n} and Lemma 9.3. $\square$

Definition 9.5. *Let X be a cyclotomic spectrum and $G \subseteq \mathbb{T}$ a finite subgroup (which is of course cyclic). We write G as $C_{p_1^{n_1}} \times \ldots \times C_{p_k^{n_k}}$ for primes $p_1, \ldots, p_k$ and define the G-fixed points as*

$$X^G := \left(\left(\left(X^{C_{p_1^{n_1}}}\right)^{C_{p_2^{n_2}}}\right)\cdots\right)^{C_{p_k^{n_k}}}.$$

Here we have used the remaining cyclotomic structures on the respective fixed points as constructed in Construction 9.4 and Definition 9.1.

Remark 9.6. An iterated application of Lemma 9.4 shows that the definition of fixed points is independent of the order of primes in the decomposition of G. Since we do not want to give a careful proof of this fact here we assume that the primes $p_1, \ldots, p_k$ are chosen in ascending order.

One can prove an analogue of Proposition 9.2: for every bounded below cyclotomic spectrum X the fixed points X^G are genuine fixed points of a $\mathbb{T}$-spectrum $\mathbb{X}$ which is genuine for the family of finite subgroups. This construction is the key to prove the fact that our category of cyclotomic spectra is equivalent to the old category of cyclotomic spectra using genuine homotopy theory as for example given in [BM15]. In fact the assignment $\big(X, (\varphi_p)_{p\in\mathbb{P}}\big) \mapsto \mathbb{X}$ defines an inverse to the obvious forgetful functor from the classical category to ours. We shall not prove this here and refer to [NS17, Sections II.5 and II.6].

We want to finish with a very beautiful result relating the fixed points of topological Hochschild homology to Witt vectors. For commutative rings this is due to Hesselholt-Madsen [HM97, Theorem 2.3 and Addendum 2.3] and for non-commutative rings and p-subgroups due to Hesselholt [Hes97, Theorem 2.2.9].

Theorem 9.7. *For every connective ring spectrum R and every $n \in \mathbb{N}_{>0}$ there is an isomorphism of abelian groups*

$$\pi_0(\mathrm{THH}(R)^{C_n}) \cong W_{\langle n\rangle}(\pi_0 R) \ .$$

This isomorphism is natural in R and compatible with the respective lax symmetric monoidal structures, in particular for commutative R it is an isomorphism of rings.

Here $W_{\langle n\rangle}(\pi_0 R)$ are the Witt vectors of the associative ring $\pi_0 R$ with respect to the truncation set $\langle n\rangle \subseteq \mathbb{N}_{>0}$ consisting of all divisors of n. For example for a commutative ring R the ring $W_{\langle p^n\rangle}(R)$ is the ring of p-typicial Witt vectors of length $n+1$. See Appendix B for a discussion of Witt vectors for non-commutative rings with respect to truncation sets $S \subseteq \mathbb{N}_{>0}$. Note that this implies in particular that for a connective ring spectrum R these groups only depend on $\pi_0(R)$.

The rest of the section is devoted to give a proof of Theorem 9.7. The idea is to exhibit as much structure on $\pi_0\mathrm{THH}(R)^{C_n}$ as possible which reflects the respective structure on the Witt vectors. For simplicity we treat the case $\pi_0\mathrm{THH}(R)^{C_{p^n}}$ and explain the modifications for the general case afterwards. This case is also probably the most important one as it connects to the p-typical Witt vectors. To this end recall that $\mathrm{THH}(R)^{C_{p^n}}$ is by definition given by

$$\mathrm{THH}(R)^{hC_{p^n}} \times_{\left(\mathrm{THH}(R)^{tC_p}\right)^{hC_{p^{n-1}}}} \mathrm{THH}(R)^{hC_{p^{n-1}}} \times \ldots \times_{\mathrm{THH}(R)^{tC_p}} \mathrm{THH}(R) \ .$$

The first piece of structure that we have are the *reduction* maps

$$R_{p^m}\colon \mathrm{THH}(R)^{C_{p^{n+m}}} \to \mathrm{THH}(R)^{C_{p^n}}$$

defined as the projection to the last $(n+1)$ factors. The *Frobenius* maps

$$F_{p^m}\colon \mathrm{THH}(R)^{C_{p^{n+m}}} \to \mathrm{THH}(R)^{C_{p^n}}$$

as defined as the projection to the first $(n+1)$ factors followed by forgetting part of the homotopy fixed points information. Finally we have the *Verschiebung* maps

$$V_{p^m}\colon \mathrm{THH}(R)^{C_{p^n}} \to \mathrm{THH}(R)^{C_{p^{n+m}}}$$

defined as the transfers into the first $(n+1)$ factors and the zero map on the remaining factors. We leave it as an exercise to see that this is a well-defined map directly using properties of the transfer. Alternatively, this map can be described as the genuine equivariant transfer of the spectrum $\mathbb{X}$ constructed in Proposition 9.2 which makes it evidently well-defined. One verifies the following properties:

Lemma 9.8. *For every ring spectrum R the map defined above satisfy the relations*

$$R_{p^m} R_{p^n} \simeq R_{p^{m+n}}, \quad F_{p^m} F_{p^n} \simeq F_{p^{m+n}}, \quad R_{p^m} F_{p^n} \simeq F_{p^n} R_{p^m}, \quad R_{p^m} V_{p^n} \simeq V_{p^n} R_{p^m}$$

and

$$F_{p^m} V_{p^m} \simeq \sum_{\sigma \in C_{p^m}} \sigma$$

where σ denotes the action by σ on the spectrum $\mathrm{THH}(R)^{C_{p^n}}$ which carries a natural $C_{p^m} = C_{p^{n+m}}/C_{p^n}$-action. □

Note that the C_{p^m}-action on $\mathrm{THH}(R)^{C_{p^n}}$ is the restriction of a $\mathbb{T}$-action, therefore it is trivial on homotopy groups and we get on the level of homotopy groups $F_{p^m} V_{p^m} = p^m$.

We finally have the *ghost map*

$$w : \mathrm{THH}(R)^{C_{p^n}} \to \prod_{k=0}^{n} \mathrm{THH}(R)$$

defined as the projection to the product followed by forgetting the homotopy fixed point information. Note that the k-th component of w can be written as $F_{p^k} R_{p^{n-k}}$. It satisfies according compatibilities with the Verschiebung, reduction and Frobenius maps which can be deduced from this formula.

The final piece of structure that we will need is the Teichmüller character. To construct this we want to first work with an abstract cyclotomic spectrum with Frobenius lifts as in Definition 8.2.

Construction 9.9. Let X be a cyclotomic spectrum with commuting Frobenius lifts $\psi_p : X \to X^{hC_p}$. Then there is a canonical map $\Psi_n : X \to X^{C_n}$ for every finite subgroup $C_n \subseteq \mathbb{T}$ such that the composition $X \xrightarrow{\Psi_n} X^{C_n} \to X^{hC_n}$ is given by the map ψ_n constructed in (4).

Proof. We construct Ψ_n by induction. First $\Psi_1 = \mathrm{id}$. Assume that Ψ_{p^n} has been constructed for some n. Then we define $\Psi_{p^{n+1}}$ as the map

$$X \to X^{C_{p^{n+1}}} = X^{hC_{p^{n+1}}} \times_{(X^{tC_p})^{hC_{p^n}}} X^{C_{p^n}}$$

given by Ψ_{p^n} in the second factor and $\psi_{p^{n+1}}$ in the first factor. These factor through the pullback since there is canonical equivalence $\varphi_p \circ \psi_{p^n} \simeq \mathrm{can} \circ \psi_{p^{n+1}}$ which follows immediately from the definition of ψ_{p^n}. For general subgroups of $\mathbb{T}$ we observe that the l-cyclotomic spectrum $X^{C_{p^n}}$ itselft has an l-Frobenius lift and thus admits maps Ψ_{l^n}. Then we define the map ψ_n for a decomposition $n = p_1^{n_1} \cdots p_k^{n_k}$ as the composition

$$X \xrightarrow{\Psi_{p_1^{n_1}}} X^{C_{p_1^{n_1}}} \xrightarrow{\Psi_{p_2^{n_2}}} \left(X^{C_{p_1^{n_1}}}\right)^{C_{p_2^{n_2}}} \xrightarrow{\Psi_{p_3^{n_3}}} \cdots \xrightarrow{\Psi_{p_k^{n_k}}} \left(\left(\left(X^{C_{p_1^{n_1}}}\right)^{C_{p_2^{n_2}}}\right)\cdots\right)^{C_{p_k^{n_k}}} .$$

which clearly has the required properties. □

Definition 9.10. *For an $\mathbb{E}_1$-ring spectrum R we define the* Teichmüller character
$$\tau_n \colon \Omega^\infty R \to \Omega^\infty \mathrm{THH}(R)^{C_n}$$
for every n as the multiplicative map adjoint to the composition
$$\mathbb{S}[\Omega^\infty R] \to \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big) \xrightarrow{\Psi_n} \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{C_n} \to \mathrm{THH}(R)^{C_n}$$
where $\Omega^\infty R$ is the multiplicative $\mathbb{E}_1$-space underlying R, which comes with a canonical $\mathbb{E}_1$ map $\mathbb{S}[\Omega^\infty R] \to R$, and Ψ_n is the map given in Construction 9.9 using that $\mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)$ has canonical Frobenius lifts (see Example 8.3).

Lemma 9.11. *We have that*
$$R_{p^m}\tau_{p^{n+m}} \simeq \tau_{p^n}, \qquad F_{p^m}\tau_{p^{n+m}} \simeq \tau_{p^n} \circ (-)^{p^m}$$
where $(-)^{p^m} : \Omega^\infty R \to \Omega^\infty R$ is the multiplicative p^m-th power map. In particular the k-component of the composition
$$\Omega^\infty R \xrightarrow{\tau_{p^n}} \Omega^\infty \mathrm{THH}(R)^{C_{p^n}} \xrightarrow{w} \prod_{k=0}^{n} \Omega^\infty \mathrm{THH}(R)$$
is given by the map $\Omega^\infty R \xrightarrow{(-)^{p^k}} \Omega^\infty R \to \Omega^\infty \mathrm{THH}(R)$.

