A modular description of the $K(2)$-local sphere at the prime 3

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Abstract

Using degree N isogenies of elliptic curves, we produce a spectrum $Q(N)$. This spectrum is built out of spectra related to tmf. At $p = 3$ we show that the $K(2)$-local sphere is built out of $Q(2)$ and its $K(2)$-local Spanier-Whitehead dual. This gives a conceptual reinterpretation a resolution of Goerss, Henn, Mahowald, and Rezk.

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Contents

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Introduction

Goerss, Henn, Mahowald, and Rezk [16] have produced at the prime 3 a tower of spectra of the form

$$\begin{aligned} TMF \to TMF \vee \Sigma^8 E(2) \to \Sigma^8 E(2) \vee \Sigma^{40} E(2) \\ \to \Sigma^{40} E(2) \vee \Sigma^{48} TMF \to \Sigma^{48} TMF. \end{aligned}$$

Here, as is the case everywhere in this paper unless indicated otherwise, we are working in the $K(2)$-local category, and everything has been implicitly $K(2)$-localized. The authors of [16] show that the tower *refines* to a resolution of the $K(2)$-local sphere at the prime 3. This means that there exists a diagram

$$\begin{tikzcd} L_{K(2)}S \arrow[d] & X_1 \arrow[l] \arrow[d] & X_2 \arrow[l] \arrow[d] & X_3 \arrow[l] \arrow[d] & X_4 \arrow[l] \arrow[d, equal] \\ TMF & \Sigma^{-1}TMF \vee \Sigma^7 E(2) & \Sigma^6 E(2) \vee \Sigma^{38} E(2) & \Sigma^{37} E(2) \vee \Sigma^{45} TMF & \Sigma^{44} TMF \end{tikzcd}$$

such that each spectrum X_i is the fiber of the ith vertical map, and such that the connecting morphisms of these fiber sequences give the maps in the tower.

This resolution helps organize the computation of Shimomura and Wang of $\pi_*(L_{K(2)}S^0)$. We shall refer to this as the GHMR resolution.

The spectrum tmf is the connective spectrum of topological modular forms. We write TMF for the non-connective spectrum $tmf[\Delta^{-1}]$, but this distinction is irrelevant, since we have

$$L_{K(2)}tmf \simeq L_{K(2)}TMF \simeq EO_2 = E_2^{hG_{24}}.$$

Here E_2 is Morava E-theory and G_{24} is a maximal finite subgroup of the extended Morava stabilizer group $\mathbb{G}_2$ at the prime 3. The spectrum EO_2 exists by the Hopkins-Miller theorem [40], and is E_∞ by the machinery of Goerss and Hopkins [17]. The equivalence $L_{K(2)}TMF \simeq EO_2$ is discussed in greater detail in Remark 1.7.3.

Remark. It has become standard to let E_2 denote the Landweber exact cohomology theory corresponding to the Lubin-Tate deformation of the Honda height 2 formal group F_2, defined over $\mathbb{F}_{p^2}$, with p-series

$$[p]_{F_2} = x^{p^2}.$$

However, for the 3-primary applications in this paper, we shall let E_2 denote the spectrum corresponding to the Lubin-Tate deformation of the height 2 formal group F_2' over $\mathbb{F}_9$ whose 3-series is given by

$$[3]_{F_2'} = -x^9.$$

Our reason for doing this is that F_2' is the formal group of the unique supersingular elliptic curve over $\mathbb{F}_9$ (see Section 1.6, in particular Remark 1.6.4).

The GHMR resolution is a generalization to chromatic level 2 of the J-spectrum fiber sequence

$$L_{K(1)}S^0 \to KO_p \xrightarrow{\psi^N - 1} KO_p.$$

That this gives the $K(1)$-local sphere was first observed in unpublished work of Adams and Baird, and an independent verification of was provided by Ravenel [38], [6]. Here N is a topological generator of $\mathbb{Z}_p^\times$.

There is a spectrum $TMF_0(2)$ which is an analog of TMF for the congruence subgroup $\Gamma_0(2) < SL_2(\mathbb{Z})$. We have equivalences

$$L_{K(2)}TMF_0(2) \simeq E_2^{hD_8} \simeq L_{K(2)}(E(2) \vee \Sigma^8 E(2))$$

where $D_8 < \mathbb{G}_2$ is a dihedral group of order 8. After inverting 6, the spectrum $TMF_0(2)$ coincides with the elliptic cohomology theory Ell of Landweber, Ravenel, and Stong [28].

The GHMR tower may be made less efficient, to take the form

$$\begin{aligned} TMF \to TMF \vee TMF_0(2) \to TMF_0(2) \vee \Sigma^{48} TMF_0(2) \\ \to \Sigma^{48} TMF_0(2) \vee \Sigma^{48} TMF \to \Sigma^{48} TMF \end{aligned}$$

by letting the extra copies of $E(2)$ kill themselves pairwise. It is this resolution that we shall endeavor to reinterpret.

The GHMR resolution was produced using the work of Devinatz and Hopkins [12], where the $K(2)$-local sphere is identified as a homotopy fixed point spectrum

$$E_2^{h\mathbb{G}_2} \simeq L_{K(2)} S^0.$$

The authors of [16] produced a resolution of the trivial $\mathbb{G}_2$-module by permutation modules. The authors then realize their resolution to produce the tower.

Mahowald and Rezk wanted a modular description of the GHMR resolution. The motivation is that the Adams operation ψ^N corresponds to the N^{th} power isogeny on the multiplicative group $\mathbb{G}_m$. By replacing the multiplicative group with an elliptic curve, one can instead consider certain degree N isogenies of elliptic curves. Mahowald and Rezk studied the corresponding map

$$TMF \xrightarrow{"\psi^N - 1"} TMF_0(N)$$

at the primes 2 and 3 [30], [29]. This setup is complicated by the fact that there is a whole moduli space of elliptic curves, and elliptic curves do not support unique degree N isogenies. The map that Mahowald and Rezk studied is a projection of the first map of the GHMR resolution. Ando, Hopkins, and Strickland have also extensively studied operations on elliptic cohomology theories arising from isogenies of elliptic curves [1], [2], [3].

The aim of this paper is expand on the work of Mahowald and Rezk to produce a tower

$$TMF \to TMF_0(N) \vee TMF \to TMF_0(N)$$

which refines to a resolution of an E_∞ ring spectrum $Q(N)$. Here N is prime to p. This tower arises conceptually from certain degree N isogenies of elliptic curves. Rezk had considered a similar setup from the point of view of flags of subgroups of an elliptic curve. For a spectrum X, let $D_{K(2)}(X)$ denote the Spanier-Whitehead dual in the $K(2)$-local category

$$D_{K(2)}(X) = F(X, L_{K(2)} S).$$

We conjecture that for appropriate N, the spectrum $Q(N)$ is half of the $K(2)$-local sphere.

Conjecture 1 *Let p be greater than 2. If N is chosen to be a topological generator of $\mathbb{Z}_p^\times$, then the natural sequence*

$$D_{K(2)}Q(N) \xrightarrow{D\eta} L_{K(2)}S \xrightarrow{\eta} Q(N)$$

is a cofiber sequence. Here η is the unit of the $K(2)$-local ring spectrum $Q(N)$.

If $p = 2$, and N is a topological generator of an index 2 subgroup of $\mathbb{Z}_2^\times$, then there is an index 2 subgroup $\widetilde{\mathbb{G}}_2$ of $\mathbb{G}_2$ and a cofiber sequence

$$D_{K(2)}Q(N) \xrightarrow{D\eta} E_2^{h\widetilde{\mathbb{G}}_2} \xrightarrow{\eta} Q(N).$$

Note that the group $\mathbb{Z}_2^\times$ is not topologically cyclic, but it does have topologically cyclic index 2 subgroups.

The main result of this paper is to verify Conjecture 1 at the prime 3, with $N = 2$ (Theorem 2.0.1). The author also knows Conjecture 1 to hold in the case $p = 5$ and $N = 2$. The author has no evidence for the invariance of Conjecture 1 under different choices of N.

This approach to understanding the $K(2)$-local sphere has several advantages. The maps building $Q(N)$ are quite computable using well known formulas for Weierstrass curves and have very natural descriptions in terms of isogenies of elliptic curves. The cofiber sequence of Conjecture 1 explains the self-duality of the GHMR resolution, and an identification of $D_{K(2)}TMF$ explains the appearance of the suspension Σ^{48}. The very difficult computations of Shimomura and Wang of the homotopy of $L_{K(2)}S$ should be verified independently using our decomposition. As a side-effect of our work we also reproduce the short tower (Proposition 2.9.1)

$$TMF \to \Sigma^8 E(2) \to \Sigma^{40} E(2) \to \Sigma^{48} TMF \tag{0.0.1}$$

which refines to the spectrum $\overline{S} = E^{h\mathbb{G}_2^1}$ given in [16]. Here $\mathbb{G}_2^1$ is the kernel of the reduced norm

$$\mathbb{G}_2 \to \mathbb{Z}_3.$$

We are not able to describe the connecting map

$$Q(N) \to \Sigma D_{K(2)}Q(N)$$

of Conjecture 1, nor are we able to describe the middle map of the short tower (0.0.1).

Our approach to proving Theorem 2.0.1 is computational, and this is the reason for our specialization to the prime 3 and $N = 2$. It would be nice to have a conceptual and elegant proof of Conjecture 1 for all primes and all N. The 2-primary applications with $N = 3$ should be rewarding. There is essentially nothing to be gained computationally from Conjecture 1 if $p \geq 5$.

This paper is organized into two parts. A detailed outline of the content is given at the beginning of each part. In Part 1 we give a construction of $Q(N)$, and quickly specialize to the case $N = 2$ and $p = 3$. In this case we give a computation of the $V(1)$-homology groups $V(1)_*Q(2)$ where $V(1)$ is the Smith-Toda complex. In Part 2 we prove Conjecture 1 in the case $N = 2$ and $p = 3$ (Theorem 2.0.1). The proof uses the $V(1)$ homology computations given in Part 1 as well as the computation of the $V(1)$ homology of $L_{K(2)}S^0$ — and thus does not generalize to arbitrary N and p.

Everything in this paper is implicitly $K(2)$-localized unless specifically specified otherwise. Sections for which this convention does not hold are indicated as such in their beginnings.

We highlight a few aspects of this paper which may be of independent interest. The duality decomposition of Conjecture 1 may be interpreted as a Lagrangian decomposition for a hyperbolic duality pairing on the $K(2)$-local sphere $L_{K(2)}S$. The abstract framework of duality pairings and Lagrangians in a triangulated symmetric monoidal category is given in Section 2.1.

In Section 2.3, we show that for arbitrary n, at an arbitrary prime p, the canonical pairing on $L_{K(n)}S$ is hyperbolic, and $\overline{S} = E_n^{h\mathbb{G}_n^1}$ is a Lagrangian. We identify the $K(n)$-local Spanier-Whitehead dual of $\overline{S}$ by proving there is an equivalence

$$D_{K(n)}(\overline{S}) \simeq \Sigma^{-1}\overline{S}.$$

Finally, for computational reasons, we need a cellular decomposition of $\overline{S}$ in the $K(n)$-local category. This task is relegated to Appendix A, where we prove that there is a ($K(n)$-local) cellular decomposition

$$\overline{S} \simeq S^0 \cup_\zeta e^0 \cup_\zeta e^0 \cup_\zeta \cdots .$$

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Part 1: The spectrum $Q(N)$

This part is organized as follows. In Section 1.1 we produce a simplicial stack based on certain degree N isogenies.

In Section 1.2 we topologically realize our simplicial stack to a cosimplicial spectrum whose totalization is our spectrum $Q(N)$. We give two approaches to this realization problem. The first is based on a sheaf of E_∞ ring spectra produced by Hopkins and his collaborators. The second, specialized to $p = 3$ and $N = 2$, uses the Goerss-Hopkins-Miller theory and a supersingular elliptic curve.

In Section 1.3 we give a description of the ring of $\Gamma_0(2)$ modular forms in terms of Weierstrass equations. The results of this section are probably well known.

Our computations of the homotopy of $Q(2)$ are based on the ANSS. In Section 1.4, we describe a chain complex which computes the Adams-Novikov E_2 term for $Q(2)$. This ANSS E_2-term is the hypercohomology of the simplicial stack we constructed in Section 1.1.

Section 1.5 is devoted to explicitly computing the maps on rings of modular forms induced by the face maps of our simplicial stack. From these maps we get the differentials in the chain complex of Section 1.4. These explicit formulas are also necessary to complete the construction of $Q(2)$ given in Section 1.2.1.

In Section 1.6 we describe explicitly the automorphism groups of a supersingular elliptic curve C and its formal group $C^\wedge$. In Section 1.7 we use this description to describe the effects that our maps have on E_2-homology. These formulas will be used in many technical lemmas in Part 2, and also may be used in the construction of $Q(2)$ given in Section 1.2.2.

Our approach to Theorem 2.0.1 is computational and based on $V(1)$-homology computations. In Section 1.8, we compute the $V(1)$-homology of $Q(2)$. This computation is compared to computations of Goerss, Henn, Mahowald, and

Shimomura.

1.1 *A simplicial stack*

Let $\mathcal{M}$ be the moduli stack of non-singular elliptic curves over $\mathbb{Z}_{(p)}$. Let ω be the line bundle of invariant differentials over $\mathcal{M}$, so that the sections of $\omega^{\otimes k}$ give weight k weakly modular forms.

$$H^0(\mathcal{M}, \omega^{\otimes k}) = MF_k$$

MF_* is the ring of weakly modular forms over $\mathbb{Z}_{(p)}$ [8].

$$MF_* = \mathbb{Z}_{(p)}[c_4, c_6, \Delta^{\pm 1}]/(c_4^3 - c_6^2 = 1728\Delta)$$

We must invert Δ because we are taking sections over the complement of the singular locus of the moduli space of generalized elliptic curves. We shall refer to the ring of modular forms (without inverting the discriminant) as mf_*.

Given an elliptic curve C, we shall mean by a $\Gamma_0(N)$ structure a chosen discrete subgroup of C which is isomorphic to C_N, the cyclic group of order N, but we do *not* fix the isomorphism. $\Gamma_0(N)$ structures are in one to one correspondence with degree N isogenies with cyclic kernels. Given a $\Gamma_0(N)$ structure H of C, one has an isogeny

$$\phi_H : C \to C/H$$

and given such an isogeny, one recovers the $\Gamma_0(N)$ structure by taking the kernel. We define $\mathcal{M}_0(N)$ to be the moduli stack of non-singular elliptic curves over $\mathbb{Z}_{(p)}$ with $\Gamma_0(N)$ structure. We shall always be considering N prime to p.

Moduli stacks of elliptic curves with additional structure have been widely studied by arithmetic algebraic geometers. See, for example, [9], [27].

Given an integer N, every elliptic curve has an N^{th} power endomorphism

$$[N] : C \to C$$

given (for positive N) by

$$[N](P) = \underbrace{P + \cdots + P}_{N}.$$

It is an isogeny of degree N^2.

Given any degree N isogeny ϕ, there exists a dual isogeny $\widehat{\phi}$. The dual isogeny has the property that

$$\widehat{\phi} \circ \phi = [N].$$

See, for instance, [45, III.6].

We shall now describe a simplicial stack $\mathcal{M}_\bullet$. Actually $\mathcal{M}_\bullet$ is a semi-simplicial stack since we do not use degeneracies. It has only 0, 1, and 2 simplices. We will describe this simplicial object first in conceptual terms, and then in precise stack theoretic language. The simplices will be given as follows:

- The 0-simplices are elliptic curves.
- The 1-simplices are certain isogenies. There will be two types:
  - · Degree N isogenies ϕ_H with cyclic kernel H. These are in one-to-one correspondence with elliptic curves C with a $\Gamma_0(N)$ structure H.
  - · The endomorphisms $[N]$. These are in one-to-one correspondence with elliptic curves, since every elliptic curve possesses a unique such endomorphism.
- The 2-simplices correspond to relations of the form $\widehat{\phi}_H \circ \phi_H = [N]$. These relations are in one-to-one correspondence with elliptic curves with a $\Gamma_0(N)$ structure H, since there is one such relation for every isogeny ϕ_H.

The semi-simplicial stack $\mathcal{M}_\bullet$ is a diagram

$$\mathcal{M} \overset{d_0}{\underset{d_1}{\leftleftarrows}} \mathcal{M} \coprod \mathcal{M}_0(N) \begin{array}{c}\xleftarrow{d_0}\\[-4pt] \xleftarrow{d_1}\\[-4pt] \xleftarrow{d_2}\end{array} \mathcal{M}_0(N)$$

in the category of stacks. Given a $\mathbb{Z}_{(p)}$-algebra R, the R-points of $\mathcal{M}$ is a groupoid of elliptic curves C over R, and the R-points of $\mathcal{M}_0(N)$ is a groupoid of pairs (C, H) of elliptic curves C over R with $\Gamma_0(N)$ structure $H \leq C(R)$. The simplicial stack $\mathcal{M}_\bullet$ should be viewed on the level of R-points as consisting of moduli of diagrams of the following form.

$$\{C\} \begin{array}{c}\xleftarrow{d_0}\\ \xleftarrow{d_1}\end{array} \begin{array}{c} \left\{(C,H) \xrightarrow{\phi_H} C/H\right\} \\ \coprod \\ \left\{C \xrightarrow{[N]} C\right\}\end{array} \begin{array}{c}\xleftarrow{d_0}\\ \xleftarrow{d_1} \\ \xleftarrow{d_2}\end{array} \left\{ \begin{tikzcd} (C,H) \arrow[rr,"{[N]}"] \arrow[dr,"\phi_H"'] & & C \\ & C/H \arrow[ur,"\widehat{\phi}_H"'] & \end{tikzcd} \right\}$$

The face maps are what one might expect from taking the nerve of a category. To give them we must define some maps of stacks. These maps of stacks are given on the level of R-points by the formulas below.

- Forget $\Gamma_0(N)$ structure:

$$\begin{aligned} \phi_f : \mathcal{M}_0(N) &\to \mathcal{M} \\ (C, H) &\mapsto C \end{aligned}$$

- Quotient out by level $\Gamma_0(N)$ structure:

$$\begin{aligned}\phi_q : \mathcal{M}_0(N) &\to \mathcal{M}\\ (C,H) &\mapsto C/H\end{aligned}$$

- Codomain of the N^{th} power endomorphism:

$$\begin{aligned}\psi_{[N]} : \mathcal{M} &\to \mathcal{M}\\ C &\mapsto C/C[N]\end{aligned}$$

- Dual $\Gamma_0(N)$ structure (the dual $\Gamma_0(N)$ structure $\widehat{H}$ is defined such that $\phi_{\widehat{H}} = \widehat{\phi}_H$):

$$\begin{aligned}\psi_d : \mathcal{M}(N) &\to \mathcal{M}(N)\\ (C,H) &\mapsto (C/H, \widehat{H})\end{aligned}$$

We comment that there is a canonical isomorphism

$$r_C : C/C[N] \cong C.$$

This isomorphism is *not* an equality, and this distinction is important in the context of stacks. We may use this isomorphism to define $\psi_{[N]}$ on $\mathcal{M}_0(N)$ by

$$\begin{aligned}\psi_{[N]} : \mathcal{M}_0(N) &\to \mathcal{M}_0(N)\\ (C,H) &\mapsto (C/C[N], r_C^{-1}(H)).\end{aligned}$$

We note that we have the following relations

$$\phi_q(C) = \phi_f(\psi_d(C)) \tag{1.1.1}$$
$$\psi_d \circ \psi_d = \psi_{[N]}. \tag{1.1.2}$$

The first relation indicates that if we have defined ϕ_f and ψ_d, then ϕ_q is automatically determined.

The careful reader will be bothered that the maps ϕ_f, $\psi_{[N]}$, and ψ_d are not defined above in a precise manner. In Section 1.5, we give explicit formulas for $N = 2$ and $p = 3$ for these maps on the level of Hopf algebroids. The stackifications of these Hopf algebroids are $\mathcal{M}$ and $\mathcal{M}_0(2)$, so we get induced maps of stacks. We can then check relations 1.1.1 and 1.1.2 explicitly. Since we are only concerned with the case $p = 3$, $N = 2$ in this paper, we choose to not elaborate further on the general case.

We use these maps of stacks to give the face maps of $\mathcal{M}_\bullet$. The face maps $d_i : \mathcal{M} \coprod \mathcal{M}_0(N) \to \mathcal{M}$ are defined by

$$\begin{aligned}d_0 &= \psi_{[N]} \coprod \phi_q\\ d_1 &= Id_{\mathcal{M}} \coprod \phi_f\end{aligned}$$

The face maps $d_i : \mathcal{M}_0(N) \to \mathcal{M} \coprod \mathcal{M}_0(N)$ are defined by

$$\begin{aligned} d_0 &= \psi_d \\ d_1 &= \phi_f \\ d_2 &= Id_{\mathcal{M}_0(N)} \end{aligned}$$

The simplicial identities are verified by relations 1.1.1 and 1.1.2.

1.2 Realizing the simplicial stack

We wish to topologically realize the simplicial stack $\mathcal{M}_\bullet$ as a cosimplicial E_∞ ring spectrum $Q(N)^\bullet$. Actually, $Q(N)^\bullet$ is a semi-cosimplicial E_∞ ring spectrum, since we have no codegeneracies.

$$\begin{tikzcd} TMF \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & TMF \times TMF_0(N) \arrow[r, shift left=2, "d_2"] \arrow[r, "d_1"] \arrow[r, shift right=2, "d_0"'] & TMF_0(N) \end{tikzcd}$$

We then define the spectrum $Q(N)$ by

$$Q(N) = Tot(Q(N)^\bullet).$$

The spectrum $Q(N)$ is an E_∞ ring spectrum since the coface maps are maps of E_∞ ring spectra.

We give two approaches to constructing $Q(N)^\bullet$. The first is based on a construction of tmf due to Hopkins and his collaborators as the global sections of a sheaf of E_∞ ring spectra. Since this work is unpublished, we give an alternative construction for $p = 3$ and $N = 2$ using the Goerss-Hopkins refinement of the Hopkins-Miller theorem. The latter approach is sufficient for our purposes since we are working $K(2)$-locally.

