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The infinite loop Adams conjecture via classification theorems for $\mathscr{F}$-spaces

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We prove the following generalized version of the complex Adams conjecture (see Theorem 10·4), as announced in (**5**).

THEOREM. *For any integer $r > 0$, the triangle of spaces localized away from r*

$$\begin{array}{ccc} BU & \xrightarrow{\psi^r} & BU \\ & {\scriptstyle J}\searrow \quad \swarrow{\scriptstyle J} & \\ & BSG & \end{array}$$

is the first stage of a homotopy commutative triangle of connected Ω-spectra, where $\psi^r: BU \to BU$ is the r-th Adams operation on complex K-theory and $J: BU \to BSG$ is the complex J-homomorphism.

Our proof also applies to prove the quaternionic analogue of this conjecture (see Theorem 10·5). These theorems assume added interest in view of the fact that J, $J \circ \psi^r: BSO \to BSG$ are not homotopic as maps of H-spaces localized away from r for any odd integer r. The reader can find certain applications of this theorem in (**12**).

Our interest in this infinite loop Adams conjecture is based to some extent upon the fact that it is a natural (but apparently difficult) sharpening of the usual Adams conjecture. More significantly, we view this conjecture as an excellent challenge to the methods used in the various proofs of the Adams conjecture and to the various approaches to infinite loop space theory. Furthermore, there are few techniques available to prove that two maps between infinite loop spaces are homotopic as maps of Ω-spectra: any new techniques developed to prove the existence of such a homotopy in a particular situation should have more general interest.

Our proof of the infinite loop Adams conjecture follows the following slightly modified outline of D. Sullivan's proof (16) of the complex Adams conjecture. After completing away from r, Sullivan obtains a model for the universal sphere bundle over $BU(n)$ using algebraic geometry, and he verifies that the Adams operation ψ^r can be realized algebraically in such a way that it is covered by a self-map of this universal sphere bundle. This leads to a diagram of spaces completed away from r consisting of cartesian squares

$$\begin{array}{ccccc} B(U(n), S^{2n}) & \longrightarrow & B(U(n), S^{2n}) & \longrightarrow & B(SG(n), S^{2n}) \\ \downarrow & & \downarrow & & \downarrow \\ BU(n) & \xrightarrow{\psi^r} & BU(n) & \xrightarrow{J} & BSG(n) \end{array}$$

Thus, $B(U(n), S^{2n}) \to BU(n)$ represents the pull-back of $B(SG(n), S^{2n}) \to BSG(n)$ by both J and $J \circ \psi^r$. Because $B(SG(n), S^{2n}) \to BSG(n)$ is a universal S^{2n} fibration, J and $J \circ \psi^r$ are homotopic after completing away from r. The complex Adams conjecture follows upon stabilization.

In providing an infinite loop space analogue of this outline, we replace spaces by $\mathscr{F}$-spaces (formerly called Γ-spaces). The basic properties of $\mathscr{F}$-spaces are given in Section 1, recalling the foundations provided by G. Segal in (15) and extended in (2). Sections 2–7 are devoted to the development of the requisite classification theory. With the expectation of additional applications, we develop this theory in full generality with arbitrary pointed simplicial sets as fibre. The classification theory relates equivalence classes of X-fibrations (as defined in Section 3) to homotopy classes of maps into a universal $\mathscr{F}$-space constructed from the monoids of self-equivalences of the iterated smash products of X (see Section 5). The somewhat awkward requirement that an X-fibration be a sectioned map as defined in Section 2 is necessary for the construction of associated principal fibrations given in Section 4. Section 6 proves the classification theorem for X-fibrations, whereas Section 7 provides the completed version we require. Section 8 sets the J-homomorphism into the context of S^2-fibrations of $\mathscr{F}$-spaces. We exhibit an algebro-geometric model of the completion of the S^2-fibration associated to the J-homomorphism in Section 9. Our proof of the infinite loop Adams conjecture is concluded in Section 10.

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1. $\mathscr{F}$-SPACES

In this section, we recall the definitions and properties of $\mathscr{F}$-spaces which we shall employ in subsequent sections. These $\mathscr{F}$-spaces are due to Segal, although the point of view we adopt is a modification of Segal's due to D. Anderson. Details not presented here can be found in (2), where $\mathscr{F}$-spaces are called Γ-spaces.

Definition 1·1. An *$\mathscr{F}$-space* is a functor

$$\mathscr{B}:\mathscr{F}\to\mathscr{S}_*$$

from the category $\mathscr{F}$ of finite pointed sets to the category $\mathscr{S}_*$ of pointed simplicial sets such that $\mathscr{B}$ sends each singleton set to the point simplicial set. The category of $\mathscr{F}$-spaces, denoted $\mathscr{F}[\mathscr{S}_*]$, is the category whose objects are $\mathscr{F}$-spaces and whose maps are natural transformations. More generally, if $\mathscr{C}$ is a category with a given final–cofinal object 0, an $\mathscr{F}$-object of $\mathscr{C}$ is a functor $\mathscr{B}:\mathscr{F}\to\mathscr{C}$ sending each singleton set to 0.

We let $\mathbf{n}\in\mathrm{Obj}(\mathscr{F})$ denote the set $\{0,1,\ldots,n\}$ with base point 0. The $\mathscr{F}$-object $\mathscr{B}:\mathscr{F}\to\mathscr{C}$ is canonically equivalent to the functor whose value on $S\in\mathrm{Obj}(\mathscr{F})$ is defined to equal $\mathscr{B}(\mathbf{n})$ for that n such that $|S| = n$ (i.e. $S\simeq\mathbf{n}$ in $\mathscr{F}$). Consequently, we shall usually view $\mathscr{B}:\mathscr{F}\to\mathscr{C}$ as a functor defined on the full subcategory of $\mathscr{F}$ consisting of objects $\mathbf{n}$ for $n\geqslant 0$. For notational convenience, we let $\mathscr{B}_{\mathbf{n}}$ denote $\mathscr{B}(\mathbf{n})$ and $a:\mathscr{B}_{\mathbf{m}}\to\mathscr{B}_{\mathbf{n}}$ denote the morphism determined by $a:\mathbf{m}\to\mathbf{n}$ in $\mathscr{F}$ for any $\mathscr{F}$-object $\mathscr{B}$.

All $\mathscr{F}$-spaces we consider will be provided with a structure map to the $\mathscr{F}$-space $\mathscr{N}$ introduced in Example 1·2.

Example 1·2. Let N denote the set of non-negative integers. For any $n>0$, let $\mathscr{N}_{\mathbf{n}} = N^{\times n}$, the discrete simplicial set given as the n-fold cartesian product of N with itself. For $a:\mathbf{n}\to\mathbf{k}$ in $\mathscr{F}$, define $a:\mathscr{N}_{\mathbf{n}}\to\mathscr{N}_{\mathbf{k}}$ by sending the n-tuple $I=(i_1,\ldots,i_n)\in\mathscr{N}_{\mathbf{n}}$ to the k-tuple $a(I) = (j_1,\ldots,j_k)$, where $j_t = \sum_{a(s)=t} i_s$.

If $\mathscr{B}\to\mathscr{N}$ is a map of $\mathscr{F}$-spaces, $\mathscr{B}_I\subset\mathscr{B}_{\mathbf{n}}$ will be used to denote the inverse image of $I\in\mathscr{N}_{\mathbf{n}}$; in particular, $\mathscr{B}_i$ denotes the inverse image of $(i)\in\mathscr{N}_{\mathbf{1}}$.

The following proposition is proved in (2). The purpose of a closed model category structure is that it provides a good homotopy category defined by inverting weak equivalences (13).

Proposition 1·3. The following definitions determine a simplicial closed model category structure on the category of $\mathscr{F}$-spaces.

(*a*) $f:\mathscr{A}\to\mathscr{B}$ in $\mathscr{F}[\mathscr{S}_*]$ is a *weak equivalence* if $f_{\mathbf{n}}:\mathscr{A}_{\mathbf{n}}\to\mathscr{B}_{\mathbf{n}}$ is a weak equivalence for each $n>0$.

(*b*) $f:\mathscr{A}\to\mathscr{B}$ in $\mathscr{F}[\mathscr{S}_*]$ is a *cofibration* if for each $n>0$ the induced map

$$f_{(n)}:\operatorname*{colim}_{r<n}(\mathscr{A}_{\mathbf{n}}\bigcup_{\mathscr{A}_{\mathbf{r}}}\mathscr{B}_{\mathbf{r}})\to\mathscr{B}_{\mathbf{n}}$$

is a Σ_n-equivariant cofibration (i.e. $f_{(n)}$ is injective and the permutation group Σ_n acts freely on any simplex of $\mathscr{B}_{\mathbf{n}}$ not in the image of $f_{(n)}$), where $\operatorname*{colim}_{r<n}$ is indexed by all maps $\mathbf{r}\to\mathbf{n}$ in $\mathscr{F}$ with $r<n$.

(*c*) $f:\mathscr{A}\to\mathscr{B}$ in $\mathscr{F}[\mathscr{S}_*]$ is a *fibration* if for each $n>0$ the induced map

$$f^{(n)}:\mathscr{A}_{\mathbf{n}}\to\lim_{n>t}(\mathscr{A}_{\mathbf{t}}\times_{\mathscr{B}_{\mathbf{t}}}\mathscr{B}_{\mathbf{n}})$$

is a fibration (i.e. a Kan fibration), where $\lim_{n>t}$ is indexed by all maps $\mathbf{n}\to\mathbf{t}$ with $n>t$.

An object $\mathscr{B}$ in $\mathscr{F}[\mathscr{S}_*]$ is said to be *cofibrant* if $* \to \mathscr{B}$ is a cofibration and *fibrant* if $\mathscr{B} \to *$ is a fibration. The $\mathscr{F}$-space $\mathscr{N}$ of Example 1·2 is neither cofibrant (the Σ_n-equivariance condition fails), or fibrant. The *homotopy category* of $\mathscr{F}[\mathscr{S}_*]$ (with respect to the structure specified in Proposition 1·3) is the category obtained by formally inverting the weak equivalences defined in 1·3(a)). If $\mathscr{A}$ is cofibrant and $\mathscr{B}$ fibrant, then any map from $\mathscr{A}$ to $\mathscr{B}$ in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]$ (the homotopy category of $\mathscr{F}[\mathscr{S}_*]$) is represented by a map $\mathscr{A} \to \mathscr{B}$ in $\mathscr{F}[\mathscr{S}_*]$.

The category $\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ admits finite limits and finite colimits. Therefore, the forgetful functor

$$\mathscr{F}[\mathscr{S}_*]/\mathscr{N} \to \mathscr{F}[\mathscr{S}_*]$$

is a map of simplicial closed model categories provided that weak equivalences, cofibrations, and fibrations in $\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ are defined by 1·3(a), (b) and (c) via the inclusion $\mathrm{Maps}(\mathscr{F}[\mathscr{S}_*]/\mathscr{N}) \to \mathrm{Maps}(\mathscr{F}[\mathscr{S}_*])$. We denote by

$$\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$$

the homotopy category $\mathrm{Ho}(\mathscr{F}[\mathscr{S}_*]/\mathscr{N})$, which equals $\mathrm{Ho}(\mathscr{F}[\mathscr{S}_*])/\mathscr{N}$.

An $\mathscr{F}$-space determines a simplicial pre-spectrum in the following manner (this formulation was explained to me by A. K. Bousfield). Let Σ be the finite simplicial set representing the circle with exactly two non-degenerate simplices, and let Σ^n be the n-fold smash product of Σ with itself. For any $\mathscr{F}$-space $\mathscr{B}: \mathscr{F} \to \mathscr{S}_*$ and any finite pointed simplicial set S, let $\mathscr{B}(S)$ denote the diagonal of the bi-simplicial set $k, l \to \mathscr{B}(S_k)_l$. We define

$$\Phi(\mathscr{B}) = \{\mathscr{B}(\Sigma^n),\ \Sigma \wedge \mathscr{B}(\Sigma^n) \to \mathscr{B}(\Sigma^{n+1})\}_{n>0},$$

where $\Sigma \wedge \mathscr{B}(\Sigma^n) \to \mathscr{B}(\Sigma^{n+1})$ is defined by the maps $\{x\} \wedge \mathscr{B}((\Sigma^n)_k) \to \mathscr{B}((\Sigma^{n+1})_k)$ given by applying $\mathscr{B}$ to $x \wedge (\ ): \Sigma^n_k \to \Sigma^{n+1}_k$ for all $k \geq 0$, $x \in (\Sigma)_k$.

The following proposition, due to G. Segal, is proved in (2).

Proposition 1·4. As defined above, Φ determines a functor

$$\Phi: \mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*] \to \mathrm{Ho}\,\mathrm{Sp},$$

where $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]$ is the *homotopy category of $\mathscr{F}$-spaces* determined by the structure of Proposition 1·3 and where $\mathrm{Ho}\,\mathrm{Sp}$ is the *homotopy category of spectra*. Furthermore, $\Phi(\mathscr{B})$ is an Ω-spectrum (i.e. the adjoints of the maps $\Sigma \wedge \mathscr{B}(\Sigma^n) \to \mathscr{B}(\Sigma^{n+1})$ are weak equivalences) whenever $\mathscr{B}$ satisfies the condition that for all $n > 0$

$$\prod_{i=1}^{n} p_i : \mathscr{B}_{\mathbf{n}} \to \prod_{i=1}^{n} \mathscr{B}_{\mathbf{1}}$$

is a weak equivalence, where $p_i: \mathbf{n} \to \mathbf{1}$ is the projection sending $i \in \mathbf{n}$ and $j \neq i$ to 0.

2. Sectioned maps

In Definition 2·1 we introduce the definition of a sectioned map of $\mathscr{F}$-spaces. The remainder of this section is devoted to introducing various forms of 'good' sections and to showing that any sectioned map may be modified so as to possess such 'good' properties. The categorically inclined reader might prefer to view this discussion as

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providing various verifications necessary to show that the fibred category of sectioned maps over $\mathscr{F}[\mathscr{S}_*]$ admits a closed model category structure.

Definition 2·1. Let $f:\mathscr{E}\to\mathscr{B}$ be a map of $\mathscr{F}$-spaces. A *section* for f consists of pointed sub-simplicial sets $\mathscr{E}_{\mathbf{n}}^S \subset \mathscr{E}_{\mathbf{n}}$ for each $n > 0$ and each pointed $S \subset \mathbf{n}$ such that

(*a*) If $S = \{0\}$, then $\mathscr{E}_{\mathbf{n}}^S = \mathscr{B}_{\mathbf{n}}$ and f restricted to $\mathscr{E}_{\mathbf{n}}^S$ is the identity; if $S = \mathbf{n}$, then $\mathscr{E}_{\mathbf{n}}^S = \mathscr{E}_{\mathbf{n}}$.

(*b*) If T is a pointed subset of S, then $\mathscr{E}_{\mathbf{n}}^T \subset \mathscr{E}_{\mathbf{n}}^S$.

(*c*) For any map $d:\mathbf{n}\to\mathbf{t}$ in $\mathscr{F}$, let $(d:S)\subset\mathbf{t}$ be the pointed subset defined by $0 \neq j \in (d:S)$ if and only if $d^{-1}(j) \subset S$. Then $d:\mathscr{E}_{\mathbf{n}}\to\mathscr{E}_{\mathbf{t}}$ restricts to $\mathscr{E}_{\mathbf{n}}^S\to\mathscr{E}_{\mathbf{t}}^{(d:S)}$.

A map $(g,h):f'\to f$ between sectioned maps $f':\mathscr{E}'\to\mathscr{B}'$ and $f:\mathscr{E}\to\mathscr{B}$ is said to be section preserving if $g_n:\mathscr{E}'_{\mathbf{n}}\to\mathscr{B}_{\mathbf{n}}$ restricts to $g_n^s:\mathscr{E}'^S_{\mathbf{n}}\to\mathscr{E}^S_{\mathbf{n}}$ for each $n > 0$ and each pointed $S \subset \mathbf{n}$. Here, $g:\mathscr{E}'\to\mathscr{E}$ and $h:\mathscr{B}'\to\mathscr{B}$ satisfy $fg = hf'$.

The examples of sectioned maps $f:\mathscr{E}\to\mathscr{B}$ which we shall consider are maps over $\mathscr{N}$ (cf. Example 1·2) having the property that $f_I:\mathscr{E}_I\to\mathscr{B}_I$ has fibres homotopy equivalent to a product $X^I = X^{i_1}\times\ldots\times X^{i_n}$ for $I = (i_1,\ldots,i_n)$, where X^i is the i-fold smash product of some simplicial set X. The section $\mathscr{E}_{\mathbf{n}}^s \subset \mathscr{E}_{\mathbf{n}}$ restricted to $\mathscr{E}_I^s = \mathscr{E}_{\mathbf{n}}^s \cap \mathscr{E}_I \subset \mathscr{E}_I$ singles out subfibres of $f_I:\mathscr{E}_I\to\mathscr{B}_I$ homotopy equivalent to $X^{S(I)}$, where $X^{S(I)}\subset X^I$ consists of simplices whose jth factor is the base point whenever $i \notin S$. A structure map $d:\mathscr{E}_I\to\mathscr{E}_{d(I)}$ induces a map on fibres (over $d:\mathscr{B}_I\to\mathscr{B}_{d(I)}$) which is to be identified with the smash product map $X^I\to X^{d(I)}$. Condition 2·1(*c*) requires that a subfibre of $f_I:\mathscr{E}_I\to\mathscr{B}_I$ corresponding to $X^{S(I)}\subset X^I$ be mapped to the appropriate (with respect to smash product) subfibre of $f_{d(I)}:\mathscr{E}_{d(I)}\to\mathscr{B}_{d(I)}$.

A most important aspect of this definition is that it requires no actual product structure on the fibres of $f_I:\mathscr{E}_I\to\mathscr{B}_I$. This enables us to obtain new sectioned maps from previously defined ones by applying a functor from $\mathscr{S}_*$ to $\mathscr{S}_*$ which does not commute with products. For the examples arising from algebraic geometry, this generality is crucial.

If $\Sigma^S \subset \Sigma_n$ denotes the subgroup of Σ_n consisting of those permutations which stabilize $S \subset \mathbf{n}$, then the action of Σ_n on $\mathscr{E}_{\mathbf{n}}$ restricts to an action of Σ^s on $\mathscr{E}_{\mathbf{n}}^S$ (as can be seen by applying 2·1(*c*) with $d \in \Sigma^S$).

The following proposition describes our standard technique of constructing section-preserving maps.

Proposition 2·2. Let $f':\mathscr{E}'\to\mathscr{B}'$ and $f:\mathscr{E}\to\mathscr{B}$ be sectioned maps of $\mathscr{F}$-spaces, let $h:\mathscr{B}'\to\mathscr{B}$ be a map of $\mathscr{F}$-spaces, and assume $\operatorname{colim}_{|S|<n} \mathscr{E}'^S_{\mathbf{n}}\to\mathscr{E}'_{\mathbf{n}}$ is a cofibration for all $n > 0$ (where colim is indexed by the pointed, proper subsets S of $\mathbf{n}$). A section-preserving map

$$(g,h):f'\to f$$

is equivalent to the following inductive data: for each $n > 0$ and each $k \leqslant n$, a choice of Σ^S-equivariant lifting

$$\begin{array}{ccc}
\operatorname*{colim}_{m<n} \mathscr{E}'^{b^{-1}(S)}_{\mathbf{m}} \cup \operatorname*{colim}_{T\subset S} \mathscr{E}'^T_{\mathbf{n}} & \longrightarrow & \mathscr{E}^S_{\mathbf{n}} \\
\big\downarrow & \nearrow & \big\downarrow \\
\mathscr{E}'^S_{\mathbf{n}} & \longrightarrow & \lim_{n>t} \mathscr{E}_{\mathbf{t}}^{(d:S)} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}}
\end{array}$$

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for some choice of pointed subset $S \subset \mathbf{n}$ with $|S| = k$, where $\operatorname{colim}_{m<n}$ is indexed by maps $b: \mathbf{m} \to \mathbf{n}$ in $\mathscr{F}$ with $m < n$ and lim is indexed by maps $d: \mathbf{n} \to \mathbf{t}$ in $\mathscr{F}$ with $n > t$.

Proof. A section-preserving map $(g, h): f' \to f$ determines such liftings: condition 2·1 (*b*) implies the condition that $g_{\mathbf{n}}^{s}: \mathscr{E}'^{s}_{\mathbf{n}} \to \mathscr{E}^{s}_{\mathbf{n}}$ restricts to $\operatorname{colim}_{T \subsetneq S} g_{\mathbf{n}}^{T}$ on $\operatorname{colim}_{T \subsetneq S} \mathscr{E}_{\mathbf{n}}^{T}$; condition 2·1 (*c*) implies that $g_{\mathbf{n}}^{s}$ restricts to $\operatorname{colim}_{m<n} g_{\mathbf{m}}^{b^{-1}(S)}$ on $\operatorname{colim}_{m<n} \mathscr{E}'^{b^{-1}(S)}_{\mathbf{m}}$ and covers

$$\lim_{n>t} g_{\mathbf{t}}^{(d:S)} \circ d: \mathscr{E}'^{S}_{\mathbf{n}} \to \lim_{n>t} \mathscr{E}'^{(d:S)}_{\mathbf{t}} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}} \to \lim_{n>t} \mathscr{E}_{\mathbf{t}}^{(d:S)} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}}.$$

Conversely, we define $g_{\mathbf{n}}^{\sigma(S)}: \mathscr{E}'^{\sigma(S)}_{\mathbf{n}} \to \mathscr{E}_{\mathbf{n}}^{\sigma(S)}$ for $\sigma \in \Sigma_n$ and $g_{\mathbf{n}}^{S}$ already defined by

$$g_{\mathbf{n}}^{\sigma(S)} = \sigma \circ g_{\mathbf{n}}^{S} \circ \sigma^{-1}: \mathscr{E}'^{\sigma(S)}_{\mathbf{n}} \to \mathscr{E}'^{S}_{\mathbf{n}} \to \mathscr{E}_{\mathbf{n}}^{S} \to \mathscr{E}_{\mathbf{n}}^{\sigma(S)}.$$

Since $g_{\mathbf{n}}^{S}$ is Σ^{S}-equivariant, $g_{\mathbf{n}}^{\sigma(S)}$ depends only on $\sigma(S) \subset \mathbf{n}$ (i.e. only the equivalence class of σ in Σ_n/Σ^{S}). The condition that $\operatorname{colim}_{S \subsetneq \mathbf{n}} \mathscr{E}'^{S}_{\mathbf{n}} \to \mathscr{E}'_{\mathbf{n}}$ is a cofibration guarantees that $\mathscr{E}'^{S}_{\mathbf{n}}$ and $\mathscr{E}'^{T}_{\mathbf{n}}$ have intersection inside $\mathscr{E}'_{\mathbf{n}}$ equal to $\mathscr{E}'^{S \cap T}_{\mathbf{n}}$. Thus, the above definition of $g_{\mathbf{n}}^{\sigma(S)}$ extends $g_{\mathbf{n}}^{S}$ to a Σ_n-equivariant map on $\bigcup_{|T| \leqslant |S|} \mathscr{E}'^{T}_{\mathbf{n}}$. Consequently, the inductive construction provides a section-preserving map as asserted.