Proof. It is enough to check both equivalences on the level of adjoints, thus we can reduce to the case $R = \mathbb{S}[\Omega^\infty R]$. We can also assume that $m = 1$ by iteratively applying the statement. Then the first equivalence immediately follows from the inductive definition of $\Psi_{p^{n+1}}$ in Construction 9.9. For the second equivalence we have to show that the diagram

$$\begin{tikzcd}
\mathbb{S}[\Omega^\infty R] \arrow[r, "\Psi_{p^{n+1}}"] \arrow[d, "(-)^p"] & \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{C_{p^{n+1}}} \arrow[r, "\simeq"] \arrow[d, "F_p"] & \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{hC_{p^{n+1}}} \times_{(\ldots)} \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{C_{p^n}} \arrow[d, "\mathrm{res} \times_{(\ldots)} F_p"] \\
\mathbb{S}[\Omega^\infty R] \arrow[r, "\Psi_{p^n}"] & \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{C_{p^n}} \arrow[r, "\simeq"] & \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{hC_{p^n}} \times_{(\ldots)} \mathrm{THH}\big(\mathbb{S}[\Omega^\infty R]\big)^{C_{p^{n-1}}}
\end{tikzcd}$$

commutes for every n (where we abusively write Ψ_{p^n} for the composition of Ψ_{p^n} with the bottom cell inclusion and where (...) is the Tate term that we suppress). The horizontal maps are given on the first factor by $\psi_{p^{n+1}}$ resp. ψ_{p^n} and on the second factor by Ψ_{p^n} resp. $\Psi_{p^{n-1}}$. We will provide equivalences $\mathrm{res} \circ \psi_{p^{n+1}} \simeq \psi_{p^n} \circ (-)^p$ for every n that are compatible under can and Frobenius maps φ_p. To obtain these, note that there are homotopy coherently commutative cubes, for each n:

$$\begin{tikzcd}
\mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_p} \arrow[rr, "\psi_{p^n}^{hC_p}"] \arrow[dr, "\psi_{p^{n-1}}^{hC_p}"] \arrow[dd, "\mathrm{res}"] & & \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_{p^{n+1}}} \arrow[dr, "\mathrm{can}"] \arrow[dd, "\mathrm{res}"] & \\
& \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_{p^n}} \arrow[rr, "\varphi_p^{hC_p^n}"] \arrow[dd, "\mathrm{res}"] & & \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{tC_{p^{n+1}}} \arrow[dd, "\mathrm{res}"] \\
\mathrm{THH}(\mathbb{S}[\Omega^\infty R]) \arrow[rr, "\psi_{p^n}"] \arrow[dr, "\psi_{p^{n-1}}"] & & \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_{p^n}} \arrow[dr, "\mathrm{can}"] & \\
& \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_p^{n-1}} \arrow[rr, "\varphi_p^{hC_p^{n-1}}"] & & \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{tC_{p^n}}
\end{tikzcd}$$

Now to construct the equivalences $\mathrm{res} \circ \psi_{p^{n+1}} \simeq \psi_{p^n} \circ (-)^p$, we first consider the $n = 1$ case. In that case, we want a filler for the diagram

$$\begin{tikzcd} \mathbb{S}[\Omega^\infty R] \arrow[r] \arrow[d, "(-)^p"] & \mathrm{THH}(\mathbb{S}[\Omega^\infty R]) \arrow[r, "\psi_p"] & \mathrm{THH}(\mathbb{S}[\Omega^\infty R])^{hC_p} \arrow[d, "\mathrm{res}"] \\ \mathbb{S}[\Omega^\infty R] \arrow[rr] & & \mathrm{THH}(\mathbb{S}[\Omega^\infty R]) \end{tikzcd}$$

which is directly obtained from the definition of the Frobenius lift for $\mathrm{THH}(\mathbb{S}[\Omega^\infty R]$.

Now for higher n, we use the definition of $\psi_{p^{n+1}}$ (see (4)) to obtain our equivalences by composition from the $n = 1$ case: By precomposing the cube diagram above with the commutative square for the $n = 1$ case, we obtain the desired equivalences, compatible with the Frobenius and can maps. $\square$

Now the two key properties for the proof of Theorem 9.7 are the following:

Lemma 9.12. (1) *The functor*

$$\mathrm{Ring}_{\mathbb{E}_1} \to \mathrm{Sp} \qquad R \mapsto \mathrm{THH}(R)^{C_{p^n}}$$

preserves geometric realizations.

(2) *For a connective ring spectrum R the spectrum $\mathrm{THH}(R)^{C_{p^n}}$ is connective and the sequence*

$$\pi_0 \mathrm{THH}(R) \xrightarrow{V_{p^{n+1}}} \pi_0\big(\mathrm{THH}(R)^{C_{p^{n+1}}}\big) \xrightarrow{R_p} \pi_0\big(\mathrm{THH}(R)^{C_{p^n}}\big) \to 0$$

is exact. The first map $V_{p^{n+1}}$ above is injective if $\pi_0 R/[\pi_0 R, \pi_0 R]$ is p-torsion free.

Proof. For the first part let $R_\bullet : \Delta^{\mathrm{op}} \to \mathrm{Ring}_{\mathbb{E}_1}$ be a diagram of ring spectra with colimit R. Now we have that THH commutes by definition with sifted colimits, i.e. the canonical map

$$\mathrm{colim}_{n \in \Delta^{\mathrm{op}}} \mathrm{THH}(R_n) \simeq \mathrm{THH}(R).$$

is an equivalence. Inductively we use that there are fibre sequences

$$\mathrm{THH}(R)_{hC_{p^{n+1}}} \to \mathrm{THH}(R)^{C_{p^{n+1}}} \xrightarrow{R_p} \mathrm{THH}(R)^{C_{p^n}} \tag{5}$$

which follows from the fact that by Definition 9.1 we have

$$\mathrm{THH}(R)^{C_{p^{n+1}}} \simeq \mathrm{THH}(R)^{hC_{p^{n+1}}} \times_{\left(\mathrm{THH}(R)^{tC_p}\right)^{hC_{p^n}}} \mathrm{THH}(R)^{C_{p^n}}$$

and since $\big(\mathrm{THH}(R)^{tC_p}\big)^{hC_{p^n}} \simeq \mathrm{THH}(R)^{tC_{p^{n+1}}}$ (the Tate Orbit Lemma). From this fibre sequence (5) we see that $\mathrm{THH}(R)^{C_{p^{n+1}}}$ commutes with sifted colimits in R since orbits commute with all colimits.

The connectivity of $\mathrm{THH}(R)$ for connective R also follows inductively from the fibre sequence (5) and the connectivity of $\mathrm{THH}(R)$. The exact sequence immediately follows by passing to π_0 of the fibre sequence. Finally we have $F_{p^{n+1}} V_{p^{n+1}} = p^{n+1}$, thus $V_{p^{n+1}}$ is injective if multiplication by p^{n+1} is injective on $\pi_0 R/[\pi_0 R, \pi_0 R]$. $\square$

Proof of Theorem 9.7 for p-groups. Let R be a connective ring spectrum. We define a map

$$I\colon \prod_{k=0}^{n} \Omega^\infty R \to \Omega^\infty\big(\mathrm{THH}(R)^{C_{p^n}}\big)$$

as follows:

$$(\alpha_k) \mapsto \sum_{k=0}^{n} V_{p^k}\tau_{p^{n-k}}(\alpha_k).$$

Note that for convenience we have defined this map on elements, but one can easily write this as as map of spaces using the addition of the spectrum $\mathrm{THH}(R)^{C_{p^n}}$. The reader should think of a sequence (α_k) as the product $\prod(1+\alpha_k t^{p^k})$ in $\pi_0 R[[t]]$. Then on π_0 the map I gives rise to a map (of sets)

$$\pi_0(I)\colon \prod_{k=0}^{n} \pi_0 R \to \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \ .$$

We consider the diagram

$$\begin{tikzcd}
\prod_{k=0}^n \pi_0 R \arrow[rr, "\pi_0(I)"] \arrow[d, "q"] & & \pi_0(\mathrm{THH}(R)^{C_{p^n}}) \arrow[d, "\pi_0(w)"] \\
W_{\langle p^n\rangle}(\pi_0 R) \arrow[r, "w_{\langle p^n\rangle}"] & \prod_{k=0}^n \pi_0 R/[\pi_0 R, \pi_0 R] \arrow[r, equal] & \prod_{k=0}^n \pi_0 \mathrm{THH}(R)
\end{tikzcd}$$

where q is the quotient map

$$\prod_{k=0}^{n} \pi_0 R \to \pi_0 R[[t]] \to W_{\langle p^n\rangle}(\pi_0 R) \ .$$

We again refer the reader to Appendix B and in particular Lemma B.11 for the notation. Now we claim the following:

(1) the diagram commutes;
(2) the map $\pi_0(w)$ is injective if $R/[R,R]$ is p-torsion free;
(3) the map $\pi_0(I)$ is a surjective group homomorphism if $R/[R,R]$ is p-torsion free.

Together these claims first imply that $\pi_0(I)$ factors to an isomorphism

$$W_{\langle p^n\rangle}(\pi_0 R) \to \pi_0(\mathrm{THH}(R)^{C_{p^n}})$$

for R a connective ring spectrum with $R/[R,R]$ torsion free because in this case $w_{\langle p^n\rangle}$ is injective as shown in Lemma B.11 in Appendix B and thus the equivalence relations induced by q and $\pi_0(I)$ agree. A general connective ring spectrum R can be written as a geometric realization of connective ring spectra R_n with $\pi_0 R_n$ a free associative ring. For example one can use the Bar resolution associated to the monadic adjunction induced by the forgetful functor

$$\Omega^\infty : \mathrm{Ring}_{\mathbb{E}_1} \to \mathcal{S}$$

(where $\mathcal{S}$ is the ∞-category of spaces).

