Equivariant nonabelian Poincaré duality and equivariant factorization homology of Thom spectra

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Abstract

In this paper, we study genuine equivariant factorization homology and its interaction with equivariant Thom spectra, which we construct using the language of parametrized higher category theory. We describe the genuine equivariant factorization homology of Thom spectra, and use this description to compute several examples of interest. A key ingredient for our computations is an equivariant nonabelian Poincaré duality theorem, in which we prove that factorization homology with coefficients in a G-space is given by a mapping space. We compute the Real topological Hochschild homology (THR) of the Real bordism spectrum $MU_{\mathbb{R}}$ and of the equivariant Eilenberg–MacLane spectra $H\underline{\mathbb{F}}_2$ and $H\underline{\mathbb{Z}}_{(2)}$, as well as factorization homology of the sphere $S^{2\sigma}$ with coefficients in these Eilenberg–MacLane spectra. In the appendix, Jeremy Hahn and Dylan Wilson compute $\mathrm{THR}(H\underline{\mathbb{Z}})$.

1 Introduction

In this paper, we study the equivariant factorization homology of Thom spectra. Factorization homology has emerged as a fruitful topic of research in recent years; its roots lie in the study of configuration spaces and their relation to mapping spaces, but it has also proven valuable in studying topological field theories, and as a unified way to treat Hochschild homology theories. Here we primarily take the axiomatic perspective on factorization homology, introduced by Ayala–Francis [AF15]. Ayala–Francis describe factorization homology with coefficients in an E_n-algebra A, $\int_{\_} A$, as a homology theory for n-manifolds: it satisfies a version of the Eilenberg–Steenrod axioms, including functoriality and excision, and is determined by these axioms.

In the case $n = 1$, $\int_{S^1} A$ agrees with Hochschild homology of a ring A. Furthermore, If A is a commutative ring spectrum, the factorization homology $\int_M A$ agrees with the Loday construction, which gives higher Hochschild homology [Pir00] for $M = S^n$ and iterated Hochschild homology for $M = \mathbb{T}^n$. As such, it is reasonable to expect that factorization homology might be of use in understanding invariants related to algebraic K-theory, and recently, Ayala–Mazel-Gee–Rozenblyum used factorization homology to obtain a new description of the cyclotomic trace from K-theory to topological cyclic homology (see, e.g. [AMGR17].) In this paper, we consider genuine equivariant factorization homology, introduced by the first named author [Hor19] in his thesis; other definitions of equivariant factorization homology have also been introduced independently by Weelinck [Wee18] and by the third named author [Zou20]. The paper [Hor19] uses parametrized higher category theory to define equivariant factorization homology axiomatically as a homology theory for G-manifolds, where G is a finite group. For 1-dimensional manifolds, this construction recovers Real topological Hochschild homology [DMPR17] and C_n-relative topological Hochschild homology [ABG$^+$14b]. Therefore, equivariant factorization homology provides a new perspective from which to study these invariants, and other invariants of equivariant ring spectra.

A particularly nice class of structured ring spectra is given by Thom spectra, for which properties of the base space of a spherical fibration can be used to deduce spectrum-level results. Lewis [LMSM86] showed that the Thom spectrum of an n-fold loop map is an E_n-ring spectrum. Blumberg–Cohen–Schlichtkrull [BCS10] showed that the Thom spectrum functor respects the cyclic bar construction, and used this to describe the topological Hochschild homology of Thom spectra and compute several examples. Schlichtkrull [Sch11] generalized this to higher Hochschild homology of commutative ring spectra. The second named author [Kla18] showed that the Thom spectrum functor respects factorization homology, and used this to describe the factorization homology and E_n topological Hochschild cohomology of Thom spectra and to compute examples.

In this paper, we apply this philosophy to equivariant Thom spectra and equivariant factorization homology. We give a construction of equivariant Thom spectra reminiscent of that of Ando–Blumberg–Gepner–Hopkins–Rezk [ABG$^+$14a], as a colimit in parametrized higher categories. We also show that it is a symmetric monoidal functor, and that its properties ensure that it respects equivariant factorization homology. This is Theorem 5.20:

Theorem. Let X be a pointed G-space and $\Omega^V f\colon \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ be a map of $\mathbb{E}_V$-algebras. Then for every V-framed G-manifold M, there is an equivalence of genuine G-spectra

$$\int_M \mathbf{Th}(\Omega^V f) \simeq \mathbf{Th}\left(\int_M \Omega^V X \xrightarrow{(\Omega^V f)_*} \int_M \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right).$$

Here, $\mathbf{Th}\colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \to \underline{\mathbf{Sp}}^G$ is the parametrized Thom G-functor in Construction 3.10.

Thus we can leverage knowledge of equivariant factorization homology on the level of spaces to determine equivariant factorization homology of Thom spectra. This is made particularly useful by the equivariant nonabelian Poincaré duality theorem of the third named author [Zou20], which we improve here in the axiomatic context. This is Theorem 2.2:

Theorem. For M a V-framed G-manifold and X a pointed G-space satisfying connectivity hypotheses, there is a natural equivalence of G-spaces

$$\int_M \Omega^V X \simeq \mathrm{Map}_*(M^+, X).$$

Here, M^+ is the one-point-compactification of M.

This theorem generalizes the nonabelian Poincaré duality theorem of Salvatore, Lurie, and Ayala–Francis. It describes equivariant factorization homology of an equivariant algebra in the category of G-spaces as a compactly supported mapping space. From our equivariant nonabelian Poincaré duality theorem, we recover equivariant Atiyah duality and equivariant Poincaré duality for V-framed G-manifolds.

We use our structural results on the equivariant Thom spectrum functor, along with the equivariant nonabelian Poincaré duality theorem, to make several computations of interest. For example, we compute the factorization homology of representation spheres with coefficients in the Real bordism spectrum $MU_{\mathbb{R}}$, the Real topological Hochschild homology of $H\underline{\mathbb{F}}_2$ and $H\underline{\mathbb{Z}}_{(2)}$, and the equivariant factorization homology of the representation spheres $S^{2\sigma}$ with coefficients in $H\underline{\mathbb{F}}_2$ and $H\underline{\mathbb{Z}}_{(2)}$. The appendix, written by Jeremy Hahn and Dylan Wilson, computes the Real topological Hochschild homology of $H\underline{\mathbb{Z}}$.

The computations in this paper rely on the two main theorems quoted above: equivariant nonabelian Poincaré duality (Theorem 2.2) and the behavior of the Thom spectrum functor under equivariant factorization homology (Theorem 5.20). The reader mainly interested in computations can keep these theorems in mind while focusing on Section 6 and the appendix.

Structure of the paper. In Section 2, we prove the equivariant nonabelian Poincaré duality theorem, using results from Section 5, and recover equivariant Atiyah duality for V-framed G-manifolds. In Section 3, we study the G-Thom spectrum functor, with preliminaries on parametrized ∞-category theory given in Section 7. In Section 4, we show that our G-Thom spectrum functor respects equivariant factorization homology. In Section 5, we explain how the Thom spectrum of a V-fold loop map gives rise to an $\mathbb{E}_V$-algebra, and give a description of the equivariant factorization homology of the G-Thom spectrum of a V-fold loop map. In Section 6, we give computations of the equivariant factorization homology of certain Thom spectra using results from the previous sections. The appendix, by Jeremy Hahn and Dylan Wilson, gives a computation of $\mathrm{THR}(H\underline{\mathbb{Z}})$.

Notation. We use Joyal's quasi-categories as a theory of ∞-categories, developed in [Lur09] and [Lur12]. We make extensive use of the theory of paramaretrized-∞-categories of Barwick–Dotto–Glasman–Nardin–Shah, developed in [BDG+16a], [BDG+16b], [Sha18], [Nar16], [Sha17], [Nar17]. We use underlines to indicate parametrized notions and constructions. For example, we write $\underline{\mathbf{Top}}^G$ for the G-∞-category of G-spaces and $\underline{\mathbf{Fin}}_*^G$ for the G-∞-category of finite pointed G-spaces. Following [Lur12], we write $\mathbf{Sp}$ for the ∞-category of spectra, and use the symbol $\otimes$ to denote the smash product symmetric monoidal structure on $\mathbf{Sp}$. We use the notation $E \otimes \Sigma^\infty_+ X$ or $E \otimes X$ for the smash product of a spectrum E and a space X (exhibiting $\mathbf{Sp}$ as tensored over spaces). We denote the wedge product of spectra by $\oplus$. We write $\mathbf{Top}_G$ for the ∞-category $\mathrm{Fun}(\mathcal{O}_G^{op}, \mathcal{S})$ of presheaves on the orbit category $\mathcal{O}_G$, and refer to its objects as G-spaces. We use the notation $\mathbf{Sp}_G$ for the ∞-category of genuine G-spectra, and denote smash products of genuine G-spectra by $\otimes$. Finally, we denote the smash product of $E \in \mathbf{Sp}_G$ and $X \in \mathbf{Top}_G$, traditionally written $E \wedge X_+$, by $E \otimes \Sigma^\infty_+ X$ or $E \otimes X$.

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2 Equivariant nonabelian Poincaré duality

Let G be a finite group and V a finite-dimensional real representation of G. In this section we prove the equivariant version of nonabelian Poincaré duality regarding equivariant factorization homology (Theorem 2.2). Our approach is similar to the one taken by Ayala–Francis (see [AF15, sec. 4]).

Definition 2.1. *A V-framing of a smooth G-manifold is an equivariant isomorphism of vector bundles*

$$TM \cong M \times V.$$

Let M be a V-framed G-manifold, and let A be an $\mathbb{E}_V$ algebra in $\underline{\mathbf{Top}}^G$. The equivariant factorization homology of M with coefficients in A, denoted as $\int_M A$, is a homotopy colimit of a diagram of G-spaces indexed by V-disks with V-framed embeddings in M,

$$\int_M A = \underrightarrow{colim} \left(\mathbf{Disk}^{G,V-fr}_{/M} \to \mathbf{Disk}^{G,V-fr} \xrightarrow{A} \mathrm{Top}^G \right).$$

The construction also works when A is an $\mathbb{E}_V$ algebra in $\underline{\mathbf{Sp}}^G$. In that case, $\int_M A$ is a G-spectrum rather than a G-space. See [Hor19, def. 3.9.7] for the definition of $\mathbb{E}_V$-algebras[1] and [Hor19, def. 4.1.2] for the construction of equivariant factorization homology.[2]

Theorem 2.2 (Equivariant nonabelian Poincaré duality)**.** *For a V-framed G-manifold M and $X \in \mathbf{Top}^G_*$ such that $\pi_k(X^H) = 0$ for all subgroups $H < G$ and $k < \dim(V^H)$, there is a natural equivalence of G-spaces*

$$\int_M \Omega^V X \simeq \mathrm{Map}_*(M^+, X).$$

Here, M^+ is the one-point-compactification of M.

We now give an outline of the proof of Theorem 2.2, using some constructions in Section 5 and lemmas proven in the rest of this section. The idea is to use the uniqueness of genuine equivariant factorization homology theories in [Hor19].

Outline of proof. We first explain the functors in the statement. In Remark 5.10, we construct the ∞-functor $\mathrm{Map}_*((-)^+, X)$ as the underlying functor of the G-symmetric monoidal ∞-functor

$$\underline{\mathrm{Map}}_*((-)^+, X) \in \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Mfld}}^{G,\sqcup}, \underline{\mathbf{Top}}^G_*)$$

at the fiber over G/G. Precomposing with the forgetful G-functor $\underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Mfld}}^{G,\sqcup}$, we get

$$\underline{\mathrm{Map}}_*((-)^+, X) \in \mathrm{Fun}^{\otimes}_G(\underline{\mathbf{Mfld}}^{G,V-fr,\sqcup}, \underline{\mathbf{Top}}^G_*), \tag{2.3}$$

whose underlying functor over G/G we still denote by $\mathrm{Map}_*((-)^+, X)$. Furthermore, restricting $\underline{\mathrm{Map}}_*((-)^+, X)$ to V-framed V-disks gives rise to an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Top}}^G$, which is exactly $\Omega^V X$ as defined in Eq. (5.9).

We claim that the functor in Eq. (2.3) is a G-factorization homology theory of V-framed G-manifolds. That is, it satisfies G-$\otimes$-excision in the sense of [Hor19, Definition 5.2.2] and respects G-sequential unions in the sense of [Hor19, Definition 5.3.2]. We prove this later in Proposition 2.13 and Proposition 2.20.

By the axiomatization [Hor19, Theorem 6.0.2], we can recover a G-factorization homology theory F by

$$F \simeq \int_- ((\iota^{\otimes})^* F),$$

[1] See also Definition 4.1 for an equivalent definition.

[2] The formula above is compatible with [Hor19, def. 4.1.2]. To see this, use the colimit formula of [Wee18, def. 4.14], and combine [Wee18, thm. 4.33] with [Hor19, prop. 5.2.3, 5.3.3] to compare the two constructions.

where $\iota^{\otimes}\colon \underline{\mathbf{Disk}}^{G,V-fr} \to \underline{\mathbf{Mfld}}^{G,V-fr}$ is the inclusion of G-∞-categories. We take the G-factorization homology theory F to be the functor $\underline{\mathrm{Map}}_*((-)^+, X)$ in Eq. (2.3). We can identify the coefficient $(\iota^{\otimes})^*(\underline{\mathrm{Map}}_*((-)^+, X))$ with $\Omega^V X$ as defined in Eq. (5.9) and get an equivalence of G-functors

$$\underline{\mathrm{Map}}_*((-)^+, X) \simeq \int_{-} \Omega^V X.$$

The conclusion follows from taking the underlying functor over the orbit G/G. $\square$

We will prove the claims used in the outline soon. Before that, we take a detour to deduce from Theorem 2.2 a version of the equivariant Atiyah duality theorem (Corollary 2.4) and the equivariant Poincaré duality theorem (Corollary 2.7) for V-framed G-manifolds. Atiyah duality for G-manifolds has previously been studied in [LMSM86, III.5].

Corollary 2.4. *Suppose M is a V-framed G-manifold and E is G-spectrum such that $\pi_k^H(\Sigma^V E) = 0$ for $k < \dim(V^H)$. Then there is a G-equivalence:*

$$\Omega^\infty(\Sigma^\infty_+ M \otimes E) \simeq \mathrm{Map}_*(M^+, \Omega^{\infty-V} E).$$

In particular, taking $E = \Sigma^\infty S^0$ to be the G-sphere spectrum, we recover Atiyah duality for M:

$$\Omega^\infty(\Sigma^\infty_+ M) \simeq \mathrm{Map}_*(M^+, \Omega^\infty \Sigma^\infty S^V).$$

Proof. We consider the G- infinite loop space of the G-spectrum E, $\Omega^\infty E$. Thus, we have an equivalence of G-E_∞ spaces

$$\Omega^\infty E \cong \Omega^V colim_W \Omega^W E_{V+W}$$

The two are equivalent as G-infinite loop spaces, and in particular as $\mathbb{E}_V$-algebras. Denote $colim_W \Omega^W E_{V+W}$ by X. Thus $\Omega^\infty E \simeq \Omega^V X$ as $\mathbb{E}_V$-algebras. Note that $X \simeq \Omega^\infty(\Sigma^V E)$, and we have $\pi_k^H(X) \cong \pi_k^H(\Sigma^V E)$ for $k \geq 0$. Therefore by assumption, X satisfies the connectivity hypotheses in Theorem 2.2, and we obtain

$$\int_M \Omega^\infty E \simeq \mathrm{Map}_*(M^+, X). \tag{2.5}$$

We claim that there is a G-equivalence:

$$\int_M \Omega^\infty E \simeq \Omega^\infty(\Sigma^\infty_+ M \otimes E). \tag{2.6}$$

First, $\Omega^\infty(\Sigma^\infty_+(-) \otimes E)$ is a G-factorization homology on V-framed G-manifolds, as we show later in Lemma 2.8. Second, the factorization homology theories $\int_{-} \Omega^\infty E$ and $\Omega^\infty(\Sigma^\infty_+(-) \otimes E)$ have the same coefficients: their coefficients are the $\mathbb{E}_V$-algebras Ω^∞ and $\Omega^\infty(\Sigma^\infty_+ V \otimes E)$. The map which contracts V to a point, $\Omega^\infty(\Sigma^\infty_+ V \otimes E) \to \Omega^\infty E$, is a map of $\mathbb{E}_V$ -algebras, as it is a map of G-infinite loop spaces. It is an equivalence of G-spaces, and therefore an equivalence of $\mathbb{E}_V$-algebras. So the two factorization homology theories in Eq. (2.6) agree and we obtain the equivalence.

The desired equivalence follows from combining (2.5) and (2.6). To apply to $E = \Sigma^\infty S^0$, we check when $k < \dim(V^H)$:

$$\begin{aligned}
\pi_k^H(\Sigma^\infty S^V) \cong& \pi_0 \mathrm{Map}_G(G/H_+ \wedge S^k, \Omega^\infty \Sigma^\infty S^V) \\
\cong& \pi_0 \mathrm{Map}_{\mathbf{Sp}_G}(\Sigma^\infty_+(G/H) \otimes \Sigma^\infty S^k, \Sigma^\infty S^V) \\
\cong& \pi_0 \mathrm{Map}_{\mathbf{Sp}_H}(\Sigma^\infty S^k, \Sigma^\infty S^V) \\
\cong& \pi_0 \mathrm{Map}_{\mathbf{Sp}}(\Sigma^\infty S^k, \Sigma^\infty S^{V^H}) = 0.
\end{aligned}$$

$\square$

Corollary 2.7. *Suppose M is a V-framed G-manifold and $\underline{B}$ is a Mackey functor. Then*

$$\mathrm{H}_\star(M;\underline{B}) \cong \widetilde{\mathrm{H}}^{V-\star}(M^+;\underline{B}).$$

In particular, if M is closed, then $\mathrm{H}_\star(M;\underline{B}) \cong \mathrm{H}^{V-\star}(M;\underline{B})$.

Proof. We can give S^V an H-CW decomposition with the lowest cells other than the base point in dimension $\dim(V^H)$. So we have $\pi_k^H(\Sigma^V \mathrm{H}\underline{B}) \cong \tilde{H}_k^H(S^V;\underline{B}) = 0$ when $k < \dim(V^H)$. Therefore we can take E in Corollary 2.4 to be the Eilenberg–MacLane spectrum $\mathrm{H}\underline{B}$ and get

$$\Omega^\infty(\Sigma^\infty_+ M \otimes \mathrm{H}\underline{B}) \simeq \mathrm{Map}_*(M^+, K(\underline{B}, V)).$$

The desired Poincaré duality follows from taking homotopy groups on both sides and identifying:

$$\pi_\star\Omega^\infty(\Sigma^\infty_+ M \otimes \mathrm{H}\underline{B}) \cong \mathrm{H}_\star(M,\underline{B});$$
$$\pi_\star(\mathrm{Map}_*(M^+, K(\underline{B}, V))) \cong \widetilde{\mathrm{H}}^{V-\star}(M^+;\underline{B}).$$

□

Lemma 2.8. *Let E be a G-spectrum. Then $\Omega^\infty(\Sigma^\infty_+(-) \otimes E)$ is a G-factorization homology theory on V-framed G-manifolds.*

Proof. One can express $\Omega^\infty(\Sigma^\infty_+(-) \otimes E)$ as the composition of G-functors

$$\Omega^\infty(\Sigma^\infty_+(-) \otimes E) \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Mfld}}^G \xrightarrow{fgt} \underline{\mathbf{Top}}^G \xrightarrow{\Sigma^\infty_+} \underline{\mathbf{Sp}}^G \xrightarrow{-\otimes E} \underline{\mathbf{Sp}}^G \xrightarrow{\Omega^\infty} \underline{\mathbf{Top}}^G.$$

Each G-functor in the composition extends to a G-symmetric monoidal functor:

1. The G-functor $\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Mfld}}^G$, forgetting the V-framing, is G-symmetric monoidal by construction.
2. The functor $fgt\colon \underline{\mathbf{Mfld}}^G \to \underline{\mathbf{Top}}^G$ is G-symmetric monoidal, as it can be defined by the following construction. As a functor of topological categories, the forgetful functor $\mathbf{Mfld}_n \to \mathbf{Top}$ is symmetric monoidal (takes disjoint unions to coproducts). Construct the forgetful functor $fgt\colon \underline{\mathbf{Mfld}}^G \to \underline{\mathbf{Top}}^G$ by applying the genuine operadic nerve construction (see also Section 7.8).
3. The G-functor $\Sigma^\infty_+\colon \underline{\mathbf{Top}}^G \to \underline{\mathbf{Sp}}^G$ is a G-left adjoint, hence strongly prereserves G-colimits. In particular, it extends to a G-symmetric monoidal functor with respect to the G-coCartesian monoidal structure on both categories.
4. Similarly, $\underline{\mathbf{Sp}}^G \xrightarrow{-\otimes E} \underline{\mathbf{Sp}}^G$ strongly preserves G-colimits, and therefore extends to a G-symmetric monoidal functor.
5. The G-functor $\underline{\mathbf{Sp}}^G \xrightarrow{\Omega^\infty} \underline{\mathbf{Top}}^G$ is a G-right adjoint, and therefore extends to a G-symmetric monoidal functor with respect to the G-*Cartesian* monoidal structures.
6. Finally, since $\underline{\mathbf{Sp}}^G$ is a G-semi-additive G-∞-category, the G-Cartesian and G-coCartesian monoidal structure are canonically equivalent.

In fact, this decomposition also shows that $\Omega^\infty(\Sigma^\infty_+(-) \otimes E)$ preserves sifted colimits fiberwise.

