MODEL CATEGORIES IN ALGEBRAIC TOPOLOGY

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Dedicated to Prof. Heinrich Kleisli on the occasion of his 70th birthday

Abstract. This survey of model categories and their applications in algebraic topology is intended as an introduction for non homotopy theorists, in particular category theorists and categorical topologists. We begin by defining model categories and the homotopy-like equivalence relation on their morphisms. We then explore the question of compatibility between monoidal and model structures on a category. We conclude with a presentation of the Sullivan minimal model of rational homotopy theory, including its application to the study of Lusternik-Schnirelmann category.

Introduction

Model category theory, first developed in the late 1960's by Quillen [17], has become very popular among algebraic topologists in the past five years. A model category is a category endowed with three distinguished classes of morphisms, called fibrations, cofibrations and weak equivalences, satisfying axioms that are properties of the topological category and its usual fibrations, cofibrations and homotopy equivalences. In any model category there is a notion of homotopy of morphisms, based upon the definiton of homotopy of continuous maps.

The primary source of topologists's current interest in model categories is probably their application to formalizing the underlying structure of stable homotopy theory. For more than 30 years the framework of stable homotopy theory was Boardman's stable category or one of its variants, due to Adams or to Lewis and May. The stable homotopy category, which is a closed, symmetric, monoidal category, is in some ways a topological version of the derived category obtained from the category of chain complexes by inverting all quasi-isomorphisms. The analogy is not perfect, however, as the "tensor product" defined on the category underlying the stable homotopy category is neither associative nor commutative. Thus, when topologists applied algebraic methods to spectra, products and actions were specified only up to homotopy, which often led to highly delicate computations.

Only in the past five years or so have topologists discovered model categories with appropriately compatible monoidal structure such that the associated homotopy category is equivalent as a monoidal category to the stable homotopy category. Examples of such monoidal model categories include the category of S-modules of

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Elmendorf, Kriz, Mandell and May [5], the category of symmetric spectra of Hovey, Shipley and Smith [13] and the category of Γ-spaces of Lydakis [16].

Model categories have also been essential to the development of algebraic homotopy theory. Given a topological problem to solve, such as the problem of lifting, extending or factoring a given continuous map, the algebraic homotopy theorist first chooses an appropriate algebraic model, i.e., an "algebraic" category $\mathcal{C}$ endowed with a reasonable notion of "homotopy" of morphisms, together with a functor $F : \mathcal{TOP} \longrightarrow \mathcal{C}$, that preserves the homotopy relation. He then translates the topological problem via the functor F into algebraic terms and studies the resulting algebraic problem, with the aim of obtaining information about the topological situation.

The algebraic homotopy theorist chooses $\mathcal{C}$ and F based on "economic" considerations, endeavoring to acheive equilibrium between the information lost in translation and the ease of computation. He will therefore select $\mathcal{C}$ and F in function of the structure to be examined and the requisite depth of detail.

The methods of algebraic homotopy theory have proved quite fruitful, notably in rational homotopy theory, as we indicate in the final section of this article.

The aim of this article, which is based on the author's lectures at the CATOP2000 conference in honor of Prof. Kleisli, is to introduce model categories and their applications in algebraic topology to non homotopy theorists, in particular to category theorists and to categorical topologists. We have chosen therefore to omit most proofs, focussing instead on examples and applications. The orientation and scope of this article is thus rather different from that of the survey article [4] of Dwyer and Spalinski, which the author highly recommends as complementary reading.

The first section of the article consists in a very concise refresher course in the homotopy theory of topological spaces. Model categories and the definition of a homotopy-like equivalence relation therein are the subject of Section 2. In Section 3 we examine conditions of compatibility between model category structure and monoidal structure. We conclude in Section 4 with a brief presentation of one of the best-known algebraic models, the Sullivan model of rational homotopy theory, including an example of a topological problem to which it was successfully applied.

1. The homotopy theory of topological spaces

Given two topological spaces A and X, the homotopy equivalence relation on the set $\mathcal{TOP}(A, X)$ of continuous maps from A to X can be defined in two equivalent ways: in terms of the cylinder on A or in terms of the path space on X.

Let I denote the interval $[0, 1]$. The *cylinder* on A is the product space $A \times I$, which fits into the following diagram

$$
\begin{tikzcd}
A \arrow[r, hook, "i_0"] \arrow[dr, "1_A"'] & A \times I \arrow[d, "\pi"] & A \arrow[l, hook', "i_1"'] \arrow[dl, "1_A"] \\
& A &
\end{tikzcd}
$$

where $i_t(a) = (a, t)$ and $\pi(a, t) = a$. Two continuous maps $f, g : A \longrightarrow X$ are *left homotopic* if there is a continuous map $H : A \times I \longrightarrow X$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & A \times I \arrow[d, "H"] & A \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

The *path space* on X, denoted X^I, is the space $\{\lambda : I \longrightarrow X \mid \lambda \text{ continuous}\}$ endowed with the compact-open topology. The path space fits into the following commutative diagram

$$\begin{tikzcd} X & X^I \arrow[l, two heads, "p_0"'] \arrow[r, two heads, "p_1"] & X \\ & X \arrow[ul, "1_X"] \arrow[u, "e"] \arrow[ur, "1_X"'] & \end{tikzcd}$$

where $p_t(\lambda) = \lambda(t)$ and $e(y)$ is the constant path at y. Two continuous maps $f, g : A \longrightarrow X$ are *right homotopic* if there is a continous map $K : A \longrightarrow X^I$ such that the following diagram commutes.

$$\begin{tikzcd} X & X^I \arrow[l, two heads, "p_0"'] \arrow[r, two heads, "p_1"] & X \\ & A \arrow[ul, "f"] \arrow[u, "K"] \arrow[ur, "g"'] & \end{tikzcd}$$

It is not difficult to see that both right and left homotopy define equivalence relations on $\mathcal{TOP}(A, X)$, and that their sets of equivalence classes are the same. If f and g are (right or left) homotopic, we write $f \simeq g$.

The research of homotopy theorists often consists of seeking the solution to topological problems *up to homotopy*, i.e., it is the homotopy class of a continuous map, rather than the map itself, that is important.

There are three distinguished classes of continuous maps that together determine the homotopy theory of topological spaces: the classes of homotopy equivalences, fibrations and cofibrations. A map $f : A \longrightarrow X$ is a *homotopy equivalence* if there exists a map $g : X \longrightarrow A$ such that $gf \simeq 1_A$ and $fg \simeq 1_X$. A (Hurewicz) *fibration* is a continuous map $p : E \longrightarrow B$ that has the *homotopy lifting property*, i.e., given any commutative diagram of continuous maps

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, hook, "i_0"'] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "H"] \arrow[ur, dotted, "\widehat{H}"] & B \end{tikzcd}$$

there is a continuous map $\widehat{H} : Y \times I \longrightarrow E$ such that $\widehat{H} i_0 = h$ and $p\widehat{H} = H$. The homotopy $\widehat{H}$ thus lifts H through p and extends h over i_0. An inclusion of a closed subspace $i : A \hookrightarrow X$ is a (closed, Hurewicz) *cofibration* if i has the homotopy extension property, i.e., given any commutative diagram of continuous maps

$$\begin{tikzcd} A \arrow[r, "K"] \arrow[d, "i"'] & Y^I \arrow[d, two heads, "p_0"] \\ X \arrow[r, "k"] \arrow[ur, dotted, "\widehat{K}"] & Y \end{tikzcd}$$

there is a continuous map $\widehat{K} : X \longrightarrow Y^I$ such that $p_0\widehat{K} = k$ and $\widehat{K}i = K$. The homotopy $\widehat{K}$ thus extends K over i and lifts k through p_0.

The axioms of a model category, stated in Section 2, codify the properties of these three classes that are essential to the definition of a reasonable homotopy-like equivalence relation in an abstract category.

2. Model categories

In this section we first present the precise definition of model categories, as well as a few of their elementary properties. We then explain how to define a homotopy relation on the sets of morphisms $\mathcal{C}(A, X)$ in a model category $\mathcal{C}$, at least for A and X satisfying additional, usually mild hypotheses. Given the definition of the homotopy relation, we construct the homotopy category of a model category $\mathcal{C}$, which is a localization of $\mathcal{C}$ with respect to a certain distinguished class of morphisms, then provide conditions under which a functor between model categories induces an equivalence on their homotopy categories.

Cofibrantly generated model categories, which are model categories in which the entire model structure can be generated in a natural manner by two distinguished classes of maps, are the next topic of this section. Though their definition is somewhat technical, it is often much easier to prove theorems about cofibrantly generated model categories than about general model categories. Fortunately, many familiar model categories are cofibrantly generated. The notion of cofibrant generation is essential to understanding the compatibility between model and monoidal structures in Section 3.

