Homology, Homotopy and Applications, vol. 10(1), 2008, pp.1–13

AN ALGEBRAIC GENERALIZATION OF IMAGE J

HIROFUMI NAKAI

(*communicated by Month Day, Year*)

Abstract

As is well known, the image of J-homomorphism in the stable homotopy groups of spheres is described in terms of the first line of Adams-Novikov E_2-term. In this paper we consider an algebraic analogue of the image J using the spectrum $T(m)_{(j)}$ defined by Ravenel and determine the Adams-Novikov first line for small values of j.

1. Introduction

As usual, let BP be the Brown-Peterson spectrum and $BP_*(-)$ be the BP-homology functor from the category of spectra to that of abelian groups. The stable homotopy groups and the BP-homology groups of BP are known to be polynomial algebras

$$\pi_*(BP) = \mathbf{Z}_{(p)}[v_1, \ldots, v_n \ldots]$$
$$\text{and} \quad BP_*(BP) = BP_*[t_1, \ldots, t_n \ldots]$$

with $|v_i| = |t_i| = 2(p^i - 1)$.

In [**2**] Ravenel has shown the existence of p-local spectra $T(m)$ and $T(m)_{(j)}$ for non-negative integers m and j, each of which satisfies

$$BP_*(T(m)) = BP_*[t_1, \ldots, t_m]$$
$$\text{and} \quad BP_*(T(m)_{(j)}) = BP_*(T(m))\{t_{m+1}^{\ell} \colon 0 \leqslant \ell < p^j\}.$$

In [**2**] and [**5**] these spectra are extensively used for analyzing the stable homotopy groups of spheres by *the method of infinite descent in homotopy theory*, which is first introduced in [**1**]: Once we have information on $T(m)_{(j)}$, then we can obtain information on $T(m)_{(j-1)}$ using *the small descent spectral sequence* induced in [**2**] Theorem 1.17 and 1.21. Iterated use of this spectral sequence gives information on $T(m)_{(0)} = T(m)$ and finally $T(0) = S^0$. In this sense the Adams-Novikov E_2-term

$$\mathrm{Ext}^{i,*}_{BP_*(BP)}(BP_*, BP_*(T(m)_{(j)})) \tag{1}$$

can be regarded as the start point for doing calculations on the stable homotopy groups of spheres.

Received Month Day, Year, revised Month Day, Year; published on Month Day, Year.
2000 Mathematics Subject Classification: 55Q45
Key words and phrases: Stable homotopy of spheres, Adams-Novikov spectral sequence
This article is available at `http://intlpress.com/HHA/v10/n1/a?`

In this paper we present computations of the Ext groups (1) in lower dimensions. For example, the $j=0$ case (4), the $i=0$ case (Theorem 4.1) and the $(i,j)=(1,1)$ case (Theorem 5.2).

Even for $i=1$ the computations are complicated for higher j in general; For example, we exhibit calculations of the $j=2$ case in Theorem 5.4 (for $p=2$) and Proposition 5.5 (for $p>2$). Note that the result for $(j,m)=(0,0)$ is not new and is well known: the group (1) is equal to $\mathrm{Ext}^1_{BP_*(BP)}(BP_*, BP_*)$ which is isomorphic to the p-primary component of the image of J-homomorphism $\pi_*(SO) \to \pi_*(S)$ for odd prime p (cf. [**4**] Theorem 2.2).

The main task in this paper is to analyze the complicated structure of the comodule W_{m+1} (Definition 2.3) by defining elements $\widehat{\alpha}_{i,p^j-1}$ (Lemma 3.5). The computations for the second or higher lines of (1) will be devoted in [**5**] and those require quite independent calculations. *That is the reason why we separate this paper from [**5**].*

Acknowledgements

The author would like to thank Doug Ravenel for allowing him to use the machineries of our collaboration freely. Through personal communications I know that he obtained Ext^0 (Theorem 4.1). The author is also grateful for valuable comments of the anonymous referee.

2. Notations and method

We will use the following notations (cf. [**5**]).

$$A(k) = \mathbf{Z}_{(p)}[v_1, \ldots, v_k], \qquad \Gamma(k) = BP_*(BP)/(t_i \colon 0 < i < k),$$
$$G(k) = A(k)[t_k] \quad \text{and} \quad T_m^{(j)} = BP_*\left\{t_{m+1}^{\ell} \colon 0 \leqslant \ell < p^j\right\}.$$

The pairs $(BP_*, \Gamma(k))$ and $(A(k), G(k))$ form Hopf algebroids. We will omit the first variable of the associated Ext groups like $\mathrm{Ext}_{\Gamma(m+i)}(-)$ or $\mathrm{Ext}_{G(m+1)}(-)$. It is shown that $T_m^{(j)}$ is in fact a $\Gamma(m+1)$-comodule and we have

Theorem 2.1 ([**2**] Lemma 1.15)**.** *The Adams-Novikov E_2-term for $T(m)_{(j)}$ (1) is isomorphic to* $\mathrm{Ext}^{i,*}_{\Gamma(m+1)}(T_m^{(j)})$. □

We will abbreviate v_{m+i} by $\widehat{v}_i$ and t_{m+i} by $\widehat{t}_i$ for short. In [**2**] (3.10) Ravenel defined elements $\widehat{\lambda}_n \in p^{-1}BP_*$ for $n > 0$ satisfying

$$\widehat{v}_1 = p\widehat{\lambda}_1, \quad \widehat{v}_2 = p\widehat{\lambda}_2 + (1-p^{p-1})v_1\widehat{\lambda}_1^p - \begin{cases} v_1^{p^{m+1}}\widehat{\lambda}_1 & (m>0) \\ 0 & (m=0) \end{cases}, \tag{2}$$

and so on. He also defined the subcomodule D^0_{m+1} of $p^{-1}BP_*$ by

$$D^0_{m+1} = A(m)[\widehat{\lambda}_1, \ldots, \widehat{\lambda}_n, \ldots]$$

and E^1_{m+1} by the short exact sequence of comodules

$$0 \longrightarrow BP_* \rightarrowtail D^0_{m+1} \xrightarrow{\varphi} E^1_{m+1} \longrightarrow 0.$$

