ON DISTRIBUTIVITY IN HIGHER ALGEBRA I: THE UNIVERSAL PROPERTY OF BISPANS

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ABSTRACT. Structures where we have both a contravariant (pullback) and a covariant (pushforward) functoriality that satisfy base change can be encoded by functors out of (∞-)categories of spans (or correspondences). In this paper we study the more complicated setup where we have two pushforwards (an "additive" and a "multiplicative" one), satisfying a distributivity relation. Such structures can be described in terms of bispans (or polynomial diagrams). We show that there exist $(\infty, 2)$-categories of bispans, characterized by a universal property: they corepresent functors out of ∞-categories of spans where the pullbacks have left adjoints and certain canonical 2-morphisms (encoding base change and distributivity) are invertible. This gives a universal way to obtain functors from bispans, which amounts to upgrading "monoid-like" structures to "ring-like" ones. For example, symmetric monoidal ∞-categories can be described as product-preserving functors from spans of finite sets, and if the tensor product is compatible with finite coproducts our universal property gives the canonical semiring structure using the coproduct and tensor product. More interestingly, we encode the additive and multiplicative transfers on equivariant spectra as a functor from bispans in finite G-sets, extend the norms for finite étale maps in motivic spectra to a functor from certain bispans in schemes, and make $\mathrm{Perf}(X)$ for X a spectral Deligne–Mumford stack a functor of bispans using a multiplicative pushforward for finite étale maps in addition to the usual pullback and pushforward maps. Combining this with the polynomial functoriality of K-theory constructed by Barwick, Glasman, Mathew, and Nikolaus, we obtain norms on algebraic K-theory spectra.

CONTENTS

1. Introduction

This paper is the first part of a project aimed at better understanding certain sophisticated ring-like structures that occur in "homotopical mathematics". By this we mean not just the theory of E_∞-rings, where additions and multiplications are indexed over finite sets, but also more exotic structures occuring in equivariant and motivic homotopy theory where operations can be indexed over finite G-sets and finite étale morphisms, respectively. Such structures are also relevant to derived algebraic geometry and algebraic K-theory.

In the present paper we construct equivariant and motivic versions of the canonical semiring structure on a symmetric monoidal ∞-category whose tensor product commutes with finite coproducts.
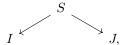
In the G-equivariant case this structure encodes the compatibility of additive and multiplicative transfers (or norms) along maps of finite G-sets. In the case of genuine G-spectra such multiplicative transfers were defined by Hill, Hopkins, and Ravenel [HHR16] (extending a construction on the level of cohomology groups due to Greenlees–May [GM97, Boh14]) and played a key role in their solution of the Kervaire invariant one problem; more recently, they have been considered as the defining structure of an equivariant symmetric monoidal ∞-category in ongoing work of Barwick, Dotto, Glasman, Nardin, and Shah [BDG$^+$16].

In the motivic version, we have multiplicative transfers along finite étale morphisms and additive transfers along smooth morphisms of schemes. Such multiplicative transfers were constructed for motivic spectra (and in a number of related examples) by Bachmann and Hoyois [BaHo21]. These generalize, among other constructions, Fulton–Macpherson's norms on Chow groups [FM87] and Joukhovitski's norms on K_0 [Jou00].

We will also show that the ∞-categories $\mathrm{Perf}(X)$ of perfect quasicoherent sheaves on a spectral Deligne–Mumford stack X have a similar structure given by a multiplicative pushforward for finite étale maps in addition to the usual pushforward and pullback functors. In all these cases we will obtain the canonical "semiring" structures using a universal property of $(\infty, 2)$-categories of *bispans*, which is the main result of this paper.

1.1. **Spans and commutative monoids.** Before we explain what we mean by bispans, it is helpful to first recall the relation between commutative monoids and spans: if $\mathbb{F}$ denotes the category of finite sets, then we can define a (2,1)-category $\mathrm{Span}(\mathbb{F})$ where
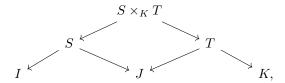
- objects are finite sets,

- morphisms from I to J are *spans* (or correspondences)

$$\begin{array}{ccc} & S & \\ & \swarrow \quad \searrow & \\ I & & J, \end{array}$$

- composition is given by pullback: the composite

$$\left(\begin{array}{ccc} & T & \\ & \swarrow \quad \searrow & \\ J & & K, \end{array} \right) \circ \left(\begin{array}{ccc} & S & \\ & \swarrow \quad \searrow & \\ I & & J, \end{array} \right)$$

is the outer span in the diagram

$$\begin{array}{ccccc} & & S \times_K T & & \\ & \swarrow & & \searrow & \\ & S & & T & \\ \swarrow & & \searrow \quad \swarrow & & \searrow \\ I & & J & & K, \end{array}$$

- 2-morphisms are isomorphisms of spans.

If M is a commutative monoid in Set, we can use the monoid structure to define a functor

$$\mathrm{Span}(\mathbb{F}) \to \mathrm{Set}$$

which takes $I \in \mathbb{F}$ to $M^I := \prod_{i \in I} M$ and a span $I \xleftarrow{f} S \xrightarrow{g} J$ to the composite $g_\otimes f^*$ where $f^* \colon M^I \to M^S$ is given by composition with f (so $f^* \phi(s) = \phi(fs)$) and $g_\otimes$ is defined using the product on M by

$$g_\otimes(\phi)(j) = \prod_{s \in g^{-1}(j)} \phi(s).$$

This is compatible with composition of spans, since a pullback square gives a canonical isomorphism of fibres and we have $(gg')_\otimes = g_\otimes g'_\otimes$ as the multiplication is associative.

It can be shown that if $\mathbf{C}$ is any category with finite products, every functor $\Phi \colon \mathrm{Span}(\mathbb{F}) \to \mathbf{C}$ such that $\Phi(I) \cong \Phi(*)^{\times|I|}$ via the canonical maps arises in this way from a commutative monoid in $\mathbf{C}$. More precisely, we can identify commutative monoids in $\mathbf{C}$ with product-preserving functors $\mathrm{Span}(\mathbb{F}) \to \mathbf{C}$. (In other words, the homotopy category of $\mathrm{Span}(\mathbb{F})$ is precisely the *Lawvere theory* for commutative monoids.) This is also true homotopically:

Theorem 1.1.1. *Let $\mathcal{C}$ be an ∞-category with finite products. There is a natural equivalence of ∞-categories between commutative monoids in $\mathcal{C}$ and product-preserving functors $\mathrm{Span}(\mathbb{F}) \to \mathcal{C}$.*

The earliest proof of this seems to be the in thesis of Cranch [Cra10, Cra11]; other proofs (as special cases of different generalizations) are due to Bachmann–Hoyois [BaHo21, Appendix C] and Glasman [Gla17, Appendix A]. In addition, it appears in Harpaz [Har20, Section 5.2] as the bottom case of his theory of m-commutative monoids.

1.2. **Bispans and commutative semirings.** We can ask for a similar description for commutative semirings. In this case, we have two operations — addition and multiplication — so we want a (2,1)-category $\mathrm{Bispan}(\mathbb{F})$ whose objects are again

finite sets, with a morphism from I to J given by a *bispan* (or polynomial diagram)

$$(1) \qquad \begin{array}{ccc} & X \xrightarrow{\ f\ } Y & \\ {}^{p}\nearrow & & \searrow^{q} \\ I & & J. \end{array}$$

If R is a commutative semiring in Set, we want a functor

$$\mathrm{Bispan}(\mathbb{F}) \to \mathrm{Set}$$

that takes a set I to R^I and the bispan (1) to $q_\oplus f_\otimes p^*$ where
- $p^* \colon R^I \to R^X$ is defined by composing with p,
$$p^*(\phi)(x) = \phi(px),$$
- $f_\otimes \colon R^X \to R^Y$ is defined by multiplying in R fibrewise,
$$f_\otimes(\phi)(y) = \prod_{x \in f^{-1}(y)} \phi(x),$$
- $q_\oplus$ is defined by adding in R fibrewise,
$$q_\oplus(\phi)(j) = \sum_{y \in q^{-1}(j)} \phi(y).$$

The question is then whether there is a way to define composition of bispans so that this gives a functor. Given a pullback square

$$(2) \qquad \begin{array}{ccc} I' & \xrightarrow{\ g\ } & J' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ I & \xrightarrow{\ f\ } & J \end{array}$$

in $\mathbb{F}$, we have identities $g_\otimes i^* = j^* f_\otimes$ and $g_\oplus i^* = j^* f_\oplus$ as before, but now we also need to deal with compositions of the form $v_\otimes u_\oplus$ for $u \colon I \to J$ and $v \colon J \to K$. Using the distributivity of addition over multiplication, for $\phi \colon I \to R$ and $k \in K$ we can write

$$(3) \qquad v_\otimes u_\oplus(\phi)(k) = \prod_{j \in J_k} \sum_{i \in I_j} \phi(i) = \sum_{(i_j) \in \prod_{j \in J_k} I_j} \prod_{t \in J_k} \phi(i_t).$$

We can interpret this in terms of a *distributivity diagram* in $\mathbb{F}$: if we let $h \colon X \to K$ be the family of sets $X_k = \prod_{j \in J_k} I_j$ (so that $h = v_* u$ where v_* is the right adjoint to pullback along v), then the pullback $v^* X$ has a canonical map to I over J: on the fibre $(v^* X)_j$, which is the product $\prod_{j' \in J_{v(j)}} I_{j'}$, we take the projection to the factor I_j. This gives a commutative diagram

$$(4) \qquad \begin{array}{ccc} & v^*X \xrightarrow{\ \tilde{v}\ } X & \\ {}^{\epsilon}\swarrow \ \ \downarrow \ \ {\scriptstyle \lrcorner} & & \downarrow{\scriptstyle h} \\ I & & \\ {}_{u}\searrow \ \ \downarrow & & \downarrow \\ & J \xrightarrow{\ v\ } K & \end{array}$$

where the square is cartesian, and we can rewrite the distributivity relation (3) as

$$v_\otimes u_\oplus = h_\oplus \tilde{v}_\otimes \epsilon^*.$$

This means we will get a functor $\mathrm{Bispan}(\mathbb{F}) \to \mathrm{Set}$ from the commutative semiring R if we define the composition of two bispans

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J,$$
$$J \xleftarrow{u} F \xrightarrow{q} C \xrightarrow{v} K,$$

as the outer bispan in the following diagram:

(5)

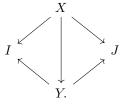Here we have used a distributivity diagram for q and the pullback π.

An explicit construction of a (2,1)-category Bispan($\mathbb{F}$) with this composition is given in the thesis of Cranch [Cra10], where it is also proved that this has products (given by the disjoint union of sets), so that we can define commutative semirings in $\mathcal{S}$ as functors Bispan($\mathbb{F}$) $\to \mathcal{S}$ that preserve finite products. Alternatively, one can relate bispans of finite sets to *polynomial functors*, which gives an easier definition of Bispan($\mathbb{F}$) (as the complicated composition law (5) corresponds to the ordinary composition of such functors); this approach was carried out by Gambino–Kock [GK13], who also show that the homotopy category of Bispan($\mathbb{F}$) is the Lawvere theory for commutative semirings, so that commutative semirings in an ordinary category **C** with finite products are equivalent to product-preserving functors

$$\text{Bispan}(\mathbb{F}) \to \mathbf{C}.$$

We expect that the homotopical analogue of this statement[1] is also true, but this has not yet been proved.

1.3. **The universal property of spans.** If $\mathcal{C}$ is a symmetric monoidal ∞-category such that $\mathcal{C}$ has finite coproducts and the tensor product preserves these in each variable, then we expect that $\mathcal{C}$ has a canonical semiring structure in Cat_∞ with multiplication and addition given by the tensor product and coproduct, respectively. This follows[2] from the universal property of an $(\infty, 2)$-category of bispans in $\mathbb{F}$, which is a special case of our main result. Before we state this, it is convenient to first recall the simpler universal property of the $(\infty, 2)$-category SPAN($\mathbb{F}$) of spans in $\mathbb{F}$, which can be used to prove that an ∞-category with finite coproducts has a canonical symmetric monoidal structure.

Here SPAN($\mathbb{F}$) has finite sets as objects, spans as morphisms, and morphisms of spans as 2-morphisms, i.e. 2-morphisms are commutative diagrams

Suppose $\mathcal{X}$ is an $(\infty, 2)$-category. A functor $\Phi \colon \mathbb{F}^{\text{op}} \to \mathcal{X}$ is called *left adjointable* if for every morphism $f \colon I \to J$ in $\mathbb{F}$, the morphism $f^\circledast := \Phi(f) \colon \Phi(J) \to \Phi(I)$

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[1] Specifically, the definition of commutative semirings in an ∞-category with finite products in terms of Bispan($\mathbb{F}$) should be equivalent to that of Gepner–Groth–Nikolaus [GGN15].

[2] This semiring structure is also constructed in [GGN15] by a different method.

in $\mathfrak{X}$ has a left adjoint $f_\oplus$, and for every pullback square (2) in $\mathbb{F}$, the canonical (Beck–Chevalley or mate) transformation

$$g_\oplus i^\circledast \to j^\circledast f_\oplus$$

is an equivalence.

Theorem 1.3.1. *Restricting along the inclusion $\mathbb{F}^{\mathrm{op}} \to \mathrm{SPAN}(\mathbb{F})$ (of the subcategory containing only the backwards maps and no non-trivial 2-morphisms) gives a natural equivalence between functors $\mathrm{SPAN}(\mathbb{F}) \to \mathfrak{X}$ and left adjointable functors $\mathbb{F}^{\mathrm{op}} \to \mathfrak{X}$.*

This is a special case of a recent result of Macpherson [Mac20], which we review in more generality below in §2.2. Another proof is sketched in the book of Gaitsgory and Rozenblyum [GR17] where this universal property is used to encode a "six-functor formalism" for various categories of quasicoherent sheaves on derived schemes. Lastly, we note that the analogous result for ordinary 2-categories seems to have been first proved by Hermida [Her00, Theorem A.2].

1.4. **The universal property of bispans.** We now want to consider a 2-category $\mathrm{BISPAN}(\mathbb{F})$ whose objects are finite sets, with morphisms given by bispans and 2-morphisms by commutative diagrams of the form

(6)
$$
\begin{array}{ccc}
E & \longrightarrow & B \\
& & \\
I & & J \\
& & \\
E' & \longrightarrow & B'
\end{array}
$$

where the middle square is cartesian. If we look at the subcategory where the morphisms are bispans whose rightmost leg is invertible and with no non-trivial 2-morphisms, we get an inclusion $\mathrm{Span}(\mathbb{F}) \to \mathrm{BISPAN}(\mathbb{F})$. A special case of our main result gives a universal property of $\mathrm{BISPAN}(\mathbb{F})$ in terms of this subcategory:

Theorem 1.4.1. *Let $\mathfrak{X}$ be an $(\infty, 2)$-category. Restricting along the inclusion $\mathrm{Span}(\mathbb{F}) \to \mathrm{BISPAN}(\mathbb{F})$ gives an equivalence between functors $\mathrm{BISPAN}(\mathbb{F}) \to \mathfrak{X}$ and distributive functors $\mathrm{Span}(\mathbb{F}) \to \mathfrak{X}$.*

Here a functor $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathfrak{X}$ is *distributive* if

- for every morphism $f\colon I \to J$ in $\mathbb{F}$, the morphism $f^\circledast := \Phi(J \xleftarrow{f} I = I)$ in $\mathfrak{X}$ has a left adjoint $f_\oplus$,
- for every pullback square (2) in $\mathbb{F}$, the Beck–Chevalley transformation $g_\oplus i^\circledast \to j^\circledast f_\oplus$ is an equivalence,
- for every distributivity diagram (4), the *distributivity transformation*

$$h_\oplus \tilde{v}_\otimes \epsilon^\circledast \to v_\otimes u_\oplus,$$

  which is defined as a certain composite of units and counits, is an equivalence in $\mathfrak{X}$.

Note that the only property of $\mathbb{F}$ we have used in the definition of distributive functors is the existence of distributivity diagrams. These exist in any locally cartesian closed ∞-category, and more generally we can consider triples $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consisting of an ∞-category $\mathcal{C}$ with a pair of subcategories $\mathcal{C}_F$ and $\mathcal{C}_L$ such that

- pullbacks along morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ exist in $\mathcal{C}$, and both subcategories are preserved under base change,
- there exist suitable distributivity diagrams in $\mathcal{C}$ for any composable pair of morphisms $l\colon x \to y$ in $\mathcal{C}_L$, $f\colon y \to z$ in $\mathcal{C}_F$.

We can then generalize the notion of distributive functors above to that of L-*distributive functors* $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$, where $\mathrm{Span}_F(\mathcal{C})$ is the ∞-category of spans in $\mathcal{C}$ whose forward legs are required to lie in $\mathcal{C}_F$. Our main result in this paper is then the following generalization of Theorem 1.4.1:

Theorem 1.4.2. *For* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ *as above, there exists an* $(\infty, 2)$-*category*

$$\mathrm{BISPAN}_{F,L}(\mathcal{C})$$

such that

- *objects are objects of* $\mathcal{C}$,
- *morphisms are bispans*

$$x \xleftarrow{p} e \xrightarrow{f} b \xrightarrow{l} y$$

  where f *is in* $\mathcal{C}_F$ *and* l *is in* $\mathcal{C}_L$,
- *2-morphisms are diagrams of the form* (6),
- *morphisms compose as in* (5).

The $(\infty, 2)$-*category* $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ *has the universal property that restricting to the subcategory* $\mathrm{Span}_F(\mathcal{C})$ *gives for any* $(\infty, 2)$-*category* $\mathcal{X}$ *an equivalence between functors* $\mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{X}$ *and* L-*distributive functors* $\mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$.

The analogue of this result for ordinary 2-categories (at least in the case where $\mathcal{C} = \mathcal{C}_F = \mathcal{C}_L$) is due to Walker [Wal19].

1.5. **Equivariant and algebro-geometric bispans.**

We will now look briefly at some examples of Theorem 1.4.2 beyond the case of finite sets, coming from equivariant and motivic homotopy theory and derived algebraic geometry. These examples are discussed in more detail in §3.

Let us first consider the equivariant setting, over a finite group G. In all of our discussion above it is straightforward to replace the category $\mathbb{F}$ of finite sets with the category $\mathbb{F}_G$ of *finite G-sets*. The analogue of a commutative monoid is then a functor

$$M \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Set}$$

that preserves products. This is (essentially[3]) the same thing as a *Mackey functor* [Dre71], an algebraic structure where for a subgroup $H \subseteq G$ we have restrictions $M^G \to M^H$ and transfers $M^H \to M^G$ satisfying a base change property that can be interpreted in terms of double cosets. Mackey functors play an important role in group theory, and every genuine G-spectrum E has an underlying Mackey functor $\pi_0 E$.

Similarly, the G-analogue of a commutative semiring is a product-preserving functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathrm{Set}$, which is essentially[4] a *Tambara functor* [Tam93, Str12, BlHi18]. This is a structure that has both an "additive" and a "multiplicative" transfer, satisfying a distributivity relation. If E is a genuine G-E_∞-ring spectrum, then $\pi_0 E$ has the structure of a Tambara functor [Bru07].

If we replace the category of sets with the ∞-category of spaces, a theorem of Nardin[5] [Nar16, Corollary A.4.1] shows that connective G-spectra are equivalently

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[3]Mackey functors are usually viewed as taking values in Ab rather than Set; since the functor induces commutative monoid structures on its values, this amounts to asking for these monoid structures to be grouplike. The relation between Mackey functors in Ab and Set is thus analogous to that between abelian groups and commutative monoids.

[4]Again, the usual notion of a Tambara functor takes values in Ab, which gives the equivariant version of a commutative ring rather than a semiring.

[5]Building on the description of G-spectra as "spectral Mackey functors", originally due to Guillou and May [GM17].

product-preserving functors $\mathrm{Span}(\mathbb{F}_G) \to \mathcal{S}$ that are grouplike, generalizing the classical description of connective spectra as grouplike commutative monoids in $\mathcal{S}$.[6]

The analogue for ring spectra is also expected to hold: connective genuine G-E_∞-ring spectra should be equivalent to product-preserving functors $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$.

Now we turn to the "categorified" versions of these structures: For H a subgroup of G, the (additive) transfer from H-spectra to G-spectra is classical[7], but the multiplicative transfer or *norm* was only introduced fairly recently by Hill–Hopkins–Ravenel as part of the foundational setup for [HHR16]. This inspired a plethora of work on equivariant symmetric monoidal structures [HH16, GMMO20, Rub17] and its relation to equivariant homotopy-coherent commutativity (in particular [BlHi15] and subsequent work on N_∞-operads), culminating from our point of view in the approach of Barwick, Dotto, Glasman, Nardin, and Shah [BDG$^+$16], where a G-symmetric monoidal ∞-category can be viewed as a product-preserving functor

$$\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty,$$

i.e. a "categorified Mackey functor".

If the contravariant (restriction) functors have left adjoints that satisfy base change and distributivity, Theorem 1.4.2 allows us to upgrade such G-symmetric monoidal structures to functors from $\mathrm{BISPAN}(\mathbb{F}_G)$, which encodes the distributive compatibility of multiplicative and additive transfers. We will see that this applies in particular to genuine G-spectra, giving a "categorified Tambara functor" structure on G-spectra.

Next, we look at the motivic setting, where it is more instructive to first work in the categorified context. By this we mean Ayoub's construction of a functor from schemes to categories $X \mapsto \mathrm{SH}(X)$ which satisfies a full six functors formalism [Ayo07], vastly expanding Voevodsky's notes in [Voe99]; we also refer the reader to the book of Cisinski and Déglise [CD19] for another exposition, [Hoy17] for an ∞-categorical enhancement of this construction in the more general motivic-equivariant setting, as well as the more recent [DG20] for a universal property of this construction. Here $\mathrm{SH}(X)$ denotes the ∞-category of motivic spectra over a scheme X.

In this context, given a smooth morphism of schemes $f \colon X \to Y$ over a base S, the pullback functor $f^* \colon \mathrm{SH}(Y) \to \mathrm{SH}(X)$ admits a left adjoint, $f_\sharp \colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$. This is a categorified version of the *additive pushforward*: if f is the fold map $\nabla \colon Y^{\amalg I} \to Y$, then $\nabla_\sharp$ computes the I-indexed direct sum. The compatibility of $f_\sharp$ with pullbacks yields a functor

$$(7) \qquad\qquad \mathrm{SH} \colon \mathrm{Span}_{\mathrm{sm}}(\mathrm{Sch}_S) \to \mathrm{Cat}_\infty.$$

An important additional functoriality of SH was recently discovered by Bachmann and Hoyois in [BaHo21]: given a finite étale morphism $f \colon X \to Y$ we have the *multiplicative pushforward* or *norm* $f_\otimes \colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$, which in the case when f is the fold map computes the I-indexed tensor product. This also satisfies base change, and so can be encoded by a functor

$$(8) \qquad\qquad \mathrm{SH} \colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{Sch}_S) \to \mathrm{Cat}_\infty,$$

which leads to the correct notion of a coherent multiplicative structure in motivic homotopy theory — a *normed motivic spectrum* — as a section of the unstraightening of (8) that is cocartesian over the backwards maps in $\mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{Sch}_S)$.

The motivic bispan category should combine these two structures, giving an additive pushforward along smooth morphisms and a multiplicative pushforward

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[6]This can be seen as an ∞-categorical version of more classical descriptions of equivariant infinite loop spaces, cf. [Shi89, Ost16, MMO17, GMMO19].

[7]See e.g. [LMMS86, §II.4]

along finite étale morphisms. For technical reasons (due to the non-existence of Weil restriction of schemes in general), for our motivic bispan categories we either have to restrict to morphisms between schemes that are smooth and quasiprojective or work with algebraic spaces. Thus we consider 2-categories of the form $\mathrm{BISPAN}_{\mathrm{fét,sm}}(\mathrm{AlgSpc}_S)$ where AlgSpc_S means the category of algebraic spaces over S, and we promote SH to a functor

(9) $$\mathrm{SH}\colon \mathrm{BISPAN}_{\mathrm{fét,sm}}(\mathrm{AlgSpc}_S) \to \mathrm{Cat}_\infty;$$

see Theorem 3.5.9.

The decategorification of the above structure has been studied by Bachmann in [Bac21]. Working over a field, and restricting to a category of bispans between smooth schemes, $\mathrm{Bispan}_{\mathrm{fét,sm}}(\mathrm{Sm}_k)$, Bachmann proved that the structure of a normed algebra in the abelian category of homotopy modules (the heart of the so-called homotopy t-structure on motivic spectra) is encoded by certain functors out of this bispan category to abelian groups (appropriately christened *motivic Tambara functors*), at least after inverting the exponential characteristic of k.

We also note that there is a discrepancy with the classical and finite-equivariant story: finite étale transfers are *a priori* not sufficient to encode the structure of a motivic spectrum. Instead, the correct kind of transfers are framed transfers in the sense of [EHK+21]. In particular, the category of framed correspondences, where the backward maps encode framed transfers, is manifestly an ∞-category. In other words, the additive and multiplicative transfers are rather different in the motivic story; for example we do not know if the space of units of a normed motivic spectrum has framed transfers (see [BaHo21, Section 1.5] for a discussion). For us, this means that a more robust theory of bispans in the motivic setting which encodes framed transfers is open for future investigations.

Finally, we consider an example in the context of derived algebraic geometry: If $\mathrm{Perf}(X)$ denotes the ∞-category of perfect quasicoherent sheaves on a spectral Deligne–Mumford stack, Barwick [Bar14] has shown that the pullback and pushforward functors extend to a functor

$$\mathrm{Perf}\colon \mathrm{Span}_{\mathcal{FP}}(\mathrm{SpDM}) \to \mathrm{Cat}_\infty$$

where SpDM is the ∞-category of spectral Deligne–Mumford stacks and $\mathcal{FP}$ is a certain class of morphisms (for which pushforwards preserve perfect objects and base change is satisfied). We promote this to a functor of $(\infty, 2)$-categories

$$\mathrm{BISPAN}_{\mathrm{fét},\mathcal{FP}'}(\mathrm{SpDM})^{\text{2-op}} \to \mathrm{CAT}_\infty,$$

using a multiplicative pushforward functor for finite étale maps (which exists by results of Bachmann–Hoyois [BaHo21]), where $\mathcal{FP}'$ is a certain subclass of $\mathcal{FP}$ for which Weil restrictions exist.

1.6. **Norms in algebraic K-theory.** A combination of the present work and [BGMN21] produces concrete examples of Tambara functors valued in $\mathcal{S}$, the ∞-category of spaces, via (connective) algebraic K-theory, which we discuss in §4. The motivation for our results traces back to classical representation theory: given a finite group G, the classical representation ring of G is a certain Grothendieck ring:

(10) $$\mathrm{Rep}(G, \mathbb{C}) \cong K_0(\mathrm{Fun}(BG, \mathrm{Vect}_{\mathbb{C}}^{\mathrm{fd}})).$$

In particular, the tensor product of representations induces the multiplicative structure on $\mathrm{Rep}(G, \mathbb{C})$. It is natural to consider the formation of representation rings as a functor in G, where the functoriality encodes various operations in representation theory such as induction and restrictions. From this viewpoint, one can enhance

the multiplicative structure on $\mathrm{Rep}(G, \mathbb{C})$ to one parametrized by cosets: if $K \subset G$ is a subgroup, then we have a map given by the operation of *tensor induction*:

$$\mathrm{Rep}(K, \mathbb{C}) \to \mathrm{Rep}(G, \mathbb{C}) \qquad V \mapsto \otimes_{G/K} V$$

As reviewed in Example 3.4.18, all of this functoriality can be encoded as a functor out of a bispan category formed out of finite G-sets.
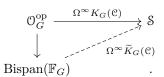
Representation theory with "fancy coefficients" entails replacing the category $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{C}}$ with a more sophisticated symmetric monoidal (∞-)category $\mathcal{C}$. This line of investigation arguably began with the subject of modular representation theory, which takes $\mathcal{C}$ to be $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$. This is especially subtle when G is a p-group because of the failure of the category $\mathrm{Fun}(BG, \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p})$ to be semisimple. More recently, Treumann has also considered replacing vector spaces with KU-module spectra, thus taking $\mathcal{C}$ to be $\mathrm{Perf}_{\mathrm{KU}}$ where KU is the $\mathbb{E}_\infty$-ring spectrum representing complex topological K-theory [Tre15] and suggests that, up to p-completion, representation theory over KU is, in a precise way, a smooth deformation of representation theory over the p-adic integers [Tre15, 1.7].

In light of the last example, which is homotopical in nature, it is natural to consider the K-theory *space* $\Omega^\infty K(\mathrm{Fun}(BG, \mathcal{C}))$ for $\mathcal{C}$ a small stable ∞-category; its homotopy groups are the higher K-groups. Using the universal property of bispans, coupled with the main result of [BGMN21], we offer the following result concerning its functoriality in the variable G:

Theorem 1.6.1. *Let G be a finite group and $\mathcal{C}$ a symmetric monoidal ∞-category, then the presheaf on G-orbits:*

$$\Omega^\infty K_G(\mathcal{C}) \colon \mathcal{O}^{\mathrm{op}}_G \to \mathcal{S} \qquad G/H \mapsto \Omega^\infty K(\mathrm{Fun}(BH, \mathcal{C})),$$

extends canonically as

$$
\begin{array}{ccc}
\mathcal{O}^{\mathrm{op}}_G & \xrightarrow{\ \Omega^\infty K_G(\mathcal{C})\ } & \mathcal{S} \\
\downarrow & \nearrow{\scriptstyle \Omega^\infty \widetilde{K}_G(\mathcal{C})} & \\
\mathrm{Bispan}(\mathbb{F}_G) & &
\end{array}.
$$

The proof of Theorem 1.6.1 is quite simple given our main theorem: we use this to deduce that $\mathrm{Fun}(BG, \mathcal{C})$ upgrades to a functor out of bispans into the ∞-category of ∞-categories. Using a general criterion which we detail in §4.2, we prove that the multiplicative pushforward enjoys the property of being a *polynomial functor* in the sense of Goodwillie calculus. A recent breakthrough of Barwick, Glasman, Mathew and Nikolaus [BGMN21] proves that the formation of algebraic K-theory spaces is functorial in polynomial functors, which then gives Theorem 1.6.1.

We believe that Theorem 1.6.1 could have computational significance. Using an equivariant analog of [BaHo21, Example 7.25], Theorem 1.6.1 produces genuine-equivariant *power operations* on these representation-theoretic gadgets. In particular, the coherence afforded by the language of bispans results in relations between these operations.

By exactly the same methods we also show that the algebraic K-theory of a genuine G-E_∞-ring spectrum is also a Tambara functor valued in $\mathcal{S}$. The latter are expected to be precisely the *connective G-E_∞-ring spectra*, and given this we prove:

Theorem 1.6.2 (see Theorem 4.3.7)**.** *The formation of algebraic K-theory preserves G-E_∞-rings.*

We believe this is a completely new structure on algebraic K-theory in equivariant homotopy theory that significantly extends several recent results in the literature; see Remark 4.3.8.

In the algebro-geometric context the same method also shows that, for instance the algebraic K-theory of schemes (or more generally spectral Deligne–Mumford stacks) has multiplicative transfers along finite étale morphisms.

1.7. **Notation.** This paper is written in the language of ∞-categories. We use the following reasonably standard notation:

- $\mathcal{S}$ is the ∞-category of spaces, i.e. ∞-groupoids.
- Cat_∞ is the ∞-category of ∞-categories.
- CAT_∞ is the $(\infty, 2)$-category of ∞-categories.
- $\mathrm{Cat}_{(\infty,2)}$ is the ∞-category of $(\infty, 2)$-categories.
- If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories or $(\infty, 2)$-categories, we write $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ for the ∞-category of functors from $\mathcal{C}$ to $\mathcal{D}$.
- If $\mathcal{C}$ and $\mathcal{D}$ are $(\infty, 2)$-categories, we write $\mathrm{FUN}(\mathcal{C}, \mathcal{D})$ for the $(\infty, 2)$-category of functors from $\mathcal{C}$ to $\mathcal{D}$.
- We write $(-)^{(1)} \colon \mathrm{Cat}_{(\infty,2)} \to \mathrm{Cat}_\infty$ for the functor taking an $(\infty, 2)$-category to its underlying ∞-category. (Thus $(-)^{(1)}$ is right adjoint to the inclusion of Cat_∞ into $\mathrm{Cat}_{(\infty,2)}$.)
- We write $(-)^{\simeq}$ for the functors $\mathrm{Cat}_\infty \to \mathcal{S}$ and $\mathrm{Cat}_{(\infty,2)} \to \mathcal{S}$ taking an ∞-category or $(\infty, 2)$-category to its underlying ∞-groupoid.
- If $\mathcal{C}$ is an ∞-category and x and y are objects of $\mathcal{C}$, we write $\mathrm{Map}_{\mathcal{C}}(x, y)$ for the space of maps from x to y in $\mathcal{C}$.
- If $\mathcal{C}$ is an $(\infty, 2)$-category and x and y are objects of $\mathcal{C}$, we write $\mathrm{MAP}_{\mathcal{C}}(x, y)$ for the ∞-category of maps from x to y in $\mathcal{C}$.
- If $\mathcal{X}$ is an $(\infty, 2)$-category we write $\mathcal{X}^{1\text{-op}}$ for the $(\infty, 2)$-category obtained by reversing the morphisms in $\mathcal{X}$ and $\mathcal{X}^{2\text{-op}}$ for that obtained by reversing the 2-morphisms.

We also adopt the following standard notation for functors between slices of an ∞-category $\mathcal{C}$:

- If $f \colon x \to y$ is a morphism in $\mathcal{C}$, we have a functor

$$f_! \colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$$

such that $f_!(t \to x) = t \to x \to y$, i.e. given by composition with f.
- If pullbacks along f exist in $\mathcal{C}$ then $f_!$ has a right adjoint

$$f^* \colon \mathcal{C}_{/y} \to \mathcal{C}_{/x}.$$

- If f^* has a further right adjoint, this will be denoted by

$$f_* \colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}.$$

(This right adjoint exists for all f precisely when $\mathcal{C}$ is locally cartesian closed, for example if $\mathcal{C}$ is an ∞-topos.)

2. Bispans and distributive functors

2.1. $(\infty, 2)$-**categories and adjunctions.** Throughout this paper we work with $(\infty, 2)$-categories, and in this section we review some basic results we use from the theory of $(\infty, 2)$-categories, particularly regarding adjunctions. There are several equivalent ways to define these objects and their homotopy theory, including Rezk's Θ_2-spaces [Rez10] and Barwick's 2-fold Segal spaces [Bar05]. We can also view $(\infty, 2)$-categories as ∞-categories enriched in ∞-categories, which can be rigidified to categories strictly enriched in quasicategories (see [Hau15]); the latter is the model used in the papers of Riehl and Verity. We will not review the details of any of these constructions here, as we do not need to refer to any particular model of $(\infty, 2)$-categories in this paper.