We conclude this section with several examples of model categories.

We refer the reader to either the survey article [4] of Dwyer and Spalinski or the monograph [11] of Hovey for further details and missing proofs. In particular, our presentation of cofibrantly generated categories is based on that in [11].

2.1 Definition and elementary properties of model categories.

Definition. Let $\mathcal{I}$ be a subset of $Mor\ \mathcal{C}$. A morphism $f : A \longrightarrow B$ in $\mathcal{C}$ satisfies the *left lifting property* with respect to $\mathcal{I}$, denoted $f \in LLP(\mathcal{I})$, if for every commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "k"'] \arrow[ur, dotted, "\hat{k}"] & D \end{tikzcd}$$

of morphisms in $\mathcal{C}$ with $g \in I$, there exists a morphism $\hat{k} : B \longrightarrow C$ such that $g\hat{k} = k$ and $\hat{k}f = h$.

Dually, we say that f has the *right lifting property* with respect to $\mathcal{I}$, denoted $f \in RLP(\mathcal{I})$, if for every commutative diagram

$$\begin{tikzcd} C \arrow[r, "h"] \arrow[d, "g"'] & A \arrow[d, "f"] \\ D \arrow[r, "k"'] \arrow[ur, dotted, "\hat{k}"] & B \end{tikzcd}$$

of morphisms in $\mathcal{C}$ with $g \in I$, there exists a morphism $\hat{k} : D \longrightarrow A$ such that $f\hat{k} = k$ and $\hat{k}g = h$.

Recall that a morphism f in a category $\mathcal{C}$ is a *retract* of a morphism g if there is a commutative diagram of morphisms in $\mathcal{C}$

$$\begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d, "f"] & \bullet \arrow[r, "r"] \arrow[d, "g"] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "j"] & \bullet \arrow[r, "s"] & \bullet \end{tikzcd}$$

such that ri and sj are identity morphisms.

Definition. A *model category* consists of a category $\mathcal{C}$, together with classes of morphisms $WE, Fib, Cof \subseteq Mor\ \mathcal{C}$ that are closed under composition and contain all identities, such that the following axioms are satisfied.

(M1) All finite limits and colimits exist.
(M2) Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be morphisms in $\mathcal{C}$. If two of f, g, and gf are in WE, then so is the third.
(M3) If f is a retract of g and g belongs to WE (respectively Fib, respectively Cof), then f also belongs to WE (respectively Fib, respectively Cof).
(M4) $Cof \subseteq LLP(Fib \cap WE)$ and $Fib \subseteq RLP(Cof \cap WE)$.
(M5) If $f \in Mor\ \mathcal{C}$, then there exist
 (a) $i \in Cof$ and $p \in Fib \cap WE$ such that $f = pi$;
 (b) $j \in Cof \cap WE$ and $q \in Fib$ such that $f = qj$.

By analogy with the homotopy structure in the category of topological spaces, the morphisms belonging to the classes WE, Fib and Cof are called *weak equivalences*, *fibrations*, and *cofibrations* and are denoted by decorated arrows $\xrightarrow{\sim}$, $\twoheadrightarrow$, and $\rightarrowtail$. The elements of the classes $Fib \cap WE$ and $Cof \cap WE$ are called, respectively, *acyclic fibrations* and *acyclic cofibrations*. Since WE, Fib and Cof are all closed under composition and contain all isomorphisms, we can and sometimes do view them as subcategories of $\mathcal{C}$, rather than simply as classes of morphisms.

Axiom (M1) implies that any model category has an initial object ϕ and a terminal object e. An object A in a model category is *cofibrant* if the unique morphism $\phi \longrightarrow A$ is a cofibration. Similarly, A is *fibrant* if the unique morphism $A \longrightarrow e$ is a fibration.

Since the axioms of a model category imply that the classes Fib and WE determine Cof, while the classes Cof and WE determine Fib, it is clear that the above set of axioms is not minimal. One definite esthetic and practical advantage to this choice of axioms, however, is the symmetry they express between cofibrations and fibrations, which is the basis of Eckmann-Hilton duality in homotopy theory.

There are numerous variations on the model category theme, different ways of endowing a category with additional structure enabling one to define a homotopy-like equivalence relation. The cofibration categories of Baues [1], in which there are only two distinguished class of morphisms, weak equivalences and cofibrations, are probably among the best known and most widely applied of these variations. If $\mathcal{C}$ is a model category in the sense of this paper, then its full subcategory consisting of cofibrant objects, together with the corresponding subclasses of weak equivalences and cofibrations, is a cofibration category. We recommend the article [3] of Doeraene to the reader interested in a comparative study of different types of categorical structure leading to a reasonable definition of a homotopy-like equivalence relation.

The three elementary but useful properties of model categories stated in the proposition below are easy consequences of the axioms of a model category.

Proposition 2.1.1. *Let $(\mathcal{C}, WE, Fib, Cof)$ be a model category.*

(1) $Cof = LLP(Fib \cap WE)$ *and* $Fib = RLP(Cof \cap WE)$.
(2) Cof *and* $Cof \cap WE$ *are preserved under push-out.*
(3) Fib *and* $Fib \cap WE$ *are preserved under pull-back.*

2.2 The homotopy relation in a model category.

Motivated by the definition of the homotopy relation in $\mathcal{TOP}$, we obtain the following two possible definitions of homotopy of morphisms in an arbitrary model category $\mathcal{C}$. Unless A and X are chosen according the criteria we establish below, the two definitions are not necessarily equivalent and do not necessarily determine equivalence relations on $\mathcal{C}(A, X)$.

Throughout this section $(\mathcal{C}, WE, Fib, Cof)$ denotes a model category.

Definition. Given $A \in Ob\ \mathcal{C}$, consider the push-out of the morphism $\phi \longrightarrow A$ with itself

$$\begin{tikzcd} \phi \arrow[r] \arrow[d] & A \arrow[d, "j_0"] \arrow[ddr, bend left, "1_A"] & \\ A \arrow[r, "j_1"] \arrow[drr, bend right, "1_A"'] & A \vee A \arrow[dr, "\nabla"] & \\ & & A \end{tikzcd}$$

where $\nabla : A \vee A \longrightarrow A$ denotes the “folding” map, i.e., the morphism induced by the identity on each copy of A. Observe that $A \vee A$ is a coproduct.

A *cylinder* on A consists of a factorization of ∇.

$$\begin{tikzcd} A \vee A \arrow[rr, "\nabla"] \arrow[dr, "i"'] & & A \\ & Cyl(A) \arrow[ur, "p"', "\sim"] & \end{tikzcd}$$

Let $i_0 = ij_0$ and $i_1 = ij_1$.

The cylinder is *good* if $i \in Cof$ and *very good* if, in addition, $p \in Fib \cap WE$.

Let $f, g : A \longrightarrow X$ be morphisms in $\mathcal{C}$. A *left homotopy* from f to g consists of a morphism $H : Cyl(A) \longrightarrow X$ such that the diagram

$$\begin{tikzcd} A \arrow[r, hook, "i_0"] \arrow[dr, "f"'] & Cyl(A) \arrow[d, "H"] & A \arrow[l, hook', "i_1"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

commutes, where $A \vee A \xrightarrow{i} Cyl(A) \xrightarrow{p} A$ is any cylinder on A. We denote the existence of a left homotopy from f to g by $f \underset{\ell}{\sim} g$.

The next proposition lists the elementary properties of left homotopy that are essential to our purposes here.

Proposition 2.2.1.

(1) *If A is a cofibrant object in $\mathcal{C}$, then $\underset{\ell}{\sim}$ is an equivalence relation on $\mathcal{C}(A, X)$ for all objects X. The quotient set of left homotopy equivalence classes is denoted $\pi^{\ell}(A, X)$.*

(2) *If A is cofibrant and $p : Y \overset{\sim}{\twoheadrightarrow} X$, then p induces an isomorphism*

$$p_* : \pi^{\ell}(A, Y) \longrightarrow \pi^{\ell}(A, X).$$

(3) *If X is fibrant, then*

$$f \underset{\ell}{\sim} g : B \longrightarrow X,\ h \underset{\ell}{\sim} k : A \longrightarrow B \Longrightarrow fh \underset{\ell}{\sim} gk : A \longrightarrow X.$$

The definition of path objects and right homotopy is dual (in the Eckmann-Hilton sense) to the definition of cylinders and left homotopy.