We now verify the $\otimes$-excision axiom. Let $M = M' \cup_{M_0 \times \mathbb{R}} M''$ be a G-collar decomposition of V-framed G-manifolds. After applying the forgetful functor

$$\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Mfld}}^G \xrightarrow{fgt} \underline{\mathbf{Top}}^G$$

the G-space $M = M' \coprod_{M_0\times\mathbb{R}} M''$ is equivalent to the geometric realization

$$\big|\ M'\coprod M'' \longleftarrow M'\coprod M_0\times\mathbb{R}\coprod M'' \overleftarrow{\overleftarrow{\longleftarrow}} M'\coprod M_0\times\mathbb{R}\coprod M_0\times\mathbb{R}\coprod M'' \overleftarrow{\overleftarrow{\overleftarrow{\longleftarrow}}} \cdots\ \big|.$$

Since $\Omega^\infty(\Sigma^\infty_+(-)\otimes E)$ preserves geometric realizations, we have an equivalence

$$\Omega^\infty(\Sigma^\infty_+(M)\otimes E)\simeq \mathbf{B}\big(\Omega^\infty(\Sigma^\infty_+(M')\otimes E),\Omega^\infty(\Sigma^\infty_+(M_0\times\mathbb{R})\otimes E),\Omega^\infty(\Sigma^\infty_+(M')\otimes E)\big),$$

hence $\Omega^\infty(\Sigma^\infty_+(-)\otimes E)$ satisfies $\otimes$-excision. A similar approach verifies $\Omega^\infty(\Sigma^\infty_+(-)\otimes E)$ respects G-sequential unions, hence it is a G-factorization homology theory. $\square$

In the remainder of this section, we complete the proof outline of Theorem 2.2 by showing that $\underline{\mathrm{Map}}_*((-)^+,X)$ in Eq. (2.3) respects G-sequential unions (Proposition 2.13) and satisfies G-$\otimes$-excision (Proposition 2.20). Notice that both properties are fiberwise for the fiber over the orbit G/H. Without loss of generality we may work with $H=G$. We think of $\mathrm{Map}_*((-)^+,X)$ as a topological functor between topological categories.

Remark 2.9. Examining the proof of [Hor19, thm. 6.0.2], we see that it is enough to verify the axioms of a G-factorization homology theory for an equivariant handle decomposition of a V-framed G-manifold M arising from a G-invariant Morse function. It is therefore enough to verify $\otimes$-excision under the assumption that M is the interior of a compact manifold with boundary $\partial\overline{M}$. Similarly, when verifying the G-sequential union property we may assume that the sequential union given by a sequence of regular values of a G-equivariant Morse function on M.

We start with proving the G-sequential union property. We find it convenient to replace the space $\mathrm{Map}_*(M^+,X)$ with the space of compactly supported maps.

Definition 2.10. *For a G-manifold M and a based G-space X, let*

$$\mathrm{Map}_c(M,X)=\{f\in\mathrm{Map}(M,X)|\mathrm{supp}(f)\ \textit{is compact}\}$$

be the space of compactly supported maps. Here, the support of a map f is the closure of the preimage of the compliment of the base point.

Remark 2.11. Let $M_1\subset M_2$ be an open inclusion of G-manifolds. Extending $f\in\mathrm{Map}_c(M_1,X)$ by the base point on M_2-M_1 gives a G-map $\mathrm{Map}_c(M_1,X)\to\mathrm{Map}_c(M_2,X)$. There is a G-equivalence $\mathrm{Map}_c(M,X)\xrightarrow{\sim}\mathrm{Map}_*(M^+,X)$, which is natural for the variable M.

The following a lemma is a geometric observation.

Lemma 2.12. *Let $f\colon M\to\mathbb{R}$ be an equivariant Morse function and $t<s$ be two regular values. Denote $M_0=f^{-1}(-\infty,t)$, $M_1=f^{-1}(-\infty,s)$, and let X be any based G-space. Then $\mathrm{Map}_c(M_0,X)\to\mathrm{Map}_c(M_1,X)$ is a G-cofibration.*

Proof. Assume $t=0$ without loss of generality. We show that $\mathrm{Map}_c(M_0,X)\to\mathrm{Map}_c(M_1,X)$ is an G-NDR (neighborhood deformation retract) pair, thus a G-cofibration. Assume $\epsilon>0$ is small enough such that $(-\epsilon,\epsilon)$ are all regular values for f. We prepare several functions to construct the G-NDR data (h,u).

Let $u'\colon\ \mathrm{Map}_c(M_1,X)\to[0,s]$ be

$$u'(-)=\sup\{f(\mathrm{supp}(-)),0\}$$

and $u\colon \mathrm{Map}_c(M_1, X) \to [0,1]$ be

$$u(-) = \min\{u'(-)/\epsilon, 1\}.$$

Then $\mathrm{Map}_c(M_0, X) = u^{-1}(0)$ and $u^{-1}(t) = \mathrm{Map}_c(f^{-1}(-\infty, \epsilon t), X)$ for $t \in (0,1)$. The map u is equivariant because f is.

Since $(-\epsilon, \epsilon)$ are all regular values of the Morse function, we can construct an equivariant flow $F \in \mathrm{Map}(I, \mathrm{Diff}(M_1, M_1))$ such that

$$F(0) = \mathrm{id}_{M_1} \text{ and } F(t)(M_0) = f^{-1}(-\infty, t\epsilon).$$

Now take $h\colon \mathrm{Map}_c(M_1, X) \times I \to \mathrm{Map}_c(M_1, X)$ to be

$$h(-, t) = - \circ F(u(-)t).$$

It is easy to verify that (h, u) represents $\mathrm{Map}_c(M_0, X) \to \mathrm{Map}_c(M_1, X)$ as a G-NDR pair. $\square$

Proposition 2.13. *Given G-manifolds $M_1 \subset M_2 \subset \cdots \subset M$ with $M = \cup_i M_i$, there is a G-equivalence*

$$\mathrm{Map}_*(M^+, X) \simeq \mathrm{hocolim}_i \mathrm{Map}_*((M_i)^+, X).$$

Proof. By Remark 2.9 we may assume that there is an equivariant Morse function $f\colon M \to \mathbb{R}$ such that for each $s \in \mathbb{R}$, $f^{-1}(-\infty, s]$ is compact, and that $M_i = f^{-1}(-\infty, s_i)$ for regular values $s_1 < s_2 < \cdots$. Then $\overline{M_i} \subset f^{-1}(-\infty, s_i] \subset M_{i+1}$, so

$$\mathrm{Map}_c(M, X) \simeq \mathrm{colim}_i \mathrm{Map}_c(M_i, X).$$

Since $\mathrm{Map}_c(M_i, X) \to \mathrm{Map}_c(M_{i+1}, X)$ is a G-cofibration (Lemma 2.12),

$$\mathrm{colim}_i \mathrm{Map}_c(M_i, X) \simeq \mathrm{hocolim}_i \mathrm{Map}_c(M_i, X).$$

Via the functorial identification $\mathrm{Map}_c(M, X) \simeq \mathrm{Map}_*(M^+, X)$, we have

$$\mathrm{Map}_*(M^+, X) \simeq \mathrm{hocolim}_i \mathrm{Map}_*((M_i)^+, X). \qquad \square$$

Next, we prove the G-$\otimes$-excision property. We begin by fixing a G-collar decomposition.

Notation 2.14. In the rest of this section, X is a pointed G-space and M is an n-dimensional G-manifold. We fix a G-collar decomposition $M = M' \bigcup_{M_0 \times \mathbb{R}} M''$, where M', M'' are open G-submanifolds and M_0 is a closed G-submanifold of codimension 1. We abuse notation to write $M - M'$ for $M - (M' - M_0 \times [0, +\infty))$ and $M - M''$ for $M - (M'' - M_0 \times (-\infty, 0])$. They are diffeomorphic as manifolds with boundaries, but we gain better control of the boundary and collar gluing. Both $M - M'$ and $M - M''$ have boundaries $M_0 \times \{0\} \cong M_0$.

Lemma 2.15. *The diagram*

$$\begin{tikzcd}
\mathrm{Map}_*(M^+, X) \arrow[r] \arrow[d] & \mathrm{Map}_*((M - M')^+, X) \arrow[d] \\
\mathrm{Map}_*((M - M'')^+, X) \arrow[r] & \mathrm{Map}_*(M_0^+, X)
\end{tikzcd}$$

is a homotopy pullback diagram of G-spaces.

Proof. All of the embeddings of submanifolds are proper, therefore the natural maps

$$M_0^+ \to (M - M')^+, (M - M')^+ \to M^+ \text{ (and similarly with } M'')$$

are defined and are pointed maps. The maps in the diagram are restrictions along those. M_0 is a closed submanifold, so M_0^+ is a closed subspace of $(M - M')^+$ and of $(M - M'')^+$, and is the intersection of the two. Similarly, $(M - M')^+$ and $(M - M'')^+$ are closed subspaces of M^+. Thus, defining a pointed map $M^+ \to X$ is equivalent to defining pointed maps $(M - M')^+ \to X$ and $(M - M'')^+ \to X$ which agree on the overlap, M_0^+. This shows that the square is a pullback square. It is also a homotopy pullback since the restriction maps to $\mathrm{Map}_*((M_0)^+, X)$ are G-fibrations, as shown in the following Lemma 2.16. $\square$

Lemma 2.16. *The restriction map* $\mathrm{Map}_*((M - M')^+, X) \to \mathrm{Map}_*((M_0)^+, X)$ *is a G-fibration with fiber* $\mathrm{Map}_*((M'')^+, X)$. *The corresponding statement in which M' and M'' are switched also holds.*

Proof. For the first part, it suffices to show that $M_0^+ \to (M - M')^+$ is a G-cofibration in the Hurewicz sense, as mapping out of it would give a Hurewicz fibration, which is in particular a Serre fibration.

By Remark 2.9 we may assume that either M is closed or M is the interior of a compact manifold with boundary $\partial\overline{M}$. In the first case denote $\partial\overline{M} = \emptyset$. We further assume that $\overline{M}$ is embedded in some orthogonal G-representation W (This is possible by [Mos57]). Since both M_0 and $M - M'$ are submanifolds of M which are closed, $\partial\overline{M_0}$ and $D = \partial(\overline{M - M'}) \cap \partial\overline{M}$ are both submanifolds of $\partial\overline{M}$ which are closed. (See Fig. 1 for illustration.)

All of $\partial\overline{M_0}, \overline{M_0}, D, \partial(\overline{M - M'}), \overline{M - M'}$ are also close submanifolds of W, consequently equivariantly embedded as a retract of an open subspace of W ([IK00, Theorem 1.4]), showing that they are all G-ENRs (Euclidean neighborhood retract).

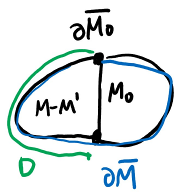

Figure 1: Illustration

By [LMSM86, III.4], an inclusion of G-ENRs is a G-cofibration. So all maps in the following pushout square are G-cofibrations:

$$\begin{tikzcd}
\partial\overline{M_0} \arrow[r] \arrow[d] & \overline{M_0} \arrow[d] & \\
D \arrow[r] & \partial(\overline{M - M'}) \arrow[r] & \overline{M - M'}
\end{tikzcd}$$

Therefore, $M_0^+ = \overline{M_0}/\partial\overline{M_0} \longrightarrow \partial(\overline{M - M'})/D \longrightarrow \overline{M - M'}/D = (M - M')^+$ is a composite of G-cofibrations. This proves the first part.

To find the fiber, we take the preimage of the constant map in $\mathrm{Map}_*((M_0)^+, X)$. Because $\partial(\overline{M - M'}) = \overline{M_0} \cup_{\partial\overline{M_0}} D$, a map from $(M - M', D)$ to $(X, *)$ that map $\overline{M_0}$ to the base point $*$ is the same as a map from $(M - M', \partial(\overline{M - M'}))$ to $(X, *)$. So the fiber can be identified with $\mathrm{Map}_*(\overline{M - M'}/\partial(\overline{M - M'}), X) \cong \mathrm{Map}_*(\overline{M''}/\partial\overline{M''}, X) \simeq \mathrm{Map}_*((M'')^+, X)$. $\square$

Let B be a based G-space. To set up for a bar construction, we use the Moore path space and loop space of B, such that ΩB is a monoid and acts on PB:

$$PB = \{(l, \alpha) \in \mathbb{R}_{\geq 0} \times \mathrm{Map}(\mathbb{R}_{\geq 0}, B) | \alpha(0) = *, \alpha(t) = \alpha(l) \text{ for } t \geq l\},$$
$$\Omega B = \{(l, \alpha) \in \mathbb{R}_{\geq 0} \times \mathrm{Map}(\mathbb{R}_{\geq 0}, B) | \alpha(0) = *, \alpha(t) = * \text{ for } t \geq l\} \subset PB.$$

They are homotopy equivalent respectively to the ordinary path space and loop space, and they have varying lengths of path recorded by l. Given two path elements (l_1, α_1) and (l_2, α_2) in PB, the concatination of them is defined to $(l_1 + l_2, \alpha_1.\alpha_2)$ where

$$(\alpha_1.\alpha_2)(t) = \begin{cases} \alpha_1(t), & 0 \le t \le l_1; \\ \alpha_2(t - l_1), & l_1 < t. \end{cases}$$

This strictifies ΩB to a strict monoid and gives a strict left action of ΩB on PB by concatenation. The unit for ΩB is $(0, *)$ where $*(t) = *$ and a homotopy inverse is the reverse of loops. The reverse of $(l, \alpha) \in \Omega B$ is defined to be $(l, \bar{\alpha})$ where

$$\bar{\alpha}(t) = \begin{cases} \alpha(l - t), & 0 \le t \le l; \\ *, & l < t. \end{cases}$$

The reverse of loops is a monoid homomorphism $\Omega B \to (\Omega B)^{op}$. Composing it with the left action gives the right action of ΩB on PB. There is also the evaluation map that records the endpoint of a path

$$ev \colon PB \to B, (l, \alpha) \mapsto \alpha(l).$$

Lemma 2.17. *Suppose* $\mathrm{Map}_*((M_0)^+, X)$ *is* G*-connected. Then there is an equivalence between the bar construction*

$$\mathbf{B}(\mathrm{Map}_*((M')^+, X),\ \Omega\mathrm{Map}_*((M_0)^+, X),\ \mathrm{Map}_*((M'')^+, X))$$

and the homotopy pullback

$$\mathrm{Map}_*((M - M'')^+, X) \times_{\mathrm{Map}_*((M_0)^+, X)} \mathrm{Map}_*((M - M')^+, X).$$

Proof. For brevity, we write B_0 for $\mathrm{Map}_*((M_0)^+, X)$, B'' for $\mathrm{Map}_*((M - M'')^+, X)$ and B' for $\mathrm{Map}_*((M - M')^+, X)$. Then B_0 is a based G-space with base point the constant map to the base point of X and G acts by conjugation.

We first describe the bar construction in the statement. Denote by $\times_{B_0}$ the homotopy pullback of spaces over B_0. Consider the restriction

$$B' = \mathrm{Map}_*((M - M')^+, X) \to \mathrm{Map}_*((M_0)^+, X) = B_0.$$

By Lemma 2.16, we have G-equivalences between the fiber and the homotopy fiber of this restriction:

$$\mathrm{Map}_*((M'')^+, X) \simeq B' \times_{B_0} PB_0,$$

and similarly,

$$\mathrm{Map}_*((M')^+, X) \simeq B'' \times_{B_0} PB_0.$$

We have a right action of ΩB_0 on $\mathrm{Map}_*((M')^+, X)$ and a left action on $\mathrm{Map}_*((M'')^+, X)$ by ΩB_0 acting on PB_0 through these equivalences. Geometrically, this action is by gluing together a map in $\Omega\mathrm{Map}_*((M_0)^+, X) \cong \mathrm{Map}_*((M_0 \times \mathbb{R}_{>0})^+, X)$ and a map in $\mathrm{Map}_*((M')^+, X)$ via a choice of identification of $\overline{M'} \cup_{\overline{M_0}} \overline{M_0 \times \mathbb{R}_{>0}} \cong \overline{M'}$. The two sided bar construction in the statment is equivalent to the geometric realization of the simplicial G-space

$$\mathbf{B}(B'' \times_{B_0} PB_0, \Omega B_0, B' \times_{B_0} PB_0).$$

Since geometric realization commutes with pullbacks of simplicial spaces ([May72, Corollary 11.6] or [Lur12, 5.5.6.17]), we have G-equivalence:

$$\begin{aligned}\mathbf{B}(\mathrm{Map}_*((M')^+, X), \Omega B_0, \mathrm{Map}_*((M'')^+, X)) &\simeq \mathbf{B}(B'' \times_{B_0} PB_0, \Omega B_0, B' \times_{B_0} PB_0) \\ &\simeq B'' \times_{B_0} \mathbf{B}(PB_0, \Omega B_0, PB_0) \times_{B_0} B'.\end{aligned}$$

We claim that there is an equivalence $\mathbf{B}(PB_0, \Omega B_0, PB_0) \simeq B_0$ as G-spaces when B_0 is G-connected. Moreover, it is an equivalence as G-spaces over B_0 on both sides. Here, the two augmentation maps of B_0 are identity on both sides; the two augmentation maps of $\mathbf{B}(PB_0, \Omega B_0, PB_0)$ over B_0 are induced by $ev\colon PB_0 \to B_0$ on either of the PB_0 in the bar construction, which we denote them by ev_l and ev_r. These two maps are actually G-homotopic, as one can construct an explicit homotopy $ev_l \simeq ev_r$ by evaluating along the concatenated paths from the left endpoint to the right endpoint in each simplicial level. We skip the details here. Now, we take the geometric realization of the simplicial levelwise G-fibration:

$$\mathbf{B}_*(\Omega B_0, \Omega B_0, PB_0) \to \mathbf{B}_*(PB_0, \Omega B_0, PB_0) \overset{ev_l}{\to} B_0$$

and to obtain a sequence

$$* \simeq \mathbf{B}(\Omega B_0, \Omega B_0, PB_0) \to \mathbf{B}(PB_0, \Omega B_0, PB_0) \overset{ev_l}{\to} B_0. \tag{2.18}$$

It suffices to show that ev_l (equivalently, ev_r) is a G-weak equivalence. This is because for each subgroup $H < G$, when $(B_0)^H$ is connected, Eq. (2.18) is known to be a quasifibration after taking H-fixed points.

Consequently, we have

$$\begin{aligned}\mathbf{B}(\mathrm{Map}_*((M')^+, X), \Omega B_0, \mathrm{Map}_*((M'')^+, X)) &\simeq B'' \times_{B_0} B_0 \times_{B_0} B' \\ &\simeq B'' \times_{B_0} B'.\end{aligned}$$ ◻

Lemma 2.19. *Let M be a smooth G-manifold and N be a closed sub-G-manifold. Let X be a based G-space such that X^H is $\dim(M^H)$-connected for all subgroups $H < G$. Then $\mathrm{Map}_*(M/N, X)$ is G-connected.*

Here, since M is a smooth G-manifold, M^H is also a manifold, but possibly empty or with components of different dimensions. We define $\dim(\varnothing) = -1$ and $\dim(M^H)$ to be the biggest dimension of the components.

Proof. Take a triangulation of (M, N), which exists by [Ill78, Theorem 3.6]. It gives (M, N) a relative G-CW structure. Denote $M^{-1} = N$ and $S^{-1} = \varnothing$. Then $\mathrm{Map}_*(M^{-1}/N, X) = *$ is G-connected. We induct on the G-CW skeleton of M. For $k \geq 0$, we have the pushout:

$$\begin{tikzcd} \coprod_{H_i} G/H_i \times S^{k-1} \arrow[r, "f"] \arrow[d, hook] & M^{k-1} \arrow[d, hook] \\ \coprod_{H_i} G/H_i \times D^k \arrow[r] & M^k \end{tikzcd}$$

It gives a cofiber sequence:

$$M^{k-1}/N \to M^k/N \to \bigvee_{H_i} (G/H_i)_+ \wedge S^k.$$

Mapping into X gives a fiber sequence:

$$\prod_{H_i} \mathrm{Map}_*((G/H_i)_+ \wedge S^k, X) \to \mathrm{Map}_*(M^k/N, X) \to \mathrm{Map}_*(M^{k-1}/N, X).$$

For any subgroups H and H_i, by the double coset formula, $G/H_i \cong \coprod_j H/K_{ij}$ as H-sets, where each of the K_{ij} is H intersecting some conjugate of H_i. Therefore,

$$\begin{aligned}\mathrm{Map}_*((G/H_i)_+ \wedge S^k, X)^H &\cong \mathrm{Map}_*((\bigvee_j (H/K_{ij})_+) \wedge S^k, X)^H \\ &\cong \prod_j \mathrm{Map}_*(S^k, X^{K_{ij}}).\end{aligned}$$

Since $k \leq dim(M^{H_i}) \leq dim(M^{K_{ij}})$, $X^{K_{ij}}$ is k-connected by assumption. So $\mathrm{Map}_*(S^k, X^{K_{ij}})$ is connected. By long exact sequence of homotopy groups and induction, $\mathrm{Map}_*(M^k/N, X)^H$ is connected for all k.

This implies that $\mathrm{Map}_*(M/N, X)$ is G-connected. □

Proposition 2.20. *Let M be a V-framed G-manifold with the collar decomposition in Notation 2.14, and let X be a G-space as in Theorem 2.2. Then there is a G-equivalence:*

$$\mathrm{Map}_*((M')^+, X) \otimes_{\mathrm{Map}_*((\mathbb{R}\times M_0)^+, X)} \mathrm{Map}_*((M'')^+, X) \to \mathrm{Map}_*(M^+, X).$$

Proof. Note that $(\mathbb{R} \times M_0)^+ \simeq \Sigma(M_0^+)$. The homotopy coherent quotient on the left-hand side, over $\mathrm{Map}_*((\mathbb{R} \times M_0)^+, X) \simeq \Omega\mathrm{Map}_*(M_0^+, X)$, can be identified with the bar construction

$$\mathbf{B}(\mathrm{Map}_*((M')^+, X), \Omega\mathrm{Map}_*(M_0^+, X), \mathrm{Map}_*((M'')^+, X)).$$

Since $M_0 \times \mathbb{R}$ is V-framed, $(M_0 \times \mathbb{R})^H \cong M_0^H \times \mathbb{R}$ is either empty or a manifold of dimension V^H, as we can find local charts using the exponential maps. So $\dim(M_0^H)+1 \leq \dim(V^H)$. By our assumption on the connectivity of X, Lemma 2.19 applied to the pair $(M, N) = (M_0, \varnothing)$ shows that $\mathrm{Map}_*(M_0^+, X)$ is G-connected. So we can use Lemma 2.17 to identify the bar construction with the homotopy pullback $\mathrm{Map}_*((M - M'')^+, X) \times_{\mathrm{Map}_*((M_0)^+, X)} \mathrm{Map}_*((M - M')^+, X)$, then use Lemma 2.15 to identify it with $\mathrm{Map}_*(M^+, X)$. □

3 Equivariant Thom spectra

In this section, we will define the G-Thom spectrum functor, and show that it respects G-colimits and is G-symmetric monoidal. This will allow us to conclude, in the next section, that it respects equivariant factorization homology.