We will, however, use the Yoneda lemma for $(\infty, 2)$-categories, which is a special case of Hinich's Yoneda lemma for enriched ∞-categories [Hin20]:

Theorem 2.1.1 (Hinich). *For any* $(\infty, 2)$-*category* $\mathfrak{X}$ *there exists a fully faithful functor of* $(\infty, 2)$-*categories*

$$\mathsf{y}_{\mathfrak{X}} \colon \mathfrak{X} \to \mathrm{FUN}(\mathfrak{X}^{\mathrm{op}}, \mathrm{CAT}_\infty)$$

such that for any functor $\Phi \colon \mathfrak{X}^{\mathrm{op}} \to \mathrm{CAT}_\infty$ *there is a natural equivalence of* ∞-*categories*

$$\Phi(d) \simeq \mathrm{MAP}_{\mathrm{FUN}(\mathfrak{X}^{\mathrm{op}}, \mathrm{CAT}_\infty)}(\mathsf{y}_{\mathfrak{X}}(d), \Phi).$$

Remark 2.1.2. Hinich's work does use a specific model for $(\infty, 2)$-categories, namely a certain definition of enriched ∞-categories specialized to enrichment in Cat_∞. Hinich's definition has been compared to the original one of Gepner–Haugseng [GH15] by Macpherson [Mac21], and for enrichment in Cat_∞ the latter is equivalent to complete 2-fold Segal spaces [Hau15], which in turn is known by work of Barwick and Schommer-Pries [BSP21] to be equivalent to most other approaches to $(\infty, 2)$-categories (including the complicial sets of Verity by recent work of Gagna–Harpaz–Lanari [GHL20]).

Recall that there exists a *universal adjunction*. This is a 2-category ADJ with two objects $-$ and $+$ and generated by 1-morphisms $L \colon \Delta^1 = \{- \to +\} \to \mathrm{ADJ}$ and $R \colon \Delta^1 = \{+ \to -\} \to \mathrm{ADJ}$ such that L is left adjoint to R; see [RV16] for an explicit combinatorial definition of this 2-category. An adjunction in a 2-category can then equivalently be described as a functor from ADJ. This universal property also holds in $(\infty, 2)$-categories, where we can formulate it more precisely as follows:

Theorem 2.1.3 (Riehl–Verity). *Given an* $(\infty, 2)$-*category* $\mathfrak{X}$, *the induced maps of spaces*

$$L^*, \, R^* \colon \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ}, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}),$$

are both inclusions of components whose images are the subspaces

$$\mathrm{Map}^L_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}), \, \mathrm{Map}^R_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X}) \, \subset \, \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathfrak{X})$$

spanned by those functors that are left and right adjoints, respectively.

For details, see [RV16, Theorem 4.3.11 and 4.4.18]. See also [HNP19] for an alternative proof, using the cotangent complex of $(\infty, 2)$-categories.

We need to upgrade this to a statement about $(\infty, 2)$-categories rather than just ∞-groupoids. This follows from the next observation, which identifies the morphisms and 2-morphisms in the $(\infty, 2)$-category of adjunctions using some results from [Hau20]; to state this we need some terminology that will be important throughout the paper:

Definition 2.1.4. Let $\mathcal{X}$ be an $(\infty, 2)$-category and consider a commutative square

$$
\begin{array}{ccc}
x' & \xrightarrow{\ g'\ } & y' \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\
x & \xrightarrow{\ g\ } & y
\end{array}
$$

in $\mathcal{X}$. If g and g' are left adjoints, with corresponding right adjoints h and h', then we can use the units and counits of the adjunctions to define a *mate* (or *Beck–Chevalley*) *transformation* $\xi h' \to h\eta$ as the composite

$$
\xi h' \to hg\xi h' \simeq h\eta g'h' \to h\eta.
$$

We say the square is *right adjointable* if this mate transformation is an equivalence. Dually, if g and g' are right adjoints, with left adjoints f and f', we say the square is *left adjointable* if the mate transformation

$$
f\eta \to f\eta g'f' \simeq fg\xi f' \to \xi f'
$$

is an equivalence.

Proposition 2.1.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be $(\infty, 2)$-categories. A 1-morphism in the $(\infty, 2)$-category $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$, i.e. a natural transformation $\eta \colon F \to G$ of functors $F, G \colon \mathcal{X} \to \mathcal{Y}$, is a right (left) adjoint if and only if*

(1) for every object $x \in \mathcal{X}$, the morphism $\eta_x \colon F(x) \to G(x)$ is a right (left) adjoint in $\mathcal{Y}$,

(2) for every morphism $f \colon x \to x'$ in $\mathcal{X}$, the commutative square

$$
\begin{array}{ccc}
F(x) & \xrightarrow{\ \eta_x\ } & G(x) \\
\downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle G(f)} \\
F(x') & \xrightarrow{\ \eta_{x'}\ } & G(x')
\end{array}
$$

is left (right) adjointable.

Proof. We consider the case of right adjoints; the left adjoint case can be proved similarly, and also follows by duality. Let $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$ denote the $(\infty, 2)$-category of functors from $\mathcal{X}$ to $\mathcal{Y}$ with lax natural transformations as morphisms (see [Hau20, §3] for a precise definition). We can view the natural transformation η as a morphism in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$. By [Hau20, Theorem 4.6] it has a right adjoint here if and only if η_x has a right adjoint in $\mathcal{Y}$ for every $x \in \mathcal{X}$; this right adjoint is given on morphisms by taking mates. If η has a right adjoint ρ in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$ then by uniqueness this is also a right adjoint in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$; hence η must be given objectwise by left adjoints and the naturality squares of ρ are the corresponding mate squares — in particular, these mate squares must commute, so conditions (1) and (2) hold. Conversely, if these conditions hold for η then η has a right adjoint ρ in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$ and the lax naturality squares of ρ actually commute. By [Hau20, Corollary 3.17] this means that ρ is in the image of the canonical functor $\mathrm{FUN}(\mathcal{X}, \mathcal{Y}) \to \mathrm{FUN}(\mathcal{X}, \mathcal{Y})_{\mathrm{lax}}$; moreover, this functor is locally fully faithful so the unit and counit of the adjunction also lie in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$, and so η has a right adjoint in $\mathrm{FUN}(\mathcal{X}, \mathcal{Y})$, as required. $\square$

Notation 2.1.6. Let $\mathcal{X}$ be an $(\infty, 2)$-category. We write $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{ladj}}$ for the locally full sub-$(\infty, 2)$-category of $\mathrm{FUN}(\Delta^1, \mathcal{X})$ whose objects are the morphisms that are left adjoints and whose morphisms are the right adjointable squares. Similarly, we write $\mathrm{FUN}(\Delta^1, \mathcal{X})^{\mathrm{radj}}$ for the sub-$(\infty, 2)$-category of right adjoints and left adjointable squares.

Corollary 2.1.7. *The functors* $L^*, R^* \colon \mathrm{FUN}(\mathrm{ADJ}, \mathfrak{X}) \to \mathrm{FUN}(\Delta^1, \mathfrak{X})$ *identify the* $(\infty, 2)$*-category* $\mathrm{FUN}(\mathrm{ADJ}, \mathfrak{X})$ *with the sub-*$(\infty, 2)$*-categories* $\mathrm{FUN}(\Delta^1, \mathfrak{X})^{\mathrm{ladj}}$ *and* $\mathrm{FUN}(\Delta^1, \mathfrak{X})^{\mathrm{radj}}$, *respectively.*

Proof. We consider the case of left adjoints; the proof for right adjoints is the same. For any $(\infty, 2)$-category $\mathcal{Y}$ we have a natural commutative square

$$
\begin{array}{ccc}
\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\mathrm{ADJ}, \mathfrak{X})) & \overset{\sim}{\longrightarrow} & \mathrm{Map}(\mathrm{ADJ}, \mathrm{FUN}(\mathcal{Y}, \mathfrak{X})) \\
\downarrow{\scriptstyle L^*} & & \downarrow{\scriptstyle L^*} \\
\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\Delta^1, \mathfrak{X})) & \overset{\sim}{\longrightarrow} & \mathrm{Map}(\Delta^1, \mathrm{FUN}(\mathcal{Y}, \mathfrak{X})).
\end{array}
$$

Here Theorem 2.1.3 implies that the right vertical map is a monomorphism of ∞-groupoids with image the components of $\mathrm{Map}(\Delta^1, \mathrm{FUN}(\mathcal{Y}, \mathfrak{X}))$ that correspond to left adjoints in $\mathrm{FUN}(\mathcal{Y}, \mathfrak{X})$. Using Proposition 2.1.5 we can identify these as precisely those in the image of the subspace $\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\Delta^1, \mathfrak{X})^{\mathrm{ladj}})$ under the bottom horizontal equivalence. By the Yoneda Lemma it follows that $L^* \colon \mathrm{FUN}(\mathrm{ADJ}, \mathfrak{X}) \to \mathrm{FUN}(\Delta^1, \mathfrak{X})^{\mathrm{ladj}}$ is an equivalence. $\square$
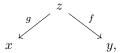
2.2. **Adjointable functors and spans.** In this section we introduce (left and right) adjointable functors and review the universal property of $(\infty, 2)$-categories of spans in terms of these.

Definition 2.2.1. A *span pair* $(\mathcal{C}, \mathcal{C}_F)$ consists of an ∞-category $\mathcal{C}$ together with a wide subcategory $\mathcal{C}_F$ (i.e. one containing all objects and equivalences) such that given morphisms $x \xrightarrow{f} y$ in $\mathcal{C}_F$ and $z \xrightarrow{g} y$ in $\mathcal{C}$, the pullback

$$
\begin{array}{ccc}
x \times_y z & \xrightarrow{f'} & z \\
{\scriptstyle g'}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\
x & \xrightarrow{f} & y
\end{array}
$$

exists in $\mathcal{C}$, and moreover f' is also in $\mathcal{C}_F$. If $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_F)$ are span pairs, then a morphism of span pairs $(\mathcal{C}, \mathcal{C}_F) \to (\mathcal{C}', \mathcal{C}'_F)$ is a functor $\phi \colon \mathcal{C} \to \mathcal{C}'$ such that $\phi(\mathcal{C}_F) \subseteq \mathcal{C}'_F$ and ϕ preserves pullbacks along morphisms in $\mathcal{C}_F$. We write Pair for the ∞-category of span pairs, which can be defined as a subcategory of $\mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$.

Given a span pair $(\mathcal{C}, \mathcal{C}_F)$ we can, as in [Bar14, Section 3], define an ∞-category $\mathrm{Span}_F(\mathcal{C})$ whose objects are the objects of $\mathcal{C}$, with morphisms from x to y given by spans

$$
\begin{array}{ccc}
 & z & \\
{\scriptstyle g}\swarrow & & \searrow{\scriptstyle f} \\
x & & y,
\end{array}
$$

where f is in $\mathcal{C}_F$; we compose spans by taking pullbacks. Following [Hau18, Section 5] we can upgrade this to an $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$ whose 2-morphisms are morphisms of spans, i.e. diagrams

$$
\begin{array}{ccccc}
 & & z & & \\
 & \swarrow & \downarrow & \searrow & \\
x & & & & y \\
 & \searrow & \downarrow & \nearrow & \\
 & & z' & &
\end{array}
$$

in $\mathcal{C}$, where $z \to z'$ can be any morphism in $\mathcal{C}$.

Warning 2.2.2. In [Hau18], the $(\infty, 2)$-category of spans in $\mathcal{C}$ was denoted $\mathrm{Span}_1^+(\mathcal{C})$, while $\mathrm{SPAN}_n(\mathcal{C})$ was used for an n-uple ∞-category of spans.

Remark 2.2.3. In the $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$, every morphism of the form

$$[f]_B := x \xleftarrow{f} z \xrightarrow{\mathrm{id}} z$$

with f in $\mathcal{C}_F$ has a left adjoint, namely the reversed span

$$[f]_F := z \xleftarrow{\mathrm{id}} z \xrightarrow{f} x.$$

The counit is the 2-morphism

and the unit is

where the fibre product $x \times_y x$ is over two copies of f and Δ is the corresponding diagonal.

The $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$ enjoys a universal property: roughly speaking, it is obtained from $\mathcal{C}^{\mathrm{op}}$ by freely adding left adjoints for morphisms in $(\mathcal{C}_F)^{\mathrm{op}}$. To state this more precisely, we need some definitions:

Definition 2.2.4. Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathfrak{X}$ an $(\infty, 2)$-category. A functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \to \mathfrak{X}$ is *left F-preadjointable* if for every morphism $f \colon x \to y$ in $\mathcal{C}_F$ the 1-morphism $f^{\circledast} := \Phi(f) \colon \Phi(y) \to \Phi(x)$ in $\mathfrak{X}$ has a left adjoint $f_{\oplus}$ in $\mathfrak{X}$.

Definition 2.2.5. We say that Φ is *left F-adjointable* if it is left F-preadjointable and for every cartesian square

$$(11) \qquad \begin{array}{ccc} x \times_y z & \xrightarrow{f'} & z \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & y, \end{array}$$

in $\mathcal{C}$ with f in $\mathcal{C}_F$, the commutative square

$$\begin{array}{ccc} \Phi(y) & \xrightarrow{f^{\circledast}} & \Phi(x) \\ {\scriptstyle g^{\circledast}}\downarrow & & \downarrow{\scriptstyle g'^{\circledast}} \\ \Phi(z) & \xrightarrow{f'^{\circledast}} & \Phi(x \times_y z) \end{array}$$

in $\mathfrak{X}$ is left adjointable. We write $\mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathfrak{X})$ for the subspace of $\mathrm{Map}(\mathcal{C}^{\mathrm{op}}, \mathfrak{X})$ whose components are the left F-adjointable functors.

Remark 2.2.6. In other words, Φ is left F-adjointable if for every cartesian square (11) with f in $\mathcal{C}_F$, the Beck–Chevalley transformation

$$(12) \qquad\qquad f'_\oplus g'^\circledast \to g^\circledast f_\oplus$$

is an equivalence.

Theorem 2.2.7 (Gaitsgory-Rozenblyum [GR17], Macpherson [Mac20])**.** *Let* $(\mathcal{C}, \mathcal{C}_F)$ *be a span pair, and let* $\mathcal{X}$ *be an* $(\infty, 2)$-*category. The inclusion of the backwards maps* $\mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ *gives a monomorphism of* ∞-*groupoids*

$$\mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X}) \to \mathrm{Map}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$$

with image $\mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$.

Remark 2.2.8. In [GR17], Gaitsgory-Rozenblyum make use of this universal property of spans in order to encode the functoriality of various ∞-categories of coherent sheaves on derived schemes. They also sketch a proof of Theorem 2.2.7 using a particular construction of $\mathrm{SPAN}_F(\mathcal{C})$. Macpherson [Mac20] has recently given an alternative, model-independent (and complete) proof. Roughly speaking, Macpherson's approach is to first show there exists an $(\infty, 2)$-category that represents left adjointable functors and then use the universal property to prove that this representing object has the expected description in terms of spans. The universal property has also been extended to higher categories of iterated spans by Stefanich [Ste20].

Variant 2.2.9. If $\mathcal{X}$ is an $(\infty, 2)$-category, then we have an equivalence of underlying ∞-categories

$$\mathcal{X}^{(1)} \simeq (\mathcal{X}^{2\text{-op}})^{(1)},$$

while a 1-morphism in $\mathcal{X}$ is a right adjoint if and only if it is a left adjoint in $\mathcal{X}^{2\text{-op}}$. We therefore say that a functor $\Phi\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$ is *right F-adjointable* if the 2-opposite functor

$$\mathcal{C}^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})^{2\text{-op}} \xrightarrow{\Phi^{2\text{-op}}} \mathcal{X}^{2\text{-op}}$$

is left F-adjointable. Theorem 2.2.7 then tells us that the right F-adjointable functors correspond to functors $\mathrm{SPAN}_F(\mathcal{C})^{2\text{-op}} \to \mathcal{X}$.

Variant 2.2.10. We say a functor $\Phi\colon \mathcal{C} \to \mathcal{X}$ is *right F-coadjointable* if the 1-opposite functor

$$\Phi^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}^{\mathrm{op}}$$

is left F-adjointable. Theorem 2.2.7 then tells us that the right F-coadjointable functors correspond to functors $\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op}} \to \mathcal{X}$. We can also combine both variants, and say that $\Phi\colon \mathcal{C} \to \mathcal{X}$ is *left F-coadjointable* if

$$\mathcal{C}^{\mathrm{op}} \simeq \mathcal{C}^{\mathrm{op},2\text{-op}} \xrightarrow{\Phi^{\mathrm{op},2\text{-op}}} \mathcal{X}^{\mathrm{op},2\text{-op}}$$

is left F-adjointable. The left F-coadjointable functors then correspond to functors $\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op},2\text{-op}} \to \mathcal{X}$.

Remark 2.2.11. Unpacking the definition, and recalling that reversing the 1-morphisms in an $(\infty, 2)$-category swaps left and right adjoints, we see that a functor $\Phi\colon \mathcal{C} \to \mathcal{X}$ is right F-coadjointable if for every morphism $f\colon x \to y$ in $\mathcal{C}_F$ the 1-morphism $f_\circledast := \Phi(f)\colon \Phi(x) \to \Phi(y)$ in $\mathcal{X}$ has a right adjoint $f^\oplus$, and for every pullback square (11) the square

$$\begin{array}{ccc}
\Phi(x \times_y z) & \xrightarrow{f'_\circledast} & \Phi(z) \\
{\scriptstyle g'_\circledast}\downarrow & & \downarrow{\scriptstyle g_\circledast} \\
\Phi(x) & \xrightarrow{f_\circledast} & \Phi(y)
\end{array}$$

is right adjointable. Note that this is *not* the same as Φ being right F-adjointable in the previous sense: this condition would involve adjointability for *pushout* squares in $\mathcal{C}$ — indeed, to even be defined this condition would require $(\mathcal{C}^{\mathrm{op}}, \mathcal{C}^{\mathrm{op}}_F)$ to be a span pair, which may well not be the case.

2.3. **The $(\infty, 2)$-category of spans.** For the sake of completeness, in this subsection we include a proof of Theorem 2.2.7. However, our argument is at most a minor variation of the proof of Macpherson [Mac20] and we make no claims to originality. We start by observing that the presentability of $\mathrm{Cat}_{(\infty,2)}$ implies that left F-adjointable functors are corepresented by some $(\infty, 2)$-category:

Proposition 2.3.1. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair, with $\mathcal{C}$ a small ∞-category. Then the functor*

$$\mathrm{Map}_{F\text{-}\mathrm{ladj}}(\mathcal{C}, -) \colon \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}$$

is corepresentable by a small $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$, so that there is a natural equivalence

$$(13) \qquad \mathrm{Map}_{F\text{-}\mathrm{ladj}}(\mathcal{C}, \mathcal{X}) \simeq \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X})$$

for any $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$.

Proof. The ∞-category $\mathrm{Cat}_{(\infty,2)}$ is presentable, for instance because it can be described as presheaves on Θ_2 satisfying Segal and completeness conditions, which gives an explicit presentation as an accessible localization of an ∞-category of presheaves. To prove that a copresheaf on $\mathrm{Cat}_{(\infty,2)}$ is corepresentable it therefore suffices by [Lur09, Proposition 5.5.2.7] to show that it is accessible and preserves limits.

We first show that this holds for the copresheaf $\mathrm{Map}_{F\text{-}\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, -)$ of left F-preadjointable functors. By definition, a functor $\mathcal{C} \to \mathcal{X}$ is left F-preadjointable if it takes every morphism in $\mathcal{X}$ to a left adjoint in $\mathcal{X}$. We can therefore write $\mathrm{Map}_{F\text{-}\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ as the pullback

$$
\begin{array}{ccc}
\mathrm{Map}_{F\text{-}\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) & \longrightarrow & \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \\
\downarrow & & \downarrow {\scriptstyle (f^*)_{f \in S}} \\
\prod_{f \in S} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ}, \mathcal{X}) & \xrightarrow{R^*} & \prod_{f \in S} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1, \mathcal{X})
\end{array}
$$

where the product is over the set S of equivalence classes of morphisms in $\mathcal{C}_F$ and $R \colon \Delta^1 \to \mathrm{ADJ}$ is the inclusion of the right adjoint of the universal adjunction. From this description it is immediate that $\mathrm{Map}_{F\text{-}\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ preserves limits in $\mathcal{X}$, since this is clear for the other three corners of the square. Moreover, since $\mathrm{Cat}_{(\infty,2)}$ is presentable we can choose a regular cardinal κ such that S is κ-small and the objects $\mathcal{C}^{\mathrm{op}}$, ADJ, and Δ^1 are all κ-compact in $\mathrm{Cat}_{(\infty,2)}$. Then we see that $\mathrm{Map}_{F\text{-}\mathrm{lpreadj}}(\mathcal{C}^{\mathrm{op}}, -)$ preserves κ-filtered colimits, since the other corners of the pullback square do so (as κ-filtered colimits in $\mathcal{S}$ commute with κ-small limits, such as our product over S).

For $\mathrm{Map}_{F\text{-}\mathrm{ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ we impose the additional requirement that every cartesian square (11) in $\mathcal{C}$ where the horizontal maps are in $\mathcal{C}_F$ is taken to a left adjointable square in $\mathcal{X}$. By Proposition 2.1.5 the left adjointable squares are the right adjoints in $\mathcal{X}^{\Delta^1}$, so if S' denotes the set of equivalence classes of relevant cartesian squares

in $\mathcal{C}$, we can write $\mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ as a pullback

$$
\begin{array}{ccc}
\mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) & \longrightarrow & \mathrm{Map}_{F\text{-lpreadj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \\
\downarrow & & \downarrow \\
\prod_{S'} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{ADJ} \times \Delta^1, \mathcal{X}) & \xrightarrow{(R \times \mathrm{id})^*} & \prod_{S'} \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\Delta^1 \times \Delta^1, \mathcal{X}).
\end{array}
$$

The same argument as for left F-preadjointable maps now implies that the presheaf $\mathrm{Map}_{F\text{-radj}}(\mathcal{C}, -)$ is also accessible and preserves limits, and hence is corepresentable. □

Remark 2.3.2. The identity of $\mathrm{SPAN}_F(\mathcal{C})$ corresponds under (13) to a left F-adjointable functor

$$
i \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C}),
$$

such that the equivalence (13) arises by restriction along i.

Remark 2.3.3. From the universal property we immediately obtain a functor from span pairs to $(\infty, 2)$-categories: For any $(\infty, 2)$-category $\mathcal{X}$, composition with a morphism of span pairs $\phi \colon (\mathcal{C}, \mathcal{C}_F) \to (\mathcal{C}', \mathcal{C}'_{F'})$ restricts to a morphism

$$
\mathrm{Map}_{F'\text{-ladj}}(\mathcal{C}', \mathcal{X}) \to \mathrm{Map}_{F\text{-ladj}}(\mathcal{C}, \mathcal{X}),
$$

natural in $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$. We obtain a functor

$$
\mathrm{Map}_{(-)\text{-ladj}}(-, -) \colon \mathrm{Pair}^{\mathrm{op}} \times \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}.
$$

Proposition 2.3.1 says that the corresponding functor $\mathrm{Pair}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Cat}_{(\infty,2)}, \mathcal{S})$ takes values in corepresentable copresheaves, and so by the Yoneda lemma factors through a canonical functor $\mathrm{SPAN} \colon \mathrm{Pair} \to \mathrm{Cat}_{(\infty,2)}$.

We can upgrade the equivalence of Proposition 2.3.1 to a statement at the level of $(\infty, 2)$-categories, rather than just ∞-groupoids. To state this we first need some notation:

Definition 2.3.4. Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathcal{X}$ an $(\infty, 2)$-category. We say that a natural transformation $\eta \colon \mathcal{C}^{\mathrm{op}} \times \Delta^1 \to \mathcal{X}$ is *left F-adjointable* if it corresponds to a left F-adjointable functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{FUN}(\Delta^1, \mathcal{X})$. From Proposition 2.1.5 it follows that η is left F-adjointable if and only if the components η_0, η_1 are both left F-adjointable, and for every morphism $f \colon x \to y$ in $\mathcal{C}_F$, the naturality square

$$
\begin{array}{ccc}
\eta_0(y) & \xrightarrow{f^{\circledast}} & \eta_0(x) \\
\downarrow{\scriptstyle \eta_y} & & \downarrow{\scriptstyle \eta_x} \\
\eta_1(y) & \xrightarrow{f^{\circledast}} & \eta_1(x)
\end{array}
$$

is left adjointable. Let $\mathrm{Fun}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ denote the subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ whose objects are the left F-adjointable functors and whose morphisms are the left F-adjointable transformations. From Proposition 2.1.5 we also know that a morphism $\mathcal{C} \times \mathsf{C}_2 \to \mathcal{X}$ (where C_2 is the 2-cell) corresponds to a left F-adjointable morphism $\mathcal{C} \to \mathrm{FUN}(\mathsf{C}_2, \mathcal{X})$ if and only if the component functors and natural transformations are left F-adjointable. We therefore write $\mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ for the locally full sub-$(\infty, 2)$-category of $\mathrm{FUN}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ whose underlying ∞-category is $\mathrm{Fun}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$.

Corollary 2.3.5. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair and $\mathcal{X}$ an $(\infty, 2)$-category. Composition with $i \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ gives an equivalence of $(\infty, 2)$-categories*

$$
\mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X}) \xrightarrow{\sim} \mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}).
$$

Proof. For any $(\infty, 2)$-category $\mathcal{Y}$ we have a natural equivalence

$$\mathrm{Map}(\mathcal{Y}, \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X})) \simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X}))$$
$$\simeq \mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathrm{FUN}(\mathcal{Y}, \mathcal{X}))$$
$$\simeq \mathrm{Map}(\mathcal{Y}, \mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})),$$

where the last equivalence follows from the description of adjoints in functor $(\infty, 2)$-categories in Proposition 2.1.5. $\qquad\square$

Variant 2.3.6. Using analogous notation for right F-coadjointable functors, we have a natural equivalence

$$\mathrm{FUN}_{F\text{-rcoadj}}(\mathcal{C}, \mathcal{X}) \simeq \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op}}, \mathcal{X}).$$

As a first step toward getting a handle on the $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$ we have the following observation:

Lemma 2.3.7. *The functor $i \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ (corresponding to the identity under (13)) is essentially surjective.*

Proof. Let $\mathcal{I}$ denote the full sub-$(\infty, 2)$-category of $\mathrm{SPAN}_F(\mathcal{C})$ spanned by the objects in the image of i. Then i factors through $i' \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{I}$, and i' is again left F-adjointable (since the relevant adjoints and 2-morphisms all live in $\mathcal{I}$). Hence i' corresponds to a functor $\mathrm{SPAN}_F(\mathcal{C}) \to \mathcal{I}$ such that the composite

$$\mathrm{SPAN}_F(\mathcal{C}) \to \mathcal{I} \to \mathrm{SPAN}_F(\mathcal{C})$$

is the identity. It follows that the inclusion of $\mathcal{I}$ must be essentially surjective, which means that i is also essentially surjective. $\qquad\square$

To get a handle on the ∞-categories of morphisms in $\mathrm{SPAN}_F(\mathcal{C})$, we will use two further ingredients: (1) the Yoneda Lemma for $(\infty, 2)$-categories (Theorem 2.1.1) due to Hinich, and (2) the construction of the free (co)cartesian fibrations due to Gepner, Nikolaus, and the second author [GHN17]. Applying the Yoneda lemma to $\mathrm{SPAN}_F(\mathcal{C})$ we get a canonical family of left F-adjointable functors to CAT_∞:

Proposition 2.3.8. *There is a left F-adjointable functor*

$$\mathsf{Y} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{FUN}_{F\text{-rcoadj}}(\mathcal{C}, \mathrm{CAT}_\infty)$$

such that for any right F-coadjointable functor (in the sense of Variant 2.2.10) $\Phi \colon \mathcal{C} \to \mathrm{CAT}_\infty$ there is a natural equivalence

$$\Phi(c) \simeq \mathrm{MAP}_{F\text{-rcoadj}}(\mathsf{Y}(c), \Phi),$$

where the latter denotes the ∞-category of right F-coadjointable natural transformations.

Proof. Applying Hinich's Yoneda embedding (Theorem 2.1.1) to $\mathrm{SPAN}_{F,L}(\mathcal{C})$ we get a functor

$$\mathsf{y} \colon \mathrm{SPAN}_F(\mathcal{C}) \to \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty).$$

By Proposition 2.3.1 this corresponds to a left F-adjointable functor

$$\mathsf{Y} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op}}, \mathrm{CAT}_\infty) \simeq \mathrm{FUN}_{F\text{-rcoadj}}(\mathcal{C}, \mathrm{CAT}_\infty)$$

via the equivalence of Variant 2.3.6. Translating the universal property of representable presheaves through the latter equivalence now gives the result. $\qquad\square$

To proceed further, we will work unstraightened, i.e. with the cocartesian fibrations corresponding to $\mathsf{Y}(c) \colon \mathcal{C} \to \mathrm{Cat}_\infty$.

Lemma 2.3.9. *The straightening equivalence*

$$\mathrm{Fun}(\mathcal{C}, \mathrm{Cat}_\infty) \simeq \mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}}$$

identifies the subcategory $\mathrm{Fun}_{F\text{-rcoadj}}(\mathcal{C}, \mathrm{CAT}_\infty)$ *with a full subcategory*

$$\mathrm{Cat}^{F\text{-rcoadj}}_{\infty/\mathcal{C}} \subseteq \mathrm{Cat}^{\mathrm{cocart}+F\text{-cart}}_{\infty/\mathcal{C}} := \mathrm{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}} \times_{\mathrm{Cat}_{\infty/\mathcal{C}_F}} \mathrm{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}_F},$$

which is the ∞-category of cocartesian fibrations over $\mathcal{C}$ that have cartesian morphisms over $\mathcal{C}_F$, and whose morphisms are functors over $\mathcal{C}$ that preserve cocartesian morphisms as well as cartesian morphisms over $\mathcal{C}_F$.

Proof. By [Lur09, Corollary 5.2.2.4] a cocartesian fibration to $\mathcal{C}$ has cartesian morphisms over $\mathcal{C}_F$ if and only if it has locally cartesian morphisms over $\mathcal{C}_F$, which is equivalent to the corresponding morphisms in Cat_∞ having right adjoints (cf. [Lur09, Definition 5.2.2.1]). Thus the unstraightening of a right F-coadjointable functor gives an object of $\mathrm{Cat}^{\mathrm{cocart}+F\text{-cart}}_{\infty/\mathcal{C}}$. Moreover, by [Lur17, Proposition 4.7.4.17] a morphism over $\mathcal{C}$ that preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_F$ corresponds to a natural transformation whose naturality squares over $\mathcal{C}_F$ are right adjointable, which is precisely the requirement for coadjointable natural transformations. □

Definition 2.3.10. For $c \in \mathcal{C}$, let $\mathcal{Y}_c \to \mathcal{C}$ denote the cocartesian fibration classified by the right F-coadjointable functor $\mathsf{Y}(c) \colon \mathcal{C} \to \mathrm{Cat}_\infty$ that corresponds via Proposition 2.3.8 to the functor $\mathrm{SPAN}_F(\mathcal{C})^{\mathrm{op}} \to \mathrm{CAT}_\infty$ represented by $i(c)$.

The idea is now to construct a "candidate" for $\mathcal{Y}_c$ using a result from [GHN17] together with the following observation:

Lemma 2.3.11. *If $p \colon \mathcal{E} \to \mathcal{C}$ is a functor such that $\mathcal{E}$ has p-cartesian morphisms over morphisms in $\mathcal{C}_F$, then the functor*

$$\mathrm{Fun}^{F\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^F_{/x}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(\{x\}, \mathcal{E}) \simeq \mathcal{E}_x$$

given by restriction along the inclusion

$$\{x\} \simeq \{\mathrm{id}_x\} \hookrightarrow \mathcal{C}^F_{/x},$$

is an equivalence, where $\mathcal{C}^F_{/x}$ denotes the full subcategory of $\mathcal{C}_{/x}$ spanned by morphisms to x in $\mathcal{C}_F$, and $\mathrm{Fun}^{F\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^F_{/x}, \mathcal{E})$ is the full subcategory of $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^F_{/x}, \mathcal{E})$ spanned by functors that preserve cartesian morphisms over $\mathcal{C}_F$.

Proof. We use the results on relative Kan extensions from [Lur09, §4.3.2]. For every object $f \colon y \to x$ of $\mathcal{C}^F_{/x}$ the ∞-category

$$\{x\}_{f/} := \{x\} \times_{\mathcal{C}^F_{/x}} (\mathcal{C}^F_{/x})_{f/} \simeq \mathrm{Map}_{\mathcal{C}^F_{/x}}(f, \mathrm{id}_x)$$

is contractible, since id_x is a terminal object in $\mathcal{C}^F_{/x}$. Hence a morphism

$$\{x\}^\triangleleft_{f/} \simeq \Delta^1 \to \mathcal{E}$$

is a p-limit if and only if it's a cartesian morphism by [Lur09, Example 4.3.1.4]. It follows that a functor $\Phi \colon \mathcal{C}^F_{/x} \to \mathcal{E}$ over $\mathcal{C}$ is a p-right Kan extension from $\{x\}$ if and only if for every $f \colon y \to x$ in $\mathcal{C}_F$ it takes the unique morphism $f \to \mathrm{id}_x$ (which is cartesian over f) to a cartesian morphism in $\mathcal{E}$. By the 3-for-2 property of cartesian morphisms this is equivalent to Φ preserving cartesian morphisms over $\mathcal{C}_F$. Hence [Lur09, Proposition 4.3.2.15] implies that, since $\mathcal{E}$ has p-cartesian morphisms over $\mathcal{C}_F$, the functor $\mathrm{Fun}_{/\mathcal{C}}(\mathcal{C}^F_{/x}, \mathcal{E}) \to \mathrm{Fun}_{/\mathcal{C}}(\{x\}, \mathcal{E})$ restricts to an equivalence from the full subcategory $\mathrm{Fun}^{F\text{-cart}}_{/\mathcal{C}}(\mathcal{C}^F_{/x}, \mathcal{E})$. □

Construction 2.3.12. Since $\mathcal{Y}_c$ corresponds to a right F-coadjointable functor, it has cartesian morphisms over $\mathcal{C}_F$. Applying Lemma 2.3.11 to $\mathcal{Y}_c$ and $x = c$, we see that there is a unique commutative square

$$
\begin{array}{ccc}
\mathcal{C}^F_{/c} & \xrightarrow{\ \alpha_c^-\ } & \mathcal{Y}_c \\
& \searrow \quad \swarrow & \\
& \mathcal{C} &
\end{array}
$$

such that the top horizontal functor preserves cartesian morphisms over $\mathcal{C}_F$ and takes id_c in $\mathc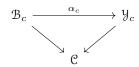al{C}^F_{/c}$ to the identity morphism $\mathrm{id}_{i(c)}$ in $\mathrm{SPAN}_F(\mathcal{C})(i(c), i(c)) \simeq \mathcal{Y}_{c,c}$.

Now since $\mathcal{Y}_c \to \mathcal{C}$ is also a cocartesian fibration, we can extend this to a unique functor from the free cocartesian fibration [GHN17, Theorem 4.5]

$$
(14) \qquad\qquad \mathcal{B}_c := \mathcal{C}^F_{/c} \times_{\mathcal{C}} \mathcal{C}^{[1]} \to \mathcal{C},
$$

giving a unique commutative triangle

$$
\begin{array}{ccc}
\mathcal{B}_c & \xrightarrow{\ \alpha_c\ } & \mathcal{Y}_c \\
& \searrow \quad \swarrow & \\
& \mathcal{C} &
\end{array}
$$

where the horizontal functor preserves cocartesian morphisms and restricts to α_c^- on $\mathcal{C}^F_{/c}$.