1.2.1 Sheaf theoretic construction

Hopkins and his collaborators have constructed a sheaf $\mathcal{O}_{ell}$ of E_∞ ring spectra on $\mathcal{M}$ in the étale topology. Paul Goerss has written a useful survey of this point of view [14]. The sheaf $\mathcal{O}_{ell}$ has the property that if C is any elliptic curve over R which is étale over $\mathcal{M}$, then the homotopy groups of the sections over C are given by

$$\pi_{2k}(\mathcal{O}_{ell}(C)) \cong \omega_C^{\otimes k}$$

where ω_C is the free R-module of holomorphic 1-forms on C. One recovers the spectra TMF and $TMF_0(N)$ as the sections of this sheaf over the appropriate moduli stacks.

$$
\begin{aligned}
TMF &= \mathcal{O}_{ell}(\mathcal{M}) \\
TMF_0(N) &= \mathcal{O}_{ell}(\mathcal{M}_0(N))
\end{aligned}
$$

Since $\mathcal{O}_{ell}$ is a sheaf in the étale topology, we need the map

$$\phi_f : \mathcal{M}_0(N) \to \mathcal{M}$$

to be étale to take sections over $\mathcal{M}_0(N)$. This is why we insist that p does not divide N.

Our requirement that p does not divide N also implies that the maps

$$
\begin{aligned}
\psi_{[N]} &: \mathcal{M}_0(N) \to \mathcal{M}_0(N) \\
\psi_{[N]} &: \mathcal{M} \to \mathcal{M}
\end{aligned}
$$

are isomorphisms. Relation 1.1.2 implies that

$$\psi_d : \mathcal{M}_0(N) \to \mathcal{M}_0(N)$$

is an isomorphism. We have already indicated that the map ϕ_f is étale. Relation 1.1.1 indicates that ϕ_q is étale. We therefore get induced maps

$$
\begin{aligned}
\phi_f^* :& TMF = \mathcal{O}_{ell}(\mathcal{M}) \to \mathcal{O}_{ell}(\mathcal{M}_0(N)) = TMF_0(N) \\
\phi_q^* :& TMF = \mathcal{O}_{ell}(\mathcal{M}) \to \mathcal{O}_{ell}(\mathcal{M}_0(N)) = TMF_0(N) \\
\psi_{[N]}^* :& TMF = \mathcal{O}_{ell}(\mathcal{M}) \to \mathcal{O}_{ell}(\mathcal{M}) = TMF \\
\psi_d^* :& TMF_0(N) = \mathcal{O}_{ell}(\mathcal{M}_0(N)) \to \mathcal{O}_{ell}(\mathcal{M}_0(N)) = TMF_0(N).
\end{aligned}
$$

These maps induce the coface maps d_i of $Q(N)^\bullet$.

1.2.2 Supersingular construction

We now give an alternative construction of $Q(2)^\bullet$ at $p = 3$ that uses computations of endomorphisms of the supersingular elliptic curve of Section 1.6. This approach to TMF mirrors the approach in [21]. The Goerss-Hopkins-Miller theorem [40], [17] states that there is a contravariant functor

$$E : \mathcal{FGL}_n \to E_\infty \textit{ ring spectra}$$

where $\mathcal{FGL}_n$ is the category of pairs (k, F) where k is a perfect field of characteristic p and F is a formal group law of height n over k. Consider the supersingular elliptic curve

$$C : y^2 = x^3 - x$$

over $\mathbb{F}_9$. The Weierstrass curve C has a canonical coordinate, and gives rise to a formal group law $C^\wedge$ of height 2. The curve C has a $\Gamma_0(2)$ structure H generated by the point $(x, y) = (0, 0)$ (see Section 1.3). In Section 1.7, we produce an explicit degree 2 isogeny

$$\psi_H : C \to C$$

with $\ker \psi_H = H$. We have extended automorphism groups (including automorphisms of the ground field $\mathbb{F}_9$) of elliptic curves (respectively, elliptic curves with $\Gamma_0(2)$ structures) given by

$$\begin{aligned} \mathrm{Aut}_{/\mathbb{F}_3}(C) &= G_{24} \\ \mathrm{Aut}_{/\mathbb{F}_3}(C, H) &= D_8. \end{aligned}$$

Here G_{24} and D_8 correspond to certain subgroups of $\mathbb{G}_2$. These subgroups are described explicitly in Sections 1.6 and 1.7. The map ψ_H is invariant under D_8, since every element of D_8 fixes the $\Gamma_0(2)$ structure H.

The value of the Goerss-Hopkins-Miller functor E on $(\mathbb{F}_9, C^\wedge)$ is the version of Morava E-theory that we will be using.

$$E_{(\mathbb{F}_9, C^\wedge)} = E_2$$

The map ψ_H induces an isomorphism of formal group laws

$$\widehat{\psi}_H : C^\wedge \to C^\wedge$$

invariant under the action of D_8, and the Goerss-Hopkins-Miller theorem implies that it induces a map of E_∞ ring spectra

$$\psi_d^* : TMF_0(2) = E_2^{hD_8} \to E_2^{hD_8} = TMF_0(2).$$

We verify directly in Section 1.7 that $(\psi_d^*)^2 = \psi_{[2]}^*$, where $\psi_{[2]}^*$ is the Adams operation corresponding to the action of the element $2 \in \mathbb{W}_{\mathbb{F}_9}^\times \subset \mathbb{S}_2$ of the Morava stabilizer group. Define ϕ_f^* to be the restriction

$$\phi_f^* = Res_{D_8}^{G_{24}} : TMF = E_2^{hG_{24}} \to E_2^{hD_8}.$$

We get ϕ_q^* for free by defining it to be the composite

$$\phi_q^* : TMF \xrightarrow{\phi_f^*} TMF_0(2) \xrightarrow{\psi_d^*} TMF_0(2).$$

We now can define the coface maps d_i in terms of ϕ_f^*, $\psi_{[2]}^*$, ψ_d^*, and ϕ_q^* as before, and we get our cosimplicial spectrum $Q(2)^\bullet$.

The idea is that $K(2)$-locally, we only need to consider sections of $\mathcal{O}_{ell}$ in a formal neighborhood of the supersingular locus, and that is precisely what the Goerss-Hopkins-Miller theorem is implicitly doing at chromatic level 2.

1.3 $\Gamma_0(2)$ modular forms

1.3.1 Moduli description of $mf_0(2)$

In order to compute the maps in the resolution for $p = 3$ and $N = 2$ we need to know something about $\Gamma_0(2)$ modular forms. Most of the computations in this section have been carried out by Mark Mahowald and Charles Rezk, or are just in the literature on modular forms. It is well known (see for instance, the appendix of [18]) that the ring of $\Gamma_0(2)$ modular forms is given (with 2 inverted) by

$$mf_0(2)_*[1/2] = \mathbb{Z}[1/2][\delta, \epsilon]$$

where the weights are given by

$$|\delta| = 2$$
$$|\epsilon| = 4.$$

The discriminant is given by

$$\Delta = -64\epsilon^2(\epsilon + \delta^2).$$

Denote the ring with Δ inverted by

$$MF_0(2)_*[1/2] = mf_0(2)_*[1/2, \Delta^{-1}].$$

Let $\mathcal{M}_0(2)$ be the moduli stack of non-singular elliptic curves with $\Gamma_0(2)$ structure. The ring $MF_0(2)$ should be interpreted as sections of tensor powers of the line bundle ω over this stack.

$$MF_0(2)_k = H^0(\mathcal{M}_0(2), \omega^{\otimes k})$$

We can eliminate our use of the line bundle ω if we add the data of a non-zero tangent vector to the structures on elliptic curves we are taking moduli of. Namely, let $\mathcal{M}_0^1(2)$ be the moduli stack of non-singular elliptic curves with the data of a $\Gamma_0(2)$ structure and a non-zero tangent vector at the identity. Then we have

$$MF_0(2)_* = H^0(\mathcal{M}_0^1(2), \mathcal{O})$$

where $\mathcal{O}$ is the structure sheaf of $\mathcal{M}_0^1(2)$. There is an action of the multiplicative group $\mathbb{G}_m$ on $\mathcal{M}_0^1(2)$; the action is given by the multiplication action on the chosen tangent vector. Under this action, the sections above break up into a direct sum as the weights of the $\mathbb{G}_m$ representations. These coincide with the weights of the modular forms.

1.3.2 Calculation of $MF_0(2)_*[1/2]$ using Weierstrass curves

We shall give an alternate description of the generators of $MF_0(2)_*[1/2]$ using Weierstrass equations. For the rest of this section we will be implicitly working with 2 inverted. Under this condition, every Weierstrass curve is isomorphic to one of the form [45]

$$C_{\mathbf{b}} : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6.$$

This curve is nonsingular if the discriminant

$$\Delta = -27b_6^2 + (9b_2b_4 - \frac{1}{4}b_2^3)b_6 - 8b_4^3 + \frac{1}{4}b_2^2b_4^2$$

is a unit. We shall only consider non-singular curves. We implicitly think of these curves as coming with the data of the tangent vector $\partial/\partial z$ where $z = x/y$. The only transformations which leave this form of equation and tangent vector invariant are those of the form

$$\chi_r : x \mapsto x + r, y \mapsto y.$$

A $\Gamma_0(2)$ structure is the choice of a point of exact order 2 in $C_{\mathbf{b}}$. These points coincide with the points (x, y) of the curve $C_{\mathbf{b}}$ with $y = 0$. Thus a $\Gamma_0(2)$ structure consists of a chosen root e_1 of the cubic $4x^3 + b_2x^2 + 2b_4x + b_6$. The curves $C_{\mathbf{b}}$ with this additional data may be written in the form

$$C_{\gamma,e_1} : y^2 = 4(x - e_1)(x^2 + \gamma_2x + \gamma_4)$$

where we have

$$\begin{aligned} b_2 &= 4(\gamma_2 - e_1) \\ b_4 &= 2(\gamma_4 - e_1\gamma_2) \\ b_6 &= -4\gamma_4e_1. \end{aligned}$$

Given the root e_1 and the coefficients b_2, b_4, b_6, the coefficients γ_2 and γ_4 may be expressed as

$$\begin{aligned} \gamma_2 &= \frac{b_2 + e_1}{4} \\ \gamma_4 &= \frac{4b_4 + e_1b_2 + e_1^2}{8}. \end{aligned}$$

The data of the $\Gamma_0(2)$ structure allows us to remove the automorphism χ_r. We simply shift the root e_1 to be 0 by the transformation χ_{e_1}. This puts C_{γ,e_1} into the canonical form

$$C_{\mathbf{q}} : y^2 = 4x(x^2 + q_2x + q_4).$$

The curve $C_{\mathbf{q}}$ is regarded as implicitly carrying the data of the tangent vector $\partial/\partial z$ and the $\Gamma_0(2)$ structure generated by the point $(x, y) = (0, 0)$. After the transformation χ_{e_1}, one has

$$q_2 = 2e_1 + \gamma_2 \tag{1.3.1}$$
$$q_4 = e_1^2 + \gamma_2 e_1 + \gamma_4. \tag{1.3.2}$$

The discriminant of the curve $C_{\mathbf{q}}$ is given by

$$\Delta = q_4^2(16q_2^2 - 64q_4).$$

There are no non-trivial automorphisms of the curves $C_{\mathbf{q}}$ preserving the $\Gamma_0(2)$ structure and the tangent vector, thus we have shown that the stack $\mathcal{M}_0^1(2)$ is an affine scheme.

$$\mathcal{M}_0^1(2) = \operatorname{spec}(\mathbb{Z}[1/2, q_2, q_4, \Delta^{-1}])$$

The ring $MF_0(2)_*[1/2]$ is just the ring of functions.

$$MF_0(2)_*[1/2] = \mathbb{Z}[1/2, q_2, q_4, \Delta^{-1}]$$

1.3.3 Relation to δ and ϵ

We now relate our generators q_2 and q_4 to the classical generators δ and ϵ that appear in the Jacobi quartic. Assume we are working over an algebraically closed field k of characteristic not equal to 2 or 3. An elliptic curve can be expressed by a Weierstrass equation

$$C_{\mathbf{e}} : y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

with $e_1 + e_2 + e_3 = 0$. (This last condition is equivalent to the Weierstrass equation taking the form $y^2 = 4x^3 - g_2 x - g_3$.) We can take e_1 to be the specified $\Gamma_0(2)$ structure. In [18], the quantities δ and ϵ are then given by

$$\delta = -\frac{3}{2}e_1$$
$$\epsilon = (e_1 - e_2)(e_1 - e_3).$$

By multiplying out the factors containing e_2 and e_3, we see that $C_{\mathbf{e}}$, with $\Gamma_0(2)$ structure e_1, is given by the curve C_{γ, e_1} with

$$\gamma_2 = -e_2 - e_3$$
$$\gamma_4 = e_2 e_3.$$

Using Equations 1.3.1 and 1.3.2, and the relation $e_1 + e_2 + e_3 = 0$, we see that

$$q_2 = -2\delta$$
$$q_4 = \epsilon.$$

Thus our generators q_2 and q_4 of $mf_0(2)[1/2]$ are, up to scaler multiple, identical to the classical generators δ and ϵ.

1.4 The Adams-Novikov E_2-term of $Q(2)$

There are several approaches to computing the homotopy groups of the spectrum $Q(2)$. One could compute the maps that ϕ_q, ϕ_f, ψ_d, and $\psi_{[2]}$ induce on the homotopy groups of TMF and $TMF_0(2)$, and then use the Bousfield-Kan spectral sequence for $Tot(Q(2)^\bullet)$, but it is actually easier to compute the E_2-term of the Adams-Novikov spectral sequence (ANSS) for $Q(2)$ and then add in the Adams-Novikov differentials using the differentials in the ANSS for TMF. The discussion of the ANSS for TMF in this section follows [41] and [21]. Everything in this section is implicitly 3-local, though most of what we say holds if we simply invert 2.

1.4.1 The ANSS E_2 term for TMF.

We explain how to compute the ANSS E_2-term for TMF using an étale cover of $\mathcal{M}^1$. This material has appeared elsewhere, see [21], [41], and [4]. As mentioned in Section 1.3.2, over a $\mathbb{Z}_{(3)}$-algebra every elliptic curve is isomorphic to one of the form

$$C_{\mathbf{b}} : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6.$$

The automorphisms of this elliptic curve which preserve the tangent vector are of the form

$$\chi_r : x \mapsto x + r.$$

Consider the elliptic curve Hopf algebroid

$$\begin{aligned} B &= \mathbb{Z}_{(3)}[b_2, b_4, b_6, \Delta^{-1}] \\ \Gamma_B &= B[r] \end{aligned}$$

which represents the groupoid of such elliptic curves and isomorphisms. The right unit is determined by how the coefficients of the Weierstrass equation for $C_{\mathbf{b}}$ transform. Since every isomorphism class of elliptic curve is represented, the stackification (in the flat topology) of the pre-stack associated to the Hopf algebroid (B, Γ_B) is $\mathcal{M}^1$.

The Hopf algebroid (B, Γ_B) suffers from the drawback that the natural map

$$\mathrm{spec}(B) \to \mathcal{M}^1$$

is not étale. We instead consider the curves

$$C_{\mathbf{b}} : y^2 = 4x(x^2 + q_2 x + q_4).$$

The automorphisms of these curves which preserve the tangent vector are those of the form $x \mapsto x + r$ where r has the property that

$$r^3 + q_2 r^2 + q_4 r = 0.$$

This groupoid of elliptic curves is represented by the Hopf algebroid $(\overline{B}, \Gamma_{\overline{B}})$ given by

$$\begin{aligned} \overline{B} &= \mathbb{Z}_{(3)}[q_2, q_4, \Delta^{-1}] \\ \Gamma_{\overline{B}} &= \overline{B}[r]/(r^3 + q_2 r^2 + q_4 r). \end{aligned}$$

Proposition 1.4.1 *The map*

$$f_{\mathbf{q}} : \mathrm{spec}(\overline{B}) \to \mathcal{M}^1$$

which classifies the curve $C_{\mathbf{q}}$ is an étale cover.

Proof. We first point out that the natural map of stackifications

$$\mathrm{stack}(\overline{B}, \Gamma_{\overline{B}}) \to \mathrm{stack}(B, \Gamma_B)$$

is an equivalence of stacks. This follows since given a curve C of the form $C_{\mathbf{b}}$ over a $\mathbb{Z}_{(p)}$-algebra R, there is a finite faithfully flat extension $R' = R[t]/(4t^3 + b_2 t^2 + 2b_4 t + b_6)$ of R so that over R', C is isomorphic to a curve of the form $C_{\mathbf{q}}$ under the transformation $x \mapsto x + t$.

Therefore we simply must check that the natural map

$$\mathrm{spec}(\overline{B}) \to \mathrm{stack}(\overline{B}, \Gamma_{\overline{B}})$$

is an étale cover. It suffices to check that the pullback

$$\begin{tikzcd} \mathrm{spec}(\Gamma_{\overline{B}}) \arrow[r] \arrow[d] & \mathrm{spec}(\overline{B}) \arrow[d] \\ \mathrm{spec}(\overline{B}) \arrow[r] & \mathrm{stack}(\overline{B}, \Gamma_{\overline{B}}) \end{tikzcd}$$

is an étale cover, that is, that $\Gamma_{\overline{B}}$ is an étale extension. This follows from the fact that since Δ is invertible, q_4 is invertible, and therefore the derivative of $f(r) = r^3 + q_2 r^2 + q_4 r$ is nonzero modulo all maximal ideals in $\overline{B}$. $\square$

In particular, $f_{\mathbf{q}}$ is a flat cover. Thus we have the following corollary.

Corollary 1.4.2 *The ANSS E_2-term for TMF may be computed as*

$$H^*(\mathcal{M}^1, \mathcal{O}) \cong \mathrm{Ext}_{\Gamma_{\overline{B}}}(\overline{B}, \overline{B}) \cong H^*(C^*(\Gamma_{\overline{B}}))$$

where $C^*(\Gamma_{\overline{B}})$ is the cobar complex of the Hopf algebroid $(\overline{B}, \Gamma_{\overline{B}})$.

Since the stack $\mathcal{M}_0^1(2)$ is an affine scheme after inverting 2, the ANSS for $TMF_0(2)$ is concentrated in the zero line. The ANSS collapses to give

$$\pi_{2k}(TMF_0(2)) = MF(2)_k.$$

1.4.2 The ANSS E_2-term for $Q(2)$

The ANSS for $Q(2)$ may be obtained by totalizing a cosimplicial Adams-Novikov resolution for $Q(2)$. Therefore, its E_2-term is the hypercohomology of the simplicial stack $\mathcal{M}_\bullet$. For convenience we choose to work with the simplicial stack $\mathcal{M}_\bullet^1$ where all of the instances of $\mathcal{M}$ and $\mathcal{M}_0(2)$ are replaced by $\mathcal{M}^1$ and $\mathcal{M}_0^1(2)$, respectively. The resulting spectral sequence takes the form

$$\mathbb{H}^{s,t}(\mathcal{M}_\bullet^1, \mathcal{O}) \Rightarrow \pi_{2t-s}(Q(2)).$$

The E_2 term may be computed via a hypercohomology spectral sequence

$$H^*_{cosimp}(H^*(\mathcal{M}_\bullet^1, \mathcal{O})) \Rightarrow \mathbb{H}^*(\mathcal{M}_\bullet^1, \mathcal{O}). \tag{1.4.1}$$

Here H^*_{cosimp} is the cohomology of the resulting cosimplicial abelian group.

The hypercohomology group $\mathbb{H}^*(\mathcal{M}_\bullet^1, \mathcal{O})$ is the cohomology of the totalization of a double cochain complex $C^{*,*}(Q(2))$ whose horizontal differentials are given by

$$\begin{tikzcd} C^*(\Gamma_{\overline{B}}) \arrow[r, "d_0-d_1"] \arrow[d, equal] & MF_0(2) \oplus \overline{C}^*(\Gamma_{\overline{B}}) \arrow[r, "d_0-d_1+d_2"] \arrow[d, equal] & MF_0(2) \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & {} \\ C^{0,*}(Q(2)) & C^{1,*}(Q(2)) & C^{2,*}(Q(2)) & C^{3,*}(Q(2)) & \end{tikzcd}$$

and whose vertical differentials are given by the differentials of the cobar complex. Here $MF_0(2)$ should be regarded as a cochain complex concentrated in degree zero. The complex $\overline{C}^*(\Gamma_{\overline{B}})$ is the cobar complex where the differentials have been given the opposite sign. The coface maps d_i are lifts of the maps induced by the simplicial face maps of $\mathcal{M}_\bullet^1$. In Section 1.5 we shall compute these maps explicitly.

We shall let the cochain complex $C^*(Q(2))$ be the totalization of the double complex $C^{*,*}(Q(2))$. Then we have

$$\mathbb{H}^*(\mathcal{M}_\bullet^1, \mathcal{O}) = H^*(C^*(Q(2)))$$

and the hypercohomology spectral sequence is simply the spectral sequence of the double complex.

1.5 Computation of the maps

In this section we compute the effects of the maps ϕ_q, ϕ_f, ψ_d, and $\psi_{[2]}$ on the appropriate rings of modular forms. Actually, since $\mathcal{M}^1$ is not a scheme, we will lift the maps involving $\mathcal{M}^1$ to the prestack associated to the Hopf algebroid $(\overline{B}, \Gamma_{\overline{B}})$. Our computations of these maps will show, as a side-effect, that ϕ_f is étale, and $\psi_{[2]}$ is an isomorphism. This was what was required to topologically realize the maps on the spectra of sections of $\mathcal{O}_{ell}$ in Section 1.2.1. We shall also see that we have the relation

$$\psi_d^2 = \psi_{[2]}.$$

This appeared as relation 1.1.2, and it is required for the simplicial identities to hold in the semi-simplicial stack $\mathcal{M}_\bullet$.

