Peking Mathematical Journal
https://doi.org/10.1007/s42543-025-00098-y

ORIGINAL ARTICLE

Classical Stable Homotopy Groups of Spheres via $\mathbb{F}_2$-Synthetic Methods

Robert Burklund[1] · **Daniel C. Isaksen**[2] · **Zhouli Xu**[3]

Received: 6 August 2024 / Revised: 31 January 2025 / Accepted: 16 February 2025

Abstract
We study the $\mathbb{F}_2$-synthetic Adams spectral sequence. We obtain new computational information about $\mathbb{C}$-motivic and classical stable homotopy groups.

Keywords Stable homotopy group · Synthetic homotopy theory · Adams spectral sequence

Mathematics Subject Classification Primary 55Q45; Secondary 55T15

1 Introduction

The topic of this manuscript is the computation of the stable homotopy groups

$$\pi_n = \operatorname{colim}_k \pi_{n+k}(S^k),$$

where $\pi_{n+k}(S^k)$ is the group of based homotopy classes of maps from a sphere S^{n+k} to another sphere S^k. These groups are a central object of study in topology. For $n > 0$, each stable homotopy group is a finite group [19]. Therefore, we can compute π_n by studying its p-primary components for each prime p. In this manuscript, we consider only the 2-primary components.

Building on earlier work of many authors, the 2-primary stable homotopy groups are mostly computed for $n \leq 90$ in [13, 14]. However, there are a few gaps in those

✉ Robert Burklund
rb@math.ku.dk

Daniel C. Isaksen
isaksen@wayne.edu

Zhouli Xu
xuzhouli@ucla.edu

1 Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark

2 Department of Mathematics, Wayne State University, Detroit, MI 48202, USA

3 Department of Mathematics, UCLA, Los Angeles, CA 90095, USA

Published online: 11 April 2025

computations; for example, π_{82} and π_{83} are computed only up to a factor of $\mathbb{Z}/2$. Our work completes the computation of both π_{82} and π_{83}.

Theorem 1.1 (1) *The* 2*-primary component of* π_{82} *is* $(\mathbb{Z}/2)^6 \oplus \mathbb{Z}/8$.
(2) *The* 2*-primary component of* π_{83} *is* $(\mathbb{Z}/2)^3 \oplus (\mathbb{Z}/8)^2$.

The proof of Theorem 1.1 is to show that a particular element $h_6 g + h_2 e_2$ in the Adams spectral sequence is a permanent cycle. We construct a certain six-cell complex that allows us to identify a homotopy class that must be detected by $h_6 g + h_2 e_2$. The structure of the argument is similar to the proof that the Kervaire class h_5^2 survives [3], but the detailed supporting computations are different.

From [14, Lemma 5.24], we already know that $h_6 g + h_2 e_2$ survives to the E_9-page of the Adams spectral sequence, but there is a possible non-zero value for $d_9(h_6 g + h_2 e_2)$.

Remark 1.2 Mark Behrens suggested an argument using height 2 chromatic homotopy theory that would rule out this differential, which would provide a completely different argument that $h_6 g + h_2 e_2$ is a permanent cycle.

A distinguishing feature of our argument is the use of the $\mathbb{F}_2$-synthetic Adams spectral sequence. Motivic stable homotopy theory is useful for classical computations because it is an enhancement of classical stable homotopy theory. This can be made precise using the language of deformations [8, 9, 18]. In this manuscript, we use $\mathbb{F}_2$-synthetic stable homotopy theory, which is a different enhancement of classical stable homotopy theory and provides access to different pieces of information. Here we are using $\mathbb{F}_2$ as shorthand for the Eilenberg–MacLane spectrum $H\mathbb{F}_2$.

In addition to the study of $h_6 g + h_2 e_2$, this manuscript also includes the proofs of a number of less prominent computations about stable homotopy groups. These results fill in several gaps left by the motivic techniques of [14]. Individually, these facts have little direct connection, but their proofs all use the same basic idea of exploiting the additional structure of $\mathbb{F}_2$-synthetic stable homotopy theory. Tables 1 and 2 summarize the precise results that we prove.

Theorem 1.3 (1) *Table 1 lists some differentials in the* $\mathbb{F}_2$*-synthetic Adams spectral sequence.*
(2) *Table 2 lists some hidden extensions in the* $\mathbb{F}_2$*-synthetic Adams spectral sequence.*

Not all of the results in Tables 1 and 2 are new. In a few cases, we have provided simpler proofs of facts that were previously known to be true by more intricate arguments. Very recently announced machine computations [15] independently verify some of our results. However, current machine results do not establish that $h_6 g + h_2 e_2$ is a permanent cycle, so our argument regarding that particular element is the only known proof.

Most of the notation in Tables 1 and 2 is the same as in [14]. The exceptions are λ, which is the $\mathbb{F}_2$-synthetic parameter analogous to the motivic parameter τ; and $\widetilde{2}$, which is our notation for an $\mathbb{F}_2$-synthetic homotopy class such that $\lambda\widetilde{2} = 2$.

Regarding degrees in these tables, s and f represent the familiar topological stem and Adams filtration, respectively, while d is the "synthetic degree". In short, if x is a classical element in Adams filtration f, then $\lambda^r x$ is a synthetic element whose

Table 1 Some Adams differentials

(s, f, d)	Element	r	d_r	Proof
(56, 9, 9)	Ph_5e_0	5	$\lambda^4 il$	Proposition 3.1
(61, 6, 6)	A'	5	$\lambda^4 Mh_1d_0$	Proposition 3.8
(70, 4, 4)	$p_1 + h_0^2h_3h_6$	5	$\lambda^4 h_2^2C' + \lambda^4 h_3(\Delta e_1 + C_0)$	[14, Lemma 5.61], [15]
(71, 5, 5)	h_1p_1	5	$\lambda^4 h_1h_3(\Delta e_1 + C_0)$	[15]
(83, 5, 5)	$h_6g + h_2e_2$	9	0	Theorem 2.17
(93, 13, 13)	$P^2h_6d_0$	6	0	Proposition 3.29
(93, 13, 13)	tQ_2	6	$\lambda^5 MP\Delta h_1d_0$	Proposition 3.31

Table 2 Some hidden extensions

(s, f, d)	Source	Type	Target	Proof
(71, 5, 5)	h_1p_1	ν	0	Proposition 3.10
(74, 6, 6)	h_3n_1	$\widetilde{2}$	$\lambda x_{74,8}$	Proposition 3.11
(77, 7, 7)	m_1	ν	0	Proposition 3.13
(77, 7, 5)	$\lambda^2 m_1$	η	0	Proposition 3.14
(77, 12, 12)	$M\Delta h_1h_3$	ν	$\lambda^2 Mh_1e_0^2$	Proposition 3.6
(79, 7, 7)	$h_2x_{76,6}$	η	0	Lemma 3.26
(81, 12, 12)	$\Delta^2 p$	ν	0	Proposition 3.21
(84, 10, 10)	$Px_{76,6}$	$\widetilde{2}$	0	Proposition 3.28

synthetic degree is $f - r$. See Sects. 1.1 and 1.4 below for a more careful discussion of degrees and notation.

To carry out the computations in Tables 1 and 2, we need a number of technical supporting details, including some Toda bracket computations as well as some careful choices of stable homotopy elements.

Theorem 1.4 (1) *There exist elements in the $\mathbb{F}_2$-synthetic stable homotopy groups with the properties shown in Table 3.*

(2) *Table 4 lists some Toda brackets in the $\mathbb{F}_2$-synthetic stable homotopy groups.*

1.1 The Synthetic Adams Spectral Sequence

For simplicity, we use the more concise term "synthetic" instead of "$\mathbb{F}_2$-synthetic". The only two synthetic homotopy theories under consideration in this manuscript are the $\mathbb{F}_2$-synthetic and BP-synthetic ones. To avoid confusion, we use the term "motivic" to describe the latter homotopy theory.

We have made no particular effort to write a self-contained document. Rather, this manuscript is a companion to [14], and it is just one part of a larger program to compute stable homotopy groups in a range. We refer to [6, Section 9 and Appendix A] and [18] for foundational material on the construction and properties of synthetic spectra

Table 3 Some elements in the $\mathbb{F}_2$-synthetic stable homotopy groups

(s, d)	Element	Properties	Defined in
$(77, 7)$	μ	Detected by m_1	Lemma 3.15
		$\lambda^2\eta\mu = 0$	
		$\nu\mu = 0$	
$(79, 8)$	β	$\lambda\nu\beta = 0$	Lemma 3.19
		$\lambda^3\beta \in \langle\widetilde{2}, \lambda^2\eta, \lambda\mu\rangle$	
$(82, 8)$	α	$2\alpha = 0$	Lemma 3.24
		$\lambda^3\alpha + \lambda\overline{\kappa}\theta_5 \in \langle\lambda^2\eta, \lambda\mu, \nu\rangle$	

Table 4 Some Toda brackets

(s, d)	Bracket	Detected by	Indeterminacy	Proof
$(46, 7)$	$\langle\eta, \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}\rangle$	Mh_1	$\lambda^4 d_0 l$	Lemma 3.3
$(77, 12)$	$\langle\widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle$	$M\Delta h_1 h_3$	0	Lemma 3.5
$(79, 5)$	$\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$	0 or $\lambda^9 M e_0^2$	$\lambda^3 h_0 h_2 x_{76,6}$	Lemma 3.16
$(82, 5)$	$\langle\nu, \lambda\mu, \lambda^2\eta\rangle$	$\lambda h_5^2 g$	?	Lemma 3.22
$(84, 8)$	$\langle\nu, \eta, \{h_2 x_{76,6}\}\rangle$	$\lambda^2 P x_{76,6}$	?	Lemma 3.26

and synthetic Adams spectral sequences. Also, [11, Chapter 6] foreshadows many of these ideas in the motivic setting.