Remark 2.3.13. To understand the $(\infty, 2)$-category $\mathrm{SPAN}_F(\mathcal{C})$, we are going to show that the functor α_c is an equivalence. The explicit construction of the ∞-category $\mathcal{B}_c$ in (14) allows us to unpack it easily, revealing the expected definition of spans. We carry this out:

- An object of $\mathcal{B}_c$ consists of an object $x \xrightarrow{f} c$ in $\mathcal{C}^F_{/c}$, i.e. a morphism f to c in $\mathcal{C}_F$, together with a morphism from x in $\mathcal{C}$; in other words, it is precisely a *span*

$$
y \xleftarrow{g} x \xrightarrow{f} c
$$

  with f in $\mathcal{C}_F$. The functor (14) to $\mathcal{C}$ takes this to the object y.
- A morphism from this object to another object

$$
y \xleftarrow{g'} x' \xrightarrow{f'} c,
$$

  in the fibre $\mathcal{B}_{y,c}$ consists of a morphism

$$
\begin{array}{ccc}
x & & \\
\big\downarrow^{\xi} & \searrow^{f} & \\
& & c \\
\big\downarrow & \nearrow_{f'} & \\
x' & &
\end{array}
$$

  in $\mathcal{C}_{/c}$ and a commutative triangle

$$
\begin{array}{ccc}
x & & \\
\big\downarrow^{\xi} & \searrow^{g} & \\
& & y \\
\big\downarrow & \nearrow_{g'} & \\
x' & &
\end{array}
$$

in $\mathcal{C}$, i.e. precisely a morphism of spans

$$
\begin{array}{c}
x \\
g \swarrow \quad \downarrow \xi \quad \searrow f \\
y \qquad \quad \qquad c \\
g' \nwarrow \quad \downarrow \quad \nearrow f' \\
x'.
\end{array}
$$

- Moreover, the cocartesian morphism over $y \xrightarrow{\eta} y'$ in $\mathcal{C}$ is given by composition in $\mathcal{C}^{[1]}$ and so takes the span

$$y \xleftarrow{g} x \xrightarrow{f} c$$

to

$$y' \xleftarrow{\eta g} x \xrightarrow{f} c.$$

To prove that α_c is an equivalence we want to use the universal property of $\mathcal{Y}_c$ (i.e. the Yoneda lemma) to produce a functor $\beta_c \colon \mathcal{Y}_c \to \mathcal{B}_c$, which will be the inverse of α_c. This requires knowing the following:

Proposition 2.3.14. *The functor $B_c \colon \mathcal{C} \to \mathrm{Cat}_\infty$ classifying the cocartesian fibration $\mathcal{B}_c \to \mathcal{C}$ is right F-coadjointable.*

Proof. For $f \colon x \to y$ in $\mathcal{C}$, let us denote the value of B_c at f by

$$f_\circledast \colon \mathcal{B}_{x,c} \to \mathcal{B}_{y,c};$$

here $\mathcal{B}_{x,c}$ can be identified with the fibre product $\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}^F_{/c}$, and the functor $f_\circledast$ is given by composing with f in the first factor, i.e.

$$x \xleftarrow{g} z \to c \quad \mapsto \quad y \xleftarrow{f \circ g} z \to c.$$

We first prove that the functor $\mathcal{B}_c \to \mathcal{C}$ is cartesian over $\mathcal{C}_F$. In other words, we must show that for every morphism $\phi \colon y' \to y$ in $\mathcal{C}_F$, the functor

$$\phi_\circledast \colon \mathcal{B}_{y',c} \to \mathcal{B}_{y,c},$$

has a right adjoint, which we will denote by $\phi^\oplus$. To see that $\phi_\circledast$ has a right adjoint it suffices (by a reformulation of [Lur09, Lemma 5.2.4.1]) to show that for any span $\sigma = (y \xleftarrow{g} x \xrightarrow{f} c)$ (f in $\mathcal{C}_F$), the ∞-category $\mathcal{B}_{y',c/\sigma} := \mathcal{B}_{y',c} \times_{\mathcal{B}_{y,c}} \mathcal{B}_{y,c/\sigma}$ has a terminal object.

An object of $\mathcal{B}_{y',c/\sigma}$ is a commutative diagram

(15)
$$
\begin{array}{ccc}
y' & \xleftarrow{g'} & x' \\
\downarrow{\phi} & & \downarrow \quad \searrow^{f'} \\
& & \qquad c \\
\downarrow & & \downarrow \quad \nearrow_{f} \\
y & \xleftarrow{g} & x.
\end{array}
$$

More formally, we can identify $\mathcal{B}_{y,c/\sigma}$ with the full subcategory $\mathcal{C}_{/x} \times_{\mathcal{C}_{/c}} \mathcal{C}^F_{/c}$ of $\mathcal{C}_{/x}$ spanned by morphisms $x' \to x$ such that $x' \to x \to c$ is in $\mathcal{C}_F$. The fibre product $\mathcal{B}_{y',c/\sigma}$ we can then identify with the full subcategory of $\mathcal{C}_{/p}$, where p is the diagram $y' \to y \leftarrow x$, spanned by commutative squares

$$
\begin{array}{ccc}
x' & \longrightarrow & x \\
\downarrow & & \downarrow \\
y' & \longrightarrow & y
\end{array}
$$

such that the composite $x' \to x \to c$ lies in $\mathcal{C}_F$. A terminal object in $\mathcal{C}_{/p}$ is precisely a fibre product $x \times_y y'$, which exists since by assumption $\mathcal{C}$ admits all pullback along ϕ. Moreover, this terminal object lies in the full subcategory $\mathcal{B}_{y',c/\sigma}$ since the projection $x \times_y y' \to x$ is a base change of ϕ and so lies in $\mathcal{C}_F$.
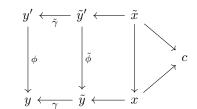
To complete the proof we must show that given a pullback square

$$(16) \qquad \begin{array}{ccc} \tilde{y}' & \xrightarrow{\tilde{\phi}} & \tilde{y} \\ \downarrow{\scriptstyle\tilde{\gamma}} & & \downarrow{\scriptstyle\gamma} \\ y' & \xrightarrow{\phi} & y \end{array}$$

in $\mathcal{C}$ with ϕ in $\mathcal{C}_F$, the Beck-Chevalley transformation

$$\tilde{\gamma}_{\circledast} \tilde{\phi}^{\oplus} \to \phi^{\oplus} \gamma_{\circledast}$$

is an equivalence. Evaluating at a span $\tilde{y} \leftarrow x \to c$ this transformation is given by the canonical dashed map in the diagram

where the two coloured squares are cartesian. Since the square (16) is by assumption cartesian, it follows from the pasting lemma for pullback squares that this is indeed an equivalence. $\qquad\square$

Translating Proposition 2.3.8 through the equivalence of Lemma 2.3.9, we see that Proposition 2.3.14 implies that any object $X \in \mathcal{B}_c$ over $c' \in \mathcal{C}$ corresponds to a morphism $\mathcal{Y}_{c'} \to \mathcal{B}_c$. In particular, we have:

Corollary 2.3.15. *There is a canonical functor $\beta_c \colon \mathcal{Y}_c \to \mathcal{B}_c$ over $\mathcal{C}$ corresponding to the identity span of c; this preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_F$.* $\qquad\square$

We now need to prove that the functor α_c has the same property:

Proposition 2.3.16. *The functor $\alpha_c \colon \mathcal{B}_c \to \mathcal{Y}_c$ over $\mathcal{C}$ preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_F$.*

Remark 2.3.17. For the proof we first need to discuss the naturality of Beck–Chevalley transformations in the following situation: Suppose we have a commutative triangle of ∞-categories

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad F \quad} & \mathcal{F} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{B}, & \end{array}$$

where p and q have both cartesian and cocartesian morphisms over $f \colon a \to b$ in $\mathcal{B}$, but F does not necessarily preserve these. We write $f_!$ for the cocartesian pushforward and f^* for the cartesian pullback along f for both p and q (so the functor $f_!$ is left adjoint to f^*). Then we can make the following commutative diagrams for $x \in \mathcal{E}_a$, $y \in \mathcal{E}_b$:

$$(17) \qquad
\begin{array}{ccc}
Fx & \longrightarrow & Ff^*f_!x \\
\downarrow & & \downarrow \\
f^*f_!Fx & \longrightarrow & f^*Ff_!x \\
\downarrow & & \downarrow \\
f_!Fx & \longrightarrow & Ff_!x,
\end{array}
\qquad
\begin{array}{ccc}
Ff^*y & \longrightarrow & f^*Fy \\
\downarrow & & \downarrow \\
f_!Ff^*y & \longrightarrow & f_!f^*Fy \\
\downarrow & & \downarrow \\
Ff_!f^*y & \longrightarrow & Fy.
\end{array}
$$

In particular, the top left and bottom right squares here encode the compatibility of F with the units and counits of the two adjunctions $f_! \dashv f^*$. Now suppose we have a commutative square

$$
\begin{array}{ccc}
a' & \xrightarrow{\ f'\ } & b' \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
a & \xrightarrow{\ f\ } & b,
\end{array}
$$

where p and q have cocartesian morphisms over f, f' and both cartesian and co-cartesian morphisms over g, g'. Then we claim that the two Beck–Chevalley transformations $f'_! g'^* \to g^* f_!$, intertwined by F, are related by a commutative diagram

$$(18) \qquad
\begin{array}{ccccc}
 & & f'_! F(g'^*X) & & \\
 & \swarrow & & \searrow & \\
f'_! g'^* F(X) & & & & F(f'_! g'^* X) \\
\downarrow & & & & \downarrow \\
g^* f_! FX & & & & F(g^* f_! X) \\
 & \searrow & & \swarrow & \\
 & & g^* F(f_! X). & &
\end{array}
$$

This can be extracted from the following diagram, where we have used (17) together with naturality:

$$f'_! F g'^* X$$

$$f'_! g'^* F X \qquad\qquad F f'_! g'^* X$$

$$g^* g_! f'_! g'^* F X \longleftarrow g^* g_! f'_! F g'^* X$$

$$g^* f_! g'_! g'^* F X \longleftarrow g^* f_! g'_! F g'^* X \qquad\qquad g^* g_! F f'_! g'^* X \quad F g^* g_! f'_! g'^* X$$

$$g^* f_! F X \longleftarrow g^* f_! F g'_! g'^* X \qquad\qquad g^* F g_! f'_! g'^* X \quad F g^* f_! g'_! g'^* X$$

$$g^* F f_! g'_! g'^* X \qquad\qquad F g^* f_! X$$

$$g^* F f_! X.$$

Proof of Proposition 2.3.16. The universal property we used to define α_c implies that it preserves cocartesian morphisms. Moreover, since α_c was extended from a functor $\mathcal{C}^F_{/c} \to \mathcal{Y}_c$ that preserved cartesian morphisms over $\mathcal{C}_F$, we know α_c preserves cartesian morphisms in the image of $\mathcal{C}^F_{/c}$. In other words, for $f: x \to c$ in $\mathcal{C}_F$, the map $\alpha_c(\phi^\oplus [h]_F) \to \phi^\oplus \alpha_c([h]_F)$ is an equivalence for all $\phi: x' \to x$ in $\mathcal{C}_F$.

More generally, for a span $\sigma = (y \xleftarrow{g} x \xrightarrow{f} c)$, we need to show that $\alpha_c(\phi^\oplus \sigma) \simeq \phi^\oplus \alpha_c(\sigma)$ for any morphism $\phi: y' \to y$ in $\mathcal{C}_F$. To proceed, let us view σ as $g_\circledast [f]_F$.

Forming the pullback square

$$\begin{array}{ccc} x' & \xrightarrow{\xi} & x \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ y' & \xrightarrow{\phi} & y, \end{array}$$

the Beck-Chevalley transformation yields an equivalence:

$$g'_\circledast \xi^\oplus [f]_F \xrightarrow{\sim} \phi^\oplus g_\circledast [f]_F.$$

Moreover, from (18) we get a natural commutative diagram

$$g'_\circledast \alpha_c(\xi^\oplus [f]_F)$$

$$\text{(1)} \qquad\qquad \text{(2)}$$

$$g'_\circledast \xi^\oplus \alpha_c([f]_F) \qquad\qquad \alpha_c(g'_\circledast \xi^\oplus [f]_F)$$

$$\text{(3)} \downarrow \qquad\qquad \downarrow \text{(3)}$$

$$\phi^\oplus g_\circledast \alpha_c([f]_F) \qquad\qquad \alpha_c(\phi^\oplus g_\circledast [f]_F)$$

$$\text{(2)} \qquad\qquad$$

$$\phi^\oplus \alpha_c(g_\circledast [f]_F),$$

where the map labelled (1) is an equivalence since α_c preserves cartesian morphisms from $\mathcal{C}^F_{/c}$, those labelled (2) are equivalences since α_c preserves cocartesian

morphisms, and those labelled (3) are equivalences because the Beck–Chevalley transformations are invertible. Hence the last morphism in the diagram is also an equivalence, which shows that α_c preserves the cartesian morphism $\phi^{\oplus} g_{\circledast}[f]_F \to g_{\circledast}[f]_F$. $\qquad\square$

Corollary 2.3.18. *The functors* $\beta_c \colon \mathcal{Y}_c \to \mathcal{B}_c$ *and* $\alpha_c \colon \mathcal{B}_c \to \mathcal{Y}_c$ *satisfy*

$$\beta_c \alpha_c \simeq \mathrm{id}_{\mathcal{B}_c}, \qquad \alpha_c \beta_c \simeq \mathrm{id}_{\mathcal{Y}_c}.$$

Thus α_c *is an equivalence with inverse* β_c.

Proof. By construction α_c takes the identity span of c to

$$\mathrm{id}_c \in \mathcal{Y}_{c,c} \simeq \mathrm{MAP}_{\mathrm{SPAN}_{F,L}(\mathcal{C})}(i(c), i(c)).$$

The composite $\beta_c \alpha_c$ is a functor $\mathcal{B}_c \to \mathcal{B}_c$ that preserves cocartesian morphisms, hence it is determined by its restriction to $\mathcal{C}^F_{/c} \to \mathcal{B}_c$. This restriction preserves cartesian morphisms over $\mathcal{C}_F$ and so by Lemma 2.3.11 it is determined by its value at id_c, which is the identity span in $\mathcal{B}_c$. The same holds for the identity of $\mathcal{B}_c$ and so $\mathrm{id}_{\mathcal{B}_c} \simeq \beta_c \alpha_c$. Conversely, $\alpha_c \beta_c$ is a functor $\mathcal{Y}_c \to \mathcal{Y}_c$ that preserves cocartesian morphisms and cartesian morphisms over $\mathcal{C}_F$ by Proposition 2.3.16. By Proposition 2.3.8, interpreted in terms of fibrations, this functor is determined by where it sends the identity of c; since we know this is taken to itself, this functor must be the identity $\mathrm{id}_{\mathcal{Y}_c}$. $\qquad\square$

The equivalence $\mathcal{Y}_c \simeq \mathcal{B}_c$ allows us to identify morphisms in $\mathrm{SPAN}_F(\mathcal{C})$ with spans, and 2-morphisms with morphisms of spans. We now check that composition of spans works as expected:

Proposition 2.3.19. *Composition of spans in* $\mathrm{SPAN}_F(\mathcal{C})$ *is given by taking pullbacks.*

Proof. By construction, the cocartesian morphisms in $\mathcal{Y}_c$ encode precomposition with the images of morphisms in $\mathcal{C}^{\mathrm{op}}$ under the functor i: Given a morphism $g \colon x \to y$ in $\mathcal{C}$ we have

$$\sigma \circ i(g) \simeq g_{\circledast} \sigma$$

for any span σ. In particular, from our description of the right-hand side we have

$$i(g) \simeq \mathrm{id}_x \circ i(g) \simeq g_{\circledast}(\mathrm{id}_x) \simeq (y \xleftarrow{g} x = x) \simeq [g]_B,$$

and more generally

$$x \xleftarrow{g} y \xrightarrow{f} z \simeq g_{\circledast}[f]_F \simeq [f]_F \circ [g]_B.$$

This means that to describe an arbitrary composition in $\mathrm{SPAN}_F(\mathcal{C})$ it suffices (by associativity of composition) to understand compositions of the form $[g]_B \circ [f]_F$. Note that $[f]_B$ is in the image of $\mathcal{C}^{\mathrm{op}}_F$ under i, and therefore admits a left adjoint in $\mathrm{SPAN}_F(\mathcal{C})$ since i is left F-adjointable; let us denote this by $[f]^{\ell}_B$. We claim that

$$[f]^{\ell}_B \simeq [f]_F.$$

To see this, we note that precomposition with $[f]_B$ is the functor $f_{\circledast}$, which admits a right adjoint $f^{\oplus}$. Therefore, by uniqueness of adjoints we conclude that precomposition with $[f]^{\ell}_B$ must coincide with $f^{\oplus}$. Therefore the span $[f]^{\ell}_B$ is computed as

$$[f]^{\ell}_B \simeq f^{\oplus}(\mathrm{id}) \simeq [f]_F.$$

Furthermore,

$$[g]_B \circ [f]_F \simeq [g]_B \circ [f]^{\ell}_B \simeq f^{\oplus}[g]_B,$$

which we saw above is computed by taking the pullback of g along f, as required. $\qquad\square$

Proposition 2.3.20. *A span* $\sigma = (y \xleftarrow{g} x \xrightarrow{f} z)$ *is invertible as a morphism in* $\mathrm{SPAN}_F(\mathcal{C})$ *if and only if the components* f *and* g *are both invertible in* $\mathcal{C}$.

Proof. Since we now know composition is given by taking pullbacks, this follows as in the proof of [Hau18, Lemma 8.2]. $\square$

Corollary 2.3.21. *The functor* $i\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ *gives an equivalence on underlying* ∞-*groupoids*

$$\mathcal{C}^{\mathrm{op},\simeq} \xrightarrow{\sim} \mathrm{SPAN}_F(\mathcal{C})^{\simeq}.$$

Proof. We know the functor i is essentially surjective on objects, so it is enough to show that for any objects x, y the map

$$\mathrm{Map}_{\mathcal{C}}(x,y)^{\mathrm{eq}} \to \mathrm{Map}_{\mathrm{Span}_F(\mathcal{C})}(ix, iy)^{\mathrm{eq}}$$

is an equivalence, where we are taking the components of the mapping spaces that correspond to equivalences. This is immediate from Proposition 2.3.20 and our description of the mapping spaces in $\mathrm{Span}_F(\mathcal{C})$. $\square$

Combining the results of this section, we have shown:

Theorem 2.3.22. *Let* $(\mathcal{C}, \mathcal{C}_F)$ *be a span triple. Then left* F-*adjointable functors out of* $\mathcal{C}^{\mathrm{op}}$ *are corepresented by an* $(\infty, 2)$-*category* $\mathrm{SPAN}_F(\mathcal{C})$ *via a left* F-*adjointable functor* $i\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{SPAN}_F(\mathcal{C})$ *with the following properties:*

(i) i *gives an equivalence*

$$\mathrm{SPAN}_F(\mathcal{C})^{\simeq} \simeq \mathcal{C}^{\simeq}$$

on underlying ∞-*groupoids,*

(ii) morphisms from $i(x)$ *to* $i(y)$ *can be identified with spans*

$$y \xleftarrow{g} x \xrightarrow{f} z$$

where f *is in* $\mathcal{C}_F$,

(iii) 2-morphisms correspond to morphisms of spans diagrams,

(iv) and composition of morphisms is given by taking pullbacks. $\square$

We end this section by deducing a description of the functor of $(\infty, 2)$-categories corresponding to a left F-adjointable functor:

Proposition 2.3.23. *For a left* F-*adjointable functor* $\phi\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{X}$, *the corresponding functor* $\Phi\colon \mathrm{SPAN}_F(\mathcal{C}) \to \mathcal{X}$ *can be described as follows:*

(1) On objects, $\Phi(c) \simeq \phi(c)$ *for* $c \in \mathcal{C}$.

(2) On morphisms, Φ *takes a span*

$$\sigma = (y \xleftarrow{g} x \xrightarrow{f} z)$$

to the composite $f_{\oplus} g^{\circledast}\colon \phi(x) \to \phi(y)$, *where* $f_{\oplus}$ *is the left adjoint to* $f^{\circledast} := \phi(f)$.

(3) On 2-morphisms, Φ *takes the 2-morphism* $\beta\colon \sigma \to \sigma'$ *given by the commutative diagram*

to the composite

$$(19) \qquad f_{\oplus} g^{\circledast} \simeq f_{\oplus} h^{\circledast} g'^{\circledast} \to f_{\oplus} h^{\circledast} f'^{\circledast} f'_{\oplus} g'^{\circledast} \simeq f_{\oplus} f^{\circledast} f'_{\oplus} g'^{\circledast} \to f'_{\oplus} g'^{\circledast},$$

where the first noninvertible arrow is an adjunction unit and the second noninvertible arrow is a counit.

Proof. We know that $\Phi \circ i \simeq \phi$ and that i is an equivalence on underlying ∞-groupoids by Corollary 2.3.21, which gives (1)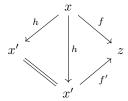. To prove (2), observe that the bispan σ is the composite $[f]_F \circ [g]_B$ in $\mathrm{SPAN}_F(\mathcal{C})$. Here $[g]_B$ is $i(g)$, and so

$$g^\circledast := \Phi([g]_B) \simeq \Phi(i(g)) \simeq \phi(g).$$

Moreover, the span $[f]_F$ is left adjoint to $[f]_B$, hence its image $\Phi([f]_F)$ is the left adjoint $f_\oplus$ to $f^\circledast$. In other words, we have $\Phi(\sigma) \simeq \Phi([f]_F) \circ \Phi([g]_B) \simeq f_\oplus g^\circledast$. To prove (3), first observe that the 2-morphism β is the composite ("whiskering") of the morphism $[g']_B$ with the 2-morphism λ given by
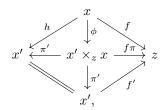

and this whiskering corresponds to the first equivalence in (19).

It thus suffices to show that Φ takes λ to the composite

$$f_\oplus h^\circledast \to f_\oplus h^\circledast f'^\circledast f'_\oplus \simeq f_\oplus f^\circledast f'_\oplus \to f'_\oplus$$

using the unit for $f'_\oplus \dashv f'^\circledast$ and the counit for $f_\oplus \dashv f^\circledast$. To show this we will check that the morphism λ has the corresponding decomposition in $\mathrm{SPAN}_F(\mathcal{C})$. Indeed, we can decompose λ as the composite

where π, π' are the projections from $x' \times_z x$ to x and x', respectively, and ϕ is the unique morphism such that $\pi\phi \simeq \mathrm{id}_x$, $\pi'\phi \simeq h$. Now unpacking the description of units and counits in $\mathrm{SPAN}_F(\mathcal{C})$ implies that the top morphism in this decomposition is the composite of the unit for $[f']_F \dashv [f']_B$ with the morphism $x' \xleftarrow{h} x \xrightarrow{f} z$ and the bottom is the composite of $[f']_F$ with the counit for $[f]_F \dashv [f]_B$. $\square$

2.4. Distributive functors.

We now start our discussion of distributivity. In this section we introduce the notion of a distributive functor, which we will prove in the next subsection is corepresented by $(\infty, 2)$-categories of bispans. For the definition we first need to introduce the notion of a distributivity diagram, which dictates how the multiplicative and additive pushforwards for a distributive functor should interact:

Definition 2.4.1. Let $x \xrightarrow{l} y \xrightarrow{f} z$ be morphisms in an ∞-category $\mathcal{C}$. A *distributivity diagram* for l and f is a commutative diagram

(20)

where the square is cartesian, with the property that for any morphism $\phi\colon u \to z$, the composite map

$$(21) \qquad \mathrm{Map}_{/z}(\phi, g) \to \mathrm{Map}_{/y}(f^*\phi, \tilde{g}) \xrightarrow{\epsilon_*} \mathrm{Map}_{/y}(f^*\phi, l)$$

is an equivalence. The distributivity diagram for l and f is necessarily unique if it exists.

Remark 2.4.2. Consider the ∞-category of diagrams of shape (20) (with the square cartesian). If all pullbacks along f exist in $\mathcal{C}$, then this is equivalent an object of the fibre product of ∞-categories

$$\mathcal{C}_{/z} \times_{\mathcal{C}_{/y}} \mathcal{C}_{/x},$$

with the functors in the pullback being $f^*\colon \mathcal{C}_{/z} \to \mathcal{C}_{/y}$ and $l_!\colon \mathcal{C}_{/x} \to \mathcal{C}_{/y}$. The universal property of the distributivity diagram can then be reformulated as that of being a terminal object in this ∞-category.

Definition 2.4.3. A *bispan triple* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consists of an ∞-category $\mathcal{C}$ together with two subcategories $\mathcal{C}_F, \mathcal{C}_L$ such that the following assumptions hold:

(a) $(\mathcal{C}, \mathcal{C}_F)$ is a span pair.
(b) $(\mathcal{C}, \mathcal{C}_L)$ is a span pair.
(c) For $l\colon x \to y$ in $\mathcal{C}_L$ and $f\colon y \to z$ in $\mathcal{C}_F$ there exists a distributivity diagram (20) where g is in $\mathcal{C}_L$ (and hence $\tilde{f}$ is in $\mathcal{C}_F$ by (a) and $\tilde{g}$ is in $\mathcal{C}_L$ by (b)).

Notation 2.4.4. For $x \in \mathcal{C}$ we write $\mathcal{C}_{/x}^L$ for the full subcategory of $\mathcal{C}_{/x}$ spanned by morphisms $y \to x$ in $\mathcal{C}_L$.

Remark 2.4.5. For a fixed $f\colon y \to z$ in $\mathcal{C}_F$, if the distributivity diagram (20) exists for all l in $\mathcal{C}_L$ then the functor $f^*\colon \mathcal{C}_{/z}^L \to \mathcal{C}_{/y}^L$ given by pullback along f has a right adjoint f_*. Indeed, from (21) we see that we have

$$(w \xrightarrow{g} z) \simeq f_*(x \xrightarrow{l} y)$$

which determines the rest of the diagram. Note, however, that if the ∞-category $\mathcal{C}_L$ is not all of $\mathcal{C}$ then the property of (21) is slightly stronger than the existence of the right adjoint: for this to exist it suffices to consider maps from ϕ in $\mathcal{C}_{/z}^L$, while (21) asks for an equivalence on maps from any ϕ in $\mathcal{C}_{/z}$. We can characterize the additional assumption on these right adjoints needed to have a bispan triple in terms of a base change property:

Lemma 2.4.6. *Suppose we have a triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ consisting of an ∞-category $\mathcal{C}$ together with two subcategories $\mathcal{C}_F, \mathcal{C}_L$ such that*

(a) *$(\mathcal{C}, \mathcal{C}_F)$ is a span pair,*
(b) *$(\mathcal{C}, \mathcal{C}_L)$ is a span pair,*
(c) *for any $f\colon x \to y$ in $\mathcal{C}_F$ the functor $f^*\colon \mathcal{C}_{/y}^L \to \mathcal{C}_{/x}^L$ given by pullback along f has a right adjoint f_*.*

Then $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple if and only if for every cartesian square

$$
\begin{array}{ccc}
x' & \xrightarrow{f'} & y' \\
\xi \downarrow & & \downarrow \eta \\
x & \xrightarrow{f} & y
\end{array}
$$

with f in $\mathcal{C}_F$ the commutative square

$$
\begin{array}{ccc}
\mathcal{C}^L_{/y} & \xrightarrow{f^*} & \mathcal{C}^L_{/x} \\
\eta^* \downarrow & & \downarrow \xi^* \\
\mathcal{C}^L_{/y'} & \xrightarrow{f'^*} & \mathcal{C}^L_{/x'}
\end{array}
$$

is right adjointable, i.e. the mate transformation

$$\eta^* f_* \to f'_* \xi^*$$

is invertible.

Proof. First suppose we have a bispan triple. Then for $l \in \mathcal{C}^L_{/x}$ and $l' \in \mathcal{C}^L_{/y'}$ we have natural equivalences

$$\mathrm{Map}_{/y'}(l', f'_*\xi^* l) \simeq \mathrm{Map}_{/x'}(f'^* l', \xi^* l) \simeq \mathrm{Map}_{/x}(\xi_! f'^* l', l)$$
$$\simeq \mathrm{Map}_{/x}(f^* \eta_! l', l) \simeq \mathrm{Map}_{/y}(\eta_! l', f_* l)$$
$$\simeq \mathrm{Map}_{/y}(l', \eta^* f_* l),$$

using the functors $\eta_!$ and $\xi_!$ given by composition with η and ξ, respectively, which act as left adjoints to η^* and ξ^* when pullbacks along η and ξ exist, and the full strength of condition (21) for distributivity diagrams, which implies that we have the second-to-last equivalence even though $\eta_! l'$ is not necessarily in $\mathcal{C}_L$.

Now suppose our triple satisfies the assumption on Beck–Chevalley transformations. To check that it is a bispan triple we must show that for $l \colon c \to x$ in $\mathcal{C}_L$ and $f \colon x \to y$ in $\mathcal{C}_F$ the pushforward $f_* l$ has the universal property (21), i.e. that for every morphism $\eta \colon y' \to y$ we have a natural equivalence

$$\mathrm{Map}_{/y}(\eta, f_* l) \simeq \mathrm{Map}_{/x}(f^* \eta, l).$$

Denoting the pullback square containing η and f as above, we have

$$\mathrm{Map}_{/y}(\eta, f_* l) \simeq \mathrm{Map}_{/y}(\eta_! \mathrm{id}_{y'}, f_* l) \simeq \mathrm{Map}_{/y'}(\mathrm{id}_{y'}, \eta^* f_* l)$$
$$\simeq \mathrm{Map}_{/y'}(\mathrm{id}_{y'}, f'_* \xi^* l) \simeq \mathrm{Map}_{/x'}(f'^* \mathrm{id}_{y'}, \xi^* l)$$
$$\simeq \mathrm{Map}_{/x'}(\mathrm{id}_{x'}, \xi^* l) \simeq \mathrm{Map}_{/x}(\xi_! \mathrm{id}_{x'}, l)$$
$$\simeq \mathrm{Map}_{/x}(f^* \eta, l),$$

where the fourth equivalence holds because $\mathrm{id}_{y'}$ is in $\mathcal{C}_L$. $\qquad\square$

Remark 2.4.7. If $\mathcal{C}$ is locally cartesian closed, then all distributivity diagrams exist in $\mathcal{C}$ for any choice of $\mathcal{C}_L$; this is the case of $\mathcal{C}$ is an ∞-topos, for example.

Notation 2.4.8. We use the following notation for a functor $\Phi \, \mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}$: For any morphism $f \colon x \to y$ in $\mathcal{C}$, we write

$$f^{\circledast} := \Phi([f]_B) \colon \, \colon \Phi(y) \to \Phi(x),$$

and if f lies in $\mathcal{C}_F$ we also write

$$f_{\otimes} := \Phi([f]_F) \colon \Phi(x) \to \Phi(y).$$

For the next definitions we fix a bispan triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$, an $(\infty, 2)$-category $\mathfrak{X}$, and a functor

$$\Phi \colon \mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}.$$

Definition 2.4.9. Given $l \colon x \to y$ in $\mathcal{C}_L$ and $f \colon y \to z$ in $\mathcal{C}_F$, we have by assumption a distributivity diagram as in (20) in $\mathcal{C}$. If $\Phi|_{\mathcal{C}^{\mathrm{op}}}$ is left L-adjointable we define the *distributivity transformation* for l and f as the composite

$$(22) \qquad g_{\oplus} \tilde{f}_{\otimes} \epsilon^{\circledast} \to g_{\oplus} \tilde{f}_{\otimes} \epsilon^{\circledast} l^{\circledast} l_{\oplus} \simeq g_{\oplus} \tilde{f}_{\otimes} \tilde{g}^{\circledast} l_{\oplus} \simeq g_{\oplus} g^{\circledast} f_{\otimes} l_{\oplus} \to f_{\otimes} l_{\oplus},$$

where the first map uses the unit for the adjunction $l_{\oplus} \dashv l^{\circledast}$, the second equivalence uses the functoriality of Φ for compositions of spans, and the last map uses the counit of the adjunction $g_{\oplus} \dashv g^{\circledast}$.

Definition 2.4.10. We say the functor Φ is *L-distributive* if $\Phi|_{\mathcal{C}^{\mathrm{op}}}$ is left L-adjointable, and the distributivity transformation (22) is an equivalence for all l in $\mathcal{C}_L$ and f in $\mathcal{C}_F$. If the context is clear, we simply call Φ *distributive*. We write

$$\mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathfrak{X}) \subset \mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathrm{Span}_F(\mathcal{C}), \mathfrak{X}),$$

for the subspace spanned by the L-distributive functors.

Remark 2.4.11. The distributivity transformation (22) is precisely the mate transformation for the commutative square

$$(23) \qquad
\begin{array}{ccc}
\Phi(y) & \xrightarrow{\;l^{\circledast}\;} & \Phi(x) \\
\downarrow{\scriptstyle f_{\otimes}} & & \downarrow{\scriptstyle \epsilon^{\circledast}} \\
 & \Phi(w \times_z y) & \\
 & \downarrow{\scriptstyle \tilde{f}_{\otimes}} & \\
\Phi(z) & \xrightarrow{\;g^{\circledast}\;} & \Phi(w).
\end{array}$$

We can therefore reformulate the condition for the functor Φ to be L-distributive as: for every distributivity diagram (20), the square (23) is left adjointable.

Variant 2.4.12. Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple and $\mathfrak{X}$ an $(\infty, 2)$-category. We say that a functor $\Phi\colon \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}} \to \mathfrak{X}$ is *L-codistributive* if the opposite functor

$$\Phi^{\mathrm{op}}\colon \mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}^{\mathrm{op}}$$

is L-distributive. Similarly, we say that a functor $\Phi\colon \mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}$ is *right L-distributive* if the functor

$$\mathrm{Span}_F(\mathcal{C}) \simeq \mathrm{Span}_F(\mathcal{C})^{2\text{-op}} \xrightarrow{\;\Phi^{2\text{-op}}\;} \mathfrak{X}^{2\text{-op}}$$

is L-distributive. Since left adjoints in $\mathfrak{X}^{2\text{-op}}$ correspond to right adjoints in $\mathfrak{X}$, this amounts to: for every morphism $l \in \mathcal{C}_L$ the morphism $l^{\circledast}$ has a *right* adjoint $l_{\circledast}$, and the restriction of Φ to $\mathcal{C}^{\mathrm{op}} \to \mathfrak{X}$ is right L-adjointable. Moreover, given a distributivity diagram as in (20) with l in $\mathcal{C}_L$ and f in $\mathcal{C}_F$, we have a right distributivity transformation

$$f_{\otimes} l_{\circledast} \to g_{\circledast} g^{\circledast} f_{\otimes} l_{\circledast} \simeq g_{\circledast} \tilde{f}_{\otimes} \epsilon^{\circledast} l^{\circledast} l_{\circledast} \to g_{\circledast} \tilde{f}_{\otimes} \epsilon^{\circledast},$$

which is required to be an equivalence.

Just as for adjointability, there is a natural notion of an L-distributive transformation:

Definition 2.4.13. By Proposition 2.1.5, an L-distributive functor $\mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}^{\Delta^1}$ corresponds to a natural transformation

$$\eta\colon \mathrm{Span}_F(\mathcal{C}) \times \Delta^1 \to \mathfrak{X}$$

such that both η_0 and η_1 are L-distributive functors, and the mate square for the required left adjoints commutes. We call such a natural transformation an L-*distributive transformation*. We let $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathfrak{X})$ denote the subcategory of $\mathrm{Fun}(\mathrm{Span}_F(\mathcal{C}), \mathfrak{X})$ consisting of L-distributive functors and L-distributive transformations. From Proposition 2.1.5 we also know that a functor $\mathrm{Span}_F(\mathcal{C}) \to \mathfrak{X}^{\mathsf{C}_2}$ is L-distributive if and only if its underlying functors and natural transformations to $\mathfrak{X}$ are L-distributive, i.e. if and only if the adjoint morphism

$$\mathsf{C}_2 \to \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathfrak{X})$$

factors through $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ on underlying ∞-categories. We therefore write $\mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ for the locally full subcategory of $\mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ whose underlying ∞-category is $\mathrm{Fun}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$.