Definition. Given $X \in Ob\ \mathcal{C}$, consider the pull-back of the morphism $X \longrightarrow e$ with itself

$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, bend left, "1_X"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times X \arrow[r, "p_1"] \arrow[d, "p_0"] & X \arrow[d] \\
& X \arrow[r] & e
\end{tikzcd}$$

where $\Delta : X \longrightarrow X \times X$ denotes the "diagonal" map, i.e., the morphism induced by the identity into each copy of X. Observe that $X \times X$ is a product.

A *path object* on X consists of a factorization of Δ.

$$\begin{tikzcd}
X \arrow[rr, "\Delta"] \arrow[dr, "\sim", "j"'] & & X \times X \\
& PX \arrow[ur, "q"'] &
\end{tikzcd}$$

Let $q_0 = p_0 q$ and $q_1 = p_1 q$.

The path object is *good* if $p \in Fib$ and *very good* if, in addition, $j \in Cof \cap WE$.

Let $f, g : A \longrightarrow X$ be morphisms in $\mathcal{C}$. A *right homotopy* from f to g consists of a morphism $K : A \longrightarrow PX$ such that the diagram

$$\begin{tikzcd}
X & PX \arrow[l, two heads, "p_0"'] \arrow[r, two heads, "p_1"] & X \\
& A \arrow[ul, "f"] \arrow[u, "K"] \arrow[ur, "g"'] &
\end{tikzcd}$$

commutes, where $X \xrightarrow{j} PX \xrightarrow{q} X \times X$ is any path object on X. We denote the existence of a right homotopy from f to g by $f \underset{r}{\sim} g$.

The elementary properties of right homotopy and their proofs are strictly dual to those for left homotopy.

Proposition 2.2.2.

(1) *If X is a fibrant object in $\mathcal{C}$, then $\underset{r}{\sim}$ is an equivalence relation on $\mathcal{C}(A,X)$ for all objects A. The quotient set of right homotopy equivalence classes is denoted $\pi^r(A,X)$.*

(2) *If X is fibrant and $i : A \overset{\sim}{\rightarrowtail} B$, then i induces an isomorphism*

$$i^* : \pi^r(B,X) \longrightarrow \pi^r(A,X).$$

(3) *If A is cofibrant, then*

$$f \underset{r}{\sim} g : A \longrightarrow X,\ h \underset{r}{\sim} k : X \longrightarrow Y \Longrightarrow hf \underset{r}{\sim} kg : A \longrightarrow Y.$$

We are now prepared to study the relationship between right and left homotopy.

Lemma 2.2.3. *Let $f, g : A \longrightarrow X$ be morphisms in $\mathcal{C}$.*

(1) *If A is cofibrant and $f \underset{\ell}{\sim} g$, then $f \underset{r}{\sim} g$.*
(2) *If X is fibrant and $f \underset{r}{\sim} g$, the $f \underset{\ell}{\sim} g$.*

The following key definition is an easy corollary of the preceding lemma and Propositions 2.2.1(1) and 2.2.2(1).

Definition/Corollary 2.2.4. Suppose that A is a cofibrant object and X is a fibrant object in a model category $(\mathcal{C}, WE, Fib, Cof)$. There is an equivalence relation $\sim$ on $\mathcal{C}(A,X)$ such that $f \sim g$ if and only if $f \underset{\ell}{\sim} g$ or, equivalently, if and only if $f \underset{r}{\sim} g$. When $f \sim g$, we say that f and g are *homotopic*.
The set of homotopy classes of morphisms from A to X is denoted $\pi(A,X)$.
If A and X are both fibrant and cofibrant, then a morphism $f : A \longrightarrow X$ is a *homotopy equivalence* if there is a morphism $g : X \longrightarrow A$ such that $gf \sim 1_A$ and $fg \sim 1_X$.

The property of the homotopy relation stated in the next proposition is an easy but important consequence of Propositions 2.2.1(3) and 2.2.2(3).

Proposition 2.2.5. *If A is cofibrant, X is fibrant and cofibrant, and Y is fibrant, then*

$$f \sim g : A \longrightarrow X, h \sim k : X \longrightarrow Y \Longrightarrow hf \sim kg : A \longrightarrow Y.$$

Thus, in particular, composition preserves the homotopy relation on morphisms such that the source and target are both fibrant and cofibrant.

The next proposition, when applied to the second model category structure given for $\mathcal{TOP}$ (Example 2.5.1), yields the famous Whitehead Theorem, which states that a weak homotopy equivalence between CW-complexes is actually a homotopy equivalence.

Proposition 2.2.6. *Suppose that A and X are objects that are both fibrant and cofibrant in a model category $(\mathcal{C}, WE, Fib, Cof)$. If $f : A \longrightarrow X$ is a morphism in $\mathcal{C}$, then f is a weak equivalence if and only if it is a homotopy equivalence.*

2.3 The homotopy category of a model category.

The next step in our development of the theory of model categories is the construction of the homotopy category of a model category $(\mathcal{C}, WE, Fib, Cof)$. We begin by supposing that we have fixed for each object A a *cofibrant model* $\phi \rightarrowtail QA \xrightarrow{\sim} A$ and a *fibrant model* $A \overset{\sim}{\rightarrowtail} RA \twoheadrightarrow e$, where we require that $QA = A$ if A is cofibrant and that $RA = A$ if A is fibrant.

Definition. The *homotopy category* $Ho\,\mathcal{C}$ of a model category $(\mathcal{C}, WE, Fib, Cof)$ is the category with $Ob\ Ho\,\mathcal{C} = Ob\,\mathcal{C}$ and $Ho\,\mathcal{C}(A, X) = \pi(RQA, RQX)$.

Observe that Proposition 2.2.5 implies that the composition in $Ho\ \mathcal{C}$ is well defined.

There is a natural quotient functor $\gamma : \mathcal{C} \longrightarrow Ho\,\mathcal{C}$ that is the identity on objects and that is defined on morphisms in two stages, as follows. Let $f : A \longrightarrow X$ be any morphism in $\mathcal{C}$. First consider the commutative diagram

$$\begin{tikzcd}
\phi \arrow[rr, tail] \arrow[d, tail] & & QX \arrow[d, two heads, "\sim"', "p_X"] \\
QA \arrow[r, two heads, "\sim", "p_A"'] \arrow[urr, dotted, "Qf"] & A \arrow[r, "f"'] & X
\end{tikzcd}$$

By axiom (M4) we can lift fp_A through p_X to a morphism $Qf : QA \longrightarrow QX$. Furthermore, Proposition 2.2.1(2) implies that all such lifts are left homotopic, since QA is cofibrant. Then, by Lemma 2.2.3(1), they are all right homotopic as well, also because QA is cofibrant.

Next consider the commutative diagram

$$\begin{tikzcd}
QA \arrow[r, "Qf"] \arrow[d, tail, "j_{QA}"', "\sim"] & QX \arrow[r, tail, "j_{QX}", "\sim"'] & RQX \arrow[d, two heads] \\
RQA \arrow[rr, two heads] \arrow[urr, dotted, "RQf"'] & & e
\end{tikzcd}$$

Applying axiom (M4) again, we obtain a extension RQf of $j_{QX}Qf$ over j_{QA}, which is unique up to homotopy for reasons strictly dual to those applicable to the construction of Qf.

The functor γ provides us with an alternate way of characterizing the homotopy category. We first motivate this characterization by the following observation.

Proposition 2.3.1. *Let f be a morphism in a model category $(\mathcal{C}, WE, Fib, Cof)$. Then $\gamma(f)$ is an isomorphism if and only if f is a weak equivalence.*

Recall that a *localization* of a category $\mathcal{C}$ with respect to a class of morphisms $S \subseteq Mor\ \mathcal{C}$ consists of a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ such that

(1) $Ob\mathcal{C} = Ob\mathcal{D}$, and F is the identity on objects;
(2) $F(s)$ is an isomorphism in $\mathcal{D}$ for all $s \in S$; and
(3) if $G : \mathcal{C} \longrightarrow \mathcal{E}$ is any other functor such that $G(s)$ is an isomorphism for all $s \in S$, then there is a unique functor $\widehat{G} : \mathcal{D} \longrightarrow \mathcal{E}$ such that $\widehat{G}F = G$.

Theorem 2.3.2. *The functor $\gamma : \mathcal{C} \longrightarrow Ho\,\mathcal{C}$ is a localization of $\mathcal{C}$ with respect to the class WE.*

Thus the homotopy category of a model category depends only on its class of weak equivalences.

We next explore the question of when a functor between two model categories respects the homotopy relation, i.e., when it induces a functor on homotopy categories. It is particularly interesting to determine when such a functor induces an equivalence of homotopy categories.