1.5.1 The map ϕ_f

The stack $\mathcal{M}_0^1(2)$ is the affine scheme spec$(\overline{B})$. There are no nontrivial automorphisms. Forgetting the $\Gamma_0(2)$ structure generated by $(x, y) = (0, 0)$ is the same as allowing the automorphisms of the curve $C_{\mathbf{q}}$ which do not preserve this level $\Gamma_0(2)$ structure. Thus the map

$$\phi_f : \mathcal{M}_0^1(2) \to \mathcal{M}^1$$

is induced by the map of Hopf algebroids

$$(\overline{B}, \Gamma_{\overline{B}}) \to (\overline{B}, \overline{B})$$

which is the identity on $\overline{B}$ and maps r to zero. Proposition 1.4.1 implies that the map ϕ_f is étale.

1.5.2 The map $\psi_{[2]}$

The map $\psi_{[2]} : \mathcal{M}^1 \to \mathcal{M}^1$ takes an elliptic curve C and returns the quotient $C/C[2]$ where $C[2]$ is the subgroup of all points of order 2 of C. The quotient map $C \to C/C[2]$ is equivalent to the second power map [2] on the elliptic curve. The second power map on the curve $C_{\mathbf{q}}$ could be regarded as an endomorphism of the curve, but the tangent vector $\partial/\partial z$ changes, where $z = x/y$. Therefore we shall instead regard the map [2] as being a map from $C_{\mathbf{q}}$ to $C_{\mathbf{q}'}$

where the tuple $\mathbf{q}'$ is given by

$$\begin{aligned} q_2' &= 2^2 q_2 \\ q_4' &= 2^4 q_4. \end{aligned}$$

We also must adjust the automorphism $x \mapsto x + r$ to $x' \mapsto x' + r'$ where

$$r' = 2^2 r.$$

Therefore, the map on Hopf algebroids

$$\psi_{[2]}^* : (\overline{B}, \Gamma_{\overline{B}}) \to (\overline{B}, \Gamma_{\overline{B}})$$

is given by

$$\begin{aligned} \psi_{[2]}^*(q_2) &= 2^2 q_2 \\ \psi_{[2]}^*(q_4) &= 2^4 q_4 \\ \psi_{[2]}^*(r) &= 2^2 r. \end{aligned}$$

The map

$$\psi_{[2]}^* : \mathcal{M}_0^1(2) \to \mathcal{M}_0^1(2)$$

is the map above restricted to $\overline{B}$. Each of these instances of $\psi_{[2]}$ is an isomorphism since we can explicitly define the inverse $\psi_{[1/2]}$ on the Hopf algebroids, since we are working 3-locally.

1.5.3 The map ψ_d

The dual isogeny map $\psi_d : \mathcal{M}_0^1(2) \to \mathcal{M}_0^1(2)$ takes a curve with $\Gamma_0(2)$ structure (C, H), identifies H as the kernel of a degree 2 isogeny

$$\phi_H : C \to C/H,$$

and returns the elliptic curve with $\Gamma_0(2)$ structure $(C/H, \widehat{H})$ corresponding to the dual isogeny

$$\widehat{\phi}_H : C/H \to C.$$

In our case we are given the curve $C_{\mathbf{q}}$ with $\Gamma_0(2)$ structure H generated by $e_1 = (x, y) = (0, 0)$. We wish to find a Weierstrass equation for the quotient curve $C_{\mathbf{q}}/H$. Given an elliptic curve C, a Weierstrass equation for C is generated by choosing a function x_1 with a pole of order 2 at the identity and a function y_1 with a pole of order 3 at the identity. We need to find such functions x_1 and y_1 for C/H.

Define the functions x_1 and y_1 on $C_{\mathbf{q}}/H$ as follows. Given a point $P \in C_{\mathbf{q}}$, define

$$x_1(P) = x(P) + x_t(P)$$
$$y_1(P) = y(P) + y_t(P).$$

Here the functions x_t and y_t are defined by

$$x_t(P) = x(P + e_1)$$
$$y_t(P) = y(P + e_1)$$

where e_1 is the generator of the $\Gamma_0(2)$ structure of $C_{\mathbf{q}}$ and $+$ denotes addition in the elliptic curve. It is immediate that the functions x_1 and y_1 descend to the quotient C/H.

One may use the formulas for the group law of a Weierstrass curve given in [45] to obtain

$$x_t = \frac{q_4}{x}$$
$$y_t = -\frac{q_4 y}{x^2}.$$

We warn the reader that the formulas of [45] must be modified slightly since those formulas are for an elliptic curve in Weierstrass form

$$C_{\mathbf{a}} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

whereas our curves

$$C_{\mathbf{q}} : y^2 = 4x(x^2 + q_2x + q_3)$$

have a coefficient of 4 instead of 1 in front of the x^3.

One then checks that the new coordinate functions x_1 and y_1 satisfy the Weierstrass equation

$$y_1^2 = 4x_1^3 + 4q_2x_1^2 - 16q_4x_1 - 16q_2q_4. \tag{1.5.1}$$

The canonical tangent vector associated to the Weierstrass equation 1.5.1 is the image of the tangent vector of $C_{\mathbf{q}}$. The right-hand side of equation 1.5.1 has a canonical root $x = -q_2$. The curve $C_{\mathbf{q}}/H$ with this $\Gamma_0(2)$ structure $e_1 = -q_2$ corresponds to the dual isogeny to $C_{\mathbf{q}} \to C_{\mathbf{q}}/H$. We must put the Weierstrass equation 1.5.1 into the form of $C_{\mathbf{q}'}$ for some tuple $\mathbf{q}'$. This is accomplished, as seen in Section 1.3.2, by application of the transformation $x \mapsto x - q_2$. The curve transforms to

$$C_{\mathbf{q}'} : y^2 = 4x(x^2 - 2q_2x - 4q_4 + q_2^2).$$

We conclude that the induced map

$$\psi_d^* : \overline{B} \to \overline{B}$$

is given by

$$\begin{aligned}\psi_d^*(q_2) &= -2q_2\\ \psi_d^*(q_4) &= -4q_4 + q_2^2.\end{aligned}$$

One sees immediately that relation 1.1.2

$$(\psi_d)^2 = \psi_{[2]} : \mathcal{M}_0^1(2) \to \mathcal{M}_0^1(2)$$

is satisfied. Since $\psi_{[2]}$ is an isomorphism, it follows that ψ_d is as well.

1.5.4 The map ϕ_q

Relation 1.1.1 forces us do define ϕ_q to be the composite

$$\phi_q : \mathcal{M}_0^1(2) \xrightarrow{\psi_d} \mathcal{M}_0^1(2) \xrightarrow{\phi_f} \mathcal{M}^1.$$

Therefore, on Hopf algebroids, the map

$$\phi_q^* : (\overline{B}, \Gamma_{\overline{B}}) \to (\overline{B}, \overline{B})$$

is given by

$$\begin{aligned}\phi_q^*(q_2) &= -2q_2\\ \phi_q^*(q_4) &= -4q_4 + q_2^2\\ \phi_q^*(r) &= 0.\end{aligned}$$

1.6 Extended automorphism groups

In this section we explain how to extend both the automorphism group of an elliptic curve, and its associated formal group, by the Galois group. We then explicitly determine the extended automorphism group of the unique supersingular elliptic curve over $\mathbb{F}_9$, and the extended automorphism group of its formal group.

In characteristic p, there are many different morphisms which all deserve to be called the Frobenius morphism, and this can lead to confusion. We first recall some standard material to clarify which one of these versions of the Frobenius we want to employ.

Let q be a power of p. Let $X = X' \times_{\mathrm{spec}(\mathbb{F}_q)} \mathrm{spec}(\mathbb{F}_{q^n})$ be a (formal) scheme over $\mathbb{F}_{q^n}$ which is obtained from a (formal) scheme X' over $\mathbb{F}_q$ by base change. Let π be the projection morphism

$$\pi : X \to \mathrm{spec}(\mathbb{F}_{q^n}).$$

Let $\mathrm{Frob}_q : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n}$ be the qth power Frobenius automorphism, the generator of the Galois group $Gal(\mathbb{F}_{q^n}/\mathbb{F}_q)$. Associated to Frob_q are several different Frobenius morphisms on X:

- Frob_q: the qth Frobenius on the base.
- Frob_q^{rel}: the qth relative Frobenius.
- Frob_q^{tot}: the qth total Frobenius.

These are given by the following diagram.

$$\begin{tikzcd}
X \arrow[dr, "\mathrm{Frob}_q^{rel}"] \arrow[drr, "\mathrm{Frob}_q^{tot}"] \arrow[ddr, "\pi"'] & & \\
& X \arrow[r, "\mathrm{Frob}_q"'] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\
& \operatorname{spec} \mathbb{F}_{q^n} \arrow[r, "\mathrm{Frob}_q"'] & \operatorname{spec} \mathbb{F}_{q^n}
\end{tikzcd}$$

The square in the above diagram is a pullback square. If $X' = \mathrm{spec}(A)$ is affine, then X is the affine scheme $\mathrm{spec}(A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n})$, and Frob_q (respectively Frob_q^{rel}) is the qth Frobenius on $\mathbb{F}_{q^n}$ (respectively on A), while Frob_q^{tot} is the Frobenius on $A \otimes_{\mathbb{F}_q} \mathbb{F}_{q^n}$.

Let $\mathrm{Aut}(X)$ denote the automorphism group of X, regarded as a (formal) scheme over $\mathrm{spec}(\mathbb{F}_{q^n})$ There is an induced action of Frob_q on $\alpha \in \mathrm{Aut}(X)$ given by the following pullback.

$$\begin{tikzcd}
X \arrow[r, "\mathrm{Frob}_q^* \alpha"] \arrow[d, "\mathrm{Frob}_q"'] & X \arrow[d, "\mathrm{Frob}_q"] \\
X \arrow[r, "\alpha"'] & X
\end{tikzcd}$$

This action of Frob_q gives rise to an action of $Gal(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on $\mathrm{Aut}(X)$. We define the extended automorphism group $\mathrm{Aut}_{/\mathbb{F}_q}(X)$ to be the semidirect product

$$\mathrm{Aut}_{/\mathbb{F}_q}(X) = \mathrm{Aut}(X) \rtimes Gal(\mathbb{F}_{q^n}/\mathbb{F}_q)$$

associated to this action. The group $\mathrm{Aut}_{/\mathbb{F}_q}(X)$ consists of automorphisms of the (formal) scheme X which do not cover the identity on $\mathrm{spec}(\mathbb{F}_{q^n})$.

At the prime 3 there is one isomorphism class of supersingular elliptic curve over $\mathbb{F}_9$, admitting the Weierstrass presentation

$$C : y^2 = x^3 - x.$$

For the remainder of this section we will focus our attention on the computation of the endomorphism ring $\mathrm{End}(C)$, and the automorphism groups

$\mathrm{Aut}_{/\mathbb{F}_3}(C)$ and $\mathrm{Aut}_{/\mathbb{F}_3}(C^\wedge)$. We shall see that all of the endomorphisms of C defined over $\overline{\mathbb{F}}_3$ are defined over $\mathbb{F}_9$.

Let $F = \mathrm{Frob}_3^{rel}$ be the relative Frobenius endomorphism of C. The map F is given by

$$F : (x, y) \mapsto (x^3, y^3).$$

Define automorphisms $t, s \in \mathrm{Aut}(C)$ by

$$\begin{aligned} t :&(x, y) \mapsto (\omega^4 x, \omega^6 y) \\ s :&(x, y) \mapsto (x + 1, y). \end{aligned}$$

Here, $\omega \in \mathbb{F}_9$ is a primitive 8th root of unity. Explicit computation, using the formulas for the group law on C [45], demonstrates that we have relations

$$F^2 = -3 \tag{1.6.1}$$
$$t^2 = -1 \tag{1.6.2}$$
$$Ft = -tF \tag{1.6.3}$$
$$s = \frac{1}{2}(1 + F). \tag{1.6.4}$$

From the definitions of the maps, we deduce the following additional relations.

$$s^3 = 1 \tag{1.6.5}$$
$$st = ts^2 \tag{1.6.6}$$

Proposition 1.6.1 *The automorphism group of C is the group $G_{12} = C_3 \rtimes C_4$ of order 12 generated by s and t.*

Proof. Relations (1.6.5), (1.6.2), and (1.6.6) demonstrate that s and t generate a subgroup G_{12} of $\mathrm{Aut}(C)$ of order 12 isomorphic to $C_3 \rtimes C_4$. The group of automorphisms of C defined over $\overline{\mathbb{F}}_3$ is of order 12 [45, A.1.2]. We therefore deduce that all of the automorphisms of C are defined over $\mathbb{F}_9$, and that they are contained in G_{12}. $\square$

Proposition 1.6.2 *The endomorphism ring of C (over $\mathbb{F}_9$) is a maximal order of the quaternion algebra*

$$\mathbb{Q}\langle F, t\rangle/(F^2 = -3, t^2 = -1, Ft = -tF).$$

A $\mathbb{Z}$-basis of the order $\mathcal{O} = \mathrm{End}(C)$ is given by

$$\{s, t, ts, tF\}.$$

Proof. Since C supersingular, the ring of endomorphisms defined over $\overline{\mathbb{F}}_3$ $\mathrm{End}_{\overline{\mathbb{F}}_3}(C)$ is a maximal order of a rational quaternion algebra D ramified at the places p and ∞. By Proposition 5.1 of [37], D admits the presentation

$$D = \mathbb{Q}\langle i, j\rangle/(i^2 = -3, j^2 = -1, ij = -ji).$$

By Proposition 5.2 of [37], there is a maximal order $\mathcal{O}$ of D with $\mathbb{Z}$-basis

$$\{\frac{1}{2}(1+i), j, \frac{1}{2}(j+ji), ji\}.$$

Let R be the subring of $\mathrm{End}(C)$ generated by t, s, and F. Since D is a division algebra, the algebra map

$$D \to R \otimes \mathbb{Q}$$

sending i to F and j to t is an isomorphism. We deduce that under this isomorphism, the ring R is a maximal order of D contained in $\mathrm{End}_{\overline{\mathbb{F}}_3}(C)$. Since the order $\mathrm{End}(C)$ is maximal, we see that $R = \mathrm{End}_{\overline{\mathbb{F}}_3}(C)$. Since the generators of R are defined over $\mathbb{F}_9$, we see that all of the endomorphisms of C are defined over $\mathbb{F}_9$. $\square$

The endomorphism ring $\mathrm{End}(C^\wedge)$ of the formal group $C^\wedge$ is the p-completion of $\mathrm{End}(C^\wedge)$. This follows from a theorem of Tate (see [48]), which states that the p-completion of the endomorphism ring of an abelian variety is the the endomorphism ring of its p-divisible group. However, in the case of the elliptic curve C, this may be deduced in a more explicit manner.

Recall that the endomorphism ring of a height 2 formal group over $\mathbb{F}_{p^2}$ is the unique maximal order $\mathcal{O}_p$ of the $\mathbb{Q}_p$-division algebra D_p of invariant 1/2 [39, Appendix B]. The ring $\mathcal{O}_p$ admits the presentation

$$\mathcal{O}_p = \mathbb{W}\langle S\rangle/(S^2 = p, w^\sigma S = Sw)$$

where $\mathbb{W} = \mathbb{W}(\mathbb{F}_{p^2})$ is the Witt ring of $\mathbb{F}_{p^2}$, w ranges over the elements of $\mathbb{W}$, and σ is the lift of the Frobenius Frob_p on $\mathbb{F}_{p^2}$.

Since C is supersingular, the formal group $C^\wedge$ is a height 2 formal group. The formal group law $C^\wedge$ is seen, using the formulas of [45], to have 3-series

$$[3]_{C^\wedge}(T) = -T^9.$$

The Witt ring $\mathbb{W}$ is the ring of integers of the unramified quadratic extension of $\mathbb{Q}_3$. We may therefore identify the Witt ring $\mathbb{W}$ with the subring $\mathbb{Z}_3[t]$ of $\mathcal{O} \otimes \mathbb{Z}_3$. Let $\omega \in \mathbb{F}_9$ be a primitive 8th root of unity. We shall also let ω represent the Teichmüller lift to $\mathbb{W}$. Relation 1.6.3 implies that in $\mathcal{O} \otimes \mathbb{Z}_3$ we have

$$F\omega = \omega^3 F. \tag{1.6.7}$$

Define S to be the element $\omega^{-1}F \in \mathcal{O} \otimes \mathbb{Z}_3$. Relations 1.6.7 and 1.6.1 imply that we have

$$S^2 = 3 \tag{1.6.8}$$
$$S\omega = \omega^3 S. \tag{1.6.9}$$

The following proposition follows immediately.

Proposition 1.6.3 *There is an isomorphism of $\mathbb{W}$-algebras*

$$\mathcal{O} \otimes \mathbb{Z}_3 = \mathrm{End}(C) \otimes \mathbb{Z}_3 = \mathrm{End}(C^\wedge) \to \mathcal{O}_3$$

given by sending F to ωS.

The Morava stabilizer group $\mathbb{S} = \mathbb{S}_2$ is the group of units of $\mathcal{O}_3$

$$\mathrm{Aut}(C^\wedge) = \mathcal{O}_3^\times.$$

To understand the extended automorphism groups

$$G_{24} = \mathrm{Aut}_{/\mathbb{F}_3}(C) = \mathrm{Aut}(C) \rtimes Gal$$
$$\mathbb{G} = \mathrm{Aut}_{/\mathbb{F}_3}(C^\wedge) = \mathrm{Aut}(C^\wedge) \rtimes Gal$$

we simply must understand the action of the Galois group $Gal = Gal(\mathbb{F}_9/\mathbb{F}_3)$. Let $\sigma = \mathrm{Frob}_3$ denote the generator of Gal. The Galois action on $\mathrm{End}(C)$ is governed by the equations

$$\sigma^* F = F$$
$$\sigma^* t = -t.$$

We deduce that the Galois action on $\mathrm{Aut}(C)$ is given by

$$\sigma^* s = s$$
$$\sigma^* t = -t.$$

The Galois action on $\mathbb{W} \subset \mathrm{Aut}(C^\wedge)$ is given by the lift of σ. The Galois action on $\mathrm{Aut}(C^\wedge)$ is then determined by the action of σ on S. We have

$$\sigma^* S = \omega^{-2} S.$$

Remark 1.6.4 *This formula for the Galois action on $\mathbb{S}$ differs from that that is in more common usage (see, for instance, [11]). The reason for this is that the Morava stabilizer group $\mathbb{S}$ is usually taken to be the group of automorphisms of the Honda height 2 formal group F_2 over $\mathbb{F}_{p^2}$. Since the formal groups F_2 and $C^\wedge$ are isomorphic over $\overline{\mathbb{F}}_3$, they have isomorphic automorphism groups. However, because they are not isomorphic over $\mathbb{F}_9$, their automorphism groups have non-isomorphic Galois actions.*

In this section we summarize the effects of our operations on E_2-homology. This is an adaptation of the techniques of Mahowald and Rezk [30]. Throughout this section, we freely use the notation established in Section 1.6. We shall denote the Morava E-theory spectrum E_2 by E. We are taking the spectrum E to be the Goerss-Hopkins-Miller E_∞ ring spectrum associated to the formal group $C^\wedge$ over $\mathbb{F}_9$. When we write the E-homology group E_*X, we mean the Morava module $\pi_*(L_{K(2)}(E \wedge X))$.

Recall [12] that for a closed subgroup $F \subseteq \mathbb{G}$ we have an isomorphism of Morava modules

$$E_*E^{hF} \cong \mathrm{Map}^c(\mathbb{G}/F, E_*).$$

Let C be the supersingular elliptic curve of Section 1.6. There is a lift of this curve to a curve $\widetilde{C}$ over $E^0 = \mathbb{W}[[u_1]]$.

$$\widetilde{C} : y^2 = 4x^3 + u_1x^2 + 2x.$$

Proposition 1.7.1 *The formal group law $\widetilde{C}^\wedge$ is a universal deformation of $C^\wedge$.*

Proof. We just need to see that the universal map classifying our deformation induces an isomorphism on the base ring E^0. This follows from the fact that the 3-series of $\widetilde{C}^\wedge$ is given by [41]

$$[3]_{\widetilde{C}^\wedge}(T) \equiv u_1T^3 + 2(1+u_1^2)T^9 + \cdots \pmod 3.$$

□

Remark 1.7.2 *The Serre-Tate theorem gives an equivalence between the category of deformations of C and the category of deformations of the formal group $C^\wedge$. Because $\widetilde{C}^\wedge$ is a universal deformation of $C^\wedge$, we may deduce that $\widetilde{C}$ is a universal deformation of C.*

We define finite subgroups of $\mathbb{G}$ to be the subgroups generated by the following elements.

$$\begin{aligned} G_{24} &= \langle s, t, \sigma \rangle \\ SD_{16} &= \langle \omega, \sigma \rangle \\ D_8 &= \langle t, \sigma \rangle \end{aligned}$$

Work of Hopkins-Miller [40] and Goerss-Hopkins [17] shows that the $\mathbb{G}$ action on E_* lifts to an E_∞ action on the spectrum E. We have the following spectra as homotopy fixed point spectra.

$$\begin{aligned} E^{hG_{24}} &= TMF \\ E^{hSD_{16}} &= E(2) \\ E^{hD_8} &= TMF_0(2) \end{aligned}$$

The relationship to some of these groups and the curve C is given below.

$$\begin{aligned} \mathbb{G} &= \mathrm{Aut}_{/\mathbb{F}_3}(C^\wedge) \\ G_{24} &= \mathrm{Aut}_{/\mathbb{F}_3}(C) \\ D_8 &= \mathrm{Aut}_{/\mathbb{F}_3}(C, H). \end{aligned}$$

The subgroup H is the level 2 structure generated by the point $(0,0)$ of C.