We first recall the construction of Thom spectra according to [ABG+14a, def. 2.20], an approach that leverages the equivalence of spaces and ∞-groupoids. The Thom spectrum of a stable spherical fibration E over X is defined as the colimit of

$$X \xrightarrow{E} \mathrm{Pic}(\mathbf{Sp}) \to \mathbf{Sp}. \tag{3.1}$$

Here, $\mathrm{Pic}(\mathbf{Sp})$ is the Picard space of the ∞-category of spectra, that is, the classifying space of local systems of invertible spectra; $\mathrm{Pic}(\mathbf{Sp}) \to \mathbf{Sp}$ is the inclusion of a sub-∞-groupoid; the map $X \xrightarrow{E} \mathrm{Pic}(\mathbf{Sp})$ is the classification map of the spherical fibration E, and the colimit of Eq. (3.1) is indexed by X, considered as an ∞-groupid.

Together these Thom spectra assemble to a colimit preserving functor from spaces over $\mathrm{Pic}(\mathbf{Sp})$ to spectra:

$$\mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})} \simeq \mathrm{Psh}(\mathrm{Pic}(\mathbf{Sp})) \to \mathbf{Sp}.$$

By the universal property of the presheaf category, this functor is characterized by its restriction along the Yoneda embedding $\mathrm{Pic}(\mathbf{Sp}) \to \mathrm{Psh}(\mathrm{Pic}(\mathbf{Sp}))$, see [ABG+14a, cor. 3.13], and therefore given as a left Kan extension

$$\begin{tikzcd} & \mathrm{Pic}(\mathbf{Sp}) \arrow[d, hook] \arrow[dr] & \\ \mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})} \arrow[r, "\simeq"] & \mathrm{Psh}(\mathrm{Pic}(\mathbf{Sp})) \arrow[r, dashed] & \mathbf{Sp}. \end{tikzcd}$$

We apply a similar approach to construct a G-equivariant Thom spectrum. The goal of this section is the following theorem.

Theorem 3.2. *There exists a G-symmetric monoidal functor*

$$\mathbf{Th}\colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \simeq \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \to \underline{\mathbf{Sp}}^G$$

that strongly preserves G-colimits. Moreover, let $E \in \mathbf{Sp}_G$ be an invertible genuine G-spectrum and $e\colon X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ be a G-map from a G-space X such that e is G-homotopic to the constant map with value E. Then the functor $\mathbf{Th}$ takes $e \in \left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/G]}$ to the genuine G-spectrum $E \otimes X \in \mathbf{Sp}_G$.

Let us briefly explain the notation used in Theorem 3.2. Since $\mathbf{Sp}$ is the fiber of the G-∞-category $\underline{\mathbf{Sp}}^G$ over G/e [3], we can endow $\mathrm{Pic}(\mathbf{Sp})$ with a G-action whose H-fixed points are equivalent to the Picard space of genuine H-spectra. We call the resulting G-space the *Picard G-space of $\underline{\mathbf{Sp}}^G$*, and think of it as an object in the category of G-spaces, $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \in \mathbf{Top}^G$ (see Section 3.1 for details). Since $\mathbf{Top}^G$ is the fiber of the G-∞-category $\underline{\mathbf{Top}}^G$ over G/G, the object $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \in \underline{\mathbf{Top}}^G_{[G/G]}$ defines a G-functor $\mathcal{O}_G^{op} \to \underline{\mathbf{Top}}^G$ by forgetting the G-action of $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$. Finally, the G-∞-category $\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ is the parametrized slice category of $\underline{\mathbf{Top}}^G$: its fiber $\left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/H]}$ is equivalent to the slice ∞-category $\mathbf{Top}^H_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ of H-spaces[4] over $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$. In particular, the fiber $\left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/G]}$ is equivalent to the category of G-spaces over $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$, and its objects are given by maps of G-spaces $e\colon X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$.

Remark 3.3. For Theorem 3.2 and its applications, one could also work with p-local genuine G-spectra. Let $\underline{\mathbf{Sp}}^G_{(p)} \subset \underline{\mathbf{Sp}}^G$ be the G-subcategory of (fiberwise) p-local spectra. Note that the G-symmetric monoidal structure of $\underline{\mathbf{Sp}}^G$ induces a G-symmetric monoidal structure on $\underline{\mathbf{Sp}}^G_{(p)}$, as the essential image of the G-localization

$$- \otimes \mathbb{S}_{(p)}\colon \underline{\mathbf{Sp}}^G \to \underline{\mathbf{Sp}}^G,$$

[3]One should think of $\underline{\mathbf{Sp}}^G$ as defining a nontrivial G-action on $\mathbf{Sp}$.

[4]Throughout this paper, an H-space means a space with an action of the subgroup $H < G$.

see [BDG+ar, thm. 3.2 and rem. 3.3]. Replacing $\underline{\mathbf{Sp}}^G$ with $\underline{\mathbf{Sp}}^G_{(p)}$, we obtain a p-local Thom spectrum functor

$$\mathbf{Th}\colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G_{(p)})} \simeq \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G_{(p)})) \xrightarrow{\mathbf{Th}'} \underline{\mathbf{Sp}}^G_{(p)}.$$

The entire section as well as Section 4 and Section 5 hold mutatis mutandis. In particular, the p-local G-Thom spectrum of a G-map $X \to \underline{\mathrm{Pic}}(\mathbb{S}_{(p)})$ is given by

$$\mathbf{Th}\left(X \to \underline{\mathrm{Pic}}(\mathbb{S}_{(p)}) \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G_{(p)})\right),$$

where the second map is the inclusion $\underline{\mathrm{Pic}}(\mathbb{S}_{(p)}) \subset \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G_{(p)})$.

3.1 The Picard G-space

We define the Picard G-space of a G-symmetric monoidal G-∞-category and show that it inherits a G-symmetric monoidal structure.

Let $p\colon \underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ be a G-symmetric monoidal G-∞-category and let $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$ be its underlying G-∞-category. Recall that each fiber $\underline{\mathcal{C}}_{[G/H]}$ of the underlying G-∞-category is endowed with a symmetric monoidal structure, defined by the pull back of p along

$$\mathbf{Fin}_* \to \underline{\mathbf{Fin}}^G_*, \quad I \mapsto (I \times G/H \to G/H).$$

Definition 3.4. *An object x in the G-∞-category $\underline{\mathcal{C}}$ which is over G/H is* invertible *if x is an invertible object in the ∞-category $\underline{\mathcal{C}}_{[G/H]}$, that is, the object $x \in \underline{\mathcal{C}}_{[G/H]}$ is dualizable and the evaluation map $x \otimes x^\wedge \to \mathbb{1}$ is an equivalence. The Picard G-space $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ is the maximal G-∞-groupoid of $\underline{\mathcal{C}}$ spanned by invertible objects.*

Remark 3.5. By construction, $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ is a G-∞-groupoid, that is, the map $\underline{\mathrm{Pic}}(\underline{\mathcal{C}}) \to \mathcal{O}_G^{op}$ is a left fibration. Since the ∞-category of left fibrations over $\mathcal{O}_G^{op}$ is equivalent to the ∞-category of G-spaces, there is a G-space corresponding to the G-∞-groupoid $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$. By abuse of notation, we write $\underline{\mathrm{Pic}}(\underline{\mathcal{C}}) \in \mathbf{Top}^G$ for this G-space.

Remark 3.6. The fiber of $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ over G/H can be identified with the Picard space $\mathrm{Pic}\left(\underline{\mathcal{C}}_{[G/H]}\right)$.

The G-symmetric monoidal structure of the Picard G-space. In Lemma 7.12, it is shown that the left fibration

$$\underline{\mathcal{C}}^{\otimes}_{coCart} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$$

endows the maximal G-∞-groupoid $\underline{\mathcal{C}}^{\simeq} \subseteq \underline{\mathcal{C}}$ with a G-symmetric monoidal structure. This G-symmetric monoidal structure further restricts to a G-symmetric monoidal structure

$$\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$$

on $\underline{\mathrm{Pic}}(\underline{\mathcal{C}}) \subseteq \underline{\mathcal{C}}^{\simeq}$, as we now explain.

Let $I = (U \to G/H) \in \underline{\mathbf{Fin}}^G_*$ be an object. As shown in Lemma 7.12, the G-Segal map for $\underline{\mathcal{C}}^{\otimes}_{coCart}$ gives an equivalence $(\underline{\mathcal{C}}^{\otimes}_{coCart})_{[I]} \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} \underline{\mathcal{C}}^{\simeq}_{[W]}$.

Construction 3.7. Let $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes} \subseteq \underline{\mathcal{C}}^{\otimes}_{coCart}$ be the full subcategory whose fiber over the object $I = (U \to G/H) \in \underline{\mathbf{Fin}}^G_*$,

$$\left(\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes}\right)_{[I]} \subseteq \left(\underline{\mathcal{C}}^{\otimes}_{coCart}\right)_{[I]} \simeq \prod_{W \in \mathrm{Orbit}\, U} \underline{\mathcal{C}}^{\simeq}_{[W]},$$

is spanned by tuples of invertible objects

$$(x_{[W]}) \in \prod_{W \in \mathrm{Orbit}\, U} \mathrm{Pic}(\underline{\mathcal{C}}_{[W]}).$$

Here, $\mathrm{Pic}(\underline{\mathcal{C}}_{[W]}) \subseteq (\underline{\mathcal{C}}_{[W]})^{\simeq} = \underline{\mathcal{C}}^{\simeq}_{[W]}$ is the Picard space of the fiber $\underline{\mathcal{C}}_{[W]}$.

Lemma 3.8. *The restriction of $\underline{\mathcal{C}}^{\otimes}_{coCart} \twoheadrightarrow \underline{\mathbf{Fin}}^G_*$ to $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes} \subseteq \underline{\mathcal{C}}^{\otimes}_{coCart}$ defines a G-symmetric monoidal structure on $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$.*

Proof. Let $I = (U \to G/H) \in \underline{\mathbf{Fin}}^G_*$ be an object. By construction, we have

$$(\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes})_{[I]} \simeq \prod_{W \in \mathrm{Orbit}\, U} \mathrm{Pic}(\underline{\mathcal{C}}_{[W]}).$$

We show that the restriction

$$p \colon \underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes} \subseteq \underline{\mathcal{C}}^{\otimes}_{coCart} \overset{p'}{\twoheadrightarrow} \underline{\mathbf{Fin}}^G_*$$

is a left fibration. In the language of [BDG+16b, defn. 4.4], this amounts to proving that $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes}$ is a $\underline{\mathbf{Fin}}^G_*$-subcategory. By [BDG+16b, lem. 4.5], it suffices to check the following: a p'-coCartesian edge $x \to y$ lies in $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes}$ just in case x does. These are true since in each fiber $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})_{[G/H]} \subset \underline{\mathcal{C}}^{\simeq}_{[G/H]}$, invertible objects are closed under tensor products, and for every $\varphi \colon G/K \to G/H$, the norm functor $\otimes_{\varphi} \colon \underline{\mathcal{C}}_{[G/K]} \to \underline{\mathcal{C}}_{[G/H]}$ is symmetric monoidal, and in particular preserves invertible objects.

The G-Segal conditions for $\underline{\mathcal{C}}^{\otimes}_{coCart}$ restrict to give the G-Segal conditions for $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})^{\otimes}$. □

Remark 3.9. By Example 7.16, we can think of $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ as a G-commutative algebra in $(\underline{\mathbf{Top}}^G)^{\times}$.

3.2 The Thom spectrum G-functor

We are now ready to construct the G-functor of Theorem 3.2.

Construction 3.10. Let $\mathbf{Th}' \colon \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \to \underline{\mathbf{Sp}}^G$ be the G-left Kan extension of the inclusion $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathbf{Sp}}^G$ along the parametrized Yoneda embedding $\iota \colon \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))$. Define

$$\mathbf{Th} \colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \simeq \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \xrightarrow{\mathbf{Th}'} \underline{\mathbf{Sp}}^G$$

to be $\mathbf{Th}'$ precomposed with an inverse of the natural equivalence of Corollary 7.5.

Proposition 3.11. *A G-functor $\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \to \underline{\mathbf{Sp}}^G$ is equivalent to $\mathbf{Th}$ if and only if it preserves colimits and its restriction along the parametrized Yoneda embedding*

$$\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \hookrightarrow \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \simeq \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$$

is equivalent to the inclusion $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathbf{Sp}}^G$.

Proof. This follows from [Sha18, thm. 11.5]. □

Corollary 3.12. *The restriction of the G-functor $\mathbf{Th} \colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \to \underline{\mathbf{Sp}}^G$ to the fiber over $G/e \in \mathcal{O}^{op}_G$ is equivalent to the Thom spectrum functor*

$$\mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})} \simeq \mathrm{Psh}(\mathrm{Pic}(\mathbf{Sp})) \to \mathbf{Sp}$$

of [ABG+14a].

Proof. It suffices to show the restriction of the Thom spectrum G-functor $\mathbf{Th}$ satisfies the universal property of the Thom spectrum functor as in [ABG+14a, cor. 3.13].

We can identify the fibers over the orbit G/e of the corresponding G-∞-categories as

$$\left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/e]} \simeq \left(\underline{\mathbf{Top}}^G_{[G/e]}\right)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/e]}} \simeq \mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})}, \quad \left(\underline{\mathbf{Sp}}^G\right)_{[G/e]} \simeq \mathbf{Sp}.$$

By construction, the functor

$$\mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})} \simeq \left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/e]} \xrightarrow{\mathbf{Th}_{[G/e]}} \left(\underline{\mathbf{Sp}}^G\right)_{[G/e]} \simeq \mathbf{Sp}$$

preserves colimits and its restriction along

$$\mathrm{Pic}(\mathbf{Sp}) \simeq \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/e]} \to \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))_{[G/e]} \simeq \mathcal{S}_{/\,\mathrm{Pic}(\mathbf{Sp})}$$

is equivalent to the canonical inclusion $\mathrm{Pic}(\mathbf{Sp}) \simeq \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/e]} \to (\underline{\mathbf{Sp}}^G)_{[G/e]} \simeq \mathbf{Sp}$. $\square$

The Thom spectrum of a stably trivial sphere bundle over X is given by a smash product $S^n \otimes \Sigma^\infty_+ X$ with an invertible spectrum $S^n \in \mathbf{Sp}$. Our next goal is an equivariant version of this fact.

We will use the following notation when discussing G-Thom spectra.

Notation 3.13. Consider $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ as a G-space by Remark 3.5. An invertible H-spectrum $E \in \left(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right)_{[G/H]} \simeq \mathrm{Pic}(\mathbf{Sp}_H)$ is then an H-fixed point of the G-space $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$, which corresponds to an H-equivariant map $* \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$. We denote by $f_E \colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ the H-map corresponding to E.

Proposition 3.14. *The G-functor* $\mathbf{Th}$ *sends the map of H-spaces $f_E \colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ to its corresponding invertible H-spectrum $E \in \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/H]}$.*

Proof. The parametrized Yoneda embedding $\iota \colon \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))$ is fully faithful ([BDG+16b, thm. 10.4]). By [Sha18, prop. 10.5] the composition $\mathbf{Th}' \circ \iota$ is equivalent to the inclusion $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathbf{Sp}}^G$, so $\mathbf{Th}'(\iota(E)) = E$. We will finish the proof by showing that $f_E \colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ corresponds to $\iota(E) \in \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ under the equivalence

$$\underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \simeq \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$$

in Corollary 7.5.

The equivalence of Corollary 7.5 sends the representable presheaf

$$\iota(E) \in \underline{\mathrm{Psh}}_G(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))_{[G/H]}$$

to a $\underline{G/H}$-functor of $\underline{G/H}$-∞-groupoids

$$\left(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{/\underline{E}} \twoheadrightarrow \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \underline{\times} \underline{G/H}\right) \in \left(\mathbf{Top}_{\underline{G/H}}\right)_{/(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\underline{\times}\underline{G/H})} \simeq \left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/H]}.$$

Note that the fibers of $(\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{/\underline{E}})$ are all contractible as slices of ∞-groupoids, so the natural $\underline{G/H}$-functor $\sigma_E \colon \underline{G/H} \to (\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))_{/\underline{E}}$ is a $\underline{G/H}$-equivalence. It follows that the $\underline{G/H}$-functor $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{/\underline{E}} \twoheadrightarrow \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\underline{\times}\underline{G/H}$ is equivalent to the composition

$$\underline{G/H} \xrightarrow{\sigma_E} (\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))_{/\underline{E}} \twoheadrightarrow \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\underline{\times}\underline{G/H}.$$

This composition is precisely the $\underline{G/H}$-object $E \colon \underline{G/H} \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\underline{\times}\underline{G/H}$ associated to E, which corresponds to the H-map $f_E \colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ under the isomorphism

$$\mathrm{Fun}_{\underline{G/H}}(\underline{G/H}, \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\underline{\times}\underline{G/H}) \cong \mathrm{Fun}_G(\underline{G/H}, \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \cong \mathrm{Map}_H(*, \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)). \qquad \square$$

Using Proposition 3.14 we can calculate the equivariant Thom spectrum of G-nullhomotopic maps.

Proposition 3.15. *Let $X \in \mathbf{Top}^G \simeq \underline{\mathbf{Top}}^G_{[G/G]}$ be a G-space and $E \in \mathrm{Pic}(\mathbf{Sp}_G) \simeq \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/G]}$ be an invertible G-spectrum, then*

$$\mathbf{Th}(X \to * \xrightarrow{f_E} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \simeq E \otimes \Sigma^\infty_+ X.$$

Proof. The point $* \in \mathbf{Top}^G \simeq \underline{\mathbf{Top}}^G_{[G/G]}$ is the terminal G-space. Express the G-space X as $G\text{-}\underrightarrow{colim}_X(*)$, the G-colimit of the constant G-diagram

$$X \to \underline{G/G} \to \underline{\mathbf{Top}}^G,$$

where the second functor corresponds to the terminal G-space. Postcomposition with f_E induces a G-functor

$$\begin{array}{rccccc} (f_E)_* \colon & \underline{\mathbf{Top}}^G & \simeq & \underline{\mathbf{Top}}^G_{/\underline{*}} & \to & \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}, \\ & X & \mapsto & (X \to *) & \mapsto & (X \to * \xrightarrow{f_E} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)), \end{array}$$

and this G-functor strongly preserves G-colimits. Since $(f_E)_*(*) = \left(f_E \colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right)$ we have

$$(f_E)_*(X) = (f_E)_* \left(G\text{-}\underrightarrow{colim}_X(*)\right) \simeq G\text{-}\underrightarrow{colim}_X (f_E) \in \left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)_{[G/G]}.$$

We can now apply the G-functor $\mathbf{Th}$ of Construction 3.10 to $(f_E)_*(X)$. By Proposition 3.11 and Proposition 3.14 we have

$$\mathbf{Th}\left((f_E)_*(X)\right) \simeq \mathbf{Th}\left(G\text{-}\underrightarrow{colim}_X (f_E)\right) \simeq G\text{-}\underrightarrow{colim}_X \left(\mathbf{Th}\left(f_E\right)\right) \simeq G\text{-}\underrightarrow{colim}_X(E).$$

On the other hand,

$$E \otimes \Sigma^\infty_+ X \simeq E \otimes \Sigma^\infty_+ \left(G\text{-}\underrightarrow{colim}_X(*)\right) \simeq G\text{-}\underrightarrow{colim}_X \left(E \otimes \Sigma^\infty_+ *\right) \simeq G\text{-}\underrightarrow{colim}_X(E).$$

Together we have

$$\mathbf{Th}\left(X \to * \xrightarrow{f_E} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) \simeq E \otimes \Sigma^\infty_+ X,$$

as claimed. $\square$

We end this section by extending $\mathbf{Th}$ to a G-symmetric monoidal functor.

We first describe the G-symmetric monoidal structure on $(\underline{\mathbf{Top}}^G)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$. We have seen in Section 3.1 that the G-symmetric monoidal structure of $\underline{\mathbf{Sp}}^G$ induces a G-symmetric monoidal structure on the G-∞-groupoid $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$, and that we can consider $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ as a G-commutative algebra in $(\underline{\mathbf{Top}}^G)^\times$. Therefore, we can endow the parametrized slice category $(\underline{\mathbf{Top}}^G)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ with the parametzied slice G-symmetric monoidal structure of Section 7.5.