We provide a short overview of the structure of the synthetic Adams spectral sequence from a computational perspective. The synthetic Adams E_2-page is tri-graded by stem, Adams filtration, and synthetic degree. The first two gradings are the familiar ones from the classical Adams spectral sequence.

The synthetic homology of a point is equal to $\mathbb{F}_2[\lambda]$, where the synthetic degree of λ is -1. We use the letter λ for the synthetic parameter to distinguish it from the motivic parameter τ. In the long run, we foresee computations that possess two parameters simultaneously, and this notation is convenient for that purpose.

The synthetic Adams E_2-page is a free $\mathbb{F}_2[\lambda]$-module whose generators are in one-to-one correspondence with an $\mathbb{F}_2$-basis for the classical Adams E_2-page. For each basis element x of the classical Adams spectral sequence in stem s and filtration f, there is an $\mathbb{F}_2[\lambda]$-module generator of the synthetic Adams spectral sequence in stem s and filtration f, and whose synthetic degree is also f. We use the same letter x to represent this generator of the synthetic Adams E_2-page.

Different authors have chosen different conventions regarding the synthetic degree. Our choice is based on practical convenience, but unfortunately it is incompatible with the traditional use of motivic weight. More precisely, the Thom reduction map $BP \to H\mathbb{F}_2$ induces a functor from BP-synthetic to $\mathbb{F}_2$-synthetic homotopy theory. By careful inspection of definitions, this functor takes τ to λ^2. In terms of stem, motivic weight, and synthetic degree, the functor induces a homomorphism from the motivic homotopy group $\pi_{s,w}$ to the synthetic homotopy group $\pi_{s,2w-s}$. Beware that the formal suspension has synthetic degree -1. For example, the cofiber of an element γ of the synthetic homotopy group $\pi_{s,d}$ has two cells in dimensions $(0, 0)$ and $(s+1, d-1)$.

The synthetic Adams differentials are in precise correspondence with classical Adams differentials. For each classical differential $d_r(x) = y$, we have a synthetic differential $d_r(x) = \lambda^{r-1}y$. Consequently, the class y itself is not hit by a differential, but y is annihilated by λ^{r-1} in the E_∞-page. Moreover, y detects a homotopy class that is annihilated by λ^{r-1}.

The synthetic Adams spectral sequence can be described as the classical Adams spectral sequence "with history". The Adams differentials can be reconstructed from the E_∞-page, since each element annihilated by λ^{r-1} corresponds to a non-zero Adams d_r differential.

If we invert λ in the synthetic Adams spectral sequence, then we recover the classical Adams spectral sequence tensored with $\mathbb{F}_2[\lambda^{\pm 1}]$. In more naive terms, the λ-free part of the synthetic Adams E_∞-page is identical to the classical E_∞-page.

On the other hand, the λ-power torsion elements in the Adams E_∞-page represent exotic phenomena. These torsion classes can be the targets of hidden extensions that are not seen classically. In general, the additional structure encoded in the λ-power torsion can be exploited to carry out computations. These phenomena are well illustrated in [2, 5, 7, 16].

1.2 π_{70} and π_{71}

The value of the Adams differential $d_5(p_1 + h_0^2 h_3 h_6)$ given in Table 1 is not proved in this manuscript. Very recently announced machine computations [15] led to the discovery of a mistake regarding this differential. Note that the value of $d_5(p_1 + h_0^2 h_3 h_6)$ claimed in [14, Table 8] is incorrect, although [14, Lemma 5.61] is correct.

We now know that there is a non-zero differential $d_5(h_1 p_1) = \lambda^4 h_1 h_3(\Delta e_1 + C_0)$ that was previously believed to be zero. Therefore, the values of π_{70} and π_{71} in [14, Table 1] are incorrect. The correct values for the 2-primary v_1-torsion in π_{70} and π_{71} are

$$(\mathbb{Z}/2)^6 \oplus \mathbb{Z}/4, \quad (\mathbb{Z}/2)^5 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/8$$

respectively.

Also, [14, Table 15] claims that there is a non-zero hidden 2-extension from $h_1 h_3 H_1$ to $h_1 h_3(\Delta e_1 + C_0)$ in the classical 70-stem. This is incorrect; $h_1 h_3(\Delta e_1 + C_0)$ is now known to be zero in the classical Adams E_∞-page.

1.3 Organization

The main goal of Sect. 2 is to establish Theorem 2.17, which shows that $h_6 g + h_2 e_2$ is a permanent cycle. This argument depends on several technical computations in stable homotopy groups, whose proofs are assembled in Sect. 3.

Section 3 contains a number of computations. Many of these computations are independent facts whose proofs are not directly connected. Some of the computations are used in the proofs of later computations. The results are arranged mostly in order of increasing stem. However, there are a few exceptions to this general principle so that each individual computation depends only on previous computations.

1.4 Notation

By default, we work in the synthetic Adams spectral sequence. Occasionally, we consider the classical or $\mathbb{C}$-motivic Adams spectral sequence; we will be explicit when that occurs.

We use the same notation as [14] for elements of the Adams spectral sequence and for elements in stable homotopy groups. For an element x in the Adams E_∞-page, let $\{x\}$ represent the set of all homotopy elements that are detected by x. This set possesses more than one element when there are other elements in filtrations higher than the filtration of x.

We write $\widetilde{2}$ for an element in $\pi_{0,1}$ that is detected by h_0. We choose $\widetilde{2}$ such that $2 = 1+1$ in $\pi_{0,0}$ is equal to $\lambda\widetilde{2}$ and is detected by λh_0. Another possible notation for $\widetilde{2}$ is h, which some authors have used for the motivic homotopy element that is detected by h_0. It stands for "hyperbolic" because of its relationship to the hyperbolic plane as an element of the Grothendieck–Witt group. It also stands for "Hopf" because it is the zeroth Hopf map (followed by η, ν, and σ).

Most of our results are stated with the degrees in which they occur. This makes the manuscript easier to use for readers who are seeking specific results.

2 The Permanent Cycle $h_6g + h_2e_2$

Our goal is to show that the element $h_6g + h_2e_2$ in the 83-stem is a permanent cycle. We begin by sketching a line of reasoning without supporting details. Motivated by this sketch, later in this section, we give a precise argument that relies on a number of specific computational facts about various stable homotopy groups. These computational facts appear in the tables in Sect. 1, and their proofs are given in Sect. 3.

One might expect that the element h_6g detects the Toda bracket $\langle 2, \theta_5, \overline{\kappa}\rangle$ since the relation $2\theta_5 = 0$ (proved in [21]) is suggested by the Adams differential $d_2(h_6) = h_0h_5^2$, and $\overline{\kappa}$ is detected by g. There are two problems with this hope. First, we do not expect h_6g itself to be a permanent cycle. Rather we expect the linear combination $h_6g + h_2e_2$ to survive. Second, the Toda bracket is not defined because $\theta_5\overline{\kappa}$ is non-zero and detected by h_5^2g in the 82-stem.

But there is a relation $h_5^2g = h_2x_1$ in the Adams E_2-page. Therefore, one might hope for a matric Toda bracket of the form

$$\left\langle 2, \begin{bmatrix} \theta_5 & \xi \end{bmatrix}, \begin{bmatrix} \overline{\kappa} \\ \nu \end{bmatrix} \right\rangle,$$

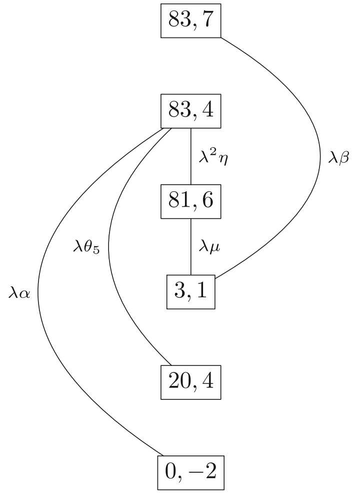


Fig. 1 The six-cell complex X

where ξ is detected by x_1. The existence of such a bracket would require that $2\xi = 0$. This relation is created by the Adams differential $d_2(e_2) = h_0x_1$. This is a promising sign because we anticipate the appearance of h_2e_2.