Remark 1.7.3 *We pause to comment on the relationship between TMF and EO_2. The pair $(E, \widetilde{C})$ is an E_∞ elliptic spectrum [19]. This E_∞ elliptic spectrum structure is classified by an E_∞ ring map*

$$\kappa : TMF \to E.$$

Because $\widetilde{C}$ is a universal deformation, the action of G_{24} on C extends to an action of G_{24} on $\widetilde{C}$ (covering the action of G_{24} on E_*). The classifying map κ therefore lifts to an E_∞ ring map

$$TMF \to E^{hG_{24}} = EO_2.$$

This map is a $K(2)$-local equivalence. The essential point is that the $K(2)$-localization of TMF will be given as the sections of the sheaf $\mathcal{O}_{ell}$ over a formal neighborhood of the unique supersingular point C of the (3-local) moduli stack $\mathcal{M}$ (see Section 1.2.1).

Given an endomorphism

$$\phi : C \to C$$

we get an induced endomorphism

$$\widehat{\phi} : C^\wedge \to C^\wedge.$$

Assuming that $\widehat{\phi}$ is an automorphism, we may regard $\widehat{\phi}$ as an element of $\mathbb{G}$, thus giving an E_∞ ring map

$$\widehat{\phi} : E \to E.$$

The induced map $E_*\widehat{\phi}$ on E-homology is given by

$$\begin{tikzcd} E_*E \arrow[r, "E_*\widehat{\phi}"] \arrow[d, equal] & E_*E \arrow[d, equal] \\ \mathrm{Map}^c(\mathbb{G}, E_*) \arrow[r, "R^*_{\widehat{\phi}}"'] & \mathrm{Map}^c(\mathbb{G}, E_*) \end{tikzcd}$$

where $R_{\widehat{\phi}}$ is right multiplication by $\widehat{\phi}$. If $\widehat{\phi}$ is in the normalizer $N_{\mathbb{G}}F$ of a closed subgroup F, then there is an induced map

$$\widehat{\phi} : E^{hF} \to E^{hF}.$$

Being in the normalizer implies that *right* multiplication descends to the right coset spaces $\mathbb{G}/F$. Thus the induced map on E-homology is given by

$$\mathrm{Map}^c(\mathbb{G}/F, E_*) \xrightarrow{R^*_{\widehat{\phi}}} \mathrm{Map}^c(\mathbb{G}/F, E_*).$$

We may apply this discussion to translate our maps ϕ_f, ψ_d, and $\psi_{[2]}$ into the language of homotopy fixed point spectra. The map ϕ_f^* is clearly the map induced from inclusion $\iota_{D_8} : D_8 \hookrightarrow G_{24}$ giving

$$\phi_f^* = Res_{D_8}^{G_{24}} : E^{hG_{24}} \to E^{hD_8}.$$

Consider the quotient isogeny

$$\phi_H : C \to C/H$$

where H is the level 2 structure on C generated by the point $(x, y) = (0, 0)$. The computations of Section 1.5.3 indicate that the curve C/H is given by the Weierstrass equation

$$C/H : y^2 = x^3 + x.$$

Note that this differs from our Weierstrass equation for C, but since there is only one isomorphism class of supersingular curve at $p = 3$, this curve must be isomorphic to C. Indeed, we find it convenient to take the isomorphism

$$\mu_{\omega^{-1}} : C/H \to C$$

given by

$$\begin{aligned} x &\mapsto \omega^{-2}x \\ y &\mapsto \omega^{-3}y. \end{aligned}$$

Thus we may consider the composite

$$\psi_H : (C, H) \xrightarrow{\phi_H} C/H \xrightarrow{\mu_{\omega^{-1}}} C.$$

The induced map $\widehat{\psi}_H$ on the formal group is readily computed from the formulas of Section 1.5.4. One finds that, regarding $\widehat{\psi}_H$ as an element of $\mathbb{G}$, we have

$$\widehat{\psi}_H = 1 + t.$$

The group element $1+t$ is in the normalizer $N_{\mathbb{G}}D_8$. Thus the map $\psi_d^* : TMF(2) \to TMF(2)$ is given by

$$\psi_d^* = [1+t] : E^{hD_8} \to E^{hD_8}.$$

Since $\psi_{[2]}$ corresponds to the second power isogeny, it is given by the element $2 \in \mathbb{G}$, so that

$$\psi_{[2]} = [2] : E^{hG_{24}} \to E^{hG_{24}}.$$

We could have used $1-t \in \mathbb{G}$ to represent the map ψ_d^*. It is irrelevant on E^{hD_8} since the cosets $(t+1)D_8$ and $(1-t)D_8$ are equal in $\mathbb{G}/D_8$. The formula

$$(t+1)(1-t) = 2$$

in $\mathbb{G}$ implies relation 1.1.2

$$(\psi_d^*)^2 = \psi_{[2]} : E^{hD_8} \to E^{hD_8}$$

required in the construction of $Q(2)$ given in Section 1.2.2.

1.8 Calculation of $V(1)_*(Q(2))$ at $p=3$

Let $V(1)$ be the 3-primary Smith-Toda complex. In this section we will compute the $V(1)$-homology group $V(1)_*(Q(2))$. We shall compare this computation to $V(1)_*(S)$, where S is the ($K(2)$-local) sphere.

1.8.1 Computation of the ANSS

We shall compute the Adams-Novikov E_2 term for $Q(2) \wedge V(1)$ using the hypercohomology spectral sequence 1.4.1 (modulo the ideal $(3, v_1)$). We first describe the E_1 term.

We recall [41] that the the [3]-series of the formal group law of the elliptic curve $C_{\mathbf{q}}$ has

$$v_1 \equiv q_2 \pmod 3.$$

Therefore, the ANSS for $TMF \wedge V(1)$ is the cohomology of the quotient Hopf algebroid of $(\overline{B}, \Gamma_{\overline{B}})$ (see Section 1.4) given by

$$(\overline{B}/I_2, \Gamma_{\overline{B}}/I_2)$$

where I_2 is the invariant regular ideal $(3, q_2) \subset \overline{B}$. The ANSS E_2 term for $TMF \wedge V(1)$ is computed [4], [41] to be

$$E_2(TMF \wedge V(1)) = H^*(\Gamma_{\overline{B}}/I_2) = \mathbb{F}_3[q_4^{\pm 1}, \beta] \otimes E[\alpha]$$

where the generators have bidegrees

$$|q_4| = (0,8)$$
$$|\beta| = (2,12)$$
$$|\alpha| = (1,4).$$

We have $v_2 = -q_4^2$. The ANSS E_2 term for $TMF_0(2) \wedge V(1)$ is concentrated on the zero line, and is given by

$$E_2(TMF_0(2) \wedge V(1)) = \mathbb{F}_3[q_4^{\pm 1}].$$

The differentials in the complex $C^*(Q(2) \wedge V(1))$ of Section 1.4 are computed from the formulas of Section 1.5. Namely, we have

$$\begin{aligned} \phi_f^*(q_4^k) &\equiv (q_4)^k \pmod{I_2} \\ \psi_{[2]}^*(q_4^k) &\equiv q_4^k \pmod{I_2} \\ \psi_d^*(q_4^k) &\equiv (-q_4)^k \pmod{I_2} \\ \phi_q^*(q_4^k) &\equiv (-q_4)^k \pmod{I_2}. \end{aligned}$$

The non-zero differentials D_i in the total complex $C^*(Q(2) \wedge V(1))$ are given by the following formulas.

$$D_0 : C^0(\Gamma_{\overline{B}}) \to C^1(\Gamma_{\overline{B}}) \oplus C^0(\Gamma_{\overline{B}}) \oplus MF_0(2)$$

$$D_0(q_4^k) \equiv (0,0,q_4^k) \pmod{I_2}, \quad k \equiv 1 \pmod 2$$

$$D_1 : C^1(\Gamma_{\overline{B}}) \oplus \overline{C}^0(\Gamma_{\overline{B}}) \oplus MF_0(2) \to C^2(\Gamma_{\overline{B}}) \oplus \overline{C}^1(\Gamma_{\overline{B}}) \oplus MF_0(2)$$

$$\begin{aligned} D_1(0,0,q_4^k) &\equiv (0,0,-q_4^k) \pmod{I_2}, \quad k \equiv 0 \pmod 2 \\ D_1(0,q_4^k,0) &\equiv (0,0,-q_4^k) \pmod{I_2}, \quad k \in \mathbb{Z} \end{aligned}$$

We therefore have the following proposition.

Proposition 1.8.1 *The E_2-term of the ANSS for $Q(2) \wedge V(1)$ is given by*

$$\mathbb{F}_3[\beta, v_2^{\pm 1}] \otimes E[\zeta]\{1, \alpha, h_1, b_1\}.$$

The generators are given in the following table. The representative is given the name of the corresponding element of the ANSS E_2-term of the layer of the tower (The brackets refer to the level of the tower that the generator lives in).

Generator	Representative	Bidegree
β	$\beta[0]$	$(2, 12)$
v_2	$-q_4^2[0]$	$(0, 16)$
α	$\alpha[0]$	$(1, 4)$
h_1	$\alpha q_4[0]$	$(1, 12)$
b_1	$\beta q_4^3[0]$	$(2, 36)$
ζ	$1_{MF}[1] - 1_{MF_0(2)}[1]$	$(1, 0)$

The differentials in the ANSS for $Q(2) \wedge V(1)$ follow from the differentials in the ANSS for $TMF \wedge V(1)$ (see [4], [41]).

Proposition 1.8.2 *The differentials in the ANSS for $Q(2) \wedge V(1)$ are given by*

$$\begin{aligned} d_5(v_2^k) &= v_2^{k-2}\beta^2 h_1, \quad k \equiv 2,3,4,6,7,8 \pmod 9 \\ d_9(v_2^k\alpha) &= v_2^{k-3}\beta^5, \quad k \equiv 3,4,8 \pmod 9 \\ d_9(v_2^k h_1) &= \beta^4 v_2^{k-4}, \quad k \equiv 3,7,8 \pmod 9 \\ d_5(v_2^k b_1) &= v_2^k \beta^3 \alpha, \quad k \equiv 0,1,2,5,6,7 \pmod 9 \end{aligned}$$

and these differentials are propagated freely by multiplication by β and ζ. This completely describes the ANSS.

1.8.2 The $V(1)$-homology groups $V(1)_*(Q(2))$ and $V(1)_*(S)$

Define patterns A, B, and C of homotopy groups as follows

$$\begin{aligned} A &= X \otimes \mathbb{F}_3\{1, b_4, v_2\} \\ B &= Y \otimes \mathbb{F}_3\{1, v_2\} \oplus Z \\ C &= Z \otimes \mathbb{F}_3\{v_2^4, v_2^5\} \oplus Y \otimes \mathbb{F}_3\{v_2^5\} \end{aligned}$$

where X, Y, and Z are the following graded $\mathbb{F}_3$-vector spaces.

$$\begin{aligned} X &= \mathbb{F}_3\{1, \alpha, \beta, \alpha\beta, \beta^2, \langle \alpha, \alpha, \beta^2 \rangle, \beta^3, \langle \alpha, \alpha, \beta^3 \rangle, \beta^4\} \\ Y &= \mathbb{F}_3\{1, \alpha, \beta, \alpha\beta, \beta^2, \alpha\beta^2, v_2 h_1, \beta^3, \beta v_2 h_1, \beta^4\} \\ Z &= \mathbb{F}_3\{h_1, v_2^{-1}b_1, \beta h_1, \beta v_2^{-1} b_1, \langle \alpha, \alpha, \beta v_2^{-1} b_1 \rangle, \beta^2 v_2^{-1} b_1, \langle \alpha, \alpha, \beta^2 v_2^{-1} b_1 \rangle, \\ &\qquad \beta^3 v_2^{-1} b_1, \langle \alpha, \alpha, \beta^3 v_2^{-1} b_1 \rangle\} \end{aligned}$$

Figure 1.1 gives a graphical description of the patterns A, B, and C, and the patterns X, Y, and Z sit inside of them as indicated. In this figure, the lines

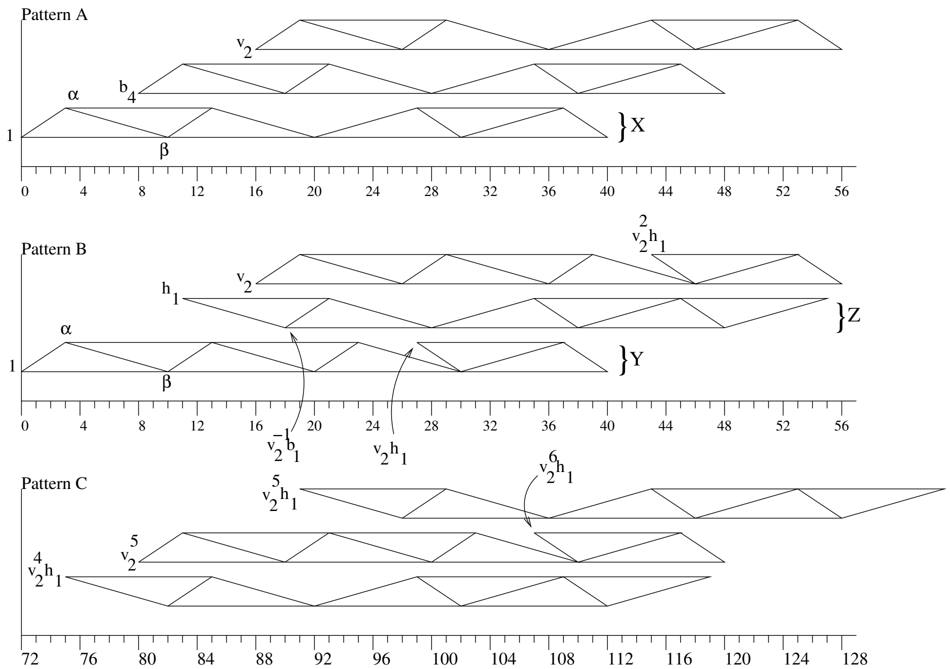

Fig. 1.1. Graphical depiction of patterns A, B, and C.

of the appropriate length depict multiplication by α, multiplication by β, and application of the Toda bracket $\langle \alpha, \alpha, - \rangle$.

Let $Q'(2)$ be the fiber of the unit

$$Q'(2) \to S \xrightarrow{\eta} Q(2).$$

With these patterns, we have the following computations. The first two computations are given in [15]. The computation of $V(1)_*(Q(2))$ is an immediate consequence of Proposition 1.8.2.

$$\begin{aligned} V(1)_*(TMF) &= (A \oplus \Sigma^{72} A) \otimes P[v_2^{\pm 9}] \\ V(1)_*(S) &= ((B \oplus C) \oplus \Sigma^{29}(B \oplus C)^\vee) \otimes P[v_2^{\pm 9}] \otimes E[\zeta] \\ V(1)_*(Q(2)) &= (B \oplus C) \otimes P[v_2^{\pm 9}] \otimes E[\zeta] \\ V(1)_*(Q'(2)) &= \Sigma^{29}(B \oplus C)^\vee \otimes P[v_2^{\pm 9}] \otimes E[\zeta] \end{aligned}$$

The computation of the $V(1)$-homology of $Q'(2)$ follows from the fact that the computations of the $V(1)$-homology of S and $Q(2)$ imply there is a short exact sequence

$$0 \to V(1)_*(Q'(2)) \to V(1)_*(S) \to V(1)_*(Q(2)) \to 0$$

Thus it appears that $V(1)$ is built out of $Q(2) \wedge V(1)$ and a 28^{th} suspension of the Brown-Comenetz dual of $Q(2) \wedge V(1)$. This is the key computation to our proof that $Q'(2) \simeq D(Q(2))$ (Theorem 2.0.1). Gross-Hopkins duality [20] implies that after smashing with $V(1)$, Spanier-Whitehead duality and

Brown-Comenetz duality agree up to a suitable suspension in the $K(2)$-local category.

Part 2: The 3-primary $K(2)$-local sphere

Let K denote $K(2)$, E denote E_2, and Q denote $Q(2)$. The (K-local) Spanier-Whitehead dual of X will be denoted by

$$DX = F(X, L_K S).$$

Recall that the nth monochromatic layer $M_n X$ is the fiber

$$M_n X \to L_n X \to L_{n-1} X.$$

The Gross-Hopkins dual $I_n X$ is the Brown-Comenetz dual of $M_n X$. We shall let $I_K X$ denote the K-localization of $I_n X$. In particular, we shall write I_K for $I_K S$. The spectrum I_K is invertible [20], [26]. Therefore, we have [46]

$$I_K X \simeq DX \wedge I_K. \tag{2.0.1}$$

In this part of the paper we wish to prove the following theorem.

Theorem 2.0.1 *There is a cofiber sequence*

$$DQ \xrightarrow{D\eta} S \xrightarrow{\eta} Q$$

where η is the unit of the ring spectrum Q.

By equation 2.0.1, we have $DQ \simeq I_K Q \wedge I_K^{-1}$. We are able to use this equation to give an explicit description of DQ. We shall prove

Theorem 2.0.2 *The spectrum DQ refines a tower of spectra of the form*

$$\Sigma^{46} TMF_0(2) \to \Sigma^{46}(TMF \vee TMF_0(2)) \to \Sigma^{46} TMF.$$

It will turn out that the spectra TMF and $TMF_0(2)$ are Gross-Hopkins self-dual, up to a suitable suspension. This will follow from the relationship of the Gross-Hopkins dual to the Mahowald-Rezk dual. The Mahowald-Rezk duals of certain fp-spectra are easily computed using the theory of [30]. The number 46 in Theorem 2.0.2 arises from $46 = 22 + 22 + 2$ where the first two 22's arise from the equations

$$I_K TMF \simeq \Sigma^{22} TMF \quad \text{(Proposition 2.4.1)}$$
$$I_K^{-1} \wedge TMF \simeq \Sigma^{22} TMF \quad \text{(Proposition 2.5.1)}$$

and the final factor of 2 arises from the fact that the tower

$$TMF \to TMF \vee TMF_0(2) \to TMF_0(2)$$

has length 2 (Lemma 2.6.2). Thus we recover, at least in form, the tower of [16]. The maps in the tower for DQ given in Theorem 2.0.2 are the Spanier-Whitehead duals of the maps in the tower for Q.

We shall first give a general abstract framework for duality pairings in triangulated closed symmetric monoidal categories, where we define the notion of a Lagrangian for a hyperbolic pairing. We shall prove that for the duality pairing for the $K(2)$-local sphere is hyperbolic, in the sense that S is built from two dual spectra. The Lagrangian for this decomposition will be $\overline{S} = E^{h\mathbb{G}^1}$, where $\mathbb{G}^1$ is the kernel of the reduced norm $\mathbb{G}^1 \to \mathbb{Z}_3$. The decomposition we consider is well known [12], but this particular interpretation of it is new. We then use this decomposition to prove that Q is also a Lagrangian for S, thus proving that S may also be built out of Q and DQ (Theorem 2.0.1).

The second part of this paper is organized as follows. In Section 2.1, we give our abstract duality framework. In Section 2.2, it is recalled that $V(1)$-homology equivalences are $K(2)$-local equivalences. In Section 2.3 we prove Proposition 2.3.1, which says that for arbitrary n, the canonical pairing on $L_{K(n)}S$ is hyperbolic with Lagrangian $\overline{S}$. Section 2.4 identifies $I_K TMF$ and Section 2.5 identifies $I_K \wedge TMF$. In Section 2.6 we use these identifications to prove Theorem 2.0.2. In Section 2.7 we define a spectrum $\overline{Q}$, and in Section 2.9 we prove that $\overline{Q}$ is a Lagrangian for $\overline{S}$. This uses some decompositions of mapping spaces of homotopy fixed point spectra that are recalled from [16] in Section 2.8. Section 2.10 describes how Q may be built from two copies of $\overline{Q}$. Theorem 2.0.1 is finally proved in Section 2.11.

2.1 Duality pairings

In this section we briefly outline the abstract framework in which the duality phenomena of this paper reside. Compare with [32], [13].

We first provide some motivation. Suppose that V is a finite dimensional vector space over a field k. A (bilinear) pairing is a linear homomorphism

$$\alpha : V \otimes V \to k.$$

It is a perfect pairing if the adjoint homomorphism

$$\widetilde{\alpha} : V \to V^*$$

is an isomorphism. A Lagrangian for such a pairing is a subspace $W \subseteq V$ such that the induced sequence

$$W \xrightarrow{\iota} V \cong V^* \xrightarrow{\iota^*} W^*$$

is a short exact sequence, where the isomorphism between V and V^* is given by $\widetilde{\alpha}$. A pairing for which a Lagrangian exists is a hyperbolic pairing. We wish to provide a framework of hyperbolic pairings in the setting of triangulated closed symmetric monoidal categories.

Let $(\mathcal{C}, \wedge, F(-,-), S)$ be a closed symmetric monoidal category with compatible triangulated structure [33]. Let DX denote the dual $F(X,S)$ of X. One says that X is *reflexive* if the natural map

$$\widetilde{\epsilon} : X \to DDX$$

is an isomorphism. This should be contrasted with the stronger notion of X being *dualizable*, where one insists that the natural map

$$DX \wedge X \to F(X,X)$$

be an isomorphism.

We shall refer to any map

$$\alpha : X \wedge X \to I$$

where I is an element of $Pic(\mathcal{C})$, as a *pairing* on X. A pairing α will be called *perfect* if the adjoint map

$$\widetilde{\alpha} : X \to F(X,I) \cong DX \wedge I$$

is an isomorphism. Note that it is immediate that if X possesses a perfect pairing α, then X is reflexive.

Suppose that X has a perfect pairing $\alpha : X \wedge X \to I$. Given a map $l : Y \to X$, we may define a dual map $l^\vee$ by the composite

$$l^\vee : X \underset{\cong}{\xrightarrow{\widetilde{\alpha}}} DX \wedge I \xrightarrow{Dl \wedge 1} DY \wedge I.$$

We shall say that $l : Y \to X$ is a *left Lagrangian* for α if the sequence

$$Y \xrightarrow{l} X \xrightarrow{l^\vee} DY \wedge I$$

extends to a distinguished triangle in $\mathcal{C}$. Alternatively, given a map $r : X \to Z$, we may define a map $r^\vee$ to be the composite

$$r^\vee : DZ \wedge I \xrightarrow{Dr \wedge 1} DX \wedge I \underset{\cong}{\xrightarrow{\widetilde{\alpha}^{-1}}} X$$

We shall say that $r : X \to Z$ is a *right Lagrangian* for α if the sequence

$$DZ \wedge I \xrightarrow{r^\vee} X \xrightarrow{r} Z$$

extends to a distinguished triangle of $\mathcal{C}$. We have the following proposition which enumerates some useful facts regarding these definitions.