Definition. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor from a model category $\mathcal{C}$ to any category $\mathcal{D}$. A *left derived functor* of F consists of a functor $\mathbf{L}F : Ho\,\mathcal{C} \longrightarrow \mathcal{D}$ together with a natural transformation $t : \mathbf{L}F \circ \gamma \longrightarrow F$ such that for each pair $(G : Ho\,\mathcal{C} \longrightarrow \mathcal{D},\ s : G \circ \gamma \longrightarrow F)$ there exists a unique natural transformation $\hat{s} : G \longrightarrow \mathbf{L}F$ such that $t\hat{s}\gamma = s$.

Dually, a *right derived functor* of F consists of a functor $\mathbf{R}F : Ho\,\mathcal{C} \longrightarrow \mathcal{D}$ together with a natural transformation $t : F \longrightarrow \mathbf{R}F \circ \gamma$ such that for each pair $(G : Ho\,\mathcal{C} \longrightarrow \mathcal{D},\ s : F \longrightarrow G \circ \gamma)$ there exists a unique natural transformation $\hat{s} : \mathbf{R}F \longrightarrow G$ such that $\hat{s}\gamma t = s$.

For example, if F sends all weak equivalences to isomorphisms in $\mathcal{D}$, then the definition of localization implies that there is a unique $\widehat{F} : Ho\,\mathcal{C} \longrightarrow \mathcal{D}$ such that $\widehat{F}\gamma = F$. Thus, in this case, $\mathbf{L}F = \widehat{F} = \mathbf{R}F$.

More generally, if F sends all weak equivalences with cofibrant source and target to isomorphisms, then the left derived functor of F exists. It is defined by $\mathbf{L}F(X) = F(QX)$ and $\mathbf{L}F(f) = F(Qf)$, where $QX \xrightarrow{\sim} X$ is the fixed cofibrant model and Qf is constructed as above. There is a similar construction in the dual case.

We now consider the case in which $\mathcal{C}$ and $\mathcal{D}$ are both model categories.

Definition. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor between model categories. The *left total derived functor* of F, denoted $\mathbb{L}F : Ho\,\mathcal{C} \longrightarrow Ho\,\mathcal{D}$, is the left derived functor of the composition $\gamma_{\mathcal{D}}F$. Dually the *right total derived functor* of F, denoted $\mathbb{R}F : Ho\,\mathcal{C} \longrightarrow Ho\,\mathcal{D}$ is the right derived functor of the composition $\gamma_{\mathcal{D}}F$.

Proving the existence of a pair of adjoint functors between two categories is a first step towards establishing an equivalence, hence the interest of the next definition.

Definition. Let $(\mathcal{C}, WE_{\mathcal{C}}, Fib_{\mathcal{C}}, Cof_{\mathcal{C}})$ and $(\mathcal{D}, WE_{\mathcal{D}}, Fib_{\mathcal{D}}, Cof_{\mathcal{D}})$ be model categories. A pair of adjoint functors $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a *Quillen pair* if $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}$ and $G(Fib_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}}$.

It is easy to see that (F, G) is a Quillen pair if and only if $G(Fib_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}}$ and $G(Fib_{\mathcal{D}} \cap WE_{\mathcal{D}}) \subseteq Fib_{\mathcal{C}} \cap WE_{\mathcal{C}}$, which is in turn equivalent to $F(Cof_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}}$ and $F(Cof_{\mathcal{C}} \cap WE_{\mathcal{C}}) \subseteq Cof_{\mathcal{D}} \cap WE_{\mathcal{D}}$. The proof of these equivalences applies the adjointness of F and G to the dual definitions of Fib and Cof in terms of lifting properties.

A simple example of a Quillen pair is the adjoint pair $W : \mathcal{C} \times \mathcal{C} \rightleftarrows \mathcal{C} : \Delta$, where $W(A \times B) = A \vee B$ and $\Delta(A) = (A, A)$. The distinguished classes of the model category $\mathcal{C} \times \mathcal{C}$ are the products of the distinguished classes of $\mathcal{C}$.

Proposition 2.3.3. *A Quillen pair* $F:\mathcal{C}\rightleftarrows\mathcal{D}:G$ *induces an adjoint pair*

$$\mathbb{L}F: Ho\,\mathcal{C}\rightleftarrows Ho\,\mathcal{D}:\mathbb{R}G.$$

Definition. A Quillen pair $F:\mathcal{C}\rightleftarrows\mathcal{D}:G$ is a *Quillen equivalence* if for all $A\in Ob\,\mathcal{C}$, $B\in Ob\,\mathcal{D}$

$$f:FA\longrightarrow B\in WE_{\mathcal{D}}\iff f^{\sharp}:A\longrightarrow GB\in WE_{\mathcal{C}},$$

where $f^{\sharp}$ denotes the adjoint of f.

Proposition 2.3.4. *The adjoint pair* $(\mathbb{L}F,\mathbb{R}G)$ *induced by a Quillen equivalence* (F,G) *is a pair of mutually inverse equivalences.*

2.4 Cofibrantly generated model categories.

In many familiar model categories it is possible to choose relatively small families of cofibrations and acyclic cofibrations that generate the entire model category structure in a natural way. In such categories many theorems about model categories are easier to prove, as it suffices to prove them for the generating families. We are also able in such a category to obtain functorial factorizations of the sort required by axiom (M5).

The definitions and results presented below are essential to understanding Section 3, where we state conditions ensuring compatability between model and monoidal category structures.

We begin with a sequence of rather technical definitions that serve to explain what it means for an object in a category to be *small* with respect to a class of morphisms. The notion of smallness is essential in the definition of cofibrant generation.

Definition. Let $\mathcal{C}$ be any cocomplete category.

(1) Let λ be an ordinal. A λ-*suite* in $\mathcal{C}$ is a functor $X:\lambda\longrightarrow\mathcal{C}$, i.e., a diagram

$$X_0\longrightarrow X_1\longrightarrow X_2\longrightarrow\cdots\longrightarrow X_\beta\longrightarrow\cdots\qquad(\beta<\lambda),$$

such that the induced morphism $colim_{\beta<\gamma}X_\beta\longrightarrow X_\gamma$ is an isomorphism for every limit ordinal γ.

(2) The *composition* of a λ-sequence is the morphism $X_0\longrightarrow colim_{\beta<\lambda}X_\beta$.

(3) Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. A transfinite composition of $\mathcal{D}$-morphisms is the composition in $\mathcal{C}$ of a λ-sequence such that $X_\beta\longrightarrow X_{\beta+1}\in Mor\,\mathcal{D}$ for all $\beta<\lambda$.

(4) Let $\mathcal{D}$ be a subcategory of $\mathcal{C}$. An object A in $\mathcal{C}$ is *small* with respect to $\mathcal{D}$ if there is a cofinal set S of ordinals such that for all $\lambda\in S$ and for all λ-sequences $X:\lambda\longrightarrow\mathcal{D}$, the induced set map

$$colim_{\beta<\lambda}\mathcal{C}(A,X_\beta)\longrightarrow\mathcal{C}(A,colim_{\beta<\lambda}X_\beta)$$

is an isomorphism.

Remarks.

(1) If $(\mathcal{C}, WE, Fib, Cof)$ is a cocomplete model category, it is relatively easy to verify that Cof and $Cof \cap WE$ are closed under transfinite composition.
(2) Definition (4) means in essence that a morphism from A into any sufficiently long composition will factor through some stage of the composition.
(3) Every set is small with respect to the entire category of sets.
(4) Every finite, pointed CW-complex is small in the category of based topological spaces, with respect to the subcategory of pointed CW-complexes and inclusions thereof.

We now introduce the three classes of morphisms naturally associated to a given class in $Mor\ \mathcal{C}$. The notion of "generation" in a cofibrantly generated category refers to creation of these three classes.

Definition. Let $\mathcal{C}$ be any cocomplete category, and let $\mathcal{I} \subseteq Mor\ \mathcal{C}$. The class of morphisms $\mathcal{I}$ gives rise to the following three other classes.

(1) $\mathcal{I} - inj := RLP(\mathcal{I})$
(2) $\mathcal{I} - cof := LLP(\mathcal{I} - inj)$
(3) $\mathcal{I} - cell$ is the class of morphisms $f : A \longrightarrow B$ in $\mathcal{C}$ for which there exist an ordinal λ and a λ-sequence $X : \lambda \longrightarrow \mathcal{C}$ such that $X_0 = A$, each $X_\beta \longrightarrow X_{\beta+1}$ is a push-out of a morphism in $\mathcal{I}$, and the composition $X_0 \longrightarrow colim_{\beta<\lambda} X_\beta$ is isomorphic to f.

Note that $\mathcal{I} - cell \subseteq \mathcal{I} - cof$.