Since both $\pi_0(\mathrm{THH}(R)^{C_n})$ and $W_{\langle n\rangle}(\pi_0(R))$ commute with geometric realizations of connective ring spectra, we thus see that a natural equivalence between the two functors over connective ring spectra with $R/[R,R]$ torsion free extends to an equivalence over all connective ring spectra.

It remains to prove properties (1)-(3) above: for the first one we compute the i-th component of the clockwise composition. Using the relations given in Lemma 9.8

and Lemma 9.11 we see that it sends the list $(\alpha_0, \ldots, \alpha_n)$ to the element

$$\begin{aligned} F_{p^i} R_{p^{n-i}} \left(\sum_{k=0}^{n} V_{p^k} \tau_{p^{n-k}}(\alpha_k) \right) &= \sum_{k=0}^{n} F_{p^i} R_{p^{n-i}} V_{p^k} \tau_{p^{n-k}}(\alpha_k) \\ &= \sum_{k=0}^{i} F_{p^i} V_{p^k} \tau_{p^{i-k}}(\alpha_k) \\ &= \sum_{k=0}^{i} p^k F_{p^{i-k}} \tau_{p^{i-k}}(\alpha_k) \\ &= \sum_{k=0}^{i} p^k \tau_1 \left(\alpha_k^{p^{i-k}} \right) = \sum_{k=0}^{i} p^k \left[\alpha_k^{p^{i-k}} \right] . \end{aligned}$$

This is the i-th Witt polynomial and thus shows the commutativity of the diagram. For the claims (2) and (3) one first checks that the diagram

$$\begin{tikzcd} \pi_0 R \arrow[r, "i_{n+1}"] \arrow[d, "\pi_0(I)"] & \prod_{k=0}^{n+1} \pi_0 R \arrow[r, "\pi_{0,\ldots,n}"] \arrow[d, "\pi_0(I)"] & \prod_{k=0}^{n} \pi_0 R \arrow[d, "\pi_0(I)"] \\ \pi_0 \mathrm{THH}(R) \arrow[r, "V_{p^{n+1}}"] \arrow[d, "p^{n+1}"] & \pi_0\big(\mathrm{THH}(R)^{C_{p^{n+1}}}\big) \arrow[r, "R_p"] \arrow[d, "\pi_0(w)"] & \pi_0\big(\mathrm{THH}(R)^{C_{p^n}}\big) \arrow[d, "\pi_0(w)"] \\ \pi_0 \mathrm{THH}(R) \arrow[r, "i_{n+1}"] & \prod_{k=0}^{n+1} \pi_0 \mathrm{THH}(R) \arrow[r, "\pi_{0,\ldots,n}"] & \prod_{k=0}^{n} \pi_0 \mathrm{THH}(R) \end{tikzcd}$$

commutes, where the upper vertical maps are only maps of sets but all other maps are group homomorphisms. By Lemma 9.12, the sequence in the middle is right exact, and if $R/[R,R]$ is p-torsion free, it is exact. The other rows are clearly exact, the first one a priori of course only as a sequence of sets.

Since $R/[R,R]$ p-torsion free also implies that the vertical map p^{n-1} is injective, we inductively see that $\pi_0(w)$ is injective.

Now for claim (3), note first that $\pi_0(I)$ is a homomorphism, since $\pi_0(w)$ is injective and $\pi_0(w) \circ \pi_0(I) = w_{\langle p^n \rangle} \circ q$ is a homomorphism. Then inductively, $\pi_0(I)$ is surjective.

Finally we note that the fact that the map $\pi_0(w)$ is lax symmetric monoidal also implies that the two functors are isomorphic as lax symmetric monoidal functors since the lax symmetric monoidal structure on Witt vectors is uniquely determined by requiring that w is lax symmetric monoidal (see Remark B.8) . $\square$

We now explain how to generalize this proof to the case of arbitrary subgroups $C_n \subseteq \mathbb{T}$. The definitions of R_{p^m}, F_{p^m} and V_{p^m} on fixed points of $\mathrm{THH}(R)$ above make sense more generally for any p-cyclotomic spectrum X instead of $\mathrm{THH}(R)$. If X has an l-cyclotomic structure for all $l \in S$ in some set of primes $S \subseteq \mathbb{P}$, then for every $p \in S$ the maps

$$\begin{aligned} R_{p^m} &\colon X^{C_{p^{n+m}}} \to X^{C_{p^n}} \\ F_{p^m} &\colon X^{C_{p^{n+m}}} \to X^{C_{p^n}} \\ V_{p^m} &\colon X^{C_{p^n}} \to X^{C_{p^{n+m}}} \end{aligned}$$

are maps of l-cyclotomic spectra for all $l \in S \setminus p$ where the spectrum $X^{C_{p^n}}$ carries the l-cyclotomic structure described in Construction 9.4. This is immediate from the definitions and Lemma 9.3. Now for any $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ and fixed p_j, this means that we can define maps

$$R_{p_j^m} \colon X^{C_{p_j^m n}} \to X^{C_n}$$
$$F_{p_j^m} \colon X^{C_{p_j^m n}} \to X^{C_n}$$
$$V_{p_j^m} \colon X^{C_n} \to X^{C_{p_j^m n}}$$

by defining them as the iterated fixed points with respect to $C_{p_{j+1}^{n_{j+1}}}, C_{p_{j+2}^{n_{j+2}}}, \dots$ of the corresponding maps on

$$\left(\left(\left(X^{C_{p_1^{n_1}}}\right)^{C_{p_2^{n_2}}}\right)\cdots\right)^{C_{p_j^{n_j}}} .$$

These commute, and for $m = \prod p_j^{m_j}$ we define

$$R_m := \prod R_{p_j^{m_j}}$$
$$F_m := \prod F_{p_j^{m_j}}$$
$$V_m := \prod V_{p_j^{m_j}}$$

These together with the Teichmüller character defined above satisfy all the relations analogous to the ones given in the p^n-case in Lemma 9.8 and Lemma 9.11. The analogue of the exact sequence (2) of Lemma 9.12 is given by

$$\pi_0 \mathrm{THH}(R)^{C_n} \xrightarrow{V_{p^{m+1}}} \pi_0\big(\mathrm{THH}(R)^{C_{np^{m+1}}}\big) \xrightarrow{R_p} \pi_0\big(\mathrm{THH}(R)^{C_{np^m}}\big) \to 0 .$$

where the left map is injective if $\pi_0 \mathrm{THH}(R)^{C_n}$ is p-torsion free. This then yields the same proof as in the p-typical situation using the inductive nature of the definitions and with the necessary changes (using especially Lemma B.11 and Example B.13 from the appendix). We omit the details.

10. TR and its universal property

In the last section we have established a close connection between Witt vectors and the fixed points of topological Hochschild homology. One abstract way to think about big Witt vectors $W(R)$ for a commutative ring R is that they form the cofree λ-ring on R [13]. Similarly the p-typical Witt vectors form the cofree p-typical λ-ring. Here a p-typical λ-ring is a commutative ring S together with a derived Frobenius lift.[14] We will show that the spectrum $\mathrm{TR}(R)$, that will be defined below, satisfies a similar universal property: it is the cofree cyclotomic spectrum with Frobenius lifts obtained from $\mathrm{THH}(R)$. This gives a conceptually satisfactory explanation why Witt vectors show up.

We first have to introduce some terminology.

[13] One can give a similar, but less elegant, universal property for non-commutative Witt vectors but we do not need this here

[14] More precisely a lift of the derived Frobenius morphism $\varphi_p : S \to S/\!\!/p$ through the canonical projection $S \to S/\!\!/p$. This is well-known to be true for torsion free rings and can then be reduced to this statement.

Definition 10.1. *For a cyclotomic spectrum (X, φ_p) we define spectra with $\mathbb{T}$ action as follows*

$$\begin{aligned}\mathrm{TR}(X,p) &:= \varprojlim \left(\ldots \xrightarrow{R} X^{C_{p^2}} \xrightarrow{R} X^{C_p} \xrightarrow{R} X \right) \\ &\simeq \left(\ldots \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X \right) .\end{aligned}$$

where the maps R are given by forgetting the first factor in the defining pullback (see Definition 9.1) and

$$\mathrm{TR}(X) := \varprojlim_{n \in \mathbb{N}_{>0}} \mathrm{THH}(R)^{C_n}$$

where $\mathbb{N}_{>0}$ is considered as a poset under the divisibility relation and the maps in the diagram are again the R-maps $\mathrm{THH}(R)^{C_{nm}} \to \mathrm{THH}(R)^{C_n}$. [15]

In this language the results of the last section can be expressed as follows.

Corollary 10.2. *For an associative ring R there are isomorphisms*

$$\pi_0 \mathrm{TR}(R,p) \cong W_p(R) \qquad \text{and} \qquad \pi_0 \mathrm{TR}(R) \cong W(R)$$

which are natural in R and compatible with external products (i.e. lax symmetric monoidal isomorphisms).

Here $W_p(R) = W_{\{1,p,p^2,\ldots\}}(R)$ are the p-typical Witt vectors and $W(R) = W_{\mathbb{N}_{>0}}(R)$ are the big Witt vectors.