Proposition 2.4.14. *For any $(\infty, 2)$-category $\mathcal{Y}$ there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})) \simeq \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X})).$$

Proof. We claim that the two sides are identified under the natural equivalence

$$\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})) \simeq \mathrm{Map}(\mathrm{Span}_F(\mathcal{C}), \mathrm{FUN}(\mathcal{Y}, \mathcal{X})).$$

Indeed, the subspace $\mathrm{Map}_{\mathrm{Cat}_{(\infty,2)}}(\mathcal{Y}, \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}))$ of the left-hand side consists of those functors $\Phi \colon \mathcal{Y} \to \mathrm{FUN}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$ such that for every object $y \in \mathcal{Y}$ the image $\Phi(y)$ is an L-distributive functor and for every morphism $f \colon y \to y'$ the image $\Phi(f)$ is an L-distributive natural transformation. Since the distributivity transformation is given pointwise by distributivity transformations in $\mathcal{X}$, and equivalences in $\mathrm{FUN}(\mathcal{Y}, \mathcal{X})$ are detected by evaluation at all objects of $\mathcal{Y}$, these conditions precisely correspond to L-distributivity for the adjoint functor $\mathrm{Span}_F(\mathcal{C}) \to \mathrm{FUN}(\mathcal{Y}, \mathcal{X})$ by Proposition 2.1.5. $\square$

2.5. **The $(\infty, 2)$-category of bispans.** Our goal in this subsection is to prove our main theorem:

Theorem 2.5.1. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple. Then there exists an $(\infty, 2)$-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ equipped with an L-distributive functor*

$$j \colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$$

such that

  (i) *Composition with j gives an equivalence*

$$\mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathcal{X}) \xrightarrow{\sim} \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})$$

  for any $(\infty, 2)$-category $\mathcal{X}$.
 (ii) *On underlying ∞-groupoids j gives an equivalence*

$$\mathcal{C}^{\simeq} \simeq \mathrm{Span}_F(\mathcal{C})^{\simeq} \xrightarrow{\sim} \mathrm{BISPAN}_{F,L}(\mathcal{C})^{\simeq}.$$

 (iii) *Morphisms in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ are bispans*

  with f in $\mathcal{C}_F$ and l in $\mathcal{C}_L$, with composition given by (5).
 (iv) *2-morphisms in $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ are given by diagrams of the form (6).*

In order to prove this we will reinterpret the notion of distributivity as a special case of left adjointability, as follows:

Theorem 2.5.2. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair.*
(1) *$(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple if and only if $(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathcal{C}_L)$ is a span pair.*
(2) *In case (1) holds, a functor $\Phi \colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$ is L-distributive if and only if it is left L-adjointable.*

Given this, the universal property we want is precisely that of the $(\infty, 2)$-category $\mathrm{SPAN}_L(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}})$ — this amounts to interpreting a bispan as a "span in spans":

The observation that the composition law for bispans can be interpreted as a pull-back in spans is due to Street [Str19]. To prove this in our ∞-categorical setting we must first describe certain pullback squares in $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$:

Notation 2.5.3. Let us write $\mathrm{Sq}(\mathcal{C}) := \mathrm{Fun}([1] \times [1], \mathcal{C})$ for the ∞-category of squares in an ∞-category $\mathcal{C}$, and given two morphisms f, g in $\mathcal{C}$ with common codomain we write $\mathrm{Sq}_{f,g}(\mathcal{C})$ for the fibre of $\mathrm{Sq}(\mathcal{C})$ at $(f, g) \in \mathrm{Fun}(\Lambda_2^2, \mathcal{C})$ (where we view $[1] \times [1]$ as $(\Lambda_2^2)^{\triangleleft}$). Then a pullback of f and g is precisely a terminal object in $\mathrm{Sq}_{f,g}(\mathcal{C})$. Note also that evaluation at the initial object in $[1] \times [1]$ gives a right fibration $\mathrm{Sq}_{f,g}(\mathcal{C}) \to \mathcal{C}$.

Notation 2.5.4. We can identify $\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}$ as the ∞-category of spans in $\mathcal{C}$ where the *backwards* map must lie in $\mathcal{C}_F$, with composition given by pullbacks in $\mathcal{C}$ as usual. To emphasize this, we will use the notation

$$\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}) := \mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}.$$

For $f : x \to y$ in $\mathcal{C}_F$, we then write $[f]_B$ for the morphism $y \xleftarrow{f} x \xrightarrow{=} x$ in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$, and for any f in $\mathcal{C}$ we write $[f]_F$ for the span $x \xleftarrow{=} x \xrightarrow{f} y$.

Remark 2.5.5. A square in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$ can then be identified with a diagram of shape

where the two indicated squares are cartesian and the backwards maps all lie in $\mathcal{C}_F$. Composing this with a span (labelled in red) we get the diagram

(24)

which exhibits a cartesian morphism in $\mathrm{Sq}_{\phi,\psi}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ where ϕ and ψ are the bottom and right sides of the square.

Remark 2.5.6. In particular, we can identify objects of $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ with diagrams of the form

which we can simplify to

(25)

$$\begin{array}{ccc} \bullet & \longleftarrow & \bullet & \longrightarrow & \bullet \\ & & \downarrow & & \downarrow{\scriptstyle g} \\ & & \bullet & \xrightarrow{\ f\ } & \bullet, \end{array}$$

where the top left arrow is required to lie in $\mathcal{C}_F$. Simplifying (24) similarly, we see that the cartesian morphism in $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ over a span is given by a diagram of the form

(26)

where the blue arrows indicate the source object, the orange ones indicate the target object, and the red ones indicate the span we compose with.

Proposition 2.5.7. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair. Given morphisms $f\colon a \to c$ and $g\colon b \to c$ in $\mathcal{C}$, the fibre product of $[f]_F$ and $[g]_F$ exists in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$ if and only if the fibre product $d := a \times_c b$ of f and g exists in $\mathcal{C}$, in which case it is given by the diagram*

$$\begin{array}{ccc} d & =\!=\!= & d & \longrightarrow & b \\ \| & & \| \quad {\scriptstyle \neg} & & \| \\ d & =\!=\!= & d & \longrightarrow & b \\ \downarrow & {\scriptstyle \llcorner} & \downarrow {\scriptstyle \lrcorner} & & \downarrow{\scriptstyle g} \\ a & =\!=\!= & a & \xrightarrow{\ f\ } & c. \end{array}$$

Proof. We first check that if the diagram

$$\begin{array}{ccc} y & \xleftarrow{\ \phi\ } & x & \xrightarrow{\ u\ } & b \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle g} \\ & & a & \xrightarrow{\ f\ } & c \end{array}$$

is a terminal object in $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$, then ϕ is necessarily an equivalence. To see this, we first observe that since the diagram

(27)

$$\begin{array}{ccc} x & =\!=\!= & x & \xrightarrow{\ u\ } & b \\ & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle g} \\ & & a & \xrightarrow{\ f\ } & c \end{array}$$

is also an object of $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$, there exists a unique span $x \xleftarrow{\alpha} y' \xrightarrow{\beta} y$ with α in $\mathcal{C}_F$, such that composing the terminal object with the span gives this object, i.e.

But then composing the terminal object with the span $y \xleftarrow{\phi\alpha} y' \xrightarrow{\beta} y$ produces the terminal object, so by uniqueness this span is equivalent to the identity of y. Thus we may take $y' = y$ and under this identification β and $\phi\alpha$ are identified with id_y. Then the pullbacks β' and ϕ' may also be identified with id_y and ϕ, respectively, which tells us that we have $\alpha\phi \simeq \mathrm{id}_x$ and $\phi\alpha \simeq \mathrm{id}_y$. Thus ϕ is an equivalence, as we wanted to show.

Now observe that composing (27) with a span $z \xleftarrow{\alpha} y \xrightarrow{\beta} x$, we obtain the object

$$
\begin{array}{ccc}
z \xleftarrow{\ \alpha\ } y & \xrightarrow{\ u\beta\ } & b \\
& \downarrow{\scriptstyle v\beta} & \downarrow{\scriptstyle g} \\
& a \xrightarrow{\ f\ } & c.
\end{array}
$$

Thus an object

$$
\begin{array}{ccc}
z \xleftarrow{\ \alpha\ } y & \xrightarrow{\ \gamma\ } & b \\
& \downarrow{\scriptstyle \delta} & \downarrow{\scriptstyle g} \\
& a \xrightarrow{\ f\ } & c
\end{array}
$$

of $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ has a unique map to (27) if and only if there is a unique diagram

in $\mathcal{C}$. This is true for all objects of $\mathrm{Sq}_{[f]_F,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ if and only if the square in (27) is cartesian in $\mathcal{C}$, which is what we wanted to prove. $\qquad\square$

Remark 2.5.8. Returning to Remark 2.5.5, an object of $\mathrm{Sq}_{[f]_B,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ can be identified with a diagram

which we can simplify to

(28)

Composing this with a span gives, by simplifying (24), a diagram of the form

(29)

$$\begin{array}{c} \text{[diagram]} \end{array}$$

with the blue arrows indicating the source object, the orange ones the target object, and the red ones the span we compose with.

Proposition 2.5.9. *Let $(\mathcal{C}, \mathcal{C}_F)$ be a span pair. Given morphisms $g \colon a \to b$ in $\mathcal{C}$ and $f \colon b \to c$ in $\mathcal{C}_F$, the fibre product of $[f]_B$ and $[g]_F$ exists in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$ if and only if there exists a distributivity diagram*

$$\begin{array}{ccc} & d \xrightarrow{\ f'\ } e & \\ {}^{\epsilon}\swarrow & \Big\downarrow{\scriptstyle h'} & \Big\downarrow{\scriptstyle h} \\ a & & \\ & {}_{g}\searrow & \\ & b \xrightarrow{\ f\ } c & \end{array}$$

in $\mathcal{C}$, in which case it is given by the diagram

$$\begin{array}{ccccc} e & \xleftarrow{\ f'\ } & d & \xrightarrow{\ \epsilon\ } & a \\ \Big\| & & \Big\| & & \Big\| \\ e & \xleftarrow{\ f'\ } & d & \xrightarrow{\ \epsilon\ } & a \\ \Big\downarrow{\scriptstyle h} & & \Big\downarrow{\scriptstyle h'} & & \Big\downarrow{\scriptstyle g} \\ c & \xleftarrow{\ f\ } & b & = & b, \end{array}$$

Proof. Suppose

$$\begin{array}{ccc} & x \longrightarrow y \xrightarrow{\ \phi\ } z & \\ \swarrow & \Big\downarrow & \Big\downarrow \\ a & & \\ & {}_{g}\searrow & \\ & b \xrightarrow{\ f\ } c & \end{array}$$

is a terminal object in $\mathrm{Sq}_{[f]_B,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$. By the same argument as in the proof of Proposition 2.5.7 we see that ϕ must be an equivalence. Moreover, if we compose an object of the form

(30)

$$\begin{array}{ccc} & x \longrightarrow y = y & \\ \swarrow & \Big\downarrow & \Big\downarrow \\ a & & \\ & {}_{g}\searrow & \\ & b \xrightarrow{\ f\ } c & \end{array}$$

with a span $y \xleftarrow{\beta} z \xrightarrow{\gamma} w$ we see from (26) that we get the the outer part of the diagram

$$
\begin{array}{ccccc}
z' & \longrightarrow & z & \xrightarrow{\gamma} & w \\
& & \downarrow^{\lrcorner} & & \downarrow^{\beta} \\
& & x & \longrightarrow & y \\
a & & \downarrow^{\lrcorner} & & \downarrow \\
& \searrow^{g} & & & \downarrow \\
& & b & \xrightarrow{f} & c.
\end{array}
$$

Thus from Remark 2.4.2 we see that every object of $\mathrm{Sq}_{[f]_B,[g]_F}(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ admits a unique morphism to an object (30) if and only if this is given by a distributivity diagram in $\mathcal{C}$, as required. $\qquad\square$

Proof of Theorem 2.5.2. Every morphism in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$ is a composite of morphisms of the form $[f]_B$ (with f in $\mathcal{C}_F$) and $[g]_F$. Thus $(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}), \mathcal{C}_L)$ is a span pair if and only if for all l in $\mathcal{C}_L$ the pullbacks of $[l]_F$ along $[g]_F$ and $[f]_B$ exist for all g in $\mathcal{C}$ and f in $\mathcal{C}_F$, and these pullbacks lie in $\mathcal{C}_L$. From Proposition 2.5.7 and Proposition 2.5.9 we see that these conditions are equivalent to $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ being a bispan triple, which proves (1). To prove (2), observe that since adjointable squares compose, the functor Φ is left L-adjointable if and only if we get left adjointable squares for both types of pullbacks along morphisms in $\mathcal{C}_L$ separately. From Proposition 2.5.7 adjointability for the first type corresponds to $\Phi|_{\mathcal{C}^{\mathrm{op}}}$ being left L-adjointable, and adjointability for the second type then corresponds to L-distributivity by Remark 2.4.11. $\qquad\square$

Definition 2.5.10. In light of Theorem 2.5.2, if $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple it makes sense to define

$$
\mathrm{BISPAN}_{F,L}(\mathcal{C}) := \mathrm{SPAN}_L(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}).
$$

Notation 2.5.11. In keeping with our conventions so far, we will denote the underlying ∞-category of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ by

$$
\mathrm{Bispan}_{F,L}(\mathcal{C}) := \mathrm{BISPAN}_{F,L}(\mathcal{C})^{(1)}.
$$

In examples we will often have $\mathcal{C}_L \simeq \mathcal{C}$, in which case we abbreviate $\mathrm{BISPAN}_F(\mathcal{C}) := \mathrm{BISPAN}_{F,L}(\mathcal{C})$. If we also have $\mathcal{C}_F \simeq \mathcal{C}$, we write $\mathrm{BISPAN}(\mathcal{C})$ for $\mathrm{BISPAN}_F(\mathcal{C})$. We also adopt the same conventions for the ∞-category of bispans $\mathrm{Bispan}_{F,L}(\mathcal{C})$.

Applying Corollary 2.3.5, we get:

Corollary 2.5.12. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple. The ∞-category $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ satisfies*

$$
\mathrm{FUN}(\mathrm{BISPAN}_{F,L}(\mathcal{C}), \mathcal{X}) \simeq \mathrm{FUN}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X})
$$

for any $(\infty, 2)$-category $\mathcal{X}$. $\qquad\square$

This proves part (i) of Theorem 2.5.1. We now prove the remainder:

Proof of Theorem 2.5.1. Part (ii) is immediate from Corollary 2.3.21. A morphism in $\mathrm{SPAN}_L(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$ is a "span of spans"

$$
\begin{array}{ccccccc}
& & X & & & Y & \\
& \swarrow^{p} & & \searrow^{f} & \parallel & & \searrow^{l} \\
I & \dashleftarrow & & & Y & & \dashrightarrow J,
\end{array}
$$

with f in $\mathcal{C}_F$ and l in $\mathcal{C}_L$, which we can think of as a bispan by contracting the identity. To compose these we take pullbacks in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$, which we can unpack to give the expected composition law for bispans using Proposition 2.5.7 and Proposition 2.5.9. Unpacking the definition of a 2-morphism in $\mathrm{SPAN}_L(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C}))$, we get a diagram

where the upward-pointing map is necessarily an equivalence, as indicated. Contracting the invertible edges, we get a diagram of shape (6), as required. $\square$

The universal property also implies that our $(\infty, 2)$-categories of bispans are functorial for morphisms of bispan triples, in the following sense:

Definition 2.5.13. A morphism of bispan triples $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L) \to (\mathcal{C}', \mathcal{C}'_{F'}, \mathcal{C}'_{L'})$ is a functor $\phi \colon \mathcal{C} \to \mathcal{C}'$ such that $\phi(\mathcal{C}_F) \subseteq \mathcal{C}'_{F'}$, $\phi(\mathcal{C}_L) \subseteq \mathcal{C}'_{L'}$, and ϕ preserves pullbacks along $\mathcal{C}_F$ and $\mathcal{C}_L$ as well as distributivity diagrams. We define Trip to be the subcategory of $\mathrm{Fun}(\Lambda^2_2, \mathrm{Cat}_\infty)$ containing the bispan triples and the morphisms thereof.

Proposition 2.5.14. *There is a functor* $\mathrm{BISPAN} \colon \mathrm{Trip} \to \mathrm{Cat}_{(\infty,2)}$ *that takes* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ *to* $\mathrm{BISPAN}_{F,L}(\mathcal{C})$.

Proof. Composition with a morphism of bispan triples $\phi \colon (\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L) \to (\mathcal{C}', \mathcal{C}'_{F'}, \mathcal{C}'_{L'})$ restricts for any $(\infty, 2)$-category $\mathcal{X}$ to a morphism

$$\mathrm{Map}_{L'\text{-dist}}(\mathrm{Span}_{F'}(\mathcal{C}'), \mathcal{X}) \to \mathrm{Map}_{L\text{-dist}}(\mathrm{Span}_F(\mathcal{C}), \mathcal{X}),$$

so we get a functor

$$\mathrm{Map}_{(-)\text{-dist}}(\mathrm{Span}_{(-)}(-), -) \colon \mathrm{Trip}^{\mathrm{op}} \times \mathrm{Cat}_{(\infty,2)} \to \mathcal{S}.$$

By Theorem 2.5.1 the associated functor $\mathrm{Trip}^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{Cat}_{(\infty,2)}, \mathcal{S})$ factors through the full subcategory of corepresentable copresheaves, and so by the Yoneda lemma arises from a functor $\mathrm{BISPAN} \colon \mathrm{Trip} \to \mathrm{Cat}_{(\infty,2)}$, as required. $\square$

Applying Proposition 2.3.23 to $\mathrm{SPAN}_L(\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$, we obtain the following description of the functor of $(\infty, 2)$-categories corresponding to a distributive functor:

Corollary 2.5.15. *For an L-distributive functor* $\phi \colon \mathrm{Span}_F(\mathcal{C}) \to \mathcal{X}$, *the corresponding functor* $\Phi \colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathcal{X}$ *can be described as follows:*

(1) On objects, $\Phi(c) \simeq \phi(c)$ *for* $c \in \mathcal{C}$.
(2) On morphisms, Φ *takes a bispan*

$$B = (x \xleftarrow{s} e \xrightarrow{f} b \xrightarrow{l} y)$$

to the composite $l_\oplus f_\otimes s^\circledast \colon \phi(x) \to \phi(y)$, *where* $l_\oplus$ *is the left adjoint to* $l^\circledast$.

(3) On 2-morphisms, Φ takes the 2-morphism $\beta\colon B \to B'$ given by the commutative diagram

to the composite

$$l_{\oplus} f_{\otimes} s^{\circledast} \simeq l_{\oplus} f_{\otimes} g^{\circledast} s'^{\circledast} \simeq l_{\oplus} h^{\circledast} f'_{\otimes} s'^{\circledast} \to l_{\oplus} h^{\circledast} l'^{\circledast} l'_{\oplus} f'_{\otimes} s'^{\circledast} \simeq l_{\oplus} l^{\circledast} l'_{\oplus} f'_{\otimes} s'^{\circledast} \to l'_{\oplus} f'_{\otimes} s'^{\circledast},$$

where the first noninvertible arrow is a unit and the second noninvertible arrow is a counit.

We end this section with some useful observations about distributivity diagrams that follow easily from Proposition 2.5.9:

Lemma 2.5.16. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose given morphisms $l_1\colon x \to y$ and $l_2\colon y \to z$ in L and $f\colon z \to w$ in F. Then we can make the following diagram:*

(31)

Here all three squares are cartesian and the two rightmost give distributivity diagrams (so $g_2 = f_* l_2$, $g_1 = f'_* l'_1$). Then the outer diagram is a distributivity diagram for $l_2 l_1$ and f.

Proof. This follows immediately from the pasting lemma for pullback squares applied to $[l_2 l_1]_F \simeq [l_2]_F \circ [l_1]_F$ and $[f]_B$ in $\mathrm{Span}_{F,\mathrm{all}}(\mathcal{C})$. $\square$

Lemma 2.5.17. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose given morphisms $l\colon x \to y$ in L and $f_1\colon y \to z, f_2\colon z \to w$ in F. Then we can make the following diagram:*

(32)

Here all three squares are cartesian and two give distributivity diagrams (exhibiting $g_1 = f_{1,*}l$ and $g_2 = f_{2,*}g_1$. Then the outer diagram is a distributivity diagram for l and $f_2 f_1$. (Note that the composite square is indeed cartesian since the two left squares are cartesian and the middle triangle just exhibits g_2' as a composite.)

Proof. This is the pasting lemma for pullback squares applied to $[f_2 f_1]_B \simeq [f_1]_B \circ [f_2]_B$ and $[l]_F$.   □

Lemma 2.5.18. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose we have morphisms $l\colon x \to y$ in $\mathcal{C}_L$, $f\colon y \to z$ in $\mathcal{C}_F$ and an arbitary morphism $\zeta\colon z' \to z$. Then we can form the following diagram*

(33)

where the front face is a distributivity diagram for l and f, and the rest of the diagram is obtained by pulling this back along ζ. Then the back face in (33) is a distributivity diagram for l' and f'.

Proof. This follows yet again from the pasting lemma for pullback squares, now applied to the pullback of $[l]_F$ along $[f]_B \circ [\zeta]_F \simeq [\eta]_F \circ [f']_B$.   □

The following proposition can be interpreted as saying that distributivity transformations are compatible with base change:

Proposition 2.5.19. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be a bispan triple, and suppose we have the diagram (33). Then the distributivity transformations for (l, f) and (l', f') are related by a commutative square*

(34)
$$
\begin{array}{ccc}
g'_\oplus h'_\otimes \epsilon'^\circledast \xi^\circledast & \longrightarrow & f'_\otimes l'_\oplus \xi^\circledast \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
g'_\oplus \omega^\circledast h_\otimes \epsilon^\circledast & & f'_\otimes \eta^\circledast l_\oplus \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
\zeta^\circledast g_\oplus h_\otimes \epsilon^\circledast & \longrightarrow & \zeta^\circledast f_\otimes l_\oplus
\end{array}
$$

Proof. We have the following commutative cube:

$$
\begin{array}{c}
\Phi(y) \xrightarrow{\ l^{\circledast}\ } \Phi(x) \\
\end{array}
$$

This means the following pair of diagrams are identified under the equivalences $f'_{\otimes}\eta^{\circledast} \simeq \zeta^{\circledast}f_{\otimes}$, $h'_{\otimes}\epsilon'^{\circledast}\xi^{\circledast} \simeq \omega^{\circledast}h_{\otimes}\epsilon^{\circledast}$ given by the left and right faces of the cube:

We get a corresponding identification of the mates of the outer squares in these two diagrams, which in turn decompose (since mate transformations are compatible with vertical pasting of squares) into the mates of the two smaller squares; according to Remark 2.4.11 these are the base change transformation $g'_{\oplus}\omega^{\circledast} \to \zeta^{\circledast}g_{\oplus}$ and the distributivity transformation for (l, f) on the left, and the distributivity transformation for (l', f') and the base change transformation $l'_{\oplus}\xi^{\circledast} \to \eta^{\circledast}l_{\oplus}$ on the right. This identification gives precisely a commutative diagram

$$
\begin{array}{ccc}
g'_{\oplus}h'_{\otimes}\epsilon'^{\circledast}\xi^{\circledast} & \longrightarrow f'_{\otimes}l'_{\oplus}\xi^{\circledast} \longrightarrow & f'_{\otimes}\eta^{\circledast}l_{\oplus} \\
\downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\
g'_{\oplus}\omega^{\circledast}h_{\otimes}\epsilon^{\circledast} & \longrightarrow \zeta^{\circledast}g_{\oplus}h_{\otimes}\epsilon^{\circledast} \longrightarrow & \zeta^{\circledast}f_{\otimes}l_{\oplus},
\end{array}
$$

which we can reorganize into the commutative diagram in the statement. $\qquad\square$

2.6. **Symmetric monoidal structures.** In this subsection we will prove that the functor SPAN: Pair $\to$ Cat$_{(\infty,2)}$ preserves products. Applying this in the special case of bispans, we will see that in some cases a symmetric monoidal structure on $\mathcal{C}$ induces a symmetric monoidal structure on BISPAN$_{F,L}(\mathcal{C})$, and also show that BISPAN($-$) is a functor of $(\infty,2)$-categories. The construction of symmetric monoidal structures on spans is also discussed in [GR17, Part III, Chapter 9] (where it is used to encode Serre duality for Ind-coherent sheaves) and [Mac20, §3.2]. An

explicit construction (not relying on the universal property) of symmetric monoidal structures on ∞-categories of spans is also given in [BGS20].

Proposition 2.6.1. *The ∞-categories* Pair *and* Trip *have finite products, given by*

$$(\mathcal{C}, \mathcal{C}_F) \times (\mathcal{C}', \mathcal{C}'_{F'}) \simeq (\mathcal{C} \times \mathcal{C}', \mathcal{C}_F \times \mathcal{C}'_{F'}),$$

$$(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L) \times (\mathcal{C}', \mathcal{C}'_{F'}, \mathcal{C}'_{L'}) \simeq (\mathcal{C} \times \mathcal{C}', \mathcal{C}_F \times \mathcal{C}'_{F'}, \mathcal{C}_L \times \mathcal{C}'_{L'}).$$

Proof. It follows from the definition of Pair that the morphism

$$\mathrm{Map}_{\mathrm{Pair}}((\mathcal{C}, \mathcal{C}_F), (\mathcal{C}', \mathcal{C}'_{F'})) \to \mathrm{Map}_{\mathrm{Cat}_\infty}(\mathcal{C}, \mathcal{C}')$$

is a monomorphism (and similarly for Trip), so it suffices to show that for any span pair $(\mathcal{D}, \mathcal{D}_F)$ (or bispan triple $(\mathcal{D}, \mathcal{D}_F, \mathcal{D}_L)$) a functor $\mathcal{D} \to \mathcal{C} \times \mathcal{C}'$ is a morphism of span pairs (or bispan triples) if and only if the functors $\mathcal{D} \to \mathcal{C}$ and $\mathcal{D} \to \mathcal{C}'$ are both morphisms of span pairs (or bispan triples). This is clear, since a pair of cartesian squares in $\mathcal{C}$ and $\mathcal{C}'$ gives a cartesian square in $\mathcal{C} \times \mathcal{C}'$ and vice versa, and similarly for distributivity diagrams. $\square$

Proposition 2.6.2. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_{F'})$ are span pairs. Then a functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ is left (F, F')-adjointable if and only if it is left adjointable in each variable, i.e.*

- *for every $c \in \mathcal{C}$, the functor $\Phi(c, -)$ is left F'-adjointable,*
- *for every $c' \in \mathcal{C}'$, the functor $\Phi(-, c')$ is left F-adjointable,*
- *for every morphism $c_1 \to c_2$ in $\mathcal{C}$, the transformation $\Phi(c_2, -) \to \Phi(c_1, -)$ is left F'-adjointable,*
- *for every morphism $c'_1 \to c'_2$ in $\mathcal{C}'$, the transformation $\Phi(-, c'_2) \to \Phi(-, c'_1)$ is left F-adjointable.*

Proof. We first observe that Φ is left (F, F')-*preadjointable* if and only if $(f, \mathrm{id}_{c'})^{\circledast}$ has a left adjoint for all f in F and c' in $\mathcal{C}'$, and $(\mathrm{id}_c, f')^{\circledast}$ has a left adjoint for all c in $\mathcal{C}$ and f' in F', since $(f, f')^{\circledast} \simeq (f, \mathrm{id})^{\circledast}(\mathrm{id}, f')^{\circledast}$ and left adjoints compose. Thus Φ is left (F, F')-preadjointable if and only if $\Phi(c, -)$ is left F'-preadjointable for all $c \in \mathcal{C}$ and $\Phi(-, c')$ is left F-preadjointable for all c' in $\mathcal{C}'$.

A pair of cartesian squares gives a cartesian square in $\mathcal{C} \times \mathcal{C}'$, so if Φ is left (F, F')-preadjointable then it is left (F, F')-adjointable if and only if for cartesian squares

$$\begin{array}{ccc} w & \xrightarrow{\;v\;} & z \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{\;f\;} & y, \end{array} \qquad \begin{array}{ccc} w' & \xrightarrow{\;v'\;} & z' \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle g'} \\ x' & \xrightarrow{\;f'\;} & y', \end{array}$$

with f in $\mathcal{C}_F$ and f' in $\mathcal{C}'_{F'}$, the square

$$(35) \qquad \begin{array}{ccc} \Phi(y, y') & \xrightarrow{(f, f')^{\circledast}} & \Phi(x, x') \\ {\scriptstyle (g, g')^{\circledast}}\downarrow & & \downarrow{\scriptstyle (u, u')^{\circledast}} \\ \Phi(z, z') & \xrightarrow{(v, v')^{\circledast}} & \Phi(w, w') \end{array}$$

is left adjointable. Taking $f = g = \mathrm{id}_c$ this implies that $\Phi(c, -)$ is left F'-adjointable, while taking $f = \mathrm{id}_y$ we see that the transformation $\Phi(g, -) \colon \Phi(y, -) \to \Phi(z, -)$ is left F'-adjointable. The same goes in the other variable, so if Φ is left (F, F')-adjointable then the four given conditions hold.

Conversely, if these conditions hold then we want to show that the square (35) is left adjointable. We can decompose this square into the commutative diagram

$$
\begin{array}{ccccc}
\Phi(y,y') & \xrightarrow{(f,\mathrm{id})^\circledast} & \Phi(x,y') & \xrightarrow{(\mathrm{id},f')^\circledast} & \Phi(x,x') \\
\downarrow{\scriptstyle(g,\mathrm{id})^\circledast} & & \downarrow{\scriptstyle(u,\mathrm{id})^\circledast} & & \downarrow{\scriptstyle(u,\mathrm{id})^\circledast} \\
\Phi(z,y') & \xrightarrow{(v,\mathrm{id})^\circledast} & \Phi(w,y') & \xrightarrow{(\mathrm{id},f')^\circledast} & \Phi(w,x') \\
\downarrow{\scriptstyle(\mathrm{id},g')^\circledast} & & \downarrow{\scriptstyle(\mathrm{id},g')^\circledast} & & \downarrow{\scriptstyle(\mathrm{id},u')^\circledast} \\
\Phi(z,z') & \xrightarrow{(v,\mathrm{id})^\circledast} & \Phi(w,z') & \xrightarrow{(\mathrm{id},v')^\circledast} & \Phi(w,w').
\end{array}
$$

Here all four squares are left adjointable:
- the top left square since $\Phi(-,y')$ is left F-adjointable,
- the top right square since $\Phi(u,-)$ is a left F'-adjointable transformation,
- the bottom left square since $\Phi(-,g')$ is a left F-adjointable transformation,
- the bottom right square since $\Phi(w,-)$ is left F'-adjointable.

Since mate transformations are compatible with horizontal and vertical compositions of squares, left adjointable squares are closed under both horizontal and vertical compositions. Thus the outer square (35) is left adjointable, which completes the proof. $\qquad\square$

Corollary 2.6.3. *Suppose* $(\mathcal{C}, \mathcal{C}_F)$ *and* $(\mathcal{C}', \mathcal{C}'_{F'})$ *are span pairs. Then there is a natural equivalence*

$$
\mathrm{Map}_{(F,F')\text{-ladj}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathrm{FUN}_{F'\text{-ladj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X}))
$$

for $\mathcal{X} \in \mathrm{Cat}_{(\infty,2)}$.

To prove this we also need the following lemma:

Lemma 2.6.4. *Suppose* $(\mathcal{C}, \mathcal{C}_F)$ *is a span pair. A morphism* $\eta\colon \phi \to \psi$ *in the* $(\infty, 2)$-*category* $\mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ *has a left adjoint if and only if*

(1) the morphism $\eta_c\colon \phi(c) \to \psi(c)$ *has a left adjoint in* $\mathcal{X}$ *for every* $c \in \mathcal{C}$,
(2) the commutative square

$$
\begin{array}{ccc}
\phi(c) & \xrightarrow{\eta_c} & \psi(c) \\
\downarrow & & \downarrow \\
\phi(c') & \xrightarrow{\eta_{c'}} & \psi(c')
\end{array}
$$

is left adjointable for every morphism $c \to c'$ *in* $\mathcal{C}$.

Moreover, a commutative square

$$
\begin{array}{ccc}
\phi & \xrightarrow{\eta} & \psi \\
\downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} \\
\phi' & \xrightarrow{\eta'} & \psi'
\end{array}
$$

in $\mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X})$ *is left adjointable if and only if the commutative square in* $\mathcal{X}$ *obtained by evaluation at* c *is left adjointable in* $\mathcal{X}$ *for every* $c \in \mathcal{C}$.

Proof. By Corollary 2.3.5 we have an equivalence

$$
\mathrm{FUN}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{FUN}(\mathrm{SPAN}_F(\mathcal{C}), \mathcal{X}).
$$

Suppose $H\colon \Phi \to \Psi$ is the morphism corresponding to $\eta\colon \phi \to \psi$ under this equivalence. Then we know that H has a left adjoint if and only if

- $H_c\colon \Phi(c) \to \Psi(c)$ has a left adjoint for every $c \in \mathcal{C}$,

- the square

$$\begin{array}{ccc} \Phi(c_1) & \xrightarrow{H_{c_1}} & \Psi(c_1) \\ \downarrow & & \downarrow \\ \Phi(c_2) & \xrightarrow{H_{c_2}} & \Psi(c_2) \end{array}$$

  is left adjointable for every morphism $c_1 \to c_2$ in $\mathrm{SPAN}_F(\mathcal{C})$.

In terms of η, these conditions say that η_c has a left adjoint for every $c \in \mathcal{C}$, and for every span $c_1 \xleftarrow{g} x \xrightarrow{f} c_2$ with f in F, the outer square in the diagram

$$\begin{array}{ccc} \phi(c_1) & \xrightarrow{\eta_{c_1}} & \psi(c_1) \\ \downarrow{\scriptstyle g^\circledast} & & \downarrow{\scriptstyle g^\circledast} \\ \phi(x) & \xrightarrow{\eta_x} & \psi(x) \\ \downarrow{\scriptstyle f_\oplus} & & \downarrow{\scriptstyle f_\oplus} \\ \phi(c_2) & \xrightarrow{\eta_{c_2}} & \psi(c_2) \end{array}$$

is left adjointable. Since left adjointable squares compose, and the two squares here are those associated to spans where one leg is the identity, it is equivalent to require these two squares to be left adjointable. For the top square this is the condition we want, while the bottom square is automatically left adjointable since its mate is obtained by passing to left adjoints everywhere in the commutative square

$$\begin{array}{ccc} \phi(c_2) & \xrightarrow{\eta_{c_2}} & \psi(c_2) \\ \downarrow{\scriptstyle f^\circledast} & & \downarrow{\scriptstyle f^\circledast} \\ \phi(x) & \xrightarrow{\eta_x} & \psi(x). \end{array}$$

Since the mate of a square of natural transformations is given by taking mates objectwise, the characterization of left adjointable squares is immediate. $\square$

Proof of Corollary 2.6.3. Unpacking definitions, a functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ corresponds to a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}_{F'\text{-ladj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X})$ if and only if $\Phi(c, -)$ is a left F'-adjointable functor for every $c \in \mathcal{C}$, and $\Phi(c_2, -) \to \Phi(c_1, -)$ is a left F'-adjointable transformation for every morphism $c_1 \to c_2$ in $\mathcal{C}$.