Proposition 2.1.1 *Suppose that X has a perfect pairing $\alpha : X \wedge X \to I$.*

(1) If Y is reflexive, then given $l : Y \to X$, we have $l^{\vee\vee} \cong l$ under the isomorphism $D(DY \wedge I) \wedge I \cong Y$.
(2) If Z is reflexive, then given $r : X \to Z$, we have $r^{\vee\vee} \cong r$ under the isomorphism $D(DZ \wedge I) \wedge I \cong Z$.
(3) If $l : Y \to X$ is a left Lagrangian, then Y is reflexive, and $l^\vee : X \to DY \wedge I$ is a right Lagrangian.
(4) If $r : X \to Z$ is a right Lagrangian, then Z is reflexive, and $r^\vee : DZ \wedge I \to X$ is a left Lagrangian.

Proof. We shall only prove (3). The other parts use similar methods. By hypothesis, there exists a map δ so that the top row of the following diagram is a distinguished triangle.

$$\begin{tikzcd}
Y \arrow[r, "l"] \arrow[d, "\epsilon_1"] \arrow[dd, bend right=60, "\widetilde{\epsilon}"'] & X \arrow[r, "l^\vee"] \arrow[d, "\cong"', "\widetilde{\alpha}"] & DY \wedge I \arrow[r, "\delta"] \arrow[d, equal] & \Sigma Y \arrow[d, "\Sigma\epsilon_1"] \\
D(DY \wedge I) \wedge I \arrow[r, "Dl^\vee \wedge 1"'] \arrow[d, "\cong"] & DX \wedge I \arrow[r, "Dl \wedge 1"'] & DY \wedge I \arrow[r, "D\Sigma^{-1}\delta \wedge 1"'] & \Sigma D(DY \wedge I) \wedge I \\
DDY & & &
\end{tikzcd}$$

The second row is distinguished, since it is what one gets when one applies the functor $D(-) \wedge I$ to the first row, after we have turned the triangle once. The sign on the map $D\Sigma^{-1}\delta \wedge 1$ is correct: we get one factor of (-1) from turning the triangle and one factor of (-1) from axiom TC2 of [33]. One concludes that ϵ_1 is an isomorphism, hence $\widetilde{\epsilon}$ is. $\square$

We conclude with a degenerate example.

Example. Given an element I of $Pic(\mathcal{C})$, there is a unique pairing

$$* \wedge * \to I$$

which is perfect. Suppose that $* \to Z$ is a right Lagrangian. Then there exists a map δ so that the sequence

$$DZ \wedge I \to * \to Z \xrightarrow{\delta} \Sigma DZ \wedge I$$

is a distinguished triangle, which means that there is an isomorphism

$$\tilde{\alpha} = \delta : Z \to DZ \wedge \Sigma I.$$

The adjoint

$$\alpha : Z \wedge Z \to \Sigma I$$

is a perfect pairing on Z. Conversely, given Z with a perfect pairing taking values in ΣI, we see that the unique map $* \to Z$ is a right Lagrangian. The choice of perfect pairing on Z coincides with the choice of how to complete the sequence $DZ \wedge I \to * \to Z$ to a distinguished triangle. Indeed, given that Z admits a perfect pairing, the collection of perfect pairings on Z is (non-canonically) isomorphic to $\mathrm{Aut}(Z)$, which is also the automorphism group of the triangle

$$\Sigma^{-1} Z \to * \to Z \xrightarrow{=} Z.$$

2.2 $K(2)$-local equivalences

In what follows we will often want to deduce certain maps are $K(2)$-local equivalences. Of course, these are precisely maps which are $K(2)$-homology equivalences. However, it is sometimes useful to use homotopy group calculations instead. A $K(1)$ version of the following lemma was used to identify the J spectrum with $L_{K(1)}S$ in [6].

Lemma 2.2.1 *Suppose that $f : X \to Y$ is a map for which the induced map*

$$f \wedge V(1) : X \wedge V(1) \to Y \wedge V(1)$$

induces an isomorphism of homotopy groups. Then f is a $K(2)$-equivalence.

Proof. Since $f \wedge V(1)$ is an equivalence, $f \wedge V(1) \wedge E(2)$ is an equivalence. But $V(1) \wedge E(2)$ is equivalent to $K(2)$. Thus f is a $K(2)$ equivalence. $\square$

2.3 $L_{K(n)}S$ is hyperbolic

In this section we work at an arbitrary prime p, and prove that the canonical duality pairing for the $K(n)$-local sphere is hyperbolic. For the purposes of this section everything is implicitly $K(n)$-local. We abbreviate K for $K(n)$ and E for E_n. Let $\mathbb{S}$ be the nth Morava stabilizer group. There is a reduced norm map

$$\mathbb{S} \to \mathbb{Z}_p$$

whose kernel we shall identify as as $\mathbb{S}^1$. Let $\mathbb{G}^1$ denote the Galois extension $\mathbb{S}^1 \rtimes Gal(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let $\overline{S}$ be the homotopy fixed point spectrum $E^{h\mathbb{G}^1}$. The spectrum $\overline{S}$ is an E_∞ ring spectrum [17]. The group $\mathbb{Z}_p$ acts on $\overline{S}$ via E_∞ maps. Let τ be a topological generator of $\mathbb{Z}_p$. For convenience of group-ring notation, we shall use multiplicative notation for the additive group $\mathbb{Z}_p$. In [12] it is proved that there is a homotopy fiber sequence

$$\Sigma^{-1}\overline{S} \xrightarrow{\delta} S \xrightarrow{r} \overline{S} \xrightarrow{\tau-1} \overline{S}. \tag{2.3.1}$$

Define a pairing γ by the composite

$$\gamma : \overline{S} \wedge \overline{S} \xrightarrow{\mu} \overline{S} \xrightarrow{\delta} S^1 \tag{2.3.2}$$

where μ is the product on $\overline{S}$. We shall refer to its adjoint as $\widetilde{\gamma}$.

$$\widetilde{\gamma} : \overline{S} \to \Sigma D\overline{S}$$

The following proposition simultaneously states that γ is a perfect pairing, and that the map $S \to \overline{S}$ is a (right) Lagrangian for S.

Proposition 2.3.1 *The map $\widetilde{\gamma}$ is an equivalence which makes the left square of the following diagram commute.*

$$\begin{tikzcd}
\Sigma^{-1}\overline{S} \arrow[r, "\delta"] \arrow[d, "\widetilde{\gamma}"', "\simeq"] & S \arrow[r, "r"] \arrow[d, equal] & \overline{S} \arrow[d, equal] \\
D\overline{S} \arrow[r, "Dr"'] & S \arrow[r, "r"'] & \overline{S}
\end{tikzcd}$$

In particular, the bottom row is a cofiber sequence.

For a spectrum X and a profinite set $K = \varprojlim_i K_i$, let $X[[K]]$ be defined to be the homotopy inverse limit

$$X[[K]] = \varprojlim_i X \wedge (K_i)_+.$$

Note that the natural map

$$X \wedge S[[K]] \to X[[K]]$$

is not in general an equivalence.

We shall first need the following theorem, which is a proven in [5].

Theorem 2.3.2 *Suppose that H and K are closed subgroups of $\mathbb{G}$. Then there is an equivalence*

$$\alpha : (E[[\mathbb{G}/H]])^{hK} \xrightarrow{\simeq} F(E^{hH}, E^{hK}).$$

Corollary 2.3.3 *Suppose that N is a closed normal subgroup of $\mathbb{G}$. Then there is an equivalence*

$$\alpha : E^{hN}[[\mathbb{G}/N]] \xrightarrow{\simeq} F(E^{hN}, E^{hN}).$$

Corollary 2.3.4 *Suppose that N is a closed normal subgroup of $\mathbb{G}$. Then there is an action map in the $K(n)$-local stable homotopy category*

$$\xi : S[[\mathbb{G}/N]] \wedge E^{hN} \to E^{hN}$$

with the property that for every coset $gN \in \mathbb{G}/N$, the composite

$$\{gN\}_+ \wedge E^{hN} \to S[[\mathbb{G}]] \wedge E^{hN} \xrightarrow{\xi} E^{hN}$$

coincides with the action of gN on E^{hN}.

Proof. Let $\widetilde{\alpha}$ be the adjoint of the map α of Corollary 2.3.3

$$\widetilde{\alpha} : E^{hN}[[\mathbb{G}/N]] \wedge E^{hN} \to E^{hN}.$$

Then the map ξ is given by the composite

$$\xi : S[[\mathbb{G}/N]] \wedge E^{hN} \xrightarrow{\eta \wedge 1 \wedge 1} E^{hN}[[\mathbb{G}/N]] \wedge E^{hN} \xrightarrow{\widetilde{\alpha}} E^{hN}$$

where η is the unit of the ring spectrum E^{hN}. $\square$

Lemma 2.3.5 *For any closed subgroup H of $\mathbb{G}$, and profinite set $K = \varprojlim_i K_i$ for which all of the maps in the tower $\{K_i\}$ are surjections, the natural map*

$$S[[K]] \wedge E^{hH} \to [[K]]E^{hH}$$

is an equivalence.

The proof of Lemma 2.3.5 will requires the following finiteness result.

Lemma 2.3.6 *There exists a dualizable spectrum Y so that*

(1) Y-homology isomorphisms are equivalences.
(2) For every closed subgroup H of G, the Y-homology groups Y_*E^{hH} are finite in each degree.

Proof. Let M be a type n complex and let U be an open normal subgroup of $\mathbb{G}$ of finite cohomological dimension (such a subgroup exists since $\mathbb{G}$ is compact p-adic analytic). Let Y be the spectrum $E^{hU} \wedge M$. The spectrum Y is dualizable since the spectrum E^{hU} is dualizable. In fact, the spectrum E^{hU} is self-dual [42].

Proof of (1). It suffices to show that for any spectrum X, if $Y \wedge X$ is null, then X must be null. In [42] it is shown that the norm map is an equivalence

$$N : (E^{hU})_{h\mathbb{G}/U} \xrightarrow{\simeq} (E^{hU})^{h\mathbb{G}/U}.$$

Combined with the fact that E^{hU} is self-dual, we have

$$\begin{aligned}
(Y \wedge X)^{h\mathbb{G}/U} &\simeq F(E^{hU}, X)^{h\mathbb{G}/U} \wedge M \\
&\simeq F((E^{hU})_{h\mathbb{G}/U}, X) \wedge M \\
&\simeq F((E^{hU})^{h\mathbb{G}/U}, X) \wedge M \\
&\simeq F(E^{h\mathbb{G}}, X) \wedge M \\
&\simeq M \wedge X.
\end{aligned}$$

Therefore $M \wedge X$ is null, but this implies that X is null.

Proof of (2). Using Theorem 2.3.2, and the fact that E^{hU} is dualizable and self-dual, we have

$$\begin{aligned}
Y \wedge E^{hH} &= E^{hU} \wedge M \wedge E^{hH} \\
&\simeq F(E^{hU}, E^{hH}) \wedge M \\
&\simeq (E[\mathbb{G}/U])^{hH} \wedge M.
\end{aligned}$$

As a left H-set, the finite $\mathbb{G}$-set $\mathbb{G}/U$ breaks up into a finite set of H-orbits

$$\mathbb{G}/U \cong \coprod_{\mathbb{G}/UH} H/(H \cap U).$$

Since $H/(H \cap U)$ is finite, there is an H-equivariant equivalence

$$E[\mathbb{G}/U] \simeq \bigvee_{\mathbb{G}/UH} \mathrm{Map}(H/(H \cap U), E).$$

By Shapiro's lemma (see, for example, [5]), we deduce that the H-homotopy fixed points are given by

$$(E[\mathbb{G}/U])^{hH} \simeq \bigvee_{\mathbb{G}/UH} E^{hH \cap U}.$$

The homotopy fixed point spectral sequence gives a spectral sequence

$$E_2^{s,t} = \bigoplus_{\mathbb{G}/UH} H_c^s(H \cap U; E_t M) \Rightarrow Y_{t-s}(E^{hH}).$$

The group $H \cap U$, being a subgroup of U, has finite cohomological dimension [47, 4.1.2], so the spectral sequence has a horizontal vanishing line. Since M is a type n complex, the E-homology of M is finite in each degree. Since the group $H \cap U$ is a closed subgroup of the p-adic analytic group $\mathbb{G}$, we may conclude that the cohomology groups $H_c^s(H \cap E_t M)$ are finite for each s and t (combine Theorem 5.1.2, Lemma 5.1.4, and Proposition 4.2.2 of [47]). We deduce that the Y-homology $Y_* E^{hH}$ is finite in each degree. $\square$

Proof of Lemma 2.3.5. It suffices to show that the map

$$S[[K]] \wedge E^{hH} \wedge Y \to [[K]]E^{hH} \wedge Y$$

is an equivalence, where Y is the dualizable spectrum of Lemma 2.3.6. The homotopy fixed point spectrum E^{hH} is a colimit of homotopy fixed point spectra

$$E^{hH} = \varinjlim_{H \leq V \leq_o \mathbb{G}} E^{hV}$$

where the groups V are open [12]. The spectra E^{hV} are dualizable [26]. Since homotopy inverse limits commute with smash products with dualizable spectra, we are reduced to showing that the natural map

$$\varinjlim_{H \leq V \leq_o \mathbb{G}} \varprojlim_i ([K_i]E^{hV} \wedge Y) \to \varprojlim_i \varinjlim_{H \leq V \leq_o \mathbb{G}} ([K_i]E^{hV} \wedge Y)$$

is an equivalence. By Theorem 3.7 of [35], it suffices to show that the induced maps

$$\varinjlim_{H \leq V \leq_o \mathbb{G}} \varprojlim_i{}^s \pi_*(E^{hV} \wedge Y)[K_i] \to \varprojlim_i{}^s \varinjlim_{H \leq V \leq_o \mathbb{G}} \pi_*(E^{hV} \wedge Y)[K_i] \tag{2.3.3}$$

are isomorphisms. Since the inverse systems are Mittag-Leffler, we only need to investigate $s = 0$. By Lemma 2.3.6, the homotopy groups $\pi_*(E^{hV} \wedge Y)$ and $\pi_*(E^{hH} \wedge Y)$ are finite in each degree. The map 2.3.3 for $s = 0$ is therefore the composite of the following sequence of isomorphisms.

$$\begin{aligned}
\varinjlim_{H \leq V \leq_o \mathbb{G}} \varprojlim_i \pi_*(E^{hV} \wedge Y)[K_i] &= \varinjlim_{H \leq V \leq_o \mathbb{G}} \pi_*(E^{hV} \wedge Y)[[K]] \\
&\cong \varinjlim_{H \leq V \leq_o \mathbb{G}} (\pi_*(E^{hV} \wedge Y) \otimes \mathbb{Z}[[K]]) \\
&\cong (\varinjlim_{H \leq V \leq_o \mathbb{G}} \pi_*(E^{hV} \wedge Y)) \otimes \mathbb{Z}[[K]] \\
&\cong \pi_*(E^{hH} \wedge Y) \otimes \mathbb{Z}[[K]] \\
&\cong \pi_*(E^{hH} \wedge Y)[[K]] \\
&\cong \varprojlim_i \pi_*(E^{hH} \wedge Y)[K_i] \\
&\cong \varprojlim_i \varinjlim_{H \leq V \leq_o \mathbb{G}} \pi_*(E^{hV} \wedge Y)[K_i]
\end{aligned}$$

$\square$

Corollary 2.3.7 *The action map*

$$\xi : S[[\mathbb{G}/N]] \wedge E^{hN} \to E^{hN}$$

of Corollary 2.3.4 lifts to an action map

$$\xi : [[\mathbb{G}/N]]E^{hN} \to E^{hN}.$$

We shall be employing Corollary 2.3.7 in the case of $N = \mathbb{G}^1$. The group $\mathbb{G}/\mathbb{G}^1 = \mathbb{Z}_p$ acts on the homotopy fixed point spectrum $\overline{S} = E^{h\mathbb{G}^1}$. We shall need an additional lemma before proceeding to the proof of Proposition 2.3.1.

Lemma 2.3.8 *Let X be a p-complete spectrum, and let τ be the canonical topological generator of the profinite group $\mathbb{Z}_p$. Then the sequence*

$$X[[\mathbb{Z}_p]] \xrightarrow{[\tau]-1} X[[\mathbb{Z}_p]] \xrightarrow{\epsilon} X$$

is a cofiber sequence.

Proof. The map ϵ is the homotopy inverse limit of the fold maps

$$X \wedge \mathbb{Z}/p^n{}_+ \to X.$$

Since the composite is null, it will suffice to show that the sequence induces a short exact sequence on homotopy groups. Since the inverse system

$$\{\pi_*(X \wedge \mathbb{Z}/p^n{}_+)\} = \{\pi_*(X)[\mathbb{Z}/p^n]\}$$

is Mittag-Leffler, applying π_* gives the sequence

$$\pi_*(X)[[\mathbb{Z}_p]] \xrightarrow{[\tau]-1} \pi_*(X)[[\mathbb{Z}_p]] \to \pi_*(X).$$

This sequence is right exact for any X, and since $\pi_*(X)$ is p-complete, this sequence is left exact. $\square$

Proof of Proposition 2.3.1 By Corollary 2.3.3, we have a canonical equivalence

$$\alpha : \overline{S}[[\mathbb{Z}_p]] \xrightarrow{\simeq} F(\overline{S}, \overline{S})$$

using the fact that $\overline{S} = E^{h\mathbb{G}^1}$ and $\mathbb{G}/\mathbb{G}^1 \cong \mathbb{Z}_p$. Using the action of the Morava stabilizer group, given an element $\lambda \in \mathbb{Z}_p$, we get an induced map

$$[\lambda] : \overline{S} \to \overline{S}.$$

The effect of the post- and pre-composition maps under α are given respectively by

$$\begin{aligned}[\lambda]_* =& L^{\Delta}_{\lambda} : \overline{S}[[\mathbb{Z}_p]] \to \overline{S}[[\mathbb{Z}_p]] \\ [\lambda]^* =& R_{\lambda} : \overline{S}[[\mathbb{Z}_p]] \to \overline{S}[[\mathbb{Z}_p]]\end{aligned}$$

where L^{Δ}_{λ} is the diagonal left action of λ and R_{λ} us the right action on the $\mathbb{Z}_p$ factor.

We shall first prove that $\widetilde{\gamma}$ is an equivalence. Define a map

$$\widetilde{\gamma}_1 : \overline{S}[[\mathbb{Z}_p]] \to F(\overline{S}, \overline{S})$$

whose adjoint is the pairing given by the composite

$$\gamma_1 : \overline{S}[[\mathbb{Z}_p]] \wedge \overline{S} \xrightarrow{\xi_{\Delta}} \overline{S} \wedge \overline{S} \xrightarrow{\mu} \overline{S}.$$

Here ξ_{Δ} is given by allowing the $\mathbb{Z}_p$ to act diagonally on the two $\overline{S}$'s (using the action maps of Corollaries 2.3.7 and 2.3.4), and μ is the product. We emphasize that $\widetilde{\gamma}_1$ is a different map from α.

Consider the following diagram:

$$\begin{tikzcd} \overline{S}[[\mathbb{Z}_p]] \arrow[r, "R_\tau - 1"] \arrow[d, "\simeq"', "\widetilde{\gamma}_1"] \arrow[dr, phantom, "(1)"] & \overline{S}[[\mathbb{Z}_p]] \arrow[r, "\epsilon"] \arrow[d, "\simeq"', "\widetilde{\gamma}_1"] \arrow[dr, phantom, "(2)"] & \overline{S} \arrow[d, "\widetilde{\gamma}"] \\ F(\overline{S}, \overline{S}) \arrow[r, "{[\tau]_* - 1}"'] & F(\overline{S}, \overline{S}) \arrow[r, "\delta_*"'] & F(\overline{S}, S^1) \end{tikzcd} \tag{2.3.4}$$

The bottom row is a cofiber sequence, since it arises from the cofiber sequence 2.3.1. The top row is a cofiber sequences by Lemma 2.3.8. It will follow that $\widetilde{\gamma}$ is an equivalence once we show that $\widetilde{\gamma}_1$ is an equivalence and we show that squares (1) and (2) of Diagram 2.3.4 commute.