We continue to employ the notation introduced in Proposition 2·2. The following definition is introduced to facilitate lifting arguments suggested by Proposition 2·2.

Definition 2·3. Let $f: \mathscr{E} \to \mathscr{B}$ be a sectioned map of $\mathscr{F}$-spaces. If for each $n > 0$ and each pointed $S \subset \mathbf{n}$ the natural map

$$\operatorname{colim}_{m<n} \mathscr{E}_{\mathbf{m}}^{b^{-1}(S)} \cup \operatorname{colim}_{T \subsetneq S} \mathscr{E}_{\mathbf{n}}^{T} \to \mathscr{E}_{\mathbf{n}}^{S}$$

is a $\Sigma_{\mathbf{n}}^{S}$ equivariant cofibration then f is said to be *cofibrantly sectioned*. Moreover, f is said to be *fibrantly sectioned* if, for each $n > 0$ and each pointed $S \subset \mathbf{n}$, the natural map

$$\mathscr{E}_{\mathbf{n}}^{S} \to \lim_{n>t} \mathscr{E}_{\mathbf{t}}^{(d:S)} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}}$$

is a fibration.

A map $a: \mathbf{n} \to \mathbf{k}$ in $\mathscr{F}$ is said to be a *projection* if $a^{-1}(j)$ is a singleton set for all j with $0 \neq j \in \mathbf{k}$. Such a projection corresponds to a pointed subset of $\mathbf{n}$ (i.e. a subset containing $0 \in \mathbf{n}$) with k non-zero elements together with a permutation of those non-zero elements. If $a: \mathbf{n} \to \mathbf{k}$ and $b: \mathbf{n} \to \mathbf{r}$ are projections, then an *intersection* of a and b is a projection $c: \mathbf{n} \to \mathbf{s}$ with the property that $c(i) \neq 0$ if and only if $a(i) \neq 0$ and $b(i) \neq 0$. The projections under $\mathbf{n}$ form a category (isomorphic to the partially ordered set of pointed subsets of $\mathbf{n}$).

We shall require the following property of fibrantly sectioned maps when we consider principalizations in Section 4.

Proposition 2·4. Let $f: \mathscr{E} \to \mathscr{B}$ be a fibrantly sectioned map of $\mathscr{F}$-spaces. Let I be some non-empty full subcategory of the category of projections under $\mathbf{n}$ with the property that some intersection of any two projections in I is again in I. For any pointed $S \subset \mathbf{n}$, the natural map

$$\mathscr{E}^S_{\mathbf{n}} \to \lim_{a \in I} \mathscr{E}^{(a:S)}_{\mathbf{k}} \times_{\mathscr{B}_{\mathbf{k}}} \mathscr{B}_{\mathbf{n}}$$

is a fibration.

Proof. Because $\lim_{a\in I} \mathscr{E}^{(a:S)}_{\mathbf{k}} \times_{\mathscr{B}_{\mathbf{k}}} \mathscr{B}_{\mathbf{n}}$ is naturally isomorphic to $\lim_{a\in J} \mathscr{E}^{(a:S)}_{\mathbf{k}} \times_{\mathscr{B}_{\mathbf{k}}} \mathscr{B}_{\mathbf{n}}$, where J is the full subcategory of the category of projections under $\mathbf{n}$ which factor through a projection in I, we may assume that every projection factoring through a projection of I is actually in I. Since $\mathscr{E}^S_{\mathbf{n}} \to \lim_{n>t} \mathscr{E}^{(d:S)}_{\mathbf{t}} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}}$ is a fibration, it suffices to prove the existence of a lifting (as dotted) for the following square

$$\begin{array}{ccc} \Lambda[r] & \to & \lim_{n>t} \mathscr{E}^{(d:S)}_{\mathbf{t}} \times_{\mathscr{B}_{\mathbf{t}}} \mathscr{B}_{\mathbf{n}} \\ {\scriptstyle \epsilon_i}\downarrow & \nearrow & \downarrow \\ \Delta[r] & \to & \lim_{a\in I} \mathscr{E}^{(a:S)}_{\mathbf{k}} \times_{\mathscr{B}_{\mathbf{k}}} \mathscr{B}_{\mathbf{n}} \end{array}$$

for all $i > 0$, $0 \leqslant i \leqslant r$ (where $\epsilon_i : \Lambda[r] \to \Delta[r]$ is the inclusion of the $(r-1)$-skeleton of $\Delta[r]$ minus its ith face into $\Delta[r]$). The lifting into any factor indexed by $b : \mathbf{n} \to \mathbf{1}$ in $\mathscr{F}$ which does not factor through a projection in I is achieved using the facts that $\mathscr{E}^{(b:S)}_{\mathbf{1}} \to \mathscr{B}_{\mathbf{1}}$ (and thus $\mathscr{E}^{(b:S)}_{\mathbf{1}} \times_{\mathscr{B}_{\mathbf{1}}} \mathscr{B}_{\mathbf{n}} \to \mathscr{B}_{\mathbf{n}}$) is a fibration and that only $\mathbf{n} \to \mathbf{0}$ and b itself factor through $b : \mathbf{n} \to \mathbf{1}$. Proceeding inductively with respect to $m < n$, we obtain a lifting into any factor indexed by $b : \mathbf{n} \to \mathbf{m}$ in $\mathscr{F}$ which does not factor through a projection in I by using the preceding liftings and the fact that

$$\mathscr{E}^{(b:S)}_{\mathbf{m}} \times_{\mathscr{B}_{\mathbf{m}}} \mathscr{B}_{\mathbf{n}} \to \lim_{m>l} \mathscr{E}^{(c\circ b:S)}_{\mathbf{l}} \times_{\mathscr{B}_{\mathbf{l}}} \mathscr{B}_{\mathbf{n}}$$

is a fibration.

The following proposition provides a factorization of a section-preserving map which will facilitate lifting arguments.

Proposition 2·5. Let $f' : \mathscr{E}' \to \mathscr{B}'$ and $f : \mathscr{E} \to \mathscr{B}$ be sectioned maps, and let $(g, \bar{g}) : f' \to f$ be section preserving. There exists a factorization of $(g, \bar{g})$,

$$(g, \bar{g}) = (p, \bar{p}) \circ (q, \bar{q}) : f' \to f'' \to f$$

together with a section of $f'' : \mathscr{E}'' \to \mathscr{B}''$ such that

(*a*) $(j, \bar{j}) : f' \to f''$ and $(p, \bar{p}) : f'' \to f$ are section preserving,

(*b*) j and $\bar{j}$ are trivial cofibrations; moreover, for each $n > 0$ and each pointed $S \subset \mathbf{n}$, the natural maps

$$\operatorname*{colim}_{m<n} \mathscr{E}''^{b^{-1}(S)}_{\mathbf{m}} \cup \mathscr{E}'^{S}_{\mathbf{n}} \cup \operatorname*{colim}_{T \subsetneq S} \mathscr{E}''^{T}_{\mathbf{n}} \to \mathscr{E}''^{S}_{\mathbf{n}},$$

$$\mathscr{E}'^{S}_{\mathbf{n}} \to \mathscr{E}''^{S}_{\mathbf{n}}$$

are trivial Σ^S-equivariant cofibrations.

(*c*) p and $\bar{p}$ are fibrations; moreover, for each $n > 0$ and each pointed $S \leqslant \mathbf{n}$, the natural maps

$$\mathscr{E}''^{S}_{\mathbf{n}} \to \lim_{n>t} \mathscr{E}''^{(d:S)}_{\mathbf{t}} \times_{\mathscr{E}^{(d:S)}_{\mathbf{t}}} \mathscr{E}^S_{\mathbf{n}},$$

$$\mathscr{E}''^{S}_{\mathbf{n}} \to \mathscr{E}^S_{\mathbf{n}}$$

are fibrations.

Proof. The proof proceeds by induction on n and $|S| = k \leqslant n$. We first factor $\bar{g}$ as $\bar{p} \circ \bar{j} : \mathscr{B}' \to \mathscr{B}'' \to \mathscr{B}$ with $\bar{j}$ a trivial cofibration and $\bar{p}$ a fibration. Setting $\mathscr{E}_{\mathbf{n}}''^{\{0\}} = \mathscr{B}_{\mathbf{n}}''$ for $n > 0$, we satisfy 2·5 (a), (b) and (c) for $S = \{0\}$. For a given $0 < k \leqslant n$, we define $\mathscr{E}_{\mathbf{n}}''^S$ for each $S \subset \mathbf{n}$ to be the pointed simplicial set with a Σ^S-action functorially constructed to provide a Σ^S-equivariant factorization

$$\operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}''^{b^{-1}(S)} \cup \mathscr{E}_{\mathbf{n}}'^S \cup \operatorname*{colim}_{\substack{T \subset S \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T \to \mathscr{E}_{\mathbf{n}}''^S \to \lim_{n>t} \mathscr{E}_{\mathbf{t}}''^{(d:S)} \times_{\mathscr{E}_{\mathbf{n}}^{(d:S)}} \mathscr{E}_{\mathbf{n}}^S$$

with left arrow a Σ^S-equivariant trivial cofibration and right arrow a fibration (cf. (6), proposition 5·51, where such a factorization is constructed – without the Σ^S-equivariant condition – as the colimit of an infinite sequence of canonical factorizations; this construction need only be modified by attaching $k!(n-k)!$ simplices for each 'horn' at each stage with Σ^S acting freely on the new simplices). If $S' = \sigma(S)$ for some $\sigma \in \Sigma_n$, then naturality and induction imply that σ induces an isomorphism

$$\sigma : \mathscr{E}_{\mathbf{n}}^S \to \mathscr{E}_{\mathbf{n}}^{S'}$$

which is Σ^S-equivariant provided that Σ^S acts on $\mathscr{E}_{\mathbf{n}}^{S'}$ via the conjugation isomorphism $(\ )^\sigma : \Sigma^S \xrightarrow{\sim} \Sigma^{S'}$ followed by the defining action of $\Sigma^{S'}$ on $\mathscr{E}_{\mathbf{n}}^{S'}$.

By construction, the composition inclusions

$$\mathscr{E}_{\mathbf{n}}''^S \to \operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}'' \cup \mathscr{E}_{\mathbf{n}}' \cup \operatorname*{colim}_{\substack{T \subset \mathbf{n} \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T \to \mathscr{E}_{\mathbf{n}}''$$

satisfy conditions 2·1 (a), (b) and (c). (An easy inductive argument implies that these maps are inclusions.) Our construction also guarantees that the composites

$$j_n : \mathscr{E}_{\mathbf{n}}' \to \operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}'' \cup \mathscr{E}_{\mathbf{n}}' \cup \operatorname*{colim}_{\substack{T \subset \mathbf{n} \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T \to \mathscr{E}_{\mathbf{n}}'',$$

$$p_n : \mathscr{E}_{\mathbf{n}}'' \to \lim_{n>t} \mathscr{E}_{\mathbf{t}}'' \times_{\mathscr{E}_{\mathbf{t}}} \mathscr{E}_{\mathbf{n}} \to \mathscr{E}_{\mathbf{n}}$$

are section preserving.

We assume inductively that

$$\operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}'^{b^{-1}(S)} \to \operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}''^{b^{-1}(S)}$$

is a trivial cofibration for all $S \subset \mathbf{n}$. By construction,

$$\operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}''^{b^{-1}(S)} \cup \mathscr{E}_{\mathbf{n}}'^S \cup \operatorname*{colim}_{\substack{T \subset S \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T \to \mathscr{E}_{\mathbf{n}}''^S$$

is a trivial cofibration. Therefore,

$$\mathscr{E}_{\mathbf{n}}'^S \cup \operatorname*{colim}_{\substack{T \subset S \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T \to \mathscr{E}_{\mathbf{n}}''^S$$

is a trivial cofibration. Using an easy patching argument together with induction on $|S| = k$, we conclude that $\operatorname*{colim}_{\substack{T \subset S \\ \neq}} \mathscr{E}_{\mathbf{n}}'^T \to \operatorname*{colim}_{\substack{T \subset S \\ \neq}} \mathscr{E}_{\mathbf{n}}''^T$ (and thus $\mathscr{E}_{\mathbf{n}}'^S \to \mathscr{E}_{\mathbf{n}}''^S$) is a trivial cofibration. Therefore, $\operatorname*{colim}_{m<n} \mathscr{E}_{\mathbf{m}}''^{b^{-1}(S)} \cup \mathscr{E}_{\mathbf{n}}'^S \to \mathscr{E}_{\mathbf{n}}''^S$ is a trivial cofibration. The same

form of patching argument now implies that

$$\operatorname*{colim}_{m<n+1} \mathscr{E}'^{\,b^{-1}(V)}_{\mathbf{m}} \to \operatorname*{colim}_{m<n+1} \mathscr{E}''^{\,b^{-1}(V)}_{\mathbf{m}}$$

is a trivial cofibration for all pointed $V \subset \mathbf{n+1}$ so that the induction continues. In particular, we conclude that $j:\mathscr{E}' \to \mathscr{E}''$ is a trivial cofibration of $\mathscr{F}$-spaces.

The proof of 2·5(*c*) is similar.

We leave to the reader the minor modifications of Proposition 2·5 which yield a 'cofibration – trivial fibration' version. Because the special case of Proposition 2·5 in which $\bar{g}:\mathscr{B}' \to \mathscr{B}$ is a trivial cofibration and $f:\mathscr{E} \to \mathscr{B}$ is the identity map will be so frequently employed, we explicitly provide its statement as Corollary 2·6.

Corollary 2·6. Let $\bar{\jmath}:\mathscr{B}' \to \mathscr{B}''$ be a trivial cofibration in $\mathscr{F}[\mathscr{S}_*]$ and let $f':\mathscr{E}' \to \mathscr{B}'$ be a sectioned map. There exists a fibrantly sectioned map $f'':\mathscr{E}'' \to \mathscr{B}''$ and a trivial cofibration $j:\mathscr{E}' \to \mathscr{E}''$ satisfying

(*a*) $(j, \bar{\jmath}):f' \to f''$ is section preserving,

(*b*) $\operatorname*{colim}_{m<n} \mathscr{E}''^{\,b^{-1}(S)}_{\mathbf{m}} \cup \mathscr{E}'^{\,S}_{\mathbf{n}} \cup \operatorname*{colim}_{T \underset{\neq}{\subset} S} \mathscr{E}''^{\,T}_{\mathbf{n}} \to \mathscr{E}''^{\,S}_{\mathbf{n}}$ is a trivial cofibration for each $n > 0$ and $S \subset \mathbf{n}$.

Such an $f'':\mathscr{E}'' \to \mathscr{B}''$ is said to be a *compatibly sectioned mapping fibration* for $\bar{\jmath} \circ f':\mathscr{E}' \to \mathscr{B}''$.

3. X-FIBRATIONS

An X-fibration is the $\mathscr{F}$-space analogue of an oriented fibration with fibres weakly equivalent to a given simplicial set X. The main result of this section is Theorem 3·5 asserting that the functor defined in Definition 3·3 which associates to the $\mathscr{F}$-space $\mathscr{B}$ the set $X(\mathscr{B})$ of X-fibrations over $\mathscr{B}$ is a homotopy functor.

We consider a fixed (but arbitrarily chosen) pointed simplicial set X and the various smash products X^m of X with itself. For any pointed simplicial set Y (e.g. $Y = X$ or X^m), $G(Y)$ denotes the simplicial monoid of pointed self-equivalences of $|Y|$; that is, $G(Y) = \operatorname{Sin}(\mathscr{G}(|Y|))$, where $\mathscr{G}(|Y|)$ is the topological monoid of pointed self-equivalences of $|Y|$ and Sin ( ) is the singular functor. For any n-tuple $I = (i_1, \ldots, i_n)$, we let $G(X)^I$ denote the product $G(X^{i_1}) \times \ldots \times G(X^{i_n})$. Because smash product commutes with geometric realization (as Kelley spaces) (6), smash product of maps determines a map

$$d: G(X)^I \to G(X)^{d(I)}$$

associated to any $d:\mathbf{n} \to \mathbf{t}$ in $\mathscr{F}$ and any n-tuple I.

We assume in this section, and in all subsequent sections, that X is provided with a *suitable orientation*. By definition, such a suitable orientation consists of a choice of submonoid $SG(X^m)$ of $G(X^m)$ for each $m > 0$ (consisting of 'special self-equivalences') satisfying for each $m > 0$:

(*a*) $\pi_0(SG(X^m))$ is a subgroup of $\pi_0(G(X^m))$.

(*b*) $SG(X^m)$ consists of those connected components of $G(X^m)$ indexed by $\pi_0(SG(X^m))$.

(*c*) $\Sigma_m \subset SG(X^m)$.

(*d*) Smash product, $G(X^m) \times G(X^m) \to G(X^{m+n})$, restricts to

$$SG(X^m) \times SG(X^n) \to SG(X^{m+n}).$$

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With this definition of a suitable orientation in mind, we define an oriented fibration as follows.

Definition 3·1. Let (Y, y) be a pointed simplicial set and let $f: E \to B$ be a fibration with given section whose fibres are weakly equivalent to (Y, y_0). Let $G(Y)$ be the simplicial monoid of pointed equivalences of $|Y|$, let $S\pi$ be a chosen subgroup of $\pi_0(G(Y))$ and let $SG(Y)$ consist of those components of $G(Y)$ indexed by $S\pi$. *An orientation of f with respect to $SG(Y)$* is a choice of subset $(Sf)_b$ (of 'orienting equivalences') for each $b: \Delta[t] \to B$ of the set of homotopy classes of maps

$$(E \times_B \Delta[t], \Delta[t]) \to (\mathrm{Sin}\,|Y|, y)$$

such that

(*a*) $(Sf)_b$ is a simply transitive left $S\pi$-set.

(*b*) For any face $b': \Delta[t-1] \to B$ of b, composition with the inclusion

$$E \times_B \Delta[t-1] \to E \times_B \Delta[t]$$

determines a bijection from $(Sf)_b$ on to $(Sf)_{b'}$.

Let $f': E' \to B'$ and $f'': E \to B$ be pointed fibration with given sections oriented with respect to $S\pi \subset \pi_0(G(Y))$. A map $(g, \bar{g}): f' \to f$ is said to be *orientation preserving* if composition on the right with g induces a bijection $(Sf)_{\bar{g}(b')} \to (Sf)_{b'}$ for every simplex b' of B'.

If $f: E \to B$ is a pointed fibration with given section oriented with respect to $S\pi$, a map $v: (Y \times \Delta[t], y \times \Delta[t]) \to (E, B)$ is said to be *special* if $f \circ v$ factors through some t-simplex $\bar{v}: \Delta[t] \to B$ and if the composition of some orienting equivalence

$$(E \times_B \Delta[t], \Delta[t]) \to (\mathrm{Sin}\,|Y|, y)$$

and the induced map $v, \bar{v}: (Y \times \Delta[t], y \times \Delta[t]) \to (E \times_B \Delta[t], \Delta[t])$ is a t-simplex of $G(Y)$ in a connected component of $S\pi$.

We remind the reader of our *vector notation* employed for $\mathscr{F}$-spaces over $\mathscr{N}$. If $f: \mathscr{E} \to \mathscr{B}$ is a sectioned map over $\mathscr{N}$, then $f_I: \mathscr{E}_I \to \mathscr{B}_I$ is the restriction of $f_{\mathbf{n}}$ above an n-tuple $I = (i_1, \ldots, i_n) \in \mathscr{N}_{\mathbf{n}}$, and $\mathscr{E}_I^S$ denotes $\mathscr{E}_I \cap \mathscr{E}_{\mathbf{n}}^S$. We let f_I^S denote the restriction of f_I to $\mathscr{E}_I^S$. Moreover, for any pointed subset $S \subset \mathbf{n}$, $X^{S(I)} \subset X^I = X^{i_1} \times \ldots \times I^{i_n 1}$ consists of simplices whose jth factor is the base point whenever $j \notin S$.

Definition 3·2. A map $f: \mathscr{E} \to \mathscr{B}$ of $\mathscr{F}$-spaces over $\mathscr{N}$ is said to be an *X-fibration* provided that for each $n > 0$ and each n-tuple $I = (i_1, \ldots, i_n)$

(*a*) $f_I: \mathscr{E}_I \to \mathscr{B}_I$ is an oriented fibration with respect to

$$SG(X)^I = SG(X^{i_1}) \times \ldots \times SG(X^{i_n}).$$

(*b*) $\prod_{j=1}^{n} p_j: \mathscr{E}_I \to \Pi\, \mathscr{E}_{i_j} \times_{\Pi \mathscr{B}_{i_j}} \mathscr{B}_I$ is an orientation-preserving fibre homotopy equivalence over $\mathscr{B}_I$, where $p_j: \mathbf{n} \to \mathbf{1}$ is the projection sending $j \in \mathbf{n}$ to $1 \in \mathbf{1}$.