Proof. The isomorphism $\pi_0(\mathrm{THH}(R)^{C_n}) \cong W_{\langle n \rangle}(R)$ of Theorem 9.7 is compatible with the R maps on either side, i.e. the topological R map as considered above and the Witt vector reduction maps. Since these maps are surjective we get

$$\pi_0 \mathrm{TR}(R,p) = \varprojlim \pi_0(\mathrm{THH}(R)^{C_{p^n}}) = \varprojlim W_{\{1,p,\ldots,p^n\}} = W_p(R)$$

and

$$\pi_0 \mathrm{TR}(R) = \varprojlim \pi_0(\mathrm{THH}(R)^{C_n}) = \varprojlim W_{\langle n \rangle} = W(R) \ .$$

Here we have used the fact that for a colimit $S = \mathrm{colim} S_i$ of truncation sets $S_i \subseteq \mathbb{N}_{>0}$ we find that $W_S(R) = \varprojlim W_{S_i}(R)$. □

Now it turns out that the spectra $\mathrm{TR}(X)$ and $\mathrm{TR}(X,p)$ have more structure than a $\mathbb{T}$-action. They are themselves cyclotomic spectra equipped with Frobenius lifts. Let us start with the case $\mathrm{TR}(X,p)$. We define a map

$$\psi_p : \mathrm{TR}(X,p) \to \mathrm{TR}(X,p)^{hC_p}$$

as follows: first recall that we have

$$\mathrm{TR}(X,p) = \left(\ldots \times_{(X^{tC_p})^{hC_p}} X^{hC_p} \times_{X^{tC_p}} X \right)$$

and therefore

$$\mathrm{TR}(X,p)^{hC_p} = \left(\ldots \times_{(X^{tC_p})^{hC_{p^2}}} X^{hC_{p^2}} \times_{(X^{tC_p})^{hC}} X^{hC_p} \right).$$

[15] We have not really proved here that the R maps are sufficiently coherent to give a diagram indexed on the divisibility poset. But this follows for example from the existence of a genuine equivariant version of THH or can be done with some more care. Since we do not really work with TR we do not want to go into details here.

The map ψ_p is then simply defined by forgetting the last factor. Thus it is an 'equivariant' version of the F-map studied in the last section. We consider $\mathrm{TR}(X,p)$ as a p-cyclotomic spectrum with Frobenius

$$\mathrm{TR}(X,p) \xrightarrow{\psi_p} \mathrm{TR}(X,p)^{hC_p} \xrightarrow{\mathrm{can}} \mathrm{TR}(X,p)^{tC_p}$$

and by construction it has a Frobenius lift.[16] There is a $\mathbb{T}$-equivariant projection to the last factor $\mathrm{TR}(X,p) \to X$ (which is a variant of the R-map from the last section) and by definition of $\mathrm{TR}(X,p)$ this is a map of p-cyclotomic spectra.

Proposition 10.3. *The map* $\mathrm{TR}(X,p) \to X$ *exhibits* $\mathrm{TR}(X,p)$ *as the cofree p-cyclotomic spectrum with Frobenius lifts over X, i.e. the functor* TR *gives a right adjoint to the forgetful functor from p-cyclotomic spectra with Frobenius lift to p-cyclotomic spectra.*

Proof. Let Y be a p-cyclotomic spectrum with Frobenius lift $\underline{\psi}_p : Y \to Y^{hC_p}$. Let us compute the space of maps from Y to $\mathrm{TR}(X,p)$ as spectra with Frobenius lift. It is given by the equalizer of

$$\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, \mathrm{TR}(X,p)\big) \underset{(\underline{\psi}_p)^*(-)^{hC_p}}{\overset{(\psi_p)_*}{\rightrightarrows}} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, \mathrm{TR}(X,p)^{hC_p}\big) \ .$$

We use the pullback description of $\mathrm{TR}(X,p)$ to see that the source of the equalizer diagram $\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, \mathrm{TR}(X,p)\big)$ is given by the iterated pullback

$$\ldots \times_{\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\left(Y,(X^{tC_p})^{hC_p}\right)} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{hC_p}\big) \times_{\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\left(Y,X^{tC_p}\right)} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X\big) \ .$$

Similarly, the target $\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y^{hC_p}, \mathrm{TR}(X,p)^{hC_p}\big)$ is given by

$$\ldots \times_{\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\left(Y,(X^{tC_p})^{hC_{p^2}}\right)} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{hC_{p^2}}\big) \times_{\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\left(Y,(X^{tC_p})^{hC_p}\right)} \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{hC_p}\big) \ .$$

Under these identifications the two maps are given by forgetting the last factor and by $(\underline{\psi}_p)^*(-)^{hC_p}$ applied to every factor. We can write this as the total equalizer of a diagram

$$\begin{array}{ccc}
\prod_{n=0}^\infty \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{hC_{p^n}}\big) & \rightrightarrows & \prod_{n=0}^\infty \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{hC_{p^{n+1}}}\big) \\
\downdownarrows & & \downdownarrows \\
\prod_{n=0}^\infty \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, (X^{tC_p})^{hC_{p^n}}\big) & \rightrightarrows & \prod_{n=0}^\infty \mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, (X^{tC_p})^{hC_{p^{n+1}}}\big)
\end{array}$$

where the horizontal maps are given by forgetting the last factor and by $(\underline{\psi}_p)^*(-)^{hC_p}$ applied to every factor and the vertical ones by φ_p and by can applied factorwise. The horizontal equalizers are given by $\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X\big)$ and $\mathrm{Map}_{\mathrm{Sp}^{B\mathbb{T}}}\big(Y, X^{tC_p}\big)$ and the two induced maps are such that the equalizer is the mapping space in cyclotomic spectra from Y to X. □

There is a similar global statement, namely that $\mathrm{TR}(X)$ is a spectrum with commuting Frobenius lits (in the sense of Definition 8.2) with a map of cyclotomic spectra $\mathrm{TR}(X) \to X$ and this map exhibits $\mathrm{TR}(X)$ as the cofree such objects. The

[16] For a p-cyclotomic spectrum (X, φ_p) a Frobenius lift is of course just given by a $\mathbb{T}$-equivariant lift of the map $\varphi_p \colon X \to X^{tC_p}$ through $\mathrm{can}\colon X^{hC_p} \to X^{tC_p}$ without any further conditions or coherences.

proof is essentially the same as the proof of Proposition 10.3 but more tedious. Therefore we do not carry it our here. The results about TR will play an important role in forthcoming work of the second author with B. Antieau.

Appendix A. The Hopkins-Mahowald result

We consider the space $\Omega^2 S^3$ which is the free $\mathbb{E}_2$-monoid on S^1. We want to construct an $\mathbb{E}_2$ map $f_p : \Omega^2 S^3 \to \mathrm{BGl}_1(\mathbb{S}_p^\wedge)$. Such a map is the same as an element in $\pi_1(\mathrm{BGl}_1(\mathbb{S}_p^\wedge)) = \mathbb{Z}_p^\times$. We take an element of the form $1 + u \cdot p$, with u a p-adic unit. [17] Then we have the following surprising result, which is for $p = 2$ due to Mahowald and for odd p due to Hopkins. We learned the idea for the proof from Mike Mandell, see also the nice paper [Blu10].

Theorem A.1 (Hopkins-Mahowald)**.** *The Thom spectrum $Mf_p = (\mathbb{S}_p^\wedge)_{h\Omega^3 S^3}$ is equivalent as an $\mathbb{E}_2$-ring spectrum to the Eilenberg-MacLane spectrum $H\mathbb{F}_p$.*

In order to prove this result we first recall that for every $\mathbb{E}_2$-ring spectrum R there exist Dyer-Lashof operations

$$Q^i : H_k(X, \mathbb{F}_2) \to H_{k+i}(X; \mathbb{F}_2)$$

for $i \leq k+1$ and they satisfy all the relation of the usual Dyer-Lashof operations as long as they make sense. For odd p, there exist operations

$$\begin{aligned} Q^i :& H_k(X, \mathbb{F}_p) \to H_{k+2i(p-1)}(X; \mathbb{F}_p) \\ \beta Q^i :& H_k(X, \mathbb{F}_p) \to H_{k+2i(p-1)-1}(X; \mathbb{F}_p) \end{aligned}$$

for $i \leq 2k+1$. Then the following classical computation is the key to the proof of Theorem A.1.

Lemma A.2. (1) *We have the Pontryagin ring*
$$H_*(\Omega^2 S^3, \mathbb{F}_2) = \mathbb{F}_2[x_1, x_2, \ldots]$$
where $|x_i| = 2^i - 1$. The element x_{i+1} is given by $Q^{2^i} Q^{2^{i-1}} \ldots Q^8 Q^4 Q^2 x_1$. In addition, $\beta x_i = x_{i-1}^2$.

(2) *We have the Pontryagin ring*
$$H_*(\Omega^2 S^3, \mathbb{F}_p) = \mathbb{F}_p[y_0, y_1, \ldots, z_1, z_2, \ldots]/(y_i^2)$$
where $|y_i| = 2p^i - 1$, $|z_i| = 2p^i - 2$. The element y_{i+1} is given by $Q^{p^i} \ldots Q^p Q^1 y_0$, the element z_i is given by $\beta Q^{p^i} \ldots Q^p Q^1 y_0$.

(3) *There is a ring isomorphism of the Pontriyagin ring $H_*(\Omega^2 S^3, \mathbb{F}_p)$ to the one of $H_*(H\mathbb{F}_p, \mathbb{F}_p)$ for all p which is compatible with Dyer-Lashof operations that are defined on both sides.* [18]

Proof. The first part is due to Araki and Kudo [KA56, Theorem 7.1], the second part is Dyer-Lashof [DL62, Theorem 5.2] and the third is Steinberger [BMMS86, Chapter 3, Theorem 2.2 and 2.3]. The first two results are relatively straightforward computation using the Serre spectral sequence and the Kudo transgression theorem

[17] In fact one can replace the p-complete sphere $\mathbb{S}_p^\wedge$ by the p-local sphere $\mathbb{S}_{(p)}$ and everything that follows remains true. One can also chase through the proofs precisely which elements need to be inverted, we leave the details to the reader.