Then he showed

Lemma 2.2 ([**2**] Theorem 3.11)**.** *The comodule D^0_{m+1} is weak injective over $\Gamma(m+1)$ and the inclusion map $BP_* \to D^0_{m+1}$ induces an isomorphism of $\mathrm{Ext}^0_{\Gamma(m+1)}$, i.e., we have*

$$\mathrm{Ext}^i_{\Gamma(m+1)}(D^0_{m+1}) = \begin{cases} A(m) & (i=0), \\ 0 & (i>0). \end{cases}$$

□

For a given $\Gamma(m+2)$-comodule M we will denote $\mathrm{Ext}^0_{\Gamma(m+2)}(M)$ by $\overline{M}$ for short. In particular we have

$$\begin{aligned} \overline{D}^0_{m+1} &= A(m)[\widehat{\lambda}_1] \\ \text{and} \quad \overline{T}^{(j)}_m &= A(m+1)\left\{\widehat{t}^{\ell}_1 \colon 0 \leqslant \ell < p^j\right\}. \end{aligned}$$

Define elements $\widehat{\beta}'_{i/j}$ and $\widehat{\beta}_{i/j}$ by

$$\widehat{\beta}'_{i/j} = \widehat{v}^i_2/ipv^j_1 \quad \text{and} \quad \widehat{\beta}_{i/j} = \widehat{v}^i_2/pv^j_1,$$

and the $A(m+1)$-submodule B_{m+1} of $\overline{E}^1_{m+1}/(v^\infty_1)$ by

$$B_{m+1} = A(m+1)\left\{\widehat{\beta}'_{i/i} \colon i > 0\right\},$$

which turns out to be a $G(m+1)$-comodule and is invariant over $\Gamma(m+2)$ (cf. [**6**]). The following is proved in [**5**]

Proposition 2.3. *Define the subcomodule W_{m+1} of $v^{-1}_1\overline{E}^1_{m+1}$ as an extension of B_{m+1} by $\overline{E}^1_{m+1}$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{E}^1_{m+1} & \stackrel{f}{\rightarrowtail} & W_{m+1} & \stackrel{\rho}{\twoheadrightarrow} & B_{m+1} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{E}^1_{m+1} & \rightarrowtail & v^{-1}_1\overline{E}^1_{m+1} & \twoheadrightarrow & \overline{E}^1_{m+1}/(v^\infty_1) & \longrightarrow & 0 \end{array} \tag{3}$$

Then W_{m+1} is weak injective over $G(m+1)$ with

$$\mathrm{Ext}^0_{G(m+1)}(W_{m+1}) = \mathrm{Ext}^1_{\Gamma(m+1)}(BP_*)$$

i.e., the map f induces an isomorphism in Ext^0. □

Remark 2.4. The case for $m=0$ have been essentially treated in [**3**], where it is shown that there exists a weak injective comodule D^1_1 and that we may set $W_{m+1} = \mathrm{Ext}^0_{\Gamma(2)}(D^1_1)$ for $p>2$ (cf. [**3**] Lemma 7.2.1, (7.2.17) and Lemma 7.2.19).

Let $C^{*,s}(T^{(j)}_m)$ denote the cochain complex obtained by applying the functor $\mathrm{Ext}^s_{G(m+1)}(\overline{T}^{(j)}_m \otimes -)$ to the sequence

$$\overline{D}^0_{m+1} \xrightarrow{f\circ\varphi} W_{m+1} \xrightarrow{\rho} B_{m+1}$$

and $H^{*,s}(T^{(j)}_m)$ the associated cohomology group. The following is proved in [**5**].

Proposition 2.5. *For $i=0$ and 1, the cohomology group $H^{i,0}(T_m^{(j)})$ is isomorphic to $\mathrm{Ext}^i_{\Gamma(m+1)}(T_m^{(j)})$, and $H^{2,0}(T_m^{(j)})=\mathrm{Ext}^1_{G(m+1)}(\overline{T}_m^{(j)}\otimes\overline{E}^1_{m+1})$.* □

This allows us to compute the 0-th and the 1-st line of (1) in terms of the cochain complex $C^{*,s}(T_m^{(j)})$. Note that the case for $j=0$ is easy: By definition we have $T_m^{(0)}=A(m+1)$ and

$$\begin{aligned} C^{0,0}(T_m^{(0)}) &= \mathrm{Ext}^0_{G(m+1)}(\overline{D}^0_{m+1}) \ = \ A(m),\\ C^{1,0}(T_m^{(0)}) &= \mathrm{Ext}^0_{G(m+1)}(W_{m+1}) \ = \ A(m)\left\{\frac{\widehat{v}_1^\ell}{\ell p}\colon \ell>0\right\} \quad \text{for } (p,m)\neq(2,0),\\ C^{0,s}(T_m^{(0)}) &= C^{1,s}(T_m^{(0)}) \ = \ 0 \quad \text{for } s>0\\ \text{and}\quad C^{2,s}(T_m^{(0)}) &= \mathrm{Ext}^s_{G(m+1)}(B_{m+1}) \quad \text{for } s\geqslant 0. \end{aligned}$$

All differentials in the cochain complex are trivial and so $H^{i,s}(T_m^{(0)})=C^{i,s}(T_m^{(0)})$ for all i and s, which gives

$$H^{0,0}(T_m^{(0)})=A(m) \quad\text{and}\quad H^{1,0}(T_m^{(0)})=A(m)\left\{\frac{\widehat{v}_1^\ell}{\ell p}\colon \ell>0\right\}. \tag{4}$$

In particular, when $m=0$ it is isomorphic to the Adams-Novikov E_2-terms for the sphere spectrum and thus the old result for the image J is recovered.

3. Structure of the cochain complex $C^{*,0}(T_m^{(j)})$

Hereafter we assume that $m>0$. Given a $G(m+1)$-comodule M and the structure map $\psi_M\colon M\to G(m+1)\otimes M$, define Quillen operations $\widehat{r}_n\colon M\to M$ $(n\geqslant 0)$ on each element $x\in M$ by $\psi_M(x)=\sum_n \widehat{t}_1^n\otimes\widehat{r}_n(x)$. It is easy to see that

Lemma 3.1. *For $x,y\in M$, we have the following relations*

1. $\widehat{r}_i\widehat{r}_j(x)=\binom{i+j}{i}\widehat{r}_{i+j}(x)$,
2. (*Cartan formula*) $\widehat{r}_n(xy)=\sum_{i+j=n}\widehat{r}_i(x)\widehat{r}_j(y)$.