(*c*) For any pointed $S \subset \mathbf{n}$, $f_I^S: \mathscr{E}_I^S \to \mathscr{B}_I$ is a fibration. Furthermore, if $a: \mathbf{n} \to \mathbf{k}$ is a projection which restricts to an isomorphism of S onto $\mathbf{k}$, then

$$(a \times f_I): \mathscr{E}_I^S \to \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$$

is an orientation-preserving fibre homotopy equivalence over $\mathscr{B}_I$.

(*d*) Let $\mu:\mathbf{n}\to\mathbf{1}$ send every non-zero $j\in\mathbf{n}$ to $1\in\mathbf{1}$. Then any special equivalence $v: X^I\times\Delta[t]\to\mathscr{E}_I$ which restricts to $v^S: X^{S(I)}\times\Delta[t]\to\mathscr{E}_I^S$ for every pointed $S\subset\mathbf{n}$ extends to a special equivalence $X^{\mu(I)}\times\Delta[t]\to\mathscr{E}_{\mu(I)}$.

A *map of X-fibrations*, $(g,\bar{g}):f'\to f$, is a section-preserving map such that

$$(g_I,\bar{g}_I):f'_I\to f_I$$

is orientation preserving for all $n>0$ and all n-tuples I.

Even without condition 3·2(*d*), $\mu\circ v: X^I\times\Delta[t]\to\mathscr{E}_I\to\mathscr{E}_{\mu(I)}$ factors through a unique (because $X^I\to X^{\mu(I)}$ is surjective) map $X^{\mu(I)}\times\Delta[t]\to\mathscr{E}_{\mu(I)}$ thanks to condition 2·1(*c*). Thus, the content of 3·2(*d*) is that this map $X^{\mu(I)}\times\Delta[t]\to\mathscr{E}_{\mu(I)}$ is a special equivalence: this means that $\mu:\mathscr{E}_I\to\mathscr{E}_{\mu(I)}$ has the effect of smash product on fibres.

Definition 3·3 below defines the set $X(\mathscr{B})$ (by Theorem 6·1, $X(\mathscr{B})$ is indeed a set) only for cofibrant $\mathscr{F}$-spaces $\mathscr{B}$ over $\mathscr{N}$. Because $X(\mathscr{B})$ is classified by homotopy classes of maps from $\mathscr{B}$ into a classifying $\mathscr{F}$-space (by Theorem 6·1 again), it should not be surprising that $X(\mathscr{B})$ can be more readily described for cofibrant $\mathscr{B}$.

Definition 3·3. Let $\mathscr{B}$ be a cofibrant $\mathscr{F}$-space provided with a given map $\mathscr{B}\to\mathscr{N}$ (we say that $\mathscr{B}$ is a cofibrant $\mathscr{F}$-space over $\mathscr{N}$). We define $X(\mathscr{B})$ to be the set of equivalence classes of X-fibrations over $\mathscr{B}$, where the equivalence relation is generated by pairs of X-fibrations over $\mathscr{B}$ between which there is a map over $\mathscr{B}$. If $g:\mathscr{B}'\to\mathscr{B}$ is a map of cofibrant $\mathscr{F}$-spaces over $\mathscr{N}$, we define

$$g^*: X(\mathscr{B})\to X(\mathscr{B}')$$

by sending $f:\mathscr{E}\to\mathscr{B}$ to $g^*(f):\mathscr{E}'=\mathscr{E}\times_{\mathscr{B}}\mathscr{B}'\to\mathscr{B}'$ (where $\mathscr{E}'^S_{\mathbf{n}}=\mathscr{E}^S_{\mathbf{n}}\times_{\mathscr{B}_n}\mathscr{B}'_{\mathbf{n}}$ for all $n>0$ and pointed $S\subset\mathbf{n}$).

Proposition 3·4. Let $\mathscr{B}$ be a cofibrant $\mathscr{F}$-space over $\mathscr{N}$. Any element of $X(\mathscr{B})$ is represented by an X-fibration which is fibrantly and cofibrantly sectioned. If $f:\mathscr{E}\to\mathscr{B}$ and $f':\mathscr{E}'\to\mathscr{B}$ are fibrantly and cofibrantly sectioned X-fibrations which are equal in $X(\mathscr{B})$, then there exist maps $f\to f'$ and $f'\to f$ of X-fibrations over $\mathscr{B}$.

Proof. If f is an X-fibration over $\mathscr{B}$, we obtain maps of X-fibrations over $\mathscr{B}$

$$f\to h\to k,$$

where h is fibrantly sectioned (using Proposition 2·5) and k is fibrantly–cofibrantly sectioned (using the cofibration – trivial fibration version Proposition 2·5). Using Proposition 2·2 and this construction, we conclude that, if f and f' are fibrantly and cofibrantly sectioned X-fibrations over $\mathscr{B}$ equal in $X(\mathscr{B})$, then there exists a chain of maps of fibrantly and cofibrantly sectioned X-fibrations over $\mathscr{B}$ relating f and f'.

Consequently, it suffices to prove that, if $g:f'\to f$ is a map of fibrantly and cofibrantly sectioned X-fibrations over $\mathscr{B}$, then there exists a map $f\to f'$ of X-fibrations over $\mathscr{B}$. Factor g as $p\circ j: f'\to f''\to f$ using Proposition 2·5. Proposition 2·2 together with the hypothesis that f is fibrantly sectioned and f' cofibrantly sectioned implies the existence of a section-preserving right inverse for p and a section-preserving left inverse for j.

We next prove that $\mathscr{B}\mapsto X(\mathscr{B})$ determines a functor on the homotopy category.

THEOREM 3·5. *The functor* $\mathscr{B}\mapsto X(\mathscr{B})$, *defined in Definition* 3·3 *on the category of cofibrant* $\mathscr{F}$*-spaces over* $\mathscr{N}$, *induces a contravariant functor*

$$X(\ ):\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}\to(\mathrm{sets}).$$

Proof. For any closed model category $\mathscr{C}$, $\mathrm{Ho}\,\mathscr{C}$ is equivalent to the category of fractions of the category $\mathscr{C}_{\mathrm{cof}}$ (the full sub-category of $\mathscr{C}$ consisting of cofibrant objects) determined by inverting weak equivalences in $\mathscr{B}_{\mathrm{cof}}$. Consequently, it suffices to prove that $g^*: X(\mathscr{B}) \to X(\mathscr{B}')$ is a bijection whenever $g: \mathscr{B}' \to \mathscr{B}$ is a weak equivalence of cofibrant $\mathscr{F}$-spaces over $\mathscr{N}$.

We first assume that g is a trivial cofibration. Define

$$g_*: X(\mathscr{B}') \to X(\mathscr{B})$$

by sending an equivalence class of an X-fibration $f': \mathscr{E}' \to \mathscr{B}'$ to the equivalence class of a compatibly sectioned mapping fibration, $\overline{g \circ f'} = g_*(f')$, for $g \circ f': \mathscr{E}' \to \mathscr{B}$ (cf. Corollary 2·6). Using Proposition 2·2, we readily conclude that g_* is well defined. The natural inclusion $f' \to g_*(f')$ determines $f' \to g^* \circ g_*(f')$; thus, $g^* \circ g_* = 1$. If $f: \mathscr{E} \to \mathscr{B}$ is a fibrantly sectioned X-fibration, then the natural map $g^*(f) \to f$ can be extended to $g_* \circ g^*(f) \to f$ using Proposition 2·2. Therefore, $g_* \circ g^* = 1$.

Since any weak equivalence $g: \mathscr{B}' \to \mathscr{B}$ of $\mathscr{F}$-spaces factors as $p \circ j: \mathscr{B}' \to \mathscr{B}'' \to \mathscr{B}$, where j is a trivial cofibration (so that $\mathscr{B}''$ is cofibrant provided that $\mathscr{B}'$ is cofibrant) and p a trivial fibration, the theorem will be proved if we verify that $p^*: X(\mathscr{B}) \to X(\mathscr{B}'')$ is a bijection whenever $p: \mathscr{B}'' \to \mathscr{B}$ is a trivial fibration between cofibrant $\mathscr{F}$-spaces over $\mathscr{N}$. Because $\mathscr{B}$ is cofibrant, p admits a right inverse; thus, p^* is injective. If $q \circ i: \mathscr{B} \to \mathscr{B}''$ is some factorization of some right inverse of p with i a trivial cofibration and q a trivial fibration, our proceeding arguments imply that $i^* \circ q^*: X(\mathscr{B}'') \to X(\mathscr{B})$ is injective as well as left inverse to p^*. Therefore, $i^* \circ q^*$ and consequently p^* are bijective.

We conclude this section with the following simple observation concerning $X(\mathscr{B})$ for $\mathscr{F}$-spaces $\mathscr{B}$ over $\mathscr{N}$ which are not necessarily cofibrant.

Proposition 3·6. Let $\mathscr{B}$ be an $\mathscr{F}$-space over $\mathscr{N}$. An X-fibration over $\mathscr{B}$, $f: \mathscr{E} \to \mathscr{B}$, naturally determines an element of $X(\mathscr{B})$. Moreover, if $f' \to f$ is a map of X-fibrations over $\mathscr{B}$, then f' and f determine the same element in $X(\mathscr{B})$.

Proof. Let $g: \mathscr{B}' \to \mathscr{B}$ be a trivial fibration with $\mathscr{B}'$ cofibrant. Then $g^*(f)$ determines an element $[g^*f]$ in $X(\mathscr{B}')$. By Theorem 3·5, $[g^*f] \in X(\mathscr{B}')$ determines an element of $X(\mathscr{B})$ independent of the choice of g. If $f \to f'$ is a map of X-fibrations over $\mathscr{B}$, then the induced map $g^*f \to g^*f'$ implies that $[g^*f]$ equals $[g^*f']$ in $X(\mathscr{B}')$; thus, Theorem 3·5 implies that f and f' determine the same element in $X(\mathscr{B})$.

4. PRINCIPALIZATION OF X-FIBRATIONS

The usual method of classifying fibrations with a specified homotopy type F as fibre is to pass from the fibration $f: E \to B$ to the associated principal fibration

$$\tilde{f}: P(f) \to B.$$

The total space $P(f)$ is defined to be the space of maps $F \to E$ which are equivalences between F and some fibre of f. The key property of $\tilde{f}$ is that it is a fibration with fibres equivalent to the monoid of self-equivalences of f.

We provide an $\mathscr{F}$-space analogue of such a principalization. The construction is very delicate. If $f: \mathscr{E} \to \mathscr{B}$ is an X-fibration and I is an n-tuple, one considers special

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equivalences from X^I into a fibre of $f_I:\mathscr{E}_I\to\mathscr{B}_I$. In order that $SG(X)^I$ act on these equivalences, we replace f_I by $\mathrm{Sin}\,|f_I|:\mathrm{Sin}\,|\mathscr{E}_I|\to\mathrm{Sin}\,|\mathscr{B}_I|$. Moreover, conditions must be imposed on these weak equivalences in order that a map $d:\mathbf{n}\to\mathbf{t}$ induces a weak equivalence from $X^{d(I)}$ into a fibre of $\mathrm{Sin}\,|\mathscr{E}_{d(I)}|\to\mathrm{Sin}\,|\mathscr{B}_{d(I)}|$. Such conditions are prescribed using the section given $f:\mathscr{E}\to\mathscr{B}$. The delicacy arises in obtaining such conditions with the additional property that the fibres of the resultant principalization are equivalent to $SG(X)^I$ (this is the content of Theorem 4·5).

The reader should observe that $I\to P(f)_I$ as defined in Definition 4·1 is not functorial on $\mathscr{F}$. This is an intrinsic failing of any principalization: if $P(f)_{(i,j)}$ consists of certain maps $X^i\times X^j\to\mathrm{Sin}\,|\mathscr{E}_{(i,j)}|$, these maps must be sent to maps $X^{i+j}\to\mathrm{Sin}\,|\mathscr{E}_{i+j}|$ via the sum map $\mu:\mathbf{2}\to\mathbf{1}$ and to maps $X^j\times X^i\to\mathrm{Sin}\,|\mathscr{E}_{(j,i)}|$ via the flip map $\tau:\mathbf{2}\to\mathbf{2}$. One can at most require that $\mu:P(f)_{(i,j)}\to P(f)_{i+j}$ and $\mu\circ\tau:P(f)_{(i,j)}\to P(f)_{(j,i)}\to P(f)_{i+j}$ differ by an involution of X^{i+j} (the domain of maps in $P(f)_{i+j}$).

As in Section 3, X is a pointed simplicial set provided with a suitable orientation.

Definition 4·1. Let $f:\mathscr{E}\to\mathscr{B}$ be an X-fibration. For any n-tuple $I=(i_1,\dots,i_n)$, we define

$$\bar{f}_I:P(f)_I\to\mathscr{B}_I \quad \text{in} \quad \mathscr{S}_*$$

as follows. A k-simplex of $P(f)_I$ above a given k-simplex $b:\Delta[k]\to\mathscr{B}_I$ is a special equivalence

$$v:(X^I\times\Delta[k],*\times\Delta[k])\to(\mathrm{Sin}\,|\mathscr{E}_I|,\mathrm{Sin}\,|\mathscr{E}_I^{\{0\}}|)$$

projecting to $b:\Delta[k]\to\mathscr{B}_I$ which satisfies

(*a*) For every pointed $S\subseteq\mathbf{n}$, v restricts to

$$v^S:X^{S(I)}\times\Delta[k]\to\mathrm{Sin}\,|\mathscr{E}_I^S|.$$

(*b*) For every $d:\mathbf{n}\to\mathbf{t}$ with $n>t$ and every pointed $S\subseteq\mathbf{n}$, v^S projects to

$$v^{(d:S)}:X^{(d:S)(d(I))}\times\Delta[k]\to\mathrm{Sin}\,|\mathscr{E}_{d(I)}^{(d:S)}|.$$

The map $v^{(d:S)}$ of 4·1 (*b*) is necessarily unique because $X^{S(I)}\to X^{(d:S)(d(I))}$ is surjective. By 3·2 (*b*) and (*c*), the maps v^S of 4·1 (*a*) and $v^{(d:S)}$ of 4·1 (*b*) are special equivalences.

Proposition 4·2. Let $f':\mathscr{E}'\to\mathscr{B}'$ and $f:\mathscr{E}\to\mathscr{B}$ be X-fibrations, let $n>0$, and let I be an n-tuple. A map $(g,\bar{g}):f'\to f$ of X-fibrations induces a map $P(f')_I\to P(f)_I$. A map $c:\mathbf{n}\to\mathbf{r}$ in $\mathscr{F}$ determines a natural map $c:P(f)_I-P(f)_{c(I)}$ (which is not functorial with respect to compositions in $\mathscr{F}$). Furthermore, there is a natural pairing

$$P(f)_I\times SG(X)^I\to P(f)_I.$$

Proof. The map $P(f')_I\to P(f)_I$ induced by $(g,\bar{g}):f'\to f$ is defined by sending $v:X^I\times\Delta[k]\to\mathrm{Sin}\,|\mathscr{E}'_I|$ to $g\circ v$. For $c:\mathbf{n}\to\mathbf{r}$ in satisfying $c(i)\neq c(j)$ for $i\neq j$, $X^I\to X^{c(I)}$ is an isomorphism; for such $c:\mathbf{n}\to\mathbf{r}$, $v:X^I\times\Delta[k]\to\mathrm{Sin}\,|\mathscr{E}_I|$ is sent to the composition

$$c\circ v\circ c^{-1}:X^{c(I)}\times\Delta[k]\to X^I\times\Delta[k]\to\mathrm{Sin}\,|\mathscr{E}_I|\to\mathrm{Sin}\,|\mathscr{E}_{c(I)}|.$$

More generally, factor $c:\mathbf{n}\to\mathbf{r}$ as $c_1\circ c_2:\mathbf{n}\to\mathbf{t}\to\mathbf{r}$, where c_1 satisfies $c_1(i)\neq c_1(j)$ for $i\neq j$ and c_2 is the unique surjective map satisfying for each i with $c(i)\neq c(j)$ for $i>j$ the condition that $c_2(i)>c_2(j)$ for $i>j$. Define $c:P(f)_I\to P(f)_{c(I)}$ as the composition of $c_2:P(f)_I\to P(f)_{c_2(I)}$ determined by 4·1 (*b*) and $c_1:P(f)_{c_2(I)}\to P(f)_{c(I)}$ defined above.

Using the adjointness of Sin ( ) and $|\ |$, we identify a k-simplex of $G(X)^I = \mathrm{Sin}\,(\mathscr{G}(X)^I)$ with a map $X^I \times \Delta[k] \to \mathrm{Sin}\,|X^I|$. Then the pairing $P(f)_I \times SG(X)^I \to P(f)_I$ is defined to send $(v : X^I \times \Delta[k] \to \mathrm{Sin}\,|\mathscr{E}_I|, \alpha : X^I \times \Delta[k] \to \mathrm{Sin}\,|X^I|)$ to the composition

$$\psi \circ v \circ (\alpha \times 1) : X^I \times \Delta[k] \to \mathrm{Sin}\,|X^I| \times \Delta[k] \to (\mathrm{Sin} \circ |\ |)^2 (\mathscr{E}_I) \to \mathrm{Sin}\,|\mathscr{E}_I|,$$

where $\psi : (\mathrm{Sin} \circ |\ |)^2 \to \mathrm{Sin} \circ |\ |$ is induced by the adjunction map $|\ | \circ \mathrm{Sin} \to 1$ (giving $\mathrm{Sin} \circ |\ |$ the structure of a 'triple' or 'monad'). So defined, $v \circ \alpha$ satisfies 4·1 (a) and 4·1 (b) whenever v does because α is a product of maps $X^{i_j} \times \Delta[k] \to \mathrm{Sin}\,|X^{i_j}|$.

The next proposition justifies the lifting arguments in our proof of Theorem 4·5. The intuition for Proposition 4·3 arises from the associated isomorphisms

$$X^I \xrightarrow{\approx} \lim_{n \to >} X^{a(I)},$$

$$X^{S(I)} \xrightarrow{\approx} \lim_{n > t} X^{(d:S)(d(I))} \quad (S \underset{\neq}{\subset} \mathbf{n})$$

for any n-tuple I.

Proposition 4·3. Let $f : \mathscr{E} \to \mathscr{B}$ be a fibrantly sectioned X-fibration. For any n-tuple $I = (i_1, \ldots, i_n)$ and any pointed $S \underset{\neq}{\subset} \mathbf{n}$, the natural maps

$$\mathscr{E}_I \to \lim_{n \to >} \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I,$$

$$\mathscr{E}_I^S \to \lim_{n > t} \mathscr{E}_{d(I)}^{(d:S)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I$$

are trivial fibrations, where $\lim\limits_{n \to >}$ is indexed by all projections $a : \mathbf{n} \to \mathbf{k}$ with $n > k$ and $\lim\limits_{n > t}$ is indexed by all $d : \mathbf{n} \to \mathbf{t}$ in $\mathscr{F}$ with $n > t$.

Proof. By Definition 2·3 and Proposition 2·4, these maps are fibrations. To prove that

$$\mathscr{E}_I \to \lim_{n \to >} \mathscr{E}_{a(I)} \times_{\mathscr{B}a(I)} \mathscr{B}_I$$

is a weak equivalence, we first recall that $\mathscr{E}_I \to \Pi \mathscr{B}_{i_j} \times_{\mathscr{B}_{i_j}} \mathscr{B}_I$ is a trivial fibration by Definition 3·2 (a) and Proposition 2·4. We then use induction to verify that

$$\lim_{\substack{n \to > \\ k \leqslant r+1}} \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I \to \lim_{\substack{n \to > \\ k \leqslant r}} \mathscr{E}_{b(I)} \times_{\mathscr{B}_{b(I)}} \mathscr{B}_I$$

is a trivial fibration, where $\lim\limits_{\substack{n \to > \\ k \leqslant r}}$ is indexed by all projections $b : \mathbf{n} \to \mathbf{k}$ with $k \leqslant r$.

Let $a : \mathbf{n} \to \mathbf{k}$ be a projection chosen so that a restricts to an isomorphism of S on to $\mathbf{k}$. By Definition 3·2 (c) and Proposition 2·4, $\mathscr{E}_I^S \to \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$ is a trivial fibration. We proceed to prove that the projection

$$\lim_{n > t} \mathscr{E}_{d(I)}^{(d:S)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I \to \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$$

is a trivial fibration. We factor this map as a sequence

$$L_n \to L_{n-1} \to \ldots \to L_2 \to L_1,$$

where $L_j = \lim\limits_{T(j)} \mathscr{E}^{(d:S)}_{d(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$, with $\lim\limits_{T(j)}$ indexed by those $d:\mathbf{n}\to\mathbf{t}$ with $n > t$ which do not initiate a chain with j non-isomorphic maps none of whose composites factors through the chosen projection $a:\mathbf{n}\to\mathbf{k}$. Because $(d:S) \underset{\neq}{\subset} \mathbf{t}$ whenever $d:\mathbf{n}\to\mathbf{t}$ does not factor through a, we may apply induction to conclude that

$$\mathscr{E}^{(d:S)}_{d(I)} \to \lim_{t>t'} \mathscr{E}^{(d'\circ d:S)}_{(d'\circ d)(I)} \times_{\mathscr{B}_{(d'\circ d)(I)}} \mathscr{B}_{d(I)}$$

is a trivial fibration for each $d\in T(j)-T(j-1)$. Therefore, each $L_j \to L_{j-1}$ is a fibration.

We state as a corollary the factorization of the projection

$$\lim_{n>t} \mathscr{E}^{(d:S)}_{d(I)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I \to \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$$

implicitly described in the proof of Proposition 4·3.