Proposition 3.16. *The G-functor $\mathbf{Th}$ of Construction 3.10 extends to a G-symmetric monoidal functor*

$$\mathbf{Th}^\otimes \colon \left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)^\otimes \to (\underline{\mathbf{Sp}}^G)^\otimes,$$

where $\left(\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right)^\otimes$ *is the slice G-symmetric monoidal structure of Section 7.5.*

Proof. By Proposition 7.8, the G-left Kan extension $\mathbf{Th}'$ extends to a G-symmetric monoidal functor. The result follows from the fact that the equivalence of Corollary 7.5 extends to a G-symmetric monoidal equivalence, see Theorem 7.20. $\square$

Corollary 3.17. *Let X be a G-space, let $f\colon Y \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ be a G-map and*

$$A = \mathbf{Th}(Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))$$

its G-Thom spectrum. Then we have an equivalence of genuine G-spectra

$$\mathbf{Th}(X \times Y \xrightarrow{pr} Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \simeq A \otimes \Sigma^\infty_+ X.$$

Proof. The tensor product in $(\underline{\mathbf{Top}}^G)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ admits the following description

$$\begin{aligned}&\left(Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) \otimes \left(X \to * \xrightarrow{\mathbb{S}} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right)\\ &\qquad\simeq \left(X \times Y \to * \times Y \xrightarrow{f_{\mathbb{S}} \times f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \times \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \xrightarrow{\otimes} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right),\end{aligned}$$

where $\mathbb{S} \in \mathbf{Sp}_G$ is the G-sphere spectrum. Since the map $f_{\mathbb{S}}\colon * \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ is constant on $\mathbb{S}$, the unit of $\mathrm{Pic}(\mathbf{Sp}_G) = \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)_{[G/G]}$, the composition

$$Y = * \times Y \xrightarrow{f_{\mathbb{S}} \times f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \times \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \xrightarrow{\otimes} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$$

is equivalent to $f\colon Y \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$. Therefore

$$\left(Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) \otimes \left(X \to * \xrightarrow{\mathbb{S}} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) \simeq \left(X \times Y \xrightarrow{pr} Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right).$$

The G-Thom functor is G-symmetric monoidal so

$$\begin{aligned}\mathbf{Th}\left(X \times Y \xrightarrow{pr} Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) &\simeq \mathbf{Th}\left(Y \xrightarrow{f} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right) \otimes \mathbf{Th}\left(X \to * \xrightarrow{\mathbb{S}} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right)\\ &\simeq A \otimes \left(\mathbb{S} \otimes \Sigma^\infty_+ X\right) \simeq A \otimes \Sigma^\infty_+ X,\end{aligned}$$

where the second equivalence follows from Proposition 3.15. $\square$

4 Parametrized Thom spectrum and genuine equivariant factorization homology

In this section, we use the results of Section 3 to prove that our G-Thom spectrum functor respects equivariant factorization homology.

We will use both V-framed G-disk algebras and $\mathbb{E}_V$-algebras, whose definitions we now recall; these definitions are in fact equivalent.

Let V be a real n-dimensional representation of G. The representation V defines a G-map $pt \to BO_n(G)$ to the classifying G-space of rank n real G-vector bundles (see [Hor19, cor. 3.2.7, ex. 3.3.3]), which defines a notion of V-framed G-manifolds and V-framed G-disks (see [Hor19, ex. 3.3.3]). Let $\underline{\mathcal{C}}$ be a G-symmetric monoidal G-∞-category.

Definition 4.1. *A V-framed G-disk algebra (see [Hor19, def. 3.6.11, ex. 3.6.12]) in $\underline{\mathcal{C}}$ is a G-symmetric monoidal functor*

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathcal{C}}^{\otimes}.$$

One can also consider the notion of an $\mathbb{E}_V$-algebra, that is, a map of G-∞-operads

$$\mathbb{E}_V^{\otimes} \to \underline{\mathcal{C}}^{\otimes},$$

where $\mathbb{E}_V^{\otimes}$ is the G-∞-operad of [Bon19, ex. 6.5]. It is the geunine operadic nerve of the V-little disks G-operad. In fact, these two notions are equivalent, and we will use them interchangeably for the rest of this paper.

Proposition 4.2 ([Hor19, cor. 3.9.9])**.** *There is an equivalence of ∞-categories*

$$\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{C}}) \simeq \mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathcal{C}})$$

between the ∞-category of $\mathbb{E}_V$-algebras in $\underline{\mathcal{C}}$ and the ∞-category $\mathrm{Fun}_G^{\otimes}(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathcal{C}})$ of V-framed G-disk algebras in $\underline{\mathcal{C}}$.

With these definitions at hand we study the compatibility of the G-functor $\mathbf{Th}$ and genuine equivariant factorization homology. We'll use the following equivariant version of [AF15, lem. 3.25].

Lemma 4.3. *Let $\underline{\mathcal{C}}^{\otimes}, \underline{\mathcal{D}}^{\otimes}$ be presentable G-symmetric monoidal G-∞-categories, and let*

$$F \colon \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$$

be a G-symmetric monoidal G-functor whose restriction to the underlying G-∞-categories strongly preserves G-colimits (see [Sha18, def. 11.2]). Let $A \in \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathcal{C}})$ be an $\mathbb{E}_V$-algebra in $\mathcal{C}$. Then the composition

$$F \circ \int_{\_} A \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathcal{C}} \to \underline{\mathcal{D}}$$

is equivalent to $\int_{\_} F(A) \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathcal{D}}$, where $F(A)$ is the $\mathbb{E}_V$-algebra in $\mathcal{D}$ corresponding the V-framed G-disk algebra

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \xrightarrow{A} \underline{\mathcal{C}}^{\otimes} \xrightarrow{F} \underline{\mathcal{D}}^{\otimes}$$

under Proposition 4.2.

Proof. The G-functor $F \circ \int_{\_} A$ extends to a G-symmetric monoidal G-functor

$$\underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \xrightarrow{\int_{\_} A} \underline{\mathcal{C}}^{\otimes} \xrightarrow{F^{\otimes}} \underline{\mathcal{D}}^{\otimes}.$$

The G-symmetric monoidal functor $\int_{\_} A$ satisfies G-$\otimes$-excision and respects G-sequential unions (see [Hor19, prop 5.2.3, prop. 5.3.3]). Since F strongly preserves G-colimits the functors $\underline{\mathbf{Mfld}}^{G,V-fr}_{[G/H]} \xrightarrow{\int_{\_} A} \underline{\mathcal{C}}_{[G/H]} \xrightarrow{F} \underline{\mathcal{D}}_{[G/H]}$ preserve colimits, so $F \circ \int_{\_} A$ also satisfies G-$\otimes$-excision and respects G-sequential unions. By [Hor19, thm. 6.0.2] we have an equivalence

$$\mathcal{H}(\underline{\mathbf{Mfld}}^{G,V-fr}, \underline{\mathcal{D}}) \xrightarrow{\sim} \mathrm{Fun}^{\otimes}_{G}(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathcal{D}})$$

from the ∞-category of G-symmetric monoidal functors which satisfy G-$\otimes$-excision and respect G-sequential unions to the ∞-category of V-framed G-disk algebras, given by restriction to $\underline{\mathbf{Disk}}^{G,V-fr} \subset \underline{\mathbf{Mfld}}^{G,V-fr}$. It follows that $F \circ \int_{\_} A$ is equivalent to a genuine G-factorization homology $\underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \to \underline{\mathcal{D}}^{\otimes}$ with coefficients given by the restriction of $F \circ \int_{\_} A$ to $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup}$. Since $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \xrightarrow{\int_{\_} A} \underline{\mathcal{C}}^{\otimes}$ corresponds to the $\mathbb{E}_V$-algebra A, the coefficients of $F \circ \int_{\_} A$ correspond to $F(A)$. □

Proposition 4.4. *If A is an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Top}}^{G}_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$, then we have a natural equivalence*

$$\int_{\_} \mathbf{Th}(A) \simeq \mathbf{Th}\left(\int_{\_} A\right).$$

Proof. By Construction 3.10 and Proposition 3.16, the G-Thom spectrum

$$\mathbf{Th} \colon \underline{\mathbf{Top}}^{G}_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \to \underline{\mathbf{Sp}}^{G}$$

strongly preserves G-colimits, and extends to a G-symmetric monoidal G-functor. The claim now follows from Lemma 4.3 with $\underline{\mathcal{C}} = \underline{\mathbf{Top}}^{G}_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$ and $\underline{\mathcal{D}} = \underline{\mathbf{Sp}}^{G}$. □

5 V-fold loop spaces and $\mathbb{E}_V$-algebras

The coefficients for $\displaystyle\int_M -$, where M is a V-framed G-manifold, are $\mathbb{E}_V$-algebras as in Definition 4.1. The $\mathbb{E}_V$-algebras we consider in this paper typically arise in two ways: as Thom spectra of V-fold loop maps, or as G-commutative algebras. In this section, we will explain how we consider each of those as an $\mathbb{E}_V$-algebra, and give a description of the equivariant factorization homology of the G-Thom spectrum of a V-fold loop map.

5.1 G-commutative algebras as $\mathbb{E}_V$-algebras

The first examples of $\mathbb{E}_V$-algebras are given by G-commutative algebras. Recall ([Nar17, ex. 3.3]) that a G-commutative algebra is a map of G-∞-operads from the terminal G-∞-operad to a G-symmetric monoidal G-∞-category $\underline{\mathcal{C}}^{\otimes} \to \underline{\mathbf{Fin}}^{G}_{*}$. Since the terminal G-∞-operad $\underline{\mathbf{Fin}}^{G}_{*} \to \underline{\mathbf{Fin}}^{G}_{*}$ is itself a G-symmetric monoidal G-∞-category, a G-commutative algebra $A \in \mathrm{CAlg}_G(\underline{\mathcal{C}})$ is a *lax* G-symmetric monoidal functor $A \colon \underline{\mathbf{Fin}}^{G}_{*} \to \underline{\mathcal{C}}^{\otimes}$.

Note that the structure map $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Fin}}_*^G$ can itself be considered as a G-symmetric monoidal functor. Therefore we can consider any G-commutative algebra $A \in \mathrm{CAlg}_G(\underline{\mathcal{C}})$ as an $\mathbb{E}_V$-algebra by precomposition with the structure map

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Fin}}_*^G \xrightarrow{A} \underline{\mathcal{C}}^{\otimes}.$$

For $\underline{\mathcal{C}}^{\otimes} = \underline{\mathbf{Top}}^{G,\times}$ the notion of G-commutative algebra agrees with a G-symmetric monoidal structure on a G-space, considered as a G-∞-groupoid.

5.2 V-fold loop spaces as $\mathbb{E}_V$-algebras

In this subsection, we explain how to establish V-fold loop spaces as G-symmetric monoidal functors

$$\Omega^V X : \underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Top}}^{G,\times}.$$

This is in Construction 5.8.

First, we upgrade the one point compactification to a G-symmetric monoidal functor. We rely on the fact that one point compactification defines a functor of topological categories.

Construction 5.1. Let $M \in \mathbf{Mfld}_n$ be an n-dimensional manifold, and denote its one point compactification by $M^+ \in \mathbf{Top}_*$. Since morphisms in $\mathbf{Mfld}_n$ are open embeddings of manifolds, one point compactification defines a functor $(-)^+ : \mathbf{Mfld}_n \to (\mathbf{Top}_*)^{op}$. Furthermore, $(-)^+$ takes disjoint unions to wedge sums, so it defines a symmetric monoidal functor

$$(-)^+ : (\mathbf{Mfld}_n, \sqcup) \to ((\mathbf{Top}_*)^{op}, \vee).$$

We consider both categories $\mathbf{Mfld}_n$ and $\mathbf{Top}_*$ as *topological* symmetric monoidal categories using the compact open topology. By [hc], this one point compactification is a functor of topological categories.

From a symmetric monoidal topological category, one can consider the G-objects and apply the genuine operadic nerve construction of [Bon19] to obtain a G-symmetric monoidal G-∞-category. For details of this procedure, see Section 7.8. This gives another way to construct the G-∞-category $\underline{\mathbf{Mfld}}^{G,\sqcup}$.

Lemma 5.2. *Let $(\mathbf{Mfld}_n, \sqcup)$ be the topological symmetric monoidal category of [AF15, def. 2.1]. The G-symmetric monoidal G-∞-category of topological G-objects in $(\mathbf{Mfld}_n, \sqcup)$ is equivalent to the G-symmetric monoidal G-∞-category $\underline{\mathbf{Mfld}}^{G,\sqcup}$ defined in [Hor19, sec. 3.4].*

Similarly, one can use the genuine operadic nerve construction to construct the G-∞-category of pointed G-spaces with the G-coCartesian G-symmetric monoidal structure (see [BDG+ar, sec. B-coCartesian operads]).

Lemma 5.3. *The G-symmetric monoidal G-∞-category of topological G-objects in $((\mathbf{Top}_*)^{op}, \vee)$ is equivalent to the G-∞-category $(\underline{\mathbf{Top}}_*^{G,\vee})^{vop}$, where the G-∞-category of pointed G-spaces $\underline{\mathbf{Top}}_*^G$ is endowed with the G-coCartesian G-symmetric monoidal structure.*

Since the construction of the G-symmetric monoidal G-∞-category of topological G-objects is functorial, we can apply it to the one point compactification functor, and get a G-symmetric monoidal functor

$$(-)^+ : \underline{\mathbf{Mfld}}^{G,\sqcup} \to (\underline{\mathbf{Top}}_*^{G,\vee})^{vop}. \tag{5.4}$$

Next, we describe the G-symmetric monoidal functor $\underline{\mathrm{Map}}_*(-, X)$.

Construction 5.5. Let $X \in (\underline{\mathbf{Top}}_*^G)_{[G/G]}$ be a pointed topological G-space. Applying the parametrized Yoneda embedding of [BDG+16b, def. 10.2] to X we get a G-functor

$$\underline{\mathrm{Map}}_*(-,X)\colon (\underline{\mathbf{Top}}_*^G)^{vop} \to \underline{\mathbf{Top}}^G.$$

By [Sha18, cor. 11.9] the functor $\underline{\mathrm{Map}}_*(-,X)\colon (\underline{\mathbf{Top}}_*^G)^{vop} \to \underline{\mathbf{Top}}^G$ preserves G-limits. Since the G-symmetric monoidal structures on $(\underline{\mathbf{Top}}_*^G)^{vop}$ and $\underline{\mathbf{Top}}^G$ are G-Cartesian it follows that $\underline{\mathrm{Map}}_*(-,X)$ extends to a G-symmetric monoidal functor

$$\underline{\mathrm{Map}}_*(-,X)\colon (\underline{\mathbf{Top}}_*^{G,\vee})^{vop} \to \underline{\mathbf{Top}}^{G,\times}. \tag{5.6}$$

Composing Eq. (5.4) and Eq. (5.6), we get a G-symmetric monoidal functor

$$\underline{\mathrm{Map}}_*\big((-)^+,X\big) : \underline{\mathbf{Mfld}}^{G,\sqcup} \xrightarrow{(-)^+} (\underline{\mathbf{Top}}_*^{G,\vee})^{vop} \xrightarrow{\underline{\mathrm{Map}}_*(-,X)} \underline{\mathbf{Top}}^{G,\times}. \tag{5.7}$$

Construction 5.8. Fixing an n-dimensional representation V, we can precompose Eq. (5.7) with the forgetful map from V-framed G-manifolds to G-manifolds. We can further restrict to V-framed G-disks and obtain

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Mfld}}^{G,\sqcup} \xrightarrow{(-)^+} (\underline{\mathbf{Top}}_*^{G,\vee})^{vop} \xrightarrow{\underline{\mathrm{Map}}_*(-,X)} \underline{\mathbf{Top}}^{G,\times}. \tag{5.9}$$

We denote the composite by $\Omega^V X$, and it is an $\mathbb{E}_V$ algebra in $\underline{\mathbf{Top}}^G$. The underlying G-space of $\Omega^V X$ is given by evaluating the functor $\Omega^V X$ at $V \in \underline{\mathbf{Disk}}^{G,V-fr}_{[G/G]}$, which is the G-space $\mathrm{Map}_*(S^V, X)$ of pointed maps from the representation sphere $S^V = V^+$ to X. Note that the G-space $\mathrm{Map}_*(S^V,X)$ is equivalent to the V-fold loop space of X, which justifies the name $\Omega^V X$.

Remark 5.10. Restricting the G-symmetric monoidal functor of Eq. (5.7) to the fiber over $G/G \in \mathcal{O}_G^{op}$ defines a symmetric monoidal functor of ∞-categories

$$\mathrm{Map}_*\big((-)^+,X\big) : \mathbf{Mfld}^{G,\sqcup} \xrightarrow{(-)^+} (\mathbf{Top}_*^{G,\vee})^{op} \xrightarrow{\mathrm{Map}_*(-,X)} \mathbf{Top}^{G,\times}.$$

Note that, to obtain this symmetric monoidal functor alone, we can simply apply the operadic nerve construction of [Lur12, 2.1.1.23] to the corresponding symmetric monoidal *topological* categories and functors. Indeed, we can identify the ∞-categories and functors above as follows:

- The symmetric monoidal ∞-category $\mathbf{Mfld}^{G,\sqcup}$ is equivalent to the operadic nerve of the topological category of smooth n-dimensional G-manifolds and G-equivariant smooth embeddings, with symmetric monoidal structure given by disjoint unions.
- The symmetric monoidal ∞-category $\mathbf{Top}_*^{G,\vee}$ is equivalent to the operadic nerve of topological category of pointed G-CW spaces with the coCartesian monoidal structure (given by the wedge of pointed G-spaces). The operadic nerve of the opposite topological category is equivalent to $\left(\mathbf{Top}_*^{G,\vee}\right)^{op}$.
- The symmetric monoidal ∞-category $\mathbf{Top}^{G,\times}$ is equivalent to the operadic nerve of topological category of G-CW spaces with the Cartesian monoidal structure (given by the products of G-spaces).
- The symmetric monoidal functor $(-)^+\colon \mathbf{Mfld}^{G,\sqcup} \to (\mathbf{Top}_*^{G,\vee})^{op}$ can be identified with the operadic nerve of the one point compactification functor, as in Construction 5.1.

- The symmetric monoidal functor $\mathrm{Map}_*(-, X)\colon (\mathbf{Top}_*^{G,\vee})^{op} \to \mathbf{Top}^{G,\times}$ can be identified with the operadic nerve of the topological functor sending a pointed G-space Y to the space of pointed maps $\mathrm{Map}_*(Y, X)$, with G-action given by conjugation.

We can therefore identify the G-space $\mathrm{Map}_*(M^+, X)$ with the space of pointed maps $M^+ \to X$, with G acting by conjugation.

Remark 5.11. Note that Construction 5.8 is functorial in $X \in \mathbf{Top}_{G,*} \simeq (\underline{\mathbf{Top}}_*^G)_{[G/G]}$. The parametrized Yoneda embedding $j\colon \underline{\mathbf{Top}}_*^G \to \underline{\mathrm{Psh}}_G(\underline{\mathbf{Top}}^G)$ of [BDG+16b, def. 10.2] is a G-functor, and in particular defines a map between the fibers over $G/G \in \mathcal{O}_G^{op}$, which is a functor of ∞-categories

$$\mathbf{Top}_{G,*} \to \mathrm{Fun}_G((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G), \quad X \mapsto \mathrm{Map}_*(-, X)\colon (\underline{\mathbf{Top}}_*^G)^{vop} \to \underline{\mathbf{Top}}^G.$$

This functor factors through the full subcategory of $\mathrm{Fun}_G((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G)$ spanned by G-functors preserving finite G-products. Let $\mathrm{Fun}_G^\times((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G)$ denote the ∞-category of G-symmetric monoidal G-functors with respect to the G-Cartesian G-symmetric monoidal structures. Since $\mathrm{Fun}_G^\times((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G) \subset \mathrm{Fun}_G((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G)$ is equivalent to the full subcategory described above, the functor $\mathrm{Map}_*(-, X)$ lifts to

$$\mathbf{Top}_{G,*} \to \mathrm{Fun}_G^\times((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G), \quad X \mapsto \mathrm{Map}_*(-, X)\colon (\underline{\mathbf{Top}}_*^G)^{vop} \to \underline{\mathbf{Top}}^G.$$

Precomposition with the G-symmetric monoidal functor

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \subset \underline{\mathbf{Mfld}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Mfld}}^{G,\sqcup} \xrightarrow{(-)^+} (\underline{\mathbf{Top}}_*^{G,\vee})^{vop}$$

defines a functor of ∞-categories

$$\mathbf{Top}_{G,*} \to \mathrm{Fun}_G^\times((\underline{\mathbf{Top}}_*^G)^{vop}, \underline{\mathbf{Top}}^G) \to \mathrm{Fun}_G^\otimes(\underline{\mathbf{Disk}}^{G,V-fr}, \underline{\mathbf{Top}}^G), \quad X \mapsto \Omega^V X.$$

5.3 V-fold loop maps as $\mathbb{E}_V$-algebras

Let $f\colon X \to Y$ be a map of pointed G-spaces. From the functoriality of Construction 5.8 we get $\Omega^V f\colon \Omega^V X \to \Omega^V Y$, which is a map of $\mathbb{E}_V$-algebras in $\underline{\mathbf{Top}}^G$. Suppose $\Omega^V Y$ is a G-commutative algebra. That is, there is a G-commutative algebra A such that the following construction in Section 5.1

$$\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Fin}}_*^G \xrightarrow{A} \underline{\mathbf{Top}}^G$$

is equivalent as a G-symmetric monoidal functor to

$$\Omega^V Y\colon \underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \to \underline{\mathbf{Top}}^G.$$

In this case, the map $\Omega^V f$ can be considered as an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Top}}_{/A}^G$.

Definition 5.12. *Let $\underline{\mathcal{C}}^\otimes$ be a G-symmetric monoidal G-∞-category and $A\colon \underline{\mathbf{Fin}}_*^G \to \underline{\mathcal{C}}^\otimes$ be a G-commutative algebra. Define $\underline{\mathcal{C}}_{/A}^\otimes \to \underline{\mathbf{Fin}}_*^G$ by applying the construction [Lur12, def. 2.2.2.1] for $K = \Delta^0, S = \underline{\mathbf{Fin}}_*^G$ and $S \times K \to S$ the identity map.*

The following result on G-symmetric monoidal structure on parametrized slice categories is known.