Moreover, it follows from Lemma 2.6.4 that such a functor

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}_{F'\text{-ladj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X}) \simeq \mathrm{Fun}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})$$

is left F-adjointable precisely when the following conditions hold:

- for every morphism $f \colon c_1 \to c_2$ in F and every object $c' \in \mathcal{C}'$, the morphism $(f, \mathrm{id})^\circledast \colon \Phi(c_2, c') \to \Phi(c_1, c')$ has a left adjoint,
- for every morphism $f \colon c_1 \to c_2$ in F and every morphism $c_1' \to c_2'$ in $\mathcal{C}'$, the commutative square

$$\begin{array}{ccc} \Phi(c_2, c_2') & \xrightarrow{(f,\mathrm{id})^\circledast} & \Phi(c_1, c_2') \\ \downarrow & & \downarrow \\ \Phi(c_2, c_1') & \xrightarrow{(f,\mathrm{id})^\circledast} & \Phi(c_1, c_1') \end{array}$$

  is left adjointable,
- for every pullback square

$$\begin{array}{ccc} w & \xrightarrow{v} & z \\ \downarrow{\scriptstyle u} & & \downarrow{\scriptstyle g} \\ x & \xrightarrow{f} & y \end{array}$$

in $\mathcal{C}$ with f in F, the commutative square

$$
\begin{array}{ccc}
\Phi(y, c') & \xrightarrow{(f,\mathrm{id})^\circledast} & \Phi(x, c') \\
\downarrow{\scriptstyle(g,\mathrm{id})^\circledast} & & \downarrow{\scriptstyle(u,\mathrm{id})^\circledast} \\
\Phi(z, c') & \xrightarrow{(v,\mathrm{id})^\circledast} & \Phi(w, c')
\end{array}
$$

is left adjointable.

These conditions say precisely that $\Phi(-, c')$ is a left F-adjointable functor for every $c' \in \mathcal{C}'$ and $\Phi(-, c_2') \to \Phi(-, c_1')$ is a left F-adjointable transformation for every morphism $c_1' \to c_2'$ in $\mathcal{C}'$. We have thus shown that a functor $\Phi \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}} \to \mathcal{X}$ corresponds to a left F-adjointable functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}_{F'\text{-ladj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X})$ if and only if it satisfies the four conditions that we saw characterized left (F, F')-adjointable functors in Proposition 2.6.2. $\qquad\square$

From Corollary 2.6.3 we can now deduce that spans preserve products:

Corollary 2.6.5. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}', \mathcal{C}'_{F'})$ are span pairs. Then the natural morphism*

$$
\mathrm{SPAN}_{(F, F')}(\mathcal{C} \times \mathcal{C}') \to \mathrm{SPAN}_F(\mathcal{C}) \times \mathrm{SPAN}_{F'}(\mathcal{C}')
$$

is an equivalence.

Proof. For $\mathcal{X}$ an $(\infty, 2)$-category we have natural equivalences

$$
\begin{aligned}
\mathrm{Map}(\mathrm{SPAN}_{(F,F')}(\mathcal{C} \times \mathcal{C}'), \mathcal{X}) &\simeq \mathrm{Map}_{(F,F')\text{-ladj}}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}'^{\mathrm{op}}, \mathcal{X}) \\
&\simeq \mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathrm{FUN}_{F'\text{-ladj}}(\mathcal{C}'^{\mathrm{op}}, \mathcal{X})) \\
&\simeq \mathrm{Map}_{F\text{-ladj}}(\mathcal{C}^{\mathrm{op}}, \mathrm{FUN}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})) \\
&\simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}), \mathrm{FUN}(\mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X})) \\
&\simeq \mathrm{Map}(\mathrm{SPAN}_F(\mathcal{C}) \times \mathrm{SPAN}_{F'}(\mathcal{C}'), \mathcal{X}). \qquad\square
\end{aligned}
$$

Corollary 2.6.6. *Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair and $\mathcal{C}$ has a (symmetric) monoidal structure such that the tensor product functor is a morphism of span pairs*

$$
\otimes \colon (\mathcal{C} \times \mathcal{C}, \mathcal{C}_F \times \mathcal{C}_F) \to (\mathcal{C}, \mathcal{C}_F),
$$

i.e. given morphisms $f \colon x \to y$ and $f' \colon x' \to y'$ in F, the morphism $f \otimes f' \colon x \otimes x' \to y \otimes y'$ is also in F, and given a pair of pullback squares

$$
\begin{array}{ccc}
w & \xrightarrow{v} & z \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle g} \\
x & \xrightarrow{f} & y,
\end{array}
\qquad
\begin{array}{ccc}
w' & \xrightarrow{v'} & z' \\
\downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle g'} \\
x' & \xrightarrow{f'} & y',
\end{array}
$$

with f and f' in $\mathcal{C}_F$, the commutative square

$$
\begin{array}{ccc}
w \otimes w' & \longrightarrow & z \otimes z' \\
\downarrow & & \downarrow \\
x \otimes x' & \longrightarrow & y \otimes y'
\end{array}
$$

is cartesian. Then $\mathrm{SPAN}_F(\mathcal{C})$ inherits a (symmetric) monoidal structure from that on $\mathcal{C}$.

Proof. Since the functor SPAN preserves products, it takes (commutative) algebras in span pairs to (commutative) algebras in $(\infty, 2)$-categories. $\qquad\square$

Example 2.6.7. Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair where $\mathcal{C}$ has finite products and morphisms in $\mathcal{C}_F$ are closed under products. Products of cartesian squares are always cartesian, so in this case Corollary 2.6.6 implies that the cartesian product induces a symmetric monoidal structure on $\mathrm{SPAN}_F(\mathcal{C})$. This recovers the discussion in [GR17, Chapter 9, 2.1] and some cases of [BGS20, Theorem 2.15].

Definition 2.6.8. We say an ∞-category $\mathcal{C}$ is *extensive* if $\mathcal{C}$ has finite coproducts and these satisfy *descent* in the sense that the coproduct functor

$$\amalg \colon \prod_{i=1}^n \mathcal{C}_{/x_i} \to \mathcal{C}_{/\amalg_{i=1}^n x_i}$$

is an equivalence. (Equivalently, pullbacks of the component inclusions in finite coproducts always exist, and coproducts are disjoint and stable under pullback.)

Example 2.6.9. Suppose $(\mathcal{C}, \mathcal{C}_F)$ is a span pair where $\mathcal{C}$ has finite coproducts and morphisms in $\mathcal{C}_F$ are closed under coproducts. If $\mathcal{C}$ is extensive, then coproducts of cartesian squares are again cartesian. Hence in this case the coproduct induces a symmetric monoidal structure on $\mathrm{SPAN}_F(\mathcal{C})$ by Corollary 2.6.6. The descent condition is satisfied, for instance, if $\mathcal{C}$ is an ∞-topos, or in the category of sets. See also [Bar14, §4], where $\mathcal{C}$ is called "disjunctive" if $\mathcal{C}$ is extensive and has pullbacks; in this case the coproduct in $\mathcal{C}$ gives both the product and coproduct in $\mathrm{Span}(\mathcal{C})$ by [Bar14, Proposition 4.3].

Specializing the preceding discussion to bispans, we get:

Corollary 2.6.10. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ and $(\mathcal{C}', \mathcal{C}'_{F'}, \mathcal{C}'_{L'})$ are bispan triples. Then the natural morphism*

$$\mathrm{BISPAN}_{(F,F'),(L,L')}(\mathcal{C} \times \mathcal{C}') \to \mathrm{BISPAN}_{F,L}(\mathcal{C}) \times \mathrm{BISPAN}_{F',L'}(\mathcal{C}')$$

is an equivalence.

Proof. From Corollary 2.6.5 we get a product of span pairs

$$(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}, \mathcal{C}_L) \times (\mathrm{Span}_{F'}(\mathcal{C}')^{\mathrm{op}}, \mathcal{C}'_{L'}) \simeq (\mathrm{Span}_{(F,F')}(\mathcal{C} \times \mathcal{C}'), \mathcal{C}_L \times \mathcal{C}'_{L'}),$$

and hence using Corollary 2.6.5 again we have

$$\mathrm{SPAN}_{(L,L')}(\mathrm{Span}_{(F,F')}(\mathcal{C} \times \mathcal{C}')^{\mathrm{op}}) \simeq \mathrm{SPAN}_L(\mathrm{Span}_F(\mathcal{C})^{\mathrm{op}}) \times \mathrm{SPAN}_{L'}(\mathrm{Span}_{F'}(\mathcal{C}')^{\mathrm{op}}),$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 2.6.11. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple and $\mathcal{C}$ has a (symmetric) monoidal structure such that the tensor product is a morphism of bispan triples*

$$\otimes \colon (\mathcal{C} \times \mathcal{C}, \mathcal{C}_F \times \mathcal{C}_F, \mathcal{C}_L \times \mathcal{C}_L) \to (\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L),$$

i.e.

(1) both $\mathcal{C}_F$ and $\mathcal{C}_L$ are closed under $\otimes$,

(2) given a pair of pullback squares

$$
\begin{array}{ccc}
w & \xrightarrow{\;v\;} & z \\
{\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle g} \\
x & \xrightarrow[\;f\;]{} & y,
\end{array}
\qquad
\begin{array}{ccc}
w' & \xrightarrow{\;v'\;} & z' \\
{\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle g'} \\
x' & \xrightarrow[\;f'\;]{} & y',
\end{array}
$$

with f and f' either both in F or both in L, the commutative square

$$
\begin{array}{ccc}
w \otimes w' & \longrightarrow & z \otimes z' \\
\big\downarrow & & \big\downarrow \\
x \otimes x' & \longrightarrow & y \otimes y'
\end{array}
$$

is cartesian,

(3) *given morphisms* $f\colon x \to y$, $f'\colon x' \to y'$ *in* $\mathcal{C}_L$ *and* $g\colon y \to z$, $g'\colon y' \to z'$, *the diagram*

obtained as the tensor product of the distributivity diagrams for (f, g) *and* (f', g'), *is a distributivity diagram for* $(f \otimes f', g \otimes g')$.[8]

Then $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ *inherits a (symmetric) monoidal structure from that on* $\mathcal{C}$. $\quad\square$

Example 2.6.12. Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple such that $\mathcal{C}$ is extensive and morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ are closed under coproducts. Then the coproduct satisfies the conditions of Corollary 2.6.11: The descent condition implies that coproducts of cartesian squares are cartesian, and the condition on distributivity diagrams amounts to asking for the natural map

$$g_* f \amalg g'_* f' \to (g \amalg g')_* (f \amalg f')$$

to be an equivalence; this is true because by descent we have

$$
\begin{aligned}
\mathrm{Map}_{/z \amalg z'}(u \amalg u', g_* f \amalg g'_* f') &\simeq \mathrm{Map}_{/z}(u, g_* f) \times \mathrm{Map}_{/z'}(u', g'_* f') \\
&\simeq \mathrm{Map}_{/y}(g^* u, f) \times \mathrm{Map}_{/y'}(g'^* u', f') \\
&\simeq \mathrm{Map}_{/y \amalg y'}(g^* u \amalg g'^* u', f \amalg f') \\
&\simeq \mathrm{Map}_{/y \amalg y'}((g \amalg g')^*(u \amalg u'), f \amalg f') \\
&\simeq \mathrm{Map}_{/y \amalg y'}(u \amalg u', (g \amalg g')_*(f \amalg f')).
\end{aligned}
$$

for an object $u \amalg u'$ over $z \amalg z'$. The coproduct therefore induces a symmetric monoidal structure on $\mathrm{BISPAN}_{F,L}(\mathcal{C})$.[9]

Remark 2.6.13. Suppose $\mathcal{C}$ is a locally cartesian closed and extensive ∞-category. Then the symmetric monoidal structure on $\mathrm{Bispan}(\mathcal{C})$ induced by the coproduct in $\mathcal{C}$ is a cartesian product: we have

$$
\begin{aligned}
\mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, x \amalg y) &\simeq \{c \leftarrow a \to b \to x \amalg y\} \\
&\simeq \{c \leftarrow a_x \amalg a_y \to b_x \amalg b_y \to x \amalg y\} \\
&\simeq \{c \leftarrow a_x \to b_x \to x\} \times \{c \leftarrow a_y \to b_y \to y\} \\
&\simeq \mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, x) \times \mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(c, y).
\end{aligned}
$$

Moreover, we have the same identification for ∞-categories of morphisms, so this is actually an $(\infty, 2)$-categorical product. However, this is *not* a coproduct in bispans: in particular, $\emptyset$ is not an initial object, since we have

$$\mathrm{Map}_{\mathrm{Bispan}(\mathcal{C})}(\emptyset, x) \simeq \{\emptyset \leftarrow \emptyset \to b \to x\} \simeq \mathcal{C}_{/x}^{\simeq}$$

which is not in general contractible.

---

[8]Note that the previous condition implies that the square in this diagram is cartesian; the condition can therefore be interpreted as asking for the natural map

$$g_* f \otimes g'_* f' \to (g \otimes g')_*(f \otimes f')$$

arising from this cartesian square to be an equivalence.

[9]However, the cartesian product in $\mathcal{C}$ does not typically give a symmetric monoidal structure on bispans — we do *not* in general have an equivalence between $g_* f \times g'_* f'$ and $(g \times g')_*(f \times f')$.

Remark 2.6.14. For any ∞-category $\mathcal{C}$ we can consider the minimal bispan triple $\mathcal{C}^\flat := (\mathcal{C}, \mathcal{C}^\simeq, \mathcal{C}^\simeq)$ where the morphisms in $\mathcal{C}_F$ and $\mathcal{C}_L$ are just the equivalences. Any functor gives a morphism of minimial bispan triples, so we have a functor

$$(-)^\flat \colon \mathrm{Cat}_\infty \to \mathrm{Trip}$$

that is moreover fully faithful. The ∞-category Trip is then a Cat_∞-module via cartesian products with $(-)^\flat$. We also have a natural equivalence $\mathrm{BISPAN}(\mathcal{C}^\flat) \simeq \mathcal{C}$ as all functors are distributive. This means the functor $\mathrm{BISPAN} \colon \mathrm{Trip} \to \mathrm{Cat}_{(\infty,2)}$ is a morphism of Cat_∞-modules, which we can view as a functor of $(\infty,2)$-categories using the recent results of Heine [Hei20]. Moreover, the natural transformation $\mathrm{Span}_F(\mathcal{C}) \to \mathrm{BISPAN}_{F,L}(\mathcal{C})$ is a transformation of Cat_∞-modules, which means the universal property of $\mathrm{BISPAN}_{F,L}(\mathcal{C})$ is actually Cat_∞-natural.

3. Examples of distributivity

3.1. **Bispans in finite sets and symmetric monoidal ∞-categories.** In this subsection we consider the relationship between symmetric monoidal ∞-categories and bispans in finite sets. We first recall that symmetric monoidal ∞-categories can be described in terms of functors from spans of finite sets, and then show that the resulting functor is distributive if and only if the tensor product commutes with finite coproducts in each variable. Our universal property then gives a (product-preserving) functor from bispans in finite sets, which we can interpret as a semiring structure with the coproduct as addition and the tensor product as multiplication.

Notation 3.1.1. We write $\mathbb{F}$ for the category of finite sets and $\mathbb{F}_*$ for the category of finite pointed sets and base-point preserving maps; every object of $\mathbb{F}_*$ is isomorphic to one of the form $\langle n \rangle := (\{0, \dots, n\}, 0)$. For $I \in \mathbb{F}$ we write I_+ for the pointed set $(I \amalg \{*\}, *)$ obtained by adding a disjoint base point to I.

Definition 3.1.2. If $\mathcal{C}$ is an ∞-category with finite products, a *commutative monoid* in $\mathcal{C}$ is a functor $\Phi \colon \mathbb{F}_* \to \mathcal{C}$ such that for every $n = 0, 1, \dots$ the map

$$\Phi(\langle n \rangle) \xrightarrow{(\Phi(\rho_i))_{i=1,\dots,n}} \prod_{i=1}^{n} \Phi(\langle 1 \rangle)$$

is an equivalence, where $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ is defined by

$$\rho_i(j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

We write $\mathrm{CMon}(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(\mathbb{F}_*, \mathcal{C})$ spanned by the commutative monoids.

Notation 3.1.3. If $\mathcal{C}, \mathcal{D}$ are ∞-categories with finite products, we write $\mathrm{Fun}^\times(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors that preserve finite products.

Remark 3.1.4. The category $\mathbb{F}_*$ can be identified with the subcategory of $\mathrm{Span}(\mathbb{F})$ whose morphisms are the spans $I \xleftarrow{f} J \xrightarrow{g} K$ where f is injective, with this corresponding to the morphism $I_+ \to K_+$ given by

$$i \mapsto \begin{cases} g(j), & i = f(j), \\ *, & \text{otherwise;} \end{cases}$$

we write $j \colon \mathbb{F}_* \to \mathrm{Span}(\mathbb{F})$ for this subcategory inclusion.

The following description of commutative monoids in terms of spans seems to have been first proved by Cranch [Cra10, Cra11]; other proofs, as special cases of various generalizations, can be found in [Gla17, Har20, BaHo21].

Proposition 3.1.5. *Let $\mathcal{C}$ be an ∞-category with finite products. Restriction along the inclusion $j\colon \mathbb{F}_* \to \mathrm{Span}(\mathbb{F})$ gives an equivalence*

$$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}), \mathcal{C}) \xrightarrow{\sim} \mathrm{CMon}(\mathcal{C}),$$

with the inverse given by right Kan extension along the functor j.

Proof. In the stated form, this is [BaHo21, Proposition C.1]. $\qquad\square$

Remark 3.1.6. The functor $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathrm{Cat}_\infty$ corresponding to a symmetric monoidal ∞-category $\mathcal{C}$ admits the following description:

- $\Phi(I) \simeq \prod_{i \in I} \mathcal{C} \simeq \mathrm{Fun}(I, \mathcal{C})$,
- for $f\colon I \to J$, the functor $f^{\circledast}\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ is that given by composition with f,
- for $f\colon I \to J$, the functor

$$f_{\otimes}\colon \mathrm{Fun}(I, \mathcal{C}) \simeq \prod_{j \in J}\prod_{i \in I_j} \mathcal{C} \to \prod_{j \in J} \mathcal{C}$$

 is given by tensoring the components corresponding to the preimages of each $j \in J$.

In particular, for $q\colon I \to *$ the functor $q_{\otimes}\colon \mathrm{Fun}(I, \mathcal{C}) \to \mathcal{C}$ takes $\phi\colon I \to \mathcal{C}$ to $\bigotimes_{i \in I} \phi(i)$, while $q^{\circledast}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ is the diagonal functor.

Proposition 3.1.7. *Suppose $\Phi\colon \mathrm{Span}(\mathbb{F}) \to \mathrm{Cat}_\infty$ is a product-preserving functor, corresponding to a symmetric monoidal structure on $\mathcal{C} = \Phi(*)$. Then Φ is distributive if and only if $\mathcal{C}$ has finite coproducts and the symmetric monoidal structure is compatible with these, i.e. the tensor product preserves finite coproducts in each variable.*

Proof. For $I \in \mathbb{F}$, the functor $q^{\circledast}\colon \mathcal{C} \to \mathrm{Fun}(I, \mathcal{C})$ corresponding to the morphism $q\colon I \to *$ is the diagonal. This has a left adjoint if and only if $\mathcal{C}$ admits all I-indexed colimits, i.e. $\mathcal{C}$ has I-indexed coproducts. Moreover, if $\mathcal{C}$ has finite coproducts, then the functor $f^{\circledast}\colon \mathrm{Fun}(J, \mathcal{C}) \to \mathrm{Fun}(I, \mathcal{C})$ given by composition with $f\colon I \to J$ has a left adjoint for any f, since all pointwise left Kan extensions along f exist in $\mathcal{C}$. Given a cartesian square

$$
\begin{array}{ccc}
I' & \xrightarrow{\;f'\;} & J' \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
I & \xrightarrow{\;f\;} & J
\end{array}
$$

in $\mathbb{F}$, the mate transformation

$$g'_{\oplus} f'^{\circledast} \to f^{\circledast} g_{\oplus}$$

is then automatically an equivalence, since for $\phi\colon J' \to \mathcal{C}$ this is given at $i \in I$ by the natural map

$$\coprod_{x \in I'_i} \phi(f'x) \to \coprod_{y \in J'_{f(i)}} \phi(y),$$

which is an equivalence since these fibres are canonically isomorphic. This proves that Φ is left adjointable if and only if $\mathcal{C}$ admits finite coproducts.

Given morphisms $l\colon I \to J$ and $f\colon J \to K$ in $\mathbb{F}$, we have the distributivity square

$$
\begin{array}{ccc}
 & J' \xrightarrow{f'} & K' \\
\epsilon\nearrow & \downarrow & \downarrow h=f_*l \\
I & & \\
l\searrow & \downarrow & \downarrow \\
 & J \xrightarrow{f} & K,
\end{array}
$$

where $K'_k \cong \prod_{j\in J_k} I_j$. The distributivity transformation $h_\oplus f'_\otimes \epsilon^\circledast \to f_\otimes l_\oplus$ is given for $\phi\colon I \to \mathcal{C}$ at $k \in K$ by the canonical map

$$
\coprod_{(i_j)_j \in \prod_{j\in J_k} I_j} \bigotimes_{j\in J_k} \phi(i_j) \to \bigotimes_{j\in J_k} \left(\coprod_{i\in I_j} \phi(i) \right).
$$

This is an equivalence if $\otimes$ preserves finite coproducts in each variable. Conversely, for $K \in \mathbb{F}$ we have in particular the distributivity diagram

$$(36)\qquad
\begin{array}{ccc}
 & K \amalg K \xrightarrow{\nabla} & K \\
\epsilon\swarrow & \downarrow q\amalg q & \downarrow q \\
K \amalg * & & \\
q\amalg\mathrm{id}\searrow & \downarrow & \downarrow \\
 & * \amalg * \xrightarrow{\nabla} & *,
\end{array}
$$

where q is the unique map $K \to *$ and ∇ are fold maps. The corresponding distributivity transformation is given for $\alpha\colon K \amalg * \to \mathcal{C}$ by

$$
\coprod_{k\in K} (\alpha(k) \otimes \alpha(*)) \to \left(\coprod_{k\in K} \alpha(k) \right) \otimes \alpha(*).
$$

If Φ is distributive then this is an equivalence for all K and α, which is precisely the condition that $\otimes$ preserves finite coproducts in each variable. $\qquad\square$

Corollary 3.1.8. *Product-preserving functors* $\mathrm{BISPAN}(\mathbb{F}) \to \mathrm{CAT}_\infty$ *correspond to symmetric monoidal ∞-categories that are compatible with finite coproducts.*

Proof. By Theorem 2.5.1, functors $\mathrm{BISPAN}(\mathbb{F}) \to \mathrm{CAT}_\infty$ correspond to distributive functors $\mathrm{Span}(\mathbb{F}) \to \mathrm{CAT}_\infty$. Moreover, from Remark 2.6.13 we know that the product in $\mathrm{BISPAN}(\mathbb{F})$ is given by the coproduct in $\mathbb{F}$, just as in $\mathrm{Span}(\mathbb{F})$, so product-preserving functors from $\mathrm{BISPAN}(\mathbb{F})$ correspond to product-preserving distributive functors under this equivalence. By Proposition 3.1.7 and Proposition 3.1.5, the latter are equivalent to symmetric monoidal ∞-categories that are compatible with finite coproducts. $\qquad\square$

3.2. **Bispans in spaces and symmetric monoidal ∞-categories.** In this section we consider a variant of the results of the preceding one: symmetric monoidal ∞-categories can also be described in terms of spans of spaces, and we will prove that the resulting functor is distributive (with respect to all morphisms of spaces) if and only if the tensor product commutes with colimits indexed by ∞-groupoids. This applies in many examples, since most naturally occurring tensor products are compatible with all colimits.

Notation 3.2.1. We write $\mathcal{S}_{\mathrm{fin}}$ for the subcategory of $\mathcal{S}$ containing only the morphisms whose fibres are equivalent to finite sets. Then $(\mathcal{S}, \mathcal{S}_{\mathrm{fin}})$ is a span pair.

Remark 3.2.2. If $f\colon X \to I$ is a morphism in $\mathcal{S}_{\mathrm{fin}}$ and I is a finite set, then the straightening equivalence

$$\mathrm{colim}_I\colon \mathrm{Fun}(I,\mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/I}$$

implies that X is an I-indexed coproduct of finite sets, and so is itself a finite set. It follows that the functor $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ induced by the morphism of span pairs $(\mathbb{F},\mathbb{F}) \to (\mathcal{S},\mathcal{S}_{\mathrm{fin}})$ is fully faithful.

Proposition 3.2.3. *Let $\mathcal{C}$ be a complete ∞-category. Right Kan extension along the fully faithful functor $\mathrm{Span}(\mathbb{F}) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ identifies $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}),\mathcal{C})$ with the full subcategory $\mathrm{Fun}^{\mathrm{RKE}}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}),\mathcal{C})$ of $\mathrm{Fun}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}),\mathcal{C})$ spanned by functors Φ such that $\Phi|_{\mathcal{S}^{\mathrm{op}}}$ is right Kan extended from $\{*\}$.*

Proof. This is a special case of [BaHo21, Proposition C.18]. $\square$

Combining this with Proposition 3.1.5, we have:

Corollary 3.2.4. *Let $\mathcal{C}$ be a complete ∞-category. There is an equivalence*

$$\mathrm{CMon}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{RKE}}(\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}),\mathcal{C}),$$

given by right Kan extension along $\mathbb{F}_* \to \mathrm{Span}(\mathbb{F}) \hookrightarrow \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$. $\square$

Remark 3.2.5. In particular, symmetric monoidal ∞-categories can be identified with functors $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ whose restrictions to $\mathcal{S}^{\mathrm{op}}$ preserve limits. If $\mathcal{C}$ is a symmetric monoidal ∞-category, the corresponding functor $\Phi\colon \mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ admits the following description (analogous to Remark 3.1.6):

- for $X \in \mathcal{S}$, $\Phi(X) \simeq \lim_{x \in X} \Phi(\{x\}) \simeq \mathrm{Fun}(X,\mathcal{C})$,
- for $f\colon X \to Y$ in $\mathcal{S}$, the functor $f^{\circledast}\colon \mathrm{Fun}(Y,\mathcal{C}) \to \mathrm{Fun}(X,\mathcal{C})$ is given by composition with f,
- for $g\colon E \to B$ in $\mathcal{S}_{\mathrm{fin}}$, the functor $g_\otimes\colon \mathrm{Fun}(E,\mathcal{C}) \to \mathrm{Fun}(B,\mathcal{C})$ is given by tensoring fibrewise along g, i.e. for $\phi\colon E \to B$ we have

$$(g_\otimes \phi)(b) \simeq \bigotimes_{e \in E_b} \phi(e).$$

In particular, for $q\colon X \to *$ the functor $q^{\circledast}$ is the diagonal functor, while if X is a finite set the functor $q_\otimes$ is the X-indexed tensor product.

We will identify when such functors from $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$ are distributive with respect to bispan triples of the following form:

Lemma 3.2.6. *Let $\mathcal{K}$ be a full subcategory of $\mathcal{S}$ with the following properties:*

- *if $p\colon E \to B$ is a morphism in $\mathcal{S}$ such that $B \in \mathcal{K}$ and $E_b \in \mathcal{K}$ for all $b \in B$, then $E \in \mathcal{K}$,*
- *$\mathcal{K}$ is closed under finite products.*

This implies that morphisms in $\mathcal{S}$ whose fibres lie in $\mathcal{K}$ are closed under composition, giving a subcategory $\mathcal{S}_{\mathcal{K}}$ of $\mathcal{S}$. Then $(\mathcal{S},\mathcal{S}_{\mathrm{fin}},\mathcal{S}_{\mathcal{K}})$ is a bispan triple.

Proof. Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms whose fibres lie in $\mathcal{K}$. We have a morphism $X_z \to Y_z$ between the fibres of gf and g at $z \in Z$, whose fibre at $y \in Y_z$ is equivalent to X_y. Since Y_z and X_y lie in $\mathcal{K}$ for all $z \in Z, y \in Y$, it follows that X_z also lies in $\mathcal{K}$. Thus we do indeed have a subcategory $\mathcal{S}_{\mathcal{K}}$ of morphisms whose fibres lie in $\mathcal{K}$. Such morphisms are obviously preserved under base change, and so $(\mathcal{S},\mathcal{S}_{\mathcal{K}})$ is a span pair.

All distributivity diagrams exist in $\mathcal{S}$ since this is a locally cartesian closed ∞-category; see Remark 2.4.7. To show that $(\mathcal{S},\mathcal{S}_{\mathrm{fin}},\mathcal{S}_{\mathcal{K}})$ is a bispan triple it therefore

only remains to check that if $l\colon X \to Y$ is a morphism in $\mathcal{S}_{\mathcal{K}}$ and $f\colon Y \to Z$ is a morphism in $\mathcal{S}_{\mathrm{fin}}$, then $f_* l$ is also a morphism in $\mathcal{S}_{\mathcal{K}}$. But we have

$$(f_* l)_z \simeq \prod_{y \in Y_z} X_y,$$

which is a finite product of objects of $\mathcal{K}$, and so again lies in $\mathcal{K}$ by assumption. $\quad\square$

Examples 3.2.7. We can take $\mathcal{K}$ in Lemma 3.2.6 to consist of

- finite sets,
- all spaces,
- π-finite spaces[10], as follows by examining the long exact sequence in homotopy groups associated to a fibre sequence,
- κ-compact spaces[11] for any regular cardinal κ, since κ-filtered colimits in $\mathcal{S}$ commute with κ-small limits, and the κ-compact spaces are precisely the (retracts of) κ-small ∞-groupoids.

Proposition 3.2.8. *Suppose* $\Phi\colon \mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ *corresponds to a symmetric monoidal structure on* $\mathcal{C} = \Phi(*)$, *and let* $\mathcal{K}$ *be as in Lemma* 3.2.6. *Then* Φ *is* $\mathcal{K}$-*distributive if and only if* $\mathcal{C}$ *has* $\mathcal{K}$-*indexed colimits (meaning K-indexed colimits for all $K \in \mathcal{K}$), and the tensor product on* $\mathcal{C}$ *is compatible with such colimits (i.e. preserves them in each variable).*

Proof. For $K \in \mathcal{K}$, the functor $q^\circledast\colon \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$ corresponding to the morphism $q\colon K \to *$ is the diagonal. This has a left adjoint if and only if $\mathcal{C}$ admits all K-indexed colimits. Moreover, if $\mathcal{C}$ has $\mathcal{K}$-indexed colimits then the functor $f^\circledast\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ given by composition with $f\colon X \to Y$ has a left adjoint for any f in $\mathcal{S}_{\mathcal{K}}$, since all pointwise left Kan extensions along f then exist in $\mathcal{C}$. Given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{g'} & & \downarrow{g} \\ X & \xrightarrow{f} & Y \end{array}$$

in $\mathcal{S}$ with g in $\mathcal{S}_{\mathcal{K}}$, the mate transformation

$$g'_\oplus f'^\circledast \to f^\circledast g_\oplus$$

is then automatically an equivalence, since for $\phi\colon Y' \to \mathcal{C}$ this is given at $x \in X$ by the natural map

$$\mathrm{colim}_{p \in X'_x} \phi(f' p) \to \mathrm{colim}_{y \in Y'_{f(y)}} \phi(y),$$

which is an equivalence since these fibres are canonically equivalent. This proves that Φ is left adjointable if and only if $\mathcal{C}$ admits $\mathcal{K}$-indexed colimits.

Given morphisms $l\colon X \to Y$ in $\mathcal{S}_{\mathcal{K}}$ and $f\colon Y \to Z$ in $\mathcal{S}_{\mathrm{fin}}$, we have the distributivity diagram

$$\begin{array}{ccc} & Y' \xrightarrow{f'} Z' & \\ {\scriptstyle\epsilon}\swarrow & \downarrow & \downarrow{\scriptstyle h = f_* l} \\ X & & \\ {\scriptstyle l}\searrow & \downarrow & \downarrow \\ & Y \xrightarrow{f} Z, & \end{array}$$

---

[10]These are the spaces X such that (1) X is n-truncated for some n, (2) $\pi_0(X)$ is finite, and (3) for each $x \in X$, the homotopy group $\pi_k(X, x)$ is finite for each $k \geq 1$.

[11]Meaning κ-compact objects of the ∞-category $\mathcal{S}$ of spaces.

where $Z'_z \simeq \prod_{y \in Y_z} X_y$. The distributivity transformation $h_\oplus f'_\otimes \epsilon^\circledast \to f_\otimes l_\oplus$ is given for $\phi\colon X \to \mathcal{C}$ at $z \in Z$ by the canonical map

$$\mathrm{colim}_{(x_y)_y \in \prod_{y \in Y_z} X_y} \bigotimes_{y \in Y_z} \phi(x_y) \to \bigotimes_{y \in Y_z} \left(\mathrm{colim}_{x \in X_y} \phi(x) \right).$$

This is an equivalence if $\otimes$ preserves $\mathcal{K}$-indexed colimits in each variable. Conversely, for $K \in \mathcal{K}$ we have in particular the distributivity diagram

$$
\begin{array}{ccc}
K \amalg K & \xrightarrow{\nabla} & K \\
\end{array}
$$

where q is the unique map $K \to *$ and ∇ are fold maps. The corresponding distributivity transformation is given for $\phi\colon K \amalg * \to \mathcal{C}$ by

$$\mathrm{colim}_{k \in K} \left(\phi(k) \otimes \phi(*) \right) \to \left(\mathrm{colim}_{k \in K} \phi(k) \right) \otimes \phi(*).$$

If Φ is distributive then this is an equivalence for all $K \in \mathcal{K}$ and ϕ, which is precisely the condition that $\otimes$ preserves $\mathcal{K}$-indexed colimits in each variable. $\qquad\square$

Corollary 3.2.9. *Let $\mathcal{K}$ be as in Lemma 3.2.6. Then functors $\Phi\colon \mathrm{BISPAN}_{\mathrm{fin},\mathcal{K}}(\mathcal{S}) \to \mathrm{CAT}_\infty$ such that the restriction to $\mathcal{S}^{\mathrm{op}}$ preserves limits correspond to symmetric monoidal ∞-categories that are compatible with $\mathcal{K}$-indexed colimits.*

Proof. By Theorem 2.5.1, functors $\mathrm{BISPAN}_{\mathrm{fin},\mathcal{K}} \to \mathrm{CAT}_\infty$ correspond to $\mathcal{K}$-distributive functors $\mathrm{Span}_{\mathrm{f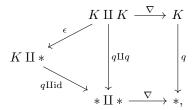in}}(\mathcal{S}) \to \mathrm{CAT}_\infty$. On the other hand, we know from Corollary 3.2.4 that such functors whose restriction to $\mathcal{S}^{\mathrm{op}}$ is right Kan extended from $\{*\}$ correspond to symmetric monoidal ∞-categories, and in this case the functor is $\mathcal{K}$-distributive if and only if the tensor product is compatible with $\mathcal{K}$-indexed colimits by Proposition 3.2.8. $\qquad\square$

3.3. **Bispans in spaces and analytic monads.** Our goal in this section is to relate bispans in the ∞-category of spaces to the polynomial and analytic functors studied in [GHK17], where it is shown that analytic monads are equivalent to ∞-operads. Using the results of the previous section we will see that there is a canonical action of every analytic monad on any symmetric monoidal ∞-category compatible with ∞-groupoid-indexed colimits.