Proof. The first relation follows from the coassociativity of the comodule M: Observe that

$$\begin{aligned} (\Delta\otimes 1)\left(\sum_{n\geqslant 0}\widehat{t}_1^n\otimes\widehat{r}_n(x)\right) &= (1\otimes\psi_M)\left(\sum_{j\geqslant 0}\widehat{t}_1^j\otimes\widehat{r}_j(x)\right)\\ \text{and so}\quad \sum_{0\leqslant i\leqslant n}\binom{n}{i}\widehat{t}_1^{n-i}\otimes\widehat{t}_1^i\otimes\widehat{r}_n(x) &= \sum_{i,j\geqslant 0}\widehat{t}_1^j\otimes\widehat{t}_1^i\otimes\widehat{r}_i(\widehat{r}_j(x)) \end{aligned}$$

where Δ is the coproduct map of $G(m+1)$. The second one follows from the equality

$\psi_M(xy) = \psi_M(x)\psi_M(y)$: Observe that

$$\sum_{n\geqslant 0} \widehat{t}_1^n \otimes \widehat{r}_n(xy) = \left(\sum_{i\geqslant 0} \widehat{t}_1^i \otimes \widehat{r}_i(x)\right)\left(\sum_{j\geqslant 0} \widehat{t}_1^j \otimes \widehat{r}_j(y)\right) = \sum_{0\leqslant i\leqslant n} \widehat{t}_1^n \otimes \widehat{r}_i(x)\widehat{r}_{n-i}(y).$$

□

We will compute Quillen operations on some BP_*-based comodules, whose structure maps are given by the right unit on Hazewinkel generators.

Lemma 3.2. *The right unit* $\eta_R\colon BP_* \to \Gamma(m+1)$ *on Hazewinkel generators* v_n $(n \leqslant m+2)$ *are given by*

$$\begin{aligned} \eta_R(v_k) &= v_k \quad \text{for } 0 \leqslant k \leqslant m,\\ \eta_R(\widehat{v}_1) &= \widehat{v}_1 + p\widehat{t}_1,\\ \eta_R(\widehat{v}_2) &\equiv \widehat{v}_2 + v_1\widehat{t}_1^p - v_1^{p^{m+1}}\widehat{t}_1 \qquad \text{mod } (p). \end{aligned}$$

This gives $\widehat{r}_\ell(\widehat{v}_1^n) = p^\ell\binom{n}{\ell}\widehat{v}_1^{n-\ell}$*. By definition we also have* $\eta_R(\widehat{\lambda}_1) = \widehat{\lambda}_1 + \widehat{t}_1$*.*

Proof. The equations of η_R follow from recursive calculations using relations [**4**] (1.1) and (1.3). Then other statements are obvious. □

To obtain generators of $\mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(j)}_m \otimes M)$, it is indeed enough to consider a certain subgroup of M.

Proposition 3.3. *Denote the subgroup* $\bigcap_{n\geqslant p^j} \ker \widehat{r}_n$ *of* M *by* $L_j(M)$*, and assume that* M *is weak injective over* $G(m+1)$*. If there is an element* $z_{\lambda,p^j-1} \in M$ *for each* $A(m)$*-module generator* $z_{\lambda,0} \in \mathrm{Ext}^0_{G(m+1)}(M)$ *satisfying* $\widehat{r}_{p^j-1}(z_{\lambda,p^j-1}) = z_{\lambda,0}$*, then* $\mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(j)}_m \otimes M)$ *is isomorphic as an* $A(m)$*-module to* $L_j(M)$ *which is spanned by elements* $z_{\lambda,k}$ $(0 \leqslant k < p^j)$ *defined by*

$$z_{\lambda,k} = \binom{p^j-1}{k}^{-1} \widehat{r}_{p^j-1-k}(z_{\lambda,p^j-1}). \tag{5}$$

Proof. By [**2**] Lemma 1.12 and 1.14, we have isomorphisms

$$\widetilde{T}^{(j)}_m \otimes \mathrm{Ext}^0_{G(m+1)}(M) \xrightarrow[\cong]{\theta} \mathrm{Ext}^0_{G(m+1)}(\overline{T}^{(j)}_m \otimes M) \xleftarrow[\cong]{(c\otimes 1)\psi_M} L_j(M)$$

where $\widetilde{T}^{(j)}_m = A(m) \otimes_{A(m+1)} \overline{T}^{(j)}_m$ and the map $c\colon G(m+1) \to G(m+1)$ is the conjugation. The left map θ is the convergence of spectral sequence based on the skeletal filtration of $T^{(j)}_m$ and $\theta(\widehat{t}_1^k \otimes z_{\lambda,0})$ corresponds to $(c\otimes 1)\psi_M(z_{\lambda,k})$ by solving the extension problem.

The elements $z_{\lambda,k}$ (5) are in $L_j(M)$ since

$$
\begin{aligned}
\psi_M(z_{\lambda,k}) &= \sum_{\ell\geqslant 0} \widehat{t}_1^{\ell} \otimes \widehat{r}_\ell(z_{\lambda,k}) \\
&= \sum_{\ell\geqslant 0} \binom{p^j-1}{k}^{-1} \widehat{t}_1^{\ell} \otimes \widehat{r}_\ell \left(\widehat{r}_{p^j-1-k}(z_{\lambda,p^j-1})\right) \\
&= \sum_{\ell\geqslant 0} \binom{p^j-1}{k}^{-1} \widehat{t}_1^{\ell} \otimes \binom{p^j-1-k+\ell}{\ell} \widehat{r}_{p^j-1-k+\ell}(z_{\lambda,p^j-1}) \\
&= \sum_{\ell\geqslant 0} \binom{k}{\ell} \widehat{t}_1^{\ell} \otimes \binom{p^j-1}{k-\ell}^{-1} \widehat{r}_{p^j-1-k+\ell}(z_{\lambda,p^j-1}) \\
&= \sum_{\ell\geqslant 0} \binom{k}{\ell} \widehat{t}_1^{\ell} \otimes z_{\lambda,k-\ell}. \qquad (6)
\end{aligned}
$$

and so we can choose $z_{\lambda,k}$ $(0 \leqslant k < p^j)$ as generators of $L_j(M)$ as desired. □