Define a map β to be the composite

$$\beta : \overline{S}[[\mathbb{Z}_p]] \xrightarrow{\Delta} \overline{S}[[\mathbb{Z}_p \times \mathbb{Z}_p]] \xrightarrow{\xi_L} \overline{S}[[\mathbb{Z}_p]]$$

where Δ is the diagonal and ξ_L is given by letting the first $\mathbb{Z}_p$ act on $\overline{S}$ through the map of Corollary 2.3.7. The map β is an isomorphism, with inverse given by the composite

$$\overline{S}[[\mathbb{Z}_p]] \xrightarrow{\overline{\Delta}} \overline{S}[[\mathbb{Z}_p \times \mathbb{Z}_p]] \xrightarrow{\xi_L} \overline{S}[[\mathbb{Z}_p]]$$

where $\overline{\Delta} : \mathbb{Z}_p \to \mathbb{Z}_p \times \mathbb{Z}_p$ is the modified diagonal given by

$$\overline{\Delta}(\lambda) = (\lambda^{-1}, \lambda).$$

(Here, as always, we are using multiplicative notation for the additive group $\mathbb{Z}_p$.) Then the map $\widetilde{\gamma}_1$ is an equivalence because it is the composite of the two equivalences

$$\widetilde{\gamma}_1 : \overline{S}[[\mathbb{Z}_p]] \xrightarrow[\beta]{\simeq} \overline{S}[[\mathbb{Z}_p]] \xrightarrow[\alpha]{\simeq} F(\overline{S}, \overline{S}).$$

Square (1) of Diagram 2.3.4 commutes since it is given by the following composite, where each of the squares clearly commutes.

$$\begin{tikzcd}
\overline{S}[[\mathbb{Z}_p]] \arrow[r, "R_\tau - 1"] \arrow[d, "\beta"'] & \overline{S}[[\mathbb{Z}_p]] \arrow[d, "\beta"] \\
\overline{S}[[\mathbb{Z}_p]] \arrow[r, "L_\tau^\Delta - 1"] \arrow[d, "\alpha"'] & \overline{S}[[\mathbb{Z}_p]] \arrow[d, "\alpha"] \\
F(\overline{S}, \overline{S}) \arrow[r, "{[\tau]_* - 1}"'] & F(\overline{S}, \overline{S})
\end{tikzcd}$$

Square (2) of Diagram 2.3.4 commutes since the adjoint maps form the following commutative diagram.

$$\begin{tikzcd}
\overline{S}[[\mathbb{Z}_p]] \wedge \overline{S} \arrow[rrr, "\epsilon \wedge 1"] \arrow[d, "\xi_\Delta"'] \arrow[drr, "{\mu[[Id]]}"] \arrow[dd, bend right=60, "\gamma_1"'] & & & \overline{S} \wedge \overline{S} \arrow[d, "\mu"] \arrow[dd, bend left=60, "\gamma"] \\
\overline{S} \wedge \overline{S} \arrow[d, "\mu"] \arrow[rr, phantom, "(3)"] & & \overline{S}[[\mathbb{Z}_p]] \arrow[r, "\epsilon"] \arrow[dll, "\xi"] \arrow[d, phantom, "(4)"'] & \overline{S} \arrow[d, "\delta"] \\
\overline{S} \arrow[rrr, "\delta"'] & & & S^1
\end{tikzcd}$$

The map ξ is the action map of $\mathbb{Z}_p$ on $\overline{S}$ given by Corollary 2.3.7. The map $\mu[[Id]]$ is the product on the two factors of $\overline{S}$. The only portions of the diagram which don't obviously commute are (3) and (4). Portion (3) commutes since $\mathbb{Z}_p$ acts by maps of E_∞ ring spectra. Since the composite given in the cofiber sequence

$$\overline{S} \xrightarrow{[\tau]-1} \overline{S} \xrightarrow{\delta} S^1$$

is null, δ coequalizes the $\mathbb{Z}_p$-action. Therefore, portion (4) commutes.

We are left with showing that the left-hand square of the diagram in the statement of the proposition commutes. It suffices to check on adjoints, and this is accomplished with the following commuting diagram.

$$\begin{tikzcd}
\overline{S} \wedge S \arrow[r, "1 \wedge r"] \arrow[dr, equal] \arrow[dd, "\delta"'] & \overline{S} \wedge \overline{S} \arrow[d, "\mu"] \\
& \overline{S} \arrow[d, "\delta"] \\
S^1 \arrow[r, equal] & S^1
\end{tikzcd}$$

$\square$

2.4 Identification of $I_{K(2)}TMF$

In this section we will prove the following proposition.

Proposition 2.4.1 *The spectra TMF and $TMF_0(2)$ are Gross-Hopkins self dual up to the suspensions given below.*

$$\begin{aligned} I_K TMF &\simeq \Sigma^{22} TMF \\ I_K TMF_0(2) &\simeq \Sigma^{22} TMF_0(2) \\ I_K E(2) &\simeq \Sigma^{22} E(2) \end{aligned}$$

For the remainder of this section, we shall work in the 3-local stable homotopy category, not localized at $K(2)$.

Recall from [30] that an fp-spectrum is a spectrum whose $\mathbb{F}_p$-cohomology is finitely presented over the Steenrod algebra A. Let $C_n^f X$ be the fiber of the finite localization map $X \to L_n^f X$. Mahowald and Rezk [31] studied a dualization functor W_n defined by

$$W_n X = I C_n^f X.$$

Their theory indicated that this dual (which we shall refer to as the Mahowald-Rezk dual) is quite computable for certain fp-spectra of type n. In particular, they show that both tmf and $BP\langle 2\rangle$ are self-dual.

Proposition 2.4.2 (Mahowald-Rezk [31]) *There are equivalences*

$$W_2 tmf \simeq \Sigma^{23} tmf$$

$$W_2 BP\langle 2\rangle \simeq \Sigma^{23} BP\langle 2\rangle.$$

Since $tmf_0(2)$ is equivalent to $BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle$, and is periodic of order 8, we have the following corollary.

Corollary 2.4.3 *The Mahowald-Rezk dual of $tmf_0(2)$ is given by an equivalence $W_2 tmf_0(2) \simeq \Sigma^{23} tmf_0(2)$.*

We would like to use these computations to identify the Gross-Hopkins duals of TMF and $TMF_0(2)$. To this end we investigate the relationship between the Gross-Hopkins dual and the Mahowald-Rezk dual. A spectrum X is said to *satisfy the $E(n)$-telescope conjecture* if the natural map

$$t_n : L_n^f X \to L_n X$$

is an equivalence. Our interest in such spectra stems from the following proposition.

Proposition 2.4.4 *Suppose that X is a spectrum which satisfies the $E(n)$ and $E(n-1)$-telescope conjectures. Then there is a cofiber sequence*

$$W_{n-1}X \to W_n X \to \Sigma I_n X.$$

Proof. Applying Verdier's axiom to the composite $X \to L_n X \to L_{n-1}X$ gives a cofiber sequence

$$\Sigma^{-1}M_n X \to C_n X \to C_{n-1}X.$$

Since X satisfies the $E(n)$-telescope conjecture, we may regard this cofiber sequence as

$$\Sigma^{-1}M_n X \to C_n^f X \to C_{n-1}^f X.$$

The proposition follows after taking the Brown-Comenetz dual. □

In [31], following a suggestion of Hal Sadofsky, it is pointed out that the chromatic tower computations of [38] imply the following proposition.

Proposition 2.4.5 *The spectra $BP\langle n\rangle$ satisfy the $E(m)$-telescope conjecture for each m.*

Proof. The spectrum BP satisfies the $E(m)$-telescope conjecture (see [30, 7.2] and [39, Theorem 6.2]). Since the localization functors L_m^f and L_m are smashing, the $E(m)$-telescope conjecture holds for any spectrum which may be obtained from BP by means of iterated cofibers and filtered homotopy colimits. In particular, the $E(m)$-telescope conjecture holds for the spectrum $BP\langle n\rangle$. □

Since $tmf_0(2)$ is equivalent to the wedge $BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle$, one has the following corollary.

Corollary 2.4.6 *The spectrum $tmf_0(2)$ satisfies the $E(m)$-telescope conjecture at the prime 3 for every m.*

The following lemma will allow us to bootstrap ourselves up from $tmf_0(2)$ to tmf.

Lemma 2 *Let T be the 3-cell complex $S^0 \cup_{\alpha_1} e^4 \cup_{\alpha_1} e^8$. Then there is an equivalence*

$$tmf \wedge T \simeq tmf_0(2).$$

Proof. The spectrum $tmf_0(2)$ is a tmf-algebra through the map that forgets the $\Gamma_0(2)$ structure

$$\phi_f^* : tmf \to tmf_0(2).$$

Since the (3-primary) homotopy groups of $tmf_0(2) \simeq BP\langle 2\rangle \vee \Sigma^8 BP\langle 2\rangle$ are concentrated in even degrees, the Hurewitz images of the attaching maps of T in $\pi_*(tmf_0(2))$ are null, and hence the map ϕ_f^* factors to give a map

$$\widetilde{\phi}_f^* : tmf \wedge T \to tmf_0(2).$$

The Adams-Novikov E_2-term of $tmf \wedge T$, and the induced map $\widetilde{\phi}_f^*$ on Adams-Novikov E_2-terms, are easily computed with an Atiyah-Hirzebruch spectral sequence, from which we deduce that the map $\widetilde{\phi}_f^*$ induces an isomorphism on the level of Adams-Novikov E_2-terms. We conclude that the map $\widetilde{\phi}_f^*$ is an equivalence. $\square$

From this, we deduce the following.

Proposition 2.4.7 *The spectrum tmf satisfies the $E(m)$-telescope conjecture at the prime 3 for every m.*

Proof. In [39], the complex T is used extensively in the following manner. One has the cofiber sequences given below.

$$S^0 \to T \to \Sigma^4 C\alpha_1$$

$$C\alpha_1 \to T \to S^8$$

Here $C\alpha_1$ is the cofiber of α_1. These splice together to give a $BP\langle 2\rangle$-Adams resolution for tmf.

$$\begin{tikzcd}
tmf \wedge S^0 \arrow[d] & tmf \wedge \Sigma^3 C\alpha_1 \arrow[l] \arrow[d] & tmf \wedge S^{10} \arrow[l] \arrow[d] & tmf \wedge \Sigma^{13} C\alpha_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
tmf_0(2) & \Sigma^3 tmf_0(2) & \Sigma^{10} tmf_0(2) & \Sigma^{13} tmf_0(2) &
\end{tikzcd}$$

Let X_i be the i^{th} term of the tower. The tower converges in the sense that there is an equivalence $\varprojlim X_i \simeq *$. Let Y_i be given by the following cofiber sequences

$$X_i \to X_0 \to Y_i.$$

Then by Verdier's axiom there exist cofiber sequences

$$Y_i \to Y_{i-1} \to \Sigma^N tmf_0(2).$$

Since localization respects cofiber sequences, the spectra Y_i inductively satisfy the $E(m)$-telescope conjecture. We may recover tmf by

$$tmf \simeq \varprojlim Y_i.$$

Localization does not respect homotopy inverse limits in general, but this inverse limit is nice, because the composite

$$S^{10} \to \Sigma^3 C\alpha_1 \to S^0$$

is the element β_1 of $\pi_{10}(S)$. Since β_1^6 is null in the stable stems, our tower X_i has the property that $X_{i+14} \to X_i$ is null. This makes the tower Y_i pro-equivalent to a constant tower, and thus localizations commute with this homotopy inverse limit. We deduce that tmf satisfies the $E(m)$-telescope conjecture. $\square$

We recall the following theorem from [26, Theorem 6.19], which also appeared without proof in [20].

Theorem 2.4.8 ([26], [20]) *For any spectrum X, the monochromatic fiber M_nX satisfies*

$$M_nX \simeq M_nL_{K(n)}X.$$

Proof of Proposition 2.4.1. Let K denote $K(2)$. We shall prove that $I_KTMF \simeq \Sigma^{-1}L_KW_2tmf$. The result will then follow from Proposition 2.4.2. The arguments for $tmf_0(2)$ and $BP\langle 2\rangle$ are similar. Applying L_K to the cofiber sequence given by Proposition 2.4.4, we get a cofiber sequence

$$L_KW_1tmf \to L_KW_2tmf \to \Sigma I_Ktmf.$$

We have an equivalence $L_Ktmf \simeq L_KTMF$. Theorem 2.4.8 implies that there is an equivalence $I_Ktmf \simeq I_KTMF$. We are left with proving that the map

$$L_KW_2tmf \to \Sigma I_KTMF$$

is an equivalence. By Lemma 2.2.1, it suffices to show that this map is an isomorphism on $V(1)$-homology. This is readily seen from computation. By Proposition 2.4.2 we have

$$V(1)_*(L_KW_2tmf) \cong \Sigma^{23}V(1)_*(TMF).$$

Since $L_1V(1) \simeq *$, we also have $I_KV(1) \simeq IL_KV(1)$, and therefore

$$\Sigma V(1)_*(I_KTMF) \cong \Sigma^7\pi_*(I(V(1) \wedge TMF)) \cong \Sigma^{23}V(1)_*TMF$$

The first isomorphism follows from the fact that the CW-complex $V(1)$ is a Spanier-Whitehead self-dual complex of dimension 6. The second isomorphism follows from the computation

$$V(1)_*(TMF) \cong \Sigma^{56}\,\mathrm{Hom}(V(1)_*(TMF), \mathbb{Q}/\mathbb{Z})$$

coupled with the fact that $V(1)_*(TMF)$ is periodic of order 72. The map $L_K W_2 tmf \to \Sigma I_K TMF$ is readily seen to induce an isomorphism between these $V(1)$-homology groups. $\square$

2.5 Identification of $I_{K(2)}^{-1} \wedge TMF$

We now resume the convention that everything is implicitly localized at $K = K(2)$. The aim of this section is to prove the following proposition.

Proposition 2.5.1 *There are equivalences*

$$\begin{aligned} I_K \wedge TMF &\simeq \Sigma^{-22} TMF \\ I_K \wedge TMF_0(2) &\simeq \Sigma^{-22} TMF_0(2) \\ I_K \wedge E(2) &\simeq \Sigma^{-22} E(2) \end{aligned}$$

Proof. The Gross-Hopkins duality theorem [20], [46] states that there is an isomorphism

$$E_*(I_K) \cong \Sigma^2 E_*[det]$$

of Morava modules. Here $[det]$ indicates a twisting with the determinant character. There is a homotopy fixed point spectral sequence

$$H^s(G_{24}; E_t(I_K)) \Rightarrow \pi_{t-s}(TMF \wedge I_K). \tag{2.5.1}$$

The E_2 term of this spectral sequence is thus identified by the Gross-Hopkins duality theorem. Since G_{24} is contained in the kernel of the determinant, we have an isomorphism

$$H^*(G_{24}; E_*(I_K)) \simeq H^*(G_{24}; \Sigma^2 E_*)$$

The spectral sequence 2.5.1 is a spectral sequence of modules over the spectral sequence of algebras

$$H^s(G_{24}; E_t) \Rightarrow \pi_{t-s}(TMF) \tag{2.5.2}$$

which computes the homotopy groups of TMF. The structure of the latter spectral sequence forces the spectral sequence 2.5.1 to be isomorphic to the spectral sequence 2.5.2 up to a suspension congruent to 2 modulo 24. Thus there exists a map

$$S^N \to TMF \wedge I_K$$

which is detected on the 0-line of spectral sequence 2.5.1 which extends (using the TMF-module structure of $TMF \wedge I_K$) to an equivalence

$$\Sigma^N TMF \xrightarrow{\simeq} TMF \wedge I_K.$$

We just need to determine the value of N, which is congruent to 2 modulo 24 and which is only unique modulo 72, the order of periodicity of TMF. Unfortunately, we are unable to avoid appealing to [15] to determine N. This computation was summarized in Section 1.8.2. It implies that

$$\pi_*(V(1)) \cong \pi_*(\Sigma^{28} IV(1)) \cong \pi_*(\Sigma^{22} I_K \wedge V(1))$$

which forces the value of N to be -22.

The identification of $I_K \wedge TMF_0(2)$ is similar. The spectrum $TMF_0(2)$ has the homotopy fixed point model E^{hD_8}. The group D_8 is in the kernel of the determinant, and the ANSS for $I_K \wedge E^{hD_8}$ is concentrated on the zero line, so there are no differentials. Thus one has an isomorphism

$$H^*(D_8; E_* I_K) \cong H^*(D_8; \Sigma^2 E_*)$$

which gives an equivalence

$$I_K \wedge TMF_0(2) \simeq \Sigma^2 TMF_0(2) \simeq \Sigma^{-22} TMF_0(2)$$

using the eightfold periodicity of $TMF_0(2)$.

The case of $E(2)$ is slightly different. The spectrum $E(2)$ is given by the homotopy fixed point spectrum $E^{hSD_{16}}$, but the subgroup SD_{16} is not contained in the kernel of the determinant. In fact, we have an isomorphism of the restricted determinant

$$det \downarrow^{\mathbb{G}}_{SD_{16}} \cong \chi$$

where χ is the non-trivial character of SD_{16}/D_8, regarded as an SD_{16} representation. Using the fact that there is an isomorphism of SD_{16}-modules

$$E_* \otimes \mathbb{Z}_3^{\chi} \cong \Sigma^8 E_*$$

it follows that there are isomorphisms

$$H^*(SD_{16}; E_* I_K) \cong H^*(SD_{16}; \Sigma^2 E_*[det]) \cong H^*(SD_{16}; \Sigma^{10} E_*)$$

which realize to equivalences

$$I_K \wedge E(2) \simeq \Sigma^{10} E(2) \simeq \Sigma^{-22} E(2)$$

using the 16-fold periodicity of $E(2)$. $\square$

2.6 Proof of Theorem 2.0.2

We begin with the an identification of $DTMF$, $DTMF_0(2)$, and $DE(2)$.

Proposition 2.6.1 *There are equivalences*

$$DTMF \simeq \Sigma^{44} TMF$$
$$DTMF_0(2) \simeq \Sigma^{44} TMF_0(2)$$
$$DE(2) \simeq \Sigma^{44} E(2).$$

Proof. Since I_K is invertible, we have for any X [46]

$$DX \wedge I_K \simeq I_K X.$$

Smashing this equation with I_K^{-1} gives

$$DX \simeq I_K X \wedge I_K^{-1}.$$

In particular, we may let X be TMF, $TMF_0(2)$, or $E(2)$, and then apply Propositions 2.4.1 and 2.5.1. $\square$

We shall make use of the following lemma, which follows from the equivalence of fibers and cofibers (up to a suspension) in the stable homotopy category.

Lemma 2.6.2 *Suppose X is a spectrum which refines the tower*

$$A_0 \to A_1 \to \cdots \to A_d.$$

Then DX is a spectrum which refines the dual tower

$$\Sigma^d DA_d \to \Sigma^d DA_{d-1} \to \cdots \Sigma^d DA_0.$$

Since Q refines a tower of the form

$$TMF \to TMF \vee TMF_0(2) \to TMF_0(2),$$

Theorem 2.0.2 follows immediately from Proposition 2.6.1 and Lemma 2.6.2.

2.7 A spectrum which is half of $\overline{S}$

In the next few sections we work our way toward proving Theorem 2.0.1. Proposition 2.3.1 in particular implies that $\overline{S} = E^{h\mathbb{G}^1}$ is self dual. In this section we introduce a spectrum $\overline{Q}$ which will turn out to be a Lagrangian for $\overline{S}$.

Following [16, Sec. 4], let χ be the the sign representation for SD_{16}/D_8 regarded as an SD_{16}-representation. (See Section 1.7 for descriptions of these subgroups.) There is a splitting

$$TMF_0(2) \simeq E(2) \vee \Sigma^8 E(2)$$

which realizes the splitting

$$\mathbb{Z}_3[[\mathbb{G}/D_8]] \cong \mathbb{Z}_3[[\mathbb{G}/SD_{16}]] \oplus \mathbb{Z}_3^{\chi} \uparrow_{SD_{16}}^{\mathbb{G}}$$

where the SD_{16}-representation χ has been induced up to a $\mathbb{G}$-representation. The first summand is generated by $[\omega]+1 \in \mathbb{Z}_3[[\mathbb{G}/D_8]]$ and the second by $[\omega]-1 \in \mathbb{Z}_3[[\mathbb{G}/D_8]]$. Here, and elsewhere, we use brackets to denote the group-like elements of our group rings when confusion may arise as to where the sums are taking place. Let

$$\nu : TMF_0(2) \to \Sigma^8 E(2)$$

be the projection map.

One deficiency of our map ϕ_q^* is that the corresponding element $1+t \in \mathbb{G}$ (see Section 1.7) has norm $det(1+t) = 2$. Since $\overline{S} = E^{h\mathbb{G}^1}$, where $\mathbb{G}^1$ is the kernel of the reduced norm, it would be more natural to consider maps corresponding to elements of reduced norm 1.

Let $\sqrt{2} \in \mathbb{W}$ be the choice of a square root of 2 which reduces to ω^2 in $\mathbb{F}_9$. Such an element exists by Hensel's lemma. We may regard $\sqrt{2}$ as an element of $\mathbb{G}$. Define a map

$$\overline{\phi}_q^* : TMF \to TMF_0(2)$$

which in the language of homotopy fixed point spectra, is given by the composite

$$E^{hG_{24}} \xrightarrow{\iota_{D_8}^*} E^{hD_8} \xrightarrow{[(1+t)/\sqrt{2}]} E^{hD_8}.$$

(The Frobenius takes $\sqrt{2}$ to $-\sqrt{2}$, so one may verify that $\sqrt{2}$ is an element of the normalizer $N_{\mathbb{G}} D_8$.)

Definition 2.7.1 *Define $\overline{Q}$ to be the homotopy equalizer*

$$\overline{Q} \longrightarrow TMF \underset{\nu \circ \overline{\phi}_q^*}{\overset{\nu \circ \phi_f^*}{\rightrightarrows}} \Sigma^8 E(2).$$

The following lemma implies that we may transfer our understanding of the effect of ϕ_q^* on $V(1)$-homology to that of $\overline{\phi}_q^*$.

Lemma 2.7.2 *The maps*

$$V(1)_*\phi_q^*, V(1)_*\overline{\phi}_q^* : V(1)_*(TMF) \to V(1)_*(TMF_0(2))$$

are identical.