[18] Under this isomorphism the generator x_i (resp. z_i) is mapped to the Milnor basis element $\overline{\xi_i}$ and y_i to $\overline{\tau_i}$.

and the last result can be obtained by dualizing the computation of the Steenrod algebra. □

Proof of Theorem A.1. First we note that Mf_p is connective and that because it is a homotopy quotient we find that

$$\pi_0((\mathbb{S}_p^\wedge)_{h\Omega^3S^3}) = \pi_0(\mathbb{S}_p^\wedge)_{\pi_0(\Omega^3S^3)}.$$

The latter are the coinvariants of the $\mathbb{Z} = \pi_0(\Omega^3S^3)$-action on $\mathbb{Z}_p = \pi_0(\mathbb{S}_p^\wedge)$ given by multiplication with $1 + u \cdot p$. Thus we get that

$$\pi_0(Mf_p) = \mathbb{Z}/(1 - (1 + u \cdot p)) = \mathbb{F}_p.$$

Therefore the 0-th Postnikov section defines an $\mathbb{E}_2$-map

$$p : Mf_p \to H\mathbb{F}_p\ .$$

Both spectra are p-torsion, in particular p-complete and connective. Thus to show that this map is an isomorphism, it suffices to prove that it induces an isomorphism in $\mathbb{F}_p$-homology:

$$p_* : H_*(Mf_p; \mathbb{F}_p) \to H_*(H\mathbb{F}_p; \mathbb{F}_p)\ .$$

In order to compute the homology $H_*(Mf_p, \mathbb{F}_p)$ we use the Thom isomorphism. More precisely by definition of (generalized) Thom spectra as a colimit we immediately find that

$$Mf_p \otimes H\mathbb{F}_p \simeq M(h \circ f_p)$$

where the latter is the Thom spectrum of the map

$$\Omega^2S^3 \xrightarrow{f_p} \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \xrightarrow{h} \mathrm{BGL}_1(H\mathbb{F}_p)\ .$$

But the composition $h \circ f_p$ is as an $\mathbb{E}_2$-map equivalent to the trivial map since $(1 + u \cdot p) = 1$ in $\mathbb{F}_p$. So we get an equivalence of $\mathbb{E}_2$-ring spectra $Mf_p \otimes H\mathbb{F}_p \simeq \Omega^2S^3 \otimes H\mathbb{F}_p$.

Now it is enough to check that p_* is an isomorphism for $* = 0, 1$. Then it automatically is an isomorphism in all degrees, since it commutes with Dyer-Lashoff operations and since both sides are generated in the same way by $\mathbb{E}_2$-Dyer-Lashof operations from elements in degree 1. To see that p_* is an isomorphism in degrees $0, 1$ we use the map $\mathbb{Z} \to \Omega^3S^3$ of $\mathbb{E}_1$-spaces induced from the map $S^1 \to \Omega^2S^3$ to get a map

$$\mathbb{S}/p = (\mathbb{S}_p^\wedge)_{h\mathbb{Z}} \to (\mathbb{S}_p^\wedge)_{h\Omega^3S^3} = Mf_p$$

which is an isomorphism in homology in degrees 0 and 1 since the map $\mathbb{Z} \to \Omega^3S^3$ is connected. But the composite

$$\mathbb{S}/p \to Mf_p \to H\mathbb{F}_p$$

is the Postnikov section which is also an homology isomorphism in degrees 0 and 1. □

Now we want to prove an extension of the Hopkins-Mahowald result, which is due to Nitu Kitchloo [Kit18]. For $0 \leq n \leq \infty$, let X_{p^n} be the p^n-fold covering of $\Omega^2S^3 =: X_1$ (for $n = \infty$ we define it to be the universal cover) and consider the composition

$$f_{p^n} : X_{p^n} \to \Omega^2S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge).$$

Theorem A.3. *For $p = 2$, assume $u \neq -1$ and let $d = v_p(u+1)$. For odd p, let $d = 0$ for any u. Then, for $n \geq 1$, the Thom spectrum Mf_{p^n} is $\mathbb{E}_2$-equivalent to $H\mathbb{Z}/p^{n+d+1}$. The Thom spectrum Mf_{p^∞} is equivalent to $H\mathbb{Z}_p$.*

Proof. In principle, one could check this on homology again, but there is an easier way to deduce this.

Observe that the fiber sequence $X_{p^n} \to X_1 \to BC_{p^n}$ exhibits X_1 as a homotopy quotient $(X_{p^n})_{hC_{p^n}}$. On Thom spectra, this means that $(Mf_{p^n})_{hC_{p^n}} \simeq Mf \simeq H\mathbb{F}_p$. As before, $\pi_0(Mf_{p^n})$ is easy to compute as $\mathbb{Z}_p/(1-(1+u\cdot p)^{p^n})$. Now note that d is defined in such a way that

$$v_p((1+u\cdot p)^{p^n} - 1) = n + d + 1 \ .$$

For odd p, this is clear. For $p = 2$, note that $(1+2u)^2 = 1+4u(u+1) = 1+u'\cdot 2^{d+2}$. So $(1+2u)^{2^n} = 1 + u'' \cdot 2^{n+d+1}$, where u' and u'' are some units.

It follows that $\pi_0(Mf_{p^n}) = \mathbb{Z}/p^{n+d+1}$.

The residual action by C_{p^n} is given by multiplication with $(1+u\cdot p)$. This gives us an equivariant map $Mf_{p^n} \to H\mathbb{Z}/p^{n+d+1}$.

We now observe that the action of C_{p^n} on $\mathbb{F}_p$-homologies of both spectra is trivial. For the second spectrum, this is clear, since the action is given by multiplication with a number which is 1 mod p. For the first spectrum we find that the homology $H_*(Mf_{p^n};\mathbb{F}_p)$ is canonically isomorphic to the homology of the space $H_*(X_{p^n};\mathbb{F}_p)$ by the Thom isomorphism (see the discussion in the proof of Theorem A.1 above). Thus it suffices to check this in the $\mathbb{F}_p$-homology of the space X_{p^n}. But this space is the fibre of a loop map $X_1 \to BC_{p^n}$ which implies the triviality of the action in homology.

Secondly, we can directly check that the C_{p^n}-action on $H\mathbb{Z}/p^{n+d+1}$ described above is acyclic, i.e. that the higher group homologies $H_i(C_{p^n};\mathbb{Z}/p^{n+d+1})$ vanish and therefore $(H\mathbb{Z}/p^{n+d+1})_{hC_{p^n}} = H\mathbb{Z}/p$. For example, a free resolution of $\mathbb{Z}$ over $\mathbb{Z}[C_{p^n}]$ is given by $x-1$ and $1+x+\ldots+x^{p^n-1}$ alternatingly, where x is a generator. The action now sends x to $(1+u\cdot p)$, and so

$$1 - x \mapsto -u\cdot p$$

$$1 + x + x^2 + \ldots + x^{p^n-1} \mapsto \frac{(1+u\cdot p)^{p^n} - 1}{(1+u\cdot p) - 1} = u' \cdot p^{n+d}$$

where u' is some unit. $\mathrm{Tor}_{\mathbb{Z}[C_{p^n}]}(\mathbb{Z}, \mathbb{Z}/p^{n+d+1})$ can therefore be computed by the chain complex

$$\mathbb{Z}/p^{n+d+1} \xleftarrow{p} \mathbb{Z}/p^{n+d+1} \xleftarrow{p^{n+d}} \mathbb{Z}/p^{n+d+1} \xleftarrow{p} \ldots$$

which is exact.

Now the map $Mf_{p^n} \to H\mathbb{Z}/p^{n+d+1}$ is an equivalence on homotopy orbits, therefore the next Lemma implies that it is an equivalence after p-completion. The computation of π_0 shows that both sides are p-power torsion, hence p-complete which finishes the proof of the first claim.

For the second part we argue similary: The fibre sequence $\tau_{\geq 2}\Omega^2 S^3 \to \Omega^2 S^3 \to B\mathbb{Z}$ shows that $(Mf_{p^\infty})_{h\mathbb{Z}} = H\mathbb{F}_p$. Now since $\pi_0(Mf_{p^\infty}) = \mathbb{Z}_p$, we obtain an equivariant map $Mf_{p^\infty} \to H\mathbb{Z}_p$, where $\mathbb{Z}$ acts on $H\mathbb{Z}_p$ by multiplication with $(1+u\cdot p)$. The action is easily seen to be acyclic, such that $(H\mathbb{Z}_p)_{h\mathbb{Z}} = H\mathbb{F}_p$ and the map $Mf_{p^\infty} \to H\mathbb{Z}_p$ induces an equivalence on homotopy orbits. As in the proof of Theorem A.3, the action on homology of both sides is trivial, and so the Lemma A.4

guarantees that Mf_{p^∞} agrees with $H\mathbb{Z}_p$ after p-completion. But M_{p^∞} is already p-complete which can be seen as follows: Since $\tau_{\geq 2}\Omega^2 S^3$ is of finite type it follows that each Postnikov section $\tau_{\leq k} M_{p^\infty}$ agrees with the corresponding Postnikov section of the Thom spectrum of some finite skeleton. The latter is p-complete as a finite colimit over the p-complete sphere. □

Lemma A.4. *Let $X \to Y$ be a G-equivariant map of connective spectra where G is a discrete group. Assume that G acts trivially on $\mathbb{F}_p$-homology of X and Y and that the induced map*

$$X_{hG} \to Y_{hG}$$

is a p-adic equivalence. Then also the initial map $X \to Y$ is a p-adic equivalence.

Proof. Let Z be the cofibre of the map $X \to Y$. Then it also carries a G-action and the homotopy orbits Z_{hG} are p-adically trivial. We want to show that Z is p-adically trivial, i.e. that the $\mathbb{F}_p$-homology of Z vanishes. Assume that it has a lowest non-trivial homology group $H_n(Z; \mathbb{F}_p)$. We get that

$$H_n(Z_{hG}, \mathbb{F}_p) = H_n(Z, \mathbb{F}_p)/G.$$

The action of G on $H_n(Z, \mathbb{F}_p)$ is not necessarily trivial, but still 2-stage nilpotent by the long exact sequence. Thus the coinvariants are non-trivial which gives a contradiction. □

Remark A.5. For odd p, Theorem A.3 allows us to describe all $H\mathbb{Z}/p^n$ as Thom spectra. However, for $p = 2$, we only get $H\mathbb{Z}/2^n$ for $n \geq 3$, as $d \geq 1$ in that case.

Now we finally want to describe a way of exhibiting $H\mathbb{Z}$ as a Thom spectrum. We take the 1-connected cover of the map $f_p : \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ which gives a map

$$\tau_{\geq 2}\Omega^2 S^3 \to \tau_{\geq 2}\mathrm{BGL}_1(\mathbb{S}_p^\wedge)$$

Taking all these maps for different p together we obtain an $\mathbb{E}_2$-map

$$f : \tau_{\geq 2}\Omega^2 S^3 \to \prod_p \tau_{\geq 2}\mathrm{BGL}_1(\mathbb{S}_p^\wedge) = \tau_{\geq 2}\mathrm{BGL}_1(\mathbb{S}) \to \mathrm{BGL}_1(\mathbb{S}).$$

The next result also seems to be well-known, see [Mah79] and [Blu10, Section 9].