Proposition 3.3 allows us to compute the cohomology group $H^{i,0}(T_m^{(j)})$ for $i \leqslant 1$ using the following commutative diagram of $A(m)$-modules:

$$
\begin{array}{ccccc}
C^{0,0}(T_m^{(j)}) & \xrightarrow{d^0} & C^{1,0}(T_m^{(j)}) & \xrightarrow{d^1} & C^{2,0}(T_m^{(j)}) \\
{\scriptstyle (c\otimes 1)\psi_{\overline{D}^0_{m+1}}} \uparrow \cong & & {\scriptstyle (c\otimes 1)\psi_{W_{m+1}}} \uparrow \cong & & {\scriptstyle (c\otimes 1)\psi_{B_{m+1}}} \uparrow \\
L_j\left(\overline{D}^0_{m+1}\right) & \xrightarrow{d^0} & L_j(W_{m+1}) & \xrightarrow{d^1} & L_j(B_{m+1}).
\end{array} \qquad (7)
$$

Note that the explicit structure of $L_j(B_{m+1})$ is not needed here since it is a subgroup of $BP_*/(p^\infty, v_1^\infty)$ and we can judge the triviality of d^1-image there.

The first cochain group $C^{0,0}(T_m^{(j)})$ is described as follows:

Corollary 3.4. $C^{0,0}(T_m^{(j)}) = A(m)\left\{\widehat{\lambda}_1^{\ell} : 0 \leqslant \ell < p^j\right\}$.

Proof. If we put $M = \overline{D}^0_{m+1}$ in Proposition 3.3, then we have

$$
\mathrm{Ext}^0_{G(m+1)}(M) = \mathrm{Ext}^0_{G(m+1)}(A(m)[\widehat{\lambda}_1]) = A(m).
$$

So we have only one $A(m)$-module generator, namely $z_0 = 1$, and we may set $z_\ell = \widehat{\lambda}_1^{\ell}$ for $0 \leqslant \ell < p^j$ which clearly satisfies the formula (6) by Lemma 3.2. □

In order to describe $C^{1,0}(T_m^{(j)})$ we put $M = W_{m+1}$. Corresponding to each $A(m)$-module generator $\widehat{v}_1^i/ip$ of $\mathrm{Ext}^0_{G(m+1)}(W_{m+1})$, we need to construct elements $\widehat{\alpha}_{i,p^j-1}$ of $L_j(W_{m+1})$ satisfying the condition of Proposition 3.3.

Lemma 3.5. *For all primes p and integers $i > 0$ and $j \geqslant 0$, define*

$$\widehat{\alpha}_{i,p^j-1} = \frac{\widehat{v}_1^{i-1}}{p^j}\left(c_{i,j}\widehat{\lambda}_1^{p^j} + \mu^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right)$$

$$\text{where } c_{i,j} = \frac{1}{\binom{i+p^j-1}{p^j}} - 1 \quad \text{and} \quad \mu = (1-p^{p-1})\widehat{\lambda}_1^p - v_1^{p^{m+1}-1}\widehat{\lambda}_1.$$

Then it is in $v_1^{-1}\overline{E}_{m+1}^1$ and it satisfies

$$\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1}) = 0$$

$$\text{and} \quad \widehat{r}_{p^j-1}(\widehat{\alpha}_{i,p^j-1}) \equiv \widehat{v}_1^i/ip \mod (v_1) \quad \text{up to unit scalar.}$$

In particular, we have $\widehat{\alpha}_{i,0} \equiv \widehat{v}_1^i/ip$ mod (v_1) up to unit scalar.

Proof. It is $\Gamma(m+2)$-invariant since the possible $\widehat{t}_2$-multiples in the comodule expansion arise from the term $-\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^{p^{j-1}}/p^j$, which is in fact $\Gamma(m+2)$-invariant.

By (2) we have $v_1^{-1}\widehat{v}_2 = pv_1^{-1}\widehat{\lambda}_2 + \mu$ and so $\mu^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}$ is divisible by p^j in $v_1^{-1}\overline{E}_{m+1}^1$. The first term is also in $v_1^{-1}\overline{E}_{m+1}^1$: For $i = 1$ it is trivial since $c_{1,j} = 0$. For $i > 1$ it is expressed as

$$\frac{(p\widehat{\lambda}_1)^{i-1}}{p^j}\left(\frac{1}{\binom{i+p^j-1}{p^j}} - 1\right)\widehat{\lambda}_1^{p^j} = \frac{p^{i-1}}{p^j}\cdot\frac{(i-1)! - c(i,j)}{c(i,j)}\widehat{\lambda}_1^{p^j+i-1}$$

where $c(i,j) = \prod_{1\leqslant k\leqslant i-1}(p^j+k)$ and the coefficient is in $\mathbf{Z}_{(p)}$ by the congruences

$$(i-1)! - c(i,j) \equiv 0 \mod (p^j) \quad \text{and} \quad c(i,j) \not\equiv 0 \mod (p^{i-1}).$$

For Quillen operations note that we can ignore elements belonging to BP_* in $v_1^{-1}\overline{E}_{m+1}$ since $E_{m+1}^1 = D_{m+1}^0/BP_*$ by definition. For example, we can ignore $\widehat{\lambda}_1$ when it is multiplied by p since $p\widehat{\lambda}_1 = \widehat{v}_1 \in BP_*$.