We first give a general construction of a functor from bispans to ∞-categories using slice ∞-categories:

Proposition 3.3.1. *Suppose $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple. Then there is a functor of $(\infty, 2)$-categories $\mathrm{Sl}\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{CAT}_\infty$ such that*

- $\mathrm{Sl}(c) \simeq \mathcal{C}^L_{/c}$, *the full subcategory of $\mathcal{C}_{/c}$ spanned by the morphisms to c that lie in $\mathcal{C}_L$.*
- *for $f\colon c \to c'$ in $\mathcal{C}$, the functor $f^\circledast$ is the functor $f^*\colon \mathcal{C}^L_{/c'} \to \mathcal{C}^L_{/c}$ given by pullback along f,*
- *for $f\colon c \to c'$ in $\mathcal{C}_L$, the functor $f_\oplus$ is the functor $f_!\colon \mathcal{C}^L_{/c} \to \mathcal{C}^L_{/c'}$ given by composition with f,*
- *for $f\colon c \to c'$ in $\mathcal{C}_F$, the functor $f_\otimes$ is the functor $f_*\colon \mathcal{C}^L_{/c} \to \mathcal{C}^L_{/c'}$ given by the partial right adjoint to $f^*\colon \mathcal{C}_{/c'} \to \mathcal{C}_{/c}$.*

Proof. Let $\mathfrak{X}$ be the full subcategory of $\mathcal{C}^{\Delta^1}$ spanned by the morphisms in $\mathcal{C}_L$. Then the restriction of $\mathrm{ev}_1\colon \mathcal{C}^{\Delta^1} \to \mathcal{C}$ to a functor $\mathfrak{X} \to \mathcal{C}$ is a cartesian fibration,

with cocartesian morphisms over morphisms in $\mathcal{C}_L \subseteq \mathcal{C}$. This corresponds to a functor $\lambda\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which takes $c \in \mathcal{C}$ to the ∞-category $\mathcal{C}^L_{/c}$ and $f\colon x \to y$ to the functor $f^*\colon \mathcal{C}^L_{/y} \to \mathcal{C}^L_{/x}$ given by pullback along f. The functor λ is right F-adjointable: since $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is a bispan triple, the functor f^* for $f\colon x \to y$ in $\mathcal{C}_F$ has a right adjoint $f_*\colon \mathcal{C}^L_{/x} \to \mathcal{C}^L_{/y}$ by Remark 2.4.5, and given a cartesian square

$$
\begin{array}{ccc}
x' & \xrightarrow{f'} & y' \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\
x & \xrightarrow{f} & y
\end{array}
$$

with f and f' in $\mathcal{C}_F$, the mate transformation

$$\eta^* f_* \to f'_* \xi^*$$

is an equivalence by Lemma 2.4.6. The functor λ therefore extends canonically to a functor

$$\Lambda\colon \mathrm{SPAN}_F(\mathcal{C})^{2\text{-op}} \to \mathrm{CAT}_\infty$$

by Theorem 2.2.7. We claim the underlying functor of ∞-categories $\lambda'\colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{Cat}_\infty$ is L-distributive. Certainly if $l\colon x \to y$ is a morphism in $\mathcal{C}_L$, then the pullback functor $l^*\colon \mathcal{C}^L_{/y} \to \mathcal{C}^L_{/x}$ has a left adjoint $l_!$ given by composition with l; to see that the restriction of λ' to $\mathcal{C}^{\mathrm{op}}$ is left L-adjointable it remains to observe that for any cartesian square

$$
\begin{array}{ccc}
x' & \xrightarrow{l'} & y' \\
\xi\downarrow & & \downarrow{\scriptstyle \eta} \\
x & \xrightarrow{l} & y
\end{array}
$$

with l and l' in $\mathcal{C}_L$, the natural transformation $l'_! \xi^* \to \eta^* l_!$ is an equivalence, since for $g\colon z \to x$ in $\mathcal{C}^L_{/x}$ in the diagram

$$
\begin{array}{ccccc}
z' & \xrightarrow{g'} & x' & \xrightarrow{l'} & y' \\
\downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \eta} \\
z & \xrightarrow{g} & y & \xrightarrow{l} & x,
\end{array}
$$

the left square is cartesian if and only if the composite square is cartesian. To see that λ' is also L-distributive, consider $l\colon x \to y$ in $\mathcal{C}_L$ and $f\colon y \to z$ in $\mathcal{C}_F$ and form a distributivity diagram (20). The distributivity transformation

$$g_! \tilde{f}_* \epsilon^* \to f_* l_!$$

evaluated at $l'\colon c \to x$ is a canonical map

$$f_* l \circ \tilde{f}_* \epsilon^* l \to f_*(l \circ l');$$

this is an equivalence by Lemma 2.5.16. It follows by Theorem 2.5.1 that λ' extends uniquely to a functor $\mathrm{Sl}\colon \mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{CAT}_\infty$, which by construction has the required properties. $\qquad\square$

Applying this to the bispan triple $(\mathcal{S}, \mathcal{S}, \mathcal{S})$ we get in particular:

Corollary 3.3.2. *There is a functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *taking* $X \in \mathcal{S}$ *to* $\mathcal{S}_{/X}$ *and a bispan*

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

to the functor $t_! p_* s^*\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$, *where* s^* *is given by pullback along* s, p_* *is the right adjoint to* p^*, *and* $t_!$ *is given by composition with* t. $\qquad\square$

Definition 3.3.3. A *polynomial functor* $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is an accessible functor that preserves weakly contractible limits. By [GHK17, Theorem 2.2.3] the polynomial functors are equivalently those functors obtained as composites $t_! p_* s^*$ for some bispan of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y.$$

Let $\mathrm{PolyFun}(\mathcal{S})$ be the sub-$(\infty, 2)$-category of CAT_∞ whose objects are the slices $\mathcal{S}_{/X}$ for $X \in \mathcal{S}$, whose 1-morphisms are the polynomial functors, and whose 2-morphisms are the cartesian natural transformations.

Remark 3.3.4. The $(\infty, 2)$-category $\mathrm{PolyFun}(\mathcal{S})$ is the underlying $(\infty, 2)$-category of the double ∞-category of polynomial functors considered in [GHK17].

Corollary 3.3.5. *The functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *restricts to an equivalence*

$$\mathrm{BISPAN}(\mathcal{S}) \xrightarrow{\sim} \mathrm{PolyFun}(\mathcal{S}).$$

Proof. We apply the description from Corollary 2.5.15 to understand Sl: On objects, Sl takes $X \in \mathcal{S}$ to the ∞-category $\mathcal{S}_{/X}$, and on morphisms it takes the bispan

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y$$

to the functor $t_! p_* s^*$. The functors of this form are precisely the polynomial functors, by [GHK17, Theorem 2.2.3]. The description of Sl on 2-morphisms from Corollary 2.5.15(3) implies that they are sent to composites of equivalences and (co)unit transformations that are cartesian by [GHK17, Lemma 2.1.5]. Hence Sl factors through $\mathrm{PolyFun}(\mathcal{S})$, and is essentially surjective on objects and morphisms. To show that Sl factors through an equivalence it then suffices to show it gives an equivalence

$$\mathrm{MAP}_{\mathrm{BISPAN}(\mathcal{S})}(X, Y) \to \mathrm{MAP}_{\mathrm{PolyFun}(\mathcal{S})}(X, Y)$$

on mapping ∞-categories for all X, Y in $\mathcal{S}$. Using Corollary 2.3.18 we can identify this with the functor shown to be an equivalence in [GHK17, Proposition 2.4.13]. $\qquad \square$

Remark 3.3.6. We expect that there is an analogue of Corollary 3.3.5 for bispans in any ∞-topos $\mathcal{X}$, but this requires working with *internal* ∞-categories in $\mathcal{X}$ (or equivalently sheaves of ∞-categories on $\mathcal{X}$): A key step in the identification of polynomial functors with bispans is the description of colimit-preserving functors between slices of $\mathcal{S}$ as spans, i.e. the equivalence

$$\mathrm{Fun}^L(\mathcal{S}_{/X}, \mathcal{S}_{/Y}) \simeq \mathrm{Fun}(X, \mathcal{S}_{/Y}) \simeq \mathrm{Fun}(X \times Y, \mathcal{S}) \simeq \mathcal{S}_{/X \times Y}.$$

This certainly fails for any other ∞-topos $\mathcal{X}$, but an analogous statement should hold if we view $\mathcal{X}$ instead as an ∞-category internal to itself.

Definition 3.3.7. An *analytic functor* $F\colon \mathcal{S}_{/X} \to \mathcal{S}_{/Y}$ is a functor that preserves sifted colimits and weakly contractible limits. By [GHK17, Proposition 3.1.9] the analytic functors are equivalently those functors obtained as composites $t_! p_* s^*$ for some bispan of spaces

$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} Y,$$

where p has finite discrete fibres.

As a consequence, we get:

Corollary 3.3.8. *The functor* $\mathrm{Sl}\colon \mathrm{BISPAN}(\mathcal{S}) \to \mathrm{CAT}_\infty$ *factors through an equivalence*

$$(37) \qquad\qquad \mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}) \xrightarrow{\sim} \mathrm{AnFun}(\mathcal{S}),$$

where $\mathrm{AnFun}(\mathcal{S})$ *is the locally full sub-$(\infty, 2)$-category of* $\mathrm{PolyFun}(\mathcal{S})$ *containing all objects, with the analytic functors as morphisms, as well as all 2-morphisms between these.* $\qquad\square$

Definition 3.3.9. An *analytic monad* is a monad in the $(\infty, 2)$-category $\mathrm{AnFun}(\mathcal{S})$, i.e. an associative algebra in the monoidal ∞-category $\mathrm{MAP}_{\mathrm{AnFun}(\mathcal{S})}(X, X)$ of endomorphisms of some object X, or a functor $\mathfrak{mnd} \to \mathrm{AnFun}(\mathcal{S})$, where $\mathfrak{mnd}$ is the universal 2-category containing a monad. In other words, it is a monad on the ∞-category $\mathcal{S}_{/X}$ whose underlying endofunctor is analytic and whose unit and multiplication transformations are cartesian. From the equivalence (37) we see that analytic monads are equivalently monads in the $(\infty, 2)$-category $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S})$.

Corollary 3.3.10. *Suppose T is an analytic monad on $\mathcal{S}_{/X}$ and $\mathcal{V}$ is a symmetric monoidal ∞-category compatible with ∞-groupoid-indexed colimits. Then T induces a canonical monad $T_{\mathcal{V}}$ on $\mathrm{Fun}(X, \mathcal{V})$.*

Proof. We can identify T with a monad on $X \in \mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S})$. By Corollary 3.2.9 $\mathcal{V}$ induces a functor $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{CAT}_\infty$ that takes $Y \in \mathcal{S}$ to $\mathrm{Fun}(Y, \mathcal{V})$. Any functor of $(\infty, 2)$-categories preserves monads, since they can be described as simply functors of $(\infty, 2)$-categories from $\mathfrak{mnd}$. Hence under the functor induced by $\mathcal{V}$ the monad T maps to a monad in CAT_∞ which indeed acts on $\mathrm{Fun}(X, \mathcal{V})$. $\qquad\square$

Remark 3.3.11. Suppose the underlying bispan of the monad T is
$$X \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} X.$$
Then the underlying endofunctor of the monad $T_{\mathcal{V}}$ is given by
$$(T_{\mathcal{V}}\phi)(x) \simeq \mathrm{colim}_{b \in B_x} \bigotimes_{e \in E_b} \phi(s(e)).$$

This has the same form as the formula for the free algebra monad of an ∞-operad, and the main result of [GHK17] is that analytic monads are equivalent to ∞-operads in the form of dendroidal Segal spaces. We therefore expect that if $\mathcal{O}$ is the ∞-operad corresponding to T, then the monad $T_{\mathcal{V}}$ is the free $\mathcal{O}$-algebra monad for $\mathcal{O}$-algebras in $\mathcal{V}$.

3.4. **Equivariant bispans and G-symmetric monoidal ∞-categories.** In this section we look at the G-equivariant version of our results from §3.1 on symmetric monoidal ∞-categories compatible with finite coproducts, where G is a finite group: we replace the category of finite sets by the category $\mathbb{F}_G$ of finite G-sets, and consider when a G-symmetric monoidal ∞-category, defined as a product-preserving functor $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$, is distributive, and so extends to a functor $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$.

Definition 3.4.1. Let G be a finite group, and BG the corresponding 1-object groupoid. We write $\mathbb{F}_G$ for the category $\mathrm{Fun}(BG, \mathbb{F})$ of finite G-sets, and $\mathcal{O}_G$ for the full subcategory of *orbits*, i.e. finite G-sets of the form G/H where H is a subgroup of G. Then $\mathbb{F}_G$ is obtained from $\mathcal{O}_G$ by freely adding finite coproducts, so that for any ∞-category $\mathcal{C}$ with finite products, restriction along the inclusion $\mathcal{O}_G \hookrightarrow \mathbb{F}_G$ gives an equivalence
$$\mathrm{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathcal{C}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{O}_G^{\mathrm{op}}, \mathcal{C}).$$
If $\mathcal{C}$ is the ∞-category of spaces, this says that the ∞-category $\mathcal{S}_G := \mathcal{P}(\mathcal{O}_G)$ of *G-spaces*[12] is equivalent to $\mathrm{Fun}^\times(\mathbb{F}_G^{\mathrm{op}}, \mathcal{S})$. By analogy with the case of G-spaces,

---

[12]By Elmendorf's theorem [Elm83] the ∞-category $\mathcal{S}_G$ is equivalent to that obtained from the category of topological spaces with G-action by inverting the maps that give weak homotopy equivalences on all spaces of fixed points.

we can think of a functor $\mathcal{F}\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{C}$ as an "object of $\mathcal{C}$ with G-action", with $\mathcal{F}^H := \mathcal{F}(G/H)$ the object of "H-fixed points" of $\mathcal{F}$. We will in particular apply this notation for functors $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$, which we will call G-∞-*categories*.

Remark 3.4.2. The category $\mathbb{F}_G$ is extensive, and so by [Bar14, Proposition 4.3] the coproduct in $\mathbb{F}_G$ gives both the product and coproduct in $\mathrm{Span}(\mathbb{F}_G)$.

Definition 3.4.3. Let $\mathcal{C}$ be an ∞-category with finite products. A G-*commutative monoid* in $\mathcal{C}$ is a product-preserving functor $\mathrm{Span}(\mathbb{F}_G) \to \mathcal{C}$. We write

$$\mathrm{CMon}_G(\mathcal{C}) := \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_G), \mathcal{C})$$

for the ∞-category of G-commutative monoids in $\mathcal{C}$. A G-*symmetric monoidal ∞-category* is a G-commutative monoid in Cat_∞.

Remark 3.4.4. When G is the trivial group this is equivalent to the usual definition of commutative monoids (in terms of functors from $\mathbb{F}_*$ satisfying a Segal condition) by Proposition 3.1.5. More generally, see [Nar16, Theorem 6.5] for an alternative description of G-commutative monoids in terms of "finite pointed G-sets" (where this must be read in a non-trivial parametrized sense).

Remark 3.4.5. A functor $\mathcal{F}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathcal{C}$ preserves products if and only if the restriction to $\mathbb{F}_G^{\mathrm{op}} \to \mathcal{C}$ preserves products, and so is determined by its restriction to $\mathcal{O}_G^{\mathrm{op}} \to \mathcal{C}$. The additional structure given by the forwards maps in $\mathrm{Span}(\mathbb{F}_G)$ can be decomposed into

- multiplication maps $\mathcal{F}^H \times \mathcal{F}^H \to \mathcal{F}^H$ for each subgroup H of G, coming from the fold map $G/H \amalg G/H \to G/H$,
- multiplicative *transfer* maps $\mathcal{F}^H \to \mathcal{F}^K$ for each inclusion $H \subseteq K$ of subgroups, coming from the quotient map $G/H \to G/K$,

together with various homotopy-coherent compatibilities that in particular make each $\mathcal{F}^H$ a commutative monoid.

Remark 3.4.6. Grouplike G-commutative monoids in $\mathcal{S}$ can be identified with connective genuine G-spectra, by [Nar16, Corollary A.4.1]. Applying π_0, such a grouplike G-commutative monoid induces a G-commutative monoid in Set, which factors through a product-preserving functor

$$\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Ab}$$

because it is grouplike — this is precisely a *Mackey functor*, which is well-known as the structure appearing as π_0 of a genuine G-spectrum.

For a functor $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ we can simplify the condition that it is left adjointable as follows:

Proposition 3.4.7. *Suppose* $\mathcal{F}\colon \mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ *is a product-preserving functor. Then* $\mathcal{F}$ *is left adjointable if and only if*

(1) for every subgroup $H \subseteq G$, the ∞-category $\mathcal{F}^H$ has finite coproducts,

(2) for every inclusion of subgroups $H \subseteq K$ the functor $(q_H^K)^\circledast\colon \mathcal{F}^K \to \mathcal{F}^H$, corresponding to the quotient map $q_H^K\colon G/H \to G/K$, has a left adjoint $(q_H^K)_\oplus$,

(3) for every inclusion of subgroups $H \subseteq K$ the functor $(q_H^K)^\circledast$ preserves finite coproducts,

(4) for subgroups $H, K \subseteq L$, let X be defined by the pullback

$$\begin{array}{ccc} X & \xrightarrow{f_K} & G/K \\ \downarrow{\scriptstyle f_H} & & \downarrow{\scriptstyle q_K^L} \\ G/H & \xrightarrow{q_H^L} & G/L; \end{array}$$

then the square

$$\begin{array}{ccc} \mathcal{F}^L & \xrightarrow{(q_H^L)^{\circledast}} & \mathcal{F}^H \\ {\scriptstyle (q_K^L)^{\circledast}}\downarrow & & \downarrow{\scriptstyle f_H^{\circledast}} \\ \mathcal{F}^K & \xrightarrow{f_K^{\circledast}} & \mathcal{F}(X) \end{array}$$

is left adjointable, i.e. the mate transformation

$$f_{K,\oplus} f_H^{\circledast} \to (q_K^L)^{\circledast}(q_H^L)_{\oplus}$$

is an equivalence.

Remark 3.4.8. The pullback X in condition (4) can be decomposed into a sum of orbits indexed by double cosets:

$$X \cong \coprod_{[g] \in H \backslash L / K} H \cap K_g,$$

where K_g denotes the conjugate gKg^{-1}. The left adjointability in (4) then amounts to the following *double coset formula*:

$$(q_K^L)^{\circledast}(q_H^L)_{\oplus} \simeq \coprod_{[g] \in H \backslash L / K} c_{g,\oplus}(q_{H \cap K_g}^{K_g})_{\oplus}(q_{H \cap K_g}^{H})^{\circledast},$$

where c_g is the isomorphism $G/K \cong G/K_g$.

Proof of Proposition 3.4.7. Since $\mathbb{F}_G$ is extensive, a morphism $\phi\colon X \to Y$ in $\mathbb{F}_G$ where $Y \cong \coprod_i G/H_i$ decomposes as a coproduct $\coprod_i \phi_i$ for $\phi_i\colon X_i \to G/H_i$. Since $\mathcal{F}$ is product-preserving, to show that $\phi^{\circledast}$ has a left adjoint it suffices to consider the case where Y is an orbit G/H. Moreover, for $\phi\colon X \to G/H$ where $X \cong \coprod_j G/K_j$ we can decompose ϕ as

$$\coprod_j G/K_j \xrightarrow{\coprod_j \phi|_{G/K_j}} \coprod_j G/H \xrightarrow{\nabla} G/H$$

where ∇ denotes the fold map. Since adjunctions compose, to prove that left adjoints exist it is enough to consider fold maps and morphisms between orbits. In the first case, the functor $\nabla^{\circledast}$ induced by the fold map $\nabla\colon \coprod_{j \in J} G/H \to G/H$ can be identified with the diagonal functor $\mathcal{F}^H \to \mathrm{Fun}(J, \mathcal{F}^H)$ and so has a left adjoint for all finite sets J if and only if $\mathcal{F}^H$ admits finite coproducts, i.e. if and only if assumption (1) holds. In the second case, a morphism $\phi\colon G/K \to G/H$ can be decomposed as $G/K \xrightarrow{\sim} G/gKg^{-1} \xrightarrow{q_{gKg^{-1}}^H} G/H$ and so $\phi^{\circledast}$ has a left adjoint for all such maps ϕ if and only if assumption (2) holds.

Now we consider the adjointability condition. Again using that $\mathbb{F}_G$ is extensive, a pullback square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$

where $W \cong \coprod_i G/H_i$ decomposes as a coproduct of pullback squares

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ Z_i & \longrightarrow & G/H_i. \end{array}$$

Since $\mathcal{F}$ preserves products and taking mate squares commutes with products, we see that $\mathcal{F}$ is left adjointable if and only if it is left adjointable for pullback squares

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Z & \longrightarrow & G/H
\end{array}
$$

over an orbit. If $Y \cong \coprod_i G/K_i$ and $Z \cong \coprod_j G/L_j$ then we can decompose our pullback square into the diagram

$$
\begin{array}{ccccc}
X & \longrightarrow & \coprod_{i,j} G/K_i & \longrightarrow & \coprod_i G/K_i \\
\downarrow & & \downarrow & & \downarrow \\
\coprod_{i,j} G/L_j & \longrightarrow & \coprod_{i,j} G/H & \longrightarrow & \coprod_i G/H \\
\downarrow & & \downarrow & & \downarrow \\
\coprod_j G/L_j & \longrightarrow & \coprod_j G/H & \longrightarrow & G/H.
\end{array}
$$

where the top left square decomposes as a coproduct of pullback squares

$$
\begin{array}{ccc}
X_{i,j} & \longrightarrow & G/K_i \\
\downarrow & & \downarrow \\
G/L_j & \longrightarrow & G/H
\end{array}
$$

of the form considered in (4) and the other squares are defined using fold maps. Since mate squares are compatible with both vertical and horizontal composition of squares, the functor $\mathcal{F}$ will be left adjointable if the images of the four squares in such decompositions are left adjointable. Using again the assumption that $\mathcal{F}$ preserves finite products, we see that this holds if and only if (4) holds and we have left adjointability for squares of the form

$$
\begin{array}{ccc}
\coprod_{i \in I} G/K & \xrightarrow{\nabla_K} & G/K \\
\downarrow{\scriptstyle \coprod_{i \in I} \phi} & & \downarrow{\scriptstyle \phi} \\
\coprod_{i \in I} G/H & \xrightarrow{\nabla_H} & G/H.
\end{array}
$$

The latter means the canonical map $\nabla_{K,\oplus}(\prod_i \phi^{\circledast}) \to \phi^{\circledast} \nabla_{H,\oplus}$ is an equivalence, i.e. the functor $\phi^{\circledast}$ preserves I-indexed coproducts where ϕ is a map between orbits in $\mathbb{F}_G$. Since such maps are composites of isomorphisms and maps coming from subgroup inclusions, this is equivalent to condition (3). $\qquad\square$

Definition 3.4.9. We say a G-∞-category $\mathcal{F}$ has *additive transfers* if it satisfies the conditions of Proposition 3.4.7 when viewed as a product-preserving functor $\mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

Remark 3.4.10. In the terminology of [Sha18, Nar16] a G-∞-category has *finite G-coproducts* if and only if it has additive transfers in our sense, cf. [Nar16, Proposition 2.11].

Proposition 3.4.11. *Suppose $\mathcal{F} \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ is a G-symmetric monoidal ∞-category whose underlying G-∞-category has additive transfers. Then $\mathcal{F}$ is distributive if and only if for all morphisms $\phi \colon X \to Y, \psi \colon Y \to G/H$ in $\mathbb{F}_G$, the distributivity transformation*

$$
g_\oplus \tilde{\psi}_\otimes \epsilon^{\circledast} \to \psi_\otimes \phi_\oplus
$$

from the distributivity square

$$
\begin{array}{ccc}
& W \xrightarrow{\tilde{\psi}} Z & \\
\epsilon \swarrow \ \ \downarrow \tilde{g} & & \downarrow g \\
X & & \\
\phi \searrow \ \ \downarrow & & \\
& Y \xrightarrow{\psi} G/H,
\end{array}
$$

is an equivalence.

Remark 3.4.12. Decomposing $Z = \psi_* X$ in such a distributivity diagram as a coproduct of orbits is often a non-trivial problem in finite group theory, so we do not expect that this distributivity condition can be simplified further in general.

Proof of Proposition 3.4.11. Since $\mathbb{F}_G$ is extensive, we know from Example 2.6.12 that finite coproducts of distributivity diagrams are again distributivity diagrams. Since $\mathcal{F}$ preserves products, distributivity transformations associated to such coproducts decompose as products, hence the distributivity condition reduces to the case where the target of the second map is an orbit. $\square$

Definition 3.4.13. We say a G-symmetric monoidal ∞-category $\mathcal{F} \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ is *compatible with additive transfers* if it satisfies the condition of Proposition 3.4.11.

Corollary 3.4.14. *Product-preserving functors* $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ *correspond to G-symmetric monoidal ∞-categories that are compatible with additive transfers.*

Proof. By Theorem 2.5.1, functors $\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$ correspond to distributive functors $\mathrm{Span}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$. Moreover, from Remark 2.6.13 we know that the product in $\mathrm{BISPAN}(\mathbb{F}_G)$ is given by the coproduct in $\mathbb{F}_G$, just as in $\mathrm{Span}(\mathbb{F}_G)$, so product-preserving functors from $\mathrm{BISPAN}(\mathbb{F}_G)$ correspond to product-preserving distributive functors under this equivalence. By Proposition 3.4.7 and Proposition 3.4.11, the latter are equivalent to G-symmetric monoidal ∞-categories that are compatible with additive transfers. $\square$

We now consider some examples of G-symmetric monoidal ∞-categories compatible with additive transfers:

Example 3.4.15 (Finite G-sets)**.** As a special case of Proposition 3.3.1 we get a functor
$$\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}$$
taking $X \in \mathbb{F}_G$ to the slice $(\mathbb{F}_G)_{/X}$. Here we can identify $(\mathbb{F}_G)_{/(G/H)}$ with $\mathbb{F}_H$, so the underlying G-category is given by $(G/H) \mapsto \mathbb{F}_H$. Since $\mathbb{F}_G$ is extensive, this is a product-preserving functor; the underlying G-symmetric monoidal category encodes the cartesian products of finite G-sets and their compatibility with the left and right adjoints to the restriction functor $\mathbb{F}_G \to \mathbb{F}_H$ for H a subgroup of G.

Example 3.4.16 (G-spaces)**.** As a variant of the previous example, we can consider the G-∞-category of G-spaces. Since $\mathcal{S}_G$ is locally cartesian closed (being a (presheaf) ∞-topos), we can apply Proposition 3.3.1 to it and then restrict to bispans in $\mathbb{F}_G$ to get a functor
$$\mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}_\infty$$
that takes $X \in \mathbb{F}_G$ to $(\mathcal{S}_G)_{/X}$; here we can identify $(\mathcal{S}_G)_{/(G/H)}$ with $\mathcal{S}_H$. Since $\mathcal{S}_G$ is extensive, this is a product-preserving functor. The underlying G-symmetric monoidal ∞-category (compatible with additive transfers) encodes the cartesian

products of H-spaces for all subgroups H of G and their compatibility with the left and right adjoints to the restriction functor $\mathcal{S}_G \to \mathcal{S}_H$.

Example 3.4.17 (G-actions in a symmetric monoidal ∞-category). Let $\mathcal{C}$ be a symmetric monoidal ∞-category. By Proposition 3.2.3 this determines a functor $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}) \to \mathrm{Cat}_\infty$ taking $X \in \mathcal{S}$ to $\mathrm{Fun}(X, \mathcal{C})$. For a finite group G we have a functor $\mathcal{G} \colon \mathbb{F}_G \to \mathcal{S}$ by restricting the colimit functor $\mathrm{Fun}(BG, \mathcal{S}) \to \mathcal{S}$; this takes a finite G-set X to the groupoid $X/\!/G = X_{hG}$. The functor $\mathcal{G}$ preserves pullbacks since the colimit functor factors as the straightening equivalence $\mathrm{Fun}(BG, \mathcal{S}) \xrightarrow{\sim} \mathcal{S}_{/BG}$ followed by the forgetful functor to $\mathcal{S}$, which preserves all weakly contractible limits. Moreover, $\mathcal{G}$ takes values in $\mathcal{S}_{\mathrm{fin}}$, and so yields a functor $\mathrm{Span}(\mathcal{G}) \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Span}_{\mathrm{fin}}(\mathcal{S})$. This functor preserves products, since $\mathcal{G}$ preserves coproducts. It follows that we can restrict along $\mathrm{Span}(\mathcal{G})$ and obtain for any symmetric monoidal ∞-category $\mathcal{C}$ a G-symmetric monoidal structure on $\mathrm{Fun}(BG, \mathcal{C})$. Moreover, if the tensor product in $\mathcal{C}$ is compatible with finite coproducts, this G-symmetric monoidal structure will be compatible with additive transfers.

Example 3.4.18 (G-representations). As a special case of the previous example, we can take $\mathcal{C}$ to be the category Vect_k of k-vector spaces with the tensor product as symmetric monoidal structure. We then obtain a functor

$$\rho_k \colon \mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}$$

such that $\rho_k(G/H)$ is the category $\mathrm{Rep}_H(k) := \mathrm{Fun}(BH, \mathrm{Vect}_k)$ of H-representations and for for subgroups $H \subseteq K \subseteq G$,

- $(q_H^K)^\circledast$ is the restriction functor $\mathrm{Res}_H^K \colon \mathrm{Rep}_K(k) \to \mathrm{Rep}_H(k)$,
- $(q_H^K)_\oplus$ is the induced representation functor $\mathrm{Ind}_H^K \colon \mathrm{Rep}_H(k) \to \mathrm{Rep}_K(k)$, left adjoint to Res_H^K, and given on objects by taking an H-representation V to $\bigoplus_{K/H} V$ with induced action of K,
- $(q_H^K)_\otimes$ is the *tensor-induction functor* $\mathrm{Rep}_H(k) \to \mathrm{Rep}_H(k)$, given on objects by taking an H-representation V to $\bigotimes_{K/H} V$ with induced action of K.

As a more sophisticated version of this construction, we can instead consider the ∞-category Perf_R of perfect (i.e. dualizable) modules over an $\mathbb{E}_\infty$-ring spectrum R. Plugging this into the previous example we see that the ∞-categories $\mathrm{Rep}_H^\infty(R) := \mathrm{Fun}(BH, \mathrm{Perf}_R)$ fit together into a functor

$$\rho_R^\infty \colon \mathrm{BISPAN}(\mathbb{F}_G) \to \mathrm{CAT}$$

such that $\rho_R^\infty(G)(G/H) = \mathrm{Rep}_H^\infty(R)$. This example will be important when we discuss Tambara functors arising from the algebraic K-theory of group actions in the next section.

Our final example of a G-symmetric monoidal ∞-category compatible with additive transfers is the ∞-category of genuine G-spectra. This is less formal than our previous examples, but the input we need is already in the literature:

Proposition 3.4.19. *The ∞-category Sp^G of genuine G-spectra is a G-symmetric monoidal ∞-category compatible with additive transfers.*

Proof. Taking fixed point spectra for subgroups of G gives a functor $\mathcal{O}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ that takes G/H to the ∞-category Sp^H of genuine H-spectra; this is the G-∞-category of G-spectra. The corresponding product-preserving functor $\mathbb{F}_G^{\mathrm{op}} \to \mathrm{Cat}_\infty$ extends to a functor $\sigma_G \colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ such that for $H \subseteq K \subseteq G$ the functor $(q_H^K)_\otimes \colon \mathrm{Sp}^H \to \mathrm{Sp}^K$ is the multiplicative norm of [HHR16]; this follows from the results of [BaHo21, §9] by restricting the functor from spans of profinite groupoids defined there. For $\phi \colon X \to Y$ in $\mathbb{F}_G$, the functor $\phi^\circledast \colon \sigma_G(Y) \to \sigma_G(X)$ has a left

adjoint by [BaHo21, Lemma 9.7(2)] — for $H \subseteq K \subseteq G$ the left adjoint $(q_H^K)_\oplus$ is the classical (additive) *transfer* or induction functor. To see that the functor σ_G is left adjointable we check the 3 remaining conditions in Proposition 3.4.7: conditions (1) and (3) hold since the ∞-categories of G-spectra are stable (hence any right adjoint functor between them automatically preserves finite colimits). To check condition (4) we use that any H-spectrum is a sifted colimit of desuspensions of suspension spectra $\Sigma_+^\infty X$ with X a finite H-set, and the functors involved preserve (sifted) colimits and desuspensions. Hence it suffices to check that the canonical map

$$f_{K,\oplus} f_H^\circledast \Sigma_+^\infty X \to (q_K^L)^\circledast (q_H^L)_\oplus \Sigma_+^\infty X$$

is an equivalence for $X \in \mathbb{F}_H$. But here all the functors are given on suspension spectra of finite H-sets by the suspension spectra on the corresponding functors for finite G-sets, so this follows from Example 3.4.15. The same argument works for distributivity, since we also have $f_\otimes \Sigma_+^\infty X \simeq \Sigma_+^\infty f_* X$. $\qquad\square$

Remark 3.4.20. The adjointability condition here reduces by Remark 3.4.8 to a double coset formula for additive transfers. This is a basic fact in equivariant stable homotopy theory that has surely long been well-known to the experts, but the only explicit references we could find are [HHR16, Proposition A.30] (applied to the direct sum in orthogonal spectra) and [Pat16, Corollary 5.2]. The distributivity condition also appears (in terms of orthogonal spectra) as [HHR16, Proposition A.37].

Variant 3.4.21. Following Blumberg and Hill [BlHi18] we can consider subcategories $\mathbb{F}_{G,\mathcal{I}}$ where $\mathcal{I}$ is an *indexing system* as in [BlHi18, Definition 1.2]; by [BlHi18, Theorem 1.4] these are precisely the subcategories of $\mathbb{F}_G$ such that $(\mathbb{F}_G, \mathbb{F}_{G,\mathcal{I}})$ is a span pair. Since $\mathbb{F}_G$ is locally cartesian closed, we then have a bispan triple $(\mathbb{F}_G, \mathbb{F}_{G,\mathcal{I}}, \mathbb{F}_G)$. Product-preserving functors out of $\mathrm{Span}_{\mathcal{I}}(\mathbb{F}_G)$ are G-symmetric monoidal ∞-categories where only some subclass of multiplicative norms exist, and we can characterize distributivity for such functors by the analogue of Proposition 3.4.11 with the map ψ restricted to lie in $\mathbb{F}_{G,\mathcal{I}}$.

Variant 3.4.22. We can also consider a G-equivariant analogue of §3.2: Using [BaHo21, Proposition C.18] a G-symmetric monoidal ∞-category determines by right Kan extension a functor $\mathrm{Span}_{\mathrm{fin}}(\mathcal{S}_G) \to \mathrm{Cat}_\infty$, where $\mathcal{S}_{G,\mathrm{fin}}$ denotes the subcategory of $\mathcal{S}_G$ containing the maps $\phi \colon X \to Y$ such that for every map $G/H \to Y$, the pullback $X \times_Y G/H$ is a finite G-set. Here $(\mathcal{S}_G, \mathcal{S}_{G,\mathrm{fin}}, \mathcal{S}_G)$ is a bispan triple, and we might say that the G-symmetric monoidal ∞-category is "compatible with G-space-indexed G-colimits" if this is distributive. We expect that this should hold for the G-symmetric monoidal ∞-category of genuine G-spectra and, by analogy with [GHK17], that monads in the $(\infty, 2)$-category $\mathrm{BISPAN}_{\mathrm{fin}}(\mathcal{S}_G)$ should be related to a notion of G-∞-*operads* [BDG$^+$16].