But the matric Toda bracket has a fatal flaw because x_1 does not survive; in fact, $d_3(x_1) = h_1m_1$. Therefore, we want to replace the entry ξ in the above matric bracket with something involving the relation $\eta\mu = 0$, where μ is detected by m_1.

At this point, the language of matric Toda brackets of maps between spheres is no longer adequate because of varying lengths of the null compositions that go into the construction. The cell complex construction that we adopt is a way of turning the above sketch into a precise argument.

2.1 The Construction of a Six-Cell Complex *X*

In this section, we will construct an explicit cell complex X that has six cells in bidegrees

$$(0, -2),\ (20, 4),\ (3, 1),\ (81, 6),\ (83, 4),\ (83, 7).$$

We also construct some closely related complexes that map to X or receive a map from X. Using standard obstruction theory, we will construct X by attaching cells to smaller complexes.

The spectrum X can be intuitively described in terms of the cell diagram in Fig. 1. See [3, 10, 20] for other uses of cell diagrams. In this figure (and throughout the construction of X), we refer to the elements α, β, and μ shown in Table 3.

Step 2.1 To begin, Lemma 3.15 implies that $\lambda\mu \cdot \lambda^2\eta$ is zero. Therefore, we can form a three-cell complex Y with cells in bidegrees (3, 1), (81, 6), and (83, 4). The attaching maps for Y are $\lambda\mu$ and $\lambda^2\eta$. The first part of Fig. 2 shows a cell diagram for Y. Note

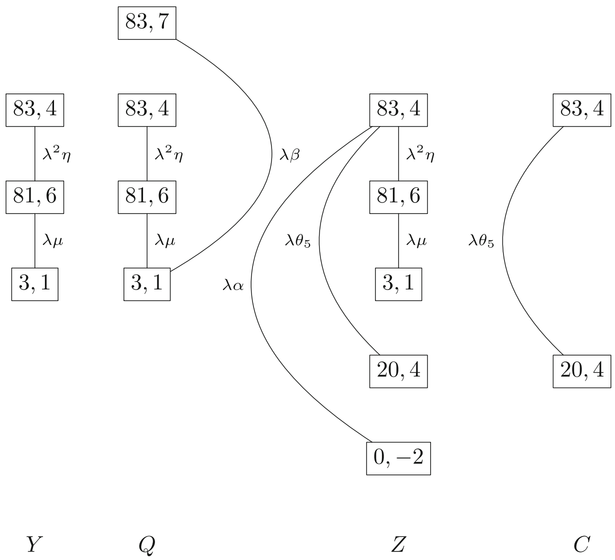


Fig. 2 Some complexes related to X

that Y comes equipped with inclusion $i_Y : S^{3,1} \to Y$ of the bottom cell and projection $p_Y : Y \to S^{83,4}$ to the top cell.

Beware that the construction of Y depends on a choice of null-homotopy for $\lambda\mu \cdot \lambda^2\eta$. For now, this choice is immaterial. However, later in Step 2.8, we will make a choice of null-homotopy with specific additional properties.

Step 2.2 Next, consider the composition $i_Y \cdot \lambda\beta : S^{82,8} \to Y$. Let Q be the cofiber of this map. The second part of Fig. 2 shows a cell diagram for Q. Note that Q comes equipped with inclusion $i_Q : S^{3,1} \to Q$ of the bottom cell.

The top cell of Q is only attached to the bottom cell in bidegree (3, 1), so the projection $p_Y : Y \to S^{83,4}$ extends to a projection $p_Q : Q \to S^{83,4}$ to the third cell of Q. More formally, the dashed arrow in the commutative diagram

$$\begin{tikzcd} S^{82,8} \arrow[r, "i_Y \cdot \lambda\beta"] & Y \arrow[r] \arrow[d, "p_Y"] & Q \arrow[dl, dashed, "p_Q"] \\ & S^{83,4} & \end{tikzcd}$$

exists because the top row is a cofiber sequence and because the composition $p_Y \cdot i_Y \cdot \lambda\beta$ is zero (since $p_Y \cdot i_Y$ is zero).

There is also a projection $p'_Q : Q \to S^{83,7} \vee S^{83,4}$ to the top two cells. The first component of p'_Q is projection to the top cell, and the second component is the map p_Q defined in the previous paragraph.

Step 2.3 Now consider the composition $(\lambda\theta_5, \lambda\alpha) \cdot p_Q : Q \to S^{21,3} \vee S^{1,-3}$. Define X to be the fiber of this map.

Note that X comes equipped with an inclusion $i_X : S^{3,1} \vee S^{20,4} \vee S^{0,-2} \to X$ of the bottom three cells because they are not attached to each other. More formally, consider the diagram

$$\begin{tikzcd}
 & & S^{3,1} \arrow[dl, dashed] \arrow[d, "i_Q"] & \\
S^{20,4} \vee S^{0,-2} \arrow[r] & X \arrow[r] & Q \arrow[r, "(\lambda\theta_5,\lambda\alpha)\cdot p_Q"'] & S^{21,3} \vee S^{1,-3},
\end{tikzcd}$$

in which the bottom row is a cofiber sequence. The last two components of i_X appear in the cofiber sequence. The first component, shown as the dashed arrow, exists because the composition $(\lambda\theta_5, \lambda\alpha) \cdot p_Q \cdot i_Q$ is zero (since $p_Q \cdot i_Q$ is zero).

Step 2.4 We define Z to be the fiber of the projection $p_X : X \to S^{83,7}$ to the top cell. The third part of Fig. 2 shows a cell diagram for Z. Note that Z comes equipped with an inclusion $i_Z : S^{3,1} \vee S^{20,4} \vee S^{0,-2} \to Z$. More precisely, consider the diagram

$$\begin{tikzcd}
 & S^{3,1} \vee S^{20,4} \vee S^{0,-2} \arrow[dl, dashed, "i_Z"'] \arrow[d, "i_X"] & \\
Z \arrow[r] & X \arrow[r, "p_X"'] & S^{83,7}
\end{tikzcd}$$

in which the bottom row is a cofiber sequence. The dashed arrow exists because the composition $p_X i_X$ is zero.

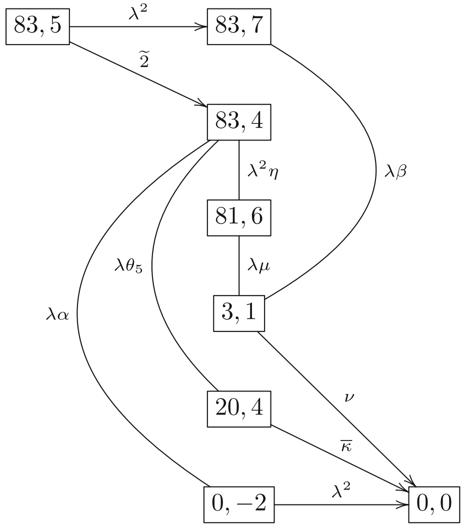
Step 2.5 We define C to be the cofiber of the map $\lambda\theta_5 : S^{82,5} \to S^{20,4}$. The fourth part of Fig. 2 shows a cell diagram for C. Note that there is an inclusion $i_C : S^{20,4} \to C$ of the bottom cell.

There is a map $\pi : X \to C$ defined as follows. Consider the diagram

$$\begin{tikzcd}
\Sigma^{-1} Q \arrow[r, "(\lambda\theta_5,\lambda\alpha)\cdot p_Q"] & S^{20,4} \vee S^{0,-2} \arrow[r] \arrow[d, "(i_C{,}0)"'] & X \arrow[dl, dashed, "\pi"] \\
 & C, &
\end{tikzcd}$$

in which the top row is the defining cofiber sequence for X, as described in Step 2.3. To show that the dashed arrow exists, we need only argue that the composition $(i_C, 0) \cdot (\lambda\theta_5, \lambda\alpha) \cdot p_Q$ is zero. This composition equals $i_C \cdot \lambda\theta_5 \cdot p_Q$, and $i_C \cdot \lambda\theta_5$ is zero because C is the cofiber of $\lambda\theta_5$.

Fig. 3 The Maps $f : S^{83,5} \to X$ and $g : X \to S^{0,0}$



2.2 The Map $f : S^{83,5} \to X$

We next construct a map $f : S^{83,5} \to X$, where X is the complex constructed in Sect. 2.1. The map f is intuitively described by the cell diagram in Fig. 3. It is represented in the left two columns of the figure.

Step 2.6 Recall from Lemma 3.19 that $\lambda\beta\cdot\lambda^2$ belongs to the Toda bracket $\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$. Therefore, we have a map $f_Q : S^{83,5} \to Q$ such that composition with the projection $p'_Q : Q \to S^{83,7} \vee S^{83,4}$ is $(\lambda^2, \widetilde{2}) : S^{83,5} \to S^{83,7} \vee S^{83,4}$. Recall that the map p'_Q was discussed in Step 2.2.