Lemma 5.13. *The map* $\underline{\mathcal{C}}^{\otimes}_{/\underline{A}} \to \underline{\mathbf{Fin}}^G_*$ *defines a G-symmetric monoidal G-∞-category, with underlying G-∞-category equivalent to the parametized slice* $\underline{\mathcal{C}}_{/\underline{A}} \to \mathcal{O}_G^{op}$ *of [Sha18, not. 4.29].*

Lemma 5.14. *Let* $\mathcal{O}^{\otimes} \to \underline{\mathbf{Fin}}^G_*$ *be a G-∞-operad. The ∞-category* $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{C}}^{\otimes}_{/\underline{A}})$ *of* $\mathcal{O}$*-algebras in* $\underline{\mathcal{C}}$ *is equivalent to the slice ∞-category* $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{C}})_{/A}$ *of* $\mathcal{O}$*-algebras over A.*

In particular, taking $\underline{\mathcal{C}}^{\otimes} = (\underline{\mathbf{Top}}^G)^{\times}$ and $A = \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$, we have an equivalence of ∞-categories

$$\mathrm{Alg}_{\mathbb{E}_V}\left((\underline{\mathbf{Top}}^G)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}\right) \simeq \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathbf{Top}}^G)_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)},$$

by which we can consider $\Omega^V f : \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ as an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$. Applying Proposition 4.4 with coefficients $A = \Omega^V f$, we get a natural equivalence

$$\int_{-} \mathbf{Th}(\Omega^V f) \simeq \mathbf{Th}\left(\int_{-} \Omega^V f\right) \tag{5.15}$$

of G-functors $\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Sp}}^G$.

5.4 Equivariant factorization homology of equivariant Thom spectra

In this subsection we describes the interaction between equivariant Thom spectra and equivariant factorization homology. Our main result is Theorem 5.20, which describes the genuine G-factorization homology theory

$$\int_{-} \Omega^V f \colon \underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)}$$

appearing at the right hand side of Eq. (5.15).

In fact, this description works when $\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ is replaced by any G-commutative algebra B in G-spaces. For the next propositions we fix $B \in \mathrm{CAlg}_G(\underline{\mathbf{Top}}^G)$ to be a G-commutative algebra in G-spaces and $\Omega^V f : \Omega^V X \to B$ to be a map of $\mathbb{E}_V$-algebras. By Section 5.3, $\Omega^V f$ can be considered as an $\mathbb{E}_V$-algebra in $\underline{\mathbf{Top}}^G_{/\underline{B}}$.

We first state some properties of the forgetful G-functor $\underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G$.

Lemma 5.16. *The forgetful G-functor* $fgt \colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G$ *preserves G-colimits.*

Lemma 5.17. *Let* $(\underline{\mathbf{Top}}^G)^{\times}$ *denote the G-Cartesian G-symmetric monoidal structure on* $\underline{\mathbf{Top}}^G$*. The forgetful G-functor* $fgt \colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G$ *extends to a G-symmetric monoidal functor*

$$fgt \colon (\underline{\mathbf{Top}}^G)^{\times}_{/\underline{B}} \to (\underline{\mathbf{Top}}^G)^{\times}.$$

Note that the composition $\underline{\mathbf{Disk}}^{G,V-fr,\sqcup} \xrightarrow{\Omega^V f} (\underline{\mathbf{Top}}^G)^{\times}_{/\underline{B}} \xrightarrow{fgt} (\underline{\mathbf{Top}}^G)^{\times}$ is equivalent to $\Omega^V X$. Combining these two lemmas with Lemma 4.3, we get:

Proposition 5.18. *Let* $\Omega^V f \colon \Omega^V X \to B$ *be a V-fold loop map. Then we have a natural equivalence*

$$fgt\left(\int_{-} \Omega^V f\right) \simeq \int_{-} \Omega^V X$$

of G-functors $\underline{\mathbf{Mfld}}^{G,V-fr} \to \underline{\mathbf{Top}}^G$*.*

We therefore know that for a V-framed G-manifold $M \in \underline{\mathbf{Mfld}}^{G,V-fr}_{[G/G]}$,

$$\int_M \Omega^V f \in (\underline{\mathbf{Top}}^G_{/\underline{B}})_{[G/G]} \simeq (\mathbf{Top}_G)_{/B}$$

is given by a map of G-spaces

$$\int_M \Omega^V X \to B. \tag{5.19}$$

Our next task is to describe this map. Consider id_B as an object of $\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathbf{Top}}^G)_{/B}$, and observe that the map $\Omega^V f\colon \Omega^V X \to B$ can be considered as a map $\epsilon : \Omega^V f \to id_B$ in $\mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathbf{Top}}^G)_{/B} \simeq \mathrm{Alg}_{\mathbb{E}_V}(\underline{\mathbf{Top}}^G_{/\underline{B}})$. This map of $\mathbb{E}_V$-algebras induces a natural transformation $\epsilon_*\colon \int_- \Omega^V f \to \int_- id_B$. Composing this with the forgetful functor $fgt\colon \underline{\mathbf{Top}}^G_{/\underline{B}} \to \underline{\mathbf{Top}}^G$, we get $fgt(\epsilon_*) = (\Omega^V f)_*\colon \int_- \Omega^V X \to \int_- B$.

In particular, for a V-framed G-manifold M, the morphism $\epsilon_*\colon \int_M \Omega^V f \to \int_M id_B$ is given by the map of G-spaces $(\Omega^V f)_*\colon \int_M \Omega^V X \to \int_M B$ over B. It follows that the G-map of Eq. (5.19) factors as

$$\int_M \Omega^V X \xrightarrow{(\Omega^V f)_*} \int_M B \to B,$$

where $\int_M B \to B$ is given by $\int_M id_B$. Specializing to the G-commutative algebra $B = \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ and combining with Eq. (5.15), we have therefore shown

Theorem 5.20. *Let X be a pointed G-space and $\Omega^V f\colon \Omega^V X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ be a map of $\mathbb{E}_V$-algebras. Then for every V-framed G-manifold M, there is an equivalence of genuine G-spectra*

$$\int_M \mathbf{Th}(\Omega^V f) \simeq \mathbf{Th}\left(\int_M \Omega^V X \xrightarrow{(\Omega^V f)_*} \int_M \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)\right).$$

Here, $\mathbf{Th}\colon \underline{\mathbf{Top}}^G_{/\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)} \to \underline{\mathbf{Sp}}^G$ *is the parametrized Thom G-functor in Construction 3.10.*

6 Some computational corollaries

Assume that G is a finite group and V is a finite dimensional G-representation. In this section, we prove Theorem 6.1, which deals with factorization homology when the algebra A is a Thom spectrum of a more highly commutative map than $\mathbb{E}_V$; it is as commutative as a representation that M embeds in. We apply Theorem 6.1 to compute the genuine equivariant factorization homology of certain Thom spectra. In Corollary 6.5, we compute the factorization homology of the Real bordism spectrum, $MU_{\mathbb{R}}$. In Corollary 6.6, we treat Eilenberg–MacLane spectra; see the appendix by Hahn–Wilson for a computation of $\mathrm{THR}(H\underline{\mathbb{Z}})$. In Corollary 6.7, we compute C_2-relative THH (see [ABG+14b]), $\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2)$.

Theorem 6.1. *Let A be the G-Thom spectrum of an $\mathbb{E}_{V\oplus W}$-map,*

$$\Omega^{V\oplus W} f : \Omega^{V\oplus W} X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G),$$

with $\pi_k(X^H) = 0$ for all subgroups $H < G$ and $k < \dim((V \oplus W)^H)$. Let M be a G-manifold of the same dimension as the representation V. Suppose that $M \times W$ embeds equivariantly in $V \times W$, and that there is an equivariant embedding $V \hookrightarrow M$ (call its image D). Then

$$\int_{M\times W} A \simeq A \otimes \Sigma^\infty_+ \mathrm{Map}_*(M^+ - D, \Omega^W X).$$

Remark 6.2. Recall that we use $\otimes$ to denote the smash product of (G-)spectra, and Map_* to denote the G-space of non-equivariant based maps.

A particularly useful corollary is obtained by setting $W = \mathbb{R}$ and $M = S^V$.

Corollary 6.3. *Let A be the G-Thom spectrum of an $\mathbb{E}_{V\oplus\mathbb{R}}$-map $\Omega^{V\oplus\mathbb{R}}X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ with $\pi_k(X^H) = 0$ for all subgroups $H < G$ and $k < \dim(V^H) + 1$. Then*

$$\int_{S^V\times\mathbb{R}} A \simeq A \otimes \Sigma^\infty_+(\Omega X).$$

Proof of Theorem 6.1. Denote the equivariant embedding by $emb : M \times W \hookrightarrow V \times W$. Let $M \times W$ be $(V \oplus W)$-framed as a submanifold. Denote by $f : X \to B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$ the map whose $\Omega^{V\oplus W}$-looping is $\Omega^{V\oplus W} f : \Omega^{V\oplus W}X \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$.

Consider the following commutative diagram, where the first horizontal map is an equivalence by Theorem 2.2. The right hand column uses the homeomorphism $(M \times W)^+ \cong \Sigma^W(M^+)$.

$$\begin{tikzcd}
\int_{M\times W} \Omega^{V\oplus W}X \arrow[r, "\sim"] \arrow[d, "(\Omega^{V\oplus W}f)_*"] & \mathrm{Map}_*(\Sigma^W(M^+), X) \arrow[d, "f_*"] \\
\int_{M\times W} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \arrow[r] \arrow[d, "emb_*"] & \mathrm{Map}_*(\Sigma^W(M^+), B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \arrow[d, "emb_*"] \\
\int_{V\times W} \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \arrow[r] \arrow[d, "\sim"] & \mathrm{Map}_*(S^{V\oplus W}, B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \arrow[d, "\sim"] \\
\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G) \arrow[r, "="] & \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)
\end{tikzcd}$$

By Theorem 5.20, $\int_{M\times W} A$ is the equivariant Thom spectrum of the left hand vertical composite; thus it is equivalent to the equivariant Thom spectrum of the right hand vertical composite. Note that this composite is also equal to

$$\mathrm{Map}_*(\Sigma^W(M^+), X) \xrightarrow{emb_*} \mathrm{Map}_*(S^{V\oplus W}, X) \xrightarrow{f_*} \mathrm{Map}_*(S^{V\oplus W}, B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \simeq \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)$$

The map emb_* above is induced by the embedding $emb : M \times W \hookrightarrow V \times W$, equivalently by the Pontryagin-Thom collapse map associated to it, $S^{V\oplus W} \to \Sigma^W(M^+)$. We have an inclusion of a small disk $D \cong V$ in M, and the cofiber sequence

$$\Sigma^W(M^+ - D) \xrightarrow{\Sigma^W i} \Sigma^W(M^+) \longrightarrow \Sigma^W S^V \cong S^{V\oplus W}$$

is split (up to homotopy) by this Pontryagin-Thom collapse map, as the composite collapse $(V \times W)^+ \to (M \times W)^+ \to (D \times W)^+ \cong (V \times W)^+$ is homotopic to the identity. Thus we have an equivalence

$$
\begin{tikzcd}
(emb_*, i^*) : \mathrm{Map}_*(\Sigma^W(M^+), X) \arrow[r, "\sim"] & \mathrm{Map}_*(S^{V\oplus W}, X) \times \mathrm{Map}_*(\Sigma^W(M^+ - D), X) \arrow[d, "\sim"] \\
& \mathrm{Map}_*(S^{V\oplus W}, X) \times \mathrm{Map}_*(M^+ - D, \Omega^W X)
\end{tikzcd}
$$

Furthermore, this equivalence fits in the following commutative diagram:

$$
\begin{tikzcd}
\mathrm{Map}_*(\Sigma^W(M^+), X) \arrow[r, "(emb_*, i^*)", "\sim"'] \arrow[d, "emb_*"] & \mathrm{Map}_*(S^{V\oplus W}, X) \times \mathrm{Map}_*(M^+ - D, \Omega^W X) \arrow[d, "pr_1"] \\
\mathrm{Map}_*(S^{V\oplus W}, X) \arrow[r, "="] \arrow[d, "f_*"] & \mathrm{Map}_*(S^{V\oplus W}, X) \arrow[d, "f_*"] \\
\mathrm{Map}_*(S^{V\oplus W}, B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G)) \arrow[r, "="] & \mathrm{Map}_*(S^{V\oplus W}, B^{V\oplus W}\underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^G))
\end{tikzcd}
$$

We have shown that $\displaystyle\int_{M\times W} A$ is equivalent to the equivariant Thom spectrum of the left hand vertical composite, thus it is also equivalent to the equivariant Thom spectrum of the right hand vertical composite, which, by Corollary 3.17, is equivalent to $A \otimes \Sigma^\infty_+ \mathrm{Map}_*(M^+ - D, \Omega^W X)$. $\square$

Our first application computes the factorization homology of $MU_\mathbb{R}$. The Real bordism spectrum $MU_\mathbb{R}$ is the Thom spectrum of a map of C_2-$\mathbb{E}_\infty$ spaces $BU_\mathbb{R} \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^{C_2})$ (for example, as in Remark 13 of [HL18]). Since $(BU_\mathbb{R})^e \simeq BU$ and $(BU_\mathbb{R})^{C_2} \simeq BO$, the C_2-space $BU_\mathbb{R}$ is C_2-connected.

Lemma 6.4. *If X is a G-connected $\mathbb{E}_V$-algebra, then $\pi_k(B^V X) = 0$ for $k \leq \dim(V^H)$. Thus, the connectivity condition in Theorem 6.1 or Corollary 6.3 is satisfied when X is G-connected.*

Proof. We say that a G-space X is V-connected if $\pi_k(X) = 0$ for $k \leq \dim(V^H)$. The V-fold delooping can be computed by the monadic bar construction $B^V X = B(\Sigma^V, \mathrm{D}_V, X)$, where D_V is the monad associated to the little V-disk operad. Since fixed points commutes with geometric realization, it suffices to show that each $\Sigma^V \mathrm{D}_V X$ is V-connected. This follows from that $(\Sigma^V \mathrm{D}_V X)^H \cong \Sigma^{V^H}(\mathrm{D}_V X)^H$ and that $\mathrm{D}_V X$ is G-connected for a G-connected X (for the proof, see [Zou20, Lemma 8.4]). $\square$

Corollary 6.3 and Lemma 6.4 combine to give

Corollary 6.5.

$$\int_{S^V\times\mathbb{R}} MU_\mathbb{R} \simeq MU_\mathbb{R} \otimes \Sigma^\infty_+(B^V BU_\mathbb{R})$$

In particular,

$$\mathrm{THR}(MU_\mathbb{R}) \simeq MU_\mathbb{R} \otimes \Sigma^\infty_+(B^\sigma BU_\mathbb{R})$$

We now use Theorem 6.1 and Corollary 6.3, along with theorems of Behrens–Wilson [BW18] and Hahn–Wilson [HW18] which show that certain equivariant Eilenberg–MacLane spectra are Thom spectra, to compute equivariant factorization homology with coefficients in these spectra.

Take $G = C_2$. Let σ be its sign representation, $\rho \cong \sigma + 1$ its 2-dimensional regular representation, and $\lambda \cong 2\sigma$ its two-dimensional rotation representation. Let THR denote Real topological Hochschild homology [DMPR17], which is equivalent to $\int_{S^\sigma}$ by [Hor19, remark 7.1.2].

Corollary 6.6. *We have*

(1) $\mathrm{THR}(H\underline{\mathbb{F}}_2) \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+(\Omega S^{\rho+1}) \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+(\Omega^\sigma S^{\lambda+1})$

(2) $\mathrm{THR}(H\underline{\mathbb{Z}}_{(2)}) \simeq H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+(\Omega^\sigma(S^{\lambda+1}\langle\lambda+1\rangle))$

(3) $\int_{S^\lambda} H\underline{\mathbb{F}}_2 \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+ S^{\lambda+1}$

(4) $\int_{S^\lambda} H\underline{\mathbb{Z}}_{(2)} \simeq H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+(S^{\lambda+1}\langle\lambda+1\rangle)$

Here, $S^{\lambda+1}\langle\lambda+1\rangle$ *is the fiber of the unit map* $S^{\lambda+1} \to K(\underline{\mathbb{Z}}, \lambda+1) = \Omega^\infty\Sigma^{\lambda+1}H\underline{\mathbb{Z}}$.

Proof. By Theorem 1.2 of [BW18], the Eilenberg–MacLane spectrum $H\underline{\mathbb{F}}_2$ is equivariantly the Thom spectrum of a ρ-fold loop map $\Omega^\rho S^{\rho+1} \to BO_{C_2}$. As the inclusion $BO_{C_2} \to \underline{\mathrm{Pic}}(\mathbf{Sp}^{C_2})$ is a map of G-symmetric monoidal G-spaces, $H\underline{\mathbb{F}}_2$ is also the Thom spectrum of a ρ-fold loop map $\Omega^\rho S^{\rho+1} \to \underline{\mathrm{Pic}}(\mathbf{Sp}^{C_2})$. Thus Corollary 6.3 yields the first equivalence of (1), with $V = \sigma$ and $W = \mathbb{R}$. Furthermore, Hahn and Wilson [HW18] have shown that $H\underline{\mathbb{F}}_2$ is equivariantly the Thom spectrum of a $(\lambda+1)$-fold loop map $\Omega^\lambda S^{\lambda+1} \to \underline{\mathrm{Pic}}(\mathbb{S}_{(2)})$, and that $H\underline{\mathbb{Z}}_{(2)}$ is equivariantly the Thom spectrum of a $(\lambda+1)$-fold loop map $\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle) \to \underline{\mathrm{Pic}}(\mathbb{S}_{(p)})$. Corollary 6.3 with Remark 3.3 yields (3) and (4), with $V = \lambda$ and $W = \mathbb{R}$.

For the second equivalence of (1) and for (2), there is an isomorphism $\lambda + 1 \cong 2\sigma + 1$ and an equivariant embedding $S^\sigma \times \mathbb{R} \hookrightarrow \sigma + 1$, thus an equivariant embedding $S^\sigma \times \rho \hookrightarrow \lambda + 1$. We intend to use Theorem 6.1 with $M = S^\sigma$, $V = \sigma$, $W = \rho$ and

$$X = B^{\lambda+1}\Omega^\lambda S^{\lambda+1} \text{ or } X = B^{\lambda+1}\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle) \text{ respectively.}$$

To check the assumptions, by Lemma 6.4 it suffices to show that $\Omega^\lambda S^{\lambda+1}$ and $\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)$ are C_2-connected. This is true by Lemma 2.19, since $\dim((S^\lambda)^e) = 2$ and $\dim((S^\lambda)^{C_2}) = 0$; it can also be verified that $S^{\lambda+1}$ and $S^{\lambda+1}\langle\lambda+1\rangle$ are C_2-connected and underlying 2-connected. So, from Theorem 6.1 we obtain

$$\int_{S^\sigma\times\rho} H\underline{\mathbb{F}}_2 \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+\mathrm{Map}_*(\sigma_+, \Omega^\rho B^{\lambda+1}\Omega^\lambda S^{\lambda+1});$$

$$\int_{S^\sigma\times\rho} H\underline{\mathbb{Z}}_{(2)} \simeq H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+\mathrm{Map}_*(\sigma_+, \Omega^\rho B^{\lambda+1}\Omega^\lambda(S^{\lambda+1}\langle\lambda+1\rangle)).$$

To simplify, we have $\Omega^\rho B^{\lambda+1}\Omega^\lambda S^{\lambda+1} \simeq B^\sigma\Omega^\lambda S^{\lambda+1} \simeq \Omega^\sigma S^{\lambda+1}$, since $\Omega^\sigma S^{\lambda+1}$ is C_2-connected. As σ is equivariantly contractible, $\mathrm{Map}_*(\sigma_+, \Omega^\sigma S^{\lambda+1}) \simeq \Omega^\sigma S^{\lambda+1}$. The second equivalence is similar. $\square$

From either of the two descriptions of $\mathrm{THR}(H\underline{\mathbb{F}}_2)$ in Corollary 6.6, one can use the Snaith splitting to compute $\mathrm{THR}(H\underline{\mathbb{F}}_2)$ as an $H\underline{\mathbb{F}}_2$-module. From either $\Sigma^\infty_+\Omega\Sigma S^\rho \simeq \oplus_{k\geq 0}\mathbb{S}^{k\rho}$ or $\Sigma^\infty_+\Omega^\sigma\Sigma^\sigma S^\rho \simeq \oplus_{k\geq 0}\mathbb{S}^{k\rho}$, we have

$$\mathrm{THR}(H\underline{\mathbb{F}}_2) \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+\Sigma^\infty_+\Omega\Sigma S^\rho \simeq \oplus_{k\geq 0}\Sigma^{k\rho}H\underline{\mathbb{F}}_2.$$

This recovers the additive part of $\mathrm{THR}(H\underline{\mathbb{F}}_2)$ in [DMPR17, Theorem 5.18]. They use this module structure and the fact that $\mathrm{THR}(H\underline{\mathbb{F}}_2)$ is an associative $H\underline{\mathbb{F}}_2$-algebra to promote this to an equivalence of C_2-ring spectra. In particular,

$$\pi_\bigstar\,\mathrm{THR}(H\underline{\mathbb{F}}_2) \cong \pi_\bigstar(H\underline{\mathbb{F}}_2)[x_\rho].$$

We can also compute $\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2)$.

Corollary 6.7.
$$\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2 \otimes \Sigma^\infty_+(\Omega S^3)$$

Note that this only computes the underlying (non-equivariant) spectrum of $\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2)$. Section 5 of [AGH+20] uses Theorem 3.2 from our paper in a somewhat different approach to compute $\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2)$ as a C_2-spectrum.

Proof. Let g denote the generator of C_2, and for a C_2-space X, let L_gX denote the twisted free loop space $\{\gamma\colon I \to X \,|\, \gamma(1) = g\gamma(0)\}$. Let S^1_{rot} denote the circle, with C_2 acting by rotation. Note that S^1_{rot} is a $\mathbb{R}$-framed C_2-manifold.

