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A UNIVERSAL PROPERTY OF THE CONVOLUTION MONOIDAL STRUCTURE

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1. Introduction

Let $\mathscr{V}$ be a complete and cocomplete locally-small symmetric monoidal closed category, with tensor product $\otimes$, unit object I, and internal-hom $[\cdot,\cdot]$. By *category, functor*, and *natural transformation* we mean *$\mathscr{V}$-category*, *$\mathscr{V}$-functor*, and *$\mathscr{V}$-natural transformation*, unless the context indicates otherwise. For a small category $\mathscr{A}$ we write $\mathscr{P}\mathscr{A}$ for the functor category $[\mathscr{A}^{\mathrm{op}},\mathscr{V}]$, and $y:\mathscr{A}\to\mathscr{P}\mathscr{A}$ for the Yoneda embedding sending A to $\mathscr{A}(-,A)$.

It is well known (see [9, Theorem 4.51]) that, for a cocomplete $\mathscr{C}$, the cocontinuous functors $\mathscr{P}\mathscr{A}\to\mathscr{C}$ are precisely the left adjoint ones; and that composition with y gives an equivalence between the category of such functors and the category of all functors $\mathscr{A}\to\mathscr{C}$. Since $\mathscr{P}\mathscr{A}$ is cocomplete, we may express this by saying that $y:\mathscr{A}\to\mathscr{P}\mathscr{A}$ exhibits $\mathscr{P}\mathscr{A}$ as the *free cocompletion* of $\mathscr{A}$. This result is particularly classical in the case $\mathscr{V}=\mathrm{Set}$ of *ordinary* categories and the case $\mathscr{V}=\mathrm{Ab}$ of *additive* categories; here Set and Ab denote as usual the categories of small sets and small abelian groups.

It was observed by Day in [2] that each monoidal structure on $\mathscr{A}$ induces a monoidal biclosed structure on $\mathscr{P}\mathscr{A}$, called the *convolution* of the structures on $\mathscr{A}$ and on $\mathscr{V}$ – this structure being symmetric monoidal closed if the monoidal structure on $\mathscr{A}$ is symmetric. Moreover $y:\mathscr{A}\to\mathscr{P}\mathscr{A}$ is now a *strong monoidal functor*, in the sense that it preserves the monoidal structure to within coherent isomorphisms.

Let us call a monoidal category $\mathscr{C}$, with tensor product $*$, *monoidally cocomplete* if $\mathscr{C}$ is cocomplete and all the endofunctors $C*-$ and $-*D$ of $\mathscr{C}$ are cocontinuous;

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which latter is surely the case if the monoidal structure on $\mathscr{C}$ is biclosed. The central observation of the present article is that $y:\mathscr{A}\to\mathscr{P}\mathscr{A}$, for a monoidal $\mathscr{A}$, exhibits Day's convolution $\mathscr{P}\mathscr{A}$ as the *free monoidal cocompletion of* $\mathscr{A}$: in the sense that, for a monoidally-cocomplete $\mathscr{C}$, composition with y gives an equivalence between the category of cocontinuous monoidal functors $\Phi:\mathscr{P}\mathscr{A}\to\mathscr{C}$ and the category of all monoidal functors $\Theta:\mathscr{A}\to\mathscr{C}$. Moreover Θ is strong monoidal exactly when the corresponding Φ is strong monoidal; and a monoidal $\Phi:\mathscr{P}\mathscr{A}\to\mathscr{C}$ is left adjoint *as a monoidal functor* precisely when it is cocontinuous and strong monoidal. There is also the analogous result for *symmetric* monoidal categories and *symmetric* monoidal functors.

Taking $\mathscr{C}$ to be the convolution $\mathscr{P}\mathscr{A}'$ for some monoidal $\mathscr{A}'$ clearly leads to a notion of *monoidal module* (or *monoidal profunctor*) $\mathscr{A}\to\mathscr{A}'$, and a monoidal version of the Morita theorems. We do not pursue these themes below, but we do apply our central result to calculate in one case the group of isomorphism classes of monoidal auto-equivalences of $\mathscr{P}\mathscr{A}$ (which, following [1], we may well call the *monoidal Picard group of* $\mathscr{A}$).

The case we examine is that where $\mathscr{P}\mathscr{A}$ is the monoidal category $\mathscr{G}$ of graded abelian groups; the Picard group turns out to have two elements, the non-identity one sending X to Y where $Y_p=X_{-p}$. It is well known (see [4, Chapter IV, Section 6]) that $\mathscr{G}$ admits exactly two different symmetries c and c', given by $c(x\otimes y)=y\otimes x$ and by $c'(x\otimes y)=(-1)^{pq}y\otimes x$, where $x\in X_p$ and $y\in Y_q$. Since neither element of the monoidal Picard group sends c to c', we have the conclusion that these two symmetric monoidal closed structures on $\mathscr{G}$ are not only different, but in fact not isomorphic and not even equivalent.