Theorem A.6. *The Thom spectrum Mf is $\mathbb{E}_2$-equivalent to $H\mathbb{Z}$.*

Proof. Since $\tau_{\geq 2}\Omega^2 S^3$ is simply-connected, $\pi_0(Mf) = \mathbb{Z}$. So there is an E_2-map $Mf \to H\mathbb{Z}$ which induces an isomorphism on π_0. To show that this map is an equivalence, it suffices to show that the rationalization and the p-completions of Mf for all p are 0-truncated. For the rationalization, this is clear, since $\tau_{\geq 2}\Omega^2 S^3$ is rationally trivial.

After p-localization, the spectrum Mf agrees with the Thom spectrum Mf_{p^∞} of the map $\tau_{\geq 2}\Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$, i.e. the other twists do not play a role anymore. This Thom spectrum is $H\mathbb{Z}_p$ as shown in Theorem A.3. □

Appendix B. (Non-commutative) Witt vectors

In this section we review and extend Hesselholt's definition [Hes97, Hes05] of Witt vectors $W(R)$ for a non-commutative ring R. In general $W(R)$ is only an abelian group but for a commutative ring it admits a canonical multiplication that turns it into a commutative ring which is isomorphic to the classical ring of big Witt vectors. There are also variants of p-typical Witt vectors $W_p(R)$ obtained as quotients of

$W(R)$. We use a slightly different construction than Hesselholt which is inspired by the well-known construction of (commutative) big Witt vectors as invertible power series.

We start with a non-commutative[19] ring R and consider the non-commutative group

$$1 + tR[[t]]$$

of power series with constant coefficient 1. The group structure is given by multiplication of power series and as in the commutative case this is a group: one can inductively compute the coefficients of the one-sided inverse to a given element $f \in 1 + tR[[t]]$.

We consider the (non-additive) map

$$\tau : R \to 1 + tR[[t]] \qquad r \mapsto 1 - rt$$

called the Teichmüller character.

Definition B.1. *The abelian group of* big Witt vectors *of a non-commutative ring R is defined as the quotient group*

$$W(R) := \frac{(1 + tR[[t]])^{\mathrm{ab}}}{\tau(rs) \sim \tau(sr)} \ .$$

Here $(1 + tR[[t]])^{\mathrm{ab}}$ is the abelianization of the multiplicative group $(1 + tR[[t]])$ in the topological sense, i.e. the quotient by the closure of the subgroup generated by commutators.

Let us try to understand this definition a little bit more explicitly. Clearly we still have a map $\tau : R \to W(R)$ which now sends rs and sr to the same element. But since τ is not additive it does not factor over $R/[R, R]$.

There is also a map

$$-\mathrm{dlog} \colon 1 + tR[[t]] \to R[[t]] \qquad f \mapsto -f^{-1} \cdot f'$$

where f' is the derivative of a power series defined as in the commutative case, f^{-1} is the multiplicative inverse and the two power series are multiplied in the ring $R[[t]]$. Now we consider $[R, R] \subseteq R$ generated as an additive subgroup (not as an ideal!) by all commutators $rs - sr$. The quotient $R/[R, R]$ is an abelian group and as such isomorphic to $\mathrm{HH}_0(R)$. We consider the additive group

$$(R/[R, R])[[t]] \cong (R/[R, R])^{\mathbb{N}}$$

with 'pointwise' addition.

Lemma B.2. *The composition*

$$1 + tR[[t]] \xrightarrow{-\mathrm{dlog}} R[[t]] \to (R/[R, R])[[t]]$$

is a group homomorphism and factors to a homomorphism

$$w\colon \quad W(R) \to (R/[R, R])[[t]] \ .$$

[19] By 'non-commutative' we mean 'not necessarily commutative' throughout.

Proof. For any pair of power series f and g in $R[[t]]$ the classes $[fg]$ and $[gf]$ in $(R/[R,R])[[t]]$ are equal. This is clear since every coefficient differs by a sum of commutators. Now in $R[[t]]$ we have the usual formula

$$(fg)' = f'g + fg' \ .$$

This can be seen by reduction to monomials (using that both sides are R-linear and commute with power series sums) where it is obvious. Now it follows that

$$\begin{aligned}\big[(fg)^{-1}(fg)'\big] &= \big[g^{-1}f^{-1}(f'g+fg')\big]\\ &= \big[g^{-1}f^{-1}f'g\big] + \big[g^{-1}f^{-1}fg'\big]\\ &= \big[f^{-1}f'\big] + \big[g^{-1}g'\big] \ .\end{aligned}$$

It is clear that the map the logarithmic derivative is a continuous map, thus factors through the abelianization of $1+tR[[t]]$. Thus it only remains to show that the images of $\tau(rs)$ and $\tau(sr)$ in $(R/[R,R])[[t]]$ agree. One gets

$$\begin{aligned}-[\tau(rs)^{-1}\tau(rs)'] &= [rs + (rs)^2t + (rs)^3t^2 + \ldots]\\ &= [sr + (sr)^2t + (sr)^3t^2 + \ldots]\\ &= -[\tau(sr)^{-1}\tau(sr)']\end{aligned}$$

where the second equality holds because $(rs)^n$ and $(sr)^n$ differ by a cyclic permutation, hence are equal in $R/[R,R]$. □

We call the map w of the last lemma the *ghost map* and the coefficients *ghost components.*

Proposition B.3. (1) *If the group $R/[R,R]$ is torsion free, then the ghost map $w : W(R) \to (R/[R,R])[[t]]$ is injective.*
(2) *If R is rational, i.e. $\mathbb{Q} \subseteq R$, then w is an isomorphism.*
(3) *The functor $R \mapsto W(R)$ as a functor from rings to abelian groups commutes with split coequalizers.*

Proof. To see (1) we will show that for every element $[f_n] \in W(R)$ in the kernel of w with representative $f_n = 1+\alpha_n t^n + \ldots \in 1+tR[[t]]$ we can also find a representative of the same class in $W(R)$ of the form $f_{n+1} \in 1+t^{n+1}R[[t]]$. Inductively we find a convergent sequence $(f_n, f_{n+1}, f_{n+2}, \ldots)$ with limit 1 representing the same class, thus $[f_n] = [1] \in W(R)$.

To construct f_{n+1} we use that we have $0 = w(f_n) = -n\alpha_n t^{n-1} + \ldots$, thus $-n\alpha_n = 0$ in $R/[R,R]$. The torsion freeness implies that $\alpha_n \in [R,R]$. So we can write $\alpha_n = \sum [x_i, y_i]$. Now we distinguish two cases: if $n = 1$, observe that

$$\tau(x_iy_i)\tau(y_ix_i)^{-1} = 1 - [x_i,y_i]t + \ldots$$

and therefore

$$f_2 := f_1 \cdot \Big(\prod \tau(x_iy_i)\tau(y_ix_i)^{-1}\Big) \in 1 + t^2R[[t]]$$

works. For $n \geq 2$ choose $k, l \geq 1$ with $k+l = n$. We compute

$$\begin{aligned}&\big[(1+x_it^k),(t+y_it^l)\big]\\ &= 1 + \big((1+x_it^k)(1+y_it^l) - (1+y_it^l)(1+x_it^k)\big)(1+x_it^k)^{-1}(1+y_it^l)^{-1}\\ &= 1 + [x_i,y_i]t^n \cdot (1+x_it^k)^{-1}(1+y_it^l)^{-1}\\ &= 1 + [x_i,y_i]t^n + \ldots \ .\end{aligned}$$

Thus

$$f_{n+1} := f_n \cdot \left(\prod \left[(1+x_i t^k), (t+y_i t^l)\right]\right)^{-1} \in 1 + t^{n+1}R[[t]]$$

works and finishes the proof of (1).

For (2) it suffices to show that w is surjective (since it is injective by (1)). But this follows since the map $1+tR[[t]] \to R[[t]]$ mapping f to $-f^{-1}f'$ is already surjective as can be seen by inductively solving the necessary equations (exercise).

For (3) we first claim that the functor $R \mapsto (1+tR[[t]])$ as a functor from rings to topological groups commutes with split coequalizers. To see this we use that split coequalizers are computed as coequalizers on underlying sets in both cases and for a surjective map $R \to S$ of rings the map $1+tR[[t]] \to 1+tS[[t]]$ is open (for the t-adic topology, which is the product topology), in particular the map is an identification map. The abelianization functor from topological groups to abelian topological groups is left adjoint and thus also commutes with split coequalizers. Finally the Witt vectors are then a split coequalizer of two maps

$$R \times R \rightrightarrows (1+tR[[t]])^{\mathrm{ab}}$$

and the source as well as the target commute with split coequalizers in R. The claim follows. $\square$

Remark B.4. Part (3) of Proposition B.3 implies that the Witt vector functor is completely determined by its value on free, associative rings. To see this note that every ring R can be written as a split coequalizer of the diagram

$$\begin{tikzcd} \mathrm{Fr}(R \sqcup I) \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & \mathrm{Fr}(R) \arrow[l] \arrow[r, "p"] & R \end{tikzcd}$$

where $\mathrm{Fr}(S)$ denotes the free associative ring on a set S, the map p is the obvious projection, $I \subseteq \mathrm{Fr}(R)$ the kernel of p, the map d_0 is adjoint to the map $R \sqcup I \to \mathrm{Fr}(R)$ that sends R and I to themselves and d_1 is adjoint to the map that sends R to itself and I to zero.

Moreover for the free associative rings $R := \mathrm{Fr}(S)$ the quotient $R/[R,R]$ is isomorphic to the free abelian group on equivalence classes of words in S under cyclic permutation. In particular it is torsion free and therefore (1) of Proposition B.3 implies that in this case $W(R)$ is a subgroup of $(R/[R,R])[[t]]$. it is also not so hard to characterize the image explicitly in terms of the image of the Teichmüller and compatibility with Verschiebung (discussed below). This approach in the commutative case is developed in [CD15].