The next theorem, which is one of the most useful tools in model category theory, explains the importance of small objects and of the classes defined above.

Theorem 2.4.1 (The Small Object Argument). *Let $\mathcal{C}$ be a cocomplete category. Suppose that $\mathcal{I} \subseteq \mathcal{C}$ is such that the source of every morphism in $\mathcal{I}$ is small with respect to $\mathcal{I}$. Then there is a functor*

$$(i, p) : Mor\ \mathcal{C} \longrightarrow \mathcal{I} - cell \times \mathcal{I} - inj$$

such that $f = p(f) \circ i(f)$ for all $f \in Mor\ \mathcal{C}$.

The definition below of cofibrantly generated model categories is inspired by the desire to apply the Small Object Argument to the construction of functorial, (M5)-type factorizations of morphisms in a model category.

Definition. A model category $(\mathcal{C}, WE, Fib, Cof)$ is *cofibrantly generated* if there exist two classes $\mathcal{I}, \mathcal{J} \subseteq Mor\ \mathcal{C}$ such that

(1) The source of every morphism in $\mathcal{I}$ is small with respect to Cof, while the source of every morphism in $\mathcal{J}$ is small with respect to $Cof \cap WE$.
(2) $Fib = \mathcal{J} - inj$ and $Fib \cap WE = \mathcal{I} - inj$.

The elements of $\mathcal{I}$ and $\mathcal{J}$ are then called *generating cofibrations* and *generating acyclic cofibrations*, respectively.

Observe that condition (2) of the definition above implies that $Cof = \mathcal{I} - cof$ and $Cof \cap WE = \mathcal{J} - cof$.

It is clear that in a cofibrantly generated model category, one can always apply the Small Object Argument to the classes of generating cofibrations and generating acyclic cofibrations, thereby obtaining functorial factorizations as in axiom (M5). Furthermore, in a cofibrantly generated model category, many results involving conditions on cofibrations can be proved by transfinite induction, via λ-sequences.

2.5 Examples of model categories.

Example 2.5.1. There are two well-known and oft-employed model category structures on the category of topological spaces, $\mathcal{TOP}$, both of which define the same homotopy theory on $\mathcal{TOP}$. On the one hand, Strøm showed in [20] that setting WE to be the class of homotopy equivalences, Fib to be the class of Hurewicz fibrations, and Cof to be the class of closed Hurewicz cofibrations defined a model category structure on $\mathcal{TOP}$. In this case, all objects are both fibrant and cofibrant.

On the other hand, according to Quillen [17], one also obtains a model category structure on $\mathcal{TOP}$ by letting WE be the class of weak homotopy equivalences, Fib the class of Serre fibrations and Cof the class of retracts of inclusions $A \hookrightarrow X$ such that (X, A) is a relative CW-complex. Recall that weak homotopy equivalences are maps inducing an isomorphism on homotopy groups, while Serre fibrations are maps with the right lifting property with respect to the class of continuous maps $\{ i_0 : D^n \longrightarrow D^n \times I \mid n \geq 0\}$, where D^n is the n-dimensional disk. With respect to this structure, all objects are fibrant, while the cofibrant objects are the CW-complexes.

Hovey showed in [11] that the second model category structure on $\mathcal{TOP}$ is cofibrantly generated. The inclusions $S^{n-1} \hookrightarrow D^n$ for $n \geq 1$ are the generating cofibrations, while the inclusions $i_0 : D^n \hookrightarrow D^n \times I$ for $n \geq 0$ are the generating acyclic cofibrations.

When it is important to have a closed, monoidal structure on the category of topological spaces in which he works, the homotopy theorist usually resorts to working in the subcategory $\mathcal{T}$ of *compactly generated spaces*. Recall that a space is compactly generated if it is weak Hausdorff and every compactly open subset is open. As Hovey explained in [11], $\mathcal{T}$ is a cofibrantly generated model category with respect to the same classes of generating cofibrations and acyclic cofibrations as $\mathcal{TOP}$. Furthermore, $\mathcal{T}$ is Quillen equivalent to $\mathcal{TOP}$.

Example 2.5.2. Let $\mathcal{C}$ be a pointed abelian category in which all finite limits and colimits exist. If $WE = Mor\ \mathcal{C}$, Fib is the class of all monomorphisms in $Mor\ \mathcal{C}$, and Cof is the class of all epimorphisms in $Mor\ \mathcal{C}$, then $(\mathcal{C}, WE, Fib, Cof)$ is a model category. The initial and terminal object of $\mathcal{C}$ is both cofibrant and fibrant, and there is no other object that is either fibrant or cofibrant.

This example serves as a reminder that our natural inclination to think of fibrations as projections and cofibrations as injections should be avoided!

Example 2.5.3. Let R be a unitary, commutative ring. A (non-negative) *chain complex* over R consists of a graded left R-module $C_* = \bigoplus_{i\in\mathbb{N}} C_i$ endowed with a R-module map $d : C_* \longrightarrow C_{*-1}$, called the *differential*, satisfying $d \circ d = 0$. The *homology* of a chain complex is a graded R-module $H_*(C_*, d)$ defined by

$$H_n(C_*, d) = ker(\, d : C_n \longrightarrow C_{n-1}\,)/Im(\, d : C_{n+1} \longrightarrow C_n\,)$$

for $n > 0$ and $H_0(C_*, d) = C_0/Im(\, d : C_1 \longrightarrow C_0\,)$.

A morphism $f : (C_*, d) \longrightarrow (C'_*, d')$ of chain complexes over R, also called a *chain map*, is a morphism of graded R-modules such that $d'f = fd$. It is easy to see that a morphism of chain complexes induces a morphism of graded modules in homology. We denote the category of chain complexes over R and their morphisms by $ChCx_*(R)$.

Let WE be the class of *quasi-isomorphisms*, i.e., of chain maps inducing isomorphisms in homology. Let Fib be the class of surjective chain maps and Cof the class of degree-wise split, injective chain maps with degree-wise projective cokernel. Hovey established in [11] that these choices determine a model category structure on $ChCx_*(R)$, with respect to which all objects are fibrant and the cofibrant chain complexes are projective in each degree. Moreover, any chain complex (C_*, d) that is projective in each degree is cofibrant.

Hovey proved furthermore that every chain complex is small with respect to class of all chain maps, which he then used in showing that $ChCx_*(R)$ is cofibrantly generated. Let $S^n(R)$ denote the chain complex with $S^n(R)_i = R$ if $i = n$ and $S^n(R)_i = 0$ otherwise. The differential is necessarily trivial in all degrees. Let $D^n(R)$ denote the chain complex with $D^n(R)_i = R$ if $i = n-1, n$ and $D^n(R)_i = 0$ otherwise. The differential $d: D^n(R)_i \longrightarrow D^n(R)_{i-1}$ is the identity if $i = n$ and 0 otherwise. The generating cofibrations of $ChCx_*(R)$ are the inclusions $S^{n-1}(R) \longrightarrow D^n(R)$ for all n, while the generating acyclic cofibrations are the inclusions of the zero complex into $D^n(R)$ for all n.

The algebraic notion of derived functor coincides in $ChCx_*(R)$ with the notion of total derived functor. In particular, if M is a right R-module, then we can define a functor

$$F = M \otimes_R - : ChCx_*(R) \longrightarrow ChCx_*(\mathbb{Z}).$$

The left total derived functor of F exists, and

$$H_i\big(\mathbb{L}F(S^0(R) \otimes_R N)\big) \cong Tor_i^R(M, N)$$

for all $i \geq 0$ and for all left R-modules N.

The model category in the next example is the target category of the well-known algebraic model we present in Section 4. When explaining this example, we work with *non-negatively graded cochain complexes* over the field $\mathbb{Q}$ of rational numbers, i.e., with graded $\mathbb{Q}$-vector spaces $C^* = \bigoplus_{i \geq 0} C^i$ endowed with a differential of degree $+1$. The definitions of cochain maps and of the cohomology of cochain complexes and their maps are analogous to the dual definitions for chain complexes.

Example 2.5.4. A commutative differential graded algebra (c.g.d.a.) over $\mathbb{Q}$ is a commutative monoid in the category of non-negatively graded cochain complexes. In other words, a c.g.d.a. (A^*, d) is a cochain complex over $\mathbb{Q}$, endowed with cochain maps

$$\eta : \mathbb{Q} \longrightarrow (A^*, d)$$

called the *unit* and

$$\mu : (A^* d) \otimes_{\mathbb{Q}} (A^*, d) \longrightarrow (A^*, d) \; : \; a \otimes b \longmapsto a \cdot b,$$

called the *product* such that

(1) μ is graded commutative, i.e., if $a \in A^p$ and $b \in A^q$, then $a \cdot b = (-1)^{pq} b \cdot a$;
(2) μ is associative; and
(3) $\mu(\eta \otimes 1_A) = 1_A = \mu(1_A \otimes \eta)$.