When $i \leqslant p$, it is enough to do calculations mod $(v_1^{p^{m+1}-1})$ by degree reason: We have

$$|\widehat{\alpha}_{i,p^j-1}| = (i-1+p^j)|\widehat{\lambda}_1| = 2(i-1+p^j)(p^{m+1}-1)$$

$$|v_1^{p^{m+1}-1}| = 2(p-1)(p^{m+1}-1)$$

$$\text{and} \quad |\widehat{t}_1^{p^j}| = 2p^j(p^{m+1}-1)$$

Since $|\widehat{\alpha}_{i,p^j-1}| - |\widehat{t}_1^{p^j}|$ is less than or equal to $|v_1^{p^{m+1}-1}|$ and there is not $v_1^{p^{m+1}-1}\widehat{t}_1^{p^j}$ in $\eta_R(\widehat{\alpha}_{i,p^j-1})$, we lose no information even if we calculate $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ mod $(v_1^{p^{m+1}-1})$. Observe that

$$\begin{aligned}\widehat{\alpha}_{i,p^j-1} &\equiv \frac{\widehat{v}_1^{i-1}}{p^j}\left(c_{i,j}\widehat{\lambda}_1^{p^j} + ((1-p^{p-1})\widehat{\lambda}_1^p)^{p^{j-1}} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right)\\ &= \frac{\widehat{v}_1^{i-1}}{p^j}\left(\bar{c}_{i,j}\widehat{\lambda}_1^{p^j} - (v_1^{-1}\widehat{v}_2)^{p^{j-1}}\right) \quad \text{where } \bar{c}_{i,j} = (c_{i,j} + (1-p^{p-1})^{p^{j-1}})\\ &= \bar{c}_{i,j}\frac{\widehat{v}_1^{i-1+p^j}}{p^{j+p^j}} - \frac{\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^{p^{j-1}}}{p^j},\end{aligned}$$

which gives

$$\begin{aligned}\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1}) &= \binom{i+p^j-1}{p^j}\bar{c}_{i,j}\frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \\ &= \binom{i+p^j-1}{p^j}(c_{i,j}+1)\frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \\ &= \frac{\widehat{v}_1^{i-1}}{p^j} - \frac{\widehat{v}_1^{i-1}}{p^j} \ = \ 0.\end{aligned}$$

When $i > p$, each $v_1^{p^{m+1}-1}$-multiple in $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ has higher p-exponent in the coefficient since its $\widehat{t}_1$-exponent almost comes from $\eta_R(\widehat{v}_1^{i-1}) = (\widehat{v}_1 + p\widehat{t}_1)^{i-1}$. Consequently, all elements in $\widehat{r}_{p^j}(\widehat{\alpha}_{i,p^j-1})$ are in fact belonging to BP_* and forgettable.
Similarly we observe

$$\eta_R(\widehat{\alpha}_{i,p^j-1}) \equiv \bar{c}_{i,j}\frac{(\widehat{v}_1+p\widehat{t}_1)^{i-1+p^j}}{p^{j+p^j}} - \frac{(\widehat{v}_1+p\widehat{t}_1)^{i-1}(v_1^{-1}\widehat{v}_2+\widehat{t}_1^p)^{p^{j-1}}}{p^j}$$

mod (v_1) and so

$$\widehat{r}_{p^j-1}(\widehat{\alpha}_{i,p^j-1}) \ \equiv \ \bar{c}_{i,j}\frac{\binom{i-1+p^j}{i}\widehat{v}_1^i}{p^{j+1}} \ \equiv \ \bar{c}_{i,j}\frac{\widehat{v}_1^i}{ip}.$$

□

Lemma 3.6. *The image of $\widehat{\alpha}_{i,p^j-1} \in v_1^{-1}\overline{E}^1_{m+1}$ under the differential d^1 (7) is $-\widehat{v}_1^{i-1}\widehat{\beta}'_{p^{j-1}/p^{j-1}} \in B_{m+1}$, and the elements $\widehat{\alpha}_{i,k}$ $(0 \leqslant k < p^j)$ defined in the same way as (5) for each i are in $L_j(W_{m+1})$.*

Proof. The first statement follows easily from definition of $\widehat{\alpha}_{i,p^j-1}$. We can conclude that $\widehat{\alpha}_{i,k} \in W_{m+1}$ for $0 \leqslant k < p^j - 1$ by (3), (6) and Lemma 3.5. □

By Proposition 3.3 and Lemma 3.6, we have

Proposition 3.7. $C^{1,0}(T_m^{(j)}) = A(m)\left\{\widehat{\alpha}_{i,k} \colon i > 0 \ \textit{and} \ 0 \leqslant k < p^j\right\}$. □

4. Computing differentials in the cochain complex

Based on the results in the previous section, we have

Theorem 4.1. *For any prime p, the 0-th line of Adams-Novikov E_2-term for $T(m)_{(j)}$ is isomorphic to the $A(m)$-module generated by $\widehat{v}_1^k$ $(0 \leqslant k < p^j)$.*

Proof. The d^0-image of $\widehat{\lambda}_1^k \in C^{0,0}(T_m^{(j)})$ $(0 \leqslant k < p^j)$ is $\widehat{v}_1^k/p^k$, which is annihilated by p^k but not by p^{k-1}. So we have

$$H^{0,0}(T_m^{(j)}) \ = \ A(m)\left\{p^k\widehat{\lambda}_1^k \colon 0 \leqslant k < p^j\right\} \ = \ A(m)\left\{\widehat{v}_1^k \colon 0 \leqslant k < p^j\right\}$$

as claimed. □

To detect elements in $\ker d^1$, we do not need to determine the explicit form of $\widehat{\alpha}_{i,k}$: By the commutativity of the diagram (7) and Lemma 3.6 we have an equality

$$d^1(\widehat{\alpha}_{i,k}) = -\binom{p^j-1}{k}^{-1} \widehat{r}_{p^j-1-k}\left(\widehat{v}_1^{i-1}\widehat{\beta}'_{p^{j-1}/p^{j-1}}\right) \tag{8}$$

i.e., the only relevant term to this calculations is the last term of $\widehat{\alpha}_{i,p^j-1}$ (Lemma 3.5) and we can ignore other terms. More precisely, we have

Lemma 4.2. *Assume that $j \leqslant m+2$. Then we have*

$$d^1(\widehat{\alpha}_{i,k}) = -\binom{p^j-1}{k}^{-1} \sum_{a+bp=p^j-1-k} p^a \binom{i-1}{a}\binom{p^{j-1}-1}{b} \widehat{v}_1^{i-1-a}\widehat{\beta}'_{p^{j-1}-b/p^{j-1}-b}.$$