Corollary 4·4. Let $f:\mathscr{E}\to\mathscr{B}$ be a fibrantly sectioned X-fibration. For any n-tuple I, any pointed $S \underset{\neq}{\subset} \mathbf{n}$, and any projection $a:\mathbf{n}\to\mathbf{k}$ which restricts to an isomorphism of S on to $\mathbf{k}$, the projection

$$\lim_{n>t} \mathscr{E}^{(d:S)}_{d(I)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I \to \mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I$$

factors as a composite of trivial fibrations each of which is the pull-back of the trivial fibration

$$\mathscr{E}^{(d:S)}_{d(I)} \to \lim_{t>t'} \mathscr{E}^{(d'\circ d:S)}_{(d'\circ d)(I)} \times_{\mathscr{B}_{(d'\circ d)(I)}} \mathscr{B}_{d(I)}$$

associated to some map $d:\mathbf{n}\to\mathbf{t}$ which does not factor through a.

The following 'Principalization Theorem' provides the core of our proof of the classification theorems.

THEOREM 4·5. *Let $f:\mathscr{E}\to\mathscr{B}$ be a fibrantly sectioned X-fibration and let I be an n-tuple. Then $\bar{f}_I : P(f)_I \to \mathscr{B}_I$ is a fibration. Moreover, for any vertex z of B_I, there exists a homotopy equivalence*

$$\phi_{z,I} : \bar{f}_I^{-1}(z) \to SG(X)^I,$$

which fits in the following commutative square

$$\begin{array}{ccc} \bar{f}_I^{-1}(z) \times SG(X)^I & \to & \bar{f}_I^{-1}(z) \\ \downarrow{\scriptstyle \phi_{z,I}\times 1} & & \downarrow{\scriptstyle \phi_{z,I}} \\ SG(X)^I \times SG(X)^I & \to & SG(X)^I \end{array}$$

whose horizontal arrows are determined by composition.

Proof. To prove $\bar{f}_I$ is a fibration, we exhibit a 'suitable' lifting (determining a k-simplex of $P(f)_I$) for commutative squares of the form

$$\begin{array}{ccc} X^I \times \Lambda[k] & \xrightarrow{u} & \operatorname{Sin}|\mathscr{E}_I| \\ \downarrow{\scriptstyle 1\times\epsilon_r} & \overset{v}{\nearrow} & \downarrow \\ X^I \times \Delta[k] & \xrightarrow{w} & \operatorname{Sin}|\mathscr{B}_I| \end{array}$$

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for any $k > 0$, any $0 \leqslant r \leqslant k$, any $u : \Lambda[k] \to P(f)_I$, and any w which factors through a k-simplex of $\mathscr{B}_I$. Using the fact that each $\mathscr{E}_{i_j} \to \mathscr{B}_{i_j}$ is a fibration, we first choose liftings for the squares

$$\begin{array}{ccc} (x^{ij} \times \Lambda[k]) \cup (* \times \Delta[k]) & \longrightarrow & \mathrm{Sin}\,|\mathscr{E}_{i_j} \times_{\mathscr{B}_{i_j}} \mathscr{B}_I| \\ \downarrow & \nearrow & \downarrow \\ X^{ij} \times \Delta[k] & \longrightarrow & \mathrm{Sin}\,|\mathscr{B}_I| \end{array} \qquad (4\cdot 5\cdot 1)$$

(using the fact that $\mathrm{Sin} \circ |\ |$ preserves fibrations (14)).

We proceed by induction on $|S|$ to obtain a suitable lifting for the square

$$\begin{array}{ccc} (\operatorname*{colim}_{T \subset S} X^{T(I)} \times \Delta[k]) \cup (X^{S(I)} \times \Lambda[k]) & \longrightarrow & \mathrm{Sin}\,|\mathscr{E}_I^S| \\ \downarrow & \nearrow & \downarrow \\ X^{S(I)} \times \Delta[k] & \longrightarrow & \lim_{n > t} \mathrm{Sin}\,|\mathscr{E}_{d(I)}^{(d:S)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I| \end{array} \qquad (4\cdot 5\cdot 2)$$

For any projection $a : \mathbf{n} \to \mathbf{r}$ which restricts to a surjection of S onto $\mathbf{r}$, $\mathscr{E}_{a(I)}^{(a:S)} = \mathscr{E}_{a(I)}$ so that a suitable lifting $X^{S(I)} \times \Delta[k] \to \mathrm{Sin}\,|\mathscr{E}_{a(I)}^{(a:S)}|$ is obtained by inductively choosing a lifting for the following square with the aid of Proposition 4·3 and the chosen liftings of (4·5·1):

$$\begin{array}{ccc} (\mathrm{colim}\, X^{(a:T)(a(I))} \times \Delta[k]) \cup (X^{(a:S)(a(I))} \times \Lambda[t]) & \longrightarrow & \mathrm{Sin}\,|\mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I| \\ \downarrow & \nearrow & \downarrow \\ X^{(a:S)(a(I))} \times \Delta[k] & \longrightarrow & \lim_{r \to >} \mathrm{Sin}\,|\mathscr{E}_{a' \circ a(I)} \times_{\mathscr{B}_{a' \circ a(I)}} \mathscr{B}_I| \end{array} \qquad (4\cdot 5\cdot 3)$$

(we implicitly use the fact that $\mathrm{Sin} \circ |\ |$ commutes with finite limits (6)). If $d : \mathbf{n} \to \mathbf{t}$ is any map with $n > t$ which factors through such a projection $a : \mathbf{n} \to \mathbf{r}$ restricting to a surjection $S \to \mathbf{r}$, then

$$X^{(a:S)(a(I))} \times \Delta[k] \to \mathrm{Sin}\,|\mathscr{E}_{a(I)} \times_{\mathscr{B}_{a(I)}} \mathscr{B}_I|$$

projects to a (uniquely defined) map

$$X^{(d:S)(d(I))} \times \Delta[k] \to \mathrm{Sin}\,|\mathscr{E}_{d(I)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I|$$

because we have inductively required that the 'axes' of $X^{(a:S)(a(I))} \times \Delta[k]$ (namely $X^{(a:T)(a(I))} \times \Delta[k]$ for $T \underset{\neq}{\subset} S$) map correctly. In particular, for $S = \mathbf{n}$, the lifting of (4·5·3) provides a lifting for (4·5·2).

For $S \underset{\neq}{\subset} \mathbf{n}$, we inductively consider maps $d : \mathbf{n} \to \mathbf{t}$ not factoring through any projection $a : \mathbf{n} \to \mathbf{r}$ which maps S on to $\mathbf{r}$ so that $(d : S) \underset{\neq}{\subset} \mathbf{t}$. Applying Proposition 4·3, we choose liftings for the following square

$$\begin{array}{ccc} (\mathrm{colim}\, X^{(d:T)(I)} \times \Delta[k]) \cup X^{(d:S)(d(I))} \times \Lambda[k] & \longrightarrow & \mathrm{Sin}\,|\mathscr{E}_{d(I)}^{(d:S)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I| \\ \downarrow & \nearrow & \downarrow \\ X^{(d:S)(d(I))} & \longrightarrow & \lim_{t > t'} \mathrm{Sin}\,|\mathscr{E}_{(d' \circ (I)}^{(d' \circ d:S)} \times_{\mathscr{B}_{d' \circ d(I)}} \mathscr{B}_I| \end{array} \qquad (4\cdot 5\cdot 4)$$

Finally, we use Proposition 4·3 again to obtain a lifting for (4·5·2). Thus, $\tilde{f}_I : P(f)_I \to \mathscr{B}_I$ is a fibration.

Because the right vertical arrows of (4·5·2), (4·5·3), and (4·5·4) are trivial fibrations, the same lifting arguments prove that

$$\Pi_{pj} : P(f)_I \to \prod_{j=1}^{n} P(f)_{i_j} \times_{\mathscr{B}_{i_j}} \mathscr{B}_I$$

is a trivial fibration. This map is clearly $SG(X)^I$-equivariant. Consequently, to construct $\phi_{z,I}$ it suffices to construct the $SG(X^{i_j})$-equivariant equivalences

$$\phi_{p_j(z), i_j} : \tilde{f}_{i_j}(p_j(z)) \to SG(X^{i_j})$$

for each j, $0 \leqslant j \leqslant n$. These $\phi_{p_j(z), i_j}$ are determined by choices of orienting equivalences $f_{i_j}^{-1}(p_j(z)) \to \mathrm{Sin}\,|X^{i_j}|$.

5. Universal X-fibrations

We continue to consider a pointed simplicial set X provided with a suitable orientation. The purpose of this section is to introduce the sectioned maps of $\mathscr{F}$-spaces which we employ in Section 6 to prove the classification theorems. Example 5·3 provides the universal X-fibration. Example 5·5 introduces the maps which will yield a classifying map. Proposition 5·6 uses these maps to relate a given X-fibration to the universal X-fibration.

We begin by constructing the classifying $\mathscr{F}$-space $\mathscr{B}SG(X)$. The reader should recall that X^i denotes the i-fold smash product of X with itself.

Example 5·1. Let $\mathscr{C}_X$ be the permutative category whose object space is the discrete set $\{|X^i|\}_{i\geqslant 0} \simeq N$ and whose morphism space equals $\coprod_{i\geqslant 0} \mathscr{S}\mathscr{G}(|X^i|)$, where $\mathscr{S}\mathscr{G}(|X^i|)$ is the topological monoid of special self equivalence of $|X^i|$

$$(\text{so that } \mathrm{Sin}\,(\mathscr{S}\mathscr{G}(|X^i|)) = SG(X^i)).$$

We define the product

$$\square : \mathscr{C}_X \times \mathscr{C}_X \to \mathscr{C}_X$$

by setting $|X^i| \square |X^j| = |X^{i+j}|$ and $\alpha \square \beta = \alpha \wedge \beta$.

As shown in (11), $\mathscr{C}_X$ determines a functor

$$\underset{\sim}{\mathscr{C}}_X : \mathscr{F} \to (\text{permutative categories})$$

such that $(\underset{\sim}{\mathscr{C}}_X)_1 = \mathscr{C}_X$. We define the $\mathscr{F}$-space

$$\mathscr{B}SG(X) : \mathscr{F} \to \mathscr{S}_*$$

by setting $\mathscr{B}SG(X)_n = B_{\mathrm{Sin}}((\underset{\sim}{\mathscr{C}}_X)_n)$, where $B_{\mathrm{Sin}}(\ )$ applied to a topological category $\mathscr{C}$ is the diagonal of the bisimplicial set defined by applying Sin ( ) to the nerve of $\mathscr{C}$.

To make this construction more concrete, we describe $\mathscr{B}SG(X)$ in terms of explicit formulas. If K is a simplicial group, we denote by $B(K, K)$ (or $B(*, K, K)$) the contractible simplicial set obtained by applying the bar construction to K; in particular, $B(K, K)_t = K^{\times(t+1)}$.

Proposition 5·2. There exists a canonical structure map $\mathscr{B}SG(X)\to\mathscr{N}$ of $\mathscr{F}$-spaces induced by functor on permutative categories. Moreover, for any n-tuple I, $\mathscr{B}SG(X)_I$ is isomorphic to

$$\prod_{1\leqslant j\leqslant n} BSG(X)^{i_j}\times \prod_{\substack{T\subset \mathbf{n}\\ |T|>1}} B(SH(X^{i_T}),SH(X^{i_T})),$$

where the product is indexed by pointed subsets $T\subseteq \mathbf{n}$ with more than one non-zero element, where $i_T=\sum_{j\in T} i_j$, and where $SH(X^i)\subset SG(X^i)$ consists of special self-equivalences which are homeomorphisms. Furthermore, $\mathscr{B}SG(X)$ is a cofibrant $\mathscr{F}$-space.

Proof. The $\mathscr{F}$-space $\mathscr{N}$ is obtained by applying the construction of Example 5·1 to the permutative category whose objects are the natural numbers, whose only maps are identities, and whose product is addition. $\mathscr{B}SG(X)\to\mathscr{N}$ is induced by the forgetful functor.

To obtain the specific formula for $\mathscr{B}SG(X)_I$, we identify (cf. (10)) Obj $((\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}})$ with $\coprod_{I\in\mathscr{N}_{\mathbf{n}}}\prod_{\substack{T\subset\mathbf{n}\\|T|>1}}\mathscr{SH}(|X^{i_T}|)$, where $\mathscr{SH}(|X^{i_T}|)$ is the topological submonoid of $\mathscr{SG}(|X^{i_T}|)$ consisting of homeomorphisms (i.e. invertible elements of $\mathscr{SG}(|X^{i_j}|)$) and $i_T=\sum_{j\in T} i_j$. We identify the morphism space of $(\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}}$ with

$$\coprod_{I\in\mathscr{N}_{\mathbf{n}}}(\Pi\mathscr{SG}(|X^{i_j}|)\times\Pi\mathscr{SH}(|X^{i_T}|)).$$

Because $\mathscr{B}SG(X)_I\subset\mathscr{B}SG(X)_{\mathbf{n}}$ is the inverse image of $I\in\mathscr{N}_{\mathbf{n}}$,

$$\operatorname*{colim}_{m<n}\mathscr{B}SG(X)_{\mathbf{m}}\to\mathscr{B}SG(X)_{\mathbf{n}}$$

is a cofibration. Because any transposition (k,l) which fixes some n-tuple $I=(i_1,\ldots,i_n)$ with each $i_j>0$ (i.e. I is not the image of some m-tuple for $m<n$) acts freely on $B(SH(X^{k+l}),SH(X^{k+l}))$ (recall that $(k,l)\in\Sigma_{k+l}\subset\mathscr{SH}(|X^{k+l}|)$), we conclude that $\operatorname*{colim}_{m<n}\mathscr{B}SG(X)_{\mathbf{m}}\to\mathscr{B}SG(X)_{\mathbf{n}}$ is a Σ_n-equivariant cofibration. Therefore, $\mathscr{B}SG(X)$ is cofibrant.

We next construct the sectioned map $\pi:\mathscr{B}(SG(X),X)\to\mathscr{B}SG(X)$ which determines the universal X-fibration. If K is a simplicial monoid acting (on the left) on a simplicial set Y, we denote by $B(K,Y)$ (or $B(*,K,Y)$) the simplicial set obtained by applying the bar construction to K and Y; in particular, $B(K,Y)_t=K^{\times t}\times Y$.

Example 5·3. We define a functor between permutative categories

$$\mathscr{C}'_X\to\mathscr{C}_X$$

as follows, where $\mathscr{C}_X$ is defined in Example 5·1. The map on object spaces

$$\text{Obj}\,(\mathscr{C}'_X)=\coprod_{i\geqslant 0}|X^i|\to\{|X^i|\}_{i\geqslant 0}=\text{Obj}\,(\mathscr{C}_X)$$

sends $x\in|X^i|$ to $|X^i|$. The map on morphism spaces

$$\text{Maps}\,(\mathscr{C}'_X)=\coprod_{i\geqslant 0}\mathscr{SG}(|X^i|)\times|X^i|\to|\coprod_{i\geqslant 0}\mathscr{SG}(|X^i|)=\text{Maps}\,(\mathscr{C}_X)$$

is the projection map (so that $\mathscr{C}'_X(x,y) \subset \mathscr{C}_X(|X^i|,|X^i|)$ consists of special self-equivalences sending $x \in |X^i|$ to $y \in |X^i|$). The product

$$\square : \mathscr{C}'_X \times \mathscr{C}'_X \to \mathscr{C}'_X$$

is defined as smash product on objects ($x \in |X^i| \,\square\, y \in |X^j| = x \wedge y \in |X^i| \wedge |X^j| = |X^{i+j}|$) and on maps.

The projection $\mathscr{C}_X \to \mathscr{C}_X$ determines a map of $\mathscr{F}$-spaces

$$\pi : \mathscr{B}(SG(X), X) \to \mathscr{B}SG(X)$$

defined by applying $B_{\mathrm{Sin}()}$ to the natural transformation $\underline{\mathscr{C}}'_X \to \underline{\mathscr{C}}_X$ obtained from $\mathscr{C}'_X \to \mathscr{C}_X$ as in (11). For any n-tuple I, $\mathscr{B}(SG(X), X)_I$ is given by the following explicit formula

$$\prod_{1 \leqslant j \leqslant n} B(SG(X^{i_j}), \mathrm{Sin}\,|X^{i_j}|) \times \prod_{\substack{T \subset \mathbf{n} \\ |T| > 1}} B(SH(X^{i_T}), SH(X^{i_T}));$$

for any pointed subset $S \subseteq \mathbf{n}$, $\mathscr{B}(SG(X), X)^S_I$ is given by

$$B(\Pi SG(X^{i_j}), \mathrm{Sin}\,|X^{S(I)}|) \times \prod_{\substack{T \subset \mathbf{n} \\ |T| > 1}} B(SH(X^{i_T}), SH(X^{i_T})).$$

This section for π is determined by the natural right inverse functor (of permutative categories) of the projection $\mathscr{C}'_X \to \mathscr{C}_X$.

Proposition 5·4 verifies that Example 5·3 does indeed yield an X-fibration.

Proposition 5·4. Any compatibly sectioned mapping fibration

$$\bar{\pi} : \bar{\mathscr{B}}\,(SG(X), X) \to \mathscr{B}SG(X)$$

of the map $\pi : \mathscr{B}(SG(X), X) \to \mathscr{B}SG(X)$ constructed in Example 5·3 is a fibrantly augmented X-fibration.

Proof. By construction, $\bar{\pi}$ is fibrantly sectioned. Condition 3·2(a) is verified by inspection. Condition 3·2(b) is implied by the weak equivalence

$$\bar{\mathscr{B}}(SG(X), X)_I \leftarrow \mathscr{B}(SG(X), X)_I = \Pi \mathscr{B}(SG(X), X)_{i_j} \underset{\Pi \mathscr{B}SG(X)_{i_j}}{\times} \mathscr{B}SG(X)_I$$

$$\to \Pi \bar{\mathscr{B}}\,(SG(X), X)_{i_j} \underset{\Pi \mathscr{B}SG(X)_{i_j}}{\times} \mathscr{B}SG(X)_I$$

and Condition 3·2(c) is similarly implied. Condition 3·2(d) is implied by the corresponding property for π.

The following construction yields maps

$$q_f : \mathscr{B}(P(f), SG(X), *) \to \mathscr{B}, \quad p_f : \mathscr{B}(P(f), SG(X), *) \to \mathscr{B}SG(X)$$

depending upon the principalization $\tilde{f} : P(f) \to \mathscr{B}$ given in Proposition 4·1. These maps will be shown (in Section 6) to determine the classifying map for the X-fibration $f : \mathscr{E} \to \mathscr{B}$. Our definition below of the categories $\mathscr{P}_\mathbf{n}$ is merely a modification of an explicit definition of the categories $(\mathscr{C}'_X)_\mathbf{n}$ of Example 5·3 in which the objects $(x_1, \ldots, x_n) \in |X^I|$ of $(\mathscr{C}'_X)_\mathbf{n}$ are replaced by objects $v : |X^I| \to |\mathscr{E}_I|$ in $\mathscr{P}_I$. Moreover, we now apply the bar construction to a simplicial monoid K and a right action of K on a simplicial set Z, $B(Z, K, *)$.

Example 5·5. For each $n > 0$, we define a functor $\mathscr{P}_{\mathbf{n}} \to (\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}}$ as follows, where $\underset{\sim}{\mathscr{C}}_X$ is defined in Example 5·1. The map on object spaces

$$\mathrm{Obj}\,(\mathscr{P}_{\mathbf{n}}) = \coprod_{I \in \mathscr{N}_{\mathbf{n}}} \mathscr{P}(f)_I \times \prod_{\substack{T \subset \mathbf{n} \\ |T| > 1}} \mathscr{S}\mathscr{H}(|X^{i_T}|)$$

$$\to \coprod_{I \in \mathscr{N}_{\mathbf{n}}} \prod_{\substack{T \subset \mathbf{n} \\ |T| > 1}} \mathscr{S}\mathscr{H}(|X^{i_T}|) = \mathrm{Obj}\,((\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}})$$

is the projection, where $\mathscr{P}(f)_I$ is the topological space of maps $v : |X^I| \to |\mathscr{E}_I|$ such that $\mathrm{Sin}\,(\mathscr{P}(f)_I) \underset{\mathrm{Sin}|\mathscr{B}_I|}{\times} \mathscr{B}_I = P(f)_I$ (as defined in Definition 4·1). The map on morphism spaces

$$\mathrm{Maps}\,(\mathscr{P}_{\mathbf{n}}) = \coprod_{I \in \mathscr{N}_{\mathbf{n}}} \mathscr{P}(f)_I \times \prod_{|T| > 1} \mathscr{S}\mathscr{H}(|X^{i_T}|) \times \prod_{1 \leqslant j \leqslant n} \mathscr{S}\mathscr{G}(|X^{i_j}|)$$

$$\to \coprod \Pi\, \mathscr{S}\mathscr{H}(|X^{i_T}|) \times \Pi \mathscr{S}\mathscr{G}(|X^{i_j}|) = \mathrm{Maps}\,((\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}})$$

is the projection: a map in $\mathscr{P}_{\mathbf{n}}$ from $\langle v, \times \theta_T\rangle$ to $\langle v', \times \theta'_T\rangle$ is an element $g \in \Pi\, \mathscr{S}\mathscr{G}(|X^{i_j}|)$ such that $v = v' \circ g$.