Proof. The map $\overline{\phi}_q^*$ differs from ϕ_q^* by a factor of $\sqrt{2} \in \mathbb{W}$. Since $\sqrt{2}$ reduces to $\omega^2 \in \mathbb{F}_9$, and $V(1)_*TMF_0(2)$ is concentrated in degrees congruent to 0 modulo 8, the effect of $\sqrt{2}$ on $V(1)_*TMF_0(2)$ is multiplication by $\omega^8 = 1$. $\square$

Let e_8 be the generator in degree 8 of $\pi_*(\Sigma^8 E(2))$. Then our computations in Section 1.5 combined with Lemma 2.7.2 imply that when we compute the effects of the map on Adams-Novikov E_2-terms

$$\nu_* \circ (\overline{\phi}_q^* - \phi_f^*) : E_2(V(1) \wedge TMF) \to E_2(V(1) \wedge \Sigma^8 E(2))$$

we have

$$\nu_* \circ (\overline{\phi}_q^* - \phi_f^*)(q_4^k) = \begin{cases} -v_2^{(k-1)/2} e_8, & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

From these formulas one easily computes the ANSS E_2-term $E_2(\overline{Q})$. The ANSS differentials are induced from the differentials in the ANSS for TMF just as in Proposition 1.8.2. One finds that (using the patterns of homotopy groups described in Section 1.8) we have an isomorphism

$$V(1)_*(\overline{Q}) = (B \oplus C) \otimes P[v_2^{\pm 9}]. \tag{2.7.1}$$

This should be compared to the computation of the $V(1)$-homology of $\overline{S}$ given in [15].

$$V(1)_*(\overline{S}) = ((B \oplus C) \oplus \Sigma^{29}(B \oplus C)^\vee) \otimes P[v_2^{\pm 9}] \tag{2.7.2}$$

We see that $V(1)_*(\overline{Q})$ is an additive summand of both $V(1)_*(\overline{S})$ and $V(1)_*(Q)$, and that $V(1)_*(\overline{S})$ is the direct sum of a copy of $V(1)_*(\overline{Q})$ and a shifted copy of the dual of $V(1)_*(\overline{Q})$.

2.8 Recollections from [16]

In [16, Prop. 2.6] the mapping spectra $F(E^{hF_1}, E^{hF_2})$ are described for F_i closed subgroups of $\mathbb{G}$, with F_2 finite. The authors of [16] prove the following proposition.

Proposition 2.8.1 (Goerss-Henn-Mahowald-Rezk, [16]) *We have*

$$F(E^{hF_1}, E^{hF_2}) \simeq \prod_{x \in F_2 \backslash \mathbb{G}/F_1} E^{hF_x}$$

where F_x is the finite subgroup $F_2 \cap xF_1x^{-1}$.

The product in Proposition 2.8.1 must be properly interpreted as an appropriate homotopy inverse limit, using the profinite structure of the double coset space $F_2 \backslash \mathbb{G}/F_1$.

We are primarily concerned with the following spectra.

$$\begin{aligned} E^{hG_{24}} &= TMF \\ E^{hD_8} &= TMF_0(2) \\ E^{hSD_{16}} &= E(2) \end{aligned}$$

The homotopy groups of E^{hF} are computed in [16] for F finite. We shall use the following observations about $\pi_* E^{hF}$ for F finite.

Lemma 2.8.2 *Suppose that one of the F_i is G_{24} and the other is either G_{24}, D_8, or SD_{16}. Then the homotopy of each of the summands $\pi_*(E^{hF_x})$ described in Proposition 2.8.1 is concentrated in degrees congruent to* 0 *modulo* 4 *and* $1, 3, 10, 13, 27, 30$ *modulo* 36.

Proof. The group $F_x = F_2 \cap xF_1x^{-1}$ is contained in a conjugate of the subgroup G_{24}, and contains the central element -1, since each of the subgroups G_{24}, D_8, and SD_{16} contains the element -1. We conclude that either: (1) the group F_x contains a conjugate of the cyclic group $C_6 = \langle s, -1 \rangle$, or (2) the order of the group F_x is prime to 3, and F_x contains the element -1. In either case F_x contains the element -1. The ring of invariants $E_*^{F_x}$ is concentrated in degrees congruent to 0 modulo 4, since this is true of the invariants $E_*^{\pm 1}$ [16, Cor. 3.15]. If we are in case (2), then there is no higher cohomology, and the homotopy fixed point spectral sequence collapses to give

$$\pi_*(E^{hF_x}) = E_*^{F_x}.$$

If we are in case (1), then the computations of [16, Thm 3.10, Rmk. 3.12] indicate that elements of $\pi_*(E^{hF_x})$ arising in the homotopy fixed point spectral sequence from group cohomology elements of cohomological degree greater than 0 may only lie in degrees congruent to 1, 3, 10, 13, 27, or 30 modulo 36. $\square$

Lemma 2.8.3 *Suppose that one of the F_i is either D_8 or SD_{16} and the other is either G_{24}, D_8, or SD_{16}. Then the homotopy of each of the summands $\pi_*(E^{hF_x})$ described in Proposition 2.8.1 is concentrated in degrees congruent to* 0 *modulo* 4.

Proof. Since one of the subgroups F_i has order prime to 3, and both of the subgroups F_i contain the central element -1, we are in case (2) of the proof of Lemma 2.8.2 $\square$

Our calculations of the $V(1)$-homology of $\overline{S} = E^{h\mathbb{G}^1}$ (2.7.2) and $\overline{Q}$ (2.7.1) suggest the following Lagrangian decomposition.

Proposition 2.9.1 *There is a map $\overline{\eta}$ such that the following is a fiber sequence*

$$\Sigma D\overline{Q} \xrightarrow{D\overline{\eta}} \overline{S} \xrightarrow{\overline{\eta}} \overline{Q}.$$

The map $D\overline{\eta}$ is the dual of the map $\overline{\eta}$, using the equivalence $D\overline{S} \simeq \Sigma^{-1}\overline{S}$ given by Proposition 2.3.1.

Remark 2.9.2 *Proposition 2.9.1 provides an alternative construction of the resolution of $\overline{S}$ given in [16].*

The inclusion $\iota_{G_{24}} : G_{24} \hookrightarrow \mathbb{G}^1$ induces a map

$$\iota_{G_{24}}^* : \overline{S} \to TMF.$$

The spectrum $\overline{Q}$ is the homotopy equalizer of the maps $\nu \circ \overline{\phi}_q^*$ and $\nu \circ \phi_f^*$ (Definition 2.7.1) where $\overline{\phi}_q^*$ and ϕ_f^* correspond to the elements $(t+1)/\sqrt{2}$ and 1 of $\mathbb{G}$, respectively. Both of these elements have norm 1. Therefore the two composites

$$\begin{tikzcd} \overline{S} \arrow[r, "\iota_{G_{24}}^*"] & TMF \arrow[r, shift left, "\overline{\phi}_q^*"] \arrow[r, shift right, "\phi_f^*"'] & TMF_0(2) \arrow[r, "\nu"] & \Sigma^8 E(2) \end{tikzcd}$$

actually agree. Thus there is a lift of $\iota_{G_{24}}^*$ to a map

$$\overline{S} \to \overline{Q}.$$

This is the map $\overline{\eta}$.

Remark 2.9.3 *The lift $\overline{\eta}$ of $\iota_{G_{24}}^*$ is unique, since Proposition 2.8.1 implies there are no nontrivial maps $\overline{S} \to \Sigma^7 E(2)$.*

Lemma 2.9.4 *The composite*

$$\Sigma D\overline{Q} \xrightarrow{D\overline{\eta}} \overline{S} \xrightarrow{\overline{\eta}} \overline{Q}$$

is null.

Proof. We shall prove that

$$[\Sigma D\overline{Q}, \overline{Q}] = 0.$$

Since $\overline{Q}$ is built from TMF and $\Sigma^7 E(2)$, by Proposition 2.6.1, $\Sigma D\overline{Q}$ is built from $\Sigma^{45}TMF$ and $\Sigma^{38}E(2)$. Lemma 2.8.2 implies that the following groups are zero.

$$\begin{aligned} \pi_{45}(F(TMF, TMF)) &= 0 \qquad & \pi_{38}(F(TMF, E(2))) &= 0 \\ \pi_{38}(F(E(2), TMF)) &= 0 \qquad & \pi_{31}(F(E(2), E(2))) &= 0 \end{aligned}$$

It follows that there are no essential maps $\Sigma D\overline{Q} \to \overline{Q}$. $\square$

Proof of Proposition 2.9.1 Let F be the fiber of the map $\overline{\eta}$. Lemma 2.9.4 implies that there exists a lift f making the following diagram commute.

$$\begin{tikzcd} F \arrow[r, "\iota"] & \overline{S} \arrow[r, "\overline{\eta}"] & \overline{Q} \\ & \Sigma D\overline{Q} \arrow[ul, dotted, "f"] \arrow[u, "D\overline{\eta}"'] & \end{tikzcd}$$

Consider the maps $V(1)_*\iota$ and $V(1)_*D\overline{\eta}$ on $V(1)$-homology. The map $V(1)_*\iota$ is an isomorphism onto $\ker V(1)_*\overline{\eta}$ since $V(1)_*\overline{\eta}$ is surjective. The map $V(1)_*D\overline{\eta}$ is seen to be an isomorphism onto $\ker V(1)_*\overline{\eta}$ from our explicit knowledge of the $V(1)$-homology groups. The effect of the map $V(1)_*D\overline{\eta}$ on $V(1)$-homology is determined from $V(1)_*\overline{\eta}$, since $V(1)_*D\overline{\eta}$ is just the Pontryagin dual of $V(1)_*\overline{\eta}$. Since both $V(1)_*\iota$ and $V(1)_*D\overline{\eta}$ induce isomorphisms onto their images, $V(1)_*f$ must be an isomorphism. By Lemma 2.2.1 the map f must therefore be an equivalence. $\square$

2.10 Building Q from $\overline{Q}$

In the first part of this section we will produce a map

$$Q \xrightarrow{r_Q} \overline{Q}$$

which will turn out to be equivalent to the unit map $Q = Q \wedge S \xrightarrow{1\wedge r} Q \wedge \overline{S}$. The map r_Q will be produced by the methods of [16, Sec. 4]. Specifically, we will prove the following lemma.

Lemma 2.10.1 *There exists a continuous map ρ of $\mathbb{Z}_3[[\mathbb{G}]]$-modules making the following diagram commute*

$$\begin{tikzcd} \mathbb{Z}_3[[\mathbb{G}/G_{24}]] \arrow[d, equal] & \mathbb{Z}_3[[\mathbb{G}/D_8]] \arrow[l, "{[(t+1)/\sqrt{2}]-1}"'] & \mathbb{Z}_3^\chi \uparrow_{SD_{16}}^{\mathbb{G}} \arrow[l, "\nu"'] \arrow[d, dotted, "\rho"] \\ \mathbb{Z}_3[[\mathbb{G}/G_{24}]] & & \mathbb{Z}_3[[\mathbb{G}/D_8]] \oplus \mathbb{Z}_3[[\mathbb{G}/G_{24}]] \arrow[ll, "{([t+1]-1)\oplus([2]-1)}"] \end{tikzcd} \tag{2.10.1}$$

We postpone the proof of Lemma 2.10.1.

Lemma 2.10.2 *The map ρ induces a map ρ^* making the following diagram commute.*

$$\begin{tikzcd}
TMF \arrow[r, "\nu\circ(\overline{\phi}_q^*-\phi_f^*)"] \arrow[d, equal] & \Sigma^8 E(2) \\
TMF \arrow[r, "d_0-d_1"'] & TMF_0(2)\vee TMF \arrow[u, "\rho^*"']
\end{tikzcd}$$

Proof. By Proposition 2.8.1, for H finite, the mapping space $F(E^{hH}, E(2))$ is equivalent to a homotopy inverse limit of products of homotopy fixed point spectra E^{hG} with G contained in SD_{16}. In particular, the groups G have order prime to 3, and thus the ANSS for the mapping space $F(E^{hH}, E(2))$ is concentrated in the zero line.

Therefore, the edge homomorphism gives a sequence of isomorphisms

$$\begin{aligned}
[E^{hH}, \Sigma^8 E(2)] &\cong \mathrm{Hom}^c_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3, \pi_8(E)[[\mathbb{G}/H]]) \\
&\cong \mathrm{Hom}^c_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3, \mathbb{Z}_3^{\chi^{-1}}\widehat{\otimes}\mathbb{Z}_3[[\mathbb{G}/H]]) \\
&\cong \mathrm{Hom}^c_{\mathbb{Z}_3[SD_{16}]}(\mathbb{Z}_3^{\chi}, \mathbb{Z}_3[[\mathbb{G}/H]]) \\
&\cong \mathrm{Hom}^c_{\mathbb{Z}_3[[\mathbb{G}]]}(\mathbb{Z}_3^{\chi}\uparrow_{SD_{16}}^{\mathbb{G}}, \mathbb{Z}_3[[\mathbb{G}/H]]).
\end{aligned}$$

It follows that the map ρ gives rise to the desired map, and the commutativity of the square in the statement of the lemma follows from the commutativity of Diagram 2.10.1. $\square$

The map of towers

$$\begin{tikzcd}
TMF \arrow[r] \arrow[d, equal] & \Sigma^8 E(2) \arrow[r] & * \\
TMF \arrow[r] & TMF_0(2)\vee TMF \arrow[r] \arrow[u, "\rho^*"'] & TMF_0(2) \arrow[u]
\end{tikzcd}$$

induces a map

$$r_Q : Q \to \overline{Q}.$$

The importance of this map for us is encoded in the following lemma and its corollary.

Lemma 2.10.3 *There is an equivalence ξ making the following diagram commute.*

$$\begin{tikzcd}
Q \arrow[r, "1\wedge\eta"] \arrow[dr, "r_Q"'] & Q\wedge\overline{S} \arrow[d, dashed, "\simeq"', "\xi"] \\
 & \overline{Q}
\end{tikzcd}$$

The proof of Lemma 2.10.3 is postponed.

Corollary 2.10.4 *There is a fiber sequence*

$$\Sigma^{-1}\overline{Q} \to Q \xrightarrow{r_Q} \overline{Q}.$$

Proof. Simply smash the fiber sequence

$$\Sigma^{-1}\overline{S} \xrightarrow{\delta} S \xrightarrow{r} \overline{S}$$

with Q, and apply Lemma 2.10.3. $\Box$

In order to prove Lemma 2.10.1, we shall need the following technical result.

Lemma 2.10.5 *The sequence*

$$\mathbb{Z}_3[[\mathbb{G}/D_8]] \oplus \mathbb{Z}_3[[\mathbb{G}/G_{24}]] \xrightarrow{f} \mathbb{Z}_3[[\mathbb{G}/G_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_3 \to 0$$

is exact, where ϵ is the augmentation and f is the map $([t+1]-1) \oplus ([2]-1)$.

Proof. The composite $\epsilon \circ f$ is clearly zero. Let $N \subset \mathbb{Z}_3[[\mathbb{G}/G_{24}]]$ be the kernel of ϵ. We must show that f surjects onto N. Lemma 4.3 of [16] implies that it suffices to show that the reduced map

$$\mathbb{F}_3 \otimes f : \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[\mathbb{G}]]} (\mathbb{Z}_3[[\mathbb{G}/D_8]] \oplus \mathbb{Z}_3[[\mathbb{G}/G_{24}]]) \to \mathbb{F}_3 \otimes_{\mathbb{Z}_3[[\mathbb{G}]]} N$$

is surjective. This is equivalent to showing that

$$f^* : \mathrm{Ext}^0_{\mathbb{Z}_3[[\mathbb{G}]]}(N, \mathbb{F}_3) \to \mathrm{Ext}^0_{\mathbb{Z}_3[[\mathbb{G}]]}(\mathbb{Z}_3[[\mathbb{G}/D_8]] \oplus \mathbb{Z}_3[[\mathbb{G}/G_{24}]], \mathbb{F}_3)$$

is injective. The group $\mathrm{Ext}^0(N, \mathbb{F}_3)$, and the map f^*, may be easily deduced from our computations of the ANSS E_2-terms for $V(1)$-homology, from which it is explicitly seen that indeed f^* is injective. $\Box$

Proof of Lemma 2.10.1. For any $\mathbb{Z}_3[[G]]$ module M, producing a map

$$\mathbb{Z}_3^\chi \uparrow_{SD_{16}}^{\mathbb{G}} \to M$$

is equivalent to specifying an element of M on which SD_{16} acts on with the sign representation. Let $x \in \mathbb{Z}_3[[\mathbb{G}/G_{24}]]$ be the element corresponding to the map

$$([(t+1)/\sqrt{2}] - 1) \circ \nu : \mathbb{Z}_3^\chi \uparrow_{SD_{16}}^{\mathbb{G}} \to \mathbb{Z}_3[[\mathbb{G}/G_{24}]].$$

The element x is in the kernel of ϵ, hence by Lemma 2.10.5 it lifts to an element $\widetilde{x}$ of $\mathbb{Z}_3[[\mathbb{G}/D_8]] \oplus \mathbb{Z}_3[[\mathbb{G}/G_{24}]]$. While it is not necessarily true that $\widetilde{x}$ generates a copy of χ, the weighted average

$$y = \frac{1}{16} \sum_{g \in SD_{16}} \chi(g)[g]\widetilde{x}$$

does have the property that $\mathbb{Z}_3 y \cong \mathbb{Z}_3^{\chi}$. The element y corresponds to the map ρ. $\square$

Proof of Lemma 2.10.3 We first must define ξ. Note that since $\nu \circ \overline{\phi}_q^*$ and $\nu \circ \phi_f^*$ are maps of $\overline{S}$-modules, the homotopy equalizer $\overline{Q}$ is a $\overline{S}$-module. Therefore there is a right action map

$$\mu : \overline{Q} \wedge \overline{S} \to \overline{Q}.$$

The map ξ is defined to be the composite

$$\xi : Q \xrightarrow{r_Q \wedge r} \overline{Q} \wedge \overline{S} \xrightarrow{\mu} \overline{Q}$$

where r is the unit $S \to \overline{S}$.

Recall that, in the notation of Section 1.8.2, we have

$$V(1)_*(Q) = (B \oplus C) \otimes P[v_2^{\pm 9}] \otimes E[\zeta]$$

$$V(1)_*(\overline{Q}) = (B \oplus C) \otimes P[v_2^{\pm 9}].$$

The definition of r_Q implies that diagram below commutes.

$$\begin{tikzcd} Q \arrow[r, "r_Q"] \arrow[d] & \overline{Q} \arrow[d] \\ TMF \arrow[r, equal] & TMF \end{tikzcd}$$

Since these are maps of S-modules, the only possibility is for the induced map $V(1)_* r_Q$ on $V(1)$-homology is to be the quotient by the ideal generated by ζ.

In Appendix A, we prove Theorem A.0.2, which says that $\overline{S}$ has a cell decomposition

$$S^0 \cup_\zeta e^0 \cup_\zeta e^0 \cdots .$$

It follows that $V(1)_*(Q \wedge \overline{S})$ is the quotient of $V(1)_*(Q)$ by the ideal generated by ζ, and the map

$$(1 \wedge r)_* : \pi_*(Q) \to \pi_*(Q \wedge \overline{S})$$

is the quotient map.

Our map ξ is easily shown to make the diagram of in the statement of Lemma 2.10.3 commute, and since $V(1)_*(r_Q)$ and $V(1)_*(1 \wedge r)$ are both surjections onto the same image, $V(1)_*(\xi)$ must be an isomorphism. Lemma 2.2.1 implies that ξ is an equivalence. $\Box$

2.11 Proof of Theorem 2.0.1

In this section we piece together our Lagrangians $\overline{Q}$ to prove Theorem 2.0.1. We first observe that the sequence of Theorem 2.0.1 looks like a fiber sequence on $V(1)$-homology.

Lemma 2.11.1 *Using the notation of Section 1.8.2, there is an isomorphism of short exact sequences.*

$$\begin{tikzcd}
V(1)_*DQ \arrow[r, "D\eta"] \arrow[d, equal] & V(1)_*S \arrow[r, "\eta"] \arrow[d, equal] & V(1)_*Q \arrow[d, equal] \\
\Sigma^{29}(B \oplus C)^\vee \otimes R \arrow[r] & \left(\substack{(B\oplus C)\oplus \\ \Sigma^{29}(B\oplus C)^\vee}\right) \otimes R \arrow[r] & (B \oplus C) \otimes R
\end{tikzcd}$$

where R is the ring $P[v_2^{\pm 9}] \otimes E[\zeta]$ and the maps in the bottom row are the obvious inclusions and projections.

Proof. The map η was essentially computed in Section 1.8.2. The map $D\eta$ is, up to suspension, just the Pontryagin dual of the map η. $\Box$

We shall need the following lemma.

Lemma 2.11.2 *There is an equivalence $DQ \wedge \overline{S} \simeq \Sigma D\overline{Q}$.*

Proof. Although $\overline{S}$ is not dualizable, we will nevertheless show that the natural map

$$\wedge : DQ \wedge D\overline{S} \to D(Q \wedge \overline{S})$$

is an equivalence. The equivalence of the statement of the lemma is then the composite

$$DQ \wedge \Sigma^{-1}\overline{S} \xrightarrow[\simeq]{1\wedge\widetilde{\gamma}} DQ \wedge D\overline{S} \xrightarrow{\wedge} D(Q \wedge \overline{S}) \xrightarrow[\simeq]{D\xi^{-1}} D\overline{Q}$$

where $\widetilde{\gamma}$ is the equivalence given in Proposition 2.3.1, and ξ is the equivalence given in Lemma 2.10.3.

Let $r : S \to \overline{S}$ be the unit. Consider the following commutative diagram.

$$\begin{tikzcd} DQ \wedge D\overline{S} \arrow[r, "1\wedge Dr"] \arrow[d, "\wedge"'] & DQ \wedge DS \arrow[d, "\simeq"', "\wedge"] \\ D(Q \wedge \overline{S}) \arrow[r, "D(1\wedge r)"'] & DQ \end{tikzcd} \tag{2.11.1}$$

We claim that when we apply $V(1)$-homology to Diagram 2.11.1, we get the following, using the notation of Section 1.8.2

$$\begin{tikzcd} \Sigma^{28}(B \oplus C)^{\vee} \otimes P \arrow[r, tail, "V(1)_*1\wedge Dr", "\cdot\zeta"'] \arrow[d, "V(1)_*\wedge"'] & \Sigma^{29}(B \oplus C)^{\vee} \otimes R \arrow[d, equal, "V(1)_*\wedge"] \\ \Sigma^{28}(B \oplus C)^{\vee} \otimes P \arrow[r, tail, "\cdot\zeta", "V(1)_*D(1\wedge r)"'] & \Sigma^{29}(B \oplus C)^{\vee} \otimes R \end{tikzcd} \tag{2.11.2}$$

where $P = P[v_2^{\pm 9}]$ and $R = P \otimes E[\zeta]$. We are claiming that the top and bottom maps are the inclusions given by multiplication by ζ. We see that the left hand map $V(1)_*\wedge$ must therefore be an isomorphism. Therefore, by Lemma 2.2.1, it is an equivalence, and the lemma is proven.