In fact, one must be careful here about the choice of null-homotopy in Step 2.1. We need that $\lambda\beta \cdot \lambda^2$ belongs not merely to the Toda bracket $\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$, but rather to the subset of this bracket consisting of representatives that are defined using the specified null-homotopy of $\lambda\mu \cdot \lambda^2\eta$.

This turns out to be no problem for us. We show in Lemma 3.19 that every element of the Toda bracket $\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$ is of the form $\lambda\beta \cdot \lambda^2$. So we can choose β to be compatible with the previously chosen null-homotopy.

Step 2.7 We construct f as the dashed map in the commutative diagram

$$\begin{tikzcd}
 & S^{83,5} \arrow[dl, dashed, "f"'] \arrow[d, "f_Q"] & \\
X \arrow[r] & Q \arrow[r, "(\lambda\theta_5, \lambda\alpha)\cdot p_Q"] & S^{21,3} \vee S^{1,-3},
\end{tikzcd}$$

in which the bottom row is the fiber sequence from Step 2.3. To do this, we just need to argue that the composition $(\lambda\theta_5, \lambda\alpha) \cdot p_Q \cdot f_Q$ is zero. This follows from two observations. First, $p_Q \cdot f_Q : S^{83,5} \to S^{83,4}$ is $\widetilde{2}$, as explained in Step 2.6. Second, $(\lambda\theta_5, \lambda\alpha)\widetilde{2}$ is zero by Lemma 3.24 and [21]; here we are using that 2 equals $\lambda\widetilde{2}$.

2.3 The Map $g : X \to S^{0,0}$

We now construct a map $g : X \to S^{0,0}$, where X is the complex constructed in Sect. 2.1. Later in Sect. 2.4, we will study the composition gf as an element of $\pi_{83,5}$.

The map g is intuitively described by the cell diagram in Fig. 3. It is represented in the right two columns of the figure.

Step 2.8 By Lemma 3.24, $\overline{\kappa} \cdot \lambda\theta_5 + \lambda^2 \cdot \lambda\alpha$ is contained in the Toda bracket $\langle \nu, \lambda\mu, \lambda^2\eta \rangle$. Therefore, we can form a map $g_Z : Z \to S^{0,0}$ such that the composition $g_Z i_Z$ is

$$(\nu, \overline{\kappa}, \lambda^2) : S^{3,1} \vee S^{20,4} \vee S^{0,-2} \to S^{0,0}.$$

One must be careful here about the choice of null-homotopy in Step 2.1. We need that $\overline{\kappa} \cdot \lambda\theta_5 + \lambda^2 \cdot \lambda\alpha$ belongs not merely to the Toda bracket $\langle \nu, \lambda\mu, \lambda^2\eta \rangle$, but rather to the subset of this bracket consisting of representatives that are defined using the specified null-homotopy of $\lambda\mu \cdot \lambda^2\eta$.

Since this is the only place in our entire argument where we have to be careful about the choice, we are free to choose a null-homotopy. In particular, we choose the null-homotopy that is relevant for the presence of $\overline{\kappa} \cdot \lambda\theta_5 + \lambda^2 \cdot \lambda\alpha$ in $\langle \nu, \lambda\mu, \lambda^2\eta \rangle$.

Step 2.9 Consider the commutative diagram

$$\begin{tikzcd}
S^{82,8} \arrow[r] & Z \arrow[r] \arrow[d, "g_Z"'] & X \arrow[r, "p_X"] \arrow[dl, dashed, "g"] & S^{83,7} \\
 & S^{0,0} & &
\end{tikzcd}$$

in which the top row is a cofiber sequence. The composition $S^{82,8} \to Z \to S^{0,0}$ is zero because $\nu \cdot \lambda\beta$ is zero. Therefore, g_Z extends to the dashed arrow $g : X \to S^{0,0}$ shown in the diagram.

2.4 The Composition $gf : S^{83,5} \to S^{0,0}$

We will detect the map $gf : S^{83,5} \to S^{0,0}$ by smashing with $S^{0,0}/\lambda$ and passing to the category of $S^{0,0}/\lambda$-modules. This latter category is entirely algebraic, i.e., is equivalent to the category of derived comodules over the classical dual Steenrod algebra. In more concrete terms, the synthetic homotopy groups of $S^{0,0}/\lambda$ are isomorphic to the classical Adams E_2-page, i.e., the cohomology of the dual Steenrod algebra. These latter homotopy groups of $S^{0,0}/\lambda$ are much more easily understood than the homotopy groups of $S^{0,0}$.

We will now show that the complex X splits completely after smashing with $S^{0,0}/\lambda$. This means that maps into and out of X/λ are easy to compute.

Proposition 2.10 *The $S^{0,0}/\lambda$-module X/λ splits as a wedge of suspensions of $S^{0,0}/\lambda$.*

Proof The $S^{0,0}/\lambda$-module Y/λ has three cells: $S^{83,4}/\lambda$, $S^{81,6}/\lambda$, and $S^{3,1}/\lambda$. The attaching map between the bottom and middle cells is the smash product of $\lambda\mu$ with $S^{0,0}/\lambda$. This smash product is zero because $\lambda\mu$ is a multiple of λ. Similarly, the attaching map between the middle and top cells is zero. The Adams E_2-page is zero in stem 79 and filtration 4; therefore, zero is the only possible attaching map between the bottom and top cells. This shows that Y/λ splits.

As shown in Step 2.2, the spectrum Q is defined by the cofiber sequence

$$S^{82,8} \xrightarrow{i_Y \cdot \lambda\beta} Y \longrightarrow Q.$$

After smashing with $S^{0,0}/\lambda$, this cofiber sequence splits because $i_Y \cdot \lambda\beta$ is a multiple of λ. Consequently, Q/λ splits as

$$(Y/\lambda) \vee \left(S^{83,7}/\lambda\right),$$

and we have already shown that Y/λ splits.

As shown in Step 2.3, the spectrum X is defined by the cofiber sequence

$$X \longrightarrow Q \xrightarrow{(\lambda\theta_5, \lambda\alpha)\cdot p_Q} S^{21,3} \vee S^{1,-3}.$$

After smashing with $S^{0,0}/\lambda$, this cofiber sequence splits because $(\lambda\theta_5, \lambda\alpha) \cdot p_Q$ is a multiple of λ. Consequently, X/λ splits as

$$(Q/\lambda) \vee \left(S^{20,4}/\lambda\right) \vee \left(S^{0,-2}/\lambda\right),$$

and we have already shown that Q/λ splits. □

Remark 2.11 Similarly to Proposition 2.10 but much easier, the $S^{0,0}/\lambda$-module C/λ splits as a wedge of two suspensions of $S^{0,0}/\lambda$. Here C is the two-cell complex discussed in Step 2.5.

After smashing f with $S^{0,0}/\lambda$, the map f/λ is of the form

$$\left(a_{0,-2}, a_{0,1}, a_{2,-1}, a_{80,4}, a_{63,1}, a_{83,7}\right),$$

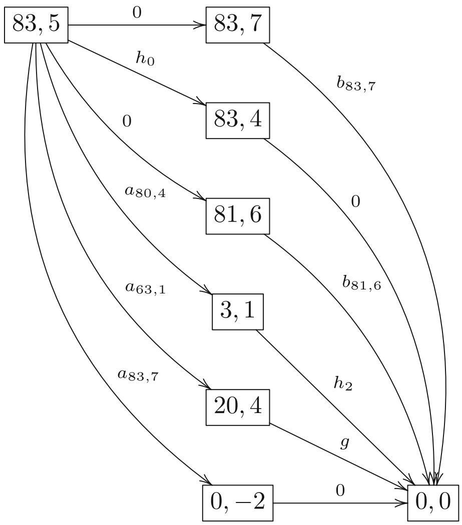
where each $a_{s,f}$ is an element of the Adams E_2-page in stem s and filtration f. Similarly, the map g/λ is of the form

$$\left(b_{83,7}, b_{83,4}, b_{81,6}, b_{3,1}, b_{20,4}, b_{0,-2}\right),$$

where each $b_{s,f}$ is an element of the Adams E_2-page in stem s and filtration f. The composition gf/λ is equal to

$$b_{83,7}a_{0,-2} + b_{83,4}a_{0,1} + b_{81,6}a_{2,-1} + b_{3,1}a_{80,4} + b_{20,4}a_{63,1} + b_{0,-2}a_{83,7}.$$

Fig. 4 The composition gf/λ



The computation of gf/λ is illustrated in Fig. 4. This figure can be interpreted as a cell diagram in the category of $S^{0,0}/\lambda$-modules. However, since X/λ splits, the cell diagram is trivial.

Lemma 2.12 *The composition* $gf/\lambda : S^{83,5}/\lambda \to S^{0,0}/\lambda$ *is equal to* $h_2 a_{80,4} + g a_{63,1}$.