2. Adjunction for monoidal functors

Let $\mathscr{A}$ be a monoidal category with tensor product $*$, unit object J, and associativity, left-identity, and right-identity isomorphisms a, l, r. Let $\mathscr{A}'$ be another such. We recall from [4, p. 473] that a *monoidal functor* $\Phi:\mathscr{A}\to\mathscr{A}'$ consists of a functor $\phi:\mathscr{A}\to\mathscr{A}'$, a natural transformation $\tilde\phi:\phi A *' \phi B\to\phi(A*B)$, and a map $\phi^0:J'\to\phi J$, satisfying the following coherence conditions:

$$\begin{array}{ccc}
(\phi A *' \phi B) *' \phi C & \xrightarrow{\;a'\;} & \phi A *' (\phi B *' \phi C)\\
\Big\downarrow{\scriptstyle \tilde\phi *' 1} & & \Big\downarrow{\scriptstyle 1 *' \tilde\phi}\\
\phi(A*B) *' \phi C & & \phi A *' \phi(B*C)\\
\Big\downarrow{\scriptstyle \tilde\phi} & & \Big\downarrow{\scriptstyle \tilde\phi}\\
\phi((A*B)*C) & \xrightarrow[\;\phi a\;]{} & \phi(A*(B*C))
\end{array} \qquad (2.1)$$

$$\begin{array}{ccc} J' *' \phi A & \xrightarrow{l'} & \phi A \\ {\scriptstyle \phi^0 *' 1}\downarrow & & \uparrow{\scriptstyle \phi l} \\ \phi J *' \phi A & \xrightarrow[\tilde{\phi}]{} & \phi(J * A) \end{array} \tag{2.2}$$

$$\begin{array}{ccc} \phi A *' J' & \xrightarrow{r'} & \phi A \\ {\scriptstyle 1 *' \phi^0}\downarrow & & \uparrow{\scriptstyle \phi r} \\ \phi A *' \phi J & \xrightarrow[\tilde{\phi}]{} & \phi(J * A) \end{array} \tag{2.3}$$

A *monoidal natural transformation* $\alpha : \Phi \to \Psi : \mathscr{A} \to \mathscr{A}'$ is just a natural transformation $\alpha : \phi \to \psi$ satisfying

$$\begin{array}{ccc} \phi A *' \phi B & \xrightarrow{\tilde{\phi}} & \phi(A * B) \\ {\scriptstyle \alpha *' \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \psi A *' \psi B & \xrightarrow[\tilde{\psi}]{} & \psi(A * B) \end{array} \tag{2.4}$$

$$\begin{array}{ccc} J' & \xrightarrow{\phi^0} & \phi J \\ & {\scriptstyle \psi^0}\searrow & \downarrow{\scriptstyle \alpha_J} \\ & & \psi J \end{array} \tag{2.5}$$

With the evident definitions of the various compositions [4, p. 474] we obtain a 2-category Mon of monoidal categories, monoidal functors, and monoidal natural transformations.

When $\mathscr{A}$ and $\mathscr{A}'$ are *symmetric* monoidal categories with symmetries c and c', a *symmetric monoidal functor* $\Phi : \mathscr{A} \to \mathscr{A}'$ is just a monoidal one satisfying the extra coherence condition

$$\begin{array}{ccc} \phi A *' \phi B & \xrightarrow{c'} & \phi B *' \phi A \\ {\scriptstyle \tilde{\phi}}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}} \\ \phi(A * B) & \xrightarrow[\phi c]{} & \phi(B * A) \end{array} \tag{2.6}$$

while a *symmetric monoidal natural transformation* is just a monoidal one. Thus we have the 2-category SMon of symmetric monoidal categories. Whatever we say about Mon below applies equally, and for the same reasons, to SMon.

Lemma 2.1. *Given a monoidal $\Phi:\mathscr{A}\to\mathscr{A}'$, a functor $\psi:\mathscr{A}\to\mathscr{A}'$, and an isomorphism $\alpha:\phi\cong\psi$, there is a unique enrichment of ψ to a monoidal $\Psi:\mathscr{A}\to\mathscr{A}'$ for which α is monoidal.*

Proof. The definition of $\tilde{\psi}$ and of ψ^0 is forced by (2.4) and (2.5), and the conditions (2.1)–(2.3) for Ψ follow easily. □

We call a monoidal (or symmetric monoidal) $\Phi:\mathscr{A}\to\mathscr{A}'$ *strong* if $\tilde{\phi}$ and ϕ^0 are isomorphisms, so that "ϕ preserves the monoidal structure to within coherent isomorphisms". Restricting to such Φ gives sub-2-categories StrMon and StrSMon of Mon and SMon.

There are 2-monads D and E on the 2-category $\mathscr{V}$-Cat such that Mon and SMon are D-Alg and E-Alg in the sense of [8, p. 95]. For the case $\mathscr{V}=$ Set of ordinary categories, this follows from the analysis of multi-variable functorial calculus in [5] and [6], and in particular Section 7 of [6]. That it is also the case for a general $\mathscr{V}$ follows from the easy verification that the analysis in Section 2 of [5] and Sections 2, 3, and 7 of [6] extends at once to $\mathscr{V}$-categories.

That being so, we have available in Mon and SMon the results of [7, Theorem 1.5], namely:

Proposition 2.2. *Let $\Phi=(\phi,\tilde{\phi},\phi^0)$ be a monoidal functor. In order that Φ be a left adjoint in* Mon, *it is necessary and sufficient that ϕ be a left adjoint in* $\mathscr{V}$-Cat *and that Φ be strong. In fact, if $\eta,\varepsilon:\phi\dashv\psi$ is an adjunction in* $\mathscr{V}$-Cat and *Φ is strong, there is a unique enrichment of ψ to a monoidal Ψ (not in general strong) that renders η and ε monoidal, so that $\eta,\varepsilon:\Phi\dashv\Psi$ in* Mon. *Hence the monoidal Φ is an equivalence in* Mon *if and only if Φ is strong and ϕ is an equivalence in* $\mathscr{V}$-Cat. *The same results hold in* SMon. □

For the reader who does not want to appeal to the long articles [5] and [6], it is also easy to give a direct but less elegant proof of Proposition 2.2 along the lines of Section 2 of [7].