We are left with showing that the top and bottom maps in Diagram 2.11.2 behave as claimed on $V(1)$-homology. The cellular model of $\overline{S}$ given in Theorem A.0.2 implies that there is an the following isomorphism of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & V(1)_*(DQ \wedge \Sigma^{-1}\overline{S}) \arrow[r, "1\wedge\delta"] \arrow[d, equal] & V(1)_*(DQ \wedge S) \arrow[r, "1\wedge r"] \arrow[d, equal] & V(1)_*(DQ \wedge \overline{S}) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \Sigma^{28}(B \oplus C)^{\vee} \otimes P \arrow[r, "\cdot\zeta"'] & \Sigma^{29}(B \oplus C)^{\vee} \otimes R \arrow[r] & \Sigma^{29}(B \oplus C)^{\vee} \otimes P \arrow[r] & 0 \end{tikzcd} \tag{2.11.3}$$

Smashing the left hand square of the diagram of Proposition 2.3.1 with DQ, we get a commutative diagram

$$\begin{tikzcd} DQ \wedge \Sigma^{-1}\overline{S} \arrow[r, "1\wedge\delta"] \arrow[d, "1\wedge\widetilde{\gamma}"', "\simeq"] & DQ \wedge S \arrow[d, equal] \\ DQ \wedge D\overline{S} \arrow[r, "1\wedge Dr"'] & DQ \wedge S \end{tikzcd}$$

from which it follows that $1 \wedge Dr$ behaves as advertised on $V(1)$-homology.

Taking the Spanier-Whitehead dual of the diagram of Lemma 2.10.3 gives the following commutative diagram.

$$\begin{tikzcd} D(Q \wedge \overline{S}) \arrow[r, "D(1\wedge r)"] \arrow[d, "D\xi"', "\simeq"] & DQ \arrow[d, equal] \\ D\overline{Q} \arrow[r, "Dr_Q"'] & DQ \end{tikzcd}$$

Gross-Hopkins duality implies that $V(1)$-cohomology is, up to a shift in degree, the Pontryagin dual of $V(1)$-homology. Therefore, the description of $V(1)_*(r_Q)$ given in the proof of Lemma 2.10.3 dualizes to give the desired behavior of $V(1)_*D(1 \wedge r)$ in Diagram 2.11.2. $\square$

We are now ready to prove a weaker version of Theorem 2.0.1 where we have smashed everything with $\overline{S}$.

Lemma 2.11.3 *The sequence*

$$DQ \wedge \overline{S} \xrightarrow{D\eta\wedge 1} S \wedge \overline{S} \xrightarrow{\eta\wedge 1} Q \wedge \overline{S}$$

is a fiber sequence.

Proof. The cellular description of $\overline{S}$ given in Theorem A.0.2 implies that on $V(1)$-homology, smashing with $\overline{S}$ corresponds to modding out by the ideal generated by ζ. Using Lemma 2.11.1, we see that we have an isomorphism of short exact sequences

$$\begin{tikzcd} V(1)_*DQ \wedge \overline{S} \arrow[r, "D\eta"] \arrow[d, equal] & V(1)_*\overline{S} \arrow[r, "\eta"] \arrow[d, equal] & V(1)_*Q \wedge \overline{S} \arrow[d, equal] \\ \Sigma^{29}(B \oplus C)^\vee \otimes P \arrow[r] & \left(\begin{smallmatrix} (B\oplus C)\oplus \\ \Sigma^{29}(B\oplus C)^\vee \end{smallmatrix} \right) \otimes P \arrow[r] & (B \oplus C) \otimes P \end{tikzcd}$$

where $P = P[v_2^{\pm 9}]$.

By Lemma 2.10.3, we have an equivalence $Q \wedge \overline{S} \simeq \overline{Q}$, and by Lemma 2.11.2 we have an equivalence $DQ \wedge \overline{S} \simeq \Sigma D\overline{Q}$. In the proof of Lemma 2.9.4, it was proven that

$$[\Sigma D\overline{Q}, \overline{Q}] = 0.$$

Thus the composite

$$DQ \wedge \overline{S} \xrightarrow{D\eta\wedge 1} \overline{S} \xrightarrow{\eta\wedge 1} Q \wedge \overline{S}$$

is null. The induced map from $DQ \wedge \overline{S}$ to the fiber of $\eta \wedge 1$ is seen to be a $V(1)$-homology equivalence, hence by Lemma 2.2.1, it is an equivalence. $\square$

We will now complete the proof of Theorem 2.0.1. The reason that Theorem 2.0.1 is more difficult to prove than Proposition 2.9.1 is that the composite

$$DQ \xrightarrow{D\eta} S \xrightarrow{\eta} Q$$

cannot be shown to be null by dimensional considerations alone. However, we have shown in Lemma 2.11.3 that after smashing the above sequence with $\overline{S}$ we get a cofiber sequence.

Lemma 2.11.4 *The map $\tau - 1 : \overline{S} \to \overline{S}$, induces a map of fiber sequences*

$$\begin{tikzcd}
\Sigma^{-1}Q \wedge \overline{S} \arrow[r, "\delta'"] \arrow[d, "1\wedge(\tau-1)"] & DQ \wedge \overline{S} \arrow[r, "D\eta\wedge 1"] \arrow[d, "1\wedge(\tau-1)"] & S \wedge \overline{S} \arrow[r, "\eta\wedge 1"] \arrow[d, "1\wedge(\tau-1)"] & Q \wedge \overline{S} \arrow[d, "1\wedge(\tau-1)"] \\
\Sigma^{-1}Q \wedge \overline{S} \arrow[r, "\delta'"] & DQ \wedge \overline{S} \arrow[r, "D\eta\wedge 1"] & S \wedge \overline{S} \arrow[r, "\eta\wedge 1"] & Q \wedge \overline{S}
\end{tikzcd}$$

where δ' is the completion of the sequence of Lemma 2.11.3 to a fiber seqeuence.

Proof. We need to show that the map δ' commutes with the map $1 \wedge (\tau - 1)$. Lemma 2.10.3, Lemma 2.11.2, and Proposition 2.6.1 combine to show that there are fiber sequences

$$\Sigma^6 E(2) \to \Sigma^{-1}Q \wedge \overline{S} \to \Sigma^{-1}TMF \to \Sigma^7 E(2)$$
$$\Sigma^5 E(2) \to \Sigma^{45}TMF \to DQ \wedge \overline{S} \to \Sigma^6 E(2).$$

By Lemma 2.8.2, the groups

$$[\Sigma^{-1}TMF, \Sigma^{45}TMF], \quad [\Sigma^{-1}TMF, \Sigma^6 E(2)], \quad \text{and} \quad [\Sigma^6 E(2), \Sigma^{45}TMF]$$

are all zero, so we conclude that the induced map

$$[\Sigma^{-1}Q \wedge \overline{S}, DQ \wedge \overline{S}] \to [\Sigma^6 E(2), \Sigma^6 E(2)]$$

is a monomorphism. The topological generator

$$\tau \in \mathbb{Z}_p = \mathbb{Z}_p^\times / \mathbb{F}_p^\times = \mathbb{G}/\mathbb{G}^1$$

may be lifted to a perfect square a^2 in $\mathbb{Z}_p^\times$, hence to a central element a in $\mathbb{G}$. The lemma therefore follows from the fact that the map $[a] - 1$ is central in endomorphism ring

$$[E(2), E(2)] = (E(2)_*[[\mathbb{G}/SD_{16}]])^{SD_{16}} \qquad \text{(Proposition 2.8.1).}$$

□

Neeman [36] defines a morphism of fiber sequences to be *good* if the induced sequence of fibers is a fiber sequence.

Lemma 2.11.5 *The morphism of fiber sequences given by Lemma 2.11.4 is good.*

Proof. There exists a morphism h such that the morphism of fiber sequences

$$\begin{tikzcd}
\Sigma^{-1}Q \wedge \overline{S} \arrow[r, "\delta'"] \arrow[d, "1\wedge(\tau-1)"'] & DQ \wedge \overline{S} \arrow[r, "D\eta\wedge 1"] \arrow[d, dotted, "h"'] & S \wedge \overline{S} \arrow[r, "\eta\wedge 1"] \arrow[d, "1\wedge(\tau-1)"'] & Q \wedge \overline{S} \arrow[d, "1\wedge(\tau-1)"'] \\
\Sigma^{-1}Q \wedge \overline{S} \arrow[r, "\delta'"] & DQ \wedge \overline{S} \arrow[r, "D\eta\wedge 1"] & S \wedge \overline{S} \arrow[r, "\eta\wedge 1"] & Q \wedge \overline{S}
\end{tikzcd}$$

is a good morphism. The morphism

$$1 \wedge (\tau - 1) : DQ \wedge \overline{S} \to DQ \wedge \overline{S}$$

differs from h by a composite $\delta' \circ \alpha \circ D\eta \wedge 1$ where α lies in

$$[\overline{S}, \Sigma^{-1}Q \wedge \overline{S}] \cong \pi_1 F(\overline{S}, \overline{Q}) \qquad \text{(Lemma 2.10.3)}.$$

Using Theorem 2.3.2, we see that the mapping spectrum $F(\overline{S}, \overline{Q})$ is built from the spectra

$$F(\overline{S}, TMF) \simeq TMF[[\mathbb{Z}_p]]$$
$$F(\overline{S}, \Sigma^7 E(2)) \simeq \Sigma^7 E(2)[[\mathbb{Z}_p]]$$

and both of these spectra have trivial π_1. We conclude that

$$[\overline{S}, \Sigma^{-1}Q \wedge \overline{S}] = 0$$

and h must equal $1 \wedge (\tau - 1)$. $\square$

Since the morphism of fiber sequences given by Lemma 2.11.4 is good, *there exist* induced maps f, g, and h below which make a 3×3 diagram of fiber sequences, as displayed below.

$$\begin{tikzcd}
\Sigma^{-2}Q \wedge \overline{S} \arrow[r] \arrow[d] \arrow[dr, phantom, "(-1)"] & \Sigma^{-1}DQ \wedge \overline{S} \arrow[r] \arrow[d] & \Sigma^{-1}\overline{S} \arrow[r] \arrow[d] & \Sigma^{-1}Q \wedge \overline{S} \arrow[d] \\
\Sigma^{-1}Q \arrow[r, dotted, "f"'] \arrow[d] & DQ \arrow[r, dotted, "g"'] \arrow[d] & S \arrow[r, dotted, "h"'] \arrow[d] & Q \arrow[d] \\
\Sigma^{-1}Q \wedge \overline{S} \arrow[r] \arrow[d] & DQ \wedge \overline{S} \arrow[r] \arrow[d] & \overline{S} \arrow[r] \arrow[d] & Q \wedge \overline{S} \arrow[d] \\
\Sigma^{-1}Q \wedge \overline{S} \arrow[r] & DQ \wedge \overline{S} \arrow[r] & \overline{S} \arrow[r] & Q \wedge \overline{S}
\end{tikzcd} \qquad (2.11.4)$$

We must show that $g = D\eta$ and $h = \eta$. Our identifications of the Spanier-Whitehead duals of TMF and $E(2)$ in Proposition 2.6.1 in particular imply that these spectra are reflexive. Therefore the spectrum $\overline{Q}$, being the fiber of a map between reflexive spectra, is itself reflexive. Given different maps g' and h' in place of g and h making Diagram 2.11.4 commute, we see that the difference $g - g'$ is in the image of the homomorphism

$$\delta_* : [DQ, \Sigma^{-1}\overline{S}] \to [DQ, S].$$

We have isomorphisms

$$\begin{aligned}
[DQ, \Sigma^{-1}\overline{S}] &\cong [DQ, D\overline{S}] && \text{(Proposition 2.3.1)} \\
&\cong [DQ \wedge \overline{S}, S] && \\
&\cong [\Sigma D(\overline{Q}), S] && \text{(Lemma 2.11.2)} \\
&\cong \pi_1(DD\overline{Q}) && \\
&\cong \pi_1(\overline{Q}) && (\overline{Q} \text{ is reflexive})
\end{aligned}$$

Similarly, the difference $h - h'$ is in the image of the homomorphism

$$(1 \wedge \delta)_* : [S, \Sigma^{-1}Q \wedge \overline{S}] \to [S, Q]$$

and we have, by Lemma 2.10.3, an isomorphism

$$[S, \Sigma^{-1}Q \wedge \overline{S}] \cong \pi_1(\overline{Q}).$$

However, $\pi_1(\overline{Q}) = 0$, since $\overline{Q}$ is built from TMF and $\Sigma^7 E(2)$. Thus g and h are uniquely determined up to homotopy having the property that they make Diagram 2.11.4 commute. Since $D\eta$ and η make Diagram 2.11.4 commute, we must have $g = D\eta$ and $h = \eta$, as desired. This completes the proof of Theorem 2.0.1.

Appendix A: A cellular model for $\overline{S}$

In this appendix we shall always be working in the $K(n)$-local category at an arbitrary prime p. Let $\mathbb{G}_n$ be the n^{th} extended Morava stabilizer group, and let $\mathbb{G}_n^1$ be the kernel of the reduced norm [12]

$$1 \to \mathbb{G}_n^1 \to \mathbb{G}_n \to \mathbb{Z}_p \to 1.$$

Let $\overline{S}$ be the homotopy fixed point spectrum $E_n^{h\mathbb{G}^1}$. Let $\tau \in \mathbb{Z}_p$ be a topological generator. In [12], the following theorem is proven as an application of the authors' continuous homotopy fixed point construction.

Theorem A.0.1 (Hopkins-Miller) *There is a cofiber sequence*

$$\Sigma^{-1}\overline{S} \xrightarrow{\delta} S \xrightarrow{r} \overline{S} \xrightarrow{[\tau]-1} \overline{S}.$$

The element ζ exists in $\pi_{-1}(S)$, and is given by the composite

$$\zeta : S^{-1} \xrightarrow{r} \Sigma^{-1}\overline{S} \xrightarrow{\delta} S.$$

We shall prove the following theorem, which gives a $K(n)$-local cellular decomposition of $\overline{S}$. We used this decomposition in the proof of Lemma 2.10.3 to compute $V(1)_*(Q(2) \wedge \overline{S})$.

Theorem A.0.2 *There exist complexes C^i_ζ with cellular decompositions*

$$C^i_\zeta = S^0 \cup_\zeta \underbrace{e^0 \cup_\zeta e^0 \cup_\zeta \cdots \cup_\zeta e^0}_{i}.$$

There is an equivalence $\varinjlim C^i_\zeta \simeq \overline{S}$.

Remark A.0.3 *The existence of the complexes C^i_ζ is equivalent to the the Toda bracket*

$$\langle\underbrace{\zeta, \ldots, \zeta}_{i}\rangle$$

being defined and containing zero, for every i.

The remainder of this appendix is dedicated to proving this theorem. Our models for the intermediate complexes C^i_ζ will be the homotopy fibers of the map $([\tau]-1)^i$, giving fiber sequences

$$\Sigma^{-1}\overline{S} \xrightarrow{\delta_i} C^i_\zeta \xrightarrow{r_i} \overline{S} \xrightarrow{([\tau]-1)^i} \overline{S}.$$

The complex C^0_ζ is just the sphere S.

We must explain why the homotopy fibers C^i_ζ have the cellular models as claimed. We shall inductively prove that there are cofiber sequences

$$S^{-1} \xrightarrow{\zeta_{i-1}} C^{i-1}_\zeta \xrightarrow{\iota_i} C^i_\zeta \xrightarrow{\nu_i} S \tag{A.1}$$

where the map ζ_{i-1} will make the following diagram commute.

$$\begin{tikzcd} S^{-1} \arrow[r, "\zeta_{i-1}"] \arrow[dr, "\zeta"'] & C^{i-1}_\zeta \arrow[d, "\nu_{i-1}"] \\ & S \end{tikzcd}$$

Thus ζ_{i-1} is a lift of ζ to the whole complex C^{i-1}_ζ.

The existence of the cofiber sequence A.1 is a direct consequence of Verdier's axiom.

$$\begin{tikzcd}
\Sigma^{-1}\overline{S} \arrow[r, equal] \arrow[d, "([\tau]-1)^{i-1}"'] & \Sigma^{-1}\overline{S} \arrow[r] \arrow[d, "([\tau]-1)^{i}"'] & * \arrow[r] \arrow[d] & \overline{S} \arrow[d, "([\tau]-1)^{i-1}"] \\
\Sigma^{-1}\overline{S} \arrow[r, "{[\tau]-1}"] \arrow[d, "\delta_{i-1}"'] & \Sigma^{-1}\overline{S} \arrow[r, "\delta"'] \arrow[d, "\delta_i"'] & S \arrow[r, "r"'] \arrow[d, equal] & \overline{S} \arrow[d, "\delta_{i-1}"] \\
C_\zeta^{i-1} \arrow[r, dotted, "\iota_i"'] \arrow[d, "r_{i-1}"'] & C_\zeta^{i} \arrow[r, dotted, "\nu_i"'] \arrow[d, "r_i"'] & S \arrow[r, "\zeta_{i-1}"'] \arrow[d] & \Sigma C_\zeta^{i-1} \arrow[d, "r_{i-1}"] \\
\overline{S} \arrow[r, equal] & \overline{S} \arrow[r] & * \arrow[r] & \Sigma\overline{S}
\end{tikzcd} \tag{A.2}$$

The commutativity of the following diagram implies that the composite of ζ_i with the projection onto the top cell is ζ.

$$\begin{tikzcd}
S^{-1} \arrow[rr, "\zeta_i"] \arrow[dr, "r"] \arrow[ddrr, bend right, "\zeta"'] & & C_\zeta^i \arrow[dd, "\nu_i"] \\
& \Sigma^{-1}\overline{S} \arrow[ur, "\delta_i"] \arrow[dr, "\delta"] & \\
& & S
\end{tikzcd}$$

In Diagram $A.2$, we saw that there are commutative diagrams

$$\begin{tikzcd}
C_\zeta^{i-1} \arrow[rr, "\iota_i"] \arrow[dr, "r_{i-1}"'] & & C_\zeta^i \arrow[dl, "r_i"] \\
& \overline{S} &
\end{tikzcd}$$

Let C_ζ^∞ denote the homotopy colimit $\varinjlim C_\zeta^i$. The compatibility of the r_i's implies that there is a map

$$r_\infty = \varinjlim r_i : C_\zeta^\infty \to \overline{S}.$$

We are left with showing that the map r_∞ is an equivalence. The left hand columns of Diagram A.2 give us a tower of cofiber sequences

$$\begin{tikzcd}
\Sigma^{-1}\overline{S} \arrow[r, "\delta_{i-1}"] \arrow[d, "{[\tau]-1}"] & C_\zeta^{i-1} \arrow[r, "r_{i-1}"] \arrow[d, "\iota_i"] & \overline{S} \arrow[d, equal] \\
\Sigma^{-1}\overline{S} \arrow[r, "\delta_i"'] & C_\zeta^i \arrow[r, "r_i"'] & \overline{S}
\end{tikzcd}$$

which, upon taking homotopy colimits gives a cofiber sequence

$$\Sigma^{-1}([\tau]-1)^{-1}\overline{S} \xrightarrow{\delta_\infty} C_\zeta^\infty \xrightarrow{r_\infty} \overline{S}.$$

Therefore, it suffices to prove the following lemma. The author is grateful to Daniel Davis for helping to streamline the proof of this lemma.

Lemma A.0.4 *The telescope* $([\tau]-1)^{-1}\overline{S}$ *is contractible.*

Proof. Let w denote the self-map $[\tau]-1$. It suffices to show that $(K_n)_*(w^{-1}\overline{S})$ is zero, where $K_n = E_n/(p, u_1, \ldots, u_{n-1})$ is 2-periodic Morava K-theory. The E_n-homology of $\overline{S}$ is given by

$$(E_n)_*(\overline{S}) \cong \mathrm{Map}^c(\mathbb{Z}_p, (E_n)_*).$$

In particular, $(E_n)_*(\overline{S})$ is pro-free over $(E_n)_*$ by Theorem 2.5 of [24]. By Corollary 5.2 of [25] we have an isomorphism

$$(K_n)_*(\overline{S}) \cong \mathrm{Map}^c(\mathbb{Z}_p, (K_n)_*).$$

Using the fact that $(K_n)_*$ is a discrete module, we have

$$\begin{aligned}
(K_n)_*(w^{-1}\overline{S}) &= \varinjlim_{w_*} (K_n)_*(\overline{S}) \\
&= \varinjlim_{w_*} \mathrm{Map}^c(\mathbb{Z}_p, (K_n)_*) \\
&= \varinjlim_{w_*} \varinjlim_{k} \mathrm{Map}(\mathbb{Z}/p^k, (K_n)_*) \\
&= \varinjlim_{k} \varinjlim_{w_*} \mathrm{Map}(\mathbb{Z}/p^k, (K_n)_*).
\end{aligned}$$

We are reduced to showing that

$$\varinjlim_{w_*} \mathrm{Map}(\mathbb{Z}/p^k, (K_n)_*) = 0.$$

Using the isomorphism

$$\mathrm{Map}(\mathbb{Z}/p^k, (K_n)_*) \cong \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_p[\mathbb{Z}/p^k], (K_n)_*),$$

the map w_* acts by right multiplication by $[\overline{\tau}]-1$ on the $\mathbb{F}_p[\mathbb{Z}/p^k]$ factor. Here $\overline{\tau}$ is a generator of the group $\mathbb{Z}/p^k$. We have, since we are now in characteristic p,

$$([\overline{\tau}]-1)^{p^k} = [\overline{\tau}]^{p^k} - 1 = 1 - 1 = 0.$$

Thus on $\mathrm{Map}(\mathbb{Z}/p^k, (K_n)_*)$, we have $w_*^{p^k} = 0$, and the colimit over w_* is zero. □

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