Proof For degree reasons, $a_{0,2}$, $a_{2,-1}$, $b_{83,4}$, and $b_{0,-2}$ must be zero. Moreover, $a_{0,1}$ must equal h_0 since h_0 detects $\widetilde{2}$ and the composition of f with the projection $X \to S^{83,4}$ is $\widetilde{2}$. Similarly, $b_{20,4}$ must equal g since g detects $\overline{\kappa}$ and the composition of g with the inclusion $S^{20,4} \to X$ is $\overline{\kappa}$. Also, $b_{3,1}$ must equal h_2 since h_2 detects ν and the composition of g with the inclusion $S^{3,1} \to X$ is ν.

Substituting these values, we find that gf/λ equals

$$b_{83,7} \cdot 0 + 0 \cdot h_0 + b_{81,6} \cdot 0 + h_2 \cdot a_{80,4} + g \cdot a_{63,1} + 0 \cdot a_{83,7},$$

which simplifies to $h_2 a_{80,4} + g a_{63,1}$. □

We still need to compute that $a_{63,1}$ and $a_{80,4}$ are equal to h_6 and e_2, respectively. We need another cell complex to accomplish this.

Step 2.13 Recall the complex C constructed as the cofiber of $\lambda\theta_5$ in Step 2.5. Let D be the cofiber of the composition $\pi f : S^{83,5} \to C$. A cell diagram for D is shown on the left side of Fig. 5.

Lemma 2.14 *The operation* Sq^{64} *is non-zero in the cohomology of* D.

Proof We work in the classical context; the synthetic result that we desire follows immediately by comparison along the realization functor, i.e., along λ-localization.

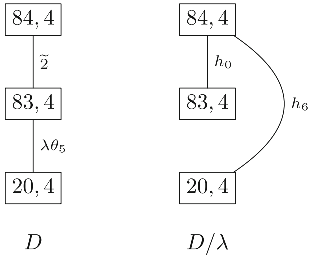


Fig. 5 The complexes D and D/λ

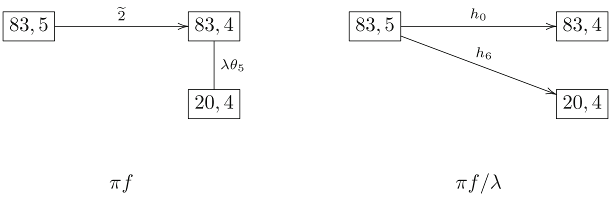


Fig. 6 The maps πf and $\pi f/\lambda$

Recall Adams decomposition

$$\mathrm{Sq}^{64} = \mathrm{Sq}^1\, \Phi_{5,5} + \cdots$$

of Sq^{64} in terms of secondary operations [1, Theorem 4.6.1]. The secondary operation $\Phi_{5,5}$ is non-zero on the cofiber of θ_5 [3, p. 536]. □

Lemma 2.15 *The map*

$$\pi f/\lambda : S^{83,5}/\lambda \to C/\lambda$$

is equal to

$$(h_0, h_6) : S^{83,5}/\lambda \to S^{83,4}/\lambda \vee S^{20,4}/\lambda.$$

Here we are using the splitting of C/λ given in Remark 2.11. Figure 6 illustrates Lemma 2.15 with cell diagrams.

Proof Unlike all of the other $S^{0,0}/\lambda$-modules under consideration, D/λ does not split. In fact, the middle cell is attached to the top cell by h_0 because $\widetilde{2}$ is detected by h_0. Lemma 2.14 implies that Sq^{64} is non-zero on D/λ. Therefore, the bottom cell is attached to the top cell by h_6. A cell diagram for D/λ is shown on the right side of Fig. 5.

The cofiber sequence

$$S^{83,5} \xrightarrow{\pi f} C \longrightarrow D$$

that defines D induces a cofiber sequence

$$S^{83,5}/\lambda \xrightarrow{\pi f/\lambda} C/\lambda \longrightarrow D/\lambda.$$

The computation of the attaching maps for D/λ is equivalent to the computation of $\pi f/\lambda$. □

Lemma 2.16 *The value of* $a_{63,1}$ *is* h_6.

Proof The map $\pi/\lambda : X/\lambda \to C/\lambda$ is the standard projection onto two wedge summands of X/λ. The computation of $\pi f/\lambda$ in Lemma 2.15 gives that $a_{63,1}$ is equal to h_6. □

Theorem 2.17 *The composition* $gf : S^{83,5} \to S^{0,0}$ *is detected by* $h_6 g + h_2 e_2$ *in the synthetic Adams spectral sequence.*

Proof Combining Lemma 2.12 with Lemma 2.16, we get that gf/λ is equal to $h_6 g + h_2 a_{80,4}$ for some value of $a_{80,4}$. Therefore, gf is detected by $h_6 g + h_2 a_{80,4}$ in the synthetic Adams E_2-page.

By inspection, $a_{80,4}$ must be a linear combination of e_2 and $h_1^2 h_4 h_6$. Since $h_6 g$ itself is known to not be a permanent cycle [14], we must have that $h_2 a_{80,4}$ is non-zero. The only possibility is that $h_2 a_{80,4}$ equals $h_2 e_2$. □

Remark 2.18 The careful reader will note that we did not compute $a_{80,4}$. It could be either e_2 or $e_2 + h_1^2 h_4 h_6$. In either case, we conclude that $h_6 g + h_2 e_2$ is a permanent cycle.

3 Computations

Proposition 3.1 $(56, 9, 9)$ $d_5(Ph_5 e_0) = \lambda^4 il$.

Remark 3.2 Proposition 3.1 immediately follows from [11, Lemma 3.92], which uses fourfold brackets. We give a simpler proof.

Proof Recall the hidden η extension from g^2 to $\lambda \Delta h_0^2 e_0$ [14, Table 18]. Multiply by d_0 to see that there is also a hidden η extension from $e_0^2 g$ to λil. Also, an immediate consequence of the main result of [5] is that there is a hidden $\widetilde{2}$ extension from $\lambda^2 h_0 h_5 i$ to $\lambda^4 e_0^2 g$.

Therefore, $\lambda^5 il$ detects a multiple of $\widetilde{2}\eta = 0$, so it must be hit by a differential. There is only one possibility. □

Lemma 3.3 $(46, 7, 7)$ *The Toda bracket* $\langle \eta, \widetilde{2}^2 \overline{\kappa}_2, \widetilde{2} \rangle$ *is detected by* Mh_1. *The indeterminacy is generated by* $\lambda^3 \eta \{\Delta h_1 g\}$, *which is detected by* $\lambda^4 d_0 l$.

Remark 3.4 The Toda bracket in Lemma 3.3 has a classical analog $\langle \eta, 4\overline{\kappa}_2, 2 \rangle$. The classical bracket contains zero because $\eta \theta_{4.5}$ is detected by Mh_1. Synthetically, the product $\eta\theta_{4.5}$ is detected by $\lambda^3 Mh_1$, but Mh_1 itself does not detect a multiple of η.

Proof Start with the Massey product $Mh_1 = \langle h_1, h_0^2 g_2, h_0 \rangle$ [14, Table 3]. Apply the Moss Convergence Theorem [4, 17] to obtain that the Toda bracket $\langle \eta, \widetilde{2}^2 \overline{\kappa}_2, \widetilde{2} \rangle$ is detected by Mh_1. The indeterminacy is computed by inspection. □

Lemma 3.5 (77, 12, 12) *The Toda bracket* $\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle$ *is detected by the element* $M\Delta h_1 h_3$*, with no indeterminacy.*

Proof The shuffle

$$\eta\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle = \langle \eta, \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}\rangle\{\Delta h_1 h_3\}$$

shows that $\eta\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle$ is detected by $Mh_1 \cdot \Delta h_1 h_3$. Here we are using the Toda bracket from Lemma 3.3. It follows that $\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle$ is detected by $M\Delta h_1 h_3$.

The indeterminacy is zero by inspection. □

Proposition 3.6 (77, 12, 12) *There is a hidden* ν *extension from* $M\Delta h_1 h_3$ *to* $\lambda^2 Mh_1 e_0^2$.

Remark 3.7 The hidden extension in Proposition 3.6 is purely synthetic; it has no classical (nor motivic) analog. In computational terms, the target $\lambda^2 Mh_1 e_0^2$ of the extension is annihilated by λ^2, so it becomes zero after λ-periodicization.

However, we do not know yet that $\lambda^2 Mh_1 e_0^2$ is annihilated by λ^2. That fact is a consequence of Proposition 3.8, whose proof depends on Proposition 3.6.

Proof Lemma 3.5 shows that the Toda bracket $\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle$ is detected by $M\Delta h_1 h_3$. We have

$$\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \{\Delta h_1 h_3\}\rangle\nu = \langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \lambda^2\eta\kappa\overline{\kappa}\rangle$$

because of the hidden ν extension from $\Delta h_1 h_3$ to $\lambda^2 h_1 e_0^2$ [14, Table 21]. Equality holds in the displayed formula because the right side has no indeterminacy by inspection.