By [Hor19, proposition 7.2.2], for A a C_n-ring spectrum, THH_{C_n} is given by the C_n-geometric fixed points of $\int_{S^1_{rot}} A$. Using the description of $H\underline{\mathbb{F}}_2$ in the proof of Corollary 6.6 and Theorem 5.20, we have
$$\int_{S^1_{rot}} H\underline{\mathbb{F}}_2 \simeq \mathbf{Th}\left(\int_{S^1_{rot}} \Omega^\rho S^{\rho+1} \to \underline{\mathrm{Pic}}(\underline{\mathbf{Sp}}^{C_2})\right).$$
By Theorem 2.2, we can identify the G-spaces:
$$\int_{S^1_{rot}} \Omega^\rho S^{\rho+1} \simeq \mathrm{Map}(S^1_{rot}, \Omega^\sigma S^{\rho+1}) \tag{6.8}$$
Moreover, the Thom spectrum $\int_{S^1_{rot}} H\underline{\mathbb{F}}_2$ has an $H\underline{\mathbb{F}}_2$-orientation given by the composite
$$H\underline{\mathbb{F}}_2 \otimes \int_{S^1_{rot}} H\underline{\mathbb{F}}_2 \xrightarrow{id\otimes i} H\underline{\mathbb{F}}_2 \otimes H\underline{\mathbb{F}}_2 \xrightarrow{mult} H\underline{\mathbb{F}}_2$$
Here, the map $i\colon \int_{S^1_{rot}} H\underline{\mathbb{F}}_2 \to H\underline{\mathbb{F}}_2$ exists because $H\underline{\mathbb{F}}_2$ is commutative. For example, we can take i to be induced on factorization homology by the embedding $S^1_{rot} \times \mathbb{R} \to \lambda$. By the Thom isomorphism and Eq. (6.8), we have
$$H\underline{\mathbb{F}}_2 \otimes \int_{S^1_{rot}} H\underline{\mathbb{F}}_2 \simeq H\underline{\mathbb{F}}_2 \otimes \Sigma^\infty_+ \mathrm{Map}(S^1_{rot}, \Omega^\sigma S^{\rho+1})$$
Upon passage to geometric fixed points, we obtain
$$\Phi^{C_2}(H\underline{\mathbb{F}}_2) \otimes \mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2) \simeq \Phi^{C_2}(H\underline{\mathbb{F}}_2) \otimes \Sigma^\infty_+ \mathrm{Map}_{C_2}(S^1_{rot}, \Omega^\sigma S^{\rho+1})$$
Because $\Phi^{C_2}(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2[t]$, where t is in degree 1, we have that
$$H\mathbb{F}_2 \otimes \mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2) \simeq H\mathbb{F}_2 \otimes \Sigma^\infty_+(L_g\Omega^\sigma S^{\rho+1})$$
By Corollary 16 of [KK10],
$$H\mathbb{F}_2 \otimes \Sigma^\infty_+(L_g\Omega^\sigma S^{\rho+1}) \simeq H\mathbb{F}_2 \otimes \Sigma^\infty_+(L\Omega S^3) \simeq H\mathbb{F}_2 \otimes H\mathbb{F}_2 \otimes \Sigma^\infty_+(\Omega S^3)$$
Note that $\mathrm{THH}_{C_2}(H\underline{\mathbb{F}}_2)$ and $H\mathbb{F}_2 \otimes \Sigma^\infty_+(\Omega S^3)$ are both $H\mathbb{F}_2$-modules (the former is in fact an algebra over $H\mathbb{F}_2$, as $H\underline{\mathbb{F}}_2$ is commutative.) They are equivalent after smashing with $H\mathbb{F}_2$, therefore they are equivalent. $\square$

7 Some results in parametrized category theory

In this section we gather the results used in Section 3, with partial proofs. Much of this section has to do with parametrized symmetric monoidal structures. However, a complete treatment of this subject is beyond the scope of this paper. We will therefore consider only G-∞-categories and G-symmetric monoidal structures (with the exception of Section 7.1).

7.1 Parametrized straightening/unstraightening

In this subsection, we state the result that we need about parameterized straightening/unstraightening. The results are stated for S-∞-categories, i.e., coCartesian fibrations over a fixed ∞-category S. Taking $S = \mathcal{O}_G^{op}$ recovers the notion of G-∞-categories, used throughout this paper.

Definition 7.1. *An S-fibration $X \twoheadrightarrow \mathcal{C}$ (see [Sha18, def. 7.1]) is an S-right fibration if $X_{[s]} \twoheadrightarrow \mathcal{C}_{[s]}$ is a right fibration for every $s \in S$.*

Definition 7.2. *Suppose $\mathcal{C}$ is an S-∞-category. Let $(\underline{\mathcal{C}\mathrm{at}}_{\infty,S})_{/\underline{\mathcal{C}}}$ denote the S-slice category (see [Sha18, not. 4.29]). Let*

$$(\underline{\mathcal{C}\mathrm{at}}_{\infty,S})^{S-right}_{/\underline{\mathcal{C}}} \subseteq (\underline{\mathcal{C}\mathrm{at}}_{\infty,S})_{/\underline{\mathcal{C}}}$$

denote the full subcategory spanned by $\underline{s}$-right fibrations

$$(X \twoheadrightarrow \mathcal{C} \times_S \underline{s}) \in \left(\mathcal{C}\mathrm{at}_{\infty,\underline{s}}\right)_{/\mathcal{C}\times_S \underline{s}} \simeq \left((\underline{\mathcal{C}\mathrm{at}}_{\infty,S})_{/\underline{\mathcal{C}}}\right)_{[s]}.$$

Theorem 7.3. *Suppose $\mathcal{C} \twoheadrightarrow S$ is an S-category. Then there is a natural equivalence of S-categories*

$$Y \colon \underline{\mathrm{Psh}}_S(\mathcal{C}) \xrightarrow{\sim} (\underline{\mathcal{C}\mathrm{at}}_{\infty,S})^{S-right}_{/\underline{\mathcal{C}}}.$$

If $x \in \mathcal{C}_{[s]}$ then Y sends the representable presheaf $\iota(x)$ to the $\underline{s}$-right fibration

$$\left(\mathcal{C}_{/\underline{x}} \twoheadrightarrow \mathcal{C} \times_S \underline{s}\right) \in \left(\mathcal{C}\mathrm{at}_{\infty,\underline{s}}\right)^{\underline{s}-right}_{/\mathcal{C}\times_S \underline{s}} \simeq \left((\underline{\mathcal{C}\mathrm{at}}_{\infty,S})^{S-right}_{/\underline{\mathcal{C}}}\right)_{[s]}.$$

For the following statements, let $B \in \underline{\mathbf{Top}}_S$ be an S-∞-groupoid and $\iota \colon B \to \underline{\mathrm{Psh}}_S(B)$ its parametrized Yoneda embedding ([BDG$^+$16b, thm. 10.5]).

Lemma 7.4. *Suppose $X \twoheadrightarrow B$ is an S-right fibration of S-∞-categories. Then X is an S-∞-groupoid.*

Proof. We have to show that the coCartesian fibration $X \twoheadrightarrow S$ is a left fibration. By [Lur09, prop. 2.4.2.4] it is enough to show that each fiber $X_{[s]}$ is a Kan complex. [Lur09, prop. 2.4.2.4] also guarantees that $B_{[s]}$ is a Kan complex. Since $X_{[s]} \to B_{[s]}$ is a right fibration over a Kan complex, we deduce that $X_{[s]}$ is indeed a Kan complex. □

Corollary 7.5. *There is a natural equivalence of S-categories*

$$Y \colon \underline{\mathrm{Psh}}_S(B) \xrightarrow{\sim} (\underline{\mathbf{Top}}_S)_{/\underline{B}}.$$

If $x \in B_{[s]}$, then Y sends the representable presheaf $\iota(x)$ to

$$\left(B_{/\underline{x}} \twoheadrightarrow B \times_S \underline{s}\right) \in \left(\mathbf{Top}_{\underline{s}}\right)_{/B\times_S \underline{s}} \simeq \left((\underline{\mathbf{Top}}_S)_{/\underline{B}}\right)_{[s]}.$$

7.2 Parametrized presheaves and Day convolution

Let $\underline{\mathcal{C}}$ be a G-∞-category. If $\underline{\mathcal{C}}$ has a G-symmetric monoidal structure $\underline{\mathcal{C}}^{\otimes}$, then $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ has a G-symmetric monoidal structure $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ given by the G-Day convolution of [BDG+ar] with respect to G-symmetric monoidal structure on $\underline{\mathcal{C}}$ and the Cartesian G-symmetric monoidal structure on $\underline{\mathbf{Top}}^G$. Our goal in this subsection is Proposition 7.8; informally, it states that parametrized left Kan extension along the Yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ takes a G-symmetric monoidal functor from $\underline{\mathcal{C}}$ to a G-symmetric monoidal functor from $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$.

We will need the following statement, which currently does not appear in the literature.

Lemma 7.6. *The parametrized Yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ extends to a G-symmetric monoidal G-functor $j^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes}$.*

We use the notion of a G-cocomplete G-∞-category from [Sha18, def. 5.12], and the notion of a distributive G-symmetric monoidal G-∞-category from [BDG+ar][5]. Note that the essentially unique G-symmetric monoidal structure of $\underline{\mathbf{Sp}}^G$ of [Nar17, cor. 3.28] is distributive by construction.

Let $F^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ be a G-symmetric monoidal functor, with underlying G-functor $F : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$. If the underlying G-∞-category $\underline{\mathcal{E}}$ is G-cocomplete, then $F^{\otimes}$ admits a G-operadic left Kan extension along $j^{\otimes}$,

$$(j^{\otimes})_! F^{\otimes} : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes} \to \underline{\mathcal{E}}^{\otimes},$$

constructed in [BDG+ar]. By construction, $(j^{\otimes})_! F^{\otimes}$ is a lax G-symmetric monoidal functor. In order to show that $(j^{\otimes})_! F^{\otimes}$ is in fact G-symmetric monoidal, we use the following proposition from [BDG+ar]:

Proposition 7.7 ([BDG+ar])**.** *Let $F : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{E}}^{\otimes}, p^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{D}}^{\otimes}$ be lax G-symmetric monoidal functors, and $(p^{\otimes})_! F^{\otimes} : \underline{\mathcal{D}}^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ the G-operadic left Kan extension of $F^{\otimes}$ along $p^{\otimes}$. Assume that the G-symmetric monoidal structure of $\underline{\mathcal{E}}^{\otimes}$ is distributive. Then the underlying G-functor of $(p^{\otimes})_! F^{\otimes}$ is equivalent to the G-left Kan extension of $F : \underline{\mathcal{C}} \to \underline{\mathcal{E}}$ along $p : \underline{\mathcal{C}} \to \underline{\mathcal{D}}$.*

If follows that if the G-symmetric monoidal structure of $\underline{\mathcal{E}}^{\otimes}$ is distributive, then the G-functor $j_! F : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \to \underline{\mathcal{E}}$ can be extended to a lax G-symmetric monoidal functor $(j_! F)^{\otimes} : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ given by the G-operadic left Kan extension $(j^{\otimes})_! F^{\otimes}$.

Proposition 7.8. *Let $F^{\otimes} : \underline{\mathcal{C}}^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ be a G-symmetric monoidal functor from a small G-symmetric monoidal ∞-category $\underline{\mathcal{C}}^{\otimes}$ to a distributive G-symmetric monoidal G-∞-category $\underline{\mathcal{E}}^{\otimes}$, with G-cocomplete underlying G-∞-category $\underline{\mathcal{E}}$. Then $(j_! F)^{\otimes} : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ is a G-symmetric monoidal functor.*

In the course of the proof we use the following notation.

Notation 7.9. For $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ a G-symmetric monoidal G-∞-category and $I \in \underline{\mathbf{Fin}}_*^G$:

1. Let $\mathcal{C}^{\otimes}_I$ denote the fiber of $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ over I.

2. For $I = (U \to G/H)$ let $\mathcal{C}^{\otimes}_{<I>}$ denote the $\underline{G/H}$-category constructed by pulling back along $\sigma_{<I>} : \underline{G/H} \to \underline{\mathbf{Fin}}_*^G$, see [Hor19, def. B.0.4].

[5] See [Nar17, def. 3.15] for a definition of a distributive parametrized functor.

Proof. We have to check that the lax G-symmetric monoidal functor $(j_!F)^{\otimes} : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes} \to \underline{\mathcal{E}}^{\otimes}$ is G-symmetric monoidal. The idea of the proof is simple: reduce to the case of parametrized representable presheaves, where the claim is clear. The argument is a bit convoluted due to the involved definition of a distributive G-symmetric monoidal structure.

Let $I \in \underline{\mathbf{Fin}}^G_*, I = (U \to G/H)$. Parametrized Day convolution defines a distributive G-symmetric monoidal structure on $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$, so the $\underline{G/H}$-functor

$$\otimes_I : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes}_{<I>} \simeq \prod_I \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{U} \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}$$

of [Hor19, def. B.0.11] is distributive (see [Nar17, def. 3.15]). Here $\prod_I : \underline{\mathcal{C}\mathrm{at}}_\infty^{\underline{U}} \to \underline{\mathcal{C}\mathrm{at}}_\infty^{\underline{G/H}}$ is the right adjoint of $(- \times_{\underline{G/H}} \underline{U}) \colon \underline{\mathcal{C}\mathrm{at}}_\infty^{\underline{G/H}} \to \underline{\mathcal{C}\mathrm{at}}_\infty^{\underline{U}}$, and the equivalence is homotopy inverse to the parametrized Segal map of [Hor19, rem. B.0.9].

We have to show that for every $X \in \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes}_I \simeq \left(\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})^{\otimes}_{<I>}\right)_{[G/H]}$ the lax structure map

$$\otimes_I \left(j_!F^{\otimes}(X)\right) \to j_!F(\otimes_I X) \tag{7.10}$$

is an equivalence. We first reduce to representable presheaves. By Corollary 7.23 we can write X as a $\underline{U}$-colimit $X \simeq \underline{U} - colim(j\chi)$ for some $\underline{U}$-diagram $\chi : K \to \underline{\mathcal{C}} \underline{\times} \underline{U}$. Inspect the following diagram:

$$\begin{tikzcd}
\otimes_I(j_!F^{\otimes}(X)) \arrow[rr] \arrow[d, "\sim"] & & j_!F(\otimes_I(X)) \arrow[d, "\sim"] \\
\otimes_I(j_!F^{\otimes}(\underline{U}\text{-}\underrightarrow{colim}(j\chi))) \arrow[rr] \arrow[d, "(1)"', "\sim"] & & j_!F(\otimes_I(\underline{U}\text{-}\underrightarrow{colim}(j\chi))) \arrow[d, "(2)"', "\sim"] \\
\otimes_I(\underline{U}\text{-}\underrightarrow{colim}(j_!F^{\otimes}(j\chi))) \arrow[d, "(2)"', "\sim"] & & j_!F(\underline{G/H}\text{-}\underrightarrow{colim}(\otimes_I(j\chi))) \arrow[d, "(1)"', "\sim"] \\
\underline{G/H}\text{-}\underrightarrow{colim}(\otimes_I(j_!F^{\otimes}(j\chi))) \arrow[rr] & & \underline{G/H}\text{-}\underrightarrow{colim}(j_!F^{\otimes}(\otimes_I(j\chi))).
\end{tikzcd}$$

The commutativity of the diagram follows from the naturality of the lax structure map (7.10). The equivalences marked (1) follow from the fact that $j_!F$ strongly preserves G-colimits, and the equivalences marked (2) follow from the distributivity of G-Day convolution.

By naturality of the lax structure map (7.10) it is therefore enough to show that the lax structure map $\otimes_I(j_!F^{\otimes}(j^{\otimes}X)) \to j_!F(\otimes_I j^{\otimes}X)$ is an equivalence for $X \in \underline{\mathcal{C}}^{\otimes}_I$. This follows from inspecting the following diagram

$$\begin{tikzcd}
 & \otimes_I j_!F(j^{\otimes}X) \arrow[dl, "(1)"'] \arrow[d, "(3)"] & \otimes_I F^{\otimes}(X) \arrow[l, "\sim"'] \arrow[d, "(4)"] \\
j_!F(\otimes_I(j^{\otimes}X)) \arrow[r, "(2)"] & j_!F(j(\otimes_I X)) & F(\otimes_I X), \arrow[l, "\sim"']
\end{tikzcd}$$

as we now explain. We wish to show that the diagonal map marked (1) is an equivalence. Since the parametrized Yoneda embedding j is G-symmetric monoidal, its lax structure map $\otimes_I(j^{\otimes}X) \to j(\otimes_I X)$ is an equivalence. It follows that it is still an equivalence after applying $j_!F$, showing that the map (2) is also an equivalence. Therefore it is enough to show that the map marked (3) is an equivalence. Note that the map marked (3) is the lax structure map of the composition $j_! F^{\otimes} \circ j^{\otimes}$. We now use the fact that the parametrized Yoneda embedding is fully faithful ([BDG+16b, thm. 10.4]) together with [Sha18, prop. 10.5] to deduce that the associated natural transformation $F \to j_! F \circ j$ is a natural equivalence. It follows that the left-pointing horizontal maps in the diagram are equivalences (the square commutes by naturality). Hence it is enough to prove that the map marked (4) is an equivalence, which is clear since it is the lax structure map of a G-symmetric monoidal functor $F^{\otimes}$. $\square$

7.3 Maximal G-∞-subgroupoid and G-symmetric monoidal structures

We recall the definition of the maximal G-∞-subgroupoid of an G-∞-category $\underline{\mathcal{C}}$, and verify that a G-symmetric monoidal structure on $\underline{\mathcal{C}}$ induces a G-symmetric monoidal on its maximal G-∞-subgroupoid. Recall that a G-∞-groupoid, or a G-space, is a G-∞-category $\underline{\mathcal{G}} \twoheadrightarrow \mathcal{O}_G^{op}$ in which every edge is coCartesian ([BDG+16b, def. 1.1]). By [Lur09, 2.4.2.4] this happens precisely when $\underline{\mathcal{G}} \twoheadrightarrow \mathcal{O}_G^{op}$ is a left fibration.

Let $\underline{\mathcal{C}} \twoheadrightarrow \mathcal{O}_G^{op}$ be a G-∞-category. The *maximal G-subgroupoid of* $\underline{\mathcal{C}}$ is the subcategory $\underline{\mathcal{C}}^{\simeq} \subset \underline{\mathcal{C}}$ spanned by all objects and all coCartesian edges. By construction, $\underline{\mathcal{C}}^{\simeq} \subset \underline{\mathcal{C}}$ is the maximal G-∞-subcategory ([BDG+16b, sec. 4]) which is a G-∞-groupoid. Note that for every orbit $W \in \mathcal{O}_G^{op}$, the morphisms in the fiber $(\underline{\mathcal{C}}^{\simeq})_{[W]}$ are coCartesian edges in $\underline{\mathcal{C}}$ over id_W, which by [Lur09, prop. 2.4.1.5] are exactly equivalences over id_W. Hence we have $(\underline{\mathcal{C}}^{\simeq})_{[W]} = (\underline{\mathcal{C}}_{[W]})^{\simeq}$ as subsets of $\underline{\mathcal{C}}_{[W]}$.

Construction 7.11. Suppose $\underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is a G-symmetric monoidal G-∞-category. Define $\underline{\mathcal{C}}^{\otimes}_{coCart} \subset \underline{\mathcal{C}}^{\otimes}$ as the full subcategory spanned by the coCartesian morphisms over $\underline{\mathbf{Fin}}_*^G$.

Lemma 7.12. *The composition $\underline{\mathcal{C}}^{\otimes}_{coCart} \subset \underline{\mathcal{C}}^{\otimes} \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is a coCartesian fibration which defines a G-symmetric monoidal structure on the full G-∞-subgroupoid of $\underline{\mathcal{C}}$.*

During the proof we use the notation $\mathcal{C}_I^{\otimes}$ for the fiber of $\mathcal{C}^{\otimes}$ over $I \in \underline{\mathbf{Fin}}_*^G$, see Notation 7.9.

Proof. The map $\underline{\mathcal{C}}^{\otimes}_{coCart} \to \underline{\mathbf{Fin}}_*^G$ is a left fibration by [Lur09, cor. 2.4.2.5]. Pulling back $\underline{\mathcal{C}}^{\otimes}_{coCart} \subset \underline{\mathcal{C}}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ over the G-functor $\sigma_{<G/G>} : \mathcal{O}_G^{op} \to \underline{\mathbf{Fin}}_*^G$, $[G/H] \mapsto (G/H \xrightarrow{=} G/H)$ we see that the underlying G-∞-category of $\underline{\mathcal{C}}^{\otimes}_{coCart}$ is the full G-subcategory of $\underline{\mathcal{C}}$ spanned by the coCartesian morphisms, i.e., the full G-∞-subgroupoid of $\underline{\mathcal{C}}$. Note that morphisms in the fiber $(\underline{\mathcal{C}}^{\otimes}_{coCart})_{[W]}$ are coCartesian edges in $\underline{\mathcal{C}}^{\otimes}$ over id_W, which by [Lur09, prop. 2.4.1.5] are exactly equivalences over id_W. Hence we have $(\underline{\mathcal{C}}^{\otimes}_{coCart})_I = (\underline{\mathcal{C}}^{\otimes}_I)^{\simeq}$ as subsets of $\underline{\mathcal{C}}^{\otimes}_I$.