An important class of c.d.g.a.'s is composed of the *KS-complexes*. For any non-negatively graded vector space V, let ΛV denote the free, commutative, graded algebra generated by V, i.e., $\Lambda V = S[V^{\text{even}}] \otimes_{\mathbb{Q}} E(V^{\text{odd}})$, tensor product of the symmetric algebra on the vectors of even degree and of the exterior algebra on the vectors of odd degree. A *KS-complex* is a c.d.g.a. $(\Lambda V, d)$ with augmentation $\varepsilon : \Lambda V \longrightarrow \mathbb{Q}$ such that

(1) V has a basis $B = \{v_\alpha \mid \alpha \in J\}$, where J is a well-ordered set, and $\varepsilon(V) = 0$;
(2) $dv_\beta \in V_{<\beta}$ for all $\beta \in J$, where $V_{<\beta}$ is the span of $\{v_\alpha \mid \alpha < \beta\}$.

A KS-complex $(\Lambda V, d)$ is *minimal* if $V^0 = 0$ and $\alpha < \beta$ implies that $\deg v_\alpha \leq \deg v_\beta$. If $V^1 = V^0 = 0$, then $(\Lambda V, d)$ is minimal if and only if $dV \subseteq \Lambda^{\geq 2} V$.

A morphism of c.g.d.a.'s $f : (A^*, d, \mu, \eta) \longrightarrow (\bar{A}^*, \bar{d}, \bar{\mu}, \bar{\eta})$ is a cochain map such that $f\mu = \bar{\mu}(f \otimes f)$ and $f\eta = \bar{\eta}$. The category of c.d.g.a.'s over $\mathbb{Q}$ and their morphisms is denoted $CDGA^*(\mathbb{Q})$.

Let WE be the class of quasi-isomorphisms in $CDGA^*(\mathbb{Q})$, Fib the class of surjective c.d.g.a. morphisms, and $Cof = LLP(Fib \cap WE)$. According to Bousfield and Gugenheim [2], $(CDGA^*(\mathbb{Q}), WE, Fib, Cof)$ is a model category. All c.d.g.a.'s are fibrant with respect to this structure, and all KS-complexes are cofibrant.

It is not difficult to see that the model category of c.d.g.a.'s is cofibrantly generated. Observe first that any c.d.g.a. is small with respect to $Mor\ CDGA^*(\mathbb{Q})$, by an argument similar to that given by Hovey to prove that any chain complex is small with respect to the class of all chain maps.

Next we define three special families of inclusions of KS-complexes. Let $(\Lambda u_m, 0)$ denote the KS-complex generated by a vector space of dimension 1, concentrated in degree m, and let $(\Lambda(u_m, v_{m-1}), d)$ denote the KS-complex generated by a vector space with one basis element of degree m and one of degree $m-1$, where $d(v) = u$. Define i_m, i'_m and j_m to be the following inclusions.

$$i_m : 0 \longrightarrow (\Lambda u_m, 0)$$

$$i'_m : (\Lambda u_m, 0) \longrightarrow (\Lambda(u_m, v_{m-1}), d)$$

$$i_m : 0 \longrightarrow (\Lambda(u_m, v_{m-1}), d)$$

Let $\mathcal{I} = \{i_m, i'_{m+1} \mid m \geq 0\}$ and $\mathcal{J} = \{j_m \mid m \geq 1\}$.

Claim. *$Fib = \mathcal{J} - inj$ and $Fib \cap WE = \mathcal{I} - inj$.*

Proof. Since $\mathcal{J} - inj = RLP(\mathcal{J})$, it is clear that $p : (A^*, d) \longrightarrow (\bar{A}^*, \bar{d}) \in \mathcal{J} - inj$ if and only if for all $m \geq 1$ and all c.d.g.a. morphisms $g : (\Lambda(u_m, v_{m-1}), d) \longrightarrow (\bar{A}^*, \bar{d})$, there exists a c.d.g.a. morphism $\hat{g} : (\Lambda(u_m, v_{m-1}), d) \longrightarrow (A^*, d)$ such that $p\hat{g} = g$. It is easy to see that this condition on p is equivalent to the surjectivity of p, so that $Fib = \mathcal{J} - inj$.

On the other hand, $p : (A^*, d) \longrightarrow (\bar{A}^*, \bar{d}) \in \mathcal{I} - inj$ if and only if

(1) for all $m \geq 0$ and for all c.d.g.a. morphisms $g : (\Lambda u_m, 0) \longrightarrow (\bar{A}^*, \bar{d})$, there exists a c.d.g.a. morphism $\hat{g} : (\Lambda u_m, d) \longrightarrow (A^*, d)$ such that $p\hat{g} = g$; and
(2) for all $m \geq 1$, given any commutative diagram of c.d.g.a. morphisms

$$\begin{tikzcd}
(\Lambda u_m, 0) \arrow[r, "f"] \arrow[d, "i'_m"'] & (A^*, d) \arrow[d, "p"] \\
(\Lambda(u_m, v_{m-1}), d) \arrow[r, "g"] \arrow[ur, dotted, "\hat{g}"] & (\bar{A}^*, \bar{d})
\end{tikzcd}$$

there exists a c.d.g.a. morphism $\hat{g} : (\Lambda(u_m, v_{m-1}), d) \longrightarrow (A^*, d)$ such that $p\hat{g} = g$ and $\hat{g}i'_m = f$.

Condition (1) is equivalent to p being surjective on the subalgebra of cocycles $ker\bar{d}$ in $\bar{A}^*$, while condition (2) is equivalent to requiring that p be a quasi-isomorphism. Since any quasi-isomorphism of cochain complexes that is surjective on cocycles must be surjective, we see that $Fib \cap WE = \mathcal{I} - inj$. $\square$

Thus, $\mathcal{I}$ and $\mathcal{J}$ are families of generating cofibrations and generating acyclic cofibrations for the model structure on $CDGA^*(\mathbb{Q})$. The elements of $\mathcal{I} - cell$ are called *KS-extensions*.

Let $\mathcal{TOP}_0^f$ denote the full subcategory of $\mathcal{TOP}$ given by those spaces that are nilpotent, rational and of finite rational type. A space X is *nilpotent* if its fundamental group is nilpotent and acts nilpotently on the higher homotopy groups, *rational* if its homotopy groups are uniquely divisible and of *finite rational type* if $\dim_{\mathbb{Q}} H_n(X,\mathbb{Q}) < \infty$ for all n.

Let $CDGA^{*,f}(\mathbb{Q})$ denote the full subcategory of $CDGA^*(\mathbb{Q})$ given by those c.d.g.a.'s (A^*,d) for which there exists a quasi-isomorphism of c.d.g.a.'s

$$\varphi : (\Lambda V, d) \xrightarrow{\sim} (A^*, d),$$

where $(\Lambda V, d)$ is a minimal KS-complex such that $\dim_{\mathbb{Q}} V^n < \infty$ for all n. The quasi-isomorphism φ, or the minimal KS-complex $(\Lambda V, d)$, is called a *minimal model* of (A^*, d).

Bousfield and Gugenheim defined a Quillen pair of contravariant functors

$$AS : \mathcal{TOP} \rightleftarrows CDGA^*(\mathbb{Q}) : |F|,$$

that induces an equivalence of homotopy categories when restricted to $\mathcal{TOP}_0^f$ and $CDGA^{*,f}(\mathbb{Q})$, i.e.,

$$Ho\,\mathcal{TOP}_0^f \cong Ho\,CDGA^{*,f}(\mathbb{Q}).$$

The algebraic model $AS : \mathcal{TOP}_0^f \longrightarrow CDGA^{*,f}(\mathbb{Q})$ is called the *Sullivan model*, as Sullivan was the first to construct it [21], while Bousfield and Gugenheim provided the firm foundation in terms of model categories some years later. In Section 4, we take a closer look at this model and describe one of the many topological problems to which it has been successfully applied.

3. Monoidal model categories

Let $(\mathcal{C}, \mathbf{1}, \otimes, Hom)$ be a closed, symmetric, monoidal category endowed with the structure of a model category. In this section we establish compatibility conditions under which there is a natural, induced monoidal structure on $Ho\ \mathcal{C}$ and examine the possibility of extending such compatibility to categories of modules and monoids. The definitions and results we present in this section are due to Hovey [12] or, in a slightly different form, to Schwede and Shipley [19].