3. Separately cocontinuous functors

We need a generalization to functors of several variables of Theorem 4.51 of [9], referred to in the introduction, on $\mathscr{P}\mathscr{A}$ as the free cocompletion of $\mathscr{A}$. Since the number of variables is irrelevant, our describing the two-variable form here will serve to remind the reader of the details of the original one-variable result.

We use $\otimes$ for the tensor product $X \otimes C$ of an X of $\mathscr{V}$ and a C of the cocomplete $\mathscr{C}$, defined by $\mathscr{C}(X \otimes C, D) \cong [X, \mathscr{C}(C, D)]$, as well as for the tensor product in $\mathscr{V}$; and for X, Y in $\mathscr{V}$ and C in $\mathscr{C}$, we write $X \otimes Y \otimes C$ for either of $X \otimes (Y \otimes C)$ and $(X \otimes Y) \otimes C$, which are canonically isomorphic. Similarly for other colimits such as coends; we suppose some definite choice made, without fruitlessly specifying which one of the canonically isomorphic choices.

We consider small categories $\mathscr{A}$ and $\mathscr{B}$, with their Yoneda embeddings $y: \mathscr{A} \to \mathscr{P}\mathscr{A}$ and $y: \mathscr{B} \to \mathscr{P}\mathscr{B}$, and a cocomplete category $\mathscr{C}$. We call a functor $\phi: \mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B} \to \mathscr{C}$ *separately cocontinuous* if all the partial functors $\phi(P, -): \mathscr{P}\mathscr{B} \to \mathscr{C}$ and $\phi(-, Q): \mathscr{P}\mathscr{A} \to \mathscr{C}$ are cocontinuous; such functors ϕ, with the natural transformations between them, form an (ordinary) category $\mathsf{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C})$. Since $\mathscr{A}$ and $\mathscr{B}$ are small, the functor category $[\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]$ exists as a $\mathscr{V}$-category; but we shall be content to consider its underlying ordinary category $[\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]_0$ (which, to use the full names for once for clarity, consists of the $\mathscr{V}$-functors $\mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$ and the $\mathscr{V}$-natural transformations between these).

Composition with $y \otimes y: \mathscr{A} \otimes \mathscr{B} \to \mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}$ gives a functor

$$R: \mathsf{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C}) \to [\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]_0,$$

the letter R suggesting 'restriction to the representables'. We define a functor $L: [\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]_0 \to \mathsf{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C})$ by

$$L(\theta)(P, Q) = \int^{A \in \mathscr{A}, B \in \mathscr{B}} PA \otimes QB \otimes \theta(A, B), \tag{3.1}$$

where $\theta: \mathscr{A} \otimes \mathscr{B} \to \mathscr{C}$, $P: \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$, and $Q: \mathscr{B}^{\mathrm{op}} \to \mathscr{V}$. (That (3.1) defines $L(\theta)$ not only on objects, but as a functor – that is, a $\mathscr{V}$-functor – is clear from Section 3.3 of [9], when we recall that PA, for instance, is a value of the evaluation $\mathscr{V}$-functor $\mathscr{P}\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}} \to \mathscr{V}$.) That $L(\theta)$ is indeed separately cocontinuous is immediate from the fact that colimits commute with colimits. The letter L is to suggest 'repeated left Kan extension'. Indeed, in the one-variable case where $\mathsf{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C})$ is replaced by the category $\mathsf{Coc}(\mathscr{P}\mathscr{A}, \mathscr{C})$ of cocontinuous functors $\mathscr{P}\mathscr{A} \to \mathscr{C}$, the corresponding formula for $L: [\mathscr{A}, \mathscr{C}]_0 \to \mathsf{Coc}(\mathscr{P}\mathscr{A}, \mathscr{C})$, namely

$$L(\theta)(P) = \int^{A \in \mathscr{A}} PA \otimes \theta A, \tag{3.2}$$

exhibits $L(\theta)$ as the left Kan extension $\mathrm{Lan}_y \theta$ of θ along y, in view of [9, (4.31)].

Taking P and Q to be yA' and yB' in (3.1) and using the Yoneda isomorphism

$$\int^{A, B} \mathscr{A}(A, A') \otimes \mathscr{B}(B, B') \otimes \theta(A, B) \cong \theta(A', B')$$

gives a natural isomorphism

$$\eta_\theta: RL\theta \cong \theta. \tag{3.3}$$

On the other hand we have, for any P in $\mathscr{P}\mathscr{A}$ and Q in $\mathscr{P}\mathscr{B}$, the Yoneda isomorphisms

$$P \cong \int^{A} PA \otimes yA, \qquad Q \cong \int^{B} QB \otimes yB,$$

so that if $\phi : \mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B} \to \mathscr{C}$ is separately cocontinuous we have

$$\phi(P,Q) \cong \int^{A,B} PA \otimes QB \otimes \phi(yA, yB),$$

giving a natural isomorphism

$$\varepsilon_{\phi} : \phi \cong LR\phi. \tag{3.4}$$

From (3.1) we get

$$\begin{aligned}\mathscr{C}(L(\theta)(P,Q),C) &\cong \int_{A} [PA, \int_{B} \mathscr{C}(QB \otimes \theta(A,B), C)] \\ &= \mathscr{P}\mathscr{A}(P, \int_{B} \mathscr{C}(QB \otimes \theta(-,B), C)),\end{aligned}$$

showing that $L(\theta)(-,Q)$ has a right adjoint; similarly each $L(\theta)(P,-)$ has a right adjoint. So we have the following result, which is at least implicit in Day's thesis [3]:

Proposition 3.1. *In the circumstances above,* $R : \mathrm{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C}) \to [\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]_0$ *and* $L : [\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]_0 \to \mathrm{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C})$ *are mutually inverse equivalences. Moreover* $\phi : \mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B} \to \mathscr{C}$ *is separately cocontinuous precisely when each* $\phi(P,-)$ *and each* $\phi(-,Q)$ *is a left adjoint.* □

We may note that, as in the original Theorem 4.51 of [9], $\mathrm{SCoc}(\mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{B}, \mathscr{C})$ has a natural $\mathscr{V}$-category structure, and that we in fact have an equivalence of this with the $\mathscr{V}$-category $[\mathscr{A} \otimes \mathscr{B}, \mathscr{C}]$. We do not go to the pains of showing this here, since our main result below is concerned with *monoidal* $\mathscr{V}$-functors $\mathscr{A} \to \mathscr{C}$, and these do *not* form a $\mathscr{V}$-category.

4. The convolution monoidal biclosed structure

What Day showed in [2] is that every monoidal biclosed structure on $\mathscr{P}\mathscr{A}$ arises from an essentially unique *promonoidal* structure on $\mathscr{A}$, of which a monoidal structure is a special case. We recall Day's construction only in this monoidal case, writing $\circ$ for the tensor product on $\mathscr{A}$ and K for the unit object. For the various isomorphisms involved in any monoidal category we shall use the letters a, l, r (and c where appropriate), undecorated, in the sense of Section 2 above.

We define the tensor product $* : \mathscr{P}\mathscr{A} \otimes \mathscr{P}\mathscr{A} \to \mathscr{P}\mathscr{A}$ to be the image under L of the composite

$$\mathscr{A} \otimes \mathscr{A} \xrightarrow{\circ} \mathscr{A} \xrightarrow{y} \mathscr{P}\mathscr{A}, \tag{4.1}$$

so that by (3.1) the explicit definition of $*$ *is*

$$P * Q = \int^{A,B} PA \otimes QB \otimes \mathscr{A}(-, A \circ B). \tag{4.2}$$

The natural isomorphism η of (3.3) here becomes a natural isomorphism

$$\tilde{y} : yA * yB \to y(A \circ B). \tag{4.3}$$

We take the unit object J of $\mathscr{P}\mathscr{A}$ to be

$$J = yK = \mathscr{A}(-, K), \tag{4.4}$$

and we set

$$y^0 = 1 : J \to yK. \tag{4.5}$$

Since $*$ is separately cocontinuous, so are the three-variable functors $(P * Q) * S$ and $P * (Q * S)$. It follows from the three-variable version of Proposition 3.1 that to give a natural isomorphism $a : (P * Q) * S \to P * (Q * S)$ is exactly to give its restriction $a : (yA * yB) * yC \to yA * (yB * yC)$ to the representables, which we define by

$$\begin{array}{ccc}
(yA * yB) * yC & \xrightarrow{\quad a \quad} & yA * (yB * yC) \\
{\scriptstyle \tilde{y} * 1} \downarrow & & \downarrow {\scriptstyle 1 * \tilde{y}} \\
y(A \circ B) * yC & & yA * y(B \circ C) \\
{\scriptstyle \tilde{y}} \downarrow & & \downarrow {\scriptstyle \tilde{y}} \\
y((A \circ B) \circ C) & \xrightarrow[\quad ya \quad]{} & y(A \circ (B \circ C))
\end{array} \tag{4.6}$$

Similarly we define a natural isomorphism $r : P * J \to P$ by giving its value on representables as

$$\begin{array}{ccc}
yA * J & \xrightarrow{\quad r \quad} & yA \\
{\scriptstyle 1} \downarrow & & \uparrow {\scriptstyle yr} \\
yA * yK & \xrightarrow[\quad \tilde{y} \quad]{} & y(A \circ K)
\end{array} \tag{4.7}$$

and define $l : J * P \to P$ correspondingly. If the monoidal structure on $\mathscr{A}$ is symmetric, we define $c : P * Q \to Q * P$ by giving it on representables as

$$\begin{array}{ccc} yA * yB & \xrightarrow{\quad c \quad} & yB * yA \\ {\scriptstyle \tilde{y}}\downarrow & & \downarrow{\scriptstyle \tilde{y}} \\ y(A \circ B) & \xrightarrow[\quad yc \quad]{} & y(B \circ A) \end{array} \qquad (4.8)$$

It remains to verify the coherence conditions [9, (1.1) and (1.2); with (1.14)–(1.16) in the symmetric case] for the a, l, r (and c where appropriate) of $\mathscr{P}\mathscr{A}$. The pentagon condition for a, for example, requires the equality of two natural transformations $((P * Q) * S) * T \to P * (Q * (S * T))$. Since the domain and the codomain here are cocontinuous in each variable, it follows from the four-variable version of Proposition 3.1 that it suffices to prove the equality when P, Q, S, T are representables; and this follows at once from (4.6). Similar arguments, using (4.6)–(4.8), establish the remaining coherence conditions.