Variant 3.4.23. In [BaHo21, Chapter 9], Bachmann and Hoyois define ∞-categories of equivariant spectra for *profinite groupoids*, and we can also consider distributivity in this setting. We can take the $(2,1)$-category of finite groupoids $\mathrm{FinGpd} \subset \mathcal{S}$ to be the full subcategory of spaces spanned by 1-truncated spaces with finite π_0, π_1. We then form the $(2,1)$-category of profinite groupoids by taking pro-objects: $\mathrm{ProfGpd} := \mathrm{Pro}(\mathrm{FinGpd})$. Let $\mathrm{ProfGpd}_{\mathrm{fp}}$ be the subcategory containing only the *finitely presented* maps as in [BaHo21, §9.1]. It then follows from [BaHo21, Lemmas 9.3 and 9.5] and Lemma 2.4.6 that we have a bispan triple

$$(\mathrm{ProfGpd}, \mathrm{ProfGpd}_{\mathrm{fp}}, \mathrm{ProfGpd}).$$

In [BaHo21, Chapter 9] equivariant spectra are defined as a functor

$$\mathrm{Span}_{\mathrm{fp}}(\mathrm{ProfGpd}) \to \mathrm{Cat}_\infty;$$

we expect that this is distributive, giving a functor of $(\infty, 2)$-categories

$$\mathrm{BISPAN}_{\mathrm{fp}}(\mathrm{ProfGpd}) \to \mathrm{CAT}_\infty.$$

3.5. **Motivic bispans and normed ∞-categories.** In this section we will relate the *normed ∞-categories* of Bachmann–Hoyois to functors from certain bispans in schemes (and more generally algebraic spaces) to CAT_∞. As an example of this, we will see that the results of [BaHo21] imply that ∞-categories of motivic spectra give such a functor. We begin by describing some bispan triples on schemes.

Warning 3.5.1. Throughout this section, schemes and algebraic spaces are always assumed to be quasi-compact and quasi-separated (qcqs).[13]

Notation 3.5.2. If S is a (qcqs) scheme, we write Sch_S for the category of (qcqs) schemes over S. (This has pullbacks since qcqs morphisms are closed under base change, see [Stacks, Tags 01KU and 01K5].)

Proposition 3.5.3. *The following are bispan triples:*
  (i) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{flf}}, \mathrm{Sch}_S^{\mathrm{qp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{flf}}$ consists of finite locally free (meaning finite, flat, and of finite presentation) morphisms of S-schemes and $\mathrm{Sch}_S^{\mathrm{qp}}$ of quasiprojective morphisms of S-schemes,*
  (ii) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{qp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{fét}}$ consists of finite étale morphisms of S-schemes,*
  (iii) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{flf}}, \mathrm{Sch}_S^{\mathrm{smqp}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{smqp}}$ consists of smooth and quasiprojective morphisms of S-schemes,*
  (iv) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{smqp}})$ *for any scheme S.*
  (v) $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{fét}}, \mathrm{Sch}_S^{\mathrm{proj}})$ *for any scheme S, where $\mathrm{Sch}_S^{\mathrm{proj}}$ consists of projective morphisms of S-schemes.*

Proof. The classes of morphisms of schemes that are finite locally free, quasiprojective, smooth, finite, and étale are all closed under base change by [Stacks, Tags 02KD,0B3G,01VB,01WL,02GO], respectively. Hence the subcategories $\mathrm{Sch}_S^{\mathrm{flf}}$, $\mathrm{Sch}_S^{\mathrm{qp}}$, $\mathrm{Sch}_S^{\mathrm{smqp}}$, and $\mathrm{Sch}_S^{\mathrm{fét}}$ of Sch_S all give span pairs.

Now the main point is the existence of *Weil restrictions* for schemes: if $f\colon S' \to S$ is a morphism of schemes and X is an S'-scheme, the Weil restriction $R_f X$, if it exists, is an S-scheme that represents the functor

$$\mathrm{Hom}_{/S'}((-) \times_{S'} S, X)\colon \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Set};$$

note that this is exactly the requirement (21) for a distributivity diagram for $X \to S'$ and f.

By [BLR90, Theorem 7.6.4], the Weil restriction $R_f X$ exists if f is a finite locally free morphism and X is quasiprojective. Moreover, $R_f X$ is quasiprojective over S by [BaHo21, Lemma 2.13]. This gives the bispan triple (i), from which (ii) is trivially obtained by restricting from finite locally free morphisms to the subclass of finite étale ones. For (iii) and (iv) the only additional input needed is that R_f takes smooth morphisms to smooth morphisms, which holds by [BLR90, Proposition 7.6.5(h)], while for (v) we use that R_f preserves proper morphisms if f is finite étale by [BLR90, Proposition 7.6.5(f)] and that a morphism is projective if and only if it is proper and quasiprojective by [Stacks, Tag 0BCL]. □

We now review the construction of a distributive functor for the bispan triple (iv) from motivic spectra, due to Bachmann and Hoyois.

---

[13]Note that every morphism between qcqs schemes is automatically a qcqs morphism (see [Stacks, Tags 01KV and 03GI]); this means we do not need to distinguish between morphisms of finite presentation and locally of finite presentation, since the additional qcqs assumption is automatic.

Notation 3.5.4. We write $\mathrm{SH}(S)$ for the ∞-category of motivic spectra over a base scheme S and $\mathrm{H}(S)$ for that of motivic spaces over S. For any morphism of schemes $f\colon S \to S'$ we have a pullback functor $f^*\colon \mathrm{SH}(S') \to \mathrm{SH}(S)$, and similarly in the unstable case. This gives functors $\mathrm{SH}, \mathrm{H}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$.

In [BaHo21], Bachmann and Hoyois promoted the contravariant functor $X \mapsto \mathrm{SH}(X)$ to include a *multiplicative pushforward* for finite étale morphisms, encoded as a functor out of a span category

$$(38) \qquad \mathrm{SH}\colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{Sch}) \to \mathrm{Cat}_\infty, \qquad X \xleftarrow{f} Z \xrightarrow{g} Y \mapsto g_\otimes f^*.$$

Given a finite étale morphism $g\colon X \to Y$, the functor

$$g_\otimes\colon \mathrm{SH}(X) \to \mathrm{SH}(Y)$$

is first constructed unstably as a functor on the level of the pointed unstable motivic homotopy ∞-category,

$$g_\otimes\colon \mathrm{H}(X)_* \to \mathrm{H}(Y)_*.$$

This functor is in turn induced by the functor of Weil restriction [BLR90, §7.6], $R_g\colon \mathrm{SmQP}_X \to \mathrm{SmQP}_Y$, where SmQP_X denotes the full subcategory of Sch_X spanned by smooth and quasiprojective X-schemes, using the fact that the inclusion $\mathrm{SmQP}_X \subset \mathrm{Sm}_X$ into the full subcategory of smooth X-schemes induces equivalent motivic unstable categories (since every smooth X-scheme is Zariski-locally also quasiprojective). We refer to [BaHo21, §1.6] for a summary of the construction and [BaHo21, §6.1] for a detailed construction of (38).

Given a smooth morphism $f\colon X \to Y$ of schemes, the pullback functor f^* admits a left adjoint

$$f_\sharp\colon \mathrm{SH}(X) \to \mathrm{SH}(Y).$$

This left adjoint should be thought of as an *additive pushforward* along f; indeed, if I is a finite set and $\nabla_I\colon \coprod_I X \to X$ is the fold map then, under the identification $\mathrm{SH}(\coprod_I X) \simeq \mathrm{SH}(X)^{\times I}$, the functor $(\nabla_I)_\sharp$ is given by

$$(X_i)_{i\in I} \mapsto \bigoplus_{i\in I} X_i.$$

The functor $f_\sharp$ is first constructed unstably as a functor

$$f_\sharp\colon \mathrm{H}(X) \to \mathrm{H}(Y),$$

which in turn is induced by the functor $\mathrm{Sm}_X \to \mathrm{Sm}_Y$ given by composition with f (i.e. the functor that sends a smooth X-scheme T to itself regarded as a Y-scheme); see [Hoy17, Section 4.1, Lemma 6.2] for a construction in the language of this paper in the more general context of equivariant motivic homotopy theory.

The importance of this additional left adjoint functoriality in formulating smooth base change was first pointed out by Voevodsky [Voe99] and worked out by Ayoub in [Ayo07]; see [Hoy17, Section 6.1, Proposition 4.2] for an ∞-categorical formulation (in the more general equivariant context). In our language, smooth base change for motivic spectra says that the functor $\mathrm{SH}\colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ is left adjointable with respect to smooth maps. Moreover, combining this with [BaHo21, Proposition 5.10(1)] we get that the functor (38) is smqp-distributive. Applying Theorem 2.5.1 these results imply:

Theorem 3.5.5. *Motivic spectra give a* smqp-*distributive functor*

$$\mathrm{SH}\colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{Sch}) \to \mathrm{Cat}_\infty,$$

and so a functor of $(\infty, 2)$-*categories*

$$\mathrm{BISPAN}_{\mathrm{f\acute{e}t},\mathrm{smqp}}(\mathrm{Sch}) \to \mathrm{CAT}_\infty.$$

As indicated already in [BaHo21, Remark 2.14], the restriction to those smooth morphisms that are quasiprojective here is an artifact of the restriction of SH to schemes instead of algebraic spaces. We will therefore extend this result by working with algebraic spaces.

Notation 3.5.6. For a (qcqs) scheme S, we write AlgSpc_S for the category of (qcqs) algebraic spaces over S.[14] This category has pullbacks since qcqs morphisms of algebraic spaces are closed under base change by [Stacks, Tags 03KL,03HF]; note also that for an S-scheme S' we have an equivalence

$$\mathrm{AlgSpc}_{S'} \simeq (\mathrm{AlgSpc}_S)_{/S'}$$

by [Stacks, Tag 04SG].

Here we again have several bispan triples:

Proposition 3.5.7. *The following are bispan triples:*
- (i) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{flf}}, \mathrm{AlgSpc}_S)$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{flf}}$ consists of finite locally free morphisms of algebraic spaces over S,*
- (ii) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{fét}}, \mathrm{AlgSpc}_S)$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{fét}}$ consists of finite étale morphisms of algebraic spaces over S,*
- (iii) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{flf}}, \mathrm{AlgSpc}_S^{\mathrm{sm}})$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{sm}}$ consists of smooth morphisms of algebraic spaces over S,*
- (iv) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{fét}}, \mathrm{AlgSpc}_S^{\mathrm{sm}})$ *for any scheme S.*
- (v) $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{fét}}, \mathrm{AlgSpc}_S^{\mathrm{prop}})$ *for any scheme S, where $\mathrm{AlgSpc}_S^{\mathrm{prop}}$ consists of proper morphisms of algebraic spaces over S.*

Proof. Morphisms of algebraic spaces that are finite locally free, finite, étale, and smooth are closed under base change by [Stacks, Tags 03ZY, 03ZS, 0466, 03ZE], respectively. Thus the subcategories $\mathrm{AlgSpc}_S^{\mathrm{flf}}$, $\mathrm{AlgSpc}_S^{\mathrm{fét}}$, $\mathrm{AlgSpc}_S^{\mathrm{sm}}$ of AlgSpc_S all give span pairs.

Suppose $f\colon Y \to Z$ is a finite locally free morphism of algebraic spaces. Then the functor

$$f^*\colon \mathrm{AlgSpc}_Z \to \mathrm{AlgSpc}_Y$$

admits a right adjoint R_f, given by Weil restriction of algebraic spaces, by a result of Rydh [Ryd11, Theorem 3.7]; note that R_f preserves the qcqs property we require by [Ryd11, Proposition 3.8(xiii,xix)]. This gives the bispan triples (i) and (ii) (since finite étale morphisms are in particular finite locally free).

To obtain (iii) and (iv) it suffices to note that R_f converts smooth morphisms to smooth morphisms. If $f\colon X \to Y$ is a finite locally free morphism of schemes this follows from [Ryd11, Proposition 3.5(i,iv)] and [Stacks, Tag 0DP0]. The extension to the general case is easy: First note that for W over X we have that $R_f W \to Y$ is smooth if and only if its pullback along any morphism $g\colon T \to Y$ with T a scheme is smooth, by [Stacks, Tag 03ZF]. In the pullback square

$$
\begin{array}{ccc}
U & \xrightarrow{g'} & X \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
T & \xrightarrow{g} & Y
\end{array}
$$

the algebraic space U is a scheme since f is finite and so by definition representable. We also have the base change equivalence $g^* R_f W \cong R_{f'} g'^* W$, and if W is smooth over X then $R_{f'} g'^* W$ is smooth over T since the base change $g'^* W$ is smooth over

---

[14]Every morphism between qcqs algebraic spaces is a qcqs morphism by [Stacks, Tag 03KR,03KS]; thus we can still ignore the distinction between morphisms of finite presentation and locally of finite presentation.

U and f' is a finite locally free morphism of schemes. Finally, (v) holds since by the same argument starting with [Ryd11, Remark 3.9] R_f preserves proper morphisms if f is finite étale. $\qquad\square$

In order to work with motivic spectra over algebraic spaces effectively we record the following lemma which amounts to saying that any Nisnevich sheaf on algebraic spaces is right Kan extended from schemes.

Lemma 3.5.8. *Let $\mathcal{C}$ be a complete ∞-category and S a scheme, and let*

$$\iota\colon \mathrm{Sch}_S \hookrightarrow \mathrm{AlgSpc}_S$$

be the inclusion. Then the restriction functor $\iota^*\colon \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C}) \to \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C})$ induces an equivalence of ∞-categories:

$$\iota^*\colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \xrightarrow{\sim} \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}),$$

with the inverse given by right Kan extension.

Proof. Let

$$\iota_*\colon \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C}) \to \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C})$$

denote the right adjoint to ι^* which is computed by right Kan extension. We first claim that ι_* preserves Nisnevich sheaves, i.e., there exists a filler in the following diagram:

$$
\begin{array}{ccc}
\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}) & \dashrightarrow & \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C}) & \xrightarrow{\ \iota_*\ } & \mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C})
\end{array}
$$

According to [GH19, Lemma 2.23], it suffices to verify that the functor ι is topologically cocontinuous[15] for the Nisnevich topology on Sch_S and on AlgSpc_S, as this implies that ι^* preserves Nis-local equivalences and thus ι_* preserves sheaves.

Unwinding definitions, this means we need to prove the following claim:

(∗) For any $X \in \mathrm{Sch}_S$ and any Nisnevich sieve $R' \hookrightarrow \iota(X)$ of algebraic spaces, the sieve on Sch_S generated by morphisms of schemes $X' \to X$ such that $\iota(X') \to \iota(X)$ factors through R' is a Nisnevich sieve of X.

This condition is verified by [Knu71, Chapter II, Theorem 6.4]; indeed for $x \in X$ and $f\colon Y \to X$ an étale morphism such that we have a lift

$$
\begin{array}{ccc}
 & & Y \\
 & \nearrow & \downarrow f \\
\mathrm{Spec}\,\kappa(x) & \xrightarrow{\ x\ } & X,
\end{array}
$$

this result tells us that we can find a completely decomposed étale morphism $U \to Y$ with U an affine scheme such that $\mathrm{Spec}\,\kappa(x) \to Y$ factors through U. Since the composite $U \to X$ is an étale morphism, the desired claim is verified.

Now we claim that ι^* also preserves sheaves. Indeed, if $\{U_\alpha \to X\}$ is a family which generates a Nisnevich covering family of schemes, then it is still a Nisnevich covering family of algebraic spaces. Therefore ι is also a morphism of sites[16], and

---

[15]In the sense of [SGA4, Éxpose III, Définition 2.1] where this is called "continuous"; to avoid confusion with the notion of a functor that preserves limits, we borrow this terminology from [Kha19].

[16]In the sense reviewed in, say, [ES21, Appendix B.1].

thus ι^* preserves sheaves, i.e., there exists a filler in the following diagram:

$$
\begin{array}{ccc}
\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) & \dashrightarrow & \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{PShv}(\mathrm{AlgSpc}_S, \mathcal{C}) & \xrightarrow{\iota^*} & \mathrm{PShv}(\mathrm{Sch}_S, \mathcal{C}).
\end{array}
$$

Therefore we have an adjunction on the level of Nisnevich sheaves

$$
\iota^* \colon \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{AlgSpc}_S, \mathcal{C}) \rightleftarrows \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sch}_S, \mathcal{C}) \colon \iota_*,
$$

where ι_* is fully faithful since it is given by right Kan extension along the fully faithful functor ι. Equivalently, the counit transformation $\iota^* \iota_* \to \mathrm{id}$ is an equivalence.

It then suffices to prove that the unit transformation $\mathrm{id} \to \iota_* \iota^*$ is also an equivalence. According to [Lur18, Theorem 3.4.2.1], any qcqs algebraic space admits a scalloped decomposition in the sense of [Lur18, Definition 2.5.3.1]. In this context, this means that any $X \in \mathrm{AlgSpc}_S$ admits a sequence of open immersions

$$
\emptyset = U_0 \hookrightarrow U_1 \hookrightarrow U_2 \hookrightarrow \cdots U_i \hookrightarrow \cdots U_n = X,
$$

such that for any $1 \leq i \leq n$ we have a bicartesian diagram in algebraic spaces

$$
\begin{array}{ccc}
W_i & \longrightarrow & V_i \\
\downarrow & & \downarrow{\scriptstyle p_i} \\
U_{i-1} & \xrightarrow{j_i} & U_i,
\end{array}
$$

where each p_i is étale. A result of Morel-Voevodsky [MV99] (in the form [Lur18, Theorem 3.7.5.1]) states that any Nisnevich sheaf converts the above square to a cartesian square. Now although W_i is not necessarily a scheme, it is an open sub-algebraic space of an affine scheme and hence is separated. Therefore, by induction and the fact that ι^* is computed on the level of presheaves, the map $F \to \iota_* \iota^* F$ is an equivalence for any Nisnevich sheaf F as soon as we know that it is an equivalence when evaluated on a separated algebraic space.

Now, if X is a separated algebraic space then, in the scalloped decomposition of X, we see that W_i is, in fact, affine, as argued in the proof of [Kha19, Proposition 2.2.13]. Therefore, by induction again, we need only check that $F \to \iota_* \iota^* F$ is an equivalence on affine schemes which is tautologically true. $\qquad\square$

We can now prove:

Theorem 3.5.9. *The functor* (38) *extends canonically to a* sm-*distributive functor*

$$
\mathrm{SH} \colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{AlgSpc}_S) \to \mathrm{Cat}_\infty
$$

and hence to a functor of $(\infty, 2)$-*categories*

$$
\mathrm{SH} \colon \mathrm{BISPAN}_{\mathrm{f\acute{e}t,sm}}(\mathrm{AlgSpc}_S) \to \mathrm{CAT}_\infty.
$$

Proof. First, by Lemma 3.5.8, the right Kan extension of the Nisnevich sheaf

$$
\mathrm{SH} \colon \mathrm{Sch}^{\mathrm{op}} \to \mathrm{CAT}_\infty
$$

to algebraic spaces defines a Nisnevich sheaf

$$
\mathrm{SH} \colon \mathrm{AlgSpc}^{\mathrm{op}} \to \mathrm{CAT}_\infty.
$$

Therefore we can apply [BaHo21, Proposition C.18] to obtain an extension

$$
\mathrm{SH} \colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{AlgSpc}) \to \mathrm{Cat}_\infty
$$

of (38).

Now in order to use Theorem 2.5.1, we need to verify that this extension satisfies the distributivity property with respect to smooth morphisms. Since the functors

involved in the adjointability and distributivity transformations are stable under
base change, we are reduced to the case of schemes we already discussed above.   □

Remark 3.5.10. In the more general setting of spectral algebraic spaces, Khan has
constructed the unstable motivic homotopy ∞-category (defined in [Kha19, Defini-
tion 2.4.1]). By Nisnevich descent, this agrees with the right Kan extended version
appearing in the proof of Theorem 3.5.9, using the uniqueness part of Lemma 3.5.8.

3.6. **Bispans in spectral Deligne-Mumford stacks and** Perf**.** In this subsec-
tion we promote the functor Perf : SpDM $\to$ Cat$_\infty$ to a functor out of an $(\infty, 2)$-
category of bispans. This extends a result of Barwick [Bar14, Example D], which
gives a functor

$$\mathrm{Perf} \colon \mathrm{Span}_{\mathcal{FP}}(\mathrm{SpDM}) \to \mathrm{Cat}_\infty$$

encoding the usual pullback f^* and pushforward f_* for a morphism in SpDM, with
$\mathcal{FP}$ a class of morphisms for which f_* restricts to perfect objects and satisfies base
change. Our version adds a multiplicative pushforward $f_\otimes$ where f is finite étale
(at the cost of restricting the class $\mathcal{FP}$ in order to guarantee the existence of Weil
restrictions). We note that the multiplicative pushforward in this situation is right
Kan extended from the symmetric monoidal structure in Perf and is thus not as
complicated as in the motivic and equivariant cases we considered above. However,
we include this section with a view towards applications in algebraic K-theory in
the following section.

We will freely use the language of spectral Deligne–Mumford (DM) stacks in-
troduced in [Lur18]. We denote by SpDM$_S$ the ∞-category of spectral Deligne-
Mumford stacks over a base S. We also adopt the following terminology from
[Bar14, Example D]:

Definition 3.6.1. Recall that for $X \in$ SpDM an object $\mathcal{E} \in \mathrm{QCoh}(X)$ is called
perfect if for every map $x \colon \mathrm{Spec}\, A \to X$, where A is an E_∞-ring spectrum, the
A-module $x^*\mathcal{E}$ is perfect (i.e. dualizable or equivalently compact in the symmetric
monoidal ∞-category Mod_A). Now suppose that $f \colon X \to Y$ is a morphism in
SpDM. We say that f is *perfect* if the pushforward functor

$$f_* \colon \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y),$$

takes perfect objects to perfect objects.

The following theorem of Lurie furnishes a large class of perfect morphisms:

Theorem 3.6.2 ([Lur18, Theorem 6.1.3.2])**.** *Let $f \colon X \to Y$ be a morphism in*
SpDM. *If f is proper, locally almost of finite presentation, and of finite Tor-
amplitude, then f is perfect.*

Notation 3.6.3. Following [Bar14, Notation D.17], we label the class of morphisms
in Theorem 3.6.8 by $\mathcal{FP}$.

We will also make use of the existence of the spectral version of Weil restriction; see
the discussion of [Lur18, Section 19.1] and note that the definitions are completely
analogous to the classical situation. There Lurie proves the following existence
theorem:

Theorem 3.6.4 ([Lur18, Theorem 19.1.0.1])**.** *Suppose that $f \colon X \to Y$ is a mor-
phism in* SpDM *that is proper, flat, and locally almost of finite presentation. Let
$p \colon Z \to X$ be a relative spectral algebraic space that is quasi-separated and locally
almost of finite presentation. Then the Weil restriction $R_f(p) \in$ SpDM$_Y$ exists.*

Notation 3.6.5. In light of this, let us write

- $\mathcal{W}$ for the class of morphisms in SpDM that are proper, flat, and locally almost of finite presentation,
- $\mathcal{Q}$ for the class of morphisms in SpDM that are relative spectral algebraic spaces, quasi-separated, and locally almost of finite presentation,
- $\mathcal{FP}' \subset \mathcal{FP}$ for the class of morphisms in $\mathcal{FP}$ which are furthermore relative spectral algebraic spaces,
- fét for the class of finite étale morphisms in SpDM.

Then Weil restrictions of morphisms in $\mathcal{Q}$ along ones in $\mathcal{W}$ exist in SpDM. Here $\mathcal{FP}' \subseteq \mathcal{Q}$ since proper morphisms are always quasi-separated, and fét $\subseteq \mathcal{W}$.

Lemma 3.6.6. *Suppose that $f \colon X \to Y$ is a morphism in* SpDM *of class $\mathcal{W}$ and $p \colon Z \to X$ is one of class $\mathcal{Q}$. Then $R_f(p)$ is again of class $\mathcal{Q}$. Assuming f is moreover finite étale, we also have:*

(a) if p is quasi-compact, then $R_f(p) \to Y$ is quasi-compact;
(b) if p is proper, then $R_f(p) \to Y$ is proper;
(c) if p is of finite Tor-amplitude, then $R_f(p) \to Y$ is of finite Tor-amplitude.

Proof. The statement that $R_f(p)$ is of class $\mathcal{Q}$ is part of [Lur18, Theorem 19.1.0.1].

Let us now verify properties (a)–(c). To verify (a) and (c), note that quasi-compactness and having finite Tor-amplitude can be detected étale locally on the target (for the former, this is [Lur18, Remark 2.3.2.5] and the equivalences of [Lur18, Proposition 2.3.2.1] and for the latter this is [Lur18, Proposition 6.1.2.2]). Therefore, we may work étale locally on Y. Since f was assumed to be finite étale, it is étale locally a fold map $f \colon X \simeq \coprod_{i=1}^{n} Y \to Y$. In this case, we can write $p \colon Z \to X$ as a coproduct $\coprod_i f_i \colon \coprod_i Z_i \to \coprod_i Y$. Therefore the Weil restriction takes the form $R_f(p) \simeq Z_1 \times_Y Z_2 \times_Y \cdots \times_Y Z_n \to Y$. To conclude (a), we note that quasi-compactness is stable under base change [Lur18, Proposition 2.3.3.1], while for (c), we note that Tor-amplitudes add up under base change [Lur18, Lemma 6.1.1.6].

To prove (b), we use the valuative criterion for properness [Lur18, Corollary 5.3.1.2], which applies since we have already verified (a) and (1)–(3), together with the functor-of-points description of the Weil restriction. $\square$

Proposition 3.6.7. *Let S be a spectral Deligne-Mumford stack. Then*

$$(\mathrm{SpDM}_S, \mathrm{SpDM}_S^{\text{fét}}, \mathrm{SpDM}_S^{\mathcal{FP}'})$$

is a bispan triple.

Proof. After Lemma 3.6.6 it suffices to note that morphisms in fét and $\mathcal{FP}'$ are stable under base change. This follows from [Lur18, Proposition 5.1.3.1, Proposition 4.2.1.6, Proposition 6.1.2.2, Proposition 1.4.1.11(2), Proposition 3.3.1.8]. $\square$

Theorem 3.6.8. *Let S be a spectral Deligne-Mumford stack. The functor*

$$\mathrm{Perf} \colon \mathrm{SpDM}_S^{\mathrm{op}} \to \mathrm{Cat}_\infty,$$

canonically extends to a functor

$$\mathrm{Perf} \colon \mathrm{Span}_{\text{fét}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty.$$

Moreover, this is right $\mathcal{FP}'$-distributive (in the sense of Variant 2.4.12), and so canonically extends further to a functor of $(\infty, 2)$-categories

$$\mathrm{Perf} \colon \mathrm{BISPAN}_{\text{fét}, \mathcal{FP}'}(\mathrm{SpDM}_S)^{2\text{-op}} \to \mathrm{CAT}_\infty.$$

Proof. We first apply [BaHo21, Proposition C.9] to extend Perf to a functor

$$\mathrm{Span}_{\text{fold}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty,$$

where $\mathrm{SpDM}_S^{\mathrm{fold}}$ consists of the finite fold maps, i.e. the maps $\coprod_I X \to \coprod_J X$ with $I \to J$ a map of finite sets. Here the pushforward

$$\nabla_\otimes \colon \mathrm{Perf}(\coprod_I X) \cong \mathrm{Perf}(X)^{\times I} \to \mathrm{Perf}(X)$$

is just the tensor product, and the base change simply encodes the fact that the pullback functors are symmetric monoidal.

Next, we use [BaHo21, Corollary C.13] for $\mathcal{C} = \mathrm{SpDM}$, t the étale topology and m the class of finite étale map to obtain a functor

$$\mathrm{Perf} \colon \mathrm{Span}_{\mathrm{f\acute{e}t}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty.$$

The content of this result is that since Perf is an étale sheaf and finite étale morphisms are étale-locally contained in the class of fold maps, we can extend the symmetric monoidal structure to norms along finite étale morphisms.

In order to show that this functor is right $\mathcal{FP}'$-distributive, we first check that its restriction $\mathrm{Perf} \colon \mathrm{SpDM}_S \to \mathrm{Cat}_\infty$ is right $\mathcal{FP}'$-adjointable. For any morphism $f \colon X \to Y$ in SpDM_S the functor $f^* \colon \mathrm{QCoh}(Y) \to \mathrm{QCoh}(Y)$ has a right adjoint f_*. Given a pullback square

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

the commutative square

$$\begin{array}{ccc} \mathrm{QCoh}(Y) & \xrightarrow{f^*} & \mathrm{QCoh}(X) \\ \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle g'^*} \\ \mathrm{QCoh}(Y') & \xrightarrow{f'^*} & \mathrm{QCoh}(X') \end{array}$$

is right adjointable by [Lur18, Corollary 3.4.2.2] provided f is quasi-compact and quasi-separated. This is true by definition [Lur18, Definition 5.1.2.1] for any proper morphism and so for any morphism in $\mathcal{FP}'$. Moreover, if f is of finite Tor amplitude then f_* preserves perfect complexes by Theorem 3.6.8, so in this case the adjunction restricts to an adjunction

$$f^* \colon \mathrm{Perf}(Y) \rightleftarrows \mathrm{Perf}(X) \colon f_*$$

on the full subcategories of perfect objects, which still satisfies the right adjointability condition if f is also quasi-compact and quasi-separated. This holds in particular if f is in $\mathcal{FP}'$, so that Perf is indeed right $\mathcal{FP}'$-adjointable.

It remains to check the (right) distributivity condition for $p \colon X \to Y$ in $\mathcal{FP}'$ and $f \colon Y \to Z$ finite étale: given a distributivity diagram

(39)

$$\begin{array}{ccc} & f^*R_f(p) \xrightarrow{\tilde{f}} R_f(p) & \\ {\scriptstyle \epsilon}\nearrow & \downarrow{\scriptstyle \tilde{g}} \quad \downarrow{\scriptstyle g} & \\ X & & \\ \downarrow{\scriptstyle p} & & \\ & Y \xrightarrow{f} Z, & \end{array}$$

the (right) distributivity transformation

$$f_\otimes p_* \to g_* \tilde{f}_\otimes \epsilon^*$$

must be invertible. Since Perf is an étale sheaf and distributivity transformations satisfy base change by Proposition 2.5.19, we may check this étale-locally on Z.

Since finite étale morphisms are étale-locally given by finite fold maps, this means we may assume that f is a fold map

$$\nabla \colon Y \simeq \coprod_{i=1}^{n} Z \to Z.$$

Since SpDM is extensive, we get a decomposition of p as

$$\coprod_{i=1}^{n} p_i \colon \coprod_{i=1}^{n} X_i \to \coprod_{i=1}^{n} Z,$$

and an equivalence

$$R_\nabla(p) \simeq X_1 \times_Z X_2 \times_Z \cdots \times_Z X_n,$$

since the universal property of $R_\nabla(p)$ is equivalent to that of this iterated fibre product:

$$\begin{aligned}
\mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(W, R_\nabla(p)) &\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(\nabla^* W, X) \\
&\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/\coprod_i Z}}(\coprod_i W, \coprod_i X_i) \\
&\simeq \prod_i \mathrm{Map}_{\mathrm{SpDM}_{S/Z}}(W, X_i) \\
&\simeq \mathrm{Map}_{\mathrm{SpDM}_{S/Z}}(W, X_1 \times_Z \cdots \times_Z X_n).
\end{aligned}$$

If π_i denotes the projection $X_1 \times_Z \cdots \times_Z X_n \to X_i$, then $\epsilon \simeq \coprod_i \pi_i$. Now given $\mathcal{F} \in \mathrm{Perf}(X)$ corresponding to $\mathcal{F}_i \in \mathrm{Perf}(X_i)$ under the equivalence $\mathrm{Perf}(X) \simeq \prod_i \mathrm{Perf}(X_i)$, we can write

$$\nabla_\otimes p_* \mathcal{F} \simeq p_{1,*} \mathcal{F}_1 \otimes \cdots \otimes p_{n,*} \mathcal{F}_n,$$

$$g_* \tilde{\nabla}_\otimes \epsilon^* \mathcal{F} \simeq g_*(\pi_1^* \mathcal{F}_1 \otimes \cdots \otimes \pi_n^* \mathcal{F}_n),$$

with the distributivity map $\nabla_\otimes p_* \mathcal{F} \to g_* \tilde{\nabla}_\otimes \epsilon^* \mathcal{F}$ given by the composite

$$\begin{aligned}
p_{1,*} \mathcal{F}_1 \otimes \cdots \otimes p_{n,*} \mathcal{F}_n &\to g_* g^*(p_{1,*} \mathcal{F}_1 \otimes \cdots \otimes p_{n,*} \mathcal{F}_n) \\
&\simeq g_*(g^* p_{1,*} \mathcal{F}_1 \otimes \cdots \otimes g^* p_{n,*} \mathcal{F}_n) \\
&\simeq g_*(\pi_1^* p_1^* p_{1,*} \mathcal{F}_1 \otimes \cdots \otimes \pi_n^* p_n^* p_{n,*} \mathcal{F}_n) \\
&\to g_*(\pi_1^* \mathcal{F}_1 \otimes \cdots \otimes \pi_n^* \mathcal{F}_n).
\end{aligned}$$

That this is an equivalence now follows from base change and the projection formula (which applies for maps in $\mathcal{FP}'$ by [Lur18, Remark 3.4.2.6]). To keep the notation bearable we spell this out only in the case $n = 2$, where it follows from Remark 2.4.11 that the distributivity condition is equivalent to the following commutative square being right adjointable (where $V := X_1 \times_Z X_2$):

$$\begin{array}{ccc}
\mathrm{Perf}(Z) \times \mathrm{Perf}(Z) & \xrightarrow{p_1^* \times p_2^*} & \mathrm{Perf}(X_1) \times \mathrm{Perf}(X_2) \\
\Big\downarrow{\scriptstyle \otimes} & & \Big\downarrow{\scriptstyle \pi_1^* \times \pi_2^*} \\
 & & \mathrm{Perf}(V) \times \mathrm{Perf}(V) \\
 & & \Big\downarrow{\scriptstyle \otimes} \\
\mathrm{Perf}(Z) & \xrightarrow{\quad g^* \quad} & \mathrm{Perf}(V).
\end{array}$$

We can decompose this diagram as follows:

$$
\begin{array}{ccccc}
\mathrm{Perf}(Z) \times \mathrm{Perf}(Z) & \xrightarrow{p_1^* \times \mathrm{id}} & \mathrm{Perf}(X_1) \times \mathrm{Perf}(Z) & \xrightarrow{\mathrm{id} \times p_2^*} & \mathrm{Perf}(X_1) \times \mathrm{Perf}(X_2) \\
& & \downarrow{\scriptstyle \mathrm{id} \times p_1^*} & & \downarrow{\scriptstyle \mathrm{id} \times \pi_2^*} \\
& & \mathrm{Perf}(X_1) \times \mathrm{Perf}(X_1) & \xrightarrow{\mathrm{id} \times \pi_1^*} & \mathrm{Perf}(X_1) \times \mathrm{Perf}(V) \\
\downarrow{\scriptstyle \otimes} & & & & \downarrow{\scriptstyle \pi_1^* \times \mathrm{id}} \\
& & \downarrow{\scriptstyle \otimes} & & \mathrm{Perf}(V) \times \mathrm{Perf}(V) \\
& & & & \downarrow{\scriptstyle \otimes} \\
\mathrm{Perf}(Z) & \xrightarrow{\;\;p_1^*\;\;} & \mathrm{Perf}(X_1) & \xrightarrow{\;\;\;\pi_1^*\;\;\;} & \mathrm{Perf}(V).
\end{array}
$$

Since horizontal and vertical pastings of right adjointable squares are again right adjointable, it suffices to check the three smaller squares in this diagram are all right adjointable. This is true since the mate of the left square is the projection formula transformation for p_1,

$$
p_{1,*}(-) \otimes - \to p_{1,*}(- \otimes p_1^*(-)),
$$

while the mate of the bottom right square is the projection formula transformation for π_1, and finally the mate of the top right square is $\mathrm{id}_{\mathrm{Perf}(X_1)}$ times the base change transformation

$$
p_1^* p_{2,*} \to \pi_{1,*} \pi_2^*
$$

corresponding to the pullback square

$$
\begin{array}{ccc}
V & \xrightarrow{\;\pi_1\;} & X_1 \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle p_1} \\
X_2 & \xrightarrow{\;p_2\;} & Z.
\end{array}
$$

We have shown that the functor

$$
\mathrm{Perf}\colon \mathrm{Span}_{\text{fét}}(\mathrm{SpDM}_S) \to \mathrm{Cat}_\infty
$$

is right $\mathcal{FP}'$-adjointable, and it therefore extends canonically to a functor of $(\infty, 2)$-categories

$$
\mathrm{BISPAN}_{\text{fét}, \mathcal{FP}'}(\mathrm{SpDM}_S) \to \mathrm{CAT}_\infty
$$

by Theorem 2.5.1. $\qquad\square$

4. Norms in algebraic K-theory

4.1. **Algebraic K-theory and polynomial functors.** Our goal in this final section is to combine our results so far with recent work of Barwick, Glasman, Mathew, and Nikolaus [BGMN21] in order to construct additional structure on algebraic K-theory spectra. In this subsection we will review the polynomial functoriality of K-theory constructed in [BGMN21].