We define the 'Verschiebung' maps

$$V_k : W(R) \to W(R)$$

as the homomorphism which sends the power series $f(t) \in 1+tR[[t]]$ to $f(t^k)$. To check that V_k is well-defined one has to show that $(1-rst^k)$ and $(1-srt^k)$ for $r, s \in R$ represent the same element in $W(R)$. This can be checked in the universal case, which is the free ring on two generators r and s. For this free ring the ghost component map $w : W(R) \to (R/[R,R])[[t]]$ is injective by (1) of Proposition B.3. Thus it suffices to show that the images $w(1-rst^k)$ and $w(1-srt^k)$ agree which is straightforward.

The following fact is well-known in the commutative case.

Lemma B.5. *Every element $f \in 1 + tR[[t]]$ can be written as an infinite product*

$$\prod_{n=1}^{\infty}(1 - \alpha_n t^n)$$

for uniquely determined elements $\alpha_n \in R$. In particular $W(R)$ is topologically generated by the elements of the form $V_k\tau(r)$ for $r \in R$ and $k \in \mathbb{N}_{>0}$.

Proof. If we expand the product we get

$$1 - \alpha_1 t - \alpha_2 t^2 + (\alpha_1\alpha_2 - \alpha_3)t^3 + \dots \quad .$$

In general the coefficient in front of t^n will always be of the form $-\alpha_n$ plus a term depending only on α_i's for $i < n$. Thus we can inductively give formulas for the α_n. □

As a result the group $1 + tR[[t]]$ is as a set canonically isomorphic to $R^{\mathbb{N}}$ and we can express the group structure under this bijection in terms of universal non-commutative polynomials. The Witt vectors are then a quotient of $R^{\mathbb{N}}$. The only relations one has to impose are that $(1 - \alpha t^n)$ commutes with $(1 - \beta t^m)$ and that $(1 - rst) = (1 - srt)$. This is the approach taken in [Hes97], where only the p-typical case is treated and not the global one. Note that there are four different conventions possible here, which amount to decomposing power series into factors of the form $(1 \pm \alpha_i)^{\pm}$. See Remark 1.15 in [Hes15] for a discussion. Hesselholt uses the signs $(-,-)$ and then defines the Teichmüller as $r \mapsto (1 - rt)^{-1}$ and then the ghost map accordingly as dlog.

Besides the Verschiebung there is another fundamental map on the Witt vectors, the *Frobenius* $F_k : W(R) \to W(R)$ for every $k \in \mathbb{N}_{>0}$. We want F_k to be a continuous homomorphism with

$$F_k(1 - \alpha t^n) = \left(1 - \alpha^{\frac{k}{\gcd(n,k)}} t^{\frac{n}{\gcd(n,k)}}\right)^{\gcd(n,k)} . \tag{6}$$

Lemma B.6. *For every k the Frobenius F_k exists, is uniquely determined by* (6) *and satisfies the relations $F_k F_l = F_{kl}$ and $F_k V_k = (-)^k$.*

Proof. We have to show that the map

$$\begin{aligned} F_k\colon \quad & W(R) \to W(R) \\ & \prod(1 - \alpha_n t^n) \mapsto \prod F_k(1 - \alpha_n t^n) \end{aligned}$$

with $F_k(1 - \alpha_n t^n)$ as in (6) is a well-defined group homomorphism for every R. It suffices to check this for free rings R in which case we have to show that the composite $W(R) \to W(R) \xrightarrow{w} (R/[R,R])[[t]]$ is a well-defined group homomorphism. We claim that the following diagram commutes (where the left vertical map is not yet well defined)

$$\begin{tikzcd} W(R) \arrow[r, "w"] \arrow[d, "F_k"] & (R/[R,R])[[t]] \arrow[d, "F_k^w"] \\ W(R) \arrow[r, "w"] & (R/[R,R])[[t]] \end{tikzcd} \tag{7}$$

where F_k^w is the map given by sending $\sum \beta_i t^{i-1}$ to $\sum \beta_{ki} t^{i-1}$. To see the commutativity of (7) we can check on linear factors. We have

$$w(1-\alpha t^n) = n\alpha t^{n-1} + n\alpha^2 t^{2n-1} + n\alpha^3 t^{3n-1} + \ldots = \sum c_i t^{i-1}$$

where the coefficients have the form

$$c_i = \begin{cases} n\alpha^{\frac{i}{n}} & \text{if } n|i \\ 0 & \text{otherwise} \end{cases}$$

Now let $g = \gcd(k,n)$. We can compute $F_k^w \circ w$ on linear factors as:

$$\begin{aligned} F_k^w w(1-\alpha t^n) &= \sum c_{ki} t^i \\ &= \sum c_{i\frac{kn}{g}} t^{i\frac{n}{g}-1} \\ &= \sum n\alpha^{i\frac{k}{g}} t^{i\frac{n}{g}-1} \\ &= gw\left(1+\alpha^{\frac{k}{g}} t^{\frac{n}{g}-1}\right) . \end{aligned}$$

This shows the claim and proves F_k is a well-defined homomorphism.

The remaining relations can be checked by direct computation. For the first one, let $g_1 = \gcd(l,n)$, and $g_2 = \gcd\left(k, \frac{n}{g_1}\right)$. Then $g_1 g_2 = \gcd(kl,n)$, and we see:

$$F_k F_l(1-\alpha t^n) = F_k\left(1-\alpha^{\frac{l}{g_1}} t^{\frac{n}{g_1}}\right)^{g_1} = \left(1-\alpha^{\frac{kl}{g_1 g_2}} t^{\frac{n}{g_1 g_2}}\right)^{g_1 g_2} = F_{kl}(1-\alpha t^n) .$$

Finally, we have

$$F_k V_k(1-\alpha t^n) = F_k(1-\alpha t^{kn}) = (1-\alpha t^n)^k .$$

$\square$

Now we want to relate our Witt vectors to the commutative case and product structures. To this end we equip the category 'Ring' of non-commutative rings with a symmetric monoidal structure given by the usual tensor product $R \otimes S$ of rings R and S (this is not the coproduct in the category Ring!). A commutative ring R then gives rise to a commutative algebra object in Ring with respect to this tensor product. In fact the category of commutative rings is equivalent to the category of commutative algebra objects in $(\text{Ring}, \otimes)$.

Our goal is to equip the functor

$$W : \text{Ring} \to \text{Ab}$$

with a canonical lax symmetric monoidal structure for the respective tensor product on both sides. This then provides $W(R)$ for R-commutative with the structure of a commutative ring and more generally all non-abelian Witt vectors with module structures over the commutative ring $W(\mathbb{Z})$. Such a lax symmetric monoidal structure is given by maps

$$* : W(R) \otimes W(S) \to W(R \otimes S)$$

for every pair of rings R and S that are natural in R and S and satisfy a certain compatiblity condition with the associators of the tensor product. One should think of this is a sort of external product.

To this end we will first define a bilinear and continuous map

$$(1 + tR[[t]]) \times (1 + tS[[t]]) \to W(R \otimes S)$$

by requiring

$$(8)\qquad (1-\alpha t^n)*(1-\beta t^m)=\left(1-\left(\alpha^{\frac{m}{\gcd(n,m)}}\otimes\beta^{\frac{n}{\gcd(n,m)}}\right)t^{\frac{nm}{\gcd(n,m)}}\right)^{\gcd(m,n)} .$$

Proposition B.7. *There is a unique bilinear continuous map*

$$*: W(R)\times W(S)\to W(R\otimes S)$$

satisfying (8). *This makes* W : Rings $\to$ Ab *into a lax symmetric monoidal functor, i.e.* $*$ *is symmetric, unital and associative.*

Moreover the transformations $\tau: R\to W(R)$ *and* $w: W(R)\to (R/[R,R])[[t]]$ *are compatible with the external product* $*$*, i.e. lax symmetric monoidal transformations where* $R[[t]]$ *has the 'pointwise' external product* τ *is a transformation of set values functors.*

Proof. Uniqueness is clear, since bilinearity forces

$$\left(\prod(1-\alpha_n t^n)\right)*\left(\prod(1-\beta_m t^m)\right)=\prod_{n,m}(1-\alpha_n t^n)*(1-\beta_m t^m)$$

and (8) determines the latter term. To check well-definedness and bilinearity, it is sufficient to do so in the universal case of a free associative ring. To that end, we first show that w relates $*$ to the pointwise external product $R/[R,R][[t]]\otimes S/[S,S][[t]]\to (R\otimes S)/[R\otimes S,R\otimes S])[[t]]$ given by

$$\sum c_i t^{i-1}\otimes\sum d_i t^{i-1}\mapsto\sum(c_i\otimes d_i)t^{i-1} .$$

When w is injective, this shows well-definedness, bilinearity, commutativity and associativity of $*$. In particular, this holds in the case of R,S free associative rings.

To see this compatibility of w and $*$, write

$$w(1-\alpha t^n)=n\alpha t^{n-1}+n\alpha^2t^{2n-1}+n\alpha^3t^{3n-1}+\dots$$

and

$$w(1-\beta t^m)=m\beta t^{m-1}+n\beta^2t^{2m-1}+n\beta^3t^{3m-1}+\dots$$

The coefficient of the pointwise external product of these terms in $(R\otimes S/[R\otimes S,R\otimes S])[[t]]$ in front of t^{l-1} is 0 unless l is divisible by n and m. In this case the coefficient is given by

$$nm(\alpha^{l/n}\otimes\beta^{l/m}) .$$

So it can be written as a sum over l of the form $i\frac{nm}{\gcd(n,m)}$

$$\sum_i nm\left(\alpha^{\frac{im}{\gcd(n,m)}}\otimes\beta^{\frac{in}{\gcd(n,m)}}\right)\cdot t^{i\frac{nm}{\gcd n,m}}$$

which is equal to

$$\gcd(n,m)\cdot w\left(1-(\alpha^{\frac{m}{\gcd(n,m)}}\otimes\beta^{\frac{n}{\gcd(n,m)}})\cdot t^{\frac{nm}{\gcd n,m}}\right) .$$

which shows the claim.