Observing that (4.6)–(4.8) are instances of (2.1)–(2.3) and (2.6), we have:

Proposition 4.1. *For a small monoidal [symmetric monoidal]* $\mathscr{A} = (\mathscr{A}, \circ, K)$, *the* $*$ *and* J *of* (4.2) *and* (4.4), *along with the isomorphisms* a, l, r *[and* c*] defined as above, constitute a monoidal biclosed [symmetric monoidal closed] structure on* $\mathscr{P}\mathscr{A}$; *and* $Y = (y, \tilde{y}, 1) : \mathscr{A} \to \mathscr{P}\mathscr{A}$ *is a strong monoidal [symmetric monoidal] functor.* □

As we said in the introduction, this structure on $\mathscr{P}\mathscr{A}$ is called the *convolution* of the monoidal structure on $\mathscr{A}$ and the symmetric monoidal closed structure on $\mathscr{V}$.

5. The universal property of the convolution monoidal structure

Recall from the introduction that a monoidal $(\mathscr{C}, *', J')$ is said to be *monoidally cocomplete* if $\mathscr{C}$ is cocomplete and $*' : \mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$ is separately cocontinuous – the latter being certainly the case if $(\mathscr{C}, *', J')$ is biclosed; and observe that the convolution $\mathscr{P}\mathscr{A}$ of Proposition 4.1 is monoidally cocomplete. For such a $\mathscr{C}$ and a small monoidal $(\mathscr{A}, \circ, K)$ we write $\mathsf{MonCoc}(\mathscr{P}\mathscr{A}, \mathscr{C})$ for the (ordinary) category of cocontinuous monoidal functors $\Phi : \mathscr{P}\mathscr{A} \to \mathscr{C}$ (meaning those for which $\phi : \mathscr{P}\mathscr{A} \to \mathscr{C}$ is cocontinuous) and of all monoidal natural transformations between these; with $\mathsf{StrMonCoc}(\mathscr{P}\mathscr{A}, \mathscr{C})$ for the full subcategory given by those Φ that are strong monoidal. Our main result is the following:

Theorem 5.1. *For a small monoidal* $\mathscr{A}$ *and a monoidally-cocomplete* $\mathscr{C}$, *the (ordinary) functor* $R : \mathsf{MonCoc}(\mathscr{P}\mathscr{A}, \mathscr{C}) \to \mathsf{Mon}(\mathscr{A}, \mathscr{C})$ *given by composition with the* $Y : \mathscr{A} \to \mathscr{P}\mathscr{A}$ *of Proposition* 4.1 *is an equivalence of categories, which restricts to an equivalence* $\mathsf{StrMonCoc}(\mathscr{P}\mathscr{A}, \mathscr{C}) \simeq \mathsf{StrMon}(\mathscr{A}, \mathscr{C})$.

Moreover, the objects of StrMonCoc($\mathcal{PA}$, $\mathcal{C}$) *are exactly those monoidal functors* $\mathcal{PA}\to\mathcal{C}$ *which are left adjoints in* Mon. *The corresponding results are true in the symmetric monoidal case.*

Proof. We give the proof in the general monoidal case, the modifications needed in the symmetric monoidal case being trivial.

We first show that R is fully faithful. Let the monoidal functors $\Phi, \Psi: \mathcal{PA}\to\mathcal{C}$ be cocontinuous, and let $\beta: \Phi Y\to\Psi Y$ be a monoidal natural transformation. We are to show that there is a unique monoidal natural $\alpha: \Phi\to\Psi$ with $\alpha Y=\beta$. By the one-variable case of Proposition 3.1, there is a unique $\alpha:\phi\to\psi$ with $\alpha y=\beta:\phi y\to\psi y$; and it remains to show that α is monoidal – that is, that α satisfies (2.4) and (2.5).

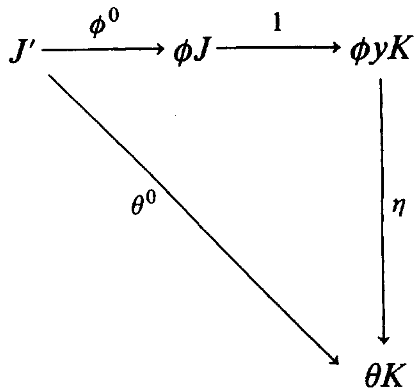
Because of the separate cocontinuity of $*$ *and* $*'$ and the cocontinuity of ϕ and ψ, the domain and the codomain of (2.4) are separately cocontinuous; so that by Proposition 3.1 it suffices to establish the commutativity of the restriction of (2.4) to the representables, which is the left square of

$$\begin{array}{ccccc} \phi yA *' \phi yB & \xrightarrow{\tilde{\phi}} & \phi(yA * yB) & \xrightarrow{\phi\tilde{y}} & \phi y(A\circ B) \\ \downarrow{\scriptstyle \alpha yA *' \alpha yB} & & \downarrow{\scriptstyle \alpha(yA * yB)} & & \downarrow{\scriptstyle \alpha y(A\circ B)} \\ \psi yA *' \psi yB & \xrightarrow[\tilde{\psi}]{} & \psi(yA * yB) & \xrightarrow[\psi\tilde{y}]{} & \psi y(A\circ B) \end{array} \tag{5.1}$$

whose right square commutes by naturality. Since $\tilde{y}$ is invertible, it suffices to prove the commutativity of the exterior of (5.1); but this is just (2.4) for the monoidal $\beta=\alpha y$. Again, since $y^0=1: J\to yK$, the diagram (2.5) for α is exactly (2.5) for $\beta=\alpha y$.