For any $d : \mathbf{n} \to \mathbf{t}$, we define $d : \mathscr{P}_{\mathbf{n}} \to \mathscr{P}_{\mathbf{t}}$ on objects by sending

$$\langle v, \times \theta_T\rangle \quad \text{to} \quad \langle d(v), d(\times \theta_T)\rangle,$$

where $d(\times \theta_T)$ is determined by $d : (\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}} \to (\underset{\sim}{\mathscr{C}}_X)_{\mathbf{t}}$ and $d(v)$ is the unique map fitting in the following commutative square

$$\begin{array}{ccc} |X^I| & \xrightarrow{\;v\;} & |\mathscr{E}_I| \\ \big\downarrow{\scriptstyle \theta_d} & & \big\downarrow{\scriptstyle d} \\ |X^{d(I)}| & \xrightarrow{\;d(v)\;} & |\mathscr{E}_{d(I)}| \end{array}$$

The map $\theta_d : |X^I| \to |X^{d(I)}|$ has jth factor defined to be the projection if

$$|d^{-1}(j) \subset \mathbf{n}| \leqslant 1$$

and to be $\theta_T \circ \mu_T$ otherwise, where $T = d^{-1}(j)$ and $\mu_T : \mathbf{n} \to \mathbf{1}$ in $\mathscr{F}$ sends $0 \neq i \in \mathbf{1}$ if and only if $i \in T$. We define $d : \mathscr{P}_{\mathbf{n}} \to \mathscr{P}_{\mathbf{t}}$ on maps by projecting to $d : (\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}} \to (\underset{\sim}{\mathscr{C}}_X)_{\mathbf{t}}$. This definition is now functorial with respect to maps d of $\mathscr{F}$ in contrast to the situation described in Proposition 4·2 for $P(f)$: the map sending $\langle v, \times \theta_T\rangle$ to $\langle d(v), d(\times \theta_T)\rangle$ differs in its first coordinate from the map $d : P(f)_I \to P(f)_{d(I)}$ of Proposition 4·2 in that the definition of $d(v)$ includes an action on the domain associated to $\times \theta_T$.

Consequently, we obtain a map of $\mathscr{F}$-spaces

$$\omega_f : \mathscr{B}(\mathscr{P}(f), SG(X), *) \to \mathscr{B}SG(X)$$

by applying the functor $B_{\mathrm{Sin}}(\;)$ to the above-defined natural transformation. If we view the space $|\mathscr{B}_{\mathbf{n}}|$ as a topological category whose only maps are identities, then the natural map $\mathscr{P}_{\mathbf{n}} \to |\mathscr{B}_{\mathbf{n}}|$ sending $\langle v, \times \theta_T\rangle$ to $f_I \circ v(|X^I|) \in |\mathscr{B}_{\mathbf{n}}|$ is a natural transformation of functors on $\mathscr{F}$. Applying $B_{\mathrm{Sin}}(\;)$, we obtain another map of $\mathscr{F}$-spaces

$$\mathscr{B}(\mathscr{P}(f), SG(X), *) \to \mathrm{Sin}\,|\mathscr{B}|.$$

We define the map of $\mathscr{F}$-spaces

$$q_f:\mathscr{B}(P(f),SG(X),*) = \mathscr{B}(\mathscr{P}(f),SG(X),*) \underset{\mathrm{Sin}\,|\mathscr{B}|}{\times} \mathscr{B} \to \mathscr{B}$$

to be the projection and the map

$$p_f:\mathscr{B}(P(f),SG(X),*) \to \mathscr{B}SG(X)$$

to be the composite $\omega_f \circ pr_1:\mathscr{B}(P(f),SG(X),*) \to \mathscr{B}(\mathscr{P}(f),SG(X),*) \to \mathscr{B}SG(X)$.

The role of the above maps p_f and q_f begins to be revealed in the following proposition.

Proposition 5·6. Let $f:\mathscr{E}\to\mathscr{B}$ be an X-fibration. Then there is a naturally defined chain of sectioned maps

$$\begin{array}{ccccccc}
\mathscr{E} & \longrightarrow & \mathrm{Sin}\,|\mathscr{E}| \underset{\mathrm{Sin}\,|\mathscr{B}|}{\times} \mathscr{B} & \xleftarrow{\tilde{p}_f} & \mathscr{B}(P(f),SG(X),X) & \xrightarrow{\tilde{q}_f} & \mathscr{B}(SG(X),X) \\
\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{Sin}|f|} & & \Big\downarrow{\scriptstyle \pi_f} & & \Big\downarrow{\scriptstyle \pi} \\
\mathscr{B} & \xrightarrow{1} & \mathscr{B} & \xleftarrow{p_f} & \mathscr{B}(P(f),SG(X),*) & \xrightarrow{q_f} & \mathscr{B}SG(X)
\end{array} \qquad (5·6·1$$

where π_f is defined to be $q_f^*(\pi)$.

Proof. To define $\tilde{p}_f$, we observe that

$$\mathscr{B}(P(f),SG(X),X) = \mathscr{B}(\mathscr{P}(f),SG(X),X) \underset{\mathrm{Sin}\,|\mathscr{B}|}{\times} \mathscr{B},$$

where $\mathscr{B}(\mathscr{P}(f),SG(X),X)_{\mathbf{n}}$ is defined by applying $B_{\mathrm{Sin}}(\;)$ to $\mathscr{P}'_{\mathbf{n}} = \mathscr{P}_{\mathbf{n}} \underset{(\underset{\sim}{\mathscr{C}}_X)_{\mathbf{n}}}{\times} (\mathscr{C}'_X)_{\mathbf{n}}$ (an object of which is an ordered set $\langle v, \times\theta_T, x\rangle$, with

$$v:|X^I|\to|\mathscr{E}_I|,\ \times\theta\in\Pi\mathscr{S}\mathscr{H}(|X^{i_T}|), x\in|X^I|.$$

If we view $|\mathscr{E}_{\mathbf{n}}|$ as a topological category whose only maps are identities, then the natural map $\mathscr{P}'_{\mathbf{n}}\to|\mathscr{E}_{\mathbf{n}}|$ sending $\langle v, \times\theta_T, x\rangle$ to $v(x)\in|\mathscr{E}_{\mathbf{n}}|$ is a natural transformation of functors on $\mathscr{F}$.

We define $\tilde{p}_f$ to be the map of $\mathscr{F}$-spaces determined by applying $B_{\mathrm{Sin}}(\;)$ to this natural transformation. Clearly, $\tilde{p}_f$ covers p_f. The map $(\tilde{p}_f, p_f):\pi_f\to\mathrm{Sin}\,|f|$ is section preserving because any $v:|X^I|\to|\mathscr{E}_I|$ in $\mathscr{P}(f)_{\mathbf{n}}$ sends $|X^{S(I)}|$ to $|\mathscr{E}_I^S|$.

6. The classification theorem for X-fibrations

Theorem 6·1 is the classification theorem asserting that $\bar{\pi}:\bar{\mathscr{B}}(SG(X),X)\to\mathscr{B}SG(X)$ is a universal X-fibration. The proof employs diagram (5·6·1) of Proposition 5·6 together with Theorem 4·5. This proof is an adaptation of May's proof of an analogous classification theorem for fibrations of topological spaces with a fixed homotopy type as fibre (9). Corollary 6·2 derives an easy necessary and sufficient condition on an X-fibration in order that it be universal. This condition is precisely the one suggested by analogy with the context of fibrations of topological spaces.

THEOREM 6·1. *Let X be a suitably oriented, pointed simplicial set and let*

$$\bar{\pi}: \bar{\mathscr{B}}\,(SG(X), X) \to \mathscr{B}SG(X)$$

be as in Proposition 5·4. *The association to a homotopy class $[g]: \mathscr{B} \to \mathscr{B}SG(X)$ in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ the equivalence class of $g^*(\bar{\pi})$ determines an isomorphism of functors*

$$\Lambda: \mathrm{Hom}_{\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\ ,\mathscr{B}SG(X)) \to X(\ ).$$

Proof. The fact that Λ is well defined follows from Theorem 3·5 and Proposition 5·4. In order to prove that Λ is an isomorphism, it suffices to prove that

$$\Lambda_{\mathscr{B}}: \mathrm{Hom}_{\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\mathscr{B}, \mathscr{B}SG(X)) \to X(\mathscr{B})$$

is a bijection for every cofibrant $\mathscr{F}$-space $\mathscr{B}$ over $\mathscr{N}$.

Let $f: \mathscr{E} \to \mathscr{B}$ be a fibrantly sectioned X-fibration. We easily verify that the maps $f_I \to \mathrm{Sin}\,(|f|)_I$, $(\pi_f)_I \to (\mathrm{Sin}\,|f|)_I$, and $(\pi_f)_I \to \pi_I$ induce oriented equivalences on homotopy fibres. Consequently, diagram (5·6·1) determines a chain of X-fibrations

$$f \to \mathrm{Sin}\,|f| \leftarrow \bar{\pi}_f \to \bar{\pi}$$

obtained as compatibly sectioned mapping fibrations (because $\mathrm{Sin} \circ |\ |$ commutes with finite limits and preserves fibrations, $\mathrm{Sin}\,|f|$ is an X-fibration whenever f is). We conclude that

$$q_f^*(\bar{\pi}) = \bar{\pi}_f = p_f^*(f) \quad \text{in} \quad X(\mathscr{B}(P(f), SG(X), *)).$$

For any n-tuple I and any k-simplex $\Delta[k] \to \mathscr{B}_I$, we readily verify that

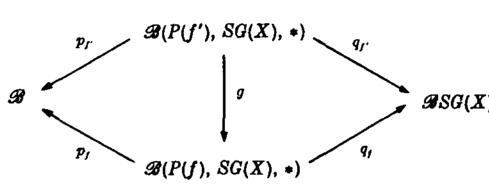
$$\mathscr{B}(P(f), SG(X), *)_I \times_{\mathscr{B}_I} \Delta[k]$$

is weakly equivalent to $B(P(f)_I \times_{\mathscr{B}_I} \Delta[k], SG(X)^I, *)$ which is contractible by Theorem 4·5. We conclude that p_f is a weak equivalence: this implies that p_f^* is an isomorphism, so that $\Lambda_{\mathscr{B}}$ is surjective.

Consider the function

$$\psi_{\mathscr{B}}: X(\mathscr{B}) \to \mathrm{Hom}_{\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\mathscr{B}, \mathscr{B}SG(X))$$

defined by $\psi_{\mathscr{B}}(f) = [q_f] \circ [p_f]^{-1}$. To verify that $\psi_{\mathscr{B}}$ is well defined, we employ Proposition 3·4 and the fact that a map $(g, \bar{g}): f' \to f$ of X-fibrations over $\mathscr{B}$ determines a commutative diagram of $\mathscr{F}$-spaces



We conclude this proof by verifying that $\psi_{\mathscr{B}} \circ \Lambda_{\mathscr{B}}$ is bijective for $\mathscr{B}$ cofibrant over $\mathscr{N}$. Let $j: \mathscr{B}SG(X) \to \mathscr{B}'$ be a trivial cofibration with $\mathscr{B}'$ fibrant over $\mathscr{N}$. For any homotopy class $[g]: \mathscr{B} \to \mathscr{B}SG(X)$ in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$, let $g': \mathscr{B} \to \mathscr{B}'$ be a map of $\mathscr{F}$-spaces representing $[g]$ (i.e. $[g] = [j]^{-1} \circ [g']$). We consider the following commutative diagram of $\mathscr{F}$-spaces over $\mathscr{N}$

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$$\begin{array}{ccccc}
\mathscr{B} & \xleftarrow{p_\tau} & \mathscr{B}(P(\tau), SG(X), *) & \xrightarrow{q_\tau} & \mathscr{B}SG(X) \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle 1} \\
\mathscr{B}' & \xleftarrow{p_{\tau'}} & \mathscr{B}(P(\tau'), SG(X), *) & \xrightarrow{q_{\tau'}} & \mathscr{B}SG(X) \\
\uparrow{\scriptstyle j} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle 1} \\
\mathscr{B}SG(X) & \xleftarrow{p_{\bar\pi}} & \mathscr{B}(P(\bar\pi), SG(X), *) & \xrightarrow{q_{\bar\pi}} & \mathscr{B}SG(X)
\end{array}$$

where $\tau' : \mathscr{E}' \to \mathscr{B}'$ is a compatibly sectioned mapping fibration for

$$j \circ \bar\pi : \bar{\mathscr{B}}(SG(X), X) \to \mathscr{B}'$$

and where $\tau = g'^*(\tau')$. Because $\Lambda_{\mathscr{B}}([g])$ is represented by τ, the top row represents $\psi_{\mathscr{B}} \circ \Lambda_{\mathscr{B}}([g])$ whereas the left column represents $[g]$.

To prove that $\psi_{\mathscr{B}} \circ \Lambda_{\mathscr{B}}$ is bijective, it now suffices to prove that $q_{\bar\pi}$ is a weak equivalence. We easily verify that the fibre of $(q_{\bar\pi})_I$ above any k-simplex $\Delta[k] \to \mathscr{B}SG(X)_I$ is weakly equivalent to $P(\bar\pi)_I$ for any n-tuple I. To show that $P(\bar\pi)_I$ is weakly contractible (thus implying that $(q_{\bar\pi})_I$ is a weak equivalence), we apply the principalization construction of Definition 4·1 to $\pi : \mathscr{B}(SG(X), X) \to \mathscr{B}SG(X)$. Because the quasi-fibration $\operatorname{Sin}|\pi_I|$ has fibre equal to $\operatorname{Sin}|X^I|$, a special equivalence

$$v : X^I \times \Delta[k] \to \operatorname{Sin}|\mathscr{B}SG(X)_I|$$

satisfies 4·1 (*a*) and 4·1 (*b*) if and only if it is given by a k-simplex of $SG(X)^I$. Therefore, $P(\pi)_I$ is equivalent to the contractible simplicial set $B(SG(X)^I, SG(X)^I)$. The inclusion $\pi \to \bar\pi$ determines $P(\pi)_I \to P(\bar\pi)_I$ over $\mathscr{B}SG(X)_I$ which induces an equivalence on fibres by Theorem 4·5. Therefore, $P(\bar\pi)_I$ is weakly equivalent to the contractible simplicial set $P(\pi)_I$.

Diagram (5·6·1) easily implies that $\Lambda_{\mathscr{B}} \circ \psi_{\mathscr{B}}([f]) = [f] \in X(\mathscr{B})$. Consequently, our proof above of Theorem 6·1 implies that $\psi_{\mathscr{B}} \circ \Lambda_{\mathscr{B}} = 1$, so that

$$q_{\bar\pi}, p_{\bar\pi} : \mathscr{B}(P(\bar\pi), SG(X), *) \to \mathscr{B}SG(X)$$

are equal to $\operatorname{Ho}\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$.

Another easy consequence of Theorem 6·1 is the following corollary.

Corollary 6·2. Let $f : \mathscr{E} \to \mathscr{B}$ be a fibrantly sectioned X-fibration. The natural transformation

$$\Lambda^f : \operatorname{Hom}_{\operatorname{Ho}\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\ , \mathscr{B}) \to X(\ )$$

sending a homotopy class $[g] : \mathscr{B}' \to \mathscr{B}$ to $[g^*f] \in X(\mathscr{B}')$ is an isomorphism of functors if and only if $P(f)_I$ is contractible for every n-tuple I. Such a fibration is said to be universal.

Proof. By Theorem 6·1, $\Lambda^f(\ )$ is an isomorphism if and only if

$$\psi_{\mathscr{B}}([f]) = [q_f] \circ [p_f]^{-1} : \mathscr{B} \to \mathscr{B}SG(X)$$

is an isomorphism in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ (because $\psi_{\mathscr{B}}$ is inverse to $\Lambda_{\mathscr{B}}$). Because $[p_f]$ is always an isomorphism and because the homotopy fibres of $(q_f)_I$ are equivalent to $P(f)_I$ (cf. argument in proof of Theorem 6·1 for $(q_{\bar{\pi}})_I$), $(q_f)_I$ is a weak equivalence if and only if $P(f)_I$ is contractible.

7. R-COMPLETED X-FIBRATIONS

In Theorem 7·8, we present an R-completed version of the classification theorem (Theorem 6·1). Theorem 7·9 then provides sufficient conditions for the R-completion of a universal X-fibration to be a universal R-completed X-fibration. Throughout this section, R will denote either a subring of the rationals or a prime field and R-completion will refer to the Bousfield–Kan functor $R_\infty(\ ):\mathscr{S}_*\to\mathscr{S}_*$. (Some technical properties of $R_\infty(\ )$ which we require are proved in the appendix.)

As defined in Definition 7·2, an R-completed X-fibration involves fibrations $f_I:\mathscr{E}_I\to\mathscr{B}_I$ with fibres equivalent to $R_\infty(X)^I$. Since smash product does not commute with the functor $R_\infty(\ )$ (not even up to homotopy when R is a prime field), some modification of the discussion of preceding sections is required. Since $R_\infty(\ )$ does not commute with fibre products (unlike $\mathrm{Sin}\circ|\ |$), care must be taken when applying lifting arguments to $R_\infty(f)$.

We require that X is pointed, connected, and *R-good* (i.e. $H_*(X,R)\xrightarrow{\sim}H_*(R_\infty(X),R)$). As discussed and proved in Proposition A 1, this implies that the simplicial set of R-equivalences from X^i to $R_\infty(X^i)$ is an equivalent submonoid of the simplicial monoid of equivalences of $R_\infty(X^i)$. (The product of $\alpha,\beta:X^i\to R_\infty(X^i)$ is defined to be the composition

$$X^i\xrightarrow{\beta}R_\infty(X^i)\xrightarrow{R_\infty(\alpha)}R^2_\infty(X^i)\xrightarrow{\psi}R_\infty(X^i)$$

where $1\to R_\infty(\ )$, $\psi:R^2_\infty(\ )\to R_\infty(\ )$ constitute the monad structure (3).) The advantage of considering maps with domain X^i is that the smash product of such maps is a similar map.

For any $i>0$, we define $RG(X^i)$ to be this simplicial monoid of R-equivalences from X^i to $R_\infty(X^i)$. For any n-tuple $I=(i_1,\ldots,i_n)$, we define $RG(X)^I$ to be the product $RG(X^{i_1})\times\ldots\times RG(X^{i_n})$. By Corollary A 2, smash product $R_\infty(X^i)\times R_\infty(X^j)\to R_\infty(X^{i+j})$ induces a monoid homomorphism $RG(X^i)\times RG(X^j)\to RG(X^{i+j})$. Thus, a map $d:\mathbf{n}\to t$ in $\mathscr{F}$ determines a monoid homomorphism

$$d:RG(X)^I\to RG(X)^{d(I)}.$$

Definition 7·1. Let X be a pointed, connected, R-good simplicial set. An *R-suitable orientation* for X is a choice of simplicial submonoid $SRG(X^i)\subset RG(X^i)$ (of 'special R-equivalences') for each $i>0$ such that

(*a*) $\pi_0(SRG(X^i))$ is a subgroup of $\pi_0(RG(X^i))$ and $SRG(X^i)$ consists of those components of $RG(X^i)$ indexed by $\pi_0(SRG(X^i))$.

(*b*) $\Sigma_i\subset SRG(X^i)$.

(*c*) $RG(X^i)\times RG(X^j)\to RG(X^{i+j})$ restricts to $SRG(X^i)\times SRG(X^j)\to SRG(X^{i+j})$ for all $j>0$.

(*d*) $\pi_0(SRG(X^i))$ acts nilpotently on $H_*(R_\infty(X^i), R)$ viewed as the homology of the homotopy fibre of the map $B(SRG(X^i), R_\infty(X^i)) \to BSRG(X^i)$.

For the remainder of this section, we assume that X is a pointed, connected R-good simplicial set provided with an R-suitable orientation. For any n-tuple $I = (i_1, \ldots, i_n)$, we let $SRG(X)^I$ denote $SRG(X^{i_1}) \times \ldots \times SRG(X^{i_n})$.

Since $\pi_0(RG(X^i)) = \pi_0(G(R_\infty(X^i))$, we may employ Definition 3·1 when considering fibrations $f_I: \mathscr{E}_I \to \mathscr{B}_I$ with given section oriented with respect to $SRG(X)^I$. For such a fibration, a map $v: X^I \times \Delta[k] \to \mathscr{E}_I$ is said to be a *special R-equivalence* provided that $f \circ r$ factors through a k-simplex of $\mathscr{B}_I$, $\Delta[k] \to \mathscr{B}_I$, and if composition with an orienting equivalence determines a k-simplex of $SRG(X)^I$,

$$X^I \times \Delta[k] \to |\mathscr{E}_I \times_{\mathscr{B}_I} \Delta[k]| \to R_\infty(X)^I = R_\infty(X^{i_1}) \times \ldots \times R_\infty(X^{i_n}).$$

Condition 7·1 (*d*) guarantees that the natural map $f_I \to R_\infty(f_I)$ induces equivalences on fibres whenever f_I is oriented with respect to $SRG(X)^I$.

Definition 7·2. A sectioned map of $\mathscr{F}$-spaces over $\mathscr{N}$, $f: \mathscr{E} \to \mathscr{B}$, is said to be an *R-completed X-fibration* provided that f_I satisfies 3·2 (*b*), 3·2 (*c*), and

(*a*) $f_I: \mathscr{E}_I \to \mathscr{B}_I$ is an oriented fibration with respect to $SRG(X)^I$;

(*d*) = 3·2 (*d*) with 'special equivalence' replaced by 'special R-equivalence' for each $n > 0$ and each n-tuple I.

A map of R-completed X-fibrations $(g, \bar{g}): f' \to f$ is a section-preserving map such that $(g_I, \bar{g}_I): f'_I \to f_I$ is orientation preserving for all n-tuples I.

The arguments in Section 3 apply without change to prove the following R-completed version of Theorem 3·5. The fact that $RX(\mathscr{B})$ is a set follows from Theorem 7·8.