Finally, the equality

$$\tau(\alpha)*\tau(\beta)=(1-\alpha t)*(1-\beta t)=(1-(\alpha\otimes\beta)t)=\tau(\alpha\otimes\beta)$$

shows that τ is lax monoidal. $\square$

Remark B.8. The compatibility of the external product with the transformation $w : W(R) \to (R/[R,R])[[t]]$ in fact uniquely determines the external product (if it is natural in R) by Proposition B.3. This can be used to define the product which is the standard definition in the commutative case.

The compatibility with the Teichmüller τ also uniquely determines the external product $*$. To see this we have to show that is it enough to know how to multiply linear factors to determine the product

$$(1-\alpha t^n) * (1-\beta t^m)$$

For $\alpha \in R$ and $\beta \in S$. We can reduce to the universal case, i.e. $R = \mathbb{Z}[\alpha]$ and $S = \mathbb{Z}[\beta]$ are polynomial rings. Now choose embeddings $R \to k$ and $S \to k$ into an algebraically closed field k. Since also $W(R \otimes S) \to W(k \otimes k)$ is injective we can thus factor $(1-\alpha t^n)$ and $(1-\beta t^m)$ into linear factors in $W(k)$.

Finally we want to define p-typical and more general versions of Witt vectors. Thus let us consider a subset $S \subseteq \mathbb{N}_{>0}$ with the property that $ab \in S$ implies that $a \in S$ and $b \in S$. We call such a set S a *truncation set.* Important examples are given by the set $\{1, p, p^2, p^3, ..\}$ for a prime p or the set $\langle n \rangle$ of all divisors of a given natural number n. For a given truncation set S we consider the subset $W^S(R) \subseteq W(R)$ consisting of all elements that can be represented by power series in $1 + tR[[t]]$ that are products

$$\prod_{n=1}^{\infty}(1-\alpha_n t^n)$$

with $\alpha_n = 0$ for $n \in S$.

Lemma B.9. *$W^S(R) \subseteq W(R)$ is a subgroup.*

Proof. It suffices to verify that the product of two elements $\prod_{n=1}^{\infty}(1+\alpha_n t^n)$ and $\prod_{n=1}^{\infty}(1+\beta_n t^n)$ with $\alpha_n = \beta_n = 0$ for $n \in S$ is of the form $\prod_{n=1}^{\infty}(1+\gamma_n t^n)$ with $\gamma_n = 0$ for $n \in S$. To see this, we more generally observe that we can rewrite a product of the form $f = \prod_{n=1}^{\infty} P_n(t^n)$ with power series P_n with constant term 1 and $P_n = 1$ for $n \in S$ to a product of the form $\prod_{n=1}^{\infty}(1+\gamma_n t^n)$ with $\gamma_n = 0$ for $n \in S$.

To do this, observe that the lowest nontrivial coefficient of f is in front of t^i, for i minimal such that $P_i \neq 1$. In particular, $f = 1 + ct^i + \ldots$ for $i \notin S$. We can write $P_i(t^i) = (1+ct^i) \cdot \prod_{d \geq 2}(1+c_d t^{di})$, and thus obtain

$$f = (1+ct^i) \cdot \prod_n P'_n(t^n)$$

with $P'_i = 1$, $P'_{di}(s) = (1+c_d s)P_{di}(s)$, and $P'_n = P_n$ for n not divisible by i. Iterating this procedure, we rewrite f as a product of the desired form.

□

Definition B.10. *For a non-commutative ring R and a truncation set $S \subseteq \mathbb{N}_{>0}$ we define an abelian group $W_S(R) := W(R)/W^S(R)$. For a prime p the* non-commutative p-typical Witt vectors *are defined by*

$$W_p(R) := W_{\{1,p,p^2,\ldots\}}(R)\ .$$

There are also truncated versions given by

$$W_{p,n}(R) := W_{\{1,p,p^2,\ldots,p^{n-1}\}}(R) = W_{\langle p^{n-1}\rangle}(R)$$

for $n \geq 1$.

For the truncated Witt vectors there is also a ghost component description. To this end consider for every S the projection

$$\pi_S : (R/[R,R])[[t]] \mapsto \prod_S R/[R,R]$$

which sends $\sum \alpha_n t^{n-1}$ to the familiy $(\alpha_n)_{n\in S}$.

Lemma B.11. *The composition $W(R) \xrightarrow{w} R[[t]] \xrightarrow{\pi_S} \prod_S R/[R,R]$ factors to a group homomorphism*

$$w_S : W_S(R) \to \prod_S R/[R,R]$$

which is injective for R torsion free and an isomorphism for $\mathbb{Q} \subseteq R$.

Proof. For the factorization we have to check that $w : 1 + tR[[t]] \to (R/[R,R])[[t]]$ sends the subgroups $W^S(R) \subseteq W(R)$ to the kernel of π_S. It sends the factor $(1 + \alpha_n t^n)$ for $n \notin S$ to the power series

$$n\alpha_n t^{n-1} - n\alpha_n^2 t^{2n-1} + n\alpha_n^3 t^{3n-1} - ...$$

which lies in the kernel of π_S by the fact that S is a truncation set.

For injectivity in the torsion free case let $f \in W(R)$ be such that $\pi_S w(f) = 0$. We want write f as an infinite product of factors $(1 - \alpha_n t^n)$ with $\alpha_n = 0$ for $n \in S$. Assume inductively that we have written f already as a product $f = \prod(1 - \alpha_n t^n)$ with $\alpha_n = 0$ for $n < N$ and $n \in S$. We can assume $N \in S$, so that we have to get rid of the factor $(1 - \alpha_N t^n)$. The coefficient of $w(f)$ in front of t^{N-1} is given by $N\alpha_N \in R/[R,R]$. By assumption this is zero and torsion freeness implies $\alpha_N \in [R,R]$. As in the proof of Proposition B.3 we can now modify the representative of f to ensure that $\alpha_N = 0$ while keeping the lower factors intact.

Finally the surjectivity in the rational case follows from surjectivity of w. □

For every inclusion $S' \subseteq S$ of truncation sets we have a natural *reduction* map

$$R\colon W_S(R) \to W_{S'}(R) \ .$$

The Verschiebung map for the natural number k gives rise to a map

$$V_k\colon W_{S/k}(R) \to W_S(R)$$

where $S/k = \{n \in \mathbb{N}_{>0} \mid kn \in S\}$. This is well defined because the map $V_k : W(R) \to W(R)$ sends $W^{S/k}(R)$ to $W^S(R)$. Similarly the Frobenius gives a map

$$F_k\colon W_S(R) \to W_{S/k}(R) \ .$$

Proposition B.12. *For a given truncation set S and a positive natural number k let $S' \subseteq S$ be the elements of S that can not be divided by k. This is a truncation set and the sequence*

$$W_{S/k}(R) \xrightarrow{V_k} W_S(R) \xrightarrow{R} W_{S'}(R) \to 0$$

is exact. Moreover the map V_k is injective if $R/[R,R]$ is torsion free or if R is commutative.

Proof. The fact that S' is a truncation set is clear and we have $S = k(S/k) \sqcup S'$. The surjectivity of R is clear since $W_{S'}(R)$ is a quotient of $W_S(R)$ by the image of

$W^{S'}(R)$. This also shows that every element in the kernel of R can be represented by a product

$$\prod_{n\in S}(1-\alpha_n t^n)$$

with $\alpha_n = 0$ for $n \in S'$. Then

$$\prod_{n\in S/k}(1-\alpha_{nk} t^n)$$

represents a preimage under V_k.

Finally we have the relation $F_k \circ V_k = (-)^k$. Thus if $R/[R,R]$ has not torsion, then by Lemma B.11 $W_{S/k}(R)$ has no torsion and thus the map $V_k(R)$ is injective. In the commutative case the map V_k is obviously injective. □

Example B.13. Let m be a natural number not divisible by a prime p. Then there is an exact sequence

$$W_{\langle p^n m\rangle}(R) \xrightarrow{V_{p^k}} W_{\langle p^{n+k} m\rangle}(R) \xrightarrow{R} W_{\langle p^{k-1} m\rangle}(R) \to 0$$

for every $k > 0$ and every n. In particular we get the usual exact sequences

$$W_{p,n}(R) \xrightarrow{V_{p^k}} W_{p,n+k}(R) \xrightarrow{R} W_{p,k}(R) \to 0\ .$$

Proposition B.14. *The functors $W_S : \mathrm{Ring} \to \mathrm{Ab}$ are lax symmetric monoidal for every truncation set S.*

Proof. To see this we claim that the external product

$$*\colon W(R)\otimes W(R') \to W(R\otimes R')$$

takes $W^S(R)\otimes W(R')$ to $W^S(R\otimes R')$. This can be checked on linear factors and follows from the fact that for $n \notin S$ the product is of the form

$$(1-\alpha t^n) * (1-\beta t^m) = \Big(1-\big(\alpha^{\frac{m}{\gcd(n,m)}}\otimes\beta^{\frac{n}{\gcd(n,m)}}\big)t^{\frac{nm}{\gcd(n,m)}}\Big)^{\gcd(m,n)}\quad .$$

We can write this as a product $\prod\big(1-\beta_i t^{\frac{nm}{\gcd(n,m)}\cdot i}\big)$. Thus all non-trivial coefficients appear in front of powers of t that are multiplies of n and therefore do not lie in S.

By symmetry we also have $W(R) * W^S(R') \subseteq W(R\otimes R')$ and therefore get a factorization

$$W_S(R)\otimes W_S(R') = \frac{W(R)\otimes W(R')}{W^S(R)\otimes W(R') + W(R)\otimes W^S(R')} \longrightarrow W_S(R\otimes R')\ .$$

□

Finally let us mention that for a colimit of truncation sets $S = \mathrm{colim} S_i$ we get that $W_S(R) = \varprojlim W_{S_i}(R)$.

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