To prove R an equivalence it remains to show that any monoidal $\Theta: \mathcal{A}\to\mathcal{C}$ is isomorphic in Mon to ΦY for some cocontinuous monoidal $\Phi: \mathcal{PA}\to\mathcal{C}$. Taking ϕ to be the $L(\theta)$ of (3.1), we have by (3.3) the isomorphism $\eta:\phi y\cong\theta$. There is in fact exactly one enrichment of ϕ to a monoidal Φ for which η is a monoidal natural isomorphism $\Phi Y\cong\Theta$. For the conditions (2.4) and (2.5) for η are

$$\begin{array}{ccccc} \phi yA *' \phi yB & \xrightarrow{\tilde{\phi}} & \phi(yA * yB) & \xrightarrow{\phi\tilde{y}} & \phi y(A\circ B) \\ \downarrow{\scriptstyle \eta *' \eta} & & & & \downarrow{\scriptstyle \eta} \\ \theta A *' \theta B & & \xrightarrow[\theta]{} & & \theta(A\circ B) \end{array} \tag{5.2}$$



(5.3)

and since η and y are invertible, (5.3) fixes ϕ^0, while (5.2) fixes the restriction of $\tilde{\phi}$ to the representables – which suffices by Proposition 3.1 to give a unique $\tilde{\phi}: \phi P *' \phi Q \to \phi(P * Q)$, because the domain and codomain of $\tilde{\phi}$ are separately cocontinuous.

We have to verify (2.1)–(2.3) for Φ; we indicate the argument for (2.1). Since the domain and codomain of (2.1) are separately cocontinuous, it suffices to verify the commutativity of the restriction of (2.1) to the representables. If we paste this restriction to ϕ of (4.6), it suffices since $\tilde{y}$ is invertible to prove the commutativity of the exterior of the resulting diagram. This diagram, however, transforms at once using the naturality of $\tilde{\phi}$ into the diagram (2.1) for ΦY; and ΦY is a monoidal functor by Lemma 2.1.

This concludes the proof that R is an equivalence. As for its restriction to strong monoidal functors, ΦY is strong monoidal when Φ is, since Y is strong monoidal; so we have only to verify that the Φ we constructed from Θ using (5.2) and (5.3) is strong when Θ is. But then ϕ^0 is invertible by (5.3), while the restriction of $\tilde{\phi}$ to the representables is invertible by (5.2), so that $\tilde{\phi}$ itself is invertible by Proposition 3.1.

The assertion in the theorem about the $\Phi: \mathscr{P}\mathscr{A} \to \mathscr{C}$ that are left adjoints in Mon follows at once from Proposition 2.2 and the one-variable version of Proposition 3.1. □

6. Graded abelian groups

We now take $\mathscr{V} = \mathrm{Ab}$, and remind the reader that in this case, in accordance with our general convention, the 'categories' and 'functors' of our results above are additive categories and additive functors, except where it has been made clear that ordinary categories and functors are intended. Since, however, authors rarely distinguish an additive category $\mathscr{C}$ from its underlying ordinary category $\mathscr{C}_0$, we shall no longer rely solely on this convention in what follows, but spell out the distinctions where necessary.

We write $\mathscr{A}$ for the free additive category on the *discrete* ordinary category Z of integers – so that $\mathscr{A}$ has the integers as objects and has $\mathscr{A}(n, n) = \mathrm{Z}$, while $\mathscr{A}(m, n) = 0$

(the zero group) for $m \neq n$. To give an additive functor θ from $\mathscr{A}$ into an additive category $\mathscr{C}$ is just to give an ordinary functor – also called θ – from Z into $\mathscr{C}_0$, and thus to give objects $\theta(n) = \theta_n$ *of* $\mathscr{C}$ indexed by $n \in \mathrm{Z}$; the value of θ on the map $k \in \mathscr{A}(n, n) = \mathrm{Z}$ is of course $k \cdot 1$, where $1 : \theta_n \to \theta_n$ is the identity. A natural transformation $\alpha : \theta \to \theta' : \mathscr{A} \to \mathscr{C}$ is just a family $\alpha_n : \theta_n \to \theta'_n$ of maps; so that $[\mathscr{A}, \mathscr{C}]$ is the additive category of ***graded*** ***$\mathscr{C}$-objects***, where 'graded' means 'Z-graded'.

In particular, since $\mathscr{A}^{\mathrm{op}} = \mathscr{A}$, the category $\mathscr{P}\mathscr{A} = [\mathscr{A}^{\mathrm{op}}, \mathrm{Ab}]$ is the additive category $\mathscr{G}$ of ***graded abelian groups***. A typical object P of $\mathscr{G}$ has *components* $P_n \in \mathrm{Ab}$, while a typical map $f : P \to Q$ has *components* $f_n : P_n \to Q_n$. The Yoneda embedding $y : \mathscr{A} \to \mathscr{P}\mathscr{A} = \mathscr{G}$ sends n to y_n where

$$(y_n)_n = \mathrm{Z}, \qquad (y_n)_m = 0 \quad \text{for } m \neq n. \tag{6.1}$$