Finally, we have to show that the Segal maps of $\underline{\mathcal{C}}^{\otimes}_{coCart}$ are equivalences. Let $I = (U \to G/H) \in (\underline{\mathbf{Fin}}_*^G)_{[G/H]}$, and consider the Segal map associated to I. Since $\underline{\mathcal{C}}^{\otimes}$ is a G-symmetric monoidal G-∞-category, the Segal map $\underline{\mathcal{C}}^{\otimes}_I \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} \underline{\mathcal{C}}_{[W]}$ is an equivalence of ∞-categories. Recall that the Segal map is defined as a product of functors $\underline{\mathcal{C}}^{\otimes}_I \to \underline{\mathcal{C}}_{[W]}$, defined by choosing coCartesian lifts of specified inert morphisms in $\underline{\mathbf{Fin}}_*^G$. The maximal groupoid functor preserves products, so applying it to the Segal map above produces an equivalence $(\underline{\mathcal{C}}^{\otimes}_I)^{\simeq} \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} (\underline{\mathcal{C}}_{[W]})^{\simeq}$. Using the equalities $(\underline{\mathcal{C}}^{\otimes}_{coCart})_I = (\underline{\mathcal{C}}^{\otimes}_I)^{\simeq}$ and $(\underline{\mathcal{C}}_{[W]})^{\simeq} = (\underline{\mathcal{C}}^{\simeq})_{[W]}$ we write the above equivalence as $(\underline{\mathcal{C}}^{\otimes}_{coCart})_I \xrightarrow{\sim} \prod_{W \in \mathrm{Orbit}(U)} (\underline{\mathcal{C}}^{\simeq})_{[W]}$, which is exactly the Segal map of $\underline{\mathcal{C}}^{\otimes}_{coCart}$. $\square$

Example 7.13. Let $\underline{\mathbf{Sp}}^G \twoheadrightarrow \mathcal{O}_G^{op}$ be the G-∞-category of genuine G-spectra, see [Nar17]. There is an essentially unique G-symmetric monoidal structure on $\underline{\mathbf{Sp}}^G$ with unit the sphere spectrum, see [Nar17, cor. 3.28]. By construction, $(\underline{\mathbf{Sp}}^G)^{\otimes}$ is a distributive G-symmetric monoidal

G-∞-category (in other words, parametrized smash products distribute over parametrized colimits). Informally, this G-symmetric monoidal structure encodes smash products and Hill-Hopkin-Ravenel norms. The G-symmetric monoidal structure on $\underline{\mathbf{Sp}}^G$ induces a G-symmetric monoidal structure on its maximal subgroupoid $(\underline{\mathbf{Sp}}^G)^{\simeq}$.

7.4 G-symmetric monoidal categories and G-commutative algebras

Recall that a G-symmetric monoidal category is an G-commutative monoid in $\underline{\mathcal{C}\mathrm{at}}_\infty^G$ (see [Nar17, sec. 3.1]).

Theorem 7.14 ([Nar17, thm. 2.32])**.** *There is an equivalence of ∞-categories*

$$\mathrm{CMon}_G(\underline{\mathcal{C}\mathrm{at}}_\infty^G) \simeq \mathrm{CAlg}_G((\underline{\mathcal{C}\mathrm{at}}_\infty^G)^\times)$$

between the ∞-category $\mathrm{CMon}_G(\underline{\mathcal{C}\mathrm{at}}_\infty^G)$ of G-symmetric monoidal G-∞-categories and the ∞-category $\mathrm{CAlg}_G((\underline{\mathcal{C}\mathrm{at}}_\infty^G)^\times)$ of G-commutative algebras in $\underline{\mathcal{C}\mathrm{at}}_\infty^G$, with respect to the G-Cartesian G-symmetric monoidal structure.

Lemma 7.15. *The equivalence of [Nar17, thm. 2.32] restricts to an equivalence*

$$\mathrm{CMon}_G(\underline{\mathbf{Top}}^G) \simeq \mathrm{CAlg}_G((\underline{\mathbf{Top}}^G)^\times)$$

between the ∞-category $\mathrm{CMon}_G(\underline{\mathbf{Top}}^G)$ of G-symmetric monoidal G-∞-groupoids and the ∞-category $\mathrm{CAlg}_G((\underline{\mathbf{Top}}^G)^\times)$ of G-commutative algebras in $\underline{\mathbf{Top}}^G$.

Example 7.16. The Picard G-space $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ admits G-symmetric monoidal structure (see Section 3.1), and therefore defines a G-commutative monoid in $\underline{\mathbf{Top}}^G$. Applying the previous lemma we can consider $\underline{\mathrm{Pic}}(\underline{\mathcal{C}})$ as a G-commutative algebra in $(\underline{\mathbf{Top}}^G)^\times$.

7.5 Slicing G-symmetric monoidal categories

Definition 7.17. *Let $\underline{\mathcal{C}}^\otimes$ be a G-symmetric monoidal G-∞-category and $A : \underline{\mathbf{Fin}}_*^G \to \underline{\mathcal{C}}^\otimes$ a G-commutative algebra. Define $\underline{\mathcal{C}}_{/A}^\otimes \to \underline{\mathbf{Fin}}_*^G$ by applying the construction [Lur12, def. 2.2.2.1] for $K = \Delta^0$, $S = \underline{\mathbf{Fin}}_*^G$ and $S \times K \to S$ the identity map.*

The following statement is a result of [Lur12, sec. 2.2.2] together with the G-Segal conditions of a G-symmetric monoidal ∞-category.

Proposition 7.18. *The map $\underline{\mathcal{C}}_{/A}^\otimes \to \underline{\mathbf{Fin}}_*^G$ defines a G-symmetric monoidal G-∞-category, with underlying G-∞-category equivalent to the parametized slice $\underline{\mathcal{C}}_{/\underline{A}} \to \mathcal{O}_G^{op}$ of [Sha18, not. 4.29].*

We will use the following description of $\mathcal{O}$-algebras in $\underline{\mathcal{C}}_{/A}^\otimes$.

Proposition 7.19. *Let $\mathcal{O}^\otimes \to \underline{\mathbf{Fin}}_*^G$ be a G-∞-operad. The ∞-category $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{C}}_{/A}^\otimes)$ of $\mathcal{O}$-algebras in $\underline{\mathcal{C}}$ is equivalent to the slice ∞-category $\mathrm{Alg}_{\mathcal{O}}(\underline{\mathcal{C}})_{/A}$ of $\mathcal{O}$-algebras over A.*

7.6 Parametrized symmetric monoidal straightening/unstraightening

Theorem 7.20. *Suppose $B^\otimes \twoheadrightarrow \underline{\mathbf{Fin}}_*^S$ is an S-symmetric monoidal S-∞-groupoid. Then the natural equivalence of Corollary 7.5 extends to an S-symmetric monoidal equivalence*

$$\underline{\mathrm{Psh}}_S(B)^\otimes \xrightarrow{\sim} (\underline{\mathbf{Top}}_S)_{/\underline{B}}^\otimes,$$

where the S-symmetric monoidal structure on the right hand side is given by Section 7.5 and the S-symmetric monoidal structure on the left hand side is given by S-Day convolution ([BDG$^+$ar]).

7.7 Parametrized presheaves and parametrized colimits

In this subsection we state some properties of the parametrized presheaf category, defined in [BDG+16b, ex. 9.9].

Let $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) = \underline{\mathrm{Fun}}_G(\underline{\mathcal{C}}^{vop}, \underline{\mathbf{Top}}^G)$ denote the parametrized presheaf G-∞-category of a small G-∞-category $\underline{\mathcal{C}} \to \mathcal{O}_G^{op}$, and let $j : \underline{\mathcal{C}} \hookrightarrow \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ be the parametrized Yoneda embedding of [BDG+16b, def. 10.2].

We will construct G-functors out of $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ using parametrized G-left Kan extension (see [Sha18, sec. 10] and [Nar16, def. 2.12]). Specifically, we will use [Sha18, thm. 11.5]. Let $\mathrm{Fun}_G^L(\underline{\mathcal{C}}, \underline{\mathcal{D}}) \subseteq \mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathcal{D}})$ denote the full subcategory of G-functors which strongly preserve G-colimits ([Sha18, def. 11.2]).

Theorem 7.21 (Shah)**.** *Let $\underline{\mathcal{C}}$ be a G-∞-category and let $\underline{\mathcal{E}}$ be a G-cocomplete G-∞-category. Then restriction along the G-Yoneda embedding $j : \underline{\mathcal{C}} \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ defines an equivalence of ∞-categories*

$$\mathrm{Fun}_G^L(\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}), \underline{\mathcal{E}}) \xrightarrow{\sim} \mathrm{Fun}_G(\underline{\mathcal{C}}, \underline{\mathcal{E}})$$

with inverse given by G-left Kan extension along j.

Unsurprisingly, every parametrized presheaf is equivalent to a parametrized colimit of representable presheaves. Before giving a formal statement we recall the relevant definition of parametrized colimits in $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$. Let $G/H \in \mathcal{O}_G^{op}$ be an orbit and $I \to \underline{G/H}$ be a $\underline{G/H}$-category. Keeping in mind the equivalence

$$\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})_{[G/H]} \simeq \mathrm{Fun}_G(\underline{G/H}, \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})) \simeq \mathrm{Fun}_{\underline{G/H}}(\underline{G/H}, \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}),$$

we define a $\underline{G/H}$-functor

$$\Delta_I : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})_{[G/H]} \simeq \mathrm{Fun}_{\underline{G/H}}(\underline{G/H}, \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}) \to \mathrm{Fun}_{\underline{G/H}}(I, \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}),$$

induced by precomposition with the structure map $I \to \underline{G/H}$. By definition $\underline{G/H}$-colimits in $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ along I-shaped diagrams are given by the left adjoint

$$\underline{G/H}\text{-}\varinjlim : \mathrm{Fun}_{\underline{G/H}}(I, \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}) \leftrightarrows \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})_{[G/H]} : \Delta_I.$$

See [Nar17, def. 1.15] for details.

Lemma 7.22. *Let $X \in \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ be a presheaf over $G/H \in \mathcal{O}_G^{op}$. Then X is equivalent to a $\underline{G/H}$-colimit of a diagram of representable presheaves*

$$K \xrightarrow{\chi} \underline{\mathcal{C}} \underline{\times} \underline{G/H} \xrightarrow{j \underline{\times} \underline{G/H}} \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H}$$

for some $\underline{G/H}$-functor $\chi : K \to \underline{\mathcal{C}} \underline{\times} \underline{G/H}$.

Proof. By the G-Yoneda lemma, [Sha18, lem. 11.1], the identity functor $Id : \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \to \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})$ is a G-left Kan extension of j along itself. By [Sha18, thm. 10.4] we can express the value of this G-left Kan extension on X as a $\underline{G/H}$-colimit

$$X = Id(X) \simeq \underline{G/H}\text{-}\varinjlim \left(\underline{\mathcal{C}}_{/\underline{X}} \to \underline{\mathcal{C}} \underline{\times} \underline{G/H} \xrightarrow{j \underline{\times} \underline{G/H}} \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}}) \underline{\times} \underline{G/H} \right),$$

where $\underline{\mathcal{C}}_{/\underline{X}} = \underline{\mathcal{C}} \times_{\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})} \underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})_{/\underline{X}}$ is the pullback of the G-slice category $\underline{\mathrm{Psh}}_G(\underline{\mathcal{C}})_{/\underline{X}}$ ([Sha18, not. 4.29]) along the G-Yoneda embedding j. $\square$

Corollary 7.23. *Let U be a finite G-set and let $X : \underline{U} \to \underline{\mathrm{Psh}_G(\mathcal{C})} \underline{\times} \underline{U}$ be $\underline{U}$-functor. Then there exists a $\underline{U}$-functor $\chi : K \to \underline{\mathcal{C}} \underline{\times} \underline{U}$, such that the $\underline{U}$-colimit of*

$$K \xrightarrow{\chi} \underline{\mathcal{C}} \underline{\times} \underline{U} \xrightarrow{j \underline{\times} \underline{U}} \underline{\mathrm{Psh}_G(\mathcal{C})} \underline{\times} \underline{U}$$

is equivalent to X.

Proof. Decompose $U = \coprod_{W \in \mathrm{Orbit}(U)} W$ into orbits. The result follows from Lemma 7.22 and the equivalence

$$\prod_W \mathcal{C}\mathrm{at}_\infty^{\underline{W}} \xrightarrow{\sim} \mathcal{C}\mathrm{at}_\infty^{\underline{U}}, \quad (\underline{\mathcal{C}_W} \twoheadrightarrow \underline{W})_{W \in \mathrm{Orbit}(U)} \mapsto \left(\coprod_W \underline{\mathcal{C}_W} \twoheadrightarrow \coprod_W \underline{W} = \underline{U} \right),$$

where coproducts and products are indexed over $W \in \mathrm{Orbit}(U)$. $\square$

7.8 The G-∞-category of topological G-objects

In this subsection, we review the genuine operadic nerve construction of [Bon19], for the case of a symmetric monoidal topological category (considered as a multi-colored topological operad).

Let $\mathcal{C}$ be a topological category, i.e., a category enriched in Top, the category of compactly generated (weak) Hausdorff topological spaces. Let $\otimes$ be an enriched symmetric monoidal structure on $\mathcal{C}$ with unit I[6]. We refer to such an enriched symmetric monoidal category as a *symmetric monoidal topological category.*

The main construction of this section, Construction 7.27, associates a topological category $\mathcal{C}^\otimes$ over $\underline{\mathbf{Fin}}_*^G$ to a symmetric monoidal topological category $\mathcal{C}$. The main theorem of this subsection allows us to quickly construct G-symmetric monoidal G-∞-categories as $N(\mathcal{C}^\otimes)$, the coherent nerve of $\mathcal{C}^\otimes$ (see [Lur09, def. 1.1.5.5]).

Theorem 7.24. *For $\mathcal{C}$ be a symmetric monoidal topological category, let $N(\mathcal{C}^\otimes)$ denote the coherent nerve of $\mathcal{C}^\otimes$, the topological category of Construction 7.27. Then*

$$N(\mathcal{C}^\otimes) \to \underline{\mathbf{Fin}}_*^G$$

is a G-symmetric monoidal G-∞-category.

We start with some preliminaries needed for Construction 7.27.

Let I be a small category. Make $\mathrm{Fun}(I, \mathcal{C})$ into a topological category by endowing the set $\mathrm{Nat}(F, G)$ of natural transformations between $F, G : I \to \mathcal{C}$ with the topology of the equalizer of $\prod_{i \in I} \mathrm{Map}_\mathcal{C}(Fi, Gi) \rightrightarrows \prod_{\phi : i \to i'} \mathrm{Map}_\mathcal{C}(Fi, Gi')$, with one map induced by precomposition with $F(\phi)$ and the other by postcomposition with $G(\phi)$.

Recall that a covering map $p : I \to J$ induces a monoidal pushforward functor $p_*^\otimes : \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C})$, see [Rub17, sec. 8.5].

Proposition 7.25. *The monoidal pushforward functor $p_*^\otimes : \mathrm{Fun}(I, \mathcal{C}) \to \mathrm{Fun}(J, \mathcal{C})$ is a topological functor.*

[6]See the following mathoverflow post: https://mathoverflow.net/questions/51783/enriched-monoidal-categories [hm]. In Kelly's book as linked in the post, the tensor product of $\mathcal{V}$-enriched categories is defined on page 12.

Proof. Let $F, G \in \mathrm{Fun}(I, \mathcal{C})$ be functors. The mapping space $\mathrm{Nat}(p_*^{\otimes}F, p_*^{\otimes}G)$ is obtained as the equalizer of

$$\prod_{j \in J} \mathrm{Map}_{\mathcal{C}}(\otimes_{pi=j}Fi, \otimes_{pi=j}Gi) \rightrightarrows \prod_{\phi: j \to j'} \mathrm{Map}_{\mathcal{C}}(\otimes_{pi=j}Fi, \otimes_{pi'=j'}Gi').$$

Because $\mathcal{C}$ is a symmetric monoidal topological category, the maps

$$\prod_{pi=j} \mathrm{Map}_{\mathcal{C}}(Fi, Gi) \to \mathrm{Map}_{\mathcal{C}}(\otimes_{pi=j}Fi, \otimes_{pi=j}Gi)$$

are continuous, and similarly with i', j'. Thus $\mathrm{Nat}(F, G) \to \mathrm{Nat}(p_*^{\otimes}F, p_*^{\otimes}G)$ is continuous. □

Let U be a G-set. The action groupoid of U, denoted $B_U G$, has objects $x \in U$ and morphisms $\mathrm{Hom}(x, y) = \{g \in G | gx = y\}$. A map of G-sets $f : U \to V$ induces a functor on the action groupoids, which we denote by $Bf : B_U G \to B_V G$.

Note that the action groupoid of a pullback of G-sets is isomorphic to the strict pullback of action groupoids,

$$P \cong X \times_Z Y \Rightarrow B_P G \cong B_X G \times_{B_Z G} B_Y G.$$

The G-category of finite pointed G-sets. Let $\underline{\mathbf{Fin}}_*^G$ be the G-∞-category of finite pointed G-sets, see [Nar17, def. 2.14]. We will use the following model of $\underline{\mathbf{Fin}}_*^G$: an object of $\underline{\mathbf{Fin}}_*^G$ is given by a map of finite G-sets $U \to O$ where O is an orbit (a transitive G-set). A morphism $\psi : I_0 \to I_1$ in $\underline{\mathbf{Fin}}_*^G$ from $I_0 = (U_0 \to O_0)$ to $I_1 = (U_1 \to O_1)$ is given by a span of arrows (a diagram of G-sets) of the form

$$\begin{tikzcd} U_0 \arrow[d] & U_0' \arrow[l, "f"'] \arrow[r, "p"] \arrow[d] & U_1 \arrow[d] \\ O_0 & O_1 \arrow[l, "\varphi"'] \arrow[r, "="] & O_1 \end{tikzcd} \tag{7.26}$$

where the induced map $U_0' \to \varphi^* U_0$ is injective (or equivalently there exists another G-set U_0'' with a G-map $U_0'' \to \varphi^* U_0$ that induces an isomorphism $U_0' \coprod U_0'' \xrightarrow{\cong} \varphi^* U_0$).

Constructing a topological category over $\underline{\mathbf{Fin}}_*^G$. Our goal is to construct a topological category $\mathcal{C}^{\otimes}$ with a functor $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$, whose coherent nerve ([Lur09, def. 1.1.5.5]) will be the G-symmetric monoidal G-∞-category of topological G-objects in $\mathcal{C}$.

Construction 7.27. Let $\mathcal{C}$ be a symmetric monoidal topological category. We construct a topological category $\mathcal{C}^{\otimes}$ over $\underline{\mathbf{Fin}}_*^G$ as follows.

- An object $x \in \mathcal{C}^{\otimes}$ over $I \in \underline{\mathbf{Fin}}_*^G, I = (U \to O)$ is a functor $x : B_U G \to \mathcal{C}$.
- Let $x \in \mathcal{C}^{\otimes}$ be an object over $I_0 = (U_0 \to O_0)$ and $\psi : I_0 \to I_1$ a morphism of $\underline{\mathbf{Fin}}_*^G$ given by the diagram (7.26). Denote the composition $B_{U_0'}G \xrightarrow{Bf} B_{U_0}F \xrightarrow{x} \mathcal{C}$ by $f^*x \in \mathrm{Fun}(B_{U_0'}G, \mathcal{C})$, and its monoidal pushforward along $p : U_0' \to U_1$ by $p_*^{\otimes} f^* x : B_{U_1} G \to \mathcal{C}$.

  Suppose we are also given $y \in \mathcal{C}^{\otimes}$ over $I_1 = (U_1 \to O_1)$. Define the space of morphisms of $\mathcal{C}^{\otimes}$ from x to y over ψ to be $\mathrm{Map}^{\psi}_{\mathcal{C}^{\otimes}}(x, y) = \mathrm{Nat}(p_*^{\otimes} f^* x, y)$.

- Define the mapping space in the topological category $\mathcal{C}^\otimes$ as $\mathrm{Map}_{\mathcal{C}^\otimes}(x,y) = \coprod_\psi \mathrm{Map}^\psi_{\mathcal{C}^\otimes}(x,y)$, where the coproduct is indexed over all $\psi \in \mathrm{Hom}_{\underline{\mathbf{Fin}}^G_*}(I_0, I_1)$.

- Let $x_0, x_1, x_2 \in \mathcal{C}^\otimes$ be object over $I_0, I_1, I_2 \in \underline{\mathbf{Fin}}^G_*$. In what follows we construct a continuous maps

$$\mathrm{Map}^{\psi_1}_{\mathcal{C}^\otimes}(x_0,x_1) \times \mathrm{Map}^{\psi_2}_{\mathcal{C}^\otimes}(x_1,x_2) \to \mathrm{Map}^{\psi_2\psi_1}_{\mathcal{C}^\otimes}(x_0,x_2),$$

for each $\psi_1 : I_0 \to I_1$, $\psi_2 : I_1 \to I_2$ in $\underline{\mathbf{Fin}}^G_*$. This allows us to define the composition map

$$\mathrm{Map}_{\mathcal{C}^\otimes}(x_0,x_1) \times \mathrm{Map}_{\mathcal{C}^\otimes}(x_1,x_2) \to \mathrm{Map}_{\mathcal{C}^\otimes}(x_0,x_2)$$

as the coproduct of these maps. In other words, we make sure that $\mathcal{C}^\otimes \to \underline{\mathbf{Fin}}^G_*$ respects compositions by definition.