Proposition 7·3. For any cofibrant $\mathscr{F}$-space $\mathscr{B}$ over $\mathscr{N}$, let $RX(\mathscr{B})$ denote the set of equivalence classes of R-completed X-fibrations over $\mathscr{B}$, where the equivalence relation is generated by pairs between which there is a map over $\mathscr{B}$. If $g: \mathscr{B}' \to \mathscr{B}$ is a map of cofibrant $\mathscr{F}$-spaces over $\mathscr{N}$, define $g^*: RX(\mathscr{B}) \to RX(\mathscr{B}')$ by sending the equivalence class of $f: \mathscr{E} \to \mathscr{B}$ to the equivalence class of $g^*(f): \mathscr{E} \times_{\mathscr{B}} \mathscr{B}' \to \mathscr{B}'$. Then this definition extends to a homotopy functor

$$RX(\ ): \mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N} \to (\mathrm{sets}).$$

Because the natural map $R_\infty(X^i) \to R_\infty(X)^i$ is a weak equivalence for R a subring of the rationals, $RX(\mathscr{B})$ equals $R_\infty(X)(\mathscr{B})$ (as defined in Definition 3·3) for such a ring R. However, if R is a prime field, then an R-completed X-fibration is not an $R_\infty(X)$-fibration.

The following is the R-completed version of Definition 4·1.

Definition 7·4. Let $f: \mathscr{E} \to \mathscr{B}$ be an R-completed X-fibration. For any n-tuple I, we define

$$\tilde{f}_I: RP(f)_I \to \mathscr{B}_I \quad \text{in } \mathscr{S}_*$$

as follows. A k-simplex of $RP(f)_I$ above a given k-simplex $b: \Delta[k] \to \mathscr{B}_I$ is a special R-equivalence

$$v: (X^I \times \Delta[k], * \times \Delta[k]) \to (R_\infty(\mathscr{E}_I), R_\infty(\mathscr{E}_I^{\{0\}}))$$

projecting to $b: \Delta[k] \to \mathscr{B}_I$ which satisfies

(a) For every pointed $S \subset \mathbf{n}$, v restricts to

$$v^S : X^{S(I)} \times \Delta[k] \to R_\infty(\mathscr{E}_I^S).$$

(b) For every $d: \mathbf{n} \to \mathbf{t}$ with $n > t$ and every $S \subset \mathbf{n}$, v^S projects to

$$v^{(d:S)} : X^{(d:S)(d(I))} \times \Delta[k] \to R_\infty(\mathscr{E}_{d(I)}^{(d:S)}).$$

We omit the easy verification of the R-completed version of Proposition 4·2. The natural pairing analogous to that of Proposition 4·2, is defined by sending

$$v: X^I \times \Delta[k] \to R_\infty(\mathscr{E}_I), \quad g: X^I \times \Delta[k] \to R_\infty(X)^I$$

to the composition

$$\psi \circ R_\infty(v) \circ (i \times 1) \circ (g \times 1): X^I \times \Delta[k] \to R_\infty(X)^I \times \Delta[k] \to R_\infty(X^I) \times \Delta[k] \to R_\infty^2(\mathscr{E}_I) \to R_\infty(\mathscr{E}_I),$$

where $i: R_\infty(X)^I \to R_\infty(X^I)$ is induced by the natural equivalence

$$R_\infty(X^i) \times R_\infty(X^j) \to R_\infty(X^i \times X^j).$$

The functor $R_\infty(\ ): \mathscr{S}_* \to \mathscr{S}_*$ in Definition 7·4 replaces the functor Sin $\circ$ | | in Definition 4·1. We require Proposition 7·5 in order to justify the lifting arguments which must be applied to $R_\infty(f)$ in proving the R-completed version of Theorem 4·5.

Proposition 7·5. Let $f: \mathscr{E} \to \mathscr{B}$ be an R-completed X-fibration. Then the natural map

$$f \to R_\infty(f)$$

admits the structure of a map of R-completed X-fibrations. Moreover, if f is fibrantly sectioned and $\mathscr{B}$ is fibrant over $\mathscr{N}$, then $R_\infty(f)$ is also fibrantly sectioned.

Proof. $R_\infty(f)$ is sectioned by defining $R_\infty(\mathscr{E})_{\mathbf{n}}^S$ to equal $R_\infty(\mathscr{E}_{\mathbf{n}}^S)$ for any $n > 0$ and any pointed $S \subseteq \mathbf{n}$. Definition 7·1 (d) assures that $R_\infty(f)$ satisfies 7·2 (a) and 7·2 (b). Condition 7·2 (c) for $R_\infty(f)$ is implied by 7·1 (d) together with the fact that $R_\infty(\ )$ preserves fibrations. Finally, 7·2 (d) for $R_\infty(f)$ is implied by 7·2 (d) for f and 7·1 (d). Therefore, $R_\infty(f)$ is an R-completed X-fibration. Clearly, $f \to R_\infty(f)$ is section preserving and orientation preserving, and hence a map of R-completed X-fibrations.

We now assume that f is fibrantly sectioned and that $\mathscr{B}$ is fibrant over $\mathscr{N}$. Then Proposition A 5 applies to prove that the canonical map

$$R_\infty(\lim_{n>t} \mathscr{E}_{d(I)}^{(d:S)} \times_{\mathscr{B}_{d(I)}} \mathscr{B}_I) \to \lim_{n>t} R_\infty(\mathscr{E}_{d(I)}^{(d:S)}) \times_{R_\infty(\mathscr{B}_{d(I)})} R_\infty(\mathscr{B}_I)$$

is a fibration for any $n > 0$, any pointed $S \subseteq \mathbf{n}$, and any n-tuple I. Because $R_\infty(\ )$ preserves fibrations, this implies that the natural maps

$$R_\infty(\mathscr{E}_I^S) \to \lim_{n>t} R_\infty(\mathscr{E}_{d(I)}^{(d:S)}) \times_{R_\infty(\mathscr{B}_{d(I)})} R_\infty(\mathscr{E}_I)$$

are fibrations. Thus $R_\infty(f)$ is fibrantly sectioned.

The following theorem is the R-completed analogue of Theorem 4·5.

THEOREM 7·6. *Let $f: \mathscr{E} \to \mathscr{B}$ be a fibrantly augmented, R-completed X-fibration with $\mathscr{B}$ fibrant over $\mathscr{N}$ and let I be an n-tuple. Then $\bar{f}_I : RP(f)_I \to \mathscr{B}_I$ is a fibration. Moreover, for any vertex z of $\mathscr{B}_I$, there exists a homotopy equivalence*

$$\phi_{z,I} : \bar{f}_I^{-1}(z) \to SRG(X)^I,$$

which fits in the following commutative square

$$\begin{array}{ccc} \bar{f}_I^{-1}(z) \times SRG(X)^I & \to & \bar{f}_I^{-1}(z) \\ \downarrow{\scriptstyle \phi_{z,I} \times 1} & & \downarrow{\scriptstyle \phi_{z,I}} \\ SRG(X)^I \times SRG(X)^I & \to & SRG(X)^I \end{array}$$

whose horizontal arrows are determined by composition.

Proof. As in Proposition 4·3, the natural maps

$$R_\infty(\mathscr{E}_I) \to \lim_{n \to >} R_\infty(\mathscr{E}_{a(I)}) \times_{R_\infty(\mathscr{B}_{a(I)})} R_\infty(\mathscr{B}_I),$$

$$R_\infty(\mathscr{E}_I^S) \to \lim_{n > t} R_\infty(\mathscr{E}_{d(I)}^{(d;S)}) \times_{R_\infty(\mathscr{B}_{d(I)})} R_\infty(\mathscr{B}_I) \quad (S \subsetneqq \mathbf{n})$$

are trivial fibrations: the proof given in Proposition 4·3 applies thanks to Proposition 7·5.

The proof of Theorem 4·5 now applies with only slight notational changes to prove Theorem 7·6.

The constructions of X-fibrations in section 5 provide R-completed X-fibrations by formally replacing $\operatorname{Sin}|X^I|$, $SG(X)^I$, $SH(X)^I$, $P(f)_I$ by $R_\infty(X)^I$, $SRG(X)^I$, $SI(X)^I$, $RP(f)_I$ (where $SI(X^i) \subset SH(X^i)$ consists of special homeomorphisms of $|X^i|$ which are realizations of isomorphisms of X^i). These notational changes also provide the following R-completed version of Proposition 5·6.

Proposition 7·7. Let $f: \mathscr{E} \to \mathscr{B}$ be an R-completed X-fibration. Then f fits in a naturally defined chain of section maps

$$\begin{array}{ccccc} \mathscr{E} \to R_\infty(\mathscr{E}) \times \mathscr{B} & \xleftarrow{\bar{p}_f} & \mathscr{B}(RP(f), SRG(X), RX) & \xrightarrow{\bar{q}_f} & \mathscr{B}(SRG(X), RX) \\ {\scriptstyle f}\downarrow \quad \downarrow{\scriptstyle R_\infty(\mathscr{B})(R_\infty(f))} & & \downarrow{\scriptstyle R\pi_f} & & \downarrow{\scriptstyle R\pi} \\ \mathscr{B} \xrightarrow{1} \mathscr{B} & \xleftarrow{p_f} & \mathscr{B}(RP(f), SRG(X), *) & \xrightarrow{q_f} & \mathscr{B}SRG(X) \end{array}$$

where $R\pi_f = q_f^*(R\pi)$.

The following classification theorem, whose statement recalls our standing hypothesis on X, is the R-completed analogue of Theorem 6·1.

THEOREM 7·8. *Let X be a pointed, connected, R-good simplicial set provided with an R-suitable orientation. Let $\overline{R\pi} : \bar{\mathscr{B}}(SRG(X), RX) \to \mathscr{B}SRG(X)$ be a compatibly augmented mapping fibration for $R\pi$. Sending a map $[g]: \mathscr{B} \to \mathscr{B}SRG(X)$ in $\operatorname{Ho}\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ to $g^*(\overline{R\pi})$ determines an isomorphism of functors*

$$\Lambda : \operatorname{Hom}_{\operatorname{Ho}\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\ , \mathscr{B}SRG(X)) \to RX(\ ).$$

Proof. The fact that $\overline{R\pi}$ is an R-completed X-fibration is verified as in Proposition 5·4. The surjectivity of $\Lambda_{\mathscr{B}}$ for $\mathscr{B}$ fibrant–cofibrant over $\mathscr{N}$ is proved exactly as in the proof of Theorem 6·1, with Theorem 7·6 and Proposition 7·7 replacing Theorem 4·5 and Proposition 5·6. Similarly, the fact that $\psi_{\mathscr{B}} \circ \Lambda_{\mathscr{B}}$ is bijective for $\mathscr{B}$ fibrant–cofibrant over $\mathscr{N}$ is proved exactly as in the proof of Theorem 6·1. The fact that $\Lambda_{\mathscr{B}}$ is an isomorphism for all $\mathscr{B}$ which are fibrant–cofibrant over $\mathscr{N}$, together with the fact that any object in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ is isomorphic to a fibrant–cofibrant object, implies that Λ is an isomorphism.

To prove the infinite loop Adams conjecture, we require that the R-completion of the universal S^2-fibration is a universal R-completed S^2-fibration.

THEOREM 7·9. *Let X be a finite, nilpotent, connected, pointed simplicial set whose homotopy groups are finitely generated and which is provided with a suitable orientation. Assume for all $i > 0$ that $\pi_0(SG(X^i))$ acts nilpotently on $H^*(X^i, R)$ viewed as the homology of the homotopy fibre of $B(*, SG(X^i), \mathrm{Sin}\,|X^i|) \to B(*, SG(X^i), *)$. Sending a map $[g]: \mathscr{B}' \to R_\infty(\mathscr{B})$ in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}$ to $g^*(R_\infty(f))$ determines an isomorphism of functors*

$$\Lambda: \mathrm{Hom}_{\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]/\mathscr{N}}(\ , R_\infty(\mathscr{B})) \to RX(\ )$$

whenever $f: \mathscr{E} \to \mathscr{B}$ is a universal X-fibration.

Proof. The nilpotency condition on the action of $\pi_0(SG(X^i))$ on $H^*(X^i, R)$ implies that $R_\infty(f)$ is an R-completed X-fibration whenever $f: \mathscr{E} \to \mathscr{B}$ is an X-fibration, where $SRG(X^i)$ is defined to consist of those components of $RG(X^i)$ in the image of $SG(X^i)$. Thus, a map $(R_\infty(\mathscr{E}_I) \times_{R(\mathscr{B}_I)} \Delta[t], \Delta[t]) \to (R_\infty(X)^I, *)$ is an orienting equivalence if and only if it is equivalent to the R-completion of an orienting equivalence

$$(\mathscr{E}_I \times_{\mathscr{B}_I} \Delta[t], \Delta[t]) \to (X^I, *).$$

Let $j: \mathscr{B} \to \overline{\mathscr{B}}$ be a trivial cofibration with $\overline{\mathscr{B}}$ fibrant over $\mathscr{N}$ and let $p: \overline{\mathscr{B}}' \to \overline{\mathscr{B}}$ be a trivial fibration with $\overline{\mathscr{B}}'$ cofibrant. Using Proposition 7·3, we shall assume $\mathscr{B}$ is fibrant–cofibrant by replacing f (if necessary) by $p^* \circ j_*(f) \in RX(\overline{\mathscr{B}}')$.

Let $n > 0$ and let I be a n-tuple. Let $RP'(R_\infty(f)))_I$ denote the simplicial set whose k-simplices are maps $X^I \times \Delta[k] \to R_\infty(\mathscr{E}_I)$ whose composite with the canonical map $i: R_\infty(\mathscr{E}_I) \to R_\infty^2(\mathscr{E}_I)$ are k-simplices of $RP(R_\infty(f))_I$. Similarly, define

$$RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I \to RP(R_\infty \circ \mathrm{Sin}\,|f|)_I.$$

The proof of Theorem 7·6 applies equally well to $RP'(R_\infty(f)) \to R_\infty(\mathscr{B})$ and

$$RP'(R_\infty \circ \mathrm{Sin}\,|f|) \to R_\infty \circ \mathrm{Sin}\,|\mathscr{B}|,$$

so that the natural inclusions $RP'(R_\infty(f))_I \to RP(R_\infty(f))_I$,

$$RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I \to RP(R_\infty \circ \mathrm{Sin}\,|f|)_I$$

are fibre homotopy equivalences over $R_\infty(\mathscr{B}_I)$ and $R_\infty \circ \mathrm{Sin}\,|\mathscr{B}_I|$. Define

$$P(f)_I \to RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I$$

by composition with the canonical map $i: \mathrm{Sin} \circ |\mathscr{E}_I| \to R_\infty \circ \mathrm{Sin} \circ |\mathscr{E}_I|$, and define $RP'(R_\infty(f))_I \to RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I$ by composition with the canonical map

$R_\infty(i): R_\infty(\mathscr{E}_I) \to R_\infty \circ \mathrm{Sin}\,|\mathscr{E}_1|$. These maps fit in the following map of fibre triples

$$\begin{array}{ccccccc}
SG(X)^I & \longrightarrow & SG(X, R_\infty \circ \mathrm{Sin}|X|)^I & \longleftarrow & SRG(X)^I & \longrightarrow & SRG(X)^I \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P(f)_I & \xrightarrow{\Theta_1} & RP(R'_\infty \circ \mathrm{Sin}|f|)_I & \xleftarrow{\Theta_2} & RP'(R_\infty(f))_I & \xrightarrow{\Theta_3} & RP(R_\infty(f))_I \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathscr{B}_I & \longrightarrow & R_\infty \circ \mathrm{Sin}|\mathscr{B}_I| & \longrightarrow & R_\infty(\mathscr{B}_I) & \xrightarrow{1} & R_\infty(\mathscr{B}_I)
\end{array}$$

whose fibres have been identified (up to homotopy) by Theorem 4·5 and 7·6, where $SG(X^i, R_\infty \circ \mathrm{Sin}\,|X^i|)$ is defined to be the function complex of special R-equivalences from X^i to $R_\infty \circ \mathrm{Sin}\,|X^i|$. By Proposition A 4, our hypotheses on X imply that the natural maps

$$SG(X^i) \to SG(X^i, R_\infty \circ \mathrm{Sin}\,|X^i|) \leftarrow SRG(X^i)$$

are R-equivalences and that $SG(X^i, R_\infty \circ \mathrm{Sin}\,|X^i|)$ and $SRG(X^i)$ are R-complete. Consequently, θ_2 is an equivalence of fibre triples and $P(f)_I \to RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I$ is an R-completion map, so that $RP'(R_\infty \circ \mathrm{Sin}\,|f|)_I$ is contractible. Because θ_3 is clearly an equivalence, we conclude that $RP(R_\infty(f))_I$ is contractible (for any n-tuple I). As argued in the proof of Corollary 6·2, this implies that $R_\infty(f)$ is universal as asserted.

8. The J-homomorphism and S^2-fibrations

In this section, we consider the complex J-homomorphism in the context of $\mathscr{F}$-spaces. Proposition 8·2 shows that the S^2-fibration introduced in Example 8·1 corresponds to the J-homomorphism. Example 8·3 introduces a modification of Example 8·1 which will be shown in Section 9 to have $\mathbb{Z}/p$-completion which is 'algebraically defined'. The close relationship between Examples 8·1 and 8·3 is described in Proposition 8·4.

Example 8·1. Let $\mathscr{U}$ be the permutative category whose object space is the discrete set $\{\mathbb{C}^i\}_{i\geq 0} \simeq N$ of finite-dimensional complex vector spaces and whose morphism space equals $\coprod_{i\geq 0} U(i)$ (where $U(i)$ is the topological group of $i\times i$ unitary complex matrices). We define

$$\Box : \mathscr{U} \times \mathscr{U} \to \mathscr{U}$$

by setting $\mathbb{C}^i \Box \mathbb{C}^j = \mathbb{C}^{i+j}$ and $\alpha \Box \beta = \alpha \oplus \beta : \mathbb{C}^i \Box \mathbb{C}^j \to \mathbb{C}^i \Box \mathbb{C}^j$ (i.e. 'block' or 'Whitney' sum).

We define a functor between permutative categories

$$\mathscr{U}' \to \mathscr{U}$$

as follows. The map on object spaces

$$\mathrm{Obj}\,(\mathscr{U}') = \coprod_{i\geq 0} (\mathbb{C}^i)^+ \to \{\mathbb{C}^i\}_{i\geq 0} = \mathrm{Obj}\,(\mathscr{U})$$

sends $x \in (\mathbb{C}^i)^+$ (the one-point compactification of $\mathbb{C}^i$ with base point at infinity) to $\mathbb{C}^i$. The map on morphism spaces

$$\text{Maps}(\mathscr{U}') = \coprod_{i\geqslant 0} (\mathbb{C}^i)^+ \times U(i) \to \coprod_{i\geqslant 0} U(i) = \text{Maps}(\mathscr{U})$$

is the projection map (so that $\mathscr{U}'(x,y) \subset \mathscr{U}(\mathbb{C}^i, \mathbb{C}^i)$ consists of the set – of at most one element – of unitary matrices sending x to y). The product

$$\square : \mathscr{U}' \times \mathscr{U}' \to \mathscr{U}'$$

is defined as Whitney sum of objects and maps.

As in Examples 5·1 and 5·3, the forgetful functor $\mathscr{U}' \to \mathscr{U}$ together with the natural right inverse determines a sectioned map of $\mathscr{F}$-spaces

$$\tau : \mathscr{B}(U, S^2) \to \mathscr{B}U$$

defined by applying the functor $B_{\mathrm{Sin}}(\;)$ to the natural transformation $\underset{\sim}{\mathscr{U}}' \to \underset{\sim}{\mathscr{U}}$ (where $\underset{\sim}{\mathscr{U}}', \underset{\sim}{\mathscr{U}} : \mathscr{F} \to$ (permutative categories) are obtained from $\mathscr{U}', \mathscr{U}$ as in (11)). In particular, for any $i \geqslant 0$, τ_i is the natural projection

$$\tau_i : B(\mathrm{Sin}(U(i)), \mathrm{Sin}((\mathbb{C}^i)^+) \to B\,\mathrm{Sin}(U(i)).$$

The reader can easily verify (as in the proof of Proposition 5·4) that any compatibly sectioned mapping fibration $\bar{\tau}$ for τ is an S^2-fibration, where S^2 is the finite simplicial set representing the 2-sphere with exactly two non-degenerate simplices, suitably oriented by choosing the connected component of the identity of each $G(S^{2m})$.

The next proposition implies that the classifying map for $\bar{\tau}$ determines the complex J-homomorphism.

Proposition 8·2. Identify $|S^{2i}|$ with $(\mathbb{C}^i)^+$, where S^2 is the above 'smallest' simplicial model for the 2-sphere. Then the natural action of $U(i)$ on $(\mathbb{C}^i)^+$ determines a map of $\mathscr{F}$-spaces

$$\mathscr{J} : \mathscr{B}U \to \mathscr{B}SG(S^2),$$

such that $\Phi(\mathscr{J})$ (cf. Proposition 1·4) is equivalent to the complex J-homomorphism

$$J : \mathsf{kU} \to \mathsf{BSG} \text{ in HoSp},$$

where kU is the Ω-spectrum of complex connective K-theory. Moreover, $\mathscr{J}^*(\bar{\pi}) = \bar{\tau}$ in $S^2(\mathscr{B}U)$, where $\bar{\pi} : \overline{\mathscr{B}}(SG(S^2), S^2) \to \mathscr{B}SG(S^2)$ is the universal S^2-fibration of Theorem 6·1 and $\bar{\tau}$ is a compatibly sectioned mapping fibration for τ of Example 8·1 (with orientations chosen so that $\bar{\tau} \to \bar{\pi}$ is orientation preserving).

Proof. Let $I = (i_1, \ldots, i_n)$ be an arbitrary n-tuple for some $n \geqslant 0$. The natural action of $U(I) = U(i_1) \times \ldots \times U(i_n)$ on

$$|S^{2I}| = |S^{2i_1}| \times \ldots \times |S^{2i_n}| \simeq (C^{i_1})^+ \times \ldots \times (C^{i_n})^+$$

determines a functor $\mathscr{U} \to \mathscr{C}_{S^2}$ (where $\mathscr{U}$ is discussed in Example 8·1 and $\mathscr{C}_{S^2}$ is discussed in Example 5·1). This functor determines a natural transformation $\underset{\sim}{\mathscr{U}} \to \underset{\sim}{\mathscr{C}}_{S^2}$, which in turn determines $\mathscr{J} : \mathscr{B}U \to \mathscr{B}SG(S^2)$. Clearly, the group completion of

$$\mathscr{J}_1 : \mathscr{B}U_1 = \coprod_{i\geqslant 0} B\,\mathrm{Sin}(U(i)) \to \coprod_{i\geqslant 0} BSG(S^{2i}) = \mathscr{B}SG(S^2)_1$$

is equivalent to the J-homomorphism $J: BU \to BSG$ in the homotopy category. Consequently, $\Phi(\mathscr{J})$ is equivalent to $J: \mathsf{kU} \to \mathsf{BSG}$ in the homotopy category of spectra (by the uniqueness theorem of (11)).