3.1 Definition and motivating theorem.

The following construction plays a very important role in elucidating the relationship between monoidal and model structure.

Definition. Let $(\mathcal{C}, \otimes)$ be a monoidal category. Let $f : A \longrightarrow B$ and $g : X \longrightarrow Y$ be morphisms in $\mathcal{C}$. The *push-out smash* of f and g, denoted $f \square g$, is the morphism induced by $f \otimes 1_Y$ and $1_B \otimes g$ in the following push-out diagram.

$$\begin{tikzcd}
A \otimes X \arrow[r, "f\otimes 1_X"] \arrow[d, "1_A\otimes g"] & B \otimes X \arrow[d] \arrow[ddr, bend left, "1_B\otimes g"] & \\
A \otimes Y \arrow[r] \arrow[drr, bend right, "f\otimes 1_Y"'] & (A\otimes Y) \underset{A\otimes X}{\vee} (B \otimes X) \arrow[dr, "f\Box g"] & \\
 & & B \otimes Y
\end{tikzcd}$$

Definition. A closed, symmetric, monoidal category $(\mathcal{C}, \mathbf{1}, \otimes, Hom)$ that is also a model category with distinguished classes WE, Fib, and Cof is a *monoidal model category* if the following axioms are satisfied.

(1) (Push-out smash axiom)

$$f, g \in Cof \Longrightarrow f\Box g \in Cof$$

and

$$f \in Cof, g \in Cof \cap WE \Longrightarrow f\Box g, g\Box f \in Cof \cap WE.$$

(2) (Unit axiom) Let $q : Q\mathbf{1} \xrightarrow{\sim} \mathbf{1}$ be a cofibrant model of the unit $\mathbf{1}$. Then $q \otimes 1_X : Q\mathbf{1} \otimes X \longrightarrow \mathbf{1} \otimes X \cong X$ and $1_X \otimes q : X \otimes Q\mathbf{1} \longrightarrow X \otimes \mathbf{1} \cong X$ are both weak equivalences.

Theorem 3.1.1(Hovey [11]**).** *The homotopy category of a monoidal model category $\mathcal{C}$ has a natural symmetric, monoidal structure, induced by the monoidal structure of $\mathcal{C}$.*

Examples. Hovey showed in [11] that $\mathcal{T}$ (cf. Example 2.5.1) is a monoidal model category. The tensor product on $\mathcal{T}$ is given by applying the k-space topology to the usual cartesian product of spaces: $X \otimes Y := k(X \times Y)$. Recall that a subset of kZ is open if and only if it is compactly open in Z.

Hovey also proved in [11] that $ChCx_*(R)$ (cf. Example 2.5.3) is a monoidal model category, where the underlying graded module of $(C_*, d) \otimes (C'_*, d')$ is defined by

$$(C_* \otimes C'_*)_n = \bigoplus_{k\in\mathbb{N}} C_k \otimes_R C'_{n-k}$$

and the tensor differential D is defined by

$$D(c \otimes_R c') = dc \otimes_R c' + (-1)^k c \otimes_R d'c'$$

if $c \in C_k$.

3.2 Model structures on categories of modules and algebras.

Given a monoidal model category $\mathcal{C}$, it is certainly natural to hope for the existence of natural model category structures on the category ${}_A\mathcal{M}od$ of (left) modules over a fixed monoid A , as well as on the category ${}_A\mathcal{A}\ell g$ of algebras over A, if A is commutative. Furthermore, supposing the existence of such structures, one would hope that they were "homotopy invariant" in some appropriate sense. For example, it seems reasonable to expect that a weak equivalence of monoids would induce an equivalence of the homotopy categories of their respective module categories.

As we see in the results below, these hopes are not foolish, at least as long as we are willing to accept a few additional constraints on the category $\mathcal{C}$ and the monoids we consider.

Throughout the remainder of this section, $(\mathcal{C}, WE, Fib, Cof)$ denotes a cofibrantly generated, monoidal model category with generating cofibrations $\mathcal{I}$ and generating acyclic cofibrations $\mathcal{J}$. Furthermore, if $A \in Ob\,\mathcal{C}$ and $\mathcal{K} \subseteq Mor\,\mathcal{C}$, then we write

$$A \otimes \mathcal{K} = \{1_A \otimes g \mid g \in \mathcal{K}\}$$

and

$$\mathcal{C} \otimes \mathcal{K} = \{f \otimes g \mid f \in Mor\,\mathcal{C}, g \in \mathcal{K}\}.$$

Theorem 3.2.1 (Hovey[12]**, Schwede/Shipley** [19]**).** *Let A be a monoid in $\mathcal{C}$ such that*

1. *the source of any morphism in $\mathcal{I}$ is small with respect to $(A \otimes \mathcal{I})$-cell;*
2. *the source of any morphism in $\mathcal{J}$ is small with respect to $(A \otimes \mathcal{J})$-cell; and*
3. *$(A \otimes \mathcal{J})$-cell $\subseteq WE$.*

Then ${}_A\mathcal{M}od$ can be endowed with a cofibrantly generated model category structure such that

$$f \in WE_{{}_A\mathcal{M}od} \Longrightarrow f \in WE_{\mathcal{C}} \cap {}_A\mathcal{M}od$$

and

$$f \in Fib_{{}_A\mathcal{M}od} \Longrightarrow f \in Fib_{\mathcal{C}} \cap {}_A\mathcal{M}od.$$

Remark. If A is a cofibrant monoid, then $A \otimes \mathcal{I} \subseteq Cof_{\mathcal{C}}$. The hypotheses (1)-(3) of the theorem above are therefore automatically satisfied in this case.

Hovey showed that the hypotheses of the theorem above are satisfied for any monoid in $\mathcal{T}$, so that the category of modules over any compactly generated topological monoid can be endowed with a cofibrantly generated model category structure.

The monoids in $ChCx_*(R)$ are the associative chain algebras over R. Since every chain complex is small with respect to the class of all chain maps, Theorem 3.2.1 and the remark above imply that the category of modules over an associative chain algebra that is bounded below and projective in each degree is a cofibrantly generated model category.

Theorem 3.2.2 (Hovey[12]**, Schwede/Shipley** [19]**).** *Suppose that*

1. *the source of any morphism in $\mathcal{I}$ is small with respect to $(\mathcal{C} \otimes \mathcal{I})$-cell;*
2. *the source of any morphism in $\mathcal{J}$ is small with respect to $(\mathcal{C} \otimes \mathcal{J})$-cell; and*
3. *$(\mathcal{C} \otimes \mathcal{J})$-cell $\subseteq WE$.*

If A is a commutative monoid in $\mathcal{C}$, then ${}_A\mathcal{A}\ell g$ can be endowed with a cofibrantly generated model category structure such that

$$f \in WE_{{}_A\mathcal{A}\ell g} \Longrightarrow f \in WE_{\mathcal{C}} \cap {}_A\mathcal{A}\ell g$$

and

$$f \in Fib_{{}_A\mathcal{A}\ell g} \Longrightarrow f \in Fib_{\mathcal{C}} \cap {}_A\mathcal{A}\ell g.$$

In particular, under the hypotheses of Theorem 3.2.2, the category of monoids in $\mathcal{C}$ is cofibrantly generated, since the monoids are the $\mathbf{1}$-algebras.

Hovey proved that the hypotheses of Theorem 3.2.2 are satisfied in the category $\mathcal{T}$ of compactly generated spaces, so that the category of compactly generated topological monoids can be endowed with a cofibrantly generated model structure.

If A is a commutative monoid, then ${}_A\mathcal{M}od$ naturally inherits a symmetric monoidal structure from $\mathcal{C}$. It seems reasonable to expect that this monoidal structure be compatible with the model category structure defined in the previous theorem.

Proposition 3.2.3. *Suppose that either*

1. *the unit $\mathbf{1}$ is cofibrant and A is a commutative monoid satisfying the hypotheses of Theorem 3.2.1; or*
2. *A is a cofibrant, commutative monoid.*

Then ${}_A\mathcal{M}od$ is a cofibrantly generated, monoidal model category.

As an application of this proposition, we obtain a cofibrantly generated, monoidal model structure on the category of modules over a commutative, associative chain algebra that is bounded below and projective in each degree.

The model category structures on ${}_A\mathcal{M}od$ and ${}_A\mathcal{A}\ell g$ satisfy two types of homotopy invariance, as expressed in the following theorems.