We first choose an explicit description of the composition $\psi_2\psi_1 : I_0 \to I_2$. Let $I_0 = (U_0 \to O_0)$, $I_1 = (U_1 \to O_1)$, $I_2 = (U_2 \to O_2)$ and

$$\psi_1 = \left(\begin{tikzcd} U_0 \arrow[d] & U_0' \arrow[l, "f_1"'] \arrow[r, "p_1"] \arrow[d] & U_1 \arrow[d] \\ O_0 & O_1 \arrow[l, "\varphi_1"'] \arrow[r, "="] & O_1 \end{tikzcd}\right), \quad \psi_2 = \left(\begin{tikzcd} U_1 \arrow[d] & U_1' \arrow[l, "f_2"'] \arrow[r, "p_2"] \arrow[d] & U_2 \arrow[d] \\ O_1 & O_2 \arrow[l, "\varphi_2"'] \arrow[r, "="] & O_2 \end{tikzcd}\right).$$

The composition $\psi_2\psi_1$ is given by

$$\psi_2\psi_1 = \left(\begin{tikzcd} U_0 \arrow[d] & U_2' \arrow[l, "f_1\overline{f}_2"'] \arrow[r, "p_2\overline{p}_1"] \arrow[d] & U_2 \arrow[d] \\ O_0 & O_2 \arrow[l, "\varphi_1\varphi_2"'] \arrow[r, "="] & O_2 \end{tikzcd}\right), \quad \begin{tikzcd} U_2' \arrow[r, "\overline{p}_1"] \arrow[d, "\overline{f}_2"'] \arrow[dr, phantom, "\ulcorner", very near start] & U_1' \arrow[r, "p_2"] \arrow[d, "f_2"] & U_2 \\ U_0' \arrow[r, "p_1"] \arrow[d, "f_1"'] & U_1 & \\ U_0 & & \end{tikzcd} \tag{7.28}$$

where the maps $\overline{f}_2 : U_2' \to U_0'$, $\overline{p}_1 : U_2' \to U_1'$ are given by the pullback square in the diagram of finite G-spaces on the right of (7.28). Note that the pullback square of diagram (7.28) induces a strict pullback square of action groupoids in the following diagram of groupoids

$$\begin{tikzcd} B_{U_2'}G \arrow[r, "\overline{p}_1"] \arrow[d, "\overline{f}_2"'] \arrow[dr, phantom, "\ulcorner", very near start] & B_{U_1'}G \arrow[r, "p_2"] \arrow[d, "f_2"] & B_{U_2}G \\ B_{U_0'}G \arrow[r, "p_1"] \arrow[d, "f_1"'] & B_{U_1}G & \\ B_{U_0}G. & & \end{tikzcd}$$

By [HHR16, prop. A.31] it follows that the following diagram commutes up to natural isomorphism (given by the symmetric monoidal structure of $\mathcal{C}$)

$$\begin{tikzcd}
\mathrm{Fun}(B_{U_2'}G, \mathcal{C}) \arrow[r, "(\overline{p}_1)_*^{\otimes}"] & \mathrm{Fun}(B_{U_1'}G, \mathcal{C}) \arrow[r, "(p_2)_*^{\otimes}"] & \mathrm{Fun}(B_{U_2}G, \mathcal{C}) \\
\mathrm{Fun}(B_{U_0'}G, \mathcal{C}) \arrow[u, "(\overline{f}_2)^*"] \arrow[r, "(p_1)_*^{\otimes}"] & \mathrm{Fun}(B_{U_1}G, \mathcal{C}) \arrow[u, "(f_2)^*"] & \\
\mathrm{Fun}(B_{U_0}G, \mathcal{C}). \arrow[u, "(f_1)^*"] & &
\end{tikzcd}$$

In particular, for $x_0 \in \mathrm{Fun}(B_{U_0}G, \mathcal{C})$ we get a natural isomorphism

$$(p_2)_*^{\otimes}(f_2)^*(p_1)_*^{\otimes}(f_1)^*x_0 \cong (p_2)_*^{\otimes}(\overline{p}_1)_*^{\otimes}(\overline{f}_2)^*(f_1)^*x_0 \cong (p_2\overline{p}_1)_*^{\otimes}(f_1\overline{f}_2)^*x_0, b \tag{7.29}$$

where the second isomorphism is given by [HHR16, prop. A.29]. Note that the mapping spaces of the topological functor categories in (7.29) are the spaces of natural transformations, so the functor $(p_2)_*^{\otimes}(f_2)^* : \mathrm{Fun}(B_{U_1}G, \mathcal{C}) \to \mathrm{Fun}(B_{U_2}G, \mathcal{C})$ induces a continuous map

$$\mathrm{Nat}\left((p_1)_*^{\otimes}(f_1)^*x_0, x_1\right) \to \mathrm{Nat}\left((p_2)_*^{\otimes}(f_2)^*(p_1)_*^{\otimes}(f_1)^*x_0, (p_2)_*^{\otimes}(f_2)^*x_1\right) \tag{7.30}$$

We now define the map $\mathrm{Map}^{\psi_1}_{\mathcal{C}^{\otimes}}(x_0, x_1) \times \mathrm{Map}^{\psi_2}_{\mathcal{C}^{\otimes}}(x_1, x_2) \to \mathrm{Map}^{\psi_2\psi_1}_{\mathcal{C}^{\otimes}}(x_0, x_2)$ as the composition

$$\begin{tikzcd}
\mathrm{Nat}\left((p_1)_*^{\otimes}(f_1)^*x_0, x_1\right) \times \mathrm{Nat}\left((p_2)_*^{\otimes}(f_2)^*x_1, x_2\right) \arrow[d] \\
\mathrm{Nat}\left((p_2)_*^{\otimes}(f_2)^*(p_1)_*^{\otimes}(f_1)^*x_0, (p_2)_*^{\otimes}(f_2)^*x_1\right) \times \mathrm{Nat}\left((p_2)_*^{\otimes}(f_2)^*x_1, x_2\right) \arrow[d, "\circ"] \\
\mathrm{Nat}\left((p_2)_*^{\otimes}(f_2)^*(p_1)_*^{\otimes}(f_1)^*x_0, x_2\right) \arrow[d, "\cong"] \\
\mathrm{Nat}\left((p_2\overline{p}_1)_*^{\otimes}(f_1\overline{f}_2)^*x_0, x_2\right),
\end{tikzcd} \tag{7.31}$$

where the first map is given by (7.30) on the first coordinate and the identity on the second, the second map is the composition in $\mathrm{Fun}(B_{U_2}G, \mathcal{C})$ and the last isomorphism is induced by (7.29).

Associativity of the composition in $\mathcal{C}^{\otimes}$ follows from [HHR16, prop. A.29].

In order to prove Theorem 7.24 we show that $N(\mathcal{C}^{\otimes}) \to \underline{\mathbf{Fin}}_*^G$ is a coCartesian fibration and verify the G-Segal conditions.

Checking that $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ is a coCartesian fibration. Our next goal is to show that the functor $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ we constructed is a coCartesian fibration (see [Lur09, def. 2.4.2.1]).

Lemma 7.32. *Let $\psi_1 : I_0 \to I_1$ be a morphism of $\underline{\mathbf{Fin}}_*^G$ given by* (7.26) *and $x \in \mathcal{C}^\otimes$ over I_0, i.e., a functor $x : B_{U_0}G \to \mathcal{C}$. Define $y : B_{U_1}G \to \mathcal{C}$ over I_1 by setting $y = (p_1)_*^\otimes (f_1)^* x$, and define $\overline{\psi_1} \in \mathrm{Map}_{\mathcal{C}^\otimes}^{\psi_1}(x, y)$ as the identity natural transformation $(p_1)_*^\otimes (f_1)^* x \xrightarrow{=} (p_1)_*^\otimes (f_1)^* x = y$. Then for every $\psi_2 : I_1 \to I_2$ in $\underline{\mathbf{Fin}}_*^G$ and $t \in \mathcal{C}^\otimes$ over I_2 the continuous map $(\overline{\psi_1})^* : \mathrm{Map}_{\mathcal{C}^\otimes}^{\psi_2}(y, t) \to \mathrm{Map}_{\mathcal{C}^\otimes}^{\psi_2\psi_1}(x, t)$ as defined in (7.31) is an isomorphism.*

Proof. Suppose ψ_1, ψ_2 are given by (7.28). Then, explicitly, $(\overline{\psi_1})^*$ is the composite

$$\mathrm{Nat}\left((p_2)_*^\otimes (f_2)^* y, z\right) \xrightarrow{\overline{\psi_1}\circ} \mathrm{Nat}\left((p_2)_*^\otimes (f_2)^* (p_1)_*^\otimes (f_1)^* x, t\right) \xrightarrow{\cong} \mathrm{Nat}\left((p_2\overline{p}_1)_*^\otimes (f_1\overline{f}_2)^* x, t\right).$$

The first map is an isomorphism by the definition of y and the second map is an isomorphism induced by (7.29) as before. □

Corollary 7.33. *The map $N(\mathcal{C}^\otimes) \to \underline{\mathbf{Fin}}_*^G$ is a coCartesian fibration.*

Proof. The lemma above implies that the square

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}^\otimes}(y, t) \arrow[r, "\overline{\psi}^*"] \arrow[d] & \mathrm{Map}_{\mathcal{C}^\otimes}(x, t) \arrow[d] \\
\mathrm{Hom}_{\underline{\mathbf{Fin}}_*^G}(I_1, I_2) \arrow[r, "\psi^*"] & \mathrm{Hom}_{\underline{\mathbf{Fin}}_*^G}(I_0, I_2)
\end{tikzcd}$$

is homotopy Cartesian, since it induces weak equivalences on the homotopy fibers of the vertical maps. Therefore by [Lur09, prop. 2.4.1.10] the morphism $\overline{\psi} : x \to y$ in $\mathcal{C}^\otimes$ is a coCartesian lift of $\psi : I_0 \to I_1$ in $\underline{\mathbf{Fin}}_*^G$. We have showed that every $\psi : I_0 \to I_1$ and $x \in \mathcal{C}^\otimes$ over I_0 has a coCartesian lift $\overline{\psi}$. Passing to the coherent nerve of $\mathcal{C}$ (see [Lur09, def. 1.1.5.5]) we know that the map $N(\mathcal{C}^\otimes) \to \underline{\mathbf{Fin}}_*^G$ is an inner fibration (again by [Lur09, prop. 2.4.1.10]), so we have showed that it is coCartesian fibration by verifying [Lur09, def. 2.4.2.1]. □

***G*-Segal conditions.** In order to prove that $N(\mathcal{C}^\otimes) \twoheadrightarrow \underline{\mathbf{Fin}}_*^G$ is a G-symmetric monoidal category (see the head of [Nar17, sec. 3.1] for a definition) we have to verify the G-Segal conditions.

Notation 7.34. For $I \in \underline{\mathbf{Fin}}_*^G$, $I = (U \to O)$, let $\mathcal{C}_I^\otimes$ be the fiber of $\mathcal{C}^\otimes \to \underline{\mathbf{Fin}}_*^G$ over I.

In other words, $C_I^\otimes$ is the topological category with objects given by functors $x : B_U G \to \mathcal{C}$ and mapping spaces $\mathrm{Map}_{\mathcal{C}_I^\otimes}(x, y) = \mathrm{Map}_{\mathcal{C}^\otimes}^{id_I}(x, y) = \mathrm{Nat}(x, y)$.

Notation 7.35. For W a G-orbit, let $\mathcal{C}_{[W]}^\otimes$ be the fiber of $\mathcal{C}^\otimes \to \underline{\mathbf{Fin}}_*^G$ over $(W \xrightarrow{=} W)$.

Remark 7.36. It is easy to see that if $W \cong G/H$ then $B_W G \simeq BH$, hence $\mathcal{C}_{[W]}^\otimes$ is equivalent to the topological category $\mathrm{Fun}(BH, \mathcal{C})$ of H-objects in $\mathcal{C}$.

If $I \in \underline{\mathbf{Fin}}_*^G, I = (U \to O)$ and $W \in \mathrm{Orbit}(U)$, then consider the following morphism in $\underline{\mathbf{Fin}}_*^G$,

$$\begin{tikzcd}
U \arrow[d] & W \arrow[l, "f"'] \arrow[r, "="] \arrow[d, "="] & W \arrow[d, "="] \\
O & W \arrow[l] \arrow[r, "="] & W,
\end{tikzcd}$$

where $f : W \to U$ is just an incusion of the orbit W into U. Since $\mathcal{C}^{\otimes} \to \underline{\mathbf{Fin}}_*^G$ is a coCartesian fiberation it induces a functor between the coherent nerves of $\mathcal{C}_I^{\otimes}$ and $\mathcal{C}_{[W]}^{\otimes}$, but our choice of coCartesian edges above implies that this functor is just the coherent nerve of

$$f^* : \mathcal{C}_I^{\otimes} \to \mathcal{C}_{[W]}^{\otimes}, \quad (x : B_U G \to \mathcal{C}) \mapsto (f^* x : B_W G \xrightarrow{Bf} B_U G \xrightarrow{x} \mathcal{C}).$$

Taking the product over all orbits $W \in \mathrm{Orbit}(U)$ we get a functor of topological categories $\mathcal{C}_I^{\otimes} \to \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}_{[W]}^{\otimes}$ whose coherent nerve is equivalent to the G-Segal map. Checking the G-Segal conditions amounts to proving

Lemma 7.37. *The functor $\mathcal{C}_I^{\otimes} \to \prod_{W \in \mathrm{Orbit}(U)} \mathcal{C}_{[W]}^{\otimes}$ is an equivalence.*

Proof. The orbit decomposition $U = \coprod_{W \in \mathrm{Orbit}(U)} W$ induces

$$\mathcal{C}_I^{\otimes} = \mathrm{Fun}(B_U G, \mathcal{C}) = \mathrm{Fun}(\coprod_W B_W G, \mathcal{C}) \xrightarrow{\sim} \prod_W \mathrm{Fun}(B_W G, \mathcal{C}) = \prod_W \mathcal{C}_{[W]}^{\otimes},$$

which is the functor described above. □

Proof of Theorem 7.24 . The map $N(\mathcal{C}^{\otimes}) \to \underline{\mathbf{Fin}}_*^G$ is a coCartesian fibration by Corollary 7.33, and by Lemma 7.37 it satisfies the G-Segal conditions. □

Appendix A The Real topological Hochschild homology of $H\underline{\mathbb{Z}}$, by Jeremy Hahn and Dylan Wilson

In this appendix we explain how the results of the main body of the paper allow one to calculate the Real topological Hochschild homology of the Eilenberg–MacLane Mackey functor $H\underline{\mathbb{Z}}$. In particular, we deduce the following theorem, which verifies a conjecture of Dotto, Moi, Patchkoria, and Reeh [DMPR17, p. 63].

Theorem A.1. *There is an equivalence of $H\underline{\mathbb{Z}}$-module spectra*

$$\mathrm{THR}(H\underline{\mathbb{Z}}) \simeq H\underline{\mathbb{Z}} \oplus \bigoplus_{k\geq 2} \Sigma^{k\rho-1} H\underline{\mathbb{Z}/k}.$$

Dotto, Moi, Patchkoria, and Reeh were able to prove that Theorem A.1 holds after localization at any odd prime, and so also after localization away from 2 [DMPR17, Theorem 5.27 & Corollary 5.28]. However, they did not have methods to calculate $\mathrm{THR}(H\underline{\mathbb{Z}})_{(2)} \simeq \mathrm{THR}(H\underline{\mathbb{Z}}_{(2)})$. On the other hand, the main body of this paper provides methods to calculate the THR of Thom spectra, and the authors of this appendix previously constructed $H\underline{\mathbb{Z}}_{(2)}$ as a C_2-equivariant Thom spectrum in [HW18]. These results were combined in Corollary 6.6(ii) of the main body to prove that

$$\mathrm{THR}(H\underline{\mathbb{Z}}_{(2)}) \simeq H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+ \Omega^\sigma (S^{\lambda+1}\langle \lambda+1\rangle).$$

The main contribution of the appendix is to observe that this can be made more explicit:

Lemma A.2. *There is an equivalence of $H\underline{\mathbb{Z}}_{(2)}$-module spectra*

$$H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+ \Omega^\sigma (S^{\lambda+1}\langle \lambda+1\rangle) \simeq H\underline{\mathbb{Z}}_{(2)} \oplus \bigoplus_{k\geq 1} \Sigma^{k\rho-1} H\underline{\mathbb{Z}/k}_{(2)}.$$

We deduce Lemma A.2 from the non-equivariant calculation of $\mathrm{THH}(H\mathbb{Z})$ together with the following C_2-equivariant fact, which we prove before 2-localization:

Lemma A.3. *There is a cofiber sequence of $H\underline{\mathbb{Z}}$-module spectra*

$$H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1}\langle \lambda+1\rangle \to \bigoplus_{k\geq 0} \Sigma^{k\rho} H\underline{\mathbb{Z}} \to \bigoplus_{s\geq 1} \Sigma^{s\rho} H\underline{\mathbb{Z}}.$$

Proof of Lemma A.3. Applying Ω^σ to the definition of $S^{\lambda+1}\langle\lambda+1\rangle$ yields a fiber sequence of C_2-equivariant spaces

$$\Omega^\sigma S^{\lambda+1}\langle \lambda+1\rangle \to \Omega^\sigma S^{\lambda+1} \to \Omega^\sigma \mathrm{K}(\lambda+1,\underline{\mathbb{Z}}),$$

where $\Omega^\sigma \mathrm{K}(\lambda+1,\underline{\mathbb{Z}}) = \Omega^\sigma \mathrm{K}(2\sigma+1,\underline{\mathbb{Z}}) \simeq \mathrm{K}(\sigma+1,\underline{\mathbb{Z}}) \simeq \mathbb{CP}^\infty_{\mathbb{R}}$.

In particular, since $\mathbb{CP}^\infty_{\mathbb{R}}$ classifies Real line bundles, it follows that the *cofiber* of the map $\Omega^\sigma S^{\lambda+1}\langle\lambda+1\rangle \to \Omega^\sigma S^{\lambda+1}$ is the Thom space of a Real line bundle $\mathcal{L}$ over $\Omega^\sigma S^{\lambda+1}$. Using the fact that $H\underline{\mathbb{Z}}$ is Real oriented, we conclude that there is a cofiber sequence of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1}\langle \lambda+1\rangle \to H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1} \to H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \left(\Omega^\sigma S^{\lambda+1}\right)^{\mathcal{L}} \simeq \Sigma^\rho H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1}.$$

By [Hil17, Theorem 4.3], there is a James splitting

$$\Sigma^\infty_+ \Omega^\sigma S^{\lambda+1} \simeq \Sigma^\infty_+ \Omega^\sigma \Sigma^\sigma S^\rho \simeq S^0 \oplus S^\rho \oplus S^{2\rho} \oplus S^{3\rho} \oplus \cdots.$$

In particular, there is a cofiber sequence of $H\underline{\mathbb{Z}}$-modules

$$H\underline{\mathbb{Z}} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1}\langle \lambda+1\rangle \to \bigoplus_{k\geq 0} \Sigma^{k\rho} H\underline{\mathbb{Z}} \to \bigoplus_{s\geq 1} \Sigma^{s\rho} H\underline{\mathbb{Z}},$$

as desired. $\square$

Proof of Lemma A.2. By 2-localizing the result of Lemma A.3, we learn that

$$\mathrm{THR}(H\underline{\mathbb{Z}}_{(2)}) \simeq H\underline{\mathbb{Z}}_{(2)} \otimes \Sigma^\infty_+ \Omega^\sigma S^{\lambda+1}\langle \lambda+1\rangle$$

may be calculated as the fiber of a map f of $H\underline{\mathbb{Z}}_{(2)}$-module spectra

$$f : \bigoplus_{k\geq 0} \Sigma^{k\rho} H\underline{\mathbb{Z}}_{(2)} \to \bigoplus_{s\geq 1} \Sigma^{s\rho} H\underline{\mathbb{Z}}_{(2)}.$$

Since the domain of f is a direct sum of free $H\underline{\mathbb{Z}}_{(2)}$-module spectra, f is determined by a sequence of elements $f_k \in \pi_{k\rho}\left(\bigoplus_{s\geq 1} \Sigma^{s\rho} H\underline{\mathbb{Z}}_{(2)}\right)$. The $RO(C_2)$-graded homotopy groups of $H\underline{\mathbb{Z}}$, as nicely displayed for example in [Gre17, p.6], show that there are no classes in $\pi_{k\rho}(\Sigma^{s\rho} H\underline{\mathbb{Z}}_{(2)})$ unless $k = s$. Furthermore, one sees that any such class is determined by its underlying non-equivariant class in $\pi_{2k}(\Sigma^{2k} H\mathbb{Z}_{(2)}) \cong \mathbb{Z}_{(2)}$, and in particular the map f is entirely determined by its underlying non-equivariant map

$$f_{\text{underlying}} : \bigoplus_{k\geq 0} \Sigma^{2k} H\mathbb{Z}_{(2)} \to \bigoplus_{s\geq 1} \Sigma^{2s} H\mathbb{Z}_{(2)}.$$

The fiber of $f_{\text{underlying}}$ must agree with the known non-equivariant calculation

$$\mathrm{THH}(H\mathbb{Z}_{(2)}) \simeq H\mathbb{Z}_{(2)} \oplus \bigoplus_{s\geq 1} \Sigma^{2s-1} (H\mathbb{Z}/s)_{(2)},$$

which determines the map $f_{\text{underlying}}$ well enough to determine the fiber of f up to equivalence. $\square$

Proof of Theorem A.1. As with any C_2-equivariant spectrum, there is a pullback square

$$\begin{array}{ccc} \mathrm{THR}(H\underline{\mathbb{Z}}) & \longrightarrow & \mathrm{THR}(H\underline{\mathbb{Z}})[\frac{1}{2}] \\ \downarrow & & \downarrow \\ \mathrm{THR}(H\underline{\mathbb{Z}})_{(2)} & \longrightarrow & \mathrm{THR}(H\underline{\mathbb{Z}}) \otimes H\underline{\mathbb{Q}}. \end{array}$$

The 2-local Lemma A.2 allows to calculate the lower left corner of the square, while the result [DMPR17, Corollary 5.28] of Dotto, Moi, Patchkoria, and Reeh calculates the upper right. From these results, we learn that the square is a direct sum of squares

$$\begin{array}{ccc} H\underline{\mathbb{Z}} & \longrightarrow & H\underline{\mathbb{Z}}[1/2] \\ \downarrow & & \downarrow \\ H\underline{\mathbb{Z}}_{(2)} & \longrightarrow & H\underline{\mathbb{Q}}, \end{array}$$

and, for all $k \geq 1$,

$$\begin{tikzcd}
\Sigma^{k\rho-1} H\underline{\mathbb{Z}/k} \arrow[r] \arrow[d] & \Sigma^{k\rho-1} H\underline{\mathbb{Z}/k}[1/2] \arrow[d] \\
\Sigma^{k\rho-1} H\underline{\mathbb{Z}/k}_{(2)} \arrow[r] & 0.
\end{tikzcd}$$

$\square$

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