We also have

$$\langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \lambda^2\eta\kappa\overline{\kappa}\rangle = \langle \widetilde{2}^2\overline{\kappa}_2, \widetilde{2}, \eta\rangle\lambda^2\kappa\overline{\kappa}.$$

Because of Lemma 3.3, this last expression is detected by $Mh_1 \cdot \lambda^2 e_0^2$. □

Proposition 3.8 (61, 6, 6) $d_5(A') = \lambda^4 Mh_1 d_0$.

Remark 3.9 Proposition 3.8 immediately follows from [20, Theorem 12.1], which requires eight pages of shuffling several Toda brackets in a delicate way. We give a simpler proof.

Proof We will show that $d_5(gA') = \lambda^4 Mh_1 e_0^2$, from which the desired formula follows immediately.

By comparison to motivic homotopy [14, Table 18], there is a hidden η extension from $x_{76,9}$ to $\lambda^2 M\Delta h_1 h_3$. Proposition 3.6 shows that there is a hidden ν extension from $\lambda^2 M\Delta h_1 h_3$ to $\lambda^4 Mh_1 e_0^2$.

Therefore, $\lambda^4 Mh_1 e_0^2$ must be hit by a differential, but there are two possibilities. By comparison to motivic homotopy, $d_3(\Delta^2 p)$ cannot equal $\lambda^2 Mh_1 e_0^2$ [14, Table 6]. The only remaining possibility is that $d_5(gA')$ equals $\lambda^4 Mh_1 e_0^2$. □

Proposition 3.10 (71, 5, 5) *There is no hidden ν extension on $h_1 p_1$.*

Proof The elements $\lambda P h_0 h_2 h_6$, $\lambda^2 P h_1^3 h_6$, $\lambda^2 x_{74,8}$, $\lambda^8 \Delta^2 h_2^2 g$, and $\lambda^{10} e_0^2 g^2$ are the possible values for a hidden ν extension on $h_1 p_1$. As shown in [14, Lemma 7.28], the element $\tau h_1 p_1$ does not support a motivic hidden 2 extension. Therefore, $h_1 p_1$ does not support a classical hidden 2 extension. Also, $h_1 p_1$ does not support a synthetic hidden $\widetilde{2}$ extension because all of the possible targets are λ-periodic. Therefore, the target of a synthetic hidden ν extension on $h_1 p_1$ must be annihilated by $\widetilde{2}$. This rules out $\lambda P h_0 h_2 h_6$ and $\lambda^2 x_{74,8}$.

The remaining three cases would imply motivic hidden ν extensions from $\tau h_1 p_1$ to $P h_1^3 h_6$, $\tau \Delta^2 h_2^2 g$, and $\tau^5 e_0^2 g^2$ respectively. The first is ruled out by comparison to $S^{0,0}/\tau$, and the last two are ruled out by comparison to mmf. □

Proposition 3.11 (74, 6, 6) *There is a hidden $\widetilde{2}$ extension from $h_3 n_1$ to $\lambda x_{74,8}$.*

Remark 3.12 Proposition 3.11 follows from [14, Lemma 7.35], which uses fourfold brackets. We give a simpler proof.

Proof Let γ be an element of $\pi_{67,5}$ that is detected by $Q_3 + n_1$. Because of the differentials $d_3(d_2) = \lambda^2 h_0(Q_3 + n_1)$ and $d_4(h_0 d_2) = \lambda^3 X_3$, projection to the top cell of $S^{0,0}/\lambda$ gives a hidden $\widetilde{2}$ extension from $h_0^2(Q_3 + n_1)$ to λX_3. Since $h_3 X_3 = h_0^2 x_{74,8}$, we conclude that $\lambda h_0^2 x_{74,8}$ detects $\widetilde{2}^3 \sigma \gamma$. This implies the desired hidden extension since $\sigma \gamma$ is detected by $h_3 n_1$. □

Proposition 3.13 (77, 7, 7) *There is no hidden ν extension on m_1.*

Proof As shown in [14], the element m_1 does not support a motivic hidden ν extension, which means that m_1 does not support a classical hidden ν extension. Therefore, a λ-free element cannot be the target of a synthetic hidden ν extension on m_1. By inspection, there are no possible λ-torsion targets for a hidden ν extension on m_1. □

Proposition 3.14 (77, 7, 5) *There is no hidden η extension on $\lambda^2 m_1$.*

Proof The product $h_1 \cdot \lambda^2 m_1$ equals zero in the synthetic Adams E_∞-page because of the differential $d_3(x_1) = \lambda^2 h_1 m_1$.

As shown in [14], the element τm_1 does not support a motivic hidden η extension, which means that m_1 does not support a classical hidden η extension. Therefore, a λ-free element cannot be the target of a synthetic hidden η extension on $\lambda^2 m_1$. By inspection, there are no possible λ-torsion targets for a hidden η extension on $\lambda^2 m_1$. □

Lemma 3.15 (77, 7, 7) *There exists an element μ in $\pi_{77,7}$ that is detected by m_1 such that $\lambda^2 \eta \mu$ and $\nu \mu$ are both zero.*

Proof Proposition 3.13 implies that there exists a choice of μ such that $\nu\mu$ is zero. Moreover, as shown in [14], there are no classical hidden ν extensions in higher filtration. Therefore, there are no synthetic hidden ν extensions in higher filtration whose targets are λ-free. By inspection, there are no λ-torsion classes that could be the targets of synthetic hidden ν extensions in higher filtration. This implies that $\nu\mu$ is zero for every choice of μ.

Proposition 3.14 implies that there exists an element μ' in $\pi_{77,5}$ that is detected by $\lambda^2 m_1$ such that $\eta\mu'$ is zero. We just need to show that μ' is of the form $\lambda^2\mu$, for some choice of μ detected by m_1.

To start, choose μ arbitrarily. The sum $\mu' + \lambda^2\mu$ is detected in higher filtration. As shown in [14], there are no classical hidden η extensions in higher filtration. Therefore, there are no synthetic hidden η extensions in higher filtration whose targets are λ-free. By inspection, there are no λ-torsion classes that could be the targets of synthetic hidden η extensions in higher filtration.

However, there is a non-hidden η extension from $\lambda^7 M\Delta h_1 h_3$ to $\lambda^7 M\Delta h_1^2 h_3$. Let μ'' be an element of $\pi_{77,7}$ that is detected by $\lambda^5 M\Delta h_1 h_3$, so $\lambda^2\mu''$ is detected by $\lambda^7 M\Delta h_1 h_3$.

If $\mu' + \lambda^2\mu$ is not equal to $\lambda^2\mu''$, then $\eta(\mu' + \lambda^2\mu)$ is zero. On the other hand, if $\mu' + \lambda^2\mu$ is equal to $\lambda^2\mu''$, then $\eta(\mu' + \lambda^2(\mu + \mu''))$ is zero. Since $\eta\mu'$ was already shown to be zero, we conclude that either $\lambda^2\eta\mu$ or $\lambda^2\eta(\mu+\mu'')$ is zero. Consequently, either μ or $\mu + \mu''$ satisfies the required conditions. □

Lemma 3.16 (79, ?, 5) *The Toda bracket $\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$ contains zero or is detected by $\lambda^9 Me_0^2$, and its indeterminacy is generated by an element that is detected by $\lambda^3 h_0 h_2 x_{76,6}$.*

Remark 3.17 The Toda bracket $\langle\lambda\mu, \lambda\eta, \widetilde{2}\rangle$ is also defined. Using the Moss Convergence Theorem [4, 17] and the differential $d_3(x_1) = \lambda^2 h_1 m_1$, it is detected by $h_0 x_1$. The element $h_0 x_1$ is not zero but is annihilated by λ.

Remark 3.18 The element $\lambda^3 h_0 h_2 x_{76,6}$ has lower Adams filtration than $\lambda^9 Me_0^2$. One must often be especially careful in situations like this when the indeterminacy is detected in lower filtration because the various elements of the Toda bracket are detected in different filtrations.

Proof By inspection, there are no λ-torsion elements that could detect the Toda bracket. Consequently, we only need to consider λ-free elements, so we can work in the classical context.

As shown in [14, Table 10], the corresponding motivic Toda bracket contains zero or is detected by $\tau^2 Me_0^2$. Therefore, the corresponding classical bracket either contains zero or is detected by Me_0^2.

The indeterminacy is generated by $\lambda\mu \cdot \lambda^3\eta^2$ and by the multiples of $\widetilde{2}$ in $\pi_{79,5}$. The first expression is zero by Lemma 3.15. As shown in [14], there are no classical hidden 2 extensions in the 79-stem. This rules out all possible synthetic hidden $\widetilde{2}$ extensions. There is only one non-hidden $\widetilde{2}$ extension in $\pi_{79,5}$, and its target is $\lambda^3 h_0 h_2 x_{76,6}$. □

Lemma 3.19 (79, ?, 8) *Every element of $\langle\lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$ is of the form $\lambda^3\beta$ for some β in $\pi_{79,8}$ such that $\lambda\nu\beta$ is zero.*

Remark 3.20 We do not specify which element of the Adams E_∞-page detects the element β in Lemma 3.19. For our purposes later, the detecting element is unimportant.