Proof. Since $j \leqslant m+2$, we have $\eta_R(\widehat{v}_2) \equiv \widehat{v}_2 + v_1\widehat{t}_1^p \bmod (v_1^{p^{j-1}})$ and

$$\psi\left(\frac{\widehat{v}_2^{p^{j-1}}}{p^j v_1^{p^{j-1}}}\right) = \frac{(\widehat{v}_2 + v_1\widehat{t}_1^p)^{p^{j-1}}}{p^j v_1^{p^{j-1}}} = \sum_{0<\ell\leqslant p^{j-1}} \binom{p^{j-1}-1}{\ell-1} \frac{\widehat{v}_2^\ell \widehat{t}_1^{p(p^{j-1}-\ell)}}{\ell p v_1^\ell}$$

which gives that $\widehat{r}_n(\widehat{\beta}'_{p^{j-1}/p^{j-1}}) = 0$ for $n \not\equiv 0 \bmod (p)$ and

$$\widehat{r}_{p(p^{j-1}-\ell)}(\widehat{\beta}'_{p^{j-1}/p^{j-1}}) = \binom{p^{j-1}-1}{\ell-1}\widehat{\beta}'_{\ell/\ell} \qquad \text{for } 0<\ell\leqslant p^{j-1}. \tag{9}$$

Then the right hand side of the equality (8) is modified using Lemma 3.1, 3.2 and the formula (9). □

Corollary 4.3. *$H^{2,0}(T_m^{(j)})$ is isomorphic to the quotient of*

$$C^{2,0}(T_m^{(j)}) = \operatorname{Ext}^0_{G(m+1)}(\overline{T}_m^{(j)} \otimes B_{m+1})$$

by $A(m+1)$-module generated by $(c\otimes 1)\psi_{B_{m+1}}(\widehat{\beta}'_{i/i})$ $(0 < i \leqslant p^{j-1})$.

Proof. By Lemma 3.6 and the formula (9) the d^1-image (7) of $L_j(W_{m+1})$ is an $A(m+1)$-module spanned by $\widehat{\beta}'_{\ell/\ell}$ $(0<\ell\leqslant p^{j-1})$. □

Proposition 4.4. *When $j = 1$, we have*

$$d^1(\widehat{\alpha}_{i,k}) = \begin{cases} -\widehat{v}_1^{i-1}\widehat{\beta}_{1/1} & \text{for } k = p-1, \\ 0 & \text{for } 0 \leqslant k < p-1. \end{cases}$$

When $j = 2$, we have

$$d^1(\widehat{\alpha}_{i,k}) = \begin{cases} (-1)^{p+1-\ell}\dfrac{\binom{p}{\ell}}{p}\widehat{v}_1^{i-1}\widehat{\beta}_{\ell/\ell} & \text{for } k = p\ell-1 \text{ with } 0<\ell\leqslant p, \\ -(i-1)\widehat{v}_1^{i-2}\widehat{\beta}_{p/p} & \text{for } k = p^2-2, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Whenever $j \leqslant 2$, we can use Lemma 4.2 with no restriction on m. The $j = 1$ case is easy. For $j = 2$ we have

$$d^1(\widehat{\alpha}_{i,(p-b)p-1}) = -\binom{p^2-1}{bp}^{-1} \frac{\binom{p}{b}}{p} \widehat{v}_1^{i-1} \widehat{\beta}_{p-b/p-b} \quad \text{for } 0 \leqslant b < p.$$

Note that $\binom{p^2-1}{bp} \equiv (-1)^b \bmod (p)$. □

By Corollary 3.4 we have

$$d^0\left(C^{0,0}(T_m^{(j)})\right) = A(m)\left\{\frac{\widehat{v}_1^\ell}{p^\ell} : 0 \leqslant \ell < p^j\right\} \tag{10}$$

and so a cocycle $\widehat{\alpha}_{i,k} \in \ker d^1$ satisfying $i + k < p^j$ is in $\operatorname{Im} d^0$ if and only if it has no $\widehat{v}_2$-multiple. We have

Proposition 4.5. *When $j = 1$, $\widehat{\alpha}_{i,k}$ does not have any non-trivial $\widehat{v}_2$-multiple except for $k = p - 1$. When $j = 2$, $\widehat{\alpha}_{i,k}$ has a non-trivial $\widehat{v}_2$-multiple if and only if $n(p-1) \leqslant k < np$ with $0 < n \leqslant p$, whose order is p for $k \neq p^2 - 1$ and p^2 for $k = p^2 - 1$,*

Proof. The statement for $j = 1$ is obvious since $\widehat{r}_{p-1-k}(\widehat{v}_1^{i-1} v_1^{-1} \widehat{v}_2/p)$ does not have any $\widehat{v}_2$-multiple except for $k = p - 1$. For $j = 2$ the comodule expansion of $-\widehat{v}_1^{i-1}(v_1^{-1}\widehat{v}_2)^p/p^2$ is

$$-(i-1)\frac{v_1^{-p}\widehat{v}_1^{i-2}\widehat{v}_2^p\widehat{t}_1}{p} - \frac{v_1^{-p}\widehat{v}_1^{i-1}}{p^2} \sum_{0 \leqslant n \leqslant \ell < p} (-1)^n \binom{p}{\ell}\binom{\ell}{n} v_1^{\ell+(p^{m+1}-1)n} \widehat{v}_2^{p-\ell} \widehat{t}_1^{p(\ell-n)+n}.$$

Replacing $\ell - n$ with t $(0 \leqslant t < p)$ and assuming that the exponent of $\widehat{v}_2$ is positive (i.e., $0 \leqslant n < p - t$), we have an inequality

$$pt \leqslant (\text{the exponent of } \widehat{t}_1) < pt + p - t.$$

This implies that the coefficient of $\widehat{t}_1^{p^2-1-k}$ (i.e., $\widehat{\alpha}_{i,k}$ up to scalar by unit) has a $\widehat{v}_2$-multiple of order p if and only if $(p-t)(p-1) \leqslant k < (p-t)p$. □

5. Structure of the cohomology $H^{1,0}(T_m^{(j)})$

In this section we finally determine the structure of $H^{1,0}(T_m^{(j)})$ for $j = 1$ and 2 using some tools arranged in the previous section.