Assume that $\mathcal{C}, \mathcal{D}$ are small ∞-categories, then a functor

$$
f\colon \mathcal{C} \to \mathcal{D}
$$

is said to be a *polynomial functor* if it is n-excisive in the sense of Goodwillie calculus [Goo91] for some n. For our purposes it is more convenient to follow the inductive definition in [BGMN21, Definition 2.4, Definition 2.11], which is based on work of Eilenberg and Maclane [EML54]:

Definition 4.1.1. Let $\mathcal{A}$ and $\mathcal{B}$ be additive ∞-categories and assume that $\mathcal{B}$ is idempotent-complete. Then we inductively define what it means for a functor $F\colon \mathcal{A} \to \mathcal{B}$ to be *polynomial of degree $\leq n$*:

- if $n = -1$ then f must be the zero functor;
- if $n = 0$ then f must be constant;
- if $n > 0$ then for each fixed $x \in \mathcal{A}$, the functor

$$D_x(f) : \mathcal{A} \to \mathcal{B}, \qquad y \mapsto \mathrm{Fib}(F(x \oplus y) \to F(y)),$$

must be polynomial of degree $\leq n - 1$. (This fibre exists since $\mathcal{B}$ is assumed to be idempotent-complete: it is the complementary summand to $F(y)$ in $F(x \oplus y)$.)

Remark 4.1.2. Via the comparison result of [BGMN21, Proposition 2.15], this definition of polynomial functors agrees with the one via Goodwillie calculus for all idempotent-complete stable ∞-categories.

Warning 4.1.3. The notion of polynomial functor from Definition 4.1.1, which makes sense in *additive* contexts, is completely unrelated to the concept with the same name that we considered in Definition 3.3.3, which only exists for slices of the ∞-category $\mathcal{S}$.

Remark 4.1.4. Any exact functor between stable ∞-categories is polynomial of degree ≤ 1. Note that this applies in particular to any functor that has a left or right adjoint.

Proposition 4.1.5. *Suppose* $F, G \colon \mathcal{C} \to \mathcal{D}$ *are polynomial of degree* $\leq n$. *Then* $F \oplus G \colon \mathcal{C} \to \mathcal{D}$ *is also polynomial of degree* $\leq n$.

Proof. This is a special case of [BaHo21, Lemma 5.24(3)]. $\qquad\square$

Definition 4.1.6. We have the (non-full) subcategory

$$\mathrm{Cat}_\infty^{\mathrm{poly}} \subset \mathrm{Cat}_\infty$$

whose objects are the small, idempotent-complete stable ∞-categories and whose morphisms are the polynomial functors between them.

Recall that (connective) algebraic K-theory can be defined as a functor from stable ∞-categories to spectra. Passing to the underlying infinite loop spaces, we get a functor

$$\Omega^\infty K \colon \mathrm{Cat}_\infty^{\mathrm{stab}} \to \mathcal{S},$$

where $\mathrm{Cat}_\infty^{\mathrm{stab}}$ denotes the ∞-category of stable ∞-categories and exact functors. This is equipped with a transformation $(-)^\simeq \to \Omega^\infty K$ that exhibits $\Omega^\infty K$ as the universal additivization of $(-)^\simeq$ in the sense of [Bar16, BGT13]. The main result of [BGMN21] shows that if we restrict to the full subcategory $\mathrm{Cat}_\infty^{\mathrm{stab,idem}}$ of idempotent-complete stable ∞-categories, then we can extend this to be functorial in all polynomial functors (rather than only the exact ones):

Theorem 4.1.7 (Barwick, Glasman, Mathew, Nikolaus)**.** *The space-valued K-theory functor extends to a functor* $\Omega^\infty K^{\mathrm{poly}} \colon \mathrm{Cat}_\infty^{\mathrm{poly}} \to \mathcal{S}$ *rendering the following diagram commutative*

$$
\begin{array}{ccc}
\mathrm{Cat}_\infty^{\mathrm{stab,idem}} & \xrightarrow{\ \Omega^\infty K\ } & \mathcal{S} \\
\big\downarrow & \nearrow{\scriptstyle \Omega^\infty K^{\mathrm{poly}}} & \\
\mathrm{Cat}_\infty^{\mathrm{poly}}. & &
\end{array}
$$

(We will usually just refer to this extension as $\Omega^\infty K$.*)*

We want to apply this to construct additional norms in K-theory, in the following way:

Corollary 4.1.8. *Suppose* $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ *is a bispan triple, and that* $\Phi\colon \mathrm{Span}_F(\mathcal{C}) \to$ Cat_∞ *is a functor such that*

(1) $\Phi(X)$ *is an idempotent-complete stable* ∞*-category for every* $X \in \mathcal{C}$,
(2) $f^\circledast\colon \Phi(Y) \to \Phi(X)$ *is a polynomial functor for every* $f\colon X \to Y$ *in* $\mathcal{C}$,
(3) $f_\otimes\colon \Phi(X) \to \Phi(Y)$ *is a polynomial functor for every* $f\colon X \to Y$ *in* $\mathcal{C}_F$,
(4) Φ *is* L*-distributive.*

Then Φ *induces a functor*

$$\mathrm{Bispan}_{F,L}(\mathcal{C}) \to \mathcal{S},$$

which takes $X \in \mathcal{C}$ *to* $\Omega^\infty K(\Phi(X))$.

Proof. Since Φ is L-distributive, it extends to a functor $\mathrm{BISPAN}_{F,L}(\mathcal{C}) \to \mathrm{CAT}_\infty$ by Theorem 2.5.1. The induced functor $\mathrm{Bispan}_{F,L}(\mathcal{C}) \to \mathrm{Cat}_\infty$ on underlying ∞-categories then factors through the subcategory $\mathrm{Cat}_\infty^{\mathrm{poly}}$: on objects and for the maps of the form $f^\circledast$ and $f_\otimes$ this is true by assumption, and for $f_\oplus$ because this is a left adjoint and so polynomial by Remark 4.1.4. We can then combine the resulting functor with that of Theorem 4.1.7 to complete the proof. $\square$

Remark 4.1.9. If $\mathcal{C}_L = \mathcal{C}$, then condition (2) is automatic, since $f^\circledast$ is a right adjoint for all f.

4.2. **Polynomial functors from distributivity.** In order to apply Corollary 4.1.8 in practice, we need to know that the functors $p_\otimes$ are polynomial for p in $\mathcal{C}_F$. Our goal in this section is to derive a convenient criterion for this using distributivity, by generalizing an argument due to Bachmann–Hoyois [BaHo21, Section 5.5]. We start by introducing some conditions and extra structure on bispan triples:

Definition 4.2.1. A span pair $(\mathcal{C}, \mathcal{C}_F)$ is said to be an *extensive span pair* if

(1) the ∞-category $\mathcal{C}$ is extensive,
(2) $\mathcal{C}_F$ is closed under coproducts,
(3) for any $x \in \mathcal{C}$ the unique morphism $\emptyset \to x$ is in $\mathcal{C}_F$,
(4) for any $x \in \mathcal{C}$ the fold map $\nabla\colon x \coprod x \to x$ is in $\mathcal{C}_F$.

Remark 4.2.2. If $(\mathcal{C}, \mathcal{C}_F)$ is an extensive span pair, then the symmetric monoidal structure on $\mathrm{Span}_F(\mathcal{C})$ induced by the coproduct in $\mathcal{C}$ via Example 2.6.9 is both cartesian and cocartesian. In other words, in this case the coproduct in $\mathcal{C}$ gives both the product and coproduct in $\mathrm{Span}_F(\mathcal{C})$.

Definition 4.2.3. Suppose $(\mathcal{C}, \mathcal{C}_F)$ is an extensive span pair. Then a *degree structure* on $(\mathcal{C}, \mathcal{C}_F)$ consists of a collection of morphisms $F_n \subseteq \mathrm{Map}([1], \mathcal{C}_F)$ for each $n \in \mathbb{N}$, called *morphisms of degree* n, such that:

(1) for each morphism $f\colon x \to y$ in $\mathcal{C}_F$, there exists a natural number $N < \infty$ and an essentially unique (finite) coproduct decomposition (called the *degree decomposition*)

$$y \simeq \coprod_{i=0}^{N} y_n^{(f)},$$

such that for each $0 \le n \le N$, the morphism

$$f_n := f \times_y y_n^{(f)}\colon x_n^{(f)} = x \times_y y_n^{(f)} \to y_n^{(f)}$$

is of degree n.
(2) The collection F_0 consists of the morphisms $\emptyset \to X$.
(3) Morphisms in F_n are stable under base change: if $f\colon x \to y$ is degree n, then for any $w \to y$, the morphism $w \times_y x \to w$ is also of degree n.

(4) Given morphisms $f\colon x \to y, g\colon z \to y$ in $\mathcal{C}_F$ which are of degrees n and m, respectively, then the morphism

$$x \coprod z \xrightarrow{f \coprod g} y \coprod y \xrightarrow{\nabla} y$$

in $\mathcal{C}_F$ is of degree $m + n$.

Furthermore, given a degree structure, we say a morphism $f\colon x \to y$ is of *degree* $\leq n$ if in the degree decomposition we have $y_i^{(f)} \simeq \emptyset$ for $i > n$. We also say that a morphism is of *degree* -1 if it is the essentially unique morphism $\emptyset \to \emptyset$.

Remark 4.2.4. Suppose that $(\mathcal{C}, \mathcal{C}_F)$ is equipped with a degree structure. We note that not all morphisms have a well-defined degree although each morphism does have a degree decomposition. Additionally, the reader is encouraged to think of $x_0^{(f)}$ as the "locus" in x where f has empty fibres, i.e. fails to be surjective.

Example 4.2.5. Here are the main examples of degree structures on the span pairs that have appeared throughout this paper:

(1) Consider the extensive span pair $(\mathbb{F}, \mathbb{F})$. There is a degree structure where a morphism of finite sets $f\colon x \to y$ is of degree n when the fibres all have cardinality exactly n.
(2) Consider the extensive span pair $(\mathcal{S}, \mathcal{S}_{\mathrm{fin}})$. There is a degree structure where a morphism $f\colon x \to y$ in $\mathcal{S}_{\mathrm{fin}}$ is of degree n if all its fibres have cardinality exactly n.
(3) Consider the extensive span pair $(\mathbb{F}_G, \mathbb{F}_G)$. Then we say that morphism of finite G-sets $f\colon x \to y$ is of degree n when (the underlying sets of) the fibres all have cardinality exactly n.
(4) Consider the extensive algebro-geometric span pairs $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{flf}})$, $(\mathrm{Sch}_S, \mathrm{Sch}_S^{\mathrm{f\acute{e}t}})$, $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{flf}})$, $(\mathrm{AlgSpc}_S, \mathrm{AlgSpc}_S^{\mathrm{f\acute{e}t}})$ and $(\mathrm{SpDM}_S, \mathrm{SpDM}_S^{\mathrm{f\acute{e}t}})$. In each of these, the degree of a morphism $f\colon X \to Y$ can be defined to be n if the sheaf of finite, locally free $\mathcal{O}_Y$-modules given by $f_*(\mathcal{O}_X)$ is of constant rank n.

Remark 4.2.6. In all the examples from Example 4.2.5, the degree 1 morphisms are precisely the equivalences. This is not assumed for a general degree structure since we do not need it for our main results.

Construction 4.2.7. We will need to compute how the degree decomposition interacts with coproducts of morphisms. So let $f\colon x \to y, g\colon z \to y$ be two morphisms. Each morphism then induces a coproduct decomposition (where the index runs through a finite set)

$$y \simeq \coprod_n y_n^{(f)} \qquad y \simeq \coprod_m y_m^{(g)}.$$

We set

$$y_{mn} := y_n^{(f)} \times_y y_m^{(g)}.$$

From this, we can form the following diagram where each square is cartesian

$$
\begin{array}{ccc}
z_{mn} & \longrightarrow & z_m \\
\downarrow & & \downarrow{\scriptstyle g_m} \\
x_{mn} \longrightarrow y_{mn} & \longrightarrow & y_m^{(g)} \\
\downarrow & \downarrow & \downarrow \\
x_n \xrightarrow{\ f_n\ } y_n^{(f)} & \lhook\joinrel\longrightarrow & y.
\end{array}
$$

Lemma 4.2.8. *Let $(\mathcal{C}, \mathcal{C}_F)$ be an extensive pair with a degree structure. Let $f \colon x \to y$, $g \colon z \to y$ be two morphisms in $\mathcal{C}_F$. Then in the degree decomposition of the map $\nabla_y \circ (f \amalg g) \colon x \amalg z \to y$, the degree-$k$ component is*

$$\coprod_{m+n=k} (x_{mn} \amalg z_{mn}) \to \coprod_{m+n=k} y_{mn},$$

where $z_{mn} \to y_{mn}$ is of degree m and $x_{mn} \to y_{mn}$ is of degree n.

Proof. Since coproduct decompositions are stable under pullbacks we have that for each $n \in \mathbb{N}$,

$$y_n^{(f)} \simeq y \times_y y_n^{(f)} \simeq (\coprod_m y_m^{(g)}) \times_y y_n^{(f)} \simeq \coprod_m y_{mn}.$$

By (3) in Definition 4.2.3 the pullback $x_{mn} \to y_{mn}$ of f is therefore also of degree n. Similarly, the pullback of g to $z_{mn} \to y_{mn}$ is of degree m, and so by (4) the composite

$$x_{mn} \amalg z_{mn} \to y_{mn} \amalg y_{mn} \to y_{mn}$$

is of degree $m + n$. Now note that we have

$$y \simeq \coprod_n y_n^{(f)} \simeq \coprod_n \coprod_m y_{mn} \simeq \coprod_k \coprod_{m+n=k} y_{mn}.$$

Here $\nabla_y \circ (f \amalg g)$ restricts over y_{mn} to the map $x_{mn} \amalg z_{mn} \to y_{mn}$ of degree $m+n$, so this is a decomposition into components of fixed degree; by uniqueness this is therefore the degree composition. $\square$

The purpose of a degree structure as in Definition 4.2.3 is to allow us to prove that certain functors are polynomial by induction on degrees using distributivity, by abstracting the arguments in [BaHo21, Section 5.5].

Definition 4.2.9. We say a bispan triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is an *extensive* bispan triple if both $(\mathcal{C}, \mathcal{C}_F)$ and $(\mathcal{C}, \mathcal{C}_L)$ are extensive span pairs.

Remark 4.2.10. If $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ is an extensive bispan triple, then the symmetric monoidal structure on $\mathrm{Bispan}_{F,L}(\mathcal{C})$ induced by the coproduct in $\mathcal{C}$ as in Example 2.6.12 is cartesian, i.e. the coproduct in $\mathcal{C}$ gives a product in $\mathrm{Bispan}_{F,L}(\mathcal{C})$.

Construction 4.2.11. Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be an extensive bispan triple, where furthermore the ∞-category $\mathcal{C}$ is idempotent-complete. Fix $p \colon x \to y$ in $\mathcal{C}_F$ and consider the distributivity diagram for $x \amalg x \xrightarrow{\nabla_x} x \xrightarrow{p} y$ in the sense of Definition 2.4.1:

(40)

$$
\begin{array}{ccc}
p^*w & \xrightarrow{\tilde{p}} & w \\
 & & \\
\end{array}
$$

Note that, according to Remark 2.4.5, $w \to y$ is equivalent to $p_*(x \amalg x \to x)$. Since $\mathcal{C}$ is idempotent-complete, we have a decomposition, over y:

$$w \simeq y \amalg c \amalg y.$$

Indeed, we can produce two sections $s_0, s_1 \colon y \to w \simeq p_*(x \amalg x \to x)$ adjoint to the two coproduct inclusions $x \to x \amalg x$. Since coproduct decompositions are preserved under pullbacks, we get $p^*w \simeq x \amalg (c \times_y x) \amalg x$. Since coproducts are disjoint, we can further decompose the restriction of ϵ to $c \times_y x \to x \amalg x$ as a coproduct of two morphisms:

$$\epsilon_L \amalg \epsilon_R : c \times_y x \simeq c_L \amalg c_R \to x \amalg x.$$
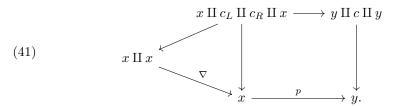
Restricting $\widetilde{p}$ to c_L and c_R gives us two maps

$$\widetilde{p}_L : c_L \to c \qquad \widetilde{p}_R : c_R \to c.$$

We also have the restriction of g to c:

$$k : c \to y.$$

All in all (40) is equivalent to:

(41)
$$
\begin{array}{ccc}
x \amalg c_L \amalg c_R \amalg x & \longrightarrow & y \amalg c \amalg y \\
& & \\
x \amalg x & & \\
& \nabla & \\
& x \xrightarrow{\ p\ } y.
\end{array}
$$

We are particularly concerned with the following diagram which we can extract from (41), where the middle square is cartesian:

(42)
$$
\begin{array}{ccc}
c_L \amalg c_R & \xrightarrow{\widetilde{p}|_{c \times_y x}} & c \\
\epsilon_L \amalg \epsilon_R & & \\
x \amalg x & & k \\
& \nabla & \\
& x \xrightarrow{\ p\ } y.
\end{array}
$$

The next lemma roughly states that if p is "surjective" then each of $\widetilde{p}_L, \widetilde{p}_R$ must be as well. This will be used to prove polynomiality of $p_\otimes$ via an inductive argument.

Lemma 4.2.12. *Keeping the notation of Construction 4.2.11, assume furthermore that $(\mathcal{C}, \mathcal{C}_F)$ is equipped with a degree structure, and $y_0^{(p)} \simeq \emptyset$. For $i = L, R$, we have $c_0^{(\widetilde{p}_i)} \simeq \emptyset$.*

Proof. To show that $\widetilde{p}_L$ has no component of degree 0, let us decompose c as

$$c \simeq c_0^{(\widetilde{p}_L)} \amalg c_{>0}^{(\widetilde{p}_L)}.$$

Then $p^* c_0^{(\widetilde{p}_L)} \to p^* c \simeq c_L \amalg c_R$ factors through c_R (since by definition its component over c_L is $\emptyset$), and hence the composite $p^* c_0^{(\widetilde{p}_L)} \to x \amalg x$ factors through the right copy of x. But then the adjoint map $c_0^{(\widetilde{p}_L)} \to p_*(x \amalg x) \simeq y \amalg c \amalg y$ factors through the right copy of y, which means $c_0^{(\widetilde{p}_L)} \simeq \emptyset$ since it also factors through c. $\qquad\square$

For the remainder of this section, we fix an extensive bispan triple $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ such that

- $\mathcal{C}$ is idempotent-complete,
- $(\mathcal{C}, \mathcal{C}_F)$ has a degree structure,

and an L-distributive functor $\Phi \colon \mathrm{Span}_F(\mathcal{C}) \to \mathrm{Cat}_\infty$ such that

(1) Φ preserves finite products,
(2) for each $x \in \mathcal{C}$, the ∞-category $\Phi(x)$ is additive,

The following computation follows [BaHo21, Corollary 5.15] closely:

Lemma 4.2.13. *For any $p \colon x \to y$ in $\mathcal{C}_F$, we have for any $E, F \in \Phi(x)$ an equivalence*

$$p_\otimes(E \oplus F) \simeq p_\otimes(E) \oplus k_\oplus \nabla_{c,\otimes}\left(\widetilde{p}_{L,\otimes}\epsilon_L^\circledast(E), \widetilde{p}_{R,\otimes}\epsilon_R^\circledast(F)\right) \oplus p_\otimes(F)$$
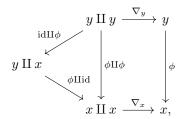
in terms of (41).

Proof. The distributivity transformation (Definition 2.4.9) for (40) gives an equivalence:

$$p_\otimes \nabla_\oplus \simeq g_\oplus \tilde{p}_\otimes \epsilon^{\circledast}.$$

The claim then follows from this equivalence, the transitivity of all the functors involved, and the identification of $\nabla_\oplus$ with the direct sum functor, which follows from the assumption that F is product-preserving. $\square$

Remark 4.2.14. For any morphism $\phi \colon y \to x$ in $\mathcal{C}_L$, it is easy to see that the following is a distributivity diagram:

$$
\begin{array}{ccc}
y \amalg y & \xrightarrow{\nabla_y} & y \\
\end{array}
$$

since for $\alpha \colon z \to x$ we have an equivalence

$$
\left\{
\begin{array}{c}
z \amalg z \longrightarrow y \amalg x \\
\alpha \amalg \alpha \searrow \quad \swarrow \phi \amalg \mathrm{id} \\
x \amalg x
\end{array}
\right\}
\simeq
\left\{
\begin{array}{c}
z \longrightarrow y \\
\alpha \searrow \quad \swarrow \phi \\
x
\end{array}
\right\}.
$$

It follows that for our L-distributive functor Φ we have an equivalence

$$\nabla_{x,\otimes}(\phi_\oplus, \mathrm{id}) \simeq \phi_\oplus \nabla_{y,\otimes}(\mathrm{id}, \phi^{\circledast})$$

of functors $\Phi(y) \times \Phi(x) \to \Phi(x)$. If we take $\phi = \nabla_x$ then $\nabla_x^{\circledast}$ is the diagonal $\Phi(x) \to \Phi(x) \times \Phi(x)$ (since Φ by assumption preserves products) and hence its left adjoint $\nabla_{x,\oplus}$ is the coproduct on $\Phi(x)$. From this the previous equivalence specializes for $E, E', F \in \Phi(x)$ to a natural equivalence

$$\nabla_{x,\otimes}(E \oplus E', F) \simeq \nabla_{x,\otimes}(E, F) \oplus \nabla_{x,\otimes}(E', F),$$

so that the functor $\nabla_{x,\otimes}$ preserves coproducts in each variable.

Proposition 4.2.15. *Suppose $p \colon x \to y$ is a morphism of degree $\leq n$ in $\mathcal{C}_F$ (for $n \geq -1$). Then the functor $p_\otimes \colon \Phi(x) \to \Phi(y)$ is polynomial of degree $\leq n$.*

Proof. By convention, a degree-(-1) morphism is given by $\emptyset \to \emptyset$ and thus defines the zero functor, a polynomial functor of degree -1. We now proceed by induction. Since any morphism $\emptyset \to y$ of degree zero induces a constant functor $\Phi(\emptyset) \simeq * \to \Phi(y)$, a degree-0 morphism gives a polynomial functor of degree ≤ 0. Now let us assume that the result has been proved for any morphism of degree $\leq n-1$ for $n \geq 1$. Consider the diagram (41). Since degree-n morphisms are stable under pullback, the morphism $\tilde{p}_L \amalg \tilde{p}_R \colon c_L \amalg c_R \to c$ is also of degree n. Since Φ preserves coproduct decompositions in $\mathcal{C}$ (which are the products in bispans) by assumption, coproducts of polynomials of degree $\leq n$ are likewise polynomial of degree $\leq n$ by Proposition 4.1.5, and we already proved the case of degree 0, we may assume that $y_0^{(p)} \simeq \emptyset$.

Now, $c_L \amalg c_R \to c$ is the composite

$$(43) \qquad\qquad c_L \amalg c_R \xrightarrow{\tilde{p}_L \amalg \tilde{p}_R} c \amalg c \xrightarrow{\nabla_c} c.$$

We claim that here $\tilde{p}_L$ and $\tilde{p}_R$ are both of degree $\leq n-1$ — indeed, since both maps have no component of degree 0 it follows from the description of the degree decomposition of (43) in Construction 4.2.7 and the additivity condition (4) in

Definition 4.2.3 that a component of degree $\geq n$ in either map would produce a component of degree $> n$ in (43), which is impossible.

Hence by the inductive hypothesis $(\widetilde{p}_L)_\otimes$ and $(\widetilde{p}_R)_\otimes$ are polynomial functors of degree $\leq n - 1$. To conclude, we fix E and note that Lemma 4.2.13 yields an equivalence:

$$D_E(p_\otimes)(-) \simeq p_\otimes(E) \oplus k_\oplus \nabla_{c,\otimes}\left(\widetilde{p}_{L,\otimes}\epsilon_L^\circledast(E), \widetilde{p}_{R,\otimes}\epsilon_R^\circledast(-)\right)$$

Here $\nabla_{c,\otimes}(X,-)$ preserves finite coproducts by Remark 4.2.14, as does $k_\oplus$ since it is a left adjoint, so the composite $k_\oplus\nabla_{c,\otimes}\left(\widetilde{p}_{L,\otimes}\epsilon_L^\circledast(E), \widetilde{p}_{R,\otimes}\epsilon_R^\circledast(-)\right)$ is polynomial of degree $\leq n - 1$ by [BaHo21, Lemma 5.24(4)]. Using Proposition 4.1.5 again we get that $D_E(p_\otimes)(-)$ is polynomial of degree $\leq n - 1$ and hence $p_\otimes$ is polynomial of degree $\leq n$. $\qquad\square$

Since any morphism in $\mathcal{C}_F$ has a degree decomposition consisting of finitely many terms and so is of degree $\leq n$ for some n, we have shown:

Corollary 4.2.16. *Let $p\colon x \to y$ be a morphism in $\mathcal{C}_F$, then $p_\otimes$ is a polynomial functor.*

Example 4.2.17. Corollary 4.2.16 applies for instance for any finite group G to the bispan triple $(\mathbb{F}_G, \mathbb{F}_G, \mathbb{F}_G)$ of finite G-sets and the G-symmetric monoidal ∞-category of G-spectra as discussed in Proposition 3.4.19. In this case we can in particular conclude that for any subgroup inclusion $H \subseteq G$, the Hill–Hopkins–Ravenel norm $\mathrm{Sp}^H \to \mathrm{Sp}^G$ is a polynomial functor. This example has already been studied in more detail by Konovalov [Kon20], though the proof of polynomiality is similar to ours.

4.3. **From bispans to Tambara functors.** Summarizing the results of the previous two subsections, we have shown

Corollary 4.3.1. *Let $(\mathcal{C}, \mathcal{C}_F, \mathcal{C}_L)$ be an extensive bispan triple such that $\mathcal{C}$ is idempotent-complete and $(\mathcal{C}, \mathcal{C}_F)$ has a degree structure. Suppose that $\Phi\colon \mathrm{Span}_F \to \mathrm{Cat}_\infty$ is an L-distributive functor such that*

- *Φ preserves finite products,*
- *for each $x \in \mathcal{C}$, the ∞-category $\Phi(x)$ is stable and idempotent-complete,*
- *for each morphism $f\colon x \to y$ in $\mathcal{C}$, the functor $f^\circledast\colon \Phi(x) \to \Phi(y)$ is polynomial.[17]*

Then Φ extends to a functor $\widetilde{\Phi}\colon \mathrm{Bispan}_{F,L}(\mathcal{C}) \to \mathrm{Cat}_\infty^{\mathrm{poly}}$, and hence induces a product-preserving functor $\mathrm{Bispan}_{F,L}(\mathcal{C}) \to \mathcal{S}$ given on an object $x \in \mathcal{C}$ by $\Omega^\infty K(\Phi(x))$.

Our goal in this final subsection is to give some applications of this result towards obtaining new structure on algebraic K-theory.

We first consider the equivariant situation, as previously discussed in §3.4. For a finite group G the category $\mathbb{F}_G$ of finite G-sets is extensive and idempotent-complete, and by Example 4.2.5(3) it also has a degree structure. We therefore get:

Corollary 4.3.2. *Let $\mathcal{C}\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty$ be a G-symmetric monoidal ∞-category that is compatible with additive transfers. If the ∞-category $\mathcal{C}^H$ is stable and idempotent-complete for every $H \subseteq G$, then $\mathcal{C}$ extends to a functor*

$$\widetilde{\mathcal{C}}\colon \mathrm{Bispan}(\mathbb{F}_G) \to \mathrm{Cat}_\infty^{\mathrm{poly}},$$

and hence induces a product-preserving functor

$$\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}, \qquad G/H \mapsto \Omega^\infty K(\mathcal{C}^H).$$

---

[17]This is automatic if $\mathcal{C}_L = \mathcal{C}$.

Remark 4.3.3. A *Tambara functor* [Tam93] is a functor $T\colon \mathrm{Bispan}(\mathbb{F}_G) \to \mathrm{Ab}$, or equivalently a functor $T\colon \mathrm{Bispan}(\mathbb{F}_G) \to \mathrm{Set}$ such that the induced commutative monoid structure on $T(G/H)$ is a group for every $H \subseteq G$. The output of Corollary 4.3.2 is an ∞-categorical analogue of this: it is a functor $T\colon \mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$ such that the commutative monoid structure on $T(G/H)$ is grouplike for every $H \subseteq G$. We will refer to this as a *homotopical Tambara functor*. Note that it is expected that connective genuine G-E_∞-ring spectra are equivalent to these homotopical Tambara functors.

Example 4.3.4. By Proposition 3.4.19 the G-symmetric monoidal ∞-category of G-spectra (given by $G/H \mapsto \mathrm{Sp}^H$) satisfies the hypotheses of Corollary 4.3.2. Hence the G-equivariant K-theory of the sphere spectrum is a homotopical Tambara functor: the functor

$$\Omega^\infty K(\mathbb{S}_G)\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{S} \qquad G/H \mapsto \Omega^\infty K(\mathrm{Sp}^H),$$

extends canonically to a product-preserving functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$.

To generalize this, we introduce some terminology:

Definition 4.3.5. Let G be a finite group. For us a *genuine G-$\mathbb{E}_\infty$-ring spectrum E* will be a section of the cocartesian fibration $\int \mathrm{Sp}_G \to \mathrm{Span}(\mathbb{F}_G)$ sending a backward arrow to a cocartesian edge.

Remark 4.3.6. This definition mimics the definition of a normed motivic spectrum in the context of motivic homotopy theory [BaHo21]; see especially [BaHo21, Definition 9.14]. As explained there, it is also equivalent to the classical definition of a genuine G-$\mathbb{E}_\infty$-ring spectrum by comparison of associated monads.

By a similar argument as in [BaHo21, Proposition 7.6(4)], the formation of modules assemble into a functor

$$\mathrm{Mod}_E\colon \mathrm{Span}(\mathbb{F}_G) \to \mathrm{Cat}_\infty, \qquad G/H \mapsto \mathrm{Mod}_{E^H}(\mathrm{Sp}^H),$$

which satisfies the hypotheses of Corollary 4.3.2. We then have:

Theorem 4.3.7. *The algebraic K-theory of a genuine G-$\mathbb{E}_\infty$-ring spectrum E is a homotopical Tambara functor: the functor*

$$\Omega^\infty K(E)\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{S} \qquad G/H \mapsto \Omega^\infty K(E^H) := \Omega^\infty K(\mathrm{Mod}_{E^H}(\mathrm{Sp}^H)),$$

extends canonically to a product-preserving functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$.

Remark 4.3.8. As already mentioned, a homotopical Tambara functor is expected to be the same thing as a connective G-$\mathbb{E}_\infty$-ring spectrum. Assuming this, Theorem 4.3.7 says that algebraic K-theory preserves G-$\mathbb{E}_\infty$-ring structures. As far as we are aware, this is a completely new structure on algebraic K-theory — indeed the recent paper [GMMO21] seems to be the first to construct even an associative ring structure valued in G-spectra, though the results of [BGS20] should also suffice to construct $K(E)$ as an ordinary E_∞-algebra in G-spectra.

Another interesting class of examples arises from the G-symmetric monoidal ∞-categories obtained from group actions as in Example 3.4.17:

Corollary 4.3.9. *Let $\mathcal{C}$ be an idempotent-complete small stable ∞-category, equipped with a symmetric monoidal structure that is compatible with finite coproducts. Then the functor*

$$\Omega^\infty K_G(\mathcal{C})\colon \mathcal{O}_G^{\mathrm{op}} \to \mathcal{S} \qquad G/H \mapsto \Omega^\infty K(\mathrm{Fun}(BG, \mathcal{C})),$$

extends canonically to a homotopical Tambara functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$. $\qquad\square$

Remark 4.3.10. In [BGS20], the authors proved that the K-theory of a "naïve" G-symmetric monoidal ∞-category admits the structure of a Green functor. The corollary above treats the case of a G-symmetric monoidal ∞-category where the action is trivial and produces a Tambara functor. We expect that a more general statement holds: the K-theory of certain "naïve" G-symmetric monoidal ∞-categories should also form Tambara functors. We leave this to the interested reader.

Example 4.3.11. As in Example 3.4.18, let R be an $\mathbb{E}_\infty$-ring spectrum and consider the G-symmetric monoidal ∞-category from that example, given by $G/H \mapsto \mathrm{Fun}(BG, \mathrm{Perf}(R))$. We obtain a spectral Tambara functor $\mathrm{Bispan}(\mathbb{F}_G) \to \mathcal{S}$ given by

$$G/H \mapsto K(\mathrm{Fun}(BG, \mathrm{Perf}(R))).$$

Lastly, we note that plugging in our algebro-geometric examples of bispans also gives additional structure on K-theory. Applying Corollary 4.3.1 to the extension of Perf to bispans as in Theorem 3.6.8 gives us the following result about the K-theory of spectral Deligne–Mumford stacks:

Theorem 4.3.12. *The K-theory presheaf*

$$\Omega^\infty K \colon \mathrm{SpDM}_S^{\mathrm{op}} \to \mathcal{S}, \qquad X \mapsto \Omega^\infty K(\mathrm{Perf}(X)),$$

canonically extends to a product-preserving functor $\mathrm{Bispan}_{\mathrm{f\acute{e}t}, \mathcal{FP}'}(\mathrm{SpDM}_S) \to \mathcal{S}$.

In other words, the K-theory of spectral Deligne–Mumford stacks has multiplicative norms along finite étale maps. On the other hand, applying Theorem 3.5.9 gives us a result about the K-theory of SH, which is in some sense a stable analogue of the secondary K-theory explored in the thesis of Röndigs [Rön16]:

Theorem 4.3.13. *The K-theory presheaf*

$$\Omega^\infty K \colon \mathrm{AlgSpc}^{\mathrm{op}} \to \mathcal{S}, \qquad X \mapsto K(\mathrm{SH}(X)),$$

canonically extends to a product-preserving functor $\mathrm{Bispan}_{\mathrm{f\acute{e}t}, \mathrm{sm}}(\mathrm{AlgSpc}_S) \to \mathcal{S}$.

Remark 4.3.14. The multiplicative pushforwards along finite étale morphisms on SH induce a kind of "Adams operations" on $K(\mathrm{SH})$. It would be interesting to explore some computational consequences of this structure.

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