In the one-variable version of Proposition 3.1, the equivalence $R : \mathrm{Coc}(\mathscr{G}, \mathscr{C}) \to [\mathscr{A}, \mathscr{C}]_0$ sends ϕ to ϕy, where $(\phi y)_n = \phi(y_n)$, while the formula (3.2) for the inverse L of R becomes

$$L(\phi)(P) = \sum_{n \in \mathrm{Z}} P_n \otimes \theta_n; \tag{6.2}$$

similarly, in the two-variable version with $\mathscr{B} = \mathscr{A}$, (3.1) reduces to

$$L(\theta)(P, Q) = \sum_{m, n \in \mathrm{Z}} P_m \otimes Q_n \otimes \theta_{mn}. \tag{6.3}$$

The discrete ordinary category Z has a monoidal structure $(\mathrm{Z}, \circ, K)$ given by $m \circ n = m + n$ and $K = 0$, the isomorphisms a, l, r being the identities. This extends to an evident monoidal structure $(\mathscr{A}, \circ, K)$ on $\mathscr{A}$, so that $\mathscr{G}$ has a monoidal biclosed structure $(\mathscr{G}, *, J)$. By (4.2) and (6.3), the explicit description of $*$ is

$$(P * Q)_n = \sum_{p + q = n} P_p \otimes Q_q; \tag{6.4}$$

while by (4.4) we have $J = y_0$, so that (6.1) gives

$$J_0 = \mathrm{Z}, \qquad J_n = 0 \quad \text{for } n \neq 0. \tag{6.5}$$

The $\tilde{y} : y_m * y_n \to y_{m+n}$ of the strong monoidal $Y = (y, \tilde{y}, 1)$ of Proposition 4.1 has, as its only non-trivial component, the canonical isomorphism $\mathrm{Z} \otimes \mathrm{Z} \cong \mathrm{Z}$ sending $1 \otimes 1$ to 1.

The monoidal biclosed structure on the additive $\mathscr{G}$ is, of course, also a monoidal biclosed structure on the underlying ordinary category $\mathscr{G}_0$. Our object is to study the monoidal auto-equivalences Φ of $\mathscr{G}_0$; but since every equivalence, and even every left adjoint, $\phi : \mathscr{G}_0 \to \mathscr{G}_0$ is automatically an additive functor, it comes to the same thing to study the monoidal auto-equivalences of $\mathscr{G}$ – which we do using Theorem 5.1.

There is an involutory isomorphism $\Pi = (\pi, \tilde{\pi}, \pi^0) : \mathscr{G} \to \mathscr{G}$ in Mon, where the functor π is given by $(\pi P)_n = P_{-n}$ and $(\pi f)_n = f_{-n}$, while $\tilde{\pi} : \pi P * \pi Q \to \pi(P * Q)$ is the obvious isomorphism and $\pi^0 = 1 : J \to J$.

Theorem 6.1. *Every equivalence $\Phi: \mathcal{G} \to \mathcal{G}$ in* Mon *is isomorphic to the identity or to Π.*

Proof. Since an equivalence Φ is certainly a left adjoint, it is by Theorem 5.1 determined to within isomorphism by the strong monoidal functor $\Theta = \Phi Y: \mathcal{A} \to \mathcal{G}$.

Write T and S for θ_1 and θ_{-1}. We have $\tilde{\theta}: \theta_1 * \theta_{-1} \cong \theta_{1+(-1)} = \theta_0$ and $\theta^0: J \cong \theta_0$, so that $T * S \cong J$; giving

$$\sum_{n \in \mathrm{Z}} T_n \otimes S_{-n} \cong \mathrm{Z}, \tag{6.6}$$

$$\sum_{n \in \mathrm{Z}} T_n \otimes S_{m-n} = 0 \quad \text{for } m \neq 0. \tag{6.7}$$

Since Z is indecomposable in Ab, it follows from (6.6) that

$$T_k \otimes S_{-k} \cong \mathrm{Z} \quad \text{for some } k \in \mathrm{Z}, \tag{6.8}$$

$$T_n \otimes S_{-n} = 0 \quad \text{for } n \neq k. \tag{6.9}$$

It is well known that abelian groups A and B with $A \otimes B \cong \mathrm{Z}$ must each be isomorphic to Z; a simple proof is as follows. There exist $a \in A$ and $b \in B$ with $a \otimes b \neq 0$. The homomorphism $- \otimes b: A \to A \otimes B \cong \mathrm{Z}$ being non-zero, its image is a subgroup of Z isomorphic to Z, so that A contains the projective Z as a direct summand; and similarly for B. If $A \cong \mathrm{Z} \oplus C$ and $B \cong \mathrm{Z} \oplus D$, we have $\mathrm{Z} \cong A \otimes B \cong Z \oplus C \oplus D \oplus (C \otimes D)$, giving $C = D = 0$ since Z is indecomposable.

Accordingly (6.8) gives $T_k \cong S_{-k} \cong \mathrm{Z}$, whereupon (6.7) gives $T_n = S_{-n} = 0$ for $n \neq k$. So $T \cong y_k$ and $S \cong y_{-k}$. The isomorphisms $\tilde{\theta}: \theta_m * \theta_n \cong \theta_{m+n}$ and $\theta^0: J \cong \theta_0$ now give

$$\theta_n \cong y_{nk}. \tag{6.10}$$

By the one-variable version of Proposition 3.1, and the formula (6.2), we have $\phi P = \sum_n P_n \otimes y_{nk}$, giving

$$(\phi P)_n \cong P_{nk}. \tag{6.11}$$

If ϕ is to be an equivalence, it must be essentially surjective on objects; so that, in view of (6.11), we must have $k = \pm 1$. If $k = -1$, we can replace Φ by $\Phi\Pi$; so we may suppose that $k = 1$, whereupon (6.10) becomes

$$\theta_n \cong y_n. \tag{6.12}$$

To complete the proof of the theorem we must show that, given (6.12), we have $\Theta \cong Y$ in Mon, so that $\Phi \cong 1$ by Theorem 5.1. By Lemma 2.1, we may as well take the isomorphism (6.12) to be an equality.