The natural action of $GL(i, \mathbb{C})$ on $|S^{2i}|$ actually determines a commutative square of permutative categories

(whose vertical arrows are the projection functors of Examples 5·3 and 8·1). This square respects the natural right inverses of $\mathscr{U}' \to \mathscr{U}$ and $\mathscr{C}'_{S^2} \to \mathscr{C}_{S^2}$; therefore, the square determines a section-preserving map $\tau \to \pi$. Since $\tau_I \to \pi_I$ is homotopy cartesian for every n-tuple I, the induced map $\bar{\tau} \to \bar{\pi}$ of compatibly sectioned mapping fibrations is a map of S^2-fibrations covering $\mathscr{J}$. This implies that $\mathscr{J}^*(\bar{\pi}) = \bar{\tau}$.

In order to employ algebraic geometry in the study of the J-homomorphism, we consider the simplicial space S^{2i}_c defined as the simplicial cone on the space $\mathbb{C}^i - 0$:

$$S^{2i}_c = * \cup (\mathbb{C}^i - 0) \times \Delta[1] \cup *.$$

Thus, S^{2i}_c in dimension k equals $* \amalg (\mathbb{C}^i - 0)^{\times k} \amalg *$. Letting $GL(i, \mathbb{C})$ act in the usual way on each component of each dimension of S^{2i}_c determines a natural action

$$\mu_i : GL(i, C) \times S^{2i}_c \to S^{2i}_c.$$

The 'addition' simplicial pairing $\Delta[k] \times \Delta[1] \to \Delta[1]$ (interpreted by viewing $\Delta[1]$ as the line from vertex 0 to vertex ∞) determines a natural 'sum' pairing

$$\rho_{i,j} : S^{2i}_c \times S^{2j}_c \to S^{2i+2j}_c,$$

which factors through the smash product, $S^{2i}_c \wedge S^{2j}_c$, and which restricts to the natural 'equatorial inclusions', $\{0\} \times S^{2i}_c \to S^{2i+2j}_c$ and $S^{2i}_c \times \{0\} \to S^{2i+2j}_c$ (sending $(0,0) \times (z', t')$ to (z', t'), and $(z, t) \times (0, 0)$ to (z, t)).

The action of μ respects the pairing ρ in the sense that the following square commutes for all $i, j \geqslant 0$:

Example 8·3. Let $\mathscr{L}$ be the permutative category whose object space is the discrete set $\{\mathbb{C}^i\}_{i \geqslant 0} \simeq N$, and whose morphism space is $\coprod_{i \geqslant 0} GL(i, \mathbb{C})$, and whose product $\square : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ is given by Whitney sum (so that $\mathscr{U}$ is a faithful subcategory of $\mathscr{L}$). We define a functor between permutative categories of simplicial spaces whose map on object simplicial spaces

$$\mathrm{Obj}\,(\mathscr{L}') = \coprod_{i \geqslant 0} S^{2i}_c \to \{\mathbb{C}^i\}_{i \geqslant 0} = \mathrm{Obj}\,(\mathscr{L})$$

is the projection (where $\mathrm{Obj}(\mathscr{L})$ is viewed as a discrete simplicial space), and whose map on morphism simplicial spaces

$$\mathrm{Maps}(\mathscr{L}') = \coprod_{i\geq 0} S_c^{2i} \times GL(i,\mathbb{C}) \to \coprod_{i\geq 0} GL(i,\mathbb{C}) = \mathrm{Maps}(\mathscr{L})$$

is also the projection. The product $\square : \mathscr{L}' \times \mathscr{L}' \to \mathscr{L}'$ is the natural one which covers $\square : \mathscr{L} \times \mathscr{L} \to \mathscr{L}$ and is given by the pairings $\rho_{i,j} : S_c^{2i} \times S_c^{2j} \to S_c^{2i+2j}$ on objects.

The projection functor $\mathscr{L}' \to \mathscr{L}$ together with the natural right inverse $\mathscr{L} \to \mathscr{L}'$ determines an augmented map of $\mathscr{F}$-spaces

$$\tau_c : \mathscr{B}(GL, S_c^2) \to \mathscr{B}GL$$

defined by applying $B_{\mathrm{Sin}}(\ )$ to $\underset{\sim}{\mathscr{L}}' \to \underset{\sim}{\mathscr{L}}$ (where $\underset{\sim}{\mathscr{L}}', \underset{\sim}{\mathscr{L}} : \mathscr{F} \to$ (permutative categories) are obtained from $\mathscr{L}', \mathscr{L}$ as in (**11**)). (The functor $B_{\mathrm{Sin}}(\ )$ applied to a category $\mathscr{C}$ of simplicial spaces is defined by $B_{\mathrm{Sin}}(\mathscr{C}) = \Delta \circ \mathrm{Sin} \circ N(\mathscr{C})$, the diagonal of the tri-simplicial set obtained by applying Sin ( ) to the nerve of $\mathscr{C}$.) In particular,

$$(\tau_c)_i : \mathscr{B}(GL, S_c^2)_i \to \mathscr{B}GL_i$$

is the projection

$$B(\mathrm{Sin} \circ GL(i,\mathbb{C}), \Delta \circ \mathrm{Sin} \circ S_c^{2i}) \to B(\mathrm{Sin} \circ GL(i,\mathbb{C})).$$

We conclude this section by establishing the following relationship between τ_c and τ.

Proposition 8·4. There is a naturally defined commutative diagram of $\mathscr{F}$-spaces of the following form:

$$\begin{array}{ccccc} \mathscr{B}(GL, S_c^2) & \to & \mathscr{B}(GL, |S_c^2|) & \leftarrow & \mathscr{B}(U, S^2) \\ \downarrow{\scriptstyle \tau_c} & & \downarrow{\scriptstyle |\tau_c|} & & \downarrow{\scriptstyle \tau} \\ \mathscr{B}GL & = & \mathscr{B}GL & \leftarrow & \mathscr{B}U \end{array}$$

constituting section-preserving maps $\tau_c \to |\tau_c| \leftarrow \tau$. Moreover, the induced maps of compatibly sectioned mapping fibrations, $\bar{\tau}_c \to |\bar{\tau}_c| \leftarrow \bar{\tau}$, are maps of S^2-fibrations (provided that $\bar{\tau}_c$ and $|\bar{\tau}_c|$ are given orientations induced from $\bar{\tau}$).

Proof. $|\tau_c|$ is defined by repeating the construction of τ_c in Example 8·1 with $U(i)$ replaced by $GL(i,\mathbb{C})$, $(\mathbb{C}^i)^+$ replaced by the geometric realization of the simplicial space S_c^{2i} (denoted $|S_c^{2i}|$), and Whitney sum on objects replaced by the geometric realizations of pairings $\rho_{i,j}$. The actions $\mu_i : GL(i, C) \times S_c^{2i} \to S_c^{2i}$ and the pairings $\rho_{i,j} : S_c^{2i} \times S_c^{2i} \to S_c^{2i+2j}$ were so defined that the natural equivalence

$$(\mathbb{C}^i)^+ \to |S_c^{2i}| = \mathbb{C}^i - 0 \times \{0\} \backslash \mathbb{C}^i - 0 \times |\Delta[1]| / \mathbb{C}^i - 0 \times \{\infty\}$$

defined by sending $z \in \mathbb{C}^i - \{0\}$ to $(z, |z|^2)$ is $U(i)$-equivariant for each $i > 0$ and compatible with products. Therefore, these equivalences and the natural inclusions $U(i) \to GL(i,\mathbb{C})$ determine a commutative square of permutative categories inducing the map $\tau \to |\tau_c|$.

Similarly, the map $\tau_c \to |\tau_c|$ is determined by the canonical weak equivalences $\Delta \circ \mathrm{Sin}(S_c^{2i}) \to \mathrm{Sin}\,|S_c^{2i}|$ (which are $\mathrm{Sin}(GL(i,\mathbb{C}))$ equivariant and compatible with products). Since the maps $(\tau_c)_I \to |\tau_c|_I \leftarrow \tau_I$ are homotopy cartesian for all n-tuples I, the induced maps of compatibly sectioned mapping fibrations are maps of S^2-fibrations.

9. R-COMPLETED S^2-FIBRATIONS FROM ALGEBRAIC GEOMETRY

Theorem 9·4 relates the $(\mathbb{Z}/p)$-completion of the S^2-fibration associated to the complex J-homomorphism to a $(\mathbb{Z}/p)$-completed S^2-fibration which arises from algebraic geometry. The latter has the valuable property that it is acted upon by discontinuous algebraic automorphisms. As the reader will see, the functorial link from algebraic geometry to simplicial sets is etale homotopy theory.

The following example is essentially a repetition of the construction of Example 8·3, except that simplicial spaces are replaced by simplicial varieties (i.e. simplicial complex algebraic varieties).

Example 9·1. The simplicial spaces of Example 8·3 have natural structures of simplicial varieties and the maps of this example are algebraic with respect to this structure. Let $\underset{\sim}{\mathscr{L}}'^{\mathrm{alg}} \to \mathscr{L}^{\mathrm{alg}}$ be the natural transformation of 'permutative categories of simplicial varieties' with natural right inverse. We define

$$\tau^{\mathrm{alg}} : \mathscr{B}^{\mathrm{alg}}(GL, S^2_c) \to \mathscr{B}^{\mathrm{alg}}_{.}GL$$

to be the sectioned map of $\mathscr{F}$-simplicial varieties (i.e. functors from the category $\mathscr{F}$ to the category of pointed, simplicial complex algebraic varieties) obtained by applying $\Delta \circ N$ (the diagonal of the nerve) to $\underset{\sim}{\mathscr{L}}'^{\mathrm{alg}} \to \underset{\sim}{\mathscr{L}}^{\mathrm{alg}}$, where $\underset{\sim}{\mathscr{L}}'^{\mathrm{alg}}, \underset{\sim}{\mathscr{L}}^{\mathrm{alg}}$ are obtained from $\mathscr{L}'^{\mathrm{alg}}, \mathscr{L}^{\mathrm{alg}}$ as in (11). In particular, for any $i \geqslant 0$ we identify τ^{alg}_i as the natural projection

$$B(GL_{i,\mathbb{C}}, S^{2i}_c) \to BGL_{i,\mathbb{C}},$$

where $GL_{i,\mathbb{C}}$ is the algebraic group whose complex points constitute $GL(i,\mathbb{C})$ and $B(GL_{i,\mathbb{C}}, S^{2i}_c)$ is the diagonal of the bi-simplicial variety obtained from the bar construction.

We briefly recall the (Cech) *etale homotopy type* functor

$$(\ )_{\mathrm{ret}} : (\text{s. cx. alg. var.}_*) \to \text{pro-}\mathscr{S}_*,$$

which associates an inverse system $(V.)_{\mathrm{ret}}$ of pointed simplicial sets to a pointed simplicial complex algebraic variety $V.$ (4). To any $V.$ in (s. cx. alg. var.$_*$), one associates a left-filtering partially ordered set $RC(V.)$ (of 'pointed, rigid etale coverings' $U. \to V.$). Then $(V.)_{\mathrm{ret}}$ is defined to be the functor $RC(V.) \to \mathscr{S}_*$ which sends $U. \to V.$ in $RC(V.)$ to $\pi_0(\Delta N_{V.}(U.))$, whose k-simplices are the connected components of the k-fold fibre product of U_k over V_k.

Proposition 9·2. Let $R = \mathbb{Z}/p$ for some prime p. There exists a commutative diagram of $\mathscr{F}$-spaces of the following form

$$\begin{array}{ccccc} R\mathscr{B}^{\mathrm{alg}}(GL, S^2_c) & \leftarrow & R\mathscr{B}^{\mathrm{eth}}(GL, S^2_c) & \rightarrow & R_\infty(\mathscr{B}(GL, S^2_c)) \\ \downarrow{\scriptstyle R\tau^{\mathrm{alg}}} & & \downarrow{\scriptstyle R\tau^{\mathrm{eth}}} & & \downarrow{\scriptstyle R_\infty\tau_c} \\ R\mathscr{B}^{\mathrm{alg}}GL & \leftarrow & R\mathscr{B}^{\mathrm{eth}}GL & \rightarrow & R_\infty(\mathscr{B}GL) \end{array}$$

whose horizontal arrows are weak equivalences, where $R\tau^{\mathrm{alg}} = (\mathrm{holim} \circ R_\infty \circ (\ )_{\mathrm{ret}})(\tau^{\mathrm{alg}})$ (with holim ( ) the Bousfield–Kan homotopy inverse limit functor).

Proof. For any $V.$ in (s. cx. alg. var.$_*$), one associates the left-filtering partially ordered set $RLH(V.)$ (of 'pointed, rigid local homeomorphism coverings' $U. \to V.$) containing $RC(V)$. The (Cech) *local homeomorphism* type functor

$$(\ )_{rlh} : (\text{s. cx. alg. var.}_*) \to \text{pro-}\mathscr{S}_*$$

associates to $V.$ the functor $(V.)_{rlh} : RLH(V.) \to \mathscr{S}_*$, which sends $U. \to V.$ in $RLH(V.)$ to $\pi_0(\Delta N_{V.}(U.))$. The inclusion $RC(V.) \to RLH(V.)$ determines a natural transformation $(\ )_{\mathrm{ret}} \leftarrow (\ )_{rlh}$. We define another functor

$$(\ )_{\Delta(\mathrm{Sin},\, rlh)} : (\text{s. cx. alg. var.}_*) \to \text{pro-}\mathscr{S}_*,$$

which associates to $V.$ the functor $(V.)_{\Delta(\mathrm{Sin},\, rlh)} : RLH(V.) \to \mathscr{S}_*$ sending $U. \to V.$ to $\Delta \circ \mathrm{Sin}\,(N_{V.}(U.))$. The natural maps

$$\pi_0(\Delta N_{V}.(U.)) \leftarrow \Delta \circ \mathrm{Sin}\,(N_{V}.(U.)) \to \Delta \circ \mathrm{Sin}\,(V.)$$

determine natural transformations $(\ )_{rlh} \leftarrow (\ )_{\Delta(\mathrm{Sin},\, rlh)} \to \Delta \circ \mathrm{Sin}\,(\ )$. The reader should recall that $\tau_c = \Delta \circ \mathrm{Sin}\,(\tau^{\mathrm{alg}})$ (cf. Example 8·3).

Since holim ( ) is functorial on maps of pro-$\mathscr{S}_*$ induced by a functorial change of indexing categories, there are natural transformations

$$\mathrm{holim} \circ R_\infty \circ (\ )_{\mathrm{ret}} \leftarrow \mathrm{holim} \circ R_\infty \circ (\ )_{\Delta(\mathrm{Sin},\, rlh)} \to R_\infty \circ \Delta \circ \mathrm{Sin}\,(\ ).$$

When applied to $\tau^{\mathrm{alg}} : \mathscr{B}^{\mathrm{alg}}(GL, S^2_c) \to \mathscr{B}^{\mathrm{alg}}GL$, these natural transformations determine the asserted commutative diagram of $\mathscr{F}$-spaces. The 'classical comparison theorem' of Artin–Mazur implies for any n-tuple I that the maps

$$R_\infty \circ (\ )_{\mathrm{ret}}(\mathscr{B}^{\mathrm{alg}}(GL, S^2_c)_I) \leftarrow R_\infty \circ (\ )_{\Delta(\mathrm{Sin},\, rlh)}(\mathscr{B}^{\mathrm{alg}}(GL, S^2_c)_I) \to R_\infty(\mathscr{B}(GL, S^2_c)_I),$$

$$R_\infty \circ (\ )_{\mathrm{ret}}(\mathscr{B}^{\mathrm{alg}}GL_I) \leftarrow R_\infty \circ (\ )_{\Delta(\mathrm{Sin},\, rlh)}(\mathscr{B}^{\mathrm{alg}}GL_I) \to R_\infty(\mathscr{B}GL_I)$$

are the R-completions of maps in pro-$\mathscr{S}_*$, which are simply connected diagonals of R-equivalences in pro-$\mathscr{S}_*$, and thus are R-equivalences. These maps become R-equivalences of R-complete simplicial sets after holim has been applied, because the pro-simplicial sets involved are either equivalent to simplicial sets or are pro-objects of pointed simply connected Kan complexes with finite homotopy groups (obtained as the R-completions of diagonals of connected bi-simplicial sets with finite homotopy groups). We conclude that holim applied to these maps yields weak equivalences of simplicial sets.

Corollary 9·3. Let S^2 be given an R-suitable orientation defined by setting $SRG(S^{2i})$ equal to the connected component of the identity of $RG(S^{2i})$ for each $i > 0$. The maps $R\tau^{\mathrm{alg}} \leftarrow R\tau^{lh} \to R_\infty \tau_c$ of Proposition 9·2 determine maps of R-completed S^2-fibrations

$$\overline{R\tau^{\mathrm{alg}}} \leftarrow \overline{R\tau^{lh}} \to \overline{R_\infty \tau_c} \to R_\infty \bar{\tau}_c$$

obtained as induced maps of compatibly sectioned mapping fibrations.

Proof. The fact that $R\tau^{\mathrm{alg}} \leftarrow R\tau^{lh} \rightarrow R_\infty \tau_c$ are section-preserving maps between sectioned maps of $\mathscr{F}$-spaces is a formal consequence of the fact that these maps were determined by natural transformations between functors applied to the sectioned map $\mathscr{B}^{\mathrm{alg}}(GL, S^2_c) \rightarrow \mathscr{B}^{\mathrm{alg}}GL$. As argued in the proof of Theorem 7·10, $R_\infty \bar{\tau}_c$ is an R-completed S^2-fibration. The natural map $R_\infty \tau_c \rightarrow R_\infty \bar{\tau}_c$ over $R_\infty \mathscr{B}GL$ extends to a section-preserving map $\overline{R_\infty \tau_c} \rightarrow R_\infty \bar{\tau}_c$ so that $\overline{R_\infty \tau_c}$ is indeed an R-completed S^2-fibration. Moreover, the equivalences of Proposition 9·2, $R\tau^{\mathrm{alg}} \leftarrow R\tau^{lh} \rightarrow R_\infty \tau_c$, extend to maps $\overline{R\tau^{\mathrm{alg}}} \leftarrow \overline{R\tau^{lh}} \rightarrow \overline{R_\infty \tau_c}$ of R-completed, S^2-fibrations.

We now summarize the relationship between the J-homomorphism and $R\tau^{\mathrm{alg}}$.

THEOREM 9·4. *Let S^2 be the simplicial model of the 2-sphere with exactly two non-degenerate simplices, and provide S^2 with the R-suitable orientation defined by setting $SRG(S^{2i})$ equal to the connected component of the identity of $RG(S^{2i})$ for each $i > 0$. Let*

$$\Theta : R\mathscr{B}^{\mathrm{alg}}GL \rightarrow R_\infty \mathscr{B}U$$

be the isomorphism in $\mathrm{Ho}\,\mathscr{F}[\mathscr{S}_*]$ *determined by the weak equivalences*

$$R\mathscr{B}^{\mathrm{alg}}GL \leftarrow R\mathscr{B}^{lh}GL \rightarrow R_\infty \mathscr{B}GL$$

of Proposition 9·2 *and the R-completion of the natural equivalence $i : \mathscr{B}U \rightarrow \mathscr{B}GL$. Then*

$$\Theta^* \circ (R_\infty \mathscr{J})^* (R_\infty \bar{\pi}) \in RS^2(R\mathscr{B}^{\mathrm{alg}}GL)$$

is represented by $\overline{R\tau^{\mathrm{alg}}}$.

Proof. Rather than verify that various $\mathscr{F}$-spaces are cofibrant, we implicitly use Proposition 3·6. Proposition 8·2 implies that

$$\mathscr{J}^*(\bar{\pi}) = \bar{\tau} \in S^2(\mathscr{B}U),$$

so that

$$(R_\infty \mathscr{J})^* (R_\infty \bar{\pi}) = R_\infty \bar{\tau} \in RS^2(R_\infty \mathscr{B}U).$$

Proposition 8·4 implies that

$$i^*(\bar{\tau}_c) = \bar{\tau} \in S^2(\mathscr{B}U),$$

so that

$$i^*(R_\infty \bar{\tau}_c) = R_\infty \bar{\tau} \in RS^2(R_\infty \mathscr{B}U).$$

Finally, Corollary 9·3 implies that

$$\Theta^*(i^*(R_\infty \bar{\tau}_c)) = \overline{R\tau}_{\mathrm{alg}} \in RS^2(R\mathscr{B}^{\mathrm{alg}}GL).$$

10. The infinite loop Adams conjecture

This conjecture is proved in Theorem 10·4 by adapting Sullivan's proof of the Adams conjecture to $\mathscr{F}$-spaces. A quaternionic analogue is given in Theorem 10·5. Our proof does not apply to the (false) real analogue because the S^1-fibration

$$\mathscr{B}(0, S^1) \rightarrow \mathscr{B}O$$

corresponding to the real J-homomorphism cannot be approximated algebraically (even though its complexification $\mathscr{B}(0, S^2) \rightarrow \mathscr{B}O$ can be so approximated).

We begin by relating R-completions of $\mathscr{F}$-spaces and spectra.