Proof The proof involves two steps. First, we will show that the Toda bracket contains an element with the desired properties. Second, we will show that the elements in the indeterminacy of the Toda bracket have the desired properties. These two steps imply that every element of the Toda bracket has the desired properties.

There are two cases in Lemma 3.16. In one case, zero is an element in the Toda bracket with the desired properties.

In the other case, let β be detected by $\lambda^6 Me_0^2$. Then $\lambda^3\beta$ and $\langle \lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$ are both detected by $\lambda^9 Me_0^2$. There are no elements in higher filtration, so in fact $\lambda^3\beta$ is contained in $\langle \lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$. Finally, $\lambda^6 Me_0^2$ cannot support a ν extension because there are no possible targets. This shows that the Toda bracket contains an element with the desired properties.

Next, we study the elements in the indeterminacy. Lemma 3.16 shows that the indeterminacy is generated by $\lambda^3\widetilde{2}\gamma$, where γ is an element of $\pi_{79,7}$ that is detected by $h_2 x_{76,6}$. We want to show that $\lambda\nu \cdot \widetilde{2}\gamma$ is zero.

Note that $\nu\gamma$ is detected by $h_2^2 x_{76,6}$. We know from [14, Lemma 7.43] that $h_2^2 x_{76,6}$ does not support a classical hidden 2 extension. Therefore, it also does not support a synthetic $\widetilde{2}$ extension because all possible targets for such a hidden extension are λ-free. This shows that $\lambda\nu \cdot \widetilde{2}\gamma$ is zero. □

Proposition 3.21 (81, 12, 12) *There is no hidden ν extension on $\Delta^2 p$.*

Proof The elements $\lambda\Delta^2 t$ and $\lambda^2 M\Delta h_1 d_0$ are the two possible targets for a hidden ν extension on $\Delta^2 p$. Both elements support h_1 extensions, so they cannot be targets of hidden ν extensions. □

Lemma 3.22 (82, 6, 5) *Every element of the Toda bracket $\langle \nu, \lambda\mu, \lambda^2\eta\rangle$ is detected by $\lambda h_5^2 g$. Here μ is the element in $\pi_{77,7}$ specified in Lemma 3.15.*

Remark 3.23 Lemma 3.22 does not compute the indeterminacy of the Toda bracket $\langle \nu, \lambda\mu, \lambda^2\eta\rangle$. In fact, the bracket does have indeterminacy because of the presence of multiples of $\lambda^2\eta$ and of ν in higher filtration. However, our later arguments do not depend on the indeterminacy.

Proof The bracket is well defined because of Lemma 3.15. Using the synthetic Adams differential $d_3(\lambda x_1) = \lambda^3 h_1 m_1$, the Moss Convergence Theorem [4, 17] implies that $\lambda h_2 x_1 = \lambda h_5^2 g$ detects the Toda bracket.

For degree reasons, all of the indeterminacy is detected in higher filtration. Consequently, every element of the bracket is detected by $\lambda h_5^2 g$. □

Lemma 3.24 (82, ?, 8) *There exists an element α of $\pi_{82,8}$ such that $\lambda^3\alpha + \lambda\overline{\kappa}\theta_5$ belongs to the Toda bracket $\langle \nu, \lambda\mu, \lambda^2\eta\rangle$, and also 2α is zero.*

Remark 3.25 We do not specify which element of the Adams E_∞-page detects the element α in Lemma 3.24. By inspection of the proof, we know that α is detected in Adams filtration 8 or higher. For our purposes later, the detecting element is unimportant. As in Remark 3.23, beware that the Toda bracket has non-zero indeterminacy.

Proof Shuffle to obtain

$$2\langle \nu, \lambda\mu, \lambda^2\eta\rangle = \langle \nu, \lambda\mu, \lambda^2\eta\rangle\lambda\widetilde{2} = \lambda\nu\langle \lambda\mu, \lambda^2\eta, \widetilde{2}\rangle.$$

In either case of Lemma 3.16, the expression $\lambda\nu\langle \lambda\mu, \lambda^2\eta, \widetilde{2}\rangle$ contains zero. Therefore, there exists an element α' in $\pi_{82,5}$ such that α' is contained in the bracket $\langle \nu, \lambda\mu, \lambda^2\eta\rangle$, and also $2\alpha'$ equals zero.

Next, let α'' equal $\alpha' - \lambda\overline{\kappa}\theta_5$ in $\pi_{82,5}$. Because of Lemma 3.22, we know that $\lambda h_5^2 g$ detects both α' and the product $\lambda\overline{\kappa}\theta_5$. Therefore, α'' is detected in higher filtration. By inspection of the possible detecting elements, we find that α'' must be a multiple of λ^3. Therefore, α'' equals $\lambda^3\alpha$ for some α in $\pi_{82,8}$. This shows that $\lambda^3\alpha$ is contained in $\langle \nu, \lambda\mu, \lambda^2\eta\rangle - \lambda\overline{\kappa}\theta_5$. We rearrange this formula and conclude that $\lambda^3\alpha + \lambda\overline{\kappa}\theta_5$ is contained in $\langle \nu, \lambda\mu, \lambda^2\eta\rangle$.

Finally, we must show that 2α is zero. We have that $2\alpha''$ is zero since 2 annihilates both α' and $\lambda\overline{\kappa}\theta_5$ (proved in [21]). Therefore, $2\lambda^3\alpha$ is zero. By inspection, every possible value of 2α in $\pi_{82,8}$ is λ-free, so 2α must also be zero. □

Lemma 3.26 (79, 7, 7) (84, 10, 8) *There exists an element γ in $\pi_{79,7}$ that is detected by $h_2x_{76,6}$ such that $\eta\gamma$ is zero. Moreover, $\lambda^2Px_{76,6}$ detects an element in the Toda bracket $\langle \nu, \eta, \gamma\rangle$.*

Remark 3.27 Beware that Lemma 3.26 does not compute the indeterminacy of the Toda bracket. In fact, the indeterminacy contains an element that is detected by $\lambda h_2c_1H_1$, whose filtration is lower than the filtration of $\lambda^2Px_{76,6}$. One must often be especially careful in situations like this when the indeterminacy is detected in lower filtration because the various elements of the Toda bracket are detected in different filtrations.

Proof We know from [14] that $h_2x_{76,6}$ does not support a classical η extension. Moreover, there are no possible targets for a hidden η extension on $h_2x_{76,6}$ that are λ-torsion. Therefore, $h_2x_{76,6}$ does not support a synthetic η extension, and it is possible to choose γ. (The element λPh_6c_0 in higher filtration supports an h_1 multiplication. This means that not all possible choices of γ are annihilated by η.)

There is a hidden $\mathbb{C}$-motivic ν extension from $h_2^2x_{76,6}$ to $Ph_1x_{76,6}$ [14, Table 21]. Therefore, there is also a classical ν extension, as well as a synthetic ν extension from $h_2^2x_{76,6}$ to $\lambda^2Ph_1x_{76,6}$. Shuffle to obtain

$$\nu^2\gamma = \langle \eta, \nu, \eta\rangle\gamma = \eta\langle \nu, \eta, \gamma\rangle.$$

Therefore, the right side is also detected by $\lambda^2Ph_1x_{76,6}$, and the Toda bracket must be detected by $\lambda^2Px_{76,6}$. □

Proposition 3.28 (84, 10, 10) *There is no hidden $\widetilde{2}$ extension on $Px_{76,6}$.*

Proof The only possible targets for a hidden $\widetilde{2}$ extension are $\lambda^3\Delta^2t$ and $\lambda^4M\Delta h_1d_0$. The first possibility is ruled out by [14] because the motivic weights are incompatible.

If there were a hidden $\widetilde{2}$ extension from $Px_{76,6}$ to $\lambda^4M\Delta h_1d_0$, then there would also be a hidden $\widetilde{2}$ extension from $\lambda^2Px_{76,6}$ to $\lambda^6M\Delta h_1d_0$. The possible differentials hitting $\lambda^8M\Delta h_1d_0$ and $\lambda^9M\Delta h_1d_0$ do not affect this argument.

Therefore, it suffices to show that there is no hidden $\widetilde{2}$ extension on $\lambda^2 Px_{76,6}$. As in Lemma 3.26, let γ be an element of $\pi_{79,7}$ that is detected by $h_2x_{76,6}$ such that $\eta\gamma$ is zero, so $\lambda^2 Px_{76,6}$ detects $\langle \nu, \eta, \gamma \rangle$.