First we take care of possibilities of linear relations among cocycles obtained by Proposition 4.4 since the leading term of each $\widehat{\alpha}_{n-\ell,\ell}$ is always a scalar multiple of $\widehat{\lambda}_1^n$. In fact, as the simplest case, we have

Proposition 5.1. *In $C^{1,0}(T_m^{(1)})$, there are relations*

$$\frac{ip}{k}\widehat{\alpha}_{i,k} = \widehat{\alpha}_{i+1,k-1} \quad \textit{for } 0 < k \leqslant p - 1. \tag{11}$$

Proof. Definition of $\widehat{\alpha}_{i,p-1}$ (Lemma 3.5) and direct calculations show that

$$\widehat{\alpha}_{i,k} = \begin{cases} \dfrac{p^{i-1}}{i\binom{i+p-2}{i}}\widehat{\lambda}_1^{i+p-2} + (i-1)\widehat{v}_1^{i-2}(v_1^{p^{m+1}-1}\widehat{\lambda}_1 + p^{p-1}\widehat{\lambda}_1^p) & \text{for } k = p-2, \\ \dfrac{p^{i-1}}{i\binom{i+k}{i}}\widehat{\lambda}_1^{i+k} & \text{for } 0 < k \leqslant p-3 \end{cases}$$

where the second term for $k = p-2$ has the order p. □

Consequently, we have the following result for $H^{1,0}(T_m^{(1)})$.

Theorem 5.2. *For any prime p, the first line of Adams-Novikov E_2-term for $T(m)_{(1)}$ is isomorphic to the $A(m)$-submodule of W_{m+1} generated by $p\widehat{\alpha}_{i,p-1}$ and $v_1\widehat{\alpha}_{i,p-1}$ for $i \geqslant 1$.*

Proof. By Proposition 4.4 the kernel of d^1 (7) consists of

$$\widehat{\alpha}_{i,k}\ (0 \leqslant k < p-1) \qquad p\widehat{\alpha}_{i,p-1} \qquad \text{and} \qquad v_1\widehat{\alpha}_{i,p-1}.$$

Proposition 5.1 tells us that the elements $\widehat{\alpha}_{i,k}$ with $k < p-1$ are needless. □

The situation is more complicated for $j = 2$. First we consider the most accessible $p = 2$ case. By Lemma 4.2 we find that

$$d^1(\widehat{\alpha}_{i,3}) = -\frac{\widehat{v}_1^{i-1}\widehat{v}_2^2}{4v_1^2}, \quad d^1(\widehat{\alpha}_{i,2}) = -\frac{(i-1)\widehat{v}_1^{i-2}\widehat{v}_2^2}{6v_1^2}, \quad d^1(\widehat{\alpha}_{i,1}) = -\frac{\widehat{v}_1^{i-1}\widehat{v}_2}{6v_1}$$

and $d^1(\widehat{\alpha}_{i,0}) = 0$. Thus we have cocycles

$$\begin{array}{rl} \widehat{\alpha}_{i,2} & \text{for odd } i, \\ \widehat{\alpha}_{i,0}, \quad 4\widehat{\alpha}_{i,3}, \quad 2\widehat{\alpha}_{i,2}, \quad 2\widehat{\alpha}_{i,1}, \quad 2i\widehat{\alpha}_{i,3} - 3\widehat{\alpha}_{i+1,2} & \text{for all } i \geqslant 1, \\ v_1^2\widehat{\alpha}_{i,3}, \quad v_1^2\widehat{\alpha}_{i,2}, \quad v_1\widehat{\alpha}_{i,1} & \text{for all } i \geqslant 1. \end{array} \tag{12}$$

For linear relations among these cocycles we have

Proposition 5.3. *For $(p,j) = (2,2)$ we have the following relations:*

1. *$\widehat{\alpha}_{i,0}$ is a scalar multiple of $\widehat{\alpha}_{i-1,1}$ if $2 \leqslant i \equiv 1, 2$ mod (4), $\widehat{\alpha}_{i-2,2}$ if $3 \leqslant i \equiv 1$ mod (2), and $\widehat{\alpha}_{i-3,3}$ if $4 \leqslant i \equiv 0, 1$ mod (4),*
2. *$2\widehat{\alpha}_{i,1}$ is a scalar multiple of $\widehat{\alpha}_{i-1,2}$ if $3 \leqslant i \equiv 3$ mod (4), and $\widehat{\alpha}_{i-2,3}$ if $4 \leqslant i \equiv 0$ mod (2).*
3. *$8\widehat{\alpha}_{i,2}$ is a scalar multiple of $\widehat{\alpha}_{i-1,3}$ for all $i \geqslant 2$.*

Proof. Tedious routine calculations give the following:

$$\begin{aligned}
\widehat{\alpha}_{i,3} &= \frac{\widehat{v}_1^{i+3}}{16i\binom{i+3}{3}} + \frac{v_1^{2^{m+1}-1}\widehat{v}_1^{i+2}}{16} + \frac{v_1^{2^{m+2}-2}\widehat{v}_1^{i+1}}{16} - \frac{v_1^{-2}\widehat{v}_1^{i-1}\widehat{v}_2^2}{4} \\
\widehat{\alpha}_{i,2} &= \frac{\widehat{v}_1^{i+2}}{24\binom{i+2}{3}} + \frac{(i+2)v_1^{2^{m+1}-1}\widehat{v}_1^{i+1}}{24} + \frac{(i+1)v_1^{2^{m+2}-2}\widehat{v}_1^{i}}{24} \\
&\quad - \frac{(i-1)v_1^{-2}\widehat{v}_1^{i-2}\widehat{v}_2^2 - v_1^{2^{m+1}-2}\widehat{v}_1^{i-1}\widehat{v}_2}{6} \\
\widehat{\alpha}_{i,1} &= \frac{\widehat{v}_1^{i+1}}{8\binom{i+1}{2}} + \frac{\binom{i+2}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i}}{12} + \frac{\binom{i+1}{2}-1}{12}v_1^{2^{m+2}-2}\widehat{v}_1^{i-1} + \frac{v_1^{-1}\widehat{v}_1^{i-1}\widehat{v}_2}{6} \\
\widehat{\alpha}_{i,0} &= \frac{\widehat{v}_1^{i}}{2i} + \frac{\binom{i+2}{3}-1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1} + \frac{\binom{i+1}{3}-(i-1)}{2}v_1^{2^{m+2}-2}\widehat{v}_1^{i-2}
\end{aligned}$$