Theorem 3.2.4 (Hovey[12], Schwede/Shipley [19]). *Suppose that the source of any morphism in $\mathcal{I}$ is cofibrant. Let $f : A \xrightarrow{\sim} A'$ be a monoid morphism with A and A' cofibrant. Let $Res : {}_{A'}\mathcal{M}od \longrightarrow {}_A\mathcal{M}od$ denote the "restriction" functor, i.e., $Res(M) = M$ with the A-action induced by f. Then*

$$A' \underset{A}{\otimes} - : {}_A\mathcal{M}od \rightleftarrows {}_{A'}\mathcal{M}od : Res$$

is a Quillen equivalence.

If, in addition $\mathbf{1}$ is cofibrant and A and A' are commutative, then

$$A' \underset{A}{\otimes} - : {}_A\mathcal{A}\ell g \rightleftarrows {}_{A'}\mathcal{A}\ell g : Res$$

is a Quillen equivalence.

Theorem 3.2.5 (Hovey[12], Schwede/Shipley [19]). *Let $\mathcal{C}$ and $\mathcal{D}$ be cofibrantly generated monoidal model categories. Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a strong monoidal functor.*

1. *If $F(q) \in WE_{\mathcal{D}}$, where $q : Q\mathbf{1} \xrightarrow{\sim} \mathbf{1}$ is the cofibrant model of $\mathbf{1}$ and A is a cofibrant monoid in $\mathcal{C}$, then the categories ${}_A\mathcal{M}od$ and ${}_{F(A)}\mathcal{M}od$ are Quillen equivalent.*
2. *If A is a cofibrant, commutative monoid, then the homotopy categories $Ho_A\mathcal{A}\ell g$ and $Ho_{F(A)}\mathcal{A}\ell g$ are equivalent.*

4. The Sullivan model and Lusternik-Schnirelmann category

Recall from the exposition of Example 3.5.4 that a minimal model of a c.g.d.a. (A^*, d) consists of a quasi-isomorphism of c.g.d.a.'s

$$\varphi : (\Lambda V, d) \xrightarrow{\sim} (A^*, d),$$

where $(\Lambda V, d)$ is a minimal KS-complex. If $H^0(A^*, d) = \mathbb{Q}$, it is straightforward to construct a minimal model of (A^*, d) inductively. Furthermore, minimal models are unique up to isomorphism.

Let X be a connected, nilpotent space of finite rational type. There is a continuous map $\ell : X \longrightarrow X_0$, called the *rationalization* of X, such that X_0 is rational and $\pi_* \ell \otimes \mathbb{Q}$ is an isomorphism. The minimal model of the Sullivan model $AS(X_0)$ of the rationalization of X is called the *Sullivan minimal model* of X. Since the c.d.g.a. $AS(X_0)$ is huge and has a complicated product, rational homotopy theorists prefer to work with the Sullivan minimal model, which has only finitely many generators in each dimension and is free as an algebra, when carrying out computations.

We refer the reader to the new, encyclopedic reference on rational homotopy theory [8] by Félix, Halperin and Thomas for an in-depth treatment of the Sullivan minimal model. We mention below only a few of its most important properties.

In the statement of the following theorem and proposition, the word "space" refers exclusively to connected, nilpotent spaces of finite rational type.

Theorem 4.1. *Let $\varphi : (\Lambda V, d) \xrightarrow{\sim} AS(X_0)$ be the Sullivan minimal model of a space X.*

(1) *The cohomologies $H^*(\Lambda V, d)$ and $H^*(X; \mathbb{Q})$ are isomorphic as graded commutative algebras.*
(2) *$H^*(V, d_1) \cong Hom_{\mathbb{Z}}(\pi_* X, \mathbb{Q})$, where d_1 denotes the composition of the restriction of d to V with the projection $\Lambda V \longrightarrow V$.*

Elementary examples.

(1) The minimal model of a sphere S^k is $(\Lambda u_k, 0)$.
(2) The minimal model of a complex projective space $\mathbb{C}P^k$ is $(\Lambda(u_2, v_{2k+1}), d)$, where $dv = u^{k+1}$.
(3) The minimal model of a product of two spaces is the tensor product of the minimal models.

The following proposition, which is essential to modelling continuous maps, can be proved in several different ways, including by inductive construction.

Proposition 4.2. *Let $\varphi : (\Lambda V, d) \xrightarrow{\sim} AS(X_0)$ and $\varphi' : (\Lambda V', d') \xrightarrow{\sim} AS(X_0')$ be Sullivan minimal models of spaces X and X'. If $f : X' \longrightarrow X$ is any continuous map, there is a morphism of c.d.g.a.'s $\psi : (\Lambda V, d) \longrightarrow (\Lambda V', d')$ such that the following diagram commutes up to homotopy.*

$$\begin{tikzcd}
(\Lambda V, d) \arrow[r, "\psi"] \arrow[d, "\varphi"'] & (\Lambda V', d') \arrow[d, "\varphi'"] \\
AS(X_0) \arrow[r, "AS(f_0)"] & AS(X_0')
\end{tikzcd}$$

It is important to note that, unlike the minimal model of a space, the morphism ψ is defined only up to homotopy.

One of the most spectacular successes of the Sullivan minimal model has been in its application to studying the numerical homotopy invariant known as *Lusternik-Schnirelmann (L.-S.) category*. The L.-S. category of a topological space X, denoted $catX$, is equal to n if the cardinality of the smallest categorical covering of X is $n+1$. A categorical covering of a space X is an open cover of X such that each member of the cover is contractible in X.

L.-S. category is in general extremely difficult to compute. It is trivial, however, to prove that $catS^n = 1$ for all n and somewhat trickier, though still not difficult, to prove that $catT = 2$, where T is the torus $S^1 \times S^1$.

One elementary property of L.-S. category is that $cat(X \times Y) \leq catX + catY$ for all spaces X and Y. At the end of the 1960's Ganea observed that in the only known examples for which $cat(X \times Y) \neq catX + catY$, the spaces X and Y had homology torsion at distinct primes. He conjectured therefore that $cat(X \times S^n) = catX + 1$ for all spaces X and all $n \geq 1$, since S^n has no homology torsion whatsoever.

The groundbreaking article [6] of Félix and Halperin, in which they established the following characterization of the L.-S. category of a rational space in terms of its Sullivan model, initiated the application of rational homotopy theory to the study of L.-S. category.

Theorem 4.3 (Félix/Halperin [6]). *Let $\varphi : (\Lambda V, d) \xrightarrow{\sim} AS(X)$ be the Sullivan minimal model of a rational, nilpotent space X of finite rational type. Let $(\Lambda V/\Lambda^{>n}V, \bar{d})$ denote the c.d.g.a. obtained by taking the quotient of $(\Lambda V, d)$ by the ideal of words of length greater than n, and let*

$$\begin{tikzcd}
(\Lambda V, d) \arrow[rr, "q"] \arrow[dr, "i"'] & & (\Lambda V/\Lambda^{>n}V, \bar{d}) \\
& (\Lambda(V \oplus W), d) \arrow[ur, "p", "\sim"'] &
\end{tikzcd}$$

be the functorial factorization of the quotient map q obtained by applying the Small Object Argument to $\mathcal{I}$. Then $catX \leq n$ if and only if there is a morphism of c.g.d.a.'s $\rho : (\Lambda(V \oplus W), d) \longrightarrow (\Lambda V, d)$ such that $\rho i = 1_{(\Lambda V, d)}$.

In [9] Halperin and Lemaire proposed the study of a weakened version of the above characterization, in which the retraction ρ is required only to be a morphism of $(\Lambda V, d)$-modules. They called the homotopy invariant of X thus obtained $Mcat_0X$.

Following Halperin and Lemaire's lead, Jessup proved in 1986 that Ganea's conjecture holds with $Mcat_0$ in the place of cat.

Theorem 4.4 (Jessup [15]). *$Mcat_0(X \times S^n) = Mcat_0X + 1$ for all simply-connected rational spaces X of finite rational type and all $n \geq 2$.*

The following theorem then completed the proof of Ganea's conjecture for rational spaces.

Theorem 4.5 (Hess [10]). *$Mcat_0X = catX$ for all simply-connected rational spaces X of finite rational type.*

Corollary 4.6. *Ganea's conjecture holds for all rational, simply-connected spaces of finite rational type and all $n \geq 2$.*

As an epilogue to this story of Sullivan minimal models and L.-S. category, we mention that in 1997 Iwase applied classical homotopy-theoretic methods to the construction of a counter-example to Ganea's conjecture [14]. We also remark that in 1996 Félix, Halperin and Lemaire generalized Ganea's conjecture for rational spaces, proving that $cat(X \times Y) = catX + catY$ for all rational, simply-connected spaces X and Y of finite rational type [7]. Finally, it is worth noting that this rational version of L.-S. category has proved to be one of the most important and useful rational homotopy invariants, as it has played a crucial role in many other significant theorems in rational homotopy theory.

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