Consider the relations

$$\langle \nu, \eta, \gamma \rangle \widetilde{2} \subseteq \langle \nu, \eta, \widetilde{2}\gamma \rangle \supseteq \langle \nu, \eta, \widetilde{2} \rangle \gamma.$$

The last expression is zero because $\langle \nu, \eta, \widetilde{2} \rangle$ is zero in $\pi_{5,2}$. Consequently, $\langle \nu, \eta, \gamma \rangle \widetilde{2}$ lies in the indeterminacy of the middle expression, which consists entirely of multiples of ν.

It remains to show that $\lambda^6 M\Delta h_1 d_0$ cannot detect a multiple of ν. This follows from inspection and Proposition 3.21. □

Proposition 3.29 (93, 13, 13) *The element $P^2h_6d_0$ is a permanent cycle.*

Proof We begin by computing $\langle \widetilde{2}, \lambda\theta_5, \{\lambda^3 P^2 d_0\} \rangle$ in $S^{0,0}/\lambda^5$. There are no crossing differentials since the differential $d_6(h_0^3 \cdot \Delta h_2^2 h_6) = \lambda^5 M\Delta h_2^2 e_0$ does not occur in $S^{0,0}/\lambda^5$. Compute the corresponding Massey product in the E_3-page using the differential $d_2(h_6) = \lambda h_0 h_5^2$. We obtain $\lambda^3 P^2 h_6 d_0$, with no indeterminacy in the E_3-page. Therefore, the Moss Convergence Theorem implies that the Toda bracket is detected by $\lambda^3 P^2 h_6 d_0$ in $S^{0,0}/\lambda^5$.

On the other hand, $\{\lambda^3 P^2 d_0\}$ equals zero since $d_4(d_0e_0 + h_0^7 h_5) = \lambda^3 P^2 d_0$. Therefore, the bracket in $S^{0,0}/\lambda^5$ consists entirely of multiples of $\widetilde{2}$. In particular, $\lambda^3 P^2 h_6 d_0$ detects a $\widetilde{2}$ multiple in $S^{0,0}/\lambda^5$. The only possibility is that there is a hidden $\widetilde{2}$ extension from $\lambda^2 h_0^4 \cdot \Delta h_2^2 h_6$ to $\lambda^3 P^2 h_6 d_0$ in $S^{0,0}/\lambda^5$.

Pull back this hidden h_0 extension along the map $S^{0,0} \to S^{0,0}/\lambda^5$. We conclude that $\lambda^2 h_0^4 \cdot \Delta h_2^2 h_6$ supports a hidden $\widetilde{2}$ extension in $S^{0,0}$. By compatibility with the extension in $S^{0,0}/\lambda^5$, the element $\lambda^3 P^2 h_6 d_0$ is the only possible value for the extension in $S^{0,0}$. In particular, $\lambda^3 P^2 h_6 d_0$ survives. In turn, this implies that $P^2 h_6 d_0$ survives. □

Remark 3.30 The proof of Proposition 3.29 shows that there is a motivic (and also classical) 2 extension from $h_0^4 \cdot \Delta h_2^2 h_6$ to $P^2 h_6 d_0$. Comparison to $S^{0,0}/\tau$ only gives that the target of the 2 extension is $P^2 h_6 d_0$ modulo the possible error term $\tau^2 M^2 h_2$.

Proposition 3.31 (93, 13, 13) $d_6(tQ_2) = \lambda^5 MP\Delta h_1 d_0$.

Proof In the motivic Adams spectral sequence, the element $MP\Delta h_1 d_0$ is hit by a differential [14, Table 9].

Adapting this argument to the synthetic context, we find that $\eta\kappa\gamma\theta_{4.5}$ is detected by $\lambda^5 MP\Delta h_1 d_0$, for some γ in $\pi_{32,6}$ that is detected by $\Delta h_1 h_3$.

The element $\lambda\eta\kappa\theta_{4.5}$ is zero because all elements of $\pi_{60,7}$ that are detected in sufficiently high filtration are also detected by tmf. (Note that $\eta\kappa\theta_{4.5}$ in $\pi_{60,8}$ is nonzero and detected by $\lambda^3 Mh_1 d_0$, but this is irrelevant.)

We now know that $\lambda\eta\kappa\gamma\theta_{4.5}$ is zero and is detected by $\lambda^6 MP\Delta h_1 d_0$. Therefore, $\lambda^6 M\Delta h_1 d_0$ must be hit by a differential. Proposition 3.29 eliminates $P^2 h_6 d_0$, $P^2 h_0 h_6 d_0$, and $P^2 h_0^2 h_6 d_0$ as possible sources for this differential. The elements $M^2 h_2$ and $\Delta^2 h_1 g_2$ are also eliminated because they are products of permanent cycles. The only remaining possibility is that $d_6(tQ_2)$ equals $\lambda^5 MP\Delta h_1 d_0$. □

Acknowledgements The first author was supported by NSF Grant DMS-2202992, by the DNRF through the Copenhagen center for Geometry and Topology (DNRF151), and by Villum Fonden through the Young Investigator Program. The second author was supported by NSF Grant DMS-1904241. The third author was partially supported by NSF Grant DMS-2105462.

Funding Open access funding provided by Copenhagen University.

Declarations

Conflict of interest Zhouli Xu is one of the associate editors of Peking Mathematical Journal and was not involved in the peer-review process of this paper. All authors declare that there are no competing interests.

References

1. Adams, J.F.: On the non-existence of elements of Hopf invariant one. Ann. of Math. (2) **72**, 20–104 (1960)
2. Baer, J.F., Johnson, M., Marek, P.: Stable comodule deformations and the synthetic Adams–Novikov spectral sequence. arXiv:2402.14274 (2024)
3. Barratt, M.G., Jones, J.D.S., Mahowald, M.E.: Relations amongst Toda brackets and the Kervaire invariant in dimension 62. J. London Math. Soc. (2) **30**(3), 533–550 (1984)
4. Belmont, E., Kong, H.J.: A Toda bracket convergence theorem for multiplicative spectral sequences. arXiv:2112.08689 (2021)
5. Burklund, R.: An extension in the Adams spectral sequence in dimension 54. Bull. Lond. Math. Soc. **53**(2), 404–407 (2021)
6. Burklund, R., Hahn, J., Senger, A.: On the boundaries of highly connected, almost closed manifolds. Acta Math. **231**(2), 205–344 (2023)
7. Burklund, R., Xu, Z.: The Adams differentials on the classes h_j^3. Invent. Math. **239**(1), 1–77 (2025)
8. Gheorghe, B., Isaksen, D.C., Krause, A., Ricka, N.: $\mathbb{C}$-motivic modular forms. J. Eur. Math. Soc. (JEMS) **24**(10), 3597–3628 (2022)
9. Gheorghe, B., Wang, G., Xu, Z.: The special fiber of the motivic deformation of the stable homotopy category is algebraic. Acta Math. **226**(2), 319–407 (2021)
10. Hopkins, M.J., Lin, J., Shi, X.D., Xu, Z.: Intersection forms of spin 4-manifolds and the Pin(2)-equivariant Mahowald invariant. Comm. Am. Math. Soc. **2**, 22–132 (2022)
11. Isaksen, D.C.: Stable stems. Mem. Am. Math. Soc. **262**(1269), viii+159 pp. (2019)
12. Isaksen, D.C., Wang, G., Xu, Z.: Classical and $\mathbb{C}$-motivic Adams charts. https://cpb-us-e1.wpmucdn.com/s.wayne.edu/dist/0/60/files/2020/01/Adamscharts.pdf (2020)
13. Isaksen, D.C., Wang, G., Xu, Z.: Stable homotopy groups of spheres. Proc. Natl. Acad. Sci. U.S.A. **117**(40), 24757–24763 (2020)
14. Isaksen, D.C., Wang, G., Xu, Z.: Stable homotopy groups of spheres: from dimension 0 to 90. Publ. Math. Inst. Hautes Études Sci. **137**, 107–243 (2023)
15. Lin, W., Wang, G., Xu, Z.: Machine proofs for Adams differentials and extension problems among CW spectra. arXiv:2412.10876 (2024)
16. Marek, P.: $H\mathbb{F}_2$-synthetic homotopy groups of topological modular forms. arXiv:2202.11305 (2022)
17. Moss, R.M.F.: Secondary compositions and the Adams spectral sequence. Math. Z. **115**, 283–310 (1970)

18. Pstragowski, P.: Synthetic spectra and the cellular motivic category. Invent. Math. **232**(2), 553–681 (2023)
19. Serre, J.-P.: Groupes d'homotopie et classes de groupes abéliens. Ann. of Math. (2) **58**, 258–294 (1953)
20. Wang, G., Xu, Z.: The triviality of the 61-stem in the stable homotopy groups of spheres. Ann. of Math. (2) **186**(2), 501–580 (2017)
21. Xu, Z.: The strong Kervaire invariant problem in dimension 62. Geom. Topol. **20**(3), 1611–1624 (2016)