LEMMA 10·1. *Let $R = \mathbb{Z}/q$ for some prime q and let $\mathscr{B}$ be an $\mathscr{F}$-space such that $\mathscr{B}_n$ is R-good (e.g. nilpotent) for all $n > 0$. The R-completion map for the spectrum $\Phi(\mathscr{B})$ associated to $\mathscr{B}$,*

$$\Phi(\mathscr{B}) \to R_\infty(\mathscr{B})$$

fits in a natural commutative square of spectra

$$\begin{array}{ccc} \Phi(\mathscr{B}) & \to & R_\infty \Phi(\mathscr{B}) \\ \downarrow & & \downarrow \\ \Phi(R_\infty \mathscr{B}) & \to & R_\infty \Phi(R_\infty \mathscr{B}) \end{array}$$

whose right vertical map is an equivalence.

Proof. The commutative square is induced by the natural map $\mathscr{B} \to R_\infty \mathscr{B}$ and the natural transformation $\Phi \to R_\infty \circ \Phi$. Because $\mathscr{B}_n$ is R-good, $\mathscr{B}_n \to R_\infty \mathscr{B}_n$ induces isomorphisms $H_*(\mathscr{B}_n, R) \xrightarrow{\sim} H_*(R_\infty \mathscr{B}_n, R)$ which imply isomorphisms

$$H_*(\Phi(\mathscr{B})_k, R) \xrightarrow{\sim} H_*(\Phi(R_\infty \mathscr{B})_k, R) \quad (k \geqslant 1),$$

where A_k denotes the kth term of the spectrum A. These isomorphisms imply the required equivalences

$$(R_\infty \Phi(\mathscr{B}))_k = R_\infty(\Phi(\mathscr{B})_k) \to R_\infty(\Phi(R_\infty \mathscr{B})_k) = (R_\infty \Phi(R_\infty \mathscr{B}))_k$$

for all $k \geqslant 1$.

Following Sullivan's approach (16), we next describe how to obtain the $(\mathbb{Z}/q)$-completion of Adams operations using Galois actions. Our use of the ring structure to identify the maps of spectra was suggested to us by J. P. May.

Proposition 10·2. Let $R = \mathbb{Z}/q$ for some prime q and let p be another prime different from q. Let $\psi \in \mathrm{Gal}(\mathbb{C}, Q)$ be an extension of the automorphism of the Witt vectors of $\overline{\mathbb{F}}_p$ (the algebraic closure of the prime field $\mathbb{R}_p$) which uniquely lifts the pth root map $(\ )^{1/p} : \overline{F}_p \to \overline{F}_p$. Let

$$R\psi^{\mathrm{alg}} : R\mathscr{B}^{\mathrm{alg}}GL \to R\mathscr{B}^{\mathrm{alg}}GL$$

be the automorphism induced by ψ on $R\mathscr{B}^{\mathrm{alg}}GL$ and let

$$\psi^\wedge = \Theta \circ R\psi^{\mathrm{alg}} \circ \Theta^{-1} : R_\infty \mathscr{B}U \to R_\infty \mathscr{B}U.$$

Then
$$R_\infty \circ \Phi(\psi^\wedge) = R_\infty \psi^p : R_\infty \mathsf{kU} \to R_\infty \mathsf{kU}$$

where kU is the spectrum of complex connective K-theory and ψ^p is the pth Adams operation.

Proof. Since the $\mathscr{F}$-simplicial variety $\mathscr{B}^{\mathrm{alg}}GL$ is defined over $\mathbb{Z}$, ψ induces an automorphism of $\mathscr{B}^{\mathrm{alg}}GL$. (If $X = X_{\mathbb{Z}} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\mathbb{C}$ is a complex variety defined over $\mathbb{Z}$, then ψ induces an automorphism of X by 'acting on the second factor'.)

$$R_\infty \circ \Phi(\psi^\wedge) : R_\infty \mathsf{kU} \to R_\infty \mathsf{kU}$$

is therefore well defined once we identify $R_\infty \mathsf{kU}$ with $R_\infty \circ \Phi(R_\infty \mathscr{B}U)$ using Lemma 10·1.

Moreover, $R_\infty \circ \Phi(\psi^\wedge)$ is a map of ringed spectra: $\mathscr{F}$-space theory permits consideration of ring structures yielding ring spectra (15), and both $R\psi^{\mathrm{alg}}$ and Θ respect the ring structures. This ring structure on $R_\infty \mathscr{B}U$ is determined by tensor product, thus corresponding to the usual ring structure on BU. Because $R_\infty \circ \Phi(\psi^\wedge)$ and $R_\infty \psi^p$ determine equivalent maps of ringed spaces as shown in (16), we conclude that $R_\infty \circ \Phi(\psi^\wedge) = R_\infty \psi^p$ as maps of ringed spectra (10).

The introduction of algebraic geometry into our study of the complex J-homomorphism was done in order to obtain the following result.

Proposition 10·3. The automorphism $R\psi^{\mathrm{alg}}: R\mathscr{B}^{\mathrm{alg}}GL \to R\mathscr{B}^{\mathrm{alg}}GL$ of Proposition 10·2 is covered by an automorphism $R\tilde{\psi}^{\mathrm{alg}}: R\mathscr{B}^{\mathrm{alg}}(GL, S^2_c) \to R\mathscr{B}^{\mathrm{alg}}(GL, S^2_c)$. More precisely, these automorphisms determine a map

$$(R\tilde{\psi}^{\mathrm{alg}}, R\psi^{\mathrm{alg}}): R\tau^{\mathrm{alg}} \to R\tau^{\mathrm{alg}}$$

of sectioned maps of $\mathscr{F}$-spaces.

Proof. As discussed in Proposition 10·2, ψ determines a commutative square of $\mathscr{F}$-simplicial varieties

$$\begin{array}{ccc} \mathscr{B}^{\mathrm{alg}}(GL, S^2_c) & \xrightarrow{\tilde{\psi}} & \mathscr{B}^{\mathrm{alg}}(GL, S^2_c) \\ \downarrow{\scriptstyle \tau^{\mathrm{alg}}} & & \downarrow{\scriptstyle \tau^{\mathrm{alg}}} \\ \mathscr{B}^{\mathrm{alg}}GL & \xrightarrow{\psi} & \mathscr{B}^{\mathrm{alg}}GL \end{array}$$

constituting an automorphism of the sectioned map τ^{alg}. The proposition now follows from the functorality of $\operatorname{holim} \circ R_\infty \circ (\ )_{\mathrm{ret}}$ (which, when applied to τ^{alg}, yields $R\tau^{\mathrm{alg}}$).

The orientation for S^2 implicit in the statement and proof of our title theorem below is that given by the choice of connected component of the identity of $G(S^{2m})$ for each $m > 0$. Our formulation of the theorem differs slightly from that given in the introduction in order to eliminate any possible ambiguity in its statement.

THEOREM 10·4. *For any integer $r > 0$,*

$$J = J \circ \psi^r: (\mathsf{kU})_{1/r} \to (\mathsf{BSG})_{1/r}$$

in the homotopy category of spectra, where $\mathsf{BSG} = \Phi(\mathscr{B}SG(S^2))$, *$J$ is the complex J-homomorphism, and $(\ )_{1/r}$ is the Bousfield–Kan $\mathbb{Z}[1/r]$-completion functor.*

Proof. Since $\psi^r = \psi^s \circ \psi^t$ whenever $r = st$, it suffices to consider r equal to a prime p. Since $(\mathsf{BSG})_{1/p}$ is naturally equivalent to the product of $(\mathbb{Z}/q)_\infty(\mathsf{BSG})$ for primes $q \neq p$, the naturality of the map $(\ )_{1/p} \to (\mathbb{Z}/q)_\infty(\ )$ implies that it suffices to prove that $(\mathbb{Z}/q)_\infty(J) = (\mathbb{Z}/q)_\infty(J \circ \psi^p)$ for each prime $q \neq p$.

Let $R = \mathbb{Z}/q$ for some prime $q \neq p$. Employing Lemma 10·1 to identify $R_\infty \Phi(\mathscr{B})$ with $R_\infty \Phi(R_\infty(\mathscr{B}))$ for $\mathscr{B} = \mathscr{B}U$ and $\mathscr{B} = \mathscr{B}SG$, we apply Proposition 8·2 to prove that $R_\infty(J) = R_\infty \circ \Phi(R_\infty \mathscr{J})$ equals $R_\infty(J) \circ R_\infty(\psi^p)$. By Proposition 10·2, $R_\infty \psi^p = R_\infty \circ \Phi(\psi^\wedge)$. Consequently, it suffices to prove that

$$R_\infty \mathscr{J} = R_\infty \mathscr{J} \circ \psi^\wedge : R_\infty \mathscr{B}U \to R_\infty \mathscr{B}SG.$$

By Theorem 7·9, it suffices to prove

$$(R_\infty \mathscr{J})^*(R_\infty \overline{\pi}) = (\psi^\wedge)^* \circ (R_\infty \mathscr{J})^* (R_\infty \overline{\pi})$$

in $RS^2(R_\infty \mathscr{B}U)$. Proposition 10.3 implies that $(R\psi^{\text{alg}})^*(\overline{R\tau}^{\text{alg}}) = \overline{R\tau}^{\text{alg}}$ which equals $\Theta^* \circ (R_\infty \mathscr{J})^*(R_\infty \overline{\pi})$ by Theorem 9.4. Consequently, $(R_\infty \mathscr{J})^*(R_\infty \overline{\pi})$ equals

$$(\Theta^{-1})^* \circ (R\psi^{\text{alg}})^* \circ \Theta^* \circ (R_\infty \mathscr{J})^*(R_\infty \overline{\pi}),$$

which equals $(\psi^\wedge)^* \circ R_\infty \mathscr{J}^*(R_\infty \overline{\pi})$ because $\psi^\wedge = \Theta \circ R\psi^{\text{alg}} \circ \Theta^{-1}$.

The following quaternionic analogue of Theorem 10.4 is proved by substituting the symplectic groups $\mathrm{Sp}(2m, \mathbb{C})$ for the linear groups $\mathrm{GL}(i, \mathbb{C})$ and by substituting S^{4i} for S^{2i} in the proof of Theorem 10.4.

In order to define $\psi^r: (\mathsf{kSp})_{1/r} \to (\mathsf{kSp})_{1/r}$ for r even, the reader should use the equivalence $(\mathsf{kSp})_{1/2} \approx (\mathsf{k}\mathscr{O})_{1/2}$.

THEOREM 10·5. *For any integer* $r > 0$,

$$J = J \circ \psi^r: (\mathsf{kSp})_{1/r} \to (\mathsf{BSG})_{1/r}$$

in the homotogy category of spectra, where kSp *is the spectrum of connective qvaternionic* K*-theory and* J *is the quaternionic* J*-homomorphism.*

APPENDIX. R-COMPLETION TECHNICALITIES

In this appendix, we consider various technical properties of the Bousfield–Kan R-completion functor which are employed in Section 7. As in that section, R denotes either subring of the rationals or a prime field and R-completion is the functor

$$R_\infty(\ ): \mathscr{S}_* \to \mathscr{S}_*$$

of (3).

Proposition A 1. Let Y be a pointed, connected, R-good simplicial set. Then the simplicial function space of pointed R-equivalences from Y to $R_\infty(Y)$, $RG(Y)$, admits a natural structure of a sub-simplicial monoid of $G(R_\infty(Y))$, where $G(R_\infty(Y))$ is the simplicial function space of pointed self-equivalences of $R_\infty(Y)$. Moreover, this inclusion is a homotopy equivalence with homotopy inverse given by restriction.

Proof. The inclusion $RG(Y) \to G(R_\infty(Y))$ is given by sending $v: Y \times \Delta[k] \to R_\infty(Y)$ in $RG(Y)_k$ to the composition

$$j(v) = \psi \circ R_\infty(v): R_\infty(Y) \times \Delta[k] \to R^2_\infty(Y) \to R_\infty(Y).$$

Because $\psi \circ R_\infty(i) = id: R_\infty(\ ) \to R_\infty(\ )$ (cf. (8)), the restriction of this map to Y equals v, $v = \psi \circ R_\infty(v) \circ i$ (where $i: id \to R_\infty$ and $\psi: R^2_\infty \to R_\infty$ constitute the monad structure on R_∞). To verify that j is a monoid inclusion, we must check that $j(v) \circ j(w) \in G(R_\infty(Y))_k$ is in the image of j for any pair of k-simplices $v, w \in RG(Y)_k$. We leave to the reader the straightforward verification that $j(v) \circ j(w) = j(v \circ w)$ where

$$v \circ w = \psi \circ R_\infty(v) \circ (w \times 1): Y \times \Delta[k] \to R_\infty(Y) \times \Delta[k] \to R^2_\infty(Y) \to R_\infty(Y)$$

(cf. (8), p. 134).

To verify that the restriction map $G(R_\infty(Y)) \to RG(Y)$ is a homotopy equivalence we employ the fact that both $G(R_\infty(Y))$ and $RG(Y)$ are Kan complexes consisting of certain components of the function complexes $\hom_\cdot(R_\infty(Y), R_\infty(Y))$ and $\hom_\cdot(Y, R_\infty(Y))$. Moreover, $G(R_\infty(Y)) \to RG(Y)$ induces a bijection on π_0 by (3), VII. 2·1. ii). Thus, the homotopy equivalence follows from the Whitehead theorem and the observation that the restriction induces a bijection on connected components of

$$\hom_\cdot(S^n \wedge R_\infty(Y), R_\infty(Y)) \to \hom_\cdot(S^n \times Y, R_\infty(Y))$$

for each $n > 0$ (*loc. cit.*).

Corollary A 2. Let X be a pointed, connected R-good simplicial set. For any $i, j > 0$, the natural map, $R_\infty(X^i) \wedge R_\infty(X^j) \to R_\infty(X^{i+j})$ determines a homomorphism

$$RG(X^i) \times RG(X^j) \to RG(X^{i+j}).$$

Proof. The map $p\colon R_\infty(X^i) \wedge R_\infty(X^j) \to R_\infty(X^{i+j})$ arises from the fact that the natural map

$$R_\infty(X^i) \vee (X^j) \to R_\infty(X^i) \times R_\infty(X^j) \to R_\infty(X^i \times X^j) \to R_\infty(X^{i+j})$$

is trivial, where $R_\infty(X^i) \times R_\infty(X^j) \to R_\infty(X^i \times X^j)$ is the canonical right inverse to the projection map. The pairing $RG(X^i) \times RG(X^j) \to RG(X^{i+j})$ is defined to send

$$v\colon X^i \times \Delta[k] \to R_\infty(X^i), \quad w\colon X^j \times \Delta[k] \to R_\infty(X^j)$$

to
$$p \circ (v \wedge w)\colon X^i \wedge X^j \times \Delta[k] \to R_\infty(X^i) \wedge R_\infty(X^j) \to R_\infty(X^{i+j}).$$

The fact that this pairing is a homomorphism is a simple consequence of the explicit definition of the product in $RG(Y)$ given in the proof of Proposition A 1 (so that for any $v, v' \in RG(X^i)_k$ and $w, w' \in RG(X^j)'_k$,

$$p \circ ((v \circ v') \wedge (w \circ w')) = (p \circ (v \wedge w)) \circ (p \circ (v' \wedge w')).$$

Proposition A 4. Let Y be a finite nilpotent, connected, pointed simplicial set with finitely generated homotopy groups. Let $G(Y)$ denote $\mathrm{Sin}\,\mathscr{G}(|Y|)$, the singular complex of the topological monoid of pointed self-equivalences of $|Y|$, and let $G(Y)_0$ be some connected component of $G(Y)$. There are natural R-equivalences

$$G(Y)_0 \to G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0 \leftarrow RG(Y)_0,$$

where $G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)$ is the function complex of R-equivalences from Y to

$$R_\infty \circ \mathrm{Sin}\,|Y|$$

and where $G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ and $RG(Y)_0$ are connected components of

$$G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)$$

and $RG(Y)$ corresponding to $G(Y)_0$.

Moreover, both $RG(Y)_0$ and $G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ are R-complete.

Proof. The adjointness of the geometric realization functor $|\;|$ and the singular functor $\mathrm{Sin}(\;)$ enables us to interpret $G(Y)$ as the subsimplicial set of the function complex $\hom_\cdot(Y, \mathrm{Sin}\,|Y|)$ consisting of components which contain weak equivalences.

The map $G(Y) \to G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)$ is defined by composition with the natural R-equivalence $\mathrm{Sin}\,|Y| \to R_\infty \circ \mathrm{Sin}\,|Y|$. By (3), VI. 7·1,

$$\pi_n(G(Y)_0) = \pi_n(\mathrm{hom}\,.\,(Y, \mathrm{Sin}\,|Y|)_0)$$
$$\to \pi_n(\mathrm{hom}\,.\,(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0) = \pi_n(G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0)$$

is R-completion of groups, and $G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ is R-complete ((3), VI. 5·4). Because the groups $\pi_n(G(Y)_0)$ are finitely generated (recall that Y is finite) and because the monoid structure on $G(Y)$ implies that $G(Y)_0$ is simple (in particular, nilpotent), we conclude that $G(Y)_0 \to G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ is an R-completion ((3), VI. 5·2). In particular, $G(Y)_0 \to G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ is an R-equivalence.

The map $RG(Y)_0 \to G(Y, R_\infty \circ \mathrm{Sin}\,|Y|)_0$ is defined by composition with the natural equivalence $R_\infty(Y) \to R_\infty(\mathrm{Sin}\,|Y|)$, and is thus an equivalence.

Proposition A 5. Let I be a finite, partial ordering satisfying the condition that $i/I \cap j/I$ has an initial element for every $i, j \in I$. Let $F: I \to \mathscr{S}_*$ be a functor satisfying the condition that for each i.e. I

$$F(i) \to \lim_{i/I} F(j)$$

is surjective, where $i/I \subset I$ consists of j with $i \gneq j$. Then

$$R_\infty(\lim_I F(i)) \to \lim_I R_\infty \circ F(i)$$

is a fibration.

Proof. The map $R_\infty(\lim_I F(i)) \to \lim_I R_\infty \circ F(i)$ is induced by a map of cosimplicial simplicial sets

$$\mathbf{R}(\lim_I F(i)) \to \lim_I \mathbf{R} \circ F(i).$$

Since this map commutes with the R-module structure in each codimension, it suffices to verify that this map is surjective in each codimension (3), X. 4·9 and X. 5·1. This is implied by the assertion that $R(\lim_I F(i)) \to \lim_I R \circ F(i)$ is surjective, where $R(\ )$ is the basic functor which provides the definition of R and thus R_∞.

For each $i \in I$ and each $t \geqslant 0$, choose a set-theoretic inverse

$$\lim_{i/I} F(j)_t \to F(i)_t$$

to the canonical (surjective) map $F(i)_t \to \lim_{i/I} F(j)_t$. For each t-simplex $f(j): \Delta[t] \to F(j)$, let $\tilde{f}(j)$ be the t-simplex of $\lim_I F(i)$ determined by these liftings (for $j' < j$, the j'th component of $\tilde{f}(j)$ is the image of $f(j)$; for other $j'' \neq j$, use the chosen inverses to inductively define the j''th component of $\tilde{f}(j)$).

Let $\{\Sigma r_{i_s} f_s(i);\ i \in I,\ s > 0\}$ be an arbitrary t-simplex of $\lim_I R\,.\,F(i)$. We define the function $l: I \to \mathbb{Z}$ by

$$l(i) = 1 - \sum_{j \in I/i} l(j).$$

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We claim that $\Sigma r_{i_s} f_s(i)$ is the image of the following t-simplex

$$\sum_{\substack{i\in I\\ s>0}} l(i)\, r_{i_s} \tilde{f}_s(i) \in R(\lim_I F(i))_t.$$

We proceed to show that the jth projection of this element, $\mathrm{pr}_j(\sum_{\substack{i\in I\\ s>0}} l(i)\, r_{i_s}\tilde{f}_s(i))$, equals $\sum_{s>0} r_{j_s} f_s(j)$. We observe that $\mathrm{pr}_j(\tilde{f}(k))$ equals $\mathrm{pr}_j(\tilde{f}(i))$, where $i = \mathrm{init}(j,k)$ (the initial element of $j/I \cap k/I$). Consequently,

$$\mathrm{pr}_j(\sum_{\substack{I/j\\ s>0}} l(i)\, r_{i_s}\tilde{f}_s(i) + \sum_{s>0} l(j)\, r_{j_s}\tilde{f}_s(j)) = \sum_{s>0} r_{j_s}\tilde{f}_s(j).$$

Moreover,

$$\mathrm{pr}_j(\sum_{\substack{i\notin I/j\\ i\neq j\\ s>0}} l(i)\, r_{i_s}\tilde{f}_s(i)) = \sum_{\substack{j/I\\ s>0}} m(i)\, r_{i_s}\tilde{f}_s(i),$$

where

$$m(i) = l(i) + \sum_{\substack{i=\mathrm{init}(j,k)\\ i\neq k}} l(k).$$

If $i\in j/I$ is maximal, then any $j'\in I/i$ either factors through j or satisfies $i = \mathrm{init}(j,j')$. Thus,

$$l(i) = 1 - l(j) - \sum_{j'\in I/j} l(j') - \sum_{\substack{i=\mathrm{init}(j,k)\\ i\neq k}} l(k) = -\sum_{i=\mathrm{init}(j,k)} l(k)$$

so that $m(i) = 0$. If $i\in j/I$ is not maximal, then

$$l(i) = 1 - l(j) - \sum_{j'\in I/j} l(j') - \sum_{\substack{j>i'>i\\ \neq\ \ \neq}} m(i') - \sum_{\substack{i=\mathrm{init}(j,k)\\ i\neq k}} l(k) = -\sum_{\substack{i=\mathrm{init}(j,k)\\ i\neq k}} l(k)$$

so that $m(i) = 0$ for all $i\in j/I$. Therefore,

$$\mathrm{pr}_j(\sum_{\substack{i\in I\\ s>0}} l(i)\, r_{i_s}\tilde{f}_s(i)) = \sum_{s>0} r_{j_s} f_s(j)$$

as required.

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