and we consequently obtain the following relations in W_{m+1}:

$$\begin{aligned}
\widehat{\alpha}_{i,0} - 2(i-1)\widehat{\alpha}_{i-1,1} &= \frac{(i-1)(i+2)}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}, \\
\widehat{\alpha}_{i,0} - 2(i-1)(i-2)\widehat{\alpha}_{i-2,2} &= \frac{i^2-1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}, \\
\widehat{\alpha}_{i,0} - \frac{4}{3}(i-1)(i-2)(i-3)\widehat{\alpha}_{i-3,3} &= \frac{3i(i-1)}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}, \\
2(\widehat{\alpha}_{i,1} - (i-1)\widehat{\alpha}_{i-1,2}) &= \frac{i+1}{4}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1} + \frac{i-1}{6}v_1^{2^{m+2}-2}\widehat{v}_1^{i-2}, \\
2\left(\widehat{\alpha}_{i,1} - \frac{2}{3}(i-1)(i-2)\widehat{\alpha}_{i-2,3}\right) &= \frac{i}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}, \\
4\left(\widehat{\alpha}_{i,2} - \frac{2}{3}(i-1)\widehat{\alpha}_{i-1,3}\right) &= \frac{1}{2}v_1^{2^{m+1}-1}\widehat{v}_1^{i-1}.
\end{aligned}$$

□

By these observations we have

Theorem 5.4. *For $p = 2$, $H^{1,0}(T_m^{(2)})$ is isomorphic to the $A(m)$-submodule of W_{m+1} generated by the elements listed in (12) with relations obtained in Proposition 5.3.* □

As showed in the Proposition 5.3, experiences of calculations tell us that it would be unhopeful to have brief formulas of relations among cocycles $\widehat{\alpha}_{n-\ell,\ell} \in H^{1,0}(T_m^{(2)})$ for general prime p, unfortunately.

We can still find all cocycles of $H^{1,0}(T_m^{(2)})$ for an arbitrary prime using Proposition 4.4 and 4.5.

Proposition 5.5. *For any prime p, $H^{1,0}(T_m^{(2)})$ consists of the following cocycles:*

1. $\widehat{\alpha}_{i,k}$ *with* $k \not\equiv -1 \bmod (p)$, $k \neq p^2-2$, $i+k \geqslant p^2$ *and* $k < p^2$,
2. $\widehat{\alpha}_{i,k}$ *with* $i+k < p^2$ *and* $n(p-1) \leqslant k < np-1$ *for an integer* $0 < n < p$,
3. $\widehat{\alpha}_{i,k}$ *with* $i+k < p^2$ *and* $p(p-1) \leqslant k < p^2-2$,
4. $\widehat{\alpha}_{i,p^2-2}$ *with* $i \equiv 1 \bmod (p)$,

5. $p^2\widehat{\alpha}_{i,p^2-1}$,
6. $p\widehat{\alpha}_{i,p\ell-1}$ *with* $0<\ell<p$ *and* $i \geqslant p(p-\ell)+1$,
7. $p\widehat{\alpha}_{i,p^2-2}$ *with* $i \not\equiv 1$ *mod* (p),
8. $ip\widehat{\alpha}_{i,p^2-1}+\widehat{\alpha}_{i+1,p^2-2}$ *with* $i \not\equiv 0$ *mod* (p),
9. $v_1^\ell\widehat{\alpha}_{i,p\ell-1}$ *with* $0<\ell\leqslant p$,
10. $v_1^p\widehat{\alpha}_{i,p^2-2}$.

Proof. By Proposition 4.4 we have the following elements of $\ker d^1$: (1) $\widehat{\alpha}_{i,k}$ with $k \not\equiv -1 \bmod (p)$ and $k \neq p^2-2$ (2) $\widehat{\alpha}_{i,p^2-2}$ with $i \equiv 1 \bmod (p)$ (3) $p^2\widehat{\alpha}_{i,p^2-1}$ (4) $p\widehat{\alpha}_{i,p\ell-1}$ with $0<\ell<p$ (5) $p\widehat{\alpha}_{i,p^2-2}$ with $i \not\equiv 1 \bmod (p)$ and (6) $ip\widehat{\alpha}_{i,p^2-1}+\widehat{\alpha}_{i+1,p^2-2}$ with $i \not\equiv 0 \bmod (p)$ (7) $v_1^\ell\widehat{\alpha}_{i,p\ell-1}$ with $0<\ell\leqslant p$ and (8) $v_1^p\widehat{\alpha}_{i,p^2-2}$.

We need to remove elements of $\operatorname{Im} d^0$ using (10), and the case (1) consequently splits into the cases (1), (2) and (3).

Notice that all possible $\widehat{v}_2$-multiples are order p, which are vanished in cases (3), (4) and (5), and that elements of these cases are not in $\operatorname{Im} d^0$ when the sum of two subscripts is larger than p^2-1. ☐

Concluding remark. We believe that it would be still hopeful to obtain all cocycles of $H^{1,0}(T_m^{(j)})$ for $j>2$ after arranging the results similar to Proposition 4.4 and 4.5. However, we don't know how we can describe all linear relations among cocycles systematically for general value of j.

References

[1] D. C. Ravenel. The Adams-Novikov E_2-term for a complex with p-cells, *Amer. J. Math.* **107** (1985), 933–968.

[2] D. C. Ravenel. The method of infinite descent in stable homotopy theory I, In D. M. Davis, editor, *Recent Progress in Homotopy Theory*, Contemp. Math., **293**, 251–284, Amer. Math. Soc., Providence, RI, 2002.

[3] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres, Second Edition.* American Mathematical Society, Providence, 2004.

[4] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in Adams-Novikov spectral sequence, *Ann. of Math.* **106** (1977), 469–516.

[5] H.Nakai and D. C. Ravenel. The method of infinite descent in stable homotopy theory II. In preparation.

[6] H.Nakai and D. C. Ravenel. On β-elements in the Adams-Novikov spectral sequence. In preparation.

Hirofumi Nakai `nakai@ma.ns.musashi-tech.ac.jp`

Department of Mathematics, Musashi Institute of Technology, Tokyo 158-8557, Japan