This being done, the isomorphism $\tilde{\theta}: \theta_n * \theta_m \to \theta_{n+m}$ is necessarily the multiple of $\tilde{y}: y_n * y_m \to y_{n+m}$ by some δ_{nm} that is ± 1, while $\theta^0: J \to \theta_0$ is $\varepsilon: y_0 \to y_0$ where $\varepsilon = \pm 1$. The conditions (2.1)–(2.3) for Θ to be a monoidal functor now reduce to

$$\delta_{nm} \delta_{n+m,k} = \delta_{mk} \delta_{n,m+k}, \tag{6.13}$$

$$\delta_{0n}\varepsilon = 1 = \delta_{n0}\varepsilon. \tag{6.14}$$

We are to exhibit a monoidal natural isomorphism $\alpha: Y \to \Theta$; if $\alpha_n : y_n \to \theta_n$ is ϱ_n times the identity map of y_n, where $\varrho_n = \pm 1$, the conditions (2.4) and (2.5) for α to be monoidal become (since $y^0 = 1$ by (4.5))

$$\delta_{nm}\varrho_n\varrho_m = \varrho_{m+n}, \tag{6.15}$$

$$\varrho_0 = \varepsilon. \tag{6.16}$$

We define ϱ_0 by (6.16) and, writing γ_n for δ_{n1}, we set

$$\varrho_1 = 1, \tag{6.17}$$

$$\varrho_n = \gamma_1\gamma_2 \cdots \gamma_{n-1} \quad \text{for } n \geq 2, \tag{6.18}$$

$$\varrho_{-n} = \varepsilon(\gamma_{-1}\gamma_{-2}\cdots\gamma_{-n})^{-1} \quad \text{for } n \geq 1. \tag{6.19}$$

It remains to verify (6.15).

We first observe that, for all n, we have

$$\varrho_n\gamma_n = \varrho_{n+1}; \tag{6.20}$$

this follows from (6.17)–(6.19), since $\gamma_0\varepsilon = 1$ by (6.14). Next, we put $k = 1$ in (6.13), getting

$$\delta_{n,m+1}\gamma_m = \delta_{nm}\gamma_{m+n}. \tag{6.21}$$

It follows at once from (6.20) and (6.21) that $\delta_{nm}\varrho_n\varrho_m/\varrho_{m+n}$ is unaltered when m is replaced by $m+1$. So it suffices to verify (6.15) for $m = 0$; but then it is true by (6.14). □

7. The two symmetries

Retaining the notation of Section 6, we now consider the two symmetries on $\mathscr{G}$ referred to in the introduction above. The discrete monoidal ordinary category Z has a symmetry c given by the identity map $m+n \to m+n$, which gives a symmetry c on the additive monoidal $\mathscr{A}$, and hence a symmetry c on $\mathscr{G}$, where $c: P * Q \to Q * P$ maps the summand $P_p \otimes Q_q$ in (6.4) to the summand $Q_q \otimes P_p$ of $(Q * P)_n$ by sending $x \otimes y$ to $y \otimes x$. We write $\mathscr{A}$ and $\mathscr{G}$ for these symmetric monoidal categories.

The monoidal $\mathscr{A}$ has another symmetry c', not induced by a symmetry on Z, given by $c'_{nm} = (-1)^{nm} \in \mathscr{A}(n+m, n+m)$; which gives a symmetry c' on $\mathscr{G}$, now sending $x \otimes y \in P_p \otimes Q_q$ to $(-1)^{pq} y \otimes x$; we write $\mathscr{A}'$ and $\mathscr{G}'$ for the symmetric monoidal categories given by the monoidal $\mathscr{A}$ and $\mathscr{G}$ with this second symmetry. It is shown in [4, Chapter IV, Section 6] that $\mathscr{G}$ – and indeed the underlying monoidal $\mathscr{G}_0$ – has only these two symmetries.

Although these two symmetries are different, it is *a priori* possible that they be *isomorphic*, in the sense that there is an isomorphism $\Phi: \mathscr{G} \to \mathscr{G}'$ in SMon; or in the even stronger sense that there is such an isomorphism with ϕ the identity. Failing this, it is *a priori* possible that they be at least *equivalent*, in the sense that there is an equivalence $\Phi : \mathscr{G} \to \mathscr{G}'$ in SMon. However we in fact have:

Theorem 7.1. *There is no equivalence* $\Phi : \mathscr{G} \to \mathscr{G}'$ *in* SMon.

Proof. Any such equivalence Φ is an equivalence in Mon and therefore, by Theorem 6.1, isomorphic in Mon to 1 or Π. It is clear from (2.4) and (2.6) that, if Φ and Ψ are isomorphic monoidal functors between symmetric monoidal categories, Ψ is a symmetric monoidal functor if Φ is symmetric. Since neither 1 nor Π is symmetric monoidal as a functor $\mathscr{G} \to \mathscr{G}'$, the result follows. □

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